Cellular structures of Hecke algebras of type A

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Born 1st September 1993 in Kirchen (Sieg)

10th June 2021

Master's Thesis Mathematics

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1. Introduction

Graham and Lehrer [GL96] first defined cellular algebras to understand the representation theory of non-semisimple specializations of the Hecke algebra $\mathscr{H} := \mathscr{H}(\mathfrak{S}_d)$ of the symmetric group. They were inspired by the work of Kazhdan and Lusztig [KL79] on what is now known as the Kazhdan–Lusztig basis $\{\underline{H}_w \mid w \in \mathfrak{S}_d\}$ of $\mathscr{H}$ (for a more accessible introduction see [Soe97]).

In this chapter we recall some results from this theory that motivated the definition of cellular algebras. Then we give an overview of another cellular basis of $\mathscr{H}$, called the Murphy basis, which will be discussed in more detail in Section 4.3. Here we describe its history and some of its features we will discuss in this thesis. Afterwards we summarize the structure of this thesis and state our main contributions.

1.1 Motivation and context

Kazhdan and Lusztig [KL79] were interested in the representation theory of $\mathscr{H}$. They constructed representations of $\mathscr{H}$ that possess a special basis by introducing a basis of $\mathscr{H}$, now known as the Kazhdan–Lusztig basis. To construct these representations they studied the action of the standard generators of $\mathscr{H}$ on Kazhdan–Lusztig basis elements and certain $\mathscr{H}$-submodules of $\mathscr{H}$ connected to them. Here we give a brief description of some of their ideas that will motivate the definition of cellular algebras below.

Relate two elements $x, y \in \mathfrak{S}_d$, denoted by $y \overset{\text{LR}}{\leftarrow} x$, if there exists an $a \in \mathscr{H}$ such that in the expression of $a\underline{H}_x$ or $\underline{H}_x a$ in terms of the Kazhdan–Lusztig basis, the coefficient of $\underline{H}_y$ is non-zero. Moreover, set $w \overset{\text{LR}}{\leq} z$ for $w, z \in \mathfrak{S}_d$, if there exists a sequence $w = x_1 \overset{\text{LR}}{\leftarrow} x_2 \overset{\text{LR}}{\leftarrow} \dots \overset{\text{LR}}{\leftarrow} x_k = z$. Finally, write $w \overset{\text{LR}}{\sim} z$ if $w \overset{\text{LR}}{\leq} z \overset{\text{LR}}{\leq} w$ and $w \overset{\text{LR}}{<} z$ if $w \overset{\text{LR}}{\leq} z$ but $z \overset{\text{LR}}{\not\leq} w$.

Kazhdan and Lusztig understood their basis well enough to make sense of these relations. They proved that for all $w \in \mathfrak{S}_d$ the submodules $\mathscr{H}(\overset{\text{LR}}{\leq} w)$ and $\mathscr{H}(\overset{\text{LR}}{<} w)$, spanned by elements $\underline{H}_x$ with $x \overset{\text{LR}}{\leq} w$ and $x \overset{\text{LR}}{<} w$ respectively, form two sided $\mathscr{H}$-submodules of $\mathscr{H}$. They called their quotients $\mathscr{H}(\overset{\text{LR}}{\leq} w)/\mathscr{H}(\overset{\text{LR}}{<} w)$ **cell representations**. Note that a basis of the cell representation is given by $\{\overline{\underline{H}_x} \mid x \overset{LR}{\sim} w\}$.

These relations also imply a partial order on partitions of size d if combined with the

Robinson–Schensted correspondence, a one-to-one correspondence

$$\begin{aligned} \mathfrak{S}_d &\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Ordered pairs of standard tableaux} \\ \text{of the same shape with } d \text{ boxes} \end{array} \right\} \\ w &\longmapsto (P(w), Q(w)) \end{aligned}$$

named after Robinson [Rob38] and Schensted [Sch61]. For two partitions $\lambda, \mu \vdash d$ set $\mu \tilde{\leq} \lambda$ if there exist $x \overset{\text{LR}}{\leq} y$ such that $\text{Shape}(P(x)) = \mu$ and $\text{Shape}(P(y)) = \lambda$. Note that we will define another partial order on partitions of size d in Section 3.1.

By this correspondence there also exists a labeling of Kazhdan–Lusztig basis elements by ordered pairs of standard tableaux. Hence, these basis elements can be clustered by the shape of their indices and are thus assigned to an element in the poset of partitions of d.

By repeating the constructions from above there is a cell representation for each partition λ of d. For an ordered pair of standard tableaux (S, T) of shape λ, the basis element $\underline{\mathrm{H}}_{ST}$ is mapped to a basis element $\overline{\underline{\mathrm{H}}_{ST}}$ of that cell representation.

Graham and Lehrer defined cellular algebras to be finite dimensional algebras with properties similar to the described properties of the relabeled Kazhdan–Lusztig basis. Roughly, a cellular algebra A is equipped with a special basis, called a cellular basis. The basis must be labeled by an ordered pair of indices, which need to be related to some poset. The action on this basis of A must follow the partial order, so A-submodules similar to $\mathscr{H}(\overset{\text{LR}}{\leq} w)$ and $\mathscr{H}(\overset{\text{LR}}{<} w)$ can be defined. Their quotients, which are the cell representations in case of the Kazhdan–Lusztig basis, are now called two-sided cell modules. For the exact definition of cellular algebras given by Graham and Lehrer [GL96] see Definition 2.1.1.

One of the most important results in the theory of cellular algebras is a classification of irreducible right modules for cellular algebras over fields (see Proposition 2.3.12). Because of its generality it is often difficult to apply this result for concrete examples. However, if A is semisimple it is easy to describe. In this case, every irreducible is a right A-submodule of precisely one two-sided cell module (cf. [GL96, Theorem 3.8]).

The Kazhdan–Lusztig basis was the motivation of Graham and Lehrer and is indeed a cellular basis, so $\mathscr{H}$ is a cellular algebra. The author of [Wil03] compiled all results necessary to prove this result. We will see a short description of the cellular structure of the Kazhdan–Lusztig basis and the Robinson–Schensted correspondence in Section 4.6.

There can be many cellular bases for a given algebra. Choosing a cellular basis to work with depends on the problem the cellular structure is applied to.

In this thesis we are, for the most part, interested in the Murphy basis, another cellular basis of $\mathscr{H}$. It was introduced by Murphy [Mur92] and is based on work done by Dipper and James [DJ86].

Dipper and James set out to study the representation theory of Hecke algebras over fields of any characteristic and were in particular interested in results for fields of positive characteristic. They approached this problem by generalizing several tools and constructions from the representation theory of the symmetric group to the Hecke algebra. Then they derived their results for the Hecke algebra by essentially repeating the classical theory.

They defined, for example, generalizations of right permutation modules M^μ for each composition $\mu \models d$. For a classical definition of permutation modules and their role in the representation theory of $\mathfrak{S}_d$ see [Sag01, Section 2.1]. They are generalized to $\mathscr{H}$ as right submodules

$$M^\mu := m_\mu \mathscr{H} \subset \mathscr{H}$$

generated by a single element $m_\mu \in \mathscr{H}$, the construction of which is connected to the subgroup $\mathfrak{S}_\mu < \mathfrak{S}_d$. Specht modules are then defined as distinguished right submodules $S^\mu \subset M^\mu$ for each μ, but do not need to be irreducible like they are in the classical theory. They rather show that a certain quotient of S^μ is either trivial or irreducible and that the non-zero quotients are an exhaustive list of pairwise non-isomorphic irreducible right modules of $\mathscr{H}(\mathfrak{S}_d)$.

Although Dipper and James achieved their goal, the study of permutation modules, Specht modules and their quotients require extensive calculations. Murphy [Mur92] sought to remove a large part of difficult computations from the ideas of Dipper and James and later presented a self-contained account of his approach in [Mur95].

He constructed the Murphy basis by transforming the generators m_μ of permutation modules into a basis labeled by pairs of standard tableaux with the same shape of size d. Graham and Lehrer had not yet defined cellular algebras, but Murphy's results show that his basis is cellular anyway. He proceeds to discuss results from Dipper and James using this cellular framework. For example, the Specht modules from Dipper and James are submodules of two-sided cell modules of the Murphy basis. The classification coming from the cellular structure is exactly the classification from Dipper and James (cf. [Mat06, Theorem 3.43]).

Consistent with the initial ideas of Dipper and James, features of the representation theory of symmetric groups can be generalized to the representation theory of $\mathscr{H}$ using Murphy's basis. We will see four of them in this thesis. Firstly, the action of generalized Jucys–Murphy elements, which were classically defined by Jucys [Juc74] and Murphy [Mur81], on cell modules associate to the Murphy basis is well understood (see Proposition 4.4.11).

Secondly, if $\mathscr{H}$ is semisimple the Murphy basis provides a basis of each irreducible representation, as they are submodules of precisely one two-sided cell module. Using Jucys–Murphy elements, this basis can be transformed into a special orthogonal basis, called Young's orthogonal form. It generalizes Young's orthogonal form for $\mathbb{C}[\mathfrak{S}_d]$, a classical result which is, for example, proved in [Jam78].

Thirdly, decomposing irreducible representations of $\mathfrak{S}_d$ into irreducible representations of $\mathfrak{S}_{d-1}$ follows a branching rule (cf. [Mur81]). Young's orthogonal form can be used to generalize this result to all semisimple Hecke algebras $\mathscr{H}$.

Lastly, the Schur algebra, an algebra arising in classical Schur–Weyl duality, is generalized as homomorphisms between certain permutation modules. It is a cellular algebra and one cellular basis, called the semistandard basis, is derived from the Murphy basis (cf. [Mat06, Chapter 4]).

1.2 Structure of the thesis

The goal of this thesis is to study the different cellular structures connected to $\mathscr{H}(\mathfrak{S}_d)$ mentioned above. Broadly, it can be split into three parts.

The first part is dedicated to preparations for our studies. In Chapter 2 we partially recall the theory of cellular algebras from [GL96] and roughly follow [Mat06]. However, our account of the proofs in this chapter are a bit more detailed and focused on certain filtrations a cellular basis entails.

We start this chapter with the definition of cellular algebras from [GL96] and state many examples. Afterwards, we construct the mentioned filtrations and study some of their properties. These are then used to prove the already mentioned classification of irreducible right module due to Graham and Lehrer (see Proposition 2.3.12).

In Chapter 3 we recall the notions of compositions, partitions and tableaux from [Ful96], [Sag01] and [Mat06]. As indicated above, these are important for both cellular bases of $\mathscr{H}$, as well as the semistandard basis of the Schur algebra.

The second part of this thesis discusses the three cellular bases connected to $\mathscr{H}$ mentioned above. For the Murphy and semistandard basis we mainly follow [Mat06]. To describe the Kazhdan–Lusztig basis we use [Soe97] and [Sag01] and refer to [Wil03] for a more detailed account.

We begin Chapter 4 by recalling the definition of Hecke algebras of type A. In Section 4.3 we describe the Murphy basis of $\mathscr{H}$ and discuss its various features in subsequent sections. In particular, Section 4.4 outlines the proof of a classification of irreducible right $\mathscr{H}$-modules, which uses the general classification result for cellular algebras.

Next, we state the cell datum associated to the Kazhdan–Lusztig basis in Proposition 4.6.9 and construct the Robinson–Schensted correspondence in Proposition 4.6.2.

Afterwards, in Chapter 5, we recall the Schur algebra and see how to upgrade the Murphy basis to the semistandard basis.

Detailed proofs for the Murphy and semistandard basis require some extensive calculations and can obfuscate the bigger picture. Proofs for the Kazhdan–Lusztig basis are even more involved. Our main contribution is a clean description of their constructions, the motivations behind them, as well as illustrating the ideas with many examples.

In the last part of this thesis, Chapter 6, we provide three explicit examples of cellular bases: the Murphy and Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ as well as the semistandard basis of $\mathscr{S}(2,3)$. We describe their constructions in full detail, state multiplication tables and verify that these bases are cellular. Moreover, we calculate all irreducible right modules explicitly, calculate their dimensions and verify the general classification result for the Murphy basis and the semistandard basis. These calculations are done for fields over any characteristic, but we only consider the Kazhdan–Lusztig basis over algebraically closed fields.

Acknowledgments

I would like to thank Dr. Catharina Stroppel for her detailed explanations during our long discussions and for the multitude of useful references she has pointed me to. I would also like to thank Dr. Daniel Tubbenhauer for his helpful comments on a preliminary version of this thesis.

2. Cellular algebras

Graham and Lehrer first defined cellular algebras in [GL96] to study non-semisimple specializations of Hecke algebras. As mentioned in the introduction, they were inspired by the construction of cell representations using the Kazhdan–Lusztig. A finite dimensional algebra is cellular if it has an exceptional basis, called a cellular basis. Vaguely, the right action on elements in the cellular basis must follow an associated poset, which can be used to generalize cell representations.

In this chapter we want to focus on the algebraic structure such basis entails, mainly the many algebra filtrations it provides us with. We will see that these filtrations lead to abstract results about the irreducible modules of a cellular algebra. However, applying these results to a given algebra with cellular basis can be difficult. It requires, for example, that we know certain coefficients appearing in the multiplication of two basis elements. Therefore, the problem is not just finding a cellular basis for a given algebra, but rather finding a cellular basis that is understood well enough to apply one of the general results.

Choosing a cellular structure to work with might also be influenced by the context it is applied in, as different cellular bases for a given algebra can possess different advantages. We will see advantages of two different cellular bases for Hecke algebras of type A in Chapter 4.

The material in this chapter is well-known and based on [Mat06] and [GL96]. Here we present it with a clear focus on the algebraic structure induced by the cellular basis. However, all results are still due to Graham and Lehrer.

We start this chapter with the definition of cellular algebras from [GL96] and give many examples of such algebras. Then we construct special bimodule filtrations of cellular algebras, which are interesting on their own, but also needed in the last and most important part of this chapter: a classification of irreducible right modules involving the cellular structure (see Proposition 2.3.12). Additionally, we will describe how to construct new cellular algebras from a given cellular algebra.

Although many examples are given throughout this chapter, more comprehensive examples with explicit computations follow in Chapter 6.

2.1 Definition and examples of cellular algebras

Definition 2.1.1. Let R be an integral domain and A an associative unital R-algebra. A **cell datum** for A is a tuple $((\Lambda, \leq), \mathcal{T}, C, *)$, where

$(\Lambda, \leq)$ is a finite poset,

$\mathcal{T} : \Lambda \to \{\text{finite sets }\}$ assigns to each $\lambda \in \Lambda$ a finite indexing set $\mathcal{T}(\lambda)$,

$C := \coprod_{\lambda\in\Lambda} C(\lambda) : \coprod_{\lambda\in\Lambda} \mathcal{T}(\lambda)\times\mathcal{T}(\lambda) \hookrightarrow A,\ (\mathfrak{s},\mathfrak{t}) \mapsto c^{\lambda}_{\mathfrak{s}\mathfrak{t}}$, which image forms a basis of A, i.e.

$$im(C) = \left\{c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \in A \mid \lambda \in \Lambda,\ \mathfrak{s},\mathfrak{t} \in \mathcal{T}(\lambda)\right\}$$

is a R-basis of A, and

$* : A \to A$ is the algebra anti-isomorphism determined by

$$(c^{\lambda}_{\mathfrak{s}\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{t}\mathfrak{s}}$$

for all $\lambda \in \Lambda$ and $\mathfrak{s},\mathfrak{t} \in \mathcal{T}(\lambda)$,

such that the right A-action on the basis elements has the **cellular property**, meaning that for all $a \in A$, $\lambda \in \Lambda$ and $\mathfrak{t},\mathfrak{v} \in \mathcal{T}(\lambda)$ there exist coefficients $r^{\mathfrak{t},a}_{\mathfrak{v}} \in R$ independent of $\mathfrak{s} \in \mathcal{T}(\lambda)$ such that

$$c^{\lambda}_{\mathfrak{s}\mathfrak{t}} a \equiv \sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r^{\mathfrak{t},a}_{\mathfrak{v}} c^{\lambda}_{\mathfrak{s}\mathfrak{v}} \mod \check{A}^{\lambda}\,, \tag{C1}$$

where for $\lambda \in \Lambda$ we define

$$\check{A}^{\lambda} := \operatorname{span}_R\left\{c^{\mu}_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Lambda,\ \ \mu > \lambda,\ \mathfrak{u},\mathfrak{v} \in \mathcal{T}(\mu)\right\}\,,$$

a right A-submodule of A. We call A a **cellular algebra** with **cellular basis** $im(C)$.

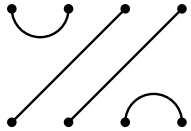
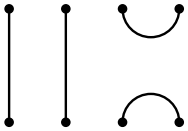
In essence, all cellular basis elements belong to an element of a poset. The right A-action onto cellular basis elements respects the poset ordering and only move upwards. We can associate a right quotient module, called a **node** of that algebra, to an element λ of the poset. It is the submodule of $A/\check{A}^{\lambda}$ spanned by the quotients of all cellular basis elements associated to λ. The basis elements of each node are related to each other via an algebra reflection map.

Note that by Definition 2.1.1, cellular algebras as defined by Graham and Lehrer [GL96] are finite dimensional, because the poset is finite and each labeling set is finite. In [KX12] the notion of cellular algebras has been generalized to an infinite dimensional setting and, for example, has been applied to affine Hecke algebras. Another generalization to the infinite dimensional case has appeared in the study of generalized highest weight categories (see [BS21] and the references therein).

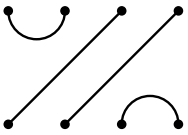
Example 2.1.2. A prototypical cellular algebra is the matrix ring $M_{n\times n}(R)$. We want to define a cell datum for $M_{n\times n}(R)$. Let $\{E_{ij} \mid i,j = 1,2,\dots,n\}$ be the standard basis of $M_{n\times n}(R)$ with $E_{ij} := (\delta_{ik}\delta_{jl})_{k,l=1,2,\dots,n}$. Let $\Lambda = \{1\}$ be the poset with one element, $\mathcal{T}(1) = \{1,2,\dots,n\}$ and C defined by assigning E_{ij} to the tuple (i,j), i.e. $c^1_{ij} := E_{ij}$ for some $i,j \in \{1,2,\dots,n\}$. Let $*$ be the usual matrix transposition.

Then acting with a matrix $B = (b_{kl})_{k,l=1,2,\dots,n}$ onto a basis element E_{ij} yields $E_{ij}B = \sum_{k=1}^{n} b_{jk}E_{ik}$. Therefore, cellular property (C1) holds, because the resulting coefficients b_{jk} are independent of i. Matrix transposition is an algebra anti-isomorphism with the property $(c^1_{ij})^* = E^*_{ij} = E_{ji} = c^*_{ji}$, so the above is indeed a cell datum for $M_{n\times n}(R)$.

Example 2.1.3. Another well-known example of cellular algebras, which was already mentioned in by Graham and Lehrer in [GL96], is the **Temperley–Lieb algebra** $\mathsf{TL}_n(R,\delta)$ for R as above, $n \in \mathbb{Z}_{>0}$ and $\delta \in R$. It is spanned, as a R-module, by **crossingless matchings** of n points, meaning diagrams of two rows with n points, each point connected by non-crossing strands to one other point. For example:

$S_1 =$  $S_2 =$ 

Multiplication of two crossingless matchings c_1 and c_2 in $\mathsf{TL}_n(R,\delta)$ is defined in two steps. First, concatenate the two diagrams, putting c_1 at the top and c_2 at the bottom. Then reduce the result to another crossingless matching by connecting the strands and removing circles that might have formed, creating a factor δ for each removed circle. For example:

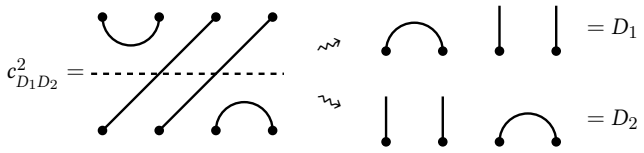
$S_1 \circ S_2 = \delta\cdot$ 

Each basis element is now labeled with two **cap diagrams** of n points. These are diagrams consisting of a single row with n points, some of them connected by non-crossing caps. The other points have a dangling strand attached to them that do not cross the caps. For example:

$D_1 =$  $D_2 =$ 

Each crossingless matching can be split horizontally into a cap diagram at the bottom and an inverted cap diagram at the top. Both have the same number of points connected via a cap. A crossingless matching is labeled by an ordered pair of cap diagrams. The first one is its top diagram, the second on is its bottom diagram.

Now we can define a cell datum for $\mathsf{TL}_n(R,\delta)$ that is associated to this basis. Let $\Lambda := \{i \mid 0 \le i \le n,\ i \text{ even}\}$, a poset with the usual ordering. For $\lambda \in \Lambda$ let $\mathcal{T}(\lambda)$ be the set of cap diagrams of n points with λ dots connected via a cap. For $D_1, D_2 \in \mathcal{T}(\lambda)$ let $c^\lambda_{D_1D_2}$ be the crossingless matching with inverted D_1 at the top and D_2 at the bottom. Finally, let $*$ be the algebra anti-isomorphism reversing each basis element, meaning $(c^\lambda_{D_1D_2})^* = c^\lambda_{D_2D_1}$.



To check the cellular property, note that acting on a basis element from the right, so preconcatinating it with another element, can not strictly reduce the number of points at the top that are connected via cups. Furthermore, the amount of circles that have to be removed in the action on such element does not depend on the top cap diagram. So the coefficient appearing is independent from D_1. It only depends on the bottom cap diagram D_2, as well as the element we are acting with.

Thus, $\mathsf{TL}_n(R,\delta)$ is a cellular algebra with the constructed cell datum. For more information about the cellular structures and the representation theory of Temperley–Lieb algebras see [Spe20].

Graham and Lehrer also considered a slightly different cell datum of $\mathsf{TL}_n(R,\delta)$ in [GL96, Chapter 6]. More concretely, they introduce a different labeling of the basis elements and a different poset.

The basis described above is not the only cellular basis of of $\mathsf{TL}_n(R,\delta)$. In [AST18] the authors construct another cellular basis of $\mathsf{TL}_n(R,\delta)$. Their construction is based on tilting modules and Schur–Weyl duality. Notably, their construction depends on a variety of choices, so more precisely, they get a whole family of cellular bases for $\mathsf{TL}_n(R,\delta)$.

We want to highlight one difference between their bases and the basis described above. Note that the construction above is independent of R and δ, as the basis elements are always crossingless matchings. In particular, there is no difference in the construction of this basis for semisimple and non-semisimple Temperley–Lieb algebras. The construction in [AST18] depends on R and δ, so they can differentiate between their bases of the semisimple and non-semisimple $\mathsf{TL}_n(R,\delta)$.

The authors of [AST18] provide a comprehensive comparison between their bases and the basis above. For example, in [AST18, Proposition 5.3] and [AST18, Proposition 5.5] they show that none of their bases are the basis described above, if $n > 1$.

Example 2.1.4. The Hecke algebra $\mathscr{H}(\mathfrak{S}_d) := \mathscr{H}_{R,v}(\mathfrak{S}_d)$ is also a cellular algebra. One cellular basis of $\mathscr{H}(\mathfrak{S}_d)$ is the famous Kazhdan–Lusztig basis, first introduced in [KL79]. A sketch of the cell datum for $\mathscr{H}(\mathfrak{S}_3)$ is pictured in Figure 2.1. The cell datum for $\mathscr{H}(\mathfrak{S}_d)$ will be stated in Section 4.6. See [Wil03] for a full proof of the cellularity of this basis.

At this point it is only important to know that the cell datum will label elements $\underline{\mathrm{H}}_s$ and $\underline{\mathrm{H}}_{st}$ with a common first label and $\underline{\mathrm{H}}_{ts}$ and $\underline{\mathrm{H}}_t$ also with another common first label. The anti-isomorphism maps $\underline{\mathrm{H}}_w$ to $\underline{\mathrm{H}}_{w^{-1}}$ for all $w \in \mathfrak{S}_3$, so in Figure 2.1 it transposes elements in each box along the diagonal from the top left to the bottom right.

Note that there is another cellular basis for $\mathscr{H}(\mathfrak{S}_d)$, called the Murphy basis. We will state its general cell datum in Chapter 4. It is distinct from the Kazhdan–Lusztig basis we considered above.

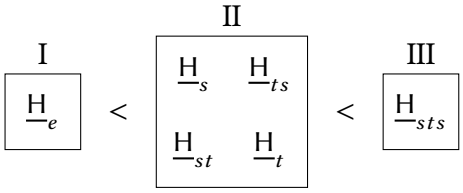


Figure 2.1: Sketch of the cell datum associated to the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$

For $\mathscr{H}(\mathfrak{S}_3)$ we will see that the Murphy basis has the following elements:

$$c^{\mathrm{III}}_{aa} := v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1$$

$$\begin{aligned} c^{\mathrm{II}}_{aa} &:= v^{-1}H_s + 1 & c^{\mathrm{II}}_{ba} &:= v^{-2}H_{ts} + v^{-1}H_t \\ c^{\mathrm{II}}_{ab} &:= v^{-2}H_{st} + v^{-1}H_t & c^{\mathrm{II}}_{bb} &:= v^{-3}H_{sts} + (v^{-2}-1)H_t + v^{-2} \end{aligned}$$

$$c^{\mathrm{I}}_{aa} := 1$$

Here we used a slightly simplified cell datum associated to this basis. The poset is again $\{\mathrm{I} < \mathrm{II} < \mathrm{III}\}$, the labeling as indicated and the anti-automorphism is defined as in Definition 2.1.1.

We verify the cellular property for this example by hand in Section 6.1. Note that there we will switch to another labeling of the Murphy basis that comes from the general construction of this basis (see Section 4.3).

Example 2.1.5. In Examples 2.1.3 and 2.1.4 we have already seen that cellular algebras can have different cellular bases. Choosing to work with one cellular basis over another then depends on the problem one wants to solve. For example, if one wants to investigate semisimplicity of Temperley–Lieb algebras from Example 2.1.3, one might want to work with a cellular basis that depends on semisimplicity.

Another, much easier example of an algebra with multiple cellular bases was given in [KX99]. For R as above consider

$$A = {}^{R\langle x,y\rangle}\!/_{\langle x^2, y^3, xyxy, yxyx, xy^2xy, yxy^2x\rangle}$$

as a right $R\langle x,y\rangle$-module, where $R\langle x,y\rangle$ is the free R-algebra in two variables. One cell datum for A is depicted in Figure 2.2. The set

$$\Lambda = \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}a, \mathrm{V}b\}$$

is partially ordered by the roman numerals. Let

$$\mathcal{T}(\mathrm{I}) = \mathcal{T}(\mathrm{II}) = \mathcal{T}(\mathrm{III}) = \mathcal{T}(\mathrm{V}a) = \mathcal{T}(\mathrm{V}b) = \{1\} \text{ and } \mathcal{T}(\mathrm{IV}) = \{1,2,3\}\ .$$

C labels basis elements in each box of Figure 2.2 with its column as its first label and its row as its second label. $*$ transpose the elements in each box along the diagonal from the top left to the bottom right corner.

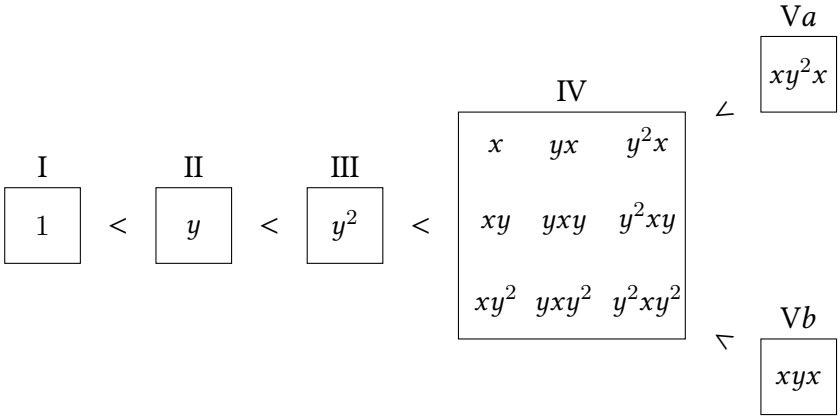


Figure 2.2: Sketch of a cellular basis of A from Example 2.1.5. Each box represents an element in the poset and contains cellular basis elements corresponding to it, for example $c_{12}^{\mathrm{IV}} = xy$.

We only need to verify the cellular property (C1) for generators x and y acting on basis elements. For x, note that acting on a basis element either yields 0 or a basis element associated to a poset element strictly above the one we started with. Additionally for y, the result can be a basis element associated to the same poset element. However, that element is then also in the same column as the element we started with and has therefore the same first label.

We will use A combined with this cell datum as an example throughout this chapter.

The other cellular basis of A presented in [KX99] is depicted in Figure 2.3. Note that the poset differs from the one above and even the nodes are completely different. We will see below that the nodes for a cell datum are of particular interest to us. It is therefore crucial to pick a cell datum that has nodes we understand well or can study effectively.

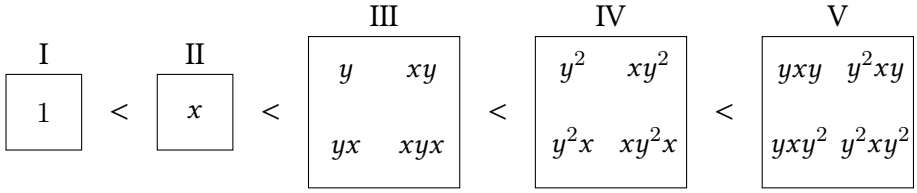


Figure 2.3: Cellular basis of A from Example 2.1.5 distinct from the one depicted in Figure 2.2.

Example 2.1.6. Every finite dimensional, commutative algebra A over an algebraically closed field k is a cellular algebra with $* = id_A$. We can argue via induction on the dimension of A.

If $A = k$ it is easy to assign a cell datum to A. For example $\Lambda = \{\mathrm{I}\}$, $\mathcal{T}(\mathrm{I}) = \{1\}$ and $c_{11}^{\mathrm{I}} = 1$.

If $\dim(A) > 1$ we can fix a one-dimensional A-submodule $N = \mathrm{span}_k\{x\}$ of A, as all irreducible A-modules are one-dimensional. Then A/N is also a commutative algebra. By induction, it is a cellular algebra with some cell datum $((\tilde{\Lambda}, \tilde{\leq}), \tilde{\mathcal{T}}, \tilde{C}, id_A)$.

To get a cell datum for A we augment $\tilde{\Lambda}$ by some μ to $\Lambda := \tilde{\Lambda} \dot{\cup} \{\mu\}$ and augment the partial order on $\tilde{\Lambda}$ by $\mu > \lambda$ for all $\lambda \in \Lambda$. Set $\mathcal{T} := \tilde{\mathcal{T}}$ on $\tilde{\Lambda}$ and $\mathcal{T}(\mu) = \{1\}$. Now lift cellular basis elements from A/N to A and let C label these elements like $\tilde{C}$ labeled the basis elements before the lift. Finally, set $C(\mu) : \mathcal{T}(\mu) \times \mathcal{T}(\mu) \hookrightarrow A,\ (1,1) \mapsto x$.

This defines a cell datum for A with $* = id_A$. Indeed, the cellular property (C1) holds for x, because it spans an irreducible A-submodule. For ever other basis element it holds, because these are lifts from cellular basis elements of A/N and the fact that x is associated μ, an element related to and above all other poset elements.

Note that if we use this inductive approach to construct a cell datum for A, in this cell datum we have $|\mathcal{T}(\lambda)| = 1$ for all $\lambda \in \Lambda$. The poset is in fact totally ordered.

Let A be a fixed cellular algebra cell with datum $((\Lambda, \leq), \mathcal{T}, C, *)$ for the rest of this chapter.

The first construction using cellular algebras is similar to the construction of $\check{A}^\lambda$ in Definition 2.1.1. Extending $\check{A}^\lambda$ by the elements of A associated to λ gives rise to another right A-submodule of A denoted by

$$A^\lambda := \operatorname{span}_R \left\{ c^\mu_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Lambda,\ \ \mu \geq \lambda,\ \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu) \right\},$$

where we use cellular property (C1) once more.

Under the quotient map $A \to A/\check{A}^\lambda$ the image of the elements $\{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ of A span the quotient module

$$A^\lambda \big/ \check{A}^\lambda = \operatorname{span}_R \left\{ c^\lambda_{\mathfrak{s}\mathfrak{t}} + \check{A}^\lambda \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda) \right\},$$

which we called the λ-node above. Note that the set $\{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ does not necessarily span a right A-submodule of A. Of course, if the set $\{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ is a right A-submodule of A, then it is isomorphic to $A^\lambda/\check{A}^\lambda$.

The result of the right A-action on an element $c^\lambda_{\mathfrak{s}\mathfrak{t}} + \check{A}^\lambda \in A^\lambda/\check{A}^\lambda$ encapsulates all the information we have from the cellular property (C1) about the right A-action on $c^\lambda_{\mathfrak{s}\mathfrak{t}}$.

The cellular structure of A manifests itself on $A^\lambda/\check{A}^\lambda$ in two more ways. For any $\mathfrak{s} \in \mathcal{T}(\lambda)$ there is a right R-submodule $C^\lambda_\mathfrak{s} \subset A^\lambda/\check{A}^\lambda$ spanned by elements $\{c^\lambda_{\mathfrak{s}\mathfrak{t}} + \check{A}^\lambda \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$. This is also a well defined right A-submodule by the cellular property (C1). Indeed, acting with $a \in A$ from the right on $c^\lambda_{\mathfrak{s}\mathfrak{t}} + \check{A}^\lambda \in C^\lambda_\mathfrak{s}$ yields

$$(c^\lambda_{\mathfrak{s}\mathfrak{t}} + \check{A}^\lambda)a \overset{(C1)}{=} \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r^{\mathfrak{t},a}_\mathfrak{v} (c^\lambda_{\mathfrak{s}\mathfrak{v}} + \check{A}^\lambda), \tag{2.1}$$

which is also contained in $C^\lambda_\mathfrak{s}$, because it is a sum over elements with $\mathfrak{s}$ as their first index.

Moreover, the coefficients appearing in (2.1) only depend on $a \in A$ and $\mathfrak{t}, \mathfrak{v} \in \mathcal{T}(\lambda)$. Most importantly, they do not depend on $\mathfrak{s} \in \mathcal{T}(\lambda)$. Therefore, the right A-module structure of $C^\lambda_\mathfrak{s}$ does not depend on the choice of $\mathfrak{s} \in \mathcal{T}(\lambda)$. In other words, for two $\mathfrak{s}, \mathfrak{s}' \in \mathcal{T}(\lambda)$ the map $C^\lambda_\mathfrak{s} \to C^\lambda_{\mathfrak{s}'}$ defined by $c^\lambda_{\mathfrak{s}\mathfrak{v}} \mapsto c^\lambda_{\mathfrak{s}'\mathfrak{v}}$ for all $\mathfrak{v} \in \mathcal{T}(\lambda)$ is a right A-module isomorphism.

There is a more abstract definition of this right A-module.

Definition 2.1.7. For $\lambda \in \Lambda$ define the corresponding **right cell module** as the R-module

$$C^\lambda := \operatorname{span}_R\left\{c^\lambda_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}(\lambda)\right\}.$$

Right cell modules are also right A-modules. The action is defined by

$$c^\lambda_{\mathfrak{t}} a := \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r^{\mathfrak{t},a}_{\mathfrak{v}} c^\lambda_{\mathfrak{v}}$$

for all $a \in A$ and $\mathfrak{t} \in \mathcal{T}(\lambda)$, where the coefficients $r^{\mathfrak{t},a}_{\mathfrak{v}}$ come from cellular property (C1).

Right cell modules are well defined right A-modules by the same arguments we used for $C^\lambda_{\mathfrak{s}}$ above. For all $\mathfrak{s} \in \mathcal{T}(\lambda)$ the R-module isomorphism

$$\begin{aligned} C^\lambda &\xrightarrow{\sim} C^\lambda_{\mathfrak{s}} \subset {}^{A^\lambda}\!/_{\check{A}^\lambda} \\ c^\lambda_{\mathfrak{t}} &\longmapsto c^\lambda_{\mathfrak{st}} + \check{A}^\lambda \end{aligned}$$

is also an A-module isomorphism, because acting with $a \in A$ from the right produces the same coefficients coming from (C1).

Example 2.1.8. To get familiar with the abstract definition of right cell modules, let's consider the algebra A of Example 2.1.5 together with its cellular basis depicted in Figure 2.2.

For $\lambda \neq \mathrm{IV}$, $A^\lambda/\check{A}^\lambda$ is spanned by one element, so $C^\lambda = A^\lambda/\check{A}^\lambda$. All cellular basis elements act by zero, except for $1 \in A$, which acts by the identity.

For $\lambda = \mathrm{IV}$ the cell module is spanned by three elements and we denote them by $C^{\mathrm{IV}} = \operatorname{span}_R\{c^{\mathrm{IV}}_1, c^{\mathrm{IV}}_2, c^{\mathrm{IV}}_3\}$. It is isomorphic to $\operatorname{span}_R\{\bar{x}, \overline{xy}, \overline{xy^2}\} \subset A^\lambda/\check{A}^\lambda$, so $1, y, y^2 \in A$ act by

$$\begin{array}{lll} c^{\mathrm{IV}}_1 \cdot 1 = c^{\mathrm{IV}}_1 & c^{\mathrm{IV}}_1 \cdot y = c^{\mathrm{IV}}_2 & c^{\mathrm{IV}}_1 \cdot y^2 = c^{\mathrm{IV}}_3 \\ c^{\mathrm{IV}}_2 \cdot 1 = c^{\mathrm{IV}}_2 & c^{\mathrm{IV}}_2 \cdot y = c^{\mathrm{IV}}_3 & c^{\mathrm{IV}}_2 \cdot y^2 = 0 \\ c^{\mathrm{IV}}_3 \cdot 1 = c^{\mathrm{IV}}_3 & c^{\mathrm{IV}}_3 \cdot y = 0 & c^{\mathrm{IV}}_3 \cdot y^2 = 0 \end{array}.$$

and all other cellular basis elements of A act by zero.

In summary, the statement of (C1) is best understood in the context of quotient modules $A^\lambda/\check{A}^\lambda$ for $\lambda \in \Lambda$. Firstly, it guarantees these quotient modules are well defined right A-modules. Secondly, it presents us with $|\mathcal{T}(\lambda)|$ many right A-submodules of $A^\lambda/\check{A}^\lambda$ we denoted by $C^\lambda_{\mathfrak{s}}$ for $\mathfrak{s} \in \mathcal{T}(\lambda)$. Lastly, it asserts that all these right A-submodules are in fact isomorphic as right A-modules, leading us to the definition of right cell modules.

At this point it is worthwhile to emphasize that the same is not true if we just consider A^λ on its own. Although it is also a right A-submodule of A, the right A-submodules spanned by elements $\{c^\lambda_{\mathfrak{st}} \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$ can differ for different $\mathfrak{s} \in \mathcal{T}(\lambda)$. , if we express $c^\lambda_{\mathfrak{st}} a$ for some

$a \in A$ in the cellular basis, then the coefficients in front of elements from $\check{A}^\lambda$ can differ for different $\mathfrak{s} \in \mathcal{T}(\lambda)$.

An example of this can be found in Section 6.1, where the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$ is computed explicitly. The action on these basis elements is summarized in Figure 6.2 and the mismatching coefficients in front of elements in $\check{A}^\lambda$ is readily observed.

Remark 2.1.9. If $(A, \cdot)$ is a cellular algebra, then so is $(A^{\mathrm{op}}, \circ)$.
Indeed, let $((\Lambda, \leq), \mathcal{T}, C, *)$ be a cell datum of A, then

$$c^\lambda_{\mathfrak{s}\mathfrak{t}} \circ a \equiv a c^\lambda_{\mathfrak{s}\mathfrak{t}} \equiv (c^\lambda_{\mathfrak{t}\mathfrak{s}} a^*)^* \overset{(C1)}{\equiv} \Big(\sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r^{\mathfrak{s}, a^*}_{\mathfrak{v}} c^\lambda_{\mathfrak{t}\mathfrak{v}}\Big)^* \overset{*}{\equiv} \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r^{\mathfrak{s}, a^*}_{\mathfrak{v}} c^\lambda_{\mathfrak{v}\mathfrak{t}} \mod \check{A}^\lambda . \tag{C2}$$

for all $a \in A, \lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$. Furthermore, the coefficients $r^{\mathfrak{s}, a^*}_{\mathfrak{v}}$ are independent of $\mathfrak{t} \in \mathcal{T}(\lambda)$ by property (C1) of A. Thus, $(A^{\mathrm{op}}, \circ)$ has cell datum $((\Lambda, \leq), \mathcal{T}, C^{\mathrm{op}}, *)$, where $C^{\mathrm{op}}(\mathfrak{s}, \mathfrak{t}) := c^\lambda_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$. In other words, the basis elements are the same as for $(A, \cdot)$, but the labels are swapped.
We denote the right cell module for $(A^{\mathrm{op}}, \circ)$ corresponding to $\lambda \in \Lambda$ by

$$C^{*\lambda} := \mathrm{span}_R\{c^\lambda_{\mathfrak{s}} \mid \mathfrak{s} \in \mathcal{T}(\lambda)\}$$

and have

$$c^\lambda_{\mathfrak{s}} \circ a = \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r^{\mathfrak{s}, a^*}_{\mathfrak{v}} c^\lambda_{\mathfrak{v}}$$

for all $a \in A$ and $\mathfrak{s} \in \mathcal{T}(\lambda)$. We will sometimes regard right A^{op}-modules as left A-modules.

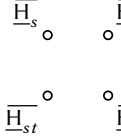
2.2 Filtrations of cellular algebras as bimodules

Combining the right A-module and right A^{op}-module structure of $A^\lambda/\check{A}^\lambda$ discussed in the previous section we can also give a complete description of $A^\lambda/\check{A}^\lambda$ as an (A, A)-bimodule. The map

$$\begin{aligned} A^\lambda/\check{A}^\lambda &\cong C^{*\lambda} \otimes_R C^\lambda \\ c^\lambda_{\mathfrak{s}\mathfrak{t}} &\mapsto c^\lambda_{\mathfrak{s}} \otimes c^\lambda_{\mathfrak{t}} \end{aligned} \tag{2.2}$$

is an (A, A)-bimodule isomorphism, implying that for each node, A and A^{op} cell modules are just multiplicity spaces for each other (see Figure 2.4).

Example 2.2.1. We continue with the cellular basis in Example 2.1.4, the Kazhdan–Lusztig basis of $\mathscr{H} := \mathscr{H}(\mathfrak{S}_3)$. This is a sketch of its $(2,1)$-node $\mathscr{H}^{(2,1)}/\check{\mathscr{H}}^{(2,1)}$:



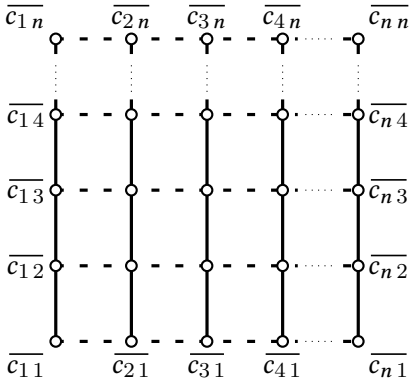
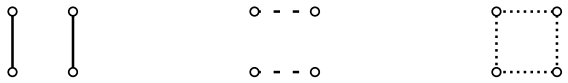


Figure 2.4: Decomposition of a node as a right A-module and as a right A^{op}-module. Here $\mathcal{T}(\lambda) = \{1, 2, ..., n\}$, the ○ are elements $\overline{c_{ij}}$ in $A^\lambda/\check{A}^\lambda$, elements connected by —— generate a right A-submodule of $A^\lambda/\check{A}^\lambda$ and elements connected by **- -** generate a right A^{op}-submodule of $A^\lambda/\check{A}^\lambda$. Moreover, all —— submodules are isomorphic and likewise all **- -** submodules are isomorphic.

