Chromatic quasi-symmetric functions and Hessenberg varieties

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Introduction

The *chromatic polynomial* of a graph is a well-studied graph invariant, whose properties are very elusive, in spite of its simple and intuitive definition. The *chromatic symmetric function* of a graph, introduced by Stanley in 1995, is a multivariate refinement of the chromatic polynomial. This graph invariant lies in the ring of *symmetric functions*, enabling its study from both a combinatorial and a representation theoretic point of view.

On the one hand, the space of symmetric functions has many distinguished bases, whose structure constants and connecting coefficients encode important invariants of partitions and permutations. On the other hand, the algebra of symmetric functions is *isomorphic* to the algebra of finite-dimensional representations of the symmetric group. Moreover, this isomorphism plays well with the distinguished bases of symmetric functions, allowing questions about positivity, which are fundamental in combinatorics, to be formulated in terms of decompositions on the representation theoretic side.

The *Stanley–Stembridge conjecture*, a long-standing conjecture in combinatorics, concerns e-positivity of chromatic symmetric functions. The *Shareshian–Wachs conjecture* from 2012 is regarded as a stepping stone towards the Stanley–Stembridge conjecture. It strives to realize the representations corresponding to chromatic symmetric functions explicitly, on the cohomology of certain closed subvarieties of the full flag variety, called *Hessenberg varieties.* This correspondence hinges on a combinatorial model for T-equivariant cohomology classes of Hessenberg varieties, which is given by the *moment graph* of the Hessenberg variety. The symmetries of the moment graph allow us to construct a suitable S_n-action on the cohomology, called *Tymoczko's dot action*, in a purely combinatorial way. This action is the geometric counterpart of chromatic (quasi-)symmetric functions.

The aim of this thesis is twofold. First, we would like to understand the correspondence between chromatic symmetric functions and Hessenberg varieties by computing a wide range of explicit examples at each stage of the construction. In Section 1, we provide the necessary background and motivating examples, involving chromatic polynomials, chromatic symmetric functions and chromatic quasi-symmetric function, focusing on *incomparability graphs of Dyck paths*, a class of graphs that play an important role in the study of e-positivity. The reasons for this will become apparent as we delve into the examples. In Section 2, we introduce moment graphs of Hessenberg varieties, and the so-called *flow-up* classes, which will be helpful to compute the dot action explicitly. We carry out these computations for some specific Hessenberg varieties, such as the *full flag variety of rank 3* and the *permutohedral variety of rank 3*, thereby verifying the Shareshian–Wach conjecture for these specific examples. The second aim of the thesis, presented in Section 3, is to go through *Guay-Paquet's Hopf algebraic proof* of the Shareshian–Wachs conjecture, and make it more palatable by interpreting each step in terms of our explicit examples from Section 2.

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1 Stanley–Stembridge Conjecture

In this section, we explore *chromatic quasi-symmetric functions*, a vast generalization of chromatic polynomials. We illustrate the celebrated Stanley–Stembridge conjecture by computing numerous examples, and describe a strategy for proving special cases of it.

Our exposition of the fundamental facts in Lemma 1.23, 1.29, 1.39, 1.42 and 1.49 follows [Sa1], while the proofs of Lemma 1.25 and 1.37 follow [St2]. In the proof of Theorem 1.69, we expand on the article [SW]. Our explicit computations featured in this section include the Frobenius character of the coinvariant algebra in Example 1.53 and the e-expansion of various families of graphs arising from Dyck paths in Example 1.54, 1.61, 1.70 and 1.71.

1.1 Chromatic Polynomials

Definition 1.1. A *proper colouring* of a graph $G = (V, E)$ is a colouring $\kappa\colon V \to \mathbb{Z}_{>0}$ such that no adjacent pair of vertices have the same colour, i.e. if u and v are adjacent, then $\kappa(u) \neq \kappa(v)$. The minimal number of colours required for a proper colouring of G is called the *chromatic number* of G, denoted by $\chi(G)$.

All the graphs that we consider will be finite and simple, i.e. graph with finitely many vertices, without loops or multiple edges, and from now on we will tacitly assume these properties.

Example 1.2. The chromatic number of the *Petersen graph* is three. Indeed, there exists a proper colouring with three colours (e.g. the colouring below), and there is no proper colouring with two colours, since the Petersen graph contains 5-cycles.

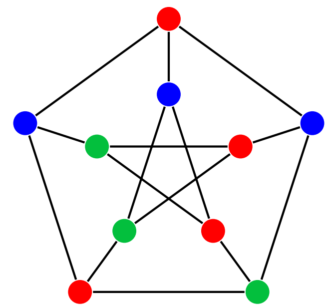


Figure 1: a 3-colouring of the Petersen graph

The notion of graph colourings gives rise to many notoriously difficult problems in combinatorics, such as the four colour problem or the classification of perfect graphs. So one would naturally be tempted to attach algebraic invariants to graph colourings.

Perhaps the most natural invariant that one can associate to proper graph colourings is the *chromatic polynomial*, introduced by George D. Birkhoff in the 1912 article [Bir]. ■

Definition 1.3. The *chromatic polynomial* enumerates the number of proper colourings of a graph G by the number of colours being used,

$$P(G; r) := \sum_{\substack{\kappa\colon V \to [r] \\ \text{proper}}} 1, \tag{1.1}$$

where $[r]$ denotes the set $\{1, \ldots, r\}$ of the first r positive integers.

Remark 1.4. (a). It is not a priori clear from the definition that $P(G; r)$ is a polynomial in r. But there is a simple *contraction-deletion algorithm*, described in Lemma 1.5, that bridges the gap between the complete graph and the edgeless graph.

(b). Note that $P(G; r) = 0$ for any $r < \chi(G)$. In particular, the constant term is zero for nonempty graphs.

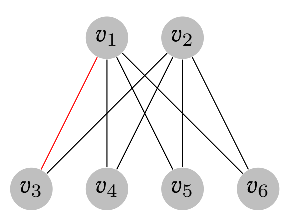
(c). The simplest examples are the edgeless graph E_n and the complete graph K_n of *order* n, i.e. with n vertices. In the case of the edgeless graph, there are no restrictions on the colourings, so $P(E_n; r) = r^n$. At the other extreme, each pair of vertices in K_n are adjacent. So each colour can be used at most once, and we have $P(K_n; r) = r(r-1)\cdots(r-n+1)$.

Lemma 1.5. *For any $e \in E$, we have*

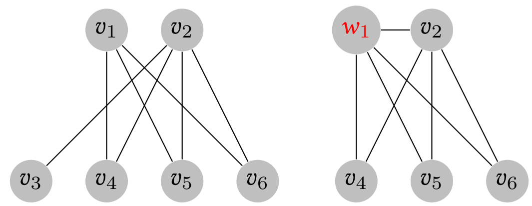
$$P(G; r) = P(G \backslash e; r) - P(G/e; r), \tag{1.2}$$

where G/e denotes the contraction *of the edge e and $G \backslash e$ denotes the* deletion *of e from the graph G.*

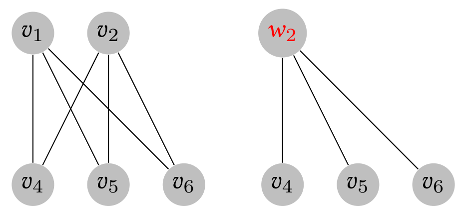
Example 1.6. Let us illustrate the contraction-deletion algorithm by a short example. Consider the complete bipartite graph $K_{2,4}$, and perform contraction-deletion on the red edge $e = v_1 v_3$



The graphs $K_{2,4}\backslash e$ and $K_{2,4}/e$ are obtained by deleting and contracting the red edge, respectively



If we then perform contraction-deletion on the edge $f = w_1v_2$, we end up with the two graphs



The second of these is the tree $K_{1,3}$, called the *claw graph*, which plays a central role in this section. ■

Proof of Lemma 1.5. Denote the endpoints of e by u and v, and note that the set of proper colourings of $G\backslash e$ is the disjoint union of the proper colourings κ of $G\backslash e$ for which $\kappa(u) = \kappa(v)$ and those for which $\kappa(u) \neq \kappa(v)$. The former corresponds bijectively to proper colourings of G/e and the latter corresponds to proper colourings of G. ■

Lemma 1.7. *The chromatic polynomial $P(G; r)$ is a polynomial in r.*

Proof. First, we induct on the number of vertices in G. For the graph K_1 with a single vertex, the chromatic polynomial is just r. If $G = E_n$ is the edgeless graph, then $P(G; r) = r^n$ is also a polynomial. We proceed by induction on the number of edges in G. The base case is K_1 for both inductions. Take an edge $e \in E$, then we can apply contraction-deletion, and write $P(G; r) = P(G\backslash e; r) - P(G/e; r)$. The right-hand side is a difference of polynomials by induction, since deletion of an edge decreases the number of edges and contraction of an edge decreases the number of vertices. ■

Remark 1.8. (a). We have seen that the constant term is always zero. Note that $P(G; r)$ is monic of degree n, where n is the order of G. This can be seen by induction via the contraction-deletion formula: eventually we end up with the edgeless graph E_n and graphs of smaller order.

(b). Note that the coefficient of the monomial r^{n-1} in $P(G; r)$ is $-m$, where m is the size of G. Indeed, take an edge e, and use induction on the size with the contraction-deletion formula. Then the coefficient of r^{n-1} in $P(G\backslash e; r)$ is $-(m-1)$ and the coefficient of r^{n-1} in $P(G/e; r)$ is 1, since G/e has order $n-1$, and the chromatic polynomial is monic. Hence, the coefficient is $-(m-1) - 1 = -m$.

(c). The coefficients of the chromatic polynomial alternate in sign. Indeed, we can proceed by induction, and write

$$P(G\backslash e; r) = r^n - a_{n-1}r^{n-1} + \cdots + (-1)^n a_0, \tag{1.3}$$

$$P(G/e; r) = r^{n-1} - b_{n-2}r^{n-2} + \cdots + (-1)^{n-1} b_0, \tag{1.4}$$

where $a_i, b_i \geq 0$. Then by contraction-deletion, we have

$$P(G; r) = P(G\backslash e; r) - P(G/e; r) \tag{1.5}$$
$$= r^n - a_{n-1}r^{n-1} + \cdots + (-1)^n a_0 - (r^{n-1} - b_{n-2}r^{n-2} + \cdots + (-1)^{n-1} b_0) \tag{1.6}$$
$$= r^n - (a_{n-1} + 1)r^{n-1} + (a_{n-2} + b_{n-2})r^{n-1} - \cdots + (-1)^n (a_0 + b_0). \tag{1.7}$$

(d). There are various open problems about chromatic polynomials, the most audacious of which is to classify those polynomials which arise as the chromatic polynomial of some graph. We have seen that the coefficients have alternating sign. However, the sequence of absolute values satisfy a regularity property, called *log-concavity*: $c_i^2 \geq |c_{i+1}c_{i-1}| > 0$. The quintessential example of a log-concave sequence is the sequence of binomial coefficients, in rows of the Pascal triangle. Log-concavity of chromatic polynomials was proved in 2012 by June Huh, at the generality of matroids, using techniques from complex geometry in the context of realisable matroids (see [Huh]).

Along with formulating general statements about chromatic polynomials of graphs, one would also like to investigate which specific properties of graphs are captured by chromatic polynomials. The simplest example is the characterization of trees (among connected graphs).

Lemma 1.9. *A connected graph is a tree if and only if its chromatic polynomial is*

$$P(T; r) = r(r-1)^{n-1}. \tag{1.8}$$

Proof. Take a tree T. To compute its chromatic polynomial, we can proceed by induction on the order of T, the base case being the complete graph K_1 on one vertex. Pick a *leaf* v, i.e. a degree one vertex (which always exists in a tree). Each colouring of the tree $T\backslash v$ with r colours permits $r-1$ colours on v, since it has only one neighbour. So we have $P(T; r) = P(T\backslash e; r) \cdot (r-1)$, as claimed. In fact, finding a leaf can be made superfluous by appealing to the contraction-deletion algorithm, since the chromatic polynomial of a disjoint union is the product of the chromatic polynomials of the components.

For the converse, note that the coefficient of r^{n-1} in $P(T; r)$ is $-m$. By assumption, this coefficient is $-(n-1)$. If a connected graph has size $n-1$ and order n, then this graph is a tree. ■

Remark 1.10. Hence, the chromatic polynomial is sophisticated enough to distinguish between trees and connected graphs with cycles, but it cannot distinguish between non-isomorphic trees of the same order. For example, the claw and a path of order 4 have the same chromatic polynomial $r(r-1)^3$,

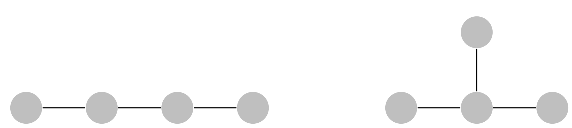


Figure 2: P_4 and the claw graph

which is a significant drawback, because the claw graph serves as a counterexample for many statements in graph theory, such as the *graph reconstruction conjecture*.

In the graph reconstruction conjecture, we associate to each graph the multiset of graphs, which are obtained by deleting an edge and all isolated vertices from the original graph. Note that the multiset associated to the claw graph $K_{3,1}$ consists of three copies of the path P_3 of order 4. The cycle C_3 of order 3 has the same associated multiset. Thus, we cannot distinguish between $K_{3,1}$ and C_3 based on these multisets. The graph reconstruction conjecture claims that this is the only counterexample, i.e. any other graph can be reconstructed from this multiset. We will see shadows of graph reconstruction in the description of certain combinatorial Hopf algebras, in Remark 3.37.

Remark 1.11. Chromatic symmetric functions are a multivariate refinement of chromatic polynomials introduced by Richard P. Stanley in 1995, in the article [St3]. This invariant was introduced in an attempt to remedy some of the deficiencies of chromatic polynomials (e.g. that chromatic polynomials cannot distinguish between trees of the same order). Although this has been checked for all trees up to order 23, in general it is still unknown whether the chromatic symmetric function distinguishes between all trees of the same order (see [MMW]).

Definition 1.12. For a graph G with vertex set $V = \{v_1, \ldots, v_n\}$, the *chromatic symmetric function* of G is defined as

$$X(G) = X(G;x) \coloneqq \sum_{\substack{c\colon V \to \mathbb{Z}_{>0} \\ \text{proper}}} x^c, \tag{1.9}$$

where the sum runs over all proper colourings of G, and where x^c denotes the monomial $x_{c(v_1)} \cdots x_{c(v_n)}$, the product of the variables indexed by the multiset of colours being used.

Remark 1.13. Note that chromatic symmetric functions are power series in infinitely many variables, and they are symmetric in the following sense: any permutation of the colours in a proper colouring produces another proper colouring.

1.2 Elementary Symmetric Functions

Definition 1.14. The *ring of symmetric functions* Sym consists of formal power series in infinitely variables $\{x_i\}_{i\in\mathbb{Z}_{>0}}$ with complex coefficients, which are fixed by any permutation of the variables and have bounded degree, i.e. there is a natural number m such that the degree of any monomial is at most m. The ring structure is inherited from the ring of formal power series.

Remark 1.15. (a). One could formulate the definition of symmetric functions over an arbitrary ring R, instead of the ground field $\mathbb{C}$, but for our present purposes it is unnecessary to take this general perspective. In Subsection 3.3, we will need $\mathbb{C}(t)$-coefficients as well as $\mathbb{C}$-coefficients, but everything in this section works analogously over any algebraically closed field k of characteristic zero.

(b). The bounded degree condition is imposed to make Sym into a graded $\mathbb{C}$-algebra, graded by degree, in other words the degree k part Sym_k consists of all symmetric functions of degree k.

If $n = |V|$ is the order of G, then the chromatic symmetric function is homogeneous of degree n, i.e. $X(G) \in \mathrm{Sym}_n$, since x^c has degree n for any colouring c.

(c). The chromatic symmetric function encodes more information than the chromatic polynomial. Indeed, if we evaluate $X(G)$ at 1 in the first r variables and at 0 elsewhere, then we recover $P(G;r)$.

Lemma 1.16. *Recall that a* partition $\lambda \vdash n$ *is a tuple* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ *of non-increasing positive integers that add up to* n. *There is a* $\mathbb{C}$*-basis of* Sym_n *consisting of* monomial symmetric functions*, indexed by partitions* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0) \vdash n$,

$$m_\lambda \coloneqq \sum_\alpha x^\alpha = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots, \tag{1.10}$$

where the sum runs over all weak compositions α *of shape* λ*, i.e. all infinite tuples* $\alpha = (\alpha_1, \alpha_2 \ldots)$ *obtained by rearranging the parts of* λ *arbitrarily, and putting zeros everywhere else.*

In particular, the dimension of Sym_n *is* $p(n)$*, the number of* partitions *of* $[n]$.

Remark 1.17. We will see in Construction 3.22 a more succinct way of describing monomial symmetric functions, with the aid of *monomial quasi-symmetric functions*, that we introduce in Subsection 3.2.

Example 1.18. For $n = 3$, there are three partitions, (3), $(2,1)$ and $(1,1,1)$. The corresponding monomial symmetric functions are given by

$$\begin{aligned} m_{(3)} &= x_1^3 + x_2^3 + x_3^3 + x_4^3 \cdots, \\ m_{(2,1)} &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + \cdots, \\ m_{(1,1,1)} &= x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots + x_2 x_3 x_4 + x_2 x_3 x_5 + \cdots. \end{aligned}$$

■

Remark 1.19. Another vector space with dimension $p(n)$ that one often encounters in representation theory, is the space of *class functions* of the symmetric group S_n, i.e. the space of $\mathbb{C}$-valued functions that are constant on conjugacy classes. We will return to this observation when we discuss the connection between symmetric functions and representations of the symmetric group in Subsection 1.3.

Proof of Lemma 1.16. The symmetric functions m_λ, for distinct partitions, are linearly independent, since they have no monomials in common. To show that they span Sym_n, take a symmetric function $f \in \mathrm{Sym}_n$, and pick a monomial $c \cdot x_{i_1}^{\lambda_1} \cdots x_{i_d}^{\lambda_d}$ with $c \in \mathbb{C}$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$ of maximal degree. Then the symmetric function $f - cm_\lambda$ has a smaller number of partitions that yield monomials of maximal degree, and there can only be finitely many of such partitions λ. Hence, we can always reduce the degree of f in finitely many steps, and obtain the monomial expansion inductively. ■

Remark 1.20. We will take the liberty to use several different notations for partitions, which is standard practice in the literature. For instance, it is sometimes more convenient to abbreviate the partition $(1,1,1)$ by (1^3). Also, when using partitions as indices, one may omit the brackets, and write m_{12} instead of $m_{(1,2)}$.

Construction 1.21. There are some other notable bases of Sym_n, and much of the theory of symmetric functions revolves around their connecting coefficients (the entries of the transition matrix), because they encode important quantities from enumerative combinatorics.

The nth *power sum symmetric function* is

$$p_{(n)} = m_{(n)} = \sum_{i \geq 1} x_i^n. \tag{1.11}$$

The nth *elementary symmetric function* is

$$e_{(n)} = m_{(1^n)} = \sum_{\substack{i_1, \ldots, i_n \in \mathbb{Z}_{>0} \\ i_1 < \cdots < i_n}} x_{i_1} \cdots x_{i_n}, \tag{1.12}$$

which is the symmetric function analogue of the nth elementary symmetric polynomial (featured, for example, in Viète's formulas, see [Gar], Chapter 15).

A similar construction yields the nth *complete homogeneous symmetric function*, given by

$$h_{(n)} = \sum_{\lambda \vdash n} m_\lambda = \sum_{\substack{i_1, \ldots, i_n \in \mathbb{Z}_{>0} \\ i_1 \leq \cdots \leq i_n}} x_{i_1} \cdots x_{i_n}. \tag{1.13}$$

To each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, one can associate a power sum symmetric function by taking the product of the power sum symmetric functions p_{λ_i} corresponding to the parts,

$$p_\lambda = p_{(\lambda_1)} p_{(\lambda_2)} \cdots p_{(\lambda_r)}, \tag{1.14}$$

and similarly for e_λ and h_λ. One can see that the sets $\{e_\lambda \mid \lambda \vdash n\}$, $\{p_\lambda \mid \lambda \vdash n\}$ and $\{h_\lambda \mid \lambda \vdash n\}$ form bases of Sym_n in many different ways.

Example 1.22. For instance, we have

$$p_{(1)} = x_1 + x_2 + x_3 + \cdots , \tag{1.15}$$
$$p_{(2)} = x_1^2 + x_2^2 + x_3^3 + \cdots , \tag{1.16}$$
$$e_{(2)} = x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_2x_3 + x_2x_4 + \cdots , \tag{1.17}$$

so we can express $e_{(2)}$ as

$$e_{(2)} = \frac{p_{(1)}^2 - p_{(2)}}{2} = \frac{p_{(1,1)} - p_{(2)}}{2}. \tag{1.18}$$

■

Lemma 1.23. *The set* $\{p_\lambda \mid \lambda \vdash n\}$ *forms a* $\mathbb{C}$*-basis of* Sym_n.

Proof. One can show that the transformation matrix from the monomial basis to power sum symmetric functions is upper-triangular and invertible via imposing two different orderings on the partitions indexing the rows and columns.

First, consider the *lexicographic order*, which is the total order on partitions defined by $\mu = (\mu_1, \ldots, \mu_m) < \lambda = (\lambda_1, \ldots, \lambda_r)$ if for some $i \geq 1$, we have $\mu_i < \lambda_i$ and $\lambda_j = \mu_j$ for any $j < i$, where we regard indices that are not present as indices of zero parts.

If $x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m}$ appears in the expansion

$$p_\lambda = \prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} + \cdots), \tag{1.19}$$

then each μ_j is a sum of some of the λ_i. Adding parts of a partition to another partition makes it larger with respect to the *dominance order*, which is the partial order on partitions defined by $\mu = (\mu_1, \ldots, \mu_m) \trianglelefteq \lambda = (\lambda_1, \ldots, \lambda_r)$ if for each $i \geq 1$, we have $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$. In our case, m_λ is the smallest term that appears with respect to the dominance order, and consequently, we have

$$p_\lambda = c_{\lambda\lambda} m_\lambda + \sum_{\mu \rhd \lambda} c_{\lambda\mu} m_\mu, \tag{1.20}$$

where $c_{\lambda\lambda} \neq 0$, which shows that the transition matrix is indeed upper-triangular and invertible. ■

Remark 1.24. When it comes to e_λ and h_λ, one can appeal to the generating functions:

$$E(t) = \sum_{n \geq 0} e_n(x) t^n = \prod_{i \geq 1} (1 + x_i t) \in \mathbb{C}[\![x, t]\!], \tag{1.21}$$
$$H(t) = \sum_{n \geq 0} h_n(x) t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t} \in \mathbb{C}[\![x, t]\!], \tag{1.22}$$

and use the identity $H(t)E(-t) = 1$. The generating functions can be computed as follows. Pick a partition λ with distinct parts and consider the *weight* associated to λ, which is the monomial $\mathrm{wt}(\lambda) := x_{\lambda_1} \cdots x_{\lambda_r}$. One can introduce a new variable t, and consider a twisted version of the weight $\widetilde{\mathrm{wt}}(\lambda) := t^{\ell(\lambda)} x_{\lambda_1} \cdots x_{\lambda_r}$. Note that we can express $E(t)$ as

$$E(t) = \sum_{n \geq 0} \sum_{\lambda \vdash n} \widetilde{\mathrm{wt}}(\lambda). \tag{1.23}$$

On the other hand, we have

$$\{\lambda \mid \text{all parts } \lambda_i \text{ distinct}\} = (\{1^0 \cup 1^1\}) \times (\{2^0 \cup 2^1\}) \times \cdots, \tag{1.24}$$

which shows the product formula. The generating function $H(t)$ can be computed by explicitly expanding

$$\prod_{i \geq 1} \frac{1}{1 - x_i t} = \prod_{i \geq 1} (1 + x_i t + x_i^2 t^2 + \cdots), \tag{1.25}$$

and realizing that the coefficient of a fixed power t^n is indeed the sum $\sum\limits_{\lambda \vdash n} m_\lambda$.

In fact, the above identity $H(t)E(-t) = 1$ yields a much stronger statement than the fact that the e_λ and the h_λ form bases of Sym_n. But first, let us describe the transformation matrices a bit more explicitly.

Lemma 1.25. *Given two partitions $\lambda, \mu \vdash n$, regarded as weak compositions, let $M_{\lambda\mu}$ denote the number of matrices $A \in \mathrm{Mat}_{n \times n}(\mathbb{F}_2)$ with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$, where $\mathrm{row}(A)$ denotes the weak composition whose parts are the row sums of A and $\mathrm{col}(A)$ denotes the weak composition whose parts are the column sums of A. Then we have*

$$e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu. \tag{1.26}$$

Remark 1.26. To clarify the notation, consider the example

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.27}$$

Then the corresponding weak compositions are $\mathrm{row}(A) = (3, 2, 3, 1, 1)$ and $\mathrm{col}(A) = (4, 0, 2, 1, 3)$.

Note that the transformation matrix $(M_{\lambda\mu})$ is symmetric. Indeed, the matrix A has $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$ if and only if $\mathrm{row}(A^t) = \mu$ and $\mathrm{col}(A^t) = \lambda$, where A^t denotes the transpose of A.

Proof. Since e_λ is a symmetric function, it suffices to show that

$$e_\lambda = \sum_{\alpha} M_{\lambda\alpha} x^\alpha, \tag{1.28}$$

where the sum runs over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n. Consider the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.29}$$

The coefficient of x^α in the expansion of e_λ can be described by the number of ways to choose λ_1 entries in the first row, λ_2 entries in the second row, and so on, such that the product of these entries is x^α. Substituting the chosen entries by 1 and the others by 0, we end up with the matrices A, counted by $M_{\lambda\mu}$, where μ is the partition corresponding to the weak composition α. ∎

Example 1.27. Let us list the expansion of elementary symmetric functions corresponding to the partitions of the first four positive integers:

$$e_{(1)} = m_{(1)}$$

$$\begin{aligned} e_{(2)} &= m_{(1^2)} \\ e_{(1^2)} &= 2m_{(1^2)} + m_{(2)} \end{aligned}$$

$$\begin{aligned} e_{(3)} &= m_{(1^3)} \\ e_{(2,1)} &= 3m_{(1^3)} + m_{(2,1)} \\ e_{(1^3)} &= 6m_{(1^3)} + 3m_{(2,1)} + m_{(3)} \end{aligned}$$

$$\begin{aligned} e_{(4)} &= m_{(1^4)} \\ e_{(3,1)} &= 4m_{(1^4)} + m_{(2,1^2)} \\ e_{(2^2)} &= 6m_{(1^4)} + 2m_{(2,1^2)} + m_{(2^2)} \\ e_{(2,1^2)} &= 12m_{(1^4)} + 5m_{(2,1^2)} + 2m_{(2^2)} + m_{(3,1)} \\ e_{(1^4)} &= 24m_{(1^4)} + 12m_{(2,1^2)} + 6m_{(2^2)} + 4m_{(3,1)} + m_{(4)} \end{aligned}$$

It is apparent from the examples above that the transition matrix from monomials to elementary symmetric functions is unipotent, which is equivalent to saying that the e_i's are algebraically independent generators of Sym, with coefficients coming from an arbitrary ring R (but as we said, we will not need this level of generality). So one can also prove that the set $\{e_\lambda \colon \lambda \vdash n\}$ is a basis of Sym_n directly, without using generating functions.

Similarly to the transformation matrix $(M_{\lambda\mu})$, one can consider the (symmetric) matrices $(N_{\lambda\mu})$ whose entries count the number of matrices A with positive integer entries with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$. Then we have

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu. \tag{1.30}$$

The proof of this identity goes by a similar double-counting argument to the expansion of e_λ. ■

Remark 1.28. The values $e_k(1^n)$ and $h_k(1^n)$ (where the first n variables are evaluated at 1 and the others at 0) are $\binom{n}{k}$ and $(-1)^k\binom{-n}{k}$, respectively. The relationship between these quantities is the archetypal example of a combinatorial duality,

$$\binom{-n}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(-n-i) \tag{1.31}$$

$$= \frac{(-1)^k}{k!}\prod_{i=0}^{k-1}(n+i) \tag{1.32}$$

$$= \frac{(-1)^k}{k!}\prod_{i=0}^{k-1}\frac{(n+k-1)!}{(n-1)!} \tag{1.33}$$

$$= (-1)^k\binom{n+k-1}{k}. \tag{1.34}$$

This duality also has a manifestation in the theory of symmetric functions, described by the following lemma.

Lemma 1.29. *There exists a* $\mathbb{C}$*-algebra involution* $\omega\colon \mathrm{Sym} \to \mathrm{Sym}$ *given by* $\omega(e_n) = h_n$ *for any* $n \in \mathbb{Z}_{>0}$*, called the* fundamental involution *of* Sym.

