Quiver Schur algebra with single vertex and no arrows

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April 11, 2024

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1 Introduction

The main objects studied in the thesis is the quiver Hecke algebra and an important generalisation, the quiver Schur algebra.

The quiver Hecke algebra was introduced by Khovanov-Lauda [KL09] and Rouquier [Rou08] independently, so it is also called KLR algebra.

Khovanov and Lauda applied a combinatorial approach. Suppose that Γ is a fixed graph with no loops and multiple edges, and let I be the set of vertices. Then for $\mathbf{d} \in \mathbb{N}[I] := \bigoplus_{i\in I} \mathbb{N}i$ of non-negative integers associated to each vertex, they define an algebra $\mathcal{R}_\mathbf{d}$ generated by braid-like plane diagrams which consist of interacting strings labelled by vertices of the graph. As vector space, the resulting algebra is given by finite linear combinations of these diagrams modulo certain relations which can also be described by linear combinations of diagrams, and the multiplication is given by concatenation.

On the other side, Rouquier treated it in a categorical way. In [Rou08], $\mathcal{R}_\mathbf{d}$ is defined by listing generators and relations explicitly, and it can also be viewed as a category, whose objects are decomposition of $\mathbf{d}$ consisting of unit vectors, and the generators of the algebra, with the same relations, generate the space of the morphisms between two objects.

Suppose that $\mathfrak{g}$ is a simply-laced Kac-Moody Lie algebra with Dynkin diagram Γ, then it admits a decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. One of the most important property of KLR algebra is that it categorifies Lusztig's integral form ${}_\mathcal{A}\mathbf{f}$ of $U_q(\mathfrak{n}^-)$. Let $K_0(\mathcal{R}_\mathbf{d})$ (resp.$G_0(\mathcal{R}_\mathbf{d})$) be the Grothendieck group of the category of finitely generated graded left projective $\mathcal{R}_\mathbf{d}$-modules (resp. finite dimensional graded left $\mathcal{R}_\mathbf{d}$-modules), and we consider

$$K_0(\mathcal{R}) = \bigoplus_{\mathbf{d}\in\mathbb{N}[I]} K_0(\mathcal{R}_\mathbf{d}), \quad G_0(\mathcal{R}) = \bigoplus_{\mathbf{d}\in\mathbb{N}[I]} G_0(\mathcal{R}_\mathbf{d}).$$

Then there exists an isomorphism of $\mathbb{N}[I]$-graded twisted bialgebras

$$\gamma : {}_\mathcal{A}\mathbf{f} \to K_0(\mathcal{R}), \quad \gamma^* : {}_\mathcal{A}\mathbf{f}^* \to G_0(\mathcal{R}),$$

where ${}_\mathcal{A}\mathbf{f}^*$ is the graded dual of ${}_\mathcal{A}\mathbf{f}$.

This result is refined by Varagnolo and Vasserot [VV11]. They proved that γ sends Lusztig's canonical basis of ${}_\mathcal{A}\mathbf{f}$ to the classes of indecomposable projective modules in $K_0(\mathcal{R})$, and γ^* sends the Lusztig's dual canonical basis of ${}_\mathcal{A}\mathbf{f}^*$ to the classes of simple modules in $G_0(\mathcal{R})$. In their paper, this algebra is constructed by geometry. The method they used is of great importance in geometric representation theory. More precisely, they used the Steinberg-type varieties and Borel-Moore homology groups to produce convolution algebras.

Steinberg-type variety

The classical Steinberg variety was introduced in the study of representations theory of semisimple Lie algebras over $\mathbb{C}$ and the corresponding Weyl groups by Springer. Let G be a semisimple complex algebraic group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$ and let $\mathcal{B}$ be the flag variety corresponding to G. We define $\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} | x \in \mathfrak{b}\}$, then the Springer resolution is the proper map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$, and $\tilde{N} \simeq T^*\mathcal{B}$ as G-equivariant vector bundles. The Steinberg variety is defined as

$$Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \simeq \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} | x \in \mathfrak{b} \cap \mathfrak{b}'\},$$

whose Borel-Moore homology group is equipped with a convolution product. If $x \in \mathcal{N}$, we define $\mathcal{B}_x$ as the fiber of μ at x, which is called the Springer fiber, then the convolution determines an $H_*^{BM}(Z)$-module structure on the Borel-Moore homology group of $\mathcal{B}_x$. It turns out that $H_*^{BM}(Z)$ is isomorphic to the group algebra of the Weyl group of G, and the Borel-Moore homology groups of these Springer fibers provide all the irreducible representations of the Weyl group.

The geometric construction in [VV11] can be viewed as a generalization of this construction. For simplicity, we just assume that $G = \mathrm{SL}_n(\mathbb{C})$. We consider a quiver with only one vertex and an arrow pointing the vertex to itself, which is called the Jordan quiver because its indecomposable representations are classified by Jordan normal forms up to isomorphism. A nilpotent element $x \in \mathcal{N}$ can now be viewed as a nilpotent representation of the quiver of dimension n. If a Borel subalgebra $\mathfrak{b}$ contains x, then the flag corresponding to $\mathfrak{b}$ in $\mathbb{C}^n$ will be compatible with x. Here a flag F such that

$$F = (0 = F^0 \subset F^1 \subset \cdots \subset F^n = \mathbb{C}^n),\ \dim F^i - \dim F^{i-1} = 1,\ \forall 1 \leq i \leq n$$

is said to be compatible with x, if it satisfies $x(F_i) \subset F_{i-1}$ for all i.

For a more general quiver Γ, we replace $\mathcal{N}$ by the space $\mathrm{Rep}_{\mathbf{d}}$ of nilpotent representations with dimension vector $\mathbf{d} = (d_j)_{j \in \mathbb{V}}$, where $\mathbb{V}$ is the set of vertices of Γ. We also replace $\tilde{N}$ by the space $\mathcal{Q}$ of pairs (f, F) satisfying some compatible conditions, where f is a representation of the quiver and $F = (F_j)_{j \in \mathbb{V}}$ where each F_j is a flag in $\mathbb{C}^{d_j}$. Roughly speaking, we can again take the fiber product $\mathcal{Q} \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}$, and this is our new Steinberg-type variety. Varagnolo and Vasserot consider an equivariant Ext-algebra on this variety, which is the same as the equivariant Borel-Moore homology group, and the resulting algebra is exactly the KLR algebra.

In order to answer the question if there is a natural graded version of the cyclotomic q-Schur algebra introduced in [DJM98], the quiver Schur algebra was introduced by Stroppel and Webster in [SW11]. They defined this algebra geometrically and via a faithful polynomial representation on equivariant (co)homology of flag varieties. The generators were interpreted diagrammatically, generalizing the KLR diagram mentioned above. A complete presentation of the algebra is however not known, and this is the motivation behind this thesis.

Hecke-Schur pattern

The study of Hecke-type algebras and their Schur version has a long history.

While studying the relationship between representations of $\mathrm{GL}_n(\mathbb{C})$ and symmetric group S_n, the classical Schur algebra was defined as the S_n-endomorphism algebra of the direct sum of permutation modules of S_n. The famous Schur-Weyl duality states that: two algebras of operators on the tensor space generated by the actions of GL_n and S_n are the full mutual centralizers in the algebra of the endomorphisms $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes n})$ and the Schur algebra can be defined as $\mathrm{End}_{S_n}((\mathbb{C}^n)^{\otimes n})$ (see [GSE07]). A Morita equivalent version of the Schur algebra is defined as the S_n-equivariant endomorphism algebra of the direct sum of permutation modules of S_n. This definition provide the idea to generalize the Schur-type algebra.

Various generalizations of this pattern play importance roles in representation theory.

1. Let q be a power of a prime p. The Hecke algebra $H_n(q)$ of finite type is a q-deformation of the group algebra of S_n, which is isomorphic to the space of functions on $\mathrm{GL}_n(\mathbb{F}_q)$ which are bi-invariant under the action of a Borel subgroup equipped with the convolution as an algebra (see [Str22b]). In this case the Hecke modules play the role of permutation modules, and Dipper and James[DJ89] define the q-Schur algebra as the $H_n(q)$-equivariant endomorphism algebra of the direct sum of Hecke modules.

2. Let p be a prime and E is a finite extension of $\mathbb{Q}_p$. Suppose that q is the cardinality of the residue field of E. The Iwahori-Matsumoto Hecke algebra $\mathcal{H}_n(q)$ is isomorphic to the space of functions on $\mathrm{GL}_n(E)$ which are bi-invariant under the action of an Iwahori subgroup equipped with the convolution as an algebra. As a vector space, $\mathcal{H}_n(q)$ is isomorphic to the tensor product of $H_n(q)$ and the polynomial algebra with n variables. Then the Hecke modules of H_n together with the polynomial algebra can generate the affine version of Hecke modules for $\mathcal{H}_n(q)$. Vignéras [Vig03] defined the affine Schur algebra as the $\mathcal{H}_n(q)$-equivariant endomorphism algebra of the direct sum of (affine) Hecke modules.

Compared with the Hecke-Schur pattern, the relationship between KLR algebra and the quiver Schur algebra has following similarities:

1. There are certain finite dimensional quotients of both algebras called the cyclotomic quotients, such that the cyclotomic quiver Schur algebra is the space of the endomorphisms over the cyclotomic KLR algebra of the direct sum of the analogues of permutation modules.

2. If we view S_n as a Coxeter group, then the permutation modules correspond to Young subgroups of S_n and Hecke modules correspond to parabolic Hecke subalgebras. The Hecke algebra itself, as a Hecke module, corresponds to the minimal parabolic Hecke subalgebra, and we may say it corresponds to the Borel type, and other Hecke modules correspond to other parabolic type. Now the geometric construction of the KLR algebra involves complete flags corresponding to Borel subalgebras, while the quiver Schur algebra uses all partial flags corresponding to parabolic subalgebras.

3. The affine Hecke algebras and the KLR algebras are isomorphic after taking certain completions. It is shown in [MS19] that we have compatible isomorphisms between the completions of affine Schur algebras and quiver Schur algebra.

Outline

The thesis has two main objectives. The first is to give the definition of two Hecke-type algebras and their Schur versions, and explain the Hecke-Schur pattern and the relationship between these two families of algebras. The second goal is to study the quiver Schur algebra in the particular case when the dimension vector is concentrated on one vertex, and present a new basis of the quiver Schur algebra in this case. We will also study other nice properties in this case related with the theory of the highest weight categories.

In Section 2 we give the definition of the affine Hecke algebra and the KLR algebra as well as their Schur versions. Various Hecke algebras and the Schur analogues are described to show the Hecke-Schur pattern, and the quiver Schur algebra is defined geometrically. We give an explicit algebraic description in case that the dimension vector is concentrated on one vertex.

In Section 3, we firstly introduce the completion of affine Hecke algebras and KLR algebras, and then we present the isomorphism between completions of affine Hecke algebras and KLR algebras. We will also give the statement for the isomorphism theorem for Schur algebras.

In Section 4 we present the main result, which is a new basis for the quiver Schur algebra in the simplest case. We give the proof, and describe why the similar construction in general cases fails to form a basis of the quiver Schur algebra.

In Section 5 we introduce the settings and results related to highest weight categories from [BS18], and show that the quiver Schur algebra in the special case fits into these settings by using the new basis given in Section 4.

Acknowledgements

First and foremost, I would like to thank Prof. Catharina Stroppel for supervising my thesis. I would also like to thank her for proofreading the draft of the thesis, providing a lot of helpful comments and answering my questions patiently. Next, I would like to thank my classmate Xiaoxiang Zhou for introducing the geometric aspects of the topic to me, and I am grateful for his inspiring insights and questions. At last, I would like to thank my parents for their unconditional support all the time.

2 Hecke type algebras and Schur analogues

In this section, we will introduce two Hecke-type algebras, as well as the Schur analogue of them. Both of them are of their own importance in representation theory, and their relationship will be discussed in Section 3.

We firstly introduce the classical Schur algebra, which is the most fundamental template of the construction of Hecke-type algebras and Schur analogues.

After that, we will define the affine version and the quiver version of this construction, and these generalized versions of Hecke algebras and Schur algebras will be the main object to study.

2.1 Schur algebra

The symmetric group is one of the most fundamental object in representation theory. The classical Schur algebra is introduced in order to study the representation of symmetric groups.

Let n be a positive integer. We denote by $\Sigma = \Sigma_n$ the symmetric group of order $n!$, with the usual Coxeter group presentation. Let $\mathbb{I}_n$ be the set of all compositions of n. For any composition λ of n, let Σ_λ be the parabolic subgroup associated to λ, and σ_λ be the longest element in Σ_λ. Also, we let ${}^\lambda\Sigma$ be the set of shortest representative elements of $\Sigma_\lambda\backslash\Sigma$, which exists by [Bou07].

Also, let $\mathbb{I} = \{1, 2, \ldots, n-1\}$, then $\mathbb{I}_n$ is bijectively corresponding to the set of subsets of $\mathbb{I}$, given by $\lambda \mapsto \mathbb{J}$ if Σ_λ is exactly the subgroup of Σ generated by s_i for all $i \in \mathbb{J}$. So we also have the notion $\Sigma_\mathbb{J} = \Sigma_\lambda$ and ${}^\mathbb{J}\Sigma = {}^\lambda\Sigma$. For example, the partition $(1, 1, \ldots, 1)$ corresponds to the empty set, while (n) corresponds to $\mathbb{I}$. Moreover, if $\mathbb{K}, \mathbb{J}, \mathbb{L}$ are subsets of $\mathbb{I}$ such that $\mathbb{K} \subset \mathbb{J}, \mathbb{L} \subset \mathbb{J}$, then we let ${}^\mathbb{K}\Sigma^\mathbb{L}_\mathbb{J}$ be the set of the shortest representative elements of $\Sigma_\mathbb{K}\backslash\Sigma_\mathbb{J}/\Sigma_\mathbb{L}$.

Suppose that V is a vector space of m dimension over $\mathbf{k}$ for some $m \geq n$, where $\mathbf{k}$ is an algebraically closed field of characteristic 0, and we have a right Σ-action on the right on $V^{\otimes n}$ by

$$v_1 \otimes \cdots \otimes v_n \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \forall \sigma \in \Sigma.$$

Then we can define the Schur algebra $S(m, n) = \mathrm{End}_\Sigma(V^{\otimes n})$.

The Schur-Weyl duality states that this algebra $S(m, n)$ is exactly the subalgebra generated by the diagonal action of natural representation of $\mathrm{GL}_m(\mathbf{k})$ on V in the endomorphism ring of $V^{\otimes n}$, and we have the following mutually centralizing property:

$$S(m, n) = \mathrm{End}_\Sigma(V^{\otimes n}), \mathbf{k}\Sigma = \mathrm{End}_{\mathrm{GL}_m}(V^{\otimes n}) = \mathrm{End}_{S(m,n)}(V^{\otimes n}).$$

According to [Hen16], there is an alternative definition of Schur algebra, which is Morita equivalent to $S(m, n)$, defined as

$$S = \mathrm{End}_\Sigma(\bigoplus_{\lambda \in \mathbb{I}_n} \mathbf{k} \otimes_{\mathbf{k}\Sigma_\lambda} \mathbf{k}\Sigma) = \mathrm{End}_\Sigma(\bigoplus_{\lambda \in \mathbb{I}_n} \mathrm{Ind}^\Sigma_{\Sigma_\lambda} \mathbf{k}_{triv}).$$

This construction uses the so called permutation modules $\mathbf{k} \otimes_{\mathbf{k}\Sigma_\lambda} \mathbf{k}\Sigma$, which is can be generalized to a quantized version.

The quantized version of the group algebra $\mathbf{k}\Sigma$ is the Hecke algebra H_n of finite type, with a basis $\{T_\sigma | \sigma \in \Sigma\}$. For any $\mathbb{J} \subset \mathbb{I}$ we can define $x_\mathbb{J} = \sum_{\sigma \in \Sigma_\mathbb{J}} T_\sigma$, and then we have a quantized analogue of the permutation module, which is called the Hecke module $H_{\mathbb{J}} := x_{\mathbb{J}} H_n$ (see [Str22b]). Also, we can also define the Hecke module H_λ for any composition of n.

In [DJ89], the q-Schur algebra is defined as

$$S_{q,n} := \mathrm{End}_{H_n}(\bigoplus_{\lambda \in \mathcal{P}(n)} H_\lambda).$$

When q is specialized to 1, this algebra is Morita equivalent to the classical Schur algebra.

So for both the classical version and the quantized version, with a Hecke type algebra, we can define the corresponding Hecke modules, and the Schur analogue of this algebra is the endomophism algebra over this Hecke-type algebra of the direct sum of Hecke modules.

2.2 Affine Hecke algebras and affine Schur algebras

The first idea is to generalize the symmetric groups to extended affine Weyl groups of type A, and we may denote it by Σ^{ext}. With this group, the resulting Hecke type algebra is called the affine Hecke algebra, which is thoroughly studied in [Lus89].

A presentation of Σ^{ext} is given by the semi-direct product $\Sigma \ltimes \mathfrak{X}$ where $\mathfrak{X}$ is the free abelian group of rank n with generators $X_1, \ldots, X_n$, and for any simple reflection $s_i \in \Sigma$ for $1 \leq i \leq n-1$, we have $s_i X_i s_i = X_{i+1}$ and $s_i X_j s_i = X_j$ if $j \notin \{i, i+1\}$.

The following definition, known as Bernstein's presentation [Lus83], gives a quantized version of the group algebra of Σ^{ext}.

Definition 2.2.1 (Affine Hecke algebra)**.** *Let* $\mathbf{k}$ *be a algebraically closed field of characteristic* 0*, and* $e \in \mathbb{Z}_{>0} \cup \{\infty\}$*. The affine Hecke algebra is the unital* $\mathbf{k}$*-algebra* $\mathcal{H} = \mathcal{H}_n$ *generated by* $T_1, \ldots, T_{n-1}$ *and* $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$*, subject to the defining relations:*

For $1 \leq i, j, k \leq n-1$ *such that* $|i-j| > 1$,

(H-1) $(T_i - q)(T_i + 1) = 0,$

(H-2) $T_i T_j = T_j T_i,$

(H-3) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$

(H-4) $X_i X_i^{-1} = 1 = X_i^{-1} X_i,$

(H-5) $X_i X_k = X_k X_i,$

(H-6) $T_i X_j = X_j T_i,$

(H-7) $T_i X_i T_i = q X_{i+1},$

where $q \in \mathbf{k}$ *is an* e*-th order primitive root of unity.*

Remark 2.2.2. *Here* q *is a root of unity. In fact, if we let* q *be the cardinality of the residue field of a finite extension* E *of* $\mathbb{Q}_p$ *for some prime* p*, with the same relations, we obtain the Iwahori-Matsumoto Hecke algebra, which is the convolution algebra of functions on* $\mathrm{GL}_n(E)$ *which are bi-invariant under the action of Iwahori subgroup with values in* $\mathbf{k}$.

Remark 2.2.3. *By construction, the Hecke algebra of finite type H_n is a subalgebra of $\mathcal{H}$, and the Laurent polynomial ring $\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, which is the group algebra of $\mathfrak{X}$, is another subalgebra. And $\mathcal{H}$ is just the quotient of the algebra generated by H_n and $\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ by relations (2.2.1.H-6) and (2.2.1.H-7). If we specialize q to 1, then it becomes the group algebra of Σ^{ext}.*

For any element $\sigma \in \Sigma$ with a reduced expression $\sigma = s_{i_1} s_{i_2} \cdots s_{i_l}$, we write $T_\sigma = T_{i_1} T_{i_2} \cdots T_{i_l}$. Via (2.2.1.H-3), this definition is independent of the choice of the reduced expression. Here we have by [Lus89, Proposition 3.7] two sets of basis of the algebra, which are both obviously bijectively corresponding to the set $\Sigma \times \mathfrak{X}$, which is exactly the same as Σ^{ext} as a set, just like in the finite type case.

Proposition 2.2.4. *The following two sets are both $\mathbf{k}$-basis of $\mathcal{H}$:*

$$\{X_1^{a_1} \cdots X_n^{a_n} T_\sigma | \sigma \in \Sigma, a_i \in \mathbb{Z}\}, \{T_\sigma X_1^{a_1} \cdots X_n^{a_n} | \sigma \in \Sigma, a_i \in \mathbb{Z}\} \tag{1}$$

There are two natural faithful representations of $\mathcal{H}$. In fact, they can be viewed as the representation induced from the trivial and sign representation of H_n.

Proposition 2.2.5 (Trivial faithful representation, [MS19])**.** *Let $U = \sum \mathcal{H}(T_i - q)$, then there exists a faithful representation of $\mathcal{H}$ on*

$$\mathcal{H}/U \simeq \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \cdot v_{\mathbb{I}} \underset{space}{\overset{vector}{\simeq}} \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

where the action of generators $X_i^{\pm 1}$ is left multiplication, and the action of T_i is given by

$$(T_i - q) \cdot f v_{\mathbb{I}} = \frac{qX_i - X_{i+1}}{X_{i+1} - X_i}(f - s_i(f)) v_{\mathbb{I}}. \tag{2}$$

where $s_i(f)$ is the Laurent polynomial f with variables X_i, X_{i+1} interchanged.

Similarly, we have a signed version of the faithful representation:

Proposition 2.2.6 (Signed faithful representation, [MS19])**.** *Let $\overline{U} = \sum \mathcal{H}(T_i + 1)$, then there exists a faithful representation of $\mathcal{H}$ on*

$$\mathcal{H}/\overline{U} \simeq \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \cdot \overline{v}_{\mathbb{I}} \underset{space}{\overset{vector}{\simeq}} \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

where the action of generators $X_i^{\pm 1}$ is left multiplication, and the action of T_i is given by

$$(T_i + 1) \cdot f \overline{v}_{\mathbb{I}} = \frac{qX_{i+1} - X_i}{X_{i+1} - X_i}(f - s_i(f)) \overline{v}_{\mathbb{I}}. \tag{3}$$

It is not difficult to check the relations of $\mathcal{H}$, and for faithfulness see [Rou12].

We have the following quantized version of the group algebra of the Young subgroup (parabolic subgroup) in $\mathcal{H}$ (In fact, it is even in H_n). Let $\mathbb{I} = \{1, 2, \dots, n-1\}$, and $\mathbb{J}$ be a subset of $\mathbb{I}$, then we define $\mathcal{H}_{\mathbb{J}}$ as the Hecke subalgebra generated by $\{T_j | j \in \mathbb{J}\}$ as $\mathbf{k}$-algebra. By Proposition 2.2.4, $\mathcal{H}$ is a free module over $\mathcal{H}_{\mathbb{J}}$, with the following $\mathbf{k}$-basis

$$\{T_\sigma X_1^{a_1} \cdots X_n^{a_n} | \sigma \in {}^{\mathbb{J}}\Sigma, a_i \in \mathbb{Z}\},$$

where ${}^{\mathbb{J}}\Sigma$ is the set of shortest representative elements in $\Sigma_{\mathbb{J}} \backslash \Sigma$. And now we can define the affine version of Hecke modules, which is the quantized and affine version of permutation modules described in the previous subsection.

Definition 2.2.7 (Hecke module). *For any $\mathbb{J} \subset \mathbb{I}$, we set*

$$v_{\mathbb{J}} = \sum_{\sigma \in \Sigma_{\mathbb{J}}} T_\sigma, \bar{v}_{\mathbb{J}} = \sum_{\sigma \in \Sigma_{\mathbb{J}}} (-q)^{\ell(\sigma)} T_\sigma$$

then the trivial (resp. signed) Hecke modules associated to $\mathbb{J}$ is the right principal ideal $v_{\mathbb{J}}\mathcal{H}$ (resp. $\bar{v}_{\mathbb{J}}\mathcal{H}$), equipped with the natural right-action of $\mathcal{H}$.

