Koszulity of type D arc algebras and type D Kazhdan–Lusztig polynomials

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Introduction

In this thesis we will study the type D arc algebras $\mathbb{D}_\Lambda$. These algebras describe in a quite explicit and elementary way important categories arising in classical and geometric representation theory, as well as in 2-representation theory.
The algebras $\mathbb{D}_\Lambda$ are a type D version of the type A arc algebras K_Λ, which ultimately go back to a construction by Khovanov [Kho]. The original algebras H_Λ constructed by Khovanov are of a topological nature; they were used to categorify the Jones polynomial.
The type A arc algebras K_Λ are quasi-hereditary covers of the algebras H_Λ. They were first introduced in [CK], and in [S], it was shown that the category of finite-dimensional K_Λ-modules (for some datum Λ) is equivalent to the category of perverse sheaves (constructible with respect to the Schubert stratification) on the Grassmannian $Gr(m, m+n)$ of m-dimensional subspaces of an $(m+n)$-dimensional complex vector space. This category is known to be equivalent to the principal block of the parabolic category $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}(m+n))$.
In a series of four articles, Brundan and Stroppel studied the algebras K_Λ and their representation theory in detail. They directly proved some properties of these algebras that before were only known via the connection to geometry: most notably the fact that the algebras K_Λ are Koszul [BS2, Theorem 5.6] in the sense of [BGS]. They also connect these algebras to the representation theory of the general linear supergroups $GL(m|n)$, see [BS4].
The type D arc algebras $\mathbb{D}_\Lambda$ were introduced by Ehrig and Stroppel in [ES1]. The construction is quite similar to the construction of the type A arc algebras as described in [BS1]. In [ES1, Section 9], it is proven that the category of finite-dimensional $\mathbb{D}_\Lambda$-modules is equivalent to the category of perverse sheaves on the isotropic Grassmannian of type D_n (constructible with respect to the Schubert stratification). This category is also equivalent to the principal block of the parabolic category $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}(2n))$, where $\mathfrak{p}$ is a maximal parabolic subalgebra of type A.
Similar to the type A case, the algebras $\mathbb{D}_\Lambda$ also describe the category of finite-dimensional representations of the Lie supergroups $OSp(m|2n)$; see [ES3], [CH] for more details.
The goal of this thesis is to explicitly study the algebras $\mathbb{D}_\Lambda$ and their representation theory, in a similar fashion to what Brundan and Stroppel did for the type A arc algebras in [BS1] and [BS2]. Although our results are very similar to the type A case, the proofs are quite different at several places.
One word of caution: the roles of the type A en type D arc algebras in 2-representation theory are quite different. The type A arc algebras categorify modules for the type A quantum groups $U_q(\mathfrak{sl}_n)$, so they are cyclotomic quotients of the type A Khovanov-Lauda-Rouquier algebras, which categorify $U_q^-(\mathfrak{sl}_n)$; see [KhoLau] and [Web]. In contrast, the type D arc algebras $\mathbb{D}_\Lambda$ do not categorify modules for type D quantum groups $U_q(\mathfrak{so}_{2n})$; they instead categorify modules for certain coideal subalgebras of $U_q(\mathfrak{gl}_{2n})$, also called quantum symmetric pairs, see [ES4] or [BSWW]. In particular this means that our algebras $\mathbb{D}_\Lambda$ are not related to type D KLR algebras.

An overview of the most important results of this thesis.
The main result of this thesis is the following theorem:

Theorem 4.3.3. The algebras $\mathbb{D}_\Lambda$ are Koszul algebras in the sense of [BGS].

This result follows indirectly from the perverse sheaves description and [BGS, Theorem 1.4.2], but we will give a direct combinatorial proof without passing through geometry. We will use the fact that the algebras $\mathbb{D}_\Lambda$ are graded quasi-hereditary, hence it suffices to prove that its standard modules have linear projective resolutions. This Koszulity criterion (Theorem 3.2.2)

was already proven in [ADL, Theorem 1.4], but we will provide a different proof which is easier and better adapted to our graded framework.
A second major goal of this thesis is a combinatorial study of the so-called *dual Kazhdan–Lusztig polynomials* $p_{\lambda,\mu}(q)$ $(\lambda, \mu \in \Lambda)$ in type D. These are in fact very closely related to the (geometric) Kazhdan–Lusztig polynomials associated to Grassmannians in the sense of [KL], but we will just define them as the decomposition numbers of the Koszul dual algebra of $\mathbb{D}_\Lambda$. Two explicit combinatorial formulas of the dual Kazhdan–Lusztig polynomials will be proved (see below for an explanation of the notation used):

Theorem 5.2.8. The dual Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ can be computed by counting *paths from* λ *to* μ, in the sense of Definition 5.2.4.

Theorem 5.3.4. The dual Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ can be computed by counting λ-*labellings of* $\mu\overline{\mu}$, in the sense of Definition 5.3.1.

We note that this last theorem agrees with the closed formula for the Kazhdan–Lusztig polynomials discovered by Lascoux and Schützenberger [LS]. Although the above theorems will be proved independently, one can in fact directly construct a bijection

$$\left\{ \begin{array}{c} \text{paths from} \\ \lambda \text{ to } \mu \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \lambda\text{-labellings} \\ \text{of } \mu\overline{\mu} \end{array} \right\}.$$

This will be done in Subsection 5.4.
In the final part of this thesis, we will study an interesting subalgebra $\mathbb{H}_\Lambda$ of $\mathbb{D}_\Lambda$, which is in fact more closely related to Khovanov's original construction. We will we explicitly show that the algebras $\mathbb{H}_\Lambda$ are symmetric by giving an explicit symmetric form (see Section 6 for a more precise statement):

Theorem 6.2.5. One can explicitly define a symmetrizing form $\tau : \mathbb{H}_\Lambda \to k$ on $\mathbb{H}_\Lambda$, making $\mathbb{H}_\Lambda$ into a symmetric algebra.

The contents of this thesis in a nutshell.
The first section is devoted to recalling the construction of the algebras $\mathbb{D}_\Lambda$, following [ES1]. Here Λ will a certain set (called *block*) of so-called *combinatorial weights*. To each such block, one can associate a set $\mathbb{B}_\Lambda$ of *circle diagrams*. The algebra $\mathbb{D}_\Lambda$ is then the vector space spanned by $\mathbb{B}_\Lambda$, together with a multiplication defined via certain surgery procedures.
In the second section, we recall the definitions and some properties of cellular and quasi-hereditary algebras. We review the cellular and quasi-hereditary structure of $\mathbb{D}_\Lambda$, as discussed in [ES1]. In particular, for each weight $\lambda \in \Lambda$, we have a simple module $L(\lambda)$, a standard module $V(\lambda)$, and a projective module $P(\lambda)$. These three classes of modules interact in a well-behaved way. We also mention how to compute the *decomposition numbers* $d_{\lambda,\mu}(q)$. Theses describe the graded composition factors of the standard modules $V(\mu)$, as well as the subquotients in a graded filtration of a projective module $P(\lambda)$ by standard modules.
In the third section, we recall the notion of a Koszul algebra. The main result of this section is the Koszulity criterion 3.2.2 mentioned before.
The fourth section is devoted to the proof of Theorems 4.3.2 and 4.3.3 (our main theorem). Our proof is based on the proof for a similar result in type A, see [BS2, Theorem 5.6]. It will make use of so-called *geometric bimodules* and *projective functors*. We note that the Koszulity criterion from Theorem 3.2.2 will play an essential role in the proof.
In the fifth section we discuss the *dual Kazhdan–Lusztig polynomials* $p_{\lambda,\mu}(q)$, where $\lambda, \mu \in \Lambda$.

These can be described as the decomposition numbers of the Koszul dual algebra of $\mathbb{D}_\Lambda$. They also describe projective resolutions of standard modules. Various explicit combinatorial descriptions of the dual Kazhdan–Lusztig polynomials are given (see above). As an application we give a combinatorial classification of the so-called *Kostant weights*.
In the last section we turn our attention to the subalgebra $\mathbb{H}_\Lambda$ of $\mathbb{D}_\Lambda$.The algebra $\mathbb{H}_\Lambda$ is no longer quasi-hereditary or Koszul, but it is still cellular. Moreover, $\mathbb{H}_\Lambda$ is a *symmetric algebra*. We give a prove of this fact by explicitly writing down a symmetrizing form.
Recently, Ehrig, Tubbenhauer and Wilbert [ETW] defined a sign-adjusted version of the type D arc algebra, which we will denote by $\overline{\mathbb{D}}_\Lambda$. Their construction of $\overline{\mathbb{D}}_\Lambda$ is based on a topological approach using certain (singular) TQFTs. The sign-adjusted version can be easier in practice. We discuss this sign-adjusted version in the appendix. In particular, we introduce the subalgebra $\overline{\mathbb{H}}_\Lambda$, and discuss its symmetric structure.

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0 Notations and conventions

The set of non-negative integers will be denoted $\mathbb{N} = \{0, 1, 2, \ldots\}$. The positive integers will be denoted by $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$.
We will always work over a fixed base field k. For simplicity we will always assume k is algebraically closed and of characteristic 0, although many of the results in this work are true for more general base fields (or even rings).
We will often be working with *positively graded algebras*. For us, a positively graded algebra is a $\mathbb{Z}$-graded k-algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ such that $A_i = 0$ for $i < 0$, and A_0 is semisimple. In practice, we will always assume that $A_0 \cong k \times k \times \ldots \times k$. If now A is a finite-dimensional positively graded algebra, then the category of finite-dimensional graded left A-modules will be denoted by A-gmod. For $M = \bigoplus_{i\in\mathbb{Z}} M_i \in A$-gmod and $d \in \mathbb{Z}$, the degree shift $M\langle d\rangle$ is defined by $M\langle d\rangle_i = M_{i-d}$ (i.e. we "shift upwards"). For $M, N \in A$-gmod, $\mathrm{Hom}_A(M, N)$ will denote the space of degree-preserving homomorphisms. The graded A-module consisting of all degree-homogeneous homomorphisms will be denoted by $\mathrm{HOM}_A(M, N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_A(M\langle i\rangle, N)$. (So $\mathrm{HOM}_A(M, N)_i = \mathrm{Hom}_A(M\langle i\rangle, N)$.) Similar for the ext-groups $\mathrm{Ext}^i_A(M, N)$ and $\mathrm{EXT}^i_A(M, N)$.

1 Definition of the type D arc algebra

In this first section, we will introduce our main object of study: the type D arc algebras $\mathbb{D}_\Lambda$, as defined in [ES1]. The elements of $\mathbb{D}_\Lambda$ are linear combinations of *circle diagrams*. In Subsections 1.1 and 1.2, we will define these circle diagrams, and in Subsection 1.3 we will describe how to define an algebra structure on the vector space $\mathbb{D}_\Lambda$ spanned by a certain set of circle diagrams, depending on the datum Λ.

1.1 Weights and blocks

Circle diagrams will be obtained by stacking a *cup diagram*, a *weight*, and a *cap diagram*. In this subsection we will introduce weights, the next one will be devoted to cup and cap diagrams, and circle diagrams. We will study the combinatorics of weights in quite some detail, because we will need this later in Section 5. A lot of this subsection could be postponed until later; only Definitions 1.1.1 and 1.1.2 and Lemma 1.1.4 will be essential for the definition of the algebra $\mathbb{D}_\Lambda$.

Definition 1.1.1.

- A *weight* is a finite sequence of the symbols $\wedge$ ("up") and $\vee$ ("down"); the *size* of a weight is the number of symbols used.
- We define an equivalence relation on the set of all weights. The equivalence classes are called *blocks*. By definition, two weights belong to the same block if and only if they have the same size, and the parity of the number of $\vee$'s is the same.

Note that a block is uniquely defined by a pair $(n, l) \in \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$, where n is the size of all its elements, and $l \in \{\overline{0}, \overline{1}\} = \mathbb{Z}/2\mathbb{Z}$ the parity of the number of $\vee$'s. We will denote this block by Λ_n^l. A block whose elements have size n will be called an *n-block*.

Definition 1.1.2. We say that a weight μ is obtained from a weight λ by a *Bruhat move*, if one of the following holds

- λ has a pair of neighbouring labels $\vee\wedge$ (say at positions $i, i+1$), and μ is obtained by replacing these by $\wedge\vee$. We will refer to this as a *type A move*, applied at position i.
- The first 2 labels of λ are $\wedge\wedge$, and μ is obtained by replacing these by $\vee\vee$. We will refer to this as a *type D move* (necessarily applied at position 1).

So in an n-block, there are potentially n different Bruhat moves available: type A moves at position $1, 2, \ldots, n-1$, and a type D move at position 1.
We will write $\lambda \leq \mu$ if there is a sequence of Bruhat moves starting in λ and ending in μ. Note that $\lambda \leq \mu$ implies that λ and μ belong to the same block.

Example 1.1.3. Let $\lambda = \wedge \wedge \wedge\wedge$ and $\mu = \wedge \vee \wedge\vee$. Then $\lambda, \mu \in \Lambda_4^{\overline{0}}$, and $\lambda \leq \mu$ since there is a sequence of Bruhat moves

$$\wedge \wedge \wedge\wedge \to \vee \vee \wedge\wedge \to \vee \wedge \vee\wedge \to \vee \wedge \wedge\vee \to \wedge \vee \wedge \vee \quad .$$

Lemma 1.1.4. *The relation $\leq$ defines a partial order on every block Λ, called* Bruhat order.

Proof. Reflexivity and transitivity are immediate from the definition.
For symmetry: observe that if $\lambda \leq \mu$ and $\mu \leq \lambda$, we get a sequence of Bruhat moves which starts in λ, passes through μ, and ends in λ. But the only sequence of Bruhat moves from λ to itself is the empty sequence: this is easily seen from the consideration that a Bruhat move will always increase the number of $\vee$'s or move $\vee$'s to the right, and can never decrease the number of $\vee$'s or move $\vee$'s to the left. So it must hold that $\lambda = \mu$. □

So $\lambda \leq \mu$ means that μ has more $\vee$'s than λ, and that the $\vee$'s in μ are further to the right than those in λ. We will make this more formal in what follows.

Definition 1.1.5. Let λ and μ be two weights, belonging the same n-block Λ. Suppose λ has m symbols $\vee$ and μ has $m+2k$ symbols $\vee$.
For $i = 1, 2, \cdots, n$, we define

$$\begin{aligned}\ell_i(\lambda,\mu) :=& 2k + |\{j \leq i \text{ s.t. } \lambda_j = \vee\}| - |\{j \leq i \text{ s.t. } \mu_j = \vee\}| \\ =& |\{j > i \text{ s.t. } \mu_j = \vee\}| - |\{j > i \text{ s.t. } \lambda_j = \vee\}|.\end{aligned}$$

In particular: $\ell_n(\lambda,\mu) = 0$ and $\ell_0(\lambda,\mu) := 2k$.
The number $\ell(\lambda,\mu) := \sum_{i=1}^{n} \ell_i(\lambda,\mu)$ will be called the *distance* from λ to μ. Note that this sum does not contain $\ell_0(\lambda,\mu)$.

The following lemma, which follows immediately from the definition, is quite useful when writing down the numbers $\ell_i(\lambda,\mu)$ in practice:

Lemma 1.1.6. *For $i = 1, 2, \cdots, n$, it holds that*

$$\ell_i(\lambda,\mu) - \ell_{i-1}(\lambda,\mu) = \begin{cases} 1, & \text{if } \lambda_i = \vee \text{ and } \mu_i = \wedge, \\ 0, & \text{if } \lambda_i = \mu_i, \\ -1, & \text{if } \lambda_i = \wedge \text{ and } \mu_i = \vee. \end{cases}$$

Example 1.1.7. Let λ and μ be as in Example 1.1.3. Then $\ell_0(\lambda,\mu) = 2$, $\ell_1(\lambda,\mu) = 2$, $\ell_2(\lambda,\mu) = 1$, $\ell_3(\lambda,\mu) = 1$, $\ell_4(\lambda,\mu) = 0$. So $\ell(\lambda,\mu) = 2+1+1+0 = 4$.

It is clear from the definition that applying a Bruhat move (of type A or D) to λ at position i will decrease $\ell_i(\lambda,\mu)$ by 1 and leave all other $\ell_j(\lambda,\mu)$, $j \neq i$, $j > 0$ invariant. Since $\lambda = \mu$ if and only if all $\ell_i(\lambda,\mu)$ $(i > 0)$ are 0, we deduce that if $\lambda \leq \mu$, it must hold that all $\ell_i(\lambda,\mu)$ are non-negative. (The converse also holds: see Proposition 1.1.8.)
Moreover, if $\lambda \leq \mu$, any sequence of Bruhat moves from λ to μ must contain, for each index $i > 0$, exactly $\ell_i(\lambda,\mu)$ moves applied at position i. In particular, the total number of moves in such a sequence is $\ell(\lambda,\mu)$.
The moves applied at position 1 can be either of type A or of type D. Our sequence needs to contain exactly k type D moves (where $2k = \ell_0(\lambda,\mu)$ is the difference in number of $\vee$'s in μ and λ), hence exactly $\ell_1(\lambda,\mu) - k$ type A moves applied at position 1. So a sequence of Bruhat moves from λ to μ is unique up to a reordering of the moves.
We will now use the numbers $\ell_j(\lambda,\mu)$ to formulate a criterion when $\lambda \leq \mu$:

Proposition 1.1.8. *For two weights λ, μ belonging to the same block Λ, we have that $\lambda \leq \mu$ if and only if all numbers $\ell_i(\lambda,\mu)$ $(i > 0)$ are non-negative.*

Proof. The "only if" implication was already proven. So it suffices to prove that if all numbers $\ell_i(\lambda, \mu)$ are non-negative, there is a sequence of Bruhat moves from λ to μ. We will prove this by induction on $\ell(\lambda, \mu)$.
If $\ell(\lambda, \mu) = 0$, then for all i, $\ell_i(\lambda, \mu) = 0$, so $\lambda = \mu$. So suppose $\ell(\lambda, \mu) > 0$.
Let j be the rightmost index (if it exists) for which $\lambda_j = \vee$ and $\ell_j(\lambda, \mu) > 0$. Then $\lambda_{j+1} = \wedge$, since if $\lambda_{j+1} = \vee$, $\ell_{j+1}(\lambda, \mu) - \ell_j(\lambda, \mu) \geq 0$ by definition of ℓ_j, so that $\ell_{j+1}(\lambda, \mu) > 0$ and we get a contradiction with the maximality of j. Now we can apply a type A Bruhat move to λ at position j to obtain a λ' for which $\ell_j(\lambda', \mu) = \ell_j(\lambda, \mu) - 1$ and $\ell_i(\lambda', \mu) = \ell_i(\lambda, \mu)$ for all $i \neq j$. By the induction hypothesis there is a sequence of Bruhat moves from λ' to μ and we are done.
We still have to deal with the case where an index j as above doesn't exist. Note that in this case $\ell_0(\lambda, \mu) = 2k > 0$:
Suppose on the contrary that $\ell_0(\lambda, \mu) = 0$, i.e. λ and μ have the same number of $\vee$'s. Then since $\lambda \neq \mu$, there is an index l for which $\lambda_l = \vee$ and $\mu_l = \wedge$. But then $\ell_l(\lambda, \mu) = \ell_{l-1}(\lambda, \mu) + 1 > 0$, contradicting our assumption that there is no j with $\lambda_j = \vee$ and $\ell_j(\lambda, \mu) > 0$.
Since $2k > 0$, in order for our assumption to hold, we must have that $\lambda_1 = \lambda_2 = \wedge$. Moreover, $\ell_1(\lambda, \mu) > 0$. So we can apply a type D Bruhat move to λ at position 1 to obtain a λ' for which $\ell_1(\lambda', \mu) = \ell_1(\lambda, \mu) - 1$ and $\ell_i(\lambda', \mu) = \ell_i(\lambda, \mu)$ for all $i > 1$. By the induction hypothesis there is a sequence of Bruhat moves from λ' to μ and we are done. $\square$

The above proof constructs a particular sequence of Bruhat moves from λ to μ, which we will call the *canonical sequence*. Intuitively, the canonical sequence is obtained by applying successively type A and type D moves as follows: If a type A move is possible then apply it to the rightmost index where it can be applied and otherwise apply a type D move.
For example, the sequence from Example 1.1.3 is the canonical sequence from λ to μ.

1.2 Cup diagrams, cap diagrams, and circle diagrams

Denote by R^- the semi-infinite strip $\mathbb{R}_{\geq 0} \times [-\frac{1}{2}, 0]$.

Definition 1.2.1.

- A *cup* is a non-selfintersecting curve γ in R^- whose endpoints are 2 distinct points on $\mathbb{Z}_{>0} \times \{0\}$, with all other points of γ lying in $\mathbb{R}_{\geq 0} \times (-\frac{1}{2}, 0)$.

- A *ray* is a non-selfintersecting curve γ in R^- with one endpoint on $\mathbb{Z}_{>0} \times \{0\}$, the other endpoint on $\mathbb{R}_{\geq 0} \times \{-\frac{1}{2}\}$, and all other points in $\mathbb{R}_{\geq 0} \times (-\frac{1}{2}, 0)$.

- An *undecorated cup diagram* c is a finite union of pairwise nonintersecting cups and rays, such that $c \cap (\mathbb{R}_{\geq 0} \times \{0\}) = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. This n will be called the *size* of the cup diagram. Two undecorated cup diagrams are *equal* if their partitioning of the vertices into subsets given by the incidence relation on cups and rays agree. (That is two diagrams are the same if the cups connect the same points, regardless of the precise shape).

- A *decorated cup diagram* is an undecorated cup diagram $c = \gamma_1, \cdots, \gamma_n$, together with a map (called *decoration*) from c to $\{0, 1\}$. Arcs that are mapped to 1 are called *dotted*, the other arcs are called *undotted*.

- A decorated cup diagram is called *admissible*, if any point of every dotted arc can be connected with the left boundary of R^- by a path not intersecting any other arc.

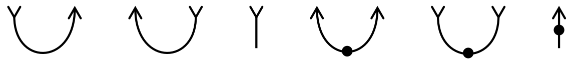
Example 1.2.2. The following figure shows two decorated cup diagrams of size 7, with one dotted cup and 2 undotted cups. The left one is admissible, while the right one is not.

Remark 1.2.3. Observe that a decorated cup diagram is admissible if and only if the following do not occur:

- A dotted cup nested inside another cup.
- A dotted cup or dotted ray to the right of a ray.

From now on, we will only consider admissible decorated cup diagrams. We will refer to them simply as *cup diagrams*.

Definition 1.2.4. An *oriented cup diagram* $c\lambda$ is a cup diagram c drawn under a weight λ of the same size, such that every arc looks like one of the following figures:



(1.1)

A cup in an oriented cup diagram is called *clockwise* if the symbol at the right end is $\vee$, and *counterclockwise* if the symbol at the right end is $\wedge$. The *degree* of an oriented cup diagram is the number of clockwise cups.

Proposition 1.2.5.
- *For every cup diagram c, there is a unique weight λ such that $c\lambda$ is an oriented cup diagram of degree* 0*.*
- *For any weight λ, there is a unique cup diagram, which we will denote by $\underline{\lambda}$, such that $\underline{\lambda}\lambda$ is an oriented cup diagram of (minimal possible) degree* 0*.*

Hence, there is a 1-1 correspondence

$$\begin{aligned} \{\textit{weights}\} &\longleftrightarrow \{\textit{cup diagrams}\}, \\ \lambda \quad &\longmapsto \quad \underline{\lambda}. \end{aligned}$$

Proof. The first claim follows immediately from Definition 1.2.4: every ray has a unique possible orientation, and every cup has a unique counterclockwise orientation.
For the second claim, given the weight λ, we define $\underline{\lambda}$ by the following procedure:

1. First connect neighboured vertices (ignoring already joint vertices) labelled $\vee\wedge$ successively by an undotted cup as long as possible.
2. Attach to each remaining $\vee$ a vertical undotted ray.
3. Connect from left to right neighboured pairs $\wedge\wedge$ by a dotted cup.
4. If a single $\wedge$ remains, attach a vertical dotted ray.

It follows immediately from the construction that $\underline{\lambda}$ is an admissible cup diagram, and that $\underline{\lambda}\lambda$ is oriented of degree 0. To check uniqueness, we go through the above procedure again and see that in every step we had no other choice:

1. If we have neighbours $\vee\wedge$ and we don't connect them with a cup, then the $\vee$ has to be either the left endpoint of a "bigger" cup, or the endpoint of an (undotted) ray. Then $\wedge$ has to be the endpoint of a dotted cup or dotted ray, but then admissibility is violated.

2. Each remaining $\vee$ has to be either the left endpoint of an undotted cup, or the endpoint of an undotted ray. But the first case is no longer possible.

3,4. Now there are only $\wedge$'s remaining, and they all need to be connected via dotted cups or rays. Clearly there is only one way to do this without violating admissibility. $\square$

If $\underline{\lambda}\mu$ is an oriented cup diagram, we write $\lambda \subset \mu$. This is related to the Bruhat order in the following way:

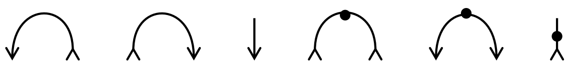
Lemma 1.2.6 ([ES1, Lemma 3.17])**.** *If $\lambda \subset \mu$, then $\lambda \leq \mu$ in the Bruhat order.*

Proof. If $\lambda \subset \mu$, then μ is obtained from $\underline{\lambda}\mu$ is obtained from $\underline{\lambda}\lambda$ by choosing some cups and reversing their orientations from counterclockwise to clockwise. Since this will only create new $\vee$'s and/or move $\vee$'s to the right, it easily follows that $\lambda \leq \mu$. $\square$

Analoguously to cup diagrams, we can also introduce cap diagrams:

Definition 1.2.7.
- A *cap diagram* b is a diagram, contained in the semi-infinite strip $\mathbb{R}_{\geq 0} \times [0, \frac{1}{2}]$, obtained by reflecting a cup diagram a around the x-axis. In this case we write $b = a^*$ and $a = b^*$.
- An *oriented cap diagram* λb is a cap diagram b drawn over a weight λ of the same size, such that $b^*\lambda$ is an oriented cup diagram.
- The *degree* of an oriented cap diagram is defined by $\deg(\lambda a^*) := \deg(a\lambda)$.

In other words a diagram μb, with μ a weight and b a cap diagram, is an oriented cap diagram if and only every cap and ray looks like one of the following:



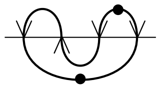
(1.2)

A cap is called *counterclockwise* if its rightmost symbol is $\wedge$, and *clockwise* if its rightmost symbol is $\vee$. Then the degree of an oriented cap diagram is the number of clockwise caps. The cap diagram $\underline{\mu}^*$ will be denoted by $\overline{\mu}$. As before, we have that $\overline{\mu}$ is the unique cap diagram making $\mu\overline{\mu}$ into an oriented cap diagram of degree 0.
For the rest of this section, we fix a block Λ.

Definition 1.2.8.
- An *unoriented circle diagram* belonging to Λ is a pair $\underline{\lambda}\overline{\nu}$, with $\lambda, \nu \in \Lambda$.
- An *oriented circle diagram* belonging to Λ is a triple $\underline{\lambda}\mu\overline{\nu}$ $(\lambda, \mu, \nu \in \Lambda)$ such that $\underline{\lambda}\mu$ is an oriented cup diagram and $\mu\overline{\nu}$ is an oriented cap diagram.

- The *degree* of an oriented circle diagram is defined as $\deg(\underline{\lambda}\mu\overline{\nu}) := \deg(\underline{\lambda}\mu) + \deg(\mu\overline{\nu})$. In other words, the degree is the number of clockwise cups plus the number of clockwise caps.

Example 1.2.9. The diagram

$\underline{\lambda}\mu\overline{\nu} =$ 

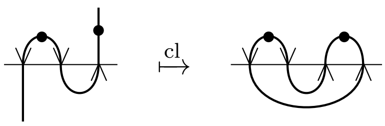
is an oriented circle diagram, with $\lambda = \wedge \vee \wedge\wedge$, $\mu = \vee \wedge \vee\vee$, and $\nu = \vee \wedge \wedge\wedge$.
The degree of $\underline{\lambda}\mu\overline{\nu}$ is 3.

Definition 1.2.10.
- We will denote the set of oriented circle diagrams belonging to Λ by $\mathbb{B}_\Lambda$. For fixed $\lambda, \nu \in \Lambda$ the subset of circle diagrams of the form $\underline{\lambda}\mu\overline{\nu}$ will be denoted by ${}_\lambda(\mathbb{B}_\Lambda)_\nu$.
- A closed arc in a circle diagram will be called a *circle*. For C a circle in some oriented circle diagram $\underline{\lambda}\mu\overline{\nu}$, the *tag* of C, denoted $t(C)$, is the index of the rightmost symbol of μ lying on the circle. If this symbol is $\wedge$, we say that the circle C is oriented *counterclockwise*; if it is $\vee$, C is oriented *clockwise*.
- A non-closed arc in a circle diagram (i.e. one that intersects the boundary of the strip $\mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]$) will be called a *line*. A line is called *propagating* is one endpoint is at the top of the strip and the other is at the bottom.