Proof. Note that the generating function identity, $H(t)E(-t) = 1$ from Remark 1.24 yields

$$\sum_{i+j=n} (-1)^i e_i h_j = \delta_{0,n}, \tag{1.35}$$

so we have $e_0 = 1 = h_0$, and for any $n \in \mathbb{Z}_{>0}$, we have

$$e_n = e_{n-1}h_1 - e_{n-2}h_2 + e_{n-3}h_3 - \cdots, \tag{1.36}$$
$$h_n = h_{n-1}e_1 - h_{n-2}e_2 + h_{n-3}e_3 - \cdots. \tag{1.37}$$

The generators are algebraically independent, so we can define a $\mathbb{C}$-algebra endomorphism ω of Sym, by $\omega(e_n) = h_n$ for $n \in \mathbb{Z}_{>0}$ and $\omega(e_0) = \omega(1) = 1 = h_0$ for $n = 0$. To show that w is an involution, we proceed by induction, and compute

$$\begin{aligned}
\omega(h_n) &= \omega(h_{n-1}e_1 - h_{n-2}e_2 + h_{n-3}e_3 - \cdots) && (1.38)\\
&= \omega(h_{n-1})\omega(e_1) - \omega(h_{n-2})\omega(e_2) + \omega(h_{n-3})\omega(e_3) - \cdots && (1.39)\\
&= e_{n-1}h_1 - e_{n-2}h_2 + e_{n-3}h_3 - \cdots && (1.40)\\
&= e_n. && (1.41)
\end{aligned}$$

Hence $\omega^2(e_n) = e_n$ for each $n \geq 0$, which agrees with id on the generators, and consequently $\omega^2 = \mathrm{id}$. ∎

Remark 1.30. The expansion of chromatic symmetric functions in the e-basis will be particularly important for us. We will be focusing on graphs whose chromatic symmetric functions have positive coefficients in the e-expansion. These chromatic symmetric functions (as well as the original graphs) are said to be *e-positive*. Note that a symmetric function f is e-positive if and only if $\omega(f)$ is h-positive (having positive coefficients when expanded in the h-basis).

1.3 Schur Functions and the Character Map

Let R_n denote the vector space of *class function* of S_n over $\mathbb{C}$, i.e. complex valued functions which are constant on conjugacy classes. The dimension of R_n is the number of conjugacy classes in S_n, which are characterized by cycle types of permutations. Cycle types of permutations are in bijection with partitions $\lambda \vdash n$. Consequently, we have

$$\dim(R_n) = p(n), \tag{1.42}$$

where $p(n)$ denotes the number of partitions of n, so that R_n and Sym_n are isomorphic as vector spaces.

Definition 1.31. Let G be a finite group, and $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ a finite dimensional *representation* of G over $\mathbb{C}$. Then the *character* of ρ is the homomorphism

$$\chi\colon G \to \mathbb{C}, \qquad \chi(g) \coloneqq \operatorname{tr}\rho(g). \tag{1.43}$$

Note that the character is independent of the choice of the matrix representation among the ones corresponding to a given G-module, since for any $T \in \mathrm{GL}_n(\mathbb{C})$, $g \in G$ and $\widetilde{\rho} = T\rho T^{-1}$, we have

$$\operatorname{tr}\widetilde{\rho}(g) = \operatorname{tr} T\rho(g)T^{-1} = \operatorname{tr}\rho(g). \tag{1.44}$$

Remark 1.32. Let us briefly recall the main properties of characters χ of finite dimensional representations $\rho\colon S_n \to \mathrm{GL}_m(\mathbb{C})$ of the symmetric group S_n over $\mathbb{C}$.

(a). For any such character, $\chi(\mathrm{id}) = m$ and χ is a class function.

(b). For any other finite dimensional representation $\widetilde{\rho}$ of G, if $\widetilde{\rho}$ is isomorphic to ρ and $\widetilde{\rho}$ has character $\widetilde{\chi}$, then $\chi = \widetilde{\chi}$. In fact, the converse also holds, which can be seen by looking at the *character relations*, listed in part (e), that arise from the following inner product.

(c). Class functions have a canonical *inner product*, given by

$$\langle \phi, \psi \rangle = \frac{1}{n!} \sum_{\pi \in S_n} \phi(\pi)\psi(\pi). \tag{1.45}$$

This stems from the more general inner product on multiplicative functions $\varphi, \psi \colon G \to A$, where G is a finite group and A is a $\mathbb{C}$-algebra, given by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\psi(g^{-1}). \tag{1.46}$$

Indeed, the two inner products coincide for class functions of the symmetric group, since π and π^{-1} have the same cycle type for any $\pi \in S_n$.

(d). *Irreducible characters* (which correspond bijectively to irreducible representations) of S_n form an orthonormal basis for R_n with respect to this inner product,

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi}. \tag{1.47}$$

(e). *Maschke's theorem* states that ρ decomposes uniquely as a direct sum of pairwise inequivalent irreducibles with some multiplicities,

$$X \cong m_1 \rho^{(1)} \oplus \cdots \oplus m_k \rho^{(k)}, \tag{1.48}$$

Denote the character of each $\rho^{(i)}$ by $\chi^{(i)}$. Then the following relations hold,

$$\chi = m_1 \chi^{(1)} + \cdots + m_k \chi^{(k)}, \tag{1.49}$$

$$\langle \chi, \chi^{(i)} \rangle = m_i \quad \text{for all } i, \tag{1.50}$$

$$\langle \chi, \chi \rangle = m_1^2 + \cdots + m_k^2. \tag{1.51}$$

Moreover, the representation ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

(f). Another important feature of this inner product is the hom-tensor adjunction, and even a reciprocity, between restriction and induction. For a subgroup $H \leq G$ and multiplicative functions $\varphi \colon H \to A$ and $\psi \colon G \to A$ as above, we have a duality

$$\langle \mathrm{Ind}_H^G(\varphi), \psi \rangle = \langle \varphi, \mathrm{Res}_H^G(\psi) \rangle, \tag{1.52}$$

called the *Frobenius reciprocity*. Recall that the *restriction* is the pullback along the inclusion, and the *induction* is given by tensoring with the group algebra $k[G]$ over $k[H]$, where we tacitly use the fact that the category of finite dimensional G-representations is equivalent to the category of $k[G]$-modules. In particular, the induction $\mathrm{Ind}_H^G(1)$ is the G-module $\mathbb{C}\mathcal{H}$, where $\mathcal{H}$ is the set of all cosets, where 1 denotes the trivial representation of G.

Remark 1.33. To construct all irreducible representations of S_n (which are indexed by partitions), the most natural thing one can do is to consider *Young subgroups* of S_n, associated to partitions $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$,

$$S_\lambda \coloneqq S_{\lambda_1} \times \cdots \times S_{\lambda_r}. \tag{1.53}$$

Recall that the shape of a Young tableau is the partition corresponding to its underlying Young diagram and the content is the weak composition that describes the numbers appearing in the entries with multiplicities, so the content (1^n) simply means that each number from the set $[n]$ is used exactly once. As an example, if $\lambda = (2,1)$ and $n = 2$, then there are six tableaux with content $1, 2, 3$, each used exactly once,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \tag{1.54}$$

Young tabloids are the equivalence classes of Young tableaux of a given shape $\lambda \vdash n$ and content (1^n), under the natural action of the Young subgroup S_λ on the rows of these tableaux. So the Young tabloids corresponding to the Young tableaux in (1.54) are

$$\begin{array}{cc}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{cc}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{cc}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \tag{1.55}$$

We shall shortly revisit this example in Construction 1.42. Note that here we did not impose any of the customary restrictions on increasing or non-decreasing entries along the rows and columns of the diagram, like in the case of semi-standard Young tableaux, in Remark 1.34.

The induction $M^\lambda \coloneqq \operatorname{Ind}_{S_\lambda}^{S_n}(1)$ can be described explicitly as

$$M^\lambda = \mathbb{C}\{\{t\} : \operatorname{sh}(t) = \lambda\}, \tag{1.56}$$

the span of the set of Young tabloids $\{t\}$ with the natural S_n-action.

In general, Young modules are not irreducible S_n-modules, but they are an important step in the construction of irreducible representations. In particular, under a suitable ordering (the dual lexicographic order) of partitions $\lambda^{(1)}, \ldots, \lambda^{p(n)} \vdash n$, the first Young module $S^{\lambda^{(1)}} \coloneqq M^{\lambda^{(1)}}$ is irreducible, and inductively, $M^{\lambda^{(i)}}$ decomposes as a direct sum of the irreducibles $S^{\lambda^{(j)}}$ with multiplicities and $j < i$, that we have already constructed, in addition to a new irreducible S_n-module that we denote by $S^{\lambda^{(i)}}$.

Remark 1.34. Recall that in a *semistandard Young tableau*, the numbers along the rows of the diagram are non-decreasing and they are increasing along the columns. The multiplicity of the Specht module S^λ in the Young module M^μ is the number of semi-standard Young tableaux of shape λ and content μ, which called the *Kostka number* and is denoted by $K_{\lambda\mu}$. This formula is called *Young's rule*. The set of all semistandard Young tableaux of shape λ, filled with numbers coming from the set $[n]$, is denoted by $\mathrm{SSYT}(\lambda, [n])$. As an example, if $\lambda = (2,1)$ and $n = 2$, then $\mathrm{SSYT}(\lambda, [n])$ consists of

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \tag{1.57}$$

For a Young tableau T of shape λ and content μ, we can define the product of the entries,

$$x^T := \prod_{(i,j)} x_{T_{i,j}}, \tag{1.58}$$

called the *weight* $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots$ of the composition μ. Recall that *compositions* are sequences of positive integers like partitions, but here the order of the parts is also fixed, i.e. compositions are not necessarily non-increasing sequences. Earlier, we have used weak-compositions, where zero parts were also permitted.

Definition 1.35. The *Schur function* associated to $\lambda \vdash n$ is defined as

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T, \tag{1.59}$$

where the sum runs over all semistandard Young tableaux of shape λ and any content from the positive integers $\mathbb{Z}_{>0}$.

Remark 1.36. Schur functions bridge the gap between elementary symmetric functions and complete homogeneous symmetric functions: $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

Lemma 1.37. *Schur functions are symmetric.*

Proof. To see that Schur functions are symmetric, we can appeal to the representation theory of S_n, and notice that for any rearrangement $\widetilde{\mu}$ of μ, the corresponding Young modules are isomorphic, and consequently they have the same multiplicities $K_{\mu\lambda}$ of Specht modules. We will further elaborate on this in the proof of Lemma 1.39.

There is also an insightful combinatorial proof. Since simple transpositions generate S_n, it suffices to show that for each $i \in [n-1]$, the Schur function s_λ is invariant under the action $(i, i+1).s_\lambda(x) = s_\lambda(x)$.

To this end, we define an involution on $\mathrm{SSYT}(\lambda, [n])$ that swaps the number of i's and an $i+1$'s in the tableau (making sure that it remains semi-standard). For each i, we fix the boxes with i if there is a box with $i+1$ strictly below, and the boxes with $i+1$ if there is a box with i strictly above. In each row, we can then swap the number of non-fixed i's and $i+1$'s, and the tableau we obtain will still be semi-standard.

To illustrate this, consider the tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\
\hline
2 & 3 & 3 & 3 & 3 & 3 \\
\cline{1-6}
3 \\
\cline{1-1}
\end{array}$$

which turns into the following tableau when we take $i = 2$,

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & {\color{red}2} & {\color{red}2} & {\color{red}2} & 2 & {\color{blue}3} & {\color{blue}3} & {\color{blue}3} & {\color{blue}3} & {\color{blue}3} & 3 \\
\hline
{\color{red}2} & {\color{blue}2} & {\color{blue}2} & {\color{red}3} & {\color{red}3} & {\color{red}3} \\
\cline{1-6}
{\color{red}3} \\
\cline{1-1}
\end{array}$$

where the red entries are fixed and the blue ones have been swapped. For instance, in the first row, originally there were six non-fixed 2s and one non-fixed 3, so we should turn five of the non-fixed 2s into 3s. ∎

Example 1.38. Let us list the Schur functions corresponding to partition of the first four positive integers in their monomial expansion. This can be done by directly computing the first few Kostka numbers: $K_{(1),(1)} = 1$, $K_{(2),(2)} = 1$, $K_{(2),(1^2)} = 1$, $K_{(1^2),(1^2)} = 1$, $K_{(1^2),(2)} = 0$, etc, and we get

$$
\begin{aligned}
s_{(1)} &= m_{(1)} \\
\\
s_{(1^2)} &= m_{(1^2)} \\
s_{(2)} &= m_{(1^2)} + m_{(2)} \\
\\
s_{(1^3)} &= m_{(1^3)} \\
s_{(2,1)} &= 2m_{(1^3)} + m_{(2,1)} \\
s_{(3)} &= m_{(1^3)} + m_{(2,1)} + m_{(3)} \\
\\
s_{(1^4)} &= m_{(1^4)} \\
s_{(2,1^2)} &= 3m_{(1^4)} + m_{(2,1^2)} \\
s_{(2^2)} &= 2m_{(1^4)} + m_{(2,1^2)} + m_{(2^2)} \\
s_{(3,1)} &= 3m_{(1^4)} + 2m_{(2,1^2)} + m_{(2^2)} + m_{(3,1)} \\
s_{(4)} &= m_{(1^4)} + m_{(2,1^2)} + m_{(2^2)} + m_{(3,1)} + m_{(4)}
\end{aligned}
$$

■

Lemma 1.39. *The set of Schur functions $\{s_\lambda \mid \lambda \vdash n\}$ forms a basis for* Sym_n.

Proof. By definition of Schur functions, we have

$$s_\lambda = \sum_{\mu} K_{\lambda\mu} x^{\mu}, \tag{1.60}$$

where the sum runs over all compositions μ of n, and $K_{\lambda\mu}$ is the Kostka number. Since Schur functions are symmetric, this sum can be rewritten as

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_{\mu}, \tag{1.61}$$

where μ runs over all partitions of n. Note that $K_{\lambda\mu} = 0$ if $\lambda \ntrianglerighteq \mu$ and 1 if $\lambda = \mu$, which follows immediately from Young's rule, in Remark 1.34.

Alternatively, there is also a combinatorial argument. If $K_{\lambda\mu} \neq 0$, then there is a λ-tableau T of content μ such that its columns are increasing and its rows are non-decreasing. All occurrences of the values $1, 2, \ldots, i$ must then be in the rows 1 through i, and consequently, for all i, we have

$$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i, \tag{1.62}$$

in other words, $\mu \trianglelefteq \lambda$. If $\lambda = \mu$, then there is only one semi-standard Young tableau of shape and content λ, since in this case row i must contain all occurrences of i.

Hence, Schur functions can be expressed in terms of monomial symmetric functions as

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_{\mu}, \tag{1.63}$$

where the sum is taken over partitions (not compositions), and $K_{\lambda\lambda} = 1$. In particular, $\{s_\lambda \mid \lambda \vdash n\}$ is another basis for Sym_n. ■

Example 1.40. (a). Since we will be exploring e-positivity and s-positivity of chromatic symmetric functions, the transformation matrix from the e-basis to the s-basis of symmetric functions plays an important role. The famous *Jacobi–Trudi determinant* expresses Schur functions as the determinant of a matrix whose entries are elementary symmetric functions, or equivalently (via the fundamental involution), in terms of complete homogeneous symmetric functions,

$$s_{\lambda'} = |e_{\lambda_i - i+j}| \quad \text{and} \quad s_\lambda = |h_{\lambda_i - i+j}|, \tag{1.64}$$

where λ' denotes the transpose of the partition λ (constructed by reflecting its Young diagram to the diagonal). Let us list the first few connecting coefficients, for later use,

$$\begin{aligned}
s_{(1)} &= e_{(1)} \\[1em]
s_{(1^2)} &= e_{(2)} \\
s_{(2)} &= e_{(1^2)} - e_{(2)} \\[1em]
s_{(1^3)} &= e_{(3)} \\
s_{(2,1)} &= e_{(2,1)} - e_{(3)} \\
s_{(3)} &= e_{(1^3)} - 2e_{(2,1)} + e_{(3)} \\[1em]
s_{(1^4)} &= e_{(4)} \\
s_{(2,1^2)} &= e_{(3,1)} - e_{(4)} \\
s_{(2^2)} &= e_{(2,2)} - e_{(3,1)} \\
s_{(3,1)} &= e_{(2,1^2)} - e_{(2,2)} - e_{(3,1)} + e_{(4)} \\
s_{(4)} &= e_{(1^4)} - 3e_{(2,1^2)} + e_{(2^2)} + 2e_{(3,1)} - e_{(4)}
\end{aligned}$$

There is an elegant combinatorial proof for Jacobi–Trudi identities. Since $\omega(s_\lambda) = s_{\lambda'}$ and $\omega(e_\lambda) = h_\lambda$, it suffices to show the first identity. The Lindström–Gessel–Viennot Lemma (see [St1], Theorem 2.7.1) expresses the determinant of a square matrix in terms of certain non-crossing lattice paths, and this is the main idea behind the proof. For the details, see the first proof of Theorem 7.16.1 in [St2]. We are not going to elaborate on this proof here, because the Jacobi–Trudi identities will not be used later on, only the handful of examples that we listed above (and these expansions can also be computed directly).

The application of the Lindström–Gessel–Viennot Lemma is reminiscent of the correspondence between *totally non-negative* invertible matrices (matrices whose minors are non-negative) and edge-weighted acyclic planar networks. Indeed, the Jacobi–Trudi matrix (being a Vandermonde-type matrix) is *totally positive*, i.e. all its minors are positive.

(b). If a symmetric function is e-positive, then it is also s-positive. This is best understood on the representation theoretic side. We have seen that Young modules decompose as a direct sum of Specht modules, in Remark 1.33. Young modules correspond to elementary symmetric functions and Specht modules to Schur functions under the character map that we will introduce in Definition 1.46.

■

Construction 1.41. We have seen that irreducible characters appear in the connecting coefficients from Schur functions to monomials. They also show up in the connecting coefficients from power sum symmetric functions to monomials.

Consider the three partitions (3), $(2,1)$ and (1^3) of 3, and expand the corresponding power sum symmetric functions in the monomial basis,

$$p_{(3)} = x_1^3 + x_2^3 + \cdots = m_{(3)}, \tag{1.65}$$
$$p_{(2,1)} = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) = m_{(3)} + m_{(2,1)}, \tag{1.66}$$
$$p_{(1^3)} = (x_1 + x_2 + \cdots)^3 = m_{(3)} + 3m_{(2,1)} + 6m_{(1^3)}. \tag{1.67}$$

The connecting coefficients may be familiar from the character theory of Young modules. The conjugacy classes of S_3 corresponding to the partitions are the following (in cycle notation),

$$K_{(1^3)} = \{e\}, \quad K_{(2,1)} = \{(12),(13),(23)\}, \quad K_{(3)} = \{(123),(132)\}. \tag{1.68}$$

Note that $M^{(3)}$ is spanned by the single Young tabloid over $\mathbb{C}$

$$\begin{array}{c} \hline 1 \quad 2 \quad 3 \\ \hline \end{array}$$

where the bars indicate that the order of the elements within each row is arbitrary, and one may choose a representative with increasing order. In particular, $M^{(3)}$ is one-dimensional, and it is isomorphic to the *trivial representation*, given by $g \mapsto 1$ for all $g \in S_3$, and consequently, the trace is 1 on all conjugacy classes. The Young module $M^{(1^3)}$ is spanned by the six Young tabloids that lie in the S_3-orbit of

$$\begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Therefore, $M^{(1^3)} \cong \mathbb{C}S^3$ as representations, which is called the *regular representation.* We fix the basis e, (12), (13), (23), (123) and (132) of $\mathbb{C}S_3$ in this order, and express the values of $M^{(1^3)}$ in terms of 6×6-matrices. Note that we only need to compute these values for representatives of conjugacy classes, since we are looking for character values. Then $M^{(1^3)}(e) = \mathrm{id}_6$, so the trace is 6 for e, and we have

$$M^{(1^3)}(12) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.69}$$

so the trace is 0 for the conjugacy class $\{(12),(13),(23)\}$. Similarly, we have

$$M^{(1^3)}(123) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.70}$$

so the trace is also 0 for the conjugacy class $\{(123),(132)\}$.

Finally, we consider the Young module $M^{(2,1)}$ with the basis

$$\begin{array}{ccc} \overline{\underline{1\ 2}} & \overline{\underline{1\ 3}} & \overline{\underline{2\ 3}} \\ \underline{3} & \underline{2} & \underline{1} \end{array} \tag{1.71}$$

which is isomorphic to the *defining representation* $\mathbb{C}\{1,2,3\}$, i.e. the canonical action of S_3 on the set $[3]$. Again, we fix an ordered basis $(1,2,3)$ and compute its matrix values on some representatives of conjugacy classes. Then $M^{(2,1)}(e) = \mathrm{id}_3$, so the trace is 3 on the conjugacy class $\{e\}$, and we have

$$M^{(2,1)}(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.72}$$

so the trace is 1 on the conjugacy class $\{(12),(13),(23)\}$, and finally,

$$M^{(2,1)}(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.73}$$

so the trace is 0 on the conjugacy class $\{(123),(132)\}$. Let us denote the character of $M^{(3)}$, $M^{(2,1)}$ and $M^{(1^3)}$ by $\phi^{(3)}$, $\phi^{(2,1)}$ and $\phi^{(1^3)}$, respectively. Then the character values are indeed the same as the connecting coefficients in equation (1.65), summarised by the following table

	$K_{(1^3)}$	$K_{(2,1)}$	$K_{(3)}$
$\phi^{(3)}$	1	1	1
$\phi^{(2,1)}$	3	1	0
$\phi^{(1^3)}$	6	0	0

Figure 3: values of Young characters of S_3

Note that $M^{(3)}$ is irreducible but $M^{(2,1)}$ and $M^{(1^3)}$ are not. This can be seen either by the character relations from Remark 1.32, or by explicitly identifying a sub-representation, for example

$$V = \{(x,y,z) \in \mathbb{C}^3 : x+y+z = 0\} \tag{1.74}$$

in the case of $M^{(1^3)}$.

Lemma 1.42. *In general, we have*

$$p_\lambda = \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu \vdash n}} \phi^\mu_\lambda m_\mu, \tag{1.75}$$

where ϕ^μ_λ denotes the character ϕ^μ of the Young module M^μ, evaluated at the cycle type (i.e. conjugacy class) corresponding to the partition λ.

Proof. To see this, let $C = (c_{\lambda\mu})$ denote the transformation matrix from the monomial basis to power sums of degree n, so that for any $\lambda \vdash n$ we have

$$\prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} + \cdots) = \sum_{\mu \vdash n} c_{\lambda\mu} m_\mu. \tag{1.76}$$

Then we need to find the coefficient of x^μ on both sides, where $\mu = (\mu_1, \ldots, \mu_m)$ is an arbitrary partition. On the right-hand side, this coefficient is simply $c_{\lambda\mu}$. On the left-hand side, we can see by expanding the bracket that the coefficient is the number of ways to distribute parts of λ into sub-partitions $\lambda^1, \ldots, \lambda^m$ such that λ is the disjoint union of the λ_i's, and for each i we have $\lambda_i \vdash \mu_i$.

Next, we need to identify the $c_{\lambda\mu}$ with the character values. For any $\pi \in S_n$ with cycle type $c(\pi) = \lambda$, $\phi^\mu_\lambda = \phi^\mu(\pi)$ is the number of fixed points of π on all standard Young tabloids T with $\mathrm{sh}(T) = \lambda$. Note that T is fixed if and only if each cycle of π has elements lying on a single row of T, which is exactly the condition for $c_{\lambda\mu}$ presented above, and this shows the formula. ■

Remark 1.43. Consider the function $p_n \colon S_n \to \mathrm{Sym}$ that sends a permutation with cycle type λ to the power sum symmetric function p_λ. Then for the character χ^λ of a Specht module S^λ, we have

$$\frac{1}{n!}\sum_{\pi\in S_n} p_n(\pi)\chi^\lambda(\pi) = \frac{1}{n!}\sum_{\pi\in S_n}\left(\sum_\mu \phi^\mu(\pi)m_\mu\right)\chi^\lambda(\pi) \tag{1.77}$$

$$= \sum_\mu m_\mu\left(\frac{1}{n!}\sum_{\pi\in S_n}\phi^\mu(\pi)\chi^\lambda(\pi)\right) \tag{1.78}$$

$$= \sum_\mu m_\mu\langle\phi^\mu, \chi^\lambda\rangle \tag{1.79}$$

$$= \sum_\mu K_{\lambda\mu}m_\mu, \tag{1.80}$$

where $K_{\lambda\mu}$ is the Kostka number, by Young's rule, since the inner product gives us the multiplicity of the Specht module S^λ in the Young module M^μ. Consequently, we have

$$s_\lambda = \frac{1}{n!}\sum_{\pi\in S_n}\chi^\lambda(\pi)p_\pi, \tag{1.81}$$

or equivalently,

$$s_\lambda = \frac{1}{n!}\sum_\mu k_\mu\chi^\lambda_\mu p_\mu, \tag{1.82}$$

where K_μ denotes the conjugacy class corresponding to the partition μ, the cardinality of K_μ is denoted by k_μ, and χ^λ_μ is the value of χ^λ on the conjugacy class K_μ. Note that we can express k_μ as

$$k_\mu = \frac{n!}{z_\mu}, \tag{1.83}$$

where z_μ is the size of the centralizer of elements in K_μ, and it can be written as

$$z_\mu = \frac{n!}{1^{\mu_1}\mu_1!\cdots n^{\mu_r}\mu_r!}. \tag{1.84}$$

Hence we can express s_λ, without factorials, as

$$s_\lambda = \sum_\mu \frac{1}{z_\lambda}\chi^\lambda_\mu p_\mu. \tag{1.85}$$

Construction 1.44. The above description of the connecting coefficients suggests that the suitable inner product on symmetric functions, that mimics the inner product of class functions, is given by

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}. \tag{1.86}$$

This is called the *Hall inner product* on Sym_n.

(a). The Hall inner product is a symmetric bilinear form: for any $f, g \in \mathrm{Sym}$, we have $\langle f, g\rangle = \langle g, f\rangle$.

(b). Note that for any $n \neq m \in \mathbb{Z}_{\geq 0}$, $f \in \mathrm{Sym}_n$ and $g \in \mathrm{Sym}_m$, we have that $\langle f, g\rangle = 0$, since in this case, $0 = \delta_{\lambda,\mu} = \langle s_\lambda, s_\mu\rangle$ and Schur functions form a basis.

(c). For any $f \in \mathrm{Sym}_n$, we have

$$\langle h_n, f\rangle = f(1) := f(1, 0, 0, \ldots). \tag{1.87}$$

Again, by $\mathbb{C}$-linearity, it suffices to show the identity for the Schur basis: take $f = s_\lambda$ for some $\lambda \vdash n$.

If $\ell(\lambda) = 1$, then $\lambda = (n)$ and $s_\lambda = s_{(n)} = h_n$. Therefore, $f(1) = h_n(1) = 1$, and $\langle h_n, f\rangle = \langle s_{(n)}, s_{(n)}\rangle = \delta_{(n),(n)} = 1$, and consequently, $\langle h_n, f\rangle = f(1)$.

If $\ell(\lambda) > 1$, i.e. if λ has multiple parts, then $m_\lambda(x_1, \ldots, x_k, 0, 0, \ldots) = 0$ for any k with $\ell(\lambda) > k$, so we have $s_\lambda(x_1, 0, 0, \ldots) = 0$, and consequently $f(1) = s_\lambda(1) = 0$. On the other and, since $\langle h_n, f\rangle = \langle s_{(n)}, s_\lambda\rangle = \delta_{(n),\lambda} = 0$, we have $\langle h_n, f\rangle = f(1)$, as desired.

Remark 1.45. It would be natural to map the orthonormal basis of irreducible characters to the orthonormal basis of Schur functions. This way, the vector space isomorphism between class functions of S_n and Sym_n automatically becomes an isometry.

Definition 1.46. The *Frobenius character map* is the linear map defined by

$$\mathrm{ch}_n \colon R_n \to \mathrm{Sym}_n, \quad \mathrm{ch}_n(\chi^\lambda) = s_\lambda. \tag{1.88}$$

Remark 1.47. (a). Consequently, $\mathrm{ch}_n \colon R_n \to \mathrm{Sym}_n$ is an isometry.

(b). One can extend this to a graded module isomorphism,

$$\mathrm{ch} := \bigoplus_{n\geq 0} \mathrm{ch}_n \colon R \to \mathrm{Sym}, \tag{1.89}$$

which is also an isometry between $R := \bigoplus_n R_n$ and Sym.

(c). According to the computations above, the character map has a simple description in terms of power sum symmetric functions too. For any $\phi \in R_n$, we can write

$$\mathrm{ch}(\phi) = \frac{1}{n!} \sum_{w\in S_n} \phi(w) p_{\lambda(w)} = \sum_{\mu\vdash n} z_\mu^{-1} \phi(\mu) p_\mu, \tag{1.90}$$

where $\lambda(w)$ denotes the cycle type of the permutation w and z_μ is the size of the centraliser of K_μ, as in equation (1.84). This formula is very convenient for explicit computations, and it is one of the main reasons why we care about power sum symmetric functions in this thesis. Another reason will be provided in Section 3, when we explore the Hopf algebra structure on symmetric functions.