Lemma 2.2.8. *The ideal $v_{\mathbb{J}}\mathcal{H}$ coincides with*

$$\{v \in \mathcal{H} | (T_i - q)v = 0, \forall i \in \mathbb{J}\},$$

and the ideal $\bar{v}_{\mathbb{J}}\mathcal{H}$ coincides with

$$\{v \in \mathcal{H} | (T_i + 1)v = 0, \forall i \in \mathbb{J}\}.$$

We can deduce from this lemma immediately that $\mathbb{K} \subset \mathbb{J} \subset \mathbb{I}$ implies $v_{\mathbb{K}}\mathcal{H} \subset v_{\mathbb{J}}\mathcal{H}$.

Now we can just follow the method of last subsection to define the Schur analogue of the affine Hecke algebra. This is firstly introduced in [Vig03] to study the representation of p-adic groups.

Definition 2.2.9 (Affine Schur algebra,[Vig03]). *Given an affine Hecke algebra $\mathcal{H}$, we define the corresponding affine Schur algebra $\mathcal{S}$ to be*

$$\mathrm{End}_{\mathcal{H}}(\bigoplus_{\mathbb{J} \subset \mathbb{I}} v_{\mathbb{J}}\mathcal{H}) = \mathrm{Hom}_{\mathcal{H}}(\bigoplus_{\mathbb{J} \subset \mathbb{I}} v_{\mathbb{J}}\mathcal{H}, \bigoplus_{\mathbb{J}' \subset \mathbb{I}} v_{\mathbb{J}'}\mathcal{H}), \tag{4}$$

where the multiplication is the composition of the maps.

Here we consider four types of maps, which appear in the basis of $\mathcal{S}$ given in [Vig03] whose action are easy to write down explicitly. For $\mathbb{K} \subset \mathbb{J} \subset \mathbb{I}$, we consider the following $\mathcal{H}$-equivariant maps

$$\begin{aligned} s_{\mathbb{K},\mathbb{J}} &: v_{\mathbb{J}}\mathcal{H} \to v_{\mathbb{K}}\mathcal{H}, \quad v_{\mathbb{J}}h \mapsto v_{\mathbb{K}}(\sum_{\sigma \in {}^{\mathbb{K}}\Sigma_{\mathbb{J}}} T_\sigma)h, \\ m_{\mathbb{J},\mathbb{K}} &: v_{\mathbb{K}}\mathcal{H} \to v_{\mathbb{J}}\mathcal{H}, \quad v_{\mathbb{K}}h \mapsto v_{\mathbb{J}}h, \\ p_{\mathbb{J},\mathbb{J}} &: v_{\mathbb{J}}\mathcal{H} \to v_{\mathbb{J}}\mathcal{H}, \quad v_{\mathbb{J}}h \mapsto v_{\mathbb{J}}ph, \\ c_{\sigma\mathbb{J},\mathbb{J}} &: v_{\mathbb{J}}\mathcal{H} \to v_{\sigma\mathbb{J}}\mathcal{H}, \quad v_{\mathbb{J}}h \mapsto v_{\sigma\mathbb{J}}T_{\sigma^{-1}}h, \end{aligned} \tag{5}$$

for $h \in \mathcal{H}$ and $p \in \mathbf{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\Sigma_{\mathbb{J}}}$. The following proposition gives another basis consisting of the product of these four types maps, so these four types maps generate the whole $\mathcal{S}$ over $\mathbf{k}$.

Proposition 2.2.10 ([MS19]). *Let $\mathbb{K}_1, \mathbb{K}_2$ be two subsets of $\mathbb{I}$, and for any subset $\mathbb{J}$ of $\mathbb{I}$, let $\mathfrak{B}_{\mathbb{J}}$ be a set of basis vector of $\mathbf{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\Sigma_{\mathbb{J}}}$, then the set*

$$\mathfrak{B}_{\mathbb{K}_2,\mathbb{K}_1} = \{m_{\mathbb{K}_2,\mathbb{L}} \cdot c_{\Sigma\mathbb{L},\mathbb{L}} \cdot p_{\mathbb{L},\mathbb{L}} \cdot s_{\mathbb{L},\mathbb{K}_1} | \mathbb{L} = \mathbb{K}_1 \cap \sigma^{-1}\mathbb{K}_2, p \in \mathfrak{B}_{\mathbb{J}}\}, \tag{6}$$

is a basis of $\mathrm{Hom}_{\mathcal{H}}(v_{\mathbb{K}_1}\mathcal{H}, v_{\mathbb{K}_2}\mathcal{H})$, and

$$\mathfrak{B}^{\mathcal{S}} = \bigcup_{(\mathbb{K}_1,\mathbb{K}_2)} \mathfrak{B}_{\mathbb{K}_2,\mathbb{K}_1} \tag{7}$$

is a basis of $\mathcal{S}$.

And we have the following faithful representation of $\mathcal{S}$, which is an enlarged version of the faithful representation of $\mathcal{H}$.

Proposition 2.2.11 (Faithful representation, [MS19]). *The affine Schur algebra $\mathcal{S}$ has a faithful representation on*

$$V_{\mathcal{S}} = \bigoplus_{\mathbb{J}\subset\mathbb{I}} \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}\cdot v_{\mathbb{I}} \underset{space}{\overset{vector}{\simeq}} \bigoplus_{\mathbb{J}\subset\mathbb{I}} \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}},$$

where the action is given by

$$\begin{aligned} s_{\mathbb{K},\mathbb{J}} &: \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}v_{\mathbb{I}} \to \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{K}}}v_{\mathbb{I}}, \quad fv_{\mathbb{I}} \mapsto fv_{\mathbb{I}},\\ m_{\mathbb{J},\mathbb{K}} &: \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{K}}}v_{\mathbb{I}} \to \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}v_{\mathbb{I}}, \quad fv_{\mathbb{I}} \mapsto \sum_{\sigma\in\Sigma_{\mathbb{J}}^{\mathbb{K}}} T_\sigma f v_{\mathbb{I}},\\ p_{\mathbb{J},\mathbb{J}} &: \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}v_{\mathbb{I}} \to \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}v_{\mathbb{I}}, \quad fv_{\mathbb{I}} \to pfv_{\mathbb{I}},\\ c_{\sigma\mathbb{J},\mathbb{J}} &: \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\mathbb{J}}}v_{\mathbb{I}} \to \mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{\Sigma_{\sigma\mathbb{J}}}v_{\mathbb{I}}, \quad fv_{\mathbb{I}} \to T_\sigma f v_{\mathbb{I}}, \end{aligned} \tag{8}$$

for $\mathbb{K}\subset\mathbb{J}\subset\mathbb{I}$ and $p\in\mathbf{k}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{W_{\mathbb{J}}}$.

It is not obvious that this representation is well-defined and faithful, and the proof can be found in [MS19].

Remark 2.2.12. *Just like the case of affine Hecke algebras, there are two versions of faithful representations, the trivial one and the signed one, and here we just present the trivial one.*

2.3 Quiver Hecke algebras and quiver Schur algebras

The second generalized version is the quiver Hecke algebra and the quiver Schur algebra. We will present the geometric definition of these two algebras, as well as the faithful representation of the quiver Schur algebra introduced in [SW11].

Let Γ be the Dynkin diagram of $\hat{\mathfrak{sl}}_e$, which is the cyclic quiver with e vertices with the fixed clockwise orientation when e is finite, and is an infinite sequence without starting and ending points equipped with a fixed orientation. Let $\mathbb{V}=\mathbb{Z}/e\mathbb{Z}$ if e is finite, and $\mathbb{V}=\mathbb{Z}$ if e is infinity.

We firstly recall the basic notion of representations of the quiver.

A finite dimensional representation (V,f) of Γ over $\mathbb{C}$ is

- V: a collection of space V_i over $\mathbb{C}$ for each $i\in\mathbb{V}$ such that $\sum_i \dim V_i<\infty$,
- f: a collection of linear maps $f_i: V_i\to V_{i+1}$.

A dimension vector $\mathbf{d}$ of the representation is an e-tuple $(d_1,\ldots,d_e)$ where $d_i=\dim V_i$. A subrepresentation of (V,f) is a representation (W,g) such that $W_i\subset V_i$, and $g_i=f_i|_{W_i}: W_i\to W_{i+1}$, for all $i\in\mathbb{V}$.

When e is finite, then a representation (V,f) is said to be nilpotent if the map $f_e\cdots f_2f_1: V_1\to V_1$ is nilpotent. If $e=\infty$, then every representation is nilpotent.

2.3.1 Vector decomposition

To define the Schur algebra used above the composition of n. We generalize this now to vector decompositions in the context of quivers with more than one vertex.

Definition 2.3.1 (Vector decomposition)**.** *A vector decomposition $\hat{\mu}$ of length r (of the dimension vector $\mathbf{d}$) is an $r \times e$ matrix $(\hat{\mu}_{ij})$ with non-negative integer entries such that the summation of the entries in j-th column is d_j and every row vector is non-zero. We denote the set of vector decompositions of $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{\mathbb{V}}$ by $I_{\mathbf{d}}$.*

Let the i-th row (resp.j-th column) of $\hat{\mu}$ be $\hat{\mu}^{(i)}$ (resp.$\hat{\mu}_{(j)}$).

Suppose that $\hat{\mu}$ is a vector decomposition of $\mathbf{d}$ of length l, and $\hat{\mu}'$ is the vector decomposition obtained by summing up the k-th and $k+1$-th row vectors for some $1 \leq k \leq l-1$. In other words, the row vectors of $(r-1) \times e$ matrix $(\hat{\mu}'_{ij})$ satisfies

$$\hat{\mu}'^{(i)} = \begin{cases} \hat{\mu}^{(i)} & i < k, \\ \hat{\mu}^{(k)} + \hat{\mu}^{(k+1)} & i = k, \\ \hat{\mu}^{(i+1)} & i > k, \end{cases}$$

then we say that $\hat{\mu}'$ is a (simple) merge of $\hat{\mu}$ at k, and $\hat{\mu}$ is said to be a (simple) split of $\hat{\mu}'$ at $\hat{\mu}$. In general, we also call $\hat{\mu}$ a merge of $\hat{\lambda}$ if we have a finite sequence of simple merges from $\hat{\lambda}$ to $\hat{\mu}$. And we also say that $\hat{\lambda}$ is a split of $\hat{\mu}$. Here the terms merge and split are just combinatorial relations between vector decompositions. Later we will define operators associated to these decompositions, and we will abuse the notation to call these operators again merges and splits.

Now we introduce some geometric object to define quiver Schur algebras.

1. The affine space of representations of the quiver of dimension vector $\mathbf{d}$ is defined as

$$\mathrm{Rep}_{\mathbf{d}} = \bigoplus_{i \in \mathbb{V}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_{i+1}}).$$

2. The second object is a generalization of the partial flags and flag varieties.

   Definition 2.3.2 (Quiver flag)**.** *Let $\hat{\mu}$ be a vector decomposition of $\mathbf{d}$. A (partial) flag F of type $\hat{\mu}$ is a collection of flags F_j of type $\hat{\mu}_{(j)}$, that is, a sequence of vector spaces:*

$$0 = F_j^0 \subset F_j^1 \subset F_j^2 \subset \cdots \subset F_j^r = \mathbb{C}^{d_j}, \dim F_j^i - \dim F_j^{i-1} = \hat{\mu}_{ij}, \forall j \in \mathbb{V}.$$

   Such a composition is called of complete type if each $\hat{\mu}^{(i)}$ is a unit vector with only one non-zero entry, and the flags of this type are called complete flags.

   For a fixed vector decomposition $\hat{\mu}$, let $\mathcal{F}(\hat{\mu})$ be the set of flags of type $\hat{\mu}$, which naturally has the structure of a variety. In case that there is only one vertex associated with a non-trivial vector space, this variety is exactly the classical partial flag variety.

   In general, this variety is the product of the classical partial flag varieties of type $\hat{\mu}_{(j)}$ for each vertex j, and it is smooth and projective. As the name suggests, when $\hat{\mu}$ has complete type, then $\mathcal{F}(\hat{\mu})$ is the product of all full flag varieties of $\mathrm{GL}_{d_j}(\mathbb{C})$ for each $j \in \mathbb{V}$.

3. If F is a flag of type $\hat{\mu}$, then a representation f is compatible with this flag if $f_j(F_j^i) \subset F_{j+1}^{i-1}$ for any i, j. For fixed $\hat{\mu}$, we denote by

$$\mathcal{Q}(\hat{\mu}) = \{((V, f), F) \in \mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\mu}) | f_j(F_j^i) \subset F_{j+1}^{i-1}, \forall i, j\}$$

the set of representations with compatible flags of type $\hat{\mu}$. Note that if a representation is compatible with a flag of certain type, then it is automatically a nilpotent representation. There is also a natural structure of a variety on this set.

Moreover, for $\hat{\mu}, \hat{\lambda} \in I_{\mathbf{d}}$, we can consider the fiber product $\mathcal{Z}(\hat{\mu}, \hat{\lambda}) := \mathcal{Q}(\hat{\mu}) \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}(\hat{\lambda})$, which is a Steinberg-type variety.

Let $G = G_{\mathbf{d}} = \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_e}$ be the automorphism group of $V = (V_j)_{j \in \mathbb{V}}$, then it can act on all of these varieties mentioned above. More explicitly, let $g = (g_j)_{j \in \mathbb{V}} \in G$, then for $(V, f) \in \mathrm{Rep}_{\mathbf{d}}$, g will send it to $(V, g(f)) \in \mathrm{Rep}_{\mathbf{d}}$ determined by $g(f)_j = g_{j+1} \circ f_j \circ g_j^{-1}$. In particular, nilpotent representations remain nilpotent under the action of G.

For a flag F, the flag $g(F)$ consists of a sequence of spaces $g(F)_j^i$ such that $g(F)_j^i = g_j(F_j^i)$. If f is compatible with the flag F, then

$$g(f)_j(g(F)_j^i) = g_{j+1} \circ f_j \circ g_j^{-1}(g_j(F_j^i)) = g_{j+1} \circ f_j(F_j^i) \subset g_{j+1}(F_{j+1}^{i-1}) = g(F)_{j+1}^{i-1},$$

so $g(f)$ is compatible with $g(F)$. Also, as g is an automorphism, it will not change the type of flags. This determines the action of G on $\mathcal{F}(\hat{\mu}), \mathcal{Q}(\hat{\mu})$ and $\mathcal{Z}(\hat{\mu}, \hat{\lambda})$. Notice that the first projection $\mathcal{Q}(\hat{\mu}) \to \mathrm{Rep}_{\mathbf{d}}$ is a proper G-equivariant map, while the second projection $\mathcal{Q}(\hat{\mu}) \to \mathcal{F}(\hat{\mu})$ is a G-equivariant affine bundle over $\mathcal{F}(\hat{\mu})$.

2.3.2 Equivariant Borel-Moore homology

Here we give the definition of the equivariant Borel-Moore homology, which will be used in the geometric construction.

Suppose that X is a locally compact topological space which has the homotopy type of a finite CW-complex and admits a closed embedding into a smooth manifold. Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X, and then the Borel-Moore homology of X is defined as $H_i^{BM}(X) := H_i(\hat{X}, \infty)$, where $H_i(-,-)$ means the relative singular homology over $\mathbb{C}$. For more detail about Borel-Moore homology, we refer to [CG09].

Now suppose that H is a Lie group and let $\{E^n H \to B^n H\}$ be an approximation of the universal bundle $EH \to BH$. Let X be a topological space satisfying the conditions in the last paragraph, and further assume that X is also a complex algebraic variety of pure dimension $x/2$. Let $r_n = \dim_{\mathbb{R}} E^n H$ and $h = \dim_{\mathbb{R}} H$. Then the H-equivariant homology of X is defined as

$$H_i^{BM,H}(X) := \varprojlim_n H_{i+r_n-h}^{BM}(E^n H \times^H X).$$

An important property of Borel-Moore homology is the convolution structure. Suppose that X_1, X_2 and X_3 are three spaces and $Y_{12} \subset X_1 \times X_2, Y_{23} \subset X_2 \times X_3$ are closed subspaces, with suitable topological assumptions (see [CG09, 2.7.8]). We define

$$Y_{13} = Y_{12} \circ Y_{23} := \{(y_1, y_3) \in X_1 \times X_3 | \exists y_2 \in X_2 \text{ such that } (y_1, y_2) \in Y_{12}, (y_2, y_3) \in Y_{23}\},$$

then we have a convolution

$$H_i^{BM}(Y_{12}) \times H_j^{BM}(Y_{23}) \to H_{i+j-\dim_{\mathbb{R}} X_2}^{BM}(Y_{13}).$$

If we put $X = X_1 = X_2 = X_3$ with a proper map $X \to Z$, and $Y = Y_{12} = Y_{23} = X \times_Z X$, then we have $Y_{13} = Y$, and then the convolution map

$$H_i^{BM}(Y) \times H_j^{BM}(Y) \to H_{i+j-\dim_{\mathbb{R}} X}^{BM}(Y),$$

determines an algebra structure on $H_*^{BM}(Y)$. With a shift of degree $\dim_{\mathbb{R}} X$, this convolution becomes a graded map. With more suitable assumptions on the group action on each spaces, the convolution map can be lifted to equivariant Borel-Moore homologies. More detail about equivariant Borel-Moore homology and convolution algebras can be found in [Bri00, CG09].

In our case, X is the space of the union of $\mathcal{Q}(\hat{\mu})$ for all vector decompositions, and Z is $\mathrm{Rep}_{\mathbf{d}}$. In this case the projection map $X \to Z$ is proper. Then we actually have the following associative convolution product

$$H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\lambda})) \otimes H_*^{BM,G}(\mathcal{Z}(\hat{\lambda}, \hat{\mu})) \to H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\mu})), \forall \hat{\mu}, \hat{\lambda}, \hat{\mu} \in I_{\mathbf{d}},$$

as $\mathcal{Z}(\hat{\mu}, \hat{\lambda}) \circ \mathcal{Z}(\hat{\lambda}, \hat{\mu}) \subset \mathcal{Z}(\hat{\mu}, \hat{\mu})$.

With this product, we have the following algebras:

$$\mathcal{R}_{\mathbf{d}} = \bigoplus_{\hat{\mu}, \hat{\lambda} \in I_{\mathbf{d}}^c} H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\lambda})),$$

which is called the quiver Hecke algebra, and

$$\mathcal{A}_{\mathbf{d}} = \bigoplus_{\hat{\mu}, \hat{\lambda} \in I_{\mathbf{d}}} H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\lambda})),$$

which is called the quiver Schur algebra.

2.3.3 Faithful representation and geometric basis

Both algebras have natural representations coming from the theory of general convolution algebras by [CG09]. Here we describe it for $\mathcal{A}_{\mathbf{d}}$.

Consider the following direct sum of equivariant Borel-Moore homology groups

$$V_{\mathbf{d}} = \bigoplus_{\hat{\mu} \in I_{\mathbf{d}}} H_*^{BM,G}(\mathcal{Q}(\hat{\mu})),$$

and $\mathcal{A}_{\mathbf{d}}$ acts on $V_{\mathbf{d}}$ via the convolution product

$$H_*^{BM,G}(\hat{\mu}, \hat{\lambda}) \otimes H_*^{BM,G}(\mathcal{Q}(\hat{\lambda})) \to H_*^{BM,G}(\mathcal{Q}(\hat{\mu})).$$

The following result emphasizes the importance of this representation, which enable us to write down this algebra explicitly.

Proposition 2.3.3 ([SW11]). *The space $V_{\mathbf{d}}$ is faithful as an $\mathcal{A}_{\mathbf{d}}$-module.*

Remark 2.3.4. *Here we only consider quiver Schur algebras over $\mathbb{C}$, but actually the statement remains valid for prime characteristic. The proof of both cases can be found in [SW11].*

In the remaining part of this subsection, we present an explicit basis over $\mathbb{C}$ of $\mathcal{A}_{\mathbf{d}}$ introduced in [SW11]. Firstly we will construct several special types of elements in $\mathcal{A}_{\mathbf{d}}$.

E1 If $\hat{\mu} \in I_{\mathbf{d}}$, then we have a closed immersion $\mathcal{Q}(\hat{\mu})$ into $\mathcal{Z}(\hat{\mu}, \hat{\mu})$ by $((V, f), F) \mapsto ((V, f), F, F)$, which induces the proper push-forward

$$H_*^{BM,G}(\mathcal{Q}(\hat{\mu})) \to H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\mu})),$$

so every class $h \in H_*^{BM,G}(\mathcal{Q}(\hat{\mu}))$ can be identified with an element in $\mathcal{A}_{\mathbf{d}}$. We call such an operator a polynomial and denote it by $h_{\hat{\mu}}$. In particular, if h is the unit of of the Borel-Moore homology group, then we call this operator the idempotent $e_{\hat{\mu}}$.

E2 If $\hat{\mu}$ is a simple merge of $\hat{\mu}'$, then there is a natural forgetful map η from $\mathcal{F}(\hat{\mu}')$ to $\mathcal{F}(\hat{\mu})$. We let

$$\mathcal{Q}(\hat{\mu}, \hat{\mu}') = \{((V, f), F, F') \in \mathcal{Z}(\hat{\mu}, \hat{\mu}'), F = \eta(F')\},$$

and its class $[\mathcal{Q}(\hat{\mu}, \hat{\mu}')] \in H_*^{BM,G}(\mathcal{Z}(\hat{\mu}, \hat{\mu}'))$ is an element in $\mathcal{A}_{\mathbf{d}}$. As it is determined by the merge of vector decompositions, we also call this operator a merge and denote it by $m_{\hat{\mu},\hat{\mu}'}$.

More generally, if $\hat{\mu}$ is a merge of $\hat{\mu}'$, then there is a sequence of simple merges from $\hat{\mu}'$ to $\hat{\mu}$, and then $m_{\hat{\mu},\hat{\mu}'}$ is defined as the composition of these simple merge operators.

An immediate question is: does the resulting operator depend on the choice of the sequence? The answer is no, which can be shown via geometry (see [Prz19, Lemma 2.9]). However, we will present this property in a special case later.

E3 With the same setting as 2.3.3.E2, but with $\hat{\mu}, \hat{\mu}'$ swapped, we let

$$\mathcal{Q}(\hat{\mu}', \hat{\mu}) = \{((V, f), F', F) \in \mathcal{Z}(\hat{\mu}', \hat{\mu}), F = \eta(F')\}.$$

Then its class $[\mathcal{Q}(\hat{\mu}', \hat{\mu})] \in H_*^{BM,G}(\mathcal{Z}(\hat{\mu}', \hat{\mu}))$ is also an element in $\mathcal{A}_{\mathbf{d}}$. In this case, $\hat{\mu}'$ is a split of $\hat{\mu}$, so we also call this operator a split, and denote it by $s_{\hat{\mu}',\hat{\mu}}$.

The split generators for general splits are defined in a similar way as the merge operators, and the resulting operators are again independent of the choice of the sequence by the same reason.