As a right $\mathscr{H}$-module it decomposes into two modules, that are both isomorphic to the right cell module $C^{(2,1)}$. Similarly, as a right $\mathscr{H}^{\mathrm{op}}$-module it decomposes into two cell modules $C^{*(2,1)}$. As a $(\mathscr{H}, \mathscr{H})$-bimodule, it does not decompose at all.



Decomposition as right $\mathscr{H}$-module | Decomposition as right $\mathscr{H}^{\mathrm{op}}$-module | Decomposition as $(\mathscr{H}, \mathscr{H})$-bimodule

Characterization (2.2) of $A^\lambda/\check{A}^\lambda$ already suggests that Λ provides many different right A-module and right A^{op}-module filtrations of A with cell modules as subquotients. Our approach is to first focus on certain (A, A)-bimodule filtrations of A involving so called poset ideals. These filtrations, while interesting on their own, can then be refined to right A-module and right A^{op}-module filtrations with the mentioned subquotients.

Before we proceed let's recall that $(\Lambda, \leq)$ is, per definition, a finite poset.

Definition 2.2.2. A subset $\Gamma \subset \Lambda$ is called a **poset ideal**, if for all $\lambda, \mu \in \Lambda$:

$$(\mu \in \Gamma \ \wedge \ \lambda > \mu) \Rightarrow \lambda \in \Gamma \ .$$

For such a Γ denote by $A(\Gamma)$ the two-sided A-submodule of A defined by

$$A(\Gamma) \coloneqq \mathrm{span}_R\{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Gamma, \ \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\} = \sum_{\lambda \in \Gamma} A^\lambda$$

A chain $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k$ of poset ideals is **maximal**, if $\Gamma_0 = \emptyset, \Gamma_k = \Lambda$ and $|\Gamma_i \backslash \Gamma_{i-1}| = 1$ for all $i = 1, 2, \ldots, k$.

$A(\Gamma)$ is a well defined two-sided A-submodule by the cellular property (C1). Maximal chains of poset ideals will be used in Proposition 2.2.5 to construct (A,A)-bimodule filtrations of A. As a preparation we first gather some properties of such maximal chains (see [Mat06, Lemma 2.14]).

Lemma 2.2.3. *Let $(\Lambda,\leq)$ be a finite poset, then:*

(i) *There exists a labeling $\Lambda = \{\lambda_1,\lambda_2,\dots,\lambda_k\}$ of this finite poset such that $\lambda_i > \lambda_j$ implies $j > i$. Then $\Gamma_i := \{\lambda_1,\lambda_2,\dots,\lambda_i\}$ for $i = 0,1,\dots,k$ defines a maximal chain of poset ideals.*

(ii) *Any poset ideal $\Gamma \subset \Lambda$ is part of a maximal chain of poset ideals that has the labeling property of (i).*

(iii) *For any maximal chain of poset ideals $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k$ there exists a labeling $\Lambda = \Gamma_k = \{\lambda_1,\lambda_2,\dots,\lambda_k\}$, such that $\lambda_i > \lambda_j$ implies $j > i$.*

Proof. We call an element μ **maximal** in Λ if $\lambda \not> \mu$ for all $\lambda \in \Lambda$.

(i) For $\lambda \in \Lambda$ denote by $\rho(\lambda)$ the maximal size of a totally ordered subset

$$\{\lambda < \eta_1 < \eta_2 < \cdots < \eta_l\} \subset \Lambda$$

starting at λ. If $\lambda > \mu$, then a maximal subset for λ can be extended to a totally ordered subset starting at μ. Therefore, $\lambda > \mu$ implies $\rho(\mu) > \rho(\lambda)$. Also, λ and μ are incomparable if $\rho(\lambda) = \rho(\mu)$ and μ is maximal if $\rho(\mu) = 0$.

Now, starting with the maximal elements, label Λ increasingly by the values $\rho(\lambda)$. Then $\lambda_i > \lambda_j \Rightarrow \rho(\lambda_j) > \rho(\lambda_i) \Rightarrow j > i$ because λ_i got labeled before λ_j. Moreover, the Γ_i define a maximal chain of poset ideals, as for some $\lambda_n \in \Gamma_i$ and $\lambda_m > \lambda_n$ we have $\rho(\lambda_n) > \rho(\lambda_m) \Rightarrow\ n > m\ \Rightarrow \lambda_m \in \Gamma_i$.

(ii) Both Γ and $\Lambda \setminus \Gamma$ are naturally posets, where the partial order agrees with the partial order on Λ, so (i) can be applied to each of them. Denote the resulting chain of poset ideals for Γ by Γ_i, where $i = 0,1,\dots,|\Gamma|$, and the chain of poset ideals for $\Lambda \setminus \Gamma$ by $\Gamma_j^{\Lambda\setminus\Gamma}$, where $j = 0,1,\dots,|\Lambda \setminus \Gamma|$. Then

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{|\Gamma|} \subset \left(\Gamma_1^{\Lambda\setminus\Gamma} \cup \Gamma\right) \subset \cdots \subset \left(\Gamma_{|\Lambda\setminus\Gamma|}^{\Lambda\setminus\Gamma} \cup \Gamma\right) = \Lambda \tag{2.3}$$

is a chain of poset ideals of Λ, as the union of two poset ideals is again a poset ideal. Additionally this chain is maximal, as Γ_i and $\Gamma_j^{\Lambda\setminus\Gamma}$ each define a maximal chain of poset ideals. Denote the chain in (2.3) by Γ_i^{Λ} for $i = 0,1,\dots,|\Lambda|$ and label elements in Λ such that $\Gamma_i^{\Lambda} = \{\lambda_1,\lambda_2,\dots,\lambda_i\}$.

Finally, to show the labeling property consider $\lambda_i > \lambda_j$. If $i,j \leq |\Gamma|$ or $i,j > |\Gamma|$ the property follows because we applied (i) before. In all other cases we must have $\lambda_j \notin \Gamma$, because otherwise $\lambda_i \in \Gamma$ as well and then $i,j \leq |\Gamma|$. In conclusion we have $i \leq |\Gamma| < j$.

(iii) For a given maximal chain of poset ideals we can apply (ii) iteratively to each poset. Because the property of (i) holds in every step it will also hold for Λ.

□

Example 2.2.4. The poset we associated to the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ in Example 2.1.4 was $\Lambda = \{\mathrm{I} < \mathrm{II} < \mathrm{III}\}$. For this poset the only maximal chain of poset ideals is

$$\emptyset \subset \{\,\mathrm{III}\} \subset \{\mathrm{II}, \mathrm{III}\,\} \subset \Lambda\,,$$

so the labeling from Lemma 2.2.3(iii) is $\lambda_1 = \mathrm{III}, \lambda_2 = \mathrm{II}$ and $\lambda_3 = \mathrm{I}$.

The following Proposition is from [Mat06, Lemma 2.14].

Proposition 2.2.5. *Let A be a cellular algebra with labeling set Λ. For any maximal chain of poset ideals $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k$ the corresponding (A,A)-bimodule filtration of A*

$$\{0\} = A(\Gamma_0) \subset A(\Gamma_1) \subset \cdots \subset A(\Gamma_k) = A \tag{2.4}$$

has subquotients

$$A(\Gamma_i)\big/_{A(\Gamma_{i-1})} \cong A^{\lambda_i}\big/_{\check{A}^{\lambda_i}} \cong C^{*\lambda_i} \otimes_R C^{\lambda_i}$$

for some $\lambda_i \in \Lambda$.

Proof. By Lemma 2.2.3(iii) there exists a labeling $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ such that $\Gamma_i = \{\lambda_1, \lambda_2, \ldots, \lambda_i\}$ and $\lambda_i > \lambda_j$ implies $j > i$. Using the definition of $A(\Gamma_i)$ we get

$$A(\Gamma_i)\big/_{A(\Gamma_{i-1})} = \sum_{\lambda\in\Gamma_i} A^\lambda \Big/ \sum_{\lambda\in\Gamma_{i-1}} A^\lambda \overset{\Gamma_i=\Gamma_{i-1}\cup\{\lambda_i\}}{\cong} A^{\lambda_i}\big/_{A^{\lambda_i} \cap \sum_{\lambda\in\Gamma_{i-1}} A^\lambda}\,. \tag{2.5}$$

The labeling clearly implies $\check{A}^{\lambda_i} \subset A^{\lambda_i} \cap \sum_{\lambda\in\Gamma_{i-1}} A^\lambda$, but suppose for contradiction that the inclusion is strict. Then, by definition of A^{λ_i}, there must exist some $\lambda_j \in \Gamma_{i-1}$ such that $\lambda_i \geq \lambda_j$. But $\lambda_i \neq \lambda_j$, because $\lambda_i \notin \Gamma_{i-1}$, and $\lambda_i > \lambda_j$ implies $j > i$, which is a contradiction to the labeling of Λ.

So $A^{\lambda_i} \cap \sum_{\lambda\in\Gamma_{i-1}} A^\lambda = \check{A}^{\lambda_i}$ and by continuing (2.5) we get

$$A(\Gamma_i)\big/_{A(\Gamma_{i-1})} \cong A^{\lambda_i}\big/_{\check{A}^{\lambda_i}} \cong C^{*\lambda_i} \otimes C^{\lambda_i}\,,$$

where the last isomorphism was already stated in (2.2). □

Example 2.2.6. Consider again the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ with poset $\Lambda = \{\,\mathrm{I} < \mathrm{II} < \mathrm{III}\}$. For the unique maximal chain of poset ideals in the poset associated to $\mathscr{H}(\mathfrak{S}_3)$, the $(\mathscr{H}(\mathfrak{S}_3), \mathscr{H}(\mathfrak{S}_3))$-bimodule filtration from Proposition 2.2.5 is:

$$\emptyset \subset \mathrm{span}_R\{\underline{\mathrm{H}}_{sts}\} \subset \mathrm{span}_R\{\underline{\mathrm{H}}_{sts}, \underline{\mathrm{H}}_s, \underline{\mathrm{H}}_{st}, \underline{\mathrm{H}}_t, \underline{\mathrm{H}}_{ts}\} \subset \mathscr{H}(\mathfrak{S}_3)$$

At the end of Section 2.3 we will come back to poset ideals and see how they can be used to construct new cellular algebras from A.

To understand why (A,A)-bimodule filtrations like in equation (2.4) of A are so useful in the theory of cellular algebras we first need to take a closer look at right A and A^{op} cell modules.

By Proposition 2.2.5, right A and A^{op} cell modules appear as multiplicity spaces in subquotients of (A,A)-bimodule filtrations of A arising from maximal chains of poset ideals. These can be refined to right A-module filtrations with right cell modules as subquotients. In this way we could build composition series of A as a right A-module and as a right A^{op} by combining composition series of right cell modules. A slight variation of this idea is used in the next section.

However, we have not yet investigated the module structure of cell modules themself, so we will need some preparation before we can combine such composition series. In the next section we will define more concepts surrounding cell modules to better understand this module structure.

Afterwards we will prove a classification result for irreducible right modules for cellular algebras using the established structures, at least if the ground ring is a field. They are labeled by a set that is connected to special submodules of right cell modules. This labeling set can be hard to determine in concrete examples, although its definition is straightforward.

2.3 Classification of irreducible representations

Consider again a cellular algebra A over R with cell datum $((\Lambda,\leq),\mathcal{T},C,*)$. Basic features of the right A and A^{op}-module structures and the (A,A)-bimodule structure of A have been described this chapter thus far. However, the cellular property of A also affects right A-modules other than A itself. These influences will be studied in this section.

As indicated above, our analysis begins by revisiting right cell modules, which were introduced in Definition 2.1.7. The main goal of this section is to classify all irreducible right A-modules in case R is a field, an additional assumption we will make later on.

To achieve this goal we also need to readdress poset ideals. We are particularly interested in quotient modules of A by two-sided A-submodules of A generated by poset ideals. These quotient modules turn out to be cellular algebras as well, just with a smaller poset than A.

We begin our analysis of the A-module structure of right cell modules with a small lemma. It identifies elements of A that have to act trivially on a given right cell module, based on the cellular structure of A. Its statement is rather technical and unassuming, its proof is not difficult and based on results of Section 2.1. However, it will simplify upcoming proofs significantly. See also [Mat06, Lemma 2.7].

Lemma 2.3.1. *Assume $\lambda,\mu\in\Lambda$. Then:*

$$\lambda\not\geq\mu \;\Rightarrow\; xa=0\ \in C^\lambda \text{ for all } x\in C^\lambda \text{ and } a\in A^\mu$$

Proof. The statement follows by combining two facts discussed before. Firstly, recall the (A,A)-bimodule isomorphism $A^\lambda/\check{A}^\lambda\cong C^{*\lambda}\otimes_R C^\lambda$ from (2.2). Using the definition of $C^{*\lambda}$ given in Remark 2.1.9 we thus also know there exists a decomposition

$$A^\lambda\!/\check{A}^\lambda\cong\bigoplus_{t\in\mathcal{T}(\lambda)}C^\lambda \tag{2.6}$$

as a right A-module into $|\mathcal{T}(\lambda)|$ copies of C^λ. Secondly, as A^λ and A^μ are both two-sided submodules of A and $\lambda \not\geq \mu$ by assumption, we have $A^\lambda \cdot A^\mu \subset A^\lambda \cap A^\mu \subset \check{A}^\lambda$. Thus

$$\left(\bigoplus_{\mathfrak{t}\in\mathcal{T}(\lambda)} C^\lambda\right)\cdot A^\mu \overset{(2.6)}{\cong} \left(A^\lambda \cdot A^\mu\right)\Big/_{\check{A}^\lambda} \cong \{0\}$$

and, in particular, $C^\lambda \cdot A^\mu = \{0\}$. □

The cellular property (C1) of A was central in defining right cell modules for A and, subsequently, for A^{op}. Additionally, it provides us with a special right A-submodule of each right cell module, which we will construct below.

Recall the cellular property (C1) for $(A,\cdot)$ and the cellular property (C2) for $(A^{\mathrm{op}},\circ)$ from Section 2.1. Specifically, let's consider some $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v} \in \mathcal{T}(\lambda)$ for some $\lambda \in \Lambda$. Then the multiplication $c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}}$ can be computed with (C1) and with (C2) respectively:

$$c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}} = \sum_{\mathfrak{m}\in\mathcal{T}(\lambda)} r_{\mathfrak{m}}^{\mathfrak{s},c^\lambda_{\mathfrak{t}\mathfrak{v}}} c^\lambda_{\mathfrak{u}\mathfrak{m}} \bmod \check{A}^\lambda$$

$$c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}} = c^\lambda_{\mathfrak{t}\mathfrak{v}} \circ c^\lambda_{\mathfrak{u}\mathfrak{s}} = \sum_{\mathfrak{n}\in\mathcal{T}(\lambda)} r_{\mathfrak{n}}^{\mathfrak{t},c^\lambda_{\mathfrak{s}\mathfrak{u}}} c^\lambda_{\mathfrak{n}\mathfrak{v}} \bmod \check{A}^\lambda$$

Because the image of C, i.e. the set $\left\{c^\lambda_{\mathfrak{s}\mathfrak{t}} \in A \mid \lambda\in\Lambda,\ \mathfrak{s},\mathfrak{t}\in\mathcal{T}(\lambda)\right\}$, forms a basis, the coefficients in both versions have to agree. So the only possible non-zero coefficient is $r_{\mathfrak{v}}^{\mathfrak{s},c^\lambda_{\mathfrak{t}\mathfrak{v}}} = r_{\mathfrak{u}}^{\mathfrak{t},c^\lambda_{\mathfrak{s}\mathfrak{u}}}$. Moreover, by (C1) and (C2) this coefficient is independent of both $\mathfrak{u}$ and $\mathfrak{v}$, which leads us to the next proposition (see [Mat06, Proposition 2.9]).

Proposition 2.3.2. *For every $\lambda\in\Lambda$ there is a unique R-bilinear map*

$$\langle .\ ,\ .\rangle : C^\lambda \times C^\lambda \longrightarrow R$$

such that

$$\langle c^\lambda_{\mathfrak{s}}, c^\lambda_{\mathfrak{t}}\rangle c^\lambda_{\mathfrak{u}\mathfrak{v}} \equiv c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}} \ \mathit{mod}\ \check{A}^\lambda$$

for all $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}\in\mathcal{T}(\lambda)$.
Basic properties of this bilinear form are:

$$\begin{array}{lll} (i) & \langle x,y\rangle = \langle y,x\rangle & \textit{(symmetry)} \\ (ii) & \langle xa,y\rangle = \langle x,ya^*\rangle & \textit{(associativity)} \\ (iii) & xc^\lambda_{\mathfrak{u}\mathfrak{v}} = \langle x,c^\lambda_{\mathfrak{u}}\rangle c^\lambda_{\mathfrak{v}} & \end{array}$$

for all $x,y\in C^\lambda$, $a\in A$ and $\mathfrak{u},\mathfrak{v}\in\mathcal{T}(\lambda)$.

Proof. By above, the element $\langle c^\lambda_{\mathfrak{s}}, c^\lambda_{\mathfrak{t}}\rangle$ is well defined and thus such a unique bilinear form exists. The other proofs are rather short and done in [Mat06, Proposition 2.9]. □

The R-bilinear map $\langle .\ ,\ . \rangle$ captures, per construction, essential information about the module structure of right cell modules. Due to its elegant definition and property (iii) from Proposition 2.3.2 it plays a central role in the construction of irreducible A-modules and in the following definition.

Definition 2.3.3. For $\lambda \in \Lambda$ let the **radical** of C^λ be defined as the A-submodule

$$\operatorname{rad} C^\lambda := \left\{ x \in C^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in C^\lambda \right\} \subset C^\lambda .$$

Denote the corresponding quotient module by

$$D^\lambda := {}^{C^\lambda}\!/_{\operatorname{rad} C^\lambda} ,$$

which also induces the labeling set

$$\Lambda_0 := \left\{ \lambda \in \Lambda \mid D^\lambda \neq \{0\} \right\} .$$

Indeed, the radical $\operatorname{rad} C^\lambda$ is a right A-submodule by Proposition 2.3.2(iii).

Remark 2.3.4. Note that $\operatorname{rad} C^\lambda \cdot A^\lambda = \{0\}$ by Proposition 2.3.2(iii), so the radical is also a right A/A^λ-module.

Example 2.3.5. Consider once again algebra A of Example 2.1.5. For $\lambda \neq \mathrm{I}$, the result of multiplying two basis elements of A associated to λ is always in $\check{A}^\lambda$. Therefore $\langle .\ ,\ . \rangle \equiv 0$ and $\operatorname{rad} C^\lambda = C^\lambda$. For $\lambda = \mathrm{I}$ the bilinear form is completely determined by $1 \cdot 1 = 1$, so $\operatorname{rad} C^{\mathrm{I}} = \{\ 0\ \}$. It follows that $\Lambda_0 = \{\mathrm{I}\}$ in this example.

The labeling set Λ_0 has an important role in the theory of cellular algebras, but in many cases it is hard to control. However, by the next proposition we know that it is always non-empty. The proof is from [Mat06, Lemma 2.15].

Proposition 2.3.6. *If λ is minimal in Λ, i.e. $\eta \neq \lambda \Rightarrow \lambda \not\geq \eta$ for all $\eta \in \Lambda$, then $C^\lambda = D^\lambda$.*

Proof. We need to show $x \in \operatorname{rad} C^\lambda \Rightarrow x = 0$. Write $1 \in A$ as a linear combination of cellular basis elements

$$1 = \sum_{\mathfrak{s},\mathfrak{t} \in \mathcal{T}(\lambda)} r_{\mathfrak{s}\mathfrak{t}} c^\lambda_{\mathfrak{s}\mathfrak{t}} + a ,$$

where $a \in \sum_{\eta \neq \lambda} A^\eta$. As λ is assumed to be minimal we know $\lambda \not\geq \eta$ and can apply Lemma 2.3.1. Therefore, $xa = 0 \in C^\lambda$ and we finish our proof by observing

$$x = x \cdot 1 \overset{2.3.1}{=} x \cdot \sum_{\mathfrak{s},\mathfrak{t} \in \mathcal{T}(\lambda)} r_{\mathfrak{s}\mathfrak{t}} c^\lambda_{\mathfrak{s}\mathfrak{t}} \overset{2.3.2(iii)}{=} \sum_{\mathfrak{s},\mathfrak{t} \in \mathcal{T}(\lambda)} r_{\mathfrak{s}\mathfrak{t}} \underbrace{\langle x, c^\lambda_{\mathfrak{s}} \rangle}_{\substack{=0, \text{ since} \\ x \in \operatorname{rad} C^\lambda}} c^\lambda_{\mathfrak{t}} = 0 .$$

□

Remark 2.3.7. For some cellular algebras it is possible to construct cell data that are independent of their ground ring R. Thus, for any choice of R there will be the same amount of right cell modules. One example of such cell datum is associated to the Murphy basis for the Hecke algebra. We will discuss this basis in detail in Section 4.3. Another example of such algebra is the Temperley–Lieb algebra from Example 2.1.3.

Even the bilinear form on right cell modules of such cellular algebras can be independent of R. However, R can influence whether the bilinear form is degenerate or not. Thus, the choice of R affects the radicals and can, for example, change their dimensions. Consequently, Λ_0 can also depend on the choice of R.

In Proposition 6.1.1 we will see this in explicit computations for the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$ over a field R. There, Λ_0 depends in part on the characteristic of R.

The main result of this section is that if R is a field, then Λ_0 labels the irreducible right A-modules, clarifying its importance. There is still some preparation needed before we can prove this statement. We begin with some properties of the quotient modules defined in Definition 2.3.3 if we work over a field (see [Mat06, Proposition 2.11, Corollary 2.13]).

Proposition 2.3.8. *Let R be a field and $\lambda, \mu \in \Lambda_0$.*

(i) *D^λ is an irreducible right A-module.*

(ii) *If $D^\lambda \cong D^\mu$ then $\lambda = \mu$.*

Proof. (i) By the assumption on λ we have $D^\lambda \neq \{0\}$ and can therefore show the equivalent statement

(i') $\operatorname{rad} C^\lambda$ is the unique maximal, strict submodule of C^λ,

which in turn follows from

(i'') $xA^\lambda = C^\lambda$ for all $x \in C^\lambda \setminus \operatorname{rad} C^\lambda$.

For such $x \in C^\lambda \setminus \operatorname{rad} C^\lambda$ there exists, by definition of the radical, an element $y \in C^\lambda$ such that $\langle x, y\rangle \neq 0$. By Definition 2.1.7

$$C^\lambda = \operatorname{span}_R\left\{c^\lambda_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{T}(\lambda)\right\}.$$

Hence, there even exists a $\mathfrak{u} \in \mathcal{T}(\lambda)$ such that $\langle x, c^\lambda_\mathfrak{u}\rangle \neq 0$, which is invertible because R is assumed to be a field. Applying property (iii) of $\langle .\ ,\ .\rangle$ from Proposition 2.3.2 we get

$$\langle x, c^\lambda_\mathfrak{u}\rangle c^\lambda_\mathfrak{v} = x c^\lambda_{\mathfrak{u}\mathfrak{v}} \ \in xA^\lambda \text{ for all } \mathfrak{v} \in \mathcal{T}(\lambda)\ .$$

So $c^\lambda_\mathfrak{v} = \langle x, c^\lambda_\mathfrak{u}\rangle^{-1} x c^\lambda_{\mathfrak{u}\mathfrak{v}} \in xA^\lambda$ for all $\mathfrak{v} \in \mathcal{T}(\lambda)$, proving (i'') and (i), because this is a basis of C^λ.

(ii) We only prove $\mu \geq \lambda$. Then simply reversing the roles of μ and λ proves (ii), because $\leq$ is antisymmetric.

Take $x \in C^\lambda \setminus \operatorname{rad} C^\lambda$ like in the proof of (i), which exists because $\lambda \in \Lambda_0$, and extend the given isomorphism $D^\lambda \cong D^\mu$ to a A-module homomorphism $\theta : C^\lambda \twoheadrightarrow D^\lambda \cong D^\mu$. Using (i'') we see that θ produces a special generating element of D^μ:

$$D^\mu = \theta(C^\lambda) \overset{(i'')}{=} \theta(x)A^\lambda \ . \tag{2.7}$$

Now, let $\overline{\theta(x)} \in C^\mu$ be a lift of $\theta(x)$. Then $\overline{\theta(x)}A^\lambda \subset C^\mu$ must be non-trivial by (2.7). Hence, $\mu \geq \lambda$ follows from Lemma 2.3.1.

□

If we work over a field the quotient modules associated to elements in Λ_0 are pairwise non-isomorphic right A-modules by Proposition 2.3.8. In fact, the main result of the theory of cellular algebras we will state in this section is, that all irreducible right A-modules are such quotient modules.

We need one final lemma to state a proof of this result. The idea is to reduce the problem for A to a problem for a cellular algebra with fewer elements in its poset. Then we use an inductive argument to finish the proof.

Poset ideals and the associated two-sided A-submodules of A, which were introduced in Definition 2.2.2, are exactly the right notions to construct such cellular algebras.

Lemma 2.3.9. *Let A be a cellular algebra with labeling set Λ.*

(i) *The quotient $A/A(\Gamma)$ is a cellular algebra for any poset ideal $\Gamma \subset \Lambda$. Its labeling poset is given by $\Lambda^\Gamma := \Lambda \setminus \Gamma \ \subset \Lambda$, where the partial order agrees with the partial order on Λ.*

(ii) *Let $\lambda \in \Lambda^\Gamma$, then the cell module of $A/A(\Gamma)$ corresponding to λ is, as a R-module, isomorphic to the cell module of A corresponding to λ. Moreover, the radical is the same for both algebras, so*

$$\Lambda_0^\Gamma \subset \Lambda_0 \ .$$

Proof. (i) We need to define a cell datum for $A/A(\Gamma)$. Consider

$\Lambda^\Gamma := \Lambda \setminus \Gamma \subset \Lambda$, partially ordered by restricting the partial order on Λ to Λ^Γ,
$\mathcal{T}^\Gamma : \Lambda^\Gamma \to \{\text{finite sets}\}$ defined as the restriction of $\mathcal{T}$ to Λ^Γ,
$C^\Gamma : \coprod_{\lambda\in\Lambda^\Gamma} \mathcal{T}^\Gamma(\lambda) \times \mathcal{T}^\Gamma(\lambda) \hookrightarrow A/A(\Gamma),\ (\mathfrak{s},\mathfrak{t}) \mapsto c^\lambda_{\mathfrak{s}\mathfrak{t}} + A(\Gamma)$ and
$*$ the algebra anti-automorphism of $A/A(\Gamma)$ determined by $(c^\lambda_{\mathfrak{s}\mathfrak{t}} + A(\Gamma))^* = c^\lambda_{\mathfrak{t}\mathfrak{s}} + A(\Gamma)$.

Then $((\Lambda^\Gamma, \leq), \mathcal{T}^\Gamma, C^\Gamma, *)$ is a cell datum for $A/A(\Gamma)$ if we can verify cellular property (C1). Note, $\left(A/\check{A}(\Gamma)\right)^\lambda = \check{A}^\lambda/A(\Gamma)$ as right A-modules, so for $\lambda \in \Lambda^\Gamma$ we have

$$(c^\lambda_{\mathfrak{s}\mathfrak{t}} + A(\Gamma))(a + A(\Gamma)) \equiv \left(\sum_{\mathfrak{v}\in\mathcal{T}^\Gamma(\lambda)} r^{\mathfrak{t},a}_{\mathfrak{v}} (c^\lambda_{\mathfrak{s}\mathfrak{v}} + A(\Gamma)) \right) \bmod \left(A/\check{A}(\Gamma)\right)^\lambda$$

for all $\lambda \in \Lambda^\Gamma$, $\mathfrak{t} \in \mathcal{T}^\Gamma(\lambda)$ and $(a + A(\Gamma)) \in A/A(\Gamma)$.

The coefficients $r_{\mathfrak{v}}^{\mathfrak{t},a}$ come from the cellular basis of A and are therefore independent of $\mathfrak{s}$. Moreover, the coefficients are independent of the coset representative, because if $a_1 = a_2 + x$ for some $x \in A(\Gamma)$, then $c_{\mathfrak{st}}^{\lambda} a_1 = c_{\mathfrak{st}}^{\lambda} a_2 + c_{\mathfrak{st}}^{\lambda} x$ with $c_{\mathfrak{st}}^{\lambda} x \in A(\Gamma) \cap A^{\lambda} \subset \check{A}^{\lambda}$.

(ii) We decorate right cell modules of $A/A(\Gamma)$ by $\sim$ to differentiate them from right cell modules of A.

For $\lambda \in \Lambda^{\Gamma}$ we have $\tilde{C}^{\lambda} = \operatorname{span}_R\{\tilde{c}_{\mathfrak{t}}^{\lambda} \mid \mathfrak{t} \in \mathcal{T}^{\Gamma}(\lambda)\}$. This basis is labeled exactly like the basis of C^{λ}, because $\mathcal{T}^{\Gamma}(\lambda) = \mathcal{T}(\lambda)$. Additionally, the bilinear forms on $\tilde{C}^{\lambda}$ and C^{λ} agree in the following way: by the proof of (i) and construction of $\langle . \ , . \rangle$ we have

$$\langle \tilde{c}_{\mathfrak{s}}^{\lambda}, \tilde{c}_{\mathfrak{t}}^{\lambda} \rangle = r_{\mathfrak{v}}^{\mathfrak{s}, c_{\mathfrak{tv}}^{\lambda}} = r_{\mathfrak{u}}^{\mathfrak{t}, c_{\mathfrak{su}}^{\lambda}} = \langle c_{\mathfrak{s}}^{\lambda}, c_{\mathfrak{t}}^{\lambda} \rangle$$

for all $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in \mathcal{T}^{\Gamma}(\lambda)$. Therefore, $\operatorname{rad} C^{\lambda} \cong \operatorname{rad} \tilde{C}^{\lambda}$ as R-modules. In particular, $\tilde{D}^{\lambda} \neq \{0\} \Rightarrow D^{\lambda} \neq \{0\}$, where $\tilde{D}^{\lambda}$ denotes the quotient module for $A/A(\Gamma)$. .

□

Example 2.3.10. $\Gamma_{\lambda} := \{\mu \in \Lambda \mid \mu \geq \lambda\}$ defines a poset ideal for all $\lambda \in \Lambda$. Its corresponding (A, A)-submodule is $A(\Gamma_{\lambda}) = A^{\lambda}$, so A/A^{λ} is a cellular algebra by Lemma 2.3.9.

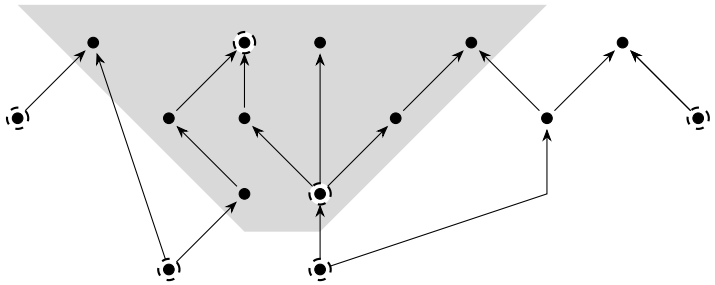


Figure 2.5: Let the dots depict elements of a poset Λ that is part of a cell datum for a cellular algebra A. Assume $a \leq b$ for $a, b \in \lambda$ if and only if there is a directed path from a to b. Let the circled dots be the elements $\Lambda_0 \subset \Lambda$.
Now assume $\Gamma \subset \Lambda$ is the subset of all dots contained in the gray area. Then Γ is a poset ideal. The poset associated to $A/A(\Gamma)$ is the set of all dots outside of the gray area and denoted by Λ^{Γ}. Circled dots outside of the gray area are also in Λ_0^{Γ}. They label non-zero quotients of right cell modules of $A/A(\Gamma)$.

Remark 2.3.11. Note that, using the decomposition

$$A^{\lambda}/\check{A}^{\lambda} \cong \bigoplus_{\mathfrak{t} \in \mathcal{T}(\lambda)} C^{\lambda} \tag{2.8}$$

of right A-modules from (2.6), the (A, A)-bimodule filtration (2.4) of A, induced by a maximal chain of poset ideals, can be refined to a right A-module filtration with subquotients isomorphic to cell modules C^{λ} for some $\lambda \in \Lambda$.

By Lemma 2.3.9(i) we can also apply this observation to $A/A(\Gamma)$. This provides a filtration of $A/A(\Gamma)$ as a right A-module with subquotients isomorphic, as a R-modules, to C^λ for some $\lambda \in \Lambda^\Gamma$, by applying Lemma 2.3.9(ii). We will see a similar result for the radical below that allows us to extract information about composition series of C^λ from the quotient $A/A(\Gamma)$.

Combining some of the preceding preparations, Graham and Lehrer [GL96] first proved the main result of this section: a classification of irreducible modules using the cellular structure of A (cf. [GL96, Theorem 3.4]). The proof below closely follows the proof of [Mat06, Theorem 2.16], which is still largely based on the ideas of Graham and Lehrer.

Proposition 2.3.12 (Classification of irreducible representations)**.** *Let R be a field. Then:*

$$\begin{aligned} \Psi : \Lambda_0 &\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Irreducible right } A\textit{-modules} \\ \textit{up to isomorphism} \end{array} \right\} \\ \lambda &\longmapsto D^\lambda \end{aligned}$$

Proof. We already know that Ψ is a well defined, injective map by Proposition 2.3.8(i), (ii), so we only need to show surjectivity.

Every irreducible right module of a cellular algebra already appears as a composition factor of a right cell module, reducing the problem of surjectivity significantly. Indeed, let

$$\{0\} = A_0 \subset A_1 \subset \cdots \subset A_k = A$$

be the right A-module filtration introduced in Remark 2.3.11, having subquotients $A_i/A_{i-1} \cong C^{\lambda_i}$ for some $\lambda_i \in \Lambda$. Any irreducible right A-module is a quotient of A and, in particular, of A_i/A_{i-1} for some $i = 1, 2, \ldots, k$, proving the stated reduction. Therefore, to prove the theorem it is enough to show that composition factors of right cell modules are isomorphic to some D^μ.

We use induction on $|\Lambda|$ and in each induction step we consider all $\lambda \in \Lambda$ individually. If $|\Lambda| = 1$ then λ is minimal and $C^\lambda = D^\lambda$ by Proposition 2.3.6. If $|\Lambda| > 1$ we differentiate between two cases. For minimal λ we again use Proposition 2.3.6. If λ is not minimal and $C^\lambda = D^\lambda$ there is nothing to show. Otherwise $\operatorname{rad} C^\lambda \neq \{0\}$ and we need to show that its composition factors are of the wanted form. Consider the poset ideal $\Gamma := \{\eta \in \Lambda \mid \lambda \not> \eta\}$. Then $A/A(\Gamma)$ is a cellular algebra with poset Λ^Γ by Lemma 2.3.9(i). Recall that, by definition, $\Lambda^\Gamma = \{\eta \in \Lambda \mid \lambda > \eta\} \neq \emptyset$ and $|\Lambda^\Gamma| < |\Lambda|$. Moreover, by Lemma 2.3.1 and Remark 2.3.4 we have

$$\operatorname{rad} C^\lambda \cdot A(\Gamma) = \operatorname{rad} C^\lambda \cdot \sum_{\lambda \not> \eta} A^\eta \overset{2.3.1}{=} \operatorname{rad} C^\lambda \cdot A^\lambda \overset{2.3.4}{=} \{0\} .$$

Hence, $\operatorname{rad} C^\lambda$ is also a right $A/A(\Gamma)$-module and, by induction, has composition factors isomorphic to some $\tilde{D}^\mu$, $\mu \in \Lambda^\Gamma$. By Lemma 2.3.9(ii) these are also composition factors for $\operatorname{rad} C^\lambda$ as a right A-module, finishing the induction step and the proof. □

We applied Lemma 2.3.1 at two important points in this proof. The base case of the induction is basically a corollary of Lemma 2.3.1 and we used it to reduce to a slightly smaller cellular algebra in the induction step.

Remark 2.3.13. In general it is not possible to determine Λ_0 only from the cellular structure of an algebra. To apply the classification result of Proposition 2.3.12 to a specific algebra in a meaningful way, one needs to first establish a cellular basis for this algebra and then develop enough theory to study right cell modules and their radicals. In this case it might even be possible to control their dimensions and thus the dimension of the irreducible modules. Only then one can hope to relabel Λ_0 and get a useful classification of the irreducibles for this algebra.

By Remark 2.3.7, Λ_0 might depend on the choice of R and, in particular, on the characteristic of R. Therefore, such further relabeling might also depend on the ground ring. This occurs, for example, for the Murphy basis of the Hecke algebra, which will be discussed in Section 4.3. There, the notion of e-restrictedness is partially based on the characteristic of R and can be used to decide which elements in Λ are in Λ_0.

Remark 2.3.14. Although it might be difficult to control Λ_0 in general, for very explicit examples, say the Hecke algebra $\mathscr{H}(\mathfrak{S}_3)$, we only need to calculate the radicals of the right cell modules. Therefore, a cellular basis provides a straightforward way of calculating the irreducibles and their dimension in explicit examples. This is a valuable aspect of the theory of cellular algebras.

For a general cellular algebra we a priori only know that minimal $\lambda \in \Lambda$ are contained in Λ_0 by Proposition 2.3.6, possibly $|\Lambda_0| = 1$. An example where this occurs is shown in Example 2.3.15 below. Still, $\Lambda_0 = \Lambda$ is a possibility, as we will see for Schur algebras over fields in Section 5. More generally, we have $\Lambda_0 = \Lambda$ for a cellular algebra over a field, if and only if it is quasi-hereditary (cf. [Mat06, Corollary 2.23]).

Example 2.3.15. Let k be an algebraically closed field and

$$A := k[x_1, x_2, \ldots, x_d]/I$$

finite dimensional with homogeneous I. Then A is a cellular algebra by Example 2.1.6. Pick a cell datum of A and let Λ be its poset. As A is local, there exists only one irreducible A-module. Therefore, by the classification in Proposition 2.3.12, we have $|\Lambda_0| = 1$ for this algebra.

Note that if we construct a cell datum for A using the inductive approach described in Example 2.1.6, then each cell module is one-dimensional. Nevertheless, because $|\Lambda_0| = 1$, all cell modules except one must have trivial bilinear form, because the radical is the whole cell module. This means the basis element associated to such cell module must have non-zero degree, as multiplication with itself is not contained in its span. The one cell module without trivial bilinear form must be the cell module associated to a basis element that has $1 \in A$ in its span.

Definition 2.3.16. For $\eta \in \Lambda_0$ and $\lambda \in \Lambda$ let $\mathbf{d}_{\lambda\eta} := [C^\lambda : D^\eta]$ be the multiplicity of D^η appearing as a subquotient of a Jordan-Hölder filtration of C^λ. Then $\mathbf{D} := \left(d_{\lambda\eta}\right)_{\lambda\in\Lambda,\, \eta\in\Lambda_0}$ is called the **decomposition matrix** of A.

Example 2.3.17. The decomposition matrix of A from Example 2.1.5 is

$$\mathbf{D} = \begin{array}{c} \mathrm{I} \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V}a \\ \mathrm{V}b \end{array} \end{array}$$

which is of course simply the dimension of each cell module because the only simple module is one dimensional by Example 2.3.15. Alternatively one can see the filtration

$$\emptyset \subset \operatorname{span}_R\left\{c_1^{\mathrm{IV}}\right\} \subset \operatorname{span}_R\left\{c_1^{\mathrm{IV}}, c_2^{\mathrm{IV}}\right\} \subset \operatorname{span}_R\left\{c_1^{\mathrm{IV}}, c_2^{\mathrm{IV}}, c_3^{\mathrm{IV}}\right\} = C^{\mathrm{IV}}$$

of C^{IV} in Example 2.1.8, which is a composition series as each subquotient is clearly isomorphic to D^{I}.

Remark 2.3.18. The proof of Proposition 2.3.12 shows $d_{\lambda\eta} = 0$ for all $\eta \not\leq \lambda$ and additionally $d_{\lambda\lambda} = 1$ if $\lambda \in \Lambda_0$. Indeed, in case of $\{0\} \neq \operatorname{rad} C^\lambda \neq C^\lambda$ the radical contributes to the composition series of C^λ. It is shown that $\operatorname{rad} C^\lambda$ is also a right module over a cellular algebra with poset $\Lambda^\Gamma = \{\eta \in \Lambda \mid \lambda > \eta\}$, so composition factors can only come from Λ^Γ.

Apart from the classification of irreducibles in Proposition 2.3.12, another major result of the theory of cellular algebras is concerned with the Cartan matrix. Here we will only state this result and refer to [GL96, Theorem 3.7] or [Mat06, Theorem 2.20] for the proof.

Recall that for each irreducible right module D^λ there exists a **principal indecomposable right A-module** P^λ with top D^λ. We are interested in composition series of P^λ as a right A-module and denote by $c_{\lambda\mu}$ the multiplicity of D^μ as a subquotient of a composition series of P^λ. Then the **Cartan matrix** is defined as $\mathbf{C} := (c_{\lambda\mu})_{\lambda,\mu\in\Lambda_0}$.

Proposition 2.3.19. *If R is a field then $\mathbf{C} = \mathbf{D}^t\mathbf{D}$, where $\mathbf{D}^t$ is the transposed of $\mathbf{D}$.*

In particular, the Cartan matrix of a cellular algebra over a field is symmetric. Thus, for $\lambda, \mu \in \Lambda_0$, the multiplicity of D^μ in a composition series of P^λ is the same as the multiplicity of D^λ in a composition series of P^μ.

Example 2.3.20. Continuing with Example 2.3.17, the cellular algebra A from Example 2.1.5 has Cartan matrix

$$\mathbf{C} = \begin{array}{c} \mathrm{I} \\ (\,14\,) \\ \mathrm{I} \end{array}$$

according to Proposition 2.3.19. Indeed, there is only one irreducible module, so A itself is its corresponding principal indecomposable right module. Moreover, the irreducible module is one-dimensional and A has dimension 14, so we have verified Proposition 2.3.19 for this example.

3. Tableaux

We want to study the two cellular bases of Hecke algebras of the symmetric group and the one cellular basis of Schur algebras mentioned in Chapter 1. By design, Murphy's basis generalizes ideas from the representation theory of the symmetric group. Irreducible representations of the symmetric group are labeled by certain tableaux, so it can be expected that they also appear in the discussion of the Murphy basis. The semistandard basis of Schur algebras is based on the Murphy basis and thus also related to tableaux. Finally, to state a cell datum for the Kazhdan–Lusztig basis the basis elements are relabeled via the Robinson–Schensted correspondence. The new labels are pairs of tableaux. Hence, the Kazhdan–Lusztig basis is related to tableaux as well.