Example 1.48. (a). Consider the trivial representation of S_3. Since it is one-dimensional, its character χ_1 has value 1 on all cycle types. Consequently, we have

$$\mathrm{ch}(\chi_1) = \frac{1}{3!} \sum_{w\in S_3} p_w = \frac{1}{6} \cdot p_{(1^3)} + \frac{1}{2} \cdot p_{(2,1)} + \frac{1}{3} \cdot p_{(3)} = s_{(3)}. \tag{1.91}$$

(b). The character value at a permutation of the sign representation of S_3 is the sign of the permutation $\chi_2(w) = \mathrm{sign}(w)$, so we have

$$\mathrm{ch}(\chi_2) = \frac{1}{3!} \sum_{w\in S_3} \mathrm{sign}(w) p_w = \frac{1}{6} \cdot p_{(1^3)} - \frac{1}{2} \cdot p_{(2,1)} + \frac{1}{3} \cdot p_{(3)} = s_{(1^3)}. \tag{1.92}$$

■

Lemma 1.49. *The character map* $\mathrm{ch} : R \to \mathrm{Sym}$ *is a graded algebra isomorphism.*

Proof. We have seen that is an isomorphism of graded vector spaces, so it suffices to show that ch is an algebra morphism.

However, the multiplication on R_n that is compatible with the multiplication of Sym inherited from the $\mathbb{C}$-algebra of formal power series is not componentwise multiplication of class functions, because we need to take the grading into account. If χ is a character of S_n and ψ a character of S_m, then $\chi \otimes \psi$ is a character of $S_n \times S_m$. To obtain a character of S_{n+m}, we can take the induced representation

$$\chi \cdot \psi := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\chi \otimes \psi), \tag{1.93}$$

and extend bilinearly. To see that the character maps respects multiplication, one can use Frobenius reciprocity, from Remark 1.32, part (f), and compute

$$\begin{aligned}
\mathrm{ch}(\chi \cdot \psi) &= \langle \chi \cdot \psi, p \rangle && (1.94)\\
&= \langle \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\chi \otimes \psi), p \rangle && (1.95)\\
&= \langle \chi \otimes \psi, \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} (p) \rangle && (1.96)\\
&= \frac{1}{n!m!} \sum_{\substack{\pi \in S_n \\ \sigma \in S_m}} (\chi \otimes \psi)(\pi \times \sigma) \cdot p_{\pi \times \sigma} && (1.97)\\
&= \frac{1}{n!m!} \sum_{\substack{\pi \in S_n \\ \sigma \in S_m}} \chi(\pi)\psi(\sigma) p_\pi p_\sigma && (1.98)\\
&= \left(\frac{1}{n!} \sum_{\pi \in S_n} \chi(\pi) p_\pi \right) \left(\frac{1}{m!} \sum_{\sigma \in S_m} \psi(\sigma) p_\sigma \right) && (1.99)\\
&= \mathrm{ch}(\chi)\, \mathrm{ch}(\psi). && (1.100)
\end{aligned}$$

■

Remark 1.50. (a). Later, when we endow Sym with the structure of a graded connected Hopf algebra, in Subsection 3.1, we will also show that the character map is in fact a Hopf algebra map.

(b). We will also need a refinement of the Frobenius character map that takes into account *graded representations*, i.e. representations on graded vector spaces where the action preserves the graded parts, because ultimately, we would like to study chromatic quasi-symmetric functions, introduced in Definition 1.60.

(c). Since the character map plays a central role in the theory of symmetric functions, it is natural to ask what kind of S_n-representations correspond to chromatic symmetric functions. Then, we can ask if these representations decompose as a direct sum of Specht modules or Young modules. In terms of base expansions of symmetric functions, this will be reflected by s-positivity or e-positivity of the corresponding chromatic symmetric functions.

Definition 1.51. The *graded Frobenius character* of a graded S_n-module $V = \bigoplus_d V_d$ is defined as

$$\mathrm{Frob}(V)(t) := \sum_d \mathrm{ch}(V_d) t^d \in \mathrm{Sym}[\![t]\!]. \tag{1.101}$$

Remark 1.52. Let $\{e_i\}_{i\in I}$ be a homogeneous basis for the graded representation V. Then the expression of the p-expansion of the Frobenius character, in Remark 1.47, yields the following p-expansion of the graded Frobenius. Let a_i denote the coefficient of e_i in $w \cdot e_i$, then we have

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{w\in S_n} \left(\sum_{i\in I} t^{\deg(e_i)} a_i \right) p_{\lambda(w)}, \tag{1.102}$$

where $\lambda(w)$ denotes the cycle type of w.

Example 1.53. The $\mathbb{C}$-algebra $\mathbb{C}[x_1, x_2, x_3]$ carries an action of S_3 given by permuting the variables

$$\sigma.f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}). \tag{1.103}$$

Invariants under this action are called *symmetric polynomials*, whose power series analogues are symmetric functions. The ideal generated by *elementary symmetric polynomials* in this ring is

$$\langle e_1, e_2, e_3 \rangle = \langle x_1 + x_2 + x_3, x_1x_2 + x_1x_2 + x_2x_3, x_1x_2x_3 \rangle, \tag{1.104}$$

and the quotient is called the *coinvariant algebra* $\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle$.

More generally, the *coinvariant algebra* for S_n is defined as $\mathbb{C}[x_1, \dots, x_n]/I_n$, where $I_n = \langle e_1, \dots, e_n\rangle$. An important basis for this space is given by the *Schubert polynomials* $\{\mathfrak{S}_w + I_n \mid w \in S_n\}$. The Schubert polynomial $\mathfrak{S}_w$ for a permutation $w \in S_n$ can be defined inductively, using the *(strong) Bruhat order*, which is the partial order on S_n defined by $u \leq v$ if some (or equivalently, every) reduced word for v contains a substring (of not necessarily consecutive letters) that forms a reduced word for u. For instance the (strong) Bruhat order for S_3 yields the following poset, using the standard notation for simple transpositions,

$$s_1 = (12), \quad s_2 = (23), \tag{1.105}$$

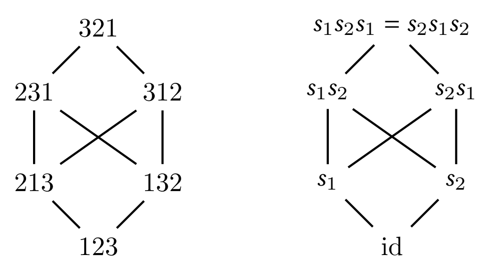


Figure 4: (strong) Bruhat order for S_3

For the longest permutation w_0 in the (strong) Bruhat order,

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1}, \tag{1.106}$$

and for any w with $w(i) > w(i+1)$, the corresponding Schubert polynomial is defined as

$$\mathfrak{S}_{ws_i} := \partial_i(\mathfrak{S}_w), \tag{1.107}$$

where ∂_i is the *divided difference operator*, given by

$$\partial_i(f(x_1, \dots, x_n)) := \frac{f(x_1, \dots, x_n) - s_i(f(x_1, \dots, x_n))}{x_i - x_{i+1}}. \tag{1.108}$$

Here are the Schubert polynomials for S_3, arranged along the vertices of the (strong) Bruhat poset.

$$\begin{tikzcd}
& \mathfrak{S}_{321} = x_1^2x_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{S}_{231} = x_1x_2 \arrow[d, dash] \arrow[drr, dash] & & \mathfrak{S}_{312} = x_1^2 \arrow[d, dash] \arrow[dll, dash] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, dash] & & \mathfrak{S}_{132} = x_1 + x_2 \arrow[dl, dash] \\
& \mathfrak{S}_{123} = 1 &
\end{tikzcd}$$

Figure 5: Schubert polynomials for S_3

Then we can compute the actions of permutations on Schubert polynomials. Note that $s_2(x_1^2x_2) = x_1^2x_3$ and $(x_1x_2 + x_1x_3 + x_2x_3)x_1 = x_1^2x_2 + x_1^2x_3 + x_1x_2x_3$, where $x_1x_2x_3 \in I_3$, which yields the base expansion $x_1^2x_3 + I_3 = -x_1^2x_2 + I_3$. Similarly, we can compute the base expansion of the other actions, summarized in the following table.

	123	132	213	231	312	321
$x_1^2x_2$	$x_1^2x_2$	$-x_1^2x_2$	$-x_1^2x_2$	$x_1^2x_2$	$x_1^2x_2$	$-x_1^2x_2$
x_1x_2	x_1x_2	$-x_1x_2 - x_1^2$	x_1x_2	x_1^2	$-x_1x_2 - x_1^2$	x_1^2
x_1^2	x_1^2	x_1^2	$-x_1x_2 - x_1^2$	$-x_1x_2 - x_1^2$	x_1x_2	x_1x_2
x_1	x_1	x_1	$-x_1 + (x_1 + x_2)$	$-x_1 + (x_1 + x_2)$	$-(x_1 + x_2)$	$-(x_1 + x_2)$
$x_1 + x_2$	$x_1 + x_2$	$x_1 - (x_1 + x_2)$	$x_1 + x_2$	$-x_1$	$x_1 - (x_1 + x_2)$	$-x_1$
1	1	1	1	1	1	1

Figure 6: S_3-action on the coinvariant algebra

Therefore, the subspaces $(x_1^2x_2)$, (x_1x_2, x_1^2), $(x_1, x_1 + x_2)$ and (1) are invariant, and the representation can be expressed in terms of matrices as follows,

	123	132	213	231	312	321
$(x_1^2x_2)$	(1)	(-1)	(-1)	(1)	(1)	(-1)
(x_1x_2, x_1^2)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$
$(x_1, x_1 + x_2)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
(1)	(1)	(1)	(1)	(1)	(1)	(1)

Figure 7: S_3-action on the coinvariant algebra, in terms of matrices

Let us start by computing the Frobenius character of the degree three part, which is in fact the sign representation, treated in Example 1.48, part (a). The conjugacy classes of S_3 are represented by id, s_1 and s_1s_2, and the classes have 1, 3 and 2 elements, respectively. The identity matrix has trace 1, while the representing matrices of the other two classes have trace -1 and 1, respectively. Therefore, the Frobenius character of the degree three part is given by

$$\frac{1}{6}(1 \cdot 1 \cdot p_{(1^3)} + 3 \cdot (-1) \cdot p_{(2,1)} + 2 \cdot 1 \cdot p_{(3)}) = s_{(1^3)}. \tag{1.109}$$

Similarly, we can compute the Frobenius character of the degree two part by reading off the traces from the representing matrices, and we have

$$\frac{1}{6}(1 \cdot 2 \cdot p_{(1^3)} + 3 \cdot 0 \cdot p_{(2,1)} + 2 \cdot (-1) \cdot p_{(3)}) = s_{(2,1)}. \tag{1.110}$$

The Frobenius character of the degree one part is the same, because the representing matrices have the same traces. The degree zero part is the trivial representation, so the Frobenius character is

$$\frac{1}{6}(1 \cdot 1 \cdot p_{(1^3)} + 3 \cdot 1 \cdot p_{(2,1)} + 2 \cdot 1 \cdot p_{(3)}) = s_{(3)}. \tag{1.111}$$

Hence, the graded Frobenius character of the coinvariant algebra is

$$\mathrm{Frob}(\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle) = s_{(1^3)}t^3 + s_{(2,1)}t^2 + s_{(2,1)}t + s_{(3)}. \tag{1.112}$$

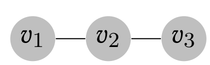
Note that the s-expansion is positive in each degree, but it is far from being e-positive. We can read off the connecting coefficients from Example 1.40, and obtain the e-expansion

$$\mathrm{Frob}(\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle) = e_{(3)}t^3 + (e_{(2,1)} - e_{(3)})t^2 + (e_{(2,1)} - e_{(3)})t + (e_{(1^3)} - 2e_{(2,1)} + e_{(3)}). \tag{1.113}$$

However, if we were to evaluate the Frobenius character at $t = 1$, then we would get $e_{(1^3)}$, which is e-positive. We will return to the covariant algebra, when we discuss the equivariant cohomology of full flag varieties, in Example 2.28. ■

1.4 Chromatic Quasi-Symmetric Functions

Example 1.54. (a). Consider the path P_3 of order 3



Then the chromatic symmetric function is

$$X(P_3; x) = 6m_{(1^3)} + m_{(2,1)}. \tag{1.114}$$

Indeed, there are six different ways to colour P_3, since any permutation yields a proper colouring, so the coefficient of the monomial symmetric function $m_{(1^3)}$ is 6. Using two colours a, b, the middle vertex v_2 must get a different colour, so the corresponding monomials are $x_a^2x_b$ and $x_ax_b^2$, and the coefficient of the term $m_{(2,1)}$ is 1. In particular, $X(P_3; x)$ is m-positive (i.e. the expansion in the monomial basis yields positive coefficients).

To find the e-expansion, we can use the transformation matrix from the monomial basis to the e-basis, presented in Lemma 1.25, to read off the following identities:

$$m_{(1^3)} = e_{(3)},$$

$$\begin{aligned} m_{(2,1)} &= e_{(2,1)} - 3m_{(1^3)} \\ &= e_{(2,1)} - 3e_{(3)}, \end{aligned}$$

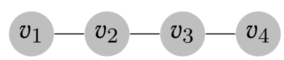
$$\begin{aligned} m_{(3)} &= e_{(1^3)} - 6m_{(1^3)} - 3m_{(2,1)} \\ &= e_{(1^3)} - 3e_{(2,1)} + 3e_{(3)}. \end{aligned}$$

Hence, the e-expansion of the chromatic symmetric function is given by

$$X(P_3; x) = 3e_{(3)} + e_{(2,1)}, \tag{1.115}$$

so this symmetric function is also e-positive. Recall from Example 1.40, part (c) that e-positivity implies s-positivity, and indeed, we can compute that $X(P_3; x) = 4s_{(1^3)} + s_{(2,1)}$.

(b). Similarly, we can compute the chromatic symmetric function of the path P_4 of order 4



whose monomial expansion is

$$24m_{(1^4)} + 6m_{(2,1^2)} + 2m_{(2^2)}.$$

Again, we can appeal to the transformation matrix to convert this to the e-basis,

$$m_{(1^4)} = e_{(4)}$$

$$\begin{aligned} m_{(2,1^2)} &= e_{(3,1)} - 4m_{(1^4)} \\ &= e_{(3,1)} - 4e_{(4)}, \end{aligned}$$

$$\begin{aligned} m_{(2^2)} &= e_{(2^2)} - 6m_{(1^4)} - 2m_{(2,1^2)}, \\ &= e_{(2^2)} - 6m_{(1^4)} - 2(e_{(3,1)} - 4m_{(1^4)}) \\ &= e_{(2^2)} + 2e_{(4)} - 2e_{(3,1)}. \end{aligned}$$

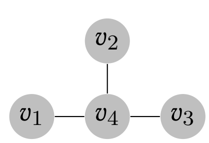
Hence, the e-expansion of the chromatic symmetric function is

$$X(P_4; x) = 4e_{(4)} + 2e_{(3,1)} + 2e_{(2^2)}. \tag{1.116}$$

By computing the Kostka numbers explicitly, we can also find its s-expansion, which is given by

$$X(P_4; x) = 8s_{(1^4)} + 4s_{(2,1^2)} + 2s_{(2^2)}. \tag{1.117}$$

(c). Now let us consider the claw graph $K_{1,3}$



Its chromatic symmetric function is

$$X(K_{1,3};x) = 24m_{(1^4)} + 6m_{(2,1^2)} + m_{(3,1)}. \tag{1.118}$$

Indeed, there are $4!$ ways to colour the four vertices with four distinct colours. Given three colours a, b, c, if a is used twice, then the vertex v_4 must be either b or c, and the other vertices must be coloured with a different colour, so we have 6 choices corresponding to each monomial with partition $(2, 1^2)$. Therefore, the coefficient of the monomial symmetric function $m_{(2,1^2)}$ is 6. If we have two colours a, b, and v_4 is coloured with a, then all the other vertices must be coloured with b, so the corresponding monomial is $x_a x_b^3$, and consequently, the coefficient of the monomial symmetric function $m_{(3,1)}$ is 1.

The salient point about this computation is that the chromatic symmetric function of the claw graph is not e-positive,

$$X(K_{1,3};x) = 4e_{(4)} + 5e_{(3,1)} - 2e_{(2,2)} + e_{(2,1^2)}. \tag{1.119}$$

In fact, its s-expansion also has a negative coefficient,

$$X(K_{1,3};x) = s_{(3,1)} - s_{(2,2)} + 5s_{(2,1^2)} + 8s_{(1^4)}. \tag{1.120}$$

We saw in this example, that unlike the chromatic polynomial, the chromatic symmetric function can distinguish between the claw graph and the path of order 4. It has been checked for all trees with order at most 23 (see [MMW]), but it remains an open problem in general, that if T_1 and T_2 are non-isomorphic trees, then $X(T_1) \neq X(T_2)$. At any rate, chromatic symmetric functions are considerably more sophisticated invariants than chromatic polynomials. ■

Remark 1.55. (a). Let $G = (V, E)$ be a graph, then the p-expansion of its chromatic symmetric function can be described as

$$X(G) = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(F)}, \tag{1.121}$$

where $\lambda(F)$ denotes the partition determined by the components spanned by F. For a bijective proof, using sign reversing involutions, see [SV].

(b). Stanley proved the following description of the expansion of chromatic symmetric functions in the elementary symmetric function basis. If we write

$$X(G;x) = \sum_{\lambda \vdash n} \alpha_\lambda e_\lambda, \tag{1.122}$$

then for any $m \in \mathbb{Z}_{>0}$ the sum

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = m}} \alpha_\lambda \tag{1.123}$$

counts the number of acyclic orientiations of G with exactly m sinks.

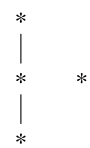
For instance, in the case of the path of order 3, we have 3 orientations with 1 sink, 1 orientation with two sinks, and it is not possible to have more than 2 sinks. Indeed, we have seen that the e-expansion of its chromatic symmetric function is $3e_{(3)} + e_{(2,1)}$.

There is a classical theorem in graph theory, called the Gallai–Hasse–Roy–Vitaver theorem, which states that the chromatic number $\chi(G)$ of a graph G equals one plus the length of the longest path(s) in an orientation of G chosen to minimize this length (for a proof, see [ChZ], Theorem 7.17). Any such orientation can be chosen to be acyclic. A corollary of this theorem is the fact that the chromatic polynomial $\chi(-1)$ evaluated at -1 yields the number of acyclic orientations of the graph. This result can be regarded as a precursor to the e-expansion of chromatic symmetric functions.

Definition 1.56. (a). For a poset P, one can consider the *incomparability graph*, whose vertices are the elements of the poset and two vertices are adjacent if and only if they are incomparable.

(b). For $a, b \in \mathbb{Z}_{>0}$, an *$(a+b)$-free poset* is a poset that does not contain an induced subposet that is isomorphic to a disjoint union of an a-chain and a b-chain.

Remark 1.57. We have seen that the claw graph is not e-positive, which is the incomparability graph of the poset given by the disjoint union of a 3-chain and a 1-chain,



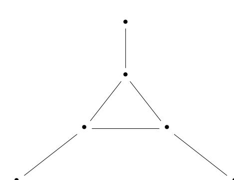
Conjecture 1.58 (Stanley–Stembridge conjecture). *The incomparability graph of a $(3+1)$-free poset P is e-positive.*

Remark 1.59. (a). A *P-tableau* is a Young diagram with a filling (a_{ij}) such that each element of P is one of the a_{ij}, and $a_{ij} <_P a_{i,j+1}$ but $a_{i+1,j} \not<_P a_{ij}$.

In 1996, Gasharov (see [Gas]) related the number c_λ of P-tableaux of a given shape λ with the chromatic symmetric function of the incomparability graph $G(P)$ of P, when P is $(3+1)$-free, and thereby, showed s-positivity for these chromatic symmetric functions.

(b). Some special cases of the Stanley–Stembridge conjecture were successfully tackled by purely combinatorial techniques, but a geometric or algebraic resolution to the Stanley–Stembridge conjecture seems more likely. To make chromatic symmetric functions more amenable to a geometric interpretation, Shareshian and Wachs introduced a graded version of this invariant in 2012, in the article [SW].

(c). It is not true, however, that *claw-free graphs*, i.e. graphs with no induced subgraphs isomorphic to the claw graph, are necessarily e-positive. For example, for the following claw-free graph



the e-expansion of the chromatic symmetric function is

$$6e_{(3,2,1)} - 6e_{(3^2)} + 6e_{(4,1^2)} + 12e_{(4,2)} + 18e_{(5,1)} + 12e_{(6)}. \tag{1.124}$$

Definition 1.60. Let $G = (V, E)$ be a graph with vertex set $V = [n]$ endowed with the natural total order, and denote by $\mathrm{PC}(G)$ the set of proper colourings $c\colon V \to \mathbb{Z}_{>0}$ of G. For any such c, denote by $\mathrm{Asc}(c)$ the set of ascents,

$$\mathrm{Asc}(c) := \{ij \in E \mid i < j \quad \text{and} \quad c(i) < c(j)\}. \tag{1.125}$$

We call the cardinality of $\mathrm{Asc}(c)$ the *ascent number* $\mathrm{asc}(c)$ of c. Let x denote the ordered set of variables $x = (x_1, x_2, \ldots)$, and we introduce another parameter t. The *chromatic quasi-symmetric function* of G is

$$X(G; x, t) := \sum_{c \in \mathrm{PC}(G)} x^c t^{\mathrm{asc}(c)}, \tag{1.126}$$

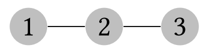
where x^c is x^λ for the partition $\lambda \vdash n$ induced by the colouring c. Sometimes we will use the following notation,

$$X_k(G; x) := \sum_{\substack{c \in \mathrm{PC}(G) \\ \mathrm{asc}(c) = k}} x^c, \tag{1.127}$$

so that the chromatic quasi-symmetric function takes the form

$$X(G; x, t) = \sum_{k \geq 0} X_k(G; x) t^k. \tag{1.128}$$

Example 1.61. Consider the path P_3 of order 3,



with the colouring $\kappa(1) = 2$, $\kappa(2) = 1$, $\kappa(3) = 2$. Then the only ascent is $(2, 3)$, so the monomial corresponding to this colouring is $t^1 x^\kappa = t x_2^2 x_1$.

Continuing in this manner, we can see that the chromatic quasi-symmetric function of P_3 with respect to this labelling is

$$X_{P_3}(x; t) = (1 + 4t + t^2) m_{(1^3)} + t m_{(2,1)}, \tag{1.129}$$

where $1 + 4t + t^2$ is the *Eulerian polynomial*, i.e. the coefficient of t^m is the number of permutations $\sigma \in S_n$ with exactly m ascents. The coefficient of m_{21} is indeed t, since the ascent set is always a singleton whether $b < a$ or $a < b$.

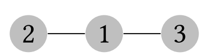
Note that this quasi-symmetric function is in fact symmetric in each degree of t, i.e. the coefficients of powers of t lie in Sym. Therefore, we may expand them in other bases of symmetric functions too.

Their e-expansion is as follows. We can read these values off from the transformation matrix between the monomial basis and the e-basis. Note that $m_{(1^3)} = e_{(3)}$ and $e_{(2,1)} = e_{(2,1)} - 3m_{(1^3)} = e_{(2,1)} - 3e_{(3)}$. Thus, we have

$$X_{P_3}(x, t) = e_{(3)} + (e_{(3)} + e_{(2,1)})t + e_{(3)} t^2, \tag{1.130}$$

which is again e-positive in each degree.

However, if we consider the path with a different labelling, for example,



then the chromatic quasi-symmetric function is far from being symmetric,

$$X_G(x; t) = (2 + 2t + 2t^2) m_{(1^3)} + \sum_{i<j} x_i^2 x_j + t^2 \sum_{i<j} x_i x_j^2. \tag{1.131}$$

The coefficient of $m_{(1^3)}$ is $2 + 2t + 2t^2 \in \mathbb{C}(t)$ since there are two colourings with no ascents: $(1, 3, 2)$ and $(2, 3, 1)$, two colourings with one ascent: $(1, 2, 3)$ and $(3, 2, 1)$, and two colourings with two ascents:

(2, 1, 3) and (3, 1, 2). The last two terms correspond to the colourings (i, j, i) and (j, i, j) with $j > i$; the first one has no ascents and the second one has two ascents.

A large class of chromatic quisi-symmetric functions which are symmetric in each degree arises from *Dyck paths*, one of the many Catalan objects in enumerative combinatorics (such as triangulations of an n-gon, binary trees with a fixed number of leaves or semi-standard Young tableaux of shape $2 \times n$). ∎

Remark 1.62. The nth Catalan number C_n is given by $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$. Recall that the square sum $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ is $\binom{2n}{n}$, which can be seen geometrically by double-counting the number of (shortest) paths along the Pascal triangle from $\binom{0}{0}$ to $\binom{2n}{n}$. Then one can interpret the Catalan number C_n as the average of such squares. For example, $C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$.

Definition 1.63. *Dyck paths* with n horizontal steps are lattice paths that do not cross the diagonal. Alternatively, a vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ is called a *Dyck path* if m is weakly increasing (i.e. for all $i < j$, we have $m_i \leq m_j$), and for all $i \in [n]$, we have $i \leq m_i \leq n$.

Example 1.64. There are five 3×3 Dyck paths, given by (1, 2, 3), (1, 3, 3), (2, 2, 3), (2, 3, 3) and (3, 3, 3):

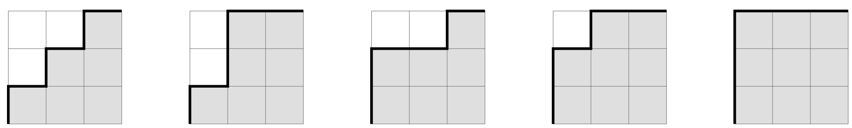
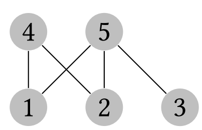


Figure 8: Dyck paths of rank 3

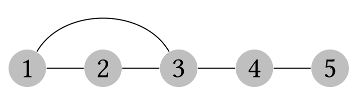
∎

Definition 1.65. One can associate a poset P_d on $[n]$ to a Dyck path d, where $i < j$ in P_d if and only if $d_i < j$. Then one can consider its *incomparability graph*, whose vertices are the elements of the poset and two vertices are adjacent if and only if they are incomparable.

Example 1.66. Consider the Dyck path $d = (3, 3, 4, 5, 5)$, then the corresponding poset is



Its incomparability graph P_d is



∎

Remark 1.67. Posets of Dyck paths P_d are $(3 + 1)$-free and $(2 + 2)$-free. In fact, this is a characterizing property of posets of Dyck paths. The following are equivalent for a poset P:

(a). P is the poset of a Dyck path,

(b). P is $(3 + 1)$-free and $(2 + 2)$-free,

(c). P is a *natural unit interval order*, i.e. a poset on a finite subset of real numbers $y_1 < \cdots < y_n$, where $y_i + 1 < y_j$ if and only if $i <_P j$ for all $i, j \in [n]$.

An explicit proof of the equivalence of these statements can be found in [SW], Proposition 4.1. Alternatively, one can show that all of these objects are Catalan objects (where the index is the number of intervals or the number of vertices of the poset or the number of labels on the grid), and then one only needs to find surjections. The $(3+1)$ and $(2+2)$-free condition is a local condition, so it suffices to show that for all 4×4 Dyck paths, there are corresponding natural unit interval orders.

Construction 1.68. (a). There is a simple description of the incomparability graph of the poset associated to the Dyck path d. Vertices are labelled by the coordinates of the grid, and two vertices i and j are adjacent if and only if the square with coordinates (i, j) lies below the Dyck path. This graph is called the *indifference graph* of the Dyck path. To see that the indifference graph of a Dyck path d is the same as the incomparability graph of the poset P_d, one simply needs to compare the two definitions.

(b). From the indifference graph description, it is clear that the complete graph corresponds to the Dyck path $(3, 3, 3)$ in Example 1.64, or in general $(n, n, \dots, n)$, whereas the empty graph corresponds to the Dyck path $(1, 2, 3)$, or in general $(1, 2, \dots, n)$.

(c). If the indifference graph of a Dyck path contains an edge (i, j) with $i < j$ and $i < i' < j' < j$, then $i'j'$ is also an edge. This is best seen from the poset point of view, since posets of Dyck paths are natural unit interval orders.

(d). The indifference graph of a Dyck path is connected if and only if the Dyck path does not touch the diagonal. We will see many examples in Example 1.70 and 1.71, but let us quickly indicate why this holds in general. This follows from the previous observation about the edges. Indeed, connectivity in such a graph implies that pairs of labels which are simple transpositions are edges, e.g. $(1, 2)$, and $(2, 3)$ in the 3×3 case. Note that the chromatic (quasi-)symmetric function of a disconnected graph is the product of the chromatic (quasi-)symmetric function of its components, which makes this construction suitable for inductive arguments.