E4 Suppose that $\hat{\mu}$ and $\hat{\lambda}$ are two vector decompositions with length r, and the matrix expressions $(\hat{\mu}_{ij})$ and $(\hat{\lambda}_{ij})$ only differs from a row action. Suppose that $u \in S_r$ has a reduced expression $u = s_{i_l} \cdots s_{i_1}$ such that the permutation u can send $(\hat{\mu}_{ij})$ to $(\hat{\lambda}_{ij})$ (Notice that there can be more than one permutation which satisfies the requirement.), then we set

$$u^{(k)} = s_{i_k} \cdots s_{i_1},$$

and we define $\hat{\mu}_{2k} = u^{(k)}(\hat{\mu})$ for $0 \le k \le l$. So $\hat{\mu} = \hat{\mu}_0$ and $\hat{\lambda} = \hat{\mu}_{2l}$, and all of these $\hat{\mu}_{2k}$ are vector decompositions of length l. Moreover, we define $\hat{\mu}_{2k+1}$ as the merge of $\hat{\mu}_{2k}$ at position

i_k for $0 \le k \le l-1$, and all of them are vector decompositions of length $l-1$. For any k, $\hat{\mu}_{2k+1}$ is a merge of $\hat{\mu}_{2k}$ and $\hat{\mu}_{2k}$ is a split of $\hat{\mu}_{2k-1}$. Then we define

$$c^{[u]}_{\hat{\lambda},\hat{\mu}} = s_{\hat{\mu}_{2l},\hat{\mu}_{2l-1}} \cdot m_{\hat{\mu}_{2l-1},\hat{\mu}_{2l-2}} \cdot \cdots \cdot s_{\hat{\mu}_2,\hat{\mu}_1} \cdot m_{\hat{\mu}_1,\hat{\mu}_0},$$

where the multiplication is convolution. We obtain an element of $H^{BM,G}_*(\mathcal{Z}(\hat{\lambda},\hat{\mu}))$, which we denote by $[u]$ because this element depends on the reduced expression. Such an operator is called a crossing.

We introduce the following notation, which is almost the same as what we do when we define affine Hecke algebras and affine Schur algebras. Let W be the symmetric group S_d, with the usual Coxeter group presentation. Let I_d be the set of all compositions of d. For any composition λ of d, let W_λ be the parabolic subgroup associated to λ, and w_λ be the longest element in W_λ. Also, we let ${}^\lambda W$ be the set of shortest representative elements of $W_\lambda \backslash W$.

Also, let $I = \{1, 2, \ldots, d-1\}$, then I_d is bijectively corresponding to the set of subsets of I, given by $\lambda \mapsto J$ if W_λ is exactly the subgroup of W generated by s_i for all $i \in J$. So we also have the notion $W_J = W_\lambda$ and ${}^J W = {}^\lambda W$. For example, the partition $(1, 1, \ldots, 1)$ corresponds to the empty set, while (d) corresponds to I. Moreover, if K, J, L are subsets of I such that $K \subset J, L \subset J$, then we let ${}^K W^L_J$ be the set of the shortest representative elements of $W_K \backslash W_J / W_L$.

For any $\hat{\mu}, \hat{\lambda} \in I_{\mathbf{d}}$ of length r, and any $w \in \prod_{j \in \mathbb{V}} {}^{\hat{\lambda}_{(j)}} W^{\hat{\mu}_{(j)}}_{\mathbf{d}_j}$, by [DJ86] there exist unique $\hat{\mu}', \hat{\lambda}' \in I_{\mathbf{d}}$ of the same length (denoted by r) such that

$$W_{\hat{\lambda}'} = W_{\hat{\lambda}} \cap w W_{\hat{\mu}} w^{-1}, W_{\hat{\mu}'} = W_{\hat{\mu}} \cap w^{-1} W_{\hat{\lambda}} w.$$

Moreover, the matrices representing $\hat{\lambda}', \hat{\mu}'$ are the same up to the permutation on row vectors determined by w, and we denote this permutation by $u \in S_r$ such that $u(\hat{\mu}') = \hat{\lambda}'$. Then the operator $c^w_{\hat{\lambda}',\hat{\mu}'}$ can be defined after fixing a reduced expression of u.

Now for any $h \in H^{BM,G}_*(\mathcal{Q}(\hat{\lambda}'))$, we can define

$$\mathbf{b}^w_{\hat{\lambda},\hat{\mu}}(h) := m_{\hat{\lambda},\hat{\lambda}'} \cdot c^w_{\hat{\lambda}',\hat{\mu}'} \cdot h_{\hat{\mu}'} \cdot s_{\hat{\mu}',\hat{\mu}} \in H^{BM,G}_*(\mathcal{Z}(\hat{\lambda},\hat{\mu})). \tag{9}$$

Finally, we can give the main theorem of this section. The operators defined as in (9) actually form a $\mathbb{C}$-basis of the quiver Schur algebra $\mathcal{A}_{\mathbf{d}}$.

Theorem 2.3.5 (Geometric basis, [SW11])**.** *Let range*

- *$\hat{\mu}, \hat{\lambda}$ over $I_{\mathbf{d}} \times I_{\mathbf{d}}$*
- *w over $\prod_{j \in \mathbb{V}} {}^{\hat{\lambda}_{(j)}} W^{\hat{\mu}_{(j)}}_{\mathbf{d}_j}$, and*
- *h over a $\mathbb{C}$-basis of $H^{BM,G}_*(\mathcal{Q}(\hat{\mu}'))$ where $W_{\hat{\mu}'} = W_{\hat{\mu}} \cap w^{-1} W_{\hat{\lambda}} w$.*

Then all these $\mathbf{b}^w_{\hat{\lambda},\hat{\mu}}(h)$ defined in (9) form a $\mathbb{C}$-basis of $\mathcal{A}_{\mathbf{d}}$. This basis is called the geometric basis of $\mathcal{A}_{\mathbf{d}}$.

The proof of this theorem in [SW11] uses geometric method.

An immediate result of Theorem 2.3.5 is that merges, splits and polynomials generate the whole $\mathcal{A}_{\mathbf{d}}$ as a $\mathbb{C}$-algebra, because the crossings can be written as the product of merges and splits.

Also, if we only let $\hat{\mu}, \hat{\lambda}$ range over $I^c_{\mathbf{d}}$, then we obtain the geometric basis of the quiver Hecke algebra. In this case, w will range over the whole $\prod_{j\in\mathbb{V}} W_{\mathbf{d}_j}$, and h ranges over a $\mathbb{C}$-basis of $H_*^{BM,G}(\mathcal{Q}(\hat{\mu}))$. Also, the element $\mathbf{b}^w_{\hat{\lambda},\hat{\mu}}(h)$ is exactly $c^w_{\hat{\lambda},\hat{\mu}} \cdot h_{\hat{\mu}}$ in this case. And this implies that the quiver Hecke algebra $\mathcal{H}_{\mathbf{d}}$ is generated by crossings and polynomial operators as a $\mathbb{C}$-algebra.

Remark 2.3.6. *As we mention before, different choices of the reduced expression while defining the operator $c^{[u]}_{\hat{\lambda},\hat{\mu}}$ can result in different geometric basis which differ by a linear combination of maps corresponding to shorter double cosets. More detail can be found in [SW11].*

2.4 Algebraic description of quiver Schur algebra with one vertex

In this section, we describe a special case of $\mathcal{A}_{\mathbf{d}}$ explicitly and introduce the diagrammatic expression when $e \geq 2$ and the dimension vector $\mathbf{d}$ is concentrated on a single vertex in Γ. Without loss of generality, we assume that $\mathbf{d} = (d, 0, \dots, 0)$, then $I_{\mathbf{d}}$ can be identified with I_d, and we may simply denote this algebra by A_d, and we will then write $\mu = \hat{\mu}_{(1)} \in I_d$ instead of $\hat{\mu} \in I_{\mathbf{d}}$.

Before we start, some algebraic constructions need to be introduced.

Definition 2.4.1 (Demazure operator). *Let $R = \mathbb{C}[x_1, \dots, x_d]$, then for any simple reflection $s_i \in W$, where $1 \leq i \leq d-1$, and any polynomial $p \in R$, we set*

$$\partial_i : p \mapsto \frac{p - s_i(p)}{x_i - x_{i+1}},$$

where s_i acts on p by interchanging variables of x_i and x_{i+1}.

By convention, $\partial_e = \mathrm{Id}$ for the identity element $e \in W$, and for any element $w \in W$ of length larger than 0, we choose an arbitrary reduced expression $w = s_{i_1} \cdots s_{i_k}$, and $\partial_w := \partial_{i_1} \circ \cdots \circ \partial_{i_k}$.

This is well-defined because this construction is independent of the choice of the reduced expression since one can show that they satisfy the following relations:

i $\partial_i \circ \partial_i = 0$

ii $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

iii $\partial_w \circ \partial_{w'} = \begin{cases} \partial_{ww'} & \ell(ww') = \ell(w) + \ell(w') \\ 0 & \ell(ww') < \ell(w) + \ell(w') \end{cases}$

For an arbitrary polynomial $f \in R$, $\partial_i(f)$ is still a polynomial in R, which follows from the fact that

$$\partial_i(x_i^{k+1}) = \frac{x_i^{k+1} - x_{i+1}^{k+1}}{x_i - x_{i+1}} = \sum_{j=0}^{k} x_i^j x_{i+1}^{k-j}, \forall k \geq 0.$$

A polynomial f satisfies that $\partial_i(f) = 0$ if and only if f is s_i-invariant. Besides, if f is of degree l, then $\partial_w(f)$ is either zero or of degree $l - \ell(w)$, as ∂_i lowers the degree by one. Moreover, for ∂_{s_i}, we have the following "Leibniz rule":

$$\partial_{s_i}(f \cdot g) = \partial_{s_i}(f) \cdot g + s_i(f) \cdot \partial_{s_i}(g) = \partial_{s_i}(f) \cdot s_i(g) + f \cdot \partial_{s_i}(g). \tag{10}$$

For a fixed integer d, and any composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in I_d$, we can consider the parabolic subgroup W_λ, and the space of invariants $R_\lambda := \mathbb{C}[x_1, x_2, \ldots, x_d]^{W_\lambda}$. An extreme case is that the composition $\lambda_0 := (n)$ has only one component, then the space of invariants is $R_d := \mathbb{C}[x_1, x_2, \ldots, x_d]^W$. All these R_λ, as a $\mathbb{C}$-algebra, are again isomorphic to the polynomial ring of d variables over $\mathbb{C}$. In fact, all of these R_λ are isomorphic to the tensor product of total invariant polynomial rings, and the total invariant case follows from the following classical result:

Theorem 2.4.2 (Fundamental theorem of symmetric polynomials,[Mac15]). *Let*

$$e_k(x_1, \ldots, x_d) = \sum_{1 \leq j_1 < \cdots < j_k \leq d} x_{j_1} \cdots x_{j_k}$$

for $1 \leq k \leq d$, then $\mathbb{C}[x_1, x_2, \ldots, x_d]^{S_d}$ is the polynomial ring generated by $e_1, e_2, \ldots, e_d$.

The space of W_μ-invariants R_μ also have the following properties:

Proposition 2.4.3. *Let $\mu \in I_d$, and let w_μ be the longest element in W_μ. Then we have*

1. *The operator ∂_{w_μ} sends R to R_μ,*
2. *If $f \in R_\mu$ and $g \in R$, then for any $w \in W_\mu$, we have $\partial_w(fg) = f\partial_w(g)$.*

Proof. In fact, a polynomial is in R_μ if and only if it is annihilated by ∂_i for all s_i appearing in W_μ. Suppose that h is an arbitrary polynomial in R. We have $\partial_i(\partial_{w_\mu}(h)) = \partial_i \circ \partial_{w_\mu}(h)$, and since $\ell(s_i w_\mu) < \ell(w_\mu)$ when $s_i \in W_\mu$, we have $\partial_i \circ \partial_{w_\mu} = 0$ by relation 2.4.**iii**, so $\partial_{w_\mu}(h)$ is killed by all ∂_i in W_μ.

Also, for any $w \in W_\mu$, its reduced expression is a product of simple reflections appearing in W_μ, and all of these simple reflections annihilate f as $f \in R_\mu$. Therefore, we have $\partial_w(fg) = f\partial_w(g)$, as the Leibniz rule (10) implies that

$$\partial_i(fg) = f\partial_i(g), \forall s_i \in W_\mu.$$

□

An important property of these invariant polynomial rings is that all R_λ are free as R_d-modules. For the space R of all polynomials, we have the following lemma:

Lemma 2.4.4. *Let $\alpha_d(x_1, x_2, \ldots, x_d) = x_1^{d-1} x_2^{d-2} \cdots x_{d-1} \in R$, then the set*

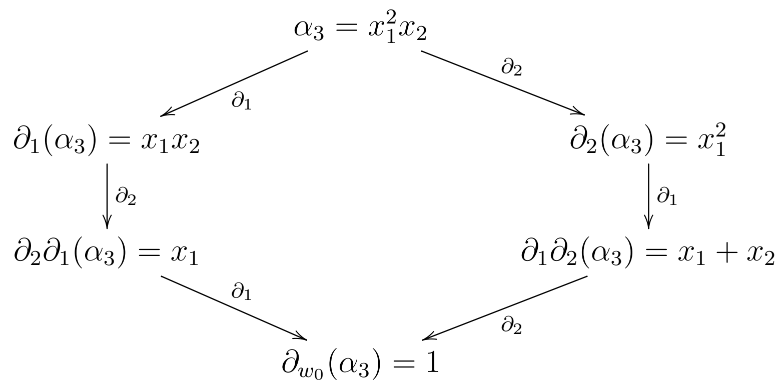
$$\boldsymbol{B} := \{\partial_w(\alpha_d) | w \in W\}$$

forms an R_d-basis of R.

Proof. See [BGG73] or [Bru14]. □

In fact, the polynomial α_d can be replaced by an arbitrary polynomial not annihilated by any ∂_s with the same degree as α_d. Moreover, we see from 2.4.**iii** and Proposition 2.4.3 that the map $\partial_{w_0} : R \to R_d$ annihilates $\partial_w(\alpha_d)$ if $w \neq 1$, and it sends $R_d \cdot \alpha_d$ bijectively to R_d. So $\partial_{w_0} : R \to R_d$ is surjective. This immediately implies that for any $\lambda \in I_d$, the map $\partial_{w_\lambda} : R \to R_\lambda$ is also surjective.

Example 2.4.5. *Let $d = 3$, then $\alpha_3(x_1, x_2, x_3) = x_1^2 x_2$, and the basis $\boldsymbol{B}$ is*



As a corollary, other R_λ are also free over R_d.

Corollary 2.4.6. *The set*

$$\boldsymbol{B}_\lambda := \{\partial_{w_\lambda w}(\alpha_d) | w \in {}^\lambda W\}$$

forms an R_d-basis of R_λ.

Proof. A polynomial is in R_λ if and only if it is invariant under the action of W_λ, which is equivalent to being invariant under the action of any simple reflection in W_λ, and which is again equivalent to being annihilated by any ∂_s for any simple reflection s in W_λ.

So all elements in the set $\{\partial_{w_\lambda w}(\alpha_d) | w \in {}^\lambda W\}$ are in R_λ. In fact, for any simple reflection $s \in W_\lambda$, we have $sw_\lambda \in W_\lambda$. As $w \in {}^\lambda W$, we have

$$\ell(w_\lambda w) = \ell(w_\lambda) + \ell(w)$$

$$\ell(sw_\lambda w) = \ell(sw_\lambda) + \ell(w) = \ell(w_\lambda) + \ell(w) - 1.$$

This, by 2.4.**iii**, implies that $\partial_s \circ \partial_{w_\lambda w} = 0$. Then all $\partial_{w_\lambda w}(\alpha_d)$ are annihilated by ∂_s for any simple reflection $s \in W_\lambda$, which implies that they are in R_λ. Also, they are obviously linearly independent by Lemma 2.4.4.

Now we need to show that they span the whole R_λ. Now we consider an element of the form

$$f = \sum_{u \in W_\lambda \backslash \{e\}} \sum_{w \in {}^\lambda W} c_{u,w} \partial_{u w_\lambda w}(\alpha_d), c_{u,w} \in R_n$$

and suppose that $f \in R_\lambda$, so it suffices to show that all these $c_{u,w} = 0$. As $f \in R_\lambda$, we have $\partial_v(f) = 0$ for all $v \in W_\lambda \backslash \{e\}$.

We will use induction on $\ell(u)$.

(1) Firstly, if $u' \in W_\lambda$ and $\ell(u') = \ell(w_\lambda)$, then $c_{u',w} = 0$.

In fact, $\ell(u') = \ell(w_\lambda)$ implies that $u' = w_\lambda$, then we put $v = w_\lambda$, and we see that

$$
\begin{aligned}
0 = \partial_{w_\lambda}(f) &= \sum_{u \in W_\lambda \backslash \{e\}} \sum_{w \in {}^\lambda W} c_{u,w} \partial_{w_\lambda} \circ \partial_{u w_\lambda w}(\alpha_d) \\
&= \sum_{u \in W_\lambda \backslash \{e\}} \sum_{w \in {}^\lambda W} c_{u,w} (\partial_{w_\lambda} \circ \partial_{u w_\lambda}) \circ \partial_w(\alpha_d) && (u w_\lambda \in W_\lambda, w \in {}^\lambda W) \\
&\overset{(2.4.\mathbf{iii})}{=} \sum_{u = u'} \sum_{w \in {}^\lambda W} c_{u,w} (\partial_{w_\lambda} \circ \partial_{u w_\lambda}) \circ \partial_w(\alpha_d) && (w_\lambda \text{ is maximal in } W_\lambda, \text{ use } (2.4.\mathbf{iii})) \\
&= \sum_{w \in {}^\lambda W} c_{u',w} \partial_{w_\lambda w}(\alpha_d)
\end{aligned}
$$

and we know that all $c_{u',w}$ has to be zero as these $\partial_{w_\lambda w}(\alpha_d)$'s are linear independent by Lemma 2.4.4.

(2) If we have already $c_{u,w} = 0$ for all $u \in W_\lambda \backslash \{e\}$ such that $k+1 \leq \ell(u) \leq \ell(w_\lambda)$ for some $k \geq 1$, now we show that for an arbitrary element $u' \in W_\lambda \backslash \{e\}$ of length k, the coefficients $c_{u',w} = 0$ for all $w \in {}^\lambda W$.

As $\ell(u') = k \geq 1$ and $u' \in W_\lambda$, we put $v = u'$ and we have

$$
\begin{aligned}
0 = \partial_{u'}(f) &= \sum_{u \in W_\lambda \backslash \{e\}} \sum_{w \in {}^\lambda W} c_{u,w} \partial_{u'} \circ \partial_{u w_\lambda w}(\alpha_d) \\
&= \sum_{u \in W_\lambda, 1 \leq \ell(u) \leq k} \sum_{w \in {}^\lambda W} c_{u,w} \partial_{u'} \circ \partial_{u w_\lambda w}(\alpha_d) \\
&= \sum_{u \in W_\lambda, 1 \leq \ell(u) \leq k} \sum_{w \in {}^\lambda W} c_{u,w} (\partial_{u'} \circ \partial_{u w_\lambda}) \circ \partial_w(\alpha_d) && (\text{Here we use } u w_\lambda \in W_\lambda, w \in {}^\lambda W) \\
&\overset{(2.4.\mathbf{iii})}{=} \sum_{u = u'} \sum_{w \in {}^\lambda W} c_{u,w} (\partial_{u'} \circ \partial_{u w_\lambda}) \circ \partial_w(\alpha_d) && (\text{Here } \ell(u') + \ell(u w_\lambda) \geq \ell(u' u w_\lambda)) \\
&= \sum_{w \in {}^\lambda W} c_{u',w} \partial_{w_\lambda w}(\alpha_d)
\end{aligned}
$$

and this implies that $c_{u',w} = 0$.

So by induciton we know that all $c_{u,w} = 0$, and this means that any element in R_λ has to be a linear combination of vectors in $\mathbf{B}_\lambda$.

In conclusion, the $\mathbf{B}_\lambda$ forms an R_d-basis of R_λ. □

2.4.1 Computing homology groups

Now we will use the faithful representation mentioned in last subsection to describe our algebra A_d as a subring of the endomorphism ring of a vector space. We need to actually write down the homology groups and the actions of generators explicitly.

Since the quiver Γ has at least two vertices and the dimension vector is concentrated on one vertex, the variety of representation $\mathrm{Rep}_{\mathbf{d}}$ has only one point, which attaches zero maps to all arrows, and the flag variety $\mathcal{F}(\hat{\mu}) = \mathcal{F}(\mu)$ is just the usual partial flag variety of $\mathbb{C}^d$ of type μ. We also have $\mathcal{Q}(\mu) = \mathrm{Red}_d \times \mathcal{F}(\mu) \simeq \mathcal{F}(\mu)$ as all flags are compatible now, and $\mathcal{Z}(\mu, \lambda)$ is isomorphic to $\mathcal{F}(\mu) \times \mathcal{F}(\lambda)$.

Proposition 2.4.7. *There exists an isomorphism $R_\mu \to H_*^{BM,G}(\mathcal{Q}(\mu))$ as algebras over $\mathbb{C}$.*

Proof. As $\mathcal{Q}(\mu) = \mathcal{F}(\mu)$ which is the quotient of G by a paraabolic subgroup of type μ, it is smooth and projective. Note that for any smooth variety X, we have by [EG98] $H_*^{BM,G}(X) \simeq H_G^{2\dim X - *}(X)$, hence it suffices to compute the G-equivariant cohomology of $\mathcal{Q}(\mu)$.

Now we see that $H_G^*(\mathcal{Q}(\mu)) = H_G^*(G/P_\mu) \simeq H_{P_\mu}^*(*)$. Let G_μ be the Levi subgroup of P_μ, then we see that $P_\mu/G_\mu \simeq U$ is contractible, so we have $H_G^*(\mathcal{Q}(\mu)) \simeq H_P^*(*) \simeq H_{G_\mu}^*(*) \simeq H^*(BG_\mu) \simeq \bigotimes_{j=1}^{l} H_{\mathrm{GL}_{\mu_j}}^*(*)$, so it suffices to show that $H_G^*(*) \simeq R_n = \mathbb{C}[x_1, \ldots, X_n]^W$, which follows from

$$H_G^*(*) \simeq H_T^*(*)^W = \mathbb{C}[x_1, \ldots, X_n]^W$$

by [Bri98]. □

So the faithful representation of $\mathcal{A}_d$ acts on $V_d = \bigoplus_{\mu \in I_d} R_\mu$. Also, the action of the polynomial generator in $H_*^{BM,G}(\mathcal{Q}(\mu))$ coincides with cap product of Borel-Moore homology (see [CG09, 2.6.16]), so its action is exactly the regular action of R_μ on itself.

Remark 2.4.8. *In fact we can also use equivariant cohomology groups instead of the equivariant Borel-Moore homology groups, but it turns out that the grading of equivariant Borel-Moore homology groups is easier to deal with. In this thesis we introduce the grading structure via algebraic definition and its relationship with geometric construction will not be discussed. More explanations can be found in [SW11].*

Now we need to know the action of merges and splits on V_d. We firstly compute the actions of operators induced by simple merges and simple splits. And it suffices to consider the following most simple case: μ is a composition of length 2 and $k = 1$.

Let $a, b \in \mathbb{Z}_{>0}$. Then $\begin{pmatrix} a \\ b \end{pmatrix}$ is a split of $\big(a + b\big)$. Now $\mathcal{Q}(a+b)$ is a point equipped with an action of GL_{a+b}, and $\mathcal{Q}(a, b)$ equals to $Gr(a, a+b)$. Let ι be the zero section of the G-equivariant fiber bundle $\pi : \mathcal{Q}(a, b) \to \mathcal{F}(a+b)$ and $q : \mathcal{F}(a, b) \to \mathcal{Q}(a+b)$ is the proper map forgetting the subspaces of dimension a, and both of two maps are G-equivariant, so we have the following diagram

$$H_*^{BM,G}(\mathcal{Q}(a,b)) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} H_*^{BM,G}(\mathcal{F}(a,b)) \underset{q^*}{\overset{q_*}{\rightleftarrows}} H_*^{BM,G}(\mathcal{Q}(a+b)),$$

and the action of merges and splits is given by the following proposition:

Proposition 2.4.9. *The following diagram commutes:*

$$\begin{array}{ccc} H_*^{BM,G}(\mathcal{Q}(a,b)) & \underset{\iota_* \circ q^*}{\overset{q_* \circ \iota^*}{\rightleftarrows}} & H_*^{BM,G}(\mathcal{Q}(a+b)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbb{C}[x_1, \ldots, x_{a+b}]^{S_a \times S_b} & \underset{\textit{inclusion}}{\overset{\mathrm{int}}{\rightleftarrows}} & \mathbb{C}[x_1, \ldots, x_{a+b}]^{S_{a+b}} \end{array}$$

and the map int*, which means integration, is given by*

$$f \mapsto \sum_{w\in S_{a+b}} (-1)^{\ell(w)} w(f) \cdot \frac{w(\prod_{1\le j<k\le a}(x_j-x_k)\cdot \prod_{a+1\le l<m\le a+b}(x_l-x_m))}{a!b!\prod_{1\le j<k\le a+b}(x_j-x_k)} \tag{11}$$

where ℓ is the length function of S_{a+b}.