Remark 1.2.11. For $n \in \mathbb{N}$ and $l \in \mathbb{Z}/2\mathbb{Z}$, we can define the *closure map*

$$\begin{aligned} \mathrm{cl}: \quad & \mathbb{B}_{\Lambda^l_n} \to \mathbb{B}_{\Lambda^l_{n+1}} \\ & \underline{\lambda}\mu\overline{\nu} \mapsto \underline{\mathrm{cl}(\lambda)}\,\mathrm{cl}(\mu)\overline{\mathrm{cl}(\mu)} \end{aligned}$$

where for $\lambda \in \Lambda$, $\mathrm{cl}(\lambda)$ is obtained by adding a $\wedge$ to the right of λ.
For example



This closure map will "close"' the rightmost line of $\underline{\lambda}\mu\overline{\nu}$ into a circle (or create a new line if $\underline{\lambda}\mu\overline{\nu}$ has no lines), and leave all other arcs unchanged. Note also that cl preserves the degree.
By iterating, we get closure maps $\mathrm{cl}^k : \mathbb{B}_{\Lambda^l_n} \to \mathbb{B}_{\Lambda^l_{n+k}}$. We can always find a k such that $\mathrm{cl}^k(\underline{\lambda}\mu\overline{\nu})$ has only circles and no lines (for example $k = n$).

Note that a *small circle* (this is a circle consisting of only 1 cap and 1 cup) contributes 0 to the degree if it's oriented counterclockwise, and 2 if it's oriented clockwise.
We can also describe how much larger circles contribute to the degree.

Proposition 1.2.12 ([ES1, Proposition 4.9])**.** *Let C be a circle in some circle diagram $\underline{\lambda}\mu\overline{\nu}$. Clearly, there is an $s \in \mathbb{N}$ such that C contains s cups, s caps and $2s$ symbols of μ.*

1. *If C is oriented counterclockwise, it contains exactly $s+1$ counterclockwise and $s-1$ clockwise cups/caps. Hence the contribution of C to the degree of $\underline{\lambda}\mu\overline{\nu}$ is $s-1$.*

2. *If C is oriented clockwise, it contains exactly $s-1$ counterclockwise and $s+1$ clockwise cups/caps. Hence the contribution of C to the degree of $\underline{\lambda}\mu\overline{\nu}$ is $s+1$.*

We can make a similar statement about lines:

Proposition 1.2.13. *Let L be a line in some circle diagram $a\mu b$. Let* $\mathrm{cups}(L)$ *resp.* $\mathrm{caps}(L)$ *denote the number of cups resp. caps in L.*

1. *If L is propagating,* $\mathrm{cups}(L) = \mathrm{caps}(L)$*, and the contribution of L to the degree of $a\mu b$ equals* $\mathrm{cups}(L)$*.*
2. *If L is not propagating and its 2 endpoints are in a, then* $\mathrm{caps}(L) = \mathrm{cups}(L) + 1$*, and the contribution of L to the degree of $a\mu b$ equals* $\mathrm{caps}(L)$*.*
3. *If L is not propagating and its 2 endpoints are in b, then* $\mathrm{cups}(L) = \mathrm{caps}(L) + 1$*, and the contribution of L to the degree of $a\mu b$ equals* $\mathrm{cups}(L)$*.*

Proof. Apply the closure map cl^k from Remark 1.2.11 for some large enough k. Then L becomes a counterclockwise circle $\mathrm{cl}(L)$, whose contribution to the degree is the same as that of L. If L is propagating, then $\mathrm{cl}(L)$ has one more cup and one more cap than L. If L is not propagating and its 2 endpoints are in a, then $\mathrm{cl}(L)$ has two more cups and one more cap than L. If L is not propagating and its 2 endpoints are in b, then $\mathrm{cl}(L)$ has one more cup and two more caps than L. In each case, the result follows from Proposition 1.2.12. □

We say that an unoriented circle diagram $\underline{\lambda}\overline{\nu}$ $(\lambda, \nu \in \Lambda)$ is *orientable* if there is a $\mu \in \Lambda$ such that $\underline{\lambda}\mu\overline{\nu}$ is an oriented circle diagram. In this case we call μ an *orientation* of $\underline{\lambda}\overline{\nu}$.

Lemma 1.2.14 ([ES1, Proposition 4.8])**.** *Let $\lambda, \nu \in \Lambda$. Then the following holds:*

1. *The circle diagram $\underline{\lambda}\overline{\nu}$ is orientable if and only if the number of dots is even on each of its circles and its propagating lines, and odd on each of its non-propagating lines.*
2. *In this case there are exactly 2^c possible orientations, where c is the number of circles. They are obtained by choosing for each of the circles one out of its two possible orientations and for each ray the unique possible orientation.*

Definition 1.2.15. We define $\mathbb{D}_\Lambda$ to be the k-vector space with basis $\mathbb{B}_\Lambda$.

In the next section we will explain how to define a multiplication on $\mathbb{D}_\Lambda$, making it a positively graded algebra.

1.3 The multiplication in $\mathbb{D}_\Lambda$

For the remainder of this section, unless stated otherwise, we will work with a fixed n-block Λ. As before, let $\mathbb{D}_\Lambda$ be the k-vector space with basis given by the set $\mathbb{B}_\Lambda$ of oriented circle diagrams. Then $\mathbb{D}_\Lambda$ can be given the structure of a positively graded associative unital algebra. In this subsection we will give a quick description how the multiplication in this algebra is defined; we refer to [ES1] for more details and proofs.
The vector subspace of $\mathbb{D}_\Lambda$ with basis ${}_\lambda(\mathbb{B}_\Lambda)_\nu$ will be denoted ${}_\lambda(\mathbb{D}_\Lambda)_\nu$.

Definition 1.3.1. Let $\lambda, \nu \in \Lambda$ and assume that the circle diagram $\underline{\lambda}\overline{\nu}$ can be oriented. Then $I_{\underline{\lambda}\overline{\nu}}$ is defined to be the ideal in $k[X_i | 1 \leq i \leq n]$ generated by:

- X_i^2 for all $1 \leq i \leq n$.
- $X_i + X_j$, if i, j are connected by an undotted cup or cap.
- $X_i - X_j$, if i, j are connected by a dotted cup or cap.
- X_i, if i is the endpoint of a ray.

Define the vector space $\mathcal{M}(\underline{\lambda}\overline{\mu}) := k[X_i | 1 \leq i \leq n]/I_{\underline{\lambda}\overline{\nu}}$.

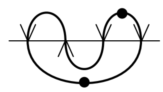
Proposition 1.3.2 ([ES1, Proposition 4.5])**.** *There is an isomorphism of vector spaces*

$$\begin{aligned} \Psi_{\underline{\lambda}\overline{\nu}} : \quad {}_{\lambda}(\mathbb{D}_\Lambda)_\nu &\xrightarrow{\sim} \mathcal{M}(\underline{\lambda}\overline{\nu}), \\ \underline{\lambda}\mu\overline{\nu} &\longmapsto \prod_{C \in \mathcal{C}_{\mathrm{clock}}(\underline{\lambda}\mu\overline{\nu})} X_{t(C)}, \end{aligned}$$

where $\mathcal{C}_{\mathrm{clock}}(\underline{\lambda}\mu\overline{\nu})$ is the set of clockwise oriented circles in $\underline{\lambda}\mu\overline{\nu}$, and $t(C)$ is the tag of C, as defined in Definition 1.2.10.

By Proposition 1.3.2, one can think of $\mathcal{M}(\underline{\lambda}\overline{\mu})$ as the vector space spanned by the possible orientations of the unoriented circle diagram $\underline{\lambda}\overline{\mu}$.

Example 1.3.3. For the circle diagram

$\underline{\lambda}\mu\overline{\nu} =$ 

from Example 1.2.9, we have that

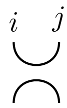
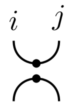
$$\Psi_{\underline{\lambda}\overline{\nu}}(\underline{\lambda}\mu\overline{\nu}) = X_4 = X_3 = -X_2 = X_1$$

Definition 1.3.4. Suppose we are given two circle diagrams ab and b^*c. We can draw the second diagram above the first one, and remove all the dots on the rays of b and b^* giving a *stacked circle diagram atc*. (Here t is the *crossingless matching* obtained by drawing the cap diagram b under the cup diagram b^*; see Definitions 4.1.1 and 4.1.3 for a precise definition of *crossingless matching* and *stacked circle diagram.*) Applying a *surgery* to atc consists of replacing a matching cup-cap pair in t by 2 vertical lines as in the following figure:

Note that the resulting stacked circle diagram $at'c$ might no longer be admissible. (As before, a crossingless matching t' is called admissible if every dotted cup or cap in it can be connected to the left side of the diagram by a path not intersecting any cups, caps, or lines. See also Definition 4.1.1.)
A surgery will either decrease the number of connected components by 1, decrease it by 1, or

keep it the same. We will refer to these respective cases as *merge*, *split* and *reconnect*. Note that a reconnect can only happen if all components involved in the surgery are lines.
We now fix a *surgery procedure* from (ab, b^*c) to ac: this is a sequence of surgery moves starting from atc and ending in aec, where e denotes the crossingless matching consisting of only vertical lines. We require every intermediate diagram in our surgery procedure to be admissible.
We will now define a map $\chi' : k[X_i|1 \le i \le n] \otimes k[X_i|1 \le i \le n] \to k[X_i|1 \le i \le n]$ associated to our surgery procedure. To obtain the image of $f \otimes g \in k[X_i|1 \le i \le n] \otimes k[X_i|1 \le i \le n]$, apply the following inductive algorithm:

1. Start by putting $y = f \cdot g \in k[X_i|1 \le i \le n]$.
2. For each surgery in the surgery procedure, consult the following table:

Merge:	leave y as it is,
Split:	multiply y with
	$(-1)^i(X_j - X_i)$, if 
	$(-1)^i(X_j + X_i)$, if 
Reconnect:	if the two lines (before the surgery) are propagating, and the resulting diagram is orientable, then leave y as it is. Else multiply y by 0.

3. We define $\chi'(f \otimes g)$ to be equal to the resulting y.

One can show (see [ES1, Section 5]) that this map $\chi' : k[X_i|1 \le i \le n] \otimes k[X_i|1 \le i \le n] \to k[X_i|1 \le i \le n]$ descends to a map $\chi : \mathcal{M}(ab) \otimes \mathcal{M}(b^*c) \to \mathcal{M}(ac)$, and that this map χ doesn't depend on our chosen surgery procedure. We will from now on write $\chi_{ab,b^*c} : \mathcal{M}(ab) \otimes \mathcal{M}(b^*c) \to \mathcal{M}(ac)$.

Now we are ready to define the multiplication on $\mathbb{D}_\Lambda$:

Definition 1.3.5. Consider the composition

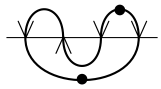
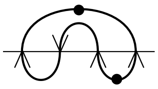
$$\Phi_{\lambda\nu\nu'} : {}_\lambda(\mathbb{D}_\Lambda)_\nu \otimes {}_\nu(\mathbb{D}_\Lambda)_{\nu'} \xrightarrow{\Psi_{\underline{\lambda}\overline{\nu}} \otimes \Psi_{\underline{\nu}\overline{\nu'}}} \mathcal{M}(\underline{\lambda}\overline{\nu}) \otimes \mathcal{M}(\underline{\nu}\overline{\nu'}) \xrightarrow{\chi_{\underline{\lambda}\overline{\nu},\underline{\nu}\overline{\nu'}}} \mathcal{M}(\underline{\lambda}\overline{\nu'}) \xrightarrow{(\Psi_{\underline{\lambda}\overline{\nu'}})^{-1}} {}_\lambda(\mathbb{D}_\Lambda)_{\nu'}$$

where the second map is the map defined above.
For two basis vectors $\underline{\lambda}\mu\overline{\nu}$ and $\underline{\lambda'}\mu'\overline{\nu'}$ of $\mathbb{D}_\Lambda$, we define their product as follows:

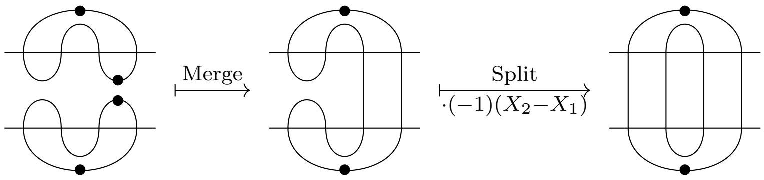
$$(\underline{\lambda}\mu\overline{\nu}) \cdot (\underline{\lambda'}\mu'\overline{\nu'}) \quad = \quad \begin{cases} \Phi_{\lambda\nu\nu'}((\underline{\lambda}\mu\overline{\nu}) \otimes (\underline{\lambda'}\mu'\overline{\nu'})), & \text{if } \nu = \lambda', \\ 0, & \text{else.} \end{cases}$$

Fact 1.3.6 ([ES1, Theorem 6.2])**.** *Extending the above product bilinearly to $\mathbb{D}_\Lambda$ equips $\mathbb{D}_\Lambda$ with the structure of an associative unital graded algebra, where the grading is the one defined in Definition 1.2.10.*

Example 1.3.7. Suppose we want to compute the product $(\underline{\lambda}\mu\overline{\nu})\cdot(\underline{\lambda'}\mu'\overline{\nu'})$, where

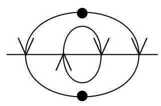
$\underline{\lambda}\mu\overline{\nu} =$  and $\underline{\lambda'}\mu'\overline{\nu'} =$ 

(Note that $\lambda' = \nu$.)
First we compute $\Psi_{\underline{\lambda}\overline{\nu}}(\underline{\lambda}\mu\overline{\nu}) = X_4 \in \mathcal{M}(\underline{\lambda}\overline{\nu})$ and $\Psi_{\underline{\lambda'}\overline{\nu'}}(\underline{\lambda'}\mu'\overline{\nu'}) = 1 \in \mathcal{M}(\underline{\lambda'}\overline{\nu'})$.
Next we need to compute $\chi(X_4 \otimes 1)$. Using the following surgery procedure



we find that $\chi(X_4\otimes 1) = X_4(X_1 - X_2) = X_4(X_4 + X_3) = X_3X_4 \in \mathcal{M}(\underline{\lambda}\overline{\nu'})$. (Note that it is not allowed to do perform the surgeries in a different order, since then the intermediate diagram would not be admissible.)
Finally, we find that

$$(\underline{\lambda}\mu\overline{\nu})\cdot(\underline{\lambda'}\mu'\overline{\nu'}) = \Psi^{-1}_{\underline{\lambda}\overline{\nu'}}(X_3X_4)$$

$=$ 

$$= \underline{\lambda}\mu\overline{\lambda}$$

Note that $\deg(\underline{\lambda}\mu\overline{\nu}) = 3$, $\deg(\underline{\lambda'}\mu'\overline{\nu'}) = 1$, and $\deg((\underline{\lambda}\mu\overline{\nu})\cdot(\underline{\lambda'}\mu'\overline{\nu'})) = 4 = 3 + 1$.

See [ES1, Section 6] for more examples.

Remark 1.3.8. Recently, Ehrig, Tubbenhauer and Wilbert [ETW] introduced a "sign-adjusted version" of the multiplication rule. See Appendix A for a discussion of this sign-adjusted algebra.

1.4 Some properties of the algebra $\mathbb{D}_\Lambda$

Proposition 1.4.1. *The degree 0 basis elements $e_\lambda := \underline{\lambda}\lambda\overline{\lambda}$ form a complete set of pairwise orthogonal primitive idempotents for $\mathbb{D}_\Lambda$.*
Moreover,

$$e_{\lambda'}\cdot(\lambda\mu\nu) = \begin{cases}\lambda\mu\nu, & \text{if } \lambda = \lambda',\\ 0, & \text{otherwise,}\end{cases} \quad \text{and } (\lambda\mu\nu)\cdot e_{\nu'} = \begin{cases}\lambda\mu\nu, & \text{if } \nu = \nu',\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Follows from the multiplication rules, see also [ES1, Theorem 6.2.]. □

Proposition 1.4.2. *The assignment $\underline{\lambda}\mu\overline{\nu} \mapsto \underline{\nu}\mu\overline{\lambda}$ on the basis $\mathbb{B}_\Lambda$ defines a graded algebra anti-automorphism $^*: \mathbb{D}_\Lambda \to \mathbb{D}_\Lambda^{\mathrm{op}}$.*

Proof. It is trivial that * is an isomorphism of graded vector spaces. The fact that it is an algebra map follows easily from the definition of the multiplication. See also [ES1, Corollary 6.4]. $\square$

Intuitively, the map * applied to an oriented circle diagram $a\mu b$ flips the unoriented circle diagram ab upside down, while keeping the weight μ unchanged.
Recall that for every $n \in \mathbb{N}$, there are two n-blocks $\Lambda_n^{\overline{0}}$ and $\Lambda_n^{\overline{1}}$; namely $\Lambda_n^{\overline{0}}$ (resp. $\Lambda_n^{\overline{1}}$) consists of all weights of size n with an even (resp. odd) number of $\vee$'s. It turns out that the algebras $\mathbb{D}_{\Lambda_k^{\overline{0}}}$ and $\mathbb{D}_{\Lambda_k^{\overline{1}}}$ are isomorphic:

Proposition 1.4.3. *Let Λ be a block, and let Λ' be the block whose elements have the same size but opposite parity of $\vee$'s. Then there is an isomorphism of graded algebras $f : \mathbb{D}_\Lambda \to \mathbb{D}_{\Lambda'}$, where $f(a\lambda b)$ is the circle diagram obtained from $a\lambda b$ by flipping the first symbol of λ, and changing in a and b the decoration of the cup/cap/ray incident with the first symbol.*

Proof. It is easy to see that f is an isomorphism of vector spaces (note that the cup/cap/ray incident with the first symbol of λ can always be connected to the left side of the diagram, so that we don't get any issues with admissibility), and that it preserves the grading. So we only need to check that f is compatible with the multiplication.
For $\lambda \in \Lambda$, let $\lambda' \in \Lambda'$ denote the weight obtained by flipping the first symbol. Note that $\overline{\lambda'}$ is obtained from $\overline{\lambda}$ by changing the decoration of the cap/ray incident with the first symbol.
Recalling Definition 1.3.1, there is an isomorphism of vector spaces $g : \mathcal{M}(\lambda\mu) \to \mathcal{M}(\lambda'\mu')$ sending X_1 to $-X_1$ and X_i to X_i for all $i > 1$. Now we will check that in the following diagram, all squares commute:

$$\begin{array}{ccccccc}
{}_\lambda(\mathbb{D}_\Lambda)_\mu \otimes {}_\mu(\mathbb{D}_\Lambda)_\nu & \xrightarrow{\Psi_{\underline{\lambda}\overline{\mu}} \otimes \Psi_{\underline{\mu}\overline{\nu}}} & \mathcal{M}(\underline{\lambda}\overline{\mu}) \otimes \mathcal{M}(\underline{\mu}\overline{\nu}) & \xrightarrow{\chi_{\underline{\lambda}\overline{\mu},\underline{\mu}\overline{\nu}}} & \mathcal{M}(\underline{\lambda}\overline{\nu}) & \xrightarrow{\Psi^{-1}_{\underline{\lambda}\overline{\nu}}} & {}_\lambda(\mathbb{D}_\Lambda)_\nu \\
\downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle g\otimes g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
{}_{\lambda'}(\mathbb{D}_\Lambda)_{\mu'} \otimes {}_{\mu'}(\mathbb{D}_\Lambda)_{\nu'} & \xrightarrow{\Psi_{\underline{\lambda'}\overline{\mu'}} \otimes \Psi_{\underline{\mu'}\overline{\nu'}}} & \mathcal{M}(\underline{\lambda'}\overline{\mu'}) \otimes \mathcal{M}(\underline{\mu'}\overline{\nu'}) & \xrightarrow{\chi_{\underline{\lambda'}\overline{\mu'},\underline{\mu'}\overline{\nu'}}} & \mathcal{M}(\underline{\lambda'}\overline{\nu'}) & \xrightarrow{\Psi^{-1}_{\underline{\lambda'}\overline{\nu'}}} & {}_{\lambda'}(\mathbb{D}_\Lambda)_{\nu'}
\end{array}$$

Indeed: To see that the first and last squares commute, just note that the first vertex can never be the tag of any circle.
Now we need to check that the second square commutes. Start by writing $p \otimes q \in \mathcal{M}(\underline{\lambda}\overline{\mu}) \otimes \mathcal{M}(\underline{\mu}\overline{\nu})$ without any X_1's. We compute $\chi(p \otimes q)$ and $\chi(g(p) \otimes g(q))$ using the same surgery procedure. If no X_1 shows up in the surgery procedure, then clearly $g(\chi(p \otimes q)) = \chi(g(p) \otimes g(q))$. Now we note that a factor X_1 only shows up in the procedure in case the surgery involving the first vertex is a split, and in this case the signs of this factor X_1 in $\chi(p \otimes q)$ and $\chi(g(p) \otimes g(q))$ will be different, since in the one case we have a dotted split and in the other case an undotted split. So also in this case $g(\chi(p \otimes q)) = \chi(g(p) \otimes g(q))$.
So f is indeed compatible with the multiplication. $\square$

The following theorem will imply that $\mathbb{D}_\Lambda$ is a *cellular algebra*, see Subsection 2.2.

Theorem 1.4.4 ([ES1, Theorem 7.1])**.** *Let $(a\lambda b)$ and $(c\mu d)$ be basis vectors of $\mathbb{D}_\Lambda$. Then*

$$(a\lambda b)(c\mu d) = \begin{cases} 0, & \textit{if } b \neq c^*, \\ s_{a\lambda b}(\mu)(a\mu d) + (\dagger), & \textit{if } b = c^* \textit{ and } a\mu \textit{ is oriented,} \\ (\dagger), & \textit{otherwise,} \end{cases}$$

where

1. (†) *denotes a linear combination of basis vectors from* $\mathbb{B}_\Lambda$ *of the form* $(a\nu d)$ *for* $\nu > \mu$.

2. *The scalar* $s_{a\lambda b}(\mu) \in \{0, 1, -1\}$ *depends on* $a\lambda b$ *and* μ *but not on* d.

In particular, the above theorem will allow us to define naturally a family of $\mathbb{D}_\Lambda$-modules, called *cell modules* or *standard modules*. The fact that $s_{a\lambda b}(\mu)$ does not depend on d will be essential for this.

Remark 1.4.5. As noted in [ES1, Remark 7.6], it holds that $s_{a\lambda\overline{\lambda}}(\lambda) = 1$. So we find that $(a\lambda\overline{\lambda}) \cdot (\underline{\lambda}\lambda d) = (a\lambda d)$: the additional terms (†) vanish for degree reasons.

Corollary 1.4.6 ([ES1, Corollary 7.2])**.** *A product* $(a\lambda b)(c\mu d)$ *of two basis vectors is a linear combination of basis vectors* $(a\nu d)$ *with* $\lambda \leq \nu \geq \mu$ *(where* $\leq$ *is the Bruhat order).*

2 Quasi-heredity and cellularity

2.1 Quasi-hereditary algebras

In this subsection we will give a brief overview about quasi-hereditary algebras and highest weight categories. An accessible introduction to the subject can be found in [KK, Section 1]. More about quasi-hereditary algebras can be found in for example [Do, Appendix], or in the original article [CPS] by Cline, Parshall and Scott. Note however that in contrast to all these references, we will be working in a graded setting.
For this subsection, let $A = \bigoplus_{i\leq 0} A_i$ be a positively graded finite-dimensional algebra over a field k. We will assume that $A_0 \cong k \times \cdots \times k$.
Let Λ be an indexing set for the pairwise orthogonal primitive idempotents e_λ, $\lambda \in \Lambda$, and let $\leq$ be a partial order on Λ. If we denote $P(\lambda) = P(\lambda) := Ae_\lambda \in A$-gmod, and $L(\lambda) := Ae_\lambda / A_{>0}e_\lambda \in A$-gmod, then the set of simple modules in A-gmod is given by $\{L(\lambda)\langle j\rangle | \lambda \in \Lambda, j \in \mathbb{Z}\}$, and $P(\lambda)$ is the projective cover of $L(\lambda)$. Note that each $L(\lambda)$ is one-dimensional as a k-vector space, and $A/A_{>0} \cong \bigoplus_{\lambda\in\Lambda} L(\lambda)$ as graded left A-modules.
We will also need the algebra A^{op}; its simple, resp. projective, modules will be denoted by $L^\circ(\lambda)$, resp. $P^\circ(\lambda)$.
We start this section with two equivalent definitions of quasi-heredity. For a proof that the definitions are indeed equivalent, see for example [KK, Proposition 1.2].

Definition 2.1.1 (Cline, Parshall, Scott)**.** We say that $(A, \leq)$ is a *positively graded quasi-hereditary algebra* if and only if for all $\lambda \in \Lambda$ there exist a left module $\Delta(\lambda)$, called *standard module*, such that:

1. There is a surjection $\varphi_\lambda : \Delta(\lambda) \to L(\lambda)$, and the composition factors $L(\mu)\langle j\rangle$ of the kernel satisfy $\mu < \lambda$ (and $j > 0$).

2. There is a surjection $\psi_\lambda : P(\lambda) \to \Delta(\lambda)$ whose kernel is filtered by modules $\Delta(\mu)\langle j\rangle$ with $\mu > \lambda$ (and $j > 0$).

From now on, we will just write "quasi-hereditary algebra" instead of "positively graded quasi-hereditary algebra".
Recall the *Nakayama functor* (sometimes simply called "duality functor") $\mathrm{Hom}_k(\_, k)$, which gives a contravariant equivalence between the categories A-gmod and A^{op}-gmod. It satisfies $\mathrm{Hom}_k(M\langle d\rangle, k) \cong \mathrm{Hom}_k(M, k)\langle -d\rangle$, sends simple modules to simple modules, projectives to injectives, and injectives to projectives. In particular, the A-module $I(\lambda) := \mathrm{Hom}_k(P^\circ(\lambda), k)$ is the injective hull of $L(\lambda)$. Now we are ready to give the second equivalent definition of quasi-heredity, which is dual to the first one:

Definition 2.1.2. We say that $(A, \leq)$ is a *quasi-hereditary algebra* if and only if for all $\lambda \in \Lambda$ there exist a left module $\nabla(\lambda)$, called *costandard module*, such that:

1. There is an injection $\varphi_\lambda : L(\lambda) \to \nabla(\lambda)$, and the composition factors $L(\mu)\langle j\rangle$ of the cokernel satisfy $\mu < \lambda$ (and $j < 0$).

2. There is an injection $\psi_\lambda : \nabla(\lambda) \to I(\lambda)$ whose cokernel is filtered by modules $\nabla(\mu)\langle j\rangle$ with $\mu > \lambda$ (and $j < 0$).

In fact, for a quasi-hereditary algebra A, A^{op} is also quasi-hereditary. We will denote its standard resp. costandard modules by $\Delta^\circ(\lambda)$ resp. $\nabla^\circ(\lambda)$. and it holds that $\nabla(\lambda) \cong \mathrm{Hom}_k(\Delta^\circ(\lambda), k)$.

Remark 2.1.3. If A is a (positively graded) quasi-hereditary algebra, then A-gmod is called a *(positively graded) highest weight category.* One can define highest weight categories in more generality (see for example [CPS]), but we will not need this.

Remark 2.1.4. We can complete the partial order $\leq$ on Λ to a total order $\leq'$. If $(A, \leq)$ is quasi-hereditary, then $(A, \leq')$ is quasi hereditary as well: this follows immediately from the definition, using that $\mu < \lambda$ implies $\mu <' \lambda$. So we can always assume that $\Lambda = \{1, \ldots, n\}$ with the usual ordering. In this case we will write $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ for $1 \leq i \leq n$, and $\varepsilon_{n+1} = 0$.

The following remark collects some basic facts about quasi-hereditary algebras:

Remark 2.1.5. Let $(A, \leq)$ be a quasi-hereditary algebra, with Λ, $\leq$ as in Remark 2.1.4.

- All subalgebras $\varepsilon_l A \varepsilon_l$ are quasi-hereditary as well, w.r.t. the induced ordering of the idempotents, see [Do, Proposition A3.11] or [KK, Corollary 1.3]. (Note that this is not true for arbitrary idempotents.)
- The standard module $\Delta(i)$ is isomorphic to $Ae_i / A\varepsilon_{i+1}Ae_i$ [DK, Definition A.1.1].
- A has finite global dimension, see [Do, Theorem A2.3] or [KK, Theorem 1.4].