Theorem 1.69. *For any Dyck path d with indifference graph G_d, we have*

$$X_{G_d}(x; t) \in \mathrm{Sym}[t] = \mathrm{Sym}_{\mathbb{C}[t]}\,. \tag{1.132}$$

Proof. For a Dyck path d, a proper colouring κ and $a \in \mathbb{Z}_{>0}$, we denote by $G_{\kappa,a}$ the subgraph of G_d induced by the subset $\kappa^{-1}(a) \cup \kappa^{-1}(a+1) \subset V$. Then we would like to show that every connected component of $G_{\kappa,a}$ is a path with consecutive vertex labels $i_1 < \cdots < i_j$.

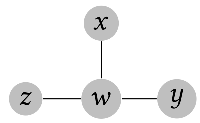
The graph $G_{\kappa,a} = (V, E)$ is bipartite with partite sets $\kappa^{-1}(a)$ and $\kappa^{-1}(a+1)$. In particular, $G_{\kappa,a}$ has no triangles, i.e. $xy, yz \in E$ implies that $xz \notin E$. Consider a path of order three with consecutive vertex labels (x, y, z), and we need to show that $x < y < z$ or $x > y > z$.

Assume that $x < y$. Since x and y are incomparable in the poset P_d, we have $d_x \geq y$ and $d_y \geq x$. On the other hand, x and z are comparable, so either $d_x < z$ or $d_z < x$. But $d_z < x$ is not possible, because $d_z < x < y$ implies that y and z are comparable in P_d (which cannot happen because yz is an edge in G_d). Therefore, we have $y \leq d_x < z$, and we end up with the chain of inequalities $x < y < z$, as desired.

Now assume that $x > y$. Since x and y are incomparable P_d, we have $d_y \geq x$ and $d_x \geq y$, and since y and z are incomparable in P_d, we also have $d_y \geq z$ and $d_z \geq y$. In particular, we have $d_x > z$. On the other hand, x, z are comparable, so we have $d_x < z$ or $d_z < x$, but we have seen that $d_x > z$, so the only remaining option is $d_z < x$. We have also seen that $x \leq d_y$, so $d_z < d_y$. Thus $z < y$, and we get the chain of inequalities $x > y > z$, as desired.

We have shown that any path in $G_{\kappa,a}$ has its labels in the desired order, since we can break it up into overlapping paths of order 3. Thus $G_{\kappa,a}$ is acyclic, i.e. a forest.

We need to show that every degree in $G_{\kappa,a}$ is at most two. If there is a vertex of degree three, then there is a subgraph isomorphic to the claw graph



Consider the path zwx, then either $x < w < z$ or $x > w > z$. Assume, without loss of generality, that $x < w < z$. Then we look at the path xwy, where we have $x < w$, so the chain of inequalities must be $x < w < y$. Thus $w < z$ and $w < z$. But in this case, the path zwy has the ordering $z < w > y$, which is impossible. This proves the claim that there is no vertex with degree greater than two.

Now we fix a proper colouring κ, and define an involution that preserves the number of ascents. For any $a \in \mathbb{Z}_{>0}$, pick a connected component of $G_{\kappa,a}$, which is a path $i_1 \cdots i_j$. If the number of vertices in this path is even, we do not change anything. If the order of this path is odd, we swap the colours a and $a+1$. Note that this involution leaves the number of ascents invariant. Since simple transpositions generate the symmetric group, this shows that the chromatic quasi-symmetric function $X_G(x;t)$ is symmetric in each degree. ∎

Example 1.70. Let us compute the e-expansion of chromatic quasi-symmetric functions of indifference graphs explicitly, for 3×3 Dyck paths.

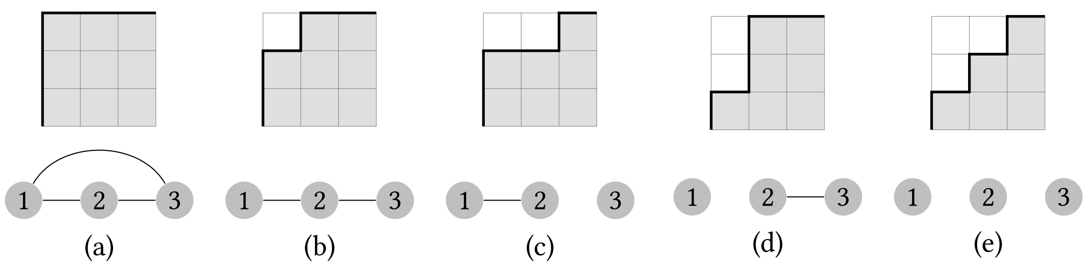


Figure 9: indifference graphs of rank 3 Dyck paths

(a). The indifference graph of the Dyck path $(3,3,3)$ is the complete graph on three vertices, since all squares in the grid lie under the path. Therefore, all proper colourings use three distinct colours, which we can assume to be $1, 2$ and 3. Then the only colouring with three ascents is 123 (by which we mean that the vertex with label 1 has colour 1, vertex 2 has colour 2 and vertex 3 has colour 3). There are two colourings with two ascents, namely 132 and 213, two colourings with one ascent, namely 231 and 312, and there is a single colouring with no ascents: 321. Hence, the chromatic quasi-symmetric function of this graph can be written as

$$(1 + 2t + 2t^2 + t^3)m_{(1^3)} = (1 + 2t + 2t^2 + t^3)e_{(3)}. \tag{1.133}$$

(b). We have already computed the chromatic quasi-symmetric function of this graph (in Example 1.61), and seen that it is e-positive,

$$(1 + 4t + t^2)m_{(1^3)} + tm_{(2,1)} = e_{(3)} + (e_{(3)} + e_{(2,1)})t + e_{(3)}t^2, \tag{1.134}$$

where the conversion to the e-basis involved some cancellations within the degree 1 part.

(c). The Dyck path $(2, 2, 3)$ touches the diagonal, so its incomparability graph is disconnected. Consequently, the chromatic quasi-symmetric function can be written as the product

$$m_{(1)} \cdot (1+t)m_{(1^2)} = (1+t)e_{(2,1)}. \tag{1.135}$$

(d). Similarly, the incomparability graph of the Dyck path $(1, 3, 3)$ has only one edge, between 2 and 3, and the chromatic quasi-symmetric function is the same as the above,

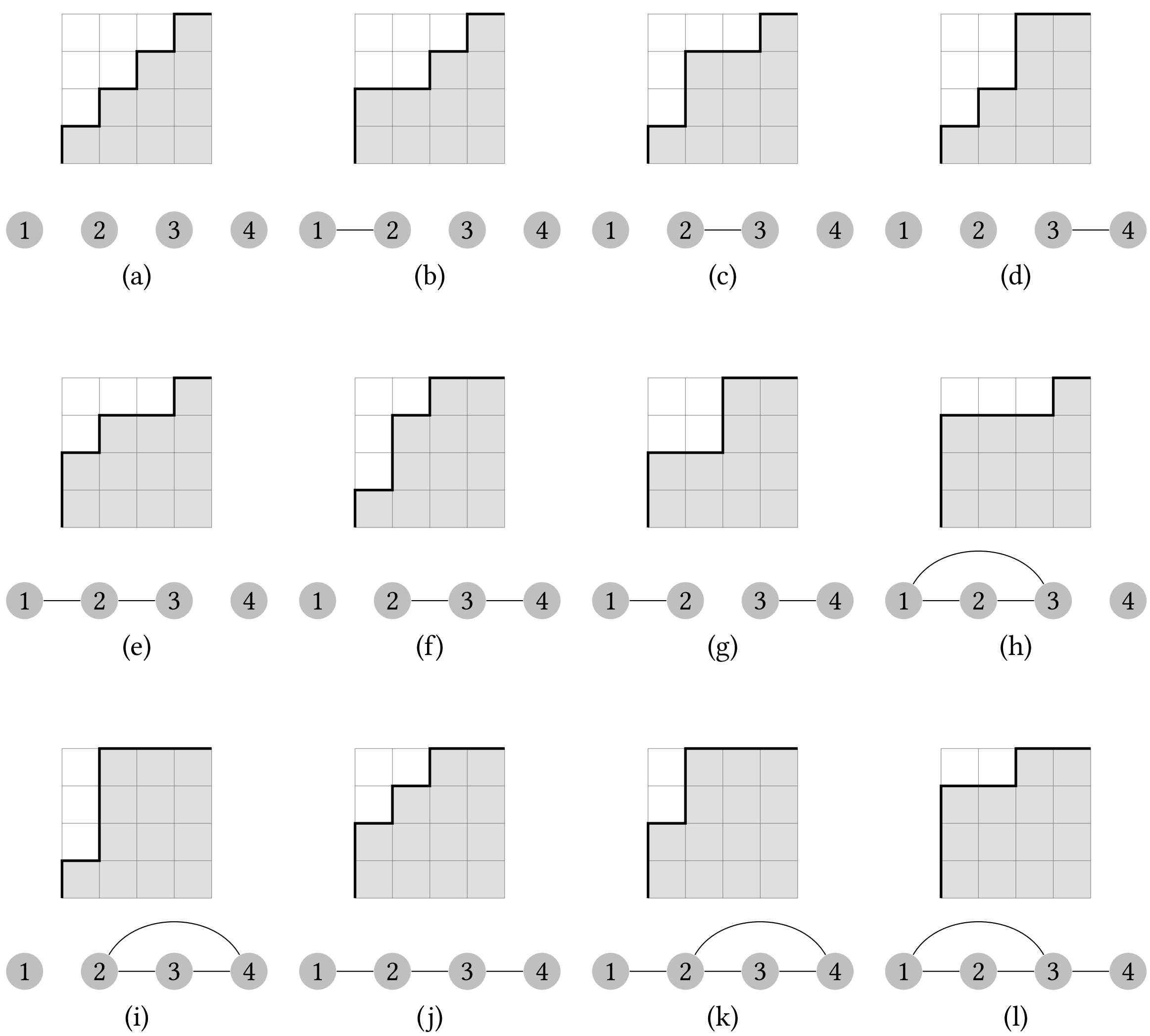
$$m_{(1)} \cdot (1+t)m_{(1^2)} = (1+t)e_{(2,1)}. \tag{1.136}$$

(e). The incomparability graph of the Dyck path $(1, 2, 3)$ is the edgeless graph of order 3, so the chromatic quasi-symmetric function is

$$m_{(1)}^3 = e_{(1^3)}. \tag{1.137}$$

■

Example 1.71. Now let us consider the indifference graphs of 4×4 Dyck paths.



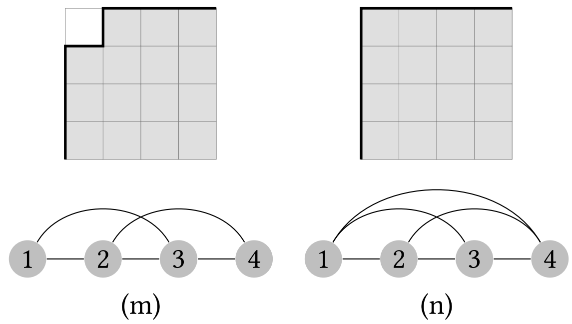


Figure 10: indifference graphs of rank 4 Dyck paths

(a). In the first nine cases, the Dyck path touches the diagonal, and the indifference graphs are disconnected. So the chromatic quasi-symmetric functions can be expressed as a product of chromatic quasi-symmetric function of smaller graphs. The edgeless graph has the chromatic quasi-symmetric function $m_{(1)}^4 = e_{(1^4)}$. Let us list the remaining eight disconnected cases:

(b)–(d). $m_{(1)}^2 \cdot (1+t)m_{(1^2)} = (1+t)e_{(2,1^2)}$,

(e)–(f). $m_{(1)} \cdot ((1+4t+t^2)m_{(1^3)} + tm_{(2,1)}) = e_{(3,1)} + (e_{(3,1)} + e_{(2,1^2)})t + e_{(3,1)}t^2$,

(g)–(i). $((1+2t+2t^2+t^3)m_{(1^3)})m_{(1)} = (1+2t+2t^2+t^3)e_{(3,1)}$,

(j). The Dyck path $(2,3,4,4)$ does not touch the diagonal, so the incomparability graph is connected. It is the 4-path with consecutive vertex labels $1,2,3,4$. The chromatic quasi-symmetric function cannot be reduced to a product of smaller ones. Let us compute it directly.

If we use four distinct colours, we may assume these to be $1,2,3,4$. Then there is only one permutation (i.e. colourings with four distinct colours) with three ascents: 1234, there are 11 permutations with two ascents: 1243, 1324, 1342, 1423, 2134, 2314, 2341, 2431, 3124, 3412 and 4123, there are 11 permutations with one ascent: 1432, 2143, 2431, 3142, 3214, 3241, 3421, 4132, 4213, 4231 and 4312, and there is one permutation with zero ascents: 4321. This gives rise to the monomial $(1+11t+11t^2+t^3)m_{(1^4)}$. Note that the coefficient of $m_{(1^4)}$ is the Eulerian polynomial of S_4.

If we use three distinct colours, we may assume that these colours are $1,2,3$. Let us first consider the case when colour 1 is used twice. There are three colourings with two ascents: 1213, 1231, 1312, and three colourings with one ascent: 1321, 2131, 3121. This gives rise to the monomial quasi-symmetric function $(3t+3t^2)M_{(2,1^2)}$. One can also compute, case-by-case, that the same set of colours $1,2,3$, when 2 or when 3 is used twice, yields three colourings with one and three colourings with two ascents. Alternatively, one can appeal to the fact that the chromatic quasi-symmetric function of the indifference graph of a Dyck path lies in $\mathrm{Sym}[t]$, which tells us that the colourings with three colours give rise to the monomial $(3t+3t^2)m_{(2,1^2)}$.

If we use two distinct colours, say 1 and 2, then there are two possible colourings 1212 and 2121, where the first one has two ascents and the second has one. Note that the colouring is symmetric in the two colours that appear. Thus, the colourings of the path P_4 with two colours gives rise to the monomial $(t+t^2)m_{(2^2)}$.

Hence, the chromatic quasi-symmetric function of P_4 (with the linear ordering of vertices) is

$$(1+11t+11t^2+t^3)m_{(1^4)} + (2t+3t^2)m_{(2,1^2)} + (t+t^2)m_{(2^2)}. \tag{1.138}$$

To find the e-expansion in each t-degree, we can read off the following identities from the transformation matrix from the monomial basis to the e-basis:

$$m_{(1^4)} = e_{(4)}$$

$$\begin{aligned} m_{(2,1^2)} &= e_{(3,1)} - 4m_{(1^4)} \\ &= e_{(3,1)} - 4e_{(4)} \end{aligned}$$

$$\begin{aligned} m_{(2^2)} &= e_{(2^2)} - 6m_{(1^4)} - 2m_{(2,1^2)} \\ &= e_{(2^2)} - 6m_{(1^4)} - 2(e_{(3,1)} - 4m_{(1^4)}) \\ &= e_{(2^2)} + 2e_{(4)} - 2e_{(3,1)}. \end{aligned}$$

Hence, the e-expansion of the chromatic quasi-symmetric function is

$$1 \cdot e_{(4)} + t \cdot (e_{(4)} + e_{(3,1)} + e_{(2^2)}) + t^2(e_{(4)} + e_{(3,1)} + e_{(2^2)}) + t^3 \cdot e_{(4)}, \tag{1.139}$$

which is (just about) e-positive. All these cancellations suggest that there should be some hidden sign-reversing cancellation lurking in the background. We shall see in Section 3, when we come to the proof of the Shareshian–Wachs conjecture, that this is indeed the case.

(l). Consider the indifference graph of the Dyck path $(2, 4, 4, 4)$, and compute its chromatic quasi-symmetric function directly.

If we use four distinct colours, say $1, 2, 3, 4$, then there is a single permutation 1234 with four ascents. There are six permutations with three ascents (recall that ascents are counted along the edges of the incomparability graph): 1243, 1324, 2134, 2314, 3124, 4123. There are ten permutations with two ascents: 1342, 1432, 2143, 2341, 2413, 3142, 3214, 3412, 4132, 4213. There are six permutations with one ascent: 1432, 2431, 3241, 3421, 4231, 4312. Finally, there is a single permutation 4321 with zero ascents. Consequently, the colourings of this graph using four different colours give rise to the monomial $(1 + 6t + 10t^2 + 6t^3 + t^4)m_{(1^4)}$.

There are three colourings using the colours, $1, 2, 3$ with colour 1 repeated twice. The colouring 1213 has three ascents, 1231 and 1312 have two ascents, and 1321 has one ascent. By a similar argument to the previous case, this means that the colourings with three colours give rise to the monomial $(t + 2t^2 + t^3)m_{(2,1^2)}$.

Hence, the chromatic quasi-symmetric function of this graph is

$$(1 + 6t + 10t^2 + 6t^3 + t^4)m_{(1^4)} + (t + 2t^2 + t^3)m_{(2,1^2)}. \tag{1.140}$$

Again, by looking at the transformation matrix from the monomial basis to the e-basis, we can find the e-expansion

$$1 \cdot e_{(4)} + t \cdot (2e_{(4)} + e_{(3,1)}) + t^2 \cdot (2e_{(4)} + 2e_{(2,1^2)}) + t^3 \cdot (2e_{(4)} + e_{(3,1)}) + t^4 \cdot e_{(4)}, \tag{1.141}$$

so this chromatic quasi-symmetric function is also e-positive in each degree.

One can also collect the coefficient of each elementary symmetric function of a given type

$$(1 + 2t + 2e^2 + 2t^3 + t^4)e_{(4)} + (t + t^3)e_{(3,1)} + 2t^2 e_{(2,1^2)}. \tag{1.142}$$

The indifference graph of the Dyck path $(3, 3, 4, 4)$ is isomorphic, so its chromatic quasi-symmetric function will be the same.

(m). Consider the indifference graph of the Dyck path $(3, 4, 4, 4)$. Note that the edges of this graph are symmetric, and thus for each colouring *abcd* (where some letters may denote the same colour), *dcba* is also a colouring. Consequently, the coefficient of monomial symmetric functions will be palindromic polynomials in t, and we only need to count half of the possible ascents (a shortcut we could have made in the case of the path P_4 too).

Looking at the 24 colourings with four distinct colours, say $1, 2, 3, 4$, we obtain one colouring 1234 with five ascents, three colourings with four ascents: 1243 1423, 2143, and eight colourings with three ascents: 1342, 1423, 2143, 2314, 2341, 3124, 3132, 3123. By reflecting these colourings, we obtain eight colourings with two ascents, three colourings with one ascent and one colouring with no ascents. There are two colourings, using the three colours $1, 2, 3$ with colour 1 repeated twice: 1231 has three ascents, and the colouring 1321 obtained by reflection has two ascents.

Hence, the chromatic quasi-symmetric function of this graph is

$$(1 + 3t + 8t^2 + 8t^3 + 3t^4 + t^5)m_{(1^4)} + (t^2 + t^3)m_{(2,1^2)}, \tag{1.143}$$

whose e-expansion is

$$1 \cdot e_{(4)} + t \cdot 3e_{(4)} + t^2(4e_{(4)} + e_{(2,1^2)}) + t^3 \cdot (4e_{(4)} + e_{(2,1^2)}) + t^4 \cdot 3e_{(4)} + t^5 \cdot e_{(4)}, \tag{1.144}$$

which is again e-positive in each coefficient. Collecting the coefficient of each elementary symmetric function yields the expression

$$(1 + 3t + 4t^2 + 4t^3 + 3t^4 + t^5)e_{(4)} + (t^2 + t^3)e_{(2,1^2)}. \tag{1.145}$$

(n). The indifference graph of the Dyck path $(4, 4, 4, 4)$ is the complete graph of order four. So all proper colourings use four distinct colours. Since the edge set is again symmetric, the coefficient of the monomial $m_{(1^4)}$ is a palindromic polynomial in t. If we take the colours $1, 2, 3, 4$, then there is one colouring 1234 with six ascents, three colourings with five ascents: 1243, 1324, 2134, five colourings with four ascents: 1342, 1423, 2143, 2314, 3124, and six colourings with three ascents: 1432, 2341, 2413, 3142, 3214, 4123. By reflecting all these colourings (since the edge set is again symmetric), we get five colourings with two ascents, three colourings with one ascent, and the one colouring with no ascents. Hence, the chromatic quisi-symmetric function is

$$(1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6)m_{(1^4)} = (1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6)e_{(4)}. \tag{1.146}$$

■

Remark 1.72. Note that for all the above chromatic quasi-symmetric functions, the coefficients in the e-expansion are all palindromic polynomials in t. Poincaré duality suggests that the corresponding geometric invariant could be the cohomology ring of a smooth variety.

2 Shareshian–Wachs Conjecture

In this section, we describe the geometric counterpart of chromatic quasi-symmetric functions, which we alluded to earlier. In particular, we discuss Tymoczko's dot action on the cohomology of regular, semisimple Hessenberg varieties, which are certain closed subvarieties of the full flag variety associated to Dyck paths.

Our main references for the constructions featured in this section, such as moment graphs of Hessenberg varieties, flow-up bases and the dot action, are [GKM], [Ty1] and [Ty2]. The construction of flow-up classes in 2.18 and 2.22 follows the article [Ty3]. Explicit computations are again a quintessential part of this section. In Example 2.13, 2.14 and 2.15 we compute a few simple examples of moment graphs. Then we compute the moment graphs of all regular semisimple Hessenberg varieties of rank 3 and 4, in Example 2.20, 2.21 and 2.23. In Example 2.26 and 2.28, we compute the dot action on the full flag variety and permutohedral variety of rank 3, which we compare with the earlier computations in Example 1.53.

2.1 Hessenberg Varieties

Definition 2.1. As a set, the *full flag variety* of rank n over $\mathbb{C}$ is the set of full flags of vector spaces

$$\mathrm{Fl}(\mathbb{C}^n) = \{(F_1 \subset \cdots \subset F_n) : \dim F_i = i\}. \tag{2.1}$$

One can endow this set with the structure of a projective variety, as a product of projective spaces (or more general Grassmannians for partial flag varieties). In fact, one can describe the full flag variety as the quotient of the group $G = \mathrm{GL}_n(\mathbb{C})$ by the (standard) Borel subgroup B, which is the smallest parabolic subgroup, consisting of upper triangular matrices. Consequently, the dimension of the full flag variety $\mathrm{Fl}(\mathbb{C}^n)$ is $\binom{n}{2}$.

Example 2.2. (a). For any $g \in \mathrm{GL}_n(\mathbb{C})$ with column vectors $v_1, \ldots, v_n$, we have

$$gB = (\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots v_n \rangle = \mathbb{C}^n). \tag{2.2}$$

Each flag has a canonical representative obtained by Gaussian elimination, where one can only subtract a larger index row or column from a smaller one. This representative is a permutation matrix with possible extra non-zero entries above and to the right of the 1s, for example:

$$\begin{pmatrix} 2 & 2 & \mathbf{1} & 0 \\ 3 & \mathbf{1} & 0 & 0 \\ 7 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix}.$$

(b). For any permutation matrix w of rank n, we have

$$wB = (\{0\} \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots e_{w(n)} \rangle = \mathbb{C}^n). \tag{2.3}$$

The flag variety G/B admits an affine cell decomposition, induced by the Bruhat decomposition of $G = \mathrm{GL}_n(\mathbb{C})$. The cell BwB/B is called the *Schubert cell* C_w associated to a permutation $w \in S_n$, which gives rise to the stratification

$$G/B = \bigsqcup_{w \in S_n} BwB/B. \tag{2.4}$$

(c). The Schubert cell C_w consists of the flags, whose canonical form has zeros to the right and under the 1s in the permutation matrix of w. In the above example, $w = 4213$, and representatives of flags in the corresponding Schubert cells are of the form

$$\begin{pmatrix} * & * & \mathbf{1} & 0 \\ * & \mathbf{1} & 0 & 0 \\ * & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix},$$

so the dimension of C_{4213} is four. In general, a Schubert cell is an affine cell, and the $\mathbb{C}$-dimension of C_w is the number of inversions of w, which is equal to the *length* of the permutation in the Bruhat order, i.e. the minimal number of simple transpositions needed to decompose w. For example, from the above diagram, which is called a *Rothe diagram*, we can see that the inversion set of the permutation $(4, 2, 1, 3)$ is $\{(4, 2), (4, 1), (4, 3), (2, 1)\}$.

(d). The closure $\overline{C_w}$ of the Schubert cell is called the *Schubert variety*, which also decomposes as a union of Schubert cells

$$\overline{C_w} = \bigsqcup_{v \leq w} C_v, \tag{2.5}$$

given explicitly by the (strong) Bruhat order on the Weyl group $W = S_n$ that we encountered in Example 1.53. Schubert cells endow the flag variety with the structure of a CW-complex, so the set of Schubert varieties $\{\overline{C_w} : w \in W\}$ induces a $\mathbb{C}$-module basis for the cohomology $H^*(G/B; \mathbb{Q})$.

Moreover, the intersection of Schubert varieties corresponds to the product of the cohomology classes that they induce.

(e). In 1990, Lakshmibai and Sandya found a combinatorial smoothness condition for Schubert varieties (see [LS]). The Schubert variety $\overline{C_w}$ is smooth if and only if the permutation w avoids the patterns $cdab$ and $dbca$ for any $a \leq b \leq c \leq d$, in the one-line notation. For example, the permutation $v = 324651$ gives rise to a smooth Schubert variety, whereas the Schubert variety corresponding to $w = 624351$ is singular, since w contains the subword 6251.

■

Remark 2.3. (a). The set of k-dimensional subspaces of an n-dimensional vector space (over $\mathbb{C}$, in this context) can also be endowed by the structure of a projective variety via the Plücker embedding. This variety is called the Grassmannian $\mathrm{Gr}(k, n)$. The Grassmannian can be described as the quotient $\mathrm{GL_n}(\mathbb{C})/P$, where P is a maximal parabolic subgroup of type A. Thus, there is a natural projection from the full flag variety to the Grassmannian, $G/B \twoheadrightarrow G/P$.

(b). It is natural to look at the images of the Schubert cells (from the Bruhat decomposition of the flag variety G/B) under this projection. The cells that do not collapse are indexed by partitions λ whose Young diagrams have at most k columns and at most $n - k$ rows. The corresponding cohomology classes are denoted by $\sigma_\lambda \in H^*(\mathrm{Gr}(k, n); \mathbb{C})$.

For example, there are only 6 Schubert cells out of the 24 Schubert cells of the rank four full flag variety, that survive the projection $\mathrm{GL}_4(\mathbb{C}) \twoheadrightarrow \mathrm{Gr}(2, 4)$. These cells are indexed by the partitions $\emptyset$, (1), (2), $(1, 1)$, $(2, 1)$ and $(2, 2)$.

(c). The structure constants of the basis elements σ_λ,

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu, \tag{2.6}$$

are the constants that appear in the tensor product decomposition of irreducible (polynomial) representation of $\mathrm{GL}_n(\mathbb{C})$, called the Littlewood-Richardson coefficients. These numbers are also pivotal in the theory of symmetric functions, and we shall discuss them in the next section, in Remark 3.9.

Remark 2.4. (a). There is another variety, that reveals a lot about the geometry of flag varieties. Let $\mathfrak{g}$ denote the Lie algebra of $G = \mathrm{GL}_n(\mathbb{C})$, which consists of all $n \times n$ matrices and let $\mathfrak{b}$ denote the Lie algebra of B, whose elements are the upper triangular matrices. The *Springer fibre* associated to a nilpotent element $X \in \mathfrak{g}$ is given by

$$\mathcal{S}_X := \{gB \in G/B : g^{-1}Xg \in \mathfrak{b}\}. \tag{2.7}$$

Springer fibres are a natural precursor to (and an example of) Hessenberg varieties, so let us briefly describe their role in representation theory.

(b). One can again consider the affine cells $C_w \cap \mathcal{S}_X$. Here, these cells do not form a CW-complex, but the closures still induce a basis for the cohomology $H^*(\mathcal{S}_X)$.

(c). Let $\lambda(X)$ be the Young diagram (or the corresponding partition) determined by the Jordan type of X, i.e. $\lambda(X)$ has as many rows as the number of Jordan boxes of X, and each row has as many squares as the size of the Jordan box. Based on this object, there is a combinatorial criterion for smoothness. The Springer variety $\mathcal{S}_X$ is not necessarily connected, but all components are smooth if $\lambda(X)$ has two rows (and there are a few other conditions that would also guarantee smoothness). In 1976, Spaltenstein proved that the top dimensional Schubert cells $C_w \cap \mathcal{S}_X$ in the Springer variety are in bijection with the standard fillings of $\lambda(X)$, in the article [Spa].

This suggests that there should be a a connection with the representation theory of S_n, which we will discuss at the end of this section. The cohomology $H^*(\mathcal{S}_X)$ carries an S_n-action, and the top-dimensional part is the irreducible S_n-module (the Specht module, that we introduced in 1.33) corresponding to the partition $\lambda(X)$. Moreover, each irreducible representation of S_n uniquely arises in this way (up to conjugation of X, which leaves the Springer variety invariant). Our goal in this section is to see how this S_n-action manifests on *regular, semisimple Hessenberg varieties*, which we shall define subsequently.