The goal of this chapter is to fix notations for the symmetric group, for compositions and tableaux, so we can use these concepts in subsequent chapters without much more explanations. We also recall basic results that will be needed later on.

A good reference for the representation theory of the symmetric group and tableaux in general is [Sag01]. There is also [Ful96]. We will mainly follow [Mat06], where the author already collected all results from this theory relevant for a cellular basis of Hecke algebras.

We begin with the definition of the symmetric group, then move to compositions and partitions of a natural number and end this chapter with a discussion of different kinds of tableaux. In particular, the interaction between tableaux and compositions is highlighted.

3.1 Symmetric group and compositions

The **symmetric group** $\mathfrak{S}_d$ of $d \geq 1$ elements consists of all permutations of $\{1, 2, \ldots, d\}$. Transposing two elements $i \neq j \in \{1, 2, \ldots, d\}$ defines an element (i, j) in $\mathfrak{S}_d$. The **simple transpositions** $s_i := (i, i+1)$ for $i = 1, 2, \ldots, d-1$ generate $\mathfrak{S}_d$ as a group and relate to each other via **braid relations**:

$$\begin{array}{rcll} s_i s_j & = & s_j s_i & \text{for all } i, j = 1, 2, \ldots, d-1 \text{ s.t. } |i-j| \geq 2 \\ s_i s_{i+1} s_i & = & s_{i+1} s_i s_{i+1} & \text{for all } i = 1, 2, \ldots, d-2 \end{array} . \tag{3.1}$$

As simple transpositions generate $\mathfrak{S}_d$ we find an **expression** $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ for each $w \in \mathfrak{S}_d$. A **subexpression** of $s_{i_1} s_{i_2} \ldots s_{i_k}$ is an expression $s_{i_{j_1}} s_{i_{j_2}} \ldots s_{i_{j_l}}$ for some $1 \leq j_1 < j_2 < \cdots < j_l \leq k$. If no strict subexpression of $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ is again an expression of w it is called a **reduced expression**. By Matsumoto's theorem, all reduced

expressions $w = s_{i_1} s_{i_2} \dots s_{i_k}$ of $w \in \mathfrak{S}_d$ are of the same **length** $\ell(w)$. Moreover, the theorem proves the existence of a well defined partial order on $\mathfrak{S}_d$. For elements $x, y \in \mathfrak{S}_d$ it is defined by

$$x \geq y \overset{\text{def.}}{\Leftrightarrow} \begin{array}{c}\text{Some reduced expression of } y \text{ is a} \\ \text{subexpression of some reduced expression of } x\end{array}$$

and called the **Bruhat order** on $\mathfrak{S}_d$.

A sequence $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers such that $\sum \mu_i = d$ is called a **composition** of d with **parts** μ_i. It is denoted by $\mu \models d$. If $\mu_l = 0$ for all $l > k$ we also write $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. If $\lambda \models d$ and $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, 2, \dots$ we call λ a **partition** of d and write $\lambda \vdash d$.

Example 3.1.1. The compositions (d) and $(1)^d := \overbrace{(1, 1, \dots, 1)}^{d-\text{times}}$ are partitions for any d. For $d = 3$ the composition $(1, 0, 2)$ is not a partition.

We will need the set of all compositions of d, the set of all partitions of d, as well as the sets only containing compositions/partitions where the last possibly non-zero part is the n-th part:

$$\begin{aligned} \Lambda(d) &:= \{\mu = (\mu_1, \mu_2, \dots) \mid \mu \models d\} \\ \Lambda^+(d) &:= \{\lambda = (\lambda_1, \lambda_2, \dots) \mid \lambda \vdash d\} \\ \Lambda(n, d) &:= \{\mu = (\mu_1, \mu_2, \dots, \mu_n) \mid \mu \models d\} \\ \Lambda^+(n, d) &:= \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \mid \lambda \vdash d\} \ . \end{aligned} \tag{3.2}$$

Let's now combine the previous two paragraphs. If we identify $\mathfrak{S}_0$ with $\emptyset$ and consider $\mu = (\mu_1, \mu_2, \dots, \mu_k) \models d$ we can define the **Young subgroup** associated to μ as

$$\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_k} < \mathfrak{S}_d \ .$$

More precisely, if $d_i := \sum_{j<i} \mu_j$ then $\mathfrak{S}_{\mu_i}$ is the subgroup of $\mathfrak{S}_d$ generated by simple transpositions $s_{d_i}, s_{d_i+1}, \dots, s_{d_i+\mu_i-1}$.

There also exists a well defined partial order $\rhd$ on all compositions, which we will mostly use to compare compositions of a given d. Relate two compositions μ and η by

$$\mu \rhd \eta \overset{\text{def}}{\Leftrightarrow} \sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} \eta_j \text{ for all } i \geq 1 \ ,$$

which defines a partial order on the set of all compositions, called the **dominance ordering**. This partial order can also be restricted to subsets of compositions, for example to a dominance ordering on partitions.

Example 3.1.2. For $d = 3$ the dominance order on partitions is

$$(3) \ \rhd \ (2, 1) \ \rhd (1, 1, 1) = (1)^3 \ .$$

For more information about the dominance ordering see [Sag01, Section 2.2].

3.2 Standard tableaux

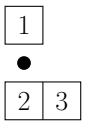
The notion of compositions is expanded by diagrams and tableaux. Formally, a **diagram** of $\mu \models d$ is the subset

$$[\mu] := \{(i,j) \mid i \geq 1,\ 1 \leq j \leq \mu_i\} \subset \mathbb{N} \times \mathbb{N}$$

and a μ-**tableau** is a bijection $\mathrm{t} : [\mu] \longrightarrow \{1, 2, \ldots, d\}$. We also call μ the **shape** of a μ-tableau t and write $\mu = \mathrm{Shape}(\mathrm{t})$.

One way to visualize diagrams is as stacked boxes, one box for each $(i,j) \in [\mu]$. A μ-tableau is then a labeling of these boxes. We use a fixed notation for stacking boxes, known as the English notation, which is demonstrated in Example 3.2.1.

Example 3.2.1. The following is an example of a $(1,0,2)$-tableau:



Notice that the 0-part of the composition is indicated by a dot.

A tableau is called **row-standard**, if the labels increase from left to right in each row. For each $\mu \models d$ there is a row-standard μ-tableau t^μ, labeling boxes increasingly from left to right and top to bottom. Moreover, for $\lambda \vdash d$ a row-standard tableaux with columns increasing from top to bottom is called a **standard tableau**. The set of standard tableaux with shape λ is denoted by $\mathrm{Std}(\lambda)$.

Example 3.2.2. The following tableaux are all standard tableaux for $d = 3$:

(3): 1 2 3

$(2,1)$: 1 2 / 3 and 1 3 / 2

$(1)^3$: 1 / 2 / 3

An example for a non-standard row-standard tableau is

,

because its first column is strictly decreasing.

These are the tableaux t^μ for all $\mu \models 3$:

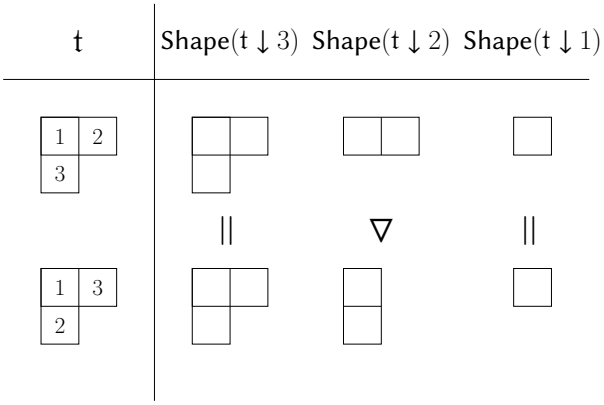
$$\mathrm{t}^{(3)} = \boxed{1}\boxed{2}\boxed{3},\quad \mathrm{t}^{(2,1)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array},\quad \mathrm{t}^{(1,2)} = \begin{array}{|c|}\hline 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array},\quad \mathrm{t}^{(1)^3} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

There exists a well defined partial order $\trianglerighteq$ on the set of all standard tableaux of a given size d, which is also called **dominance ordering**. For $m \leq d$ and a standard tableau $\mathfrak{t}$ of size d, let $\mathfrak{t} \downarrow m$ be the standard tableaux carved out of $\mathfrak{t}$ by removing all boxes labeled with integers greater than m. Now, if $\mathfrak{s}$ is also a standard tableaux of size d then

$$\mathfrak{s} \trianglerighteq \mathfrak{t} \overset{\text{def}}{\Leftrightarrow} \text{Shape}(\mathfrak{s} \downarrow m) \trianglerighteq \text{Shape}(\mathfrak{t} \downarrow m) \text{ for all } m \leq d \ .$$

Note that we use the dominance ordering on compositions to compare the shapes of $\mathfrak{s} \downarrow m$ and $\mathfrak{t} \downarrow m$.

Example 3.2.3. To check if and how $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ are related in the dominance ordering on standard tableaux of size 3 we need to compare three diagrams:



We see that these two tableaux are related in the dominance ordering and that

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \ \triangleright \ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \ .$$

For more information about this partial ordering see [Sag01, Section 2.2].

Compositions of d are in bijection with diagrams of d boxes by identifying $\mu = (\mu_1, \mu_2 \ldots, \mu_n) \models d$ with the diagram having μ_i boxes in row i for $i = 1, 2, \ldots, n$. However, there can be several μ-tableaux for each $\mu \models d$.

Recall that compositions also correspond to Young subgroups of $\mathfrak{S}_d$. We now describe the connection of Young subgroups to tableaux.

The symmetric group $\mathfrak{S}_d$ acts from the right on tableaux, which shapes correspond to a composition of d. Let $\mu \models d$, $\mathfrak{t}$ a μ-tableaux and $w \in \mathfrak{S}_d$, then $\mathfrak{t}.w$ is defined to be the μ-tableaux

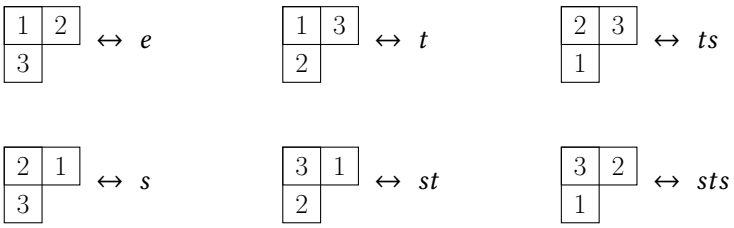
$$\mathfrak{t}.w : [\mu] \xrightarrow{\mathfrak{t}} \{1, 2, \ldots, d\} \xrightarrow{.w^{-1}} \{1, 2, \ldots, d\} \ ,$$

where $\mathfrak{S}_d$ acts on $\{1, 2, \ldots, d\}$ by applying the permutation. In other words, the right action on tableaux permutes the labels of boxes in the diagram of the tableaux. For every μ-tableau $\mathfrak{t}$ there exists a unique $w(\mathfrak{t}) \in \mathfrak{S}_d$ such that $\mathfrak{t} = \mathfrak{t}^\mu.w(\mathfrak{t})$, so there is a bijection

$$\begin{array}{rcl} \{\mu - \text{tableaux}\} & \overset{1:1}{\leftrightarrow} & \mathfrak{S}_d \\ \mathfrak{t} & \mapsto & w(\mathfrak{t}) \end{array} \ . \tag{3.3}$$

Example 3.2.4. Assume, as we will for many examples below, that $d = 3$ and the simple transpositions of $\mathfrak{S}_3$ are denoted by $s := (1,2)$ and $t := (2,3)$.

Let $\mu = (2,1)$, then the correspondence of (3.3) is:



The Young subgroup $\mathfrak{S}_\mu < \mathfrak{S}_d$ is exactly the set of elements that only permute labels in each row of $\mathfrak{t}^\mu$. While their corresponding tableaux from (3.3) are not of special interest to us, their cosets will be important.

Proposition 3.2.5. *For $\mu \models d$ the set*

$$\mathscr{D}_\mu := \{w \in \mathfrak{S}_d \mid \mathfrak{t}^\mu.w \text{ is row-standard tableau}\}$$

is a complete set of minimal length right coset representatives of $\mathfrak{S}_\mu < \mathfrak{S}_d$, so if $w \in \mathfrak{S}_\mu$ and $x \in \mathscr{D}_\mu$ then $\ell(wx) = \ell(w) + \ell(x)$.

Proof. We only sketch the idea of the proof of Proposition 3.2.5. For a full proof see [Mat06, Proposition 3.3].

As $\mathfrak{S}_\mu$ reorders labels in rows, these elements must be in different cosets. It is complete, because the rows of any tableau can be reordered such that it becomes row-standard. Minimality follows from the idea that ordered rows produce the lowest number of inversions. □

Example 3.2.6. Only tableaux in the first row of 3.2.4 are row-standard, so:

$$\mathscr{D}_{(2,1)} = \{e, t, ts\}$$

Tableaux of a given shape $\mu \models d$ can be interpreted as $\mathfrak{S}_d$-orbits of the composition μ via the action on $\mathfrak{t}^\mu$. By Proposition 3.2.5 this action enables us to study Young subgroups $\mathfrak{S}_\mu < \mathfrak{S}_d$ and their cosets.

However, this is not an orbit in the set of compositions, as the shape of a tableau does not change under the action. We therefore consider an action of compositions on tableaux in the next section to study compositions in this context.

3.3 Semistandard tableaux

The notion of μ-tableau for some $\mu \models d$ is generalized by relaxing the restrictions on labels of the underlying diagram.

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \models d$ such that $\eta_m = 0$ for all $m > n$ for a given $n \in \mathbb{N}$, so $\eta \in \Lambda(n, d)$. Then a μ-tableau of **type** η is a not necessarily bijective map

$$T : [\mu] \longrightarrow \{1, 2, \ldots, n\}$$

such that $\eta_i = \#\{x \in [\mu] \mid T(x) = i\}$ for all $i \geq 1$. Put simply, these are diagrams with labels ranging from 1 to n, where label i appears exactly η_i-times.

Similarly to the normal tableaux from before we call a μ-tableau T of type η **row-semistandard**, if labels in each row are non-decreasing. If additionally $\mu \vdash d$ and the columns of T are strictly increasing we call T **semistandard** and we use the notation

$$\mathcal{T}_0(\mu, \eta) \coloneqq \{\text{semistandard } \mu\text{-tableaux of type } \eta\} \ .$$

Example 3.3.1. For partitions (3) and $(2,1)$ of $d = 3$ the semistandard tableaux are:

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \in \mathcal{T}_0(\square\square\square, \square\square\square)$

$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \in \mathcal{T}_0\left(\square\square\square\,,\ \begin{array}{|c|}\hline\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\right)$ $\qquad$ $\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \in \mathcal{T}_0\left(\begin{array}{|c|c|}\hline\square&\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\,,\ \begin{array}{|c|}\hline\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\right)$

$\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} \in \mathcal{T}_0\left(\square\square\square\,,\ \begin{array}{|c|}\hline\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\right)$ $\qquad$ $\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \in \mathcal{T}_0\left(\begin{array}{|c|c|}\hline\square&\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\,,\ \begin{array}{|c|}\hline\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\right)$

$\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array} \in \mathcal{T}_0\left(\square\square\square\,,\ \begin{array}{|c|}\hline\bullet\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\begin{array}{|c|}\hline\square\\\hline\end{array}\right)$

In this example there is at most one element in each $\mathcal{T}_0(\mu, \eta)$. This is not true in general and thus misleading. However, we will continue the example below and this circumstance makes it easier to keep track of all semistandard tableaux, because we can label them by their combination of μ and η.

Note that in a μ-tableau of type (1^d) each label from 1 to d appears exactly once, so introducing types generalized the normal μ-tableaux from before. We can also generate a μ-tableaux of type η from a normal μ-tableaux t by replacing a label i by the row it appears in t^η. Put simply, we replace labels $1, 2, \ldots, \eta_1$ in t by 1, we replace $\eta_1 + 1, \eta_1 + 2 \ldots, \eta_2$ by 2 and so on. Hence, we get a μ-tableau of type η, which is denoted by $\eta(\mathrm{t})$.

There are three important features of this construction. Firstly, notice that replacing labels $i < j$ either keeps the ordering intact or they get replaced by the same label. Thus row-standard tableaux t generate row-semistandard tableaux $\eta(\mathrm{t})$, but standard tableaux do not have to generate semistandard tableaux.

Secondly, for any normal μ-tableau t from Section 3.2 and any $\eta \neq (1^d)$ the μ-tableau $\eta(\mathrm{t})$ will not be a normal μ-tableau. Hence, the orbits of normal μ-tableaux under the action

of compositions can not be interpreted as orbits in the set of tableaux. In this sense it is similar to the $\mathfrak{S}_d$ action on the diagram $[\mu]$ via the row-standard μ-tableau t^μ. Its orbits can not be interpreted as orbits in the set of diagrams and must instead be seen as orbits in the set of tableaux.

Thirdly, acting with η on two different μ-tableaux can yield the same μ-tableau of type η. Therefore, in contrast to the bijection (3.3) for normal tableaux, there is no sensible way to assign an element of $\mathfrak{S}_d$ to each μ-tableau of type η.

Example 3.3.2. If $\eta = (1, 2)$, then:

$$\eta\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = \eta\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)$$

Note that both actions generate the same semistandard tableau.

Now let $\eta = (3)$, then:

$$\eta\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array}$$

Note that in this example the action on a standard tableau produced a non-semistandard tableau.

We can rectify the last feature for our purposes, as we will again only be interested in row-semistandard tableaux. The action on row-standard μ-tableaux gives orbits in row-semistandard μ-tableaux. By generalizing Proposition 3.2.5 we can identify these elements with certain elements of $\mathscr{D}_\mu \subset \mathfrak{S}_d$. In fact, these turn out to be some kind of minimal coset representatives as well, but we will not discuss it in detail.

Proposition 3.3.3. *For $\mu, \eta \models d$ there is a bijection*

$$\begin{array}{rcl} \mathscr{D}_{\mu\eta} \coloneqq \mathscr{D}_\mu \cap \mathscr{D}_\eta^{-1} & \overset{1:1}{\leftrightarrow} & \{\textit{row-semistandard } \mu \textit{ -tableaux of type } \eta\} \\ w & \mapsto & \eta(\mathrm{t}^\mu . w) \end{array} .$$

A proof of this result can be found in [Mat06, Proposition 4.4]. We see that the action of η on row-standard tableaux gets translated to intersecting with $\mathscr{D}_\eta^{-1}$. For $\eta = (1^d)$ Proposition 3.3.3 reduces to Proposition 3.2.5.

This finishes our recap of the symmetric group, compositions and tableaux. We are now ready to define Hecke algebras associated to a symmetric group, construct cell data for it and discuss their implications. The concepts introduced in this chapter will appear throughout this discussion.

4. Cellular bases for Hecke algebras of type A

The definition of cellular algebras in [GL96] was motivated by the Kazhdan–Lusztig basis of Hecke algebras $\mathscr{H}$ of symmetric groups, also known as Hecke algebras of type A. The authors also briefly describe a cell datum associated to that basis. A much more extensive exploration of this cell datum can be found in [Wil03].

However, this is not the only cellular basis of Hecke algebras of type A. Murphy introduced another basis, now called the Murphy basis, in [Mur92]. His goal was to simplify the approach of Dipper and James in [DJ86], but still referred to some of their results. They were interested in the representation theory of Hecke algebras over fields of arbitrary characteristic and constructed all irreducible representations. In [Mur95], Murphy presented a self contained version of his approach. Although cellular algebras were not yet defined, his techniques are very similar to the techniques that Graham and Lehrer used. He labels his basis by ordered pairs of standard tableaux of a given size, orders them via the dominance ordering on their shapes and proves that the action on these elements satisfies a cellular property. He continues to construct one-sided cell representations, defines a bilinear form on them and uses this form to classify all irreducible representations of the Hecke algebra. There is a cell datum associated to the Murphy basis that formally puts Murphy's results into the cellular context. A detailed description of the Murphy basis as a cellular basis is given in [Mat06].

One goal of this chapter is to give an overview over the constructions of both cellular bases of Hecke algebras of type A mentioned above. We introduce all relevant notions needed to define the cell data, give many examples and state the complete cell data, but will only give a few proofs and intuitions.

Proving the cellular property for the Murphy basis requires some careful induction (cf. [Mur95, Theorem 4.18]). We also refer to [Mat06] for a complete discussion of the Murphy basis, which we will partially follow here. In [Mat06] there is also a description of the role of the Murphy basis in the construction of a cellular basis for Schur algebras.

Proving the cellular property for the Kazhdan–Lusztig basis is much more involved and we refer to [Wil03] for details, where all results needed to prove cellularity are presented and collected. Compared to the definitions from Chapter 2, the author uses a slightly different, but equivalent, definition of cellular algebras. If we cite this reference we will still be using the definitions from Chapter 2.

The Murphy basis was build around the ideas of Dipper and James [DJ86] to generalize an approach to the classification of irreducible representations of the symmetric group to Hecke algebras. One advantage of the Murphy basis is that it can also be used to generalize other constructions from the representation theory of $\mathfrak{S}_d$ to the representation theory of $\mathscr{H}$. The other goal of this chapter is to describe a few of these generalizations.

Murphy already used generalized Jucys–Murphy elements in his classification of irreducible $\mathscr{H}$-modules, so this will be the first generalization we will discuss in Section 4.4. If $\mathscr{H}$ is semisimple, the Murphy basis also defines a basis for each irreducible module. In combination with Jucys–Murphy elements, this basis can be transformed into a generalization of Young's orthogonal form, a special orthogonal basis we will see in Section 4.5. This result can then be used to describe the decomposition of restrictions of irreducible modules, classically known as the Branching theorem.

For $\mathbb{C}[\mathfrak{S}_d]$, these results can be found, for example, in [CST10] and [VO05]. References for these constructions for Hecke algebras of type A are [Mur81], [Mat06] and [Soe97]. We will closely follow their presentations in this chapter.

This chapter is organized as follows. First we recall the definition of Hecke algebras of type A. Then we study special modules, called permutation modules, in Section 4.2, that are needed to define the Murphy basis and will also be used in Chapter 5, where we consider the Schur algebra. Finally, we state the full cell datum associated to the Murphy basis in Section 4.3. In these sections we mainly follow [Mat06] and provide additional examples for all constructions.

In Section 4.4 we describe Murphy's approach to the classification of irreducible $\mathscr{H}$-modules using Jucys–Murphy elements. Afterwards we construct Young's orthogonal form for each irreducible representation in Section 4.5, if $\mathscr{H}$ is semisimple, and use it to derive the Branching theorem for $\mathscr{H}$. References for the classical case are [Sag01] and [VO05], part of the general case is also described in [Mur81].

To close this chapter we then recall the Kazhdan–Lusztig basis mentioned above and state its full cell datum, partially following [Wil03]. We are particularly interested in the labeling of this basis, which is done via a one-to-one correspondence between $\mathfrak{S}_d$ and the set of ordered pairs of standard tableaux of the same shape and of size d, called Robinson–Schensted correspondence.

4.1 Definitions and notations

Let R be an integral domain. Additionally, let $v \in R$ be invertible and $d \in \mathbb{Z}_{>0}$.

Definition 4.1.1. The associative, unital R-algebra generated by elements $H_1, H_2, \ldots, H_{d-1}$ such that

$$\begin{aligned} H_iH_j &= H_jH_i && \text{for all } i,j = 1,2,\ldots,d-1 \text{ s.t. } |i-j| \geq 2 \\ H_iH_{i+1}H_i &= H_{i+1}H_iH_{i+1} && \text{for all } i = 1,2,\ldots,d-2 \\ 0 &= (H_i - v^{-1})(H_i + v) && \text{for all } i = 1,2,\ldots,d-1 \end{aligned} \tag{4.1}$$

is called the **Hecke algebra** $\mathscr{H}_{R,v}(\mathfrak{S}_d)$ the symmetric group $\mathfrak{S}_d$.

When we fix R, v and d we will often simply write $\mathscr{H}$ for the associated Hecke algebra.

The connection of $\mathscr{H}$ to $\mathfrak{S}_d$ is hidden in Definition 4.1.1. A more elaborate notation for the generating elements H_i is H_{s_i}, where $s_i = (i, i+1) \in \mathfrak{S}_d$ the i-th simple transposition. Then the first two lines of (4.1) mimic the braid relations of corresponding simple transpositions in $\mathfrak{S}_d$. We will use these notations interchangeably.

Expanding on this idea we would like to associated to each $w \in \mathfrak{S}_d$ an element in $\mathscr{H}$ such that some structure of $\mathfrak{S}_d$ is carried over to $\mathscr{H}$.

Let $w = s_{i_1} s_{i_2} \dots s_{i_k} \neq e$ be a reduced expression for $w \in \mathfrak{S}_d$, then define

$$H_w := H_{i_1} H_{i_2} \dots H_{i_k} \in \mathscr{H} \tag{4.2}$$

and $H_e := 1$. By Matsumoto's theorem, the element $H_w \in \mathscr{H}$ does not depend on the reduced expression and is thus well defined.

Proposition 4.1.2. *The set of elements $\{H_w \mid w \in \mathfrak{S}_d\}$ is a R-module basis of $\mathscr{H}$, called the **standard basis** of $\mathscr{H}$. Acting with a generator onto this basis yields*

$$H_w H_{s_i} = \begin{cases} H_{ws_i} & \text{if } l(ws_i) > l(w) \\ H_{ws_i} + (v^{-1} - v) H_w & \text{if } l(ws_i) < l(w) \end{cases}$$

See [Mat06, Theorem 1.13] for a proof of this proposition.

By Proposition 4.1.2, H_s is invertible for any simple transposition $s \in \mathfrak{S}_d$ with inverse

$$H_s^{-1} = H_s - (v - v^{-1}) .$$

Therefore, any standard basis element H_w, $w \in \mathfrak{S}_d$ is invertible with inverse

$$H_w^{-1} = H_{i_k}^{-1} H_{i_{k-1}}^{-1} \dots H_{i_1}^{-1} .$$

Hence, there exists a well defined algebra anti-isomorphism

$$\begin{aligned} * : \mathscr{H} &\longrightarrow \mathscr{H} \quad , \\ H_w &\mapsto H_{w^{-1}} \end{aligned}$$

which is part of both cell data for $\mathscr{H}$ we consider in this chapter.

Note that Proposition 4.1.2, in combination with (4.2), completely describes the multiplication between standard basis elements.

Remark 4.1.3. Because standard basis elements are well defined by (4.2), we can associate to each H_w the length $\ell(w)$ and use this to construct two, possibly isomorphic, one-dimensional representations of $\mathscr{H}$. In the **sign representation**, each H_w acts by $(-v)^{\ell(w)}$. In the **trivial representation**, each H_w acts by $v^{-\ell(w)}$. We will take another look at these representations for the explicit example $\mathscr{H}(\mathfrak{S}_3)$ in Section 6.1.

Finally, we can associate to each Young subgroup $\mathfrak{S}_\mu < \mathfrak{S}_d$ a subalgebra of $\mathscr{H}(\mathfrak{S}_d)$. Indeed, let $\mathscr{H}(\mathfrak{S}_\mu) \subset \mathscr{H}(\mathfrak{S}_d)$ be the R-submodule spanned by H_s for all simple reflections $s \in \mathfrak{S}_\mu$. Then $\mathscr{H}(\mathfrak{S}_\mu)$ is also a subalgebra of $\mathscr{H}(\mathfrak{S}_d)$ by Proposition 4.1.2.

Remark 4.1.4. Suppose $R = \mathbb{C}$ and $v = 1$, then there exists an algebra isomorphism $\mathscr{H}_{\mathbb{C},1}(\mathfrak{S}_d) \cong \mathbb{C}[\mathfrak{S}_d]$ that identifies $H_i \in \mathscr{H}_{\mathbb{C},1}(\mathfrak{S}_d)$ with the simple transposition $(i, i+1) \in \mathbb{C}[\mathfrak{S}_d]$ for all $i = 1, 2, \ldots, d-1$. We are able to identify H_i with a transposition that has braid relations because of the three relations in (4.1). The Hecke algebra is said to be a one-parameter deformation of the group algebra of the symmetric group.

Remark 4.1.5. There exists another, almost equivalent definition of the Hecke algebra that is used in many of our references.

For R as above and $q \in R$ they define $\widetilde{\mathscr{H}}_{R,q}(\mathfrak{S}_d)$ as the unital R-algebra generated by elements $T_1, T_2, \ldots, T_{d-1}$ and the following relations:

$$\begin{aligned} T_iT_j &= T_jT_i && \text{for all } i,j = 1,2,\ldots,d-1 \text{ s.t. } |i-j| \geq 2 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && \text{for all } i = 1,2,\ldots,d-2 \\ 0 &= (T_i - q)(T_i + 1) && \text{for all } i = 1,2,\ldots,d-1 \end{aligned}$$

The two definitions are equivalent if q is invertible and $q^{-\frac{1}{2}}$ exists, for example in the case of $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Indeed, we can transition between the definitions by identifying $v = q^{-\frac{1}{2}}$ and $H_i = vT_i$ for all $i = 1, 2, \ldots, d-1$.

We will continue with the Hecke algebra from Definition 4.1.1 and translate results from our references into this convention if necessary. For another short introduction to Hecke algebras as defined in Definition 4.1.1, see the beginning of [Soe97, Section 2].

4.2 Permutation modules

One possible definition of permutation modules for the Hecke algebra is as an induced representations. Recall the Young subgroup $\mathfrak{S}_\mu < \mathfrak{S}_d$ associated to $\mu \models d$ from Section 3.1 and let $\mathscr{H}(\mathfrak{S}_\mu) \subset \mathscr{H}(\mathfrak{S}_d)$ be the corresponding subalgebra. Denote by $1_{\mathscr{H}(\mathfrak{S}_\mu)}$ the one dimensional trivial representation of $\mathscr{H}(\mathfrak{S}_\mu)$ from Remark 4.1.3. Then the permutation module associated to μ is defined to be the right $\mathscr{H}(\mathfrak{S}_d)$-module

$$M^\mu := 1_{\mathscr{H}(\mathfrak{S}_\mu)} \otimes_{\mathscr{H}(\mathfrak{S}_\mu)} \mathscr{H}(\mathfrak{S}_d) \,.$$

In the classical case of $\mathbb{C}[\mathfrak{S}_d]$, permutation modules were used to determine all irreducible representations. More precisely, these are found as submodules of permutation modules. See [Sag01] or [Jam78] for a complete discussion.

Here we are interested in two applications of permutation modules. Firstly, the Hecke algebra itself is the permutation module for the composition $(1)^d \models d$. After defining the Murphy basis of the Hecke algebra it can be generalized to bases for all permutation modules. Secondly, it is possible to define the Schur algebra using permutation modules, as we will see in Chapter 5. There are other, equivalent definitions of the Schur algebra, even in the classical case. The advantage of defining it via permutation modules is, that we can use the generalized bases of permutation modules to build a cellular basis for the Schur algebra.

One reference for this algebra is [Don98]. For its mentioned basis stemming from permutation modules see [Mat06].

Note that the construction of permutation modules from above can be generalized even further. Young subgroups of the symmetric group can be replaced by parabolic subgroups of Coxeter groups. It is also possible to consider induction of the one-dimensional sign representation from Remark 4.1.3, instead of the trivial representation. Modules constructed via these methods are called parabolic Hecke modules. We refer to [Soe97] for more information. There the author constructs, for example, a Kazhdan–Lusztig basis of parabolic Hecke modules. An even more detailed account of this theory can be found in [Str20].

We begin this section with a redefinition of permutation modules as submodules of $\mathscr{H}(\mathfrak{S}_d)$ generated by a single element. In Lemma 4.2.2 we see that both definitions are equivalent. Afterwards we use these generators to construct the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. This basis is then generalized to bases for all permutation modules.

4.2.1 Definition and row-standard basis

We continue with the setup of Section 4.1, so R is an integral domain, $v \in R$ is invertible and $d \in \mathbb{Z}_{>0}$.

Definition 4.2.1. Assume $\mu \models d$ and let $\mathfrak{S}_\mu < \mathfrak{S}_d$ be the corresponding Young subgroup. Then the element

$$m_\mu := \sum_{w \in \mathfrak{S}_\mu} v^{-\ell(w)} H_w \ \in \mathscr{H}(\mathfrak{S}_\mu) \subset \mathscr{H}(\mathfrak{S}_d) \tag{4.3}$$

spans the **right permutation module** associated to μ:

$$M^\mu := m_\mu \mathscr{H} \ \subset \mathscr{H}$$

As indicated above, permutation modules are closely related to the subalgebras $\mathscr{H}(\mathfrak{S}_\mu) \subset \mathscr{H}$. With the next lemma we see that permutation modules are, as $\mathscr{H}$-modules, just induced modules of the trivial representation of $\mathscr{H}(\mathfrak{S}_\mu)$. The following proof is given in [Mat06, Lemma 3.2].

Lemma 4.2.2. *If $\mu \models d$ and $w \in \mathfrak{S}_\mu$, then*

$$m_\mu H_w = v^{-\ell(w)} m_\mu \ . \tag{4.4}$$

Proof. By (4.2) it is enough to show (4.4) for simple reflections $s \in \mathfrak{S}_\mu$. We can reorder the sum of (4.3) by realizing that $\ell(w) < \ell(ws)$ also implies $\ell(ws) > \ell(wss) = \ell(w)$. Now

$$\begin{aligned} m_\mu H_s &\overset{(4.3)}{=} \sum_{w \in \mathfrak{S}_\mu} v^{-\ell(w)} H_w H_s = \sum_{\substack{w \in \mathfrak{S}_\mu \\ \ell(w) < \ell(ws)}} (v^{-\ell(w)} H_w + v^{-\ell(w)-1} H_{ws}) H_s \\ &= \sum_{\substack{w \in \mathfrak{S}_\mu \\ \ell(w) < \ell(ws)}} (v^{-\ell(w)-1} H_w + v^{-\ell(ws)-1} H_{ws}) = v^{-1} \cdot \sum_{w \in \mathfrak{S}_\mu} v^{-\ell(w)} H_w \quad = v^{-1} m_\mu \end{aligned}$$

finishes the proof, using the multiplication formulas in $\mathscr{H}(\mathfrak{S}_d)$ from Proposition 4.1.2 in the third equation. □

Remark 4.2.3. For composition $\mu = (1)^d$ of d the corresponding Young subgroup is $\mathfrak{S}_\mu = \{e\}$, the generating element is $m_{(1)^d} = H_e = 1$ and the permutation module is $M^{(1)^d} = \mathscr{H}$. Hence, all permutation modules are submodules of a single permutation module, the Hecke algebra itself.

There exists a R-basis of each permutation module, which uses the coset representatives of Young subgroups from Proposition 3.2.5. Recall that for a μ-tableau t we denote by $w(\mathrm{t})$ the element in $\mathfrak{S}_d$ such that $\mathrm{t} = \mathrm{t}^\mu.w(\mathrm{t})$.

Proposition 4.2.4. *Let $\mu \models d$, then M^μ has the **row-standard basis***

$$\{m_\mu H_{w(t)} \mid \mathrm{t} \text{ row-standard } \mu\text{-tableaux}\} \,. \tag{4.5}$$

Proof. By Lemma 3.2.5 we can express each $w \in \mathfrak{S}_d$ as $w = xy$ with $\ell(w) = \ell(x) + \ell(y)$ for some elements $x \in \mathfrak{S}_\mu$ and $y \in \mathscr{D}_\mu$. Then we have $m_\mu H_w = v^{-\ell(x)} m_\mu H_y$ by Lemma 4.2.2, so the elements of (4.5) span M^μ. They are linearly independent because each $m_\mu H_y$ is a sum over standard basis elements of $\mathscr{H}$ coming from disjoint cosets of $\mathfrak{S}_\mu$. □

The proof of Proposition 4.2.4 is from [Mat06, Corollary 3.4]. Here we see the advantage of extending the notion of compositions to tableaux in Section 3. Row-standard tableaux correspond to minimal length coset representatives of $\mathfrak{S}_\mu < \mathfrak{S}_d$ by Proposition 3.2.5, which help us to understand the $\mathscr{H}$-action onto the generator of M^μ.

The row-standard basis is a generalization of the standard basis of $\mathscr{H}$ from Proposition 4.1.2 to permutation modules. Indeed, both constructions agree for $\mathscr{H}$, as all $(1)^d$-tableaux are row-standard. Just like the standard basis of $\mathscr{H}$, the row-standard basis reveals a basic structure of permutation modules. However, it is not suitable for our purposes and we will modify it below. Note that there also exists a Kazhdan–Lusztig basis for all permutation modules (cf. [Soe97], [Str20]).

4.2.2 Standard basis

One example of the shortcomings of the row-standard basis of permutation modules is its interaction with the anti-isomorphism $*$. A central object for Schur algebras below will be the intersection $M^\mu \cap M^{\eta *}$ for $\mu, \eta \models d$. Note that because $\mathfrak{S}_\mu$ is a subgroup of $\mathfrak{S}_d$, so $w \in \mathfrak{S}_\mu \Leftrightarrow w^{-1} \in \mathfrak{S}_\mu$ and $\ell(w^{-1}) = \ell(w)$, we have

$$m_\mu^* = \sum_{w \in \mathfrak{S}_\mu} v^{-\ell(w)} H_w^* = \sum_{w \in \mathfrak{S}_\mu} v^{-\ell(w^{-1})} H_{w^{-1}} = m_\mu \,, \tag{4.6}$$

and thus $M^{\eta *} = \mathscr{H} m_\eta \subset \mathscr{H}$. Although Proposition 4.2.4 also gives us a basis for $M^{\eta *}$, it is unclear which row-standard basis elements are in this intersection, or which elements even span it.

These observations motivate the search for elements in $\mathscr{H}$ similar to row-standard basis elements that interact with such intersection and $*$ by construction.

For $\mu \models d$ and two row-standard μ-tableaux $\mathfrak{s}, \mathfrak{t}$ define

$$m_{\mathfrak{st}} := v^{-(\ell(w(s))+\ell(w(t)))} H^*_{w(\mathfrak{s})} m_\mu H_{w(\mathfrak{t})} \in \mathscr{H} \,. \tag{4.7}$$

We have $m^*_{\mathfrak{st}} = m_{\mathfrak{ts}}$ by (4.6). We will see concrete examples of these elements in Example 4.2.6 below.

Note that M^μ is not necessarily a two-sided module, so these elements do not have to lie in M^μ. They also do not have to be linearly independent. At first glance this seems problematic for our goal of finding another basis of permutation modules, but in reality the situation was similar for the row-standard basis. There we established a row-standard basis for $\mathscr{H}$ in Proposition 4.1.2, which elements are only contained in certain permutation modules. However, summing up elements in the same Young subgroup cosets led to row-standard bases for all permutation modules in Proposition 4.2.4.

Here we will proceed in a similar way. We start with the construction of a basis of $\mathscr{H}$ consisting of some of the $m_{\mathfrak{st}}$. For the general case we then sum up these elements using the action of compositions on tableaux.

Proposition 4.2.5. *The Hecke algebra $\mathscr{H}$ has a R-basis*

$$\{m_{\mathfrak{st}} \mid \lambda \vdash d \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathit{Std}(\lambda)\} \,,$$

*which is called the **Murphy basis** of $\mathscr{H}$.*

Proof. Here we only sketch the proof and refer to [Mat06, Theorem 3.20] for a complete proof. The set

$$B := \{m_{\mathfrak{st}} \mid \mu \models d \text{ and } \mathfrak{s}, \mathfrak{t} \text{ row-standard}\}$$

generates $\mathscr{H}$, because for $\mu = (1)^d$ and $\mathfrak{s} = \mathfrak{t}^\mu$ the elements $m_{\mathfrak{st}}$ are the basis in Proposition 4.1.2. Note that the tableaux $\mathfrak{t}$ involved in this argument are not standard, except for $\mathfrak{t} = \mathfrak{t}^\mu$.

Then one can show that those $m_{\mathfrak{st}}$ stemming from partitions of d already generate B. A careful induction shows that the remaining elements can be generated by $m_{\mathfrak{st}}$ stemming from standard tableaux.

For linear independence one then counts the elements in this set and compares this with the cardinality of the standard basis in Proposition 4.1.2. □

Example 4.2.6. We continue with the example $\mathscr{H}(\mathfrak{S}_3)$ and want to construct the Murphy basis with (4.7).

The set of all partitions of $d = 3$ is

$$\Lambda^+(3) = \left\{ \ydiagram{3} \rhd \ydiagram{2,1} \rhd \ydiagram{1,1,1} \right\},$$

their corresponding Young subgroups are

$$\mathfrak{S}_{\ydiagram{3}} = \langle s, t \rangle = \mathfrak{S}_3, \quad \mathfrak{S}_{\ydiagram{2,1}} = \langle s \rangle = \{e, s\}, \quad \mathfrak{S}_{\ydiagram{1,1,1}} = \langle t \rangle = \{e, t\}$$

and the generators of their permutation modules defined in Definition 4.2.1 are

$$m_{\ydiagram{3}} = v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1$$
$$m_{\ydiagram{2,1}} = v^{-1}H_s + 1$$
$$m_{\ydiagram{1,1,1}} = 1\ .$$

To construct elements of the Murphy basis we need the standard tableaux of all partitions. The standard tableaux of $\ydiagram{3}$ and $\ydiagram{1,1,1}$ are obvious, as rows must be increasing and columns strictly increasing. For $\ydiagram{2,1}$ the standard tableaux are included in Example 3.2.4.

$$\mathrm{Std}\left(\ydiagram{3}\right) = \left\{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right\},\ \mathrm{Std}\left(\ydiagram{2,1}\right) = \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\},\ \mathrm{Std}\left(\ydiagram{1,1,1}\right) = \left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\}$$

Finally, the elements in the Murphy basis are:

$$m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = H_e^* m_{\ydiagram{3}} H_e = v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1$$

$$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = H_e^* m_{\ydiagram{2,1}} H_e = v^{-1}H_s + 1$$

$$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = v^{-1}H_e^* m_{\ydiagram{2,1}} H_t = v^{-2}H_{st} + v^{-1}H_t$$

$$m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = v^{-1}H_t^* m_{\ydiagram{2,1}} H_e = v^{-2}H_{ts} + v^{-1}H_t$$

$$m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = v^{-2}H_t^* m_{\ydiagram{2,1}} H_t = v^{-3}H_{sts} + (v^{-2}-1)H_t + v^{-2}$$

$$m_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}} = H_e^* m_{\ydiagram{1,1,1}} H_e = 1$$

Note that m_{st} are defined for all row-standard tableaux, but only standard tableaux appear in the basis. An example of an omitted element is

$$\begin{aligned} m_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&1\\\cline{1-2}\end{array}\begin{array}{|c|c|c|}\hline 2&3&3\\\hline 1&2&2\\\cline{1-2}\end{array}} &= v^{-2}H_e^* m_{\ydiagram{2,2}} H_{ts} \\ &= v^{-3}H_{sts} + v^{-2}H_{ts} = m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} - m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} - m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}\ . \end{aligned}$$

Semistandard tableaux, which were introduced in Section 3.3, are used to generalize the Murphy basis of $\mathscr{H}$ to all permutation modules. Let $\lambda \vdash d$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$ a standard λ-tableaux. Let $\mu \models d$ and S a semistandard λ-tableaux of type μ. Then we define

$$m_{S\mathsf{t}} := \sum_{\substack{\mathsf{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathsf{s})=S}} m_{\mathsf{st}}$$

and

$$m_{\mathsf{t}S} := \sum_{\substack{\mathsf{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathsf{s})=S}} m_{\mathsf{ts}} = m^*_{S\mathsf{t}} \ .$$

Example 4.2.7. Let $\mu = (1,2) =$  , then by Example 3.3.1 there are two semistandard tableaux of this type:  and  .