Since A is a graded algebra, the Grothendieck group [A-gmod] becomes a (free) $\mathbb{Z}[q^{\pm 1}]$-module when we define $q^d[M] := [M\langle d\rangle]$ for $M \in A$-gmod, $d \in \mathbb{Z}$. It's not hard to see that all of $\{[L(\lambda)] | \lambda \in \Lambda\}$, $\{[\Delta(\lambda)] | \lambda \in \Lambda\}$, $\{[\nabla(\lambda)] | \lambda \in \Lambda\}$, and $\{[P(\lambda)] | \lambda \in \Lambda\}$ are bases of [A-gmod]. We define elements $(P(\lambda) : \Delta(\mu))\,(q) \in \mathbb{Z}[q^{\pm 1}]$ by

$$[P(\lambda)] = \sum_{\mu \in \Lambda} \Big(P(\lambda) : \Delta(\mu)\Big)(q) \cdot [\Delta(\mu)].$$

So $(P(\lambda) : \Delta(\mu))\,(q)$ is the graded multiplicity of $\Delta(\mu)$ in any filtration of $P(\lambda)$ by standard modules (such a filtration exists by Definition 2.1.1 of quasi-heredity). Note that this in particular implies that in each filtration of $P(\lambda)$ by standard modules, the graded multiplicities of the subquotients $\Delta(\mu)$ are the same.
For $X \in A$-gmod, we will write

$$[X : L(\lambda)](q) := \sum_{i \in \mathbb{Z}} [X : L(\lambda)\langle i\rangle] q^i,$$

where $[X : L(\lambda)\langle i\rangle]$ denotes the Jordan-Hölder multiplicity of $L(\lambda)\langle i\rangle$ in a composition series of X. So $[X : L(\lambda)](q)$ is the graded multiplicity of $L(\lambda)$ in a filtration of $P(\mu)$ by standard modules, and we have

$$[X] = \sum_{\lambda \in \Lambda} [X : L(\lambda)](q) \cdot [L(\lambda)].$$

For a proof of the following fact, see for example [Do, Proposition A2.2] or [DK, Corollary A.3.10].

Fact 2.1.6 (Bernstein-Gelfand-Gelfand reciprocity)**.** *For A a quasi-hereditary algebra, it holds that*

$$\Big(P(\lambda) : \Delta(\mu)\Big)(q) = [\nabla(\mu) : L(\lambda)](q^{-1})$$

Remark 2.1.7. Suppose A is equipped with an isomorphism $^* : A \to A^{\mathrm{op}}$ such that $e_\lambda^* = e_\lambda$ for all $\lambda \in \Lambda$. Together with the Nakayama functor, this gives us a contravariant equivalence ("duality") $\mathrm{D} : A\text{-gmod} \to (A\text{-gmod})^{\mathrm{op}}$ satisfying $\mathrm{D}(M\langle d\rangle) \cong \mathrm{D}(M)\langle -d\rangle$. For every $\lambda \in \Lambda$, it holds that sends $\mathrm{D}(L(\lambda)) \cong L(\lambda)$, $\mathrm{D}(P(\lambda)) \cong I(\lambda)$, and $\mathrm{D}(\Delta(\lambda)) \cong \nabla(\lambda)$. We call A a *quasi-hereditary algebra with duality.*

In particular we get that $[\nabla(\mu) : L(\lambda)](q^{-1}) = [\Delta(\mu) : L(\lambda)](q)$, so that our BGG reciprocity becomes

$$\Big(P(\lambda) : \Delta(\mu)\Big)(q) = [\Delta(\mu) : L(\lambda)](q).$$

From now on we will write $d_{\lambda,\mu}(q) := [\Delta(\mu) : L(\lambda)](q) = (P(\lambda) : \Delta(\mu))\,(q)$, and call these polynomials $d_{\lambda,\mu}(q)$ *decomposition numbers.* An immediate corollary of our BGG reciprocity is that

$$[P(\lambda) : L(\nu)](q) = \sum_{\mu\in\Lambda} \Big(P(\lambda) : \Delta(\mu)\Big)(q) \cdot [\Delta(\mu), L(\nu)](q) = \sum_{\mu\in\Lambda} d_{\lambda,\mu}(q) \cdot d_{\nu,\mu}(q).$$

If we introduce the *Cartan matrix*

$$C(q) = \Big([P(\lambda) : L(\mu)](q)\Big)_{\lambda,\mu\in\Lambda}$$

and the *decomposition matrix*

$$D(q) = \Big(d_{\lambda,\mu}(q)\Big)_{\lambda,\mu\in\Lambda}$$

we obtain

$$C(q) = D(q)D(q)^T.$$

2.2 Cellular algebras

The notion of a *cellular algebra* was originally introduced by Graham and Lehrer [GL]. Our algebra $\mathbb{D}_\Lambda$ will turn out to be both cellular and quasi-hereditary; the proof of quasi-heredity of $\mathbb{D}_\Lambda$ relies on the cellularity of $\mathbb{D}_\Lambda$. Later we will also study some subalgebras $\mathbb{H}_\Lambda$ of $\mathbb{D}_\Lambda$ which are no longer quasi-hereditary, but still cellular.

We begin by giving the original definition of cellularity.

Definition 2.2.1 (Graham, Lehrer)**.** An associative k-algebra A is called a *cellular algebra* with *cell datum* $(\Lambda, M, C, ^*)$ if the following conditions are satisfied:

1. Λ is a finite poset, and for each $\lambda \in \Lambda$ we are given a finite set $M(\lambda)$. The algebra A has a k-basis $C^\lambda_{\alpha\beta}$, where (α, β) runs through the set $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

2. * is a k-linear involutive anti-automorphism of A, sending $C^\lambda_{\alpha\beta}$ to $C^\lambda_{\beta\alpha}$.

3. For $x \in A$, $\mu \in \Lambda$, and $\gamma, \delta \in M(\mu)$, the product $xC^\mu_{\gamma\delta}$ can be written as

$$xC^\mu_{\gamma\delta} = \sum_{\gamma'\in M(\mu)} r_x(\gamma', \gamma)C^\mu_{\gamma'\delta} + r'$$

where $r_x(\gamma', \gamma) \in k$ does not depend on δ, and $r' \in A$ is a linear combination of basis elements $C^\nu_{\gamma''\delta''}$ for which $\nu > \mu$.

In the above definition, we didn't assume A to be graded. One can in fact extend the theory of cellular algebras to our graded setting, see [HM, Section 2]:

Definition 2.2.2. A graded k-algebra A is called a *graded cellular algebra* with *graded cell datum* $(\Lambda, M, C, {}^*, \deg)$ if:

1. A is a cellular algebra with cell datum $(\Lambda, M, C, {}^*)$.
2. The elements $C^\lambda_{\alpha\beta}$ are homogeneous.
3. We have $\deg : \coprod_{\lambda\in\Lambda} M(\lambda) \to \mathbb{Z}$ such that $\deg(C^\lambda_{\alpha\beta}) = \deg(\alpha) + \deg(\beta)$ for all $\alpha, \beta \in M(\lambda)$.

Let A be a cellular algebra with cell datum $(\Lambda, M, C, {}^*)$. For every $\lambda \in \Lambda$, we define the *cell module* $W(\lambda)$ with k-basis $\{C_\gamma | \gamma \in M(\lambda)\}$ and A-module structure given by

$$xC_\gamma = \sum_{\gamma'\in\Lambda} r_x(\gamma, \gamma')C_{\gamma'}.$$

In general, for A a cellular algebra, the simple A-modules (and their projective covers) are indexed by a subset Λ_0 of Λ. See [GL, Section 3] for more details.

Remark 2.2.3. The results from Remark 2.1.7 also hold for positively graded cellular algebras: we still have that $P(\mu)$, $\mu \in \Lambda_0$ is filtered by standard modules, so the graded multiplicity of $W(\lambda)$, $\lambda \in \Lambda$ in such a filtration is given by $(P(\lambda) : \Delta(\mu))\,(q)$. We also still have a version of BGG reciprocity [HM, Theorem 2.17]:

$$(P(\lambda) : \Delta(\mu))\,(q) = [\Delta(\mu) : L(\lambda)](q) =: d_{\lambda,\mu}(q).$$

If we define the (graded) Cartan matrix

$$C(q) = ([P(\mu) : L(\nu)](q))_{\mu,\nu\in\Lambda_0}$$

and the decomposition matrix (which is not necessarily a square matrix)

$$D(q) = (d_{\lambda,\mu})_{\lambda\in\Lambda,\mu\in\Lambda_0}$$

then it still holds that

$$C(q) = D(q)D(q)^T.$$

Proposition 2.2.4 ([GL, Remark 3.10])**.** *A cellular algebra A is quasi-hereditary (with duality) if and only if $\Lambda_0 = \Lambda$. In this case the cell modules $W(\lambda)$ coincide with the standard modules $\Delta(\lambda)$.*

2.3 Cellularity and quasi-heredity of $\mathbb{D}_\Lambda$

From Theorem 1.4.4, it easily follows that $\mathbb{D}_\Lambda$ is a cellular algebra:

Theorem 2.3.1. *The algebra $\mathbb{D}_\Lambda$ is a graded cellular algebra with cell datum $(\Lambda, M, C, {}^*, \deg)$, where*

- *For $\lambda \in \Lambda$, $M(\lambda)$ is the set $\{\alpha \in \Lambda | \alpha \subset \lambda\}$. (Recall that $\alpha \subset \lambda$ by definition means that $\underline{\alpha}\lambda$ is an oriented cup diagram.)*

- *For* $\lambda \in \Lambda$ *and* $\alpha, \beta \in M(\lambda)$, $C^{\lambda}_{\alpha\beta} = (\underline{\alpha}\lambda\overline{\beta})$.
- * *is the anti-automorphism from Proposition 1.4.2.*
- *For* $\alpha \in M(\lambda)$, $\deg(\alpha)$ *equals the degree of the oriented cup diagram* $\underline{\alpha}\lambda$.

Proof. Corollary of Theorem 1.4.4, see [ES1, Corollary 7.3] for the details. ☐

We can describe the projective, cell and simple modules explicitly:

Definition 2.3.2. We have, for each $\lambda \in \Lambda$,

- The indecomposable projective module

$$P(\lambda) = \mathbb{D}_\Lambda e_\lambda = \langle \{\underline{\nu}\mu\overline{\lambda} | \nu, \mu \in \Lambda \text{ s.t. } \nu \subset \mu \supset \lambda\} \rangle \ .$$

- The cell module, or standard module, $V(\lambda) := P(\lambda)/U(\lambda)$, where $U(\lambda) \subseteq P(\lambda)$ is the submodule with basis $\{\underline{\nu}\mu\overline{\lambda} | \nu, \mu \in \Lambda \text{ s.t. } \nu \subset \mu \supset \lambda \text{ and } \mu \neq \lambda\}$. (Note that by Corollary 1.4.6, $U(\lambda)$ is indeed a submodule.)
- The simple module $L(\lambda) := P(\lambda)/\mathbb{D}^+_\Lambda e_\lambda$, where $\mathbb{D}^+_\Lambda$ is the sum of all components of strictly positive degree. In other words: $L(\lambda)$ is the 1-dimensional module, concentrated in degree 0, on which e_λ acts by the identity and all other basis elements act trivially.

Now from Proposition 2.2.4 we can immediately deduce:

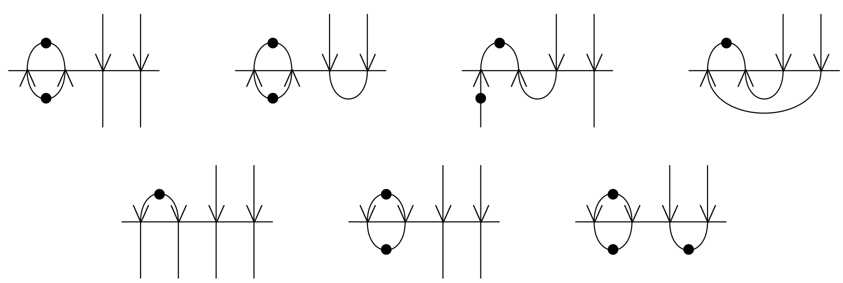
Theorem 2.3.3. $\mathbb{D}_\Lambda$ *is a quasi-hereditary algebra.*

Using Theorem 1.4.4, one can deduce the following (see [ES1, Theorem 8.3, Theorem 8.4] and [BS1, Theorem 5.1, Theorem 5.2]):

Theorem 2.3.4.
1. *For* $\lambda \in \Lambda$, $P(\lambda)$ *has a filtration (as graded* $\mathbb{D}_\Lambda$*-module) with subquotients* $V(\mu)\langle \deg(\mu\overline{\lambda})\rangle$, *where* μ *runs over the set* $\{\mu \in \Lambda | \mu \supset \lambda\}$.
2. *For* $\mu \in \Lambda$, $V(\mu)$ *has a filtration with subquotients* $L(\lambda)\langle \deg(\underline{\lambda}\mu)\rangle$, *where* λ *runs over the set* $\{\lambda \in \Lambda | \lambda \subset \mu\}$.

In fact, in [BS1] and [ES1], quasi-heredity is deduced from the above result, without needing to use Proposition 2.2.4.
Theorem 2.3.4 means that the decomposition numbers $d_{\lambda,\mu}(q)$ (sometimes also called *Kazhdan–Lusztig polynomials*) are given by $d_{\lambda,\mu}(q) = q^{\deg(\underline{\lambda}\mu)}$.

Example 2.3.5. Let Λ be the block containing $\lambda = \wedge \wedge \vee\vee$. The module $P(\lambda) = \mathbb{D}_\Lambda e_\lambda$ has a basis given by the following 7 diagrams:



The three diagrams on the bottom row form a submodule isomorphic with $V(\mu)\langle 1\rangle$, where $\mu = \vee\vee\vee\vee$. Quotienting out this submodule gives the module $V(\lambda)$, with basis given by the four diagrams on the top row. Hence, in the Grothendieck group $[\mathbb{D}_\Lambda\text{-gmod}]$, we have that

$$[P(\lambda)] = [V(\lambda)] + q[V(\mu)].$$

Note that this is consistent with

$$[P(\lambda)] = \sum_{\nu \supset \lambda} q^{(\deg \underline{\lambda}\nu)}[V(\nu)].$$

From the basis of $V(\lambda)$, it is clear that we have

$$[V(\lambda)] = [L(\lambda)] + q[L(\wedge\wedge\wedge\wedge)] + q[L(\wedge\vee\wedge\vee)] + q^2[L(\vee\vee\wedge\wedge)]$$

which agrees with

$$[V(\lambda)] = \sum_{\nu \subset \lambda} q^{(\deg \underline{\nu}\lambda)}[L(\nu)].$$

3 Koszul algebras

In this section, we first recall the notion of a *Koszul algebra*, in the sense of [BGS]. Most of this section is devoted to prove a criterion (Theorem 3.2.2) which is extremely useful for proving that a given quasi-hereditary algebra is Koszul. In the next section, this criterion will be applied to deduce the main result of this thesis: the type D arc algebra $\mathbb{D}_\Lambda$ is Koszul.
The criterion has already been proven by Ágoston, Dlab en Lukácz [ADL] in the more general framework of ungraded algebras. In our graded algebra setting, it is possible to provide a proof which is significantly easier.

3.1 Koszul algebras

Let A be a finite-dimensional positively graded k-algebra (recall that A_0 is assumed to be semisimple). With "A-module" we will always mean "finite-dimensional graded left A-module" unless stated otherwise.
We know that any A-module M admits a *minimal projective resolution*, which is the unique (up to isomorphism) projective resolution

$$\cdots \xrightarrow{d_2} P^2 \xrightarrow{d_1} P^1 \xrightarrow{d_0} P^0 \twoheadrightarrow M$$

for which $d_i P^{i+1} \subseteq A_{>0} P^i$ for all $i \in \mathbb{N}$.

Definition 3.1.1. 1. For $d \in \mathbb{Z}$, let M be a graded A-module generated by its degree d component (i.e. $M = AM_d$). A *linear projective resolution* of M is a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow M$$

such that P^i is generated by its degree $i + d$ component. (In other words, the indecomposable summands of P^i are of the form $P(\lambda)\langle i + d\rangle$ for some $\lambda \in \Lambda$.)
Note that a linear projective resolution is necessarily minimal.
The subcategory of left (resp. right) A-modules whose minimal projective resolution is linear will be denoted by $\mathcal{C}_A$ (resp. $\mathcal{C}_A^\circ$).

2. A is a *Koszul algebra* if every simple A-module has a linear projective resolution, i.e. if $\mathcal{C}_A = A$-gmod.
(Note that this is equivalent to asking that the left A-module $A/A_{>0}$, which we will simply denote by A_0, has a linear projective resolution.)

Remark 3.1.2. Let us briefly recall some properties of Koszul algebras. For more details and background about Koszul algebras, we refer to [BGS].

- A positively graded algebra A is Koszul if and only if for every $k \in \mathbb{Z}$ it holds that $\mathrm{Ext}^t_A(A_0, A_0\langle k\rangle) = 0$ unless $t = k$ [BGS, Proposition 2.1.3]. (Morally speaking, this means that A is as close to semisimple as a positively graded algebra can possibly be.)
- A Koszul algebra A is quadratic [BGS, Proposition 2.3.3], i.e. $A \cong T_{A_0}(A_1)/(R)$ for some $R \subseteq A_1 \otimes_{A_0} A_1$.
- If A is Koszul, then the *quadratic dual* $A^!$ of A is also Koszul [BGS, Proposition 2.9.2], and there is an isomorphism $(A^!)^{\mathrm{op}} = \mathrm{EXT}_A(A_0, A_0)$ [BGS, Theorem 2.10.1]. The algebra $\mathrm{EXT}_A(A_0, A_0)$ is called the *Koszul dual* of A. We can relate the representation theory of a Koszul algebra with that of its Koszul dual, see [BGS, Theorem 2.12.1].

3.2 Standard Koszul algebras

Definition 3.2.1. A positively graded algebra A is called *standard Koszul*, if the following conditions are satisfied:

- A is a (graded) quasi-hereditary algebra.
- A is generated by its degree 0 and 1 parts. In other words, A is generated by A_1 as an A_0-algebra.
- The left and right standard modules $\Delta(\lambda)$ and $\Delta^\circ(\lambda)$ belong to $\mathcal{C}_A$ and $\mathcal{C}_A^\circ$, respectively.

The goal of this section is to prove the following theorem:

Theorem 3.2.2 (Koszulity criterion)**.** *If A is standard Koszul, then A is Koszul.*

In other words: for a quasi-hereditary algebra, in order to prove that every simple module has a linear projective resolution, it suffices to check this for the standard modules instead.
This theorem was already proven by Ágoston, Dlab and Lukács: see [ADL, Theorem 1.4]. We give here a new proof, in more modern language, adapted to our special situation of positively graded algebras with semisimple part $A_0 \cong k \times \cdots \times k$. Our proof is based on the proof from [ADL], but simplifies drastically in our framework.
For the rest of this subsection, we will always assume $A_0 \cong k \times k \times \ldots \times k$. We will need some preparatory lemmas. By Remark 2.1.4 we can assume that $\Lambda = \{1, 2, \ldots n\}$ with the usual ordering, and we write $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ for $i \leq n$.
The proof uses induction on the number of idempotents n. Our first goal will be to show that if A is standard Koszul, so is the subalgebra $\varepsilon_2 A \varepsilon_2$. This will be Corollary 3.2.5. Since $\varepsilon_2 A \varepsilon_2$ has $\{e_2, \ldots, e_n\}$ as complete set of pairwise orthogonal primitive idempotents, this will allow us to apply the induction hypothesis to $\varepsilon_2 A \varepsilon_2$.

Lemma 3.2.3. *If A is standard Koszul, $\varepsilon_i A_{>1} \varepsilon_i \subseteq \varepsilon_i A_{>0} \varepsilon_i A_{>0} \varepsilon_i$ for $1 \leq i \leq n$. In particular, the algebra $\varepsilon_i A \varepsilon_i$ is generated by its degree 0 and 1 parts.*

Proof. It suffices to prove that $e_i A_{>1} e_j \subseteq e_i A_{>0} \varepsilon_{\min(i,j)} A_{>0} e_j$ for all $1 \leq i, j \leq n$. Let $x \in e_i A_{>1} e_j$. Without loss of generality, we can assume that $j \leq i$. (If $i > j$, apply the proof to the quasi-hereditary algebra A^{op}, whose left standard modules correspond to the right standard modules $\Delta^\circ(\lambda)$ of A.)
Note that $x \in P(j)$. We first claim that the surjection $\psi_j : P(j) \to \Delta(j)$ maps x to 0. If $i > j$, this is clear, since $\Delta(j) = Ae_j / A\varepsilon_{j+1} A e_j$. So we can suppose $i = j$, in other words $x = e_j x e_j$. Then $\psi_j(x) \in \Delta(j)$ is of degree greater than 0, so in the kernel K of $\varphi_j : \Delta(j) \to L(j)$. By definition of quasi-heredity, the composition factors of K are isomorphic to $L(p)\langle d \rangle$, for $p < j$. From this it easily seen that $e_j K = 0$. But since $\psi_j(x) = \psi_j(e_j x) = e_j \psi_j(x)$, this implies that $\psi_j(x) = 0$, proving our claim.
Now let

$$\cdots \to \bigoplus_r P(l_r)\langle 1 \rangle \xrightarrow{f} P(j) \twoheadrightarrow \Delta(j)$$

be a (minimal) linear projective resolution of $\Delta(j)$. Since A is quasi-hereditary, $l_r > j$ for every r. On the summand $P(l_r)\langle 1 \rangle$, the map f is given by right multiplication by some $a_r \in e_{l_r} A_1 e_j$. By the claim above, x is in the image of f, i.e. x is a linear combination of elements $e_i b_r e_{l_r} a_r e_j$. Since $a_r \in A_1$ and $x \in A_{>1}$, every b_r must be in $A_{>0}$. So $x \in e_i A_{>0} \varepsilon_j A_{>0} e_j$. □

Lemma 3.2.4. *Suppose A is standard Koszul, and let $X \in \mathcal{C}_A$ such that $\mathrm{Ext}^t_A(X, L(1)\langle k\rangle) = 0$ for all $t \geq 0$, $k \in \mathbb{Z}$. Then $\varepsilon_2 X \in \mathcal{C}_{\varepsilon_2 A \varepsilon_2}$.*

Proof. Without loss of generality, we suppose that $X = AX_0$. Consider the minimal projective resolution of X, which is linear by assumption (i.e. P^i is generated in degree i):

$$\cdots \to P^t \to \cdots \to P^1 \to P^0 \twoheadrightarrow X \ .$$

The assumption $\mathrm{Ext}^t_A(X, L(1)\langle k\rangle) = 0$ means that none of the P^t have a summand isomorphic to $P(1)\langle k\rangle$.
For $X, Y \in A$-gmod, let $\mathrm{HOM}_A(X, Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_A(X\langle i\rangle, Y) \in A$-gmod.
Then $\mathrm{HOM}_A(A\varepsilon_2, -) : A\text{-gmod} \to \varepsilon_2 A\varepsilon_2\text{-gmod}$ is an exact functor (since $A\varepsilon_2$ is projective). Applying it to the above resolution yields the exact sequence of graded $\varepsilon_2 A \varepsilon_2$-modules

$$\cdots \to \varepsilon_2 P^t \to \cdots \to \varepsilon_2 P^1 \to \varepsilon_2 P^0 \twoheadrightarrow \varepsilon_2 X \ .$$

Since all summands of P^t are (up to degree shift) isomorphic to $P(i)$ for $i > 1$, $\varepsilon_2 P^t$ is a projective $\varepsilon_2 A\varepsilon_2$-module generated in degree t. So $\varepsilon_2 X$ has a linear projective resolution, i.e. $\varepsilon_2 X \in \mathcal{C}_{\varepsilon_2 A\varepsilon_2}$. □

Corollary 3.2.5. *If A is standard Koszul, then so is the subalgebra $\varepsilon_2 A\varepsilon_2$ of A.*

Proof. It is trivial that $\varepsilon_2 A \varepsilon_2$ is again a positively graded algebra. It is generated in degree 0 and 1 by Lemma 3.2.3, and quasi-hereditary by Remark 2.1.5.
Note that the left and right standard modules of $\varepsilon_2 A\varepsilon_2$ are given by $\varepsilon_2\Delta(i)$ and $\Delta(i)^\circ\varepsilon_2$ for $2 \leq i \leq n$, respectively. We need to show that they are contained in $\mathcal{C}_{\varepsilon_2 A\varepsilon_2}$ and $\mathcal{C}^\circ_{\varepsilon_2 A\varepsilon_2}$, respectively. This follows from Lemma 3.2.4, where the assumption $\mathrm{Ext}^t_A(\Delta(i), L(1)\langle k\rangle) = 0$ is satisfied because A is quasi-hereditary. □

Definition 3.2.6. Let $X \in A$-gmod, and suppose $X = AX_d$. The module X is called *$L(1)$-Koszul*, if $\mathrm{Ext}^t_A(X, L(1)\langle k\rangle) = 0$ unless $t = k - d$.

Note that X is $L(1)$-Koszul if and only if in the minimal projective resolution $\cdots \to P^t \to P^{t-1} \to \cdots$ of X, every summand of P^t that is of the form $P(1)\langle k\rangle$ has $t = k - d$.

Definition 3.2.7. Let κ be the subclass of A-gmod consisting of modules X that satisfy the following properties:

- $X = AX_d$ for some $d \in \mathbb{Z}$ (i.e. X is generated in degree d).
- $A\varepsilon_2 X$ is also generated in degree d, i.e. $A\varepsilon_2 X = A\varepsilon_2 X_d$.
- X is $L(1)$-Koszul.
- $\varepsilon_2 X \in \varepsilon_2 A\varepsilon_2$-gmod has a linear projective resolution.

Now to prove Theorem 3.2.2, we will first prove that every module in κ has a linear projective resolution, and then conclude by showing that all the simple modules $L(i)$ are in fact in κ.

Lemma 3.2.8. *Every module in κ has a linear projective resolution.*

Proof. To every $X \in \kappa$, we attach the following integers:

$$e_X := \dim_k \left(\bigoplus_{t \geq 0, k \in \mathbb{Z}} \operatorname{Ext}^t_A(X, L(1)\langle t \rangle) \right) \qquad \text{and} \qquad p_X = \operatorname{pdim}(X)$$

(Note that both these numbers are finite since A has finite global dimension.) We order the pairs (e_X, p_X) lexicographically, i.e. $(e_Y, p_Y) \leq (e_X, p_X)$ if and only if either $e_Y < e_X$, or $e_Y = e_X$ and $p_Y \leq p_X$. We will prove the theorem by induction w.r.t. this well-ordering.
Note that if X satisfies $(e_X, p_X) = (0, 0)$, there is nothing to prove. So suppose we have proven the lemma already for all $Y \in \kappa$ satisfying $(e_Y, p_Y) < (e_X, p_X)$. We want to show that then X has a linear projective resolution as well.
We distinguish two cases: first we suppose that $A\varepsilon_2 X \neq X$.

Claim 3.2.9. *$A\varepsilon_2 X \in \kappa$, and $(e_{A\varepsilon_2 X}, p_{A\varepsilon_2 X}) < (e_X, p_X)$.*

The proof of the claim will be given later.
Since A is quasi-hereditary, it holds that $A_{>0} \subseteq A\varepsilon_2 A$. (Indeed: else we would find an element $e_1 a e_1 \in A_1$, but then the kernel of $P(1) \mapsto \Delta(1) \cong L(1)$ would have a $\Delta(1)\langle 1 \rangle$ in its standard filtration, contradicting quasi-heredity.) So $A_{>0} X \subseteq A\varepsilon_2 X$. This means that the quotient $X/A\varepsilon_2 X$ is concentrated in degree d, hence of the form $\bigoplus L(1)\langle d \rangle$. So we have a short exact sequence

$$0 \to A\varepsilon_2 X \to X \to \bigoplus L(1)\langle d \rangle \to 0.$$

By the above claim and the induction hypothesis, $A\varepsilon_2 X$ has a linear projective resolution. By assumption, $L(1) \cong \Delta(1)$ has a linear projective resolution. Then applying the horseshoe lemma [Wei, Lemma 2.2.8] yields a linear projective resolution for X.
Now we turn to the case $X = A\varepsilon_2 X$. For simplicity, we suppose $d = 0$ (i.e. $X = AX_0$).
We have a short exact sequence

$$0 \to \Omega \to P \to X \to 0,$$

where P is the projective cover of X, and Ω is the first syzygy. Note that by our assumption $X = A\varepsilon_2 X$, P has no summands of the form $P(1)\langle k \rangle$.

Claim 3.2.10. *$\Omega \in \kappa$, and $(e_\Omega, p_\Omega) < (e_X, p_X)$.*

The proof of this claim will be postponed as well.
Now Ω has a linear projective resolution by the induction hypothesis. Since Ω is generated in degree 1 (see below), it follows that X has a linear projective resolution. □

We still need to provide proofs for the two claims we made:

Proof of Claim 3.2.9. We first check the four properties of the definition of κ: The first 2 properties are trivial (note that $A\varepsilon_2 A\varepsilon_2 X = A\varepsilon_2 X$). To see that $A\varepsilon_2 X$ is $L(1)$-Koszul, note that the short exact sequence

$$0 \to A\varepsilon_2 X \to X \to \bigoplus L(1)\langle d \rangle \to 0$$

induces the long exact sequence

$$\cdots \to \operatorname{Ext}^t_A(\bigoplus L(1)\langle d \rangle, L(1)\langle k \rangle) \to \operatorname{Ext}^t_A(X, L(1)\langle k \rangle) \to \operatorname{Ext}^t_A(A\varepsilon_2 X, L(1)\langle k \rangle) \to$$
$$\operatorname{Ext}^{t+1}_A(\bigoplus L(1)\langle d \rangle, L(1)\langle k \rangle) \to \cdots .$$

Since A is quasi-hereditary, $\mathrm{Ext}^t_A(L(1)\langle l\rangle, L(1)\langle k\rangle) = 0$ for all $t \geq 0$ (this follows from [Do, Proposition A.2.2(ii)]). Together with the assumption that X is $L(1)$-Koszul, it follows that most terms in the above long exact sequence vanish, so that also $A\varepsilon_2 X$ is Koszul.
The last property is trivial, since $\varepsilon_2 A\varepsilon_2 X = \varepsilon_2 X$.
To see that $(e_{A\varepsilon_2 X}, p_{A\varepsilon_2 X}) < (e_X, p_X)$, note that the long exact sequence above yields an isomorphism $\mathrm{Ext}^t_A(X, L(1)\langle k\rangle) \cong \mathrm{Ext}^t_A(A\varepsilon_2 X, L(1)\langle k\rangle)$ for every $t \geq 1$.
Since $\mathrm{Hom}_A(A\varepsilon_2 X, L(1)\langle k\rangle) = 0$ while $\mathrm{Hom}_A(X, L(1)\langle k\rangle) \neq 0$, we find that $e_{A\varepsilon_2 X} < e_X$. □

Proof of Claim 3.2.10. To see that $(e_\Omega, p_\Omega) < (e_X, p_X)$, we simply note that $e_\Omega = e_X$ and $p_\Omega < p_X$.
Next, we check the four properties of the definition of κ:

- $\varepsilon_2\Omega$ has a linear projective resolution: applying $\mathrm{HOM}_A(A\varepsilon_2, -)$ to the short exact sequence $0 \to \Omega \to P \to X \to 0$ yields the short exact sequence

$$0 \to \varepsilon_2\Omega \to \varepsilon_2 P \to \varepsilon_2 X \to 0,$$

  where $\varepsilon_2 P$ is projective by our assumption $X = A\varepsilon_2 X$. So $\varepsilon_2\Omega$ is the first syzygy of $\varepsilon_2 X$. Since $\varepsilon_2 X$ has a linear projective resolution, so does $\varepsilon_2\Omega$.