Definition 2.5. A *Hessenberg variety* is a subvariety of the full flag variety $\mathrm{Fl}(\mathbb{C}^n)$, which generalizes Springer fibres. It is determined by a Dyck path h and a matrix $M \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ in the Lie algebra of $\mathrm{GL}_n(\mathbb{C})$, and it is defined by

$$\mathrm{Hess}(M, h) = \{F_* \in \mathrm{Fl}(\mathbb{C}^n) \mid MF_i \subseteq F_{h(i)} \, \forall i \in [n]\}. \tag{2.8}$$

Alternatively, it can be described as the set $\{gB \in G/B \mid g^{-1}Mg \in H_h\}$, where H_h is the *Hessenberg space*, the complex vector space generated by all elementary matrices E_{ij} with $h(j) \geq i$. From this description, it is clear that Hessenberg varieties are closed in G/B, and consequently they are projective varieties.

Example 2.6. (a). If we take the Dyck path $h(i) = n$, then we have $H_h = \mathfrak{g}$. Thus, for any $M \in \mathfrak{g}$, $\mathrm{Hess}(M, h)$ is the full flag variety.

(b). If we take the Dyck path $h(i) = i$ for all i, then we have $H_h = \mathfrak{b}$ is the Lie algebra of B. When $M \in \mathfrak{g}$ is nilpotent, $\mathrm{Hess}(M, h)$ is the Springer variety.

(c). If $h(i) = i + 1$ for all $i \in \{1, \ldots, n-1\}$, then the Hessenberg variety is a toric variety $\mathcal{H}_n$, called the *permutohedral variety* associated to the *Coxeter complex* of S_n, whenever $M \in \mathfrak{g}$ is *regular and semisimple*, i.e. conjugate to a diagonal matrix with distinct eigenvalues. Note that for any $g \in \mathfrak{g}$, we have

$$\mathrm{Hess}(M, h) \cong \mathrm{Hess}(gMg^{-1}, h), \tag{2.9}$$

where the isomorphism is given by $V_* \mapsto gV_*$, so we may choose a diagonal matrix S with distinct diagonal entries. From now on, we will only consider Hessenberg varieties corresponding to such a matrix S. The Dyck path $h(i) = i$, that appeared in part (b) is also important in the context of regular semisimple Hessenberg varieties, not only nilpotent one. We will see a number of examples later on, in Example 2.21 and 2.23.

(d). In 1992, De Mari, Procesi and Shayman proved, in the article [MPS], that any regular semisimple Hessenberg variety $\mathrm{Hess}(S, h)$ is smooth of $\mathbb{C}$-dimension

$$\sum_{i=1}^{n}(h(i) - i).$$

Its *Poincaré polynomial* is given by

$$\mathrm{Poin}(\mathrm{Hess}(S, h); q) := \sum_{k\geq 0} \dim_{\mathbb{C}} H^k(\mathrm{Hess}(S, h))q^k = \sum_{w\in S_k} q^{2\ell_h(w)}, \tag{2.10}$$

where the modification of the length $\ell_h(w)$, with respect to the Dyck path h, is defined as

$$\ell_h(w) := \#\{(i, j) \mid 1 \leq i < j \leq h(i),\ w(i) > w(j)\}. \tag{2.11}$$

We shall elaborate on these properties in Construction 2.20.

■

Remark 2.7. The restriction map $H^*(\mathrm{Fl}(\mathbb{C}^n)) \to H^*(X(h))$ is not surjective in general, for regular semisimple Hessenberg varieties $X(h)$. We will show this in Example 2.22, where we explicitly construct generators.

Example 2.8. We would like to exhibit a very explicit combinatorial model for the generators of the cohomology ring. To this end, one should consider *equivariant cohomology* first, and then go back to ordinary cohomology. We will be regarding Hessenberg varieties as T-*varieties*, i.e. varieties with a torus action. In order to leverage the torus action and construct a suitable contravariant functor from the category of T-varieties to the category of graded rings, it is expedient to replace X by a larger space, which is homotopy equivalent to X but which has a suitable T-action.

A possible candidate would be taking the orbit space X/T. In order to see why the orbit space is inadequate for our purposes, it is convenient to consider a more general setting. If $G = S^1$ is acting on S^2 by rotation,

$$r_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.12}$$

then the orbit space is isomorphic to the closed integral $[-1, 1]$, which is contractible, so its cohomology is isomorphic to that of a point, and it gives us no information about the action. On the other hand, if we consider the action of $\mathbb{Z}$ on $\mathbb{R}$ by translation then the orbit space $\mathbb{R}/\mathbb{Z}$ is isomorphic to S^1, whose cohomology ring with complex coefficient is $\mathbb{C}[x]/(x^2)$. The second action was *free* (i.e. if we have $g \cdot x = x$ for some $x \in X$, then $g = e$), whereas the first one wasn't. One would hope to extend the original group action to a free action on a larger, homotopy equivalent space.

The main idea behind this construction comes from the following observation. Let G be a group acting freely on a space E. Let M be another space with a G-action. Then no matter how G acts on M, the diagonal action of G on $E \times M$, given by $g \cdot (e, x) = (g \cdot e, g \cdot x)$, is free. Indeed, g lies in the stabilizer of the point $(e, x) \in E \times M$ if and only if $g \cdot e = e$ and $g \cdot x = x$, which is equivalent to saying that $g = e$, since the G-action on E is free. Thus, if take a contractible space EG with a free G action, then the diagonal action on $EG \times M$ is free and this space has the same homotopy type as M. For algebraic groups or compact Lie groups one can always find a principal G-bundle $EG \to BG$, with weakly contractible total space EG. ■

Construction 2.9. Consider the orbit space of $EG \times X$, i.e. the space

$$X /\!/ G := EG \times_G X = EG \times X/\sim, \tag{2.13}$$

where $(e, x) \sim (eg^{-1}, gx)$ for some $g \in G$.

This construction is called the *homotopy quotient* (or *Borel construction*). Then the *equivariant cohomology* of X with respect to the G-action is defined as

$$H_G^*(X; \mathbb{C}) = H^*(X /\!/ G; \mathbb{C}), \tag{2.14}$$

where we could have chosen coefficients from any ring R, but we will always stick to complex coefficients, just like we did in our treatment of symmetric functions. The fact that equivariant cohomology is independent of the choice of EG is a difficult result, that uses the theory of *approximation spaces*. By construction, $EG \times_G X \longrightarrow BG$ is a fibre bundle with fibre X, so we have a string of ring homomorphisms

$$H^*(BG) \longrightarrow H_G^*(X) \longrightarrow H^*(X). \tag{2.15}$$

Now we can return to our original setting of a rank k torus action on X. Here, the total space is $ET = ET_k \cong (S^\infty)^k$ and the classifying space is $BT = BT_k \cong \prod_k \mathbb{CP}^\infty$. The torus T acts on $ET \times_T X$ via the diagonal action $(e, x) \cdot t = (et, t^{-1}x)$.

Note that if $X = \mathrm{pt}$ is a point, then, by definition, the equivariant cohomology of X is isomorphic to the ordinary (singular) cohomology of the classifying space $BT = ET/T$. Since $ET \longrightarrow BT$ is a principal T-bundle, $H^*(BT)$ is isomorphic to the ring of characteristic classes. This observation, along with the Künneth formula for ordinary cohomology, yields the following string of isomorphisms between the equivariant cohomology of the point with respect to the (trivial) action of the rank k torus and the symmetric algebra of the cotangent space $\mathfrak{t}^*$ of T,

$$H^*(BT) \cong H^*\left(\prod_k \mathbb{CP}^\infty\right) \cong \bigotimes_k H^*(\mathbb{CP}^\infty) \cong \bigotimes_k \mathbb{C}[t_i] \cong \mathcal{S}(\mathfrak{t}^*), \tag{2.16}$$

where the variable t_i corresponds to the first Chern class $c_1(\mathcal{O}_{\mathbb{CP}^\infty}(-1))$ of the *tautological bundle*, i.e. the line bundle over BT corresponding to the projection $T \longrightarrow \mathbb{C}^*$ given by $\mathrm{diag}(x_1, \ldots, x_k) \longrightarrow x_i$. From now on, let us denote the ring $H_T^*(\mathrm{pt})$ by Λ or Λ_T, which is isomorphic to the polynomial ring $\mathbb{C}[t_1, \ldots, t_k]$.

Analogously to the ordinary cohomology, functionality is satisfied for T-equivariant cohomology, in the category of T-spaces with equivariant maps as morphisms. By functionality of equivariant cohomology, the projection $X \longrightarrow \mathrm{pt}$ induces a map $\Lambda \longrightarrow H_T^*(X)$, which allows us to regard $H_T^*(X)$ as a module over Λ, in fact, an algebra via the cup product. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & ET \times_T X \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BT \end{tikzcd} \tag{2.17}$$

Restriction induces a canonical map $H_T^*(X) \longrightarrow H^*(X)$ lying over $\Lambda \longrightarrow H^*(\mathrm{pt})$. To describe this Λ-module explicitly, we would like the following properties to hold.

(a). The map $\mathbb{C}[t_1, \ldots, t_k] \cong H^*(BT) \longrightarrow H_T^*(X)$ is injective.

(b). The induced map $H_T^*(X) \longrightarrow H^*(X)$ is surjective.

(c). The ordinary cohomology can be obtained from the equivariant cohomology by the isomorphism

$$H^*(X) \cong \frac{H_T^*(X)}{\langle t_1, \ldots, t_k \rangle \, H_T^*(X)}. \tag{2.18}$$

(d). The *localisation map*

$$\iota^* \colon H_T^*(X) \to H_T^*(X^T) = \bigoplus_{w \in X^T} \mathbb{C}[t_1, \ldots, t_k], \tag{2.19}$$

induced by the inclusion $\iota \colon X^T \hookrightarrow X$ of fixed points, is injective.

The above conditions are, in practice, fairly ubiquitous, but not true for all T-spaces. One can obtain a counterexample by gluing three Riemann spheres $\mathbb{CP}^1$ along a pair of distinguished points for each pair, and letting $\mathbb{C}^*$ act on this space by fixing the intersection points and rotating each sphere around the axes given by these intersection points.

Construction 2.10. A variety with a T-action having finitely many fixed points $\{v_i\}_{i \in V}$ and finitely many 1-dimensional orbits $\{O_i\}_{i \in E}$, in addition to satisfying the conditions listed in Construction 2.9, are called *GKM-spaces*. The closure of each O_i is an embedded copy of the Riemann sphere $\mathbb{CP}^1$, and it contains exactly two fixed points, which we will call the *south pole* N_i and the *south pole* S_i, in any particular order, for the time being. These properties allow us to construct a graph whose vertices correspond to the fixed points of the T-action and whose edges correspond to the 1-dimensional orbits. It is a difficult theorem that there are no double edges in this graph (so this graph is simple).

Moreover, for each orbit O, we have a subtorus $T' \subset T$ of codimension 1 that fixes O pointwise. The main result of GKM theory (see [GKM], Theorem 1.2.2) states that in order to identify the image of the localisation map ι^* based on the above graph, we need to assign weights for the edges and impose additional conditions on the weights. These conditions stem from character theory. If α_i denotes the annihilator in the cotangent space $\mathfrak{t}^*$ of the subtorus fixing O_i pointwise, then the torus acts on the two fixed points in $\overline{O}$ with weight $\alpha_i \colon \mathfrak{t} \to \mathbb{C}$ and $-\alpha_i$, respectively. We will call the fixed point with weight α_i the north pole, and the other fixed point the south pole. Subsequently, we label the edge by α_i, and give it a direction, going from the south pole to the north pole. We call this graph the *moment graph* or *GKM graph* $M(X)$ of X. The GKM Presentation Theorem states that the information encoded in the moment graph is sufficient to determine the equivariant cohomology.

Theorem 2.11. *Given a moment graph of order n, the equivariant cohomology can be described as*

$$H_T^*(X) \cong \left\{ (f_{v_1}, \ldots, f_{v_n}) \in \bigoplus_n \Lambda \,\middle|\, f_{N_j} - f_{S_j} \in \langle \alpha_j \rangle \quad \forall\, j \in E(M(X)) \right\}, \tag{2.20}$$

where $E(M(X))$ denotes the edge set of the moment graph $M(X)$.

Remark 2.12. A regular semisimple Hessenberg variety $X = \mathrm{Hess}(M, h)$, which we introduced in Example 2.6, is smooth. This stems from the following general statement (Lemma 4 in [MPS]). If X is a complete variety with a T-action such that every $x \in X^T$ is a smooth point, then X is smooth. As a consequence, the action of T on G/B given by $t \cdot gB = tgB$, restricted to X (which is well-defined since $T = C_G(X)$ is the centraliser) makes X into a GKM-space.

Example 2.13. Consider the action of $T = \mathbb{C}^*$ on $X = \mathbb{CP}^1$ given by $t \cdot [x_0, x_1] = [x_0, tx_1]$. There are two fixed points $v_1 = [1, 0]$ and $v_2 = [0, 1]$ and a single 1-dimensional orbit $O = [*, *] = [1, *]$, where $*$ denotes an arbitrary element of $\mathbb{C}^*$. Then the moment graph is the complete graph K_2 of order two

$$v_1 \text{ —— } 0 \in \mathfrak{t}^* \text{ —— } v_2$$

Figure 11: moment graph of $\mathbb{CP}^1$

where the edge is labelled by $0 \in \mathfrak{t}^*$ since the subtorus fixing $[*, *]$ is $1 \subset T$ with the identity action, and consequently the annihilator in the cotangent space is 0. Therefore, by Theorem 2.11, we have

$$H_T^*(X) \cong \{(f_1, f_2) \in \Lambda^{\oplus 2} \mid f_1(0) = f_2(0)\}, \tag{2.21}$$

in other words, $f_2 - f_1 \in \langle t \rangle$, and the polynomials associated to the vertices are

$$p_1 \text{ ———— } p_1 + tp_2$$

Figure 12: $H_T^*(\mathbb{CP}^1)$

where $p_2 \in \Lambda$ is arbitrary. Since p_1 and p_2 may be picked arbitrarily, by reading off their coefficients, we see that $H_T^*(X)$ is generated as a module over Λ by $(1, 1)$ and $(0, t)$ in degrees 0 and 1, as polynomials in t.

In fact the equivariant cohomology $H_T^*(X)$ of a smooth Hessenberg variety X can only be nontrivial in even degrees, and the degree of a class (i.e. homogeneous tuple of polynomials) associated to its moment graph is only half the degree of the corresponding cohomology class. However, this shift in the degrees will not be important in our computations. ■

Example 2.14. Analogously to our the previous example, consider the action of $(\mathbb{C}^*)^2$ on $\mathbb{CP}^2$ given by $(t_1, t_2) \cdot [x_0, x_1, x_2] = [x_0, t_1x_1, t_2x_2]$. This action has three fixed points $v_1 = [1, 0, 0]$, $v_2 = [0, 1, 0]$, $v_3 = [0, 0, 1]$ and three 1-dimensional orbits $O_1 = [1, *, 0]$, $O_2 = [0, 1, *]$, $O_1 = [1, 0, *]$. The codimension 1 subtorus that stabilizes O_1 is $1 \times \mathbb{C}^*$, the orbit O_2 is stabilized by $\{(t, t) : t \in \mathbb{C}^*\}$ and O_3 is stabilized by $\mathbb{C}^* \times 1$. These stabilizers are annihilated by $t_1, t_1 - t_2$ and $t_2 \in \mathfrak{t}^*$, respectively. So the moment graph associated to this action is

$$\begin{tikzcd} & v_1 \arrow[dl, dash, "t_1"'] \arrow[dr, dash, "t_2"] & \\ v_2 \arrow[rr, dash, "t_1 - t_2"'] & & v_3 \end{tikzcd}$$

Figure 13: moment graph of $\mathbb{CP}^1$

By invoking Theorem 2.11 again, we may associate an arbitrary polynomial $p_1 \in \Lambda$ to v_1, and a polynomial p_2 to v_2 subject to the condition that $p_2 - p_1 \in \langle t_1 \rangle$. Thus p_1 is of the form $p_1 + t_1q$, where $q \in \Lambda$ is again arbitrary. To compute the polynomial P_3 associated to v_3, we need to impose two conditions. We have $p_3 - p_1 \in \langle t_2 \rangle$ and $p_3 - p_2 \in \langle t_1 - t_2 \rangle$. Therefore, p_3 is of the form $p_1 + t_2r$ for some $r \in \Lambda$, and $p_3 - p_2 = t_2r - t_1p_2 \in \langle t_1 - t_2 \rangle$. Consequently, $r = p_1 + (t_2 - t_1)p_3$, and $p_3 = p_1 + t_2(p_2 + (t_2 - t_1)p_3)$. So we have the following vertex labels,

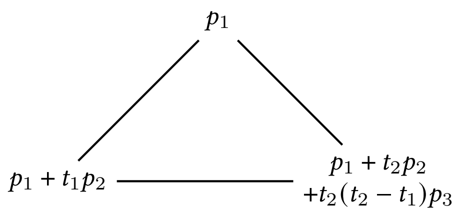


Figure 14: $H_T^*(\mathbb{CP}^1)$

Hence the equivariant cohomology can be described as

$$H_T^*(X) = \{(p_1, p_1 + t_1 p_2, p_1 + t_2 p_2 + t_2(t_2 - t_1)p_3) \in \Lambda^{\oplus 3} : p_1, p_2, p_3 \in \Lambda\}, \tag{2.22}$$

and by reading off the coefficients as before, it is generated as a module over Λ by $(1, 1, 1)$, $(0, t_1, t_2)$ and $(0, 0, t_2(t_2 - t_1))$ in degrees 0, 1 and 2, respectively, as polynomials. These degrees are in turn 0, 2 and 4 according to the grading of $H_T^*(X)$. ■

2.2 Moment Graphs of Hessenberg Varieties

Example 2.15. Consider the action $t \cdot [g] \mapsto [t \cdot g]$ of the maximal torus $(\mathbb{C}^*)^3$ on the full flag variety of rank three. Recall that we may represent a full flag $V_1 \subsetneq V_2 \subsetneq V_3 = \mathbb{C}^3$ by a matrix $(g_{ij}) = g \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ with columns c_1, c_2, c_3 such that V_i is spanned by the vectors $c_1, \dots, c_i$. Explicitly, the T-action is given by

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \cdot (g_{ij}) = \begin{pmatrix} t_1 g_{11} & t_1 g_{12} & t_1 g_{13} \\ t_2 g_{21} & t_2 g_{22} & t_2 g_{23} \\ t_3 g_{31} & t_3 g_{32} & t_3 g_{33} \end{pmatrix}. \tag{2.23}$$

Note that S_n embeds in $\mathrm{GL}_n(\mathbb{C})$ as the subgroup of permutation matrices. The fixed points of the torus action are exactly the permutations, since in a matrix representing a fixed point, each column has at most one nonzero entry (otherwise we can see from the matrix formula that the orbit contains the span of at least two of the t_i's, so it cannot be 1-dimensional) and the matrix must be of full rank since it represents an element of the full flag variety. Denote the permutations (i.e. the fixed points, which correspond to the vertices of the moment graph) by the following

$$v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By the properties of moment graphs discussed in the previous subsection (see Construction 2.10), each 1-dimensional orbit has exactly two fixed points in its closure, which is an embedded copy of the Riemann sphere. We infer that 1-dimensional orbits are spanned by a matrix that differs from a permutation matrix by one extra nonzero entry. This entry has to lie to the left and on top of the 1s in the permutation matrix. This follows from the assumption that the matrices representing elements of the quotient G/B are in echelon form. We denote the nine 1-dimensional orbits by the following

$$O_1 = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad O_3 = \begin{pmatrix} 0 & a & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad O_4 = \begin{pmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad O_5 = \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$O_6 = \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad O_7 = \begin{pmatrix} 0 & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad O_8 = \begin{pmatrix} 0 & a & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad O_9 = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We claim that O_1 connects the vertices v_2 and v_1 in the moment graph. Indeed, we have

$$\mathrm{Flag}(\langle e_2\rangle \subsetneq \langle e_1, e_2\rangle \subsetneq \langle e_1, e_2, e_3\rangle) \xleftarrow{0\leftarrow a} \mathrm{Flag}(\langle ae_1 + e_2\rangle \subsetneq \langle ae_1 + e_2, e_1\rangle \subsetneq \langle ae_1 + e_2, e_1, e_3\rangle) \tag{2.24}$$

$$= \mathrm{Flag}(\langle e_1 + a^{-1}e_2\rangle \subsetneq \langle e_1 + a^{-1}e_2, e_2\rangle \subsetneq \langle e_1, e_2, e_3\rangle) \tag{2.25}$$

$$\xrightarrow{a\to\infty} \mathrm{Flag}(\langle e_1\rangle \subsetneq \langle e_1, e_2\rangle \subsetneq \langle e_1, e_2, e_3\rangle), \tag{2.26}$$

where we used the fact that we may change the basis without altering the full flag it represents, when we multiply by an element of the Borel B. Similarly, we can find the edges corresponding to the other orbits: O_2 connects the vertices v_3 and v_1 in the moment graph. Indeed, we have

$$\mathrm{Flag}(\langle e_1\rangle \subsetneq \langle e_1, e_3\rangle \subsetneq \langle e_1, e_2, e_3\rangle) \xleftarrow{0\leftarrow a} \mathrm{Flag}(\langle e_1\rangle \subsetneq \langle e_1, ae_2 + e_3\rangle \subsetneq \langle e_1, ae_2 + e_3, e_2\rangle) \tag{2.27}$$

$$= \mathrm{Flag}(\langle e_1\rangle \subsetneq \langle e_1, e_2 + a^{-1}e_3\rangle \subsetneq \langle e_1, e_2, e_3\rangle) \tag{2.28}$$

$$\xrightarrow{a\to\infty} \mathrm{Flag}(\langle e_1\rangle \subsetneq \langle e_1, e_2\rangle \subsetneq \langle e_1, e_2, e_3\rangle), \tag{2.29}$$

By a similar computation, O_3 connects v_4 and v_2, O_4 connects v_4 and v_3, O_5 connects v_5 and v_2, O_6 connects v_5 and v_3, O_7 connects v_6 and v_4, O_8 connects v_6 and v_5 and O_9 connects v_6 and v_1.

Let us summarise these computations by drawing the (unlabeled) moment graph,

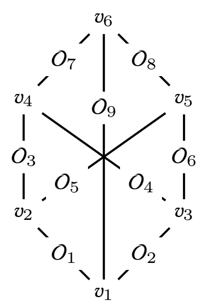


Figure 15: unlabeled moment graph of $\mathrm{Fl}(\mathbb{C}^3)$

By the matrix description of the action, the codimension 1 subtorus fixing O_1 pointwise in G/B is

$$\{(s, s, r) : s, r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3.$$

Indeed, the condition $t_1 = t_2$ must be imposed, and nothing else matters, since we have

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \cdot \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t_1 a & t_1 & 0 \\ t_2 & 0 & 0 \\ 0 & 0 & t_3 \end{pmatrix} = \begin{pmatrix} (t_1/t_2)a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.30}$$

Similarly, the subtorus fixing O_2 pointwise is $\{(r, s, s) : s, r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$ since

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 a & t_2 \\ 0 & t_3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (t_2/t_3)a & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.31}$$

Analogously, the subtorus fixing O_3 pointwise is $\{(s,r,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, the subtorus fixing O_4 pointwise is $\{(s,s,r) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, the subtorus fixing O_5 pointwise is $\{(r,s,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, the subtorus fixing O_6 pointwise is $\{(s,r,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, the subtorus fixing O_7 pointwise is $\{(r,s,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, the subtorus fixing O_8 pointwise is $\{(s,s,r) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$, and the subtorus fixing O_9 pointwise is $\{(s,r,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$.

Note that the annihilator of the subtorus $\{(s,s,r) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$ is generated by $t_1 - t_2 \in \Lambda$, the annihilator of $\{(s,r,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$ is generated by $t_1 - t_3$ and the annihilator of $\{(r,s,s) : s,r \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$ is generated by $t_2 - t_3$. Therefore, we obtain the following moment graph,

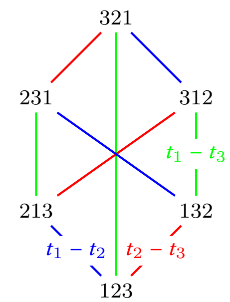
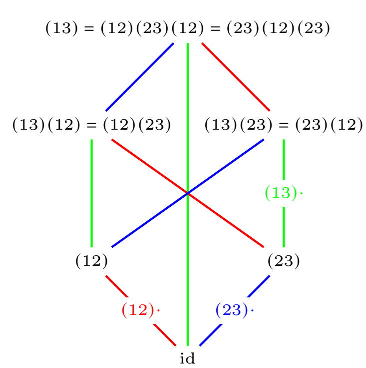


Figure 16: moment graph of $\mathrm{Fl}(\mathbb{C}^3)$

If we draw the graph in a suitable fashion (like the one above), then the linear flow of the torus action endows each edge with a direction such that the source is upwards from the target. Also, parallel edges have the same label; an observation that will be key to what follows. The reason for this becomes clear when one writes the vertices as a product of a minimal number of transpositions. Indeed, each edge which is labelled $t_i - t_j$ corresponds to multiplication of its south pole by the transposition (i, j), giving a combinatorial meaning to the annihilators.

In this context, the moment graph is the Hasse diagram of the (strong) Bruhat order on S_3 with an additional edge from 321 to 123. The length of each permutation is given by the minimal number of transposition needed. In terms of the matrix representation, the transposition in question swaps the two row indices of the extra nonzero element a and the entry 1 that lies in the same column. If the edge label is $t_i - t_j$, then these indices are (i, j). Moreover, the edge is directed towards the smaller vertex in the Bruhat order.



Similarly to our earlier examples, the polynomial f_i associated to the fixed point v_i can be computed by taking into account the conditions determined by the edges of the moment graph. We may appeal to Theorem 2.11 to find generators for the equivariant cohomology of this torus action.

Let us pick v_1 as our first vertex, and associate an arbitrary polynomial $f_1 = b_1 \in \Lambda$ to it. If the second vertex we consider is v_2, then the associated polynomial must be of the form $f_2 = b_1 + (t_1 - t_2)b_2$ for some $b_2 \in \Lambda$, since $f_2 - b_1 \in \langle t_1 - t_2 \rangle$ and no other condition is imposed. Similarly, a polynomial f_3 associated to our third vertex v_3 is of the form $b_1 + (t_2 - t_3)b_3$ for some $b_3 \in \Lambda$, since $f_3 - b_1 \in \langle t_2 - t_3 \rangle$. Now we get to v_4, where two conditions have to be taken into account simultaneously. If $f_4 \in \Lambda$ is attached to v_4, then

$$f_4 - (b_1 + (t_1 - t_2)b_2) \in \langle t_1 - t_3 \rangle \quad \text{and} \quad f_4 - (b_1 + (t_2 - t_3)b_3) \in \langle t_1 - t_2 \rangle. \tag{2.32}$$

So f_4 is of the form

$$f_4 = b_1 + (t_1 - t_2)b_2 + (t_1 - t_3)q, \tag{2.33}$$

for some $q \in \Lambda$, satisfying the condition that

$$b_1 + (t_1 - t_2)b_2 + (t_1 - t_3)q - (b_1 + (t_2 - t_3)b_3) = (t_1 - t_2)p_2 + (t_1 - t_3)q - (t_2 - t_3)b_3 \in \langle t_1 - t_2 \rangle, \tag{2.34}$$

which yields $q = (t_1 - t_2)b_4 - b_3$ for an arbitrary $b_4 \in \Lambda$. Consequently, we have

$$f_4 = b_1 + (t_1 - t_2)b_2 - (t_1 - t_3)b_3 + (t_1 - t_3)(t_1 - t_2)b_4. \tag{2.35}$$

Similarly, we impose the conditions

$$f_5 - (b_1 + (t_1 - t_2)b_2) \in \langle t_2 - t_3 \rangle \quad \text{and} \quad f_5 - (b_1 + (t_2 - t_3)b_3) \in \langle t_1 - t_3 \rangle, \tag{2.36}$$

and consequently, f_5 takes the form

$$f_5 = b_1 + (t_1 - t_3)b_2 + (t_2 - t_3)b_3 + (t_1 - t_3)(t_2 - t_3)b_5, \tag{2.37}$$

for an arbitrary $b_5 \in \Lambda$. By the time we get to v_6 we have 3 conditions to take into account. We must have

$$f_6 - b_1 \in \langle t_1 - t_3 \rangle, \tag{2.38}$$

$$f_6 - (b_1 + (t_1 - t_2)b_2 + (t_1 - t_3)b_3 + (t_1 - t_2)(t_1 - t_2)b_4) \in \langle t_2 - t_3 \rangle, \tag{2.39}$$

$$f_6 - (b_1 + (t_2 - t_3)b_3 + (t_1 - t_3)b_2 + (t_1 - t_3)(t_2 - t_3)b_5) \in \langle t_1 - t_2 \rangle, \tag{2.40}$$

and thus f_6 takes the form

$$b_1 + (t_1 - t_3)b_2 + (t_1 - t_3)b_3 + (t_1 - t_3)(t_1 - t_2)b_4 + (t_1 - t_3)(t_2 - t_3)b_5 + (t_1 - t_2)(t_2 - t_3)(t_1 - t_3)b_6, \tag{2.41}$$

for an arbitrary $b_6 \in \Lambda$. We can summarize this computation by the following diagram.

