Different approaches to Young-Jucys-Murphy elements

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1 Introduction

Schur-Weyl duality of type A makes the study of representations over $\mathbb{C}$ of two seemingly disparate objects, namely the symmetric group S_d and the Lie algebra $\mathfrak{gl}_n$ akin to exploring two reflections in the same mirror. In other words, when working over $\mathbb{C}$, one side completely determines the other side. On the one hand, in the classical approach to studying the representations of S_n, partitions and the combinatorics of tableaux appear in a rather unnatural way, as this approach is not taking into account special algebraic features that S_n enjoys, for example its structure as a Coxeter group, or that symmetric groups are naturally embedded in each other. Juxtaposingly, the classification and study of finite dimensional representations of $\mathfrak{gl}_n$ (or more generally any finite dimensional complex semisimple Lie algebra) heavily depends on the existence of a weight space decomposition i.e, a simultaneous eigenspace decomposition with respect to a maximal commutative (Cartan) subalgebra of the universal enveloping algebra $U(\mathfrak{gl}_n)$. Thus, one might expect that we can mimic this approach to study the representation theory of the symmetric groups using a spectral approach in analogy to weights and weight spaces arising from the action of a commutative subalgebra. Okounkov and Vershik [11] realized this idea, using special elements of the group algebra $\mathbb{C}[S_n]$ called the Young-Jucys-Murphy elements which were independently used by Young, Jucys and Murphy for proving various results about the structure of $\mathbb{C}[S_n]$. These elements are generators of a maximal commutative subalgebra of $\mathbb{C}[S_n]$, the Gelfand-Zetlin algebra, and dominate the study of irreducible representations over $\mathbb{C}$. Note that the field of complex numbers was chosen in order to ensure some properties exist, for example over $\mathbb{C}$ we have a generalized eigenspace decompsition.

Ever since, various attempts have been made to generalize this to other types of diagrammatic algebras arising from passing this duality to other Lie subalgebras or even the group of permutation matrices S_n inside $\mathfrak{gl}_n$ while in parallel enlarging the "mirror" player $\mathbb{C}[S_n]$. A peculiar case is the one of the partition algebras $P_{2n}(t)$, arising as the "mirror" player for the group of permutation matrices, naturally identified with S_n (where the action is different than the one coming from the classical Schur-Weyl duality, namely the first one permutting tensor factors while this one permutes basis elements), first proven by Martin, see [8]. The partition algebra first arose from problems in statistical mechanics, see [9]. Halverson and Ram [6] defined analogues of YJM elements for the partition algebra, using this duality with the symmetric groups and showed that whenever these algebras are semisimple, their representation theory is again governed by these YJM elements. However, a problem with this approach is the following: The diagrammatics available did not include YJM elements as generators. Instead, they could only be described via linear combinations of other diagrams. As a consequence, the power of YJM elements and of diagrammatics could not be combined.

Enyang [5] provided us with very complicated recursive formulas for the YJM elements of $P_{2n}(t)$, making relations in this algebra difficult to confirm. In order to solve the first problem, one first tries to find an invariant way to define these YJM elements (i.e a recursive formula completely characterizing them) and secondly find a "nice" presentation of the corresponding algebra, compatible with the definition of these YJM elements. Then one creates a new algebra that contains the original one both as a subalgebra and as a quotient, by adjoining these YJM elements as generators satisfying certain relations. In particular, for the case of S_n with its usual presentation as a Coxeter group, this new algebra is the degenerate affine Hecke algebra $\mathcal{H}_n$, originally introduced by Nazarov. For the partition algebra, recently Creedon and De Visscher in [4] defined a version of the affine partition algebra, using it to prove various results for the representation theory of $P_{2n}(t)$ both for the semisimple and for the non-semisimple case. But this still did not solve the problem of the YJM elements not being included in the diagrammatics.

Towards that, Brundan and Vargas [1] took a more modern approach to the matter, defining a new category, the affine partition category. They used a theorem from [10], which embeds the partition category $\mathcal{P}ar_t$ inside the Heisenberg category $\mathcal{H}eis$ defined by Khovanov [7]. The latter has natural candidates for YJM elements. With this approach, the YJM elements are included in the diagrammatics and moreover, one can now employ the string calculus of $\mathcal{H}eis$ to prove relations involving these complicated YJM elements, making them easier to control. This thesis takes a tour through these ideas and is structured as follows: In section 2, we give an exposition of the Okounkov-Vershik approach, mostly following [11] and [2]. In section 3 we deal with the case of the partition algebras $P_{2n}(t)$, following [6],[4] and [3]. In section 4, we take a detour to the structure of Deligne's interpolation category $\underline{Rep}(S_t;\mathbb{C})$, mostly following [3]. Finally, section 5 is devoted to the definition of the affine partition category and the retrieval of the classical YJM elements via this approach, following [1].

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2 The Okounkov-Vershik Approach

2.1 Gelfand-Zetlin Constructions

For this approach, we are working over $\mathbb{C}$, thus for any finite group G, the group algebra $\mathbb{C}[G]$ is semisimple by Maschke's theorem. Also, unless otherwise stated, all our representations are unitary and finite dimensional. By unitary we mean that any representation (V, σ) of G is equipped with a scalar product $(\ ,\ )$ making the actions of every $g \in G$ unitary operators, or equivalently $(\sigma(g)v, \sigma(g)w) = (v, w)$ for every $v, w \in V$. We can turn every representation (V, σ) into a unitary representation as follows. Pick a non-zero arbitrary scalar product $\langle\ ,\ \rangle$ for V. Then the scalar product

$$(v, w) = \sum_{g \in G} \langle \sigma(g)v, \sigma(g)w \rangle$$

endows V with the structure of a unitary representation of G. For a group G, we denote by $\hat{G}$ the set consisting of a fixed complete class of inequivalent irreducible representations of G. Moreover, if $H \leq G$ is a subgroup, we denote by $\mathcal{C}(G, H) = \{g \in G\ : hgh^{-1} = g,\ \ \forall h \in H\}$ the centralizer algebra of H in G.

Definition 2.1 (Multiplicity-free subgroup)**.** *Let $H \leq G$. We say that H is a multiplicity-free subgroup of G if for every $\sigma \in \hat{G}$ the restriction $\operatorname{Res}^G_H \sigma$ is multiplicity free. Equivalently, we can characterize the above as*

$$\dim \operatorname{Hom}_H(\rho, \operatorname{Res}^G_H \sigma) \leq 1$$

for every $\rho \in \hat{H}, \sigma \in \hat{G}$.

Definition 2.2 (Multiplicity free chain)**.** *Let G be a group. A chain of subgroups*

$$G_1 = \{1\} \leq G_2 \leq \ldots \leq G_{n-1} \leq G_n \leq \ldots$$

is called a multiplicity free chain if G_{n-1} is a multiplicity free subgroup of G_n for all $n \geq 2$.

Remark 2.3. *Note that by adjunction of restriction and induction for finite groups, the above definition can also be reformulated as*

$$\dim \operatorname{Hom}_G(\sigma, \operatorname{Ind}^G_H \rho) \leq 1$$

for every $\sigma \in \hat{G}, \rho \in \hat{H}$.

Theorem 2.4. *The following are equivalent:*

(a) The algebra $\mathcal{C}(G, H)$ is commutative.

(b) H is a multiplicity-free subgroup of G.

Proof. For a full proof, see [2]. □

Note that for the chain to be multiplicity free, we necessarily need $G_1 = \{1\}$ to be a multiplicity free subgroup in G_2, or equivalently by theorem 2.4, $\mathcal{C}(G_2, 1) = G_2$ to be abelian. Note that by theorem 2.4, an equivalent definition for the chain to be multiplicity free would be that the centralizers $\mathcal{C}(G_i, G_{i-1})$ are commutative for all i.

Proposition 2.5 (Generalized Gelfand Lemma)**.** *Let H be a subgroup of the finite group G such that every $g \in G$ is H-conjugate to g^{-1}. Then the centralizer $\mathcal{C}(G, H)$ is commutative.*

Proof. See Proposition 2.1.12 from [2]. □

Definition 2.6 (Branching graph)**.** *The branching graph of a multiplicity free chain is the oriented graph whose vertex set is* $\hat{G}_1 \sqcup \hat{G}_2 \sqcup \ldots \sqcup \hat{G}_n \sqcup \ldots$ *and whose edge set is*

$$\{(\rho, \sigma) : \rho \in \hat{G}_n, \sigma \in \hat{G}_{n-1}, m_{\sigma,\rho} = 1, \, n \geq 2\}$$

where $m_{\sigma,\rho}$ *is the multiplicity of* σ *in* $\operatorname{Res}^{G_n}_{G_{n-1}} \rho$*.*

In other words, it is the graph with vertex set the inequivalent representations of each individual group in the chain and two irreducible representations in consecutive levels are connected, if and only if we can find the one in the lower level inside the decomposition of the restriction of the one in the higher level into irreducible representations when restricting to the previous level. We shall write $\rho \to \sigma$ if (ρ, σ) is an edge in the branching graph.

CONSTRUCTION OF GELFAND-ZETLIN BASIS

Remember that by Schur's lemma, the irreducible representations of an abelian group over $\mathbb{C}$ are all 1 dimensional. Let $(V_\sigma, \sigma) \in \hat{G}_n$. If we have a multiplicity free chain then

$$\operatorname{Res}^{G_n}_{G_{n-1}} V_\sigma = \bigoplus_{\rho \in \hat{G}_{n-1} : \sigma \to \rho} V_\rho$$

is an orthogonal decomposition. Iterating this decomposition we get that for $\rho \in \hat{G}_{n-1}$ the decomposition

$$\operatorname{Res}^{G_{n-1}}_{G_{n-2}} V_\rho = \bigoplus_{\substack{\theta \in \hat{G}_{n-2}: \\ \rho \to \theta}} V_\theta$$

Iterating this process for $n-1$ steps, we get direct sums of one dimensional trivial representations. To formalize this, let $\mathcal{T}(\sigma)$ be the set of all paths in the branching graph, i.e

$$\mathcal{T}(\sigma) = \{T = (\sigma = \sigma_n \to \sigma_{n-1} \to \ldots \to \sigma_2 \to \sigma_1) : \, \sigma_i \in \hat{G}_i, \, \forall i = 1, 2, \ldots, n.\}.$$

Notice that every path in $\mathcal{T}(\sigma)$ ends in the trivial representation. We can now write

$$V_\sigma = \bigoplus_{\substack{\sigma_{n-1} \in \hat{G}_{n-1}: \\ \sigma \to \sigma_{n-1}}} V_{\sigma_{n-1}} = \ldots = \bigoplus_{T \in \mathcal{T}(\sigma)} V_{\sigma_1}$$

In the last decomposition, every space is one dimensional, thus for each $T \in \mathcal{T}(\sigma)$, we may choose v_T in V_{σ_1} such that $\|v_T\| = 1$. Then the decomposition above may be written as

$$V_\sigma = \bigoplus_{T \in \mathcal{T}(\sigma)} \langle v_T \rangle.$$

In other words, the set of v_T's defined above yields an orthonormal basis for V_σ.

Definition 2.7. *The above orthonormal basis* $\{v_T \mid T \in \mathcal{T}(\sigma)\}$ *will be called a Gelfand-Zetlin basis (or GZ basis for short) for the irreducible* $V_\sigma \in \hat{G}_n$ *with respect to the multiplicity free chain*

$$\{e\} = G_1 \leq G_2 \leq \ldots \leq G_{n-1} \leq G_n.$$

Note that in every level of the graph, for a $\theta \in \hat{G}_k$, its multiplicity in $\mathrm{Res}^{G_n}_{G_k} \sigma$ is exactly the number of paths in the graph from σ to θ. Moreover, we get an effective way of decomposing the isotypic component of V_θ in $\mathrm{Res}^{G_n}_{G_k} \sigma$ into its orthogonal irreducible copies. Indeed, with each path from σ to θ one associates a unique component V_θ and in particular, two different paths correspond to different components. For $j = 1, \ldots, n$ we denote by $\mathcal{T}_j(\rho)$ the set of paths S in the branching graph of the form

$$S = (\rho = \sigma_n \to \sigma_{n-1} \to \ldots \to \sigma_{j+1} \to \sigma_j)$$

where $\sigma_k \in \hat{G}_k$ for all k and note that $\mathcal{T}_1(\rho) = \mathcal{T}(\rho)$. For $T = (\sigma_n = \rho \to \sigma_{n-1} \to \ldots \to \sigma_1) \in \mathcal{T}(\rho)$ denote by $T_j \in \mathcal{T}_j(\rho)$ the j-truncated path of T, or in other words the path following T from ρ up to σ_j. Now for $j = 1, \ldots, n$ and $S \in \mathcal{T}_j(\rho)$ set

$$V_S = \bigoplus_{\substack{T \in \mathcal{T}(\rho): \\ T_j = S}} V_{\rho_1}$$

(That is, the space consisting of the decomposition of one of the irreducible factors in the decomposition of $\mathrm{Res}^{G_n}_{G_j} \rho$ into G_j-irreducibles in the selected GZ basis). So following this, by setting $\rho_S = (\mathrm{Res}^{G_n}_{G_j} \rho)|_{V_S}$ we have that ρ_S is a G_j-irreducible representation and in fact, $\rho_S = \rho_j$. Also, $V_T = \mathbb{C}v_T$ for all $T \in \mathcal{T}(\rho)$. Finally, the restriction of ρ to G_j yields the orthogonal decomposition into G_j irreducible representations

$$\mathrm{Res}^{G_n}_{G_j} V_\rho = \bigoplus_{S \in \mathcal{T}_j(\rho)} V_S.$$

Lastly, for every j, if $S \in \mathcal{T}_j(\rho)$ and $T \in \mathcal{T}(\rho)$, then $T_j = S$ if and only if $v_T \in V_S$. In other words, an element of the GZ basis belongs to a certain G_j irreducible summand (indexed by a truncated path) if and only if the j- truncation of the path we took is exactly the path corresponding to that summand.

Remark 2.8. *For $\sigma \in \hat{G}_n$, it holds that*

$$\dim V_\sigma = |\mathcal{T}(\sigma)|.$$

That is, the dimension of any irreducible G_n representation is equal to the number of paths starting from it in the branching graph.

Example 2.9. *For the sake of sanity, suppose we know that the chain of symmetric groups is multiplicity free and that their branching graph is isomorphic to the Young graph of partitions. We explain how the above ideas work out for $S^{(2,1)}$, the irreducible representation of S_3 corresponding to the partition $(2,1) \vdash 3$. There are two paths from it to S_1, namely*

$$T_1 = \left(\ydiagram{2,1} \to \ydiagram{2} \to \ydiagram{1} \right), \quad T_2 = \left(\ydiagram{2,1} \to \ydiagram{1,1} \to \ydiagram{1} \right)$$

and thus $S^{(2,1)}$ is two-dimensional. Moreover, as an S_2 representation it decomposes in a multiplicity free way into irreducibles as

$$S^{(2,1)} = S^{(2)} \oplus S^{(1,1)}$$

i.e it is the sum of the trivial and the alternating representation for S_2. For a GZ basis $\{v_{T_1}, v_{T_2}\}$, we have that $v_{T_1} \in S^{(2)} \subseteq S^{(2,1)}$ as its 2-truncated path $\left(\ydiagram{2,1} \to \ydiagram{2} \right)$ falls exactly onto the trivial partition (2) corresponding to the representation $S^{(2)}$. Similarly $v_{T_2} \in S^{(1,1)} \subseteq S^{(2,1)}$.

2.1.1 Gelfand-Zetlin Algebra

The classification and study of finite dimensional representations of a finite dimensional complex semisimple Lie algebra heavily depends on the existence of a weight space decomposition that is a simultaneous eigenspace decomposition of a commutative Cartan subalgebra. In analogy to this theory, one might think that we would like to find a maximal commutative subgroup of G. In the case of the symmetric group, that is completely hopeless, but as in the case of Lie algebras, having the existence of the universal enveloping algebra which entails all the information of its representations, so does the group algebra here. So we instead try to find a maximal commutative subalgebra of $\mathbb{C}[G]$. This is the role of the Gelfand-Zetlin algebra whose definition is given below.

Definition 2.10 (Gelfand-Zetlin algebra)**.** *Let G be a finite group. We denote by $\mathcal{Z}(i)$ the center of the group algebra $\mathbb{C}[G_i]$. The Gelfand-Zetlin algebra (GZ for short), denoted by $\mathcal{GZ}(n)$ associated with the multiplicity-free chain $G_1 \leq G_2 \leq \ldots \leq G_{n-1} \leq G_n \leq \ldots$ is the algebra generated by the subalgebras*

$$\mathcal{Z}(1), \mathcal{Z}(2), \ldots, \mathcal{Z}(n)$$

i.e in symbols

$$\mathcal{GZ}(n) = \langle \mathcal{Z}(1), \mathcal{Z}(2), \ldots, \mathcal{Z}(n) \rangle.$$

We want to give a characterization of this algebra as a maximal commutative subalgebra of $\mathbb{C}[G_n]$. In order to gain some insight on how this algebra might look like, we first find a maximal commutative subalgebra of $M_{n\times n}(\mathbb{C})$.

Example 2.11. *We claim that the algebra of diagonal matrices D_n is maximal commutative in $M_{n\times n}(\mathbb{C})$. Commutativity is imminent by commutativity of $\mathbb{C}$. Now for maximality, assume that $D_n \subset A$, where A is a commutative subalgebra of $M_{n\times n}(\mathbb{C})$. For $i = 1, \ldots, n$, denote by $E_{i,i}$ the matrix containing 1 in position (i,i) and 0 elsewhere and let $(x_{i,j})_{i,j=1}^{n} = x \in A$. Since A is commutative and contains diagonal matrices, it holds*

$$E_{i,i}x = xE_{i,i}$$

for all $i = 1, \ldots, n$. Equivalently, since multiplying with $E_{i,i}$ from the left (right) means nullifying every entry of x but row (column) i, we get $x_{i,j} = 0$ for all $j \neq i$. Since i was random, this yields $x \in D_n$, proving that D_n is a maximal commutative subalgebra of $M_{n\times n}(\mathbb{C})$.

Theorem 2.12. *The Gelfand-Zetlin algebra $\mathcal{GZ}(n)$ is a maximal commutative subalgebra of $\mathbb{C}[G_n]$. Moreover, it coincides with the subalgebra of elements $f \in \mathbb{C}[G_n]$ whose actions $\sigma(f)$, $\sigma \in \hat{G}_n$ are simultaneously diagonalized by a Gelfand-Zetlin basis of V_σ. In formulas,*

$$\mathcal{GZ}(n) = \{f \in \mathbb{C}[G_n] : \sigma(f)(v_T) \in \mathbb{C}v_T, \text{ for all } \sigma \in \hat{G}_n \text{ and } T \in \mathcal{T}(\sigma)\}.$$

Proof. We prove that $\mathcal{GZ}(n)$ is commutative and a set of generators is the set of all products of the form

$$f_1 f_2 \cdots f_n \tag{1}$$

where $f_i \in \mathcal{Z}(i)$. Since we are dealing with central elements, for $j \leq i$, if $f_j \in \mathcal{Z}(j)$ and $f_i \in \mathcal{Z}(i)$ then it is clear that

$$f_i f_j = f_j f_i.$$

since $\mathbb{C}[G_j] \subseteq \mathbb{C}[G_i]$ and f_i is central. This proves that $\mathcal{GZ}(n)$ is commutative and that the set proposed in eq. (1) is a generating set. Now denote by $\mathcal{A}$ the set on the right hand side of the theorem. Since the linearization of an action is an algebra morphism, $\mathcal{A}$ is an algebra. We show that it contains all the centers. Suppose $f_i \in \mathcal{Z}(i)$, $\sigma \in \hat{G}_n$, $T \in \mathcal{T}(\sigma)$ and $S = T_i$. Then for the GZ basis element v_T corresponding to this path, we have that $v_T \in V_S$, where V_S is the irreducible summand corresponding to the path S in the decomposition of $\mathrm{Res}^{G_n}_{G_i} \sigma$. Combined with $f_i \in \mathcal{Z}(i)$, we get that $\sigma(f_i) \in \mathrm{End}_{G_i}(\mathrm{Res}^{G_n}_{G_i} \sigma)$ and by Schur's lemma, its action onto v_T falls into $\mathrm{End}_{G_i}(V_S) = \mathbb{C}id_{V_S}$, thus it acts by a scalar, or in other words, there exists $\lambda_{S,f_i} \in \mathbb{C}$ such that $\sigma(f_i)v_T = \lambda_{S,f_i} v_T$. This proves $\mathcal{Z}(i) \subseteq \mathcal{A}$ for all i and consequently $\mathcal{GZ}(n) \subseteq \mathcal{A}$. We are left with proving the opposite direction and maximality. To show $\mathcal{A} \subseteq \mathcal{GZ}(n)$, let $\sigma \in \hat{G}_n$ and $T \in \mathcal{T}(\sigma)$, say

$$T = (\sigma = \sigma_n \to \sigma_{n-1} \ldots \to \sigma_2 \to \sigma_1).$$

Remember that we have the isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{\sigma \in \hat{G}} \mathrm{Hom}(V_\sigma, V_\sigma) \tag{2}$$

given by $f \mapsto \oplus \sigma(f)$. Then since $\mathrm{End}_{G_i}(V) = \mathbb{C}id_V$ for every irreducible representation V of G_i, for each $i = 1, \ldots, n$ we can choose $f_i \in \mathbb{C}[G_i]$ such that their actions satisfy

$$\rho(f_i) = \begin{cases} id_{V_\rho}, \text{if } \rho = \sigma_i \\ 0, \text{ otherwise} \end{cases}$$

for all $\rho \in \hat{G}_i$. By repeating this argument, it is easy to see that the element

$$F_T = f_1 f_2 \ldots f_n$$

satisfies $F_T \in \mathcal{GZ}(n)$ since,

$$\sigma(F_T)v_S = \begin{cases} v_T, \text{if } S = T \\ 0, \text{ otherwise} \end{cases}$$

for all $S \in \mathcal{T}(\sigma)$. It also follows, that the set $\{F_T : T \in \mathcal{T}(\sigma), \sigma \in \hat{G}_n\}$ is a basis for $\mathcal{A}$ and thus $\mathcal{A} \subseteq \mathcal{GZ}(n)$. Maximality stems from the isomorphism

$$\mathbb{C}[G_n] \cong \bigoplus_{\sigma \in \hat{G}_n} Hom(V_\sigma, V_\sigma) \cong \bigoplus_{\sigma \in \hat{G}_n} M_{d_\sigma \times d_\sigma}(\mathbb{C}).$$

where $d_\sigma = \dim V_\sigma$, together with the fact that $\mathcal{A}$ contains the algebra of diagonal matrices. □

By the last computation and the existence of the elements F_T, we get the following.

Corollary 2.13. *Every element v_T, for $T \in \mathcal{T}(\sigma)$ in the Gelfand-Zetlin basis of V_σ is a common eigenvector for all operators $\sigma(f)$ with $f \in \mathcal{GZ}(n)$. In particular, it is uniquely determined up to a scalar factor by the corresponding eigenvalues.*

We end the section by showing that the Gelfand-Zetlin algebra of a multiplicity-free chain contains the centralizer algebra of G_{n-1} in G_n.

Proposition 2.14. *Let $G_1 \leq G_2 \leq \ldots \leq G_{n-1} \leq G_n \leq \ldots$ be a multiplicity-free chain of finite groups. Then $\mathcal{C}(G_n, G_{n-1}) \subseteq \mathcal{GZ}(n)$.*

Proof. First note that $f \in \mathcal{C}(G_n, G_{n-1})$ if and only if $fh = hf$ for every $h \in G_{n-1}$. Then if $\sigma \in \hat{G}_n$, we have

$$\sigma(h)\sigma(f) = \sigma(hf) = \sigma(fh) = \sigma(f)\sigma(h).$$

which means that

$$\sigma(f) \in \mathrm{Hom}_{G_{n-1}}(\mathrm{Res}^{G_n}_{G_{n-1}} \sigma, \mathrm{Res}^{G_n}_{G_{n-1}} \sigma).$$

Since the chain is multiplicity-free, $\mathrm{Res}^{G_n}_{G_{n-1}} \sigma$ is multiplicity free, meaning that

$$\sigma(f)V_\rho \subseteq V_\rho$$

for every $f \in \mathcal{C}(G_n, G_{n-1})$, $\rho \in \hat{G}_{n-1}$ such that $V_\rho \leq V_\sigma$. Now observe that if $f \in \mathcal{C}(G_n, G_{n-1})$, then $f \in \mathcal{C}(G_n, G_k)$ for every $k = n-2, \ldots, 2, 1$. Therefore, we can iterate the above argument all the way down to when $\rho \in \hat{G}_1$, implying that every v_T in the GZ basis is an eigenvector for $\sigma(f)$. Thus $\mathcal{C}(G_n, G_{n-1}) \subseteq \mathcal{A}$ which is equal to $\mathcal{GZ}(n)$ by the theorem 2.12. $\square$

Remark 2.15. *All the considerations in this chapter naturally generalize for chains of finite dimensional semisimple algebras over any field.*

2.2 Combinatorics of Young diagrams and Multiplicity free chain of S_n

We start by reminding some basic facts about the symmetric group S_n. Remember that every $\sigma \in S_n$ can be written uniquely as a product of disjoint cycles. That is,

$$\sigma = (a_1 \ldots a_{\lambda_1})(b_1 \ldots b_{\lambda_2}) \ldots (c_1 \ldots c_{\lambda_k})$$

where the numbers above form a permutation of $1, 2, \ldots, n$. Since the cycles are disjoint, they commute and thus we can assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. Additionally, if $\pi \in S_n$ is any other permutation, then

$$\pi\sigma\pi^{-1} = (\pi(a_1) \ldots \pi(a_{\lambda_1}))\,(\pi(b_1) \ldots \pi(b_{\lambda_2})) \ldots (\pi(c_1) \ldots \pi(c_{\lambda_k})).$$

That means conjugacy classes in S_n are completely characterized by cycle structures. In other words, the permutations in the orbit of the random $\sigma \in S_n$ are exactly the elements with cycle structure $(\lambda_1, \ldots, \lambda_k)$. Note that the cycle structure by the above considerations is also completely characterized by partitions of n. Since conjugacy classes are in bijection with the irreducible representations of S_n for all $n \in \mathbb{N}$, we get that an indexing set for the set of inequivalent irreducible representations for S_n are partitions of n.

Lemma 2.16 (Indexing set for $\hat{S_n}$ is set of partitions of n.)**.** *The conjugacy classes of S_n can be parametrized by partitions of n. In particular, if $\lambda \vdash n$, the conjugacy class associated to λ consists of all permutations $\sigma \in S_n$ whose cycle decomposition is of the form*

$$\sigma = (a_1 \ldots a_{\lambda_1})(b_1 \ldots b_{\lambda_2}) \ldots (c_1 \ldots c_{\lambda_k}).$$

Consequently, an indexing set for $\hat{S_n}$ is the set of partitions of n.

For any $n \in \mathbb{N}$ we embed S_{n-1} into S_n as the subgroup stabilizing n, i.e $S_{n-1} = \{\sigma \in S_n : \sigma(n) = n\}$.

Theorem 2.17. *For any $n \in \mathbb{N}$, it holds that S_{n-1} is a multiplicity free subgroup of S_n. Consequently, the chain*

$$S_1 \leq S_2 \leq \ldots \leq S_{n-1} \leq S_n$$

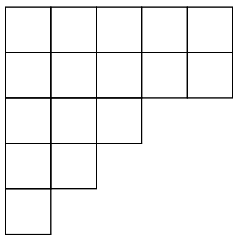
is multiplicity free.

Proof. By proposition 2.5, it is enough to show that every element $\sigma \in S_n$ is S_{n-1}-conjugate to its inverse σ^{-1}. Decomposing σ into its disjoint cycle decomposition, we have that cycles belonging in S_{n-1} are conjugate to their inverse as the inverse of a k-cycle is a k-cycle yielding that they belong in the same conjugacy class. Thus we only have to consider cycles involving n. Also since cycles are cyclic arrangements of numbers, we can assume $\sigma = (\ n\ a_k\ a_{k-1}\ \ldots\ a_1\ )$ is a k cycle with $k \leq n$ and $a_i < n$ for all i. Then $\sigma^{-1} = (\ n\ a_1\ a_2\ \ldots\ a_k\ ) = \pi\sigma\pi^{-1}$ where $\pi \in S_{n-1}$ is the permutation bringing the a_i's back in place. □

For example, the cycle $(1\ 2\ 3\ 4) = (4\ 1\ 2\ 3)$ is S_3 conjugate to its inverse $(4\ 3\ 2\ 1) = (1\ 3)(4\ 1\ 2\ 3)(1\ 3)$ with $(1\ 3) \in S_3$.

Definition 2.18 (Young diagram)**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. The Young diagram of shape λ associated to the partition is the array formed by n boxes with k left-justified rows in which every row i contains exactly λ_i boxes. In particular, every such Young diagram contains λ_1 columns.*

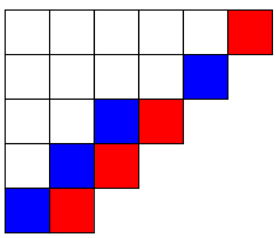
Example 2.19. *Consider the partition $(5, 5, 3, 2, 1) \vdash 16$. The Young diagram associated with it is given below*



The rows and the columns are enumerated from top left to bottom right, exactly like the rows and columns of a matrix. We can thus give coordinates to every box according to the row and column they are in.

Definition 2.20 (Removable-Addable boxes)**.** *In the Young diagram associated to $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we say that the box in position (i, j) is removable if in the positions $(i+1, j)$ and $(i, j+1)$ there is no box. Equivalently, either $i < k$ and $j = \lambda_i > \lambda_{i+1}$, or $i = k$ and $j = \lambda_k$. This means that removing this box we are again left with a Young diagram associated with a partition $\lambda' \vdash n-1$. Similarly, the position (i, j) is addable, if $\lambda_i = j - 1 < \lambda_{i-1}$ or $i = k+1$ and $j = 1$. This means that if we add a box to an addable position (i, j) we get a Young diagram associated to a partition $\lambda' \vdash n+1$.*

Pictorially, addable and removable boxes in a Young diagram are very easy to identify. Namely, removable boxes are the inner low-right corner boxes, while addable boxes are the outer high- right boxes. In example 2.19,



the boxes colored in blue are the removable boxes, while the ones colored in red are the addable ones. As usual, a standard tableaux of shape $\lambda \vdash n$ is a bijective filling of the Young diagram with the numbers $1, 2, \ldots, n$ such that the numbers are strictly increasing both along the rows and along the columns. For any partition $\lambda \vdash n$, we denote by $Tab(\lambda)$ the set of all standard tableaux of shape λ. Finally we set

$$Tab(n) = \bigsqcup_{\lambda \vdash n} Tab(\lambda).$$

Example 2.21. *The following is a standard tableaux of shape* $(5, 5, 3, 2, 1) \vdash 16$*:*

1	2	10	13	15
3	4	11	14	16
5	6	12		
7	8			
9				

Remark 2.22. *Note that in a standard tableau, number* 1 *is always placed in position* $(1, 1)$*, and the box containing the value* n *is always a removable box.*

Let T be a Young tableau of shape λ and $\sigma \in S_n$. We denote by σT the tableau obtained from T by replacing i with $\sigma(i)$ for all $i = 1, \ldots, n$.

Definition 2.23 (Admissible transpositions)**.** *Let* T *be a standard Young tableau. We say that a simple transposition* $s_i = (i\ i+1)$ *is admissible for* T *if* $s_i T$ *is again a standard tableau.*

Notice that since only the entries $i, i+1$ are affected by the action of s_i, a simple transposition is admissible for T if and only if $i, i+1$ belong neither to the same row nor to the same column of T. In example 2.21, $s_2, s_4, s_6, s_8, s_{14}$ are admissible transpositions, while s_7 is not since it results in a non standard tableaux.