The shape of $S =$  is  . The only standard tableau with this shape is  . Acting with μ on this tableau yields

$$(1,2)\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} = S \ .$$

Hence, we get the element

$$m_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} \in M^{\ydiagram{2,1}} \ .$$

The shape of $S =$  is  . The standard tableaux with this shape are  and  . Acting with μ on them yields

$$(1,2)\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} = (1,2)\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) ,$$

so we get two summands for each $\mathsf{t} \in \mathrm{Std}($$)$:

$$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} \in M^{\ydiagram{2,1}}$$

$$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} \in M^{\ydiagram{2,1}}$$

Proposition 4.2.8. *The permutation module M^μ for $\mu \models d$ has the* ***standard basis***

$$\{m_{S\mathsf{t}} \mid \lambda \vdash d,\ S \in \mathcal{T}_0(\lambda,\mu),\ \mathsf{t} \in \mathit{Std}(\lambda)\} \ .$$

Proof. We again only sketch the proof and refer to [Mat06, Theorem 4.9] for a complete proof. Its key input is an extended version of Proposition 3.3.3. It implies that for $w \in \mathscr{D}_{\lambda\mu}$ such that $S = \mu(\mathsf{t}^\lambda.w)$ we have

$$\sum_{\substack{y\in\mathscr{D}_\lambda\\ \mu(\mathsf{t}^\lambda.y)=S}} v^{-\ell(y)} H^*_y m_\lambda \ = \sum_{z\in\mathfrak{S}_\mu w^{-1}\mathfrak{S}_\lambda} v^{-\ell(z)} H_z \ \in M^\mu \tag{4.8}$$

and, by applying $*$ to (4.8),

$$m_\lambda \sum_{\substack{y\in\mathscr{D}_\lambda \\ \mu(\mathrm{t}^\lambda.y)=S}} v^{-\ell(y)}H_y \;=\; \sum_{z\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu} v^{-\ell(z)}H_z \;\in M^{\mu *}\,. \tag{4.9}$$

Then

$$m_{S\mathrm{t}} \overset{def}{=} \sum_{\substack{\mathrm{s}\in\mathrm{Std}(\lambda) \\ \mu(\mathrm{s})=S}} m_{\mathrm{st}} \overset{3.2.5}{=} v^{-\ell(w(\mathrm{t}))} \sum_{\substack{y\in\mathscr{D}_\lambda \\ \mu(\mathrm{t}^\lambda.y)=S}} v^{-\ell(y)}H_y^* m_\lambda H_{w(\mathrm{t})} \overset{4.8}{\in} M^\mu\,, \tag{4.10}$$

where the double cosets of the last sum are disjoint for different S, so these elements are linearly independent.

The argument for generation of M^μ involves expressing an element of M^μ in the Murphy basis of $\mathscr{H}$ and observing that all m_{st} with $\mu(\mathrm{s}) = S$ have the same coefficient. □

Example 4.2.9. We already computed the standard basis of M  in Example 4.2.7 using the semistandard tableaux of Example 3.3.1. Here are the permutation modules for compositions $\mu = (\mu_1, \mu_2)$ of $d = 3$:

$$M \;=\; \mathrm{span}_R\left\{ m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} \right\}$$

$$M \;=\; \mathrm{span}_R\left\{ m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}, m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}, m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} \right\}$$

$$M \;=\; \mathrm{span}_R\left\{ m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}, m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}, m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\;\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} \right\}$$

$$M \;=\; \mathrm{span}_R\left\{ m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} \right\}$$

From Proposition 4.2.8 we get a similar basis for the dual permutation modules, consisting of elements $m_{\mathrm{t}S}$. Note that if $\mathscr{H}$ is viewed as a permutation module, the basis in Proposition 4.2.8 is the Murphy basis from Proposition 4.2.5.

Remark 4.2.10. The naming convention of the basis of permutation modules in Proposition 4.2.8 is problematic. The Hecke algebra is a permutation module itself and already has a basis named standard basis from Proposition 4.1.2. Therefore, the basis of the Hecke algebra from Proposition 4.2.8 is called the Murphy basis (cf. Proposition 4.2.5), named after Murphy [Mur95].

The standard basis of permutation modules can now be used to study intersections $M^\mu \cap M^{\eta *}$. By summing up standard basis elements once more we get a R-basis. To make this precise let $\lambda \vdash d$, $S \in \mathcal{T}_0(\lambda, \mu)$ and $T \in \mathcal{T}_0(\lambda, \eta)$. Then define

$$m_{ST} := \sum_{\substack{\mathrm{s},\mathrm{t}\in\mathrm{Std}(\lambda) \\ \mu(\mathrm{s})=S,\ \eta(\mathrm{t})=T}} m_{\mathrm{st}} = \sum_{\substack{\mathrm{t}\in\mathrm{Std}(\lambda) \\ \eta(\mathrm{t})=T}} m_{S\mathrm{t}} = \sum_{\substack{\mathrm{s}\in\mathrm{Std}(\lambda) \\ \mu(\mathrm{s})=S}} m_{\mathrm{s}T}\,. \tag{4.11}$$

The following proposition is from [Mat06, Corollary 4.11].

Proposition 4.2.11. *For $\mu, \eta \models d$ the intersection $M^\mu \cap M^{\eta *}$ has a basis*

$$\{m_{ST} \mid \lambda \vdash d,\ S \in \mathcal{T}_0(\lambda, \mu) \text{ and } T \in \mathcal{T}_0(\lambda, \eta)\} \ .$$

Proof. These elements are, by construction, contained in the intersection. They are linearly independent as they are sums over disjoint sets of standard basis elements.

To see that they generate the intersection we use the Murphy basis and compare coefficients. Let $h \in M^\mu \cap M^{\eta *} \subset \mathscr{H}$ and express it in the Murphy basis as

$$h = \sum_{\lambda \vdash d} \sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)} r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}} \ . \tag{4.12}$$

Then we have for all $\mathfrak{s}, \mathfrak{s}', \mathfrak{t}, \mathfrak{t}' \in \mathrm{Std}(\lambda)$ that

$$\begin{array}{l} \mu(\mathfrak{s}) = \mu(\mathfrak{s}') \overset{4.2.8}{\Rightarrow} r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}} \\ \eta(\mathfrak{t}) = \eta(\mathfrak{t}') \overset{4.2.8}{\Rightarrow} r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}\mathfrak{t}'} \end{array} ,$$

so

$$\left.\begin{array}{l} \mu(\mathfrak{s}) = \mu(\mathfrak{s}') \\ \eta(\mathfrak{t}) = \eta(\mathfrak{t}') \end{array}\right\} \Rightarrow r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}\mathfrak{t}'} = r_{\mathfrak{s}'\mathfrak{t}'} \ .$$

Therefore, every summand of some m_{ST} has the same coefficient in (4.12). Thus, h is generated by the set $\{m_{ST}\}$. □

Example 4.2.12. We continue with the setup of Example 4.2.7 for the semistandard tableaux $\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$.

There is only one standard tableau of shape $\square\square\square$, so we get

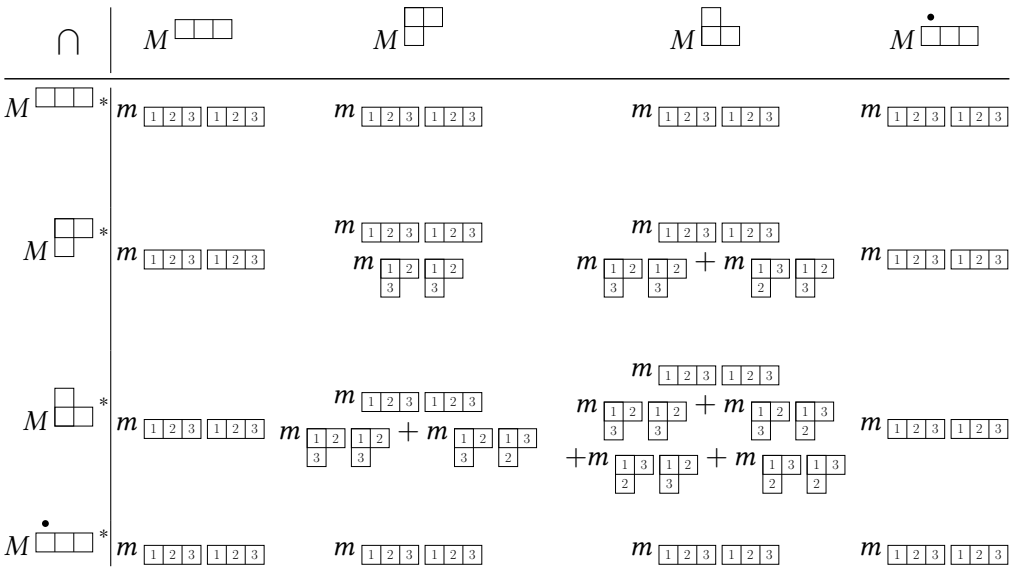
$$m_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}} = m_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} \in M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}} \cap M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}\,*} \ .$$

For $S = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$ we computed two elements $m_{S\mathfrak{t}} \in M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}}$, one for each $\mathfrak{t} \in \mathrm{Std}(\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array})$. To get m_{SS} we have to take the sum of these two elements:

$$\begin{aligned} m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}} &= m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} \\ &= m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} \in M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}} \cap M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}\,*} \end{aligned}$$

The two elements $m_{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}}$ and $m_{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}}$ form the basis of $M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}} \cap M^{\begin{array}{|c|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}\,*}$ described in Proposition 4.2.11.

Bases for all intersections between permutation modules and dual permutation modules of Example 4.2.9 are summarized in the following table:



Remark 4.2.13. It is worth highlighting that all constructions in this chapter thus far did not really depend R or v. This includes the standard basis of $\mathscr{H}$, permutation modules, row-standard bases, the Murphy basis and even the basis of the intersections in Proposition 4.2.11. These constructions are mainly based on the combinatorics of the symmetric group, partitions and different tableaux.

Of course, multiplication in $\mathscr{H}$ and the action on modules will depend on R and v. Thus, if these constructions should be part of a cell datum, we will eventually have some dependencies on R and v.

This ends the discussion of permutation modules. In Section 4.3 we will continue to deal with them in the context of a cellular basis of the Hecke algebra, called the Murphy basis. In Chapter 5 we will also use them in the construction of another cellular algebra, called the Schur algebra.

4.3 Murphy basis

We can now state a cellular basis for $\mathscr{H}$. In fact, we have already seen most of it in Proposition 4.2.5, but let's describe it in full detail. We refer to [Mat06, Theorem 3.20] for this result, but much of it was already proven in [Mur95].

Proposition 4.3.1. *The Hecke algebra $\mathscr{H} = \mathscr{H}(\mathfrak{S}_d)$ is a cellular algebra. A cell datum for $\mathscr{H}$ is given by the tuple $((\Lambda^+(d), \triangleright), Std, \mathcal{M}, *)$, where*

$(\Lambda^+(d), \triangleright)$ *is the set of all partitions of d with dominance ordering,*

$Std : \Lambda^+(d) \to \{\text{finite sets}\}$ *assigns to $\lambda \in \Lambda^+(d)$ the set of standard λ-tableaux $Std(\lambda)$,*

$\mathcal{M} : \coprod_{\lambda \in \Lambda^+(d)} Std(\lambda) \times Std(\lambda) \hookrightarrow \mathscr{H}(\mathfrak{S}_d)$, $(\mathfrak{s}, \mathfrak{t}) \mapsto m_{\mathfrak{s}\mathfrak{t}}$ *is the map with image*

$$im(\mathcal{M}) = \{m_{\mathfrak{s}\mathfrak{t}} \mid \lambda \vdash d \text{ and } \mathfrak{s}, \mathfrak{t} \in Std(\lambda)\},$$

the R-basis of $\mathscr{H}$ from Proposition 4.2.5, and

$* : \mathscr{H} \to \mathscr{H}$ *is the algebra anti-isomorphism $H_w \mapsto H_{w^{-1}}$.*

Much of what is used to prove this proposition has been discussed in Section 4.2, but we have not yet touched on the cellular property (C1) of this basis. By Proposition 4.1.2 the cellular property (C1) can be checked by verifying it for the right action of all $H_w \in \mathscr{H}$ on Murphy basis elements. The full proof involves careful induction on the length of $w \in \mathfrak{S}_d$, on partitions $\lambda \in \Lambda^+(d)$ using the dominance ordering on tableaux (cf. [Mat06, Theorem 3.20] or [Mur95, Theorem 4.18]). Instead of repeating these arguments we will verify the cellular property in an example.

Example 4.3.2. We already constructed the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$ in Example 4.2.6. The complete $\mathscr{H}$-action on Murphy basis elements can be found in Section 6.1. Here we will only consider an excerpt of two basis elements to see the cellular property in an example.

$m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	right $\mathscr{H}$-action
$(v^{-2}+1)m_{123\,123}$	$v^{-2}(v^{-2}+1)m_{123\,123}$	$\cdot m_{123\,123}$
$(v^{-2}+1)m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$(v^{-2}+1)m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$\cdot m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$
$(v^{-2}+1)m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$(v^{-2}+1)m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$\cdot m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$
$m_{123\,123} - m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$v^{-2}m_{123\,123} - m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$\cdot m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$
$v^{-2}m_{123\,123} - m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$v^{-4}m_{123\,123} - m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$\cdot m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}}$
$m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$\cdot m_{\begin{smallmatrix}1&1\\2&2\\3&3\end{smallmatrix}}$

The basis elements are chosen such that they have a common second tableau but differ in the first tableau. We have to check three points to verify the cellular property for these elements.

Firstly, tableaux of shape $\ydiagram{2,1}$ appearing as a first tableau have to have the same filling as the first tableaux of the basis element that is acted on. This is true in both columns of the table.

Secondly, partitions are ordered as $\ydiagram{3} \rhd \ydiagram{2,1} \rhd \ydiagram{1,1,1}$, so the element $m_{\begin{smallmatrix}1&1\\2&2\\3&3\end{smallmatrix}}$ should not appear in any of the two columns. Indeed, it does not.

Thirdly, basis elements associated to $\ydiagram{2,1}$ with the same second tableau should have the same coefficient for both columns in each row of the table. This is true as well, for example in the second row of the table the coefficient $(v^{-2}+1)$ appears in both columns for the correct basis elements.

Hence, we have verified the cellular property for these elements in this example.

Remark 4.3.3. As noted in Remark 4.2.13 above, the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$ is independent of R and v, in the sense that the construction of the basis elements, their labeling, the poset and the anti-isomorphism do not depend on them. Therefore, the number of right cell modules is also independent of R and v.

So if the irreducible representations of $\mathscr{H}(\mathfrak{S}_d)$ depend on R and v, which is what we would expect, then R and v have to affect the bilinear form. More precisely, whether the bilinear form is degenerate or not has to depend on R and v. As a result, we expect the radicals and Λ_0 to depend on these choices.

Having established a cellular basis of $\mathscr{H}$ we will now apply some constructions and results developed in Chapter 2 to Hecke algebras for this basis. For simplicity we will assume that R is a field throughout the rest of this section, although it is only necessary for classifying the irreducibles.

We start with the construction of cell modules of $\mathscr{H}$, which are called **Specht modules** because they generalize Specht modules from the representation theory of symmetric groups (cf. [DJ86, Chapter 4]). Hence, we denote the cell module for a partition $\lambda \vdash d$ by S^λ instead of C^λ.

By Definition 2.1.7, S^λ is the right $\mathscr{H}$-module

$$S^\lambda := \operatorname{span}_R \left\{ m_{\mathrm{t}}^\lambda \mid \mathrm{t} \in \operatorname{Std}(\lambda) \right\} \tag{4.13}$$

with right $\mathscr{H}$-action determined by the cellular property (C1). Note that $\operatorname{Shape}(\mathrm{t}) = \lambda$, so the superscript λ in the generators of S^λ is redundant and will be omitted from now on.

Example 4.3.4. For $\mathscr{H}(\mathfrak{S}_3)$ the Specht modules are:

$$S^{\ydiagram{3}} = \operatorname{span}_R \left\{ m_{\begin{smallmatrix}1&2&3\end{smallmatrix}} \right\}, \; S^{\ydiagram{2,1}} = \operatorname{span}_R \left\{ m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}, \; m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \right\}, \; S^{\ydiagram{1,1,1}} = \operatorname{span}_R \left\{ m_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \right\}$$

The right $\mathscr{H}$-action on the generators can be inferred from Figures 6.2 and 6.3 in Section 6.1. They are summarized in Figure 6.4 in Chapter 6.1. A more detailed description of these computations can be found in Section 6.1 as well.

Recall that, by Proposition 2.3.12, the set $\{D^\lambda \mid \lambda \vdash d,\ D^\lambda \neq \{0\}\}$ is a complete set of non-isomorphic irreducible $\mathscr{H}$-modules, where $D^\lambda := S^\lambda / \operatorname{rad} S^\lambda$. In principle, it is now possible to compute the bilinear form introduced in Proposition 2.3.2 on Specht modules S^λ, the corresponding radicals $\operatorname{rad} S^\lambda$ and quotients D^λ to determine the non-trivial quotients and their dimensions for fields of any characteristic. We will do so for the case $d = 3$ in Section 6.1, an excerpt of this is also stated in Example 4.3.5 below.

Although these calculations are tedious and prone to error we can make an important observation: The radical depends on the characteristic of R and the choice of unit v. This dependency does not come from the Specht modules or the construction of the bilinear form on them. It is the degeneracy of the bilinear form that is dependent on R and v. We will see how the degeneracy is affected by them in the example below.

Example 4.3.5. The bilinear form on Specht modules of $\mathscr{H}(\mathfrak{S}_3)$ can be inferred from the multiplication table in Figure 6.3 in Section 6.1, similar to the action on Specht modules in Example 4.3.4.

$$
\begin{aligned}
S^{\ydiagram{3}}: &\quad \langle m_{\ytableaushort{123}}, m_{\ytableaushort{123}} \rangle = v^{-6} + 2v^{-4} + 2v^{-2} + 1 \\
S^{\ydiagram{2,1}}: &\quad \langle m_{\ytableaushort{12,3}}, m_{\ytableaushort{12,3}} \rangle = v^{-2} + 1 \qquad \langle m_{\ytableaushort{13,2}}, m_{\ytableaushort{12,3}} \rangle = -1 \\
&\quad \langle m_{\ytableaushort{12,3}}, m_{\ytableaushort{13,2}} \rangle = -1 \qquad \langle m_{\ytableaushort{13,2}}, m_{\ytableaushort{13,2}} \rangle = v^{-4} + 1 \\
S^{\ydiagram{1,1,1}}: &\quad \langle m_{\ytableaushort{1,2,3}}, m_{\ytableaushort{1,2,3}} \rangle = 1
\end{aligned}
$$

Note that the bilinear form on $S^{\ydiagram{2,1}}$ and $S^{\ydiagram{1,1,1}}$ is never trivial, so their radical is never the whole module. The bilinear form on $S^{\ydiagram{4}}$ is trivial, if and only if $v^{-6} + 2v^{-4} + 2v^{-2} + 1 = 0$. So either $v^{-2} = 1$ and $\operatorname{char} R = 2, 3$ or $v^{-2} \neq 1$ and v^{-2} is a 2nd or $3rd$ root of unity.

We calculate the radicals and irreducible quotients explicitly in Proposition 6.1.1 of Section 6.1.

Instead of explicitly calculating each quotient D^μ it is possible to determine the set $\{\lambda \vdash d \mid D^\lambda \neq \{0\}\}$ by a simpler condition on the partitions. However, this condition will not give us the dimension or even the explicit form of the irreducible modules.

Definition 4.3.6. For a field R with unit v we define

$$e := \inf \left\{ m > 0 \mid 1 + v^{-2} + v^{-4} \cdots + v^{-2(m-1)} = 0 \right\} .$$

Then a partition $\lambda \vdash d$ is called e-**restricted**, if

$$\lambda_i - \lambda_{i+1} < e \text{ for all } i \geq 1 . \tag{4.14}$$

Note that the definition of e is just a compact way of writing

$$e = \begin{cases} \operatorname{char} R & , \text{ if } v^{-2} = 1 \\ m & , \text{ if } v^{-2} \neq 1 \text{ and } v \text{ is } m\text{-th root of unity} \\ \infty & , \text{ otherwise} \end{cases} . \tag{4.15}$$

With Definition 4.3.6 we can state a classification of irreducible right $\mathscr{H}$-modules based on Proposition 2.3.12, the general classification result for cellular algebras.

Proposition 4.3.7. *Let R be a field, $v \in R$ a unit and $d \in \mathbb{N}$. Then:*

$$\{\lambda \vdash d \mid \lambda \text{ is } e\text{-restricted}\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Irreducible right } \mathscr{H}_{R,v}(\mathfrak{S}_d)\text{-modules} \\ \text{up to isomorphism} \end{array} \right\}$$
$$\lambda \longmapsto D^{\lambda}$$

We will outline the proof of this proposition that was presented in [Mat06, Theorem 3.43 (i)] in Section 4.4. Moreover, we verify Proposition 4.3.7 for $\mathscr{H}(\mathfrak{S}_3)$ in Proposition 6.1.1 of Section 6.1 and in Example 4.3.8.

Example 4.3.8. We use Proposition 4.3.7 to classify the irreducible representations of $\mathscr{H}(\mathfrak{S}_3)$. To check that a partition $\lambda \vdash 3$ is e-restricted for a fixed e we need to compute all $\lambda_i - \lambda_{i+1}$ by (4.14):

$$\lambda = \ydiagram{3} \Rightarrow \lambda_1 - \lambda_2 = 3$$
$$\lambda = \ydiagram{2,1} \Rightarrow \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$$
$$\lambda = \ydiagram{1,1,1} \Rightarrow \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 0,\ \lambda_3 - \lambda_4 = 1$$

Because $e \geq 2$, the only partition not e-restricted is $\ydiagram{3}$ for $e = 2$ or $e = 3$. So by Proposition 4.3.7:

$$\left\{ \begin{array}{c} \text{Irreducible right } \mathscr{H}(\mathfrak{S}_3)\text{-modules} \\ \text{up to isomorphism} \end{array} \right\} = \begin{cases} D^{\ydiagram{2,1}},\ D^{\ydiagram{1,1,1}} & , \text{ if } e = 2 \text{ or } 3 \\ D^{\ydiagram{3}},\ D^{\ydiagram{2,1}},\ D^{\ydiagram{1,1,1}} & , \text{ otherwise} \end{cases}$$

By reformulation (4.15) of e-restriction we also have

$$D^{\ydiagram{3}} = \{0\} \Leftrightarrow \begin{array}{ll} & v^{-2} = 1 \ \wedge\ (\operatorname{char} R = 2 \text{ or } 3) \\ \text{or} & v^{-2} \neq 1 \ \wedge\ (v^{-2} \text{ is 2nd or 3rd root of unity}) \end{array}$$

This result agrees with our explicit analysis of the bilinear form in Example 4.3.5, verifying Proposition 4.3.7 for this example.

Note that Proposition 4.3.7 does not reveal the module structure of the irreducible representations. In particular we do not know which irreducible representation is the trivial and which is the sign representation of $\mathscr{H}$. For this we need to calculate the radicals explicitly, which we will do in Section 6.1.

Proposition 4.3.7 also makes no statements about the dimension of the irreducible modules of $\mathscr{H}$. When we outline its proof in Section 4.4 we will at least derive a lower bound on the dimension.

However, it is important to note that, for a specific example, we could get the exact dimension of all irreducibles by calculating the radicals. Providing a distinct path to calculate these dimensions is a valuable feature of the cellular structure of $\mathscr{H}$, or rather of cellular bases of all cellular algebras.

Remark 4.3.9. For the first time in the discussion of the Murphy basis, the result of Proposition 4.3.7 does depend on the choice of R and v. And, as we have seen in Example 4.3.8, the choice really matters. For R and v such that $e = 2, 3$ there is one less non-zero quotient of a right cell module than for choices such that $e \neq 2, 3$, so one less irreducible right $\mathscr{H}(\mathfrak{S}_3)$-module. Moreover, in all of these cases we can get the irreducible modules explicitly by calculating the radicals.

4.4 Classification of irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules

In this section we want to outline the proof of Proposition 4.3.7, in which irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules are classified by e-restricted partitions of d. Recall that the irreducibles are exactly the non-zero quotients of Specht modules by their radicals.

Careful computations with the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$ show, that $D^\lambda = \{0\}$ if $\lambda \vdash d$ is not e-restricted. Indeed, if $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ-tableaux, then the author of [Mat06] explicitly computes factors of $\langle m_\mathfrak{s}, m_\mathfrak{t} \rangle$ in [Mat06, Lemma 3.42] and shows that one of these factors is 0, if λ is not e-restricted. Because $\mathfrak{s}$ and $\mathfrak{t}$ are arbitrary, it follows that D^λ is trivial.

The other part of this proof, showing that $D^\lambda \neq \{0\}$ if λ is e-restricted, involves Jucys–Murphy elements of $\mathscr{H}(\mathfrak{S}_d)$, a generalization of Jucys–Murphy elements of $\mathbb{C}[\mathfrak{S}_d]$. One key result of this part of the proof is a lower bound for the dimension of quotients D^λ of right Specht modules (see Corollary 4.4.16 or [Mat06, Corollary 3.38]).

The Jucys–Murphy elements of $\mathscr{H}(\mathfrak{S}_d)$ are interesting beyond these applications. In particular, they will reappear in Section 4.5 when we discuss Young's orthogonal form, an orthogonal basis of Specht modules.

Here we focus on the second part of the proof and give examples for all constructions and the intuitions behind them. All results are due to the author of [Mur95], but we also refer to [Mat06, Section 3.3 and 3.4] for rigorous proofs of these results.

4.4.1 Jucys–Murphy elements and residue sequences

Jucys–Murphy elements originate in the representation theory of the symmetric group over $\mathbb{C}$. Classically, these are defined as elements

$$L_k := \sum_{i=1}^{k-1} (i,k) \in \mathbb{C}[\mathfrak{S}_d]$$

for $k = 2, 3, \ldots, d$ and $L_1 = 0$, where $(i,\ k) \in \mathfrak{S}_d$ is the element transposing i and k.

Jucys proved in [Juc74] that every element in the center of $\mathbb{C}[\mathfrak{S}_d]$ can be expressed as a symmetric polynomial in Jucys–Murphy elements. Murphy used these elements in [Mur81] to construct a special basis for each Specht module of $\mathbb{C}[\mathfrak{S}_d]$. The matrix representations of simple transpositions in this basis are sparse. Symmetric groups are naturally nested in each other and this special basis are also important in the study of restricted representations in this nesting structure. We will come back to this point in Section 4.5.

A good reference for Jucys–Murphy elements and the mentioned orthogonal basis in the classical case is [CST10, Chapter 3]. Their work is based on [VO05], but their explanations are a bit more detailed.

There are several generalizations of Jucys–Murphy elements to the Hecke algebra of the symmetric group. We follow the conventions used in [Mat06]. However, depending on the application, other conventions might be better suited (see [Mat06, Notes on Chapter 3]).

The other tool we introduce in this section is the residue sequence of a standard tableau, which is a specific tuple of integer we associate to a tableau. In the classical case of $\mathbb{C}[\mathfrak{S}_d]$ it is possible to recover the tableau from its residue sequence. We will see that entries of this sequence naturally appear in the action of Jucys–Murphy elements on basis elements of Specht modules.

One approach to generalize the classical Jucys–Murphy elements is to generalize the transpositions appearing. We denote by

$$H_{(l,k)} \in \mathscr{H}(\mathfrak{S}_d)$$

the standard basis element associated to the transposition

$$(l,k) = s_{k-1}s_{k-2}\ldots s_{l+1}s_l s_{l+1}\ldots s_{k-2}s_{k-1} \in \mathfrak{S}_d\ .$$

Definition 4.4.1. The **Jucys–Murphy elements** of $\mathscr{H}(\mathfrak{S}_d)$ are defined as

$$L_1 = 0$$

and

$$L_k := v\sum_{i=1}^{k-1} H_{(i,k)} = vH_{(k-1,k)} + vH_{(k-2,k)} + \cdots + vH_{(1,k)}$$

for $k = 2, 3, \ldots, d$.

Example 4.4.2. If we consider $R = \mathbb{C}$, specialize v to 1 and identify $H_{(i,k)}$ with $(i,k) \in \mathbb{C}[\mathfrak{S}_d]$ we get the classical Jucys–Murphy elements for $\mathbb{C}[\mathfrak{S}_d]$.

Example 4.4.3. The Jucys–Murphy elements of $\mathscr{H}(\mathfrak{S}_3)$ are:

$$\begin{aligned} L_1 &= 0 \\ L_2 = vH_{(1,2)} &= vH_s \\ L_3 = vH_{(2,3)} + vH_{(1,3)} &= vH_t + vH_{sts} \end{aligned}$$

The goal of Section 4.4 is to demonstrate how Jucys–Murphy elements can be used to prove the classification of irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules by e-restricted partitions from Proposition 4.3.7. Therefore, we focus on two central properties of these elements that we need for our goal.

Just like in Proposition 4.3.7 we will now assume that R is a field.

The first property is connected to the subalgebra generated by all Jucys–Murphy elements.

Proposition 4.4.4. *The Jucys–Murphy elements of $\mathscr{H}(\mathfrak{S}_d)$ commute and thus generate an abelian subalgebra*

$$\mathscr{L} := \langle L_1, L_2, \dots, L_d \rangle \subset \mathscr{H}(\mathfrak{S}_d) \ .$$

See [Mat06, Proposition 3.26 (iii)] for a proof of this property.

Example 4.4.5. For $\mathscr{H}(\mathfrak{S}_3)$ this property is easily verified, as we only need to check:

$$L_2L_3 = v(1-v^2)H_{sts} + v^2H_{st} + v^2H_{ts} = L_3L_2$$

Compared to $\mathscr{H}(\mathfrak{S}_d)$, the representation theory of $\mathscr{L}$ turns out to be quite simple. In Section 4.4.2 the Specht modules of $\mathscr{H}(\mathfrak{S}_d)$ are considered as right $\mathscr{L}$-modules. Their composition series contain enough information to deduce results about the representation theory of $\mathscr{H}(\mathfrak{S}_d)$.

The second property needed to achieve our goal is, that the right action of Jucys–Murphy elements on Specht modules is triangular, if we consider the partial order on standard tableaux of a given shape, which was introduced in Section 3.2.

Moreover, the diagonal part of this action is entirely determined by so called residue sequences.

Definition 4.4.6. Let λ be a partition of d and $\mathfrak{t}$ a standard λ-tableau. Let $k = 1, 2, \dots, d$ and (i_k, j_k) the position of the box in $\mathfrak{t}$, which is labeled by k. Then the e-**residue** of k in $\mathfrak{t}$ is defined as

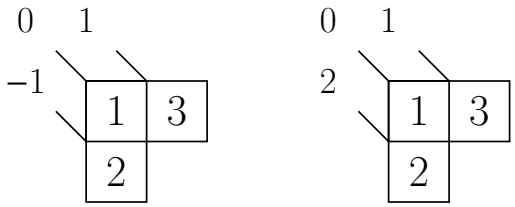
$$\mathrm{res}_{\mathfrak{t}}(k) := (j_k - i_k) \bmod e \ .$$

The **residue sequence** of $\mathfrak{t}$ is defined as the tuple of integers

$$(\mathrm{res}_{\mathfrak{t}}(1), \mathrm{res}_{\mathfrak{t}}(2), \dots, \mathrm{res}_{\mathfrak{t}}(d)) \ .$$

An equivalent definition of the e-residue is based on top-left to bottom-right diagonals of t. Label these diagonals increasingly with integers, such that box $\boxed{1}$ is on diagonal 0. Then relabel the diagonals by taking these labels mod e. The e-residue of k in t is then just the label of the diagonal on which the box labeled with k resides.

Example 4.4.7. Suppose $v = 1$, char$R = 3$, so $e = 3$. Let t be the $(2,1)$-standard tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Here is how the diagonals are labeled:



diagonals labeled by increasing integers

previous diagonals taken mod e

The residue sequence of t is $(0,2,1)$.

Remark 4.4.8. In the setup of Definition 4.4.6 we could also consider the tuple

$$(j_1 - i_1, j_2 - i_2, \ldots, j_d - i_d)\ . \tag{4.16}$$

If $e = \infty$ then 4.16 is just the residue sequence. It is possible to recover the original tableau from this sequence, so different standard tableaux can not share such sequence.

Indeed, if $(l_1, l_2, \ldots l_k)$ is sequence (4.16) of some standard tableau, then we can rebuild it by sequentially adding boxes with labels k from 1 to d to an empty tableau by placing them at diagonal l_k. Because we know the original tableau has strictly increasing columns and rows, there is only one place on diagonal l_k, where a new box can be added. It is the first spot on diagonal l_k, where no box has been placed yet.

Note that the same does not hold true for the residue sequence. Suppose R and v are chosen such that $e = 2$. Then sequence (4.16) of $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ is $(0,-1,1)$ and $(0,1,-1)$ respectively, so they are distinct. However, both have residue sequence $(0,1,1)$.

Residue sequences of standard tableaux also appear in the representation theory of $\mathbb{C}[\mathfrak{S}_d]$, for example in [Mur81]. To adapt residue sequences to Hecke algebras, they are transformed from integers to expressions in R involving v, so called quantum integers.

Definition 4.4.9. The **quantum integers** are defined as

$$[0]_v := 0$$
$$[m]_v := v^{m-1} + v^{m-3} + \cdots + v^{-(m-3)} + v^{-(m-1)}$$

for all $m \in \mathbb{Z}_{>0}$ and

$$[-m]_v := -[m]_v$$

for $m \in \mathbb{Z}_{\geq 0}$.

Remark 4.4.10. This definition is equivalent to

$$[m]_v := \frac{v^m - v^{-m}}{v - v^{-1}}$$

for all $m \in \mathbb{Z}$ if $v \neq v^{-1}$, so if $v \neq 1, -1$.

If v is specialized to 1, the quantum integers are the usual integers.
Note that for $m > 0$ we have

$$v^{-(m-1)}[m]_v = 1 + v^{-2} + v^{-4} + \cdots + v^{-2(m-1)}$$

and

$$v^{-(m-1)}[m]_v = 0 \Leftrightarrow [m]_v = 0 \ .$$

So the definition of e for e-restricted partitions in Definition 4.3.6 can be reformulated using quantum integers:

$$e := \inf \{m > 0 \mid [m]_v = 0\} = \inf \left\{m > 0 \mid v^{-(m-1)}[m]_v = 0\right\}$$

In the next proposition we will see where the residue sequence appears in the action of Jucys–Murphy elements on Specht modules. For a proof of this result see [Mat06, Theorem 3.32].

Proposition 4.4.11. *Let $\lambda \vdash d$, $\mathfrak{t}$ a standard λ-tableau and $1 \leq k \leq d$. Then there exist coefficients $a_\mathfrak{v} \in R$ for all standard λ-tableaux $\mathfrak{v} \rhd \mathfrak{t}$ such that*

$$m_\mathfrak{t} L_k = v^{-(res_\mathfrak{t}(k)-1)}[res_\mathfrak{t}(k)]_v m_\mathfrak{t} + \sum_{\substack{\mathfrak{v} \in Std(\lambda) \\ \mathfrak{v} \rhd \mathfrak{t}}} a_\mathfrak{v} m_\mathfrak{v} \ .$$

In other words, the action of Jucys–Murphy elements on a basis element $m_\mathfrak{t}$ of a Specht module S^λ follows the dominance order on $\mathrm{Std}(\lambda)$. The coefficient of $m_\mathfrak{t}$ is entirely determined by the residue sequence of $\mathfrak{t}$.

These coefficients are gathered in the **quantum residue sequence** $\rho^\mathfrak{t}$ of $\mathfrak{t}$. Set

$$\rho^\mathfrak{t} := (v^{-(\mathrm{res}_\mathfrak{t}(1)-1)}[\mathrm{res}_\mathfrak{t}(1)]_v,\ v^{-(\mathrm{res}_\mathfrak{t}(2)-1)}[\mathrm{res}_\mathfrak{t}(2)]_v,\ \ldots,\ v^{-(\mathrm{res}_\mathfrak{t}(d)-1)}[\mathrm{res}_\mathfrak{t}(d)]_v\ ) \tag{4.17}$$

and

$$\rho^\mathfrak{t}_k := v^{-(\mathrm{res}_\mathfrak{t}(k)-1)}[\mathrm{res}_\mathfrak{t}(k)]_v$$

for $1 \leq k \leq d$.

Example 4.4.12. Assume R and v are chosen such that $e > 3$. Under these assumptions the quantum residue sequences of all standard tableaux of size $d = 3$ are distinct:

$$\begin{array}{rcll}
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} & : & (v[0]_v, [1]_v, v^{-1}[2]_v) & = (0, 1, v^{-1}(v + v^{-1})) \\
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} & : & (v[0]_v, [1]_v, v^{2}[-1]_v) & = (0, 1, -v^{2}) \\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & : & (v[0]_v, v^{2}[-1]_v, [1]_v) & = (0, -v^{2}, 1) \\
\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array} & : & (v[0]_v, v^{2}[-1]_v, v^{3}[-2]_v) & = (0, -v^{2}, -v^{3}(v + v^{-1}))
\end{array}$$

We summarize the right action of Jucys–Murphy elements on basis elements of Specht modules for $\mathscr{H}(\mathfrak{S}_3)$ in the following table:

$m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$	$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$	$m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}}$	$m_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array}}$	
0	0	0	0	$\cdot L_1$
$m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$	$m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$	$-v^2 m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} - v^2 m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$	$-v^2 m_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array}}$	$\cdot L_2$
$v^{-1}(v + v^{-1}) m_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$	$-v^2 m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$	$m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + v^2 m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$	$-v^3(v + v^{-1}) m_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array}}$	$\cdot L_3$

Entries of the quantum residue sequences appear as coefficients, as predicted by Proposition 4.4.11. Furthermore, the Jucys–Murphy elements act diagonally on $m_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$, but not on $m_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}}$. This agrees with Proposition 4.4.11, because $(2, 1)$-standard tableaux are partially ordered by $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \triangleright \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$.

Note that Proposition 4.4.11 is not directly related to the cellular property of $\mathscr{H}(\mathfrak{S}_d)$, although it looks similar. Here we have a partial order on tableaux of a given shape, which label basis elements of a Specht module. For $\mathscr{H}(\mathfrak{S}_d)$ the partial order is on partitions of d and basis elements are labeled by ordered pairs of tableaux with the same shape.

4.4.2 Specht modules as right $\mathscr{L}$-modules

We are now able to describe how Murphy [Mur95] used Jucys–Murphy elements and their properties to proof Proposition 4.3.7, the classification of irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules.

Proposition 4.4.11 yields certain composition series of Specht modules as right $\mathscr{L}$-modules. Order the set $\mathrm{Std}(\lambda) = \{\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_k\}$ for $\lambda \vdash d$ following the partial order, meaning $i > j$ if $\mathrm{t}_i \rhd \mathrm{t}_j$ for $i, j = 1, 2, \ldots, k$. Then the R-submodule

$$S_i := \mathrm{span}_R \{m_{\mathrm{t}_j} \mid i \leq j \leq k\} \subset S^\lambda$$

is a well defined right $\mathscr{L}$-submodule for $i = 1, 2, \ldots, k+1$ by Proposition 4.4.11. The filtration

$$S^\lambda = S_1 > S_2 > \cdots > S_k > S_{k+1} = \{0\} \tag{4.18}$$

is a right $\mathscr{L}$-module composition series with one-dimensional composition factors S_i/S_{i+1}. Again by Proposition 4.4.11, a Jucys–Murphy element L_l acts on S_i/S_{i+1} by $\rho_l^{\mathrm{t}_i} = v^{-(\mathrm{res}_{\mathrm{t}_i}(l)-1)}[\mathrm{res}_{\mathrm{t}_i}(l)]_v$, an element of the quantum residue sequence ρ^{t_i}. Thus, this irreducible right $\mathscr{L}$-module is denoted by $\mathscr{L}_{\rho^{\mathrm{t}_i}}$.

Additionally, every irreducible right $\mathscr{L}$-module is of this form, because $\mathscr{L}$ is a subalgebra of $\mathscr{H}(\mathfrak{S}_d)$ and every irreducible right $\mathscr{H}(\mathfrak{S}_d)$-module is a composition factor of some Specht module. Therefore, irreducible right $\mathscr{L}$-modules are characterized by quantum residue sequences of standard tableaux of size d. However, by Remark 4.4.8 they are in general not in one-to-one correspondence with standard tableaux of size d themself, because different standard tableaux can yield the same quantum residue sequence.

Example 4.4.13. Let's continue with the setup of Example 4.4.12, so $e > 3$ and $d = 3$. $S^{\ytableaushort{123}}$ and $S^{\ytableaushort{1,2,3}}$ are one-dimensional and we see from Example 4.4.12 that $S^{\ytableaushort{123}} = \mathscr{L}_{(0,1,v^{-1}(v+v^{-1}))}$ and $S^{\ytableaushort{1,2,3}} = \mathscr{L}_{(0,-v^2,-v^3(v+v^{-1}))}$. For $S^{\ytableaushort{12,3}}$ recall that $\ytableaushort{12,3} \rhd \ytableaushort{13,2}$, so filtration (4.18) for $\lambda = \ytableaushort{12,3}$ is

$$S^{\ytableaushort{12,3}} = \mathrm{span}_R\{m_{\ytableaushort{12,3}}, m_{\ytableaushort{13,2}}\} > \mathrm{span}_R\{m_{\ytableaushort{12,3}}\} > \{0\}\ .$$

Its right $\mathscr{L}$-module composition factors are

$$S_1/S_2 = \mathscr{L}_{(0,-v^2,1)} \quad \text{and} \quad S_2/\{0\} = \mathscr{L}_{(0,1,-v^2)}\ .$$

Recall that by Proposition 2.3.12, the general classification result for cellular algebras, irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules are exactly the non-zero quotients of Specht modules by their radicals. Let's see how $\mathscr{L}$ is used in [Mur95] to identify non-zero quotients.

Let S^λ be the Specht module associated to $\lambda \vdash d$ and D^λ the quotient by its radical. By Remark 2.3.18 and the proof of Proposition 2.3.12 we know that a right $\mathscr{H}(\mathfrak{S}_d)$-module composition series of S^λ can only have composition factors isomorphic to D^μ for $\lambda \trianglerighteq \mu$.

Refine such composition series to a composition series of S^λ as a right $\mathscr{L}$-module and compare its composition factors to the composition factors appearing in (4.18).

Assume $\mathscr{L}_{\rho^\mathfrak{t}}$ is a composition factor from (4.18). This composition factor thus also appears in the refined composition series, but the question is: When does $\mathscr{L}_{\rho^\mathfrak{t}}$ have to appear as a composition factor of D^λ? In other words, under what condition is $\mathscr{L}_{\rho^\mathfrak{t}}$ never a composition factor of D^μ for $\lambda \rhd \mu$?