- Note that the above also implies that $\varepsilon_2\Omega$ is generated in degree 1. So $A\varepsilon_2\Omega$ is generated in degree 1.

- Ω is generated in degree 1: consider the projective cover $P' \to \Omega$, then we need to show that every summand of P' is generated in degree 1. For the summands of the from $P(i)\langle k\rangle$ with $i \geq 2$, this follows from the fact that $A\varepsilon_2\Omega$ is generated in degree 1. For the summands of the from $P(1)\langle k\rangle$, it follows from $L(1)$-koszulity of X.

- Ω is $L(1)$-Koszul: this follows from the $L(1)$-koszulity of X, using that Ω is generated in degree 1. □

This establishes Lemma 3.2.8. Now we will finish the proof of the main theorem of this section.

Proof of Theorem 3.2.2. We use induction on the number n of idempotents. If $n = 1$, there is nothing to prove, so suppose $n > 1$. By Lemma 3.2.5 and the induction hypothesis, the subalgebra $\varepsilon_2 A\varepsilon_2$ is Koszul.
By Lemma 3.2.8, it suffices to show that $A_0 \in \kappa$.
A_0 and $A\varepsilon_2 A_0$ are obviously generated in degree 0, and $\varepsilon_2 A_0 \in \varepsilon_2 A\varepsilon_2$-gmod has a linear projective resolution since $\varepsilon_2 A\varepsilon_2$ is Koszul. So the only thing left to check is that A_0 is $L(1)$-Koszul.
By assumption, the simple right module $L(1)^\circ \in \mathrm{gmod}\text{-}A$ has a linear projective resolution. This implies that for every simple module $L(i)^\circ$, $\mathrm{Ext}^t_{A^{\mathrm{op}}}(L(1)^\circ, L(i)^\circ\langle k\rangle) = 0$ unless $t = k$. Applying the dualizing functor $\mathrm{Hom}_k(-, k) : \mathrm{gmod}\text{-}A \to A\text{-gmod}$, we get that $\mathrm{Ext}^t_A(L(i)\langle -k\rangle, L(1)) = 0$ unless $t = k$. Since A_0 is the direct sum of all simple modules, it follows that $\mathrm{Ext}^t_A(A_0, L(1)\langle k\rangle) = 0$ unless $t = k$. So A_0 is $L(1)$-Koszul. □

Corollary 3.2.11. *By Corollary 3.2.5, we find that if A is standard Koszul, every subalgebra $\varepsilon_i A\varepsilon_i$ is Koszul.*

4 Koszulity of $\mathbb{D}_\Lambda$

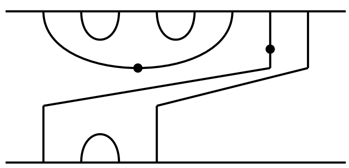
The goal of this section is to give a proof that the algebra $\mathbb{D}_\Lambda$ is Koszul in the sense of the previous section. Brundan and Stroppel proved this for the type A arc algebra (see [BS2, Theorem 5.6]); our proof for type D is inspired by their methods.
We first need to define *geometric bimodules* and *projective functors*. For 2 fixed blocks Λ and Γ, we will construct *geometric* $\mathbb{D}_\Lambda$-$\mathbb{D}_\Gamma$-*bimodules* $K^t_{\Lambda\Gamma}$. The elements of $K^t_{\Lambda\Gamma}$ will be called *stacked circle diagrams*, and the actions of Λ and Γ on $K^t_{\Lambda\Gamma}$ will be a generalization of the multiplication law from Subsection 1.3. Given a geometric bimodule $K^t_{\Lambda\Gamma} \in \mathbb{D}_\Lambda$-gmod-$\mathbb{D}_\Gamma$, we can define the *projective functor* $G^t_{\Lambda\Gamma} : \mathbb{D}_\Gamma\text{-gmod} \to \mathbb{D}_\Lambda\text{-gmod}$ by taking the tensor product with $K^t_{\Lambda\Gamma}$.
The proof of Koszulity of $\mathbb{D}_\Lambda$ will go by induction on the size of the block Λ: using the projective functor $G^t_{\Lambda\Gamma}$, we will be able to deduce Koszulity of $\mathbb{D}_\Lambda$ from the Koszulity of $\mathbb{D}_\Gamma$, where Γ is some block of shorter size. The criterion from the previous section (Theorem 3.2.2) will play an essential role in our proof.

4.1 Geometric bimodules

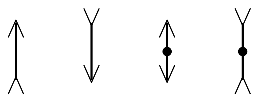
In this section, we will define the so-called *geometric bimodules* for our algebra $\mathbb{D}_\Lambda$. Geometric bimodules for the type A arc algebra were introduced by Khovanov in [Kho], and play a prominent role in [BS2], in particular, in the proof that the type A arc algebra is Koszul. This is precisely our reason for defining geometric bimodules in type D: we will need them in our proof that $\mathbb{D}_\Lambda$ is Koszul.
For this subsection we will fix an n-block Λ and an m-block Γ (for some $n, m \in \mathbb{N}$).

Definition 4.1.1. A *crossingless matching* t is a a diagram obtained by drawing a cap diagram c under a cup diagram d with the same number of rays, and connecting the rays of c with the rays of d with line segments via an order-preserving bijection. We will assume that every line segment has at most one dot. A crossingless matching t is called *admissible* if c and d are admissible. From now on we assume that all occurring crossingless matchings are admissible, unless stated otherwise.
A crossingless matching where the bottom line has n endpoints and the top line has m endpoints will be referred to as an *n-m-matching*.

Example 4.1.2. The following figure shows a 4-8-matching obtained by drawing a cap diagram of size 4 under a cup diagram of size 8.



Definition 4.1.3. • An *oriented* $\Lambda\Gamma$-*matching* $\lambda t \mu$ consists of a crossingless matching t whose bottom line resp. top line are decorated with weights $\lambda \in \Lambda$ resp. $\mu \in \Gamma$, such that all cups and caps are oriented as in an oriented cup/cap diagram (see Subsection 1.2, Figures (1.1) and (1.2)), and all line segments are oriented as in



(An intuitive way to think about this is that at a dot, the orientation of an arc reverses.)

- An *unoriented stacked circle diagram* is a diagram atb obtained by stacking a cup diagram a with n endpoints, an n-m-matching t, and a cap diagram with m endpoints.
- An *oriented stacked circle diagram* is a diagram $a\lambda t\mu b$, with $\lambda \in \Lambda$, $\mu \in \Gamma$, and unoriented stacked circle diagram, such that $a\lambda$ is an oriented cup diagram, $\lambda t\mu$ is an oriented $\Lambda\Gamma$-matching, and μb is an oriented cap diagram.
  In this case we call (λ, μ) an *orientation* of the unoriented stacked circle diagram atb.
- The *degree* $\deg(\lambda t\mu)$ of an oriented $\Lambda\Gamma$-matching $\lambda t\mu$ is given by the number of clockwise cups plus the number of clockwise caps. The *degree* of an oriented stacked circle diagram $a\lambda t\mu b$ is defined by $\deg(a\lambda t\mu b) = \deg(a\lambda) + \deg(\lambda t\mu) + \deg(\mu b)$, which is just the total number of clockwise cups and caps in the diagram.

Remark 4.1.4. More generally, one can define *stacked circle diagrams of height k* by stacking a cup diagram a, k crossingless matchings $t_1, \ldots, t_k$, and a cap diagram b. With "stacked circle diagram" we will always mean a stacked circle diagram of height 1, unless stated otherwise.

For stacked circle diagrams, we can define circles, lines and propagating lines as we did for circle diagrams. A circle (resp. line) that does not intersect the lower number line will be called an *upper circle* (resp. *upper line*).
We also have the analogue of Lemma 1.2.14:

Lemma 4.1.5. *Let atb be an unoriented stacked circle diagram. Then the following holds:*

1. *The diagram atb has an orientation if and only if the number of dots is even on each of its circles and its propagating lines, and odd on each of its non-propagating lines.*
2. *In this case there are exactly 2^c possible orientations, where c is the number of circles. They are obtained by choosing for each of the circles one out of its two possible orientations and for each line the unique possible orientation.*

Proof. The proof is similar to the proof of Lemma 1.2.14 and will be omitted. See also [ES1, Proposition 4.8]. □

For C a circle in a stacked circle diagram $a\lambda t\mu b$, the rightmost symbol of C contained in λ and the rightmost symbol of C contained in μ are connected by an undotted line, so they are the same. So also in stacked circle diagrams it makes sense to talk about counterclockwise (rightmost symbols are $\wedge$) and clockwise (rightmost symbols are $\vee$) circles.
Similar to Propositions 1.2.12 and 1.2.13, we can describe the contribution of a connected component (circle or line) to the degree of an oriented stacked circle diagram.

Proposition 4.1.6. *For C a connected component of some circle diagram $a\lambda t\mu b$, let* $\mathrm{cups}(C)$ *resp.* $\mathrm{caps}(C)$ *denote the number of cups resp. caps in C.*

1. *If C is a counterclockwise circle, then* $\mathrm{caps}(C) = \mathrm{cups}(C)$*, and the contribution of C to the degree of $a\lambda t\mu b$ equals* $\mathrm{caps}(C) - 1$.
2. *If C is a clockwise circle, then* $\mathrm{caps}(C) = \mathrm{cups}(C)$*, and the contribution of C to the degree of $a\lambda t\mu b$ equals* $\mathrm{caps}(C) + 1$.

3. *If L is a propagating line, then* $\mathrm{caps}(L) = \mathrm{cups}(L)$*, and the contribution of L to the degree of $a\lambda t\mu b$ equals* $\mathrm{caps}(L)$*.*

4. *If L is a nonpropagating line and its 2 endpoints are in a, then* $\mathrm{caps}(L) = \mathrm{cups}(L) + 1$*, and the contribution of L to the degree of $a\lambda t\mu b$ equals* $\mathrm{caps}(L)$*.*

5. *If L is nonpropagating line and its 2 endpoints are in b, then* $\mathrm{cups}(L) = \mathrm{caps}(L) + 1$*, and the contribution of L to the degree of $a\lambda t\mu b$ equals* $\mathrm{cups}(L)$*.*

Proof. Similar to the proof of [ES1, Proposition 4.9]. □

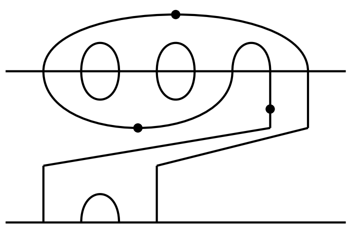
Definition 4.1.7. Let t be a crossingless n-m-matching. We define $K^t_{\Lambda\Gamma}$ to be the graded vector space with basis

$$\{a\lambda t\mu b | \lambda \in \Lambda, \mu \in \Gamma, a\lambda t\mu b \text{ is an oriented stacked circle diagram}\}.$$

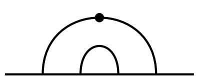
The vector space $K^t_{\Lambda\Gamma}$ can be given the structure of a graded $\mathbb{D}_\Lambda$-$\mathbb{D}_\Gamma$-bimodule. The action of $\mathbb{D}_\Lambda$ is similar to the multiplication of circle diagrams: to multiply $a\lambda b \in \mathbb{D}_\Lambda$ with $c\mu t\nu d \in K^t_{\Lambda\Gamma}$, we draw the first diagram under the second one and perform surgeries. We will make this more precise in what follows.

Definition 4.1.8. Let t be a crossingless $\Lambda\Gamma$-matching, and $\mu \in \Gamma$. We can consider the diagram $t\overline{\mu}$ obtained by drawing $\overline{\mu}$ on top of t. The *upper reduction* $\mathrm{red}(t\overline{\mu})$ of the diagram $t\overline{\mu}$, is a cap diagram (of size n) obtained in the following way:
Start with the diagram $t\overline{\mu}$, and remove the upper number line. Next, remove all connected components that don't touch the lower number line. Finally, remove from every component an even number of dots, so that in the end every cap and every ray has at most one dot.
We define the upper reduction of an unoriented stacked circle diagram by $\mathrm{red}(atb) = a\,\mathrm{red}(tb)$ (this is an unoriented circle diagram). The upper reduction of an oriented stacked circle diagram is $\mathrm{red}(a\lambda t\mu b) = a\lambda\,\mathrm{red}(tb)$ (an oriented circle diagram).

Example 4.1.9. The upper reduction of the diagram



is the diagram

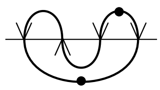
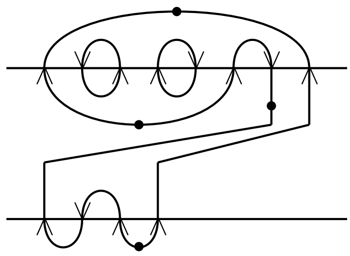


.

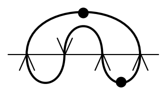
We can now define the left action of $\mathbb{D}_\Lambda$ on $K^t_{\Lambda\Gamma}$.

Definition 4.1.10. Let $a\lambda b \in \mathbb{D}_\Lambda$ and $c\mu t\nu d \in K^t_{\Lambda\Gamma}$. If $c \neq b^*$, the product $(a\lambda b) \cdot (c\mu t\nu d)$ is defined to be 0. So suppose $c = b^*$.
Compute the product $(a\lambda b) \cdot \mathrm{red}(c\mu t\nu d) = \sum_i \pm(a\lambda^i d)$ in $\mathbb{D}_\Lambda$. Then $(a\lambda b) \cdot (c\mu t\nu d)$ is defined to be the sum $\sum_i \pm(a\lambda^i t\nu^i d)$, where the weight $\nu^i \in \Gamma$ is defined as follows:
The symbols of ν^i lying on components that touch the bottom number line are uniquely determined by λ^i. The symbols that lie on a component not touching the bottom number line are defined to be the same as in ν.

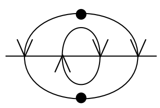
Example 4.1.11. Suppose we want to compute the product $(a\lambda b) \cdot (c\mu t\nu d)$, where

$a\lambda b =$  and $c\mu t\nu d =$  .

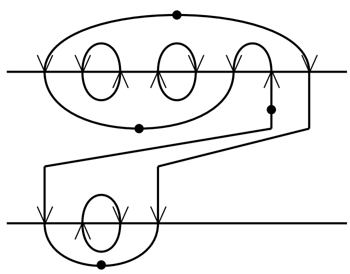
We find that

$\mathrm{red}(c\mu t\nu d) =$  .

Now we compute the product (see Example 1.3.7)

$(a\lambda b) \cdot \mathrm{red}(c\mu t\nu d) =$ 

and conclude that

$(a\lambda b) \cdot (c\mu t\nu d) =$  .

The right action of $\mathbb{D}_\Gamma$ on $K^t_{\Lambda\Gamma}$ can be defined similarly to the above. Alternatively, we can use the anti-automorphism $^* : \mathbb{D}_\Gamma \to \mathbb{D}_\Gamma^{\mathrm{op}}$ from Proposition 1.4.2:

Definition 4.1.12. Let $a\mu t\nu b \in K^t_{\Lambda\Gamma}$ and $c\lambda d \in \mathbb{D}_\Gamma$.
We define $(a\mu t\nu b) \cdot (c\lambda d) = ((c\lambda d)^* \cdot (a\mu t\nu b)^*)^*$. (Here $(a\mu t\nu b)^* := b^*\nu t^*\mu a^* \in K^{t^*}_{\Gamma\Lambda}$, where t^* is the m-n-matching obtained by vertically reflecting t.)

Proposition 4.1.13. *The above actions give $K^t_{\Lambda\Gamma}$ the structure of a graded $\mathbb{D}_\Lambda$-$\mathbb{D}_\Gamma$-bimodule.*

Proof. Follows from the definition and the associativity of $\mathbb{D}_\Lambda$. $\square$

We have the analogue of Theorem 1.4.4 for geometric bimodules.

Theorem 4.1.14. *Let $(a\lambda b) \in \mathbb{D}_\Lambda$ and $(c\mu t\nu d) \in K^t_{\Lambda\Gamma}$. Then,*

$$(a\lambda b)(c\mu t\nu d) = \begin{cases} 0, & \text{if } b \neq c^*, \\ s_{a\lambda b}(\mu)(a\mu t\nu d) + (\dagger), & \text{if } b = c^* \text{ and } a\mu \text{ is oriented}, \\ (\dagger), & \text{otherwise}, \end{cases}$$

where

1. *$(\dagger)$ denotes a linear combination of basis vectors from $\mathbb{B}_\Lambda$ of the form $(a\mu' t\nu' d)$ for $\mu' > \mu$.*
2. *The scalar $s_{a\lambda b}(\mu) \in \{0, 1, -1\}$ is the same as in Theorem 1.4.4.*

Proposition 4.1.15. *The left action of $\mathbb{D}_\Lambda$ has the following properties:*

- *$e_\gamma(\underline{\gamma}\mu t\nu d) = \underline{\gamma}\mu t\nu d$.*
- *$(a\lambda b)(b\mu t\nu d)$ is a linear combination of basis vectors $a\mu' t\nu' d$ (for varying μ' and ν'). Moreover, these μ' and ν' satisfy $\lambda \leq \mu' \geq \mu$ and $\nu' \geq \nu$.*
- *$(a\gamma\overline{\gamma})(\underline{\gamma}\gamma t\nu d) = (a\gamma t\nu d)$.*

Similar statements hold for the right action.

Proof. The first statement follows immediately from the definition of the action of $\mathbb{D}_\Lambda$. The second statement is a corollary of Theorem 4.1.14, and the third statement follows from Remark 1.4.5. $\square$

Remark 4.1.16. If $\Lambda = \Gamma$ and t is the Λ-Γ-matching containing only vertical lines, then it is easily seen that $K^t_{\Lambda\Gamma} \cong \mathbb{D}_\Lambda$ as $\mathbb{D}_\Lambda$-bimodules.

4.2 Projective functors

As in the previous subsection, fix an n-block Λ and an m-block Γ. For a crossingless n-m-matching t, let caps(t) resp. cups(t) denote the number of caps resp. cups in the diagram.

Definition 4.2.1. For t a crossingless n-m-matching, we define the *projective functor*

$$G^t_{\Lambda\Gamma} := K^t_{\Lambda\Gamma}\langle -\operatorname{caps}(t)\rangle \otimes_{\mathbb{D}_\Gamma} \_ : \mathbb{D}_\Gamma\text{-gmod} \to \mathbb{D}_\Lambda\text{-gmod}\ .$$

Let R be the graded k-vector space $k[x]/(x^2)$, with 1 in degree -1 and x in degree 1.
The next result is similar to [BS2, Theorem 4.2]. It describes how $G^t_{\Lambda\Gamma}$ acts on the projective Γ-modules $P(\gamma)$, $\gamma \in \Gamma$.

Theorem 4.2.2.

1. *$G^t_{\Lambda\Gamma}P(\gamma) \cong K^t_{\Lambda\Gamma}e_\gamma\langle -\operatorname{caps}(t)\rangle$ (as left $\mathbb{D}_\Lambda$-modules).*
2. *If in $t\overline{\gamma}$ there is an upper line with an even number of dots, or an upper circle with an odd number of dots, $G^t_{\Lambda\Gamma}P(\gamma) = 0$. (Recall that an* upper line*, resp.* upper circle*, is a line, resp. circle, that does not cross the lower number line.)*

3. *If every upper line (resp. upper circle) in $t\overline{\gamma}$ has an odd (resp. even) number of dots, define $\lambda \in \Lambda$ by declaring that $\overline{\lambda} = \mathrm{red}(t\overline{\gamma})$ (the upper reduction of $t\overline{\gamma}$), and let k be the number of upper circles removed in the reduction process. Then*

$$G^t_{\Lambda\Gamma} P(\gamma) \cong P(\lambda) \otimes_k R^{\otimes k} \langle \mathrm{cups}(t) - \mathrm{caps}(t) \rangle.$$

Proof. Note that

$$\begin{aligned} G^t_{\Lambda\Gamma} P(\gamma) &= K^t_{\Lambda\Gamma} \langle -\mathrm{caps}(t) \rangle \otimes_{\mathbb{D}_\Lambda} P(\gamma) \\ &= K^t_{\Lambda\Gamma} \otimes_{\mathbb{D}_\Lambda} \mathbb{D}_\Lambda e_\gamma \langle -\mathrm{caps}(t) \rangle \\ &\cong K^t_{\Lambda\Gamma} e_\gamma \langle -\mathrm{caps}(t) \rangle \ . \end{aligned}$$

Note that $K^t_{\Lambda\Gamma} e_\gamma \langle -\mathrm{caps}(t) \rangle$ has a basis given by the diagrams in $K^t_{\Lambda\Gamma}$ of the form $a\mu t\nu\overline{\gamma}$. So $G^t_{\Lambda\Gamma} P(\gamma)$ has a basis given by the diagrams in $K^t_{\Lambda\Gamma} \otimes_{\mathbb{D}_\Lambda} \mathbb{D}_\Lambda$ of the form $(a\mu t\nu\overline{\gamma}) \otimes e_\gamma$.
Suppose $t\overline{\gamma}$ has an upper line with an even number of dots. Since all diagrams $at\overline{\gamma}$ contain a non-propagating line with an even number of dots, none of them can be oriented (see Lemma 4.1.5). Hence $G^t_{\Lambda\Gamma} P(\gamma) = 0$. A similar argument proves that if $t\overline{\gamma}$ has an upper circle with an odd number of dots, $G^t_{\Lambda\Gamma} P(\gamma) = 0$.
Now suppose every upper line (resp. circle) in $t\overline{\gamma}$ has an odd (resp. even) number of dots. Enumerate the k upper circles in the diagram $t\overline{\gamma}$ in some fixed order. Consider the map

$$f : K^t_{\Lambda\Gamma} e_\gamma \to \mathbb{D}_\Lambda e_\lambda \otimes R^{\otimes k} : (a\mu t\nu\overline{\gamma}) \mapsto (a\mu\overline{\lambda}) \otimes x_1 \otimes \cdots \otimes x_k$$

where x_i is 1 if the ith upper circle of $t\overline{\gamma}$ is counterclockwise in $a\mu t\nu\overline{\gamma}$, and x if this circle is clockwise. It follows from the definition of $K^t_{\Lambda\Gamma}$ that f is a morphism of ungraded left $\mathbb{D}_\Lambda$-modules. Using Lemmas 1.2.14 and 4.1.5, we see that giving an orientation of $at\overline{\gamma}$ corresponds to giving an orientation of $a\overline{\lambda}$ together with an orientation of the k circles that that were removed in the reduction process. Therefore f is in fact an isomorphism of left (ungraded) $\mathbb{D}_\Lambda$-modules. We will now check that f is homogeneous of degree $(-\mathrm{cups}(t))$. By Proposition 4.1.6 an upper line in $a\mu t\nu\overline{\gamma}$ containing s cups will always contribute exactly s to the degree of $a\mu t\nu\overline{\gamma}$. The ith upper circle in $a\mu t\nu\overline{\gamma}$ containing s cups will, depending on its orientation, contribute $s+1$ or $s-1$ to the degree of $a\mu t\nu\overline{\gamma}$. The corresponding factor x_i in $f(a\mu t\nu\overline{\gamma})$ will contribute $+1$ or -1 to the degree of $f(a\mu t\nu\overline{\gamma})$, depending on the orientation of the ith upper circle. Let C be a component of $a\mu t\nu\overline{\gamma}$ which is not an upper line or circle, and C' the corresponding component of $a\mu\overline{\lambda}$. C and C' have the same orientation, and using Proposition 4.1.6, we see that the difference in degree is precisely the number of cups of t contained in C.
Putting all this together, we find that $\deg(f(a\mu t\nu\overline{\gamma})) = \deg(a\mu t\nu\overline{\gamma}) - \mathrm{cups}(t)$. Since $P(\lambda) = \mathbb{D}_\Lambda e_\lambda$ and $G^t_{\Lambda\Gamma} P(\gamma) \cong K^t_{\Lambda\Gamma} e_\gamma \langle -\mathrm{caps}(t) \rangle$, our theorem is proven. □

Corollary 4.2.3. *$K^t_{\Lambda\Gamma}$ is projective both as a left $\mathbb{D}_\Lambda$-module and as a right $\mathbb{D}_\Gamma$-module. Hence $G^t_{\Lambda\Gamma}$ is an exact functor.*

Proof. We have that $K^t_{\Lambda\Gamma} \cong \bigoplus_{\gamma \in \Gamma} K^t_{\Lambda\Gamma} e_\gamma$ as left $\mathbb{D}_\Lambda$-modules. By Theorem 4.2.2, every summand is projective; hence $K^t_{\Lambda\Gamma}$ is projective as a left $\mathbb{D}_\Lambda$-module. By the same argument, $K^{t^*}_{\Gamma\Lambda}$ is projective as a left $\mathbb{D}_\Gamma$-module. Twisting with the anti-automorphism $^* : \mathbb{D}_\Gamma \to \mathbb{D}_\Gamma^{\mathrm{op}}$, we find that $K^t_{\Lambda\Gamma}$ is projective as a right $\mathbb{D}_\Gamma$-module. Now it immediately follows that the functor $G^t_{\Lambda\Gamma} := K^t_{\Lambda\Gamma} \langle -\mathrm{caps}(t) \rangle \otimes_{\mathbb{D}_\Gamma} \_$ is exact. □

The next theorem is similar to [BS2, Theorem 4.5] and will be crucial in our proof that $\mathbb{D}_\Lambda$ is a Koszul algebra. It describes how $G^t_{\Lambda\Gamma}$ acts on the standard Γ-modules $V(\gamma)$:

Theorem 4.2.4. *$G^t_{\Lambda\Gamma}V(\gamma)$ has a filtration*

$$\{0\} = M(0) \subset M(1) \subset \cdots \subset M(m) = G^t_{\Lambda\Gamma}V(\gamma)$$

such that $M(i)/M(i-1) \cong V(\mu_i)\langle \deg(\mu_i t\gamma) - \mathrm{caps}(t)\rangle$ for each i. Here $\{\mu_1, \cdots, \mu_m\} = \{\mu \in \Lambda | \mu t\gamma$ oriented$\}$, ordered so that $\mu_i > \mu_j$ (in the Bruhat order) implies $i < j$.