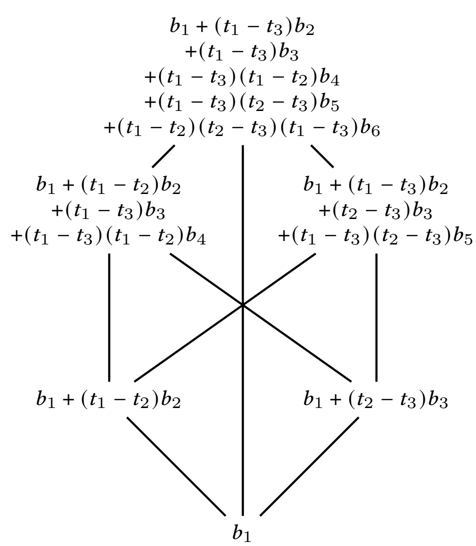


Figure 17: $H_T^*(\mathrm{Fl}(\mathbb{C}^3))$

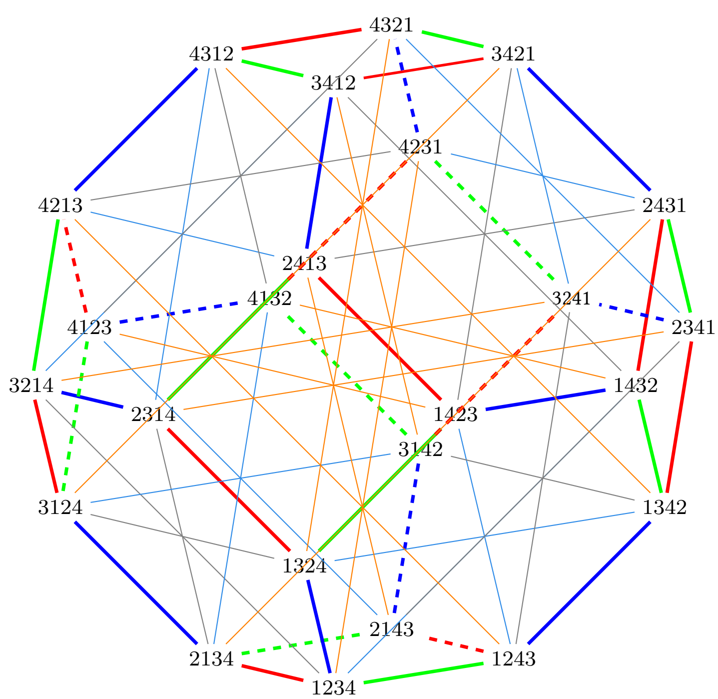
By construction, $b_i \in \Lambda$ may all be picked arbitrarily, so we get a set of generators of $H_T^*(X)$ as a free Λ-module by reading off the tuple of coefficients for each b_i

$$(1, 1, 1, 1, 1, 1), \tag{2.42}$$
$$(0, t_1 - t_2, 0, t_1 - t_2, t_1 - t_3, t_1 - t_3), \tag{2.43}$$
$$(0, 0, t_2 - t_3, t_1 - t_3, t_2 - t_3, t_1 - t_3), \tag{2.44}$$
$$(0, 0, 0, (t_1 - t_2)(t_1 - t_3), 0, (t_1 - t_2)(t_1 - t_3)), \tag{2.45}$$
$$(0, 0, 0, 0, (t_2 - t_3)(t_1 - t_3), (t_2 - t_3)(t_1 - t_3)), \tag{2.46}$$
$$(0, 0, 0, 0, 0, (t_1 - t_2)(t_2 - t_3)(t_1 - t_3)). \tag{2.47}$$

We shall illustrate these classes attached to the moment graph in Example 2.19. Later, in Example 2.28, we will revisit this basis in our explicit computations. ■

Example 2.16. Now consider the action of the maximal torus $(\mathbb{C}^*)^4$ on the full flag variety $\mathrm{Fl}(\mathbb{C}^4)$ of rank four. Analogously to the calculations in our previous example, we can see that the fixed points are represented by the matrices of the 24 permutations of $\{1, \ldots, 4\}$ and the 1-dimensional orbits have one extra nonzero element above and to the right of the 1s in a permutation matrix (recall, that the representatives of G/B are chosen to be in echelon form), which corresponds to left-multiplication by the transposition (i, j), where the extra nonzero element a is in the ith row and the row index of the entry 1 that lies in the same column as a is j. The edges of the moment graph can again be directed according to the underlying Bruhat order, such that the target of each edge has smaller length than the target.

It is easier to visualize the moment graph of by picturing the permutohedron, where the red edges have the label $t_1 - t_2$ (i.e. they correspond to multiplication by the transposition (12)), gray edges have the label $t_1 - t_3$, orange edges have the label $t_1 - t_4$, blue edges have the label $t_2 - t_3$, light blue edges have the label $t_2 - t_4$ and green edges have the label $t_3 - t_4$.



■

Remark 2.17. A crucial observation is that unicoloured edges form *perfect matchings* in the moment graphs above, i.e. sets of edges with disjoint endpoints that cover all vertices in the graph. The correspondence between transpositions and perfect matchings would suggest the existence of an S_n-representation on the moment graph, generated by assigning certain involutions to the transpositions. These involutions will somehow alter the polynomials associated to the endpoints of the corresponding edges, without violating any of conditions imposed by other edges. Later, in Subsection 2.3, we will describe this action in the more general setting of Hessenberg varieties, which will be the geometric counterpart of chromatic quasi-symmetric functions, that we are looking for.

The highlighted edges along the frame form a polytope, called the *permutohedron.* This will turns out to be a fundamental observation, that we'll revisit in Example 2.23 and later in Construction 3.43, in an attempt to interpret Guay-Paquet's Hopf algebraic construction geometrically.

Construction 2.18. We shall construct the moment graph and the *flow-up classes* of full flag varieties below. In part (a) we give an explicit description of moment graphs of $\mathrm{Fl}_n(\mathbb{C})$. In part (b) we define flow-up classes, and provide an example. In part (c), we prove the existence of flow up classes for $\mathrm{Fl}_n(\mathbb{C})$, and in part (d) we show uniqueness.

(a). The full flag variety $\mathrm{Fl}(\mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})/B$ of rank n, being a smooth variety over $\mathbb{C}$, is a GKM space. Denote its moment graph by $G = (V, E)$. The vertex set is given by $V = \{wB \,|\, w \in i(S_n)\}$, where $i\colon S_n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is the canonical embedding of S_n as permutation matrices. The explicit identification is given by sending a permutation $w \in S_n$ to the full flag $F_\bullet = (\mathrm{span}_{\mathbb{C}}\{e_{w(1)}, \dots, e_{w(i)}\})$, where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{C}^n$.

Analogously to Example 2.15, one can characterise the edges by the following condition. Two vertices v_i and v_j are connected by an edge if and only if the permutations matrices connecting them differ by a transposition. The label of the edge corresponding to left-multiplication by the transposition (ij) is $t_i - t_j \in \Lambda$. The direction of the edge goes from the higher to the lower vertex with respect to the Bruhat order, i.e. there is an edge going from the vertex $w'B$ to wB, labeled by $t_i - t_j$ if $w' = (ij)w$ and $(i < j) \in \mathrm{inv}(w)$ is an inversion.

(b). One can use the direction of the edges to obtain a considerably easier method of producing bases for $H_T^*(\mathrm{Fl}_n(\mathbb{C}))$. Pick a vertex v, assign 0s to all vertices that are either smaller than v or incompatible with v with respect to the Bruhat order, and set p_v^v to be the product of the labels of the edges protruding out of v

$$p_v^v := \prod_{(i<j)\in \mathrm{Inv}(v^{-1})} (t_i - t_j). \tag{2.48}$$

For the moment, assume that we can assign some polynomials $p_u^v \in \Lambda$ for the other vertices that respect the conditions imposed by the edge labels, such that the GKM conditions hold from Theorem from Theorem 2.11. Running over the vertex set S_n, the tuples $\{p_v\}_{v\in S_n}$ would then be linearly independent, since the support of each p_v is contained in the set $\{w \in S_n \,|\, v \leq_{\mathrm{Br},h} w\}$, while $p_v^v \neq 0$. We call the above sets of tuples $\{p_v\}$ *flow-up classes*, if they are homogeneous (with degree equal to the out-degree of v, i.e. the number of edges pointing out of v). For instance, we have already exhibited flow-up classes for the full flag variety of rank 3 has,

$$\begin{aligned}
p_{123} &= (1, 1, 1, 1, 1, 1), &(2.49)\\
p_{213} &= (0, t_1 - t_2, 0, t_1 - t_2, t_1 - t_3, t_1 - t_3), &(2.50)\\
p_{132} &= (0, 0, t_2 - t_3, t_1 - t_3, t_2 - t_3, t_1 - t_3), &(2.51)\\
p_{231} &= (0, 0, 0, (t_1 - t_2)(t_1 - t_3), 0, (t_1 - t_2)(t_1 - t_3)), &(2.52)\\
p_{312} &= (0, 0, 0, 0, (t_2 - t_3)(t_1 - t_3), (t_2 - t_3)(t_1 - t_3)), &(2.53)\\
p_{321} &= (0, 0, 0, 0, 0, (t_1 - t_2)(t_2 - t_3)(t_1 - t_3)), &(2.54)
\end{aligned}$$

and the illustrate the notation of the elements in the tuple, e.g. we have

$$p_{312}^{321} = (t_2 - t_3)(t_1 - t_3), \quad p_{123}^{231} = 1, \quad p_{132}^{132} = t_2 - t_3, \quad p_{132}^{312} = t_1 - t_3. \tag{2.55}$$

It follows from Theorem 2.11, by induction on the finite number of fixed points, that a choice of flow-up classes (if they exist) generate the equivariant cohomology $H_T^*(X)$, as a free Λ-module.

(c). The existence of flow-up classes is best formulated in the more general setting of *splines* (see [AMT], Lemma 2.7), but for us, it suffices to exhibit flow-up classes for the full flag variety $\mathrm{Fl}_n(\mathbb{C})$.

For each vertex $v \in S_n$, let us denote by $I(v)$ the ideal generated by the edge labels protruding out of v, i.e. downwards in the moment graph. Then we have

$$p_v^v = \prod_{t_i - t_j \in I(v)} (t_i - t_j). \tag{2.56}$$

If $\ell(u) = \ell(v) + 1$ and there is an edge $u \to v$, whose label we denote by $t_j - t_k$, then p_v^u can be expressed as the product

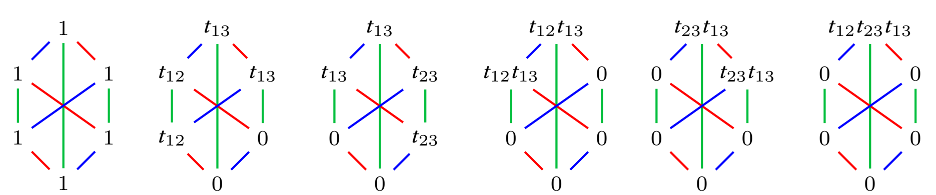
$$q := \prod_{\substack{t_i - t_{i'} \in I(u) \\ t_i - t_{i'} \neq t_j - t_k}} (t_i - t_{i'}). \tag{2.57}$$

Indeed, first note that there are $\ell(v) + 1$ edges protruding out of u; one edge points towards v and $\ell(v)$ edges point towards vertices u' that are incomparable with v, so that we have $p_v^{u'} = 0$ for all such vertices u'. Thus, the polynomial $p_v^u \in \Lambda$ lies in the ideal generated by the labels of these $\ell(v)$ edges, so p_v^u must be a scalar multiple of the product q. Moreover, since we have $u = (jk)v$, by the explicit description of the edge labels of the moment graph from part (a), we have $I(u) \cong I((jk)u)$ modulo $\langle t_j - t_k \rangle$. Therefore, $q - p_v^u \in \langle t_j - t_k \rangle$. However, $q \notin \langle t_j - t_k \rangle$, so $p_v^u = q$, as claimed.

(d). Note that for any directed edge $w \to v$ in the moment graph of the full flag variety, the *out-degree* of w, i.e. the number of vertices pointing out of w, is strictly larger than that of v. This property ensures that the flow-up classes are unique. Indeed, let $p_v = (p_v^w)_{w \in S_n}$ and $q_v = (q_v^w)_{w \in S_n}$ be two flow-up classes, corresponding to a vertex v. Then we may consider the class $p_v - q_v \in H_T^*(X)$, and note that $(p_v - q_v)^u = 0$, i.e. the value of the class $p_v - q_v$ vanishes at the vertex u if $u = v$ or if there is no directed path from u to v. Pick a vertex u_0 that is minimal in the Bruhat order such that the value $(p_v - q_v)^{u_0}$ is nonzero. Moreover, by the GKM condition in Theorem 2.11, $(p_v - q_v)^{u_0}$ lies in the ideal generated by the labels of the edges $u_0 \to v$. However, the number of edges $u_0 \to v$ is greater than $\deg(p_v) = \deg(p_v - q_v)$; a contradiction. Thus, there is no vertex u_0 with $(p_v - q_v)^{u_0} \neq 0$, i.e. we have $p_v - q_v = 0$.

We shall shortly observe, in Example 2.22, that uniqueness no longer holds for general Hessenberg varieties.

Example 2.19. We have already calculated flow-up classes for the full flag variety $\mathrm{Fl}_3(\mathbb{C})$ in Example 2.15. By Construction 2.18, these are the unique flow-up classes. Let us draw these tuples of polynomials on the vertices of the moment graph. We will abbreviate the difference $t_i - t_j$ by t_{ij}.



(2.58)

■

Construction 2.20. We shall provide an explicit description of the moment graph of regular, semisimple Hessenberg varieties, starting from the moment graph of the full flag variety, that we discussed in Construction 2.18, and throughout Example 2.15 and 2.16. In part (a), we describe the vertices of these moment graphs and in part (b), we describe the edges. In particular, we present a direct way to compute the edges solely based on the underlying Dyck path.

(a). The T-fixed points of a regular, semisimple Hessenberg variety $\mathrm{Hess}(M, h)$ with Dyck path h and matrix $M \in \mathfrak{g}$ of rank n coincides with the T-fixed points of the full flag variety $\mathrm{Fl}(\mathbb{C}^n)$. To see this, first recall from Construction 2.18 part (a), that the T-fixed points $(G/B)^T$ can be identified with the symmetric group S_n. We cannot have any other fixed point, since the T-action is given by restriction. To see the inclusion $(G/B)^T \subset \mathrm{Hess}(M, h)$, note that the canonical image of the Weyl group $W = N/T \cong S_n$ of $G = \mathrm{GL}_n(\mathbb{C})$ in the full flag variety G/B is given by $NB/B \cong (G/B)^T$, where N is the normaliser of the maximal torus T, and for any $n \in N$, we have $n^{-1}Mn \in \mathfrak{t} \subset H_h$, where H_h is the Hessenberg space, that we introduced in Definition 2.5.

(b). Similarly to the vertex set of the moment graph of $\mathrm{Hess}(M, h)$, we cannot have any 1-dimensional orbits other than those of the complete flag varieties. However, by restricting our torus action to a Hessenberg variety we may reduce the number of 1-dimensional orbits.

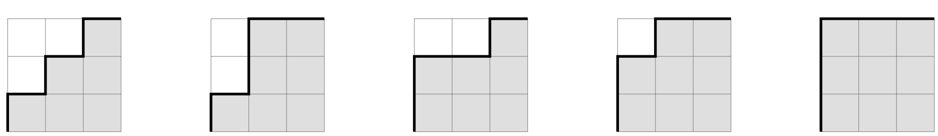
As we saw in Example 2.15 and 2.16, every 1-dimensional T-orbit of the full flag variety correspond to multiplication by the transposition that turns the permutation corresponding to one endpoints into the other, i.e. two vertices w and w' are connected by an edge if $w' = (ij)w$ for some transposition (ij). By definition of the Hessenberg variety, this 1-dimensional orbit lies in $\mathrm{Hess}(h, M)$ if and only if $w^{-1}(i) \leq h(w^{-1}(j))$. We will see this characterisation in action momentarily, in Example 2.21 and 2.23.

Equivalently, there is an edge from w to w' if and only if $w' = w(i'j')$ for some $i' < j'$ with $h(i') \geq j'$. Indeed, we have $w' = (ij)w = w(w^{-1}(j)w^{-1}(i))$, and we may take $i' = w^{-1}(j) < w^{-1}(i) = j'$.

Therefore, the moment graph of the Hessenberg variety $\mathrm{Hess}(h, M)$ can be described completely in terms of the Dyck path. It contains the edges corresponding to right-multiplication by those transpositions (ij) with $i < j$, for which the square (i, j) lies between the diagonal and the path.

This tells us that the moment graph is disconnected if and only if the Dyck path touches the diagonal. Indeed, if the Dyck path does not touch the diagonal, then all edges in the moment graph corresponding to right-multiplication by simple transpositions are drawn, and simple transpositions generate the symmetric group. On the other hand, if the Dyck path touches the diagonal, then the simple transposition corresponding to the square where the Dyck path touches the diagonal cannot be generated by any of the other permitted transpositions. Let us look at a few examples.

Example 2.21. Let us compute the moment graphs of $\mathrm{Hess}(h, M)$ for all 3×3 Dyck paths, where M is an arbitrary regular semisimple matrix of rank three.



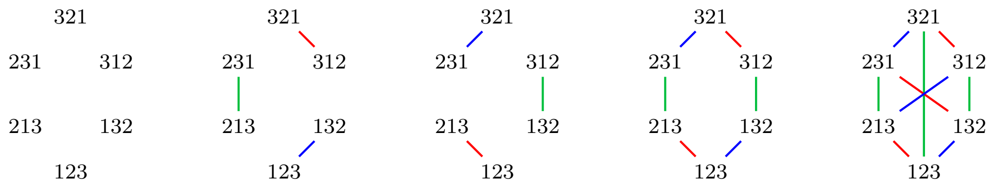


Figure 18: moment graphs corresponding to rank 3 Dyck paths

Indeed, for the Dyck path $(1, 2, 3)$, multiplication on the right by any transposition is not permitted, and consequently, the corresponding moment graph has no edges. Here we are using Construction 2.20, part (b). For the Dyck path $(1, 3, 3)$, right-multiplication by the transposition $(2, 3)$ is allowed, and the others are forbidden, so the corresponding moment graph is the disjoint union of three edges. The moment graph corresponding to the Dyck path $(2, 2, 3)$ is isomorphic to that of $(1, 3, 3)$: here, multiplication by the transposition (12) is allowed.

The Dyck path $(2, 3, 3)$ does not touch the diagonal, so the corresponding moment graph is connected. It is the 6-cycle, the Hasse diagram of the *weak Bruhat order*, where $u \leq v$ if some *initial* substring of some reduced word for v is a reduced word for u. Similarly to the (strong) Bruhat order, that we introduced in Example 1.53, using the standard notation for simple transpositions, we can illustrate the weak Bruhat order on S_3 by the following Hasse diagram

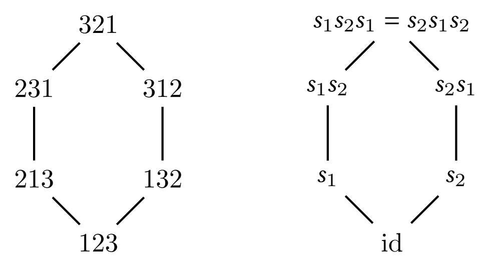


Figure 19: (strong) Bruhat order for S_3

Finally, the moment graph of the full flag variety, corresponding to the Dyck path $(3, 3, 3)$, where right-multiplication by any transposition is permitted. As we have observed before, this is the Hasse diagram of the (strong) Bruhat order with an additional edge from 321 to 123. ■

Construction 2.22. As we saw in Constriction 2.20, the moment graph of the full flag variety has more edges than the moment graph of other regular semisimple Hessenberg varieties. When computing the cohomology, one needs to take into account all edge labels, so the degrees of flow-up bases might be smaller for general (regular semisimple) Hessenberg varieties. For example, we have

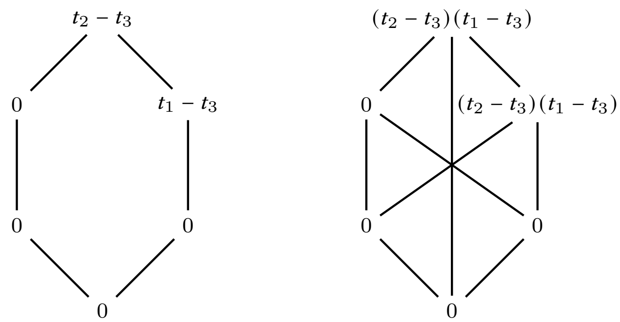


Figure 20: moment graphs of the permutohedron of rank 3

The strong Bruhat poset on S_n induces a partial order on the vertices of the moment graph of the Hessenberg variety $\mathrm{Hess}(M, h)$. For any two vertices $v, w \in S_n$, we have $v \leq_{\mathrm{Br},h} w$ if and only if there is a path

$$w = w_0 \longrightarrow w_1 \longrightarrow \cdots \longrightarrow w_k = v \tag{2.59}$$

in the moment graph of the Hessenberg variety. This partial order gives rise to flow-up classes, similarly to the case of full flag varieties that we discussed in Construction 2.18. The only difference is that the polynomial assigned to the first vertex is

$$f(v) := \prod_{\substack{(i<j)\in \mathrm{Inv}(v^{-1}) \\ v^{-1}(i) \leq h(v^{-1}(j))}} (t_i - t_j) \tag{2.60}$$

with support $\mathrm{supp}(f) \subset \{w \in S_n \mid v \leq_{\mathrm{Br},h} w\}$, where the support of f is the set $\{w \in S_n \mid f(w) \neq 0\}$.

Unlike the full flag variety, general Hessenberg varieties may have multiple flow-up classes associated to a given vertex. For instance, recall that the moment graph corresponding to the Dyck path $h = (2, 3, 3)$ is

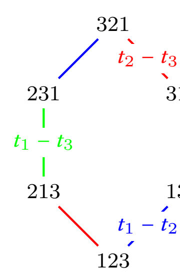


Figure 21: moment graphs of the permutohedron of rank 3

The following two are distinct flow-up classes for $v = s_2$ in $H_T^*(\mathrm{Hess}(M, h))$

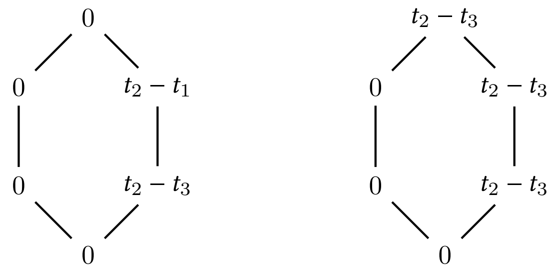


Figure 22: distinct flow-up classes

Furthermore, note that these classes are elements of $H^*(\mathrm{Hess}(M, h))$ but not of $H^*(\mathrm{Fl}(\mathbb{C}^n))$, which shows that the aforementioned restriction map from Remark 2.7 is not always surjective.

However, the moment graph of a Hessenberg variety has at most as many edges as the moment graph of the full flag variety of the same rank, so there exists at least one flow-up class corresponding to any vertex of the moment graph, for any Hessenberg variety. Consequently, by Construction 2.18, these flow-up classes form a basis for $H_T^*(X)$.

Example 2.23. Now let us compute the moment graphs corresponding to the fourteen 4×4 Dyck paths.

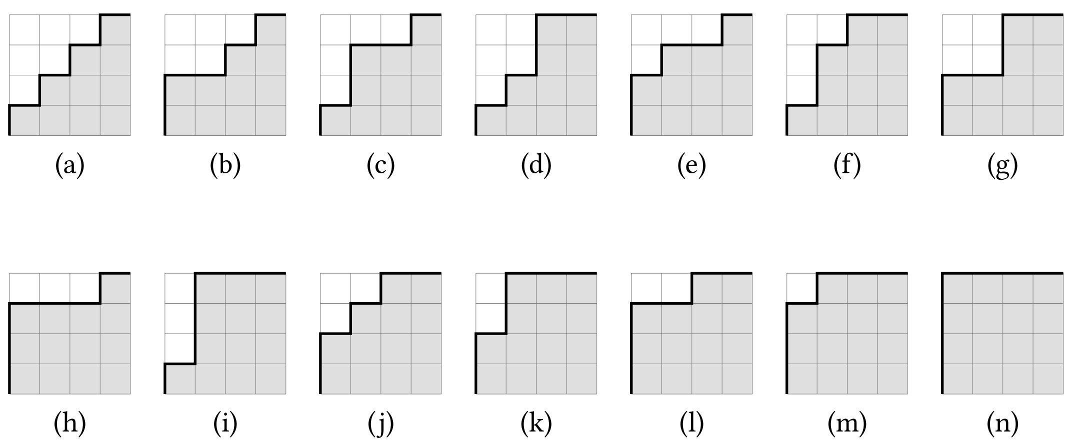
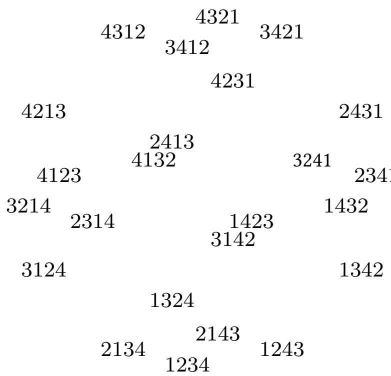