We know that every composition factor of D^μ as a right $\mathscr{L}$-module is of the form $\mathscr{L}_{\rho^\mathfrak{s}}$ for a μ-standard tableau $\mathfrak{s}$. The existence of a right $\mathscr{L}$-module isomorphism $\mathscr{L}_{\rho^\mathfrak{t}} \cong \mathscr{L}_{\rho^\mathfrak{s}}$ is equivalent to $\rho^\mathfrak{t} = \rho^\mathfrak{s}$, meaning $\mathfrak{s}$ and $\mathfrak{t}$ have the same quantum residue sequence. This motivates the following definition.

Definition 4.4.14. Let ρ be the quantum residue sequence of some standard tableaux of size d. Equip the set $\{\mathfrak{t} \mid \lambda \vdash d,\ \mathfrak{t} \in \mathrm{Std}(\lambda),\ \rho^\mathfrak{t} = \rho\}$ with a partial order defined by

$$\mathfrak{s} \leq \mathfrak{t} \overset{\text{def}}{\Leftrightarrow} \begin{cases} \mathfrak{s} = \mathfrak{t} \text{ or} \\ \mathfrak{s} \neq \mathfrak{t} \text{ and } \mathrm{Shape}(\mathfrak{s}) \lhd \mathrm{Shape}(\mathfrak{t}) \ . \end{cases}$$

Minimal elements in this poset are called e-**restricted** tableaux. For $\lambda \vdash d$ the set of e-restricted tableaux is denoted by $\mathrm{Std}_e(\lambda)$.

Example 4.4.15. Suppose $e = 3$, then the quantum residue sequences of standard tableaux of size $d = 3$ are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} : (0,1,2) \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} : (0,2,1)$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} : (0,1,2) \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} : (0,2,1)$$

By comparing their shapes we see for $(0,1,2)$ that $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ and for $(0,2,1)$ that $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} < \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ in the partial order defined above. Hence, $\mathrm{Std}_3((2,1)) = \{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\}$ and $\mathrm{Std}_3((1)^3) = \{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\}$. Note that there is no 3-restricted tableau of shape (3).

Continuing the reasoning from above we can see that if $\mathfrak{t}$ is e-restricted, then $\mathscr{L}_{\rho^\mathfrak{t}}$ can not be isomorphic to $\mathscr{L}_{\rho^\mathfrak{s}}$. Otherwise $\rho^\mathfrak{s} = \rho^\mathfrak{t}$, which, in combination with the assumption $\lambda \rhd \mu$, contradicts the minimality of $\mathfrak{t}$ in the poset stated in Definition 4.4.14.

We summarize the reasoning from above in the following corollary.

Corollary 4.4.16. *For all $\lambda \vdash d$ the dimension of D^λ is bounded by*

$$\dim D^\lambda \ \geq\ |\mathrm{Std}_e(\lambda)| \ .$$

The reasoning from above closely follows [Mat06, Section 3.4]. In particular, Corollary 4.4.16 is [Mat06, Corollary 3.38].

Example 4.4.17. If $e > d$, two quantum integers for $0 \leq m \leq d$ agree, if and only if the underlying integers agree. Thus, two quantum residue sequences agree, if and only if the underlying residue sequences agree. By Remark 4.4.8 all residue sequences of standard tableaux are distinct in this setup. Thus, every standard tableau is e-restricted.

Hence, $D^\lambda \neq \{0\}$ by Corollary 4.4.16 for all $\lambda \vdash d$. Furthermore, $D^\lambda = S^\lambda$ for all $\lambda \vdash d$, because the dimension of each Specht module is $|\mathrm{Std}(\lambda)|$.

Definition 4.4.14 already suggests that a partition λ is e-restricted, if and only if there exists an e-restricted λ-tableau. We have stated in the introduction of Section 4.4 above that $D^\lambda = \{0\}$ if λ is not e-restricted. To show the converse it is enough so construct an e-restricted λ-tableau if λ is e-restricted. This is surprisingly difficult, as the construction should hold for any e.

The author of [Mat06] constructs a so called **ladder tableau** and proves that it is indeed e-restricted if λ is e-restricted in [Mat06, Lemma 3.40]. Here is an algorithm that constructs the ladder tableau for λ. First label each box (i, j) in the diagram of λ by its **ladder number** $j - i + e(i-1)$. Then label boxes of another λ-diagram with $1, 2, \ldots, d$, increasingly by their ladder number. If two boxes have the same ladder number, label the one on the lowest row in the diagram first, that is the box with the highest row number.

Example 4.4.18. Suppose R and v are chosen such that $e = 3$. To construct the ladder tableau of shape $(2, 1)$ first label boxes of the $(2, 1)$-diagram by their ladder numbers:

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Then label boxes of a $(2, 1)$-diagram increasingly by $1, 2, 3$ according to their ladder numbers, prioritizing boxes with highest row numbers in ties:

$$\begin{array}{|c|c|}\hline 1 & \\ \hline & \\ \cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline & \\ \cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Note that by Example 4.4.15 this is the single 3-restricted tableau of shape $(2, 1)$.

If two boxes have the same ladder number, then they also have the same e-residue. So the ladder number splits the set of boxes with the same e-residue into groups that indicate, which boxes of a given e-residue should be added first.

This finishes our outline of the proof of Proposition 4.3.7, classifying irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules by e-restricted partitions of d, following [Mat06]. Careful computations show, that if $\lambda \vdash d$ is not e-restricted, then the bilinear form on S^λ is trivial, so $D^\lambda = \{0\}$. Here we focused on the reverse implication. Using the subalgebra of $\mathscr{H}(\mathfrak{S}_d)$ generated by Jucys–Murphy elements, one can bound the dimension of D^λ by the amount of e-restricted λ-tableaux. Such a tableau exists if λ is e-restricted, proving $D^\lambda \neq \{0\}$.

4.5 An orthogonal basis of Specht modules

In Section 4.3 we recalled Jucys–Murphy elements for $\mathscr{H}(\mathfrak{S}_d)$ from [Mur95] and gave an overview of their application to the classification of irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules. Now we want to discuss another application of these elements: constructing an orthogonal basis for each Specht module, which is known as Young's orthogonal form, if $\mathscr{H}$ is semisimple. There are two properties that make this orthogonal basis special. Firstly, the action of standard generators of $\mathscr{H}(\mathfrak{S}_d)$ on these elements is known explicitly and quickly computed. Secondly, matrix representations of these generators are sparse.

Their construction is a generalization of the corresponding result for the classical case of $\mathbb{C}[\mathfrak{S}_d]$. See [Mur81] or [CST10] for more details of the classical case. We will look at the general case of $\mathscr{H}(\mathfrak{S}_d)$ and explicitly compute the bases for Specht module of $\mathscr{H}(\mathfrak{S}_3)$.

Jucys–Murphy elements and the orthogonal basis have been generalized to a broader class of cellular algebras. Such algebras need to possess a set of elements that behave like Jucys–Murphy elements. The right cell modules then have an orthogonal basis with similar properties as described above. One account of this generalization is given in [MS06].

Thus, the construction presented below is rather natural and is connected to the cellular structure of $\mathscr{H}(\mathfrak{S}_d)$. For rigorous proofs of the results we once again refer to [Mat06, Section 3.3].

Afterwards we will see how to use this orthogonal basis to decompose irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules as $\mathscr{H}(\mathfrak{S}_j)$-modules for $j < d$ using ideas from [VO05]. The traditional approach to such decompositions in the case of $\mathbb{C}[\mathfrak{S}_d]$ requires a decent amount of computations and can, for example, be found in [Jam78]. The approach with the orthogonal basis is simpler and can also be applied to the general case of all semisimple $\mathscr{H}(\mathfrak{S}_d)$.

4.5.1 Young's orthogonal form

Assume, just like in Section 4.4, that R is a field. An additional assumption is $e > d$, so either $v^{-2} = 1$ and $\operatorname{char} R > d$ or, if v^{-2} is a m-th root of unity, then $m > d$.

Remark 4.5.1. These assumptions guarantee that every Specht module of $\mathscr{H}(\mathfrak{S}_d)$ is irreducible. Indeed, following the arguments in Example 4.4.17, $e > d$ implies that the residue sequences of all standard tableaux of size d are distinct. Therefore, all standard tableaux are e-restricted and the claim follows from Corollary 4.4.16. Alternatively, all partitions of d are e-restricted if $e > d$.

It is possible to show that the reverse implication is also true: if all Specht modules are irreducible, then $e > d$. Furthermore, it is also true that this assumption is equivalent to semisimplicity of $\mathscr{H}(\mathfrak{S}_d)$. See [Mat06, Corollary 3.44] or [GL96, Theorem 3.8] for details.

The construction of Young's orthogonal form begins with the definition of an intermediate element in $\mathscr{H}(\mathfrak{S}_d)$. Recall the definition of the quantum residue sequence

$$\begin{aligned}\rho^{\mathsf{t}} &:= (\rho_1^{\mathsf{t}}, \rho_2^{\mathsf{t}}, \dots, \rho_d^{\mathsf{t}}) \\ &:= (v^{-(\mathrm{res}_{\mathsf{t}}(1)-1)}[\mathrm{res}_{\mathsf{t}}(1)]_v,\ v^{-(\mathrm{res}_{\mathsf{t}}(2)-1)}[\mathrm{res}_{\mathsf{t}}(2)]_v,\ \dots,\ v^{-(\mathrm{res}_{\mathsf{t}}(d)-1)}[\mathrm{res}_{\mathsf{t}}(d)]_v\ )\end{aligned}$$

for $\lambda \vdash d$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$ from (4.17).

For such t set

$$F_{\mathsf{t}} := \prod_{k=1}^{d} \prod_{\substack{\mathsf{s} \in \mathrm{Std}(\lambda) \\ \rho_k^{\mathsf{s}} \neq \rho_k^{\mathsf{t}}}} \frac{L_k - \rho_k^{\mathsf{s}}}{\rho_k^{\mathsf{t}} - \rho_k^{\mathsf{s}}} \in \mathscr{H}(\mathfrak{S}_d)\ , \tag{4.19}$$

which is well defined, because Jucys–Murphy elements commute with each other and R is a field.

Moreover, define

$$f_{\mathsf{t}} := m_{\mathsf{t}} F_{\mathsf{t}} \in S^{\lambda}\ . \tag{4.20}$$

These definitions are straightforward generalizations of similar elements in the classical case (cf. [Mur81]). There are two important properties of these elements.

Firstly, because Jucys–Murphy elements act triangular on m_{t} by Proposition 4.4.11, following the partial order on $\mathrm{Std}(\lambda)$, so does F_{t}. Additionally, each factor of F_{t} acts non-trivially on m_{t} and gets normalized by the coefficient of m_{t}. Thus, $\{f_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$ is a basis of S^{λ} and the base change from $\{m_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$ is unitriangular. It can be shown that $\{f_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$ is orthogonal with respect to the bilinear form on S^{λ}.

Secondly, it is possible to compute

$$f_{\mathsf{t}} L_k = \rho_k^{\mathsf{t}} f_{\mathsf{t}} \tag{4.21}$$

for all $k = 1, 2, \dots, d$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$, so the transformed basis elements are eigenvectors for the Jucys–Murphy elements. For detailed proofs see [Mat06, Proposition 3.35].

Remark 4.5.2. Recall that if $v = 1$ the quantum integers are the ordinary integers. Therefore the Jucys–Murphy basis act by elements in $\mathbb{Z}$ in the classical case of $\mathbb{C}[\mathfrak{S}_d]$.

Example 4.5.3. We want to compute all elements F_{t} and f_{t} for $\mathscr{H}(\mathfrak{S}_3)$ and verify the two properties stated above. Note that the element $1 + v^2 \in R$ is invertible, because we assume $e > 3$ and R is a field.

To compute F_{t} for $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$, note that the product in (4.19) is empty. Hence, $F_{\mathsf{t}} = 1$ for these standard tableaux.

For $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ there are two factors in F_{t}. Indeed, the quantum residue sequences are

$$(v[0]_v, [1]_v, v^2[-1]_v) = (0, 1, -v^2)$$

and

$$(v[0]_v, v^2[-1]_v, [1]_v) = (0, -v^2, 1)$$

respectively. They only agree in the first entry, because we assume $e > 3$.

We get the following elements in $\mathscr{H}(\mathfrak{S}_3)$:

$$\begin{aligned} F_{\begin{smallmatrix}1&2&3\end{smallmatrix}} &= 1 \\ F_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} &= -\frac{1}{(1+v^2)^2}(L_2+v^2)(L_3-1) \\ F_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} &= -\frac{1}{(1+v^2)^2}(L_2-1)(L_3+v^2) \\ F_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} &= 1 \end{aligned}$$

For elements $f_{\mathfrak{t}}$ in the Specht modules we use (4.20) and the action of Jucys–Murphy elements on the standard basis of Specht modules, which were summarized in Example 4.4.12:

$$\begin{aligned} f_{\begin{smallmatrix}1&2&3\end{smallmatrix}} &= m_{\begin{smallmatrix}1&2&3\end{smallmatrix}} && \in S^{\square\square\square} \\ f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} &= m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} && \in S^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}} \\ f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} &= m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} + \frac{v^2}{(1+v^2)} m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} && \in S^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}} \\ f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} &= m_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} && \in S^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \end{aligned}$$

We see that for $\mathscr{H}(\mathfrak{S}_3)$ the elements $f_{\mathfrak{t}}$ indeed form a basis of each Specht module. Recall that $\begin{smallmatrix}1&2\\3\end{smallmatrix} \rhd \begin{smallmatrix}1&3\\2\end{smallmatrix}$, so the base change in $S^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}$ is unitriangular with respect to the dominance order on standard tableaux. The other Specht modules satisfy this property as well.

To compute the action of Jucys–Murphy elements on the transformed basis we once again use the results of Example 4.4.12 and express the results in the transformed basis:

$f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	
0	0	0	0	$\cdot L_1$
$f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$-v^2 f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$-v^2 f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	$\cdot L_2$
$(1+v^{-2}) f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$-v^2 f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$-v^3(v+v^{-1}) f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	$\cdot L_3$

We see that the transformed basis elements are eigenvectors for Jucys–Murphy elements. The coefficients agree with entries of the quantum residue sequences of the corresponding standard tableaux from Example 4.4.12. We have therefore verified (4.21) in the case of $d = 3$.

We are now ready to state Young's orthogonal form for $\mathscr{H}(\mathfrak{S}_d)$. The statement is taken from [Mat06, Theorem 3.36].

Proposition 4.5.4. *Let R be a field and $v \in R$ invertible such that $e > d$. Then the set $\{f_\mathfrak{t} \mid \mathfrak{t} \in Std(\lambda)\}$ is an orthogonal basis of S^λ for all $\lambda \vdash d$. Moreover, let $\mathfrak{s} \in Std(\lambda)$ and $i = 1, 2, \ldots, d-1$ and define $\alpha_{\mathfrak{s},i} := res_\mathfrak{s}(i) - res_{\mathfrak{s}(i,i+1)}(i)$ as the difference between the i-th e-residue of $\mathfrak{s}$ and $\mathfrak{s}(i, i+1)$. Then the action of the standard generator $H_i \in \mathscr{H}(\mathfrak{S}_d)$ on the orthogonal basis element $f_\mathfrak{s}$ is given by:*

$$f_\mathfrak{s} H_i = \begin{cases} -\frac{v^{\alpha_{\mathfrak{s},i}}}{[\alpha_{\mathfrak{s},i}]_v} f_\mathfrak{s} & \text{, if } \mathfrak{s}(i,i+1) \notin Std(\lambda) \\ -\frac{v^{\alpha_{\mathfrak{s},i}}}{[\alpha_{\mathfrak{s},i}]_v} f_\mathfrak{s} + v f_{\mathfrak{s}(i,i+1)} & \text{, if } \mathfrak{s}(i,i+1) \in Std(\lambda) \text{ and } \mathfrak{s} \rhd \mathfrak{s}(i,i+1) \\ -\frac{v^{\alpha_{\mathfrak{s},i}}}{[\alpha_{\mathfrak{s},i}]_v} f_\mathfrak{s} + v^{-1} \frac{[\alpha_{\mathfrak{s},i}+1]_v [\alpha_{\mathfrak{s},i}-1]_v}{[\alpha_{\mathfrak{s},i}]_v^2} f_{\mathfrak{s}(i,i+1)} & \text{, if } \mathfrak{s}(i,i+1) \in Std(\lambda) \text{ and } \mathfrak{s} \lhd \mathfrak{s}(i,i+1) \end{cases}$$

Example 4.5.5. To verify Proposition 4.5.4 for $\mathscr{H}(\mathfrak{S}_3)$ we continue with the setup of Example 4.5.3. Orthogonality can be verified with Example 4.3.5, where we stated the bilinear form on each Specht module of $\mathscr{H}(\mathfrak{S}_3)$ explicitly.

Using the constructions from Example 4.5.3 we can calculate the action of standard generators on the orthogonal basis directly:

$f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	
$v^{-1} f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$v^{-1} f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$-v f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$-v f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	$\cdot H_s$
$v^{-1} f_{\begin{smallmatrix}1&2&3\end{smallmatrix}}$	$-\frac{v^2}{(v+v^{-1})} f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} + v f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$	$\frac{v^{-2}}{(v+v^{-1})} f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} + \frac{v^{-1}(v^2+1+v^{-2})}{(v+v^{-1})^2} f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$	$-v f_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$	$\cdot H_t$

A comparison to the coefficients in Proposition 4.5.4 shows that we have verified the result for $\mathscr{H}(\mathfrak{S}_3)$.

Instead of the construction above we could have used Gram-Schmidt to deduce an orthogonal basis of each Specht module. What is special about this particular orthogonal basis are the explicit formulas for the right action of the standard generators of $\mathscr{H}(\mathfrak{S}_d)$. Note that for each combination of $\mathfrak{s} \in \mathrm{Std}(\lambda)$ and $i = 1, 2, \ldots d-1$, $f_\mathfrak{s} H_i$ expressed in the orthogonal basis is the sum of at most two elements. Hence, the matrix representation of each H_i in this basis is sparse.

4.5.2 Restrictions of irreducible representations

We want to discuss another application of Young's orthogonal form besides sparse matrix representations of the H_i's. Symmetric groups are naturally nested in each other and representations can be restricted according to this structure. One is interested in the decomposition of such restricted representations into irreducibles, classically in the semisimple case of representations over $\mathbb{C}$. With Young's orthogonal form already established it is relatively straightforward to describe the decomposition, even for all semisimple Hecke algebras.

We first recall the classical results before proving their generalizations to semisimple Hecke algebras.

For any $d > 3$ the natural injection

$$\mathfrak{S}_{d-1} \hookrightarrow \mathfrak{S}_d,\ (i, i+1) \mapsto (i, i+1)$$

yields a chain of finite groups

$$\{1\} = \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \dots . \tag{4.22}$$

If V is a representation of $\mathfrak{S}_d$ and $j < d$, then V is also a representation of $\mathfrak{S}_j$ by acting via the injection from (4.22). We call this representation the **restriction** of V to $\mathfrak{S}_j$ and denote it by $\operatorname{res}^{\mathfrak{S}_d}_{\mathfrak{S}_j} V$.

This structure naturally leads us to the following question in the semisimple case of representations of the symmetric group over $\mathbb{C}$: if V is an irreducible representation of $\mathfrak{S}_d$, which irreducible representations of $\mathfrak{S}_j$ appear in the decomposition of $\operatorname{res}^{\mathfrak{S}_d}_{\mathfrak{S}_j} V$ and what is their multiplicity? The traditional approach to this question, for example in [Jam78], involves extensive calculations using tableaux.

More recently, the authors of [VO05] presented another approach to the representation theory of $\mathfrak{S}_d$. They build the representation theory on top of the nested structure (4.22) and even introduced a more general framework to study representations for groups with a nested structure similar to (4.22). The decomposition of restricted representations is a natural part of their construction. For the symmetric group they even show how to get Young's orthogonal form from their results.

For a more elaborate explanation of the approach of [VO05] in the classical case of $\mathbb{C}[\mathfrak{S}_d]$ see [CST10].

Here we can take a much easier approach that still relies on the ideas of [VO05], because we already stated Young's orthogonal form in Proposition 4.5.4. This proposition holds for any semisimple Hecke algebra of the symmetric group, which also has a nested structure. We can therefore immediately prove the general result.

Assume that R is a field and $e > d$. Recall from Remark 4.5.1 that the Hecke algebra $\mathscr{H}(\mathfrak{S}_d)$ is semisimple under these assumptions and that every Specht module is irreducible. Under these assumptions we can apply Proposition 4.5.4, so the Specht module S^λ for $\lambda \vdash d$ has Young's orthogonal form, an orthogonal basis $\{f_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\}$.

Like for the symmetric group there is a natural injection

$$\mathscr{H}(\mathfrak{S}_{d-1}) \hookrightarrow \mathscr{H}(\mathfrak{S}_d),\ H_i \mapsto H_i$$

for any $d > 3$, which yields a chain of finite-dimensional R-modules

$$R = \mathscr{H}(\mathfrak{S}_1) \subset \mathscr{H}(\mathfrak{S}_2) \subset \mathscr{H}(\mathfrak{S}_3) \subset \dots .$$

Any right $\mathscr{H}(\mathfrak{S}_d)$-module M is also a right $\mathscr{H}(\mathfrak{S}_j)$-module for $j \leq d$. It is called the **restriction** of M to $\mathscr{H}(\mathfrak{S}_j)$. We denote this right $\mathscr{H}(\mathfrak{S}_j)$-module by $\mathrm{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} M$ and are particularly interested in the decomposition of Specht modules into irreducible right $\mathscr{H}(\mathfrak{S}_j)$-modules. The Branching theorem, a naming convention that becomes obvious momentarily, gives their exact decomposition.

If $j < d$, $\lambda \vdash d$ and $\mu \vdash j$ then we write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for any $i \in \mathbb{N}$. In other words, we write $\mu \subset \lambda$ if we get the diagram of μ by removing boxes from the end of rows in the diagram of λ. This relation is transitive.

Example 4.5.6. Consider the partitions $(1^3) \vdash 3$, $(1^2) \vdash 2$ and $(2) \vdash 2$. Then $(1^2) \subset (1^3)$ but $(2) \not\subset (1^3)$.

The proof of the general Branching theorem works by repeatedly applying the one-step case, meaning the case where $j = d - 1$. We deal with this one-step case in the following lemma.

Lemma 4.5.7. *Let R be a field, $e > d$ and $\lambda \vdash d$. The restriction of S^λ to $\mathscr{H}(\mathfrak{S}_{d-1})$ decomposes as a right $\mathscr{H}(\mathfrak{S}_{d-1})$-module as*

$$\mathrm{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_{d-1})} S^\lambda = \bigoplus_{\substack{\mu \vdash d-1 \\ \mu \subset \lambda}} S^\mu \ .$$

In particular, S^μ for $\mu \vdash d-1$ is a summand of S^λ, if and only if $\mu \subset \lambda$. In this case it has multiplicity one.

Proof. We begin by clustering basis elements $f_{\mathfrak{s}} \in S^\lambda$ based on having the same box in their diagram labeled by d. Let $\mu \vdash d-1$ and let $\hat{S}^\mu \subset S^\lambda$ be the right $\mathscr{H}(\mathfrak{S}_{d-1})$-module generated by elements $f_{\mathfrak{s}}$, such that if we remove the box labeled d from $\mathfrak{s}$, the resulting tableau has shape μ. By construction, $\hat{S}^\mu$ is non-trivial, if and only if $\mu \subset \lambda$.

Firstly, note that the generating elements of $\hat{S}^\mu$ also form a R-basis. Indeed, by Proposition 4.5.4 we know that for $i < d-2$ the element $f_{\mathfrak{s}} H_i$ lies in the span of $f_{\mathfrak{s}}$ and $f_{\mathfrak{s}(i,i+1)}$. The tableau $\mathfrak{s}(i, i+1)$ has the same shape as $\mathfrak{s}$ and moreover the same box labeled d as $\mathfrak{s}$. Thus, $f_{\mathfrak{s}(i,i+1)}$ is a generating element of $\hat{S}^\mu$ and the claim follows.

Secondly, we have $\hat{S}^\mu \cong S^\mu$ as right $\mathscr{H}(\mathfrak{S}_{d-1})$-modules. Indeed, by Proposition 4.5.4 the coefficients appearing in $f_{\mathfrak{s}} H_i$ for $i < d-2$ do not depend on the box of $\mathfrak{s}$ labeled d. Moreover, they are the same coefficients appearing in $f_{\mathfrak{t}} H_i \in S^\mu$, where the tableau $\mathfrak{t}$ is $\mathfrak{s}$ with box labeled d removed. This follows again by Proposition 4.5.4.

Lastly, we see that each $f_{\mathfrak{s}} \in S^\lambda$ is part of exactly one $\hat{S}^\mu$, proving the claimed decomposition and multiplicity of $S^\mu \in S^\lambda$. □

Example 4.5.8. Let's consider an example of the grouping done in the proof of Lemma 4.5.7. The Specht module $S^{\ydiagram{2,1}}$ of $\mathscr{H}(\mathfrak{S}_3)$ has basis elements $f_{\syt{1,2\\3}}$ and $f_{\syt{1,3\\2}}$. Removing the box labeled 3 from $\syt{1,2\\3}$ and $\syt{1,3\\2}$ yields the tableaux $\syt{1,2}$ and $\syt{2\\1}$ respectively. Thus, $\hat{S}^{\ydiagram{2}} = \operatorname{span}_R\{f_{\syt{1,2\\3}}\}$ and $\hat{S}^{\ydiagram{1,1}} = \operatorname{span}_R\{f_{\syt{1\\2\\3}}\}$.

The proof of Lemma 4.5.7 motivates the definition of a directed graph $\mathbb{Y}$ with all partitions of all $d \geq 1$ as vertices and a directed edge from $\mu \vdash j$ to $\lambda \vdash d$ if $j = d-1$ and $\mu \subset \lambda$. This graph is sketched in Figure 4.1 with partitions represented as diagrams.

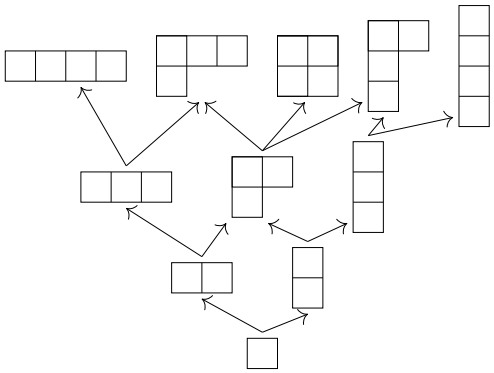


Figure 4.1: Sketch of the directed graph $\mathbb{Y}$ up to partitions of size 4. Partitions of the same size are placed on the same level of the graph. There is at most one edge between two vertices. The ending vertex of each directed edge is exactly one level above its starting edge.

Proposition 4.5.9 (Branching theorem)**.** *Let R be a field, $e > d$, $\lambda \vdash d$ and $j < d$. The restriction of S^λ to $\mathscr{H}(\mathfrak{S}_j)$ decomposes as a right $\mathscr{H}(\mathfrak{S}_j)$-module as*

$$\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda = \bigoplus_{\substack{\mu \vdash j \\ \mu \subset \lambda}} (S^\mu)^{\oplus r^\lambda_\mu},$$

where r^λ_μ is the number of directed paths in $\mathbb{Y}$ between μ and λ. In particular, S^μ for $\mu \vdash j$ is a summand of S^λ, if and only if $\mu \subset \lambda$. In this case it has multiplicity r^λ_μ.

Proof. In light of Lemma 4.5.7 the proof of this proposition is an induction over j, starting at $j = d-1$. For $j = d-1$ this proposition is exactly Lemma 4.5.7, because if $\mu \subset \lambda$, $\mu \vdash d-1$, $\lambda \vdash d$, then there is a single edge between μ and λ in $\mathbb{Y}$. For $j < d-1$ we apply induction for $j+1$ and get a decomposition as right $\mathscr{H}(\mathfrak{S}_{j+1})$ modules

$$\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_{j+1})} S^\lambda = \bigoplus_{\substack{\nu \vdash j+1 \\ \eta \subset \lambda}} (S^\eta)^{\oplus r^\lambda_\eta}.$$

Now $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda = \operatorname{res}^{\mathscr{H}(\mathfrak{S}_{j+1})}_{\mathscr{H}(\mathfrak{S}_j)}(\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_{j+1})} S^\lambda)$ as right $\mathscr{H}(\mathfrak{S}_j)$-modules. and thus

$$\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda = \bigoplus_{\substack{\eta \vdash j+1 \\ \eta \subset \lambda}} (\operatorname{res}^{\mathscr{H}(\mathfrak{S}_{j+1})}_{\mathscr{H}(\mathfrak{S}_j)} S^\eta)^{\oplus r^\lambda_\eta} .$$

Each $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_{j+1})}_{\mathscr{H}(\mathfrak{S}_j)} S^\eta$ decomposes according to Lemma 4.5.7. All S^μ for $\mu \vdash j$, $\mu \subset \eta$ appear as summands with multiplicity one and, by transitivity, we also have $\mu \subset \lambda$. Finally, if $\mu \vdash j$, $\eta \vdash j+1$ and $\mu \subset \eta \subset \lambda$, then each path from η to λ in $\mathbb{Y}$ can be extended to a path from μ to λ. For each extension, one copy of S^μ appears in the decomposition of $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda$. So, varying η and adding up all paths, the multiplicity of S^μ in $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda$ is the number of all paths from μ to λ in $\mathbb{Y}$. □

Example 4.5.10. To decompose restrictions of Specht modules of $\mathscr{H}(\mathfrak{S}_3)$ to $\mathscr{H}(\mathfrak{S}_2)$ we apply Proposition 4.5.9 and use Example 4.5.8 and Figure 4.1.

$S^{\ydiagram{3}}$ is one-dimensional, the only partition $\mu \vdash 2$ such that $\mu \subset \ydiagram{3}$ is $\mu = \ydiagram{2}$ and there is a single path from $\ydiagram{2}$ to $\ydiagram{3}$ in $\mathbb{Y}$. Hence $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_3)}_{\mathscr{H}(\mathfrak{S}_2)} S^{\ydiagram{3}} = S^{\ydiagram{2}}$ as right $\mathscr{H}(\mathfrak{S}_2)$-modules.

Similarly, $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_3)}_{\mathscr{H}(\mathfrak{S}_2)} S^{\ydiagram{1,1,1}} = S^{\ydiagram{1,1}}$ as right $\mathscr{H}(\mathfrak{S}_2)$-modules.

This is expected, because $S^{\ydiagram{2}}$ is the trivial representation of $\mathscr{H}(\mathfrak{S}_2)$ and $S^{\ydiagram{1,1}}$ is its sign representation.

By Example 4.5.8 we already know that $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_3)}_{\mathscr{H}(\mathfrak{S}_2)} S^{\ydiagram{2,1}} = S^{\ydiagram{2}} \oplus S^{\ydiagram{1,1}}$. The first summand is associated to the path in $\mathbb{Y}$ from $\ydiagram{2}$ to $\ydiagram{2,1}$, the other summand to the path in $\mathbb{Y}$ from $\ydiagram{1,1}$ to $\ydiagram{2,1}$.

Remark 4.5.11. The inductive proof of the Branching Theorem obscures the importance of Young's orthogonal form from Proposition 4.5.4 in this proof, so we want to highlight it in this remark.

The restriction $\operatorname{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_j)} S^\lambda$ for some $j < d$ and $\lambda \vdash d$ is decomposed by stepwise decomposition of S^λ using Lemma 4.5.7. First, it is decomposed into irreducible right $\mathscr{H}(\mathfrak{S}_{d-1})$-modules, then into irreducible right $\mathscr{H}(\mathfrak{S}_{d-2})$-modules and so on until $\mathscr{H}(\mathfrak{S}_j)$ is reached.

By the proof of Lemma 4.5.7, each element $f_{\mathfrak{s}} \in S^\lambda$ is part of some copy of S^μ for some $\mu \vdash j$, $\mu \subset \lambda$, if S^μ is viewed as a right $\mathscr{H}(\mathfrak{S}_j)$-submodule of S^λ. This can also be seen with the explicit formulas from Proposition 4.5.4 and a similar argument as in the proof of Lemma 4.5.7. Acting with H_i for $i < j$ on $f_{\mathfrak{s}} \in S^\lambda$ is part of the R-span of elements $f_{\mathfrak{t}}$, where $\mathfrak{t}$ has the same boxes labeled $j+1, j+2, \ldots, d$ as $\mathfrak{s}$. In fact, the right $\mathscr{H}(\mathfrak{S}_j)$-submodule $f_{\mathfrak{s}}\mathscr{H}(\mathfrak{S}_j) \subset S^\lambda$ is isomorphic to S^μ, because of the explicit formulas given in Proposition 4.5.4.

Note that, by the decomposition in (4.23) and its construction, basis elements in Young's orthogonal form can be labeled by the path from $\square$ to λ in $\mathbb{Y}$.

Remark 4.5.12. Let's consider the Branching Theorem for the extreme case $\mathrm{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_1)} S^\lambda$ for some $\lambda \vdash d$. If we decompose $\mathrm{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_1)} S^\lambda$ as described in Remark 4.5.11 we get

$$\mathrm{res}^{\mathscr{H}(\mathfrak{S}_d)}_{\mathscr{H}(\mathfrak{S}_1)} S^\lambda = (S^{\square})^{\oplus r^\lambda_{\square}}$$

as right $\mathscr{H}(\mathfrak{S}_1)$-modules, with $\mathscr{H}(\mathfrak{S}_1) = R$. As R-submodules of S^λ, each copy of $S^{\square}$ is generated by an element of Young's orthogonal form $f_{\mathfrak{s}} \in S^\lambda$. Moreover, for $j < d$ the right $\mathscr{H}(\mathfrak{S}_j)$-submodule

$$f_{\mathfrak{s}} \mathscr{H}(\mathfrak{S}_j) \subset S^\lambda \tag{4.23}$$

is an irreducible right $\mathscr{H}(\mathfrak{S}_j)$-module. By the Branching theorem it is a Specht module S^μ for $\mu \vdash j$ with $\mu \subset \lambda$.

Example 4.5.13. With our previous calculations we can verify Remark 4.5.12 for the Specht module $S^{\square\square}$ of $\mathscr{H}(\mathfrak{S}_3)$. In Example 4.5.5 we summarized the right action on Young's orthogonal form. As a subalgebra, $\mathscr{H}(\mathfrak{S}_2) \subset \mathscr{H}(\mathfrak{S}_3)$ is generated by H_s. Therefore,

$$f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \mathscr{H}(\mathfrak{S}_2) \subset \mathscr{H}(\mathfrak{S}_3)$$

is determined by $f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \cdot H_s = v^{-1} f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$. Hence, $f_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \mathscr{H}(\mathfrak{S}_2)$ is the trivial representation $S^{\square\square}$ of $\mathscr{H}(\mathfrak{S}_2)$, which is indeed a summand of $S^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}$ by Example 4.5.10. Similarly,

$$f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \mathscr{H}(\mathfrak{S}_2) \subset \mathscr{H}(\mathfrak{S}_3)$$

is determined by $f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \cdot H_s = -v f_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$. Hence, it is the sign representation $S^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ of $\mathscr{H}(\mathfrak{S}_2)$, which is the other summand of $S^{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}}$ by Example 4.5.10.

In the next section we will see another cellular basis of $\mathscr{H}$, that differs from the Murphy basis. The Branching theorem 4.5.9 can be seen as a motivation for the labeling of this cellular basis.

Remark 4.5.14. Assume that R is a field and $e > d$, then $\mathscr{H}$ is semisimple. Recall that if $\mathscr{H}$ is semisimple, all Specht modules are irreducible. Then $\mathscr{H}$ decomposes into a sum of matrix rings by the Artin-Wedderburn theorem. Their dimensions are determined by the dimension of the Specht modules.

Thus, there exists an isomorphism of algebras:

$$\mathscr{H} \cong \bigoplus_{\lambda \vdash d} \left(S^{\lambda^*} \otimes S^\lambda \right) \tag{4.24}$$

By Young's orthogonal form, S^λ for $\lambda \vdash d$ has an orthogonal basis

$$\{ f_{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}(\lambda) \} .$$

Hence, $\mathscr{H}$ has an exceptional basis

$$\{f_{\mathfrak{s}} \otimes f_{\mathfrak{t}} \mid \lambda \vdash d,\ \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)\}\ .$$

To study $\mathscr{H}$ as a right $\mathscr{H}$-module using (4.24), consider $\lambda \vdash d$ and $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)$, so

$$f_{\mathfrak{s}} \otimes f_{\mathfrak{t}} \in S^{\lambda^*} \otimes S^{\lambda} \subset \mathscr{H}\ .$$

In the decomposition of $S^{\lambda^*} \otimes S^{\lambda}$ as a right $\mathscr{H}$-module, the element $f_{\mathfrak{s}} \otimes f_{\mathfrak{t}}$ has the following interpretation. The first factor $f_{\mathfrak{s}}$ of $f_{\mathfrak{s}} \otimes f_{\mathfrak{t}}$ fixes a copy of S^{λ} in the decomposition of $S^{\lambda^*} \otimes S^{\lambda}$ as a right $\mathscr{H}$-module. By Remark 4.5.12, the second factor $f_{\mathfrak{t}}$ is an element in that copy of S^{λ} and associated to a path in $\mathbb{Y}$ from $\square$ to λ.

In summary, Young's orthogonal form and the Branching theorem yield a special basis of $\mathscr{H}$, also labeled by ordered pairs of standard tableaux of size d. Their shape determines a summand in (4.24). The first label fixes a right $\mathscr{H}$-submodule of this summand, which is also a right Specht module. Each standard tableau is associated to a different copy of this Specht module in $\mathfrak{S}$ from (4.24). The second label determines an element in Young's orthogonal form of that Specht module. It is interpreted as a path from $\square$ to λ in $\mathbb{Y}$.

Note that this basis differs from the Murphy basis in general. An element $m_{\mathfrak{st}}$ of the Murphy basis would be associated to $m_{\mathfrak{s}} \otimes m_{\mathfrak{t}}$ in (4.24). This motivates us to find another cellular basis of $\mathscr{H}$, where the second label of each basis element is interpreted as a path in $\mathbb{Y}$.

4.6 Kazhdan–Lusztig basis

Cellular algebras can have many different cellular bases. One example of such algebra was discussed in Example 2.1.5, where we highlighted the different sized posets between two cellular bases for a certain quotient of the free polynomial ring. Another example of such algebra is the Temperley–Lieb algebra from Example 2.1.3, where we described one cell datum and referred to [AST18] for a whole family of cellular bases that differ from the one we described. We also stated two cellular bases for $\mathscr{H}(\mathfrak{S}_3)$ in Example 2.1.4.

In general, cell modules for different cellular bases do not have to be pairwise isomorphic. Hence, we potentially have a variety of different cell modules for a cellular algebra. To study a particular problem it is then crucial to find a cellular structure adjusted to that problem.

In Remark 4.5.1 we mentioned that $\mathscr{H}(\mathfrak{S}_d)$ is semisimple over a field, if and only if all Specht modules are irreducible. More generally, a cellular algebra over a field is semisimple, if and only if all right cell modules are irreducible (cf. [GL96, Theorem 3.8]).

Consequently, the cell modules for a semisimple algebra of different cellular bases are pairwise isomorphic. Even in this situation, one basis of the cell modules might be preferable over another.

We conclude that it is advantageous to have several cellular bases for a cellular algebra. In light of our discussion of the Murphy basis, we are particularly interested in another cellular basis for $\mathscr{H}(\mathfrak{S}_d)$.

In this section we will consider the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_d)$, introduced by Kazhdan and Lusztig in [KL79]. In their article the authors used the convention for generators and relations of $\mathscr{H}(\mathfrak{S}_d)$ mentioned in Remark 4.1.5. See [Soe97] for a discussion of Kazhdan–Lusztig elements using the convention of Definition 4.1.1.

As mentioned in Chapter 1, the work of Kazhdan and Lusztig inspired Graham and Lehrer to define cellular algebras as in Definition 2.1.1. The first example of a cellular algebra given by them in [GL96] was the Hecke algebra of the symmetric group with Kazhdan–Lusztig basis as its cellular basis. A complete account of this result can be found in [Wil03], where the author compiled all the necessary prerequisites to proof that the Kazhdan–Lusztig basis is cellular. One advantage of the Kazhdan–Lusztig basis as a cellular basis is that it is also defined for more general Hecke algebras of Coxeter groups. It can, in particular, be used to construct a cellular basis for Hecke algebras of Coxeter groups of finite type (see [Gec07] for more details).

Even for Hecke algebras of type A, proving cellularity for the Kazhdan–Lusztig basis is non-trivial. We do not attempt to prove it here but refer to [Wil03]. Note that the author uses left action on cellular basis elements for the cellular property and a reverse ordering on the poset Λ that is easier to work with in the context of Kazhdan–Lusztig elements. Thus, results might only be applicable in our setup if the anti-isomorphism $*$ is used and the partial order on Λ reversed. Here, we will state the results using the convention from Definition 2.1.1.

We will verify the cellular property for the Kazhdan–Lusztig basis in the case $d = 3$ by explicit computations in Section 6.2.

This section is organized as follows: we first recall the definition and basic properties of the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_d)$, relabel the basis elements using the Robinson–Schensted correspondence (named after Robinson [Rob38] and Schensted [Sch61]) and then state the corresponding cell datum.

4.6.1 Construction of Kazhdan–Lusztig elements

For the remaining sections of this chapter we assume that R is an algebraically closed field and $v \in R$ invertible.

Recall from Section 4.1 that each standard basis element of $\mathscr{H}(\mathfrak{S}_d)$ is invertible. To construct the Kazhdan–Lusztig elements of $\mathscr{H}(\mathfrak{S}_d)$, we first define the **bar-involution** as the unique ring homomorphism

$$\begin{aligned} - : \mathscr{H}(\mathfrak{S}_d) &\longrightarrow \mathscr{H}(\mathfrak{S}_d) \\ H_w &\mapsto (H_{w^{-1}})^{-1} \\ v &\mapsto v^{-1} \end{aligned} .$$

Note that the bar-involution is not the anti-isomorphism $*$ of $\mathscr{H}(\mathfrak{S}_d)$ defined in Section 4.1, which maps H_w to $H_{w^{-1}}$ and v to v.

An element $H \in \mathscr{H}(\mathfrak{S}_d)$ is called **self-dual**, if $\overline{H} = H$.

Now we recall the **Kazhdan–Lusztig basis** of $\mathscr{H}(\mathfrak{S}_d)$ from [Soe97]. It consists of the elements $\underline{H}_w$ for all $w \in \mathfrak{S}_d$, where $\underline{H}_w \in \mathscr{H}(\mathfrak{S}_d)$ is defined to be the unique self-dual element in $\mathscr{H}(\mathfrak{S}_d)$ such that

$$\underline{H}_w \in H_w + \sum_{y<w} vZ[v]H_y \ .$$

The partial order appearing in the sum is the Bruhat order on $\mathfrak{S}_d$ from Section 3.1. Uniqueness of such elements is not that difficult to prove. Assume there are two elements with that property, then their difference is in $\sum_{y<w} v\mathbb{Z}[v]H_y$ and still self-dual. The coefficient of H_z in their difference, where $z < w$ is a highest length element $y < w$ such that the coefficient of H_y is non-zero, can be shown to be self-dual as well. But the only self-dual element in $v\mathbb{Z}[v]$ is 0, so the difference above must be 0. Existence of such elements is not so clear. We refer to [Soe97, Chapter 2] for a concise proof and to [Wil03, Section 3.5] for a slightly more elaborate proof.