Proof. We apply the exact functor $G^t_{\Lambda\Gamma}$ to the short exact sequence $0 \to U(\gamma) \to P(\gamma) \to V(\gamma) \to 0$, with $U(\gamma)$ as in Definition 2.3.2. From Theorem 4.2.2 we know that $G^t_{\Lambda\Gamma}P(\gamma)$ has basis given by all diagrams $(a\mu t\nu\overline{\gamma}) \otimes e_\gamma$, and using Proposition 4.1.15, we can see that the submodule $G^t_{\Lambda\Gamma}U(\gamma)$ has basis given by those diagrams $(a\mu t\nu\overline{\gamma}) \otimes e_\gamma$ for which $\nu > \gamma$ (just write $(a\mu t\nu\overline{\gamma}) \otimes e_\gamma = (a\mu t\nu\overline{\nu}) \otimes (\underline{\nu}\nu\overline{\gamma})$). So $G^t_{\Lambda\Gamma}V(\gamma)$ has basis given by all diagrams $(a\mu t\gamma\overline{\gamma}) \otimes e_\gamma$. Now we define $M(0) := \{0\}$, and for $i = 1, \ldots, m$ let $M(i)$ be the subspace of $G^t_{\Lambda\Gamma}V(\gamma)$ generated by $M(i-1)$ and the vectors

$$\{(a\mu_i t\gamma\overline{\gamma}) \otimes e_\gamma \text{ for all oriented cup diagrams } a\mu_i\}.$$

By Proposition 4.1.15, each $M(i)$ is a $\mathbb{D}_\Lambda$-submodule of $G^t_{\Lambda\Gamma}V(\gamma)$. So the $M(i)$ define a filtration of $G^t_{\Lambda\Gamma}V(\gamma)$. The subquotient $M(i)/M(i-1)$ has basis given by the images of the vectors

$$\{(a\mu_i t\gamma\overline{\gamma}) \otimes e_\gamma \text{ for all oriented cup diagrams } a\mu_i\}.$$

Now it remains to check that the map

$$g : M(i)/M(i-1) \to V(\mu_i)\langle \deg(\mu_i t\gamma) - \mathrm{caps}(t)\rangle : [a\mu_i t\gamma\overline{\gamma}] \mapsto [a\mu_i\overline{\mu_i}]$$

is an isomorphism of graded $\mathbb{D}_\Lambda$-modules.
Clearly g is an isomorphism of graded vector spaces. For $a\lambda b \in \mathbb{D}_\Lambda$ and $[c\mu_i t\gamma\overline{\gamma}] \in M(i)/M(i-1)$ we have by Theorem 4.1.14 that

$$(a\lambda b) \cdot [c\mu_i t\gamma\overline{\gamma}] = \begin{cases} s_{a\lambda b}(\mu_i)[a\mu_i t\gamma\overline{\gamma}], & \text{if } b = c^* \text{ and } a\mu \text{ is oriented,} \\ 0, & \text{otherwise.} \end{cases}$$

Comparing with Theorem 1.4.4, we see that g is compatible with the action of $\mathbb{D}_\Lambda$. □

4.3 Proof of Koszulity

In this subsection we will take the viewpoint that a projective resolution of a module M is a chain complex $P^\bullet$ of projectives, concentrated in positive homological degrees, such that $H_0(P^\bullet) \cong M$, and $H_i(P^\bullet) = 0$ for $i \neq 0$. For later use, we recall the following result from homological algebra:

Proposition 4.3.1. *Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. Suppose the chain complexes $P^\bullet$ and $Q^\bullet$ are projective resolutions of A resp. B. Then f induces a map $\tilde{f} : P^\bullet \to Q^\bullet$ on chain complexes, and the mapping cone $\mathrm{Cone}(\tilde{f})^\bullet$ (as defined in [Wei, 1.5.1]) is a projective resolution of C.*

Proof. $\mathrm{Cone}(\tilde{f})$ is a complex of projectives by definition of the mapping cone (recall that $\mathrm{Cone}(\tilde{f})_i = P^{i-1} \oplus Q^i$). From the long exact sequence (see [Wei, 1.5.2])

$$\cdots \to H_1(\mathrm{Cone}(\tilde{f})^\bullet) \to H_0(P^\bullet) \xrightarrow{f} H_0(Q^\bullet) \to H_0(\mathrm{Cone}(\tilde{f})^\bullet) \to H_{-1}(P^\bullet) \to \cdots$$

it follows that $\mathrm{Cone}(\tilde{f})$ is a projective resolution of C. □

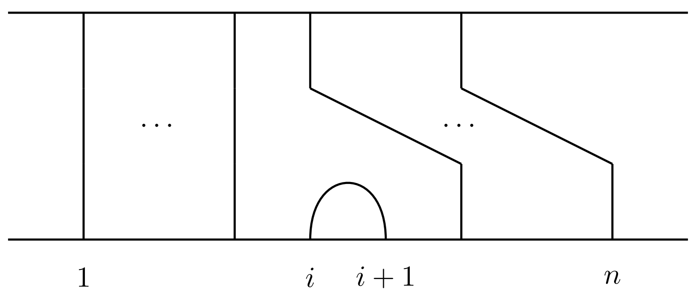
Fix a block Λ. We will now use the projective functors introduced in the last subsection to prove that every standard $\mathbb{D}_\Lambda$-module $V(\lambda)$ has a linear projective resolution. Then it will follow from Theorem 3.2.2 that $\mathbb{D}_\Lambda$ is a Koszul algebra.

Theorem 4.3.2. *For $\lambda \in \Lambda$, the standard module $V(\lambda)$ has a projective resolution*

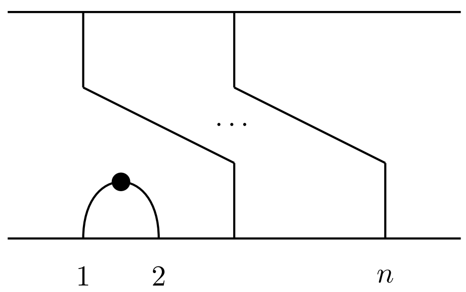
$$\cdots \xrightarrow{d_1} P^1(\lambda) \xrightarrow{d_0} P^0(\lambda) \xrightarrow{\varepsilon} V(\lambda) \to 0,$$

where $P^k(\lambda)$ is generated by its degree k part, for every $k \geq 0$. In other words: there are integers $p^{(k)}_{\lambda,\mu}$ such that $P^k(\lambda) := \bigoplus_{\mu\in\Lambda} p^{(k)}_{\lambda,\mu} P(\mu)\langle k\rangle$.

Proof. We will use two nested inductions: first of all we use induction on the size n of the block Λ. Note that if $n = 0, 1$ there is nothing to prove, since if $n = 0$ then $\mathbb{D}_\Lambda \cong 0$ and if $n = 1$ then $\mathbb{D}_\Lambda \cong k$. From now on we will fix an n-block Λ with $n > 1$, and we assume that the theorem is already proven for all n'-blocks with $n' < n$.
Next we use induction on the Bruhat order: note that if λ is maximal w.r.t. the Bruhat order on Λ, then $V(\lambda) = P(\lambda)$ by definition, so that the minimal projective resolution of $V(\lambda)$ has $P^0(\lambda) = P(\lambda)$, and $P^i(\lambda) = 0$ for $i > 0$. So our theorem is true for λ maximal w.r.t. the Bruhat order. From now on we will fix $\lambda \in \Lambda$ not maximal w.r.t. the Bruhat order, and assume that the theorem is proven for all $\lambda' \in \Lambda$ with $\lambda' > \lambda$.
Since we assumed λ to be not maximal w.r.t. the Bruhat order, it is possible to apply a Bruhat move to it. Say that the weight $\lambda' \in \Lambda$ is obtained by applying a Bruhat move B to λ at position i. Furthermore, let γ be the weight obtained from λ by removing the i'th and $(i+1)$'th symbol. Then γ belongs to a smaller block (of size $(n-2)$) which we will call Γ. Now we let t be the n-$(n-2)$-matching as in the following figure:



if B is a type A move.



if B is a type D move (hence $i = 1$).

We now use Theorem 4.2.4 to describe $G^t_{\Lambda\Gamma}V(\lambda')$: by the construction of t, there are only 2 weights $\mu \in \Lambda$ so that $\mu t \gamma$ is an oriented matching: namely $\mu = \lambda$ (corresponds to orienting the cap in t counterclockwise), and $\mu = \lambda'$ (corresponds to orienting the cap in t clockwise). Since $\lambda < \lambda'$, it follows from Theorem 4.2.4 that there is a short exact sequence

$$0 \to V(\lambda') \xrightarrow{f} G^t_{\Lambda\Gamma}V(\gamma) \to V(\lambda)\langle -1\rangle \to 0. \tag{4.1}$$

By the induction hypothesis we have already constructed a projective resolution $P(\lambda')^\bullet$ of $V(\lambda')$, and a projective resolution $P(\gamma)^\bullet$ of $V(\gamma)$, with $P^k(\lambda') = \bigoplus_{\mu\in\Lambda} p^{(k)}_{\lambda',\mu} P(\mu)\langle k\rangle$ and $P^k(\gamma) = \bigoplus_{\xi\in\Gamma} p^{(k)}_{\gamma,\xi} P(\xi)\langle k\rangle$. Applying the exact functor $G^t_{\Lambda\Gamma}$ to $P(\gamma)^\bullet$ yields a projective resolution $G^t_{\Lambda\Gamma}(P(\gamma))^\bullet$ of $G^t_{\Lambda\Gamma}V(\gamma)$, with $G^t_{\Lambda\Gamma}(P(\gamma))^k \cong \bigoplus_{\xi\in\Gamma} p^{(k)}_{\gamma,\xi} G^t_{\Lambda\Gamma}P(\xi)\langle k\rangle$.
By Theorem 4.2.2, $G^t_{\Lambda\Gamma}P(\xi) \cong P(\nu)\langle -1\rangle$, where:

- $\nu \in \Lambda$ is obtained from ξ by inserting $\vee\wedge$ after the $i-1$'th position, in the case where our fixed Bruhat move B is a type A move applied at position i.
- $\nu \in \Lambda$ is obtained from ξ by inserting $\wedge\wedge$ before the first position, in the case where our fixed Bruhat move B is a type D move applied at position 1.

Let $\Lambda_B \subseteq \Lambda$ consist of all weights $\upsilon \in \Lambda$ to which the Bruhat move B can be applied. In other words, if B is the type A move applied at position i, Λ_B consists of all weights υ with $\upsilon_i = \vee$ and $\upsilon_{i+1} = \wedge$; if B is the type D move applied at position 1, Λ_B consists of all weights υ with $\upsilon_1 = \upsilon_2 = \wedge$. For $\upsilon \in \Lambda_B$, we will write $\partial_i(\upsilon) \in \Gamma$ for the weight obtained by deleting the i'th and $i+1$'th symbol (where i is the position corresponding to the Bruhat move B).
Then from the above considerations it follows that $G^t_{\Lambda\Gamma}(P(\gamma))^k \cong \bigoplus_{\nu\in\Lambda_B} p^{(k)}_{\gamma,\partial_i(\nu)} P(\nu)\langle k-1\rangle$.
Recalling the short exact sequence (4.1), let $\tilde{f}$ be the induced map $P(\lambda')^\bullet \to G^t_{\Lambda\Gamma}(P(\gamma))^\bullet$. Then by Proposition 4.3.1 it follows that the mapping cone $\mathrm{Cone}(\tilde{f})^\bullet$ is a projective resolution of $V(\lambda)\langle -1\rangle$. Now note that

$$\mathrm{Cone}(\tilde{f})^k = P(\lambda')^{k-1} \oplus G^t_{\Lambda\Gamma}(P(\gamma))^k = \bigoplus_{\mu\in\Lambda} p^{(k-1)}_{\lambda',\mu} P(\mu)\langle k-1\rangle \oplus \bigoplus_{\nu\in\Lambda_B} p^{(k)}_{\gamma,\partial_i(\nu)} P(\nu)\langle k-1\rangle.$$

We can therefore conclude that $\mathrm{Cone}(\tilde{f})^\bullet\langle 1\rangle$ is a projective resolution of $V(\lambda)$ of the desired form. □

Theorem 4.3.3. *$\mathbb{D}_\Lambda$ is a Koszul algebra.*

Proof. By Theorem 3.2.2, it suffices to show that $\mathbb{D}_\Lambda$ is a standard Koszul algebra. We already know that $\mathbb{D}_\Lambda$ is quasi-hereditary (Theorem 2.3.3). The fact that $\mathbb{D}_\Lambda$ is generated by its degree 0 and 1 parts is proven in [ES1, Theorem 6.10]. Theorem 4.3.2 precisely says that every standard left $\mathbb{D}_\Lambda$-module has a linear projective resolution. Using the anti-automorphism $^* : \mathbb{D}_\Lambda \to \mathbb{D}_\Lambda^{\mathrm{op}}$ from Proposition 1.4.2, we see that every standard right $\mathbb{D}_\Lambda$-module has a linear projective resolution. So Theorem 3.2.2 applies, and $\mathbb{D}_\Lambda$ is a Koszul algebra. □

Corollary 4.3.4. *For $k \in \mathbb{N}$, let $\mathbb{D}^k_\Lambda \subseteq \mathbb{D}_\Lambda$ be the subalgebra spanned by all circle diagrams containing at most k cups and at most k caps. Then $\mathbb{D}^k_\Lambda$ is Koszul.*

Proof. Let $\Lambda^k \subseteq \Lambda$ consist of all weights λ for which $\underline{\lambda}$ contains at most k cups. Write $e_k = \sum_{\lambda\in\Lambda^k} e_\lambda$. Then $\mathbb{D}^k_\Lambda = e_k \mathbb{D}_\Lambda e_k$.
Now note that if $\lambda \in \Lambda^k$ and $\lambda \leq \mu$ in the Bruhat order, then $\mu \in \Lambda^k$. This implies that the Bruhat order on Λ can be refined to a total order, such that $e_k = \varepsilon_l$ for some $l \in \mathbb{Z}_{>0}$ in the sense of Remark 2.1.4. Then we are done by Corollary 3.2.11. □

In the next section we will try to understand the coefficients $p_{\lambda,\mu}^{(k)}$ appearing in Theorem 4.3.2, which describe the projective resolution of a standard module. We will group them together as the coefficients of polynomials

$$p_{\lambda,\mu}(q) = \sum_{k\in\mathbb{N}} p_{\lambda,\mu}^{(k)} q^k, \tag{4.2}$$

which will be called *dual Kazhdan–Lusztig polynomials.* Note that these are actual polynomials: $p_{\lambda,\mu}^{(k)} = 0$ for k large enough. This is clear since the quasi-hereditary algebra $\mathbb{D}_\Lambda$ has finite global dimension; it also follows more explicitly from our inductive construction of the linear projective resolutions in the proof of Theorem 4.3.2.
From the proof of 4.3.2, we get an inductive procedure to compute the coefficients $p_{\lambda,\mu}^{(k)}$: if $\lambda \in \Lambda$ is maximal w.r.t. the Bruhat order, then $p_{\lambda,\mu}^{(k)} = 0$ unless $\mu = \lambda$ and $k = 0$, in which case we get $p_{\lambda,\lambda}^{(0)} = 1$. For λ not maximal we need to pick a Bruhat move B which can be applied to λ, and we get

$$p_{\lambda,\mu}^{(k)} = \begin{cases} p_{\lambda',\mu}^{(k-1)}, & \text{if } \mu \notin \Lambda_B, \\ p_{\lambda',\mu}^{(k-1)} + p_{\gamma,\partial_i(\mu)}^{(k)}, & \text{if } \mu \in \Lambda_B, \end{cases} \tag{4.3}$$

where Λ_B, λ', γ and $\partial_i(\mu)$ are as in the proof of Theorem 4.3.2.
Using (4.3), one can check that for $\lambda \not\leq \mu$, it holds that $p_{\lambda,\mu}^{(k)} = 0$ for all k, and that for all $\lambda \in \Lambda$, we have that $p_{\lambda,\lambda}^{(k)} = 0$ for $k \neq 0$ and $p_{\lambda,\lambda}^{(0)} = 1$.
The following result is similar to [BS2, Corollary 5.5]:

Corollary 4.3.5. *For $\lambda, \mu \in \Lambda, k \in \mathbb{N}$, the graded vector space $\mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu))$ is concentrated in degree $-k$, and*

$$\dim \mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu)) = p_{\lambda,\mu}^{(k)}.$$

Proof. We compute $\mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu))$ by applying the functor $\mathrm{HOM}_{\mathbb{D}_\Lambda}(\_, L(\mu))$ to the projective resolution of $V(\lambda)$ from Theorem 4.3.2. $\mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu))$ is given by the k'th cohomology of the complex

$$0 \to \mathrm{HOM}_{\mathbb{D}_\Lambda}(P^0(\lambda), L(\mu)) \to \mathrm{HOM}_{\mathbb{D}_\Lambda}(P^1(\lambda), L(\mu)) \to \cdots .$$

Since $P^k(\lambda) = \bigoplus_{\mu\in\Lambda} p_{\lambda,\mu}^{(k)} P(\mu)\langle k\rangle$, it follows that $\mathrm{HOM}_{\mathbb{D}_\Lambda}(P^k(\lambda), L(\mu))$ is concentrated in degree $-k$, and $\dim \mathrm{HOM}_{\mathbb{D}_\Lambda}(P^k(\lambda), L(\mu)) = p_{\lambda,\mu}^{(k)}$. It follows that all differentials in the above complex are 0. So $\mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu)) \cong \mathrm{HOM}_{\mathbb{D}_\Lambda}(P^k(\lambda), L(\mu))$ and we are done. □

Define the matrix

$$P_\Lambda(q) := \left(p_{\lambda,\mu}(q)\right)_{\lambda,\mu\in\Lambda},$$

where $p_{\lambda,\mu}(q)$ are the dual Kazhdan–Lusztig polynomials from (4.2).

Corollary 4.3.6. *The matrix $P_\Lambda(-q)$ is inverse to the decomposition matrix $D_\Lambda(q) = (d_{\lambda,\mu}(q))_{\lambda,\mu\in\Lambda} = \left(q^{\deg(\underline{\lambda}\mu)}\right)_{\lambda,\mu\in\Lambda}$.*

Proof. From the long exact sequence in Theorem 4.3.2, it follows that in the Grothendieck group $[\mathbb{D}_\Lambda\text{-gmod}]$, we have that

$$\begin{aligned}
[V(\lambda)] &= \sum_{k\in\mathbb{N}} (-1)^k [P^k(\lambda)] \\
&= \sum_{\substack{k\in\mathbb{N}\\ \mu\in\Lambda}} (-1)^k q^k p^{(k)}_{\lambda,\mu} [P(\mu)] \\
&= \sum_{\mu\in\Lambda} p_{\lambda,\mu}(-q)[P(\mu)].
\end{aligned}$$

Since $P(\lambda) = \sum_{\mu\in\Lambda} d_{\lambda,\mu}(q)[V(\mu)]$, our theorem is proven. $\square$

Remark 4.3.7. The previous corollary implies that the dual Kazhdan–Lusztig polynomials give the decomposition numbers for the Koszul dual algebra $\mathbb{D}^!_\Lambda$.

5 Combinatorics of dual Kazhdan–Lusztig polynomials

In this section, we will give two explicit combinatorial descriptions of the Kazhdan–Lusztig polynomials. The first one (Theorem 5.2.8) is based on [CHR]: one can compute $p_{\lambda,\mu}(q)$ by counting so-called *paths* from λ to μ. The second description goes back to a beautiful closed formula discovered by Lascoux and Schützenberger [LS]. In our language (see [BS2, Section 5] for the type A case) we get a way of computing $p_{\lambda,\mu}(q)$ by counting *λ-labellings of $\mu\overline{\mu}$* (Theorem 5.3.4). We will connect our two descriptions by giving an explicit bijection between paths and labellings (Theorem 5.4.7). As an application of Theorem 5.3.4, we give an easy combinatorial description of the so-called *Kostant weights* (Theorem 5.5.4).
For this whole section, we will be working in a fixed block Λ.

5.1 Inductive definition of the dual Kazhdan–Lusztig polynomials

We start this section by fixing some notation:

- If $\tilde{\lambda}$ is obtained from λ by deleting positions $i, i+1$, we write $\tilde{\lambda} = \partial_i(\lambda)$ (note that λ and $\partial_i(\lambda)$ don't belong to the same block).
- $\mathrm{B}(\lambda)$ will always mean the weight obtained by applying some Bruhat move B to the weight λ. Of course this only makes sense if the Bruhat move B can actually be applied to λ.
- If we want to specify the Bruhat move, we will write $\mathrm{B} = \mathrm{A}_i$ for a type A move at position i, and $\mathrm{B} = \mathrm{D}_1$ for a type D move.
- If we write $\mathrm{B}_i(\lambda)$ this means that B_i is a Bruhat move at position i, but we don't specify whether it's type A or type D. In other words: if $i > 1$, $\mathrm{B}_i(\lambda)$ will always mean $\mathrm{A}_i(\lambda)$. $\mathrm{B}_1(\lambda)$ can either mean $\mathrm{A}_1(\lambda)$ or $\mathrm{D}_1(\lambda)$ (note that it can't happen that both A_1 and D_1 can be applied to λ, so the notation $\mathrm{B}_1(\lambda)$ is unambiguous).
- For B a Bruhat move, we let Λ_{B} be the set of weights in Λ to which B can be applied (see also the proof of Theorem 4.3.2).

Definition 5.1.1. We define the *dual Kazhdan–Lusztig polynomials* $p_{\lambda,\mu}(q)$ (for $\lambda, \mu \in \Lambda$) inductively as follows:

1. If $\lambda = \mu$, then $p_{\lambda,\mu}(q) = 1$ and if $\lambda \not\leq \mu$ then $p_{\lambda,\mu}(q) = 0$.
2. If $\lambda < \mu$, let B_i be a Bruhat move that can be applied to λ. We distinguish 2 cases:

$$p_{\lambda,\mu}(q) = \begin{cases} p_{\partial_i(\lambda),\partial_i(\mu)}(q) + q p_{\mathrm{B}_i(\lambda),\mu}(q), & \text{if } \mu \in \Lambda_{\mathrm{B}}, \\ q p_{\mathrm{B}_i(\lambda),\mu}(q), & \text{if } \mu \notin \Lambda_{\mathrm{B}}. \end{cases}$$

Note that in the above definition we had to choose a Bruhat move. So we need to check that the resulting polynomials are well-defined.
Comparing with (4.3), we find that the degree d coefficients of $p_{\lambda,\mu}(q)$ are precisely the numbers $p^{(d)}_{\lambda,\mu}$ which describe the projective resolution of the $\mathbb{D}_\Lambda$-module $V(\lambda)$ (see Theorem 4.3.2), so this definition agrees with (4.2). Since linear projective resolutions are unique, this shows that $p_{\lambda,\mu}(q)$ is well-defined.
We find it noteworthy that this fact can be proved purely combinatorially:

Proposition 5.1.2. *The polynomial* $p_{\lambda,\mu}(q)$ *is well-defined, i.e. doesn't depend on the choice of the Bruhat move* B_i.

Proof. Suppose both B_i and B_j are 2 Bruhat moves that can be applied to λ. We can assume $i+1<j$. Depending on which of the 2 Bruhat moves B_i,B_j can be applied to μ, we need to distinguish 4 cases.

1. If both B_i and B_j can be applied to λ, then the first way of computing $p_{\lambda,\mu}(q)$ yields

$$\begin{aligned}p_{\lambda,\mu}(q) =& p_{\partial_i(\lambda),\partial_i(\mu)}(q) + qp_{\mathrm{B}_i(\lambda),\mu}(q)\\ =& p_{\partial_{j-2}(\partial_i(\lambda)),\partial_{j-2}(\partial_i(\mu))}(q) + qp_{\mathrm{B}_{j-2}(\partial_i(\lambda)),\partial_i(\mu)}(q)\\ &+ qp_{\partial_j(\mathrm{B}_i(\lambda)),\partial_j(\mu)}(q) + q^2 p_{\mathrm{B}_j(\mathrm{B}_i(\lambda)),\mu}(q)\end{aligned}$$

   and the second way yields

$$\begin{aligned}p_{\lambda,\mu}(q) =& p_{\partial_j(\lambda),\partial_j(\mu)}(q) + qp_{\mathrm{B}_j(\lambda),\mu}(q)\\ =& p_{\partial_i(\partial_j(\lambda)),\partial_i(\partial_j(\mu))}(q) + qp_{\mathrm{B}_i(\partial_j(\lambda)),\partial_j(\mu)}(q)\\ &+ qp_{\partial_i(\mathrm{B}_j(\lambda)),\partial_i(\mu)}(q) + q^2 p_{\mathrm{B}_i(\mathrm{B}_j(\lambda)),\mu}(q).\end{aligned}$$

   These are clearly equal, proving the independence of choice of index.

2. If B_i can be applied to μ, but B_j cannot, then the first way of computing $p_{\lambda,\mu}(q)$ yields

$$p_{\lambda,\mu}(q) = p_{\partial_i(\lambda),\partial_i(\mu)}(q) + qp_{\mathrm{B}_i(\lambda),\mu}(q) = qp_{\mathrm{B}_{j-2}(\partial_i(\lambda)),\partial_i(\mu)}(q) + q^2 p_{\mathrm{B}_j(\mathrm{B}_i(\lambda)),\mu}(q)$$

   and the second way yields

$$p_{\lambda,\mu}(q) = qp_{\mathrm{B}_j(\lambda),\mu}(q) = qp_{\partial_i(\mathrm{B}_j(\lambda)),\partial_i(\mu)}(q) + q^2 p_{\mathrm{B}_i(\mathrm{B}_j(\lambda)),\mu}(q).$$

3. If B_j can be applied to μ, but B_i cannot, then the first way of computing $p_{\lambda,\mu}(q)$ yields

$$p_{\lambda,\mu}(q) = qp_{\mathrm{B}_i(\lambda),\mu}(q) = qp_{\partial_j(\mathrm{B}_i(\lambda)),\partial_j(\mu)}(q) + q^2 p_{\mathrm{B}_j(\mathrm{B}_i(\lambda)),\mu}(q)$$

   and the second way yields

$$p_{\lambda,\mu}(q) = p_{\partial_j(\lambda),\partial_j(\mu)}(q) + qp_{\mathrm{B}_j(\lambda),\mu}(q) = qp_{\mathrm{B}_i(\partial_j(\lambda)),\partial_j(\mu)}(q) + q^2 p_{\mathrm{B}_i(\mathrm{B}_j(\lambda)),\mu}(q).$$

4. If neither B_i nor B_j can be applied to μ, then the first way of computing $p_{\lambda,\mu}(q)$ yields

$$p_{\lambda,\mu}(q) = qp_{\mathrm{B}_i(\lambda),\mu}(q) = q^2 p_{\mathrm{B}_j(\mathrm{B}_i(\lambda)),\mu}(q)$$

   and the second way yields

$$p_{\lambda,\mu}(q) = qp_{\mathrm{B}_j(\lambda),\mu}(q) = q^2 p_{\mathrm{B}_i(\mathrm{B}_j(\lambda)),\mu}(q).$$

□

Example 5.1.3. For sake of readability, we will write the dual Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ as $p\binom{\mu}{\lambda}$.

1. The dual Kazhdan–Lusztig polynomial $p\binom{\wedge\vee\wedge\vee}{\vee\wedge\vee\wedge}$ can be computed in 2 ways:

$$p\binom{\wedge\vee\wedge\vee}{\underline{\vee\wedge}\vee\wedge} = q\cdot p\binom{\wedge\vee\wedge\vee}{\wedge\vee\underline{\vee\wedge}} = q^2\cdot p\binom{\wedge\vee\wedge\vee}{\wedge\vee\wedge\vee} = q^2,$$

or

$$p\binom{\wedge\vee\wedge\vee}{\vee\wedge\underline{\vee\wedge}} = q\cdot p\binom{\wedge\vee\wedge\vee}{\underline{\vee\wedge}\wedge\vee} = q^2\cdot p\binom{\wedge\vee\wedge\vee}{\wedge\vee\wedge\vee} = q^2.$$

2. The Kazhdan–Lusztig polynomial $p\binom{\wedge\vee\wedge\vee}{\wedge\wedge\wedge\wedge}$ can be computed as follows:

$$p\binom{\wedge\vee\wedge\vee}{\underline{\wedge\wedge}\wedge\wedge} = q\cdot p\binom{\wedge\vee\wedge\vee}{\vee\underline{\vee\wedge}\wedge} = q\cdot p\binom{\wedge\vee}{\vee\wedge} + q^2\cdot p\binom{\wedge\vee\wedge\vee}{\vee\wedge\vee\wedge} = q^2+q^4.$$

5.2 Description via paths

In this subsection we give a different description of the dual Kazhdan–Lusztig polynomials. This will be the type D analogue of the description in [CHR, Section 2.7]

Definition 5.2.1. A *labelled weight* is a weight $\lambda \in \Lambda$ together with a bijection between the set of symbols $\vee$ in λ, and the set $\{1, 2, \dots, m\}$, where $m \in \mathbb{N}$ is the number of symbols $\vee$ in λ. These labelled symbols will be denoted $\vee_1, \vee_2, \cdots, \vee_m$.
Let λ be a labelled weight. Suppose the symbol at position i is a $\vee_k$, and that it is the left end of an (undotted) cap γ in $\overline{\lambda}$. Then the *right move* R_i is defined by exchanging $\vee_k$ with the $\wedge$ at the other end of γ. We say that the move R_i *deplaces* the symbol $\vee_k$.
Suppose the symbol at position i is an $\wedge$, and that it is the left end of a (dotted) cap γ in $\overline{\lambda}$. Then the *flip move* F_i is defined as follows: replace the 2 $\wedge$'s marking the endpoints of γ by $\vee_{m+2}$ and $\vee_{m+1}$ in that order. We say that the move F_i *creates* the symbols $\vee_{m+1}$ and $\vee_{m+2}$.

Note that every type A Bruhat move that can be applied to λ is a right move, and every type D Bruhat move is a flip move. So we can think of right and flip moves as generalizations of Bruhat moves: a Bruhat move is the same as a right move or a flip move in which the 2 symbols of λ that change are consecutive.
The following lemma describes the effect of a right move or a flip move to the numbers $\ell_i(\lambda, \mu)$ (see Definition 1.1.5):

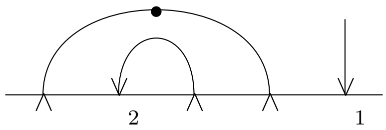
Lemma 5.2.2. *Let $\lambda, \mu \in \Lambda$, and suppose M is a right move or a flip move that can be applied to λ. Write $\lambda' = M(\lambda)$.*

- *If M is a right move that switches the $\vee$ at at position i with the $\wedge$ at position $j > i$, then $\ell_k(\lambda', \mu) = \ell_k(\lambda, \mu) - 1$ for $i \leq k < j$, and $\ell_k(\lambda', \mu) = \ell_k(\lambda, \mu)$ for all other k. So $\ell(\lambda', \mu) = \ell(\lambda, \mu) - (j - i)$.*
- *If M is a flip move that flips the $\wedge$'s at positions i and j ($i < j$), then $\ell_k(\lambda', \mu) = \ell_k(\lambda, \mu) - 2$ for $k < i$, $\ell_k(\lambda', \mu) = \ell_k(\lambda, \mu) - 1$ for $i \leq k < j$, and $\ell_k(\lambda', \mu) = \ell_k(\lambda, \mu)$ for $j \leq k$. So $\ell(\lambda', \mu) = \ell(\lambda, \mu) - (i + j - 2)$.*

In particular, if M is a Bruhat move then $\ell(\lambda', \mu) = \ell(\lambda, \mu) - 1$; if M is not a Bruhat move then $\ell(\lambda', \mu) < \ell(\lambda, \mu) - 1$.