Figure 23: Dyck paths of rank 4

The first nine Dyck paths touch the diagonal, so their moment graphs are disconnected, and they can be reduced to our previous example of $3{\times}3$ Dyck paths, while the last five cases give rise to five new, connected moment graphs, where the edge labels correspond to the same colours as in Example 2.16.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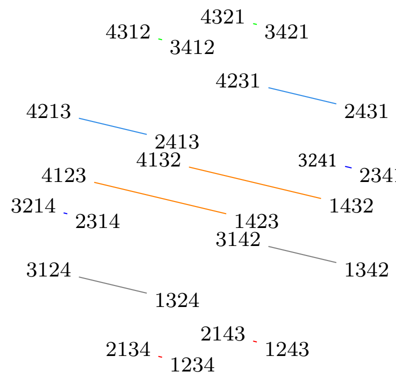
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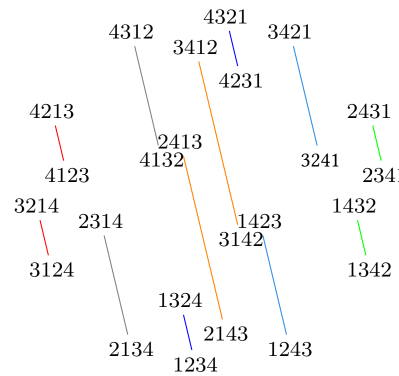
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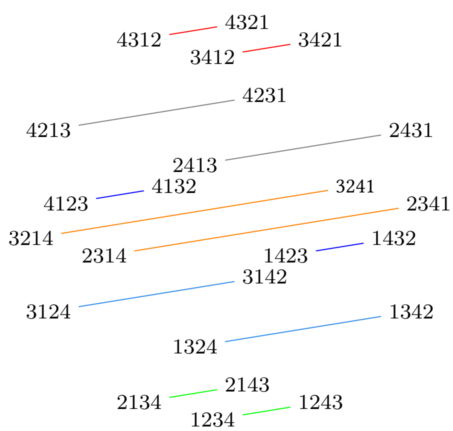
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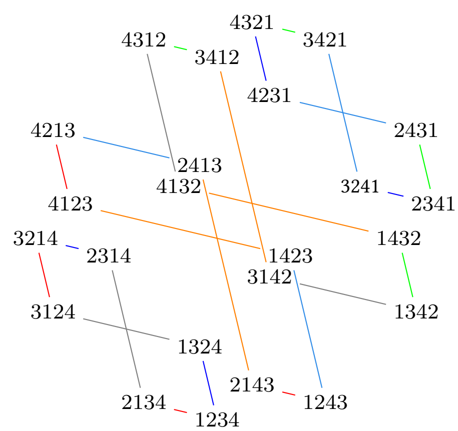
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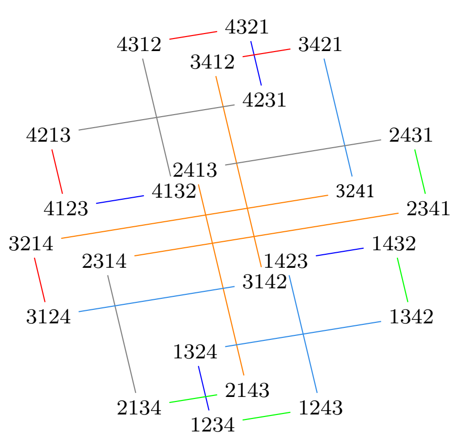
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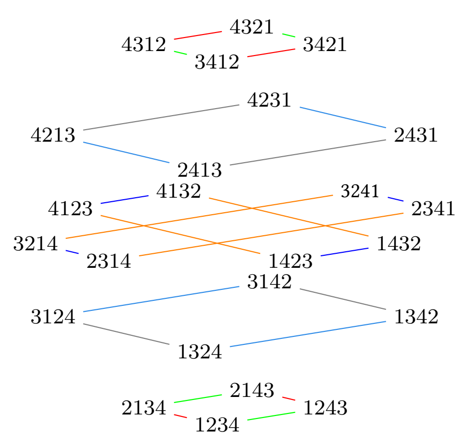
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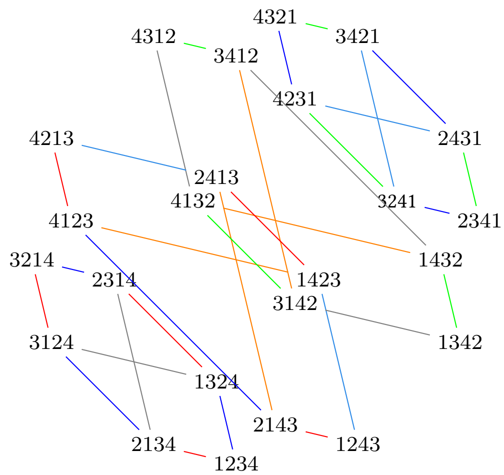
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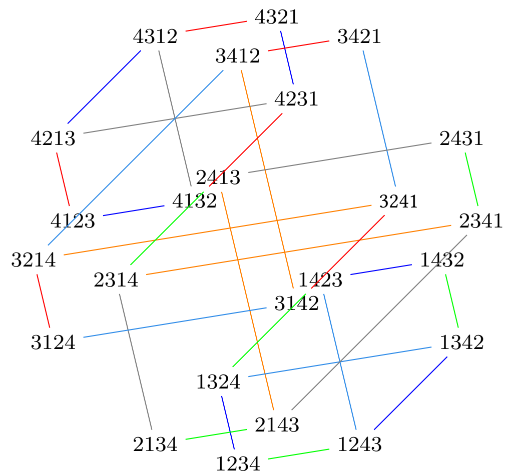
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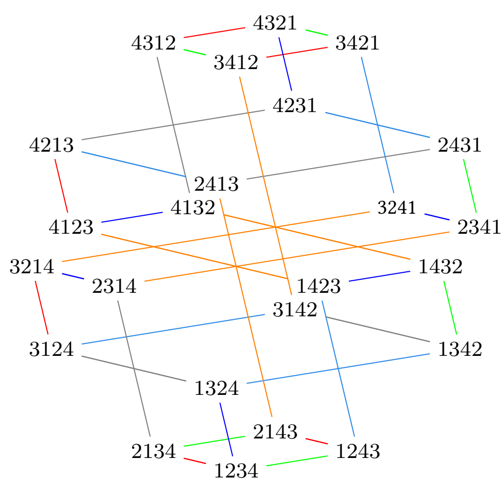
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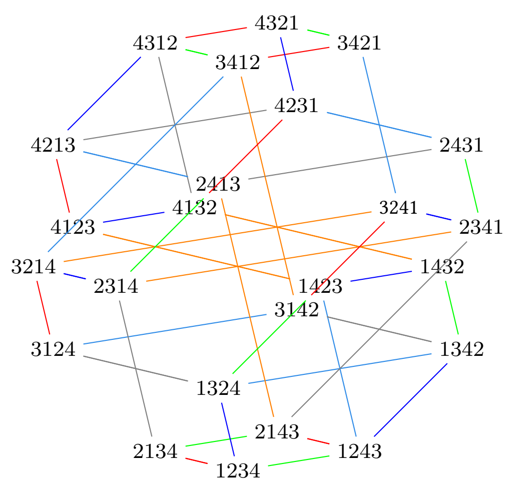
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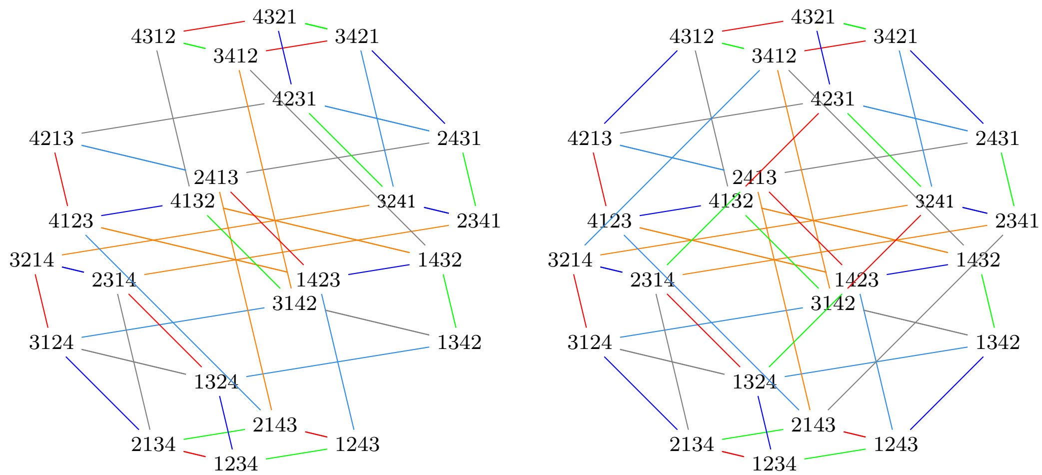


(l) (m)

Figure 24: moment graphs of rank 4 Hessenberg varieties

Finally, the Hessenberg variety in case (n) is the full flag variety of rank four, whose moment graph we have already seen in Example 2.16. Note that all these moment graphs are *regular*, i.e. each vertex has the same number of neighbours, in the undirected case, and unicoloured edges form matchings, but not necessarily perfect matchings, i.e. the matching does not necessarily cover all vertices. The first connected graph, in case (j), is a (twisted) permutohedron, isomorphic to the highlighted frame in Example 2.16, in the moment graph of the full flag variety. This is the underlying graph of (the Hasse diagram of) the weak Bruhat order of the symmetric group S_4. ■

2.3 The Dot Action on Moment Graphs

Example 2.24. The group S_n, or rather its canonical image given by permutation matrices (as in Construction 2.18) acts on the full flag variety $\mathrm{Fl}(\mathbb{C}^n)$ by left multiplication. However, Hessenberg varieties in general are not invariant under this S_n action. To give a counterexample, consider $h = 233$ and the regular semisimple matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Recall that the Hessenberg space H_h is generated by elementary matrices E_{ij} with $h(j) \geq i$. In this case, $h(1) = 2$, $h(2) = 3$, $h(3) = 3$, so $H_h = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \right\}$ Then the matrix $g = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ represents a full flag in $\mathrm{Hess}(M, h)$, since we have $g^{-1}Mg = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. However, if we take the action of the transposition (12) with matrix $w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then the result $(wg)^{-1}M(wg) = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ no longer represents a full flag in $\mathrm{Hess}(M, h)$.

Providing a suitable S_n-action on the cohomology of Hessenberg varieties is the main application of moment graphs. ■

Construction 2.25. The group S_n acts on $\mathbb{C}[t_1, \ldots, t_n]$ by permuting the indices of variables

$$w.f(t_1, \ldots, t_n) = f(t_{w(1)}, \ldots, t_{w(n)}), \tag{2.61}$$

which is a ring automorphism of $\mathbb{C}[t_1, \ldots, t_n]$. As indicated before, this action extends to an action of S_n on the moment graph of regular semisimple Hessenberg varieties. In Example 2.16, we saw that edges corresponding to a given transposition form a perfect matching of the moment graph of the full flag variety.

Let $w = (i\,j)$ be a transposition. Its action interchanges the vertex labels across those edges in the moment graph of the full flag variety, whose label is $t_i - t_j$, and it permutes the variables.

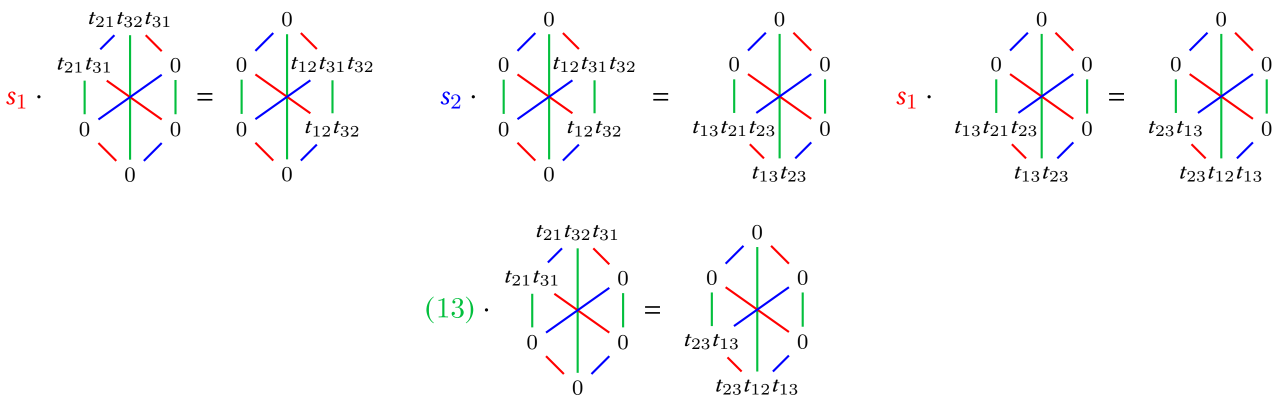
Let $f(v, t_1, \ldots, t_n)$ be the polynomial associated to the vertex $v \in S_n$ in the moment graph. Then for any $w \in S_n$, the *dot action* on $H_T^*(\mathrm{Hess}(h, M); \mathbb{C})$ is given by

$$(w \cdot f)(v, t_1, \ldots, t_n) \coloneqq f(w^{-1}v, t_{w(1)}, \ldots, t_{w(n)}). \tag{2.62}$$

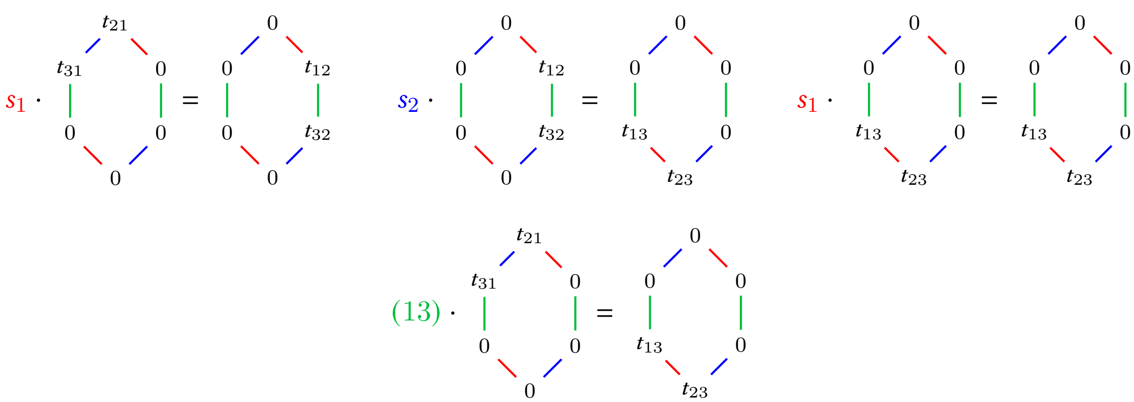
Let (v, v') be a directed edge with label $t_i - t_j$ and $p \in H_T^*(\mathrm{Hess}(h, M))$. To see that the dot action is well-defined, we need to show that $w \cdot p$ still satisfies the GKM condition $(w \cdot p)_v - (w \cdot p)_{v'} \in \langle t_i - t_j \rangle$, where the subscript denotes the polynomial at vertex v. Indeed, we can directly compute that

$$(w \cdot p)_v - (w \cdot p)_{v'} = w.(p_{w^{-1}(v)} - p_{w^{-1}(v')}) \in w.\langle t_{w^{-1}(i)} - t_{w^{-1}(j)} \rangle = \langle t_i - t_j \rangle. \tag{2.63}$$

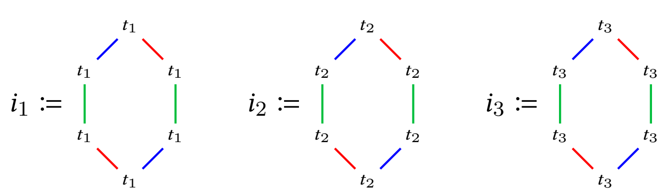
Let us consider the dot action of S_3 on the T-equivariant cohomology ring $H_T^*(\mathrm{Fl}(\mathbb{C}^3))$ of the full flag variety of rank three. Let us abbreviate the polynomials $t_i - t_j \in \Lambda$ by t_{ij}. Then we have



This induces an action on each Hessenberg variety by restricting to the relevant subset of edges, described in Construction 2.20. For example, the action on the permutohedron $\mathcal{H}_3$ of rank three is the following



Example 2.26. The equivariant cohomology of $\mathcal{H}_3$ has an ideal that is fixed by the dot action, namely



$$\tag{2.64}$$

Let us denote the invariant ideal by $I = \langle i_1, i_2, i_3 \rangle$. Consider the following flow-up classes that generate the equivariant cohomology ring R of $\mathcal{H}_3$

$$r_1 \coloneqq \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ 1 & & 1 \\ & 1 & \end{array} \qquad r_2 \coloneqq \begin{array}{ccc} & t_{21}t_{32} & \\ 0 & & 0 \\ 0 & & 0 \\ & 0 & \end{array} \qquad r_3 \coloneqq \begin{array}{ccc} & t_{21} & \\ t_{31} & & 0 \\ 0 & & 0 \\ & 0 & \end{array} \tag{2.65}$$

$$r_4 \coloneqq \begin{array}{ccc} & 0 & \\ t_{23} & & 0 \\ t_{21} & & 0 \\ & 0 & \end{array} \qquad r_5 \coloneqq \begin{array}{ccc} & t_{32} & \\ 0 & & t_{31} \\ 0 & & 0 \\ & 0 & \end{array} \qquad r_6 \coloneqq \begin{array}{ccc} & 0 & \\ 0 & & t_{12} \\ 0 & & t_{32} \\ & 0 & \end{array} \tag{2.66}$$

Then the six generators above give rise to a basis $\{r_1 + I, \dots, r_6 + I\}$ for the quotient ring R/I, which is isomorphic to the ordinary cohomology ring $H^*(\mathcal{H}_3)$, by Construction 2.9, part (c).

Since I is an invariant ideal, S_3 acts on $r_1 + I$ trivially. Its action on $r_2 + I$ is also trivial, because

$$s_1 \cdot r_2 = \begin{array}{ccc} & 0 & \\ 0 & & t_{12}t_{31} \\ 0 & & 0 \\ & 0 & \end{array} = \begin{array}{ccc} & t_{21}t_{32} & \\ 0 & & 0 \\ 0 & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & t_{12}t_{32} & \\ 0 & & t_{12}t_{31} \\ 0 & & 0 \\ & 0 & \end{array} = r_2 + (i_1 - i_2) r_5, \tag{2.67}$$

$$s_2 \cdot r_2 = \begin{array}{ccc} & 0 & \\ t_{23}t_{31} & & 0 \\ 0 & & 0 \\ & 0 & \end{array} = \begin{array}{ccc} & t_{21}t_{32} & \\ 0 & & 0 \\ 0 & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & t_{21}t_{23} & \\ t_{23}t_{31} & & 0 \\ 0 & & 0 \\ & 0 & \end{array} = r_2 + (i_2 - i_3) r_3. \tag{2.68}$$

Furthermore, $s_2 \cdot r_3 = r_3$ is fixed, while $s_1 \cdot r_3 = r_6$. Similarly, $s_1 \cdot r_5 = r_5$ and $s_2 \cdot r_4 = r_5$. The remaining two cases are a tad more complicated,

$$s_2 \cdot r_6 = \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ t_{13} & & 0 \\ & t_{23} & \end{array} = \begin{array}{ccc} & t_{21} & \\ t_{31} & & 0 \\ 0 & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & 0 & \\ t_{23} & & 0 \\ t_{31} & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & 0 & \\ 0 & & t_{21} \\ 0 & & t_{23} \\ & 0 & \end{array} + \begin{array}{ccc} & t_{12} & \\ t_{12} & & t_{12} \\ t_{12} & & t_{12} \\ & t_{12} & \end{array} = r_3 + r_4 - r_6 + (i_1 - i_2), \tag{2.69}$$

$$s_2 \cdot r_6 = \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ t_{13} & & 0 \\ & t_{23} & \end{array} = \begin{array}{ccc} & 0 & \\ t_{32} & & 0 \\ t_{12} & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & t_{32} & \\ 0 & & t_{31} \\ 0 & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & 0 & \\ 0 & & t_{12} \\ 0 & & t_{32} \\ & 0 & \end{array} + \begin{array}{ccc} & t_{23} & \\ t_{23} & & t_{23} \\ t_{23} & & t_{23} \\ & t_{23} & \end{array} = -r_4 + r_5 + r_6 + (i_2 - i_3). \tag{2.70}$$

Therefore, the representation R/I is graded, the degree 0 and degree 2 parts are the trivial representation of S_3 (of rank 1), and the degree 1 part is given by $\{r_3 + I, r_4 + I, r_5 + I, r_6 + I\}$. The conjugacy classes of S_3 are represented by id, s_1 and $s_1 s_2$, and the classes have 1, 3 and 2 elements, respectively. The identity matrix has trace 4, while the other two classes are represented by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}$$

so their traces are 2 and 1, respectively. Therefore, we can use Example 1.52 the Frobenius character of the degree 1 part is

$$\frac{1}{6}(1 \cdot 4 \cdot p_{(1^3)} + 3 \cdot 2 \cdot p_{(2,1)} + 2 \cdot 1 \cdot p_{(3)}) = 2s_{(3)} + s_{(2,1)}, \tag{2.71}$$

Hence, the graded Frobenius character of the representation R/I is

$$\mathrm{Frob}(H^*(\mathcal{H}_3)) = s_{(3)}t^2 + (2s_{(3)} + s_{(2,1)})t + s_{(3)}. \tag{2.72}$$

In the e-basis, this symmetric function takes the form

$$(e_{(1^3)} - 2e_{(2,1)} + e_{(3)})t^2 + (2e_{(1^3)} - 3e_{(2,1)} + e_{(3)})t + (e_{(1^3)} - 2e_{(2,1)} + e_{(3)}), \tag{2.73}$$

which is not e-positive. However, its image under the fundamental involution, that we introduced in Lemma 1.29, is e-positive,

$$\omega(\mathrm{Frob}(H^*(\mathcal{H}_3))) = (h_{(1^3)} - 2h_{(2,1)} + h_{(3)})t^2 + (2h_{(1^3)} - 3h_{(2,1)} + h_{(3)})t + (h_{(1^3)} - 2h_{(2,1)} + h_{(3)}) \tag{2.74}$$

$$= e_{(3)}t^2 + (e_{(3)} + e_{(2,1)})t + e_{(3)}. \tag{2.75}$$

In fact, it is the chromatic quasi-symmetric function of the path P_3, that we encountered in Example 1.70. This relationship between Hessenberg varieties and chromatic quasi-symmetric functions was first observed by Shareshian and Wach in 2012, in the article [SW]. ■

Conjecture 2.27 (Shareshian–Wachs). *For any Dyck path h, we have*

$$\omega(\mathrm{Frob}(H^*(X_h))) = X(G(h); x, t), \tag{2.76}$$

where X_h is a regular semisimple Hessenberg variety associated to h, $H^*(X_h)$ is the S_n-module given by the dot action on the cohomology ring of X_h, Frob *is the Frobenius character map, ω is the fundamental involution, $G(h)$ is the indifference graph of h and $X(G(h); x, t)$ is its chromatic quasi-symmmetric function.*

Example 2.28 (Dot action on $\mathrm{Fl}(\mathbb{C}^3)$). The cohomology ring $H^*(\mathrm{Fl}_n(\mathbb{C}))$ is isomorphic to the coinvariant algebra, which we discussed in Example 1.53. However the Frobenius character of the canonical S_3-action on $\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle$, described in the aforementioned example, is given by

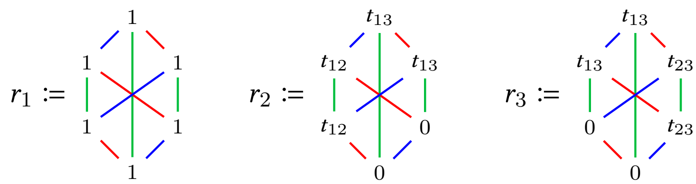
$$\mathrm{Frob}(\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle) = e_{(3)}t^3 + (e_{(2,1)} - e_{(3)})t^2 + (e_{(2,1)} - e_{(3)})t + (e_{(1^3)} - 2e_{(2,1)} + e_{(3)}). \tag{2.77}$$

When we apply the fundemental involution to this, we obtain

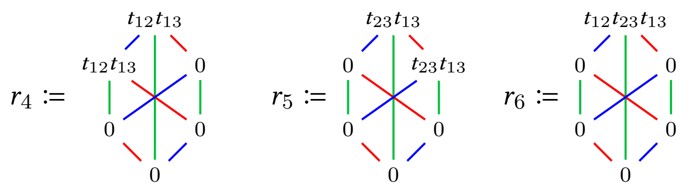
$$\omega(\mathrm{Frob}(\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle)) = (e_{(1^3)} - 2e_{(2,1)} + e_{(3)})t^3 + (e_{(2,1)} - e_{(3)})t^2 + (e_{(2,1)} - e_{(3)})t + e_{(3)}. \tag{2.78}$$

Thus, the canonical representation of S_3 on the coinvariant algebra should not coincide with the dot action.

Consider the (unique) flow-up classes that we computed in Example 2.19,

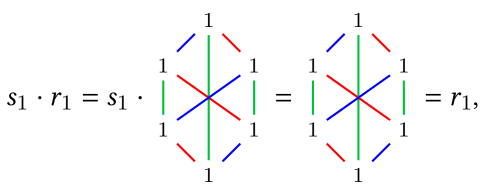


(2.79)

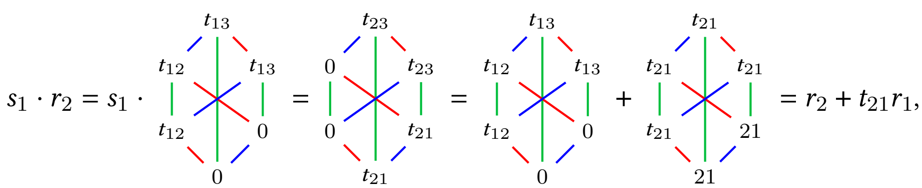


(2.80)

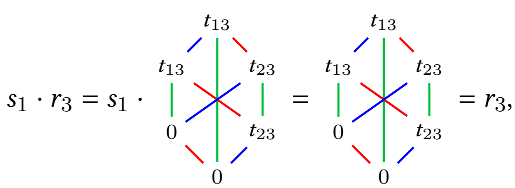
Let us describe the dot action on these elements,



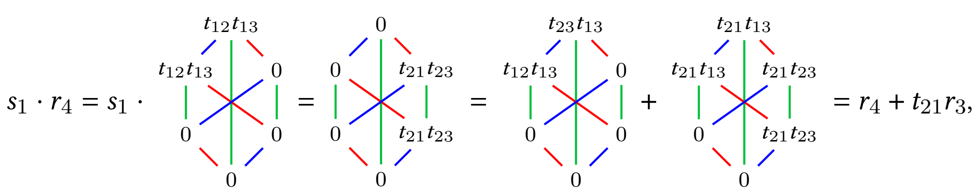
(2.81)



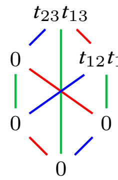
(2.82)

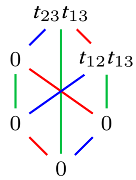


(2.83)

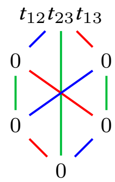


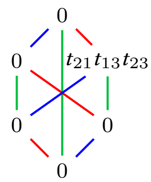
(2.84)

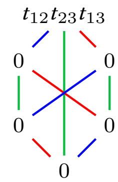
$s_1 \cdot r_5 = s_1 \cdot$ 

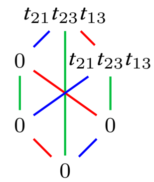
$=$ 

$= r_5,$ (2.85)

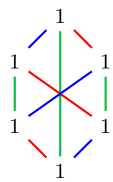
$s_1 \cdot r_6 = s_1 \cdot$ 

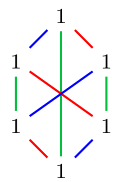
$=$ 

$=$ 

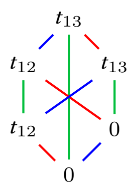
$+$ 

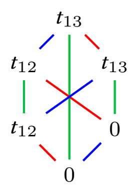
$= r_6 + t_{21}r_5,$ (2.86)

$s_2 \cdot r_1 = s_2 \cdot$ 

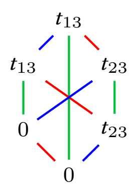
$=$ 

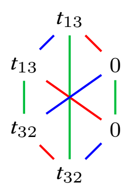
$= r_1,$ (2.87)

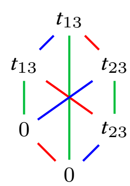
$s_2 \cdot r_2 = s_2 \cdot$ 

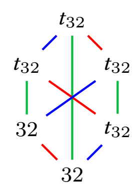
$=$ 

$= r_2,$ (2.88)

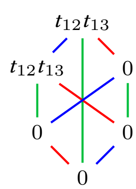
$s_2 \cdot r_3 = s_2 \cdot$ 

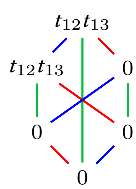
$=$ 

$=$ 

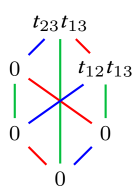
$+$ 

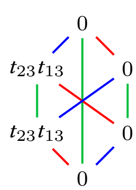
$= r_3 + t_{32}r_1,$ (2.89)

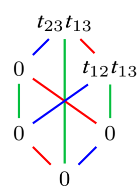
$s_2 \cdot r_4 = s_2 \cdot$ 

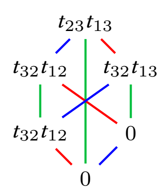
$=$ 

$= r_5,$ (2.90)

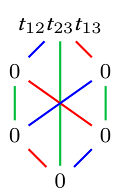
$s_2 \cdot r_5 = s_2 \cdot$ 

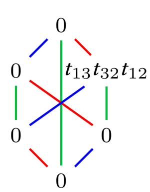
$=$ 

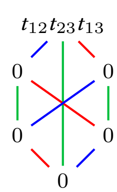
$=$ 

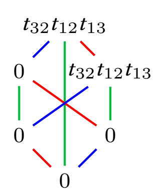
$+$ 

$= r_5 + t_{32}r_2,$ (2.91)

$s_2 \cdot r_6 = s_2 \cdot$ 

$=$ 

$=$ 

$+$ 

$= r_6 + t_{21}r_5,$ (2.92)

We can describe this action more succinctly by the formula

$$s_i \cdot r_w = \begin{cases} r_w & \text{if } s_i w > w \\ r_w + t_{i+1,i}\, r_{s_i w} & \text{if } s_i w < w, \end{cases} \tag{2.93}$$

where r_w is the flow-up class corresponding to $w \in S_3$.

Therefore, by Construction 2.9, part (c), the graded S_3-action descending to the ordinary cohomology $H^*(\mathrm{Fl}_3(\mathbb{C}))$ is the trivial representation in each degree. The conjugacy classes of S_3 are represented by id, s_1 and $s_1 s_2$, and these classes have 1, 3 and 2 elements, respectively. Any class is represented by the identity matrix in each degree $2d$ whose rank is the number of flow-up classes of degree d. The trace of this matrix is the number of flow-up classes of the given degree. Hence, the Frobenius character in degree zero is given by

$$\frac{1}{6}(1 \cdot 1 \cdot p_{(1^3)} + 3 \cdot 1 \cdot p_{(2,1)} + 2 \cdot 1 \cdot p_{(3)}) = s_{(3)}, \tag{2.94}$$

the Frobenius character in degree one is

$$\frac{1}{6}(1 \cdot 2 \cdot p_{(1^3)} + 3 \cdot 2 \cdot p_{(2,1)} + 2 \cdot 2 \cdot p_{(3)}) = 2s_{(3)}, \tag{2.95}$$

in degree two we have

$$\frac{1}{6}(1 \cdot 2 \cdot p_{(1^3)} + 3 \cdot 2 \cdot p_{(2,1)} + 2 \cdot 2 \cdot p_{(3)}) = 2s_{(3)}, \tag{2.96}$$

and finally, in degree six, the Frobenius character can be expressed as

$$\frac{1}{6}(1 \cdot 1 \cdot p_{(1^3)} + 3 \cdot 1 \cdot p_{(2,1)} + 2 \cdot 1 \cdot p_{(3)}) = s_{(3)}. \tag{2.97}$$

Thus, the Schur expression of the graded Frobenius is given by

$$\mathrm{Frob}(H^*(\mathcal{H}_3)) = s_{(3)} t^3 + 2s_{(3)} t^2 + 2s_{(3)} t + s_{(3)}. \tag{2.98}$$

In the e-basis, the symmetric function $s_{(3)}$ takes the form

$$s_{(3)} = e_{(1^3)} - 3e_{(2,1)} + e_{(3)}, \tag{2.99}$$

which is not e-positive, but again, after we apply the fundamental involution ω, we obtain an e-positive expression,

$$\omega(\mathrm{Frob}(H^*(\mathrm{Fl}_3(\mathbb{C})))) = e_{(3)} t^3 + 2e_{(3)} t^2 + 2e_{(3)} t + e_{(3)}. \tag{2.100}$$

It is the chromatic quasi-symmetric function of the path P_3, that we encountered in Example 1.70.