To summarize, notable properties of Kazhdan–Lusztig basis elements $\underline{H}_w$ are that they are self-dual, the coefficient of H_w is 1, only H_y with y comparable and below w appear in the sum and non-zero coefficients are polynomial in v with minimal degree 1. For more results regarding the coefficients appearing in the Kazhdan–Lusztig elements see [Soe97].

Most important for us is their interaction with the anti-isomorphism $*$. Let $w \in \mathfrak{S}_d$, then

$$\underline{H}^*_w = \underline{H}_{w^{-1}} \ . \tag{4.25}$$

See [Wil03, Proposition 3.6.2] for a proof of this property.

Example 4.6.1. The Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ consists of:

$$\begin{aligned}
\underline{H}_e &= 1 \\
\underline{H}_s &= H_s + v \\
\underline{H}_t &= H_t + v \\
\underline{H}_{st} &= H_{st} + vH_s + vH_t + v^2 \\
\underline{H}_{ts} &= H_{ts} + vH_s + vH_t + v^2 \\
\underline{H}_{sts} &= H_{sts} + vH_{st} + vH_{ts} + v^2H_s + v^2H_t + v^3
\end{aligned}$$

In each element the correct coefficients appear in front of the standard basis. Checking that these elements are self-dual is lengthy.

We want to state a cell datum associated to the Kazhdan–Lusztig basis and thus need to define a finite labeling sets $\mathcal{T}(\lambda)$ for each $\lambda \in \Lambda$, an anti-isomorphism $*$ and then label the Kazhdan–Lusztig elements in a way such that the cellular property (C1) holds. Here we will only describe the construction of the poset, labeling sets and the anti-isomorphism for the cell datum associated to the Kazhdan–Lusztig basis. Proving the cellular property is quite involved, using deep theorems from the Kazhdan–Lusztig theory. An exhaustive explanation of the cellular structure of the Kazhdan–Lusztig basis, including rigorous proofs, is presented in [Wil03].

4.6.2 Robinson–Schensted correspondence

To state a cell datum for the Kazhdan–Lusztig basis, it is more convenient to first state the labeling of the elements by an ordered pair of indices, which is needed for a cell datum. Afterwards we define a suitable poset and its connection to the labeling.

Rather than labeling the Kazhdan–Lusztig elements directly, we relabel the elements in $\mathfrak{S}_d$ they are associated to. This is achieved with the so called Robinson–Schensted correspondence, named after Robinson [Rob38] and Schensted [Sch61]. It is a well-known result and discussed in many references, for example in [Ful96] and [Sag01].

Proposition 4.6.2 (Robinson–Schensted correspondence). *There exists a bijection:*

$$\begin{aligned} \mathfrak{S}_d &\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Ordered pairs of standard tableaux} \\ \textit{of the same shape with } d \textit{ boxes} \end{array} \right\} \\ w &\longmapsto (P(w), Q(w)) \end{aligned}$$

This result is proven in [Sch61, Lemma 3] and another proof is in [Sag01, Theorem 3.1.1]. We focus on the algorithm that constructs the ordered pair $(P(w), Q(w))$, which is also described in [Sch61] and [Sag01].

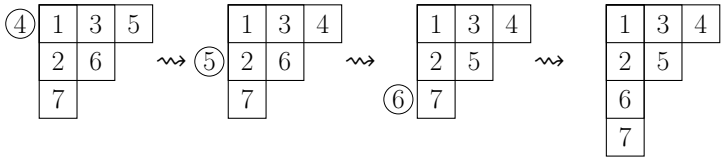
$P(w)$ is constructed by repeated application of the **row bumping algorithm**. Its inputs are a standard tableau $\mathfrak{s}$ and a natural number x that is not already a label in $\mathfrak{s}$. Its output, denoted by $\mathfrak{s} \leftarrow x$, is a standard tableau with one more box than $\mathfrak{s}$ that has label x in addition to all the labels from $\mathfrak{s}$. The algorithm works as follows. If x is greater than all labels in the first row of $\mathfrak{s}$, then add a box to the end of this row of $\mathfrak{s}$ and label it with x. Otherwise replace the smallest label larger than x in that row of $\mathfrak{s}$ and call the displaced label y. Then repeat this step with the next row and y instead of x. The algorithm ends if a box is created, meaning if a box is added to the end of some row.

By [Sch61, Lemma 1], the output $\mathfrak{s} \leftarrow x$ is again a standard tableau.

Example 4.6.3. We apply the row bumping algorithm to:

$$\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \quad \text{and} \quad x = 4$$

Then the following diagram describes how we arrive at $\mathfrak{s} \leftarrow x$.



At each step, the circled number indicates the label we want to insert into the row it is attached to. Because $4 < 5$ the 4 replaces the 5 and the 5 needs to be inserted into row two. As $5 < 6$ the 5 then replaces the 6, which in turn replaces the 7. Then 7 is added to empty row four, so a box is created and the algorithm stops.

To get $P(w)$ we first express w via its permutation action of $\{1, 2, \ldots, d\}$, meaning for $i = 1, 2, \ldots d$ we define $w_i := w(i)$. Then set

$$P^{(0)}(w) := \emptyset$$

and for $i = 1, 2, \ldots, d$ set

$$P^{(i)}(w) := P^{(i-1)}(w) \leftarrow w_i \ .$$

Finally, denote $P(w) := P^{(d)}(w)$. In other words, we get $P(w)$ by starting with the empty tableau and successively adding the w_i to the tableau using the row bumping algorithm, beginning with w_1.

Example 4.6.4. We construct $P(w)$ for $w = 32541 \in \mathfrak{S}_5$, so in total we need five applications of the row bumping algorithm. Note that in the diagram below, each $\rightsquigarrow$ indicates one pass of the row bumping algorithm, adding the circled label above to the tableau left of it.

$$\emptyset \overset{\textcircled{3}}{\rightsquigarrow} \begin{array}{|c|}\hline 3 \\ \hline \end{array} \overset{\textcircled{2}}{\rightsquigarrow} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \overset{\textcircled{5}}{\rightsquigarrow} \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \overset{\textcircled{4}}{\rightsquigarrow} \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \overset{\textcircled{1}}{\rightsquigarrow} \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Then $P(w)$ is the rightmost tableau in this sequence.

Constructing $Q(w)$ is much easier, as it only tracks the step in which each box in $P(w)$ was added. Formally, set $Q^{(0)}(w) := \emptyset$ and for $i = 1, 2, \ldots, d$ add a box to $Q^{(i-1)}(w)$ to get $Q^{(i)}(w)$, where the box is added in the same place as a box is added to $P^{(i-1)}(w)$ to get $P^{(i)}(w)$. This box is then labeled with i. Again, we set $Q(w) := Q^{(d)}(w)$.

Example 4.6.5. For $w = 32541 \in \mathfrak{S}_5$ the associated $Q(w)$ is derived from the sequence of tableaux in Example 4.6.4:

$$Q(w) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

$Q(w)$ naturally has the same shape as $P(w)$ and is also a standard tableau, because when a box is added in some step, its label is the largest label appearing. See also [Sch61, Lemma 2].

The other direction of the Robinson–Schensted correspondence, recovering an element in $\mathfrak{S}_d$ from an ordered pair (P, Q) of standard tableaux of the same shape with d boxes, builds on the idea that if we know where the last box was placed in P, then we know which label was last inserted by reversing the row bumping algorithm. We use this idea in the example below, the general case works just the same. We can read off of Q when which box was added to P by its construction. So by repeating the idea from above we build w by piecing together the labels in reverse order.

Example 4.6.6. Consider the standard tableau we build in Example 4.6.3:

1	3	4
2	5	
6		
7		

Suppose we know this tableau is the result of one pass of the row bumping algorithm and we know that the last box added is the one labeled 7. Note that this information can be read off of $Q(w)$, if we wanted to reverse the Robinson–Schensted correspondence.

As 7 is larger than the 6 in the row above the box labeled 7, we know that 6 must have displaced 7 during the row bumping algorithm. Now 6 is larger than the 5 in row two and $2 < 5$, so 5 must have displaced 6. Lastly, $5 > 4$ and 4 is the largest label in row one that is smaller than 5, so 4 must have displaced 5. Therefore, the row bumping algorithm must have inserted 4 into the standard tableau

1	3	5
2	6	
7		

.

A comparison to Example 4.6.3 shows, that we have successfully reversed the row bumping algorithm from Example 4.6.3.

Remark 4.6.7. Note that in each step in the construction of $P(w)$, the shape of $P^{(i)}(w)$ moves one layer upwards in the directed graph $\mathbb{Y}$ from Section 4.5.2. Its path in $\mathbb{Y}$ is recorded in $Q(w)$. Or in other words, its path can be reconstructed from $Q(w)$ by successively adding boxes to an empty diagram according to their label in $Q(w)$. Thus, the pair $(P(w), Q(w))$ is an ordered pair of standard tableaux of the same shape and the second entry can be interpreted as a path in $\mathbb{Y}$.

In Remark 4.5.14 we wanted to find a cellular basis with such an ordered pair as labels. Using the Robinson–Schensted correspondence, we can now relabel the Kazhdan–Lusztig basis, which labels are not suitable for a cellular algebra, with our wanted labels. This enables us to view the Kazhdan–Lusztig basis as a cellular basis.

For our purposes, the next result is an important property of the Robinson–Schensted correspondence, because knowing what happens when labels are swapped is a key property of a cellular basis.

Proposition 4.6.8. *Let $w \in \mathfrak{S}_d$. If $w \sim (P, Q)$ under the Robinson–Schensted correspondence, then $w^{-1} \sim (Q, P)$.*

There is a good amount of preparation needed to prove this result. We refer to [Ful96, Chapter 5] for a full proof. All necessary details to prove Proposition 4.6.8 are also collected in [Wil03, Section 2.5]. More concretely, the above result is [Wil03, Theorem 2.5.2].

We are finally ready to state the complete cell datum associated to the Kazhdan–Lusztig basis and refer to [Wil03, Theorem 5.6.2] for the proof.

Proposition 4.6.9. *The Hecke algebra $\mathscr{H} = \mathscr{H}(\mathfrak{S}_d)$ is a cellular algebra. A cell datum for $\mathscr{H}$ is given by the tuple $((\Lambda^+(d), \tilde{\triangleright}), Std, \mathcal{K}, *)$, where*

$(\Lambda^+(d), \tilde{\triangleright})$ *is the set of all partitions of d with reverse dominance ordering,*

$Std : \Lambda^+(d) \to \{$*finite sets*$\}$ *assigns to each $\lambda \in \Lambda^+(d)$ the set of all standard λ-tableaux $Std(\lambda)$,*

$\mathcal{K} : \coprod_{\lambda \in \Lambda^+(d)} Std(\lambda) \times Std(\lambda) \hookrightarrow \mathscr{H}(\mathfrak{S}_d)$, $(P(w), Q(w)) \mapsto \underline{H}_w$ *is the map with image*

$$im(\mathcal{K}) = \{\underline{H}_w \mid w \in \mathfrak{S}_d\},$$

the Kazhdan–Lusztig basis of $\mathscr{H}$, and

$* : \mathscr{H} \to \mathscr{H}$ *is the algebra anti-isomorphism $\underline{H}_w \mapsto \underline{H}_{w^{-1}}$.*

The reverse dominance ordering on $\Lambda^+(d)$ is defined by $\lambda \tilde{\triangleright} \mu \Leftrightarrow \lambda \triangleleft \mu$ for all partitions $\lambda, \mu \in \Lambda^+(d)$, where $\triangleleft$ is the usual dominance ordering that is part of cell datum for the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. Note that $*$ is the same anti-isomorphism we used in Section 6.1. Furthermore, the labeling sets $\mathrm{Std}(\lambda)$ for $\lambda \in \Lambda^+(d)$ are also part of the cell datum for the Murphy basis.

To emphasize the cellular structure of the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_d)$ we denote

$$H^{\lambda}_{P(w)Q(w)} := \underline{\mathrm{H}}_w \,,$$

where $w \in \mathfrak{S}_d$ and where $\lambda \in \Lambda^+(d)$ is the shape of $P(w)$ and $Q(w)$. Sometimes we omit λ, because it is the shape of $P(w)$ and $Q(w)$. By the important property of the Robinson–Schensted correspondence from Proposition 4.6.8 the following relation holds:

$$(H^{\lambda}_{P(w)Q(w)})^* = (\underline{\mathrm{H}}_w)^* \overset{(4.25)}{=} \underline{\mathrm{H}}_{w^{-1}} \overset{4.6.8}{=} H^{\lambda}_{Q(w)P(w)}$$

Recall that this is exactly the interaction of $*$ with cellular basis elements we need by Definition 2.1.1.

Remark 4.6.10. By [GL96, Theorem 3.8] a cellular algebra over a field is semisimple, if and only if all right cell modules are irreducible. Hence, if $\mathscr{H}$ is semisimple, each cell module from the Kazhdan–Lusztig basis is isomorphic to a Specht module from the Murphy basis.

The general result of Graham and Lehrer does not give us specific isomorphisms, nor does it provide us with a matching between cell modules from this section and Specht modules. In Section 6.2.2 we describe these matchings for $\mathscr{H}(\mathfrak{S}_3)$ and provide explicit isomorphisms.

5. A cellular basis for Schur algebras

Schur algebras originate from the representation theory of $\mathfrak{S}_d$ over $\mathbb{C}$ and its connection to the general linear group.

Let $n, d > 0$ and denote $V := \mathbb{C}^n$. Then GL_n naturally acts from the left on V, as well as on the tensor space $V^{\otimes d}$ by acting on each factor separately. Moreover, $\mathfrak{S}_d$ acts from the right on the tensor space $V^{\otimes d}$ by permuting the entries. This situation is usually depicted as:

$$\mathrm{GL}_n \curvearrowright V^{\otimes d} \curvearrowleft \mathfrak{S}_d \tag{5.1}$$

Classical Schur–Weyl duality states that these two actions centralize each other (see [Eti+11] for details). In particular, viewing the action $\mathrm{GL}_n \curvearrowright V^{\otimes d}$ as a map $\mathrm{GL}_n \to \mathrm{End}(V^{\otimes d})$, the image of this map is $\mathrm{End}_{\mathfrak{S}_d}(V^{\otimes d})$ and called the Schur algebra $\mathscr{S}(n, d)$.

Its importance stems from the study of polynomial representations of GL_n. We refer to [Gre80] and the introduction of [Don98] for the complete picture.

There exists a quantum version of the Schur algebra, which was first introduced in [DJ89] and will be recalled below. It appears in a quantum Schur–Weyl duality, first described in [Jim86], where the right side of (5.1) is replaced by $\mathscr{H}(\mathfrak{S}_d)$ and the left side of (5.1) by a quantum group.

A good reference for these Schur algebras is [Don98]. For an extensive review of quantum Schur–Weyl duality, the classical Schur–Weyl duality and even affine versions of Schur–Weyl duality we refer to [Ant20].

Here we only focus on one aspect of quantum Schur algebras: they are cellular algebras. Below we state a cell datum for a cellular basis, called the semistandard basis, that is deduced from the Murphy basis of $\mathscr{H}$. Compared to the Hecke algebra, classifying irreducible modules for quantum Schur algebras over a field R using the cellular basis is very simple: every quotient D^λ of a cell module C^λ is non-zero and thus irreducible. It is, in particular, independent of the choice of R and v.

We begin this chapter with the definition of quantum Schur algebras, which we simply call Schur algebras, following [Mat06]. Afterwards we state the full cell datum associated to the semistandard basis. We will see an explicit example of the semistandard basis in Section 6.3. In this example we will also verify the cellular property of the semistandard basis by hand.

5.1 Definition and notations

Let R be an integral domain and $v \in R$ invertible. Note that this is the setup from Section 4.3, where we considered the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. Recall the sets

$$\begin{aligned} \Lambda(n,d) &:= \{\mu = (\mu_1, \mu_2, \ldots, \mu_n) \mid \mu \vDash d\} \\ \Lambda^+(n,d) &:= \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda \vdash d\} \end{aligned}$$

for $n, d \in \mathbb{N}$ from (3.2). Denote by $\mathscr{H} := \mathscr{H}_{R,v}(\mathfrak{S}_d)$ the Hecke algebra associated to $\mathfrak{S}_d$.

Definition 5.1.1. The **Schur algebra** is defined as

$$\mathscr{S}_{R,v}(n,d) := \mathrm{End}_{\mathscr{H}}\left(\bigoplus_{\mu \in \Lambda(n,d)} M^\mu\right).$$

For some fixed R, v, n and d we will simply write $\mathscr{S}(n,d)$ or just $\mathscr{S}$ for the associated Schur algebra.

Example 5.1.2. The poset of compositions $\mu = (\mu_1, \mu_2)$ of 3 is

$$\Lambda(2,3) = \left\{ \ydiagram{3}, \ydiagram{2,1}, \ydiagram{1,1,1}, \ydiagram{0,3} \right\},$$

so the involved permutation modules are

$$M^{\ydiagram{3}}, \; M^{\ydiagram{2,1}}, \; M^{\ydiagram{1,1,1}}, \text{ and } M^{\ydiagram{0,3}}.$$

Remark 5.1.3. Compare Definition 5.1.1 to the definition of Schur algebras as certain endomorphisms of a tensor space from Schur–Weyl duality. The definitions are equivalent. Indeed, there is a well defined right action of $\mathscr{H}$ on this tensor space and an isomorphism $V^{\otimes d} \cong \bigoplus_{\mu \in \Lambda(n,d)} M^\mu$ as right $\mathscr{H}$-modules. See [Mat06, Exercise 4.19] or [Ant20, Lemma 2.19] for more details.

Definition 5.1.1 is advantageous for our purposes, because we can apply results from the bases of permutation modules derived from the Murphy basis of $\mathscr{H}$. Recall that these were discussed in Section 4.2.

The cellular basis of $\mathscr{S}(n,d)$ we are interested in is constructed via the decomposition

$$\mathscr{S}(n,d) = \bigoplus_{\mu,\eta \in \Lambda(n,d)} \mathrm{Hom}_{\mathscr{H}}(M^\eta, M^\mu) \tag{5.2}$$

by constructing a basis on each $\mathrm{Hom}_{\mathscr{H}}(M^\eta, M^\mu)$ and extending these basis elements trivially to $\mathscr{S}(n,d)$.

For now we fix compositions $\mu, \eta \in \Lambda(n,d)$ and want to study $\mathrm{Hom}_{\mathscr{H}}(M^\eta, M^\mu)$. The permutation module M^η is, by Definition 4.2.1, generated by m_η as a right $\mathscr{H}$-module. Hence, any $\phi \in \mathrm{Hom}_{\mathscr{H}}(M^\eta, M^\mu)$ is uniquely determined by $\phi(m_\eta)$. Conversely, elements in M^μ that determine a homomorphism can be characterized by the following lemma.

Lemma 5.1.4. *For $\mu, \eta \in \Lambda(n,d)$ there exists a R-linear isomorphism*

$$\begin{array}{ccc} Hom_{\mathscr{H}}(M^\eta, M^\mu) & \xrightarrow{\sim} & M^\mu \cap M^{\eta *} \\ \phi & \mapsto & \phi(m_\eta) \end{array} .$$

For a proof of Lemma 5.1.4 see [Mat06, Corollary 4.8]. We have already studied the intersection $M^\mu \cap M^{\eta *}$ in Proposition 4.2.11. It has a R-basis given by all m_{ST} for $S \in \mathcal{T}_0(\lambda, \mu)$ and $T \in \mathcal{T}_0(\lambda, \eta)$ for some $\lambda \vdash d$. Recall that the elements m_{ST} were defined in (4.11) as

$$m_{ST} = \sum_{\substack{\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s})=S,\ \eta(\mathfrak{t})=T}} m_{\mathfrak{s}\mathfrak{t}} .$$

We can now use these elements to define the homomorphisms

$$\phi^\lambda_{ST} : \begin{array}{ccc} M^\eta & \longrightarrow & M^\mu \\ m_\eta & \mapsto & m_{ST} \end{array} .$$

Recall that $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, because $\mu \in \Lambda(n,d)$. If a semistandard tableau $S \in \mathcal{T}_0(\lambda, \mu)$ exists, then λ has at most n non-zero parts as well, because $\mathrm{Shape}(S) = \lambda$ and columns of S must be strictly increasing.

Conversely, for all $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n) \in \Lambda^+(n,d)$ there exists the semistandard λ-tableau $\lambda(\mathfrak{t}^\lambda)$ of type $\lambda \in \Lambda^+(n,d) \subset \Lambda(n,d)$. Therefore, the set $\mathcal{T}_0(\lambda) := \bigcup_{\mu \in \Lambda(n,d)} \mathcal{T}_0(\lambda, \mu)$ of semistandard λ-tableaux of types $\Lambda(n,d)$ is non-empty.

A combination of Lemma 5.1.4 with Proposition 4.2.11 gives us a basis of $\mathrm{Hom}_{\mathscr{H}}(M^\eta, M^\mu)$.

Proposition 5.1.5. *For $\mu, \eta \in \Lambda(n,d)$ $Hom_{\mathscr{H}}(M^\eta, M^\mu)$ is a R-module with basis*

$$\left\{ \phi^\lambda_{ST} \mid \lambda \in \Lambda^+(n,d),\ S \in \mathcal{T}_0(\lambda, \mu) \text{ and } T \in \mathcal{T}_0(\lambda, \eta) \right\} .$$

By extending ϕ^λ_{ST} trivially to a map on $\bigoplus_{\mu \in \Lambda(n,d)} M^\mu$ we get an element in $\mathscr{S}(n,d)$, which we will also denote by ϕ^λ_{ST}. Combined, these elements form a basis of $\mathscr{S}(n,d)$ by the decomposition (5.2). In fact, this basis is cellular.

5.2 Semistandard basis

Proposition 5.2.1. *The Schur algebra $\mathscr{S}(n,d)$ is a cellular algebra. A cell datum for $\mathscr{S}(n,d)$ is given by the tuple $((\Lambda^+(n,d),\triangleright),\mathcal{T}_0,\Phi(n,d),*)$, where*

$(\Lambda^+(n,d),\triangleright)$ *is the set of partitions of d of form $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$,*

$\mathcal{T}_0$ *is the map assigning the finite set $\bigcup_{\mu\in\Lambda(n,d)}\mathcal{T}_0(\lambda,\mu)$ to each $\lambda\in\Lambda^+(n,d)$,*

$\Phi(n,d):\coprod_{\lambda\in\Lambda^+(n,d)}\mathcal{T}_0(\lambda)\times\mathcal{T}_0(\lambda)\hookrightarrow\mathscr{S}(n,d),\ (S,T)\mapsto\phi^\lambda_{ST}$ *is the map with image*

$$im(\Phi(n,d))=\{\phi^\lambda_{ST}\mid\lambda\in\Lambda^+(n,d)\text{ and }S,T\in\mathcal{T}_0(\lambda)\},$$

the R-basis of $\mathscr{S}(n,d)$ called ***semistandard basis*** *stemming from Proposition 5.1.5, and*

$*$ *is the anti-automorphism $\phi^\lambda_{ST}\mapsto\phi^\lambda_{TS}$ of $\mathscr{S}(n,d)$.*

Proof. These elements form a basis by Proposition 5.1.5 and decomposition (5.2) of $\mathscr{S}(n,d)$. We only sketch the proof of the cellular property here, which is based on the cellular property of the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. For a full proof of this proposition see [Mat06, Theorem 4.13].

Let $\phi^\lambda_{ST}\in\mathrm{Hom}_{\mathscr{H}}(M^\eta,M^\mu)$ and $\phi\in\mathscr{S}(n,d)$ and consider $\phi^\lambda_{ST}\circ\phi$. We can assume that ϕ is an element of the semistandard basis and $\phi\in\mathrm{Hom}_{\mathscr{H}}(M^\tau,M^\eta)$, otherwise the action is trivial. Then $\phi(m_\tau)=m_\eta h$ for some $h\in\mathscr{H}$ and thus $\phi^\lambda_{ST}\circ\phi(m_\tau)=m_{ST}h$. We want to express $\phi^\lambda_{ST}\circ\phi\in\mathrm{Hom}(M^\tau,M^\mu)$ in the semistandard basis and hence compute $m_{ST}h$ and express it in the basis of $M^\mu\cap M^{\tau *}$.

Recall from (4.11) that

$$m_{ST}=\sum_{\substack{\mathfrak{s},\mathfrak{t}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=S,\ \eta(\mathfrak{t})=T}}m_{\mathfrak{s}\mathfrak{t}}\,,$$

so by the cellular property of the Murphy basis we have

$$m_{ST}h\equiv\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=S}}\sum_{\mathfrak{v}\in\mathrm{Std}(\lambda)}r^{\mathfrak{t},h}_{\mathfrak{v}}m_{\mathfrak{s}\mathfrak{v}}+\sum_{\substack{\kappa\trianglerighteq\lambda\\ \mathfrak{p},\mathfrak{q}\in\mathrm{Std}(\kappa)}}r^h_{\mathfrak{p}\mathfrak{q}}m_{\mathfrak{p}\mathfrak{q}};\,. \tag{5.3}$$

Moreover, because $M^\mu\cap M^{\tau *}$ is a right $\mathscr{H}$-submodule we have $m_{ST}h\in M^\mu\cap M^{\tau *}$ and thus

$$m_{ST}h=\sum_{\substack{\alpha\in\Lambda^+(n,d)\\ U\in\mathcal{T}_0(\alpha,\mu),\ V\in\mathcal{T}_0(\alpha,\tau)}}r_{UV}m_{UV}\,. \tag{5.4}$$

We compare (5.3) with (5.4) and see that only r_{UV} associated to $\alpha\trianglerighteq\lambda$ can be non-zero. Moreover, because of the first sum of (5.3), r_{UV} associated to λ can only be non-zero, if $U=S$. Summarized, we get

$$m_{ST}h=\sum_{V\in\mathcal{T}_0(\lambda,\tau)}r_Vm_{SV}+\sum_{\substack{\alpha\in\Lambda^+(n,d)\ \alpha\triangleright\lambda\\ U'\in\mathcal{T}_0(\alpha,\mu),\ V'\in\mathcal{T}_0(\alpha,\tau)}}r_{U'V'}m_{U'V'}$$

and hence verify the cellular property: $\phi_{ST}\circ\phi\equiv\sum_{V\in\mathcal{T}_0(\lambda,\tau)}r_V\phi_{SV}\bmod\check{\mathscr{S}}^\lambda(n,d)$. □

Note that $\mathcal{T}_0(\lambda)$ is never empty, as it contains the canonical element $T^\lambda := \lambda(\mathrm{t}^\lambda)$, the unique semistandard λ-tableau of type λ.

Example 5.2.2. All elements of the semistandard basis of $\mathscr{S}(2,3)$ are gathered in Figure 6.8 of Section 6.3.

Remark 5.2.3. The construction of elements m_{ST}, and subsequently the construction of ϕ^λ_{ST}, is largely independent of the choice of R and v. In particular, the number of right cell modules of $\mathscr{S}(n,d)$ is independent of R and v. Similarly to the Murphy basis of $\mathscr{H}(\Xi_d)$, we would therefore expect that R and v influence the radicals of the cell modules, because we suspect that the irreducible right $\mathscr{S}(n,d)$-modules depend on R and v.

Having established a cellular basis of $\mathscr{S}(n,d)$ in Proposition 5.2.1 we can classify the irreducible representations using Proposition 2.3.12, the general classification result for cellular algebras. It is much less involved than the classification of irreducible $\mathscr{H}$-modules in Section 4.

Firstly, consider the right cell modules of $\mathscr{S}(n,d)$, which are called **Weyl modules** W^λ for some $\lambda \in \Lambda^+(n,d)$. Instead of using the abstract construction of cell modules from Section 2.1 we will work with a specific realization of them. This lets us compute the $\mathscr{S}(n,d)$-action more naturally.

From the theory of cellular algebras we know that $\mathscr{S}^\lambda(n,d)/\check{\mathscr{S}}^\lambda(n,d)$, as a right $\mathscr{S}(n,d)$-module, decomposes into $|\mathcal{T}_0(\lambda)|$ copies of the right cell module for λ. We define the Weyl module W^λ to be the copy associated to the canonical element $T^\lambda \in \mathcal{T}_0(\lambda)$. Explicitly, it is the $\mathscr{S}(n,d)$-submodule

$$W^\lambda := \mathrm{span}_R\left\{\overline{\phi^\lambda_{T^\lambda T}} \mid T \in \mathcal{T}_0(\lambda)\right\} \subset {}^{\mathscr{S}(n,d)}\!/_{\check{\mathscr{S}}^\lambda(n,d)}\,, \tag{5.5}$$

which is actually generated by $\overline{\phi^\lambda_{T^\lambda T^\lambda}}$ as a right $\mathscr{S}(n,d)$-module, because $\phi^\lambda_{T^\lambda T^\lambda}$ is the identity on M^λ. To simplify the notation we can define $\phi^\lambda_T := \overline{\phi^\lambda_{T^\lambda T}}$.

Now, by Section 2.3 we know that the right $\mathscr{S}(n,d)$-module $L^\lambda := W^\lambda/\operatorname{rad} W^\lambda$ for $\lambda \in \Lambda^+(n,d)$ is either irreducible or $\{0\}$. Recall that by definition of the bilinear form on W^λ in Proposition 2.3.2 we have

$$\langle \phi^\lambda_{T^\lambda}, \phi^\lambda_{T^\lambda}\rangle \phi^\lambda_{T^\lambda T^\lambda} \equiv \phi^\lambda_{T^\lambda T^\lambda} \circ \phi^\lambda_{T^\lambda T^\lambda} \bmod \check{\mathscr{S}}^\lambda(n,d)\,,$$

so $\langle \phi^\lambda_{T^\lambda}, \phi^\lambda_{T^\lambda}\rangle = 1$, as $\phi^\lambda_{T^\lambda T^\lambda}$ is the identity on M^λ.

Therefore, $\phi^\lambda_{T^\lambda}$ is not in the radical and thus every L^λ is irreducible and we get the following classification by Proposition 2.3.12.

Proposition 5.2.4. *Let R be a field, $v \in R$ a unit and $n, d \in \mathbb{N}$. Then:*

$$\begin{aligned} \Psi : \Lambda^+(n,d) &\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Irreducible right } \mathscr{S}(n,d)\textit{-modules} \\ \textit{up to isomorphism} \end{array} \right\} \\ \lambda &\longmapsto L^\lambda \end{aligned}$$

A concrete calculation of all irreducible representations of $\mathscr{S}(2,3)$ can be found in Section 6.3, in particular in Proposition 6.3.1.

Remark 5.2.5. Like the construction of the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$, the construction of the semistandard basis of $\mathscr{S}_{R,v}(n,d)$ does not really depend on R and v. Moreover, the classification of irreducible right $\mathscr{S}_{R,v}(n,d)$-modules is also independent of R and v. Hence, compared to the classification of irreducible right $\mathscr{H}(\mathfrak{S}_d)$-modules, this result is much easier to apply in practice.

Note that this result does not give us the dimensions of the irreducible representations. For this we still need to compute the radical explicitly. We have done so for $\mathscr{S}(2,3)$ in Section 6.3.

6. Explicit examples

One caveat of the theory of cellular algebras is, that it can be difficult to construct a cell datum for a given algebra. Usually, the algebra has to be well understood before a cellular basis can be defined. We gave an overview over three cellular bases in Chapters 4 and 5, but the complete proofs either computationally involved or, in the case of the Kazhdan–Lusztig basis, require hard results from its theory.

Given a cell datum it might also be difficult to apply the classification result from Proposition 2.3.12, as Λ_0 can be elusive. We have seen so in Section 4.4, where we outlined a reformulation of the general classification result Proposition 2.3.12 for the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. Moreover, these classification results do not give us the dimension of the irreducible right modules.

In this chapter we explicitly construct an example for each of the cellular bases discussed in Chapters 4 and 5. We verify their cellularity in these explicit examples by providing multiplication tables. Furthermore, we compute the associated right cell modules and state the right $\mathscr{H}$-action on them. The multiplication tables also give us the bilinear form on each cell module, which we use to calculate their radicals. Finally, we compute all irreducible right modules.

All explicit computations are done over fields of arbitrary characteristic and for any non-zero $v \in R$. For the Kazhdan–Lusztig basis we assume, like in Section 4.6, that the field is algebraically closed.

We begin this chapter by discussing the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$ in Section 6.1, then we construct the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ in Section 6.2 and finish with the semistandard basis of $\mathscr{S}(2,3)$ in Section 6.3.

6.1 Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$

In Section 4.3 we described the general construction of the Murphy basis of $\mathscr{H}(\mathfrak{S}_d)$. In this section we will do the whole construction for $d = 3$, considering both the generic and non-generic case of v. We give detailed explanations on how to apply the results of Section 4.3 in practice and state all computations explicitly.

In our explicit calculations we will see, how and why the results depend on v and on the characteristic of R.

We first recall the standard basis of $\mathscr{H}(\mathfrak{S}_3)$, calculate the Murphy basis elements and verify the cellular property for them. Then we compute the right A-action on Specht modules and calculate the irreducible right A-modules without Proposition 4.3.7. Finally, we will identify the trivial and sign representation in our results.

6.1.1 Constructing the cell datum

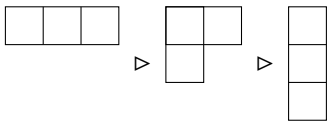
Let's assume, similar to Section 4.3, that R is a field and $v \in R$ is a unit and denote $\mathscr{H} := \mathscr{H}(\mathfrak{S}_3)$. The simple transpositions that generate $\mathfrak{S}_3$ are denoted by $s := (1,2)$ and $t := (2,3)$.

By Proposition 4.1.2 we know that $\mathscr{H}$ has the standard basis

$$\{H_e, H_s, H_t, H_{st}, H_{ts}, H_{sts}\}$$

and also know how to multiply standard basis elements. The complete multiplication table is given in Figure 6.1.

By Example 3.1.2 we know that the partitions of $d = 3$ are totally ordered by the dominance ordering:



In this chapter we will fix a notation for standard tableaux to make the labeling of Murphy basis elements more compact.

A standard tableau $\mathfrak{t}$ is denoted by A decorated with a subscript, containing the shape λ of the tableau, and a superscript, containing the element in $w(\mathfrak{t}) \in \mathfrak{S}_3$ such that $\mathfrak{t} = \mathfrak{t}^\lambda.w(\mathfrak{t})$. By bijection (3.3) the tableau is uniquely determined by this information.

$$A^e_{(3)} := \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ , \quad A^e_{(2,1)} := \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ , \quad A^t_{(2,1)} := \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ , \quad A^e_{(1)^3} := \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \tag{6.1}$$

Recall the Murphy basis of $\mathscr{H}$ calculated in Example 4.2.6:

$$\begin{aligned}
m_{A^e_{(3)}A^e_{(3)}} &= H_e^* m_{\square\square\square} H_e &&= v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1\\
m_{A^e_{(2,1)}A^e_{(2,1)}} &= H_e^* m_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} H_e &&= v^{-1}H_s + 1\\
m_{A^e_{(2,1)}A^t_{(2,1)}} &= v^{-1}H_e^* m_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} H_t &&= v^{-2}H_{st} + v^{-1}H_t\\
m_{A^t_{(2,1)}A^e_{(2,1)}} &= v^{-1}H_t^* m_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} H_e &&= v^{-2}H_{ts} + v^{-1}H_t\\
m_{A^t_{(2,1)}A^t_{(2,1)}} &= v^{-2}H_t^* m_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} H_t &&= v^{-3}H_{sts} + (v^{-2}-1)H_t + v^{-2}\\
m_{A^e_{(1)^3}A^e_{(1)^3}} &= H_e^* m_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} H_e &&= 1
\end{aligned}$$

H_e	H_s	H_t	H_{st}	H_{ts}	H_{sts}	
H_e	H_s	H_t	H_{st}	H_{ts}	H_{sts}	$\cdot H_e$
H_s	$(\nu^{-1}-\nu)H_s$ $+H_e$	H_{ts}	H_{sts}	$(\nu^{-1}-\nu)H_{ts}$ $+H_t$	$(\nu^{-1}-\nu)H_{sts}$ $+H_{st}$	$\cdot H_s$
H_t	H_{st}	$(\nu^{-1}-\nu)H_t$ $+H_e$	$(\nu^{-1}-\nu)H_{st}$ $+H_s$	H_{sts}	$(\nu^{-1}-\nu)H_{sts}$ $+H_{ts}$	$\cdot H_t$
H_{st}	$(\nu^{-1}-\nu)H_{st}$ $+H_t$	H_{sts}	$(\nu^{-1}-\nu)H_{sts}$ $+H_{ts}$	$(\nu^{-1}-\nu)H_{sts}$ $+(\nu^{-1}-\nu)H_t+H_e$	$(\nu^{-1}-\nu)^2H_{sts}$ $+(\nu^{-1}-\nu)H_{ts}$ $+(\nu^{-1}-\nu)H_{st}+H_s$	$\cdot H_{st}$
H_{ts}	H_{sts}	$(\nu^{-1}-\nu)H_{ts}$ $+H_s$	$(\nu^{-1}-\nu)H_{sts}$ $+(\nu^{-1}-\nu)H_s+H_e$	$(\nu^{-1}-\nu)H_{sts}$ $+H_{st}$	$(\nu^{-1}-\nu)^2H_{sts}$ $+(\nu^{-1}-\nu)H_{ts}$ $+(\nu^{-1}-\nu)H_{st}+H_t$	$\cdot H_{ts}$
H_{sts}	$(\nu^{-1}-\nu)H_{sts}$ $+H_{ts}$	$(\nu^{-1}-\nu)H_{sts}$ $+H_{st}$	$(\nu^{-1}-\nu)^2H_{sts}$ $+(\nu^{-1}-\nu)H_{st}$ $+(\nu^{-1}-\nu)H_{ts}+H_t$	$(\nu^{-1}-\nu)^2H_{sts}$ $+(\nu^{-1}-\nu)H_{st}$ $+(\nu^{-1}-\nu)H_{ts}+H_s$	$(\nu^{-3}-2\nu^{-1}+2\nu-\nu^3)H_{sts}$ $+(\nu^{-1}-\nu)^2H_{ts}$ $+(\nu^{-1}-\nu)^2H_{st}$ $+(\nu^{-1}-\nu)H_t$ $+(\nu^{-1}-\nu)H_s+H_e$	$\cdot H_{sts}$

Figure 6.1: Multiplication table for the standard basis of $\mathscr{H}(\mathfrak{S}_3)$.

To verify the cellular property of the Murphy basis in this example it is enough to check it for the action of some basis elements of $\mathscr{H}$. The right action of the standard basis and Murphy basis on the Murphy basis itself is given in Figure 6.2 and Figure 6.3 respectively. One can check the cellular property using both figures. We will use Figure 6.2 to check it for the action on $m_{A^e_{(1)^3}A^e_{(1)^3}}$ and $m_{A^e_{(3)}A^e_{(3)}}$ and Figure 6.3 for the action on the other basis elements.

Like in Example 4.3.2 we have to check that only certain elements appear in each column, that the dominance ordering on partitions is followed and that the same coefficients appear in both columns for each row of the table.

For $m_{A^e_{(1)^3}A^e_{(1)^3}}$ and $m_{A^e_{(3)}A^e_{(3)}}$ we only need to check that acting on them only produces results that are associated to poset elements above them, because there is only one standard tableaux of shape (3) and $(1)^3$ respectively. This is always true for $m_{A^e_{(1)^3}A^e_{(1)^3}}$, because $(1)^3$ is minimal in the poset, and also hold true for $m_{A^e_{(3)}A^e_{(3)}}$, as we can observe in the first column of Figure 6.3.

There are four elements associated to $(2,1)$, making verifying the cellular property a bit more involved. Compare, for example, the results of acting from the right with $m_{A^e_{(2,1)}A^e_{(2,1)}}$ on $m_{A^e_{(2,1)}A^e_{(2,1)}}$ and $m_{A^t_{(2,1)}A^e_{(2,1)}}$, which have the same tableau as second label:

$$m_{A^e_{(2,1)}A^e_{(2,1)}} \cdot m_{A^e_{(2,1)}A^e_{(2,1)}} \overset{\text{Figure 6.3}}{=} (v^{-2}+1)m_{A^e_{(2,1)}A^e_{(2,1)}}$$
$$m_{A^t_{(2,1)}A^e_{(2,1)}} \cdot m_{A^e_{(2,1)}A^e_{(2,1)}} \overset{\text{Figure 6.3}}{=} (v^{-2}+1)m_{A^t_{(2,1)}A^e_{(2,1)}}$$

Both results are a scaled Murphy basis element and both have the same first tableau as as the element that was acted on. In both results the tableaux in the second label is the same and these elements also have the same coefficient $(v^{-2}+1)$. No other basis elements appear in the expression. In particular, $m_{A^e_{(1)^3}A^e_{(1)^3}}$, which is associated to $(1)^3$ below $(2,1)$, does not appear in the expression. Hence, the property we verified for $m_{A^e_{(1)^3}A^e_{(1)^3}}$ and $m_{A^e_{(3)}A^e_{(3)}}$ is also valid here.

Similarly, the cellular property can be verified for all other elements associated to $(2,1)$ and thus for the whole basis.

6.1.2 Right Specht modules and their irreducible quotients

Now we can compute the right A-action on Specht modules of $\mathscr{H}$. They are defined as

$$S^{\ydiagram{3}} = \mathrm{span}_R\left\{m_{\ytableaushort{123}}\right\},\ S^{\ydiagram{2,1}} = \mathrm{span}_R\left\{m_{\ytableaushort{12,3}},\ m_{\ytableaushort{13,2}}\right\},\ S^{\ydiagram{1,1,1}} = \mathrm{span}_R\left\{m_{\ytableaushort{1,2,3}}\right\},$$

where the right $\mathscr{H}$-action on the generators can be inferred from Figures 6.2 and 6.3.

We give an informal description of this process. For an element $m_{\mathfrak{t}} \in S^\lambda$, where $\mathfrak{t} \in \mathrm{Std}(\lambda)$, one picks some $m_{\mathfrak{s}\mathfrak{t}} \in \mathscr{H}^\lambda$ and computes the right action of $\mathscr{H}$ on this element.