Proof. In our setting, the map π and ι are just identity maps, so π^* and ι^* are isomorphisms. Also, the map ι_* is also the isomorphism.

As we know that $H_*^{BM,G}(\mathcal{F}(a,b)) \simeq H_*^{BM,T}(\mathcal{F}(a,b))^{S_{a+b}}$ and $H_*^{BM,G}(\mathcal{Q}(a+b)) \simeq H_*^{BM,T}(\mathcal{Q}(a+b))^{S_{a+b}}$, and the fact that there exists T-fixed points in $\mathcal{Q}(a,b)$, so the natural map $H_*^{BM,T}(\mathcal{Q}(a+b)) \to H_*^{BM,T}(\mathcal{F}(a,b))$ is injective, and hence q^* has to be injective, which turns out to the inclusion of the total invariants by considering the map $H_*^{BM,T}(*) \to H_*^{BM,T}(\mathcal{F}(a,b)) \to H_*^{BM,T}(\mathcal{F}(a,b)^T)$.

Now it suffices to analyze the map q_*, which is the equivariant integration:

$$\begin{aligned}\int_{\mathcal{F}(a,b)} f &= \int_{\mathcal{F}(a,b)} (\int_{P_{a,b}/B_{a+b}} \frac{\prod_{1\le j<k\le a}(x_j-x_k)\prod_{a+1\le l<m\le a+b}(x_l-x_m)}{a!b!}) \cdot f \\ &= \int_{G_{a+b}/B_{a+b}} \frac{\prod_{1\le j<k\le a}(x_j-x_k)\prod_{a+1\le l<m\le a+b}(x_l-x_m)}{a!b!} \cdot f\end{aligned} \tag{12}$$

and we apply the formula for full flag variety

$$\int_{G_c/B_c} g = \frac{\sum_{S_c}(-1)^{\ell(w)} w\cdot g}{\prod_{1\le j<k\le c}(x_j-x_k)} \tag{13}$$

we will get the formula in the statement. A more detailed treatment of the integration formula of equivariant cohomology classes can be found in [TA20]. □

Let $w_0[a,b]$ be the minimal representative of w_0 in $S_b\times S_a\backslash S_{a+b}/S_a\times S_b$. In other words,

$$w_0[a,b](j) = \begin{cases} b+j & 1\le j\le a, \\ j-a & a<j\le a+b, \end{cases} \tag{14}$$

then we have the following lemma:

Lemma 2.4.10. *The map* int *is the same as*

$$\partial_{w_0[a,b]} : \mathbb{C}[x_1,\dots,x_{a+b}]^{S_a\times S_b} \to \mathbb{C}[x_1,\dots,x_{a+b}]^{S_{a+b}}$$

Proof. By [Ful96], for w_0 the longest element in S_r for any non-negative integer r, we have the following formula

$$\partial_{w_0} : f \mapsto \sum_{w\in S_r}(-1)^{\ell(w)} w(f) \cdot \frac{1}{\prod_{1\le i<j\le r}(x_i-x_j)}. \tag{15}$$

We denote by $w_{(a,b)}$ the longest element in $S_a \times S_b$, which is the product of longest elements in S_a and S_b, and w_{a+b} the longest element in S_{a+b}.

For any $f \in \mathbb{C}[x_1, \dots, x_{a+b}]^{S_a \times S_b}$, we denote by

$$g = f \cdot \prod_{1 \le j < k \le a} (x_j - x_k) \cdot \prod_{a+1 \le l < m \le a+b} (x_l - x_m) \in R.$$

Then we have

$$\partial_{w_{(a,b)}}(g) = f \cdot \partial_{w_{(a,b)}}\Big(\prod_{1 \le j < k \le a} (x_j - x_k) \cdot \prod_{a+1 \le l < m \le a+b} (x_l - x_m)\Big),$$

and by applying the formula (15) for $r = a$ and $r = b$ respectively, and the fact that

$$w\Big(\prod_{1 \le j < k \le a} (x_j - x_k) \cdot \prod_{a+1 \le l < m \le a+b} (x_l - x_m)\Big) = (-1)^{\ell(w)} \prod_{1 \le j < k \le a} (x_j - x_k) \cdot \prod_{a+1 \le l < m \le a+b} (x_l - x_m),$$

we see $\partial_{w_{(a,b)}}(g) = f \cdot |S_a \times S_b| = (a!b!)f$. Then we have

$$\partial_{w_0[a,b]}(f) = \frac{1}{a!b!} \partial_{w_0[a,b]} \partial_{w_{(a,b)}}(g) = \frac{1}{a!b!} \partial_{w_{a+b}}(g),$$

and we apply the formula (15), so

$$\partial_{w_0[a,b]}(f) = \frac{1}{a!b!} \sum_{w \in S_{a+b}} (-1)^{\ell(w)} w(g) \frac{1}{\prod_{1 \le i < j \le a+b} (x_i - x_j)},$$

which is exactly $\mathrm{int}(f)$.

□

This lemma shows that the merge operators satisfy associativity and answer the question arising when we define 2.3.3.E2 in this special case. More precisely, if μ_1, μ_2, μ_3 are compositions of d such that μ_1 is a merge of μ_2 and μ_2 is a merge of μ_3, then the merge operators defined by them satisfy

$$m_{\mu_1, \mu_3} = m_{\mu_1 \mu_2} \cdot m_{\mu_2, \mu_3}.$$

Indeed, if we write this equality in terms of Demazure operators, it follows from the relation 2.4.**iii** of Demazure operators. It is obvious that the split operators also satisfy this property, because the composition of inclusions is still the inclusion, so we have

$$s_{\mu_3, \mu_1} = s_{\mu_3, \mu_2} \cdot s_{\mu_2, \mu_1}.$$

As every merge (resp. split) of vector decompositions is a sequence of simple merges (resp. splits), the algebra $\mathcal{A}_{\mathbf{d}}$ is generated by simple merge operators, simple split operators and polynomials.

Remark 2.4.11. *The associativity of these operators remains true for the quiver Schur algebra $\mathcal{A}_{\mathbf{d}}$ for an arbitrary quiver involving general dimension vectors and more vertices. The geometric proof of this property can be found in [Prz19, §2].*

Remark 2.4.12. *In this one-vertex case, the crossing operator defined in 2.3.3.E4 does not depend on the choices of the reduced expression. In fact, for any reduced expression, the corresponding operator $c^{[w]}_{\lambda,\mu}$ just acts by ∂_w for $w \in {}^{\lambda}W^{\mu}$ in the faithful representation.*

2.4.2 Explicit actions and the grading

Now we give the explicit description of generators as maps between summands of $V_d = \bigoplus_{\mu \in I_d} R_\mu$.

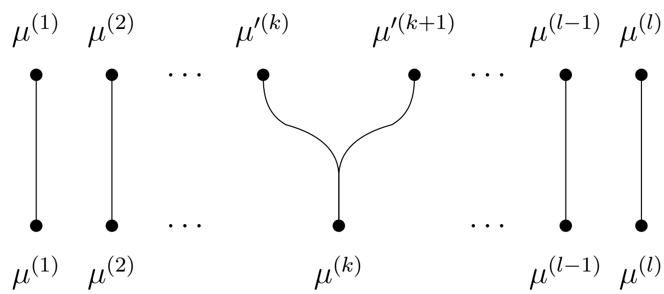
1° For μ and μ' in I_d such that μ' is a split of μ' at k, which means

$$\mu^{(i)} = \begin{cases} \mu'^{(i)} & i < k \\ \mu'^{(k)} + \mu'^{(k+1)} & i = k \\ \mu'^{(i+1)} & i > k \end{cases},$$

then we have split operator

$$\begin{aligned} s_{\mu',\mu} : R_\mu &\to R_{\mu'} \\ f &\mapsto f, \end{aligned} \tag{16}$$

We use the following diagram to represent this element:

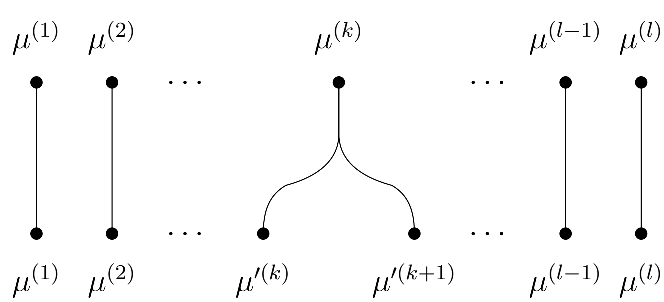


This operator is of degree $-\mu'^{(k)}\mu'^{(k+1)}$. In general, if λ is a split of μ, then the degree of $s_{\lambda,\mu}$ equals $\ell(w_\lambda) - \ell(w_\mu)$.

2° With the same setting as 2.4.2.1° , we have a map in the opposite direction, the merge operator, given by

$$\begin{aligned} m_{\mu,\mu'} : R_{\mu'} &\to R_\mu \\ f &\mapsto \partial_{w_0^{\mu,\mu'}}(f), \end{aligned} \tag{17}$$

where $w_0^{\mu,\mu'}$ is the longest element in $W_\mu^{\mu'}$. In case that $k = 1$ and the length of μ' is 2, then this element is exactly $w_0[\mu'^{(1)}, \mu'^{(2)}]$ defined in (14). We use the following diagram to represent this element:

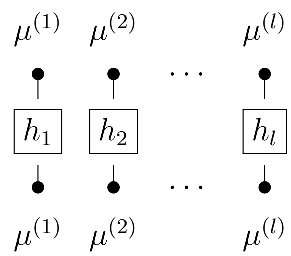


This operator is again of degree $-\mu'^{(k)}\mu'^{(k+1)}$. Also, if μ is a merge of λ, then the degree of $m_{\mu,\lambda}$ equals to $\ell(w_\lambda) - \ell(w_\mu)$.

3° The third map is the polynomial. Let $h \in H_*^{BM,G}(\mu) \simeq R_\mu$, which is now indeed a polynomial, then we have

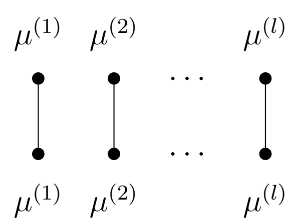
$$\begin{aligned} h_\mu : R_\mu &\to R_\mu \\ f &\mapsto h \cdot f \end{aligned} \tag{18}$$

and we use the following diagram to represent this element:



where each h_k is a symmetric polynomial with $\mu^{(k)}$ variables, and h is a linear combination of the products of the form $h_1 \cdots h_l$. If h is a monomial, then the degree of h_μ is $2\deg(h)$, where $\deg(h)$ is the degree of h as a polynomial in $x_1, \ldots, x_d$.

4° In particular, in the case of 2.4.2.3° , if h is just the identity, we call the map as idempotents e_μ, with the following diagrammatic expression:



As the name and the diagram suggest, this element is indeed an idempotent, and $\sum_{\mu \in I_d} e_\mu$ is exactly the identity of $\mathcal{A}_d$. Every idempotent is of degree 0.

All these diagrams should be read from bottom to top. We represent the product $v_1 \cdot v_2$ of two elements v_1 and v_2 by stacking the diagram of v_1 on the diagram v_2, so every element in $\mathcal{A}_d$ is a linear combination of these diagrams.

Now we have an alternative definition for the algebra $\mathcal{A}_d$.

Definition 2.4.13 (Alternative definition)**.** *The quiver Schur algebra $\mathcal{A}_d$ is the $\mathbb{C}$-subalgebra of* $\mathrm{End}_{\mathbb{C}}(V_d)$*, where*

$$V_d = \bigoplus_{\mu \in I_d} R_\mu,$$

generated by simple merge operators, simple split operators and polynomial operators presented in 2.4.2.

The following proposition states that $\mathcal{A}_{\mathbf{d}}$ is a free unital algebra over R_d.

Proposition 2.4.14. *The quiver Schur algebra $\mathcal{A}_d$ is a free algebra over R_d.*

Proof. For any $f \in R_d$, we can define the map

$$\begin{aligned} \mathcal{A}_d &\to \mathcal{A}_d \\ \mathbf{b}^w_{\lambda,\mu}(h) &\mapsto \mathbf{b}^w_{\lambda,\mu}(f \cdot h). \end{aligned}$$

where λ, μ range over I_d, w ranges over ${}^{\lambda}W^{\mu}$ and $h \in \mathbb{C}[x_1, \dots, x_d]^{w^{-1}W_\lambda w \cap W_\mu}$. This map immediately determines an R_d-module structure on $\mathcal{A}_d$.

By [DJ86], for arbitrary $\lambda, \mu \in I_d$ and $w \in {}^{\lambda}W^{\mu}$, there exists unique $\mu_w \in I_d$ such that $W_{\mu_w} = w^{-1}W_\lambda w \cap W_\mu$. This implies that the vector space $\mathbb{C}[x_1, \dots, x_d]^{w^{-1}W_\lambda w \cap W_\mu}$ is a free module over R_d with a basis $\mathbf{B}_{\mu_w}$ given in Corollary 2.4.6. This implies that

$$\mathcal{A}_d = \bigoplus_{\mu,\lambda \in I_d} \bigoplus_{w \in {}^{\lambda}W^{\mu}} \bigoplus_{p \in \mathbf{B}_{\mu_w}} R_d \cdot \mathbf{b}_{\lambda,\mu}(p).$$

In particular, $\mathcal{A}_d$ is a free R_d-module.

So the map, which is an algebra homomorphism,

$$\begin{aligned} R_d \to \mathcal{A}_d &= \bigoplus_{\lambda,\mu \in I_d} e_\lambda \mathcal{A}_d e_\mu, \\ f \mapsto f \cdot \mathbf{1}_{\mathcal{A}_d} &= \sum_{\mu \in I_d} \mathbf{b}^1_{\mu,\mu}(f) \end{aligned} \tag{19}$$

makes $\mathcal{A}_d$ a free R_d-algebra. □

Let λ_d be the composition $(1, 1, \dots, 1)$, which is the only decomposition of complete type, then $\mathcal{R}_d = e_{\lambda_d}\mathcal{A}_d e_{\lambda_d}$, and in this case it is called the nil-Hecke algebra, and the geometric basis of $\mathcal{R}_d$ is exactly the set

$$\{\mathbf{b}^w_{\lambda_d,\lambda_d}(p) = \partial_w \circ p | w \in W, p \text{ runs over a basis of } \mathbb{C}[x_1, \dots, x_d]\},$$

considered as a subring of $\mathrm{End}_{\mathbb{C}}[x_1, \dots, x_n]$.

Warning: Note that the notations of nil-Hecke algebra $\mathcal{R}_d$ and the ring of symmetric polynomials in d variables R_d are similar, but they are different.

The following proposition concerns the structure of the nil-Hecke algebra.

Proposition 2.4.15 ([Rou12])**.** *The ring $\mathcal{R}_d$ is isomorphic to the matrix algebra $\mathrm{End}_{R_d}(R)^{\mathrm{opp}}$, as $R \simeq R_d^{\oplus d!}$, and the center of $\mathcal{R}_d$ is isomorphic to R_d.*

Via this result, we have:

Proposition 2.4.16. *The center of $\mathcal{A}_d$ is isomorphic to $R_d = \mathbb{C}[x_1, \dots, X_n]^{S_d}$*

Proof. The image of the map in (19), which is isomorphic to R_d, lies in the center of $\mathcal{A}_d$. It is obvious that elements in the image commute with polynomials and splits. Moreover, multiplying by a symmetric polynomial commutes with all Demazure operators by Proposition 2.4.3, so the elements in the image also commute with all merges.

Now we show that the center of $\mathcal{A}_d$ is exactly the image of (19).

Suppose that $a = \bigoplus_{\mu,\lambda \in I_d} a_{\mu,\lambda} = \sum a_{\mu,\lambda}$ is in the center of $\mathcal{A}_d$, where $a_{\mu,\lambda} \in e_\mu \mathcal{A}_d e_\lambda$.

Firstly, a should commute with any idempotent e_{λ_1} whenever $\lambda_1 \in I_d$. This means we have

$$\sum_\mu a_{\mu,\lambda_1} = ae_{\lambda_1} = e_{\lambda_1} a = \sum_\lambda a_{\lambda_1,\lambda}, \quad \forall \lambda_1 \in I_d,$$

which immediately implies that $a_{\mu,\lambda_1} = a_{\lambda_1,\lambda} = 0$ for any $\mu \neq \lambda_1$ and $\lambda \neq \lambda_1$. So we may assume that $a = \sum_\mu a_{\mu,\mu}$.

Next, a should commute with all splits. Now we consider the split $s_{\lambda_d,\mu}$, and we should have

$$a_{\lambda_d,\lambda_d} \circ s_{\lambda_d,\mu} = a \circ s_{\lambda_d,\mu} = s_{\lambda_d,\mu} \circ a = s_{\lambda_d,\mu} \circ a_{\mu,\mu}.$$

Via the faithful representation of $\mathcal{A}_d$, this implies that for any $f \in R_\mu$ we have

$$a_{\lambda_d,\lambda_d}(s_{\lambda_d,\mu}(f)) = s_{\lambda_d,\mu} a_{\mu,\mu}(f),$$

which means $a_{\lambda_d,\lambda_d}(f) = a_{\mu,\mu}(f)$.

It is obvious that a_{λ_d,λ_d} lies in the center of $e_{\lambda_d} \mathcal{A}_d e_{\lambda_d}$, and by Proposition 2.4.15, it acts by multiplying by some $g_a \in R_d$, so we see that $a_{\mu,\mu}(f) = g_a \cdot f$, which means that $a_{\mu,\mu} = e_\mu g_a e_\mu = (g_a)_\mu$, a polynomial operator, and hence

$$a = \sum_\mu e_\mu g_a e_\mu = \sum_\mu (g_a)_\mu.$$

So we have completed the proof. ☐

As a result, we see that for any $\mu \in I_d$, $e_\mu R_d e_\mu$ is contained in the center of subalgebra $e_\mu \mathcal{A}_d e_\mu$. In fact, we have the following stronger result:

Proposition 2.4.17. *The center of the subalgebra $e_\mu \mathcal{A}_d e_\mu$ is $e_\mu R_d e_\mu$, which is isomorphic to R_d.*

Proof. Suppose that a lies in the center of $e_\mu \mathcal{A}_d e_\mu$. By faithful representation of $\mathcal{A}_d$, it is obvious that $e_\mu \mathcal{A}_d e_\mu$ acts on R_μ faithfully, and it suffices to show that a acts on R_μ by multiplying with a symmetric polynomial.

For any $f \in R_\mu$, then we have $a(f) = a(f \cdot 1)$, and this is the composition of a and the polynomial operator f_μ. As a is in the center, we have $a(f) = f \cdot a(1) = a(1) \cdot f$, as $a(1) \in R_\mu$. This already implies that a is a polynomial operator. So it suffices to show that $a(1)$ is a symmetric polynomial, and this is equivalent to that $a(1)$ is annihilated by ∂_i for $1 \leq i \leq d-1$.

If $s_i \in W_\mu$, then there is nothing to prove as $a(1) \in R_\mu$. Now if $s_i \notin W_\mu$, we can consider the following operator b defined as the composition of m_{μ,λ_d}, $\mathbf{b}^{s_i}_{\lambda_d,\lambda_d}(g)$ and $s_{\lambda_d,\mu}$, where g is an arbitrary polynomial in R. Then a commutes with b is equivalent to

$$\partial_{w_\mu}(\partial_i(a(1) \cdot fg)) = a(1) \cdot \partial_{w_\mu}(\partial_i(fg)), \forall f \in R_\mu, g \in R. \tag{20}$$

We apply the Leibniz rule (10). Then the left hand side of (20) equals to

$$\partial_{w_\mu}(\partial_i(a(1)) \cdot s_i(fg)) + \partial_{w_\mu}(a(1) \cdot \partial_i(fg)), .$$

Using Proposition 2.4.3, we see that $\partial_{w_\mu}(a(1) \cdot \partial_i(fg)) = a(1) \cdot \partial_{w_\mu}(\partial_i(fg))$, which is exactly the right hand side of (20). This means $\partial_{w_\mu}(\partial_i(a(1)) \cdot s_i(fg)) = 0$ for arbitrary $f \in R_\mu$ and $g \in R$, so we actually have

$$\partial_{w_\mu}(\partial_i(a(1))h) = 0$$

for any polynomial $h \in R$. This implies that $\partial_i(a(1)) = 0$ because $\partial_{w_0} : R \times R \to R$ is a non-degenerated bilinear form over R_d (see [BGG73] or Remark 4.2.3), and ∂_{w_0} factors through ∂_{w_μ}.

Then we have shown that $a(1)$ is annihilated by any ∂_i, which means $a(1) \in R_d \subset R_\mu$ and hence $a \in e_\mu R_d e_\mu$. So we finish the proof. □

2.4.3 Examples

The following examples show the diagrammatic expression of the geometric basis and we write down the action explicitly.

Example 2.4.18. *Let $n = 3$, $\mu = (2,1), \lambda = (1,2)$. Then there are two double cosets in $(S_1 \times S_2)\backslash S_3/(S_2 \times S_1)$, that is,*

$$\{1, s_1, s_2, s_2s_1\}, \{s_1s_2, s_1s_2s_1\},$$

and the set of shortest representatives is

$${}^\lambda W^\mu = \{1, s_1s_2\}.$$

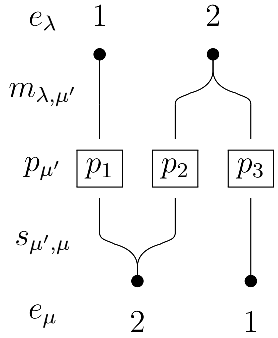
1. *Let w be identity, then we have that $W_{\mu'} = W_\mu \cap W_\lambda = W_{\lambda'}$ is the trivial group, and the space $R_{\mu'} = R$.*

   This means $\lambda' = \mu'$, and we can let the polynomial range over R. Moreover, the crossing operator $c^1_{\lambda',\mu,}$ induced by $w = 1$ is trivial.

   So the basis $\mathbf{b}^1_{\lambda,\mu}(p)$ is

   $$e_\lambda m_{\lambda,\mu'} p_{\mu'} s_{\mu'.\mu} e_\mu,$$

   or simply $m_{\lambda,\mu'} p_{\mu'} s_{\mu',\mu}$. The diagram of $\mathbf{b}^1_{\lambda,\mu}(p)$ is given by



where p ranges over a basis of $\mathbb{C}[x_1, x_2, x_3] \simeq \mathbb{C}[x_1] \otimes \mathbb{C}[x_2] \otimes \mathbb{C}[x_3]$.