Proof. Follows immediately from the definitions. □

Example 5.2.3. Let $\lambda = \wedge \vee_2 \wedge \wedge \vee_1$. Then

$\lambda\overline{\lambda} =$  .

So there are 2 moves that can be applied to λ: the right move R_2 would turn λ into $\wedge \wedge \vee_2 \wedge \vee_1$, and the flip move F_1 would turn λ into $\vee_4 \vee_2 \wedge \vee_3 \vee_1$.

Definition 5.2.4. Let $\lambda \in \Lambda$. We can label λ in a canonical way by numbering the $\vee$'s from right to left in ascending order. The resulting labelled weight will also be denoted by λ.
A *path* P starting at λ is a sequence

$$M_1, M_2, \dots, M_r$$

where M_k is either a right move or a flip move, for which the following four conditions hold:

- For every $1 \leq k \leq r$, the move M_k can be applied to the weight $(M_{k-1} \circ \cdots \circ M_2 \circ M_1)(\lambda)$.
- For $1 \leq k \leq l \leq r$, if M_k and M_l are both right moves: say M_k deplaces $\vee_\alpha$ and M_l deplaces $\vee_\beta$. Then $\alpha \leq \beta$.
- For $1 \leq k \leq l \leq r$, if M_k is a flip move and M_l is a right move: say M_k creates $\vee_{\alpha+1}, \vee_\alpha$ and M_l deplaces $\vee_\beta$. Then $\alpha \leq \beta$.
- For $1 \leq k \leq l \leq r$, if M_k and M_l are both flip moves: say $M_k = F_i$ and $M_l = F_j$ (i.e. M_k is at position i, and M_l at position j), then $i < j$.

The first condition ensures that $\mu := (M_r \circ \cdots M_2 \circ M_1)(\lambda)$ is a well-defined weight in Λ. We call μ the *endpoint* of the path, and say that P is a path *from λ to μ*.
The *length* $\ell(P)$ of a path is the number of moves it consists of.

Remark 5.2.5. The second and third conditions in the definition of a path mean that a path does the following to λ:
First we put $\vee_1$ in its final position using a number of right moves. Then similarly we put $\vee_2$, $\vee_3$, ..., $\vee_m$ in their final positions.
Now we create 2 new $\vee$-symbols ($\vee_{m+1}$ and $\vee_{m+2}$) with a flip move, and put them in their final positions using right moves (first $\vee_{m+1}$, then $\vee_{m+2}$). This process of creating 2 new $\vee$-symbols and deplacing them is repeated until we reach the weight μ.
The fourth condition ensures that, whenever we do a flip move at position i, no further moves will be done to the left of i.

Proposition 5.2.6. *If $\lambda \leq \mu$, there is a unique path from λ to μ that consists of Bruhat moves only. This path is the unique path from λ to μ of maximal length. We will call it the* trivial path *from λ to μ.*

Proof. Since $\lambda \leq \mu$, there is a sequence of Bruhat moves from λ to μ. We claim that the "canonical sequence", introduced in the proof of Proposition 1.1.8, is the only sequence of Bruhat moves from λ to μ which is also a path. Recall that this canonical sequence is constructed by at each step applying to λ' a type A Bruhat move at the rightmost index it can be applied to (i.e. to the largest i for which $\lambda'_i = \vee$ and $\ell_i(\lambda', \mu) > 0$), or a type D move if no type A move can be applied to λ'. Note that in each intermediate step, the labels of the $\vee$'s are in descending order. This ensures that the second and third condition in the definition of a path are satisfied. Since a type D Bruhat move is always applied to the first index, the fourth condition is satisfied as well.

Now let P be any sequence of Bruhat moves from λ to μ which is also a path, and suppose P is not the canonical sequence. Look at the first move where P differs from the canonical sequence. Before applying a Bruhat move there, it holds that the labels in λ' are in descending order. Let i be the largest index for which $\lambda'_i = \vee$ and $\ell_i(\lambda', \mu) > 0$; write $\lambda'_i = \vee_\alpha$. Then the next Bruhat move is not at position i. This means that it either it is a type A move applied to a $\vee_\beta$ with a label $\beta > \alpha$, or it is a type D move. In both cases the definition of a path implies that no further Bruhat moves can be made at index i. But since $\ell_i(\lambda', \mu) > 0$, this gives a contradiction. So the only sequence of Bruhat moves from λ to μ which is also a path is the canonical sequence.

For P any path from λ to μ, it follows from Lemma 5.2.2 that every move M will decrease the value of $\ell(\lambda', \mu)$ by exactly 1 if M is a Bruhat move, and by strictly more than 1 if M is not a Bruhat move. So the trivial path has length $\ell(\lambda, \mu)$, and any other path has strictly smaller length. □

Example 5.2.7. Let $\lambda = \wedge \wedge \wedge\wedge$ and $\mu = \wedge \vee \wedge\vee$. Then there are 2 paths from λ to μ: the trivial path

$$P_1 : \wedge \wedge \wedge\wedge \xrightarrow{F_1} \vee_2 \vee_1 \wedge\wedge \xrightarrow{R_2} \vee_2 \wedge \vee_1\wedge \xrightarrow{R_3} \vee_2 \wedge \wedge\vee_1 \xrightarrow{R_1} \wedge \vee_2 \wedge\vee_1$$

which has length 4, and the path

$$P_2 : \wedge \wedge \wedge\wedge \xrightarrow{F_1} \vee_2 \vee_1 \wedge\wedge \xrightarrow{R_1} \wedge \vee_1 \wedge\vee_2$$

of length 2.

Note that for example the following sequence of moves

$$\wedge \wedge \wedge\wedge \xrightarrow{F_1} \vee_2 \vee_1 \wedge\wedge \xrightarrow{R_2} \vee_2 \wedge \vee_1\wedge \xrightarrow{R_1} \wedge \vee_2 \vee_1\wedge \xrightarrow{R_3} \wedge \vee_2 \wedge\vee_1$$

is NOT a valid path, since we are not allowed to deplace any $\vee_1$ after $\vee_2$ has been deplaced.

Theorem 5.2.8. *The dual Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ can be computed as*

$$p_{\lambda,\mu}(q) = \sum_{P \text{ path from } \lambda \text{ to } \mu} q^{\ell(P)}.$$

Proof. If $\lambda \not\leq \mu$ there is no path from λ to μ, and if $\lambda = \mu$ there is exactly one path, of length 0. We can suppose that $\lambda < \mu$, and that the last symbol of λ is an $\wedge$ (since if λ ends with l $\vee$'s, so does μ, and we can throughout the whole proof ignore the last l symbols).

We will assume by induction that the theorem is already proven for all smaller blocks, and for all $\lambda' \in \Lambda$ with $\lambda' > \lambda$. If λ contains no symbols $\vee$, we distinguish 2 cases:

1. $\mu \notin \Lambda_{\mathrm{D}_1}$ (i.e. μ does not start with $\wedge\wedge$.)
   Then $p_{\lambda,\mu}(q) = qp_{\mathrm{D}_1(\lambda),\mu}(q)$. Note that every path from λ to μ has to start with F_1. (Else we can't change the first 2 symbols anymore.) So applying the induction hypothesis we find

$$\begin{aligned} p_{\lambda,\mu}(q) =& qp_{\mathrm{D}_1(\lambda),\mu}(q) \\ =& \sum_{P'' \text{ path from } \mathrm{D}_1(\lambda) \text{ to } \mu} q^{\ell(P'')+1} \\ =& \sum_{P \text{ path from } \lambda \text{ to } \mu} q^{\ell(P)}. \end{aligned}$$

2. $\mu \in \Lambda_{\mathrm{D}_1}$ (i.e. μ starts with $\wedge\wedge$.)
   Then $p_{\lambda,\mu}(q) = p_{\partial_1(\lambda),\partial_1(\mu)}(q) + qp_{\mathrm{D}_1(\lambda),\mu}(q)$. Now

$$\begin{aligned} \{\text{paths from } \lambda \text{ to } \mu\} =& \{\text{paths from } \lambda \text{ to } \mu \text{ starting with } F_1\} \\ & \sqcup \{\text{paths from } \lambda \text{ to } \mu \text{ starting with } F_i \text{ for some } i > 1\}. \end{aligned}$$

   By the fourth condition in Definition 5.2.4, the second set is in bijection with the set of paths from $\partial_1(\lambda)$ to $\partial_1(\mu)$. The first one is in bijection with the set of paths from $\mathrm{D}_1(\lambda)$ to μ. So

$$\begin{aligned} p_{\lambda,\mu}(q) =& p_{\partial_1(\lambda),\partial_1(\mu)}(q) + qp_{\mathrm{D}_1(\lambda),\mu}(q) \\ =& \sum_{P' \text{ path from } \partial_1(\lambda) \text{ to } \partial_1(\mu)} q^{\ell(P')} + \sum_{P'' \text{ path from } \mathrm{D}_1(\lambda) \text{ to } \mu} q^{\ell(P'')+1} \\ =& \sum_{P \text{ path from } \lambda \text{ to } \mu} q^{\ell(P)}. \end{aligned}$$

If λ contains at least 1 symbol $\vee$, suppose the rightmost one ($\vee_1$) is at position i. We distinguish 2 cases:

1. $\mu \notin \Lambda_{\mathrm{A}_i}$
   Then $p_{\lambda,\mu}(q) = qp_{\mathrm{A}_i(\lambda),\mu}(q)$. Note that all paths from λ to μ have to start with R_i: if a path does not start with R_i, the symbols at positions $i, i+1$ will stay $\vee\wedge$ throughout the whole procedure. But then the resulting weight cannot equal μ by our assumption $\mu \notin \Lambda_{\mathrm{A}_i}$. So

$$\begin{aligned} p_{\lambda,\mu}(q) =& qp_{\mathrm{A}_i(\lambda),\mu}(q) \\ =& \sum_{P'' \text{ path from } \mathrm{A}_i(\lambda) \text{ to } \mu} q^{\ell(P'')+1} \\ =& \sum_{P \text{ path from } \lambda \text{ to } \mu} q^{\ell(P)}. \end{aligned}$$

2. $\mu \in \Lambda_{\mathrm{A}_i}$
   Then $p_{\lambda,\mu}(q) = p_{\partial_i(\lambda),\partial_i(\mu)}(q) + qp_{\mathrm{A}_i(\lambda),\mu}(q)$. Now

$$\begin{aligned} \{\text{paths from } \lambda \text{ to } \mu\} =& \{\text{paths from } \lambda \text{ to } \mu \text{ starting with } R_i\} \\ & \sqcup \{\text{paths from } \lambda \text{ to } \mu \text{ not starting with } R_i\}. \end{aligned}$$

Recalling the second and third conditions in Definition 5.2.4, we find that the second set is in bijection with the set of paths from $\partial_i(\lambda)$ to $\partial_i(\mu)$. The first one is in bijection with the set of paths from $\mathrm{A}_i(\lambda)$ to μ. So

$$\begin{aligned} p_{\lambda,\mu}(q) =& p_{\partial_i(\lambda),\partial_i(\mu)}(q) + q p_{\mathrm{A}_i(\lambda),\mu}(q) \\ =& \sum_{P' \text{ path from } \partial_i(\lambda) \text{ to } \partial_i(\mu)} q^{\ell(P')} + \sum_{P'' \text{ path from } \mathrm{A}_i(\lambda) \text{ to } \mu} q^{\ell(P'')+1} \\ =& \sum_{P \text{ path from } \lambda \text{ to } \mu} q^{\ell(P)}. \end{aligned}$$

□

Example 5.2.9. From Example 5.2.7 it follows that $p\binom{\wedge\vee\wedge\vee}{\wedge\wedge\wedge\wedge} = q^4 + q^2$. Note that this agrees with Example 5.1.3.

5.3 Description via labelled diagrams

Lascoux and Schützenberger [LS] discovered a beautiful closed formula for the computing the dual Kazhdan–Lusztig polynomials of the type A arc algebra. In [BS2, Section 5], Brundan and Stroppel reformulate this formula in terms of so-called *labelled cap diagrams*. In this section we will discuss a version of this formula for our type D arc algebra $\mathbb{D}_\Lambda$.

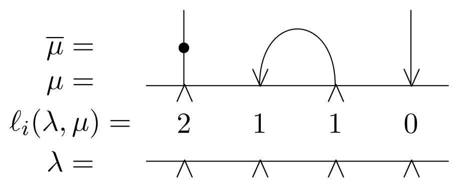
Definition 5.3.1. For a cap diagram $\overline{\mu}$, we say that a cap γ is *D-nested* inside a cap γ, if either γ lies under γ', or γ' is dotted and γ lies to the left of γ'.
Now let $\lambda \leq \mu$ be 2 weights in Λ; suppose λ has m symbols $\vee$ and μ has $m + 2k$ symbols $\vee$ (so $\ell_0(\lambda, \mu) = 2k$). Consider the oriented cap diagram $\mu\overline{\mu}$. A λ*-labelling* C of this oriented cap diagram consists of assigning to every cap a natural number, such that the following properties are satisfied:

1. If the left end of an undotted cap is at position i, its label is at most $\ell_i(\lambda, \mu)$.
2. The label of any dotted cap is even and at most $\ell_0(\lambda, \mu)$.
3. If a cap γ is D-nested inside another cap γ', the label of γ is greater than or equal to the label of γ'.
4. A cap may only have an odd label if there is some other cap above it or to the left of it which has a strictly smaller label, or if there is a ray to the left of it.

Denote the set of λ-labellings of $\mu\overline{\mu}$ by $D(\lambda, \mu)$. The *value* of a labelling $C \in D(\lambda, \mu)$, denoted $|C|$, is defined to be the sum of the labels in C.

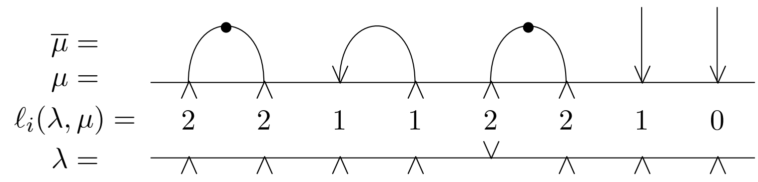
Note that the "0-labelling", in which each cap gets label 0, is always a valid λ-labelling of $\mu\overline{\mu}$.

Example 5.3.2. Let $\lambda = \wedge\wedge\wedge\wedge$ and $\mu = \wedge\vee\wedge\vee$. We want to find the λ-labellings of $\mu\overline{\mu}$. For this it will be helpful to draw the following figure:

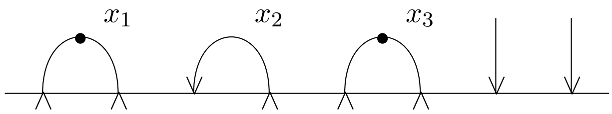


From the definition, we see that $\mu\overline{\mu}$ has exactly 2 λ-labellings: one labelling C_1 in which the unique cap of $\mu\overline{\mu}$ has label 0, and the labelling C_2 where this cap has label 1.

Example 5.3.3. Let's try to find $D(\lambda,\mu)$, with λ and μ as in the following figure:



We will write a labelling



of $\mu\overline{\mu}$ simply as (x_1, x_2, x_3).
The first 2 conditions in Definition 5.3.1 imply that x_1 can only be 0 or 2, x_2 can only be 0 or 1, and x_3 can only be 0 or 2. Since the second cap is D-nested inside the third one, and x_2 is at most 1, it must hold that $x_3 = 0$ (third condition in Definition 5.3.1). This leaves us with 4 possible labellings: $(0,0,0)$, $(0,1,0)$, $(2,0,0)$ and $(2,1,0)$. The first three are valid labellings, but the fourth one is not: the fourth condition in Definition 5.3.1 is violated.
So we conclude that $\mu\overline{\mu}$ has exactly three λ-labellings: $C_1 = (0,0,0)$, $C_2 = (0,1,0)$, and $C_3 = (2,0,0)$. We have $|C_1| = 0$, $|C_2| = 1$, and $|C_3| = 2$.

Theorem 5.3.4. *For $\lambda \leq \mu$, the dual Kazhdan–Lusztig polynomial $p_{\lambda,\mu}(q)$ can be computed as*

$$p_{\lambda,\mu}(q) = q^{\ell(\lambda,\mu)} \sum_{C \in D(\lambda,\mu)} q^{-2|C|}.$$

Proof. If $\lambda = \mu$ then is exactly one λ-labelling of $\mu\overline{\mu}$, of value 0.
We can suppose that $\lambda < \mu$, and that the last symbol of λ is an $\wedge$ (since if λ ends with l $\vee$'s, so does μ, and we can throughout the whole proof ignore the last l symbols).
We will use induction: suppose the theorem has been proven for all smaller blocks, and for all $\lambda' \in \Lambda$ for which $\lambda < \lambda' \leq \mu$.
If λ contains no symbols $\vee$, we distinguish 2 cases:

1. $\mu \in \Lambda_{\mathrm{D}_1}$ (i.e. μ starts with $\wedge\wedge$.)
   Then $p_{\lambda,\mu}(q) = p_{\partial_1(\lambda),\partial_1(\mu)}(q) + q p_{\mathrm{D}_1(\lambda),\mu}(q)$. Let γ be the dotted cap in $\overline{\mu}$ which connects the first 2 symbols. Now

$$\begin{aligned} D(\lambda,\mu) = &\{C \in D(\lambda,\mu) | \gamma \text{ has label } < \ell_0(\lambda,\mu)\} \\ &\sqcup \{C \in D(\lambda,\mu) | \gamma \text{ has label } \ell_0(\lambda,\mu)\}. \end{aligned}$$

   We claim that $\{C \in D(\lambda,\mu) | \gamma \text{ has label } < \ell_0(\lambda,\mu)\} = D(\mathrm{D}_1(\lambda),\mu)$. This follows immediately from the observation that $\ell_0(\mathrm{D}_1(\lambda),\mu) = \ell_0(\lambda,\mu) - 2$, and $\ell_i(\mathrm{D}_1(\lambda),\mu) = \ell_i(\lambda,\mu)$ for $i > 1$.
   Next, we claim that there is a bijection

$$D(\partial_1(\lambda),\partial_1(\mu)) \leftrightarrow \{C \in D(\lambda,\mu) | \gamma \text{ has label } \ell_0(\lambda,\mu)\},$$

constructed in the following (obvious) way: from a $\partial_1(\lambda)$-labelling of $\partial_1(\mu)\overline{\partial_1(\mu)}$ we obtain a λ-labelling of $\mu\overline{\mu}$ by labelling the first cap $\ell_0(\lambda,\mu)$. We need to check that the properties (1-4) in Definition 5.3.1 are preserved under our bijection. For this, note that $\ell_i(\partial_1(\lambda),\partial_1(\mu)) = \ell_{i+2}(\lambda,\mu)$ for all $i > 0$, so that property 1 is preserved. Property 2 is preserved since $\ell_0(\partial_1(\lambda),\partial_1(\mu)) = \ell_0(\lambda,\mu)$. Property 3 is preserved since if γ is D-nested inside some other cap γ', then γ' is dotted hence its label is at most $2k$. To check that property 4 is preserved, note that, since λ has no symbols $\vee$, all occurring $\ell_i(\lambda,\mu)$'s are at most $\ell_0(\lambda,\mu)$, so no odd label with value $> \ell_0(\lambda,\mu)$ can occur in $C \in D(\lambda,\mu)$.
We find by induction that

$$\begin{aligned}p_{\lambda,\mu}(q) =& p_{\partial_1(\lambda),\partial_1(\mu)}(q) + q p_{\mathrm{D}_1(\lambda),\mu}(q)\\ =& q^{\ell(\partial_1(\lambda),\partial_1(\mu))}\sum_{C\in D(\partial_1(\lambda),\partial_1(\mu))} q^{-2|C|} + q^{\ell(\mathrm{D}_1(\lambda),\mu)+1}\sum_{C\in D(\mathrm{D}_1(\lambda),\mu)} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)-2\ell_0(\lambda,\mu)}\sum_{\substack{C\in D(\lambda,\mu)\\ \gamma \text{ has label } \ell_0(\lambda,\mu)}} q^{-2(|C|-\ell_0(\lambda,\mu))} + q^{\ell(\lambda,\mu)}\sum_{\substack{C\in D(\lambda,\mu)\\ \gamma \text{ has label } <\ell_0(\lambda,\mu)}} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)}\sum_{C\in D(\lambda,\mu)} q^{-2|C|}.\end{aligned}$$

2. $\mu \notin \Lambda_{\mathrm{D}}$ (i.e. μ does not start with $\wedge\wedge$.)
   Then $p_{\lambda,\mu}(q) = q p_{\mathrm{D}_1(\lambda),\mu}(q)$. We will prove that $D(\mathrm{D}_1(\lambda),\mu) = D(\lambda,\mu)$. Note that $\ell_1(\mathrm{D}_1(\lambda),\mu) = \ell_1(\lambda,\mu) - 1$ and $\ell_i(\mathrm{D}_1(\lambda),\mu) = \ell_i(\lambda,\mu)$ for $i > 1$. So it suffices to show that for no λ-labelling of $\mu\overline{\mu}$, the cap γ that starts at position 1 (if it exists) is undotted and has label $\ell_1(\lambda,\mu)$. If μ starts with $\vee$ this follows from property 4 (note that $\ell_1(\lambda,\mu) = \ell_0(\lambda,\mu) - 1$ is odd), and if μ starts with $\wedge\vee$ then γ is a dotted cap.
   Now

$$\begin{aligned}p_{\lambda,\mu}(q) =& q p_{\mathrm{D}_1(\lambda),\mu}(q)\\ =& q^{\ell(\mathrm{D}_1(\lambda),\mu)+1}\sum_{C\in D(\mathrm{D}_1(\lambda),\mu)} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)}\sum_{C\in D(\lambda,\mu)} q^{-2|C|}.\end{aligned}$$

If λ contains at least 1 symbol $\vee$, suppose the rightmost one is at position i. We distinguish 2 cases:

1. $\mu \notin \Lambda_{\mathrm{A}_i}$
   Then $p_{\lambda,\mu}(q) = q p_{\mathrm{A}_i(\lambda),\mu}(q)$. We will prove that $D(\mathrm{A}_i(\lambda),\mu) = D(\lambda,\mu)$. Note that $\ell_i(\mathrm{A}_i(\lambda),\mu) = \ell_i(\lambda,\mu) - 1$ and $\ell_j(\mathrm{A}_i(\lambda),\mu) = \ell_j(\lambda,\mu)$ for $j \neq i$. So it suffices to show that for no λ-labelling of $\mu\overline{\mu}$, the cap γ that starts at position i (if it exists) is undotted and has label $\ell_i(\lambda,\mu)$. If μ has $\vee$ at position i, then γ contains a cap starting at $i+1$ whose label is at most $\ell_{i+1}(\lambda,\mu) = \ell_i(\lambda,\mu) - 1$, so γ has label at most $\ell_i(\lambda,\mu) - 1$ by property 3. If μ has $\wedge$ at position i then γ is a dotted cap.

Now

$$\begin{aligned}p_{\lambda,\mu}(q) =& q p_{\mathrm{A}_i(\lambda),\mu}(q)\\ =& q^{\ell(\mathrm{A}_i(\lambda),\mu)+1}\sum_{C\in D(\mathrm{A}_i(\lambda),\mu)} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)}\sum_{C\in D(\lambda,\mu)} q^{-2|C|}.\end{aligned}$$

2. $\mu \in \Lambda_{\mathrm{A}_i}$

Then $p_{\lambda,\mu}(q) = p_{\partial_i(\lambda),\partial_i(\mu)}(q) + q p_{\mathrm{A}_i(\lambda),\mu}(q)$. Let γ be the dotted cap in $\overline{\mu}$ which connects i and $i+1$. Now

$$\begin{aligned}D(\lambda,\mu) =& \{C \in D(\lambda,\mu) | \gamma \text{ has label } < \ell_i(\lambda,\mu)\}\\ & \sqcup \{C \in D(\lambda,\mu) | \gamma \text{ has label } \ell_i(\lambda,\mu)\}.\end{aligned}$$

We claim that $\{C \in D(\lambda,\mu)|\gamma \text{ has label } < \ell_i(\lambda,\mu)\} = D(\mathrm{A}_i(\lambda),\mu)$. This follows immediately from the observation that $\ell_i(\mathrm{A}_i(\lambda),\mu) = \ell_i(\lambda,\mu) - 1$, and $\ell_j(\mathrm{A}_i(\lambda),\mu) = \ell_j(\lambda,\mu)$ for $j \neq i$.

Next, we claim that there is a bijection

$$D(\partial_i(\lambda),\partial_i(\mu)) \leftrightarrow \{C \in D(\lambda,\mu)|\gamma \text{ has label } \ell_i(\lambda,\mu)\},$$

constructed in the following (obvious) way: from a $\partial_i(\lambda)$-labelling of $\partial_i(\mu)\overline{\partial_i(\mu)}$ we obtain a λ-labelling of $\mu\overline{\mu}$ by giving γ the label $\ell_i(\lambda,\mu)$. We need to check that the properties (1-4) in Definition 5.3.1 are preserved under our bijection. Suppose we have $C \in D(\partial_i(\lambda),\partial_i(\mu))$. We need to show that, if we extend C to $C' \in D(\lambda,\mu)$ by giving γ label $\ell_i(\lambda,\mu)$, C' satisfies the properties (1-4). 1 and 2 are trivial.

To check 3, we need to show that if γ is D-nested inside a cap β, it can't happen that β has label $> \ell_i(\lambda,\mu)$. For this, look at the closest $\vee$ in μ to the left of i, and denote its position by j. j is either the left end of β, or the left end of another cap D-nested in β. Since $\ell_j(\lambda,\mu) \leq \ell_i(\lambda,\mu)$, and by using that C satisfies property 3 it follows that β has label at most $\ell_i(\lambda,\mu)$. This argument fails in the case that the leftmost $\vee$ in μ is at position i, but in this case β is a dotted cap and $\ell_i(\lambda,\mu) \geq \ell_0(\lambda,\mu)$ so we're also done.

To check property 4: suppose $\ell_i(\lambda,\mu)$ is odd. We need to check that in C, there is a cap lying above or to the left of γ, whose label is strictly smaller than $\ell_i(\lambda,\mu)$. If i is not the leftmost $\vee$ in μ, let the closest $\vee$ in μ to the left of i be at position j. By the arguments from above, its corresponding cap has label at most $\ell_i(\lambda,\mu)$, and we are done by applying that C satisfies property 4. If i is the the leftmost $\vee$ in μ, then $\ell_i(\lambda,\mu) > \ell_0(\lambda,\mu)$ (recall that $\ell_i(\lambda,\mu)$ is odd), and to the left of i there is either a dotted ray, or a dotted cap with label at most $\ell_0(\lambda,\mu)$.

We find by induction that

$$\begin{aligned}p_{\lambda,\mu}(q) =& p_{\partial_i(\lambda),\partial_i(\mu)}(q) + q p_{\mathrm{A}_i(\lambda),\mu}(q)\\ =& q^{\ell(\partial_i(\lambda),\partial_i(\mu))}\sum_{C\in D(\partial_i(\lambda),\partial_i(\mu))} q^{-2|C|} + q^{\ell(\mathrm{A}_i(\lambda),\mu)+1}\sum_{C\in D(\mathrm{A}_i(\lambda),\mu)} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)-2\ell_i(\lambda,\mu)}\sum_{\substack{C\in D(\lambda,\mu)\\ \gamma \text{ has label } \ell_i(\lambda,\mu)}} q^{-2(|C|-\ell_i(\lambda,\mu))} + q^{\ell(\lambda,\mu)}\sum_{\substack{C\in D(\lambda,\mu)\\ \gamma \text{ has label } <2\ell_i(\lambda,\mu)}} q^{-2|C|}\\ =& q^{\ell(\lambda,\mu)}\sum_{C\in D(\lambda,\mu)} q^{-2|C|}.\end{aligned}$$

□

From this description of the dual Kazhdan–Lusztig polynomials, it immediately follows that $p^{(k)}_{\lambda,\mu} = 0$ unless $k \equiv \ell(\lambda,\mu) \pmod 2$.

Example 5.3.5.
1. From Example 5.3.2, it follows that $p\binom{\wedge\vee\wedge\vee}{\wedge\wedge\wedge\wedge} = q^4(1+q^{-2}) = q^4+q^2$. Note that this is consistent with Example 5.1.3.
2. For λ,μ as in Example 5.3.3, we get that $p_{\lambda,\mu}(q) = q^{11}(1+q^{-2}+q^{-4}) = q^{11}+q^9+q^7$.