For $\lambda \vdash n$ we denote by T^λ the canonical standard tableau of shape λ shown below

1	2	3	$\ldots$	$\lambda_1{-}1$	λ_1
$\lambda_1{+}1$	$\lambda_1{+}2$	$\ldots$	$\lambda_1{+}\lambda_2$		
$\vdots$	$\vdots$	$\vdots$			
$\bar{\lambda}_{k-1}$	$\ldots$	n			

where $\bar{\lambda}_{k-1} = \lambda_1 + \lambda_2 + \ldots + \lambda_{k-1} + 1$. If $T \in Tab(\lambda)$, we denote by $\sigma_T \in S_n$ the unique permutation such that $\sigma_T T = T^\lambda$.

Example 2.24. *For example, the canonical standard tableau* $T^{(3,2,2)}$ *is illustrated below:*

1	2	3
4	5	
6	7	

Theorem 2.25. *Let $T \in Tab(\lambda)$ and set $l = l(\sigma_T)$. Then there exists a sequence of admissible simple transpositions $s_{i_1}s_{i_2}\dots s_{i_l}$ such that*

$$s_{i_1}s_{i_2}\dots s_{i_l}T = T^\lambda.$$

Proof. Let j denote the entry in the right bottom box of T. If $j = n$ then the box is removable, so we can consider the standard tableau T' of shape $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_k - 1) \vdash n - 1$ obtained by removing that box. Then we can apply induction to T' to find a sequence of $l' = l(\sigma_{T'})$ admissible transpositions transforming T' into $T^{\lambda'}$. Then it is clear that since the box removed had the value n, the same sequence transforms T into T^λ and that $l = l'$. Suppose now that $j \neq n$. Then since j is sitting in the right bottom corner of a standard tableau, it is clear that s_j is an admissible transposition for T. Similarly, s_{j+1} is an admissible transposition for s_jT. Iterating this process, s_{n-1} is an admissible transposition for $s_{n-2}s_{n-3}\dots s_{j+1}s_jT$. But now the standard tableau $s_{n-1}s_{n-2}s_{n-3}\dots s_{j+1}s_jT$ contains the value n in the bottom right corner box and thus the situation is reduced to the previous case. $\square$

Example 2.26. *We illustrate how the algorithm of theorem 2.25 works for the standard tableau*

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array}.$$

Since 7 *is in the right bottom removable box, we can remove it and continue the process with*

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Since 5 *is in the right bottom box, $s_5 \in S_7$ is admissible and such that* 6 *goes to the right bottom removable box. Thus we continue with*

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

but 5 *is in the right box, so we continue to*

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Here the transposition s_2 is admissible and right after s_3 is admissible, finally bringing everything in order. Thus $s_3s_2s_5 \in S_7$ is the permutation transforming the original tableau to the canonical standard tableau.

Corollary 2.27. *Let $S, T \in Tab(\lambda)$. Then S can be obtained from T by applying a sequence of admissible simple transpositions.*

Proof. If σ, π are the permutations transforming S, T respectively to T^λ, then $\pi^{-1}\sigma$ is the permutation transforming T to S. $\square$

Remark 2.28. *theorem 2.25 provides us with a standard procedure to decompose σ_T as a product of $l(\sigma_T)$ admissible simple transpositions.*

2.2.1 Content of a tableau

Let T be a Young tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. We denote by $i : [n] \longrightarrow [k]$ and $j : [n] \longrightarrow [\lambda_1]$ the functions defined by setting $i(t)$ and $j(t)$ to be the row, respectively column of T containing the value t. Set $c(t) = j(t) - i(t)$ and call it the content of the box with value t. For example, in the canonical standard tableau $T^{(5,5,3,2,1)}$ it holds that $i(10) = 2, j(10) = 5$.

Definition 2.29 (Content of a tableau)**.** *Let $T \in Tab(\lambda)$. The content of T is the vector in $\mathbb{Z}^n$ given by*

$$C(T) = (c(1), c(2), \ldots, c(n))$$

For example, the content vector for the canonical standard tableau $T^{(5,5,3,2,1)}$ is the vector $v \in \mathbb{Z}^{16}$, with coordinates

$$v = (0, 1, 2, 3, 4, -1, 0, 1, 2, 3, -2, -1, 0, -3, -2, -4).$$

Note that for any two standard tableau of the same shape, the values appearing in the content vector are going be the same modulo a permutation in the position of the coordinates. Below is an illustration of the contents of the canonical standard tableau $T^{(5,5,3,2,1)}$.

0	1	2	3	4
−1	0	1	2	3
−2	−1	0		
−3	−2			
−4				

The choice of a particular standard tableau T determines the order in which the contents are going to appear in the content vector. Note also that the Young diagram of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ can be divided into diagonals, enumerated by

$$-k + 1, -k + 2, \ldots, 0, 1, \ldots, \lambda_1 - 1.$$

The t diagonal consists exactly of the boxes for which $c(i, j) = t$. Moreover, the lengths of the diagonals completely determine the shape of the tableau.

Example 2.30. *For the standard tableau from example 2.26, the content vector is*

$$C(T) = (0, -1, 1, 2, -2, 0, -1).$$

The way to recover the shape of T from the content vector is the following: Since the biggest entry in $C(T)$ is 2, we know that the first row has length 3. Similarly from the lowest entry, namely -2 we recover that the diagram has 3 columns. Then counting the number of times each content number appears, we fill in that many boxes in the corresponding diagonal to recover that the shape is $(3, 2, 2)$.

In order for a vector to be a content vector of some standard tableau, it has to satisfy certain properties. We will prove that these properties are the ones stated in the following definition.

Definition 2.31. *Let $Cont(n)$ be the set of all vectors $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that*

(1) $a_1 = 0$.

(2) $\{a_q + 1, a_q - 1\} \cap \{a_1, a_2, \dots, a_{q-1}\} \neq \emptyset$ *for all* $q \geq 1$.

(3) if $a_p = a_q$ *for some* $p < q$ *then* $\{a_q - 1, a_q + 1\} \subseteq \{a_{p+1}, \dots, a_{q-1}\}$

Inductively from the definition, it is clear that $Cont(n) \subseteq \mathbb{Z}^n$. Given two vectors $u, v \in Cont(n)$, we write $u \approx v$ if v can be obtained from u by permuting its entries. In other words, if there exists $\sigma \in S_n$ such that $\sigma u = v$. The relation $\approx$ is an equivalence relation in $Cont(n)$. However, it does not necessarily hold that given $u \in Cont(n)$ and $\sigma \in S_n$ that $\sigma u \in Cont(n)$. For instance, $u = (0, 1) \in Cont(2)$ and $s = (12) \in S_2$ but $su = (1, 0) \notin Cont(2)$ as it does not satisfy condition (1).

Proposition 2.32. *Suppose $u = (u_1, u_2, \dots, u_n) \in Cont(n)$. Then for $p, q \in [n]$ the following hold:*

(a) If $u_q > 0$, *then* $u_q - 1 \in \{u_1, u_2, \dots, u_{q-1}\}$. *Analogously, if* $u_q < 0$ *then* $u_q + 1 \in \{u_1, u_2, \dots, u_{q-1}\}$.

(b) If $p < q$, $a_p = a_q$ *and* $a_r \neq a_q$ *for all* $r = p+1, p+2, \dots, q-1$ *then there exist unique* $s_-, s_+ \in \{p+1, p+2, \dots, q-1\}$ *such that* $u_{s_-} = u_q - 1$ *and* $u_{s_+} = u_q + 1$

Proof. (a) Suppose that $u_q > 0$. Then using (2) in the definition of $Cont(n)$ we can construct a sequence $u_{s_0} = u_q, u_{s_1}, \dots, u_{s_k} = 0$ with $s_0 = q > s_1 > s_2 > \dots > s_k \geq 1$ such that $u_{s_h} > 0$ and $|u_{s_h} - u_{s_{h+1}}| = 1$ for all $h = 0, 1, \dots, k-1$. Then as h varies, u_{s_h} attains all values in $[u_q]$ and thus it attains the value $u_q - 1$. For negative u_q the argument is completely analogous.

(b) The existence of such s_-, s_+ is guaranteed by (3) in the definition of $Cont(n)$. Their uniqueness stems from the fact that if there is another s'_- such that $u_{s'_-} = u_q - 1$, and without loss of generality $s_- < s'_-$, then again by (3) there exists an s between these two numbers such that $u_s = u_q - 1 + 1 = u_q$ contradicting the assumptions.

□

Theorem 2.33. *The map*

$$Tab(n) \to Cont(n)$$

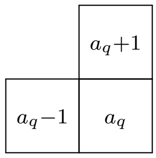
mapping

$$T \mapsto C(T)$$

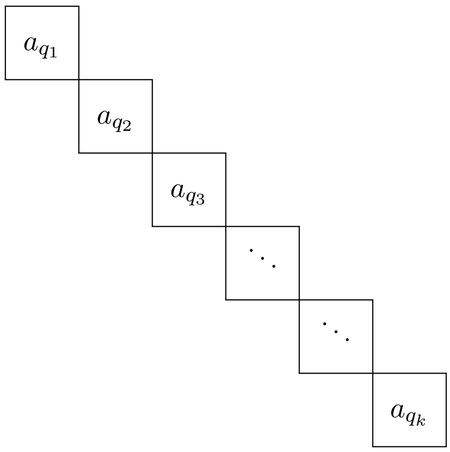
is a bijection. Moreover, if $a, b \in Cont(n)$ such that $a = C(T)$ and $b = C(S)$ then $a \approx b$ if and only if T, S are of the same shape.

Proof. First of all we have to show that the map is well defined. That is, we have to show that for a standard tableaux T of shape λ, its content $C(T)$ satisfies the three conditions imposed on the set $Cont(n)$. Suppose the content of T is $C(T) = (a_1, a_2, \dots, a_n)$. Clearly by construction since 1 is always on box $(1, 1)$ we have $a_1 = 0$. So (1) is satisfied. Now if $q \in \{2, 3, \dots, n\}$ is placed in position (i, j) so that $a_q = j - i$ then we have $i > 1$ or $j > 1$. In the first case, i.e $i > 1$, consider the number p in the box right above from (i, j), that is $(i-1, j)$. Then since T is standard, $p < q$ and $a_p = j - i + 1 = a_q + 1$. Similarly, if $j > 1$ we consider the value p' in the box on the left of (i, j) namely $(i, j-1)$. Then, since T is standard, $p' < q$ and $a_{p'} = j - i - 1 = a_q - 1$. Thus condition (2) is also satisfied. For condition

(3) suppose that $a_p = a_q$ with $p < q$. This means that p, q are placed in the same diagonal. Thus, if (i, j) is the box containing q, then $i, j > 1$ and denoting by q_-, q_+ the values (in $\{p+1, p+2, \dots, q-1\}$ since T standard) in the boxes with coordinates $(i-1, j)$ and $(i, j-1)$, by the same argument as above we have $a_{q_+} = a_q - 1$ and $a_{q_-} = a_q + 1$. See the picture below



Thus (3) is also satisfied, proving that $C(T) \in Cont(n)$.
We now prove that the map is injective. Suppose $C(T) = (a_1, a_2, \dots, a_n)$. Then the h-diagonal of T is filled in with the numbers $q \in [n]$ such that $a_q = h$ from up-left to bottom right, as shown below.



where $q_1 < q_2 < q_3 < \dots < q_k$ are such that $a_{q_j} = h$ for all $j = 1, 2, \dots, k$ and $a_q \neq h$ for $q \notin \{q_1, q_2, \dots, q_k\}$. Thus the diagonals completely determine the whole standard tableau, meaning that if T_1, T_2 are standard tableau such that $C(T_1) = C(T_2)$ then they have the same diagonals and therefore they must coincide.
We are left to show that the map $T \longmapsto C(T)$ is surjective. For that we do induction on n. For $n = 1, 2$ this is trivial. Suppose that the map $Tab(n-1) \longrightarrow Cont(n-1)$ is surjective and let $a = (a_1, a_2, \dots, a_n) \in Cont(n)$. Then by truncating the last entry, we get that $a' = (a_1, a_2, \dots, a_{n-1}) \in Cont(n-1)$. By the induction hypothesis, there exists $T' \in Tab(n-1)$ such that $C(T') = a'$. We show that by adding a box on the lower-rightmost box in the a_n diagonal and placing n in this box, we get a standard tableau $T \in Tab(n)$ such that $C(T) = a$. We distinguish two cases:
Case 1: $a_n \notin \{a_1, \dots, a_{n-1}\}$. In this case, we know that either $a_n - 1$ or $a_n + 1$ is one of the numbers $\{a_1, \dots, a_{n-1}\}$. If $a_n - 1$ is among them, we can add a box to the first row, while in the other case we add a box to the first column. (Pictorially, in the first case a_n is on the rightmost box of the first row and thus the position next to it is addable, while in the second case, a_n is sitting alone on the end of the first column and thus the position below it is addable.)
Case 2: $a_n \in \{a_1, \dots, a_{n-1}\}$. In this case, denote by p the largest index among these numbers such that $a_p = a_n$. If the coordinates of the box containing p are (i, j), then the box with coordinates $(i+1, j+1)$ is addable. That is because from proposition 2.32, we have the existence of unique $r, s \in \{p+1, p+2, \dots, n\}$ such that $a_r = a_n + 1$ and $a_s = a_n - 1$, as is shown in the figure below

p	r
s	n

We thus place the value n in the box $(i+1, j+1)$, yielding us a standard tableau T such that $C(T) = a$. Finally, note that if $a = C(T)$ and $b = C(S)$ then b may be obtained from a by permutation of its entries, if and only if T, S are of the same shape. That is because the shape of a standard tableau is completely determined by the length of the diagonals. $\square$

Given $a \in Cont(n)$ we say that a simple transposition $s_i = (i\ i+1)$ is admissible for a, if it is admissible for the unique standard tableau T such that $C(T) = a$. This is equivalent to $a_{i+1} \neq a_i \pm 1$. We thus get the following two corollaries.

Corollary 2.34. *Given $a, b \in Cont(n)$ we have that $a \approx b$ if and only if there exists a sequence of admissible transpositions which transforms a to b.*

Proof. By the bijection of theorem 2.33, we can assume $a = C(T)$ and $b = C(T')$. Then $a \approx b$ if and only if T, T' are of the same shape, which is equivalent by corollary 2.27 to being able to obtain T' from T by applying a sequence of admissible simple transpositions. $\square$

Corollary 2.35. *The cardinality of the quotient $Cont(n)/\approx$ is equal to $p(n) = |\{\lambda : \lambda \vdash n\}|$.*

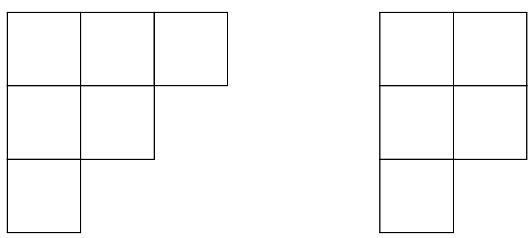
Proof. By theorem 2.33, an equivalence class of $Cont(n)/\approx$ consists of all standard tableau corresponding to a partition $\lambda \vdash n$. Altogether, we have as many equivalence classes as partitions of n. $\square$

2.2.2 The Young graph

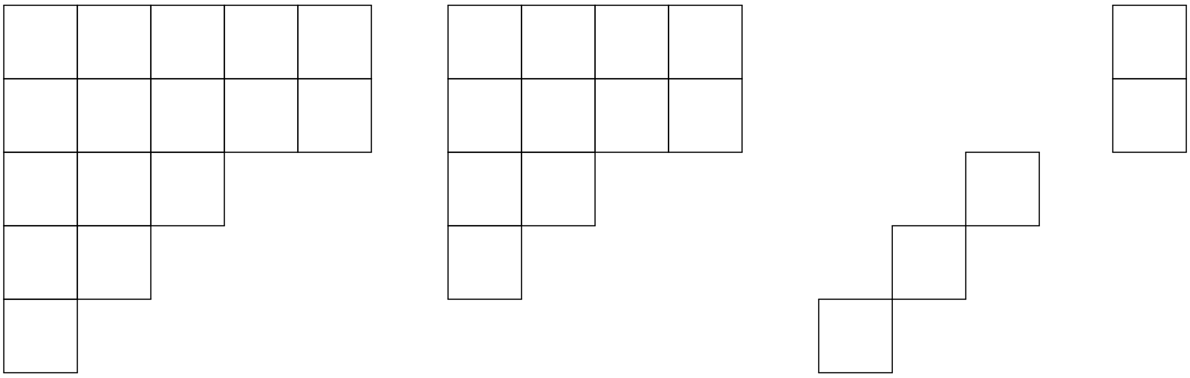
Denote by $\mathbb{Y} = \{\lambda : \lambda \vdash n, n \in \mathbb{N}\}$ the set of all partitions. Equivalently, we can also regard $\mathbb{Y}$ as the set of all Young diagrams. We endow $\mathbb{Y}$ with the structure of a poset, by setting for $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash m$,

$$\mu \preceq \lambda$$

if $m \geq n$, $l \geq k$ and $\lambda_j \geq \mu_j$ for all $j = 1, 2, \ldots, k$. From the perspective of Young diagrams, $\mu \preceq \lambda$ if and only if the Young diagram of μ is contained in the Young diagram of λ, i.e if μ contains a box in the position (i, j) then so does λ. For instance, consider $\lambda = (3, 2, 1)$ and $\mu = (2, 2, 1)$. Then it is clear that $\mu \preceq \lambda$ as depicted below.



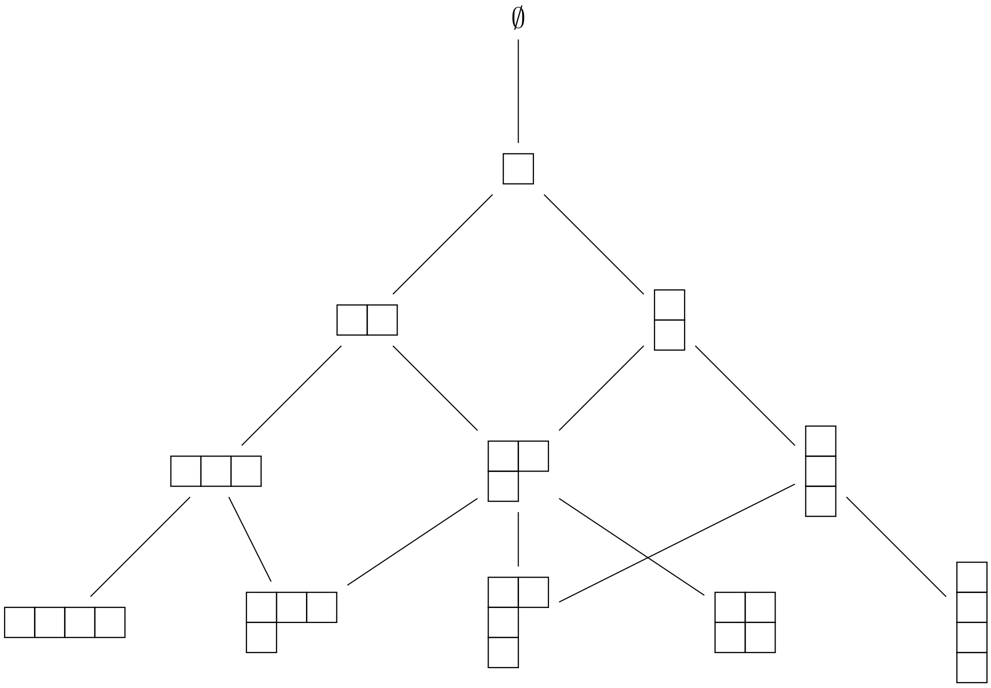
Now if $\mu \preceq \lambda$, we denote by λ/μ the array obtained by removing from the Young diagram of λ the boxes of the Young diagram of μ. Notice that λ/μ is a skew-diagram, as shown in the example below. Consider the partitions $\lambda = (5, 5, 3, 2, 1)$ and $\mu = (4, 4, 2, 1)$. Then



For $\mu, \lambda \in \mathbb{Y}$, we say that λ covers μ if $\mu \preceq \lambda$ and

$$\mu \preceq \nu \preceq \lambda,\ \nu \in \mathbb{Y} \implies \nu = \mu \,\text{or}\, \nu = \lambda.$$

In the language of diagrams, λ covers μ if and only if λ/μ consists of a single box. We write $\lambda \to \mu$ to denote that λ covers μ. We define the Young graph to be the oriented graded graph with levels indexed by nonnegative integer, whose vertex set on level n is $\mathbb{Y}_{\ltimes}$ (partitions of n) and we draw an arrow between partitions λ and μ in consecutive levels, if and only if $\lambda \to \mu$. Below is an illustration of the first 4 levels of the Young graph. We read the diagram from bottom to top, thus abbreviating the arrows.



A path in the Young graph is a sequence $p = (\lambda^n \to \lambda^{n-1} \to \ldots \to \lambda^1)$ of partitions $\lambda^k \vdash k$ such that λ^k covers λ^{k-1} for all $k = 1, 2, \ldots, n$. Notice that a path always ends at the trivial partition $(1) \vdash 1$. The number $l(p) = n$ is called the length of the path p. We denote the set of all paths of length n in the Young graph by $\mathrm{Path}_n(\mathbb{Y})$ and we set

$$\mathrm{Path}(\mathbb{Y}) = \bigcup_{n=1}^{\infty} \mathrm{Path}_n(\mathbb{Y})$$

to be the set of all paths in the Young graph.
With a partition $\lambda \vdash n$ and a path $(\lambda = \lambda^{(n)} \to \lambda^{(n-1)} \to \ldots \to \lambda^{(1)})$ we associate the standard tableau T of shape λ obtained by placing the value $k \in [n]$ in the box $\lambda^{(k)}/\lambda^{(k-1)}$.
As an example, the standard tableau T of shape $\lambda = (4, 3, 2, 1)$ pictured below

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 \\ \cline{1-3} 8 & 10 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

is associated with the path $\lambda \to (4,3,1,1) \to (4,3,1) \to (4,3) \to (3,3) \to (3,2) \to (2,2) \to (2,1) \to (1,1) \to (1)$. This way, we obtained a natural bijection

$$\text{Path}_n(\mathbb{Y}) \longleftrightarrow Tab(n)$$

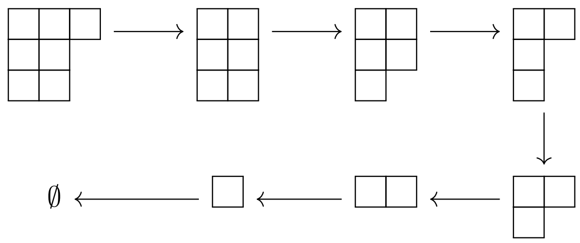
which extends to a bijection

$$\text{Path}(\mathbb{Y}) \longleftrightarrow \bigcup_{n=1}^{\infty} Tab(n).$$

By the bijection in theorem 2.33, we also get a bijection

$$\text{Path}_n(\mathbb{Y}) \longleftrightarrow Cont(n).$$

Example 2.36. *Suppose we have the content vector* $c = (0,1,-1,-2,0,-1,2) \in Cont(7)$. *This corresponds to a standard tableau of shape* $(3,2,2)$. *Moreover, the path it corresponds to is*



where in every step we remove the unique removable box with content the last coordinate of our content vector. Now the standard tableau corresponding to this sequence is obtained by filling in the numbers $7, 6, \ldots, 1$ *in this order, to the box removed in each step, thus*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array}$$

Corollary 2.37. *Let* $a, b \in Cont(n)$. *Suppose they correspond to the paths* $\lambda^{(n)} \to \lambda^{(n-1)} \to \ldots \to \lambda^{(1)}$ *and* $\mu^{(n)} \to \mu^{(n-1)} \to \ldots \to \mu^{(1)}$ *respectively. Then* $a \approx b$ *if and only if* $\lambda^{(n)} = \mu^{(n)}$.

2.3 The Young Jucys Murphy elements and a Gelfand-Zetlin basis for S_n.

We study the GZ algebra associated to the multiplicity free chain of symmetric groups. The main tool to study this algebra are the so called Young-Jucys-Murphy elements defined below.

Definition 2.38 (YJM-elements)**.** *The Young-Jucys-Murphy (YJM) elements of* $\mathbb{C}[S_n]$ *are defined by* $\xi_1 = 0$, *and for* $k = 2, 3, \ldots, n$ *we let*

$$\xi_k = (1,k) + (2,k) + \ldots + (k-1,k) \in \mathbb{C}[S_k].$$

Some observations on these elements to start highlighting their importance. First notice that ξ_k comprises exactly of the transpositions in S_k that are not transpositions in S_{k-1}. With this observation, if we denote the sum of transpositions in S_k by T_k, we can write $\xi_k = T_k - T_{k-1}$. Since T_k is central in $\mathbb{C}[S_k]$ as the sum of elements in a conjugacy class, we get $\xi_k \in \mathcal{Z}(k) - \mathcal{Z}(k-1)$ yielding $\xi_k \in \mathcal{C}(S_k, S_{k-1})$. As a consequence, these elements commute $\xi_i \xi_j = \xi_j \xi_i$ for all $i, j = 1, 2, \ldots, n$, but note that they are not central themselves. There is also an invariant way to define these elements which will be more useful for future endeavors.

$$s_k \xi_k s_k = \sum_{i=1}^{k-1} (i\ k+1) = \xi_{k+1} - s_k$$

and thus we can redefine these elements recursively by setting $\xi_1 = 0$ and

$$\xi_{k+1} = s_k \xi_k s_k + s_k, \text{ for all } k = 1, 2, \ldots, n-1.$$

For $l, k \geq 1$, we denote by S_{l+k}, S_l, the symmetric groups on the sets $[l+k], [l]$ and set

$$\mathcal{Z}(l,k) = \mathcal{C}(S_{l+k}, S_l).$$

That is, $\mathcal{Z}(l,k)$ is the algebra of all S_l-conjugacy invariant elements in $\mathbb{C}[S_{l+k}]$.

Theorem 2.39 (Olshanskii)**.**

$$\mathcal{Z}(l,k) = \langle \xi_{l+1}, \xi_{l+2}, \ldots, \xi_{l+k}, S_k, \mathcal{Z}(l) \rangle$$

Proof. For a complete proof, see [2], Theorem 3.2.6. The proof heavily relies on calculus of marked permutations, a characterization of the S_l-conjugacy classes in S_{l+k}. □

It is now imminent that the GZ algebra is generated by the YJM elements.

Corollary 2.40 (GZ algebra generated by YJM-elements)**.** *The GZ algebra $\mathcal{GZ}(n)$ of the multiplicity-free chain $S_1 \leq S_2 \leq \ldots \leq S_{n-1} \leq S_n \leq \ldots$ is generated by the YJM-elements $\xi_1, \xi_2, \ldots, \xi_n$. In formulae*

$$\mathcal{GZ}(n) = \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$$

Proof. As we have already noticed, $\xi_k = T_k - T_{k-1}$ and $T_i \in \mathcal{Z}(i)$, thus $\xi_k \in \mathcal{GZ}(n)$ for all $k = 1, 2, \ldots, n$. Moreover,

$$\mathcal{Z}(n) = \mathcal{C}(S_n, S_n) \subseteq \mathcal{C}(S_n, S_{n-1}) = \mathcal{Z}(n-1, 1) = \langle \mathcal{Z}(n-1), \xi_n \rangle$$

where the last equality follows from theorem 2.39. Suppose now by induction that

$$\mathcal{GZ}(n-1) = \langle \xi_1, \xi_2, \ldots, \xi_{n-1} \rangle.$$

Then

$$\mathcal{GZ}(n) = \langle \mathcal{GZ}(n-1), \mathcal{Z}(n) >= \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$$

completing the proof. □

Remark 2.41. *We now have a different way of proving that the centralizer $\mathcal{C}(S_n, S_{n-1})$ is commutative. Indeed, $\mathcal{Z}(n-1,1) = \langle \mathcal{Z}(n-1), \xi_n \rangle$ and $\xi_n = T_n - T_{n-1}$ commutes with $\mathcal{Z}(n-1)$ as it commutes with $\mathbb{C}[S_{n-1}]$.*

Now that we've established an explicit set of generators for the Gelfand-Zetlin algebra, we can start characterizing the irreducible representations of the symmetric groups by understanding what the corresponding eigenvalues of the YJM elements are with respect to a GZ basis.

2.4 Spectrum of YJM elements and Branching graph of S_n

2.4.1 The weight of a Young basis vector

Let $(V_\sigma, \sigma) \in \hat{S}_n$ and consider the Gelfand-Zetlin basis $\{v_T : T \in \mathcal{T}(\sigma)\}$ associated with the multiplicity-free chain $S_1 \leq S_2 \leq \ldots \leq S_{n-1} \leq S_n \leq \ldots$. In this setting, the basis is also called a Young basis for V_σ, which Young used to prove his seminormal and orthogonal forms, but could not prove them a global basis since this required the notion of the GZ algebra. Since the Gelfand-Zetlin algebra is generated by the YJM-elements $\xi_1, \xi_2, \ldots, \xi_n$, we have that every v_T is an eigenvector for $\sigma(\xi_i)$ for all $i = 1, 2, \ldots, n$. Thus for every v_T, we define

$$a(T) = (a_1, a_2, \ldots, a_n)$$

where a_i is the eigenvalue of $\sigma(\xi_i)$ corresponding to the eigenvector v_T, i.e $\xi_i.v_T = a_i v_T$ for all $i = 1, 2, \ldots, n$. Moreover, the elements v_T are completely determined up to a scalar by the vector $a(T)$ by corollary 2.13. We call the vector $a(T)$ the weight of v_T. We now study the action of the Coxeter generators on the Young basis.

Proposition 2.42. *For every $\sigma \in \hat{S}_n$ and $T = (\sigma = \sigma_n \to \sigma_{n-1} \to \ldots \to \sigma_2 \to \sigma_1) \in \mathcal{T}(\sigma)$, the vector $s_k.v_T$ is a linear combination of the vectors $v_{T'}$ with $T' = (\sigma = \sigma'_n \to \sigma'_{n-1} \to \ldots \to \sigma'_2 \to \sigma'_1) \in \mathcal{T}(\sigma)$ such that $\sigma'_i = \sigma_i$ for all $i \neq k$.*

Proof. Let V_j be the representation space of σ_j for $j = 1, 2, \ldots, n$. Note that

$$V_j = \mathbb{C}[S_j]v_T.$$

Indeed, the right hand side is a nontrivial S_j sub-representation and since σ_j is irreducible, it is the whole space.
Now notice that for $j > k$ we have $s_k \in S_j$, thus $\sigma_j(s_k)v_T \in V_j$, for all $j = k+1, k+2, \ldots, n$. This implies that $\sigma_j = \sigma'_j$ for all $j = k+1, k+2, \ldots, n$. On the other hand, if $j < k$ then by setting $W_j = \{\sigma_j(f)\sigma(s_k)v_T : f \in \mathbb{C}[S_j]\} = \sigma(s_k)V_j$, we see that the map

$$\begin{aligned} V_j &\longrightarrow W_j \\ \sigma_j(f)v_T &\longmapsto \sigma_j(f)\sigma(s_k)v_T \end{aligned}$$

is morphism of S_j-representations since s_k commutes with S_j. Moreover, it is an isomorphism, with inverse given by

$$\begin{aligned} W_j &\longrightarrow V_j \\ \sigma(s_k)\sigma_j(f)v_T &\longmapsto \sigma_j(f)v_T. \end{aligned}$$

Thus the vector $\sigma(s_k)v_T$ belongs to the σ_j-isotypic component of $\mathrm{Res}^{S_n}_{S_j} \sigma$ and therefore $\sigma_j = \sigma'_j$ for all $j = 1, 2, \ldots, k-1$. $\square$

proposition 2.42 informs us that the action of a simple transposition s_k on the GZ basis affects only the k-th level of the branching graph and depends only on the levels $k-1, k, k+1$.

2.4.2 The Spectrum of the YJM elements

Set

$$Spec(n) = \{a(T) : T \in \mathcal{T}(\sigma), \sigma \in \hat{S}_n\}$$

where $a(T)$ is the weight of v_T. That is, $Spec(n)$ is the spectrum of the YJM elements of S_n. Since this spectrum determines the elements of a Young basis, we have

$$|Spec(n)| = \sum_{\sigma \in \hat{S}_n} \dim V_\sigma$$

That is, $Spec(n)$ is in natural bijection with the set of all paths in the branching graph of $S_1 \leq S_2 \leq \ldots \leq S_n$. Denote this bijection by $a \mapsto T_a$. Also denote by v_a the Young basis vector corresponding to T_a. Notice that for any $a \in Spec(n)$, we have that $v_a \in \bigsqcup_{\lambda \vdash n} S^\lambda$, where S^λ is the irreducible S_n representation corresponding to λ.