Since $s_{\mu',\mu}$ is just the inclusion, $p_{\mu'}$ acts by multiplying with p and $m_{\lambda,\mu'}$ acts by ∂_2, the action of $\mathbf{b}^1_{\lambda,\mu}$ is

$$\mathbb{C}[x_1, x_2, x_3]^{S_2\times S_1} \to \mathbb{C}[x_1, x_2, x_3]^{S_1\times S_2}$$
$$f \mapsto \partial_2(p \cdot f).$$

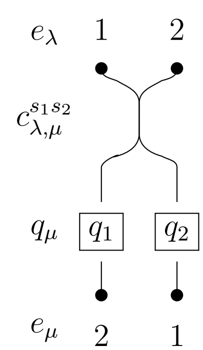
Suppose that p is a monomial of degree $\deg(p)$, then this element is homogeneous. Now the degree of this element is the sum of the degrees of each step. We have

$$\deg(s_{\mu',\mu}) = -1,$$
$$\deg(p_{\mu'}) = 2\deg(p),$$
$$\deg(m_{\lambda,\mu'}) = -1,$$

so the degree of $\mathbf{b}^1_{\lambda,\mu}(p)$ equals the sum $(-1) + 2\deg(p) + (-1)$, which is $2\deg(p) - 2$.

2. *Let w be $s_1 s_2$, then $W_{\mu'} = W_\mu \cap w^{-1}W_\lambda w = W_\mu$, and $W_{\lambda'} = W_\lambda \cap wW_\mu w^{-1} = W_\lambda$, so we have $\mu' = \mu, \lambda' = \lambda$, and the polynomial operator is in R_μ. In this case, the merging part and splitting part are trivial, and the crossing determined by w is given by permuting two row vectors, which acts by $\partial_1\partial_2$.*

   So for $q \in R_\mu$, the element $\mathbf{b}^{s_1s_2}_{\lambda,\mu}(q)$ is $e_\lambda \cdot c^{s_1s_2}_{\lambda,\mu} \cdot q_\mu e_\mu$. The diagram of $\mathbf{b}^{s_1s_2}_{\lambda,\mu}(q)$ is



   where q ranges over a basis of $\mathbb{C}[x_1, x_2, x_3]^{S_2\times S_1} \simeq \mathbb{C}[x_1, x_2]^{S_2} \otimes \mathbb{C}[x_3]$.

   The action of $\mathbf{b}^{s_1s_2}_{\lambda,\mu}(q)$ is

$$\mathbb{C}[x_1, x_2, x_3]^{S_2\times S_1} \to \mathbb{C}[x_1, x_2, x_3]^{S_1\times S_2}$$
$$f \mapsto \partial_1\partial_2(q \cdot f).$$

   Suppose the degree of q is $\deg(q)$, then $\deg(q_\mu) = 2\deg(q)$, and the degree of $c^{s_1s_2}_{\lambda,\mu} = 2\cdot(-2\cdot 1) = -4$, so the degree of $\mathbf{b}^{s_1s_2}_{\lambda,\mu}(q)$ equals $2\deg(q) - 4$.

3 Isomorphism theorem of completions

In this section, we will present the isomorphism theorem of completions of affine Hecke algebras and quiver Hecke algebras, as well as their Schur analogue. Although both of these two families of algebras have natural origins, this theorem demonstrates the relation between these two family of algebras.

Various versions of this kind of isomorphisms can be found in [BK09, Web19, MS19].

3.1 Completions of affine Hecke algebras and quiver Hecke algebras

We begin with introducing the completion of both types of algebras.

3.1.1 Completion of affine Hecke algebras

We denote by $\mathscr{O}$ the subalgebra $\mathbf{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Recall from [Lus89, Proposition 3.11] that the center $\mathscr{Z}$ of $\mathcal{H}$ is given by

$$\mathscr{Z} = Z(\mathcal{H}) = \mathscr{O}^{\Sigma} = \mathbf{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\Sigma},$$

which is a subalgebra of $\mathscr{O}$.

For $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbf{k}^*)^n$ we define the corresponding central character

$$\chi_{\mathbf{a}} : \mathscr{Z} \to \mathbf{k}$$

by restriction from the algebra homomorphism from $\mathscr{O}$ to $\mathbf{k}$ sending X_i to a_i. It is obvious that this character only depends on the Σ-orbit of $\mathbf{a}$. If M is a finite-dimensional representation of $\mathcal{H}$, then we can decompose it as $M = \oplus_{\chi} M_{\chi}$ where χ runs over Σ-orbits on $\mathbf{k}^n$ and M_{χ} is the generalized eigenspace of $\mathscr{Z}$ with eigen-character χ. We denote by $\mathfrak{m}_{\chi}$ the kernel of χ in $\mathscr{Z}$.

Recall that in Definition 2.2.1 there is a parameter q in the affine Hecke algebra, which is a root of unity in $\mathbf{k}$. The interesting cases we care about are those where $\mathbf{a} \in \{(q^{i_1}, q^{i_2}, \ldots, q^{i_n})|(i_1, \ldots, i_n) \in \mathbb{Z}^n\}$. So now we use $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$ to represent the n-tuple $\mathbf{a}$ such that $a_j = q^{i_j}$, and the corresponding central character is denoted by $\chi_{\mathbf{i}}$. Since in our setting q is a e-th primitive root of unity, so we may also view $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$.

Definition 3.1.1 (Completion of $\mathcal{H}$, [Lus89])**.** *For fixed* $\mathbf{i}$*, we set* $\chi = \chi_{\mathbf{i}}$*. Then we define* $\hat{\mathscr{Z}}$ *as the completion with respect to* $\mathfrak{m}_{\chi}$*, and we define* $\hat{\mathscr{O}} = \mathscr{O} \otimes_{\mathscr{Z}} \hat{\mathscr{Z}}$*,* $\hat{\mathcal{H}} = \mathcal{H} \otimes_{\mathscr{Z}} \hat{\mathscr{Z}}$*.*

For $\mathbf{u} \in \Sigma\mathbf{i}$, we define the ideal $J_{\mathbf{u}} = (X_1 - q^{u_1}, \ldots, X_n - q^{u_n})$, and $\hat{\mathscr{O}}_{\mathbf{u}}$ is defined as the $J_{\mathbf{u}}$-adic completion of $\mathscr{O}$.

By [Lus89, §7], we have decompositions $\hat{\mathcal{H}} = \underset{\sigma\in\Sigma}{\oplus} T_{\sigma}\hat{\mathscr{O}} = \underset{\sigma\in\Sigma}{\oplus} \hat{\mathscr{O}} T_{\sigma}$, and $\hat{\mathscr{O}} = \prod_{\mathbf{u}\in\Sigma\mathbf{i}} \hat{\mathscr{O}}_{\mathbf{u}}$. We also have that each $\hat{\mathscr{O}}_{\mathbf{u}}$ is isomorphic to the ring of formal power series of n variables.

So we have $\hat{\mathcal{H}} = \underset{\mathbf{u}\in\Sigma\mathbf{i}}{\oplus} \hat{\mathcal{H}}_{\mathbf{u}}$ as vector spaces, where by [MS19, Section 3.3]

$$\hat{\mathcal{H}}_{\mathbf{u}} := \underset{\sigma\in\Sigma}{\oplus} T_{\sigma}\mathscr{O}_{\mathbf{u}} = \{h \in \hat{\mathcal{H}} | \forall m \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \forall j \in \{1, \ldots, n\}, h(X_j - q^{u_j})^N \in \mathfrak{m}_{\chi}^m\}.$$

We denote by $e_{\mathbf{u}}$ the idempotent in $\hat{\mathcal{H}}$ which projects to the summand $\hat{\mathcal{H}}_{\mathbf{u}}$. Then the set $\{e_{\mathbf{u}} | \mathbf{u} \in \Sigma\mathbf{i}\}$ forms a complete set of orthogonal idempotents. Combining these results, we obtain by [MS19, Lemma 3.8] the following result on basis:

Lemma 3.1.2. *The following sets*

$$\{T_\sigma X_1^{b_1}\cdots X_n^{b_n}e_{\mathbf{u}}|\sigma\in\Sigma, b_i\in\mathbb{Z}_{\geq 0}, \mathbf{u}\in\Sigma\mathbf{i}\}$$

$$\{T_\sigma X_1^{b_1}\cdots X_n^{b_n}e_{\mathbf{u}}|\sigma\in\Sigma, b_i\in\mathbb{Z}_{\leq 0}, \mathbf{u}\in\Sigma\mathbf{i}\}$$

both form a topological basis of $\hat{\mathcal{H}}$.

Just as in the case of affine Hecke algebras (see Subsection 2.2), we have the analogue of Hecke modules and faithful representations for the completion.

The Hecke modules are defined as $v_{\mathbb{J}}\hat{\mathcal{H}}$. Similarly to Lemma 3.1.2, we have

Corollary 3.1.3. *The following sets form a topological* $\mathbf{k}$*-basis of* $v_{\mathbb{J}}\hat{\mathcal{H}}$*:*

$$\{v_{\mathbb{J}}e_{\mathbf{u}}T_\sigma X_1^{b_1}\cdots X_n^{b_n}|\mathbf{u}\in\Sigma\mathbf{i}, \sigma\in{}^{\mathbb{J}}\Sigma, b_i\in\mathbb{Z}_{\geq 0}\},$$

$$\{v_{\mathbb{J}}e_{\mathbf{u}}T_\sigma X_1^{b_1}\cdots X_n^{b_n}|\mathbf{u}\in\Sigma\mathbf{i}, \sigma\in{}^{\mathbb{J}}\Sigma, b_i\in\mathbb{Z}_{\leq 0}\}.$$

As a corollary of Proposition 2.2.5 and Proposition 2.2.6, we have the following corollaries ([MS19, Corollary 3.13]), which are faithful representations for the completion.

Corollary 3.1.4 (Faithful representation of $\hat{\mathcal{H}}$)**.** *1 There is a faithful representation of* $\hat{\mathcal{H}}$ *on*

$$\bigoplus_{\mathbf{u}\in\Sigma\mathbf{i}}\mathbf{k}[[X_1,\ldots,X_n]]e_{\mathbf{u}}\overline{v}_{\mathbb{I}}$$

by completing the representation from Proposition 2.2.6 with respect to the maximal ideal generated by

$$(X_r-q^{i_r})e_{\mathbf{u}},\ 1\leq r\leq n,\ \mathbf{u}\in\Sigma\mathbf{i},$$

2 There is a faithful representation of $\hat{\mathcal{H}}$ *on*

$$\bigoplus_{\mathbf{u}\in\Sigma\mathbf{i}}\mathbf{k}[[X_1^{-1},\ldots,X_n^{-1}]]e_{\mathbf{u}}v_{\mathbb{I}}$$

by completing the representation from Proposition 2.2.5 with respect to the maximal ideal generated by

$$(X_r^{-1}-q^{-i_r})e_{\mathbf{u}},\ 1\leq r\leq n,\ \mathbf{u}\in\Sigma\mathbf{i}.$$

There is an important type of elements called intertwining elements in $\hat{\mathcal{H}}$, which is a slightly modified version of the usual intertwining elements as in [Kle05, §5.1]. Now we consider the following intertwining elements: for $r\in\mathbb{I}$,

$$\Phi_r=T_r+\sum_{u_{r+1}\neq u_r}\frac{1-q}{1-X_rX_{r+1}^{-1}}e_{\mathbf{u}}+\sum_{u_{r+1}=u_n}e_{\mathbf{u}},$$

which is studied in [BK09]. It is obviously that $\mathcal{H}$ can also be generated by all these Φ_r and X_i for $1\leq i\leq n$ and $r\in\mathbb{I}$. If we fix a reduced expression $\sigma\in\Sigma$ such that $\sigma=s_{i_1}\cdots s_{i_r}$, then we define $\Phi_{[\sigma]}=\Phi_{i_1}\cdots\Phi_{i_r}$. This does depend on the choice of reduced expression, so we use the notation $[\sigma]$ here.

Direct computation gives us the following lemma.

Lemma 3.1.5. *For $1 \leq r \leq n-1$ and $\mathbf{u} \in \Sigma\mathbf{i}$, we have*

1. $e_{s_r\cdot\mathbf{u}}\Phi_r\overline{v}_{\mathbb{I}} = \begin{cases} \dfrac{X_r - qX_{r+1}}{X_{r+1-X_r}} e_{s_r\cdot\mathbf{u}}\overline{v}_{\mathbb{I}} & \text{if } u_{r+1} \neq u_r, \\ 0 & \text{if } u_{r+1} = u_r. \end{cases}$

2. $e_{s_r\cdot\mathbf{u}}\Phi_r(X_{r+1} - X_r)\overline{v}_{\mathbb{I}} = \begin{cases} (qX_{r+1} - X_r)e_{s_r\cdot\mathbf{u}}\overline{v}_{\mathbb{I}} & \text{if } u_{r+1} \neq u_r, \\ 2(qX_{r+1} - X_r)e_{\mathbf{u}}\overline{v}_{\mathbb{I}} & \text{if } u_{r+1} = u_r. \end{cases}$

3. $e_{s_r\cdot\mathbf{u}}\Phi_r v_{\mathbb{I}} = \begin{cases} \dfrac{X_{r+1} - qX_r}{X_{r+1} - X_r} e_{s_r\cdot\mathbf{u}} v_{\mathbb{I}} & \text{if } u_{r+1} \neq u_r, \\ (q+1)v_{\mathbb{I}} & \text{if } u_{r+1} = u_r. \end{cases}$

4. $e_{s_r\cdot\mathbf{u}}\Phi_r(X_{r+1} - X_r)v_{\mathbb{I}} = \begin{cases} (qX_r - X_{r+1})e_{s_r\cdot\mathbf{u}} v_{\mathbb{I}} & \text{if } u_{r+1} \neq u_r, \\ (q-1)(X_{r+1} + X_r)v_{\mathbb{I}} & \text{if } u_{r+1} = u_r. \end{cases}$

3.1.2 Completion of quiver Hecke algebras

Now we come to the quiver Hecke algebra side. This time, we will consider the quiver Hecke algebra related with the fixed data $\mathbf{i}$ and e.

Let $\Gamma = \Gamma_e$ be the Dynkin diagram of the affine Kac-Moody Lie algebra $\hat{\mathfrak{sl}}_e$ with the vertices numbered clockwise from 1 to e, and we identify these vertices with fixed representatives $1, \ldots, e$ of $\mathbb{Z}/e\mathbb{Z}$. Recall our $\mathbf{i} \in \mathbb{Z}^n$. Since $q \in \mathcal{H}$ is a primitive e-th root of unity, then we may assume that all the factors of $\mathbf{i}$ are in $\{1, \ldots, e\}$.

For $j \in \{1, \ldots, e\}$, we let d_j be the multiplicity how often j appears in $\mathbf{i}$. Then any $\mathbf{u} \in \Sigma\mathbf{i}$ corresponds to a unique vector decomposition $\widehat{\mu(\mathbf{u})}$ with a matrix expression $(\widehat{\mu(\mathbf{u})_{kl}})$ of $\mathbf{d} = (d_1, \ldots, d_e)$ of length n. This vector decomposition is of complete type. More explicitly, the k-th row row vector is a unit vector whose only nonzero term is in the u_k-th column. Moreover, the orbit $\Sigma\mathbf{i}$ is exactly the set of all vector decomposition of $\mathbf{d}$ of complete type.

We give an alternative definition of $\mathcal{R}_{\mathbf{d}}$ in terms of generators and relations ([MS19, Definition 7.1]), which turns out to be equivalent to the geometric definition presented in Section 2.

Definition 3.1.6. *The quiver Hecke algebra $\mathcal{R}_{\mathbf{d}}$ is the unital $\mathbb{C}$-algebra generated by elements*

$$\{e(\mathbf{u}) | \mathbf{u} \in \Sigma\mathbf{i}\} \cup \{\psi_1, \ldots, \psi_{n-1}\} \cup \{x_1, \ldots, x_n\}$$

subject to the relations

$$e(\mathbf{u})e(\mathbf{u'}) = \delta_{\mathbf{u},\mathbf{u'}} e(\mathbf{u}); \quad \sum_{\mathbf{u}\in\Sigma\mathbf{i}} e(\mathbf{u}) = 1;$$

$$x_r e(\mathbf{u}) = e(\mathbf{u})x_r; \quad \psi_r e(\mathbf{u}) = e(s_r \cdot \mathbf{u})\psi_r; \quad x_r x_s = x_s x_r;$$

$$\psi_r\psi_s = \psi_s\psi_r, \ \text{if } |r-s| > 1;$$

$$\psi_r x_s = x_s\psi_r, \ \text{if } s \neq r, r+1;$$

$$\psi_r x_{r+1} e(\mathbf{u}) = (x_r\psi_r + \delta_{u_r,u_{r+1}})e(\mathbf{u}); \quad x_{r+1}\psi_r e(\mathbf{u}) = (\psi_r x_r + \delta_{u_r,u_{r+1}})e(\mathbf{u});$$

$$\psi_r^2 e(\mathbf{u}) = \begin{cases} 0 & \textit{if } u_r = u_{r+1}, \\ e(\mathbf{u}) & \textit{if } u_{r+1} \neq u_r \pm 1, u_r, \\ (x_{r+1} - x_r) & \textit{if } u_{r+1} = u_r + 1, e \neq 2, \\ (x_r - x_{r+1}) & \textit{if } u_{r+1} = u_r - 1, e \neq 2, \\ (x_{r+1} - x_r)(x_r - x_{r+1}) e(\mathbf{u}) & \textit{if } u_{r+1} = -u_r, e = 2. \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r e(\mathbf{u}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1) e(\mathbf{u}) & \textit{if } u_{r+2} = u_r = u_{r+1} - 1, e \neq 2, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1) e(\mathbf{u}) & \textit{if } u_{r+2} = u_r = u_{r+1} + 1, e \neq 2, \\ (\psi_{r+1}\psi_r\psi_{r+1} - x_r - x_{r+2} + 2x_{r+1}) e(\mathbf{u}) & \textit{if } u_{r+2} = u_r = -u_{r+1}, e = 2, \\ \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{u}) & \textit{otherwise.} \end{cases}$$

Remark 3.1.7. *Though we will not prove it, the relationship between generators and the operators mentioned in 2.3.3 is as follows.*

1. *The idempotent* $e(\mathbf{u})$ *here is exactly the idempotent operator, which should be denoted by* $e_{\widehat{\mu(\mathbf{u})}}$.
2. *The suitable products of these* x_s *and idempotents* $e(\mathbf{u})$ *are exactly polynomial operators.*
3. *As all vector decompositions have the complete type and are of length* n*, the* ψ_r *is exactly the crossing operator corresponding to the block permutation interchanging the* r*-th row vector with the* $r+1$*-th row vector. These* ψ_r *generate all crossing operators, just like all simple transports generate the whole symmetric group* Σ.

We denote by $\mathcal{O}$ the subalgebra of $\mathcal{R}_{\mathbf{d}}$ generated by $\{e(\mathbf{u}) | \mathbf{u} \in \Sigma\mathbf{i}\} \cup \{x_1, \ldots, x_n\}$.

We now write explicitly down the natural faithful representation of $\mathcal{R}_{\mathbf{d}}$ which is defined and arises from the general theory of convolution algebra. Again, we will not explain why they are the same representation.

Proposition 3.1.8. *The algebra* $\mathcal{R}_{\mathbf{d}}$ *has a faithful representation on*

$$\mathbf{F}_{\mathbf{d}} = \bigoplus_{\mathbf{u} \in \Sigma\mathbf{i}} e(\mathbf{u}) \mathbb{C}[x_1, \ldots, x_n] \cdot \mathbb{1}$$

where $e(\mathbf{u})$ and x_s act by the ordinary multiplication, while

$$\psi_r e(\mathbf{u}) \cdot \mathbb{1} \begin{cases} 0 & \text{if } u_r = u_{r+1}, \\ (x_r - x_{r+1}) e(s_r \cdot \mathbf{u}) \cdot \mathbb{1} & \text{if } u_{r+1} = u_r + 1, \\ e(s_r \cdot \mathbf{u}) \cdot \mathbb{1} & \text{if } u_{r+1} \neq u_r, u_r + 1. \end{cases}$$

Definition 3.1.9. *Now we let* $J_m = \mathcal{R}_{\mathbf{d}}(x_1, \ldots, x_n)^m \mathcal{R}_{\mathbf{d}}$*, and we denote by* $\hat{\mathcal{R}}_{\mathbf{d}}$ *the completed algebra with respect to the sequence of ideals* $(J_m)_{m \geq 1}$.

We denote by $\hat{\mathcal{O}}$ the subalgebra generated by $\{e(\mathbf{u}) | \mathbf{u} \in \Sigma\mathbf{i}\} \cup \{x_1, \ldots, x_n\}$ in $\hat{\mathcal{R}}_{\mathbf{d}}$. The completed algebra $\hat{\mathcal{R}}_{\mathbf{d}}$ has also a completed version of the faithful representation, denoted by $\hat{\mathbf{F}}_{\mathbf{d}} = \hat{\mathcal{R}}_{\mathbf{d}} \otimes_{\mathcal{R}_{\mathbf{d}}} \mathbf{F}_{\mathbf{d}}$.

3.2 Isomorphism between completions

Now we let $\mathbf{k} = \mathbb{C}$, and we present an isomorphism between $\hat{\mathcal{H}}$ and $\hat{\mathcal{R}}_{\mathbf{d}}$.

Firstly we have an isomorphism

$$\gamma : \hat{\mathscr{O}} \to \hat{\mathcal{O}} : (X_i - q^{u_i})e_{\mathbf{u}} \mapsto -q^{u_i} x_i e(\mathbf{u}), \forall 1 \leq i \leq n, \mathbf{u} \in \Sigma\mathbf{i}. \tag{21}$$

In fact, both of $\hat{\mathscr{O}}$ and $\hat{\mathcal{O}}$ are the direct product of $|\Sigma\mathbf{i}|$ copies of formal power series ring, and γ just sends the variables from the left hand side subtracted by some elements in $\mathbb{C}$ to the variables from the right hand side multiplied by some elements in $\mathbb{C}^\times$, so it is an isomorphism.

We denote by $\hat{\mathcal{F}}$ the first representation in Corollary 3.1.4 of $\hat{\mathcal{H}}$. As $\hat{\mathscr{O}}$ (resp. $\hat{\mathcal{O}}$) acts on $\hat{\mathcal{F}}$ (resp. $\hat{\mathbf{F}}_{\mathbf{d}}$) via regular action, γ induces an isomorphism

$$\hat{\mathcal{F}} \to \hat{\mathbf{F}}_{\mathbf{d}} : \prod_{i=1}^{n} X_i^{b_i} e_{\mathbf{u}} \overline{v}_{\mathbb{I}} \mapsto \prod_{i=1}^{n} (q^{u_i}(1 - x_i))^{b_i} e(\mathbf{u}) \cdot \mathbb{1},$$

as we require that $\overline{v}_{\mathbb{I}} \mapsto \mathbb{1}$.

Finally, we can draw the following conclusion, see [MS19, Theorem 7.3]:

Theorem 3.2.1. *The isomorphism γ in* (21) *can be extended to an isomorphism of algebras $\gamma : \hat{\mathcal{H}} \to \hat{\mathcal{R}}_{\mathbf{d}}$, via $\gamma(e_{s_r \cdot \mathbf{u}} \Phi_r) = A_r^{\mathbf{u}} \psi_r e(\mathbf{u})$ for $\mathbf{u} \in \Sigma\mathbf{i}$ and $1 \leq r \leq n-1$, where*

$$A_r^{\mathbf{u}} = \begin{cases} 1 - q - x_r + q x_{r+1} & \text{if } u_{r+1} = u_r, \\ \dfrac{-q}{1 - q + x_{r+1} + q x_r} & \text{if } u_{r+1} = u_r + 1, \\ \dfrac{q^{u_{r+1}}(1 - x_r) - q^{u_r + 1}(1 - x_{r+1})}{q^{u_r}(1 - x_{r+1}) - q^{u_{r+1}}(1 - x_r)} & \text{if } u_{r+1} \neq u_r, u_r + 1. \end{cases}$$

The fact that map γ is a well-defined algebra homomorphism follows from the equations

$$\gamma(s_r \cdot e_{\mathbf{u}} \Phi_r \overline{v}_{\mathbb{I}}) = A_r^{\mathbf{u}} \psi_r e(\mathbf{u}) \cdot \mathbb{1},$$

$$\gamma(e_{s_r \cdot \mathbf{u}} \Phi_r (X_{r+1} - X_r) \overline{v}_{\mathbb{I}}) = A_r^{\mathbf{u}} \psi_r \gamma((X_{r+1} - X_r) e_{\mathbf{u}}) \cdot \mathbb{1},$$

for $1 \leq r \leq n-1$ and all $\mathbf{u} \in \Sigma\mathbf{i}$. These equations can be checked via a direct computation just using the definition, and we omit it. As all coefficients $A_r^{\mathbf{u}}$ are invertible in $\hat{\mathcal{O}}$ (so γ^{-1} sends them to invertible elements in $\hat{\mathscr{O}}$), this map turns out to be an isomorphism.