5.4 Connection between paths and labellings

If we compare Theorems 5.2.8 and 5.3.4, it follows that there must be a bijection

$$\left\{ \begin{array}{c} \text{paths from} \\ \lambda \text{ to } \mu \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \lambda\text{-labellings} \\ \text{of } \mu\overline{\mu} \end{array} \right\} = D(\lambda,\mu),$$

where $C \in D(\lambda,\mu)$ corresponds to a path of length $\ell(\lambda,\mu) - 2|C|$. The goal of this subsection is to explicitly construct such a bijection. From now on, we will denote the set op paths from λ to μ by $P(\lambda,\mu)$.
The bijection we want, can be defined as follows:

Definition 5.4.1. For λ and μ two weights, we define a map $\chi : P(\lambda,\mu) \to D(\lambda,\mu)$ as follows: For a path P, we get a λ-labelling of $\mu\overline{\mu}$ by the following procedure:

- Start with labelling every cap of $\overline{\mu}$ with 0.
- For every right move R_i in P, if it deplaces a $\vee$ from i to j, increase the label of every cap in $\mu\overline{\mu}$ that lies strictly between i and j by 1.
- For every flip move F_i in P, if it makes a $\vee$ appear at positions $i < j$, increase the label of every cap in $\mu\overline{\mu}$ that lies strictly to the left of i by 2, and the label of every cap strictly in between i and j by 1.

Remark 5.4.2. Note that at the moment we make a right move R_i that deplaces a $\vee$ from i to j, the intermediate diagram λ' has to agree with μ between i and j. In particular, since in λ' i and j are connected by a cap, the portion of $\overline{\mu}$ that lies strictly between i and j consists only of caps. We can make a similar observation for flip moves.

The procedure we just described assigns a label to every cap of $\mu\overline{\mu}$. We need to check that this assignment actually is a λ-labelling in the sense of Definition 5.3.1:

Theorem 5.4.3. *The map χ is well-defined, i.e. the above procedure really gives a λ-labelling of $\mu\overline{\mu}$.*

Proof. We check that the labelling $\chi(P)$ satisfies the four conditions of Definition 5.3.1:

1. Suppose λ has x $\vee$'s strictly to the right of position i. If the cap starting at i has label ℓ, this implies that μ has at least $x+\ell$ $\vee$'s strictly to the right of position i. By the definition of $\ell_i(\lambda,\mu)$, it follows that $\ell \leq \ell_i(\lambda,\mu)$.
2. This is clear since the label of a dotted cup can only increase if an F-move is made to the right of it, and the total number of F-moves made is equal to k.

3. By construction, when we increase the label of a cap, the label of every cap lying under it is also increased. When we increase the label of a dotted cap, this happens by applying an F-move to the right of it, hence every label to the left is also increased.

4. Suppose the cap γ of $\overline{\mu}$ that starts at position i has odd label. This means that there is $x < i < y$ such that there is a certain move M of P which is either a right move deplacing a $\vee$ from x to y, or a flip move creating $\vee$'s at x and y. Note that in both cases, after M no further moves can be applied strictly between x and y. Now look at the position x in $\mu\overline{\mu}$. If there is a ray at that position we are done. If it is the end of a cap β, we claim that β has a strictly smaller label than γ. Indeed: after the move M, β has label 0 and γ has label at least 1, and every subsequent move that increases the label of β also increases the label of γ. $\square$

To prove that χ is a bijection, we will need to introduce the auxiliary notion of a *vertex numbering*:

Definition 5.4.4. Fix a weight $\lambda \in \Lambda$.

1. A *vertex numbering* of λ is an assignment of a natural number to each position in λ. We will denote the set of vertex numberings of λ by $N(\lambda)$. If y is a vertex numbering, $|y|$ will denote the sum $\sum_i y(i)$.

2. We define a map $\Phi : \{\text{Paths starting at } \lambda\} \to N(\lambda)$ as follows: given a path P, we assign to the i'th position of λ the number of moves that takes place at position i (i.e. the number of moves M_k in the path $P = M_1, \dots, M_r$ such that $M_k = R_i$ or $M_k = F_i$).

3. For μ another weight, we define a map $\Psi : D(\lambda, \mu) \to N(\lambda)$ as follows: given a labelling C, we assign to the i'th position of λ the number $\ell_i(\lambda, \mu) - x_i$, where x_i is the label (in C) of the cap that ends in i (and $x_i = 0$ if a ray ends in i).

Remark 5.4.5. We have that $|\Phi(P)|$ is equal to the length of the path P, and $|\Psi(C)| = \ell(\lambda, \mu) - 2|C|$.

The idea is that the inverse map to $\chi : P(\lambda, \mu) \to D(\lambda, \mu)$ can be computed as the following composition:

$$D(\lambda, \mu) \xrightarrow{\Psi} N(\lambda) \xrightarrow{\Phi^{-1}} P(\lambda, \mu).$$

Note that Φ can never be a bijection. However we will see that Φ is an injection with $\operatorname{im}\Phi = \operatorname{im}(\Psi)$, so that the above composition makes sense.
To make this precise, we prove the following lemma:

Lemma 5.4.6. *1. The map Φ is injective.*

2. The following diagram commutes:

$$\begin{tikzcd} P(\lambda,\mu) \arrow[rr, "\chi"] \arrow[dr, "\Phi"'] & & D(\lambda,\mu) \arrow[dl, "\Psi"] \\ & N(\lambda) & \end{tikzcd}$$

Proof. To prove the first statement: Suppose we are given a vertex numbering y of λ which is in the image of Φ. We will describe an algorithm for finding an inverse image of y, and then argue that every inverse image of y has to be given by this algorithm.
The algorithm is the following:

- If all numbers of the vertex numbering y are 0, stop.
- If there is a $\vee$ so that the corresponding number is not 0, consider the rightmost one (say it is at position i), and apply the right move R_i to it. Then decrease the number $y(i)$ by 1.
- If for every $\vee$ the corresponding number is 0, pick the leftmost $\wedge$ whose label is not 0 (say it is at position i), and apply the flip move F_i to it. Then decrease the number $y(i)$ by 1.

We note that this algorithm is a generalization of the procedure described in the proof of Proposition 1.1.8: if we start from the vertex numbering $y_i = \ell_i(\lambda, \mu)$, the algorithm will produce the trivial path/canonical sequence from λ to μ.
Now let P be any path such that $\Phi(P) = y$. Suppose we have already proven that the first k steps of the above algorithm correspond to the first k moves of P. Consider the situation after k steps of the algorithm.
If there is a $\vee$ so that the corresponding number is not 0, consider the rightmost one (say it is at position i). Then strictly less then $y(i)$ moves have been applied to position i in the first k moves of P, so P needs to contain another move at i. But by the definition of a path, if the next move is not at i, no moves will be applied to this $\vee$ again, and we get a contradiction. Hence the $k+1$'th move in P is an R-move at position i.
If for every $\vee$ the corresponding number is 0, pick the leftmost $\wedge$ whose label is not 0 (say it is at position i). Then strictly less then $y(i)$ moves have been applied to position i in the first k moves of P, so P needs to contain another move at i. But by the definition of a path, if the next move is not at i, no moves will be applied to this $\wedge$ again, and we get a contradiction. Hence the $k+1$'th move in P is an F-move at position i.
To prove the second statement: let $P \in P(\lambda, \mu)$ be a path, and write $\Phi(P) = y$. Then $y(i)$ is the number of moves in P that is made at position i. Note that the moves of P need to increase the number of symbols $\vee$ in λ that lie strictly to the right of i by $\ell_i(\lambda, \mu)$. Since every move at i will increase this number by 1, we get that $\ell_i(\lambda, \mu) = y(i) + c_i$, where c_i is the contribution of right moves that deplace a $\vee$ from strictly to the left of i to strictly to the right of i, and of flip moves not applied at i that create one or two $\vee$'s strictly to the right of i. Now it suffices to show that $c_i = x_i$, where x_i is the number appearing in the definition of Ψ (see Definition 5.4.4). But this follows easily from Definition 5.4.1. $\square$

Now our desired result easily follows:

Theorem 5.4.7. *1. The map χ defined in Definition 5.4.1 is a bijection.*

2. *For $C \in D(\lambda, \mu)$, its inverse image $\chi^{-1}(C)$ cap be computed as $\Phi^{-1}(\Psi(C))$.*
3. *The bijection χ is compatible with path length and value of labellings in the following sense: for $P \in P(\lambda, \mu)$, the length of P is equal to $\ell(\lambda, \mu) - 2|\chi(P)|$.*

Proof. Applying Lemma 5.4.6, we find that χ is injective since Φ is. By Theorems 5.2.8 and 5.3.4, $P(\lambda, \mu)$ and $D(\lambda, \mu)$ have the same cardinality. So χ is a bijection.
The second statement now follows immediately from Lemma 5.4.6, and the last one follows from Remark 5.4.5. $\square$

Note that the proof of the first part of Lemma 5.4.6 provides us with an explicit algorithm to compute $\chi^{-1}(C) = \Phi^{-1}(\Psi(C))$.

Example 5.4.8. As in Examples 5.2.7 and 5.3.2, let $\lambda = \wedge\wedge\wedge\wedge$ and $\mu = \wedge\vee\wedge\vee$. Then $P(\lambda,\mu)$ consists of the two paths P_1 and P_2 from Example 5.2.7, and $D(\lambda,\mu)$ consists of the two labellings C_1 and C_2 from Example 5.3.2. We find that $\chi(P_i) = C_i$ for $i \in \{1,2\}$. Note that the equation $\ell(P_i) = \ell(\lambda,\mu) - 2|C_i|$ holds for $i \in \{1,2\}$. We can also compute the corresponding vertex numberings: we get that $\Phi(P_1) = \Psi(C_1) = \wedge_2\wedge_1\wedge_1\wedge_0$, and $\Phi(P_2) = \Psi(C_2) = \wedge_2\wedge_0\wedge_0\wedge_0$.

5.5 Kostant weights

Definition 5.5.1. A weight $\mu \in \Lambda$ is called a *Kostant weight*, if

$$\sum_{k\geq 0} \dim \mathrm{EXT}^k_{\mathbb{D}_\Lambda}(V(\lambda), L(\mu)) \leq 1$$

for all $\lambda \in \Lambda$.

In other words, μ is a Kostant weight if and only for every weight λ, the linear projective resolution of $V(\lambda)$ contains at most one term $P_k(\lambda)$ which has a summand $P(\mu)\langle k\rangle$.

Proposition 5.5.2. *$\mu \in \Lambda$ is a Kostant weight if and only if $p_{\lambda,\mu}(q) = q^{\ell(\lambda,\mu)}$ for all $\lambda \leq \mu$.*

Proof. This follows immediately from Corollary 4.3.5, since we know that for $\lambda \leq \mu$, $p_{\lambda,\mu}(q)$ will always have a term $q^{\ell(\lambda,\mu)}$. □

Similar to [BS2, Lemma 7.2], we will now give an explicit combinatorial description of the Kostant weights. Our proof will use the description of dual Kazhdan–Lusztig polynomials via labelled diagrams (see Subsection 5.3).

Definition 5.5.3. Let χ be a sequence of $\wedge$'s and $\vee$'s. We say that a weight $\mu \in \Lambda$ is *χ-avoiding*, if the symbols of χ don't occur in Λ as a subsequence.
For example, μ is $\wedge\vee\wedge\vee$-avoiding if we can't find vertices $i < j < k < l$ whose labels in μ are $\wedge$, $\vee$, $\wedge$, $\vee$, respectively.

Theorem 5.5.4. *For a weight $\mu \in \Lambda$, the following are equivalent:*

1. *μ is a Kostant weight.*
2. *μ is $\wedge\vee\wedge\vee$-avoiding, $\wedge\wedge\vee\vee$-avoiding, $\vee\vee\wedge\vee$-avoiding and $\vee\wedge\vee\vee$-avoiding.*

Proof. Suppose that the second condition is satisfied. Let $\lambda \leq \mu$. We need to prove that there is only one λ-labelling of $\mu\overline{\mu}$, namely the one where all labels are 0. So let $C \in D(\lambda,\mu)$. Note that it suffices to prove that every small cap of $\mu\overline{\mu}$ has label 0 (by small cap we mean a cap for which no other cap is D-nested inside of it, see Definition 5.3.1). So let $i, i+1$ be vertices which are the endpoints of a small cap γ in $\overline{\mu}$.
We first consider the case where γ is undotted; then $\mu_i = \vee$ and $\mu_{i+1} = \wedge$. If in μ all symbols to the right of $i+1$ are $\wedge$'s, then since $\lambda \leq \mu$, also in λ all symbols to the right of $i+1$ (including $i+1$ itself) are $\wedge$'s, so that $\ell_i(\lambda,\mu) = 0$, and γ has label 0. If in μ there is a $\vee$ to the right of $i+1$, then by our hypothesis, $i = 1$ and μ looks like $\vee\wedge\cdots\vee\cdots$, where the dots mean a number of $\wedge$'s. Since μ has only 2 $\vee$'s and $\lambda \leq \mu$, λ has either exactly 2 $\vee$'s, or none at all. In the former case, we find that $\ell_1(\lambda,\mu) = 0$, so that γ has label 0. In the latter case we have $\lambda = \wedge\wedge\cdots\wedge$,

and $\ell_1(\lambda, \mu) = 1$. Now use use condition 4 from Definition 5.3.1: since there no no other ray or cap above or to the left of γ, it can't have an odd label, so γ has label 0.
Now consider the case where γ is dotted, so that i and $i+1$ have label $\wedge$. Since γ is a small cap, we have $i = 1$. By our hypothesis, μ has at most one symbol $\vee$. Since $\lambda \leq \mu$, λ must have the same number (0 or 1) of symbols $\vee$, so that all dotted caps are labelled 0.
For the other direction, suppose μ contains one of the sequences $\wedge\vee\wedge\vee$, $\vee\vee\wedge\vee$, $\vee\wedge\vee\vee$, $\wedge\wedge\vee\vee$. We need to find a $\lambda \leq \mu$ and a λ-labelling $C \in D(\lambda, \mu)$ such that C has at least one non-zero label.
First suppose μ contains $\wedge\vee\wedge\vee$. Clearly it's possible to find $i < j < j+1 < l$ such that $\mu_i = \wedge$, $\mu_j = \vee$, $\mu_{j+1} = \wedge$ and $\mu_l = \vee$. We pick i, j, l with this property such that $l - i$ is minimal. Let λ be the weight obtained from μ by interchanging the labels on the ith and lth vertices. Then $\ell_j(\lambda, \mu) = 1$, and $\overline{\mu}$ has a small cap γ with endpoints $j, j+1$. We claim that labelling this cap 1 and every other cap 0 gives a valid λ-labelling of $\mu\overline{\mu}$: the first three conditions are clearly satisfied; for the fourth one note that i is either the endpoint of a ray, or the endpoint of a cap with label 0.
Now suppose μ contains $\vee\vee\wedge\vee$. As before, pick $i < j < j+1 < l$ with these labels, such that $l - i$ is minimal with this property. Let λ be the weight obtained from μ by replacing the $\vee$'s on vertices i and l by $\wedge$'s. Then $\ell_j(\lambda, \mu) = 1$, and $\overline{\mu}$ has a small cap γ with endpoints $j, j+1$. As before, labelling this cap 1 and every other cap 0 gives a λ-labelling of $\mu\overline{\mu}$ as desired.
Next, suppose μ contains $\vee\wedge\vee\vee$. Pick $i < i+1 < j < l$ with these labels. Let λ be the weight obtained from μ by replacing the $\vee$'s on vertices j and l by $\wedge$'s. Then $\ell_i(\lambda, \mu) = 2$, and $\overline{\mu}$ has a small cap γ with endpoints $i, i+1$. Labelling this cap 2 and every other cap 0 clearly gives a λ-labelling of $\mu\overline{\mu}$ as desired.
Finally, suppose μ contains $\wedge\wedge\vee\vee$. We can suppose the first 2 symbols of μ are $\wedge$'s, since else μ also contains $\vee\wedge\vee\vee$ and we are done by the above. Let λ be the weight obtained from μ by replacing the $\vee$'s on vertices 1 and 2 by $\wedge$'s. Then $\overline{\mu}$ has a small (dotted) cap γ with endpoints $1, 2$. Labelling this cap 2 and every other cap 0 clearly gives a λ-labelling of $\mu\overline{\mu}$ as desired. □

Remark 5.5.5. If μ is a Kostant weight, we can imitate [BS2, Theorem 7.3] to obtain a so-called *BGG-resolution*

$$\cdots \to V_2 \to V_1 \to V_0 \to L(\mu) \to 0$$

of $L(\mu)$, where

$$V_k := \bigoplus_{\substack{\lambda \leq \mu \\ \ell(\lambda,\mu)=k}} V(\lambda)\langle k \rangle.$$

This construction goes back to Bernstein, Gelfand and Gelfand [BGG] and Lepowsky [L].

6 The subalgebra $\mathbb{H}_\Lambda$

6.1 Definition of $\mathbb{H}_\Lambda$

Let Λ be an n-block, and let $\Lambda^\circ \subseteq \Lambda$ be the subset of all weights λ for which $\overline{\lambda}$ has at most one ray (i.e. in the case that n is even $\overline{\lambda}$ has no rays, in the case that n is odd $\overline{\lambda}$ has exactly one ray). Define the idempotent $e_0 := \sum_{\lambda\in\Lambda^\circ} e_\lambda \in \mathbb{D}_\Lambda$. The algebra $\mathbb{H}_\Lambda$ is defined to be the idempotent truncation $e_0\mathbb{D}_\Lambda e_0$. Explicitly, it is the subalgebra $\mathbb{H}_\Lambda \subseteq \mathbb{D}_\Lambda$ with basis given by $\{\underline{\mu}\lambda\overline{\nu} | \mu, \nu \in \Lambda^\circ, \lambda \in \Lambda, \mu \subset \lambda \supset \nu\}$.
$\mathbb{H}_\Lambda$ is the type D analogue of Khovanov's original arc algebra (see for example [BS1]). It arises for example when studying type D Springer fibers, see [ES2].
$\mathbb{H}_\Lambda$ can also be described in the following way:

Fact 6.1.1 ([ES1, Corollary 9.3])**.** *Let $\Lambda = \Lambda_n^{\overline{0}}$. Then the algebra $\mathbb{H}_\Lambda$ is the endomorphism algebra of the sum of all indecomposable projective-injective $\mathbb{D}_\Lambda$-modules.*

From the definition of $\mathbb{H}_\Lambda$, we immediately deduce:

Proposition 6.1.2 ([ES1, Corollary 7.5])**.** *$\mathbb{H}_\Lambda$ is a cellular algebra.*

Proof. Since $\mathbb{H}_\Lambda$ is an idempotent truncation of the cellular algebra $\mathbb{D}_\Lambda$, this follows from [KX, Proposition 4.3]. $\square$

In the language of section 2.2, it holds that the cell modules of $\mathbb{H}_\Lambda$ are indexed by Λ, while the simple and projective modules are indexed by Λ_0. In particular, for $n > 1$, $\mathbb{H}_\Lambda$ is no longer a quasi-hereditary algebra (see Proposition 2.2.4).
From now on, we will for sake of simplicity assume that Λ is an n-block with n even; we will write $n = 2k$. Note that this means that all appearing circle diagrams consist of circles only (and no lines).

6.2 Symmetric algebra structure

In this section we will prove that the algebra $\mathbb{H}_\Lambda$ can be given the structure of a symmetric algebra. Let us first recall the definition:

Definition 6.2.1. A *symmetric algebra* (or *symmetric Frobenius algebra*) is a finite-dimensional k-algebra A together with a linear functional $\tau : A \to k$ such that the induced bilinear form $\beta : A \otimes A \to k : a \otimes b \mapsto \tau(ab)$ is nondegenerate and symmetric.

For a basis vector $a\lambda b \in \mathbb{H}_\Lambda$, let $(a\lambda b)^\#$ denote the basis vector of $\mathbb{H}_\Lambda$ whose diagram is obtained by reversing the orientation of every circle in the diagram $b^*\lambda a^*$. Note that the top degree component of $\mathbb{H}_\Lambda$ lives in degree n, and has basis $\{e_\lambda^\# | \lambda \in \Lambda^\circ\}$.
In the following lemma, we want to compute the product $(a\lambda b)(a\lambda b)^\#$. It is not hard to see that, up to a sign, this product equals $a\mu a^*$ (where μ is the unique weight such that all circles in $a\mu a^*$ are clockwise). To write down what this sign precisely is, we need the following auxiliary definitions:

Definition 6.2.2. Let b be a cap diagram with no rays. For γ a cap in b, we will write $i(\gamma)$ resp. $j(\gamma)$ for its left resp. right endpoint. Define $\operatorname{sgn}(b) := \prod_{\gamma\in\operatorname{caps}(b)}(-1)^{i(\gamma)}$.
For ab a circle diagram, define $T(ab) \subseteq \{1,\cdots,n\}$ to be the set of indices i such that if we walk from i to $t(C_i)$ along cups and caps, we use an even number of cups/caps. (Here C_i is the circle in ab to which i belongs, and $t(C)$ is the tag of C, as defined in Definition 1.2.10.)
Furthermore, we define $T'(ab) \subseteq T(ab)$ to be the elements of $T(ab)$ which are the left end of an undotted cap.
Finally, we define

$$\operatorname{Sgn}(ab) := \left(\prod_{C \text{ circle in } ab}(-1)^{t(C)-1}\right)(-1)^{|T'(ab)|}(-1)^{|T'(b^*a^*)|}.$$

It follows immediately from the definition that $\operatorname{Sgn}(ab) = \operatorname{Sgn}(b^*a^*)$.

Lemma 6.2.3. *Let $a\lambda b$ be a basis vector of $\mathbb{H}_\Lambda$. Then*

$$(a\lambda b)(a\lambda b)^\# = \operatorname{sgn}(b)\operatorname{Sgn}(ab)(a\mu a^*)$$

where μ is the unique weight such that every circle in $a\mu a^*$ is oriented clockwise.

Proof. Recalling Definition 1.3.5, we first need to compute

$$y := \chi_{ab,b^*a^*}\left(\Psi_{ab}(a\lambda b)\otimes\Psi_{b^*a^*}((a\lambda b)^\#)\right) \in \mathcal{M}(aa^*).$$

First note that

$$\Psi_{ab}(a\lambda b)\cdot\Psi_{b^*a^*}((a\lambda b)^\#) = \prod_{C \text{ circle in } ab} X_{t(C)}.$$

(This follows directly from the definition of $(a\lambda b)^\#$.)
Now we need to compute $y \in \mathcal{M}(aa^*)$ using some surgery procedure (see Definition 1.3.4). Suppose ab has m circles. We consider the surgery procedure where we write ab under b^*a^* and then always glue the rightmost cup/cap pair that hasn't been glued already (with "rightmost cup/cap" we mean more precisely the cup/cap with the rightmost right end). It's easy to see this is a well-defined surgery procedure (i.e. all intermediate steps are admissible diagrams).
For every circle in ab, the first surgery that involves this circle is necessarily a merge. So of our k surgeries, there are at least m merges. Since at the start we have $2m$ circles and at the end we have k circles, the remaining $k-m$ surgeries are necessarily splits.
For a cap γ, let $i(\gamma)$ resp. $j(\gamma)$ denote its left end resp. right end. Let X be the set of caps γ in b for which $j(\gamma) \neq t(C_\gamma)$ (where C_γ is the circle where γ belongs to). Then we have that

$$y = \left(\prod_{C \text{ circle in } ab} X_{t(C)}\right)\left(\prod_{\gamma\in X}\left((-1)^{i(\gamma)}(X_{j(\gamma)} \pm X_{i(\gamma)})\right)\right),$$

where the signs depend on whether or not γ is dotted. Now note that

$$\operatorname{sgn}(b) = \prod_{\gamma\in\operatorname{caps}(b)}(-1)^{i(\gamma)} = \left(\prod_{\gamma\in X}(-1)^{i(\gamma)}\right)\left(\prod_{C \text{ circle in } ab}(-1)^{t(C)-1}\right).$$

(Here we used the obvious observation that, for every cap γ, $i(\gamma)$ and $j(\gamma)$ have different parity.) So this becomes

$$y = \operatorname{sgn}(b)\left(\prod_{C \text{ circle in } ab} (-1)^{t(C)-1}\right)\left(\prod_{C \text{ circle in } ab} X_{t(C)}\right)\left(\prod_{\gamma\in X}\left(X_{j(\gamma)} \pm X_{i(\gamma)}\right)\right).$$

If we expand the product $(\prod_{C \text{ circle in } ab} X_{t(C)}) \cdot (\prod_{\gamma\in X}(X_{j(\gamma)} \pm X_{i(\gamma)}))$, we find that exactly one term survives, which equals $(-1)^{|T'(ab)|}\prod_{i\in T(ab)} X_i$. So we get

$$y = \operatorname{sgn}(b)(-1)^{|T'(ab)|}\left(\prod_{C \text{ circle in } ab} (-1)^{t(C)-1}\right)\left(\prod_{i\in T(ab)} X_i\right).$$

Finally, we note that

$$\prod_{i\in T(ab)} X_i = (-1)^{|T'(b^*a^*)|}\left(\prod_{i \text{ right end of a cap in } a^*} X_i\right) \in \mathcal{M}(aa^*)$$

so that

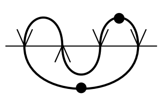
$$\begin{aligned} y &= \operatorname{sgn}(b)\left(\prod_{C \text{ circle in } ab} (-1)^{t(C)-1}\right)(-1)^{|T'(ab)|}(-1)^{|T'(b^*a^*)|}\left(\prod_{i \text{ right end of a cap in } a^*} X_i\right) \\ &= \operatorname{sgn}(b)\operatorname{Sgn}(ab)\left(\prod_{i \text{ right end of a cap in } a^*} X_i\right). \end{aligned}$$

Now we can compute that

$$\begin{aligned} (a\lambda b)(a\lambda b)^{\#} &= \Psi_{aa^*}^{-1}(y) \\ &= \operatorname{sgn}(b)\operatorname{Sgn}(ab)\Psi_{aa^*}^{-1}\left(\prod_{i \text{ right end of a cap in } a^*} X_i\right) \\ &= \operatorname{sgn}(b)\operatorname{Sgn}(ab)(a\mu a^*), \end{aligned}$$

where μ is the unique weight such that every circle in $a\mu a^*$ is oriented clockwise. □

Example 6.2.4. Consider the diagram

$a\lambda b =$  .

We find that $T(ab) = \{2, 4\}$, $T'(ab) = \emptyset$, and $T'(b^*a^*) = \{2\}$. So we get $\operatorname{sgn}(b) = (-1)^1 \cdot (-1)^3 = +1$, and $\operatorname{Sgn}(ab) = (-1)^{4-1} \cdot (-1)^0 \cdot (-1)^1 = +1$.
So $(a\lambda b)(a\lambda b)^{\#} = a\mu a^*$, where $\mu = \vee\wedge\vee\vee$ is the unique weight such that all circles in $a\mu a^*$ are clockwise. Note that this is precisely the product we computed in Example 1.3.7.

We will introduce one more notation: write $\widetilde{\operatorname{Sgn}}(ab) := \operatorname{sgn}(a^*)\operatorname{sgn}(b)\operatorname{Sgn}(ab)$. Note that $\widetilde{\operatorname{Sgn}}(ab) = \widetilde{\operatorname{Sgn}}(b^*a^*)$.

Theorem 6.2.5. *Define $\tau : \mathbb{H}_\Lambda \to k$ to be the linear map sending every basis vector $(a\mu a^*)$ of degree n to $\operatorname{sgn}(a^*)$, and every other basis vector to 0. Then*

$$\tau((a\lambda b)(c\mu d)) = \begin{cases} \widetilde{\operatorname{Sgn}}(ab), & \text{if } (c\mu d) = (a\lambda b)^\#, \\ 0, & \text{else.} \end{cases}$$

Since $\widetilde{\operatorname{Sgn}}(ab) = \widetilde{\operatorname{Sgn}}(b^* a^*)$, this implies that the induced bilinear form $\beta : \mathbb{H}_\Lambda \otimes \mathbb{H}_\Lambda \to k$ is nondegenerate and symmetric. Hence τ is a symmetrizing form making $\mathbb{H}_\Lambda$ into a symmetric algebra.

Proof. It's clear that $\tau((a\lambda b)(c\mu d)) = 0$ unless $d = a^*$ and $c = b^*$. If a circle in $a\lambda b$ and its corresponding circle in $b^*\mu a^*$ are both clockwise oriented, the product will be 0. On the other hand, if a circle in $a\lambda b$ and its corresponding circle in $b^*\mu a^*$ are both counterclockwise oriented, the product will not have maximal degree. So we conclude that $\tau((a\lambda b)(c\mu d)) = 0$ unless $(c\mu d) = (a\lambda b)^\#$.
By Lemma 6.2.3, we find that

$$\tau((a\lambda b)(a\lambda b)^\#) = \operatorname{sgn}(b)\operatorname{Sgn}(ab)\tau(a\mu a) = \operatorname{sgn}(b)\operatorname{Sgn}(ab)\operatorname{sgn}(a^*) = \widetilde{\operatorname{Sgn}}(ab),$$

so our proof is finished. □

Remark 6.2.6. The fact that $\mathbb{H}_\Lambda$ is a symmetric algebra was already known: it follows from [MS, Theorem 5.4(1)] (see also [HL]), taking into account Fact 6.1.1 and the isomorphism $\mathbb{D}_\Lambda \cong \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}(2n))$ mentioned in the introduction. However, our proof is much more explicit, and provides us with an explicit symmetrizing form.