Definition 2.43. *For $a, b \in Spec(n)$, we say $a \sim b$ if v_a and v_b belong to the same irreducible S_n-representation. In terms of the branching graph, that means that the corresponding paths have the same starting point.*

It is clear that the relation defined above is an equivalence relation. Also the space of equivalence classes essentially identify the irreducible representations of S_n by picking one representative for each class. In other words,

$$|Spec(n)/\sim| = |\hat{S}_n|$$

We now try to explore the properties of $Spec(n)$ as well as the equivalence relation $\sim$. When $j \neq i, i+1$, the YJM elements commute with simple transpositions, i.e,

$$s_i \xi_j = \xi_j s_i. \tag{3}$$

Moreover,

$$s_i \xi_i s_i + s_i = \xi_{i+1} \tag{4}$$

for all $i = 1, \ldots, n-1$. We are ready to give some basic properties of $Spec(n)$.

Proposition 2.44 (Properties of $Spec(n)$)**.** *Let $a = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n) \in Spec(n)$. Then*

(a) $a_i \neq a_{i+1}$ for $i = 1, 2, \ldots, n-1$.

(b) $a_{i+1} = a_i \pm 1$ if and only if $s_i u_a = \pm u_a$

(c) if $a_{i+1} \neq a_i \pm 1$ then

$$a' = s_i a = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, \ldots, a_n) \in Spec(n),$$

with $a \sim a'$, and the Young basis vector associated to a' up to a scalar factor, is

$$v_{a'} = s_i v_a - \frac{1}{a_{i+1} - a_i} v_a.$$

Moreover, the space $\langle v_a, v_{a'} \rangle$ is invariant for the actions of ξ_i, ξ_{i+1} and s_i, and in the basis $\{v_a, v_{a'}\}$ these operators are represented by the matrices

$$\begin{bmatrix} a_i & 0 \\ 0 & a_{i+1} \end{bmatrix}, \quad \begin{bmatrix} a_{i+1} & 0 \\ 0 & a_i \end{bmatrix} \quad and \quad \begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{bmatrix}$$

respectively.

Proof. From the definition of v_a, we have that $\xi_i.v_a = a_i v_a$ and $\xi_{i+1}.v_a = a_{i+1}v_a$. Moreover, using eq. (3) and eq. (4), we get that $\{v_a, s_i v_a\}$ is invariant under the action of ξ_i, ξ_{i+1} and s_i. Assuming that the two vectors are linearly dependent, i.e $s_i v_a = \lambda v_a$ then since $s_i^2 = 1$ we have $\lambda^2 = 1$ i.e $\lambda = \pm 1$. Using eq. (4) yields

$$a_i s_i v_a + v_a = a_{i+1} s_i v_a.$$

Thus, $s_i v_a = \pm v_a$ if and only if $a_{i+1} = a_i \pm 1$. This proves (b).
Suppose now that $a_{i+1} \neq a_i \pm 1$. Then by (b), v_a and $s_i v_a$ are linearly independent, or equivalently

$$\dim\langle v_a, s_i v_a\rangle = 2$$

and the restrictions of s_i, ξ_i, ξ_{i+1} to $\langle v_a, s_i v_a\rangle$ are represented with respect to the basis $\langle v_a, s_i v_a\rangle$ by the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} a_i & -1 \\ 0 & a_{i+1} \end{bmatrix}, \quad \begin{bmatrix} a_{i+1} & 1 \\ 0 & a_i \end{bmatrix}$$

respectively. We know that the restriction of the operators ξ_i are diagonalizable. But a matrix of the form

$$\begin{bmatrix} a & \pm 1 \\ 0 & b \end{bmatrix}$$

is diagonalizable, if and only if $a \neq b$ and in that case, the eigenvalues are a, b with corresponding eigenvectors $(1, 0)$ and $(\pm\frac{1}{b-a}, 1)$. Thus it holds $a_i \neq a_{i+1}$, yielding (a). Applying this to our context, we get that $v' = s_i v_a - \frac{1}{a_{i+1}-a_i} v_a$ is an eigenvector of ξ_i, ξ_{i+1} with corresponding eigenvalues a_{i+1}, a_i respectively. Moreover, for any $j \neq i, i+1$ by eq. (3), we get that $\xi_j v' = a_j v'$ for all other j. Therefore, we conclude that $a' = s_i a = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, \ldots, a_n) \in Spec(n)$ and $v' = v_{a'}$ is a vector in the Young basis (thus $a \sim a'$). Computing the matrix representing s_i in the basis $\{v_a, v_{a'}\}$ is trivial. □

Let $a = (a_1, a_2, \ldots, a_n) \in Spec(n)$. If $a_{i+1} \neq a_i \pm 1$, so that by proposition 2.44

$$a' = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, \ldots, a_n) \in Spec(n)$$

we say that s_i is an admissible transposition for a. In order to prove that $Spec(n) = Cont(n)$ we will need the following observation.

Lemma 2.45. *Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. If $a_i = a_{i+2} = a_{i+1} - 1$ for some $i \in \{1, 2, \ldots, n-2\}$ then $a \notin Spec(n)$.*

Proof. On the contrary, suppose that $a \in Spec(n)$. Then by (b) of proposition 2.44, we have $s_i v_a = v_a$ and $s_{i+1} v_a = -v_a$. By the braid relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, we get that

$$v_a = s_{i+1} s_i s_{i+1} v_a = s_i s_{i+1} s_i v_a = -v_a,$$

a clear contradiction. □

Lemma 2.46. *Suppose $a \in Spec(n)$. The following hold:*

(a) For every $a = (a_1, a_2, \ldots, a_n) \in Spec(n)$ we have $a_1 = 0$.

(b) If $a = (a_1, a_2, \ldots, a_n) \in Spec(n)$ then $a' = (a_1, a_2, \ldots, a_{n-1}) \in Spec(n-1)$.

(c) $Spec(2) = \{(0, 1), (0, -1)\}$.

Proof. (a) is imminent, since $\xi_1 = 0$. (b) follows automatically from the fact that $\xi_1, \ldots, \xi_{n-1} \in \mathbb{C}[S_{n-1}]$ and that $\xi_i v_a = a_i v_a$. For (c), note that the irreducible representations of S_2 are ι, ϵ, the trivial and the alternating representation. As $\xi_2 = (1\ 2) = s_1$, we get that for $v \in V_\iota$ it holds $\xi_2 v = v$ and that for $v \in V_\epsilon$, it holds $\xi_2 v = -v$. □

We are ready to give the inductive step.

Lemma 2.47. *(a) For every $n \geq 1$ we have $Spec(n) \subseteq Cont(n)$.*

(b) If $a \in Spec(n), b \in Cont(n)$ and $a \approx b$ then $b \in Spec(n)$ and $a \sim b$.

Proof. We show (a) by induction on n. For $n = 1$ it is trivial while for $n = 2$ it was proven in lemma 2.46. Suppose that $Spec(n-1) \subseteq Cont(n-1)$ and let $a = (a_1, a_2, \ldots, a_n) \in Spec(n)$. By lemma 2.46, we have that $a_1 = 0$ which corresponds to Condition (1) in the definition of $Cont(n)$. Since $a' = (a_1, a_2, \ldots, a_{n-1}) \in Spec(n-1)$, we have to check conditions (2) and (3) just for $q = n$. Suppose by contradiction that

$$\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \ldots, a_{n-1}\} = \emptyset \tag{5}$$

Then, the transposition $s_{n-1} = (n-1\ n)$ is admissible for a, that is, $(a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1}) \in Spec(n)$. Thus $(a_1, a_2, \ldots, a_{n-2}, a_n) \in Spec(n-1) \subseteq Cont(n-1)$. From eq. (5) we deduce that $\{a_n - 1, a_n + 1\} \cap \{a_1, a_2, \ldots, a_{n-2}\} = \emptyset$ which contradicts (2) in the definition of $Cont(n-1)$. Thus (2) holds. Now assume again by contradiction that a doesn't satisfy (3) for $q = n$, that is $a_p = a_n = x$ for some $p < n$ and for instance,

$$x - 1 \notin \{a_{p+1}, a_{p+2}, \ldots, a_{n-1}\}.$$

We can also assume that p is maximal, that is $x \notin \{a_{p+1}, a_{p+2}, \ldots, a_{n-1}\}$. Since $(a_1, a_2, \ldots, a_{n-1}) \in Cont(n-1)$ by the inductive hypothesis, $x+1$ can appear among $\{a_{p+1}, a_{p+2}, \ldots, a_{n-1}\}$ at most once. Suppose that it does not appear. Then $(a_p, a_{p+1}, \ldots, a_n) = (x, *, *, \ldots, *, x)$ where every $*$ represents a number not equal to $x, x-1, x+1$. In this case, by using a sequence of $n - p - 1$ admissible transpositions, we get that

$$a \sim a' = (\ldots, x, x, \ldots) \in Spec(n).$$

which is a contradiction to (a) from proposition 2.44. Similarly, assume that $x+1$ appears exactly once among $\{a_{p+1}, a_{p+2}, \ldots, a_{n-1}\}$. Then $(a_p, a_{p+1}, \ldots, a_n) = (x, *, \ldots, x+1, * \ldots, *, x)$, where again every $*$ is a number different than $x, x-1, x+1$. Now again by using a sequence of admissible transpositions we get

$$a \sim a' = (\ldots, x, x+1, x, \ldots) \in Spec(n)$$

which contradicts lemma 2.45. Therefore, condition (3) is also satisfied and thus $Spec(n) \subseteq Cont(n)$. Now (b) is an immediate consequence of (a) with our previous results on the equivalence relations. □

Theorem 2.48. *It holds that $Spec(n) = Cont(n)$. Moreover the equivalence relations $\sim$ and $\approx$ coincide. Finally the Young graph $\mathbb{Y}$ is isomorphic to the branching graph of the multiplicity-free chain*

$$S_1 \leq S_2 \leq \ldots \leq S_{n-1} \leq S_n \leq \ldots$$

Proof. We know that $|Cont(n)/\approx| = p(n)$ that is, the number of equivalence classes is equal to the number of partitions of n. Moreover, we know that this is equal to the conjugacy classes in S_n (by cycle structure) and the latter is equal to the number of irreducible representations of the symmetric group, that is $|\hat{S}_n|$ which in turn is equal to $|Spec(n)/\sim|$. Additionally, by lemma 2.47, an equivalence class in $Cont(n)/\approx$ is either disjoint from $Spec(n)$ or is contained in an equivalence class of $Spec(n)/\sim$. Rephrasing, the partition of $Spec(n)$ induced by $\approx$ is finer than the one induced by $\sim$. Collecting all of the above, we get

$$|Spec(n)/\sim| \leq |Spec(n)/\approx| \leq |Cont(n)/\approx| = |Spec(n)/\sim|$$

and thus $Spec(n) = Cont(n)$ and the two equivalence relations coincide. This equality gives a natural correspondence between the set of all paths in the branching graph (parametrized by $Spec(n)$) and the set of all paths in $\mathbb{Y}$ parametrized by $Cont(n)$. This yields a bijective correspondence between the vertices of the two graphs which is the required graph isomorphism. □

From the above theorem, we get a natural correspondence between $\hat{S}_n$ and the n-th level of the Young graph $\mathbb{Y}$, that is the set of all partitions of n. We can now recover some of the standard results for the irreducible representations S^λ.

Proposition 2.49. *It holds that the dimension of the irreducible representation corresponding to λ is equal to the number of standard tableau's of shape λ, that is $\dim S^\lambda = |Tab(\lambda)|$.*

Another immediate consequence is the following.

Corollary 2.50. *Let $0 \leq k < n, \lambda \vdash n$ and $\mu \vdash k$. Then the multiplicity $m_{\mu,\lambda}$ of S^μ in $\operatorname{Res}^{S_n}_{S_k} S^\lambda$ is equal to zero if $\mu \not\nearrow \lambda$ and it is equal to the number of paths in $\mathbb{Y}$ from λ to μ otherwise. In any case, $m_{\mu,\lambda} \leq (n-k)!$ and this estimate is sharp.*

Proof. We have

$$\operatorname{Res}^{S_n}_{S_k} S^\lambda = \operatorname{Res}^{S_{k+1}}_{S_k} \operatorname{Res}^{S_{k+2}}_{S_{k+1}} \dots \operatorname{Res}^{S_n}_{S_{n-1}} S^\lambda$$

where in each step of the consecutive restrictions the decomposition is multiplicity free and according to the Young diagram $\mathbb{Y}$. This way, the multiplicity of S^μ in $\operatorname{Res}^{S_n}_{S_k} S^\lambda$ is equal to the number of paths in $\mathbb{Y}$ that start from λ and end at μ. It is also equal to the number of ways in which we can obtain the diagram of λ from the diagram of μ by adding $n-k$ addable boxes to μ. In particular, the number of ways in which this can happen is bounded above by $(n-k)!$ and this estimate is sharp when the boxes can be added to different rows and columns. □

Corollary 2.51 (Branching Rule). *For every $\lambda \vdash n$ we have*

$$\operatorname{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\substack{\mu \vdash n-1: \\ \lambda \to \mu}} S^\mu$$

that is, the direct sum runs over all partitions of $n-1$ that can be obtained from λ by removing one box. Equivalently, by adjunction of induction and restriction for finite groups, for every $\mu \vdash n-1$, it holds

$$\operatorname{Ind}^{S_n}_{S_{n-1}} S^\mu = \bigoplus_{\substack{\lambda \vdash n: \\ \lambda \to \mu}} S^\lambda.$$

Proof. Follows from corollary 2.50, for $k = n-1$. □

Remark 2.52. *Since YJM elements generate the GZ algebra, for any irreducible representation $S^\lambda \in \mathbb{C}[S_n]-mod$ with Young basis $\{v_T \mid T \in Tab(\lambda)\}$ we have that this basis is a simultaneous eigenvector basis for the action of $\xi_1, \dots, \xi_n$. Moreover, the corresponding eigenvalue vectors are the content vectors.*

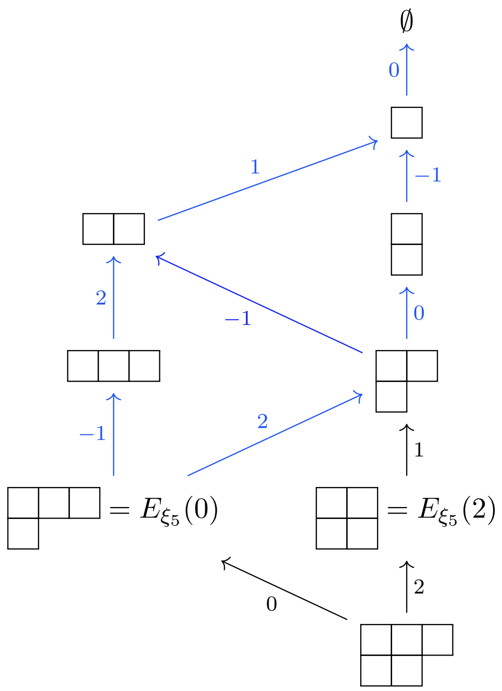
In particular, every content vector determines an eigenspace and vice versa. Exactly because $\xi_n \in \mathcal{C}(S_n, S_{n-1})$, these eigenspaces are $\mathbb{C}[S_{n-1}]$ modules and by induction they are irreducible, completely determined by the $n-1$ paths in the Young graph.

We illustrate the power of this approach with the following example.

Example 2.53. *Consider the simple $\mathbb{C}[S_5]$ module $S^{(3,2)}$. Restricting the action down to S_4 we notice the following. Since $\xi_5 \in \mathcal{C}(S_5, S_4)$, we have $\xi_5 \in End_{S_4}(Res^{S_5}_{S_4} S^{(3,2)})$. Moreover, this endomorphism is diagonalizable with respect to the GZ basis of $S^{(3,2)}$. The corresponding eigenvalues are the contents of the removable boxes in $(3,2)$ so $0, 2$ respectively. We thus have*

$$Res^{S_5}_{S_4} S^{(3,2)} = E_{\xi_5}(0) \oplus E_{\xi_5}(2)$$

By the relations we have from both $Spec(5)$ and $Cont(5)$, we inductively have that $E_{\xi_5}(0)$ is the span of elements of the Young basis $\{v_a \mid a \in Cont(5),\ a_5 = 0\}$ which is identified inductively with the span of the Young vectors corresponding to the 3 blue paths in the graph below, i.e to the content vectors $(0,1,2,-1), (0,1,-1,2), (0,-1,0,2)$.



Notice that this is exactly the irreducible representation corresponding to the partition $(3,1) \vdash 4$ and thus $E_{\xi_5}(0) = S^{(3,1)}$. The same applies for the two dimensional $E_{\xi_5}(2)$ which we can identify with $S^{(2,2)}$ thus justifying

$$Res^{S_5}_{S_4} S^{(3,2)} = S^{(3,1)} \oplus S^{(2,2)}.$$

The space $S^{(3,2)}$ is then identified with the span of Young vectors corresponding to the content vectors created by the above graph.

Remark 2.54. *We have not completely characterized S^λ, since we only know the action of YJM elements. In the next section we will find formulas for the action of the Coxeter generators in a GZ basis, thus giving a complete characterization for all the irreducibles.*

2.5 Young's normal forms

2.5.1 Young's seminormal form

Remember the canonical standard tableau T^λ. Also recall that the chain $S_1 \leq S_2 \leq \ldots \leq S_n$ determines a decomposition of every irreducible representation into one dimensional subspaces and that the GZ-basis is obtained by choosing a non-trivial vector from each of these subspaces. If the vectors are normalized we call them an orthonormal basis, while if they are not we will refer to it as an orthogonal basis (the decomposition into these subspaces is orthogonal with respect to a scalar product making them unitary representations). In both cases, the vectors are completely determined up to a scalar factor. For $T \in Tab(\lambda)$ we denote the corresponding GZ element by v_T.

Proposition 2.55. *It is possible to choose the scalar factors of the GZ-basis $\{v_T : T \in Tab(n)\}$ in such a way that, for every $T \in Tab(n)$, one has*

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{R \in Tab(\lambda): l(\pi_R) < l(\pi_T)} \gamma_R v_R$$

where $\gamma_R \in \mathbb{C}$ and π_T is the permutation transforming T to T^λ.

Proof. We use induction on $l(\pi_T)$. In each step, we choose the scalar factor for all T with $l(\pi_T) = l$. If $l(\pi_T) = 1$, then π_T is an admissible simple transposition for T^λ and the result follows from proposition 2.42 (where v_T corresponds to $v_{a'}$ and the corresponding coefficient for v_{T^λ} can also be chosen accordingly). So suppose now that $\pi_T = s_{i_1} s_{i_2} \ldots s_{i_{l-1}} s_j$ is the standard decomposition of π_T into a product of admissible simple transpositions. Set $l = l(\pi_T)$ and $j = i_l$ for simplicity of notation. Then $\pi_T = \pi_{T_1} s_j$, where $T_1 = s_j T$ is a standard tableau. It is clear that $l(\pi_{T_1}) = l(\pi_T) - 1$. By the inductive hypothesis we can write

$$\pi_{T_1}^{-1} v_{T^\lambda} = v_{T_1} + \sum_{R \in Tab(\lambda): l(\pi_R) < l(\pi_{T_1})} \gamma_R^{(1)} v_R. \tag{6}$$

Since $T = s_j T_1$, we can choose the scalar factor of v_T such that

$$s_j v_{T_1} = v_T + \frac{1}{a_{j+1} - a_j} v_{T_1} \tag{7}$$

where $a = (a_1, a_2, \ldots, a_n) = C(T_1) \in Cont(n)$ is the content of T_1. Combining eq. (6) with eq. (7), we get the required formula (by keeping in mind how to compute $s_j v_R$ for $R \in Tab(\lambda)$ such that $l(\pi_R) < l(\pi_{T_1})$. □

As is already hinted, the matrix coefficients will actually turn out to be rational numbers. The next theorem finally gives us how Coxeter generators act on a GZ basis of an irreducible representation of S_n.

Theorem 2.56 (Young's seminormal form)**.** *Choose the vectors in the Young basis of S_n according to proposition 2.55. If $T \in Tab(\lambda)$ and $C(T) = (a_1, a_2, \ldots, a_n)$ its content vector, then the simple transposition s_j acts on v_T as follows:*

(a) If $a_{j+1} = a_j \pm 1$ then $s_j v_T = \pm v_T$.

(b) If $a_{j+1} \neq a_j \pm 1$, then setting $T' = s_j T$ (s_j is admissible for T in this case) yields

$$s_j v_T = \begin{cases} \frac{1}{a_{j+1}-a_j} v_T + v_{T'} & \text{if } l(\pi_{T'}) > l(\pi_T) \\ \frac{1}{a_{j+1}-a_j} v_T + [1 - \frac{1}{(a_{j+1}-a_j)^2}] v_{T'} & \text{if } l(\pi_{T'}) < l(\pi_T). \end{cases}$$

Proof. (a) is already proven in proposition 2.44. (b) follows from proposition 2.55. The only thing to check is that when choosing the basis according to proposition 2.55, $s_j v_T$ has exactly this expression. So suppose first that $l(\pi_{T'}) > l(\pi_T)$. We have

$$\pi_{T'} = \pi_T s_j$$

and by proposition 2.55,

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{R \in Tab(\lambda): l(\pi_R) < l(\pi_T)} \gamma_R v_R$$

We thus deduce that

$$\pi_{T'}^{-1} v_{T^\lambda} = v_{T'} + \sum_{R' \in Tab(\lambda): l(\pi_{R'}) < l(\pi_{T'})} \gamma_{R'} v_{R'} =$$
$$= s_j v_T + s_j \sum_{R \in Tab(\lambda): l(\pi_R) < l(\pi_T)} \gamma_R v_R$$

therefore the formula holds exactly in the same form as proposition 2.55 (notice that the coefficient of $v_{T'}$ in $s_j v_T$ is equal to 1). The other case is analogous to the first one by starting from $\pi_T = \pi_{T'} s_j$ and taking $a = C(T')$ when applying proposition 2.55. $\square$

Corollary 2.57. *Fixing a GZ-basis, the matrix coefficients of the irreducible representations of S_n are rational numbers. In particular, the coefficients γ_R in proposition 2.55 are rational numbers. So one can define all irreducible representations of S_n over $\mathbb{Q}$.*

2.5.2 Young's orthogonal form

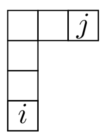
We normalize the GZ-basis $\{v_T : T \in Tab(\lambda)\}$ of S^λ by taking

$$w_T = \frac{v_T}{\|v_T\|}$$

where the norm is associated with an arbitrary scalar product making S^λ a unitary representation of S_n. Let T be a standard tableau and $C(T) = (a_1, a_2, \ldots, a_n)$ be its content.

Definition 2.58 (Axial distance)**.** *For $i, j \in [n]$ the axial distance from j to i is the integer $a_j - a_i$.*

This has a clear meaning when we translate everything to Young diagrams. Suppose that we want to move from j to i. Each step to the left or downwards counts as $+1$, while each step to the right or upwards counts for -1. Then the resulting integer is exactly $a_j - a_i$ and this is independent of the chosen path.



In the above example, the axial distance is $a_j - a_i = 5$. Remember that the content a_j counts the boxes on the left of the box with entry j minus the entries upwards from the same box. Now in our picture to reach from j to i, we first need to bring them in the same column, so we count that number. Suppose their coordinates are $(x(i), y(i))$ and $(x(j), y(j))$. Then that number is clearly equal to (in our case) $y(j) - y(i)$ and then bring them to the same box, making $x(i) - x(j)$ moves downwards. Altogether, we account for $y(j) - y(i) + x(i) - x(j) = a_j - a_i$. All the other cases work similarly to give the same result.

Theorem 2.59 (Young's orthogonal form). *Given the orthonormal basis* $\{w_T : \ T \in Tab(n)\}$*, we have*

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}} w_{s_j T}$$

where for $C(T) = (a_1, a_2, \ldots, a_n)$ *the quantity* $r = a_{j+1} - a_j$ *is the axial distance from* $j + 1$ *to* j*. In particular, if* $a_{j+1} = a_j \pm 1$ *we have* $r = \pm 1$ *and* $s_j w_T = \pm w_T$*.*

Proof. Set $T' = s_j T$ and suppose that $l(\pi_{T'}) > l(\pi_T)$. Then from Young's seminormal form, we have

$$\begin{aligned}
&\|v_{T'}\|^2 = \|s_j v_T - \frac{1}{r} v_T\|^2 = \\
&= \|v_T\|^2 - \frac{1}{r}\langle s_j v_T, v_T\rangle - \frac{1}{r}\langle v_T, s_j v_T\rangle + \frac{1}{r^2}\|v_T\|^2 = \\
&(\text{remember } s_j v_T = \frac{1}{r} v_T + v_{T'} \text{ and } v_T \perp v_{T'}) = (1 - \frac{1}{r^2})\|v_T\|^2.
\end{aligned}$$

where $\|s_j v_T\|^2 = \|v_T\|^2$ since s_j is unitary. Then Young's seminormal form in the orthonormal basis $\{\frac{v_T}{\|v_T\|}, \frac{v_{T'}}{\sqrt{1-\frac{1}{r^2}}\|v_T\|}\}$ reads

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}} w_{T'}$$

In the case $l(\pi_{T'}) < l(\pi_T)$ the proof is similar. □

Why is this theorem called Youngs orthogonal form? The reason is the following. It is clear that the space generated by $w_T, w_{T'}$ is invariant under the action of s_j. Youngs formulas in this basis, read

$$\begin{cases} s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}} w_{s_j T} \\ s_j w_{s_j T} = -\frac{1}{r} w_{s_j T} + \sqrt{1 - \frac{1}{r^2}} w_T \end{cases}$$

and thus the operator s_j in this basis is represented by the orthogonal matrix

$$\begin{bmatrix} \frac{1}{r} & \sqrt{1 - \frac{1}{r^2}} \\ \sqrt{1 - \frac{1}{r^2}} & -\frac{1}{r} \end{bmatrix}$$

Let's give some applications. Young's normal forms yield explicit formulas for the actions of the Coxeter generators in the GZ basis. Thus, we have completely characterized the action of the symmetric groups on any of their respective irreducible modules!

Example 2.60. *(a) Let* $\lambda = (n) \vdash n$ *be the trivial partition. Then there exists only one standard tableau of shape* λ*, namely*

1	2	3	.	.	.	n

The corresponding content is $C(T) = (0, 1, 2, \ldots, n-1)$ *and clearly since* $a_{j+1} = a_j + 1$ *we have* $s_j w_T = w_T$ *for all* $j = 1, 2, \ldots, n-1$*. Since these generate* S_n*, we deduce that* $S^{(n)}$ *is the trivial representation of* S_n*.*

(b) Let $\lambda = (1, 1, 1, \ldots, 1) \vdash n$*. In this case, we again have exactly one standard tableau of shape* λ*, namely*

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline . \\ \hline . \\ \hline . \\ \hline n \\ \hline \end{array}$$

The corresponding content is $C(T) = (0, -1, -2, -3, \dots, -n+1)$. Since in this case for all j we have $a_{j+1} = a_j - 1$, we get that $s_j w_T = -w_T$, $j = 1, 2, \dots, n-1$. It follows that $S^{(1,1,\dots,1)}$ is the alternating representation of S_n.

(c) *Consider $S^{(3,1)}$. All standard tableau's of this shape are shown below*

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array},\ T_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array},\ T_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

with corresponding content vectors $C(T_1) = (0, 1, 2, -1)$, $C(T_2) = (0, -1, 1, 2)$, $C(T_3) = (0, 1, -1, 2)$. Then for the orthogonal Young basis $\{w_{T_1}, w_{T_2}, w_{T_3}\}$ of $S^{(3,1)}$ and the Coxeter generators s_1, s_2, s_3 of S_4 we have

$$s_1 w_{T_1} = w_{T_1},\ s_1 w_{T_2} = -w_{T_2}\ s_1 w_{T_3} = w_{T_3}$$

since in each of these tableau's $a_2 = a_1 \pm 1$. For the action of s_2,

$$s_2 w_{T_1} = w_{T_1},\ s_2 w_{T_2} = \frac{1}{2} w_{T_2} + \frac{\sqrt{3}}{2} w_{T_3},\ s_2 w_{T_3} = \frac{1}{2} w_{T_3} + \frac{\sqrt{3}}{2} w_{T_2}$$

and for the action of s_3

$$s_3 w_{T_1} = -\frac{1}{3} w_{T_1} + \frac{2\sqrt{2}}{3} w_{T_3},\ s_3 w_{T_2} = w_{T_2},\ s_3 w_{T_3} = \frac{1}{3} w_{T_3} + \frac{2\sqrt{2}}{3} w_{T_1}.$$

We end the Okounkov-Vershik approach with some comments and observations. Even though the YJM elements provide very important information about the structure of $\mathbb{C}[S_n]$, this algebra does not really see them as generators together with the Coxeter generators s_i in the sense that they are superfluous. One can thus consider the following algebra

$$\mathcal{H}_n = \langle s_i,\ x_j \ \mid i = 1, \dots, n-1,\ j = 1, \dots, n \rangle$$

called the degenerate affine Hecke algebra, where the Coxeter generators are subject to the usual Coxeter relations in order to have $\mathbb{C}[S_n]$ as a subalgebra and the x_j are subject to the defining relations for YJM elements, namely

$$x_{i+1} s_i = s_i x_i + 1, \forall i\ x_i s_j = s_j x_i, \forall i \neq j, j+1$$

The s_i then generate a copy of the group algebra $\mathbb{C}[S_n]$ inside this algebra. Moreover, as a vector space $\mathcal{H}_n$ is readily seen to be isomorphic to $\mathbb{C}[x_1, x_2, \dots, x_n] \otimes \mathbb{C}[S_n]$. There is also a natural surjective algebra map (a projection)

$$\mathcal{H}_n \to \mathbb{C}[S_n]$$

mapping $s_i \to s_i$ and $x_1 \mapsto 0$. Note that by the defining relations it is then clear that $x_i \mapsto \xi_i$. Thus we can recover the group algebra as the quotient of $\mathcal{H}_n$ by the ideal generated by x_1, i.e

$$\mathbb{C}[S_n] \cong \mathcal{H}_n / \langle x_1 \rangle.$$

This algebra essentially freed up the YJM elements. One can then use this algebra to reformulate most of the results in this chapter in a modern way. In particular, in $\mathcal{H}_n$ it is easier to prove relations and thus find central elements compared to $\mathbb{C}[S_n]$. This is one way to prove for example that the center of the group algebra $\mathbb{C}[S_n]$ is the algebra of symmetric polynomials in YJM elements.

3 Partition algebras and their YJM elements.

3.1 Construction of partition algebras and connection to symmetric groups

In this chapter, we will deal with the case of partition algebras. We mostly follow [3],[6],[9],[8] as well as [4]. We start by giving some motivation for the partition algebra through Schur-Weyl duality with S_n. For d a nonnegative integer, consider the tensor category $Rep(S_d;\mathbb{C})$ of finite dimensional representations of S_d over $\mathbb{C}$. Let $V_d = \mathbb{C}^d$ be the natural $d-$dimensional permutation module of $\mathbb{C}[S_d]$ with ordered basis $\{v_1, \ldots, v_d\}$, the action just permuting basis elements. Then we can extend this action diagonally on tensor powers, or in other words we have

$$\sigma.(v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_n}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \ldots \otimes v_{\sigma(i_n)},$$

for all $i_1, \ldots, i_n \in \{1, \ldots, d\}$. Setting $V_d^{\otimes 0} = \mathbb{C}$, for all $d \geq 0$, since the natural representation is faithful, the following holds.