Remark 3.2.2. *Although this isomorphism can be checked by computation, the idea behind it involves analyzing finite dimensional module of $\hat{\mathcal{H}}$, which can be found in [BK09, §3].*

Suppose that M is a finite dimensional module of $\hat{\mathcal{H}}$ such that $\forall z \in \mathscr{Z}$, the action of $z - \chi(z)$ is nilpotent, where $\chi = \chi_{\mathbf{i}}$. Then we have a decomposition $M = \bigoplus_{\mathbf{u} \in \Sigma\mathbf{i}} M_{\mathbf{u}}$, where

$$M_{\mathbf{u}} = \{m \in M | \exists N > 0, \text{ such that } (X_i - q^{u_i})^N m = 0\}.$$

Then the idempotent $e_{\mathbf{u}}$, as well as $e(\mathbf{u}) = \gamma(e_{\mathbf{u}})$, is exactly the projection to $M_{\mathbf{u}}$.

By (2.2.1.H-7), we have $T_r(M_{\mathbf{u}}) \subset M_{\mathbf{u}} + M_{\mathbf{s_r \cdot u}}$, and one can check that the intertwining element Φ_r sends $M_{\mathbf{u}}$ to $M_{s_r \cdot \mathbf{u}}$. In fact, if $u_r = u_{r+1}$, then $M_{s_r \cdot \mathbf{u}} = M_{\mathbf{u}}$ and T_r is an endomorphism of $M_{\mathbf{u}}$.

In this case, Φ_r acts on $M_{\mathbf{u}}$ by $T_r + 1$, and there is nothing to prove. If $u_r \neq u_{r+1}$, then Φ_r acts by $T_r + \dfrac{1-q}{1-X_rX_{r+1}^{-1}}$, and by definition we have:

$$X_r(T_r + \frac{1-q}{1-X_rX_{r+1}^{-1}}) = (T_r + \frac{1-q}{1-X_rX_{r+1}^{-1}})X_{r+1},$$
$$X_{r+1}(T_r + \frac{1-q}{1-X_rX_{r+1}^{-1}}) = (T_r + \frac{1-q}{1-X_rX_{r+1}^{-1}})X_r.$$

This means Φ_r interchange the generalized eigenvalues of X_r and X_{r+1}. On the other hand, the formula in Proposition 3.1.8 implies that ψ_r also sends $M_{\mathbf{u}}$ to $M_{s_r\cdot\mathbf{u}}$. In fact, we have

$$\gamma(\Phi_r) = (\sum_{\mathbf{u}\in\Sigma\mathbf{i}} A_r^{s_r\cdot\mathbf{u}} e(\mathbf{u})) \cdot \psi_r,$$

which means Φ_r is just the product of a "weight funtion" and ψ_r.

The relationship between the two families of polynomial operators is similar. Each $M_{\mathbf{u}}$ is stable under the action of X_i and x_i, and the actions of $X_i - q^{u_i}$ and x_i are nilpotent on $M_{\mathbf{u}}$. In fact, we have

$$\gamma^{-1}(x_i) = -\sum_{\mathbf{u}\in\Sigma\mathbf{i}} q^{-u_i} e_{\mathbf{u}}(X_i - q^{u_i}) = 1 - \sum_{\mathbf{u}\in\Sigma\mathbf{i}} q^{-u_i} e_{\mathbf{u}} X_i,$$

which implies that x_i has the same action as $-q^{-u_i}(X_i - q^{u_i})$ on $M_{\mathbf{u}}$.

3.3 Isomorphism theorem for the Schur version

The isomorphism in Theorem 3.2.1 can extend to the Schur version. This is a special case of the main result of [MS19], in which the ground field $\mathbb{C}$ is generalized.

With fixed $\mathbf{i}$, we can define the completion $\hat{\mathcal{S}}$ of the affine Schur algebra $\mathcal{S}$, see [MS19, §5]. This construction is compatible with the completion of the affine Hecke algebra. In fact, the completed version also fits into the Hecke-Schur pattern, because we have

$$\hat{\mathcal{S}} \simeq \mathrm{Hom}_{\hat{\mathcal{H}}}(\bigoplus_{\mathbb{J}\subset\mathbb{I}} v_{\mathbb{J}}\hat{\mathcal{H}}).$$

As for the quiver side, there is an algebra $\mathcal{C}_{\mathbf{d}}$ called the modified quiver Schur algebra [MS19, Definition 8.4], which turns out to be isomorphic to $\mathcal{A}_{\mathbf{d}}$. In their article, this algebra is generated by three types of operators, which are the modified versions of merges, splits and polynomials.

For instance, there are two different ways to embed $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ into $\mathbb{C}[x_1, \ldots, x_n]$. One can either send $f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ to f directly, or send it to $f \cdot \prod_{i<j}(x_j - x_i)$. Working with either of the two inclusions defines two versions of split operators. This can result in two presentations of the quiver Schur algebra. In fact, the first inclusion gives us $\mathcal{C}_{\mathbf{d}}$, while the second inclusion gives us $\mathcal{A}_{\mathbf{d}}$.

We can take the completion of $\mathcal{C}_{\mathbf{d}}$ with respect to the ideal generated by all polynomial operators whose constant terms are zero, and the completed algebra is denoted as $\widehat{\mathcal{C}_{\mathbf{d}}}$. We also call this algebra as the completion of the quiver Schur algebra $\mathcal{A}_{\mathbf{d}}$, and we also denote it by $\widehat{\mathcal{A}_{\mathbf{d}}}$. We have the following theorem [MS19, Theorem 9.7]:

Theorem 3.3.1. *There is an isomorphism of algebras $\hat{\mathcal{S}} \to \widehat{\mathcal{A}_{\mathbf{d}}}$, which extends the isomorphism in Theorem 3.2.1.*

4 A new basis theorem for one vertex case

In this section, we will present the main result of the thesis. We continue focusing on the special case in Subsection 2.4, and find a new basis for the quiver Schur algebra $\mathcal{A}_d$.

4.1 Reminder and motivation

The quiver we consider now is a cyclic quiver with e vertices labelled by $1, \dots, e$ such that $e \geq 2$, with the fixed clockwise orientation, and the dimension vector is concentrated at vertex 1 of dimension d.

Let us recall the notion in the previous section. Let $V_d = \bigoplus_{\mu \in I_d} R_\mu$, then the quiver Schur algebra $\mathcal{A}_d$ is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(V_d)$ generated by merges, splits, idempotents and polynomials mentioned in 2.4.2.

Combining Theorem 2.3.5 and Proposition 2.4.14, we have the following corollary, which gives an R_d-basis of $\mathcal{A}_d$. We still call it the geometric basis, although no geometric meaning is used here.

Corollary 4.1.1 (Geometric R_d-basis)**.** *The algebra $\mathcal{A}_d$ has an R_d-basis*

$$\{\mathbf{b}^w_{\lambda,\mu}(h) | \mu, \lambda \in I_d, w \in {}^\lambda W^\mu, h \in \boldsymbol{B}_{\mu'}, W_{\mu'} = W_\mu \cap w^{-1} W_\lambda w\}. \tag{22}$$

An open question is: what is, for general quiver Schur algebras, a complete set of relations among merges, splits, idempotents and polynomials.

In the special case for one vertex, a complete set of relations is given in [Sei17]. This set of relations is related with the combinatorics of the symmetric group $W = S_d$, but the relations appearing in this set are complicated. They turned out to be the same relations as the intertwiners of tensor product of exterior powers of natural representations of $\mathfrak{gl}_n$ for $n \to \infty$ (see [TVW17]).

Another idea of finding a complete set of relations is to compute the structure coefficients of the geometric basis of $\mathcal{A}_d$. However, this is even more complicated. The product of two elements in the geometric basis will be a linear combination of geometric basis elements, and the coefficients are hard to determine.

A possible solution is to find another basis, whose structure coefficients are easier to determine, and in the special case for one vertex, we found such a new basis. The next task is to explain this basis.

4.2 Technical lemma

Before we start, here is a technical lemma which we will use later.

Lemma 4.2.1. *For $u, v \in W$ such that $\ell(u) \geq \ell(v)$, we have*

$$\partial_{w_0}(\partial_u(\alpha_d) \cdot \partial_{vw_0}(\alpha_d)) = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v \end{cases}$$

Proof. Notice that $\partial_{w_0}(\partial_u(\alpha_d) \cdot \partial_{vw_0}(\alpha_d))$ is either zero or of degree $\ell(v) - \ell(u)$. If $\ell(u) > \ell(v)$, the result is obvious by considering the degree of the polynomial. So it suffices to check the case that $\ell(u) = \ell(v)$.

When $\ell(u)=\ell(v)=0$, which means that $u=v$ are the identity element, the statement is true, because we always have $\partial_{w_0}(\alpha_d)=1$ by [BGG73].

Now suppose that the statement is already true for $\ell(u)=\ell(v)\leq k-1$ for $k\geq 1$, then we consider the case that $\ell(u)=\ell(v)=k$. Then we may assume that $u=s_{i_1}s_{i_2}\cdots s_{i_k}$, and we have

$$\begin{aligned}&\partial_{w_0}(\partial_u(\alpha_d)\cdot\partial_{vw_0}(\alpha_d))\\ \overset{(2.4.\mathbf{iii})}{=}&\partial_{w_0s_{i_1}}\circ\partial_{s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{vw_0}(\alpha_d))\\ \overset{(10)}{=}&\partial_{w_0s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{s_{i_1}}\circ\partial_{vw_0}(\alpha_d)+\partial_{s_{i_1}}(\partial_u(\alpha_d))\cdot s_{i_1}(\partial_{vw_0}(\alpha_d)))\end{aligned}$$

As $\ell(s_{i_1}u)<\ell(u)$, we know that $\partial_{s_{i_1}}(\partial_u(\alpha_d))=0$ by (2.4.**iii**) and hence we have

$$\partial_{w_0}(\partial_u(\alpha_d)\cdot\partial_{vw_0}(\alpha_d))=\partial_{w_0s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{s_{i_1}}\circ\partial_{vw_0}(\alpha_d)). \tag{23}$$

Notice that $\ell(s_{i_1}vw_0)>\ell(vw_0)$ if and only if $\ell(s_{i_1}v)<\ell(v)$.

1. If $\ell(s_{i_1}v)>\ell(v)$, which implies that $\partial_{s_{i_1}}\circ\partial_{vw_0}=0$ and $u\neq v$, by (2.4.**iii**) we immediately have $\partial_{w_0s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{s_{i_1}}\circ\partial_{vw_0}(\alpha_d)=0$, which satisfies the statement.

2. If $\ell(s_{i_1}v)<\ell(v)$, and we assume that $v'=s_{i_1}v$ and $u'=s_{i_1}u$, so both of them are of length $k-1$. Then we have $\partial_u=\partial_{s_{i_1}}\circ\partial_{u'}$ and $\partial_{s_{i_1}}\circ\partial_{vw_0}=\partial_{s_{i_1}vw_0}=\partial_{v'w_0}$. So by (23) we have

$$\partial_{w_0}(\partial_u(\alpha_d)\cdot\partial_{vw_0}(\alpha_d))=\partial_{w_0s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d)) \tag{24}$$

As $\ell(s_{i_1}v'w_0)=\ell(vw_0)=\ell(v'w_0)-1$, we know that $\partial_{v'w_0}(\alpha_d)$ is vanished by $\partial_{s_{i_1}}$, and hence invariant under s_{i_1}, so by (10), we have

$$\begin{aligned}&\partial_{s_{i_1}}(\partial_{u'}(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d))\\ =&\partial_{s_{i_1}}(\partial_{u'}(\alpha_d))\cdot s_{i_1}(\partial_{v'w_0}(\alpha_d)))+\partial_{u'}(\alpha_d)\cdot\partial_{s_{i_1}}(\partial_{v'w_0}(\alpha_d))\\ =&\partial_{s_{i_1}}(\partial_{u'}(\alpha_d))\cdot\partial_{v'w_0}(\alpha_d)\\ =&\partial_u(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d).\end{aligned}$$

Putting this into (24), we obtain

$$\begin{aligned}\partial_{w_0}(\partial_u(\alpha_d)\cdot\partial_{vw_0}(\alpha_d))&=\partial_{w_0s_{i_1}}(\partial_u(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d))\\ &=\partial_{w_0s_{i_1}}(\partial_{s_{i_1}}(\partial_{u'}(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d)))\\ &\overset{(2.4.\mathbf{iii})}{=}\partial_{w_0}(\partial_{u'}(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d))\end{aligned}$$

By the induction hypothesis, we know

$$\partial_{w_0}(\partial_{u'}(\alpha_d)\cdot\partial_{v'w_0}(\alpha_d))=\begin{cases}0 & \text{if } u'\neq v'\\ 1 & \text{if } u'=v'\end{cases}$$

and $u'=v'$ if and only if $u=v$.

So we know that the statement is true for elements of length k, and by induction on $\ell(u)$. We completed the proof of the lemma. $\square$

Remark 4.2.2. *This result can be related to cohomology groups of the flag varieties (see [BGG73]), but we just prove it without any geometry. Also, the cohomology of the flag variety is a very interesting algebra, which is useful in the study of the representations of Lie algebras, but we will not discuss this in the thesis. More detail we refers to [Str22b].*

Remark 4.2.3. *This result can rephrased as follows: if $\ell(u) + \ell(w) \geq \ell(w_0)$, then we have*

$$\partial_{w_0}(\partial_u(\alpha_d) \cdot \partial_w(\alpha_d)) = \begin{cases} 0 & \text{if } w^{-1}u \neq w_0 \\ 1 & \text{if } w^{-1}u = w_0 \end{cases}$$

Moreover, this implies that

$$\langle \ , \ \rangle : R \otimes_{R_d} R \to R_d, (f, g) \mapsto \partial_{w_0}(f \cdot g)$$

is a non-degenerated bilinear form over R_d.

4.3 Main theorem: new basis

In order to parameterize the geometric basis, we need to compute the double cosets of W. Among all double cosets, the following types of cosets are particularly simple: let λ_0 be the decomposition of d represented by 1×1 matrix (d), then for arbitrary $\mu, \lambda \in I_d$, the double cosets ${}^{\lambda}W^{\lambda_0}, {}^{\lambda_0}W^{\mu}$ are both trivial.

By Corollary 4.1.1, we know that the space $e_{\lambda_0}\mathcal{A}_d e_\mu$ is spanned by $\{\mathbf{b}^1_{\lambda_0,\mu}(p) = m_{\lambda_0,\mu} \circ p | p \in \mathbf{B}_\mu\}$ over R_d. Similarly, we know that the space $e_\lambda \mathcal{A}_d e_{\lambda_0}$ is spanned by $\{q \circ s_{\lambda,\lambda_0} | q \in \mathbf{B}_\lambda\}$ over R_d. Now we have a morphism of R_d-modules:

$$\circ : e_\mu \mathcal{A}_d e_{\lambda_0} \otimes_{R_d} e_{\lambda_0} \mathcal{A}_d e_\lambda \to e_\mu \mathcal{A}_d e_\lambda \tag{25}$$

via taking the composition, and we claim that this is in fact an isomorphism.

In fact, a basis over R_d of the left hand side of (25) is $\{\mathbf{b}^1_{\lambda,\lambda_0}(q) \otimes \mathbf{b}^1_{\lambda_0,\mu}(p) | q \in \mathbf{B}_\lambda, p \in \mathbf{B}_\mu\}$, and the image of this set is $\{\mathbf{b}^1_{\lambda,\lambda_0}(q) \cdot \mathbf{b}^1_{\lambda_0,\mu}(p) | q \in \mathbf{B}_\lambda, p \in \mathbf{B}_\mu\}$. So the statement that the composition map $\circ$ is an isomorphism is equivalent to the following statement:

Theorem 4.3.1. *Let $\mathbf{v}_{\lambda,\mu}(q,p) = \mathbf{b}^1_{\lambda,\lambda_0}(q) \cdot \mathbf{b}^1_{\lambda_0,\mu}(p)$ for $\lambda, \mu \in I_d$ and $p \in \boldsymbol{B}_\mu, q \in \boldsymbol{B}_\lambda$. Then the set*

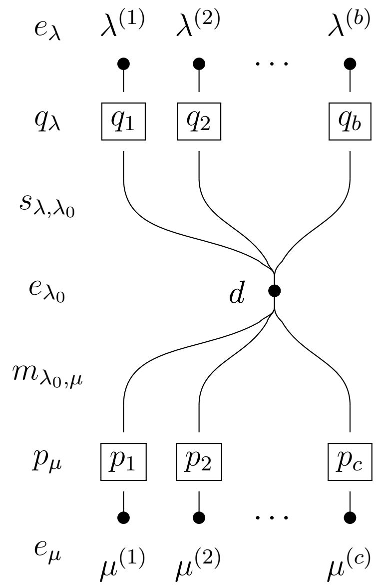
$$\{\mathbf{v}_{\lambda,\mu}(q,p) | q \in \boldsymbol{B}_\lambda, p \in \boldsymbol{B}_\mu\}$$

which is the same as

$$\{\partial_{w_\lambda u}(\alpha_d) \circ s_{\lambda,\lambda_0} \circ m_{\lambda_0,\mu} \circ \partial_{w_\mu v}(\alpha_d) | u \in {}^{\lambda}W, v \in {}^{\mu}W\},$$

forms a basis of $e_\lambda \mathcal{A}_d e_\mu$ over R_d.

The diagrammatic expression of the element $\mathbf{v}_{\lambda,\mu}(q,p)$ is:



(From bottom to top, we can read $\mathbf{v}_{\lambda,\mu}(q,p) = q_\lambda \cdot s_{\lambda,\lambda_0} \cdot m_{\lambda_0,\mu} \cdot p_\mu$.)

Before we give the proof of Theorem 4.3.1, we discuss a nice property of this new basis, which answers the question mentioned in Subsection 4.1. In fact, the multiplication becomes much easier to compute in terms of our new basis.

By Theorem 4.3.1, we have an R_d-basis for $\mathcal{A}_d$ given by vectors of the form $\mathbf{v}_{\lambda,\mu}(q,p) = \mathbf{b}^1_{\lambda,\lambda_0}(q) \cdot \mathbf{b}^1_{\lambda_0,\mu}(p)$ for $q \in \mathbf{B}_\lambda, p \in \mathbf{B}_\mu$, and now we consider the multiplication

$$\mathbf{v}_{\lambda_1,\mu_1}(q_1,p_1) \cdot \mathbf{v}_{\lambda_2,\mu_2}(q_2,p_2).$$

If $\lambda_2 \neq \mu_1$, this will be zero. So we should study the case $\lambda_2 = \mu_1$.

Now suppose that $f \in R_{\mu_2}$, then we have

$$\mathbf{v}_{\lambda_1,\mu_1}(q_1,p_1) \cdot \mathbf{v}_{\lambda_2,\mu_2}(q_2,p_2)(f) = q_1 \cdot m_{\lambda_0,\mu_1}(p_1 q_2 \cdot m_{\lambda_0,\mu_2}(p_2 \cdot f)) \in R_{\lambda_1}.$$

Here the splits are not shown in the formula because they are just inclusions and do not change the polynomial itself. The only role of splits is sending the polynomial from one summand of V_d to another.

As we know that $m_{\lambda_0,\mu_2}(p_2 f) \in R_d$, we have

$$q_1 \cdot m_{\lambda_0,\mu_1}(p_1 q_2 \cdot m_{\lambda_0,\mu_2}(p_2 f)) = m_{\lambda_0,\mu_1}(p_1 q_2) \cdot q_1 \cdot m_{\lambda_0,\mu_2}(p_2 f) = m_{\lambda_0,\mu_1}(p_1 q_2) \cdot \mathbf{v}_{\lambda_1,\mu_2}(q_1,p_2)(f),$$

and here $m_{\lambda_0,\mu_1}(p_1 q_2) \in R_d$ is the coefficient. This coefficient is determined by μ_1, p_1 and q_2. Recall that $\mathbf{B}_\mu$ is bijectively corresponding to ${}^\mu W$. We define

$$c_{\mu,v_1,v_2} = m_{\lambda_0,\mu}(\partial_{w_\mu v_1}(\alpha_d)\partial_{w_\mu v_2}(\alpha_d)), \forall \mu \in I_d, v_1, v_2 \in {}^\mu W,$$

and we also let $c_{\mu,v_1,v_2} = 0$ if $(v_1,v_2) \notin {}^\mu W \times {}^\mu W$.

Then we have the following result, which gives a presentation of the algebra:

Corollary 4.3.2. *The algebra $\mathcal{A}_d$ is isomorphic to the unital R_d-algebra generated by the set*

$$X = \{(\lambda,\mu,u,v) | \lambda,\mu \in I_d, u \in {}^\lambda W, v \in {}^\mu W\},$$

with relations

$$(\lambda,\mu,u,v) \cdot (\lambda',\mu',u',v') = \delta_{\mu,\lambda'} \cdot c_{\mu,v,u'}(\lambda,\mu',u,v'), \forall (\lambda,\mu,u,v), (\lambda',\mu',u',v') \in X.$$

4.4 Proof of the theorem

4.4.1 Linear independence

We firstly prove that (25) is injective by showing that the set $\{\mathbf{b}^1_{\lambda,\lambda_0}(q)\cdot\mathbf{b}^1_{\lambda_0,\mu}(p)|q\in\mathbf{B}_\lambda, p\in\mathbf{B}_\mu\}$ forms an R_d-linear independent set.

Proposition 4.4.1. *The set $\{q\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ p|q\in \boldsymbol{B}_\lambda, p\in \boldsymbol{B}_\mu\}$ is an R_d-linear independent subset.*

Proof. By definition, we can rewrite the set in the statement as

$$\{\partial_{w_\lambda u}(\alpha_d)\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ\partial_{w_\mu v}(\alpha_d)|u\in {}^\lambda W, v\in {}^\mu W\}$$

Now if we assume that there exists $c_{u,v}\in R_d$ for $u,v\in {}^\lambda W, {}^\mu W$ respectively, such that

$$\sum_{u\in {}^\lambda W, v\in {}^\mu W} c_{u,v}\partial_{w_\lambda u}(\alpha_d)\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ\partial_{w_\mu v}(\alpha_d)=0\in e_\lambda\mathcal{A}_d e_\mu\subset \mathrm{Hom}_{\mathbb{C}}(R_\mu,R_\lambda),$$

then we claim that all $c_{u,v}$ must be zero.