$m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(2,1)}A^e_{(2,1)}}$	$m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^e_{(1)^3}A^e_{(1)^3}}$	
$m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(2,1)}A^e_{(2,1)}}$	$m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_e$
$\nu^{-1}m_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-1}m_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}m_{A^t_{(2,1)}A^e_{(2,1)}}$	$\begin{array}{c}\nu m_{A^e_{(3)}A^e_{(3)}}\\ -\nu m_{A^e_{(2,1)}A^e_{(2,1)}}\\ -\nu m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^{-1}m_{A^e_{(3)}A^e_{(3)}}\\ -\nu m_{A^t_{(2,1)}A^e_{(2,1)}}\\ -\nu m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\nu m_{A^e_{(2,1)}A^e_{(2,1)}} - \nu m_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_s$
$\nu^{-1}m_{A^e_{(3)}A^e_{(3)}}$	$\nu m_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu m_{A^t_{(2,1)}A^t_{(2,1)}}$	$\begin{array}{c}\nu^{-1}m_{A^e_{(2,1)}A^e_{(2,1)}}\\ +(\nu^{-1}-\nu)m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^{-1}m_{A^t_{(2,1)}A^e_{(2,1)}}\\ +(\nu^{-1}-\nu)m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^3\Big(-m_{A^e_{(3)}A^e_{(3)}}+m_{A^t_{(2,1)}A^t_{(2,1)}}\\ +m_{A^t_{(2,1)}A^e_{(2,1)}}+m_{A^e_{(2,1)}A^t_{(2,1)}}\\ +m_{A^e_{(2,1)}A^e_{(2,1)}}-\nu^{-2}m_{A^e_{(1)^3}A^e_{(1)^3}}\Big)\end{array}$	$\cdot H_t$
$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$\begin{array}{c}m_{A^e_{(3)}A^e_{(3)}}\\ -m_{A^e_{(2,1)}A^e_{(2,1)}}\\ -m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}\\ -m_{A^t_{(2,1)}A^e_{(2,1)}}\\ -m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^4\Big(m_{A^e_{(3)}A^e_{(3)}}-m_{A^t_{(2,1)}A^t_{(2,1)}}\\ -m_{A^t_{(2,1)}A^e_{(2,1)}}+(\nu^{-2}-1)m_{A^e_{(2,1)}A^t_{(2,1)}}\\ -m_{A^e_{(2,1)}A^e_{(2,1)}}+\nu^{-2}m_{A^e_{(1)^3}A^e_{(1)^3}}\Big)\end{array}$	$\cdot H_{st}$
$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$	$\begin{array}{c}\nu^2 m_{A^e_{(3)}A^e_{(3)}}\\ -\nu^2 m_{A^e_{(2,1)}A^e_{(2,1)}}\\ -\nu^2 m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}m_{A^e_{(3)}A^e_{(3)}}\\ -\nu^2 m_{A^t_{(2,1)}A^e_{(2,1)}}\\ -\nu^2 m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}(1-\nu^2)m_{A^e_{(3)}A^e_{(3)}}\\ +(\nu^{-2}-1+\nu^2)m_{A^e_{(2,1)}A^e_{(2,1)}}\\ -(1-\nu^2)m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}(\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}\\ +(\nu^{-2}-1+\nu^2)m_{A^t_{(2,1)}A^e_{(2,1)}}\\ -(1-\nu^2)m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^4\Big(m_{A^e_{(3)}A^e_{(3)}}-m_{A^t_{(2,1)}A^t_{(2,1)}}\\ +(\nu^{-2}-1)m_{A^t_{(2,1)}A^e_{(2,1)}}-m_{A^e_{(2,1)}A^t_{(2,1)}}\\ -m_{A^e_{(2,1)}A^e_{(2,1)}}+\nu^{-2}m_{A^e_{(1)^3}A^e_{(1)^3}}\Big)\end{array}$	$\cdot H_{ts}$
$\nu^{-3}m_{A^e_{(3)}A^e_{(3)}}$	$\begin{array}{c}\nu m_{A^e_{(3)}A^e_{(3)}}\\ -\nu m_{A^e_{(2,1)}A^e_{(2,1)}}\\ -\nu m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^{-1}m_{A^e_{(3)}A^e_{(3)}}\\ -\nu m_{A^t_{(2,1)}A^e_{(2,1)}}\\ -\nu m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}(\nu^{-1}-\nu)m_{A^e_{(3)}A^e_{(3)}}\\ -(\nu^{-1}-\nu)m_{A^e_{(2,1)}A^e_{(2,1)}}\\ +\nu m_{A^e_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}(\nu^{-3}-\nu^{-1})m_{A^e_{(3)}A^e_{(3)}}\\ -(\nu^{-1}-\nu)m_{A^t_{(2,1)}A^e_{(2,1)}}\\ +\nu m_{A^t_{(2,1)}A^t_{(2,1)}}\end{array}$	$\begin{array}{c}\nu^5\Big((\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}+m_{A^t_{(2,1)}A^t_{(2,1)}}\\ -(\nu^{-2}-1)m_{A^t_{(2,1)}A^e_{(2,1)}}-(\nu^{-2}-1)m_{A^e_{(2,1)}A^t_{(2,1)}}\\ +(\nu^{-2}-1)m_{A^e_{(2,1)}A^e_{(2,1)}}-\nu^{-2}m_{A^e_{(1)^3}A^e_{(1)^3}}\Big)\end{array}$	$\cdot H_{sts}$

Figure 6.2: Multiplication table for Murphy with standard basis of $\mathscr{H}(\mathfrak{S}_3)$.

$m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(2,1)}A^e_{(2,1)}}$	$m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^e_{(1)^3}A^e_{(1)^3}}$	
$(\nu^{-6}+2\nu^{-4}+2\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-2}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-2}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-4}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(3)}A^e_{(3)}}$	$\cdot m_{A^e_{(3)}A^e_{(3)}}$
$(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$(\nu^{-2}+1)m_{A^e_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-2}+1)m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(2,1)}A^e_{(2,1)}}$	$\cdot m_{A^e_{(2,1)}A^e_{(2,1)}}$
$\nu^{-2}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$(\nu^{-2}+1)m_{A^e_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-2}+1)m_{A^t_{(2,1)}A^t_{(2,1)}}$	$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-4}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$\cdot m_{A^e_{(2,1)}A^t_{(2,1)}}$
$\nu^{-2}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^t_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}$ $+(\nu^{-4}+1)m_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-2}(\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}$ $+(\nu^{-4}+1)m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^t_{(2,1)}A^e_{(2,1)}}$	$\cdot m_{A^t_{(2,1)}A^e_{(2,1)}}$
$\nu^{-4}(\nu^{-2}+1)m_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-2}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-4}m_{A^e_{(3)}A^e_{(3)}}$ $-m_{A^t_{(2,1)}A^t_{(2,1)}}$	$\nu^{-2}(\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}$ $+(\nu^{-4}+1)m_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-4}(\nu^{-2}-1)m_{A^e_{(3)}A^e_{(3)}}$ $+(\nu^{-4}+1)m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$\cdot m_{A^t_{(2,1)}A^t_{(2,1)}}$
$m_{A^e_{(3)}A^e_{(3)}}$	$m_{A^e_{(2,1)}A^e_{(2,1)}}$	$m_{A^t_{(2,1)}A^e_{(2,1)}}$	$m_{A^e_{(2,1)}A^t_{(2,1)}}$	$m_{A^t_{(2,1)}A^t_{(2,1)}}$	$m_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot m_{A^e_{(1)^3}A^e_{(1)^3}}$

Figure 6.3: Multiplication table for the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$.

Then one applies the quotient map $\mathscr{H} \to \mathscr{H}/\check{\mathscr{H}}^{\lambda}$ and transforms non-zero quotients of Murphy basis elements back to elements in S^{λ} by dropping the first tableau of each Murphy basis element.

Consider, for example, the result of $m_{A^t_{(2,1)}} \cdot H_{st} \in S^{\ydiagram{2,1}}$. We pick $m_{A^e_{(2,1)}A^t_{(2,1)}} \in \mathscr{H}(\mathfrak{S}_3)$ and compute

$$m_{A^e_{(2,1)}A^t_{(2,1)}} \cdot H_{st} = m_{A^e_{(3)}A^e_{(3)}} - m_{A^e_{(2,1)}A^e_{(2,1)}} - m_{A^e_{(2,1)}A^t_{(2,1)}} \in \mathscr{H}$$

with Figure 6.2. Applying the quotient map $\mathscr{H} \to \mathscr{H}/\check{\mathscr{H}}^{\ydiagram{2,1}}$ kills $m_{A^e_{(3)}A^e_{(3)}}$, so we get

$$-\overline{m_{A^e_{(2,1)}A^e_{(2,1)}}} - \overline{m_{A^e_{(2,1)}A^t_{(2,1)}}} \in \mathscr{H}/\check{\mathscr{H}}^{\ydiagram{2,1}}.$$

This element corresponds, by construction of cell modules in Section 2.1, to $-m_{A^e_{(2,1)}} - m_{A^t_{(2,1)}}$ in the Specht module. Therefore,

$$m_{A^t_{(2,1)}} \cdot H_{st} = -m_{A^e_{(2,1)}} - m_{A^t_{(2,1)}} \in S^{\ydiagram{2,1}}.$$

Using this process we can describe the complete action on each cell module. The results are summarized in Figure 6.4.

The bilinear form on each Specht module is defined in Proposition 2.3.2 as

$$\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle m_{\mathfrak{u}\mathfrak{v}} \equiv m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \bmod \check{\mathscr{H}}^{\lambda}$$

for all $\lambda \vdash 3$, $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)$ and can hence be computed from Figure 6.3. As stated in Example 4.3.5 we have:

$$S^{\ydiagram{3}}: \quad \langle m_{\begin{smallmatrix}1&2&3\end{smallmatrix}}, m_{\begin{smallmatrix}1&2&3\end{smallmatrix}} \rangle = v^{-6} + 2v^{-4} + 2v^{-2} + 1$$

$$S^{\ydiagram{2,1}}: \quad \langle m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}, m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \rangle = v^{-2} + 1 \quad \langle m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}, m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} \rangle = -1$$
$$\langle m_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}, m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \rangle = -1 \quad \langle m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}, m_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} \rangle = v^{-4} + 1$$

$$S^{\ydiagram{1,1,1}}: \quad \langle m_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}, m_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \rangle = 1$$

Using e-restriction we know which Specht modules have irreducible quotients by Proposition 4.3.7. Here we want to verify the classification of irreducibles for $\mathscr{H}(\mathfrak{S}_3)$ by calculating the radicals explicitly. Moreover, we want to compute the irreducible representations and their dimensions in this example.

$m_{A^e_{(3)}}$	$S^{\square\square\square}$
$m_{A^e_{(3)}}$	$\cdot H_e$
$v^{-1}m_{A^e_{(3)}}$	$\cdot H_s$
$v^{-1}m_{A^e_{(3)}}$	$\cdot H_t$
$v^{-2}m_{A^e_{(3)}}$	$\cdot H_{st}$
$v^{-2}m_{A^e_{(3)}}$	$\cdot H_{ts}$
$v^{-3}m_{A^e_{(3)}}$	$\cdot H_{sts}$

$m_{A^e_{(1)^3}}$	$S^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$
$m_{A^e_{(1)^3}}$	$\cdot H_e$
$-vm_{A^e_{(1)^3}}$	$\cdot H_s$
$-vm_{A^e_{(1)^3}}$	$\cdot H_t$
$+v^2m_{A^e_{(1)^3}}$	$\cdot H_{st}$
$+v^2m_{A^e_{(1)^3}}$	$\cdot H_{ts}$
$-v^3m_{A^e_{(1)^3}}$	$\cdot H_{sts}$

$m_{A^e_{(2,1)}}$	$m_{A^t_{(2,1)}}$	$S^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$
$m_{A^e_{(2,1)}}$	$m_{A^t_{(2,1)}}$	$\cdot H_e$
$v^{-1}m_{A^e_{(2,1)}}$	$-vm_{A^e_{(2,1)}} - vm_{A^t_{(2,1)}}$	$\cdot H_s$
$vm_{A^t_{(2,1)}}$	$v^{-1}m_{A^e_{(2,1)}} + (v^{-1}-v)m_{A^t_{(2,1)}}$	$\cdot H_t$
$m_{A^t_{(2,1)}}$	$-m_{A^e_{(2,1)}} - m_{A^t_{(2,1)}}$	$\cdot H_{st}$
$-v^2m_{A^e_{(2,1)}} - v^2m_{A^t_{(2,1)}}$	$(v^{-2}-1+v^2)m_{A^e_{(2,1)}} - (1-v^2)m_{A^t_{(2,1)}}$	$\cdot H_{ts}$
$-vm_{A^e_{(2,1)}} - vm_{A^t_{(2,1)}}$	$-(v^{-1}-v)m_{A^e_{(2,1)}} + vm_{A^t_{(2,1)}}$	$\cdot H_{sts}$

Figure 6.4: Right $\mathscr{H}(\mathfrak{S}_3)$-action on Specht modules.

Proposition 6.1.1. *The quotients of Specht modules of $\mathscr{H}(\mathfrak{S}_3)$ by their radicals are:*

$$D^{\square\square\square} = \begin{cases} \{0\} & , \text{ if } (v^{-2}=1) \wedge (\text{char}R = 2 \text{ or } 3) \\ & \text{or } (v^{-2}\neq 1) \wedge (v^{-2} \text{ is 2nd or 3rd root of unity}) \\ S^{\square\square\square} & , \text{otherwise} \end{cases}$$

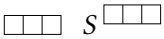
$$D^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \begin{cases} S^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \diagup \text{span}_R\{-v^{-2}m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}}\} & , \begin{array}{l}\text{if } (v^{-2}=1)\wedge \\ (\text{char}R=3) \\ \text{or } (v^{-2}\neq 1)\wedge \\ (v^{-2} \text{ is 3rd root of unity})\end{array} \\ S^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} & , \text{otherwise} \end{cases}$$

$$D^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = S^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$$

Proof. Recall that by Definition 2.3.3 the radical is the submodule

$$\operatorname{rad} S^\lambda = \left\{ x \in S^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in S^\lambda \right\} \subset S^\lambda$$

for some $\lambda \in \Lambda^+(3)$. We consider each partition separately.

 $S^{\ydiagram{3}}$ is one-dimensional, so we have to determine when $v^{-6}+2v^{-4}+2v^{-2}+1$ is 0:

$\underline{v^{-2}=1}$: $6=0 \Leftrightarrow$ char$R=2$ or 3

$\underline{v^{-2}\neq 1}$: Then $v^{-2}-1\neq 0$, so:

$$\begin{aligned} & v^{-6}+2v^{-4}+2v^{-2}+1=0 \\ \Leftrightarrow\quad & (v^{-2}-1)(v^{-6}+2v^{-4}+2v^{-2}+1)=0 \\ \Leftrightarrow\quad & (v^{-6}-1)(v^{-2}+1)=0 \\ \Leftrightarrow\quad & v^{-2} \text{ is } 2nd \text{ or } 3rd \text{ root of unity} \end{aligned}$$

 The Specht module is $S^{\ydiagram{2,1}} = \operatorname{span}_R\{m_{A^e_{(2,1)}}, m_{A^t_{(2,1)}}\}$.

Assume $x = a \cdot m_{A^e_{(2,1)}} + b \cdot m_{A^t_{(2,1)}} \in \operatorname{rad} S^{\ydiagram{2,1}}$, then we have for all $y = c \cdot m_{A^e_{(2,1)}} + d \cdot m_{A^t_{(2,1)}} \in S^{\ydiagram{2,1}}$:

$$\langle x, y\rangle = (v^{-2}+1)ac - ad - bc + (v^{-4}+1)bd = 0 \tag{6.2}$$

Firstly, notice that if $x \neq 0$ then $a \neq 0$ and $b \neq 0$. Indeed, if $a = 0$ then $b \neq 0$ and if we choose $c = 1$ and $d = 0$ then $\langle x, y\rangle = -b \neq 0$. The case $b = 0$ is similar. Thus we have already established that $\operatorname{rad} S^{\ydiagram{2,1}} \neq S^{\ydiagram{2,1}}$.

Secondly, we test (6.2) with element $y = m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}}$, so $c = d = 1$, to get information about a, b and v. In this case, (6.2) is

$$(v^{-2}+1)a - a - b + (v^{-4}+1)b = 0\,,$$

which reduces to $a = -v^{-2}b$. This already gives us an element spanning the radical, should a non-zero radical exist. Plug $a = -v^{-2}b$ back into (6.2), rearrange the terms and get that

$$(v^{-4}+v^{-2}+1)d = (v^{-4}+v^{-2}+1)c$$

has to hold for all $c, d \in R$. Therefore, a non-zero radical exists if and only if $v^{-4}+v^{-2}+1 = 0$, so either $v^{-2} = 1$ and char$R = 3$, or $v^{-2} \neq 1$ and v^{-2} is a 3rd root of unity. In these cases, the radical is spanned by $-v^{-2}m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}}$.

 $S^{\ydiagram{1,1,1}}$ is one-dimensional and $\langle m_{A^e_{(1)^3}}, m_{A^e_{(1)^3}}\rangle = 1$, so $\operatorname{rad} S^{\ydiagram{1,1,1}} = \{0\}$.

□

By Section 2.3 the non-zero quotients in Proposition 6.1.1 are the irreducible representations of $\mathscr{H}(\mathfrak{S}_3)$. This result agrees with the classification in Example 4.3.8, where we used the criterion of e-restriction. Additionally, we have a complete description of the $\mathscr{H}$-action on the irreducible representations and their dimension.

In particular, if $(v^{-2}=1)\wedge(\mathrm{char}R=3)$ or $(v^{-2}\neq 1)\wedge(v^{-2}$ is 3rd root of unity$)$, there is no two-dimensional representation of $\mathscr{H}(\mathfrak{S}_3)$.

6.1.3 Identifying the trivial and sign representation

Before we end this section about the Murphy basis we use Proposition 6.1.1 to identify which quotients correspond to known irreducible representations of $\mathscr{H}$.

The trivial and sign representation defined in Remark 4.1.3 are one-dimensional and thus irreducible $\mathscr{H}$-modules. By the classification of irreducible $\mathscr{H}$-modules in Proposition 4.3.7 they must be isomorphic to some irreducible quotient of some Specht module.

This leads to a natural question: to which quotients do they correspond exactly? We need our explicit calculations from above, the general classification result is not enough to determine this identification.

We approach this question for our example using the action on Specht modules from Figure 6.4 and Proposition 6.1.1.

The irreducible module $D^{\ydiagram{1,1,1}}$ is the sign representation. Indeed, $S^{\ydiagram{1,1,1}}$ is always irreducible and by Figure 6.4 H_s and H_t act by $-v$ on the basis element of $D^{\ydiagram{1,1,1}}$.

Identifying the trivial representation seems straightforward as well. By Figure 6.4 we expect $D^{\ydiagram{3,1}}$ to be the trivial representation. But we know from Proposition 6.1.1 that $D^{\ydiagram{3}}$ is not always irreducible. For these cases we have to identify the trivial representation with another quotient.

We start with the case of $(v^{-2}=1)\wedge(\mathrm{char}R=2 \text{ or } 3)$. If $(v=-1)$, then the trivial representation is obviously the sign representation, so $D^{\ydiagram{1,1,1}}$. This also includes the case of $(v=1)\wedge(\mathrm{char}R=2)$, because then also $(v=-1)$.

For $(v=1)\wedge(\mathrm{char}R=3)$ or $(v^{-2}\neq 1)\wedge(v^{-2}$ is 2nd or 3rd root of unity$)$ we have $D^{\ydiagram{2,1}}=S^{\ydiagram{2,1}}/\mathrm{span}_R\{-v^{-2}m_{A^e_{(2,1)}}+m_{A^t_{(2,1)}}\}$ by Proposition 6.1.1. Using the relation in $D^{\ydiagram{2,1}}$, denoted by $\diamond$, we see for the action of the standard generators on the basis element $\overline{m_{A^e_{(2,1)}}}$ of $D^{\ydiagram{2,1}}$:

$$\begin{aligned}\overline{m_{A^e_{(2,1)}}}\cdot H_s &= v^{-1}\overline{m_{A^e_{(2,1)}}}\\ \overline{m_{A^e_{(2,1)}}}\cdot H_t = v\overline{m_{A^t_{(2,1)}}} &\overset{\diamond}{=} v^{-1}\overline{m_{A^e_{(2,1)}}}\end{aligned}$$

Thus, in these cases the trivial representation is $D^{\ydiagram{2,1}}$.

We summarize our results in the following table:

	trivial representation	sign representation
$(v=-1)\wedge(\mathrm{char}R=2 \text{ or } 3)$ including $(v=1)\wedge(\mathrm{char}R=2)$	D 	D 
$(v=1)\wedge(\mathrm{char}R=3)$ or $(v^{-2}\neq 1)\wedge(v^{-2}$ 2nd or 3rd root of unity)	D 	D 
otherwise	D 	D 

6.2 Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$

Like in Section 6.1 we can verify the cellular property of the Kazhdan–Lusztig basis for $\mathscr{H}(\mathfrak{S}_3)$ using explicit calculations. This is also a good opportunity to demonstrate the Robinson–Schensted correspondence from Section 4.6.2 once more. Our computations show that, in general, the Kazhdan–Lusztig basis differs from the Murphy basis, although parts of the cell data agree.

We begin this section with the cell datum associated to the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$. In particular, we calculate the complete Robinson–Schensted correspondence from Proposition 4.6.2 for $\mathfrak{S}_3$. Using this result we relabel the basis elements, which were already stated in Example 4.6.1. Afterwards we compute the irreducible right $\mathscr{H}(\mathfrak{S}_3)$-modules once more, this time using the Kazhdan–Lusztig cell datum. We finish this section with a brief comparison between the Kazhdan–Lusztig and Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$.

6.2.1 Constructing the cell datum

Recall that simple transpositions $s=(1,2)$ and $t=(2,3)$ generate $\mathfrak{S}_3$ as a group. The standard basis of $\mathscr{H}(\mathfrak{S}_3)$ consists of elements

$$\{H_e,\ H_s,\ H_t,\ H_{st},\ H_{ts},\ H_{sts}\}\ .$$

The Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ consists of elements

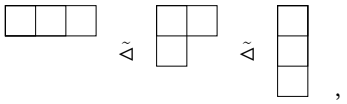
$$\left\{\underline{H}_e,\ \underline{H}_s,\ \underline{H}_t,\ \underline{H}_{st},\ \underline{H}_{ts},\ \underline{H}_{sts}\right\}\ ,$$

which were stated in Example 4.6.1 and will be recalled below.

We will now gather the complete cell datum associated to the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$. Its poset $(\Lambda^+(3),\tilde{\triangleright})$ consists of three elements

$$(3)\ \tilde{\triangleleft}\ (2,1)\ \tilde{\triangleleft}(1,1,1)=(1)^3\quad , \tag{6.3}$$

or as diagrams



that are partially ordered by the reverse dominance ordering as indicated.

The standard tableaux with these shapes also appeared as labels in the Murphy basis. Recall the compact notation for them from (6.1) in Section 6.1:

$$A^e_{(3)} := \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ , \quad A^e_{(2,1)} := \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ , \quad A^t_{(2,1)} := \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ , \quad A^e_{(1)^3} := \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

Recall that this notation was based on the action of $\mathfrak{S}_3$ on the tableau $\mathfrak{t}^\lambda$ that produces the standard tableau for $\lambda \in \Lambda^+(3)$.

To label the Kazhdan–Lusztig elements with ordered pairs of standard tableaux of the same shape we apply the Robinson–Schensted correspondence 4.6.2 to each element in $\mathfrak{S}_3$.

For another example of the construction of the ordered pair of standard tableaux $(P(w), Q(w))$ for a $w \in \mathfrak{S}_3$, consider the element $w = st \in \mathfrak{S}_3$. First, we need to express w as a word $w_1w_2w_3$ with $w_i = w(i)$ for $i = 1, 2, 3$. For our example it is $st = 231$, because:

$$\begin{aligned} 1 &\xrightarrow{t(\cdot)} 1 \xrightarrow{s(\cdot)} 2\\ 2 &\xrightarrow{t(\cdot)} 3 \xrightarrow{s(\cdot)} 3\\ 3 &\xrightarrow{t(\cdot)} 2 \xrightarrow{s(\cdot)} 1 \end{aligned}$$

Here is the stepwise construction of the pair $(P(w), Q(w))$:

$$P^{(0)}(w) = \emptyset \overset{②}{\rightsquigarrow} \begin{array}{|c|}\hline 2\\\hline\end{array} \overset{③}{\rightsquigarrow} \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \overset{①}{\rightsquigarrow} \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = P(w)$$

$$Q^{(0)}(w) = \emptyset \rightsquigarrow \begin{array}{|c|}\hline 1\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = Q(w)$$

The circled number indicates the label added to the standard tableau left of it using the row bumping algorithm. First, 2 is added in a new box to the first line of the trivial standard tableau. Then, because $3 > 2$, 3 is added in a new box at the end of the first row. Lastly, because $1 < 2$, 1 displaces 2 to the second row, adding a new box. The resulting standard tableau is $P(w)$. By construction, $Q(w)$ tracks when each box in $P(w)$ is created and labels them accordingly.

Here are the ordered pairs of standard tableaux of the same shape corresponding to each element in $\mathfrak{S}_3$:

$$\begin{aligned}
e &\sim \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) &&= \left(A^e_{(3)}, A^e_{(3)}\right)\\
s &\sim \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) &&= \left(A^t_{(2,1)}, A^t_{(2,1)}\right)\\
t &\sim \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) &&= \left(A^e_{(2,1)}, A^e_{(2,1)}\right)\\
st &\sim \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) &&= \left(A^t_{(2,1)}, A^e_{(2,1)}\right)\\
ts &\sim \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) &&= \left(A^e_{(2,1)}, A^t_{(2,1)}\right)\\
sts &\sim \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array}\,,\,\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\cline{1-1}\end{array}\right) &&= \left(A^e_{(1)^3}, A^e_{(1)^3}\right)
\end{aligned}$$

We can immediately observe the results of Proposition 4.6.8 in our example $\mathfrak{S}_3$. The elements $e, s, t, sts \in \mathfrak{S}_3$ are self-inverse, so they correspond to a pair with the same tableau in both entries. The elements $st, ts \in \mathfrak{S}_3$ are inverse to each other, so the pairs they correspond to are mirrored to each other, meaning we get one pair by swapping the entries of the other pair.

With the established Robinson–Schensted correspondence we now state the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ with the labeling used to highlight its cellular structure:

$$\begin{aligned}
H_{A^e_{(3)}A^e_{(3)}} &= H_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} &&= \underline{H}_e &&= 1\\
H_{A^t_{(2,1)}A^t_{(2,1)}} &= H_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} &&= \underline{H}_s &&= H_s + v\\
H_{A^e_{(2,1)}A^e_{(2,1)}} &= H_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} &&= \underline{H}_t &&= H_t + v\\
H_{A^t_{(2,1)}A^e_{(2,1)}} &= H_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} &&= \underline{H}_{st} &&= H_{st} + vH_s + vH_t + v^2\\
H_{A^e_{(2,1)}A^t_{(2,1)}} &= H_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} &&= \underline{H}_{ts} &&= H_{ts} + vH_s + vH_t + v^2\\
H_{A^e_{(1)^3}A^e_{(1)^3}} &= H_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\cline{1-2}\end{array}} &&= \underline{H}_{sts} &&= H_{sts} + vH_{st} + vH_{ts} + v^2H_s + v^2H_t + v^3
\end{aligned}$$

Our next goal is to verify the cellular property of the Kazhdan–Lusztig basis in this example. It is enough to check it for the right action of a basis of $\mathscr{H}(\mathfrak{S}_3)$ on the Kazhdan–Lusztig elements. Of course, the result must be expressed in terms of the Kazhdan–Lusztig basis.

Recall the multiplication table for the standard basis from Figure 6.1. With this table we can compute the action of the standard basis on the Kazhdan–Lusztig basis. It is summarized in Figure 6.5. Multiplication of two Kazhdan–Lusztig basis elements is summarized in Figure 6.6.

Now we can check the cellular property in our example. For $H_{A^e_{(3)}A^e_{(3)}}$ and $H_{A^e_{(1)^3}A^e_{(1)^3}}$ we only need to check that elements from poset elements above the respective element appear, because there is only one standard tableau each with shape (3) and $(1)^3$. From (6.3) we know that partition (3) is minimal in the reversed Bruhat ordering, so the above is always

$H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^t_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$	
$H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^t_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_e$
$H_{A^t_{(2,1)}A^t_{(2,1)}} - \nu H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}} - \nu H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}} + H_{A^t_{(2,1)}A^t_{(2,1)}} - \nu H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-1}H_{A^t_{(2,1)}A^t_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_s$
$H_{A^e_{(2,1)}A^t_{(2,1)}} - \nu H_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-1}H_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^e_{(2,1)}A^t_{(2,1)}} + H_{A^e_{(2,1)}A^e_{(2,1)}}$	$-\nu H_{A^t_{(2,1)}A^t_{(2,1)}} + H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_t$
$H_{A^t_{(2,1)}A^e_{(2,1)}} - \nu H_{A^t_{(2,1)}A^t_{(2,1)}} - \nu H_{A^e_{(2,1)}A^e_{(2,1)}} + \nu^2 H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^t_{(2,1)}A^t_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - H_{A^e_{(2,1)}A^t_{(2,1)}} + \nu^{-1}H_{A^e_{(2,1)}A^e_{(2,1)}}$	$-H_{A^t_{(2,1)}A^t_{(2,1)}} + \nu^{-1}H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-2}H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{st}$
$H_{A^e_{(2,1)}A^t_{(2,1)}} - \nu H_{A^t_{(2,1)}A^t_{(2,1)}} - \nu H_{A^e_{(2,1)}A^e_{(2,1)}} + \nu^2 H_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-1}H_{A^e_{(2,1)}A^t_{(2,1)}} - H_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} + \nu^{-1}H_{A^t_{(2,1)}A^t_{(2,1)}} - H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-2}H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{ts}$
$H_{A^e_{(1)^3}A^e_{(1)^3}} - \nu H_{A^e_{(2,1)}A^t_{(2,1)}} - \nu H_{A^t_{(2,1)}A^e_{(2,1)}} + \nu^2 H_{A^t_{(2,1)}A^t_{(2,1)}} + \nu^2 H_{A^t_{(2,1)}A^e_{(2,1)}} - \nu^3 H_{A^e_{(3)}A^e_{(3)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - H_{A^e_{(2,1)}A^t_{(2,1)}}$	$\nu^{-2}H_{A^e_{(1)^3}A^e_{(1)^3}} - H_{A^t_{(2,1)}A^t_{(2,1)}}$	$\nu^{-2}H_{A^e_{(1)^3}A^e_{(1)^3}} - H_{A^e_{(2,1)}A^e_{(2,1)}}$	$\nu^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - H_{A^t_{(2,1)}A^e_{(2,1)}}$	$\nu^{-3}H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{sts}$

Figure 6.5: Multiplication table for Kazhdan–Lusztig with standard basis of $\mathscr{H}(\mathfrak{S}_3)$.

$H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^t_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$	
$H_{A^e_{(3)}A^e_{(3)}}$	$H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^t_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{A^e_{(3)}A^e_{(3)}}$
$H_{A^e_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^e_{(2,1)}A^e_{(2,1)}}$	$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)^2H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{A^e_{(2,1)}A^e_{(2,1)}}$
$H_{A^t_{(2,1)}A^e_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^e_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^t_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+(\nu^{-1}+\nu)H_{A^e_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^t_{(2,1)}A^e_{(2,1)}}$	$(\nu^{-1}+\nu)^2H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{A^t_{(2,1)}A^e_{(2,1)}}$
$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+(\nu^{-1}+\nu)H_{A^t_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^t_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)^2H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{A^e_{(2,1)}A^t_{(2,1)}}$
$H_{A^t_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(2,1)}A^t_{(2,1)}}$	$H_{A^e_{(1)^3}A^e_{(1)^3}}$ $+H_{A^t_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^t_{(2,1)}A^t_{(2,1)}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\cdot H_{A^t_{(2,1)}A^t_{(2,1)}}$
$H_{A^e_{(1)^3}A^e_{(1)^3}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$	$(\nu^{-1}+\nu)^2H_{A^e_{(1)^3}A^e_{(1)^3}}$	$(\nu^{-1}+\nu)^2H_{A^e_{(1)^3}A^e_{(1)^3}}$	$(\nu^{-1}+\nu)H_{A^e_{(1)^3}A^e_{(1)^3}}$	$\begin{matrix}(\nu^{-3}+2\nu^{-2}\\ +2\nu^{-1}+1)H_{A^e_{(1)^3}A^e_{(1)^3}}\end{matrix}$	$\cdot H_{A^e_{(1)^3}A^e_{(1)^3}}$

Figure 6.6: Multiplication table for the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$.

satisfied for $H_{A^e_{(3)}A^e_{(3)}}$. As $(1)^3$ is maximal, for the above property the action on $H_{A^e_{(1)^3},A^e_{(1)^3}}$ must be linear. This is easily seen in Figures 6.5 and 6.6.

This property also holds for the other four elements, because the single element $H_{A^e_{(3)}A^e_{(3)}}$ from the poset element below $(2,1)$ does not appear in columns two to five in Figures 6.5 and 6.6.

Additionally, acting on those elements and expressing the result in the Kazhdan–Lusztig basis only produces elements from the poset element $(2,1)$, which have the same first label as the element we started with. For example, consider

$$H_{A^t_{(2,1)}A^e_{(2,1)}} \cdot H_{ts} = v^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} + v^{-1}H_{A^t_{(2,1)}A^t_{(2,1)}} - H_{A^t_{(2,1)}A^e_{(2,1)}} \tag{6.4}$$

from Figure 6.5. Only elements $H_{A^t_{(2,1)}A^t_{(2,1)}}$ and $H_{A^t_{(2,1)}A^e_{(2,1)}}$ from the poset element $(2,1)$ appear in the expression. These have the same first label as the element $H_{A^t_{(2,1)}A^e_{(2,1)}}$ we started with.

Finally, if we fix the second label, the coefficients appearing in the expression should not depend on the first label. Compare the example in (6.4) above to

$$H_{A^e_{(2,1)}A^e_{(2,1)}} \cdot H_{ts} = v^{-1}H_{A^e_{(2,1)}A^t_{(2,1)}} - H_{A^e_{(2,1)}A^e_{(2,1)}}$$

from Figure 6.5. Coefficients v^{-1} and -1 appear in both expressions. The elements they are associated to have the same second label, which verifies the cellular property for these elements and this one action. Note that we do not compare the coefficients in front of $H_{A^e_{(1)^3}A^e_{(1)^3}}$, because this element is not from the poset element $(2,1)$ and thus vanishes in the quotient.

With Figures 6.5 and 6.6 one can verify the cellular property for all elements in the poset element $(2,1)$ in a similar fashion, thus for the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$.

6.2.2 Right cell modules and their irreducible quotients

Having established the cell datum associated to the Kazhdan–Lusztig basis we can now construct the right cell modules for this basis. We have already done this for the Murphy basis in Section 6.1, where cell modules were called Specht modules. The approach is the same in both cases. Note that the cell modules below were called cell representations by Kazhdan and Lusztig in [KL79].

Cell modules are, as R-modules, defined as:

$$C^{\ydiagram{3}} = \mathrm{span}_R\{H_{A^e_{(3)}}\},\ C^{\ydiagram{2,1}} = \mathrm{span}_R\{H_{A^e_{(2,1)}}, H_{A^t_{(2,1)}}\},\ C^{\ydiagram{1,1,1}} = \mathrm{span}_R\{H_{A^e_{(1)^3}}\}$$

The right $\mathscr{H}(\mathfrak{S}_3)$-action can be inferred from Figure 6.6 or 6.5. This process was already described in Section 6.1 for the Murphy basis. Here we do one more example of this process.

$H_{A^e_{(3)}}$	$C\square\square\square$
$H_{A^e_{(3)}}$	$\cdot H_e$
$-vH_{A^e_{(3)}}$	$\cdot H_s$
$-vH_{A^e_{(3)}}$	$\cdot H_t$
$+v^2H_{A^e_{(3)}}$	$\cdot H_{st}$
$+v^2H_{A^e_{(3)}}$	$\cdot H_{ts}$
$-v^3H_{A^e_{(3)}}$	$\cdot H_{sts}$

$H_{A^e_{(1)^3}}$	$C\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$
$H_{A^e_{(1)^3}}$	$\cdot H_e$
$v^{-1}H_{A^e_{(1)^3}}$	$\cdot H_s$
$v^{-1}H_{A^e_{(1)^3}}$	$\cdot H_t$
$v^{-2}H_{A^e_{(1)^3}}$	$\cdot H_{st}$
$v^{-2}H_{A^e_{(1)^3}}$	$\cdot H_{ts}$
$v^{-3}H_{A^e_{(1)^3}}$	$\cdot H_{sts}$

$H_{A^e_{(2,1)}}$	$H_{A^t_{(2,1)}}$	$C\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$
$H_{A^e_{(2,1)}}$	$H_{A^t_{(2,1)}}$	$\cdot H_e$
$H_{A^t_{(2,1)}} - vH_{A^e_{(2,1)}}$	$v^{-1}H_{A^t_{(2,1)}}$	$\cdot H_s$
$v^{-1}H_{A^e_{(2,1)}}$	$H_{A^e_{(2,1)}} - vH_{A^t_{(2,1)}}$	$\cdot H_t$
$-vH_{A^t_{(2,1)}}$	$v^{-1}H_{A^e_{(2,1)}} - H_{A^t_{(2,1)}}$	$\cdot H_{st}$
$v^{-1}H_{A^t_{(2,1)}} - H_{A^e_{(2,1)}}$	$-vH_{A^e_{(2,1)}}$	$\cdot H_{ts}$
$-H_{A^t_{(2,1)}}$	$-H_{A^e_{(2,1)}}$	$\cdot H_{sts}$

Figure 6.7: Right $\mathscr{H}(\mathfrak{S}_3)$-action on cell modules from the Kazhdan–Lusztig basis.

Say we want to compute

$$H_{A^e_{(2,1)}} \cdot H_{st} \ .$$

We pick a representative of $H_{A^e_{(2,1)}}$ in $\mathscr{H}(\mathfrak{S}_3)$, say $H_{A^t_{(2,1)}A^e_{(2,1)}}$, and compute the action on this element instead. We can look this computation up in Figure 6.5:

$$H_{A^t_{(2,1)}A^e_{(2,1)}} \cdot H_{st} = v^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}} - vH_{A^t_{(2,1)}A^t_{(2,1)}}$$

First, we drop $v^{-1}H_{A^e_{(1)^3}A^e_{(1)^3}}$ from this expression, because it is not associated to $(2,1)$ and will vanish in the quotient. Then we drop the first label of the remaining $-vH_{A^t_{(2,1)}A^t_{(2,1)}}$ and arrive at:

$$H_{A^e_{(2,1)}} \cdot H_{st} = \ -vH_{A^t_{(2,1)}}$$

The complete right $\mathscr{H}(\mathfrak{S}_3)$-action on the cell modules is summarized in Figure 6.7.

We can now state the bilinear form on right cell modules by using the multiplication table in Figure 6.6 for the Kazhdan–Lusztig elements or the explicitly given right action on the cell modules in Figure 6.7. Recall that by Proposition 2.3.2 the bilinear form on a cell module C^λ for $\lambda \in \Lambda^+(3)$ is defined by

$$\langle H_{\mathfrak{s}}, H_{\mathfrak{t}} \rangle H_{\mathfrak{u}\mathfrak{v}} \equiv H_{\mathfrak{u}\mathfrak{s}} H_{\mathfrak{t}\mathfrak{v}} \bmod \check{\mathscr{H}}^\lambda(\mathfrak{S}_3)$$

for all $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)$. For example, to get $\langle H_{A^e_{(2,1)}}, H_{A^t_{(2,1)}} \rangle$ we compute

$$H_{A^e_{(2,1)}, A^e_{(2,1)}} \cdot H_{A^t_{(2,1)}, A^e_{(2,1)}} \equiv H_{A^e_{(2,1)}, A^e_{(2,1)}} \bmod \check{\mathscr{H}}^{(2,1)}(\mathfrak{S}_3)$$

with Figure 6.6 and get

$$\langle H_{A^e_{(2,1)}}, H_{A^t_{(2,1)}} \rangle = 1 \,.$$

Alternatively, we could have taken these from certain coefficients in Figure 6.7, but this approach is more error prone.

Here is the bilinear form on each cell module:

$C^{\ydiagram{3}}$: $\langle H_{A^e_{(3)}}, H_{A^e_{(3)}} \rangle = 1$

$C^{\ydiagram{2,1}}$: $\langle H_{A^e_{(2,1)}}, H_{A^e_{(2,1)}} \rangle = v^{-1} + v$ $\quad \langle H_{A^t_{(2,1)}}, H_{A^e_{(2,1)}} \rangle = 1$

$\langle H_{A^e_{(2,1)}}, H_{A^t_{(2,1)}} \rangle = 1$ $\quad \langle H_{A^t_{(2,1)}}, H_{A^t_{(2,1)}} \rangle = v^{-1} + v$

$C^{\ydiagram{1,1,1}}$: $\langle H_{A^e_{(1)^3}}, H_{A^e_{(1)^3}} \rangle = v^{-3} + 2v^{-2} + 2v^{-1} + 1$

Note that we already see from this overview that $D^{\ydiagram{2,1}}$ will never be trivial, because $H_{A^e_{(2,1)}}$ and $H_{A^t_{(2,1)}}$ are not in the radical.

Finally, we can state the irreducible representations of $\mathscr{H}(\mathfrak{S}_3)$ once more, this time using the cell modules coming from the Kazhdan–Lusztig basis. To apply the classification result from Proposition 2.3.12, we need to take the quotient of each cell module by its radical and identify the non-zero quotients. These are decorated with $\sim$ to differentiate them from the quotients from the Murphy basis in Section 6.1.

The computations are summarized in the following proposition.

Proposition 6.2.1. *The quotients of cell modules associated to the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_3)$ by their radicals are:*

$$\tilde{D}^{\ydiagram{3}} = C^{\ydiagram{3}}$$

$$\tilde{D}^{\ydiagram{2,1}} = \begin{cases} C^{\ydiagram{2,1}} \big/ \mathit{span}_R\{H_{A^e_{(2,1)}} - H_{A^t_{(2,1)}}\} & , \begin{array}{l}\mathit{if}\ (-v^{-1}=1)\wedge(\mathit{charR}=3)\\ \mathit{or}\ (-v^{-1}\neq 1)\wedge(-v^{-1}\ \mathit{is\ 3rd\ root\ of\ unity})\end{array} \\ C^{\ydiagram{2,1}} \big/ \mathit{span}_R\{H_{A^e_{(2,1)}} + H_{A^t_{(2,1)}}\} & , \begin{array}{l}\mathit{if}\ (v^{-1}=1)\wedge(\mathit{charR}=3)\\ \mathit{or}\ (v^{-1}\neq 1)\wedge(v^{-1}\ \mathit{is\ 3rd\ root\ of\ unity})\end{array} \\ C^{\ydiagram{2,1}} & , \mathit{otherwise} \end{cases}$$

$$\tilde{D}^{\ydiagram{1,1,1}} = \begin{cases} \{0\} & , \begin{array}{l}\mathit{if}\ (v^{-1}=1)\wedge(\mathit{charR}=2\ \mathit{or}\ 3)\\ \mathit{or}\ (v^{-1}\neq 1)\wedge(v^{-1}\ \mathit{is\ 2nd\ or\ 3rd\ root\ of\ unity})\end{array} \\ C^{\ydiagram{3}} & , \mathit{otherwise} \end{cases}$$

Proof. Recall that by Definition 2.3.3 the radical is the submodule

$$\operatorname{rad} C^\lambda = \left\{x \in C^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in C^\lambda\right\} \subset C^\lambda$$

for some $\lambda \in \Lambda^+(3)$. We consider each partition separately, just like we did for the Murphy basis in Proposition 6.1.1.