Proposition 3.1. *Any irreducible representation of S_d is a direct summand of $V_d^{\otimes n}$ for some nonnegative integer n.*

Proof. Let W be an irreducible representation of S_d and fix $\langle\ ,\ \rangle$ a non-degenerate bilinear form on W. Then for any element $0 \neq w \in W$, we have the nonzero linear form $\langle w,\ \rangle : W \to \mathbb{C}$. We construct a map $i : W \to {V_d}^{\otimes d}$ by setting

$$i(w') = \sum_{\sigma \in S_d} \langle w,\ \sigma^{-1}.w' \rangle v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(d)}.$$

Notice that every element in the sum gives a different basis vector of $V_d^{\otimes d}$. Now this is an S_d morphism, since

$$\sigma' i(w') = \sum_{\sigma \in S_d} \langle w,\ (\sigma'\sigma)^{-1}(\sigma'.w') \rangle v_{\sigma'\sigma(1)} \otimes v_{\sigma'\sigma(2)} \otimes \ldots \otimes v_{\sigma'\sigma(d)} = i(\sigma' w')$$

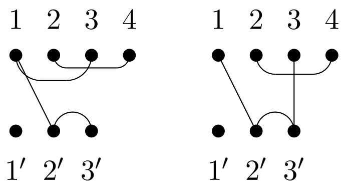
by making the change of variables $\sigma'\sigma \leftrightarrow \sigma$, since this map is bijective. We thus found a nonzero morphism in $\mathrm{Hom}_{S_d}(W, V_d^{\otimes d})$, concluding the proof. $\square$

Thus, a way to understand the category of finite dimensional representations of the symmetric groups is to study the category with objects tensor products of the natural representation $V_d^{\otimes n}$ and morphisms between them. This category will be missing direct sums as well as summands in order to include all representations, but luckily some categorical constructions will fix these issues.

We first want to control morphisms between these objects. For that, we will use set partitions as a diagrammatic way to help us have combinatorial rules to construct such morphisms. Recall that a partition π of a finite set S is a collection $\pi_1, \ldots, \pi_n$ of disjoint subsets of S such that $S = \sqcup_{i=1}^{n} \pi_i$. We call the sets π_i the parts of π. Given a partition π of $\{1, \ldots, n, 1', \ldots, m'\}$, a partition diagram of π is any graph with vertices labelled $\{1, \ldots, n, 1', \ldots, m'\}$ whose connected components are exactly the parts of π. We always draw partition diagrams using the following conventions:

- Vertices $1, \ldots, n$ (resp. $1', \ldots, m'$) are aligned horizontally and increasing from left to right with i directly above i'.
- Edges lie entirely below the vertices labelled $1, \ldots, n$ and above the vertices labelled $1', \ldots, m'$.

Example 3.2. *The partition diagrams*



both represent the set partition $\pi = \{\{1, 3, 2', 3'\}, \{2, 4\}, \{1'\}\}$ *of the set* $\{1, 2, 3, 4, 1', 2', 3'\}$.

So the diagram representing a set partition is not unique, but the connected components are. We will associate a morphism of representations $V_d^{\otimes m} \to V_d^{\otimes n}$ to each partition of the set $\{1, 2, \ldots, n, 1', 2', \ldots, m\}$. To prepare the ground for this assignment, we setup the following notations:

- For $n, m \in \mathbb{Z}_{\geq 0}$, we let $P_{n,m}$ be the set of set partitions of the set $\{1, \ldots, n, 1', \ldots, m'\}$.
- For n, d nonnegative integers $[n, d] = [d]^{[n]} = \{f : [n] \to [d]\}$. Given $i \in [n, d],\ j \in [n]$, we write $i_j = i(j)$.
- For $i \in [n, d]$ and $i' \in [m, d]$ the (i, i') coloring of a partition $\pi \in P_{n,m}$ is obtained by coloring the vertices of a partition diagram representing π labelled j (reps. j') by the integer i_j (resp. i'_j).

Remark 3.3. *In order to create a map* $f \in \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$*, we first have to fix a basis of* $V_d^{\otimes n}$ *and* $V_d^{\otimes m}$*, describe when a matrix coefficient is non-zero and then find an explicit rule yielding the matrix describing* f *in the respective bases of the two spaces. So for any* k*, fix the basis* $\{v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_k} : i_1, \ldots, i_k \in \{1, \ldots, d\}\}$ *of* $V_d^{\otimes k}$*. We first describe what is the diagrammatic rule describing when a matrix coefficient is non-zero, for any* $\pi \in P_{n,m}$*.*

Definition 3.4 (Good/Perfect colorings)**.** *Let* n, m, d *be nonnegative integers,* $\pi \in P_{n,m}$ *and an* (i, i')*-coloring of* π*.*

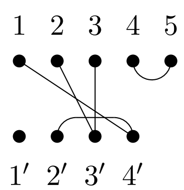
- *We say that the* (i, i')*-coloring of* π *is good, if vertices in the same connected component have the same color.*
- *We say that the* (i, i')*-coloring of* π *is perfect, when vertices have the same color if and only if they are in the same connected component.*

For $n \neq 0$ and $i \in [n, d]$, let

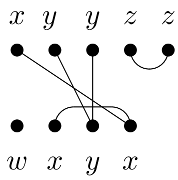
$$v_i := v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_n} \in {V_d}^{\otimes n}$$

and set $v_\emptyset = 1 \in {V_d}^{\otimes 0} = \mathbb{C}$.

Example 3.5. *As examples of good and perfect colorings, consider the following. Let*

$\pi =$ 

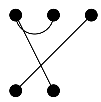
Clearly, there is only "one" choice for a perfect coloring, namely the one assigning a distinct color to every connected component, as shown below (notice that every permutation of the colors is also a perfect coloring).

$\pi =$ 

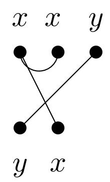
where x, y, z, w distinct nonnegative integers. In particular we can see that in order for a partition diagram to have a perfect $d-$ coloring, we need d to be as big, as the number of parts of the partition. A perfect coloring is always a good coloring, but the opposite doesn't hold. For example, the trivial coloring (assigning one color to the entire diagram) is a good coloring, but it is only perfect when the diagram consists of only one part.

Remark 3.6. *The definitions of good and perfect colorings are exactly what we need to describe a "nice" map $f : \mathbb{C}P_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$. In particular, for $\pi \in P_{n,m}$ and $v_i, v_{i'}$ basis elements of $V_d^{\otimes n}$ and $V_d^{\otimes m}$ respectively, the coefficient of $v_{i'}$ in $f(\pi)(v_i)$ is non-zero, if and only if the (i, i')-coloring of π is good. We make the pairing of basis elements clear with the following example.*

Example 3.7. *Assume we have*

$\pi =$  $\in P_{3,2}$

When we assign numbers on the top nodes of the diagram, we are assigning a basis element of $V_d^{\otimes 3}$ and assigning numbers on the bottom corresponds to a certain basis element of $V_d^{\otimes 2}$. Then the diagrammatics provide the following information. Good colorings correspond to

$\pi =$  ,

where $x, y \in \{1, 2, 3\}$. This means that the only non-zero matrix coefficients are between basis elements of the form $v_x \otimes v_x \otimes v_y$ and $v_y \otimes v_x$. For example, automatically under π, the image of $v_1 \otimes v_2 \otimes v_3$ is zero, while $v_1 \otimes v_1 \otimes v_2$ is a multiple scalar of $v_2 \otimes v_1$.

Remark 3.8. *We can assign to every coloring of the set $\{1, \ldots, n, 1', \ldots, m'\}$ a unique partition diagram, namely the one whose parts consist of the vertices with the same color. Notice that this is exactly the partition diagram having this coloring as a perfect coloring.*

We are now ready to define the Schur-Weyl map.

Definition 3.9 (Schur-Weyl Map)**.** *For nonnegative integers n, m, d we define the $\mathbb{C}$-linear map*

$$f : \mathbb{C}P_{n,m} \to \mathrm{Hom}(V_d^{\otimes n}, V_d^{\otimes m})$$

by setting on a basis element

$$f(\pi)(v_i) = \sum_{i' \in [m,d]} f(\pi)^i_{i'} v_{i'}$$

where

$$f(\pi)^i_{i'} = \begin{cases} 1, \ (i, i') - \textit{coloring of } \pi \textit{ is good,} \\ 0 \text{ , } \textit{otherwise.} \end{cases}$$

and then extending linearly (both to $\mathbb{C}P_{n,m}$ and to $V_d^{\otimes n}$).

Lemma 3.10. *It holds that*

$$f : \mathbb{C}P_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m}).$$

Proof. We have to check that indeed, $f(\pi)$ is a morphism of S_d representations. Let $\sigma \in S_d$. We want to prove that for any basis element v_i, it holds that

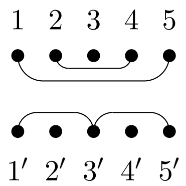
$$\sigma.f(\pi)(v_i) = f(\pi)(\sigma.v_i).$$

This is equivalent to proving for the coefficients that $f(\pi)^{\sigma.i}_{\sigma.i'} = f(\pi)^{i}_{i'}$ for every $i' \in [m,d]$. But this would mean that the coloring $(\sigma.i, \sigma.i')$ is good if and only if the coloring (i,i') is good which is true since σ will just permute the colors, giving elements in the same connected component the same color. $\square$

Remark 3.11. *Let us unpack the definition of f. For a partition $\pi \in P_{n,m}$ and $i \in [n,d]$, we know that in order for a coefficient $f(\pi)^i_{i'}$ to survive, we first have to check if $i \in [n,d]$ satisfies $i_j = i_k$ whenever j,k are in the same connected component. Then the only "good" choices for $i' \in [m,d]$ are the ones for which $i'_k = i_j$ for every pair of vertices k', j in the same connected component.*

Example 3.12. *1. As a first example, consider the trivial partition $\emptyset \in P_{0,0}$. Then $f(\emptyset) : \mathbb{C} \to \mathbb{C}$ is trivially the identity map, $f(\emptyset) = id_{\mathbb{C}}$.*

2. Suppose $d > 0$ and consider the partition diagram

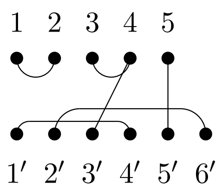
$\pi =$ 

Then $f(\pi) : V_d^{\otimes 5} \to V_d^{\otimes 5}$ is given by

$$f(\pi)(v_i) = \sum_{x,y,z \in [d]} \delta_{i_1,i_5}\delta_{i_2,i_4} v_x \otimes v_y \otimes v_x \otimes v_z \otimes v_x.$$

Since good colorings only occur when the vertices $1', 3', 5'$ have the same color and to even consider a nonzero map, we need the vertices $1,5$ and $2,4$ to have the same colour.

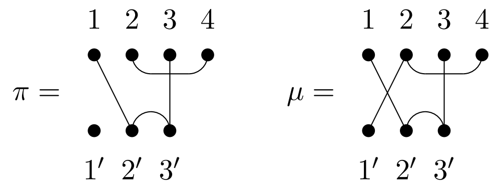
Let again $d > 0$ and π be the partition represented by the diagram



We see from the above that a choice of $i \in [5,d]$, able to yield a good coloring needs $i_1 = i_2$, $i_3 = i_4$. Now for $i' \in [6,d]$, we need thus to have $i'_3 = i_3 = i_4$ $i'_5 = i_5$ and then for the lower traversal edges $i'_1 = i'_4$ $i'_2 = i'_6$. Thus for a basis element $v_i \in V_d^{\otimes 5}$ we have that $f(\pi) : V_d^{\otimes 5} \to V_d^{\otimes 6}$ is given by

$$f(\pi)(v_i) = \sum_{j,k \in [d]} \delta_{i_1,i_2}\delta_{i_3,i_4} v_j \otimes v_k \otimes v_{i_3} \otimes v_j \otimes v_{i_5} \otimes v_k.$$

Define $\leq$ to be the partial ordering on $P_{n,m}$ given by $\mu \leq \pi$ if and only if every part of π is contained in a part of μ. In the example below, $\pi \leq \mu$, where



Definition 3.13 (Orbit basis)**.** *For every $\pi \in P_{n,m}$ define the elements $O(\pi)$ by the recursive relation*

$$\pi = \sum_{\mu \geq \pi} O(\mu).$$

It is easy to see that the set $\{O(\pi) : \pi \in P_{n,m}\}$, forms a basis of $\mathbb{C}P_{n,m}$ as with every extension of $\leq$ to a total ordering, the change of basis matrix is unitriangular.

Remark 3.14. *The orbit basis will give an even better combinatorial rule for the matrix coefficients, namely the one corresponding to perfect colorings.*

Lemma 3.15. *For the orbit elements $f(O(\pi))$, the matrix coefficients with respect to the diagrammatic basis are given by*

$$f(O(\pi))^i_{i'} = \begin{cases} 1, & (i,i') - \text{coloring of } \pi \text{ is perfect,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We do induction on the number of parts of π. If π has one part, then a good coloring is also perfect (since the only good coloring is the trivial one, which is perfect). Thus it holds for a connected diagram. Suppose we know it for diagrams with at most t parts and consider π with $t+1$ parts. By definition

$$O(\pi) = \pi - \sum_{\mu > \pi} O(\mu)$$

where every $\mu > \pi$ has less than or equal to t parts. Consider a perfect coloring of π. Then it is not a good coloring of any of $\mu > \pi$ and thus $f(O(\pi))^i_{i'} = 1$. Moreover, if a coloring of π is not good, then two vertices in the same part have the same colour and thus this is definitely not a perfect coloring for any $\mu > \pi$, yielding $f(O(\pi))^i_{i'} = 0$. Lastly, for a good but not perfect coloring of π, exactly one of the $\mu > \pi$ is going to have this coloring as perfect, yielding $f(O(\pi))^i_{i'} = 1 - 1 = 0$. □

Theorem 3.16. *For n, m, d nonnegative integers, the map f enjoys the following properties:*

1. *f is surjective.*

2. *$ker f = span\{O(\pi) : \pi$ has more than d-parts$\}$.*

Proof. For $d = 0$, the map is identically zero, thus both assertions hold. Suppose $d > 0$, and let $g \in \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$. Then for every $i \in [n,d]$ we write

$$g(v_i) = \sum_{i' \in [m,d]} g^i_{i'} v_{i'}.$$

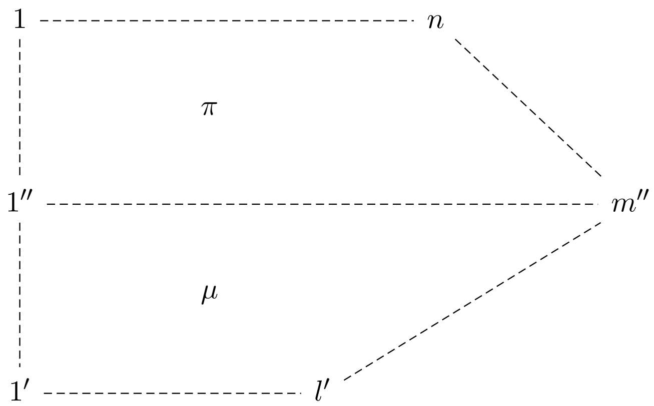
Since g is an S_d-morphism, the matrix entries $g^i_{i'}$ are constant on the S_d orbits of the matrix coordinates $\{(i,i') : \ i \in [n,d], \ i' \in [m,d]\}$. Notice that every orbit corresponds to a partition as follows: (i,i') is in the orbit corresponding to π if and only if the (i,i')-coloring of π is perfect. It is clear that if $\mu > \pi$ and the (i,i')-coloring of π is perfect, then it is not even a good coloring of μ as in order to make μ coarser than π we connected two different connected components (and thus there is at least a pair of vertices in the same connected component with a different color). Consequently, using lemma 3.15, g is a $\mathbb{C}$-linear combination of the $f(O(\pi))$, proving surjectivity of f.

As for the kernel, notice that a partition $\pi \in P_{n,m}$ has a perfect coloring if and only if it has at most d-components. By lemma 3.15, we see that $f(O(\pi)) = 0$ if and only if π has more than $d-$parts. Thus $span\{O(\pi) : \pi \text{ has more than d-parts}\} \subseteq ker f$ and by the first isomorphism theorem together with counting dimensions we conclude that equality holds. $\square$

Corollary 3.17. *If $d \geq n+m$ then f is an isomorphism of vector spaces, yielding $\mathbb{C}P_{n,m} \cong \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$.*

We want to find a way to turn $\mathbb{C}P_{n,n}$ into an algebra, such that this map turns into an algebra morphism. For that we equip $P_{n,n}$ with the structure of a monoid.

Definition 3.18. *Let $\pi \in P_{n,m}$ and $\mu \in P_{m,l}$. We construct a new diagram $\mu * \pi$ by identifying the vertices $1', 2', \ldots, m'$ of π with the vertices $1, 2, \ldots, m$ of μ and renaming them $1'', 2'', \ldots, m''$, by vertical concatenation with μ below π as illustrated below.*



Now let $l(\mu, \pi)$ denote the number of connected components of $\mu * \pi$ whose vertices are involving only $1'', \ldots, m''$. Finally, let $\mu \cdot \pi \in P_{n,l}$ be the partition obtained by restricting $\mu * \pi$ to $\{1, \ldots, n, 1', \ldots, l'\}$ (r, s in the same part of $\mu \cdot \pi$ if and only if they are in the same part of $\mu * \pi$).

Proposition 3.19.

$$f(\mu)f(\pi) = d^{l(\mu,\pi)} f(\mu \cdot \pi)$$

for $\pi \in P_{n,m}$ and $\mu \in P_{m,l}$.

Proof. Let $\pi \in P_{n,m}$ and $\mu \in P_{m,l}$ for n, m, l nonnegative integers. By definition of f, the matrix coefficients of $f(\mu)f(\pi) : V_d^{\otimes n} \to V_d^{\otimes l}$ are given by

$$(f(\mu)f(\pi))^i_{i'} = \sum_{i'' \in [m,d]} f(\mu)^{i''}_{i'} f(\pi)^i_{i''}.$$

In concrete terms, it is equal to the number of $i'' \in [m, d]$ such that the (i, i'')-coloring of π and the (i'', i')-coloring of μ are simultaneously good. Notice that these are equal to the number of $i'' \in [m, d]$ such that coloring the vertices j, j', j'' of $\mu * \pi$ with the integers i_j, i'_j, i''_j yields a good coloring of $\mu * \pi$. Obviously, any good coloring of $\mu * \pi$ gives rise to a good coloring of $\mu \cdot \pi$ and any good coloring has to arise in that way. Additionally, any two good colorings of $\mu * \pi$ yield the same good coloring of $\mu \cdot \pi$ if and only if they differ in connected containing only the vertices $1'', \ldots, m''$. Since there are d-choices for the color of each such component, the number of $i'' \in [m, d]$ such that the two colorings are simultaneously good is

$$d^{l(\mu,\pi)} f(\mu \cdot \pi)^i_{i'}.$$

$\square$

Remark 3.20. *The above proposition informs us that the structure constants of $f(\mu)f(\pi)$ are polynomial in the integer d. We will exploit this to create a new category by freeing up this integer d to be any element of our field.*

Definition 3.21 (Partition category)**.** *For d a nonnegative integer, let $Rep_0(S_d;\mathbb{C})$ denote the full subcategory of $Rep(S_d;\mathbb{C})$ whose objects are of the form $V_d^{\otimes n}$, $n \geq 0$. Clearly objects can be indexed by nonnegative integers and since the map f is surjective, morphisms in this category can be represented (not uniquely!) by $\mathbb{C}$-linear combinations of the maps $f(\pi)$. Moreover, the structure constants are polynomials in d. We then define the partition category $\mathcal{P}ar_t$, to be the category with*

1. *Objects of the form $[n]$, for $n \geq 0$.*

2. *Morphism spaces given by $\mathrm{Hom}_{\mathcal{P}ar}([n],[m]) = \mathbb{C}P_{n,m}$.*

3. *Composition by $\mathbb{C}P_{m,l} \times \mathbb{C}P_{n,m} \to \mathbb{C}P_{n,l}$ by the bilinear map satisfying*

$$\mu \circ \pi = t^{l(\mu,\pi)}\mu \cdot \pi.$$

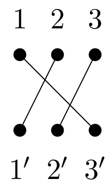
It is easy to see that $\nu\circ(\mu\circ\pi) = (\nu\circ\mu)\circ\pi$, since the numbers $l(\mu,\pi)+l(\nu,\mu\cdot\pi)$ and $l(\nu,\mu)+l(\nu\cdot\mu,\pi)$ both represent the number of connected components of $\nu * \mu * \pi$ with vertices only among $\{1'', 2'', \ldots\}$. The identity morphism $id_n : [n] \to [n]$ is the partition with parts $\{i, i' \ : \text{ for all } i = 1,\ldots,n\}$.

Definition 3.22. *The partition algebra $P_{2n}(t)$ is defined to be the endomorphism algebra $\mathrm{End}_{\mathcal{P}ar}([n])$.*

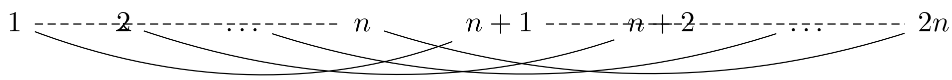
Remark 3.23. *Notice that by twisting the monoid structure of $P_{n,n}$ in this way, we made f an algebra morphism. Consequently, for large nonnegative integer values of t, we have $P_{2n}(t) \cong \mathrm{End}_{S_t}(V_t^{\otimes n})$ as algebras.*

Remark 3.24. *The group algebra of the symmetric group $\mathbb{C}[S_n]$ can be naturally embedded in the partition algebra $P_{2n}(t)$ with its usual diagrammatic representation. In concrete terms, we identity $\sigma \in S_n$ with the partition whose connected components are all of the form $\{i, \sigma(i)'\}$. Denote this by $i : \mathbb{C}[S_n] \hookrightarrow P_{2n}(t)$.*

Example 3.25. *The cycle $\sigma = (1\ 3\ 2)$ in $S_3 \subseteq P_3(t)$ is depicted by the following diagram:*



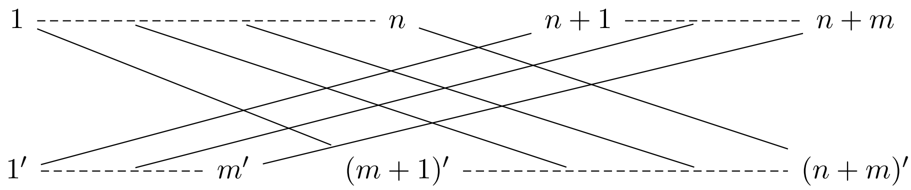
By the connection of this category to the category containing n-fold tensor products of the natural representation, we can naturally endow it with a tensor product and appropriate morphisms to give it the structure of a tensor category. In particular, we set $[n] \otimes [m] = [n+m]$, tensor products of diagrams is horizontal stacking, every object is self dual with evaluation and coevaluation maps defined as follows:

$\mathrm{ev_n} =$ 

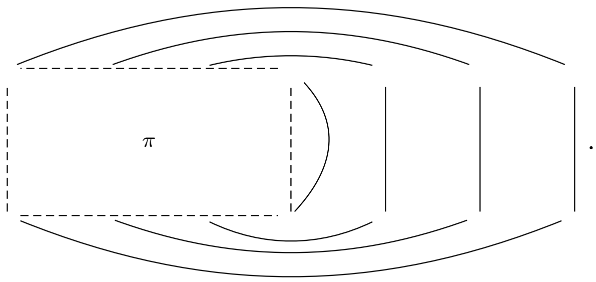
i.e the partition with parts $\{i, n+i\}$ for all i and

$$\mathrm{coev_n}$$

being the flip of $\mathrm{ev_n}$ by 180°. Also, the commutator $\beta_{n,m} : [n] \otimes [m] \to [m] \otimes [n]$ is given by



so that the trace of a morphism $\pi \in P_{2n}(t)$, is equal to $\mathrm{tr}(\pi) = t^a$ where a is the number of connected components in the diagram



In other words, $tr(\pi)$ is the number of connected components in the diagram connecting the components containing i, i', for every i. We first plan to classify idempotents in this algebra, using the following lemma.

Lemma 3.26. *Suppose A is a finite dimensional algebra over $\mathbb{C}$ and e is an idempotent in A. Let (e) be the two sided ideal of A generated by e. There is a bijective correspondence*

$$\left\{\begin{matrix}\textit{primitive idempotents}\\ \textit{in } A \textit{ up to conjugation}\end{matrix}\right\} \leftrightarrow \{\textit{p.i. in } A/(e) \textit{ up to conj.}\} \sqcup \{\textit{p.i. in } eAe \textit{ up to conj.}\}$$

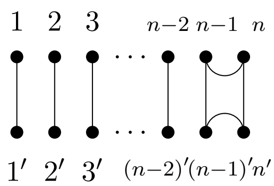
satisfying the following property:
Suppose that p is a primitive idempotent in A. Then p corresponds to a primitive idempotent in eAe if and only if $p \in (e)$. Moreover, if $p \notin (e)$, then p corresponds to its image under the quotient map $A \to A/(e)$.

We have a natural embedding

$$i : P_{2n-2}(t) \hookrightarrow P_n(t)$$

sending a partition $\pi \in P_{2n-2}(t)$ to the partition connecting n with $n-1$ and n' with $(n-1)'$ respectively, or equivalently sending π to $e(\pi \otimes 1)e$, where e is the idempotent defined below lemma 3.27.

Lemma 3.27. *For $n > 1$, let e denote the following idempotent in $P_{2n}(t)$:*

$e =$ 

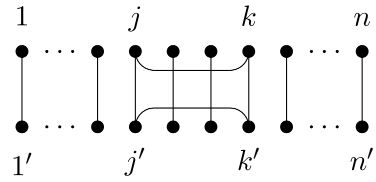
Then we have the following algebra isomorphisms:

- $eP_{2n}(t)e \cong P_{2n-2}(t)$.

- $P_{2n}(t)/(e) \cong \mathbb{C}[S_n]$.

Proof. The image of the embedding $i : P_{2n-2}(t) \hookrightarrow P_{2n}(t)$ is the algebra $eP_{2n}(t)e$ as the e on the left draws $n-1$ in the same part as n and the right one does the same for $(n-1)'$ and n'. For the second isomorphism, observe that the ideal (e) always has a part containing either both $n-1$ and n or $(n-1)'$ and n'. By that and the embedding of $\mathbb{C}[S_n]$ in $P_n(t)$ it is clear that $\mathbb{C}[S_n] \cap (e) = (0)$. It thus suffices to show that a partition $\pi \in P_{n,n}$ satisfies $\mu \in (e)$ whenever $\mu \notin \mathbb{C}[S_n]$. To that direction, we have that for fixed j and k, the partition

$\pi_{j,k} = \sigma e \sigma =$

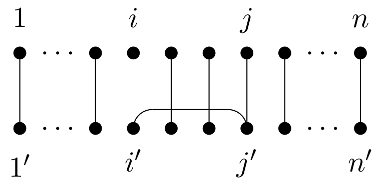


is in the ideal (e), where $\sigma = (j\ n-1)(k\ n) \in S_n \subseteq P_{n,n}$. Let $\mu \in P_{n,n} \backslash S_n$. That means that a part of μ either is of the form $\{i\}$ for some $i \in [n]$, or there exist $j, k \in [n]$ in the same part. In the first case, we have that

$$\mu = \mu \pi_{i,j} \nu_{i,j} \in (e)$$

where $j \neq i$ and

$\nu_{i,j} =$



.

In the second case, it is clear that $\mu = \mu \pi_{j,k} \in (e)$ which completes the proof. $\square$

Remark 3.28. *In fact, the proof shows that the following short sequence is exact*

$$0 \to \mathbb{C}[S_n] \xrightarrow{i} P_{2n}(t) \xrightarrow{p} P_{2n}(t)/(e) \cong \mathbb{C}[S_n] \to 0$$

and thus the group algebra of the symmetric group is both a subalgebra and a quotient of the partition algebra.

We are able to now classify primitive idempotents in partition algebras, using lemma 3.27 and lemma 3.26.

Theorem 3.29. 1. *When $t \neq 0$ we have the following bijection*

$$\left\{\begin{matrix}\text{primitive idempotents} \\ \text{in } P_{2n}(t) \text{ up to conjugation}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Young diagrams } \lambda \\ \text{with } |\lambda| \leq n\end{matrix}\right\}$$

2. *When $n > 0$ the bijection becomes*

$$\left\{\begin{matrix}\text{primitive idempotents} \\ \text{in } P_{2n}(0) \text{ up to conjugation}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Young diagrams } \lambda \\ \text{with } 0 < |\lambda| \leq n\end{matrix}\right\}$$

Proof. The first statement is trivial for $n = 0$, as $P_0(t) = \mathbb{C}$. For $n = 1$, let

$$\pi = \begin{array}{c} 1 \\ \bullet \\ \\ \bullet \\ 1' \end{array}$$

and $f = \frac{1}{t}\pi$. It is obvious that f is idempotent and since $P_2(t) = span\{1, f\}$ we immediately get that $1 = f + (1 - f)$ is a decomposition of 1 into primitive idempotents. Thus the result holds for $n = 1$. The result follows by induction on n, considering the lemma 3.27 together with the fact that irreducibles up to isomorphism (and thus also primitive idempotents up to conjugation) of $\mathbb{C}[S_n]$ are indexed by partitions of n.
The proof of 2 is the same, keeping in mind that for $t = 0$ and $n = 1$, it holds that 1 is the only primitive idempotent in $P_2(0)$ since $\pi^2 = 0$. □

Remark 3.30. *By this, a primitive idempotent $f \in P_{2n}(t)$ corresponds to a partition of n if and only if $f \notin (e)$. Moreover, if $f \notin (e)$, then the image of f under the quotient map $P_{2n}(t) \twoheadrightarrow \mathbb{C}[S_n]$ is a primitive idempotent corresponding to λ in S_n. Moreover, since primitive idempotents are in correspondence with indecomposable modules, the above discussion shows that an indexing set for indecomposable $P_{2n}(t)$-modules is $\mathbb{Y}_{\leq n}$, the set of Young diagrams with $|\lambda| \leq n$. Thus, whenever $P_{2n}(t)$ is semisimple, this is an indexing set for its irreducibles.*

3.2 Jucys Murphy elements for Partition Algebras

Halverson and Ram originally defined analogues of the classic Jucys Murphy elements for the partition algebras. Later Enyang worked with these elements and provided us with a 5 term recursive formula, together with a presentation of the partition algebra introducing new generators σ_i that resemble the Coxeter generators of the symmetric group in their properties. These Jucys Murphy elements, even though immensely complicated compared to their classic analogues, provide a spectral approach to the representation theory of the partition algebras. As in the Okounkov-Vershik approach, we would like to have a multiplicity free chain of semisimple algebras

$$\mathbb{C} = P_0(t) \leq P_2(t) \leq \ldots \leq P_{2n-2}(t) \leq P_{2n}(t)$$

but unfortunately this chain is not multiplicity free. An intuitive reason for that is the following: Using Schur Weyl duality, we have seen that the centralizer for the action of S_n as a subgroup of $GL_n(\mathbb{C})$ for the natural representation is $P_n(t)$. What if we further restrict to S_{n-1}? Ideally, to follow the structure of the symmetric groups, we would like the centralizer to be $P_{2n-2}(t)$, which is not the case.

Definition 3.31. *The subalgebra of $P_{2n}(t)$ generating the centralizer of the restriction to S_{n-1} is denoted by $P_{2n-1}(t)$ and is defined as the subalgebra of $P_{2n}(t)$ consisting of partitions having n, n' in the same part.*

Including these into the chain, we have refined it as follows:

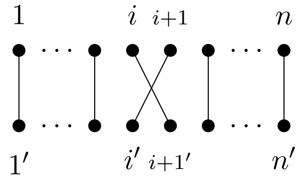
$$\mathbb{C} = P_0(t) \subseteq P_1(t) \subseteq P_2(t) \subseteq \ldots \subseteq P_n(t).$$

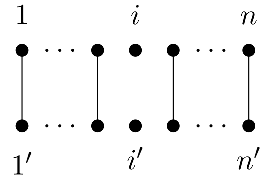
Martin proved that it is a multiplicity free chain, whenever $t \notin \mathbb{Z}_{\geq 0}$. In fact, it can be proven that the partition algebra $P_n(t)$ is semisimple, if and only if $t \notin \{0, 1, \ldots, 2\lfloor \frac{n}{2} \rfloor - 1\}$. We start by giving a presentation for the partition algebra $P_{2n}(t)$. For more details, see [6].