By assumption, for any $g\in R$ we can set $f=\partial_{w_\mu}(g)\in R_\mu$, and obtain

$$\begin{aligned}0=&\sum_{u,v} c_{u,v}\partial_{w_\lambda u}(\alpha_d)\cdot m_{\lambda_0,\mu}(\partial_{w_\mu v}(\alpha_d)\cdot f)\\ =&\sum_{u,v} c_{u,v}\partial_{w_\lambda u}(\alpha_d)\cdot m_{\lambda_0,\mu}\circ\partial_{w_\mu}(\partial_{w_\mu v}(\alpha_d)\cdot g).\end{aligned}$$

Notice that $m_{\lambda_0,\mu}\circ\partial_{w_\mu}$ is the composition of two merges, which equals to the merge from R to R_d, so it equals to ∂_{w_0} by (17), so we have

$$\sum_{u,v} c_{u,v}\partial_{w_\lambda u}(\alpha_d)\cdot\partial_{w_0}(\partial_{w_\mu v}(\alpha_d)\cdot g)=0.$$

Now we let $\beta_x:=\partial_{w_\mu x w_0}(\alpha_d)$ for each $x\in {}^\mu W$. We will test this equation by substituting g to these β_x.

We firstly let $x=1$, then $\partial_{w_0}(\partial_{w_\mu v}(\alpha_d)\cdot\partial_{w_\mu w_0}(\alpha_d))$ is zero for each $v\in {}^\mu W\backslash\{e\}$ by considering the degree, while when $v=1$, $\partial_{w_0}(\partial_{w_\mu}(\alpha_d)\cdot\partial_{w_\mu w_0}(\alpha_d))=1$ by Lemma 4.2.1. So we have

$$\sum_{u\in {}^\lambda W} c_{u,e}\partial_{w_\lambda u}(\alpha_d)\cdot 1=0.$$

This implies that $c_{u,e}=0$ because these $\partial_{w_\lambda u}(\alpha_d)$ are R_d-linear independent by Lemma 2.4.4.

Now suppose that $c_{u,v}=0$ for all $\ell(v)\leq k-1$, then for v_1 of length k in ${}^\mu W$, we set $x=v_1$ and $g=\beta_{v_1}$. Now we have

$$\sum_{u\in {}^\lambda W, v\in {}^\mu W, \ell(v)\geq k} c_{u,v}\partial_{w_\lambda u}(\alpha_d)\cdot\partial_{w_0}(\partial_{w_\mu v}(\alpha_d)\cdot\beta_{v_1})=0$$

As $\beta_{v_1} = \partial_{w_\mu v_1 w_0}(\alpha_d)$ and $\ell(w_\mu v_1) = \ell(w_\mu) + k \leq \ell(w_\mu v)$ if $\ell(v) \geq k$ and $v \in {}^\mu W$, we can apply Lemma 4.2.1, and obtain

$$\sum_{u \in {}^\lambda W} c_{u,v_1} \partial_{w_\lambda u}(\alpha_d) \cdot 1 = 0,$$

which implies that all c_{u,v_1} have to be zero by Lemma 2.4.4. So by induction, we know that all $c_{u,v}$ have to be zero, which means that

$$\{\partial_{w_\lambda u}(\alpha_d) \circ s_{\lambda,\lambda_0} \circ m_{\lambda_0,\mu} \circ \partial_{w_\mu v}(\alpha_d) | u \in {}^\lambda W, v \in {}^\mu W\}$$

is a linear independent set. □

4.4.2 Graded dimension

Now we prove that (25) is surjective. This can be shown by comparing $e_\lambda \mathcal{A}_d e_\mu$ and its R_d-module spanned by

$$\{\mathbf{b}^1_{\lambda,\lambda_0}(q) \cdot \mathbf{b}^1_{\lambda_0,\mu}(p) | q \in \mathbf{B}_\lambda, p \in \mathbf{B}_\mu\}.$$

We know that the second space is a subspace of the first one. However, both of them are infinite dimensional over $\mathbb{C}$, so we cannot compare the dimension of them directly. Also, as R_d is not a field, having the same rank over R_d does not imply that two spaces are the same.

The solution is decompose both spaces into a direct sum of finite dimensional vector spaces over $\mathbb{C}$. More precisely, we introduce the grading structure of $\mathcal{A}_d$, so we can compare the dimension of the homogeneous components of each degree.

For a $\mathbb{Z}$-graded $\mathbb{C}$-space E, let E_m be the homogeneous component of degree m. Assume that for any fixed m, E_m is of finite dimension over $\mathbb{C}$, then we define the *graded dimension* of E as

$$\mathrm{grdim}(E) = \sum_{m \in \mathbb{Z}} \dim_{\mathbb{C}}(E_m) t^m.$$

If E and E' are both $\mathbb{Z}$-graded with finite dimensional homogeneous components, and assume that $E_m = 0 = E'_m$ for m sufficiently small, then $E \otimes_{\mathbb{C}} E'$ turns into a $\mathbb{Z}$-graded vector space, whose homogeneous components are still finite dimensional, via

$$(E \otimes_{\mathbb{C}} E')_m = \bigoplus_{n+l=m} E_n \otimes E'_m,$$

and we have

$$\mathrm{grdim}(E \otimes E') = \mathrm{grdim}(E) \cdot \mathrm{grdim}(E').$$

For a homogeneous vector $v \in E$, we define the graded dimension of v as the graded dimension of the one-dimensional space $\mathbb{C}v$, that is,

$$\mathrm{grdim}(v) = \mathrm{grdim}(\mathbb{C}v) = t^{\deg(v)}.$$

Example 4.4.2. *Let us consider R as a graded ring such that each x_i is of degree 2. Moreover, for any decomposition μ, the W_μ-invariant polynomial ring R_μ can be viewed as a graded subring of R.*

1. *The first example is the graded dimensions of the polynomial ring and the symmetric polynomial ring. They are infinite-dimensional spaces, but their homogeneous components are all finite dimensional.*

Lemma 4.4.3. *Then we have the following equalities in formal power series.*

$$\mathrm{grdim}(R) = \prod_{i=1}^{d} \mathrm{grdim}(\mathbb{C}[x_i]) = \prod_{i=1}^{d} \sum_{k=0}^{\infty} t^{2k} = \prod_{i=1}^{d} \frac{1}{1-t^2}.$$

Also, by Theorem 2.4.2, we have

$$\mathrm{grdim}(R_d) = \prod_{i=1}^{d} \mathrm{grdim}(\mathbb{C}[e_i]) = \prod_{i=1}^{d} \sum_{k=0}^{\infty} t^{2\deg(e_i)} = \prod_{i=1}^{d} \frac{1}{1-t^{2i}}.$$

Using Lemma 4.4.3, we can compute the graded dimension of the subspace of R spanned by the set $\boldsymbol{B}$ given in Lemma 2.4.4 over $\mathbb{C}$. As we have

$$R \simeq R_d \otimes_{\mathbb{C}} \bigoplus_{v \in \boldsymbol{B}} \mathbb{C}v,$$

we immediately know that

$$\sum_{v \in \boldsymbol{B}} t^{\deg(v)} = \sum_{v \in \boldsymbol{B}} \mathrm{grdim}(v) = \mathrm{grdim}(R)/\mathrm{grdim}(R_d) = \prod_{i=1}^{d} \frac{1-t^{2i}}{1-t^2}.$$

2. *By construction of polynomials in $\boldsymbol{B}$, we know that*

$$\sum_{w \in W} t^{2\ell(w)} = \sum_{w \in W} t^{2\ell(w_0)-2\ell(w)} = \sum_{w \in W} t^{\deg(\partial_w(\alpha_d))} = \sum_{v \in \boldsymbol{B}} \mathrm{grdim}(v),$$

which is also equal to $\mathrm{grdim}(R)/\mathrm{grdim}(R_d)$.

We can also use this method to compute $\sum_{w \in W_\mu} t^{2\ell(w)}$ for $\mu \in I_d$. If $\mu = (\mu^{(1)}, \ldots, \mu^{(c)})$, then we have

$$\sum_{w \in W_\mu} t^{2\ell(w)} = \prod_{k=1}^{c} \prod_{i=1}^{\mu^{(k)}} \frac{1-t^{2i}}{1-t^2}. \tag{26}$$

Also, notice that $W_\mu \times {}^\mu W \to W$ is a bijection, and if $(u, v) \mapsto w = uv$, then we have $\ell(u) + \ell(v) = \ell(w)$. Using this, we obtain

$$\sum_{w \in {}^\mu W} t^{2\ell(w)} = \frac{\sum_{w \in W} t^{2\ell(w)}}{\sum_{w \in W_\mu} t^{2\ell(w)}}.$$

Now we apply (26)*, then we obtain the following formula:*

$$\sum_{w \in {}^\mu W} t^{2\ell(w)} = \frac{\prod_{i=1}^{d}(1-t^{2i})}{\prod_{k=1}^{c}\prod_{i=1}^{\mu^{(k)}}(1-t^{2i})} = t^{\ell(w_0)-\ell(w_\mu)} \cdot \begin{bmatrix} d \\ \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(c)} \end{bmatrix}_t.$$

Here we use the following notion of quantized multinomial coefficients

$$\begin{bmatrix} d \\ \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(c)} \end{bmatrix}_t := \frac{\prod_{i=1}^{d}(t^i - t^{-i})}{\prod_{k=1}^{c}\prod_{i=1}^{\mu^{(k)}}(t^i - t^{-i})},$$

and for more details we refer to [Str22a].

By a similar method, we can show that

$$\sum_{w\in{}^{\mu}W} t^{-2\ell(w)} = \frac{\prod_{i=1}^{d}(1-t^{2i})}{\prod_{k=1}^{c}\prod_{i=1}^{\mu^{(k)}}(1-t^{2i})} = t^{-\ell(w_0)+\ell(w_\mu)}\cdot \begin{bmatrix} d \\ \mu^{(1)},\mu^{(2)},\dots,\mu^{(c)}\end{bmatrix}_t. \tag{27}$$

Now we return to our task to prove the surjectivity.

As $e_\lambda\mathcal{A}_d e_\mu$ is a free R_d-module, the graded dimension of $e_\lambda\mathcal{A}_d e_\mu$ equals to the product of the graded dimension of R_d and the sum of the graded dimensions of R_d-basis vectors as in Corollary 4.1.1.

On the other hand, as in Proposition 4.4.1 we know that vectors in $\{q\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ p | q\in \mathbf{B}_\lambda, p\in\mathbf{B}_\mu\}$ are R_d-linearly independent, so the graded dimension of R_d-submodule spanned by these vectors equals to the product of the graded dimension of R_d and the sum of the graded dimensions of vectors in

$$\{q\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ p | q\in \mathbf{B}_\lambda, p\in\mathbf{B}_\mu\}.$$

In order to show that the R_d-submodule spanned by these vectors is exactly the whole $e_\lambda\mathcal{A}_d e_\mu$, it suffices to show that the sum of the graded dimensions of vectors in $\{q\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ p | q\in \mathbf{B}_\lambda, p\in\mathbf{B}_\mu\}$ equals to the sum of the graded dimensions of an R_d-basis of $e_\lambda\mathcal{A}_d e_\mu$, that is,

$$\sum_{q\in\mathbf{B}_\lambda,p\in\mathbf{B}_\mu} \mathrm{grdim}(q\circ s_{\lambda,\lambda_0}\circ m_{\lambda_0,\mu}\circ p) = \sum_{w\in{}^{\lambda}W^{\mu},h\in\mathbf{B}_{\mu'},W_{\mu'}=W_\mu\cap w^{-1}W_\lambda w} \mathrm{grdim}(\mathbf{b}^{w}_{\lambda,\mu}(h)). \tag{28}$$

Now we compute both sides of (28). We begin with the left hand side.

Proposition 4.4.4. *The sum of the graded dimensions of the vectors in* $\{\mathbf{v}_{\lambda,\mu}(q,p)|q\in\boldsymbol{B}_\lambda, p\in\boldsymbol{B}_\mu\}$ *equals to*

$$\begin{bmatrix} d \\ \lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(b)}\end{bmatrix}_t\cdot\begin{bmatrix} d \\ \mu^{(1)},\mu^{(2)},\dots,\mu^{(c)}\end{bmatrix}_t$$

for compositions $\lambda=(\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(b)})$ *and* $\mu=(\mu^{(1)},\mu^{(2)},\dots,\mu^{(c)})$.

Proof. Notice that the sum of graded dimensions of $\mathbf{v}_{\lambda,\mu}(q,p)$ for $q\in\mathbf{B}_\lambda, p\in\mathbf{B}_\mu$ equals to the product of the sum of graded dimensions of $\partial_{w_\lambda u}(\alpha_d)\circ s_{\lambda,\lambda_0}$ for $u\in{}^{\lambda}W$ and the sum of graded dimensions of $m_{\lambda_0,\mu}\circ\partial_{w_\mu v}(\alpha_d)$ for $v\in{}^{\mu}W$.

As the graded dimensions of the split s_{λ,λ_0} and the merge $m_{\lambda_0,\lambda}$ are the same, so the computation of the two parts is almost the same. Then we only give the details for the merge.

The graded dimension of $m_{\lambda_0,\mu}\circ\partial_{w_\mu v}(\alpha_d)$ is equal to the product of the graded dimension of $m_{\lambda_0,\mu}$ and that of $\partial_{w_\mu v}(\alpha_d)$. The graded dimension of $m_{\lambda_0,\mu}$ equals $t^{\ell(w_\mu)-\ell(w_0)}$, and by definition,

$$\mathrm{grdim}(\partial_{w_\mu v}(\alpha_d)) = t^{2(\ell(w_0)-\ell(w_\mu)-\ell(v))}.$$

By (27) in Example 4.4.2, we have

$$\sum_{v\in{}^{\mu}W} t^{-2\ell(v)} = \sum_{w\in W} t^{-2\ell(w)}/\sum_{w\in W_\mu} t^{-2\ell(w)} = t^{-\ell(w_0)+\ell(w_\mu)}\cdot\begin{bmatrix} d \\ \mu^{(1)},\mu^{(2)},\dots,\mu^{(c)}\end{bmatrix}_t.$$

Now we take the summation over $v \in {}^{\mu}W$, we have

$$\sum_{v\in {}^{\mu}W} \mathrm{grdim}(m_{\lambda_0,\mu} \circ \partial_{w_\mu v}(\alpha_d)) = \mathrm{grdim}(m_{\lambda_0,\mu}) \cdot \sum_{v\in {}^{\mu}W} t^{2\ell(w_0)-2\ell(w_\mu)} \cdot t^{-2\ell(v)} = \begin{bmatrix} d \\ \mu^{(1)}, \mu^{(2)}, \dots, \mu^{(c)} \end{bmatrix}_t .$$

In conclusion, we have

$$\sum_{u\in {}^{\lambda}W, v\in {}^{\mu}W} \mathrm{grdim}(\partial_{w_\lambda u}(\alpha_d) \circ s_{\lambda,\lambda_0} \circ m_{\lambda_0,\mu} \circ \partial_{w_\mu v}(\alpha_d)) = \begin{bmatrix} d \\ \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(b)} \end{bmatrix}_t \cdot \begin{bmatrix} d \\ \mu^{(1)}, \mu^{(2)}, \dots, \mu^{(c)} \end{bmatrix}_t ,$$

which completes the proof. □

Now we compute sum of the graded dimensions of the geometric basis over R_d of $e_\lambda \mathcal{A}_d e_\mu$, which is parameterized by pairs (σ, h) such that σ runs over ${}^{\lambda}W^{\mu}$, and h runs over $\mathbf{B}_{\mu_\sigma}$ where $W_{\mu_\sigma} = W_\mu \cap \sigma^{-1} W_\lambda \sigma$.

For a fixed σ, the graded dimension corresponding to the submodule spanned by all (σ, h) is computed as follows:

The graded dimension of $s_{\mu_\sigma,\mu}$ is $t^{-\ell(w_\mu)+\ell(w_{\mu_\sigma})}$, and the graded dimension of $m_{\lambda,\sigma(\mu_\sigma)}$ is

$$t^{-\ell(w_\lambda)+\ell(w_{\sigma(\mu_\sigma)})} = t^{-\ell(w_\lambda)+\ell(w_{\mu_\sigma})}.$$

The graded dimension of the crossing given by σ is $t^{-2\ell(\sigma)}$, and the sum of the graded dimensions given by h ranging over $\mathbf{B}_{\mu_\sigma}$ is

$$\sum_{w\in {}^{\mu_\sigma}W} t^{2\ell(w_0)-2\ell(w_{\mu_\sigma})-2\ell(w)} = t^{2\ell(w_0)-2\ell(w_{\mu_\sigma})} \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{w\in W_\mu \cap \sigma^{-1} W_\lambda \sigma} t^{-2\ell(w)}},$$

so the sum of graded dimensions of the geometric basis equals

$$\begin{aligned}
&\sum_{\sigma\in {}^{\lambda}W^{\mu}} t^{-\ell(w_\lambda)-\ell(w_\mu)+2\ell(w_{\lambda_\sigma})} \cdot t^{-2\ell(\sigma)} \cdot t^{2\ell(w_0)-2\ell(w_{\lambda_\sigma})} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{w\in W_\lambda \cap \sigma W_\mu \sigma^{-1}} t^{-2\ell(w)}} \\
=& t^{2\ell(w_0)-\ell(w_\lambda)-\ell(w_\mu)} \cdot \sum_{\sigma\in {}^{\lambda}W^{\mu}} t^{-2\ell(\sigma)} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{w\in W_\mu \cap \sigma^{-1} W_\lambda \sigma} t^{-2\ell(w)}} \\
=& t^{2\ell(w_0)-\ell(w_\lambda)-\ell(w_\mu)} \cdot (\sum_{w\in W} t^{-2\ell(w)}) \cdot \sum_{\sigma\in {}^{\lambda}W^{\mu}} \frac{t^{-2\ell(\sigma)}}{\sum_{w\in W_\mu \cap \sigma^{-1} W_\lambda \sigma} t^{-2\ell(w)}}
\end{aligned} \tag{29}$$

In order to compute the last part of (29), we need the following lemma studying the structure of the double coset $W_\lambda \sigma W_\mu$.

Lemma 4.4.5. *For a fixed σ, there exists bijection*

$$\begin{aligned} W_\lambda \times {}^{\mu_\sigma}W_\mu &\to W_\lambda \sigma W_\mu \\ (r, u) &\mapsto r\sigma u \end{aligned}$$

and $\ell(r\sigma u) = \ell(r) + \ell(\sigma) + \ell(u)$.

Proof. As σ is the shortest representative element in $W_\lambda\sigma W_\mu$, any element $w \in W_\lambda\sigma W_\mu$ has a reduced expression $w = r_1\sigma u_1$ for $r_1 \in W_\lambda$ and $u_1 \in W_\mu$ by [DJ86] or [Bou07].

If there is another reduced expression $w = r_2\sigma u_2$ such that $r_2 \in W_\lambda$ and $u_2 \in W_\mu$, then we have $r_1^{-1}r_2 = \sigma u_1 u_2^{-1}\sigma^{-1} \in W_\lambda \cap \sigma W_\mu \sigma^{-1}$ and $u_1 u_2^{-1} \in W_\mu \cap \sigma^{-1}W_\lambda\sigma = W_{\mu_\sigma}$, so we can choose $u_w \in {}^{\mu_\sigma}W_\mu$ satisfies $u_1 \in W_{\mu_\sigma}u_w$, which is uniquely determined by w and independent with the choice of u_1.

Now we see that $\ell(u_1) = \ell(u_w) + \ell(u_1u_w^{-1})$, and we have $\ell(w) = \ell(u_w) + \ell(wu_w^{-1})$. We have $wu_w^{-1} = r_1\sigma u_1 u_w^{-1} \in W_\lambda\sigma$, so we let $r_w := wu_w^{-1}\sigma^{-1} \in W_\lambda$, which is also uniquely determined by w, and we have $\ell(wu_w^{-1}) = \ell(\sigma) + \ell(r_w)$ as $\sigma \in {}^\lambda W^\mu \subset {}^\lambda W$, so we have $\ell(w) = \ell(r_w) + \ell(\sigma) + \ell(u_w)$.

One can check that the map $w \mapsto (r_w, u_w)$ is the inverse of the map in the statement. □

Lemma 4.4.5 immediately implies the following corollary:

Corollary 4.4.6. *For $\sigma \in {}^\lambda W^\mu$, we have*

$$t^{-2\ell(\sigma)} \cdot \sum_{r\in W_\lambda} t^{-2\ell(r)} \cdot \sum_{u\in {}^{\mu_\sigma}W_\mu} t^{-2\ell(u)} = \sum_{w\in W_\lambda\sigma W_\mu} t^{-2\ell(w)}. \tag{30}$$

If we apply a similar argument as (27), we have

$$t^{-2\ell(\sigma)} \cdot \sum_{r\in W_\lambda} t^{-2\ell(r)} \cdot \frac{\sum_{v\in W_\mu} t^{-2\ell(v)}}{\sum_{w\in W_\mu\cap\sigma^{-1}W_\lambda\sigma} t^{-2\ell(w)}} = \sum_{w\in W_\lambda\sigma W_\mu} t^{-2\ell(w)}. \tag{31}$$

Then we take the summation over $\sigma \in {}^\lambda W^\mu$, we have

$$\sum_{u\in W_\mu} t^{-2\ell(u)} \cdot \sum_{v\in W_\lambda} t^{-2\ell(v)} \cdot \sum_{\sigma\in{}^\lambda W^\mu} \frac{t^{-2\ell(\sigma)}}{\sum_{w\in W_\lambda\cap\sigma W_\mu\sigma^{-1}} t^{-2\ell(w)}} = \sum_{w\in W} t^{-2\ell(w)}. \tag{32}$$

Now we come back to (29), we see that the sum of graded dimensions of an R_d-basis equals to

$$\begin{aligned}
&t^{2\ell(w_0)-\ell(w_\lambda)-\ell(w_\mu)} \cdot (\sum_{w\in W} t^{-2\ell(w)}) \cdot \sum_{\sigma\in{}^\lambda W^\mu} \frac{t^{-2\ell(\sigma)}}{\sum_{w\in W_\lambda\cap\sigma W_\mu\sigma^{-1}} t^{-2\ell(w)}}\\
=&t^{2\ell(w_0)-\ell(w_\lambda)-\ell(w_\mu)} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{u\in W_\mu} t^{-2\ell(u)}} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{v\in W_\lambda} t^{-2\ell(v)}}\\
=&\underbrace{(t^{\ell(w_0)-\ell(w_\mu)} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{u\in W_\mu} t^{-2\ell(u)}})}_{A_1} \cdot \underbrace{(t^{\ell(w_0)-\ell(w_\lambda)} \cdot \frac{\sum_{w\in W} t^{-2\ell(w)}}{\sum_{v\in W_\lambda} t^{-2\ell(v)}})}_{A_2}
\end{aligned} \tag{33}$$

As we computed in (27) in Example 4.4.2, we have

$$A_1 = \begin{bmatrix} d \\ \mu^{(1)}, \mu^{(2)}, \dots, \mu^{(c)} \end{bmatrix}_t,$$

$$A_2 = \begin{bmatrix} d \\ \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(b)} \end{bmatrix}_t.$$

Then we see that the graded dimension of $e_\lambda\mathcal{A}_d e_\mu$ equals to the submodule spanned by $\{q \circ s_{\lambda,\lambda_0} \circ m_{\lambda_0,\mu} \circ p | q \in \mathbf{B}_\lambda, p \in \mathbf{B}_\mu\}$.

So we obtain the following result.

Proposition 4.4.7. *The sub-R_d-module spanned by*

$$\{\partial_{w_\lambda u}(\alpha_d) \circ s_{\lambda,\lambda_0} \circ m_{\lambda_0,\mu} \circ \partial_{w_\mu v}(\alpha_d) | u \in {}^{\lambda}W, v \in {}^{\mu}W\}$$

is the whole $e_\lambda \mathcal{A}_d e_\mu$.

So after combining Proposition 4.4.1 and Proposition 4.4.7, we finally complete the proof of Theorem 4.3.1.