6.3 The comultiplication

Let A be a symmetric algebra, with symmetrizing form τ and pairing β. The requirement that β is nondegenerate can be equivalently formulated as the existence of a linear map $\gamma : k \to A \otimes A$ such that the composition

$$A \xrightarrow{\gamma\otimes\mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes\beta} A \tag{6.1}$$

is the identity. (To verify this: note that clearly such a γ cannot exist if β is degenerate. If β is nondegenerate, we can find an orthonormal basis $\{v_i | i \in I\}$ for A, and define γ by mapping 1 to $\sum v_i \otimes v_i$.)
We can use this to define the *comultiplication map* $\Delta : A \to A \otimes A$ as the composition

$$\Delta : A \xrightarrow{\gamma\otimes\mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes\mu} A \otimes A. \tag{6.2}$$

(Here $\mu : A \otimes A \to A$ is the multiplication map.) The maps μ and Δ satisfy the so-called *Frobenius relation*: the following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\mathrm{id}"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] & A \otimes A \end{tikzcd}$$

Remark 6.3.1. All of the above can be generalized to arbitrary *Frobenius algebras*. These are algebras A equipped with an associative nondegenerate pairing β, which doesn't need to be symmetric. See [Ko] for more about Frobenius algebras, including a connection 2-dimensional TQFTs.

We will now explicitly describe the comultiplication map for our case $A = \mathbb{H}_\Lambda$.
We first need to find the map $\gamma : k \to \mathbb{H}_\Lambda \otimes \mathbb{H}_\Lambda$ satisfying (6.1).

Claim 6.3.2. *The linear map $\gamma : k \to \mathbb{H}_\Lambda \otimes \mathbb{H}_\Lambda$ defined by sending 1 to*

$$\sum_{c\mu d \in \mathbb{H}_\Lambda} \widetilde{\mathrm{Sgn}}(cd)(c\mu d) \otimes (c\mu d)^\#$$

satisfies (6.1).

Proof. This follows from Theorem 6.2.5: for $a\lambda b \in \mathbb{H}_\Lambda$, we have that

$$\sum_{c\mu d \in \mathbb{H}_\Lambda} \widetilde{\mathrm{Sgn}}(cd)(c\mu d)\tau\Big((c\mu d)^\#(a\lambda b)\Big) = a\lambda b. \qquad \square$$

Now we can explicitly write down the comultiplication as in (6.2): we get that

$$\Delta(x\nu y) = \sum_{c\mu d \in \mathbb{H}_\Lambda} \widetilde{\mathrm{Sgn}}(cd)(c\mu d) \otimes \Big((c\mu d)^\#(x\nu y)\Big).$$

This can be rewritten as follows:

$$\begin{aligned}
\Delta(x\nu y) &= \sum_{a\lambda b \in \mathbb{H}_\Lambda} \widetilde{\mathrm{Sgn}}(ab)(a\lambda b)^\# \otimes \Big((a\lambda b)(x\nu y)\Big) \\
&= \sum_{\substack{a\lambda b \in \mathbb{H}_\Lambda \\ c\mu d \in \mathbb{H}_\Lambda}} \widetilde{\mathrm{Sgn}}(cd)\,\widetilde{\mathrm{Sgn}}(ab)(a\lambda b)^\# \otimes (c\mu d)^\#\tau\Big((c\mu d)(a\lambda b)(x\nu y)\Big) \\
&= \sum_{\substack{a\lambda b, c\mu d \in \mathbb{H}_\Lambda \\ (c\mu d)(a\lambda b) = (x\nu y)^\#}} \widetilde{\mathrm{Sgn}}(xy)\,\widetilde{\mathrm{Sgn}}(cd)\,\widetilde{\mathrm{Sgn}}(ab)(a\lambda b)^\# \otimes (c\mu d)^\# \\
&= \sum_{\substack{a\lambda b, c\mu d \in \mathbb{H}_\Lambda \\ (c\mu d)(a\lambda b) = (x\nu y)^\#}} \mathrm{Sgn}(cb)\,\mathrm{Sgn}(ca^*)\,\mathrm{Sgn}(ab)(a\lambda b)^\# \otimes (c\mu d)^\#.
\end{aligned}$$

We note that $\deg(\Delta(x\nu y)) = \deg(x\nu y) + n$.
In the appendix we will describe the comultiplication for the sign-adjusted version of $\mathbb{H}_\Lambda$, as defined by Ehrig, Tubbenhauer and Wilbert [ETW]. The formula will look a lot nicer since no signs will show up.

Outlook and further questions

There is a connection between type D arc algebras, and the representation theory of the orthosymplectic Lie supergroup $OSp(r|2n)$. In [ES3], it is shown that the blocks of the category of finite-dimensional representations of $OSp(r|2n)$ can be described by certain algebras $A_{\mathfrak{B}}$, which can be described as a subquotient of a certain limit of type D arc algebras $\mathbb{D}_\Lambda$. One can think of $A_{\mathfrak{B}}$ as an infinite-dimensional version of the type D arc algebra, where the occurring weights are infinite sequences (indexed by $\mathbb{Z}_{>0}$) of $\wedge$'s and $\vee$'s, and the occurring cup and cap diagrams are infinite, but with a fixed (finite) number of cups/caps. As one can explicitly check in small examples, $A_{\mathfrak{B}}\text{- mod}$ is not a highest weight category.
It was conjectured in [ES3, Conjecture D] that the algebras $A_{\mathfrak{B}}$ are Koszul. A similar result is true in the type A case: here the category of finite-dimensional representations of $GL(m|n)$ is described by certain infinite-dimensional type A arc algebras [BS4], and it was already proven in [BS2, Corollary 5.13] that these are Koszul. The proof essentially follows from describing the infinite-dimensional algebra as a limit of finite-dimensional ones, for which Koszulity has already been proven. However, the methods from [BS2] do not seem to suffice to show that our type D algebras $A_{\mathfrak{B}}$ are Koszul.

A Connection to the sign-adjusted algebra

A.1 The sign-adjusted multiplication rule

In [ETW, 5.2], an alternative multiplication rule is introduced, leading to a *sign-adjusted version* $\overline{\mathbb{D}}_\Lambda$ of our algebra. The multiplication rule in $\overline{\mathbb{D}}_\Lambda$ is the same as the one for $\mathbb{D}_\Lambda$, and there is an isomorphism $\mathbb{D}_\Lambda \cong \overline{\mathbb{D}}_\Lambda$ of algebras ([ETW, Proposition 5.11]). In practice, the sign-adjusted version is often more convenient to work with since there are fewer minus-signs during computations. Perhaps more importantly, $\overline{\mathbb{D}}_\Lambda$ has a topological interpretation in terms of certain TQFTs (see [ETW]).
We will now describe how one can define the sign-adjusted multiplication. The definition will be very similar to the one for the ordinary multiplication, see Subsection 1.3.

Definition A.1.1. Let $\lambda, \nu \in \Lambda$ and assume that the circle diagram $\underline{\lambda}\overline{\nu}$ can be oriented. Then $\tilde{I}_{\underline{\lambda}\overline{\nu}}$ is defined to be the ideal in $k[X_i|1 \leq i \leq n]$ generated by:

- X_i^2 for all $1 \leq i \leq n$.
- $X_i - X_j$ if i, j are connected by an undotted cup or cap.
- $X_i + X_j$ if i, j are connected by a dotted cup or cap.
- X_i if i is the endpoint of a ray.

Define the vector space $\tilde{\mathcal{M}}(\underline{\lambda}\overline{\mu}) := k[X_i|1 \leq i \leq n]/\tilde{I}_{\underline{\lambda}\overline{\nu}}$.

Similar to Proposition 1.3.2, we have:

Proposition A.1.2. *There is an isomorphism of vector spaces*

$$\begin{aligned} \Psi_{\underline{\lambda}\overline{\nu}} : \quad {}_\lambda(\mathbb{D}_\Lambda)_\nu &\longrightarrow \tilde{\mathcal{M}}(\underline{\lambda}\overline{\nu}), \\ \underline{\lambda}\mu\overline{\nu} &\longmapsto \prod_{C \in \mathcal{C}_{\text{clock}}(\underline{\lambda}\mu\overline{\nu})} X_{t(C)} \end{aligned}$$

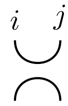
where $\mathcal{C}_{\text{clock}}(\underline{\lambda}\mu\overline{\nu})$ is the set of clockwise oriented circles in $\underline{\lambda}\mu\overline{\nu}$.

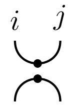
Definition A.1.3. Suppose we are given two circle diagrams ab and b^*c. As before, we stack them on top of each other yielding a stacked circle diagram atc, and we fix a surgery procedure from atc to aec.
In contrast to the ordinary multiplication rule, we don't require every intermediate diagram to be admissible. (It turns out every sequence of surgery moves will give the same result, even if some of the intermediate diagrams are not admissible.)
Recall that every surgery move was either a split, a merge, or a reconnect. We will need to distinguish between so-called *nested splits* and *non-nested splits.* We know that in a split one connected component is split in two. In the case where both of these resulting components are circles one of which is contained in the other one, the split is called a *nested split* (and else it is *non-nested*).
As before, we define a map $\tilde{\chi}' : k[X_i|1 \leq i \leq n] \otimes k[X_i|1 \leq i \leq n] \to k[X_i|1 \leq i \leq n]$ associated to our surgery procedure. To obtain the image of $f \otimes g \in k[X_i|1 \leq i \leq n] \otimes k[X_i|1 \leq i \leq n]$, apply the following algorithm:

1. Start by putting $y = f \cdot g \in k[X_i | 1 \leq i \leq n]$.
2. For each surgery in the surgery procedure, consult the following table:

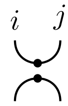
Merge: leave y as it is,

Split: multiply y with

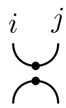
$(X_j + X_i)$ if 

$(X_j + X_i)$ if 

and the split is not nested

$(X_j - X_i)$ if 

and the split is nested with i on the outer circle

$(-X_j + X_i)$ if 

and the split is nested with j on the outer circle

Reconnect: if the two lines (before the surgery) are propagating, and the resulting diagram is orientable, leave y as it is.
Else multiply y by 0.

3. We define $\tilde{\chi}'(f \otimes g)$ to be equal to the resulting y.

One can show that this map $\tilde{\chi}' : k[X_i | 1 \leq i \leq n] \otimes k[X_i | 1 \leq i \leq n] \to k[X_i | 1 \leq i \leq n]$ descends to a map $\tilde{\chi} : \tilde{\mathcal{M}}(ab) \otimes \tilde{\mathcal{M}}(b^*c) \to \tilde{\mathcal{M}}(ac)$, and that this map $\tilde{\chi}$ doesn't depend on our chosen surgery procedure. we will from now on write $\tilde{\chi}_{ab,b^*c} : \tilde{\mathcal{M}}(ab) \otimes \tilde{\mathcal{M}}(b^*c) \to \tilde{\mathcal{M}}(ac)$.

Now the sign-adjusted multiplication is defined in exactly the same way as the ordinary multiplication:

Definition A.1.4. Consider the composition

$$\Phi_{\lambda\nu\nu'} : {}_\lambda(\mathbb{D}_\Lambda)_\nu \otimes {}_\nu(\mathbb{D}_\Lambda)_{\nu'} \xrightarrow{\Psi_{\underline{\lambda}\overline{\nu}} \otimes \Psi_{\underline{\nu}\overline{\nu'}}} \tilde{\mathcal{M}}(\underline{\lambda}\overline{\nu}) \otimes \tilde{\mathcal{M}}(\underline{\nu}\overline{\nu'}) \xrightarrow{\tilde{\chi}_{\underline{\lambda}\overline{\nu},\underline{\nu}\overline{\nu'}}} \tilde{\mathcal{M}}(\underline{\lambda}\overline{\nu'}) \xrightarrow{(\Psi_{\underline{\lambda}\overline{\nu'}})^{-1}} {}_\lambda(\mathbb{D}_\Lambda)_{\nu'},$$

where the second map is the map defined above.
For two basis vectors $\underline{\lambda}\mu\overline{\nu}$ and $\underline{\lambda'}\mu'\overline{\nu'}$ of $\mathbb{D}_\Lambda$, we define their product as follows:

$$(\underline{\lambda}\mu\overline{\nu}) \cdot (\underline{\lambda'}\mu'\overline{\nu'}) \quad = \quad \begin{cases} \Phi_{\lambda\nu\nu'}((\underline{\lambda}\mu\overline{\nu}) \otimes (\underline{\lambda'}\mu'\overline{\nu'})), & \text{if } \nu = \lambda', \\ 0, & \text{else.} \end{cases}$$

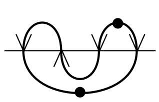
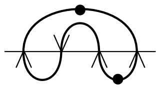
Fact A.1.5. *[ETW, Proposition 5.11] Extending the above product bilinearly to $\mathbb{D}_\Lambda$ equips $\mathbb{D}_\Lambda$ with the structure of an associative unital graded algebra, where the grading is the one defined in Definition 1.2.10. We will denote the algebra with this sign-adjusted multiplication by $\overline{\mathbb{D}}_\Lambda$. Moreover, there is an isomorphism of algebras*

$$\mathbb{D}_\Lambda \xrightarrow{\cong} \overline{\mathbb{D}}_\Lambda$$

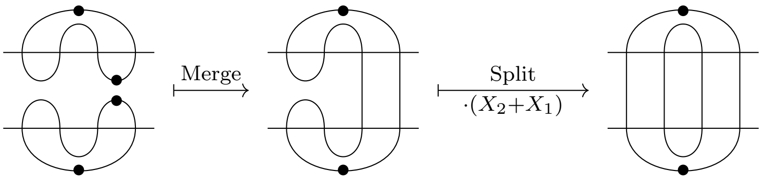
given by linearly extending the map

$$\begin{aligned} \mathrm{sign} : \quad \mathbb{B}_\Lambda \quad &\longrightarrow \quad \mathbb{B}_\Lambda, \\ (a\lambda b) &\longmapsto \left(\prod_{\substack{C \text{ clockwise} \\ \text{circle in } (a\lambda b)}} (-1)^{t(C)+1} \right) (a\lambda b). \end{aligned}$$

Example A.1.6. We again compute the product $(\underline{\lambda}\mu\overline{\nu}) \cdot (\underline{\lambda'}\mu'\overline{\nu'})$, where

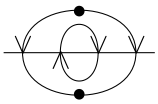
$\underline{\lambda}\mu\overline{\nu} =$  and $\underline{\lambda'}\mu'\overline{\nu'} =$ 

but now using the sign-adjusted multiplication rule.
First we compute $\Psi_{\underline{\lambda}\overline{\nu}}(\underline{\lambda}\mu\overline{\nu}) = X_1 \in \tilde{\mathcal{M}}(\underline{\lambda}\overline{\nu})$ and $\Psi_{\underline{\lambda'}\overline{\nu'}}(\underline{\lambda'}\mu'\overline{\nu'}) = 1 \in \tilde{\mathcal{M}}(\underline{\lambda'}\overline{\nu'})$.
Next we need to compute $\tilde{\chi}(X_4 \otimes 1)$. Using the following surgery procedure



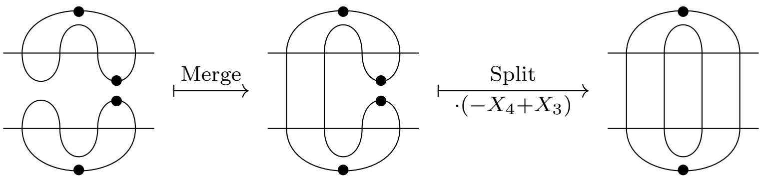
we find that $\tilde{\chi}(X_4 \otimes 1) = X_4(X_2 + X_1) = X_4(X_3 - X_4) = X_3X_4 \in \tilde{\mathcal{M}}(\underline{\lambda}\overline{\nu'})$.
Finally, we find that

$$(\underline{\lambda}\mu\overline{\nu}) \cdot (\underline{\lambda'}\mu'\overline{\nu'}) = \Psi^{-1}_{\underline{\lambda}\overline{\nu'}}(X_3X_4)$$

$=$ 

$$= \underline{\lambda}\mu\overline{\lambda}.$$

If we would have chosen the other possible surgery procedure



we would have gotten the same result.
We will now check that at least in this example, the isomorphism sign is compatible with our two multiplication rules. We find that

$$\begin{aligned} \mathrm{sign}(\underline{\lambda}\mu\overline{\nu}) &= (-1)^{4+1}(\underline{\lambda}\mu\overline{\nu}) = -(\underline{\lambda}\mu\overline{\nu}) \\ \mathrm{sign}(\underline{\lambda'}\mu'\overline{\nu'}) &= (\underline{\lambda'}\mu'\overline{\nu'}) \\ \mathrm{sign}(\underline{\lambda}\mu\overline{\lambda}) &= (-1)^{3+1}(-1)^{4+1}(\underline{\lambda}\mu\overline{\lambda}) = -(\underline{\lambda}\mu\overline{\lambda}). \end{aligned}$$

Now, comparing this example with Example 1.3.7, we see that

$$\begin{aligned}
\operatorname{sign}\left((\underline{\lambda}\mu\overline{\nu}) \cdot_{\mathbb{D}_\Lambda} (\underline{\lambda'}\mu'\overline{\nu'})\right) &= \operatorname{sign}(\underline{\lambda}\mu\overline{\lambda}) \\
&= -(\underline{\lambda}\mu\overline{\lambda}) \\
&= -(\underline{\lambda}\mu\overline{\nu}) \cdot_{\overline{\mathbb{D}}_\Lambda} (\underline{\lambda'}\mu'\overline{\nu'}) \\
&= \operatorname{sign}(\underline{\lambda}\mu\overline{\nu}) \cdot_{\overline{\mathbb{D}}_\Lambda} \operatorname{sign}(\underline{\lambda'}\mu'\overline{\nu'})
\end{aligned}$$

as desired.

Remark A.1.7. The sign-adjusted version has some advantages: In the original multiplication rule, some surgery moves give rise to signs that depend on the whole diagram. This *non-locality* makes some of the proofs in [ES1] (for example the proof that $\mathbb{D}_\Lambda$ is an associative algebra) significantly more complicated. This non-locality is no longer an issue in the sign-adjusted algebra $\overline{\mathbb{D}}_\Lambda$. Also in explicit computations in $\overline{\mathbb{D}}_\Lambda$, fewer signs tend to show up. For example if there are no dotted cups or caps there will be no minus signs at all.
On the other hand, the original arc algebra $\mathbb{D}_\Lambda$ is more closely related to our original motivation of studying these algebras in the first place: for example the equivalence $\mathbb{D}_\Lambda\text{- mod} \cong \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}(2k))$ is clearer when we use the original multiplication rule instead of the sign-adjusted one.

A.2 Symmetric algebra structure of the sign-adjusted algebra

We repeat Section 6 for the sign-adjusted multiplication rule.
As before, let $\Lambda^\circ \subseteq \Lambda$ be the subset of all weights λ for which $\overline{\lambda}$ has at most one ray. Define the idempotent $e_0 := \sum_{\lambda\in\Lambda^\circ} e_\lambda \in \overline{\mathbb{D}}_\Lambda$. The algebra $\overline{\mathbb{H}}_\Lambda$ is defined to be the idempotent truncation $e_0\overline{\mathbb{D}}_\Lambda e_0$. Explicitly, it is the subalgebra $\overline{\mathbb{H}}_\Lambda \subseteq \overline{\mathbb{D}}_\Lambda$ with basis given by $\{\underline{\mu}\lambda\overline{\nu} | \mu, \nu \in \Lambda^\circ, \lambda \in \Lambda, \mu \subset \lambda \supset \nu\}$.
As in Section 6, we find that $\overline{\mathbb{H}}_\Lambda$ is a cellular algebra. In fact, the isomorphism $\mathbb{D}_\Lambda \cong \overline{\mathbb{D}}_\Lambda$ from Fact A.1.5 restricts to an isomorphism $\mathbb{H}_\Lambda \cong \overline{\mathbb{H}}_\Lambda$.
From now on, we will for sake of simplicity assume that Λ is an n-block with n even; we will write $n = 2k$. Note that this means that all appearing circle diagrams consist of circles only (and no lines).
For a basis vector $a\lambda b \in \overline{\mathbb{H}}_\Lambda$, let $(a\lambda b)^\#$ denote the basis vector of $\overline{\mathbb{H}}_\Lambda$ whose diagram is obtained by reversing the orientation of every circle in the diagram $b^*\lambda a^*$. Note that the top degree component of $\overline{\mathbb{H}}_\Lambda$ lives in degree n, and has basis $\{e_\lambda^\# | \lambda \in \Lambda^\circ\}$.
In the following lemma, we want to compute the product $(a\lambda b)(a\lambda b)^\#$. It is not hard to see that, up to a sign, this product equals $a\mu a^*$ (where μ is the unique weight such that all circles in $a\mu a^*$ are clockwise). If we would be using the ordinary multiplication of $\mathbb{D}_\Lambda$, this sign would be rather complicated to write down. However, in the sign-adjusted algebra $\overline{\mathbb{D}}_\Lambda$ the sign is always positive, as we will now prove.
During this section, whenever we have a cup or cap γ, we will denote its left end resp. right end with $i(\gamma)$ resp. $j(\gamma)$.
We start with some auxiliary definitions.

Definition A.2.1. For ab a circle diagram, define $T(ab) \subseteq \{1, \cdots, n\}$ to be the set of indices i such that if we walk from i to $t(C_i)$ along cups and caps, we use an even number of cups/caps. (Here C_i is the circle in ab to which i belongs.)

Note that $i \in T(ab)$ if and only if $i \equiv t(C_i) \pmod 2$.

Definition A.2.2. Let ab be a circle diagram (without rays). To every point x in the semi-infinite strip $R := \mathbb{R}_{\geq 0} \times [-\frac{1}{2}, \frac{1}{2}]$ that does not belong to a a cup or cap, the *degree of nestedness of* x is defined to be the number of circles C in ab for which x is contained in the interior of x.

We can use this topological notion to prove the following auxiliary lemma (of which we will only need the 3rd part):

Lemma A.2.3. *Let ab be a circle diagram.*

1. *Let $x \in [0, n] \setminus \mathbb{Z} \subset R$. Then the degree of nestedness of x is even whenever $\lfloor x \rfloor$ is even, and odd whenever $\lfloor x \rfloor$ is odd.*
2. *Let C be a circle that is not contained in the interior of another circle. Then $t(C)$ is even.*
3. *Let γ be a cup in a that is not nested inside another cup. Then $j(\gamma) \in T(ab)$ and $i(\gamma) \notin T(ab)$.*

Proof. The first part follows from the observation that passing through an arc will increase or decrease the degree of nestedness by 1.
For the second part: if $t(C)$ is odd, the points directly to the right of $t(C)$ will have an odd degree of nestedness, i.e. they are contained in at least one circle (necessarily different from C), hence C is contained in the interior of another circle.
For the third part: it is easy to see that $i(\gamma)$ is odd and $j(\gamma)$ is even. So we are done by the second part. □

Theorem A.2.4. *Let $a\lambda b$ be a basis vector of $\overline{\overline{\mathbb{H}}}_\Lambda$. Then*

$$(a\lambda b)(a\lambda b)^\# = (a\mu a^*)$$

where μ is the unique weight such that every circle in $a\mu a^*$ is oriented clockwise.

Proof. Recalling Definition A.1.4, we first need to compute

$$y := \tilde{\chi}_{ab,b^*a^*}\left(\Psi_{ab}(a\lambda b) \otimes \Psi_{b^*a^*}((a\lambda b)^\#)\right) \in \mathcal{M}(aa^*).$$

First note that

$$\Psi_{ab}(a\lambda b) \otimes \Psi_{b^*a^*}\left((a\lambda b)^\#\right) = \prod_{C \text{ circle in } ab} X_{t(C)}$$

(This follows directly from the definition of $(a\lambda b)^\#$.)
Now we need to compute $y \in \tilde{\mathcal{M}}(aa^*)$ using some surgery procedure. Suppose ab has m circles. Now consider the surgery procedure where we write ab under b^*a^* and then always glue the rightmost cup/cap pair that hasn't been glued already (with "rightmost cup/cap" we mean more precisely the cup/cap with the rightmost right end).
For every circle in ab, the first surgery that involves this circle is necessarily a merge. So of our k surgeries, there are at least m merges. Since at the start we have $2m$ circles and at the end we have k circles, the remaining $k - m$ surgeries are necessarily splits.
Let X be the set of caps γ in b for which $j(\gamma) \neq t(C_\gamma)$ (where C_γ is the circle where γ belongs to). Then we have that

$$y = \left(\prod_{C \text{ circle in } ab} X_{t(C)}\right)\left(\prod_{\gamma \in X} (\pm X_{j(\gamma)} \pm X_{i(\gamma)})\right), \tag{A.1}$$

where the signs are always positive, except for the case of a dotted nested split.
The product $(a\lambda b)(a\lambda b)^{\#}$ is just equal to $\Psi^{-1}_{aa^*}(y)$.
We now need the following auxiliary lemma, whose proof will be postponed:

Lemma A.2.5. *In the expression (A.1), any term X_i with $i \in T(ab)$ has a positive sign.*

If we expand the product $(\prod_{C \text{ circle in } ab} X_{t(C)}) \cdot (\prod_{\gamma \in X} (\pm X_{j(\gamma)} \pm X_{i(\gamma)}))$, we find that exactly one term survives, which equals $\prod_{i \in T(ab)} X_i$, where the sign is positive because of Lemma A.2.5. Now we can compute that

$$\begin{aligned}(a\lambda b)(a\lambda b)^{\#} &= \Psi^{-1}_{aa^*}(y) \\ &= \Psi^{-1}_{aa^*}\left((\prod_{C \text{ circle in } ab} X_{t(C)}) \prod_{\gamma \in X} (\pm X_{j(\gamma)} \pm X_{i(\gamma)}) \right) \\ &= \Psi^{-1}_{aa^*}\left(\prod_{i \in T(ab)} X_i \right).\end{aligned}$$

The above product contains one factor X_γ for every cup γ in a, where X_γ is either $X_{i(\gamma)}$ or $X_{j(\gamma)}$. By Lemma A.2.3, for any dotted cup γ we have $X_\gamma = X_{j(\gamma)}$. For an undotted cup γ, it holds that $X_{j(\gamma)} = X_{i(\gamma)} = X_\gamma$. So we find

$$\begin{aligned}(a\lambda b)(a\lambda b)^{\#} &= \Psi^{-1}_{aa^*}\left(\prod_{i \in T(ab)} X_i \right) \\ &= \Psi^{-1}_{aa^*}\left(\prod_{i \text{ right end of a cup in } a} X_i \right) \\ &= (a\mu a^*),\end{aligned}$$

where μ is the unique weight such that every circle in $a\mu a^*$ is oriented clockwise. □

We still need to prove Lemma A.2.5:

Proof of Lemma A.2.5. We give a proof by contradiction:
First case: suppose we get a factor $X_{j(\gamma)} - X_{i(\gamma)}$ in the expression A.1, with $i(\gamma) \in T(ab)$. This would in particular imply that γ is a dotted cap, contradicting Lemma A.2.3.
Second case: suppose we get a factor $-X_{j(\gamma)} + X_{i(\gamma)}$ in the expression A.1, with $j(\gamma) \in T(ab)$. This would in particular imply that the surgery applied to γ is a nested split, where $i(\gamma)$ belongs to the inner circle and $j(\gamma)$ belongs to the outer circle. But since we are performing our surgeries from right to left and γ is a dotted cap, this cannot happen. □

Theorem A.2.6. *Let $\tilde{\tau} : \overline{\mathbb{H}}_\Lambda \to k$ be the linear map sending every basis vector $(a\mu a^*)$ of degree n to 1, and every other basis vector to 0.*
Then

$$\tilde{\tau}((a\lambda b)(c\mu d)) = \begin{cases} \tilde{\tau}((c\mu d)(a\lambda b)) = 1, & \text{if } (c\mu d) = (a\lambda b)^{\#}, \\ 0, & \text{else.} \end{cases}$$

Hence $\tilde{\tau}$ is a symmetrizing form, making $\overline{\mathbb{H}}_\Lambda$ a symmetric algebra.

Proof. The proof that $\tilde{\tau}((a\lambda b)(c\mu d)) = 0$ unless $(c\mu d) = (a\lambda b)^{\#}$ is exactly the same as in Theorem 6.2.5.
By Theorem A.2.4, we find that $\tilde{\tau}((a\lambda b)(a\lambda b)^{\#}) = 1 = \tilde{\tau}((a\lambda b)^{\#}(a\lambda b))$, so the proof is finished. $\square$

We can now also write down the comultiplication, as in Subsection 6.3. The arguments are exactly the same, except there are no signs showing up. So in the end we get

$$\tilde{\Delta}(x\nu y) = \sum_{\substack{a\lambda b, c\mu d \in \overline{\mathbb{H}}_\Lambda \\ (c\mu d)(a\lambda b) = (x\nu y)^{\#}}} (a\lambda b)^{\#} \otimes (c\mu d)^{\#}.$$

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