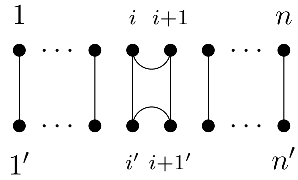
Theorem 3.32. *For $t \in \mathbb{C}$,the partition algebra $P_{2n}(t)$, has a presentation with generating set*

$$\{s_i, e_j \mid i \in [n-1], j \in [2n-1]\}$$

where

$s_i =$ 

$, e_{2i-1} =$ 

$e_{2i} =$ 

and relations

1. *Coxeter relations:*
   - $s_i^2 = 1$.
   - $s_i s_j = s_j s_i$ *for* $|j - i| \neq 1$.
   - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.
2. *Idempotent relations:*
   - $e_{2i-1}^2 = t e_{2i-1}$.
   - $e_{2i}^2 = e_{2i}$.
   - $s_i e_{2i} = e_{2i} s_i = e_{2i}$.
   - $s_i e_{2i-1} e_{2i+1} = e_{2i-1} e_{2i+1} s_i = e_{2i-1} e_{2i+1}$.
3. *Commutation relations:*
   - $e_{2i-1} e_{2j-1} = e_{2j-1} e_{2i-1}$.
   - $e_{2i} e_{2j} = e_{2j} e_{2i}$.
   - $e_{2i-1} e_{2j} = e_{2j} e_{2i-1}$.
   - $s_i e_{2j-1} = e_{2j-1} s_i$ *for* $j \neq i, i+1$.
   - $s_i e_{2j} = e_{2j} s_i$ *for* $|j - i| \neq 1$.
   - $s_i e_{2i-1} s_i = e_{2i+1}$.
   - $s_i e_{2i-2} s_i = s_{i-1} e_{2i} s_{i-1}$.
4. *Contraction relations:*
   - $e_i e_{i+1} e_i = e_i$.
   - $e_{i+1} e_i e_{i+1} = e_{i+1}$.

From the symmetry of all the relations in theorem 3.32, the map $^* : P_{2n}(t) \to P_{2n}(t)$ flipping the corresponding diagrams though a horizontal axis is an involution, fixing the generators s_i, e_j. Notice that the restriction to S_n is given by the classic involution, i.e $\sigma^* = \sigma^{-1}$, for $\sigma \in S_n$. We can now define YJM elements and Enyang's generators for the algebra $P_{2n}(t)$.

Definition 3.33. *Let $x_1 = 0, x_2 = e_1, \sigma_2 = 1$ and $\sigma_3 = s_1$. Then for $i = 1, 2, \ldots,$ define*

$$x_{2i+2} = s_i x_{2i} s_i - s_i x_{2i} e_{2i} - e_{2i} x_{2i} s_i + e_{2i} x_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1} \tag{8}$$

where for $i = 2, 3, \ldots,$ we have

$$\begin{aligned} \sigma_{2i+1} &= s_{i-1} s_i \sigma_{2i-1} s_i s_{i-1} + s_i e_{2i-2} x_{2i-2} s_i e_{2i-2} s_i + e_{2i-2} x_{2i-2} s_i e_{2i-2} \\ &- s_i e_{2i-2} x_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} - e_{2i-2} e_{2i-1} e_{2i} s_{i-1} x_{2i-2} e_{2i-2} s_i. \end{aligned}$$

Additionally, define

$$x_{2i+1} = s_i x_{2i-1} s_i - x_{2i} e_{2i} - e_{2i} x_{2i} + (t - x_{2i-1}) e_{2i} + \sigma_{2i} \tag{9}$$

where

$$\begin{aligned} \sigma_{2i} &= s_{i-1} s_i \sigma_{2i-2} s_i s_{i-1} + e_{2i-2} x_{2i-2} s_i e_{2i-2} s_i + s_i e_{2i-2} x_{2i-2} s_i e_{2i-2} \\ &- e_{2i-2} x_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} - s_i e_{2i-2} e_{2i-1} e_{2i} s_{i-1} x_{2i-2} e_{2i-2} s_i. \end{aligned}$$

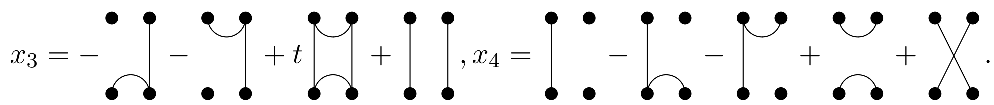
The elements σ_i resemble the Coxeter generators in their properties and the elements x_i are the Jucys Murphy elements for the partition algebras. By induction, one can show that $x_i \in P_i(t)$ and $\sigma_i \in P_{i+1}(t)$. The two recursive formulas eq. (8) and eq. (9) above are the analogues of the equation $\xi_{i+1} = s_i \xi_i s_i + s_i$. Some of the main properties of these elements are listed below.

Proposition 3.34. *For all i, the following hold:*

1. $x_i^* = x_i$ *and* $\sigma_i^* = \sigma_i$.
2. x_i *commutes with* $P_{i-1}(t)$.
3. σ_{i+1} *commutes with* $P_{i-1}(t)$.

Remark 3.35. *We have changed the notation from [5]. In particular, the elements x_{2i} (x_{2i+1} respectively) correspond to x_i ($x_{i+\frac{1}{2}}$ respectively) and the same holds for Enyang generators.*

Example 3.36. *The first non trivial YJM elements are presented below*



The projection we created for the group algebras respects these YJM elements, as well as Enyang's generators.

Lemma 3.37. *The projection $p : P_{2n}(t) \twoheadrightarrow \mathbb{C}[S_n]$ projects the even YJM elements to their corresponding counterparts in the symmetric group, and the odd ones contain no information. In other words,*

$$p(x_{2j}) = \xi_j, \quad p(x_{2j-1}) = (j-1) \cdot id$$

As for Enyang's generators

$$p(\sigma_{2j}) = id \quad p(\sigma_{2j-1}) = s_{j-1}.$$

Proof. We proceed by induction on j. For $j = 1$, the assertion is trivial. Since p is an algebra morphism killing the generators e_i, it follows by the inductive hypothesis that

$$p(x_{2j+2}) = s_j p(x_{2j}) s_j + p(\sigma_{2j+1}) = s_j \xi_j s_j + s_j = \xi_{j+1}$$
$$p(x_{2j+1}) = s_j p(x_{2j-1}) s_j + p(\sigma_{2j}) = (j-1) \cdot id + id = j \cdot id$$

and similarly for Enyang's generators. $\square$

3.3 Representation theory of $P_{2n}(t)$.

In this section, we provide a quick exposition for how the Okounkov-Vershik approach from Chapter 1 generalizes to the case of the partition algebras, whenever they are semisimple. We describe the branching graph of the partition algebras as well as the spectrum of the Jucys Murphy elements in the Gelfand-Tsetlin basis.

Theorem 3.38. *For $n \geq 0$, the partition algebra $P_n(t)$ is semisimple if and only if $t \notin \{0, 1, \ldots, 2\lfloor \frac{n}{2} \rfloor - 2\}$.*

Theorem 3.39. *For values making $P_n(t)$ semisimple, the chain of $\mathbb{C}$-algebras*

$$\mathbb{C} = P_0(t) \leq P_1(t) \leq P_2(t) \leq \ldots \leq P_n(t)$$

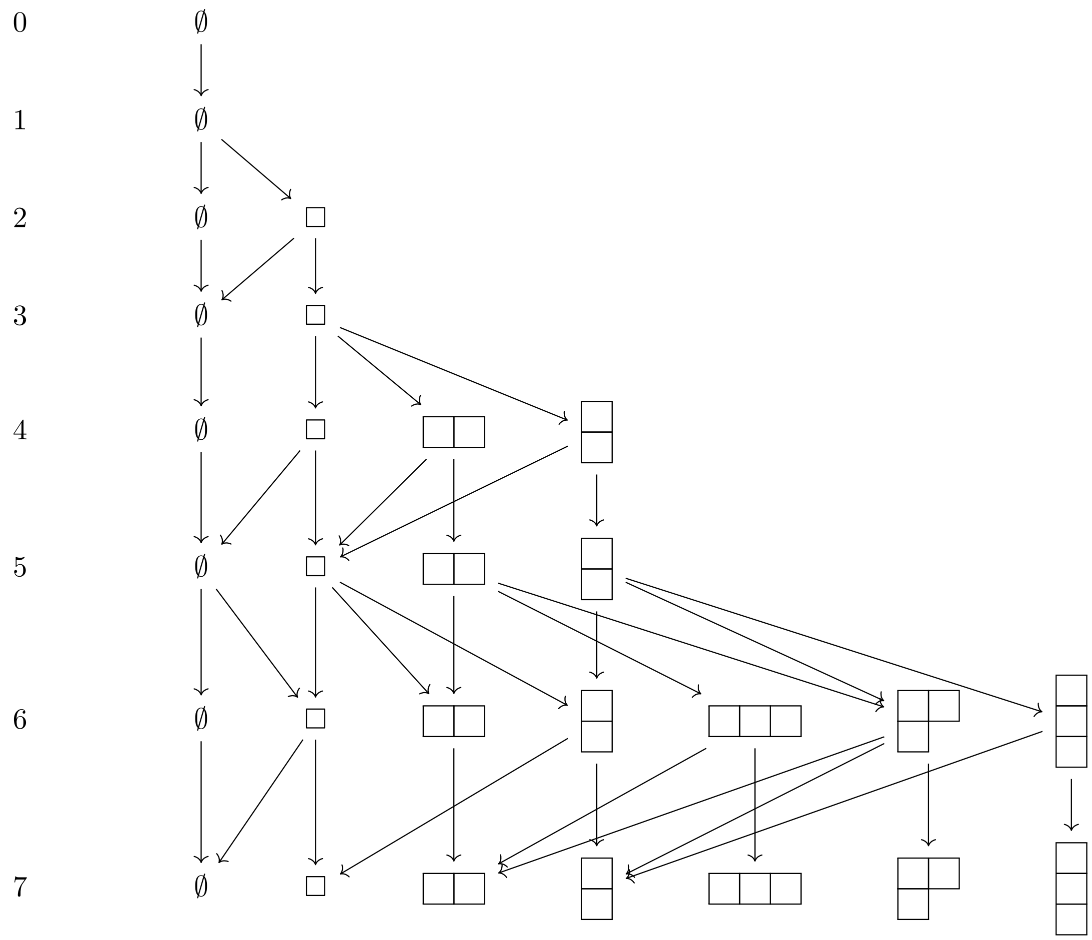
is multiplicity free.

In accordance to the case of the symmetric groups, we can thus define the branching graph for this chain, as well as its GZ basis.

Definition 3.40. *We let $\hat{B}$ denote the graded directed graph, with levels indexed by nonnegative integers $n \geq 0$ such that:*

1. *The vertices on level i are indexed by $\mathbb{Y}_{\leq \lfloor \frac{i}{2} \rfloor} \times \{i\}$*
2. *For i even, an edge $(\lambda, i) \to (\mu, i+1)$ exists if and only if $\lambda = \mu$ or $\mu = \lambda - \square$.*
3. *For i odd, an edge $(\lambda, i) \to (\mu, i+1)$ exists if and only if $\lambda = \mu$ or $\mu = \lambda + \square$.*

Example 3.41. *The first 8 levels of the graph are illustrated below.*



Denote the set of paths in $\hat{B}$ with endpoint λ by $\mathrm{Path}(\lambda)$. The above graph, truncated at level n gives the branching graph of the multiplicity free chain from above. This is described by the following theorem.

Theorem 3.42. *Let n be a nonnegative integer and $t \notin \{0, 1, 2, \ldots, 2\lfloor \frac{n}{2} \rfloor - 2\}$. Then for all $i \leq n$:*

1. *The $i-th$ level of $\hat{B}$ provides an indexing set for the set of inequivalent simple modules of $P_i(t)$. We denote by $L(\lambda, i)$ the simple module corresponding to $(\lambda, i) \in \mathbb{Y}_{\leq \lfloor \frac{i}{2} \rfloor}$.*

2. *The Branching rule is given by the graph $\hat{B}$. In other words, for a simple $P_i(t)$-module $L(\lambda, i)$, the following holds:*
$$Res^{P_i(t)}_{P_{i-1}(t)} L(\lambda, i) = \bigoplus_{(\mu, i-1) \to (\lambda, i)} L(\mu, i-1).$$

It is clear from the above theorem and the consideration of the analogue for the symmetric groups, that $\dim L(\lambda, i) = \mid \mathrm{Path}(\lambda, i) \mid$. Moreover, we can use the branching rule to obtain a decomposition of $L(\lambda, i)$ in one dimensional vector spaces indexed by the set $\mathrm{Path}(\lambda, i)$. Choosing a non-zero vector for each of these summands, we obtain a basis $\{v_T \ : \ T \in \mathrm{Path}(\lambda, i)\}$ of $L(\lambda, i)$ to which we refer again as a GZ basis. We now describe how the GZ basis of any $L(\lambda, i)$ simultaneously diagonalizes the action of the Jucys Murphy elements and describe the corresponding eigenvalues.

Definition 3.43. *For $(\lambda, i) \in \hat{B}$, and a path $T = (\lambda^0, 0) \to (\lambda^1, 1) \to \ldots \to (\lambda^i, i)$ to λ we define the following for $j \leq i$:*

1. *If j odd, we set*

$$cont(T,j) = \begin{cases} t-|\lambda^i|, \text{ if } \lambda^i = \lambda^{i-1} \\ c(\Box), \text{ if } \lambda^i = \lambda^{i-1} + \Box. \end{cases}$$

2. *If j even, we set*

$$cont(T,j) = \begin{cases} |\lambda^i|, \text{ if } \lambda^i = \lambda^{i-1} \\ t - c(\Box), \text{ if } \lambda^i = \lambda^{i-1} - \Box. \end{cases}$$

We call the above numbers the content of path T at step j.

The above numbers play exactly the same role as contents do for the Jucys Murphy elements of the symmetric groups. In particular, they are the eigenvalues for the action of the JM elements in the GZ basis of an irreducible $P_n(t)$ module as highlighted below.

Theorem 3.44. *Suppose we have a nonnegative integer n and $t \notin \{0, 1, \ldots, 2\lfloor \frac{n}{2} \rfloor - 2\}$. Also let $L(\lambda, n)$ be any simple $P_n(t)$ module with GZ-basis $\{v_T \mid T \in \mathrm{Path}(\lambda, n)\}$. The GZ basis simultaneously diagonalizes the action of the YJM elements $x_1, x_2, \ldots, x_n$. In particular, we have that*

$$x_i v_T = cont_T(t, i) v_T$$

for any $i = 1, \ldots, n$ and $T \in \mathrm{Path}(\lambda, n)$.

For proofs of the results stated in this section, see [6]. Thus we see that in the semisimple case, the representation theory of $P_n(t)$ is again controlled by YJM elements.

Remark 3.45. *In fact, even in the non-semisimple case, Creedon and De Visscher [4] proved that YJM elements provide a big enough commutative subalgebra of elements that seperate blocks via the use of central characters.*

4 Deligne's interpolation category $\underline{Rep}(S_t;\mathbb{C})$.

4.1 Construction and basis properties.

Definition 4.1. *Let $\underline{Rep}_1(S_t;\mathbb{C})$ denote the additive envelope of $\mathcal{P}ar_t$. Deligne's interpolation category $\underline{Rep}(S_t;\mathbb{C})$ is then defined as the Karoubian envelope (or pseudo-abelian completion) of $\underline{Rep}_1(S_t;\mathbb{C})$. This category has objects pairs (A,e), where A is an object in $\underline{Rep}_1(S_t;\mathbb{C})$ and $e \in \mathrm{End}_{\underline{Rep}_1(S_t;\mathbb{C})}(A)$ is an idempotent. Morphisms are given by $\mathrm{Hom}_{\underline{Rep}(S_t;\mathbb{C})}((A,e),(B,f)) = f\,\mathrm{Hom}_{\underline{Rep}_1(S_t;\mathbb{C})}(A,B)e$. The composition is the induced composition from the additive envelope $\underline{Rep}_1(S_t;\mathbb{C})$.*

The following proposition stems from general properties of the additive and Karoubian envelopes.

Proposition 4.2. *(a) $\underline{Rep}(S_t;\mathbb{C})$ is a tensor category with the obvious extensions of the tensor structure of $\mathcal{P}ar$.*

(b) Given a nonnegative integer n and $e \neq 0$ an idempotent in $P_{2n}(t)$, the object $([n],e)$ in $\underline{Rep}(S_t;\mathbb{C})$ is indecomposable if and only if e is primitive. Moreover, every indecomposable object in $\underline{Rep}(S_t;\mathbb{C})$ is isomorphic to one of the form $([n],e)$.

(c) Two primitive idempotents $e, e' \in P_{2n}(t)$ give isomorphic objects $([n],e)$ and $([n],e')$ if and only if they are conjugate in $P_{2n}(t)$.

(d) $\underline{Rep}(S_t;\mathbb{C})$ is a Krull-Schmidt category in the sense that every object can be decomposed as a finite direct sum of indecomposable objects in an essentially unique way up to permutations.

Remark 4.3. *The category $\mathcal{P}ar_t$ should be thought of as the category studying all the partition algebras $P_{2n}(t)$ simultaneously. Via Schur Weyl duality combined with the fact that every irreducible $\mathbb{C}[S_n]$ module appears as a summand of a high enough power of the natural representation, we created this category to interpolate the representations of S_n. In particular, the category $\mathcal{P}ar_t$, did not have direct sums and more importantly, direct summands. Through this construction we endowed it the properties it was missing and thus now we can somewhat safely say that $\underline{Rep}(S_t;\mathbb{C})$ interpolates the category of finite dimensional $\mathbb{C}[S_n]$-modules. This idea will become clear once we have defined the interpolation functor. Notice that whenever we choose nonnegative integer values for t, naively one might expect that the category we made is going to be $Rep(S_t,\mathbb{C})$. That is not the case, since we made our category slightly bigger starting with $\mathcal{P}ar_t$, as by theorem 3.16, morphisms between tensor products of the natural representation do not have a unique presentation as $f(\pi)$ for π a partition diagram.*

We are now in a position to classify the indecomposable objects in the Deligne category $\underline{Rep}(S_t;\mathbb{C})$. By theorem 3.29, if λ is a Young diagram of arbitrary size, then it corresponds to a primitive idempotent $e_\lambda \in P_{2|\lambda|}(t)$, where for $t=0$ we set $e_\emptyset = id_0 \in P_0(0)$. Remember that e_λ is unique up to conjugation and thus the object $L(\lambda) = ([|\lambda|], e_\lambda)$ is an indecomposable object in $\underline{Rep}(S_t;\mathbb{C})$, well defined up to isomorphism.

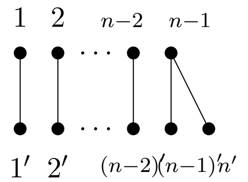
Lemma 4.4. *For a fixed $n \geq 0$, the assignment $\lambda \mapsto L(\lambda)$ induces a bijection*

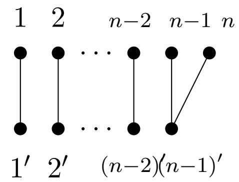
$$\left\{\begin{array}{l}\textit{nonzero indecomposable objects in}\\ \textit{in } \underline{Rep}(S_t;\mathbb{C})\textit{ of the form } ([m],e)\\ \textit{with } m \leq n, \textit{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{Young diagrams } \lambda\\ \textit{with } 0 \leq |\lambda| \leq n\end{array}\right\}.$$

For the above bijection, we also have:

- *If λ is a Young diagram with $n \geq |\lambda| > 0$, then there exists an idempotent $e \in P_n(t)$ such that $([n], e) \cong L(\lambda)$.*
- *If $t \neq 0$, then there exists an idempotent $e \in P_n(t)$ such that $([n], e) \cong L(\emptyset)$.*
- *If $t = 0$, then $([0], id_0)$ is the unique object of the form $([m], e)$, isomorphic to $L(\emptyset)$.*

Proof. We first assume that $t \neq 0$ and proceed by induction on n. For $n = 0$ the statement becomes trivial. For $n = 1$, write a, a' for the unique objects in $P_{1,0}$ and $P_{0,1}$ respectively and let $\frac{1}{t}\pi = f \in P_2(t)$ be the idempotent from section 3.1. Then we have that $\{f, 1 - f\}$ is a complete set of pairwise non-conjugate primitive idempotents in $P_2(t)$. Thus the indecomposable objects $([1], f), ([1], 1 - f)$ are not isomorphic. Additionally, $fa'id_0 : L(\emptyset) \cong ([0], id_0) \to ([1], f)$ is an isomorphism with inverse $\frac{1}{t}id_0af$. Therefore, the objects $([1], f) \cong L(\emptyset)$ and $([1], 1 - f) \cong L(\square)$ form a complete list of nonzero pairwise non-isomorphic indecomposable objects in the Deligne category with $m \leq 1$. Suppose now that $n > 1$ and let λ be a Young diagram with $0 \leq |\lambda| < n$. By induction we can find a primitive idempotent $f_\lambda \in P_{2n-2}(t)$ such that $([n-1], f_\lambda) \cong L(\lambda)$ such that $\{([n-1], f_\lambda) \mid 0 \leq |\lambda| < n\}$ is a complete set of nonzero pairwise non-isomorphic indecomposable objects in $\underline{Rep}(S_t; \mathbb{C})$ of the form $([m], e)$ with $m < n$. We elevate the above idempotents to $P_{2n}(t)$ as follows. Let

$\phi_n =$ 

, $\phi'_n =$ 

and for every such λ, set $\hat{f}_\lambda = \phi_n f_\lambda \phi'_n$. Then by noticing that $\phi'_n \phi_n = id_{n-1}$ we get that

$$\hat{f}_\lambda^{\,2} = \phi_n f_\lambda \phi'_n \phi_n f_\lambda \phi'_n = \hat{f}_\lambda \in P_{2n}(t)$$

is an idempotent and that $f_\lambda \phi'_n \hat{f}_\lambda : ([n], \hat{f}_\lambda) \to ([n-1], f_\lambda)$ is an isomorphism, with inverse $\hat{f}_\lambda \phi_n f_\lambda$. But this means $([n], \hat{f}_\lambda)$ is indecomposable, or equivalently $\hat{f}_\lambda \in P_{2n}(t)$ is a primitive idempotent. Moreover, by passing through ϕ'_n, ϕ_n we connected n and $n-1$, so that $\hat{f}_\lambda = e\hat{f}_\lambda \in (e)$. Therefore, $\hat{f}_\lambda$ is not conjugate to e_μ for any Young diagram μ of size n by lemma 3.26 and theorem 3.29. It follows that the set $\{\hat{f}_\lambda \mid 0 \leq |\lambda| < n\} \cup \{e_\lambda \mid |\lambda| = n\}$ is a set of pairwise non conjugate primitive idempotents in $P_{2n}(t)$. As this is indexed by Young diagrams of size at most n, it follows from theorem 3.29 that it is a complete set of pairwise non-conjugate primitive idempotents in $P_{2n}(t)$, thus yielding that the objects $([n], \hat{f}_\lambda) \cong L(\lambda)$ for $0 \leq |\lambda| < n$ together with $([n], e_\lambda) \cong L(\lambda)$ for $|\lambda| = n$ are a complete list of nonzero pairwise non-isomorphic indecomposable objects in $\underline{Rep}(S_t; \mathbb{C})$ of the form $([m], e)$ with $m \leq n$. This completes the proof for $t \neq 0$.
Suppose now that $t = 0$. Then every composition $([0], id_0) \to ([m], e) \to ([0], id_0)$ is equal to the zero map unless $m = 0$ and $e = id_0$ and thus $([0], id_0) \cong L(\emptyset)$ is unique. The remainder of the lemma follows again by induction on n with the same arguments, this time using the second statement of theorem 3.29 for primitive idempotents in $P_{2n}(0)$. $\square$

Theorem 4.5. *The assignment $\lambda \mapsto L(\lambda)$ induces an isomorphism*

$$\left\{\begin{matrix} \textit{nonzero indecomposable objects in} \\ \textit{in } \underline{Rep}(S_t; \mathbb{C}) \textit{ up to isomorphism} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{Young diagrams } \lambda \\ \textit{of arbitrary size.} \end{matrix}\right\}.$$

4.2 The interpolation functor

In this section, we always assume that d is a nonnegative integer. We make the connection between the Deligne category $\underline{Rep}(S_d;\mathbb{C})$ and $Rep(S_d;\mathbb{C})$ clear. In particular, we will show that we can recover $Rep(S_d;\mathbb{C})$ by the semisimplification of $\underline{Rep}(S_d;\mathbb{C})$, yielding that for nonnegative integers d, the category $\underline{Rep}(S_d;\mathbb{C})$ is not a semisimple category. Remember the map $f : \mathbb{C}P_{n,m} \to \mathrm{Hom}_{S_d}(V_d^{\otimes n}, V_d^{\otimes m})$ from definition 3.9.

Definition 4.6 (Interpolation functor)**.** *We define the functor $\mathcal{F} : \underline{Rep}(S_d;\mathbb{C}) \longrightarrow Rep(S_d;\mathbb{C})$ on indecomposable objects by $\mathcal{F}([n],e) = f(e)(V_d^{\otimes n})$ and on morphisms $a : ([n],e) \to ([n'],e')$ by $\mathcal{F}(a) = f(a)$.*

This functor preserves the tensor structures of the above categories and thus is a tensor functor.

Proposition 4.7. *$\mathcal{F}$ is surjective on both objects and morphisms.*

Proof. The fact that it is surjective on morphisms comes from the fact that the map f is always surjective. Moreover, for primitive idempotent $e \in P_n(t)$ we have that $f(e)(V_d^{\otimes n})$ is an indecomposable (and thus irreducible) direct summand of $V_d^{\otimes n}$. This combined with the fact that every irreducible $\mathbb{C}[S_n]$-module is a direct summand of a high enough tensor power of the natural representation concludes the proof that it is also surjective on objects, since $\underline{Rep}(S_t;\mathbb{C})$ has finite direct sums (and is Krull-Schmidt, so that every object decomposes as a finite direct sum of indecomposables). $\square$

Remark 4.8. *As we have already hinted, $\mathcal{F}$ is not going to be an equivalence of categories, see remark 4.3. The amount to which this functor fails to be an equivalence is measured by the so called negligible morphisms, defined below.*

Definition 4.9 (Negligible Morphisms)**.** *A morphism $f : X \to Y$ in a tensor category is called negligible, if $tr(fg) = 0$, for every $g : Y \to X$. We set $\mathcal{N}(X,Y) := \{f : X \to Y \mid f \text{ is negligible}\}$.*

In partition algebras, to find negligible morphisms, we just find morphisms that have zero trace when we compose them with any of the diagrammatic basis elements. By the characterization of the kernel of f by spans of orbit basis elements, they are natural candidates for negligible morphisms.

Example 4.10.

1. *For $t = 0$, we have that the only non-negligible morphisms in $\mathcal{P}ar_0$ are nonzero scalars of id_0.*
2. *Suppose $\pi : [1] \to [1]$ is the partition having only singletons as parts. Then $O(\pi) : [1] \to [1]$ is equal to $O(\pi) = \pi - id_1$. Since $tr(\pi) = t = tr(id_1)$ it holds that $tr(O(\pi)) = 0$. Moreover, since $O(\pi)\pi = (t-1)\pi$, we have $tr(O(\pi)\pi) = t(t-1)$. Thus $O(\pi)$ is negligible if and only if $t = 0, 1$.*

Proposition 4.11. *On any tensor category the following hold:*

1. *$\mathcal{N}$ is a tensor ideal.*
2. *The image under a full tensor functor of a morphism f is negligible if and only if f is negligible.*

Since the category $Rep(S_d;\mathbb{C})$ is semisimple, it does not contain nonzero negligible morphisms and thus the functor $\mathcal{F}$ sends every negligible morphism to zero. Thus $\mathcal{F}$ induces a functor $\overline{\mathcal{F}} : \underline{Rep}(S_d;\mathbb{C})/\mathcal{N} \longrightarrow Rep(S_d;\mathbb{C})$. Additionally, for nonnegative integer values of t, the category $\underline{Rep}(S_t;\mathbb{C})$ is not semisimple since it contains negligible morphisms.

Remark 4.12. *More generally, in $\mathcal{P}ar_t$ it holds that*

$$\mathcal{N}([n],[m]) = \begin{cases} span\{O(\pi) \mid \pi \text{ has more than } t \text{ parts}\}, \ t \geq 0, \\ 0, \ otherwise. \end{cases}$$

For $t \geq 0$, this follows by the definition of $\mathcal{F}$, together with proposition 4.11, since $\mathcal{F}$ is full. If $t \notin \mathbb{Z}_{\geq 0}$, we will eventually show that the Deligne category is semisimple, implying that it does not contain any negligible morphisms.

Theorem 4.13. *$\overline{\mathcal{F}}$ is faithful and thus induces an equivalence of categories $\underline{Rep}(S_d;\mathbb{C})/\mathcal{N} \cong Rep(S_d;\mathbb{C})$.*

Proof. Suppose that two morphisms a, b have the same image under $\mathcal{F}$. Then $a - b$ is in the kernel of f, so by remark 4.12 we get that $a - b \in \mathcal{N}$, yielding $a = b$ in $\underline{Rep}(S_d;\mathbb{C})/\mathcal{N}$. □

We end by investigating how the functor $\mathcal{F}$ interacts with indecomposable objects.

Definition 4.14. *For arbitrary $t \in \mathbb{C}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ a Young diagram, we define the $t-$completion of λ to be*

$$\lambda(t) = (t - |\lambda|, \lambda_1, \lambda_2, \ldots).$$

Proposition 4.15. *Suppose d is a nonnegative integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a Young diagram. If the d-completion of λ is again a Young diagram, then $\mathcal{F}(L(\lambda)) = L_{\lambda(d)}$, otherwise $\mathcal{F}(L(\lambda)) = 0$.*

Proof. Idea : One can prove that the indecomposable objects $L(\lambda)$ in $\underline{Rep}(S_d;\mathbb{C})$ of nonzero categorical dimension are exactly those for which the d-completion of λ is again a Young diagram. □

4.3 Study of Blocks and YJM like elements for $\underline{Rep}(S_t;\mathbb{C})$.

Comes and Ostrik defined morphisms in $\underline{Rep}(S_t;\mathbb{C})$ interpolating the action of the sum of all $r-$cycles on representations of the symmetric groups. These elements have a striking resemblance to the classic Jucys Murphy elements from Chapter 1, in the sense that they form a large commutative subalgebra and their actions completely characterize blocks in $\underline{Rep}(S_t;\mathbb{C})$. However, there is a significant difference. Rather than these elements acting diagonally, they act with a pure Jordan block, yielding another indication of the non-semisimplicity.

Let $\Omega_{r,d} \in \mathbb{C}[S_d]$ denote the sum of all $r-$cycles in S_d. Notice that these elements are in the center of the group algebra as we are summing up all the elements in a conjugacy class. Thus their action on $V_d^{\otimes n}$ yields an element of $\mathrm{End}_{S_d}(V_d^{\otimes n})$. Since the map f is an isomorphism whenever $d \geq n + m$, the following is well defined.

Definition 4.16. *For nonnegative integers r, n and d with $r \leq d$ and $2n \leq d$, let $C_n^r(d)$ be the unique element of $P_{2n}(d)$ such that $f(C_n^r(d))$ is given by the action of $\Omega_{r,d}$.*

Our goal is to define elements in $P_{2n}(t)$ that agree with the elements $C_n^r(d)$ for large integer values of t. We are able to do so because as we will see, $C_n^r(d)$ depends polynomially on d.