4.5 Discussion on general quiver Schur algebras

A natural question is: can we generalize this construction to obtain a basis of $\mathcal{A}_{\mathbf{d}}$ for general dimension vector $\mathbf{d}$? Or we may ask whether the map

$$e_{\hat{\lambda}} \mathcal{A}_{\mathbf{d}} e_{\hat{\lambda}_0} \otimes_{R_{\mathbf{d}}} e_{\hat{\lambda}_0} \mathcal{A}_{\mathbf{d}} e_{\hat{\mu}} \to e_{\hat{\lambda}} \mathcal{A}_{\mathbf{d}} e_{\hat{\mu}} \tag{34}$$

is an isomorphism. The answer to both questions is negative. In fact, we can still obtain a linear independent set whose cardinality equals to the rank of $\mathcal{A}_{\mathbf{d}}$ over $R_{\mathbf{d}}$, but the space spanned by this set is not necessarily the whole $\mathcal{A}_{\mathbf{d}}$. In other words, the map (34) is injective but not surjective.

A significant difference between the one-vertex case and general cases is that the actions of split operators are different. For one-vertex quiver Schur algebra $\mathcal{A}_d$, the split is the same as the inclusion map between spaces of invariant polynomials, but in general, the split acts by multiplying with a polynomial corresponding to the Euler class of certain vector bundles. When the dimension vector is concentrated on a single vertex, we have that $\mathrm{Rep}_{\mathbf{d}}$ is just a point so that the vector bundle in this case is of rank 0 and the Euler class is 1. As the elements appearing on the left hand side of (34) always involve multiplying with the Euler class, the elements on the right hand side may not. For example, when $\hat{\mu} = \hat{\lambda}$, then the idempotent does not involve an Euler class. The counter example can be constructed even when the quiver only "has two vertices".

Example 4.5.1 (Counter example)**.** *In this example, the map in* (34) *is no longer surjective.*

Let $\mathbb{V} = \{1, 2\}$ and $\mathbf{d} = (1, 1)$. In this case, $R_{\mathbf{d}} \simeq \mathbb{C}[x_1, y_1]$, and $I_{\mathbf{d}}$ has three elements: Let $\lambda_0 = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $\lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\lambda_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then we see that $e_{\lambda_0} \mathcal{A}_{\mathbf{d}} e_{\lambda_1} = R_{\mathbf{d}} \cdot m_{\lambda_0,\lambda_1}$, and actually the map m_{λ_0,λ_1} is the identity from $\mathbb{C}[x_1, y_1]$ to $\mathbb{C}[x_1, y_1]$ in the faithful representation, while $e_{\lambda_1} \mathcal{A}_{\mathbf{d}} e_{\lambda_0} = R_{\mathbf{d}} \cdot s_{\lambda_1,\lambda_0}$, where s_{λ_1,λ_0} acts by multiplying with $y_1 - x_1$ in the faithful representation, hence we see that the image of $e_{\lambda_1} \mathcal{A}_{\mathbf{d}} e_{\lambda_0} \otimes e_{\lambda_0} \mathcal{A}_{\mathbf{d}} e_{\lambda_1}$ in $e_{\lambda_1} \mathcal{A}_{\mathbf{d}} e_{\lambda_1} \simeq R_{\mathbf{d}}$ is $R_{\mathbf{d}} \cdot (y - x)$.

However, by using the proof of the injectivity for the one vertex case, we can still obtain the following statement:

Lemma 4.5.2. *The multiplication map* (34) *is injective.*

In fact, we can still use the faithful representation to that the composition of multiplication map (34) and $e_{\hat{\lambda}} \mathcal{A}_{\mathbf{d}} e_{\hat{\mu}} \to \mathrm{Hom}_{\mathbb{C}}(R_{\hat{\mu}}, R_{\hat{\lambda}})$ is injective.

The set

$$\{p \circ s_{\hat{\lambda},\hat{\lambda_0}} | p \text{ runs over a basis of } R_{\hat{\mu}}\}$$

is a basis of $e_{\hat{\lambda}}\mathcal{A}_{\mathbf{d}}e_{\hat{\lambda_0}}$, and the set

$$\{m_{\hat{\lambda_0},\hat{\mu}} \circ q | q \text{ runs over a basis of } R_{\hat{\lambda}}\}$$

is a basis of $e_{\hat{\lambda_0}\mathcal{A}_{\mathbf{d}}e_{\hat{\mu}}}$. Their composition sends $f \in R_{\hat{\mu}}$ to $p \cdot E_{\hat{\lambda}} \cdot m_{\hat{\lambda_0},\hat{\lambda}}fq$, where $E_{\hat{\lambda}}$ is a polynomial in $R_{\hat{\lambda}}$ determined by $\hat{\lambda}$ only. So it suffices to show that the set

$$\{p \circ m_{\hat{\lambda}_0,\hat{\mu}} \circ q\}$$

forms an $R_{\mathbf{d}}$-linear independent set, and this is true because polynomial operators and merge operators can be computed locally. In other words, the action of polynomial operators and merges can be decomposed into the tensor product of polynomial operators and merges at each vertex, so it is reduced to the case for one vertex.

5 Application: the based quasi-hereditary algebra structure

In this section, we firstly put our quiver Schur algebra aside and introduce some definition and result about highest weight categories and quasi-hereditary algebras. Then we will show that the algebra $\mathcal{A}_d$ fits into the setting of this theory, and use these results to study the representation theory of the algebra arising from $\mathcal{A}_d$.

5.1 Highest weight category and quasi-hereditary algebra

Highest weight categories were introduced in [CPS88], in order to set up the axiomatic framework for a type of categories arising in many situations in representation theory. Abstracting from the representation theory of semisimple groups, it turns out that the theory of highest weight categories is related to the theory of quasi-hereditary algebras due to Ringel [DR92] and Scott[Sco87].

This notion is generalized to semi-infinite situations and its connection to highest weight categories is streamlined by Brundan and Stroppel in [BS18]. They give an alternative characterization of these categories in terms of *based* quasi-hereditary algebras and *based* stratified algebras, and we apply their definitions in the thesis.

A stratification of an Abelian category $\mathscr{R}$ is a quintuple $(\mathscr{B}, L, \rho, \Lambda, \leq)$ consisting of a set $\mathscr{B}$, a function L labelling a full set $\{L(b)|b \in \mathscr{R}\}$ of pairwise inequivalent irreducible objects in $\mathscr{R}$, and a function $\rho : \mathscr{B} \to \Lambda$ for the poset $(\Lambda, \leq)$ whose fibers are all finite. The stratification of $\mathscr{R}$ is finite if $\mathscr{R}$ is a finite Abelian category, which means that $\mathscr{R}$ is equivalent to the category of finite dimensional modules of a finite dimensional algebra (see [EGNO16, Definition 1.8.5]).

We denote by $\mathscr{B}_\lambda$ the fiber $\rho^{-1}(\lambda)$, and $\mathscr{B}_{\leq\lambda} := \bigcup_{\mu\leq\lambda} \mathscr{B}_\mu, \mathscr{B}_{<\lambda} := \bigcup_{\mu<\lambda} \mathscr{B}_\mu$. Let $\mathscr{R}_{\leq\lambda}$ (resp., $\mathscr{R}_{<\lambda}$) be the Serre subcategories of $\mathscr{R}$ associated to the subsets $\mathscr{B}_{\leq\lambda}$ (resp., $\mathscr{B}_{<\lambda}$), and $\mathscr{R}_\lambda := \mathscr{B}_{\leq\lambda}/\mathscr{B}_{<\lambda}$ is called a stratum. We further assume:

$$\text{the irreducible object } L(b) \text{ has both a projective cover and an injective hull in } \mathscr{R}_{\leq\rho(b)} \text{ for all } b \in \mathscr{B}, \text{ and each stratum is equivalent to the category of finite-dimensional vector spaces.} \tag{35}$$

Definition 5.1.1 (Standard object, costandard object)**.** *With assumption* (35)*, for $b \in \mathscr{B}$ such that $\rho(b) = \lambda$ we define the standard object $\Delta(b)$ as the projective cover of $L(b)$ in $\mathscr{R}_{\leq\lambda}$, and the costandard object $\nabla(b)$ as the injective hull of $L(b)$ in $\mathscr{R}_{\leq\lambda}$.*

For $V \in \mathscr{R}$, a Δ-flag of V means a filtration $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ with sections $V_m/V_{m-1} \cong \Delta(b_m)$ for some $b_m \in \mathscr{B}$. We denote by $\Delta(\mathscr{R})$ the exact subcategory consisting of all objects with a Δ-flag. We also define ∇-flags and $\nabla(\mathscr{R})$ in a similar way.

Definition 5.1.2 (Highest weight category)**.** *Let $\mathscr{R}$ be an Abelian category equipped with a finite stratification $(\mathscr{B}, L, \rho, \Lambda, \leq)$ satisfying assumption* (35)*.*

Then we say $\mathscr{R}$ is a finite highest weight category if for each $\lambda \in \Lambda$, there exists a projective object P_λ admitting a Δ-flag with $\Delta(\lambda)$ at the top and other sections of the form $\Delta(\mu)$ for $\mu \in \Lambda$ with $\mu \geq \lambda$.

Remark 5.1.3. *There are many important and classical examples of highest weight categories, such as blocks of the BGG category $\mathcal{O}$ for a semisimple Lie algebra [Hum08], the category of modules of classical Schur algebras [Mat99] and categories arising from perverse sheaves and singular spaces [BBDG18]. More recent examples, like categories defined via certain diagrams, can be found in [BS18].*

Now suppose that $\mathscr{R}$ is a highest weight category, we define $\mathscr{T}(\mathscr{R}) = \Delta(\mathscr{R}) \cap \nabla(\mathscr{R})$. Then for any $b \in \mathscr{B}_\lambda$ there is a unique indecomposable object $T_b \in \mathscr{T}(\mathscr{R})$ such that such that $[T_b : L(b)] = 1$ and $\rho(b') \leq \rho(b)$ whenever $[T_b : L(b')] \neq 0$ by [Don98, Theorem A4.2] (see also [BS18]). We also have that any object $T \in \mathscr{T}(\mathscr{R})$ is a direct sum of these T_b. If $T \in \mathscr{T}(\mathscr{R})$ has a summand isomorphic to T_b for all $b \in \mathscr{B}$, we call it a tilting generator.

With such a tilting generator, we let $B = \mathrm{End}_{\mathscr{R}}(T)^{\mathrm{op}}$. Then the finite Abelian category $\mathscr{R}' := B\text{-Mod}_{\mathrm{fd}}$ is called the *Ringel duality* of $\mathscr{R}$ relative to T. It turns out that $\mathscr{R}'$ is also finitely stratified, and more detail can be found in [BS18, Theorem 4.10].

To give an elementary characterization of highest weight categories, we need the notion of based quasi-hereditary algebras. We apply the following definition in [BS18], which is equivalent to the one given in [KM20, Definition 2.4]. For more details we refer to [BS18, §5].

Definition 5.1.4 (Based quasi-hereditary algebras)**.** *A finite based quasi-hereditary algebra is a locally unital algebra over a field such that $A = \bigoplus_{i,j \in I} e_i A e_j$ which is free of finite rank over the ring, with the following data:*

QH1 *A subset $\Lambda \subset I$ indexing special idempotents $\{e_\lambda | \lambda \in \Lambda\}$.*

QH2 *A partial order $\leq$ making Λ into a poset.*

QH3 *Sets $Y(i, \lambda) \subset e_i A e_\lambda, X(\lambda, j) \subset e_\lambda A e_j$ for $i, j \in I, \lambda \in \Lambda$.*

Let $Y(\lambda) = \cup_{i \in I} Y(i, \lambda), X(\lambda) = \cup_{j \in I} X(\lambda, j)$, and we impose the following three axioms:

QH4 *Products $\{yx | (y, x) \in \bigcup_{\lambda \in \Lambda} Y(\lambda) \times X(\lambda)\}$ form a basis for A.*

QH5 *For $\lambda, \mu \in \Lambda$, $Y(\mu, \lambda)$ and $X(\lambda, \mu)$ are empty unless $\mu \leq \lambda$.*

QH6 *For each $\lambda \in \Lambda$, we have $X(\lambda, \lambda) = Y(\lambda, \lambda) = \{e_\lambda\}$.*

If there is some given algebra anti-involution $\sigma : A \to A$ with $\sigma(e_i) = e_i$ and $Y(i, \lambda) = \sigma(X(\lambda, i))$ for all $i \in I, \lambda \in \Lambda$, then A is said to be symmetrically based.

The basis given by 5.1.4.QH4 is called the triangular basis of A (which obviously depends on the choice of elements in $X(\lambda, j)$ and $Y(i, \lambda)$).

Suppose that A is already a finite based quasi-hereditary algebra, then for $\lambda \in \Lambda$ we let $A_{\leq \lambda}$ be the quotient of A be the sum of $Ae_\mu A$ such that $\mu \not\leq \lambda$. We write $\overline{e_\lambda}$ for the image of e_λ in $A_{\leq \lambda}$, then the standard and costandard modules associated to λ is defined as

$$\Delta(\lambda) := A_{\leq \lambda} \overline{e_\lambda}, \nabla(\lambda) := (\overline{e_\lambda} A_{\leq \lambda})^*$$

respectively.

Remark 5.1.5. *The notion above also makes sense if we replace the field by a commutative ring, and replace the finite dimensional condition by being free of finite rank over the ground ring. Then in the symmetrically based case, it is equivalent to the notion of an object-adapted cellular category introduced by Elias and Lauda. See [EL16, Definition 2.1, Lemmas 2.6-2.8] for more explanations.*

The following theorem [BS18, Theorem 5.9, Theorem 5.10] gives the preicse relationship between highest weight categories and based quasi-hereditary algebras.

Theorem 5.1.6. *1. Let A be a finite based quasi-hereditary algebra over a field. The modules*

$$\{L(\lambda) := \mathrm{hd}\Delta(\lambda) \cong \mathrm{soc}\nabla(\lambda) | \lambda \in \Lambda\}$$

give a complete set of pairwise inequivalent irreducible left A-modules. Moreover, the category $\mathscr{R} := A\text{-mod}_{\mathrm{fd}}$ is a finite highest weight category with the given weight poset $(\Lambda, \leq)$. Its standard and costandard objects $\Delta(\lambda)$ and $\nabla(\lambda)$ are as defined by (5.2).

2. Let $\mathscr{R}$ be a finite highest weight category with weight poset $(\Lambda, \leq)$ and labelling function L. Suppose we are given $\Lambda \subset I$ and a tilting generator $T = \bigoplus_{t\in I} T_i$ for $\mathscr{R}$ such that each T_λ for $\lambda \in \Lambda$ is a direct sum of $T(\lambda)$ and other $T(\mu)$ for $\mu < \lambda$. Let

$$A := (\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathscr{R}}(T_i, T_j))^{\mathrm{op}},$$

and for $i, j \in I, \lambda \in \Lambda$ we pick morphisms

$$Y(i, \lambda) \subset \mathrm{Hom}_{\mathscr{R}}(T_i, T_\lambda), X(\lambda, j) \subset \mathrm{Hom}_{\mathscr{R}}(T_\lambda, T_j)$$

lifting basis for $\mathrm{Hom}_{\mathscr{R}}(T_i, \nabla(\lambda))$ and $\mathrm{Hom}_{\mathscr{R}}(\Delta(\lambda), T_j)$ such that $Y(\lambda, \lambda) = X(\lambda, \lambda) = \{\mathrm{id}_{T_\lambda}\}$, then

$$\{yx | (y, x) \in \bigcup_{i,j\in I} \bigcup_{\lambda\in\Lambda} Y(i, \lambda) \times X(\lambda, j)\}$$

is a triangular basis making A into a finite based quasi-hereditary algebra with respect to the opposite poset $(\Lambda, \geq)$.

5.2 The algebra $\mathcal{A}_d$ as a based quasi-hereditary algebra

Back to our algebra $\mathcal{A}_d$, which is free over R_d. We have the decomposition of R_d-module $\mathcal{A}_d \simeq \bigoplus_{\mu,\lambda\in I_d} e_\mu \mathcal{A}_d e_\lambda$ where μ, λ runs over I_d, the set of all compositions of d. All these summands are free of finite rank over R_d because $e_\mu \mathcal{A}_d e_\lambda \simeq R_d^{|^\mu W| \cdot |^\lambda W|}$ as R_d-modules.

Now we show that $\mathcal{A}_d$ is a based quasi-hereditary algebra over R_d.

The subset $\Lambda \subset I_d$ has only one element, which is the composition $\lambda_0 = (d)$ with only one component, and for any $\mu, \lambda \in I_d$, we set

$$Y(\lambda, \lambda_0) = \{\mathbf{b}^1_{\lambda,\lambda_0}(p) | p \in \mathbf{B}_\lambda\}, \quad X(\lambda_0, \mu) = \{\mathbf{b}^1_{\lambda_0,\mu}(q) | q \in \mathbf{B}_\mu\}. \tag{36}$$

And by Theorem 4.3.1, one can check that all axioms in Definition 5.1.4 are satisfied, so we have the following result:

Theorem 5.2.1. *The algebra* $\mathcal{A}_d \simeq \bigoplus_{\lambda,\mu\in I_d} e_\mu \mathcal{A}_d e_\lambda$ *is a finite based quasi-hereditary algebra over* R_d *with* $\Lambda = \{\lambda_0\}$ *and* $Y(\lambda,\lambda_0), X(\lambda_0,\mu)$ *defined as in* (36).

Moreover, the algebra $\mathcal{A}_d$ actually is symmetrically based.

We consider the R_d-linear map $\sigma : \mathcal{A}_d \to \mathcal{A}_d$ such that

$$\mathbf{v}_{\lambda,\mu}(p,q) \mapsto \mathbf{v}_{\mu,\lambda}(q,p),$$

for any $\mu, \lambda \in I_d, p \in \mathbf{B}_\lambda, q \in \mathbf{B}_\mu$, then it is obvious that $\sigma^2 = 1$. Alsp, it follows from definition that $\sigma(X(\lambda_0,\mu)) = Y(\mu,\lambda_0)$ for $\mu \in I_d$.

Now we show that σ is an anti-algebra homomorphism. Then for arbitrary $\mu_1,\mu_2,\mu_3,\mu_4 \in I_d$ and $q_i \in \mathbf{B}_{\mu_i}$ for $i = 1,2,3,4$, we have

$$\begin{aligned}
&\sigma(\mathbf{v}_{\mu_1,\mu_2}(q_1,q_2)) \cdot \sigma(\mathbf{v}_{\mu_3,\mu_4}(q_3,q_4))\\
=&\mathbf{v}_{\mu_2,\mu_1}(q_2,q_1) \cdot \mathbf{v}_{\mu_4,\mu_3}(q_4,q_3)\\
=&\delta_{\mu_1,\mu_4} m_{\mu_1,\lambda_0}(q_1 \cdot q_4) \cdot \mathbf{v}_{\mu_2,\mu_3}(q_2,q_3)\\
=&\delta_{\mu_1,\mu_4} m_{\mu_1,\lambda_0}(q_1 \cdot q_4) \cdot \sigma(\mathbf{v}_{\mu_3,\mu_2}(q_3,q_2))\\
=&\sigma(\delta_{\mu_4,\mu_1} m_{\mu_1,\lambda_0}(q_4 \cdot q_1) \cdot \mathbf{v}_{\mu_3,\mu_2}(q_3,q_2))\\
=&\sigma(\mathbf{v}_{\mu_3,\mu_4}(q_3,q_4) \cdot \mathbf{v}_{\mu_1,\mu_2}(q_1,q_2)),
\end{aligned}$$

so σ is an anti-algebra involution over R_d.

Moreover, we have $\sigma(e_\lambda) = e_\lambda$ for any $\lambda \in I_d$, and hence $\sigma(1) = 1$. In fact, we know that e_λ is an idempotent and it is in the center of $e_\lambda \mathcal{A}_d e_\lambda$, which is $e_\lambda R_d e_\lambda$, by Proposition 2.4.17. So we know that e_λ is the only idempotent contained in the center of $e_\lambda \mathcal{A}_d e_\lambda$, and the same is true for $\sigma(e_\lambda)$, because $\sigma(e_\lambda)\cdot\sigma(e_\lambda) = \sigma(e_\lambda^2) = \sigma(e_\lambda)$, and for any $\beta \in e_\lambda \mathcal{A}_d e_\lambda$, we have

$$\sigma(e_\lambda)\cdot\beta = \sigma(e_\lambda)\cdot\sigma\sigma^{-1}(\beta) = \sigma(\sigma^{-1}(\beta)\cdot e_\lambda) = \sigma(e_\lambda\cdot\sigma^{-1}(\beta)) = \sigma\sigma^{-1}(\beta)\cdot\sigma(e_\lambda) = \beta\cdot\sigma(e_\lambda),$$

so we immediately see that $\sigma(e_\lambda) = e_\lambda$, and under σ, $\mathcal{A}_d$ becomes a symmetrically based quasi-hereditary algebra over R_d.

Then we draw the following conclusion:

Proposition 5.2.2. *The algebra* $\mathcal{A}_d$*, together with the quasi-hereditary algebra structure as in Theorem 5.2.1, can be equipped with an anti-algebra involution* σ *which sends* $\mathbf{v}_{\lambda,\mu}(p,q)$ *to* $\mathbf{v}_{\mu,\lambda}(q,p)$*, making* $\mathcal{A}_d$ *a symmetric based quasi-hereditary algebra.*

Theorem 5.2.1 implies that $\mathcal{A}_d = \mathcal{A}_d e_{\lambda_0} \mathcal{A}_d$, we immediately have $\mathcal{A}_d$ is Morita equivalent to $e_{\lambda_0}\mathcal{A}_d e_{\lambda_0} = R_d$ (see [AF92, §21]). More precisely, the following functors give the equivalence:

$$e_{\lambda_0}\mathcal{A}_d \otimes_{\mathcal{A}_d} (-) : \mathcal{A}_d - \mathrm{Mod} \longrightarrow e_{\lambda_0}\mathcal{A}_d e_{\lambda_0} - \mathrm{Mod} = R_d - \mathrm{Mod},$$

$$\mathcal{A}_d e_{\lambda_0} \otimes_{e_{\lambda_0}\mathcal{A}_d e_{\lambda_0}} (-) : R_d - \mathrm{Mod} = e_{\lambda_0}\mathcal{A}_d e_{\lambda_0} - \mathrm{Mod} \longrightarrow \mathcal{A}_d - \mathrm{Mod}.$$

And since Λ in this case has only one element λ_0, we have $(\mathcal{A}_d)_{\leq\lambda_0} = \mathcal{A}_d$, and the standard (resp., costandard) module $\Delta(\lambda_0)$ (resp., $\nabla(\lambda_0)$) is $\mathcal{A}_d e_{\lambda_0}$ (resp., $(e_{\lambda_0}\mathcal{A}_d)^*$). Note that

$$\mathcal{A}_d e_{\lambda_0} \simeq \bigoplus_{\mu\in I_d} R_\mu$$

is exactly the faithful representation of $\mathcal{A}_d$ given in the previous section.

Since R_d is not a field, we cannot apply Theorem 5.1.6 to $\mathcal{A}_d$. But we can let K_d be the fraction field of R_d, i.e the field of symmetric rational functions of d variables, and we let $\mathcal{A}_d^K := K_d \otimes_{R_d} \mathcal{A}_d$.

Obviously, this is again a symmetrically based quasi-hereditary algebra over K_d, with exactly the same $\Lambda = \{\lambda_0\}$, $X(\lambda, \lambda_0)$ and $Y(\lambda_0, \mu)$. Also, the category of finite (K_d-)dimensional $\mathcal{A}_d^K$-modules is a highest weight category. Via Theorem 5.1.6, there is only one isomorphism class of simple objects in this category, given by the head of $\mathcal{A}_d^K e_{\lambda_0}$.

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