$\ydiagram{3}$ The cell module $C^{\ydiagram{3}}$ is one-dimensional and $\langle H_{A^e_{(3)}}, H_{A^e_{(3)}}\rangle = 1$, so the radical only contains 0.

$\ydiagram{2,1}$ Assume that $x = a \cdot H_{A^e_{(2,1)}} + b \cdot H_{A^t_{(2,1)}} \in \operatorname{rad} C^{\ydiagram{3}}$ for some $a, b \in R$. Then for all $y = c \cdot H_{A^e_{(2,1)}} + d \cdot H_{A^t_{(2,1)}} \in C^{\ydiagram{3}}$, so for all $c, d \in R$, we have:

$$\langle x, y\rangle = (v^{-1}+v)ac + ad + bc + (v^{-1}+v)bd = 0 \tag{6.5}$$

As in the proof of Proposition 6.1.1, we first notice that if $x \neq 0$, then $a \neq 0$ and $b \neq 0$. Indeed, if $a = 0$ then $b \neq 0$ as $x \neq 0$. But if we test (6.5) with $c = 1$ and $d = 0$ we get $b = 0$, a contradiction. So $a \neq 0$, and similarly $b \neq 0$. Therefore, we already know that $\operatorname{rad} C^{\ydiagram{2,1}} \neq C^{\ydiagram{2,1}}$, so the quotient module $D^{\ydiagram{2,1}}$ is always irreducible.

To get more information about a, b and v we test (6.5) with the element $y = H_{A^e_{(2,1)}} + H_{A^t_{(2,1)}}$, so $c = d = 1$, rearrange the terms and get:

$$(v^{-1}+v+1)(a+b) = 0$$

Thus, a non-zero radical can only exist, if either $v^{-1} + v + 1 = 0$ or $a = -b$. We consider each case individually.

$\underline{a=-b}$: We can plug $a=-b$ into (6.5):

$$\begin{aligned}&(v^{-1}+v)(-b)c-bd+bc+(v^{-1}+v)bd=0\\ \overset{b\neq 0}{\Leftrightarrow}&(v^{-1}+v-1)(d-c)=0\end{aligned}$$

This needs to hold for all $c,d\in R$, so $(v^{-1}+v-1)=0$, which is equivalent to $(v^{-2}-v^{-1}+1)=0$, as v is invertible. Hence, either $-v^{-1}=1$ and $\mathrm{char}R=3$, or $-v^{-1}\neq 1$ and $-v^{-1}$ is a 3rd root of unity. In these cases the radical is spanned by $H_{A^e_{(2,1)}}-H_{A^t_{(2,1)}}$ as a R-module.

$\underline{\substack{v^{-1}+v\\+1=0}}$: This assumption is equivalent to $(v^{-2}+v^{-1}+1)=0$, so either $v^{-1}=1$ and $\mathrm{char}R=3$, or $v^{-1}\neq 1$ and v^{-1} is a 3rd root of unity.
To find a non-zero element of the radical in these cases, we once again test (6.5), this time with $c=0$ and $d=1$:

$$a+b(v^{-1}+v)=0$$

By our assumption $v^{-1}+v=-1$, so we must have $a=b$. Thus, the radical is spanned by $H_{A^e_{(2,1)}}+H_{A^t_{(2,1)}}$ as a R-module.

$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$: The cell module $C^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ is one-dimensional, so we only have to check when $v^{-3}+2v^{-2}+2v^{-1}+1$ is 0. This works exactly like a case in Proposition 6.1.1:

$\underline{v^{-1}=1}$: $6=0\Leftrightarrow \mathrm{char}R=2$ or 3

$\underline{v^{-1}\neq 1}$: Then $v^{-1}-1\neq 0$, so:

$$\begin{aligned}&v^{-3}+2v^{-2}+2v^{-1}+1=0\\ \Leftrightarrow&(v^{-1}-1)(v^{-3}+2v^{-2}+2v^{-1}+1)=0\\ \Leftrightarrow&(v^{-3}-1)(v^{-1}+1)=0\\ \Leftrightarrow&v^{-1}\text{ is 2nd or 3rd root of unity}\end{aligned}$$

□

Note that, just like we saw in the explicit computations for the Hecke algebra in Proposition 6.1.1, there is no two-dimensional irreducible module if $(v^{-2}=1)\wedge(\mathrm{char}R=3)$ or $(v^{-2}\neq 1)\wedge(v^{-2}$ is 3rd root of unity$)$.

In Remark 4.6.10 we referred to result [GL96, Theorem 3.8] of Graham and Lehrer that characterizes semisimple cellular algebras by their cell modules. In particular, if $\mathscr{H}$ is semisimple then all right cell modules are irreducible. From this abstract result we know that cell modules from the Kazhdan–Lusztig basis must be isomorphic to cell modules from the Murphy basis.

Before ending this section we want to give explicit isomorphisms between these cell modules in our example.

Proposition 6.2.2. *Let R be an algebraically closed field and assume that $\mathscr{H}(\mathfrak{S}_3)$ is semisimple. Then there are the following isomorphisms of right $\mathscr{H}(\mathfrak{S}_3)$-modules between cell modules from the Kazhdan–Lusztig basis and cell modules from the Murphy basis:*

$$
\begin{aligned}
C^{\ydiagram{3}} &\xrightarrow{\sim} S^{\ydiagram{1,1,1}}, & H_{A^e_{(3)}} &\mapsto m_{A^e_{(1)^3}}\\
C^{\ydiagram{1,1,1}} &\xrightarrow{\sim} S^{\ydiagram{3}}, & H_{A^e_{(1)^3}} &\mapsto m_{A^e_{(3)}}\\
C^{\ydiagram{2,1}} &\xrightarrow{\sim} S^{\ydiagram{2,1}}, & H_{A^e_{(2,1)}} &\mapsto v(m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}})\\
& & H_{A^t_{(2,1)}} &\mapsto m_{A^e_{(2,1)}}
\end{aligned}
$$

Proof. For the first two maps we can see in Figures 6.4 and 6.7 that the modules on both sides are the sign and the trivial representation respectively.

For the last map note that H_s acts on $H_{A^t_{(2,1)}} \in C^{\ydiagram{2,1}}$ by v^{-1} and H_t acts on $H_{A^e_{(2,1)}} \in C^{\ydiagram{2,1}}$ also by v^{-1}. The idea is to find elements in $S^{\ydiagram{2,1}}$ on which H_s and H_t act similarly.

For H_s they are of the form $a_1 m_{A^e_{(2,1)}}$ and for H_t they are of the form $a_2(m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}})$ for all $a_1, a_2 \in R$. Hence, our candidate for the isomorphism assigns

$$H_{A^t_{(2,1)}} \mapsto a_1 m_{A^e_{(2,1)}} \text{ and } H_{A^e_{(2,1)}} \mapsto a_2(m_{A^e_{(2,1)}} + m_{A^t_{(2,1)}}) \ .$$

Now a_1 and a_2 need to be chosen such that this map is a right $\mathscr{H}(\mathfrak{S}_3)$-module isomorphism. We check the action of H_s and H_t on both sides and conclude, after some calculation, that $a_1 = 1$ and $a_2 = v$. □

6.3 Semistandard basis of $\mathscr{S}(2,3)$

In Section 5.2 we recalled the semistandard basis of Schur algebras from [Mat06]. There we did not provide many examples for this construction, which we want to rectify in this section.

Here we give a detailed description of the semistandard basis of $\mathscr{S}(2,3)$, including the construction of all basis elements, their multiplication, verifying the cellular property and calculating the irreducible representations of $\mathscr{S}(2,3)$. Like in Section 6.1 for the Murphy basis of $\mathscr{H}(\mathfrak{S}_3)$, we will see how the results depend on the choices of R and v in our explicit calculations.

The semistandard basis of $\mathscr{S} := \mathscr{S}(2,3)$ is build on top of the Murphy basis of $\mathscr{H} := \mathscr{H}(\mathfrak{S}_3)$, so we will be using results from Section 6.1.

6.3.1 Constructing the cell datum

Assume, like we did in Section 6.1, that R is a field and $v \in R$ a unit. Note that we will be deviating from the notation used in Section 5 to make it more compact.

By (5.2) the Schur algebra $\mathscr{S}(2,3)$ has a natural decomposition

$$\mathscr{S}(2,3) = \bigoplus_{\mu,\eta \in \Lambda(2,3)} \mathrm{Hom}_{\mathscr{H}}(M^{\eta}, M^{\mu}) \ .$$

The poset of compositions $\mu = (\mu_1, \mu_2)$ of 3 is

$$\Lambda(2,3) = \left\{ \ydiagram{3}, \ydiagram{2,1}, \ydiagram{1,1}, \ydiagram{0,3} \right\} ,$$

so the involved permutation modules are

$$M^{\ydiagram{3}}, \ M^{\ydiagram{2,1}}, \ M^{\ydiagram{1,1}}, \ \text{and} \ M^{\ydiagram{0,3}} .$$

By Definition 4.2.1 they are, as a right $\mathscr{H}$-module, generated by

$$\begin{aligned}
m_{\ydiagram{3}} &= m_{A^e_{(3)}\, A^e_{(3)}} &&= v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1 \\
m_{\ydiagram{2,1}} &= m_{A^e_{(2,1)}\, A^e_{(2,1)}} &&= v^{-1}H_s + 1 \\
m_{\ydiagram{1,1}} & &&= v^{-1}H_t + 1 \\
m_{\ydiagram{0,3}} &= m_{A^e_{(3)}\, A^e_{(3)}} &&= v^{-3}H_{sts} + v^{-2}H_{ts} + v^{-2}H_{st} + v^{-1}H_t + v^{-1}H_s + 1 \ .
\end{aligned}$$

First, we need the poset of the semistandard basis of $\mathscr{S}$, which are all partitions in $\Lambda(2,3)$ with dominance ordering:

$$\Lambda^+(2,3) = \left\{ \ydiagram{3} \rhd \ydiagram{2,1} \right\}$$

The labeling set associated to each of these consists of all semistandard tableaux of this shape. Recall that we have already stated them in Example 3.3.1. Here we will use a modified notation:

$$B^{(3,0)}_{(3,0)} := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{3}, \ydiagram{3}\right)$$

$$B^{(2,1)}_{(3,0)} := \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{3} \ , \ydiagram{2,1}\right) \qquad B^{(2,1)}_{(2,1)} := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{2,1} \ , \ydiagram{2,1}\right)$$

$$B^{(1,2)}_{(3,0)} := \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{3} \ , \ydiagram{1,1}\right) \qquad B^{(1,2)}_{(2,1)} := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{2,1} \ , \ydiagram{1,1}\right)$$

$$B^{(0,3)}_{(3,0)} := \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array} \quad \in \mathcal{T}_0 \left(\ydiagram{3} \ , \ydiagram{0,3}\right)$$

A semistandard tableau is denoted by B decorated with its shape as subscript and its type as superscript. Note that this notation only makes sense in this example where $n = 2$ and $d = 3$, as there is at most one semistandard tableau for each combination of shape and type.

The labeling sets of the semistandard basis of $\mathscr{S}$ are:

$$\mathcal{T}_0\left(\ydiagram{3}\right) = \left\{B^{(3,0)}_{(3,0)},\ B^{(2,1)}_{(3,0)},\ B^{(1,2)}_{(3,0)},\ B^{(0,3)}_{(3,0)}\right\}$$
$$\mathcal{T}_0\left(\ydiagram{2,1}\right) = \left\{B^{(2,1)}_{(2,1)},\ B^{(1,2)}_{(2,1)}\right\}$$

To construct the semistandard basis elements we first need to construct the basis of intersections of permutation modules with dual intersection modules from Proposition 4.2.11. Recall that we have already done it for the intersection $M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*}$ in Example 4.2.12:

$$m_{B^{(1,2)}_{(3,0)}B^{(1,2)}_{(3,0)}} = m_{A^e_{(3)}A^e_{(3)}}$$
$$m_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}} = m_{A^e_{(2,1)}\,A^e_{(2,1)}} + m_{A^t_{(2,1)}\,A^e_{(2,1)}} + m_{A^e_{(2,1)}\,A^t_{(2,1)}} + m_{A^t_{(2,1)}\,A^t_{(2,1)}}$$
$$\Rightarrow M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*} = \mathrm{span}_R\left\{m_{B^{(1,2)}_{(3,0)}B^{(1,2)}_{(3,0)}},\ m_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}\right\}$$

By similar calculations we get bases for all intersections, which are summarized in the table of Example 4.2.12. Here is the table using the notation of this chapter:

$\cap$	$M^{\ydiagram{3}}$	$M^{\ydiagram{2,1}}$	$M^{\ydiagram{2,1}}$	$M^{\ydiagram{1,1,1}\bullet}$
$M^{\ydiagram{3}\,*}$	$m_{B^{(3,0)}_{(3,0)}B^{(3,0)}_{(3,0)}}$	$m_{B^{(2,1)}_{(3,0)}B^{(3,0)}_{(3,0)}}$	$m_{B^{(1,2)}_{(3,0)}B^{(3,0)}_{(3,0)}}$	$m_{B^{(0,3)}_{(3,0)}B^{(3,0)}_{(3,0)}}$
$M^{\ydiagram{2,1}\,*}$	$m_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}}$	$m_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}}$ $m_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$m_{B^{(1,2)}_{(3,0)}B^{(2,1)}_{(3,0)}}$ $m_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$m_{B^{(0,3)}_{(3,0)}B^{(2,1)}_{(3,0)}}$
$M^{\ydiagram{2,1}\,*}$	$m_{B^{(3,0)}_{(3,0)}B^{(1,2)}_{(3,0)}}$	$m_{B^{(2,1)}_{(3,0)}B^{(1,2)}_{(3,0)}}$ $m_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$m_{B^{(1,2)}_{(3,0)}B^{(1,2)}_{(3,0)}}$ $m_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$m_{B^{(0,3)}_{(3,0)}B^{(1,2)}_{(3,0)}}$
$M^{\ydiagram{1,1,1}\bullet\,*}$	$m_{B^{(3,0)}_{(3,0)}B^{(0,3)}_{(3,0)}}$	$m_{B^{(2,1)}_{(3,0)}B^{(0,3)}_{(3,0)}}$	$m_{B^{(1,2)}_{(3,0)}B^{(0,3)}_{(3,0)}}$	$m_{B^{(0,3)}_{(3,0)}B^{(0,3)}_{(3,0)}}$

For $\lambda = \ydiagram{3}$ the basis elements are

$$m_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}} = m_{A^{e}_{(3)}A^{e}_{(3)}} \in M^{\mu} \cap M^{\eta *}$$

for all $\mu, \eta \in \Lambda(2,3)$, because there is exactly one semistandard tableau of shape $\ydiagram{3}$ for each type in $\Lambda(2,3)$.

For $\lambda = \ydiagram{2,1}$ these basis elements are:

$$\begin{aligned}
m_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}} &= m_{A^{e}_{(2,1)}A^{e}_{(2,1)}} && \in M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*} \\
m_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}} &= m_{A^{e}_{(2,1)}A^{e}_{(2,1)}} + m_{A^{e}_{(2,1)}A^{t}_{(2,1)}} && \in M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*} \\
m_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}} &= m_{A^{e}_{(2,1)}A^{e}_{(2,1)}} + m_{A^{t}_{(2,1)}A^{e}_{(2,1)}} && \in M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*} \\
m_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}} &= m_{A^{e}_{(2,1)}A^{e}_{(2,1)}} + m_{A^{e}_{(2,1)}A^{t}_{(2,1)}} && \in M^{\ydiagram{2,1}} \cap M^{\ydiagram{2,1}\,*} \\
&\quad + m_{A^{t}_{(2,1)}A^{e}_{(2,1)}} + m_{A^{t}_{(2,1)}A^{t}_{(2,1)}}
\end{aligned}$$

Now we are able to construct the semistandard basis elements of $\mathscr{S}(2,3)$. For $\mu, \eta \in \Lambda(2,3)$, $\lambda \in \Lambda^{+}(2,3)$, $S \in \mathcal{T}_0(\lambda, \mu)$ and $T \in \mathcal{T}_0(\lambda, \eta)$ they were defined in Chapter 5 as

$$\begin{array}{rccc} \phi^{\lambda}_{ST} : & M^{\eta} & \longrightarrow & M^{\mu} \\ & m_{\eta} & \mapsto & m_{ST} \end{array},$$

viewed as elements in $\mathscr{S}$ by trivial extension to $\bigoplus_{\mu \in \Lambda(2,3)} M^{\mu}$. All basis elements are summarized in Figure 6.8. Note that we do not need λ in the superscript with our notation, because it is the shape of S and T, which is the subscript of each B.

For easier multiplication of the semistandard basis elements we have already expressed $\phi^{\lambda}_{ST}(m_{\eta}) = m_{ST} \in M^{\mu}$ via the $\mathscr{H}$-action on the generator m_{μ} of M^{μ} in Figure 6.8.

Because we extend these elements trivially to $\bigoplus_{\mu \in \Lambda(2,3)} M^{\mu}$ a multiplication $\phi^{\lambda'}_{S'T'} \circ \phi^{\lambda}_{ST}$ can only be non-zero, if S and T' are of the same type. For example, one possibly non-zero multiplication is $\phi_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}} \circ \phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}}$. Using Figure 6.8 we get

$$\phi_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}} \circ \phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{3}},\ m_{\ydiagram{2,1}} \mapsto m_{\ydiagram{3}} \cdot (v^{-1}H_{sts} + H_{st} + vH_s)\ .$$

Now, because we know $m_{A^{e}_{(3)}A^{e}_{(3)}} = H^{*}_{e} \cdot m_{\ydiagram{3}} \cdot H_e = m_{\ydiagram{3}}$ from Section 6.1, we can use Figure 6.2 to calculate

$$m_{\ydiagram{3}} \cdot (v^{-1}H_{sts} + H_{st} + vH_s) = (v^{-4} + v^{-2} + 1) m_{\ydiagram{3}}$$

and so

$$\phi_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}} \circ \phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}} = (v^{-4} + v^{-2} + 1)\phi_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}}\ .$$

$$\phi_{B^{(3,0)}_{(3,0)}B^{(3,0)}_{(3,0)}} : M^{\ydiagram{3}} \to M^{\ydiagram{3}}, \quad m_{\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{3}}$$

$$\phi_{B^{(2,1)}_{(3,0)}B^{(3,0)}_{(3,0)}} : M^{\ydiagram{3}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_{sts} + H_{st} + \nu H_s)$$

$$\phi_{B^{(1,2)}_{(3,0)}B^{(3,0)}_{(3,0)}} : M^{\ydiagram{3}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-2}H_{st} + \nu^{-1}H_s + H_e)$$

$$\phi_{B^{(0,3)}_{(3,0)}B^{(3,0)}_{(3,0)}} : M^{\ydiagram{3}} \to M^{\bullet\ydiagram{3}}, \quad m_{\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\bullet\ydiagram{3}}$$

$$\phi_{B^{(3,0)}_{(3,0)}B^{(2,1)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{3}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{3}}$$

$$\phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_{sts} + H_{st} + \nu H_s)$$

$$\phi_{B^{(1,2)}_{(3,0)}B^{(2,1)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-2}H_{st} + \nu^{-1}H_s + H_e)$$

$$\phi_{B^{(0,3)}_{(3,0)}B^{(2,1)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\bullet\ydiagram{3}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\bullet\ydiagram{3}}$$

$$\phi_{B^{(3,0)}_{(3,0)}B^{(1,2)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{3}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{3}}$$

$$\phi_{B^{(2,1)}_{(3,0)}B^{(1,2)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_{sts} + H_{st} + \nu H_s)$$

$$\phi_{B^{(1,2)}_{(3,0)}B^{(1,2)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-2}H_{st} + \nu^{-1}H_s + H_e)$$

$$\phi_{B^{(0,3)}_{(3,0)}B^{(1,2)}_{(3,0)}} : M^{\ydiagram{2,1}} \to M^{\bullet\ydiagram{3}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\bullet\ydiagram{3}}$$

$$\phi_{B^{(3,0)}_{(3,0)}B^{(0,3)}_{(3,0)}} : M^{\bullet\ydiagram{3}} \to M^{\ydiagram{3}}, \quad m_{\bullet\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{3}}$$

$$\phi_{B^{(2,1)}_{(3,0)}B^{(0,3)}_{(3,0)}} : M^{\bullet\ydiagram{3}} \to M^{\ydiagram{2,1}}, \quad m_{\bullet\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_{sts} + H_{st} + \nu H_s)$$

$$\phi_{B^{(1,2)}_{(3,0)}B^{(0,3)}_{(3,0)}} : M^{\bullet\ydiagram{3}} \to M^{\ydiagram{2,1}}, \quad m_{\bullet\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-2}H_{st} + \nu^{-1}H_s + H_e)$$

$$\phi_{B^{(0,3)}_{(3,0)}B^{(0,3)}_{(3,0)}} : M^{\bullet\ydiagram{3}} \to M^{\bullet\ydiagram{3}}, \quad m_{\bullet\ydiagram{3}} \mapsto m_{A^e_{(3)}A^e_{(3)}} = m_{\bullet\ydiagram{3}}$$

$$\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(2,1)}A^e_{(2,1)}} = m_{\ydiagram{2,1}}$$

$$\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(2,1)}A^e_{(2,1)}} + m_{A^t_{(2,1)}A^e_{(2,1)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_s + H_e)$$

$$\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} \mapsto m_{A^e_{(2,1)}A^e_{(2,1)}} + m_{A^e_{(2,1)}A^t_{(2,1)}} = m_{\ydiagram{2,1}} \cdot (\nu^{-1}H_t + H_e)$$

$$\begin{aligned}\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}} : M^{\ydiagram{2,1}} \to M^{\ydiagram{2,1}}, \quad m_{\ydiagram{2,1}} &\mapsto m_{A^e_{(2,1)}A^e_{(2,1)}} + m_{A^e_{(2,1)}A^t_{(2,1)}} \\ &\quad + m_{A^t_{(2,1)}A^e_{(2,1)}} + m_{A^t_{(2,1)}A^t_{(2,1)}} \\ &= m_{\ydiagram{2,1}} \cdot (m_{A^e_{(2,1)}A^e_{(2,1)}} + m_{A^e_{(2,1)}A^t_{(2,1)}}) \\ &= m_{\ydiagram{2,1}} \cdot (\nu^{-2}H_{st} + \nu^{-1}H_t + \nu^{-1}H_s + H_e)\end{aligned}$$

Figure 6.8: All semistandard basis elements of $\mathscr{S}(2,3)$.

Our next goal is to verify the cellular property of this basis. By Definition 2.1.1 of cellular algebras we need to show that for all $\lambda \in \Lambda^+(2,3)$, $\eta, \gamma \in \Lambda(2,3)$, $a \in \mathscr{S}$ there exist coefficients $r_\gamma^{\eta,a} \in R$ independent of $\mu \in \Lambda(2,3)$ such that

$$\phi_{B_\lambda^\mu B_\lambda^\eta} \circ a \equiv \sum_{\gamma \in \Lambda(2,3)} r_\gamma^{\eta,a} \phi_{B_\lambda^\mu B_\lambda^\gamma} \mod \mathscr{S}^{\check{\lambda}}.$$

Of course it is enough to show this for all a in the semistandard basis. Calculating all 20^2 multiplications to verify the cellular property is too much work. We can reduce that number by using the anti-automorphism $*$ and only calculating the possibly non-zero multiplications. Even then there are many cases left, in particular for basis elements associated to $\square\square\square$. Therefore we will use another argument to verify a large part of the cellular property for $\square\square\square$, reduce the calculations needed for $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ and then calculate the remaining cases by hand.

Let $\lambda = \square\square\square$, $\mu, \eta \in \Lambda(2,3)$ and ϕ_{ST} a semistandard basis element, where T is of type $\gamma \in \Lambda(2,3)$. We want to calculate $\phi_{B_\lambda^\mu B_\lambda^\eta} \circ \phi_{ST}$, so we assume that it is non-zero and thus S of type η. By Figure 6.8 we have $\phi_{B_\lambda^\mu B_\lambda^\eta}(m_\eta) = m_{A^e_{(3)}A^e_{(3)}}$ for any μ, so if we express $\phi_{ST}(m_\gamma)$ in terms of m_η, like we did in Figure 6.8, then $\phi_{B_\lambda^\mu B_\lambda^\eta}$ just replaces m_η by $m_{A^e_{(3)}A^e_{(3)}}$ in that expression:

$$\begin{array}{rccl} \phi_{B_\lambda^\mu B_\lambda^\eta} \circ \phi_{ST}: & M^\gamma & \longrightarrow & M^\mu \\ & m_\gamma & \mapsto & \phi_{ST}(m_\gamma)_{|m_\eta = m_{A^e_{(3)}A^e_{(3)}}} \end{array}$$

Notice that $\phi_{ST}(m_\gamma)_{|m_\eta = m_{A^e_{(3)}A^e_{(3)}}}$ is independent of μ. Moreover, it is the result of some element of $\mathscr{H}$ acting on $m_{A^e_{(3)}A^e_{(3)}}$ from the right. In Section 6.1 we saw that $m_{A^e_{(3)}A^e_{(3)}}$ is part of the Murphy basis of $\mathscr{H}$, that $\square\square\square$ is maximal in the dominance ordering on $\Lambda^+(3)$ and that $A^e_{(3)}$ is the only standard tableau of shape $\square\square\square$.

Therefore, $\phi_{ST}(m_\gamma)_{|m_\eta = m_{A^e_{(3)}A^e_{(3)}}} \in \operatorname{span}_R\{m_{A^e_{(3)}A^e_{(3)}}\}$, say

$$\phi_{ST}(m_\gamma)_{|m_\eta = m_{A^e_{(3)}A^e_{(3)}}} = \alpha_{ST} m_{A^e_{(3)}A^e_{(3)}}$$

with $\alpha_{ST} \in R$ independent of μ.

Finally, note that the basis element $\phi_{B_\lambda^\mu B_\lambda^\gamma}$ maps m_γ to $m_{A^e_{(3)}A^e_{(3)}} \in M^\mu$, so

$$\phi_{B_\lambda^\mu B_\lambda^\eta} \circ \phi_{ST} = \alpha_{ST} \phi_{B_\lambda^\mu B_\lambda^\gamma} \tag{6.6}$$

with α_{ST} independent of μ. This proves the cellular property for all elements associated to $\lambda = \square\square\square$.

Furthermore, the coefficient α_{ST} is the same for all T of shape $\square\square\square$, because we see in Figure 6.8 that after replacing m_η by $m_{A^e_{(3)}A^e_{(3)}}$ the expression is the same for all γ. This allows us to state the $\mathscr{S}$-action on semistandard basis elements associated to $\square\square\square$ in Figure 6.9.

For standard basis elements associated to $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ we can do a similar reduction by applying $*$ and using the results from above. We want to consider $\mu, \eta \in \{(2,1),(1,2)\}$, $S, T \in \mathcal{T}_0(\square\square\square)$

such that $\phi_{B^{\mu}_{(2,1)}B^{\eta}_{(2,1)}} \circ \phi_{ST} \neq 0$. Then S must be of type η, so $S = B^{\eta}_{(3,0)}$. Let $\gamma \in \Lambda(2,3)$ be the type of T, so $T = B^{\gamma}_{(3,0)}$.

By applying $*$ twice we get

$$\phi_{B^{\mu}_{(2,1)}B^{\eta}_{(2,1)}} \circ \phi_{B^{\eta}_{(3,0)}B^{\gamma}_{(3,0)}} = \left(\phi_{B^{\gamma}_{(3,0)}B^{\eta}_{(3,0)}} \circ \phi_{B^{\eta}_{(2,1)}B^{\mu}_{(2,1)}}\right)^* \overset{6.6}{=} \left(\alpha_{B^{\eta}_{(2,1)}B^{\mu}_{(2,1)}}\phi_{B^{\gamma}_{(3,0)}B^{\mu}_{(3,0)}}\right)^* = \alpha_{B^{\eta}_{(2,1)}B^{\mu}_{(2,1)}}\phi_{B^{\mu}_{(3,0)}B^{\gamma}_{(3,0)}}$$

with $\alpha_{B^{\eta}_{(2,1)}B^{\mu}_{(2,1)}}$ independent of γ, so we only need to calculate one of these coefficients. In fact, these coefficients already appear in Figure 6.9. Moreover, we see the cellular property holds, because $\phi_{B^{\mu}_{(3,0)}B^{\gamma}_{(3,0)}}$ is associated to $\ydiagram{3} \rhd \ydiagram{2,1}$, so

$$\phi_{B^{\mu}_{(2,1)}B^{\eta}_{(2,1)}} \circ \phi_{B^{\eta}_{(3,0)}B^{\gamma}_{(3,0)}} \equiv 0 \bmod \check{\mathscr{S}}^{\ydiagram{2,1}}$$

for all μ.

Although we have reduced the amount of calculations needed to verify the cellular property for elements associated to $\ydiagram{2,1}$ there are still a few that need to be done by hand using Figure 6.8. We have gathered the results in Figure 6.10 to check the cellular property for these cases.

Consider, for example, the non-zero actions of $\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$:

$$\begin{aligned}
\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}} \circ \phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}} &= \phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}} + \nu^{-2}\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}} \\
\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}} \circ \phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}} &= (\nu^{-2}+1)\phi_{B^{(1,2)}_{(3,0)}B^{(2,1)}_{(3,0)}} + \nu^{-2}\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}
\end{aligned}$$

The coefficients of $\phi_{\bullet B^{(2,1)}_{(2,1)}}$ agree and $\phi_{\bullet B^{(2,1)}_{(3,0)}} \in \check{\mathscr{S}}^{\ydiagram{2,1}}$, so their coefficients do not have to agree. Hence, the cellular property holds for this action. Similarly, the cellular property hold for actions of all semistandard basis elements associated to $\ydiagram{2,1}$.

We have finally shown that the semistandard basis of $\mathscr{S}(2,3)$ is cellular and thus verified Proposition 5.2.1 for $n = 2$ and $d = 3$.

$\phi_{B^{\mu}_{(3,0)}B^{(3,0)}_{(3,0)}}$	$\phi_{B^{\mu}_{(3,0)}B^{(2,1)}_{(3,0)}}$	$\phi_{B^{\mu}_{(3,0)}B^{(1,2)}_{(3,0)}}$	$\phi_{B^{\mu}_{(3,0)}B^{(0,3)}_{(3,0)}}$	
$\phi_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}}$				$\circ\phi_{B^{(3,0)}_{(3,0)}B^{\eta}_{(3,0)}}$
	$(\nu^{-4}+\nu^{-2}+1)\phi_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}}$			$\circ\phi_{B^{(2,1)}_{(3,0)}B^{\eta}_{(3,0)}}$
		$(\nu^{-4}+\nu^{-2}+1)\phi_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}}$		$\circ\phi_{B^{(1,2)}_{(3,0)}B^{\eta}_{(3,0)}}$
			$\phi_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}}$	$\circ\phi_{B^{(0,3)}_{(3,0)}B^{\eta}_{(3,0)}}$
	$\phi_{B^{\mu}_{(3,0)}B^{(2,1)}_{(3,0)}}$			$\circ\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$
	$(\nu^{-2}+1)\phi_{B^{\mu}_{(3,0)}B^{(1,2)}_{(3,0)}}$			$\circ\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$
		$(\nu^{-2}+1)\phi_{B^{\mu}_{(3,0)}B^{(2,1)}_{(3,0)}}$		$\circ\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$
		$(\nu^{-2}+1)^2\phi_{B^{\mu}_{(3,0)}B^{(1,2)}_{(3,0)}}$		$\circ\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$

Figure 6.9: $\mathscr{S}(2,3)$-action on semistandard basis elements associated to $\square\square\square$.
Here μ, η are arbitrary elements in $\Lambda^+(2,3)$. Empty entries illustrate that the action is zero.

$\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	
				$\circ\phi_{B^{\mu}_{(3,0)}B^{\eta}_{(3,0)}}$
$\phi_{B^{(2,1)}_{(3,0)}B^{\gamma}_{(3,0)}}$	$\phi_{B^{(1,2)}_{(3,0)}B^{\gamma}_{(3,0)}}$			$\circ\phi_{B^{(2,1)}_{(3,0)}B^{\gamma}_{(3,0)}}$
		$(\nu^{-2}+1)\phi_{B^{(2,1)}_{(3,0)}B^{\gamma}_{(3,0)}}$	$(\nu^{-2}+1)^2\phi_{B^{(1,2)}_{(3,0)}B^{\gamma}_{(3,0)}}$	$\circ\phi_{B^{(1,2)}_{(3,0)}B^{\gamma}_{(3,0)}}$
$\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$			$\circ\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$
$\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$			$\circ\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$
		$\phi_{B^{(2,1)}_{(3,0)}B^{(2,1)}_{(3,0)}} + \nu^{-2}\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$(\nu^{-2}+1)\phi_{B^{(1,2)}_{(3,0)}B^{(2,1)}_{(3,0)}} + \nu^{-2}\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$	$\circ\phi_{B^{(1,2)}_{(2,1)}B^{(2,1)}_{(2,1)}}$
		$(\nu^{-2}+1)\phi_{B^{(2,1)}_{(3,0)}B^{(1,2)}_{(3,0)}} + \nu^{-2}\phi_{B^{(2,1)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$(\nu^{-2}+1)^2\phi_{B^{(1,2)}_{(3,0)}B^{(1,2)}_{(3,0)}} + \nu^{-2}\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$	$\circ\phi_{B^{(1,2)}_{(2,1)}B^{(1,2)}_{(2,1)}}$

Figure 6.10: $\mathscr{S}(2,3)$-action on semistandard basis elements associated to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$.

Here $\gamma \in \{(2,1),(1,2)\}$, $\mu \in \{(3,0),(0,3)\}$ and $\eta \in \Lambda(2,3)$. Empty entries illustrate that the action is zero.

6.3.2 Right cell modules and their irreducible quotients

Let's derive the irreducible representations of $\mathscr{S}(2,3)$ from the calculations above. Recall that there is a canonical element $T^\lambda = \lambda(t^\lambda) \in \mathcal{T}_0(\lambda)$ for each $\lambda \in \Lambda^+(2,3)$. With the notation of this chapter these are $B^{(3,0)}_{(3,0)} \in \mathcal{T}_0(\ydiagram{3})$ and $B^{(2,1)}_{(2,1)} \in \mathcal{T}_0(\ydiagram{2,1})$.

By (5.5) there are two Weyl modules for $\mathscr{S}(2,3)$:

$$W^{\ydiagram{3}} = \operatorname{span}_R\left\{\overline{\phi_{B^{(3,0)}_{(3,0)}T}} \mid T \in \mathcal{T}_0(\ydiagram{3})\right\} \subset {}^{\mathscr{S}}\!/_{\check{\mathscr{S}}^{\ydiagram{3}}} \cong \mathscr{S}$$

$$W^{\ydiagram{2,1}} = \operatorname{span}_R\left\{\overline{\phi_{B^{(2,1)}_{(2,1)}T}} \mid T \in \mathcal{T}_0(\ydiagram{2,1})\right\} \subset {}^{\mathscr{S}}\!/_{\check{\mathscr{S}}^{\ydiagram{2,1}}}$$

We abbreviate these generators by ϕ_T, where the shape of T indicates which Weyl modules it is a generator of. The right $\mathscr{S}(2,3)$-actions can be easily inferred from Figure 6.9 and Figure 6.10.

Next we use the canonical semistandard tableaux to calculate the bilinear form on both Weyl modules:

$$\langle\phi_S,\phi_T\rangle\phi_{B^{(3,0)}_{(3,0)}B^{(3,0)}_{(3,0)}} = \phi_{B^{(3,0)}_{(3,0)}S} \circ \phi_{TB^{(3,0)}_{(3,0)}} \bmod \check{\mathscr{S}}^{\ydiagram{3}} \text{ for all } S,T \in \mathcal{T}_0(\ydiagram{3})$$

$$\langle\phi_S,\phi_T\rangle\phi_{B^{(2,1)}_{(2,1)}B^{(2,1)}_{(2,1)}} \equiv \phi_{B^{(2,1)}_{(2,1)}S} \circ \phi_{TB^{(2,1)}_{(2,1)}} \bmod \check{\mathscr{S}}^{\ydiagram{2,1}} \text{ for all } S,T \in \mathcal{T}_0(\ydiagram{2,1})$$

By Figures 6.9 and 6.10 the non-zero pairings are:

$$\langle\phi_{B^{(3,0)}_{(3,0)}},\phi_{B^{(3,0)}_{(3,0)}}\rangle = 1$$

$$\langle\phi_{B^{(2,1)}_{(3,0)}},\phi_{B^{(2,1)}_{(3,0)}}\rangle = \nu^{-4}+\nu^{-2}+1 \qquad \langle\phi_{B^{(2,1)}_{(2,1)}},\phi_{B^{(2,1)}_{(2,1)}}\rangle = 1$$

$$\langle\phi_{B^{(1,2)}_{(3,0)}},\phi_{B^{(1,2)}_{(3,0)}}\rangle = \nu^{-4}+\nu^{-2}+1 \qquad \langle\phi_{B^{(1,2)}_{(2,1)}},\phi_{B^{(1,2)}_{(2,1)}}\rangle = \nu^{-2}$$

$$\langle\phi_{B^{(0,3)}_{(3,0)}},\phi_{B^{(0,3)}_{(3,0)}}\rangle = 1$$

We are now ready to find all irreducible representations of $\mathscr{S}(2,3)$. By Proposition 5.2.4 all $L^\lambda = W^\lambda/\operatorname{rad} W^\lambda$, $\lambda \in \Lambda^+(2,3)$ are non-zero and thus irreducible.

Proposition 6.3.1. *The irreducible representations of $\mathscr{S}(2,3)$ are*

$$\left\{L^{\ydiagram{3}}, L^{\ydiagram{2,1}}\right\}$$

with

$$L^{\ydiagram{3}} = \begin{cases} W^{\ydiagram{3}}/\mathit{span}_R\left\{\phi_{B^{(2,1)}_{(3,0)}},\phi_{B^{(1,2)}_{(3,0)}}\right\} & \text{, if } \nu^{-4}+\nu^{-2}+1=0 \\ W^{\ydiagram{3}} & \text{, otherwise} \end{cases}$$

$$L^{\ydiagram{2,1}} = W^{\ydiagram{2,1}}.$$

Proof. We proceed like in the proof of Proposition 6.1.1.

$\ydiagram{3}$: Assume $x \in \operatorname{rad} W^{\ydiagram{3}}$ and express x in the basis of $W^{\ydiagram{3}}$, say $a_x, b_x, c_x, d_x \in R$ and

$$x = a_x \phi_{B^{(3,0)}_{(3,0)}} + b_x \phi_{B^{(2,1)}_{(3,0)}} + c_x \phi_{B^{(1,2)}_{(3,0)}} + d_x \phi_{B^{(0,3)}_{(3,0)}} .$$

Then $\langle x, y \rangle = 0$ for all $y \in W^{\ydiagram{3}}$, so if we also express y in this way and use the calculations from above we get

$$y = a_y \phi_{B^{(3,0)}_{(3,0)}} + b_y \phi_{B^{(2,1)}_{(3,0)}} + c_y \phi_{B^{(1,2)}_{(3,0)}} + d_y \phi_{B^{(0,3)}_{(3,0)}} .$$

and

$$a_x a_y + (v^{-4} + v^{-2} + 1) b_x b_y + (v^{-4} + v^{-2} + 1) c_x c_y + d_x d_y = 0$$

for all $a_y, b_y, c_y, d_y \in R$.

Testing with $a_y = 1, b_y = 0, c_y = 0, d_y = 0$ we see $a_x = 0$ and by similar arguments also $d_x = 0$. If $(v^{-4} + v^{-2} + 1) \neq 0$ the same applies for b_x and c_x and therefore $\operatorname{rad} W^{\ydiagram{3}} = \{0\}$. Otherwise $\langle x, y \rangle = 0$ for any b_y and c_y, so

$$\operatorname{rad} W^{\ydiagram{3}} = \operatorname{span}_R \left\{ \phi_{B^{(2,1)}_{(3,0)}}, \phi_{B^{(1,2)}_{(3,0)}} \right\} .$$

$\ydiagram{2,1}$: Assume $x \in \operatorname{rad} W^{\ydiagram{2,1}}$, so $\langle x, y \rangle = 0$ for all $y \in W^{\ydiagram{2,1}}$. Express x and y again in the basis of $W^{\ydiagram{2,1}}$:

$$x = e_x \phi_{B^{(2,1)}_{(2,1)}} + f_x \phi_{B^{(1,2)}_{(2,1)}} \text{ and } y = e_y \phi_{B^{(2,1)}_{(2,1)}} + f_y \phi_{B^{(1,2)}_{(2,1)}}$$

By the calculations above we get:

$$e_x e_y + v^{-2} f_x f_y = 0$$

for all $e_y, f_y \in R$.

Because $v \neq 0$ we can use similar argument as for $\ydiagram{3}$ and get $\operatorname{rad} W^{\ydiagram{2,1}} = \{0\}$.

□

Before ending this section about explicit computations for $\mathscr{S}(2,3)$ we use our results to calculate the decomposition matrix of $\mathscr{S}(2,3)$. Decomposition matrices were introduced in Definition 2.3.16. If $v^{-4}+v^{-2}+1 \neq 0$, both Weyl modules are irreducible by Proposition 6.3.1. Otherwise $\operatorname{rad} W^{\ydiagram{3}} \subset W^{\ydiagram{3}}$ is a proper submodule and there exists a R-linear isomorphism

$$\begin{array}{ccc} \operatorname{rad} W^{\ydiagram{3}} & \xrightarrow{\sim} & L^{\ydiagram{2,1}} \\ \phi_{B^{(2,1)}_{(3,0)}} & \mapsto & \phi_{B^{(2,1)}_{(2,1)}} \\ \phi_{B^{(1,2)}_{(3,0)}} & \mapsto & \frac{1}{1+v^{-2}}\phi_{B^{(1,2)}_{(2,1)}} \end{array},$$

which is also $\mathscr{S}(2,3)$-linear. Hence, the decomposition matrix of $\mathscr{S}(2,3)$ is:

$$v^{-4}+v^{-2}+1 \neq 0 \quad \Rightarrow \mathbf{D} = \begin{array}{c} \begin{array}{cc} \ydiagram{3} & \ydiagram{2,1} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \ydiagram{3} \\ \ydiagram{2,1} \end{array}$$

$$v^{-4}+v^{-2}+1 = 0 \quad \Rightarrow \mathbf{D} = \begin{array}{c} \begin{array}{cc} \ydiagram{3} & \ydiagram{2,1} \end{array} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \ydiagram{3} \\ \ydiagram{2,1} \end{array}$$

Moreover, by Proposition 2.3.19 the Cartan matrix of $\mathscr{S}(2,3)$ is $\mathbf{C} = \mathbf{D}^t\mathbf{D}$, so:

$$v^{-4}+v^{-2}+1 \neq 0 \quad \Rightarrow \mathbf{C} = \begin{array}{c} \begin{array}{cc} \ydiagram{3} & \ydiagram{2,1} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \ydiagram{3} \\ \ydiagram{2,1} \end{array}$$

$$v^{-4}+v^{-2}+1 = 0 \quad \Rightarrow \mathbf{C} = \begin{array}{c} \begin{array}{cc} \ydiagram{3} & \ydiagram{2,1} \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{array} \begin{array}{c} \ydiagram{3} \\ \ydiagram{2,1} \end{array}$$

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