Proposition 4.17. *Let $n \geq 0$ and $\pi \in P_{n,n}$. Fix the following notation*

- *Let a denote the number of parts of π.*
- *Let b denote the number of parts π_k of π such that $j, j' \in \pi_k$ for some $j \in [n]$.*
- *Let c denote the number of connected components in the trace diagram of π.*

- *Suppose r, d are positive integers with $d \geq r$ and $i, i' \in [n, d]$ such that the $(i, i')-$coloring of π is perfect. Let $S(\pi, r, d)$ denote the number of $r-$cycles $\sigma \in S_d$ such that $\sigma(i_j) = i'_j$ for all $j \in [n]$.*

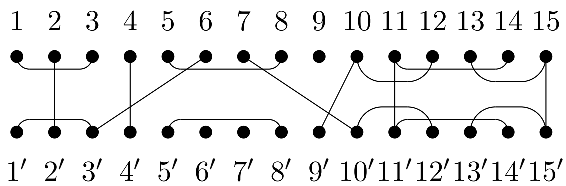
Then for a positive integer r the following hold:

1. *If $S(\pi, r, d) \neq 0$ for some integer $d \geq r$, then $S(\pi, r, d') \neq 0$ for all $d' \geq d$.*
2. *If $S(\pi, r, d) \neq 0$, then*

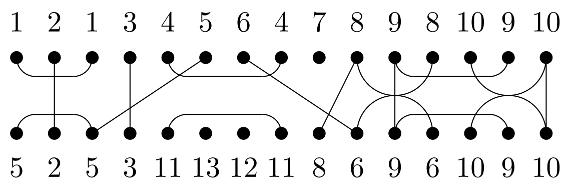
$$S(\pi, r, d) = \frac{(r-a+c-1)!}{(r-a+b)!} \prod_{k=1}^{r+b-a} (d-r-b+k).$$

Proof. The first assertion is clear since if an $r-$cycle in S_d satisfies the assertion, then the same cycle considered as an element in $S_{d'}$ also satisfies the assertion. For the second, since $S(\pi, r, d)$ does not depend on the choice of perfect coloring of π, thus we may assume $\{i_j, i'_j \mid j \in [n]\} = \{1, 2, \ldots, a\}$. Then for $x, y \in [a]$ we write $x \to y$ if $x = i_j$ and $y = i'_j$ for some $j \in [n]$. Generate the weakest equivalence relation on $[a]$ such that $x \to y$ implies $x \sim y$. The equivalence classes correspond to the connected components in the trace diagram of π. Thus there are exactly c equivalence classes. Now assume that $S(\pi, r, d) \neq 0$. Then the following two implications are clear. If $x \to y$ and $x \to y'$ then $y = y'$. If $x \to y$ and $x' \to y$ then $x = x'$. In particular, there are precisely b equivalence classes with exactly one element. Thus the $c - b$ equivalence classes with more than one element must account for $a - b$ elements of $[a]$. Suppose that an $r-$cycle $\sigma \in S_d$ satisfies $\sigma(i_j) = i'_j$ for all j. Then among the elements in $[a]$, the cyclic arrangement corresponding to σ contains precisely the $a - b$ elements in equivalence classes with more than one element. Additionally, if $x \to y$ for $x \neq y$, then x, y must be adjacent in the cyclic arrangement corresponding to σ. Thus σ is determined by a cyclic arrangement of the following $r - a + c$ items: $c - b$ equivalence classes with more than one element and $r - a + b$ elements of $\{a+1, \ldots, d\}$. Finally, if such a cycle exists, then any choice of $r - a + b$ elements of $\{a + 1, \ldots, d\}$ arranged cyclically with the equivalence classes containing more than one element determines an $r-$cycle in S_d interchanging i_j, i'_j. Since there are $(r - a + c - 1)!$ cyclic arrangements of $r - a + c$ elements and $\binom{d-a}{r-a+b}$ different $r - a + b$ element subsets of $\{a + 1, \ldots, d\}$, it follows that $S(\pi, r, d) = (r - a + c - 1)!\binom{d-a}{r-a+b}$ which is equivalent to the desired formula. $\square$

Example 4.18. *Consider*



Then $a = 13, b = 4, c = 7$. For $d \geq 13$, let (i, i') be the perfect coloring of π shown below:



In order for an $r-$cycle $\sigma \in S_d$ to satisfy $\sigma(i_j) = i'_j$ for all j, it must first fix the $b = 4$ numbers $2, 3, 9, 10$ and map $1 \mapsto 5 \mapsto 13$, $4 \mapsto 11$, $7 \mapsto 8 \mapsto 6 \mapsto 12$. Such a cycle exists only if $r \geq 9 = a - b$

and $d \geq r+4 = r+b$. In that case, we count the cycles as follows. We consider the 4 non trivial equivalence classes as 4 different objects and thus in order to count the cycles, we have to count the cyclic arrangements of these 4 elements together with the rest $r-9$ elements in the cycle. Thus we have $(r-9+4-1)! = (r-6)!$ such cyclic arrangements. Since for these $r-9$ elements we have $d-9-3 = d-12$ choices, we can choose them in $\binom{d-12}{r-9}$ ways. In total we have

$$S(\pi, r, d) = (r-6)!\binom{d-12}{r-9}$$

which after rearrangement is the same with the formula given in the proposition above.

With the above in mind, we can define elements in the partition algebras $P_{2n}(t)$ interpolating $\Omega_{r,d}$.

Definition 4.19. *For $t \in \mathbb{C}$ and integers $r > 0, n \geq 0$, define $\omega_n^r(t) \in P_n(t)$ as follows.*

$$\omega_n^r(t) = \sum_{\pi \in P_{n,n}} q_{\pi,r,t} O(\pi)$$

where $O(\pi)$ are the elements of the orbit basis, and

$$q_{\pi,r,t} = \begin{cases} 0, \ \textit{if} \quad S(\pi,r,d) = 0 \quad \textit{for all} \quad d > 1. \\ \frac{(r-a+c-1)!}{(r-a+b)!} \prod_{k=1}^{r+b-a} (t-r-b+k), \ \textit{otherwise}. \end{cases}$$

The idea for the above elements is that the coordinates in the orbit basis can be found by the action of the image under f of $C_n^r(d)$, as the following proposition justifies.

Proposition 4.20. *1. Fix integers $r > 0, n \geq 0$. Whenever d is a sufficiently large integer, $\omega_n^r(d) = C_n^r(d)$. In other words, for large integers d the map $f(\omega_n^r(d))$ is given by the action of $\Omega_{r,d} \in \mathbb{C}[S_d]$.*

2. Fix $t \in \mathbb{C}$ and an integer $r > 0$. The morphisms $\omega_n^r(t) : [n] \to [n]$ for each nonnegative integer n form an endomorphism of the identity functor in $\mathcal{P}ar$. In particular, the elements $\omega_n^r(t)$ are central in $P_{2n}(t)$ for every $t \in \mathbb{C}$ and $n \geq 0$.

Proof. For $i, i' \in [n,d]$, let $\pi(i,i') \in P_{2n}(t)$ denote the partition having (i,i') as a perfect coloring. Then the action of $\Omega_{r,d}$ on $V_d^{\otimes n}$ maps the basis vector v_i to $\sum_{i' \in [n,d]} S(\pi(i,i'), r, d) v_{i'}$. On the other hand, since $f(O(\pi))_{i'}^{i} = 1$ if (i,i') is a perfect coloring of π and is equal to zero otherwise, we have that $f(\omega_n^r(t))$ maps v_i to $\sum_{i' \in [n,d]} q_{\pi(i,i'),r,d} v_{i'}$. By proposition 4.17, for sufficiently large integers d it holds that $S(\pi(i,i'), r, d) = q_{\pi(i,i'),r,d}$ proving the first claim.
For the second assertion, let $\mu \in P_{n,m}$. For an integer $d > r$, since $\Omega_{r,d}$ is a central element we know that $f(\mu) : V_d^{\otimes n} \to V_d^{\otimes m}$ commutes with the action of $\Omega_{r,d}$. Hence, for large integers d, it holds that

$$f(\mu \omega_n^r(d)) = f(\omega_m^r(d)\mu)$$

Since f is an isomorphism for sufficiently large integers, we have

$$\omega_m^r(d)\mu = \mu\omega_n^r(d)$$

Thus, if we set $\omega_m^r(t)\mu = \sum_{\pi \in P_{n,m}} a_\pi(t)\pi$ and $\mu\omega_n^r(t) = \sum_{\pi \in P_{n,m}} a'_\pi(t)\pi$, then we have shown that for sufficiently large integers t the polynomials $a_\pi(t)$ and $a'_\pi(t)$ are equal. Thus they are always equal. $\square$

We are in a position to study how these elements $\omega_n^r(t)$ interact with indecomposable objects in $\underline{Rep}(S_t;\mathbb{C})$.

Proposition 4.21. *Fix $t \in \mathbb{C}$ along with integers $r > 0, n \geq 0$. If e is a primitive idempotent in $P_n(t)$, then there exists $\xi \in \mathbb{C}$ and positive integer m such that*

$$(\omega_n^r(t) - \xi)^m e = 0.$$

Proof. Since $P_{2n}(t)$ is finite dimensional, there exists a minimal polynomial $a \in \mathbb{C}[x]$ such that $a(\omega_n^r(t))e = 0$. We first show that a is a power of an irreducible polynomial in $\mathbb{C}[x]$. Suppose we can factorize $a(x) = b(x)c(x)$ for relatively prime monic polynomials b, c. Then by Bezout, there exist polynomials g, h with $deg(g) < deg(c)$ and $deg(h) < deg(b)$, such that

$$g(x)b(x) + h(x)c(x) = 1.$$

Hence $g(\omega_n^r(t))b(\omega_n^r(t))e + h(\omega_n^r(t))c(\omega_n^r(t))e = e$ is a decomposition of e into orthogonal idempotents since $\omega_n^r(t)$ is central and $a(\omega_n^r(t))e = 0$. Since e is primitive, this implies that either $g(\omega_n^r(t))b(\omega_n^r(t))e = 0$ or $h(\omega_n^r(t))c(\omega_n^r(t))e = 0$. By minimality of a, this implies $g(x) = 0$ or $h(x) = 0$ which in turn yields $b(x) = 1$ or $c(x) = 1$. So indeed, a is a power of an irreducible polynomial in $\mathbb{C}[x]$. Thus there exist $\xi \in \mathbb{C}$ and positive integer m such that $a(x) = (x - \xi)^m$. $\square$

Theorem 4.22 (Frobenius formula)**.** *Fix integers $d \geq r > 1$. Given a Young diagram $\lambda = (\lambda_0, \lambda_1, \dots)$ of size d, set $\mu_i = \lambda_i - i$ for every $i \geq 0$. Then*

$$\Omega_{r,d}.S^\lambda = \xi_{r,k}^\lambda S^\lambda$$

where

$$\xi_{r,k}^\lambda = \frac{1}{r}\sum_{k=0}^{r}(\mu_i + k - 1)(\mu_i + k - 2)\cdots(\mu_i + k - r)\prod_{\substack{0 \leq j \leq k \\ j \neq i}} \frac{\mu_i - \mu_j - r}{\mu_i - \mu_j}.$$

We end this section by showing that the scalar ξ from proposition 4.21 is given by the Frobenius formula.

Theorem 4.23. *Fix $t \in \mathbb{C}$, a positive integer r and a Young diagram λ with positive integer k such that $\lambda_{k+1} = 0$. Then for the indecomposable $L(\lambda) = ([n], e) \in \underline{Rep}(S_t;\mathbb{C})$ we have that*

$$(\omega_n^r(t) - \xi_{r,k}^{\lambda(t)})^m e = 0$$

for some positive integer m.

Proof. By the classification of indecomposables, we may assume $n = |\lambda|$. Let ξ and m be as in proposition 4.21, so that $(\omega_n^r(t) - \xi)^m e = 0$. We push this through the quotient map $p_n : P_{2n}(t) \twoheadrightarrow \mathbb{C}[S_n]$ to get $(p(\omega_n^r(t)) - \xi)^m c_\lambda = 0$ in $\mathbb{C}[S_n]$, where c_λ is the primitive idempotent corresponding to λ. Since $p(\omega_n^r(t))$ is central, it holds that

$$p(\omega_n^r(t))c_\lambda = \xi c_\lambda.$$

By the definition of these elements, it becomes imminent that ξ depends polynomially on t.
Now let d be a positive integer such that $d - |\lambda| \geq \lambda_1$. Passing the equation $(\omega_n^r(d) - \xi)^m e = 0$ through the interpolation functor $\mathcal{F}$ yields $(\Omega_{r,d} - \xi)^m c_{\lambda(d)} = 0$. Using the Frobenius formula now yields $\xi = \xi_{r,k}^{\lambda(t)}$ whenever $t = d$ is a sufficiently large integer. Since ξ depends polynomially on t and $\xi_{r,k}^{\lambda(t)}$ is a rational function in t such that they agree in infinitely many values of t, they must be equal for all $t \in \mathbb{C}$. $\square$

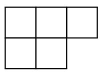
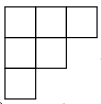
4.4 Description of Blocks

Suppose $\mathcal{C}$ is an arbitrary $\mathbb{C}$-linear Krull-Schmidt category. We consider the weakest equivalence relation on the set of isomorphism classes of indecomposable objects in $\mathcal{C}$, where two indecomposable objects are equivalent whenever there is a nonzero morphism between them. We call the equivalence classes in this relation blocks. We will also refer to this term for a full subcategory whose objects are direct sums of indecomposable objects in a single block. We say that a block is trivial, if it contains only one indecomposable object and the endomorphism ring of that object is $\mathbb{C}$. For example, in any semisimple category over any algebraically closed field, all blocks are trivial. Non trivial blocks essentially show how far our category falls from being semisimple. In this section, we describe the blocks of $\underline{Rep}(S_t;\mathbb{C})$.

Definition 4.24. *For $t \in \mathbb{C}$ and a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$, set*

$$\mu_\lambda(t) = (t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \ldots).$$

For Young diagrams λ and λ' write $\mu_\lambda(t) = (\mu_0, \mu_1, \ldots)$ and $\mu_{\lambda'}(t) = (\mu'_0, \mu'_1, \ldots)$. We say $\lambda \overset{t}{\sim} \lambda'$ whenever there exists a bijection $a : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\mu_i = \mu'_{a(i)}$ for all $i \geq 0$.

Example 4.25. *Let $\lambda =$*  *and $\lambda' =$*  *. Then $\mu_\lambda(t) = (t-5, 2, 0, -3, -4, -5, -6, \ldots)$ and $\mu_{\lambda'}(t) = (t-6, 2, 0, -2, -3, -4, -5, -6, \ldots)$. Thus we see that for $t = 3$ one is a permutation of the other, or equivalently $\lambda \overset{3}{\sim} \lambda'$.*

For each $t \in \mathbb{C}$ the relation $\overset{t}{\sim}$ defines an equivalence relation on the set of all Young diagrams and therefore on the indecomposable objects of $\underline{Rep}(S_t;\mathbb{C})$. We can now see how the action of these $\omega_n^r(t)$ seperates blocks.

Lemma 4.26. *Suppose λ, λ' are Young diagrams and $k > 0$ is such that $\lambda_{k+1} = \lambda'_{k+1} = 0$. Then the following hold:*

(a) If $L(\lambda)$ and $L(\lambda')$ are in the same block in $\underline{Rep}(S_t;\mathbb{C})$, then $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for all $r > 0$.

(b) If $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for all $r > 0$, then $\lambda \overset{t}{\sim} \lambda'$.

Proof. For the first part, let n, n' be nonnegative integers and $e \in P_{2n}(t), e' \in P_{(2n)'}(t)$ be idempotents with $L(\lambda) \cong ([n], e)$ as well as $L(\lambda') \cong ([n'], e')$. Moreover, fix $r > 0$ and write $\omega = \omega_n^r(t)$, $\omega' = {\omega'}_n^r(t)$, as well as $\xi = \xi_{r,k}^{\lambda(t)}$, $\xi' = \xi_{r,k}^{\lambda'(t)}$. Additionally let m a positive integer such that $(\omega - \xi)^m e = (\omega' - \xi')^m e = 0$. Now suppose that $\xi \neq \xi'$. Then there exist polynomials $p, q \in \mathbb{C}[x]$ such that

$$p(x)(x-\xi)^m + q(x)(x-\xi')^m = 1.$$

Thus given any morphism $\phi : ([n'], e') \to ([n], e)$ while remembering that the $\omega_n^r(t)$ form an endomorphism of the identity functor, together with $\phi e' = \phi = e\phi$ we get

$$\phi = p(\omega)(\omega-\xi)^m \phi + q(\omega)(\omega-\xi')^m \phi = p(\omega)(\omega-\xi)^m e\phi + q(\omega)(\omega-\xi')^m \phi e'$$

which in turn is equal to

$$p(\omega)(\omega-\xi)^m e\phi + \phi q(\omega')(\omega'-\xi')^m e' = 0$$

Thus if there exists a nonzero morphism between the two indecomposable objects, then $\xi = \xi'$.
For the second part, notice that $\xi_{r,k}^{\lambda(t)}$ is symmetric in $\mu_0, \ldots, \mu_k$. Thus by multiplying with $\prod_{0 \leq i < j \leq k} (\mu_i -$

μ_j) we get an antisymmetric polynomial in $\mu_0, \ldots, \mu_k$. But every antisymmetric polynomial is divisible by the Vandermonde product $\prod_{0 \le i < j \le k} (\mu_i - \mu_j)$, yielding that $\xi_{r,k}^{\lambda(t)}$ is a symmetric polynomial in $\mu_0, \ldots, \mu_k$. Moreover, by opening up the formula for $\xi_{r,k}^{\lambda(t)}$, we see that as a polynomial in $\mu_0, \ldots, \mu_k$, it holds that

$$\xi_{r,k}^{\lambda(t)} = \frac{1}{r} \sum_{i=0}^{k} \mu_i^r + (\text{lower order terms}).$$

This in turn implies that the ring generated by the $\xi_{r,k}^{\lambda(t)}$ for every $r > 0$ contains the power sums $\sum_{i=0}^{k} \mu_i^r$ and hence is equal to the ring of all symmetric polynomials in $\mu_0, \ldots, \mu_k$. But then $\xi_{r,k}^{\lambda(t)} = \xi_{r,k}^{\lambda'(t)}$ for all $r > 0$, implies that μ is a permutation of μ' or $\lambda \overset{t}{\sim} \lambda'$. □

We finish this section by stating the main result by Comes and Ostrik [3] together with some results and examples to understand the relation $\overset{t}{\sim}$.

Theorem 4.27. *Two indecomposable objects $L(\lambda)$ and $L(\lambda')$ are in the same block of $\underline{Rep}(S_t;\mathbb{C})$ if and only if $\lambda \overset{t}{\sim} \lambda'$.*

We have already proven the forward implication. For the other implication one needs to first understand the relation $\overset{t}{\sim}$, as well as how lifts of indecomposables of $\underline{Rep}(S_t;\mathbb{C})$ decompose in its generic version $\underline{Rep}(S_T; Frac(\mathbb{C}[T-t]))$, which is semisimple. The following proposition characterizes when a transformation of $\mu_\lambda(t)$ is again of the form $\mu_{\lambda'}(t)$ for a Young diagram λ.

Proposition 4.28. *Let $\lambda \in \mathbb{Y}$ and write $\mu_\lambda(t) = (\mu_0, \mu_1, \ldots)$. Suppose that $\tau : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is a bijection and set $\mu' = (\mu'_0, \mu'_1, \ldots)$ where $\mu'_i = \mu_{\tau^{-1}(i)}$. There exists a Young diagram λ' such that $\mu' = \mu_{\lambda'}(t)$ if and only if $\mu'_i \in \mathbb{Z}$ with $\mu'_i > \mu'_{i+1}$ for all $i > 0$.*

Proof. Suppose λ' satisfies $\mu' = \mu_{\lambda'}(t)$. Then $\mu'_i = \lambda'_i - i > \lambda'_{i+1} - i - 1 = \mu'_{i+1}$ for all $i > 0$. For the converse, assume $\mu'_i > \mu'_{i+1}$ for all $i > 0$ and set $\lambda'_i = \mu_i + i$ for all $i \ge 0$. Since both $\mu_i > \mu_{i+1}$ and $\mu'_{i+1} = \mu_{\tau^{-1}(i+1)} > \mu_{\tau^{-1}(i)} = \mu'_i$ for all $i > 0$, τ^{-1} must be increasing for values $i > \tau^{-1}(0)$, or else if for some i we have $\tau^{-1}(i) > \tau^{-1}(i+1)$, then $\lambda_{\tau^{-1}(i)} - \tau^{-1}(i) > \lambda_{\tau^{-1}(i+1)} - \tau^{-1}(i+1)$ but since λ is a Young diagram, $\lambda_{\tau^{-1}(i)} < \lambda_{\tau^{-1}(i+1)}$, yielding that $\tau^{-1}(i+1) > \tau^{-1}(i)$ a contradiction. But then τ is also increasing, thus for values $i > \max\{\tau(0), \tau^{-1}(0)\}$ it holds that $\tau(i) \ge i$ so that $\tau^{-1}(i) \le i$, finally yielding $\tau(i) = i$, for all $i > \max\{\tau(0), \tau^{-1}(0)\}$. Consequently, $\lambda'_i = \lambda_i$ for all $i > \max\{\tau(0), \tau^{-1}(0)\}$, yielding $\lambda'_i = 0$ for large values. Moreover, $\lambda_i - i > \lambda_{i+1} - i - 1$ implies $\lambda'_i \ge \lambda'_{i+1}$ for all $i > 0$, proving that λ' is a Young diagram. By choosing $k > \max\{\tau(0), \tau^{-1}(0)\}$ with $\lambda_k = 0$ (and thus also $\lambda'_k = 0$), we get that

$$t = \sum \lambda_i = \sum \mu_i + \frac{k(k+1)}{2} = \sum \mu'_i + \frac{k(k+1)}{2} = \sum \lambda'_i.$$

yields $\lambda'_0 = t - |\lambda'|$. □

Corollary 4.29. *(a) The $\overset{t}{\sim}$ equivalence classes are all trivial, unless $t \in \mathbb{Z}_{\ge 0}$.*

(b) Suppose d is a nonnegative integer and λ a Young diagram. The $\overset{d}{\sim}$ equivalence class containing λ is non trivial if and only if the coordinates of $\mu_\lambda(d)$ are pairwise distinct.

Proof. From proposition 4.28, we get that the class containing λ is nontrivial, if and only if all the coordinates of $\mu_\lambda(t)$ are integers and pairwise distinct. Thus if $t \notin \mathbb{Z}$ the classes are all trivial. Now if t is a negative integer, then $\lambda_{|\lambda|-t} = 0$ (since we passed the length) and thus μ has repetitions as $\mu_0 = \mu_{|\lambda|-t}$. □

Proposition 4.30. *Let d be a nonnegative integer and λ a Young diagram in a non-trivial $\overset{d}{\sim}$ equivalence class. By the proposition 4.28, we can label the coordinates of $\mu_\lambda(d)$ by $\mu_0, \mu_1, \ldots$ such that $\mu_i < \mu_{i-1}$ for all $i > 0$. For each integer $i \geq 0$, set $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots)$, where*

$$\lambda_j^{(i)} = \begin{cases} \mu_{j-1} + j, & \text{if } j \leq i, \\ \mu_j + j & j > i. \end{cases}$$

for all $j > 0$. Then $\lambda^{(0)} \leq \lambda^{(1)} \leq \ldots$ is a complete list of the Young diagrams in the $\overset{d}{\sim}$ equivalence class containing λ.

Proof. $\lambda^{(i)}$ is a Young diagram for which $\mu_{\lambda^{(i)}}(d)$ has coordinates

$$\mu_j^{(i)} = \begin{cases} \mu_i & \text{if } j = 0, \\ \mu_{j-1} & \text{if } 0 < j \leq i, \\ \mu_j & \text{if } j > i. \end{cases}$$

Thus $\lambda^{(i)} \overset{d}{\sim} \lambda$ for all $i \geq 0$. Moreover from proposition 4.28, the list is complete. Lastly, for all $j \neq i$ we have $\lambda_j^{(i-1)} = \lambda_j^{(i)}$ and $\lambda_i^{(i-1)} = \mu_i + i > \mu_{i-1} + i = \lambda_i^{(i)}$, yielding $\lambda^{(i-1)} \leq \lambda^{(i)}$. for all $i > 0$. $\square$

We can now fully characterize non-trivial blocks in a combinatorial manner.

Corollary 4.31. *A Young diagram is the minimal element in a nontrivial $\overset{d}{\sim}$ equivalence class, if and only if its d-completion is again a Young diagram. In particular, the non-trivial d equivalence classes are parametrized by partitions of d. Additionally, if $\{\lambda^{(0)} \leq \lambda^{(1)} \leq \ldots\}$ is a nontrivial d equivalence class, then for each $i \geq 0$, the Young diagram $\lambda^{(i)}$ is created from $\lambda^{(0)}(d)$ by removing row i and adding one box to row j for all $j = 0, 1, \ldots, i-1$.*

Proof. If $\lambda(d)$ is again a Young diagram, then $d - |\lambda| > \lambda_1 - 1 > \lambda_2 - 2 > \ldots$. By corollary 4.29 λ belongs to a nontrivial equivalence class and with the convention used there $\lambda = \lambda^{(0)}$ is the minimal element in its equivalence class. Conversely, let $\{\lambda^{(0)} \leq \lambda^{(1)} \leq \ldots\}$ be a nontrivial $\overset{d}{\sim}$ equivalence class and $\mu_0, \mu_1, \ldots$ be the coordinates of $\mu_\lambda(d)$ labelled as in proposition 4.30. Then $d - |\lambda^{(0)}| = \mu_0 \geq \mu_1 + 1 = \lambda_1^{(0)}$ proving that $\lambda^{(0)}(d)$ is a Young diagram. We have $\lambda_1^{(i)} = \mu_0 + 1 = d - |\lambda^{(0)}| + 1$ and the j-th row of $\lambda^{(i)}$ is $\lambda_j^{(i)} = \mu_{j-1} + j = \lambda_{j-1}^{(0)} + 1$ for $j \leq i$ and for $j > i$ $\lambda_j^{(i)} = \lambda_j^{(0)}$. Thus we obtain $\lambda^{(i)}$ from $\lambda^{(0)}$ by removing row i and adding one box to row j for $j \leq i - 1$. $\square$

Example 4.32. *Let us consider the non trivial $\overset{4}{\sim}$ equivalence classes. They are parametrized by partitions of 4, thus we have 5 non trivial equivalence classes, and each one of them is given by the chain proposed in corollary 4.31. Moreover, to obtain the minimal elements in equivalence classes we need λ Young diagram such that its 4-completion is again a Young diagram, thus $|\lambda| \leq 4$. Out of all those partitions, the ones whose 4-completion is again a Young diagram are* $\emptyset, \square, \ydiagram{1,1}, \ydiagram{2}, \ydiagram{1,1,1}$ *with respective 4-completions* $\ydiagram{4}, \ydiagram{3,1}, \ydiagram{2,1,1}, \ydiagram{2,2}, \ydiagram{1,1,1,1}$. *The corresponding equivalence classes are presented below.*

$$C_\emptyset = \left\{ \emptyset \leq \ydiagram{5} \leq \ydiagram{5,1} \leq \ydiagram{5,1,1} \leq \ydiagram{5,1,1,1} \leq \ldots \right\},$$

$$C_{\ydiagram{1}} = \left\{ \ydiagram{1} \leq \ydiagram{4} \leq \ydiagram{4,2} \leq \ydiagram{4,2,1} \leq \ydiagram{4,2,1,1} \leq \dots \right\},$$

$$C_{\ydiagram{2}} = \left\{ \ydiagram{2} \leq \ydiagram{3} \leq \ydiagram{3,3} \leq \ydiagram{3,3,1} \leq \ydiagram{3,3,1,1} \leq \dots \right\},$$

$$C_{\ydiagram{1,1}} = \left\{ \ydiagram{1,1} \leq \ydiagram{3,1} \leq \ydiagram{3,2} \leq \ydiagram{3,2,2} \leq \ydiagram{3,2,2,1} \leq \dots \right\},$$

$$C_{\ydiagram{1,1,1}} = \left\{ \ydiagram{1,1,1} \leq \ydiagram{2,1,1} \leq \ydiagram{2,2,1} \leq \ydiagram{2,2,2} \leq \ydiagram{2,2,2,1} \leq \dots \right\}.$$

Lastly, we give the dimensions of morphism spaces between indecomposable objects. As a corollary, we get the result originally secured by Deligne, characterizing when $\underline{Rep}(S_t;\mathbb{C})$ is semisimple.

Proposition 4.33. *(a) Whenever λ is in a trivial $\overset{t}{\sim}$ equivalence class, $\dim_{\mathbb{C}} End_{\underline{Rep}(S_t;\mathbb{C})}(L(\lambda)) = 1$. Thus the block corresponding to a trivial t-equivalence class is trivial.*

(b) For a non-trivial block $\{\lambda^{(0)} \leq \lambda^{(1)} \leq \dots\}$ in $\underline{Rep}(S_t;\mathbb{C})$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_t;\mathbb{C})}(L(\lambda^{(i)}), L(\lambda^{(j)})) = \begin{cases} 2 \textit{ if } & i = j > 0, \\ 0 \textit{ if } & |i-j| \geq 2, \\ 1 \textit{ if } & |i-j| = 1. \end{cases}$$

For a proof of the above result, see [3].

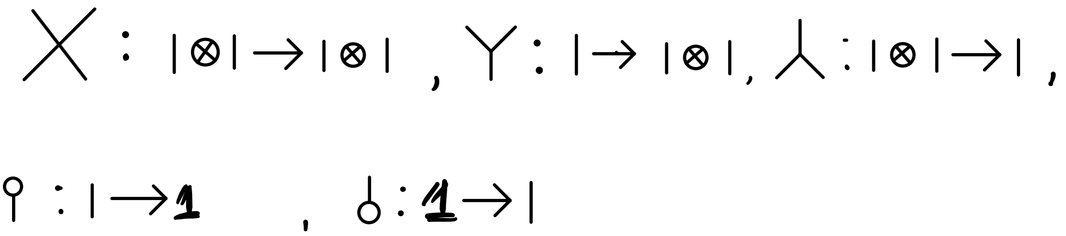
Corollary 4.34. *$\underline{Rep}(S_t;\mathbb{C})$ is semisimple if and only if t is a nonnegative integer.*

Proof. We have a non trivial block if and only if t is a nonnegative integer. Otherwise all blocks are trivial, yielding that $\underline{Rep}(S_t;\mathbb{C})$ is semisimple. ∎

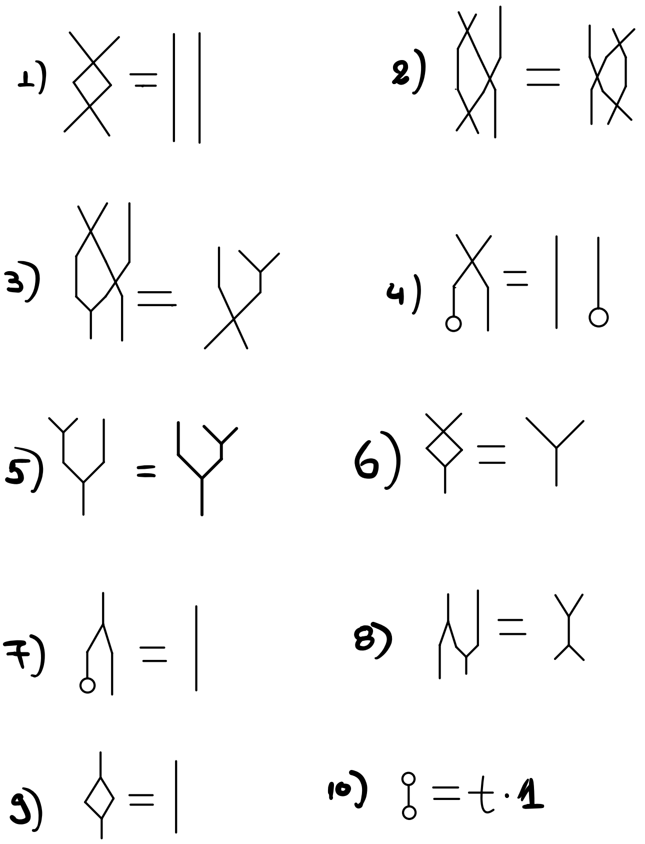
5 The Affine partition category

In this chapter we give the modern approach to YJM elements for the partition algebras, via a new category defined by Brundan and Vargas [1]. In order to define this category which somewhat easily recovers the classic YJM elements defined in definition 3.33, they took advantage of a result by Likeng and Savage [10], namely an embedding of Deligne's category $\underline{Rep}(S_t;\mathbb{C})$(and thus also $\mathcal{P}ar_t$) in the Heisenberg category. The main advantage with the Heisenberg category is that it has a categorical action to the symmetric category, giving it natural candidates for YJM elements, justyfying their definition in the affine partition category. Another advantage of working in the Heisenberg category is that we can employ monoidal calculus, making calculations more manageable compared to working directly in the partition algebra.

Definition 5.1 (Partition Category). *The partition category $\mathcal{P}ar_t$ is the strict $\mathbb{C}$-linear monoidal category generated by one object $|$, with generating morphisms*



which we call $cross, split, merge, dl, ul$ respectively. subject to the following relations, as well as the ones obtained from these by horizontal and vertical flips:

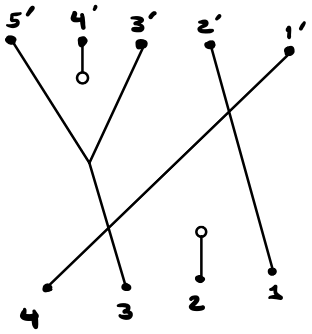


where $\mathbf{1}$ is the unit object of $\mathcal{P}ar_t$. Note that the object set of $\mathcal{P}ar_t$ is $\{|^{\otimes n} : n \in \mathbb{N}\}$, so we will identify it with the natural numbers $\mathbb{N}$.

Remark 5.2. *In order to avoid confusion with later notation, when a circle is attached to the bottom (top) of a strand, that means the source (target) of that strand is the unit object* $\mathbf{1}$.

Relations (1-4) imply that $\mathcal{P}ar_t$ is a symmetric monoidal category, while (5)-(8) imply that the generating object is a commutative Frobenius object, (9) implies it is a special Frobenius object (i.e product is left inverse of coproduct). Note that the category is rigid with every object being self-dual. We can write the evaluation and co-evaluation maps as

which are easily seen to satisfy the snake identities by relations 7 and 8. Thus, relations 9 and 10 imply that the object $|$ is of categorical dimension t. An $m \times n$ partition diagram is a string diagram f in $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ consisting of vertical and horizontal compositions of the above generating morphisms such that every connected component of f has at least one endpoint, as floating bubbles of the form $\mathrm{ev} \circ \mathrm{coev}$ can be removed giving the result a factor of $t \in \mathbb{C}$. Then $\mathrm{Hom}_{\mathcal{P}ar_t}(m, n)$ is $\mathbb{C}$-linear combinations of partition diagrams as defined above. We also have the anti-isomorphism $flip : \mathcal{P}ar_t \to (\mathcal{P}ar_t)^{op}$ being identity on objects and sending morphisms to their flips in a horizontal axis. By labelling the endpoints of an $m \times n$ partition diagram f from right to left by $1, \ldots, n$ on the bottom boundary and by $1', \ldots, m'$ on the top boundary, a partition diagram defines a partition of the set $\{1, \ldots, n, 1', \ldots, m'\}$ as in the following example:



The above partition diagram determines the partition

$$\{1, 2'\} \cup \{1', 4\} \cup \{2\} \cup \{3, 3', 5'\} \cup \{5'\}$$

of the set $\{1, 2, 3, 4, 1', 2', 3', 4', 5'\}$. We thus get a nice functor from the partition category defined here to the partition category defined in definition 3.21, which is an isomorphism of categories. We can now identify $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m) = \mathbb{C}P_{n,m}$, with the convention that we are now working with flipped partition diagrams (and thus composition also goes from bottom to top). We next reformulate Schur-Weyl duality categorically.

Theorem 5.3. *Let t be a nonnegative integer. Viewing $\mathbb{C}[S_t]-mod$ as a symmetric monoidal category using the Kronecker tensor product, there is a full $\mathbb{C}$-linear symmetric monoidal functor $\psi_t : \mathcal{P}ar_t \to \mathbb{C}[S_t]-mod$ sending the generating object $|$ to the natural t-dimensional representation V_t and defined* *on generating morphisms by*

$$
\begin{aligned}
&\psi_t(\times) : V_t \otimes V_t \to V_t \otimes V_t \ , && u_i \otimes u_j \mapsto u_j \otimes u_i \ , \\
&\psi_t(\curlywedge) : V_t \otimes V_t \to V_t, && u_i \otimes u_j \mapsto \delta_{i,j} u_i \ , \\
&\psi_t(\curlyvee) : V_t \to V_t \otimes V_t, && u_i \mapsto u_i \otimes u_i \ , \\
&\psi_t(\text{⫯}) : V_t \to \mathbb{C} \ , && u_i \mapsto 1 \ , \\
&\psi_t(\text{⫰}) : \mathbb{C} \to V_t && 1 \mapsto u_1 + \cdots + u_t \ .
\end{aligned}
$$

Moreover, the linear map $\mathbb{C}P_{n,m} \to \mathrm{Hom}_{S_t}(V_t^{\otimes n}, V_t^{\otimes m})$ *sending* $f \mapsto \psi_t(f)$ *is an isomorphism whenever* $t \geq n+m$.

We also define the generic partition category $\mathcal{P}ar$ to be the strict $\mathbb{C}$-linear monoidal category with the same generating objects and morphisms as in $\mathcal{P}ar_t$ subject to all the relations but the last one which is ommited. We denote the floating morphism by

$$T = dl \circ ul$$

which is strictly central in $\mathcal{P}ar$, so that it can be viewed as a $\mathbb{C}[T]$-linear monoidal category. For $t \in \mathbb{C}$ we get the canonical evaluation functor

$$ev_t : \mathcal{P}ar \to \mathcal{P}ar_t$$

taking T to $t1_{\mathbf{1}}$. We can use the basis theorem for $\mathcal{P}ar_t$ infinitely many times through ev_t to obtain a basis theorem for $\mathcal{P}ar$. Each morphism space $\mathrm{Hom}_{\mathcal{P}ar}(n,m)$ is free as a $\mathbb{C}[T]$-module, with basis given by a set of representatives for the equivalence classes of $m \times n$ partition diagrams. This category is useful since whenever we want to check if a relation holds in $\mathcal{P}ar_t$ for all values of t, we just check that they hold for all large enough positive integers. Moreover, using the functor ψ_t from above, we can reduce the original question to one for symmetric groups. In particular, for $t \in \mathbb{C}$, let

$$\phi_t = \psi_t \circ ev_t : \mathcal{P}ar \to \mathbb{C}[S_t] - mod$$

assuming that t is a nonnegative integer.

Lemma 5.4. *Suppose* $f \in \mathrm{Hom}_{\mathcal{P}ar}(n,m)$ *satisfies* $\phi_t(f) = 0$ *for infinitely many values of* $t \in \mathbb{Z}_{\geq 0}$. *Then* $f = 0$.

Proof. By the basis theorem, we write $f = \sum_i p_i(T) f_i$ for polynomials $p_i \in \mathbb{C}[T]$ and f_i running over a set of representatives for the equivalence classes of $m \times n$ partition diagrams. Since $\phi_t(f) = 0$, it holds that

$$\sum_i p_i(t) \phi_t(f_i) = 0$$

for infinitely many values of t. But since ψ_t is an isomorphism for large integers, we get that

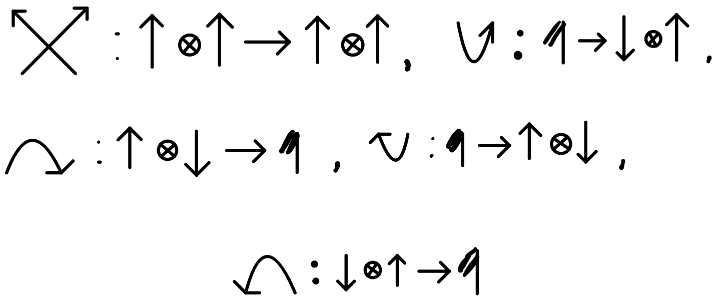
$$\sum_i p_i(t) ev_t(f_i) = 0$$

for infinite values $t \geq m+n$. As the set of representatives form a basis for the Hom spaces of $\mathcal{P}ar$, for each i, we have $p_i(t) = 0$ for infinitely many values of t. Since our field is of characteristic zero, it thus holds that the polynomials are identically zero, i.e $p_i(T) = 0$, for all i. $\square$

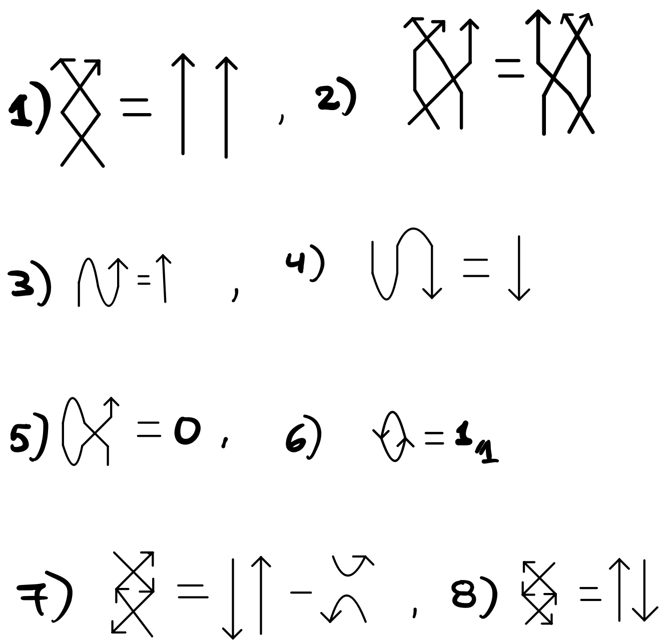
Remark 5.5. *By lemma 5.4, whenever we want to prove a relation in $\mathcal{P}ar$, it suffices to prove that it holds asymptotically for their images through ϕ_t.*

We can now define the category that will lead us to the definition of the Affine partition category.

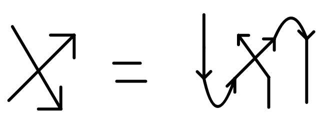
Definition 5.6. *The Heisenberg category $\mathcal{H}eis$ is the strict $\mathbb{C}$-linear monoidal category with two generating objects $\uparrow$ and $\downarrow$ and five generating morphisms:*



subject to the following relations:

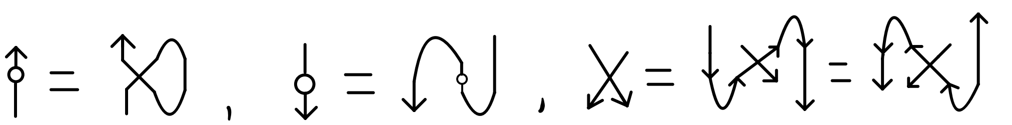


where we have used the sideways crossings, defined by



and the other one being its $180°$ flip. In order to make $\mathcal{H}eis$ a strictly pivotal category, we also require the $180°$ flips of the snake relations 3 and 4 to hold.

This way, we made $\mathcal{H}eis$ a strictly pivotal category, with duality functor rotating diagrams through $180°$. We will also need the following shorthand notations for morphisms:

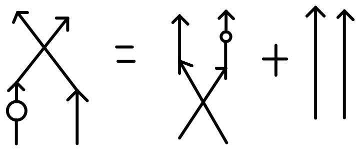


In particular, using the above relations, one can prove that the braid relation holds no matter the orientation of the arrows. The two decorated arrows play the role of YJM elements, as we will shortly see.

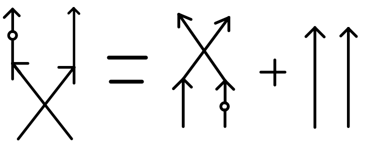
Remark 5.7. *Note that now the circle attached to the middle of a strand does not have anything to do with the unit object as in definition 5.1. It is just the way to add YJM elements in the diagrammatics.*

As a first indication, we show that they satisfy the degenerate Hecke algebra relation.

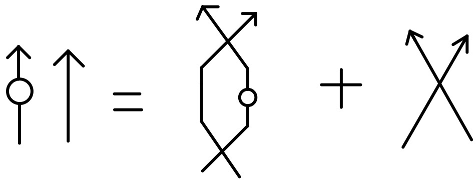
Lemma 5.8. *The decorated up-arrow morphism satisfies the defining degenerate Hecke algebra relation:*

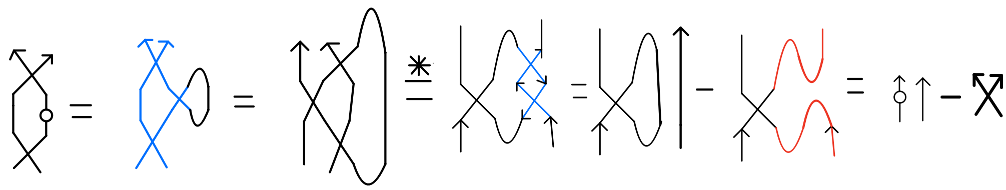


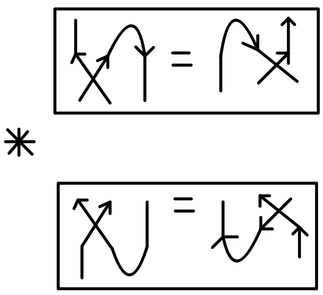
or its equivalent by composing both up and down with the upwards crossing



Proof. We prove its equivalent by composing on top with the up crossing.



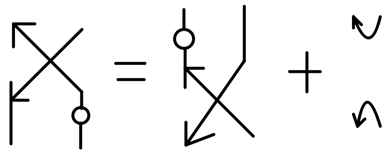




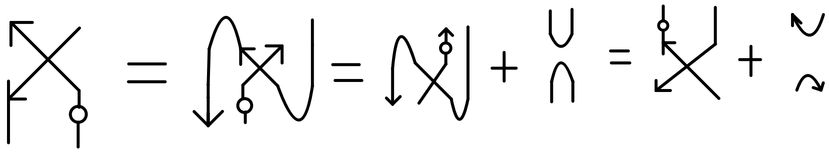
where we first used the braid relation, after that the two relations provided in $*$ and in the last equality the two snake identities. $\square$

There exists a symmetric monoidal functor $\sigma : \mathcal{H}eis \to (\mathcal{H}eis)^{op}$ which reverts the generating objects and reflects a morphism along a horizontal axis, together with reverting the orientation of the arrows. We can use this symmetry to obtain further variations of the defining relations of the Heisenberg category or even get other versions of the degenerate affine Hecke algebra relation. One which we will need later on is the following.

Lemma 5.9. *In $\mathcal{H}eis$, the following relation holds:*



Proof. This is a direct calculation



where in the second equality we used the classic degenerate affine Hecke relation. □

Remark 5.10. *Notice how this relation is exactly what one would expect to obtain from the rotation of the degenerate affine Hecke algebra relation lemma 5.8 by* $90°$*. We can use this to obtain other variations of relations in* $\mathcal{H}eis$*.*

Khovanov [7] constructed a categorical action of this category on the category of finite dimensional modules over the symmetric category, denoted by $\mathcal{S}ym - mod = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[S_n] - mod$. In other words, he constructed a strict $\mathbb{C}$-linear monoidal functor

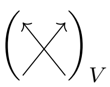
$$\Theta : \mathcal{H}eis \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}ym - mod) \tag{10}$$

and proved that this functor is faithful over fields of characteristic zero.

Theorem 5.11. *There exists a faithful strict* $\mathbb{C}$*-linear monoidal functor*

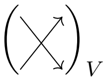
$$\Theta : \mathcal{H}eis \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}ym - mod)$$

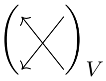
sending the generating objects $\uparrow, \downarrow$ *to the induction and restriction functors* I, R *respectively. Moreover,* Θ *sends the generating morphisms of* $\mathcal{H}eis$ *to the natural transformations defined on a* $\mathbb{C}[S_n]$*-module* V *as presented below:*

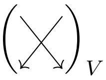


$$: \mathbb{C}[S_{n+2}] \otimes_{\mathbb{C}[S_{n+1}]} \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} V \to \mathbb{C}[S_{n+2}] \otimes_{\mathbb{C}[S_{n+1}]} \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} V,$$

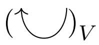
$$g \otimes 1 \otimes v \mapsto g s_{n+1} \otimes 1 \otimes v,$$

 $: \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} V \to \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} V, \quad g \otimes v \mapsto g s_n \otimes v,$

 $: \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} V \to \mathbb{C}S_n \otimes_{\mathbb{C}[S_{n-1}]} V, \quad g \otimes v \mapsto \begin{cases} g_2 \otimes g_1 v & \textit{if } g = g_2 s_n g_1 \textit{ for } g_i \in S_n, \\ 0 & \textit{otherwise}, \end{cases}$

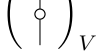
 $: V \to V, \quad v \mapsto s_{n-1} v,$

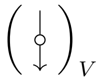
 $: \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} V \to V, \quad g \otimes v \mapsto gv,$

 $: V \to \mathbb{C}[S_n \otimes_{\mathbb{C}[S_{n-1}]} V \quad v \mapsto \sum_{i=1}^{n} (i\, n) \otimes (i\, n) v,$

 $: \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n} V \to V, \quad g \otimes v \mapsto \begin{cases} gv & \textit{if } g \in S_n, \\ 0 & \textit{otherwise}, \end{cases}$

 $: V \to \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n} V \quad v \mapsto 1 \otimes v,$

 $: \mathbb{C}[S_{n+1}] \otimes_{\mathbb{C}[S_n]} V \to \mathbb{C}S_{n+1} \otimes_{\mathbb{C}[S_n]} V, \quad g \otimes v \mapsto g \xi_{n+1} \otimes v,$

 $: V \to V, \quad v \mapsto \xi_n v.$

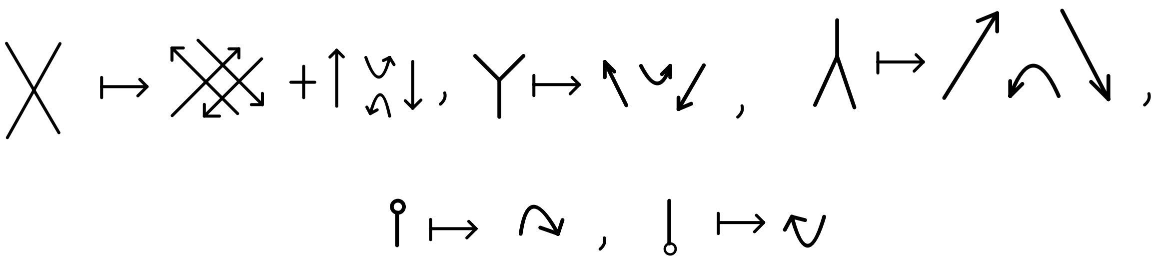
Proof. For a proof, see [7]. ☐

Remark 5.12. *By theorem 5.11, we can think of the Heisenberg category as the category controlling inductions and restrictions of representations of the symmetric groups, together with the necessary natural transformations between them. Additionally, the dotted up and down arrows are acting as YJM elements, making them "natural" candidates for YJM elements in $\mathcal{H}eis$. Later, Likeng and Savage [10] embedded the partition category into $\mathcal{H}eis$. This result is not completely unexpected, as both categories control representations of the symmetric groups, with $\mathcal{H}eis$ clearly being larger than $\mathcal{P}ar$, so in order to connect the two, one tries to find which morphisms in $\mathcal{H}eis$ have compatible actions with the generating morphisms of $\mathcal{P}ar$.*

Theorem 5.13. *There is a strict $\mathbb{C}$-linear monoidal functor*

$$i : \mathcal{P}ar \to \mathcal{H}eis \tag{11}$$

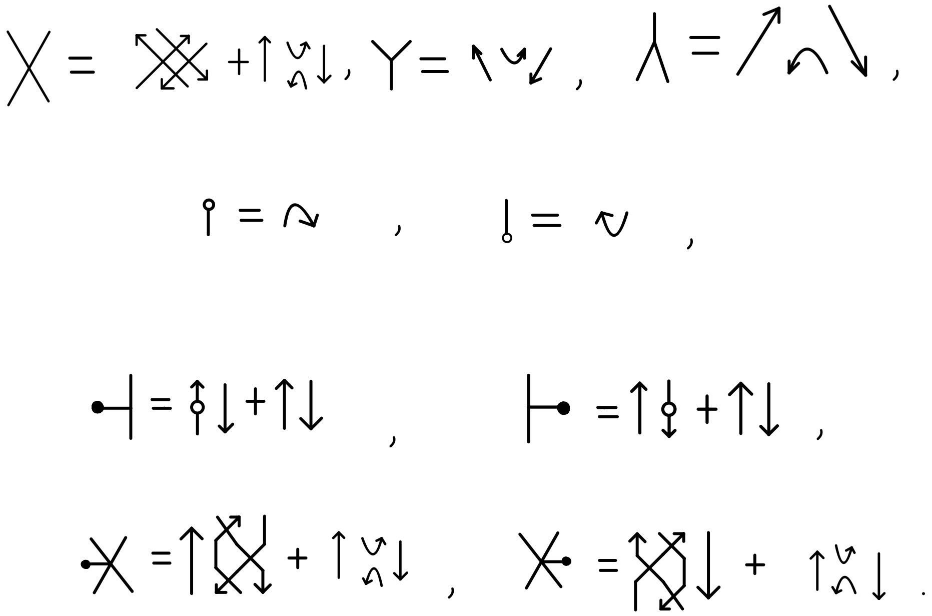
sending the generating object $|$ of $\mathcal{P}ar$ to the object $(\uparrow \otimes \downarrow)$ and defined on generating morphisms by



Proof. See [10], Theorem 4.1. ☐

We are now ready to give the definition of the affine partition category.

Definition 5.14. *The affine partition category $\mathcal{AP}ar$ is the monoidal subcategory of $\mathcal{H}eis$ generated by the object $| = \uparrow \otimes \downarrow$ and the following morphisms:*



We will refer to the last four generators as the left dot, right dot, left and right crossing respectively.

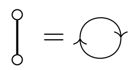
Remark 5.15. *Brundan and Vargas adjointed these* 4 *new generators following the idea of Schur-Weyl duality between partition algebras and symmetric groups. In particular, we will see that the left and right dots resemble YJM elements and the left and right crossings are closely related to the Enyang generators. Moreover, theorem 5.13 now can be reformulated as follows.*

Corollary 5.16. *There is a strict* $\mathbb{C}$*-linear monoidal functor*

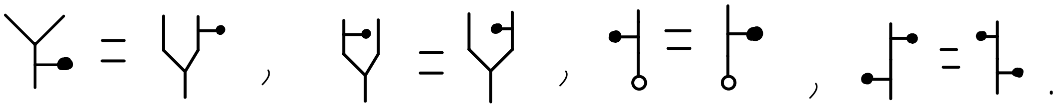
$$i : \mathcal{P}ar \to \mathcal{AP}ar \tag{12}$$

sending the generating object and generating morphisms of $\mathcal{P}ar$ *to the generating object and generating morphisms of* $\mathcal{AP}ar$ *corresponding to the same diagrams.*

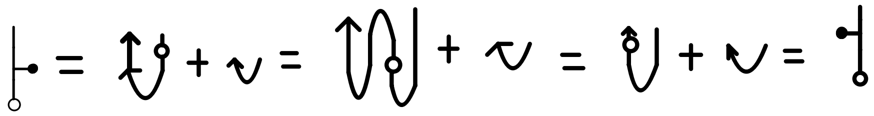
Remark 5.17. *Notice that there is a symmetry of the generators of* $\mathcal{AP}ar$ *under rotation through* 180°*, enabling the restriction of the strictly pivotal structure of* $\mathcal{H}eis$ *to a strictly pivotal structure of* $\mathcal{AP}ar$*. This way, the left and right dots are duals, as are the left and right crossings. The evaluations and co-evaluations making* $|$ *a self-dual object are given by the same formulas as in* $\mathcal{P}ar$*, making the functor i pivotal. Additionally, the symmetry* σ *on* $\mathcal{H}eis$ *restricts to* $\sigma : \mathcal{AP}ar \to (\mathcal{AP}ar)^{op}$ *again by the symmetry of the generators. This reflects morphisms in* $\mathcal{AP}ar$ *in a horizontal axis. Note also that*

$T =$  . (13)

Lemma 5.18. *The following commutation relations hold in* $\mathcal{AP}ar$*:*

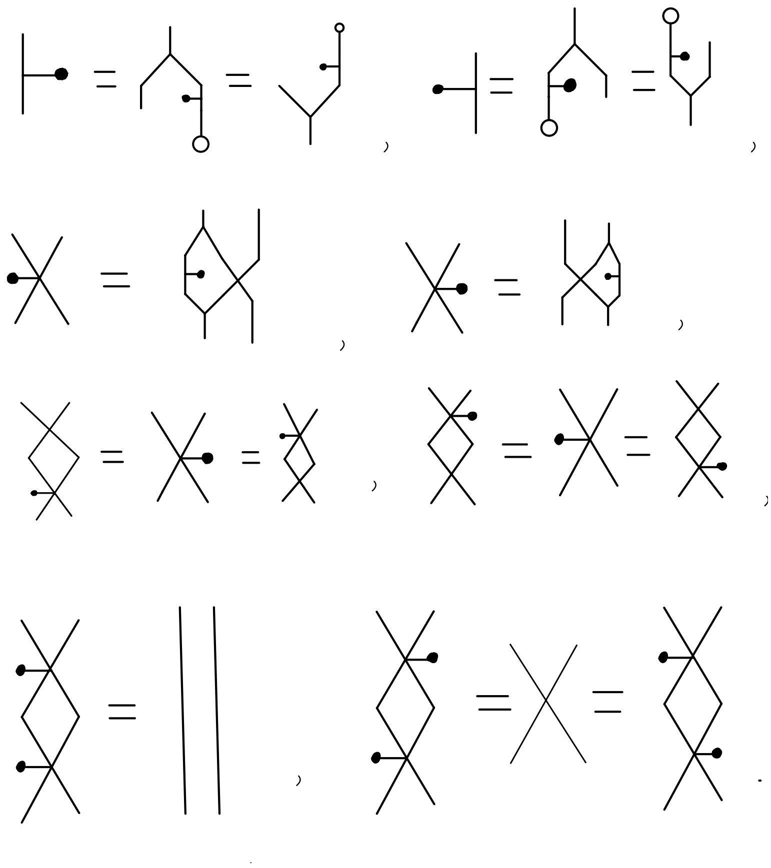


Proof. The only non-trivial relation is the third one, since the other ones all follow since any morphism commutes with the identity. For the third one, we calculate in $\mathcal{H}eis$:

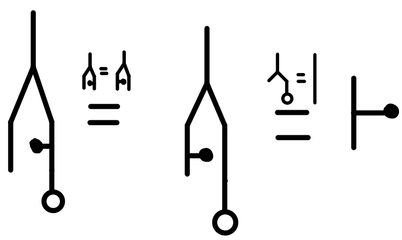


where the second to last equality follows from the snake identities in $\mathcal{H}eis$. $\square$

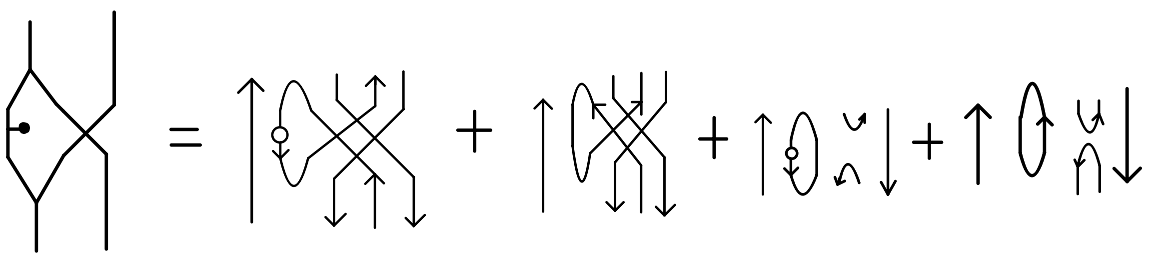
Lemma 5.19. *The following relations hold in $\mathcal{AP}ar$:*



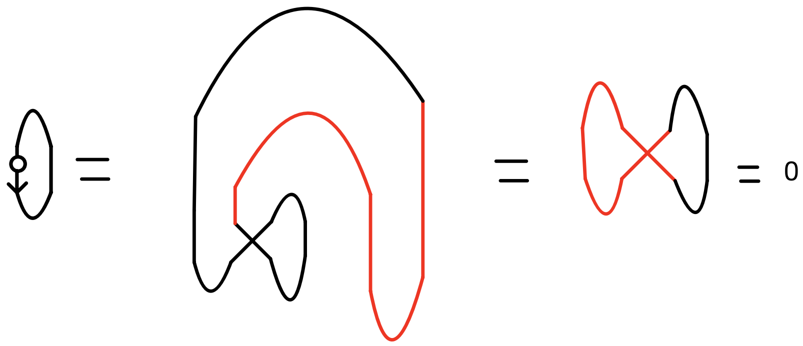
Proof. For every set of relations, we prove only the first one (and specifically the first equality) since the other ones follow by using the symmetry σ and the duality functor to reflect in horizontal and vertical axes. For the first one, we have



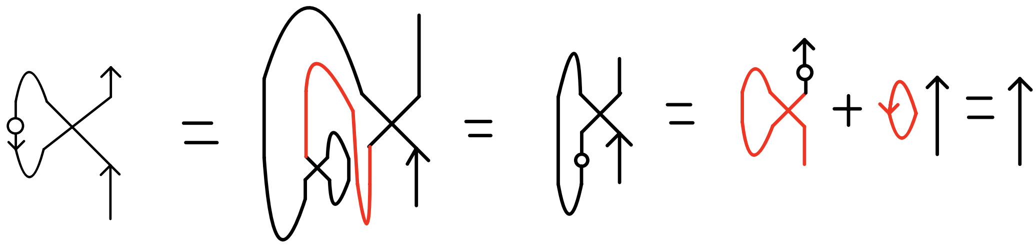
where the first equality comes from lemma 5.18 and the second one from the defining relations in $\mathcal{P}ar$. For the second one we expand as morphisms in $\mathcal{H}eis$ to get



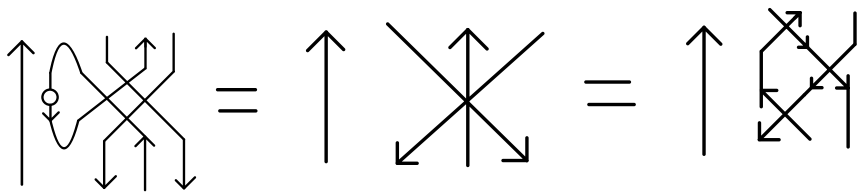
we treat each term seperately. First note that the last term is easy to handle, since by the defining relations of $\mathcal{H}eis$, clockwise bubbles are equal to 1. We show that the third term is equal to zero, since it contains a left curl:



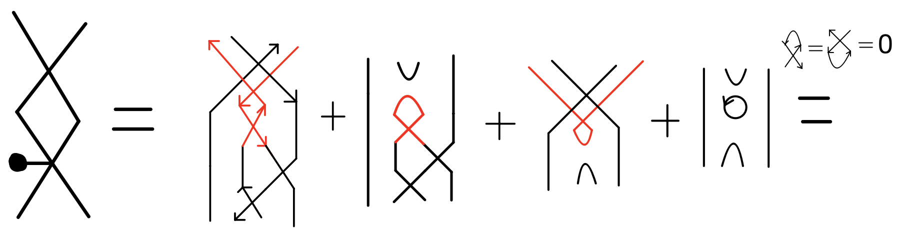
where the first equality stems from the snake identities and the second from the defining relation in $\mathcal{H}eis$. The second term is also zero by containing a left curl. Thus we are left with the first term, for which we calculate

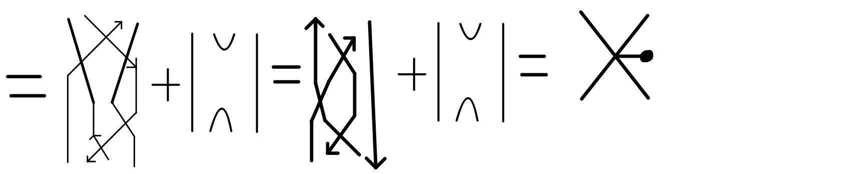


and thus it is equal to



yielding the desired result. For the third relation,





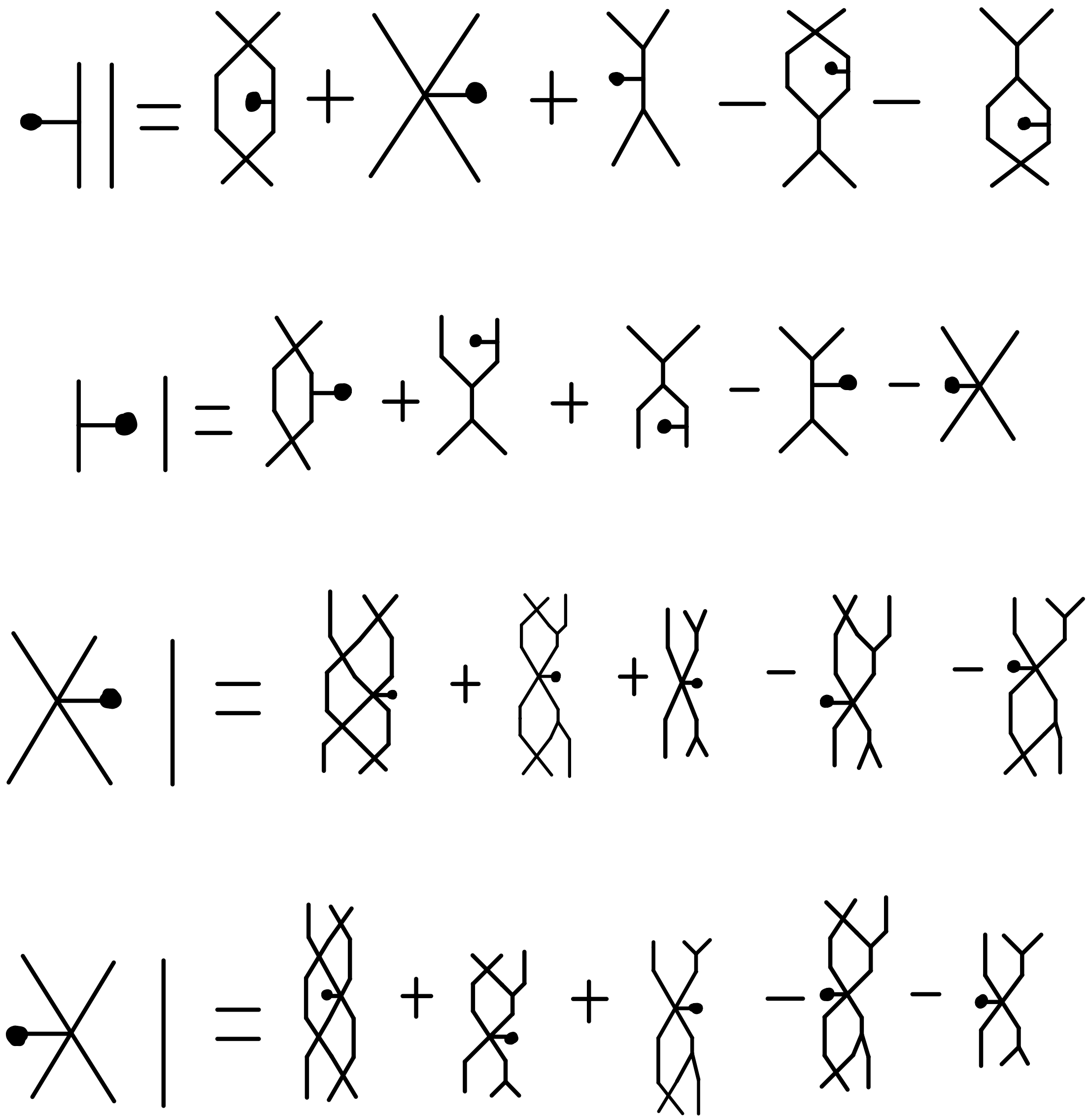
The fourth relation follows analogously by expanding in $\mathcal{H}eis$. $\square$

Remark 5.20. *By lemma 5.19, every dotted generator of $\mathcal{AP}ar$ can be recovered from the undotted generators together with the left dot. Consequently, the other generators are superfluous, yielding the following corollary.*

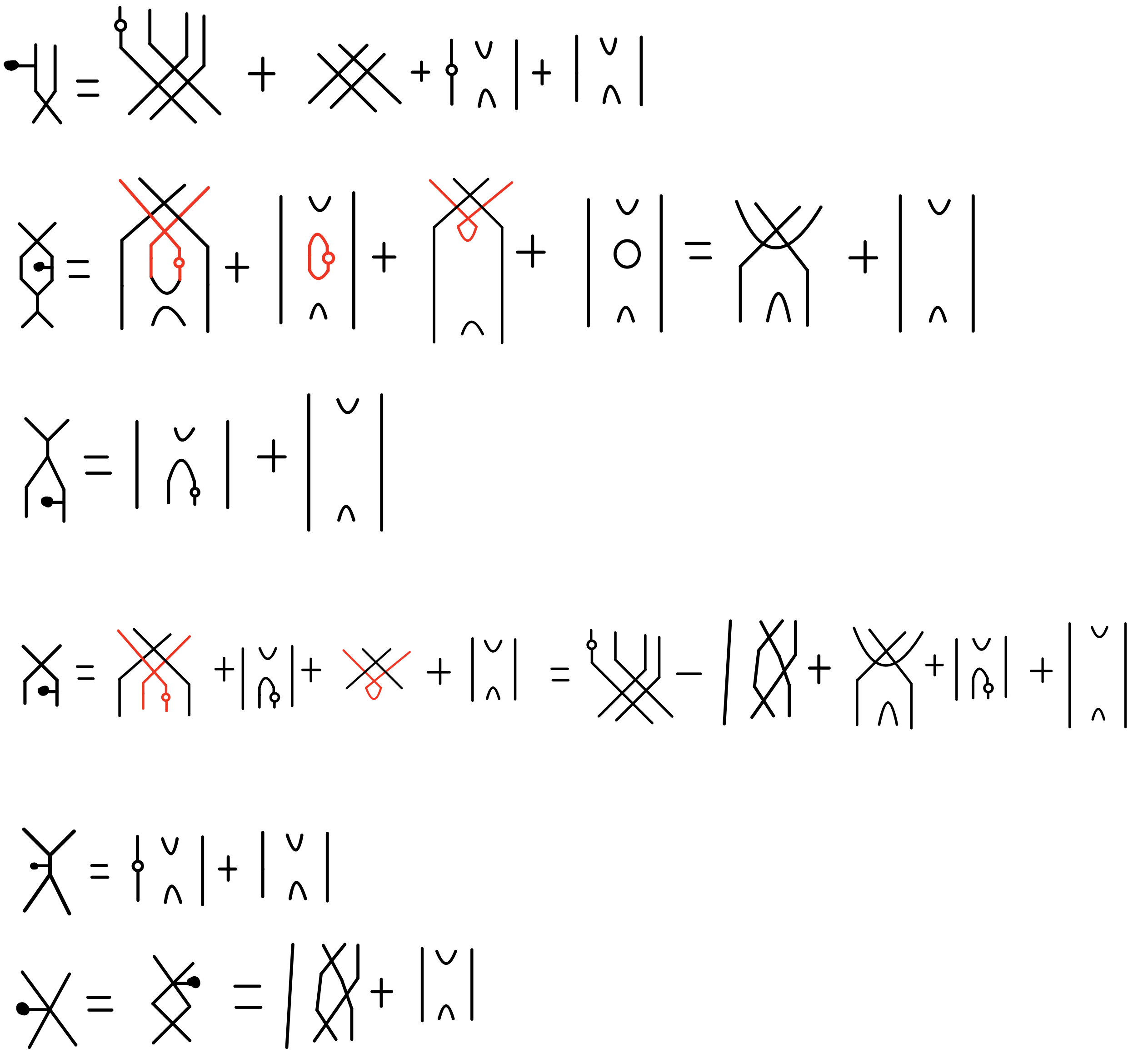
Corollary 5.21. *As a $\mathbb{C}$-linear category, $\mathcal{AP}ar$ is generated by the object $|$, together with the five undotted generators and the left dot.*

We can now prove the equivalent of the five term recurrence relations given by Enyang in [5].

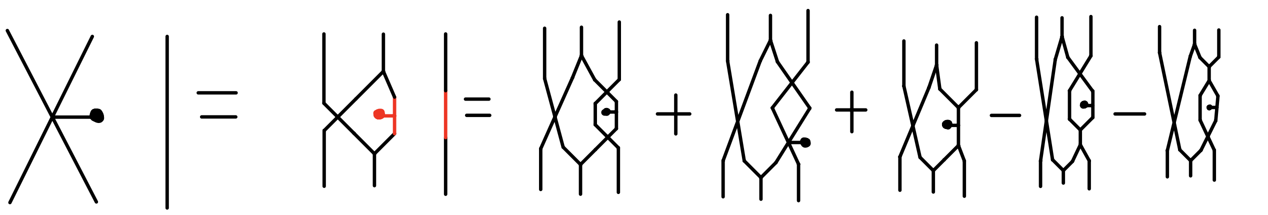
Lemma 5.22. *The following recursive formulas hold in $\mathcal{AP}ar$:*



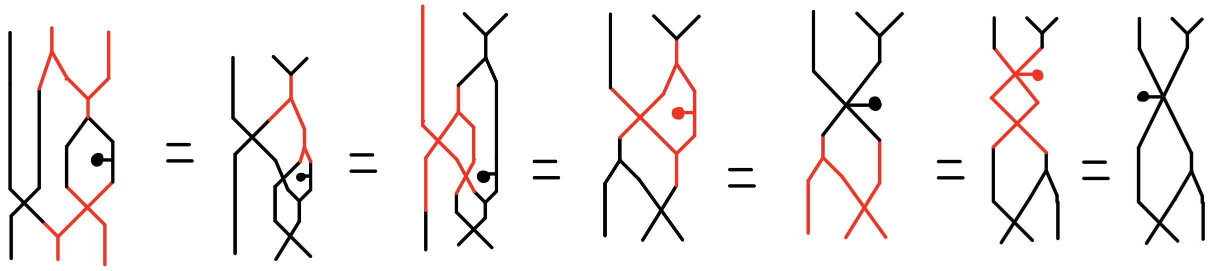
Proof. For the first one, we instead prove its equivalent by composing with the crossing in the bottom. We explore each term seperately by expanding as morphisms in $\mathcal{H}eis$.



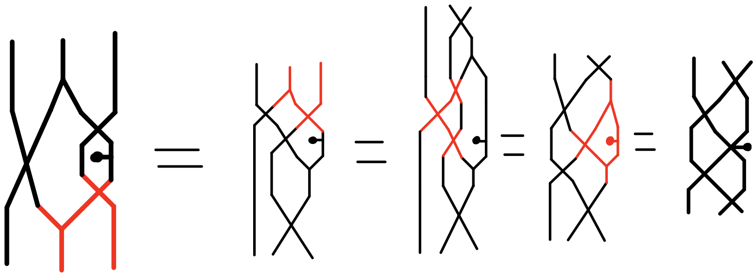
Substitution now yields the first relation. For the second relation, rotate the first one by 180° and compose both on the top and the bottom with the crossing. Then the defining relations of $\mathcal{P}ar$ combined with lemma 5.19 yield the result. To treat the third relation, we rewrite as follows:



Since term 5 is a horizontal flip of 4, it suffices to treat one of the two terms. In particular, using just the defining relations in $\mathcal{P}ar$ together with lemma 5.19 yields



For the first term, we get



The third term is readily seen to be equal to what we want just by uniting merges with splittings. Term 2 remains a mystery to me. The last relation follows by composing on the bottom with a crossing on the left two strands, using lemma 5.19. $\square$

Corollary 5.23. *As a $\mathbb{C}$-linear category, $\mathcal{AP}ar$ has object set identified with $\mathbb{N}$ and morphisms that are linear combinations of vertical compositions of morphisms in the image of $i : \mathcal{P}ar \to \mathcal{AP}ar$ together with the morphism having the left dot in the first strand and identity on all other strands.*

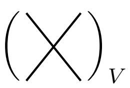
Proof. Induction on the length of the morphisms, using lemma 5.22. $\square$

Remark 5.24. *The above relations are equivalent to Enyang's recursive relations. If we manage to produce a nice functor projecting elements of $\mathcal{AP}ar$ to $\mathcal{P}ar_t$, we will see that they project onto the correct YJM elements. In order to do that, we see how the categorical action Θ from theorem 5.11 of $\mathcal{H}eis$ restricts to $\mathcal{AP}ar$.*

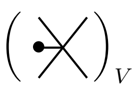
Notice that Θ now sends the generating object $|$ of $\mathcal{AP}ar$ to the functor $\mathrm{I} \circ \mathrm{R}$, thus it sends $\mathbb{C}[S_t]$ modules to $\mathbb{C}[S_t]$ modules. Consequently, for $t \in \mathbb{N}$ we get a functor

$$\Theta_t : \mathcal{AP}ar \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod).$$

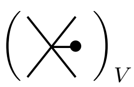
Below are the explicit actions of the generators of $\mathcal{AP}ar$ for $V \in \mathbb{C}[S_t]$-mod.

 $: \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V,$

$$g \otimes h \otimes v \mapsto gh \otimes h^{-1} \otimes hv,$$

 $: \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V,$

$$g \otimes h \otimes v \mapsto g \otimes h \otimes (h^{-1}(t)\ t)v,$$

 $: \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V,$

$$g \otimes h \otimes v \mapsto gh \otimes (h^{-1}(t)\ t) \otimes v,$$

$$
\begin{aligned}
&(\quad)_V : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}S_t \otimes_{\mathbb{C}[S_{t-1}]} V, && g \otimes h \otimes v \mapsto \delta_{h(t),t} gh \otimes v,\\
&(\quad)_V : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V, && g \otimes v \mapsto g \otimes 1 \otimes v,\\
&(\quad)_V : \mathbb{C}[S_t \otimes_{\mathbb{C}[S_{t-1}]} V \to V, && g \otimes v \mapsto gv,\\
&(\quad)_V : V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V, && v \mapsto \sum_{i=1}^{t} (i\ t) \otimes (i\ t)v,\\
&(\quad)_V : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V, && g \otimes v \mapsto \sum_{j=1}^{t} g(j\ t) \otimes v,\\
&(\quad)_V : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V \to \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} V, && g \otimes v \mapsto \sum_{j=1}^{t} g \otimes (j\ t)v.
\end{aligned}
$$

Remark 5.25. *If we denote the trivial representation of S_t with* triv_t, *then we have an isomorphism of representations $\mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} triv_{t-1} \cong V_t$, by sending $g \otimes 1 \mapsto gv_t$. We also have an endofunctor of $\mathbb{C}[S_t]$-mod, namely the one sending objects W to $V_t \otimes W$ and morphisms given by Kronecker product.*

Lemma 5.26. *The functor Θ_t is monoidally isomorphic to the strict $\mathbb{C}$-linear monoidal functor*

$$\Phi_t : \mathcal{AP}ar \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod)$$

sending the generating object to the endofunctor $V_t \otimes$ and the generating morphisms of $\mathcal{AP}ar$ to the natural transformations defined on a module W as follows:

$$
\begin{aligned}
&(\quad)_V : V_t \otimes V_t \otimes W \to V_t \otimes V_t \otimes W, && u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes v,\\
&(\quad)_V : V_t \otimes V_t \otimes W \to V_t \otimes V_t \otimes W, && u_i \otimes u_j \otimes v \mapsto u_i \otimes u_j \otimes (i\ j)v,\\
&(\quad)_W : V_t \otimes V_t \otimes W \to V_t \otimes V_t \otimes W, && u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes (i\ j)v,\\
&(\quad)_W : V_t \otimes V_t \otimes W \to V_t \otimes W, && u_i \otimes u_j \otimes v \mapsto \delta_{i,j} u_i \otimes v,\\
&(\quad)_W : V_t \otimes W \to V_t \otimes V_t \otimes W, && u_i \otimes v \mapsto u_i \otimes u_i \otimes v,\\
&(\quad)_W : V_t \otimes W \to W, && u_i \otimes v \mapsto v,\\
&(\quad)_W : W \to V_t \otimes W, && v \mapsto \sum_{i=1}^{t} u_i \otimes v,\\
&(\quad)_W : V_t \otimes W \to V_t \otimes W, && u_i \otimes v \mapsto \sum_{j=1}^{t} u_j \otimes (i\ j)v,\\
&(\quad)_W : V_t \otimes W \to V_t \otimes W, && u_i \otimes v \mapsto \sum_{j=1}^{t} u_i \otimes (i\ j)v.
\end{aligned}
$$

Proof. As mentioned in remark 5.25, we have the isomorphism $\mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} triv_{S_{t-1}} \cong V_t$. Tensoring over $\mathbb{C}$ with any module W yields $\mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} W \cong V_t \otimes W$. We thus obtain a natural isomorphism

$$(a_1^t)_W : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} W \to V_t \otimes W.$$

mapping $g \otimes v \mapsto gv_t \otimes gv$ for any W a module over $\mathbb{C}[S_t]$. In turn, this defines an isomorphism of functors

$$(a_1^t) : \mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]} \to V_t \otimes .$$

Denote by a_n^t its n-fold horizontal composition. This is a natural isomorphism

$$a_n^t : (\mathbb{C}[S_t] \otimes_{\mathbb{C}[S_{t-1}]})^{\circ n} \Rightarrow (V_t \otimes)^{\circ n}.$$

Explicitly, the action on a module W is given as follows

$$\left(a_n^t\right)_W : g_n \otimes \ldots \otimes g_1 \otimes v \longmapsto g_n v_t \otimes g_n g_{n-1} v_t \otimes \ldots \otimes g_n g_{n-1} \ldots g_1 v_t \otimes g_n g_{n-1} \ldots g_1 v_t.$$

We define $\Phi_t : \mathcal{AP}ar \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod)$ to be the strict $\mathbb{C}$-linear functor sending the object n to $(V_t \otimes)^{\circ n}$ and defined on a morphism $f \in \mathrm{Hom}_{\mathcal{AP}ar}(n, m)$ by

$$\Phi_t(f) = a_m^t \circ \Theta_t(f) \circ (a_n^t)^{-1}.$$

By definition, $a^t = (a_t^n)_{n \in \mathbb{N}} : \Theta_t \Rightarrow \Phi_t$ is an isomorphism of strict $\mathbb{C}$-linear monoidal functors. It remains to prove that the actions of Φ_t are as proposed. We calculate for a couple of generators below.

1. Suppose f is the split. We calculate

$$\begin{aligned}
&(a_2^t)_W \circ (\Theta_t(f))_W \circ ((a_1^t)^{-1})_W(v_i \otimes v) = (a_2^t)_W \circ (\Theta_t(f))_W((i\,t) \otimes (i\,t)v) \\
&= (a_2^t)_W((i\,t) \otimes 1 \otimes (i\,t)v) = (i\,t)v_t \otimes (i\,t)v_t \otimes (i\,t)(i\,t)v = v_i \otimes v_i \otimes v.
\end{aligned}$$

2. Suppose f is the merge. We then have to consider four cases because of the nature of $(a_2^t)^{-1}$.

   (a) $t \neq i \neq j \neq t$. Then, it holds

   $$(a_1^t)_W \circ (\Theta_t(f))_W \circ ((a_2^t)^{-1})_W(v_i \otimes v_j \otimes v) = (a_2^t)_W \circ (\Theta_t(f))_W((i\,t) \otimes (j\,t) \otimes (j\,t)(i\,t)v) = 0$$

   since $j \neq t$.

   (b) $i = t \neq j$ then

   $$(a_2^t)_W^{-1}(v_t \otimes v_j \otimes v) = 1 \otimes (j\,t) \otimes (j\,t)$$

   so we have again

   $$(a_1^t)_W \circ (\Theta_t(f))_W \circ ((a_2^t)^{-1})_W(v_i \otimes v_j \otimes v) = 0$$

   (c) $i \neq t = j$. Then

   $$(a_1^t)_W \circ (\Theta_t(f))_W \circ ((a_2^t)^{-1})_W(v_i \otimes v_j \otimes v) = (a_1^t)_W \circ (\Theta_t(f))_W((i\,t) \otimes (i\,t) \otimes v) = 0$$

   (d) Lastly, if $i = j$ we get

   $$\begin{aligned}
   &(a_1^t)_W \circ (\Theta_t(f))_W \circ ((a_2^t)^{-1})_W(v_i \otimes v_j \otimes v) = (a_1^t)_W \circ (\Theta_t(f))_W((i\,t) \otimes 1 \otimes (i\,t)v) \\
   &= (a_1^t)_W((i\,t) \otimes (i\,t)v) = v_i \otimes v.
   \end{aligned}$$

The other cases follow analogously. □

We also define the functor $\mathrm{Act} : \mathbb{C}[S_t] \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod)$ as the $\mathbb{C}$-linear monoidal functor induced by the Kronecker product. In other words, $\mathrm{Act}(V) = V \otimes$ and $\mathrm{Act}(f) = f \otimes$ for a morphism $f : V \to V'$.

Lemma 5.27. *For $t \in \mathbb{N}$, the following diagram commutes up to canonical isomorphisms of monoidal functors:*

$$\begin{tikzcd} \mathcal{AP}ar \arrow[r, "\Phi_t"] & \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod) \\ \mathcal{P}ar \arrow[u, "i"] \arrow[r, "\phi_t"'] & \mathbb{C}[S_t] - mod \arrow[u, "\mathrm{Act}"'] \end{tikzcd}$$

where ϕ_t is the functor section 5 and i the inclusion functor.

Proof. On objects, $\Phi_t \circ i$ takes the n-th object of $\mathcal{AP}ar$ to $(V_t\otimes)^{\circ n}$ while $\mathrm{Act} \circ \phi_t$ takes it to $V_t^{\otimes n}\otimes$. Let

$$\beta_n^t : (V_t\otimes)^{\circ n} \Rightarrow V_t^{\otimes n}\otimes$$

be the canonical isomorphism given by compositions of the associators. One notices that tensoring the formulas defining ϕ_t with a vector v on the right are exactly the same as the ones defining Φ_t, thus yielding a natural isomorphism of monoidal functors

$$\beta^t = \left(\beta^t\right)_{n\in\mathbb{N}} : \Phi_t \circ i \Rightarrow \mathrm{Act} \circ \phi_t.$$

□

Let $\mathrm{Ev} : \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod)$ be the non-monoidal $\mathbb{C}$-linear functor evaluating on the trivial module. We then have a canonical isomorphism of functors $\mathrm{Ev} \circ \mathrm{Act} \Rightarrow Id_{\mathbb{C}[S_t]-mod}$ defined on a module V by the trivial isomorphism $V \otimes triv_t \cong V$, mapping $v \otimes 1 \mapsto v$. We can now prove that the partition category $\mathcal{P}ar$ can be retrieved as a quotient of $\mathcal{AP}ar$. Denote by $\gamma_n^t : V_t^{\otimes n} \otimes triv_t \to V_t^{\otimes n}$ the trivial isomorphism defined by $v_{i_n} \otimes \ldots \otimes v_{i_1} \otimes 1 \mapsto v_{i_n} \otimes \ldots \otimes v_{i_1}$.

Definition 5.28. *We say a morphism $f \in \mathrm{Hom}_{\mathcal{AP}ar}(n,m)$ is good, if there exists $\bar{f} \in \mathrm{Hom}_{\mathcal{P}ar}(n,m)$ such that*

$$\phi_t(\bar{f}) = \gamma_m^t \circ \mathrm{Ev}(\Phi_t(f)) \circ (\gamma_n^t)^{-1}.$$

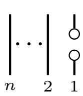
Before stating the main theorem, we give some basic properties of good morphisms. Eventually we will prove that all morphisms in $\mathcal{AP}ar$ are good and this is how we will project down to $\mathcal{P}ar$. Firstly, if f is good, it comes from a unique $\bar{f}$. Towards that end, suppose $\bar{f}, \bar{g}$ both satisfy the condition for f being good. Then we would have $\phi_t(\bar{f}) = \phi_t(\bar{g})$ for all $t \in \mathbb{N}$, yielding $\bar{f} = \bar{g}$ by lemma 5.4. It is also imminent that sums and compositions of good morphisms are good, respecting the operation, or in other words, $\overline{f \circ g} = \bar{f} \circ \bar{g}$ and $\overline{f \circ g} = \bar{f} \circ \bar{g}$.

Lemma 5.29. *Every morphism in $\mathcal{AP}ar$ is good.*

Proof. Since $\mathcal{AP}ar$ is generated by the images of the generators of $\mathcal{P}ar$ under i together with the left dot by corollary 5.23 and being good respects sums and compositions, it is enough to prove that the generating family is good. For all generators of the form $i(f)$, they are good with $\overline{i(f)} = f$ by the following computation:

$$\gamma_m^t \circ \mathrm{Ev}(\Phi_t(i(f))) \circ (\gamma_n^t)^{-1} = \gamma_m^t \circ \mathrm{Ev}(\mathrm{Act}(\phi_t(f))) \circ (\gamma_n^t)^{-1} = \phi_t(f)$$

using commutativity of lemma 5.27. We are thus left to prove that the left dot is a good morphism. We will show that

$\bar{f} =$  .

By definition of ϕ_t, we have that on basis vectors

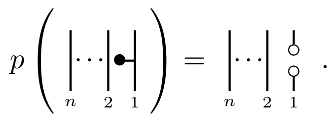
$$\phi_t(\bar{f})(v_{i_n} \otimes \ldots \otimes v_{i_1}) = \sum_{j=1}^{t} v_{i_n} \otimes \ldots \otimes v_j.$$

On the other hand, $\mathrm{Ev}(\Phi_t(f))(v_{i_n} \otimes \ldots \otimes v_{i_1} \otimes 1) = \sum_{j=1}^{t} v_{i_n} \otimes \ldots \otimes v_j \otimes 1$, thus going through γ_n^t omits the final $\otimes 1$, making the two equal. $\square$

Theorem 5.30. *There is a unique non-monoidal $\mathbb{C}$-linear functor*

$$p : \mathcal{APar} \to \mathcal{Par}$$

such that $p \circ i = Id_{\mathcal{Par}}$ and

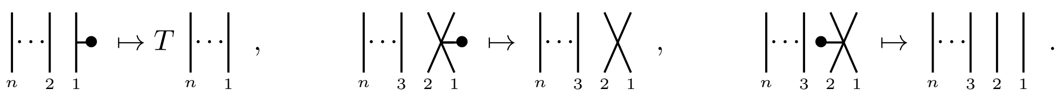


(14)

Moreover, for $t \in \mathbb{N}$ the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{APar} \arrow[r, "\Phi_t"] & \mathrm{End}_{\mathbb{C}}(\mathbb{C}[S_t] - mod) \arrow[d, "\mathrm{Ev}"] \\ \mathcal{Par} \arrow[u, "p"] \arrow[r, "\phi_t"'] & \mathbb{C}[S_t] - mod \end{tikzcd} .$$

Additionally, p maps



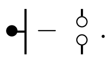
(15)

Proof. We define $p(n) = n$ on objects and on morphisms $f \in \mathrm{Hom}_{\mathcal{APar}}(n, m)$ by $p(f) = \bar{f}$. By section 5, this is a well defined $\mathbb{C}$-linear functor satisfying eq. (14). By definition 5.28, we have that $\gamma^t = (\gamma_n^t)_{n \in \mathbb{N}} : \mathrm{Ev} \circ \Phi_t \Rightarrow \phi_t \circ p$ is a natural isomorphism. Also, $p \circ i(f) = \overline{i(f)} = f$ yielding $p \circ i = Id_{\mathcal{Par}}$. Uniqueness follows from being defined on generators as a $\mathbb{C}$-linear category by corollary 5.23. The three remaining properties follow by commutativity of the diagram, comparing the actions through ϕ_t. $\square$

Corollary 5.31. *The functor $i : \mathcal{Par} \to \mathcal{APar}$ is faithful and the functor $p : \mathcal{APar} \to \mathcal{Par}$ is full.*

Proof. We already established $p \circ i = Id$. $\square$

Corollary 5.32. *The functor p induces an isomorphism $\mathcal{APar}/\mathcal{I} \to \mathcal{Par}$, where $\mathcal{I}$ is the left tensor ideal of $\mathcal{APar}$ generated by the morphism*



Proof. By the image of the left dot under p, it is clear that any morphisms in $\mathcal{I}$ are mapped to zero. This way, p induces $\bar{p} : \mathcal{APar}/\mathcal{I} \to \mathcal{Par}$. This is still surjective on objects and morphisms. For faithfulness, suppose $f + \mathcal{I}(n, m)$ is such that $\bar{p}(f + \mathcal{I}(n, m)) = 0$. Then $p(f) = 0$. By factoring out the left dots, f is a linear combination of compositions in the image of i, and we may thus assume $f = i(\bar{f})$ for $\bar{f} \in \mathrm{Hom}_{\mathcal{Par}}(n, m)$. But this yields

$$\bar{f} = p(i(\bar{f})) = p(f) = 0$$

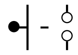
so that $f = i(\bar{f}) = 0$. $\square$

We are finally in a position to project morphisms all the way down to $\mathcal{P}ar_t$. In particular, for $t \in \mathbb{N}$, we have the full $\mathbb{C}$-linear functor

$$p_t = ev_t \circ p : \mathcal{AP}ar \to \mathcal{P}ar$$

which exactly as in corollary 5.32 induces an isomorphism

$$\mathcal{AP}ar/\mathcal{I}_t \to \mathcal{P}ar_t$$

where $\mathcal{I}_t$ is the left tensor ideal of $\mathcal{AP}ar$ generated by $T - t1_{\mathbf{1}}$ and  .

Definition 5.33. *We define the affine partition algebra to be*

$$AP_{2n} = \mathrm{End}_{\mathcal{AP}ar}(n)$$

We now denote the elements of AP_{2n} defined by the left and right dots on the j-th string by X_j^L and X_j^R as well as the left and right crossings on the k-th and $k+1$-th strings by S_k^L and S_k^R. Taking their images under p_t gives us elements of $P_{2n}(t)$ which we denote as

$$x_j^L = p_t(X_j^L), \;\; x_j^R = p_t(X_j^R)$$

and

$$s_k^L = p_t(S_k^L) \;\; s_k^R = p_t(S_k^R).$$

Theorem 5.34. *For $t \in \mathbb{N}$ let $\psi_t : P_{2n}(t) \to \mathrm{End}_{\mathbb{C}[S_t]}(V_t^{\otimes n})$ be the morphism induced by ϕ_t. Then the elements $x_j^L, x_j^R, s_k^L, s_k^R$ act on basis elements by the following formulas :*

$$\begin{aligned}
\psi_t(x_j^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) &= \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_{j+1}} \otimes (i \;\; i_j) \left[u_{i_j} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \\
\psi_t(x_j^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) &= \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_j} \otimes (i \;\; i_j) \left[u_{i_{j-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \\
\psi_t(s_k^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) &= u_{i_n} \otimes \cdots \otimes u_{i_k} \otimes (i_k \;\; i_{k+1}) \left[u_{i_{k-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \\
\psi_t(s_k^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) &= u_{i_n} \otimes \cdots \otimes u_{i_{k+2}} \otimes (i_k \;\; i_{k+1}) \left[u_{i_{k+1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right].
\end{aligned}$$

Proof. Follows by commutativity of theorem 5.30 and lemma 5.26. □

These actions are the same with the ones corresponding to the classic YJM elements and the Enyang generators of the partition algebra $P_{2n}(t)$ as in [5]. By the fact that ψ_t is an isomorphism for large t, we can finally say that we recovered the YJM elements with this approach. Notice that the only thing to take care is that we are now identifying $P_{2n}(t)$ with the algebra of diagrams reflected through a vertical axis, to account for enumerating vertices from right to left instead of from left to right.

Corollary 5.35. *The following equalities hold*

$$\begin{aligned}
x_j^L = x_{2j}, \;\; x_j^R = t - x_{2j-1}, \\
s_k^L = \sigma_{2k}, \;\; s_k^R = \sigma_{2k+1}
\end{aligned}$$

where x_j, σ_j are as in definition 3.33.

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