In [Ty3], Tymoczko proved that the dot action on a full flag variety of arbitrary rank is trivial in each degree, using a recursive construction on flow-up classes that generalise the example above, building on the work of Billey (see [Bil]) on *Kostant polynomials* and the cohomology of full flag varieties of any type, not only type A. ■

3 Proof of the Shareshian–Wachs Conjecture

In this section, we discuss the Hopf algebraic proof of the Shareshian–Wachs conjecture, due to Guay-Paquet. The Shareshian–Wachs conjecture is regarded as a stepping stone towards the Stanley–Stembridge conjecture, as the proof of this correspondence allows us to interpret e-positivity in terms of combinatorial Hopf algebras and geometry, simultaneously.

In the first subsection, the proofs of Lemma 3.11, 3.12, 3.18, 3.19 and Construction 3.15 follow [GR]. In the second subsection, the proof of the universal property in Theorem 3.35, as well as the two auxiliary statements in Lemma 3.31 and 3.32 follow [ABS], and [GR]. In the final subsection, in the proof of Lemma 3.41, 3.44, 3.46, 3.49 and 3.50 leading up to Theorem 3.51, we follow and expand on the article [GP]. Our goal is to revisit our earlier computations, e.g. in Example 2.26 and 2.28, and to present this Hopf algebraic construction in a more transparent and accessible manner.

3.1 Hopf Algebra of Symmetric Functions

Definition 3.1. Recall the (diagrammatic) definition of an algebra over a field k. A *k-algebra* is a k-vector space A together with a linear map $m\colon A\otimes A \to A$, called *multiplication*, and a linear map $u\colon k \to A$, called *unit*, such that the following diagrams commute

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} A\otimes k = A = k\otimes A \arrow[r, "\mathrm{id}\otimes u"] \arrow[d, "u\otimes \mathrm{id}"'] \arrow[dr, equal] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd} \tag{3.1}$$

A *k-coalgebra* is a k-vector space C together with linear maps $\Delta\colon C \to C\otimes C$, called the *comultiplication*, and a linear map $\epsilon\colon C \to k$, called the *counit*, such that the dual diagrams commute

$$\begin{tikzcd} C\otimes C\otimes C & C\otimes C \arrow[l, "\Delta\otimes \mathrm{id}"'] \\ C\otimes C \arrow[u, "\mathrm{id}\otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} C\otimes k = C = k\otimes C & C\otimes C \arrow[l, "\mathrm{id}\otimes \epsilon"'] \\ C\otimes C \arrow[u, "\epsilon\otimes \mathrm{id}"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, equal] \end{tikzcd} \tag{3.2}$$

A *bialgebra* is a k-vector space which is both an algebra and a coalgebra at the same time, such that Δ and ϵ are algebra morphisms, or equivalently, m and u are coalgebra morphisms, i.e. the algebra and the coalgebra structures are compatible. This can be formulated equivalently by imposing the following conditions,

$$\epsilon\circ u = \mathrm{id}, \tag{3.3}$$
$$\epsilon\circ m = m\circ(\epsilon\otimes\epsilon), \tag{3.4}$$
$$\Delta\circ u = (u\otimes u)\circ\Delta, \tag{3.5}$$
$$\Delta\circ m = (m\otimes m)\circ(\mathrm{id}\otimes T\otimes \mathrm{id})\circ(\Delta\otimes\Delta), \tag{3.6}$$

where the coalgebra isomorphism $T\colon C\otimes C \to C\otimes C$ is given by $T(x\otimes y) = y\otimes x$.

Remark 3.2. There is a shorthand, called the *Sweedler notation*, for describing the comultiplication. We will write

$$\Delta c = \sum_{(c)} c_1\otimes c_2, \tag{3.7}$$

by which we mean the sum

$$\Delta c = \sum_{i=1}^{m} d_i\otimes e_i, \tag{3.8}$$

where is some choice of $d_1,\ldots,d_m,e_1,\ldots,e_m \in C$. Then, for any bilinear form $f\colon C\times C \to M$ from the coalgebra $C\times C$ to a $\mathbb{C}$-module M, the value of

$$\sum_{k=1}^{m} f(d_i,e_i) \tag{3.9}$$

is independent of the choice of the d_i and e_i above. So we may rewrite this sum as

$$\sum_{(c)} f(c_1, c_2). \tag{3.10}$$

Let C be a coalgebra and A an algebra over $\mathbb{C}$. Then the $\mathbb{C}$-module $\mathrm{Hom}(C, A)$ becomes an associative algebra with the following (associative) multiplication. For any $f, g \in \mathrm{Hom}(C, A)$, we define the *convolution product* $f * g$ to be the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A, \tag{3.11}$$

or written in Sweedler notation,

$$(f * g)(c) = \sum_{(c)} f(c_1)g(c_2). \tag{3.12}$$

Definition 3.3. A bialgebra A is called a *Hopf algebra* if there is a two-sided inverse S, called the *antipode*, for the identity map id_A under the convolution product $*$, i.e. the following diagram commutes

$$\begin{tikzcd}
& A \otimes A \arrow[rr, "S\otimes\mathrm{id}"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "u"] & & A \\
& A \otimes A \arrow[rr, "\mathrm{id}\otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd} \tag{3.13}$$

Using the Sweedler notation, we can abbreviate it by

$$\sum_{(a)} S(a_1)a_2 = u(\epsilon(a)) = \sum_{(a)} a_1 S(a_2). \tag{3.14}$$

It follows from the defining diagrams that the antipode is unique, when it exists. Moreover, it is always an algebra anti-endomorphism.

Example 3.4. The *symmetric algebra* $\mathcal{S}(V)$ on a vector space V over $\mathbb{C}$ is the quotient of the tensor algebra $\mathcal{T}(V)$ by the commutator ideal. An element x in a bialgebra is called *primitive* if its comultiplication is given by $\Delta x = 1 \otimes x + x \otimes 1$. The tensor algebra $\mathcal{T}(V)$ is a Hopf algebra, where the comultiplication is determined by the primitive elements $x \in V^{\otimes 1}$, the counit is zero on $V^{\otimes n}$ for $n > 0$ and the identity on the zero degree part $V^{\otimes 0}$, which is isomorphic to the ground field $\mathbb{C}$. The Hopf algebra structure of $\mathcal{T}(V)$ descends to $\mathcal{S}(V)$, since the counit is zero on any commutator, and the commutator of primitive elements in $\mathcal{T}(V)$ is also a primitive element,

$$\begin{aligned}
\Delta[x, y] &= \Delta(xy - yx) && (3.15)\\
&= \Delta x \Delta y - \Delta y \Delta x && (3.16)\\
&= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1) && (3.17)\\
&= 1 \otimes xy - 1 \otimes yx + xy \otimes 1 - yx \otimes 1 + x \otimes y + y \otimes x - x \otimes y - y \otimes x && (3.18)\\
&= 1 \otimes (xy - yx) + (xy - yx) \otimes 1 && (3.19)\\
&= 1 \otimes [x, y] + [x, y] \otimes 1. && (3.20)
\end{aligned}$$

The polynomial ring $\mathbb{C}[x]$ can also be endowed with a Hopf algebra structure, since $\mathcal{S}(\mathbb{C}^1) \cong \mathbb{C}[x]$ by sending the basis element of the vector space $\mathbb{C}^1$ to the variable x. Thus, x is primitive, i.e. the comultiplication is given by $\Delta x = 1 \otimes x + x \otimes 1$ and the antipode yields $S(x) = -x$. Indeed, any primitive element x satisfies the identity $S(x) \cdot 1 + S(1) \cdot x = u\epsilon(x) = u(0) = 0$ by the defining diagram (in Definition 3.3), and hence we have $S(x) = -x$. Moreover, for any polynomial $p \in \mathbb{C}[x]$, we have $S(p(x)) = p(-x)$, since $\mathbb{C}[x]$ is commutative and S is an algebra anti-endomorphism. ■

Construction 3.5. Our choice of *comultiplication* on the ring of symmetric functions Sym will be the usual construction, motivated by splitting the variables: $(x, y) := (x_1, x_2, \ldots, y_1, y_2, \ldots)$. Consider the ring of formal power series of bounded degree on these variables, denoted by $R(x, y)$, with the action of the group $S_{(\infty,\infty)}$ permuting all variables. Denote the $\mathbb{C}$-algebra of invariants by $\mathrm{Sym}(x, y)$, which is isomorphic to $\mathrm{Sym} = \mathrm{Sym}(x)$ by construction. The subgroup $S_{(\infty)} \times S_{(\infty)} \leq S_{(\infty,\infty)}$ acts, via the canonical inclusion. by permuting the sets of x_i's and y_i's separately. Thus, the $\mathbb{C}$-vector space of invariants under this action has a basis $\{m_\lambda(x)m_\mu(y)\}$, where the indices run over all partitions λ and μ.

This suggests that we should consider the k-algebra homomorphism

$$R(x) \otimes R(x) \to R(x, y), \quad f(x) \otimes g(y) \mapsto f(x)g(y), \tag{3.21}$$

which restrict to an isomorphism

$$\mathrm{Sym}(x) \otimes \mathrm{Sym}(y) = R(x)^{S_{(\infty)}} \otimes R(x)^{S_{(\infty)}} \cong R(x, y)^{S_{(\infty)} \times S_{(\infty)}}. \tag{3.22}$$

since the basis $\{m_\lambda \otimes m_\mu\}$ is sent to $\{m_\lambda(x)m_\mu(y)\}$.

Note that $\mathrm{Sym}(x, y) = R(x, y)^{S_{(\infty,\infty)}}$ injects into $R(x, y)^{S_{(\infty)} \times S_{(\infty)}}$, which is just $\mathrm{Sym}(x) \otimes \mathrm{Sym}(y)$ under the above isomorphism. This yields the suitable comultiplication on the $\mathbb{C}$-algebra of symmetric functions

$$\Delta\colon \mathrm{Sym}(x) \to \mathrm{Sym}(x, y) \hookrightarrow \mathrm{Sym}(x) \otimes \mathrm{Sym}(y) \simeq \mathrm{Sym}(x) \otimes \mathrm{Sym}(x), \tag{3.23}$$

where the first map is given by $f(x) \mapsto f(x, y)$.

Example 3.6. Let us explicitly describe comultiplication on different bases of Sym. Along the way, we will see that this operation satisfies the coassociativity and counit axioms.

(a). The coproduct of a monomial symmetric function m_λ can be expressed as

$$\Delta m_\lambda = \sum_{(\mu,\nu)} m_\mu \otimes m_\nu, \tag{3.24}$$

where the sum is over all ordered pairs (μ, ν) of sub-partitions whose disjoint union is λ. For example, we can compute

$$\begin{aligned}
\Delta m_{(3,1)} &= m_{(3,1)}(x_1, x_2, \ldots, y_1, y_2, \ldots) \\
&= x_1^3 x_2 + x_1 x_2^3 \cdots + x_1^3 y_1 + x_1^3 y_2 + \cdots + x_1 y_1^3 + x_1 y_2^3 + \cdots + y_1 y_2^3 + y_1 y_2^3 + \cdots \\
&= m_{(3,1)}(x) + m_{(3)}(x)m_{(1)}(y) + m_{(1)}(x)m_{(3)}(y) + m_{(3,1)}(y) \\
&\cong m_{(3,1)} \otimes 1 + m_{(3)} \otimes m_{(1)} + m_{(1)} \otimes m_{(3)} + 1 \otimes m_{(3,1)}.
\end{aligned}$$

On the other hand, multiplication on the monomial basis can be expressed as a kind of *shuffle*, for example, we have

$$m_{(4,2^2)} m_{(3,1)} = m_{(4,3,2,2,1)} + 2m_{(4,3,3,2)} + m_{(5,3,2,2)} + m_{(5,4,2,1)} + m_{(5,4,3)} + 2m_{(5,5,2)} + m_{(7,2,2,1)} + m_{(7,3,2)}.$$

(b). Power sum symmetric functions are primitive, i.e. $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$ for every $n \geq 1$. Indeed, for any symmetric function $f(x)$, Δ sends $f(x)$ to $f(x, y)$, so we get

$$p_n(x, y) = \sum_i x_i^n + \sum_i y_i^n = p_n(x) \cdot 1 + 1 \cdot p_n(y) \cong 1 \otimes p_n + p_n \otimes 1. \tag{3.25}$$

(c). Similarly, for elementary symmetric functions, we have

$$\Delta e_n = \sum_{i+j=n} e_i \otimes e_j \tag{3.26}$$

for every $n \in \mathbb{Z}_{>0}$, and we have

$$\Delta h_n = \sum_{i+j=n} h_i \otimes h_j \tag{3.27}$$

for every $n \in \mathbb{Z}_{>0}$.

■

Lemma 3.7. *The comultiplication* Δ *in* Sym *is coassociative.*

Proof. The description of comultiplication on power sum symmetric functions can be used to show the coassociativity axiom. On the one hand, we have

$$(\Delta \otimes 1)\Delta p_n = \Delta(p_n \otimes 1) + 1 \otimes p_n \tag{3.28}$$

$$= (p_n \otimes 1) \otimes 1 + (1 \otimes p_n) \otimes 1 + (1 \otimes 1) \otimes p_n. \tag{3.29}$$

On the other hand,

$$(1 \otimes \Delta)\Delta p_n = \Delta(p_n \otimes 1) + 1 \otimes p_n \tag{3.30}$$

$$= p_n \otimes (1 \otimes 1) + 1 \otimes (p_n \otimes 1) + 1 \otimes (1 \otimes p_n). \tag{3.31}$$

The two expressions above are equal by associativity of the tensor product. ■

Example 3.8. For any partition λ, we have

$$\Delta s_\lambda = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu}, \tag{3.32}$$

where the sum runs over all sub-partitions of λ, i.e. all partitions comprised of some parts of λ. To see this, we can rewrite the explicit formula,

$$s_\lambda(x, y) = \sum_T (x, y)^{\mathrm{cont}(T)}, \tag{3.33}$$

where the sum runs over all semi-standard Young tableaux of shape λ with content coming from the ordered set $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$, and by T_y the restriction to the alphabet y. The restriction T_x of any such tableau to the boxes with entries from the alphabet x yields a skew-tableau of shape λ/μ, and the restriction T_y yields the other sub-tableau, of shape μ. So we can express the comultiplication as

$$s_\lambda(x, y) = \sum_T x^{\mathrm{cont}(T_x)} \cdot y^{\mathrm{cont}(T_y)} \tag{3.34}$$

$$= \sum_{\mu \subseteq \lambda} \left(\sum_{T_x} x^{\mathrm{cont}(T_x)} \right) \left(\sum_{T_y} y^{\mathrm{cont}(T_y)} \right) \tag{3.35}$$

$$= \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda/\mu}(y). \tag{3.36}$$

■

Remark 3.9. There is an alternative description of this formula via the *Littlewood-Richardson coefficients*, which are the structure constants of both multiplication and comultiplication of Schur functions

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda, \qquad \Delta s_\lambda = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu. \tag{3.37}$$

A *graded connected Hopf algebra* is a graded Hopf algebra whose degree zero part is isomorphic to the ground field. A graded connected Hopf with a distinguished basis with respect to which the structure constants of multiplication and comultiplication coincide is called a *self-dual Hopf algebra.*

Recall that the for a $\mathbb{C}$-module V, the dual is given by $V^* := \mathrm{Hom}(V, \mathbb{C})$, where a morphism $\phi\colon V \to W$ is sent to the adjoint $\phi^*\colon W^* \to V^*$, determined by $(f, \phi(v)) = (\phi^*(f), v)$. If V is a graded module, then one can define the *graded dual* by

$$V^o := \bigoplus_{n\geq 0} (V_n)^* \subset \prod_{n\geq 0} (V_n)^* = V^*. \tag{3.38}$$

If V is finite type, i.e. each V_n is a finite free module, then the dualizing functor is fully faithful, i.e. the map $\phi \mapsto \phi^*$ is bijective. Thus, in this situation duals of graded bialgebras and graded Hopf algebras are graded bialgebras and graded Hopf algebras, respectively.

The famous *Littlewood-Richardson rule* gives an explicit combinatorial formula for the above structure constants by counting certain skew-tableaux, and thereby showing *positivity* of the structure constants. The construction is somewhat lengthy, and since we are not using positivity later on, we shall not elaborate on the Littlewood-Richardson rule (see [Sa1], Theorem 4.9.4). However, let us briefly delineate the main properties of positive self-dual Hopf algebras. A connected graded Hopf algebra with a distinguished basis consisting of homogeneous elements, is called a *positive self-dual Hopf algebra (PSH)* if the structure constants are positive, and the Hopf algebra is *self-dual.*

(a). The ring of symmetric functions is the only indecomposable PSH up to isomorphism.

(b). The Hall inner product from Construction 1.44 induces an isomorphism of Hopf algebras

$$\mathrm{Sym} \to k, \quad g \mapsto (f, g).$$

To see that this is an isomorphism of bialgebras, tantamount to showing *self-duality.*

(c). A close relative of the Hall inner product is the Frobenius character map. So it does not come as a surprise that the Frobenius character map is also a PSH-isomorphism. We have seen that it is an isometry in Remark 1.47, and an isomorphism of algebras in Lemma 1.49. We will not use positivity, but we will need the fact that the character map respects the Hopf algebra structure.

Remark 3.10. Positivity of Sym will not be essential to what follows, but self-duality will be needed to see that the counit axiom holds (in Lemma 3.11). So, let us indicate why these structure constants coincide. There is a combinatorial identity, called the *Cauchy product identity*, in the ring $\mathbb{C}[x_1, x_2, \ldots, y_1, y_2, \ldots] = \mathbb{C}[x, y]$, that yields a useful product expansion of comultiplication of Schur functions,

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} s_\lambda(x) s_\lambda(y), \tag{3.39}$$

where the sum runs over all partitions λ. There is an equivalent formulation of the Cauchy identity, given by

$$\prod_{i,j=1}^{\infty} (1 - t x_i y_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} t^{|\lambda|} s_\lambda(x) s_\lambda(y). \tag{3.40}$$

This last equality is a consequence of the *Robinson-Schensted-Knuth (RSK) algorithm*, which recursively constructs a bijection between permutations $\binom{i}{j} = \binom{i_1 \cdots i_\ell}{j_1 \cdots j_\ell}$ written in two-line notation and pairs of standard Young tableaux (P, Q) of shape λ, such that the content of Q is i, the content of P is j. In particular, the RSK algorithm can be used to show the following famous identity, relating permutations and partitions:

$$n! = \sum_{\lambda \vdash n} t_\lambda^2, \tag{3.41}$$

where t_λ denotes the number of standard Young tableaux of shape λ. The proof of this bijection (which is not essential to what follows), as well as an explicit description of the RSK algorithm, can be found in [Sa2], Subsection 3.1. To see how the RSK algorithm implies the Cauchy identity, we need to expand the terms on the right-hand side,

$$(1 - txy)^{-1} = 1 + txy + (txy)^2 + \cdots, \tag{3.42}$$

and note that the left-hand side can be written as

$$\sum_{\binom{i}{j}} t^\ell x^{\mathrm{cont}(i)} y^{\mathrm{cont}(j)}, \tag{3.43}$$

where the sum runs over all biwords $\binom{i}{j}$, where $i = (i_1, \ldots, i_\ell)$ and $j = (j_1, \ldots, j_\ell)$ are non-increasing tuples of nonnegative integers. The RSK algorithm easily generalises from permutations in two-line notation and standard Young tableaux to a correspondence between biwords as above and semistandard Young tableaux, and this is where Schur functions enter the fray.

It is also worth noting that the Cauchy product can be expanded in terms of monomials and complete homogeneous symmetric functions,

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} h_\lambda(x) m_\lambda(y), \tag{3.44}$$

which implies (after a short computation) that the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ are dual bases of Sym with respect to the Hall inner product. This duality has already appeared tacitly, during the construction of the Frobenius character map in Construction 1.44.

Recall from Remark 1.45 that $\{s_\lambda\}$ forms an orthonormal basis of Sym, so $\{s_\mu(x)s_\nu(y)\}$ forms an orthonormal basis of $\mathrm{Sym} \otimes \mathrm{Sym}$. As a consequence, multiplication and comultiplication are adjoint, i.e. we have $\langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle$ for any $f, g, h \in \mathrm{Sym}$. Let us elaborate on this phenomenon in the proof of the following lemma.

Lemma 3.11. *The structure constants of multiplication and comultiplication in* Sym *coincide with respect to the Schur basis.*

Proof. Denote the structure constants of multiplication by $c_{\mu,\nu}^\lambda$ and the structure constants of comultiplication by $\bar{c}_{\mu,\nu}^\lambda$. Since the Littlewood-Richardson coefficients are indexed by triples of partitions, consider the ring $\mathbb{C}[x, y, z] := \mathbb{C}[x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots]$, and expand the Cauchy product in two different ways to relate $c^{\lambda}_{\mu,\nu}$ and $\overline{c}^{\lambda}_{\mu,\nu}$. On the one hand, we have

$$\prod_{i,j=1}^{\infty}(1-x_iz_j)^{-1}\prod_{i,j=1}^{\infty}(1-y_iz_j)^{-1}=\Big(\sum_{\mu}s_\mu(x)s_\mu(z)\Big)\Big(\sum_{\nu}s_\nu(y)s_\nu(z)\Big) \tag{3.45}$$

$$=\sum_{\mu,\nu}s_\mu(x)s_\nu(y)\cdot s_\mu(z)s_\nu(z) \tag{3.46}$$

$$=\sum_{\mu,\nu}s_\mu(x)s_\nu(y)\Big(\sum_{\lambda}c^{\lambda}_{\mu,\nu}s_\lambda(z)\Big), \tag{3.47}$$

where the sums run over all partitions μ and ν. On the other hand,

$$\prod_{i,j=1}^{\infty}(1-x_iz_j)^{-1}\prod_{i,j=1}^{\infty}(1-y_iz_j)^{-1}=\sum_{\lambda}s_\lambda(x,y)s_\lambda(z) \tag{3.48}$$

$$=\sum_{\lambda}\Delta s_\lambda(x)s_\lambda(z) \tag{3.49}$$

$$=\sum_{\lambda}\Big(\sum_{\mu,\nu}\overline{c}^{\lambda}_{\mu,\lambda}s_\mu(x)s_\nu(y)\Big)s_\lambda(z). \tag{3.50}$$

Therefore, both $c^{\lambda}_{\mu,\nu}$ and $\overline{c}^{\lambda}_{\mu,\nu}$ arise as the coefficient of $s_\mu(x)s_\nu(y)s_\lambda(z)$ in the above Cauchy product, and consequently, they are equal. ■

Lemma 3.12. *The* counit ϵ *in* Sym *is defined as the identity on* $\mathrm{Sym}_0\cong\mathbb{C}$ *and the zero map zero elsewhere, i.e. it is the evaluation map* $f\mapsto f(0,0,\ldots)$*. The counit* ϵ *in* Sym *satisfies the counit axiom.*

Proof. Recall from diagram 3.2, that we need to show the identity

$$(\mathrm{id}\otimes\epsilon)\circ\Delta=\mathrm{id}=(\epsilon\otimes\mathrm{id})\circ\Delta. \tag{3.51}$$

This fact hinges on self-duality from Lemma 3.11. On the one hand, we have

$$(\epsilon\otimes 1)\Delta s_\lambda=(\varepsilon\otimes 1)\sum_{\mu,\nu}c^{\lambda}_{\mu,\nu}s_\mu\otimes s_\nu \tag{3.52}$$

$$=\sum_{\mu,\nu}\epsilon(s_\mu)\otimes c^{\lambda}_{\mu,\nu}s_\nu \tag{3.53}$$

$$=1\otimes s_\lambda. \tag{3.54}$$

On the other hand,

$$(1\otimes\epsilon)\Delta s_\lambda=(1\otimes\varepsilon)\sum_{\mu,\nu}c^{\lambda}_{\mu,\nu}s_\mu(x)s_\nu(y) \tag{3.55}$$

$$=\sum_{\mu,\nu}c^{\lambda}_{\mu,\nu}s_\mu\otimes\epsilon(s_\nu) \tag{3.56}$$

$$=s_\lambda\otimes 1, \tag{3.57}$$

so the counit axiom is also satisfied, and Sym can be endowed with a coalgebra structure. The compatibility conditions in Definition 3.1 can directly checked by the definition of the operations. The twist operation from equation 3.6 T is easily dealt with, because Sym is cocommutative, which we will show subsequently in Remark 3.14. Hence, we can endow Sym with a structure of a graded connected Hopf algebra over $\mathbb{C}$. ■

Example 3.13. Recall from Example 3.6, part (b) that power sum symmetric functions are primitive elements of Sym, i.e. $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$. Therefore, we obtain the identity $1 \cdot \epsilon(p_n) + \epsilon(p_n) \cdot p_n = p_n$ by the defining diagram (in Definition 3.1), and consequently $\epsilon(p_n) = 0$. ■

Remark 3.14. Comultiplication in Sym is cocommutative, i.e. the following diagram commutes,

$$\begin{tikzcd} & \mathrm{Sym} \otimes \mathrm{Sym} \\ \mathrm{Sym} \otimes \mathrm{Sym} \arrow[ur, "T"] & \\ & \mathrm{Sym} \arrow[uu, "\Delta"'] \arrow[ul, "\Delta"] \end{tikzcd} \tag{3.58}$$

where T denotes the twist map $f \otimes g \mapsto g \otimes f$. Here, it is expedient to consider the $\mathbb{C}$-algebra generating set $\{h_n\}$ of complete homogeneous symmetric functions. Using the explicit description of comultiplication from Example 3.6 (b), we can compute

$$(T \circ \Delta)(h_n) = T(\Delta(h_n)) \tag{3.59}$$

$$= T\left(\sum_{i+j=n} h_i \otimes h_j\right) \tag{3.60}$$

$$= \sum_{i+j=n} h_j \otimes h_i = \sum_{i+j=n} h_i \otimes h_j \tag{3.61}$$

$$= \Delta h_n. \tag{3.62}$$

Therefore, $T \circ \Delta = \Delta$ on the generators, and these maps are $\mathbb{C}$-algebra homomorphisms, so they must be equal on the entire bialgebra Sym.

Construction 3.15. So far, we have seen that Sym is a graded bialgebra over $\mathbb{C}$, and $\mathrm{Sym}_0 \cong \mathbb{C}$, i.e. it is also a *connected* bialgebra. According to a general statement about graded connected bialgebras (see [GR], Proposition 1.4.16), the antipode comes for free, and it is completely determined. Moreover, a morphism of bialgebras between Hopf algebras is in fact a morphism of Hopf algebras.

There is an explicit description of the antipode in a graded connected Hopf algebra A, called *Takeuchi's formula*, given by

$$S = \sum_{k \geq 0} (-1)^k m^{(k-1)} f^{\otimes k} \Delta^{(k-1)} \tag{3.63}$$

$$= u\epsilon - f + m \circ f^{\otimes 2} \circ \Delta - m^{(2)} \circ f \circ \Delta^{(2)} + \cdots, \tag{3.64}$$

where $f = \mathrm{id}_A - u\epsilon \in \mathrm{End}(A)$.

To see this, note that f sends the zero degree part A_0 to 0. Then for every $x \in A_n$ and $m > n$, each summand of $\Delta^{(m-1)}$ contains a factor lying in A_0. Thus, $f^{\otimes m}$ sends each summand to zero, and consequently, $f^{*m}(a) = 0$, which means that the graded map f is locally $*$-nilpotent. Hence, $u\epsilon + f = \mathrm{id}_A$ has a two-sided inverse, which is by definition the antipode S. Explicitly, we can write

$$S = (\mathrm{id}_A)^{*(-1)} = (u\epsilon + f)^{*(-1)} = \sum_{k \geq 0} (-1)^k f^{*k},$$

which yields the desired formula.

Remark 3.16. There is an equivalence of categories between affine group schemes (group objects in the category of affine schemes) and commutative Hopf algebras. Hopf algebras graded by the integers correspond to affine group schemes with $\mathbb{G}_m$-action, where $\mathbb{G}_m$ denotes the multiplicative group scheme $\operatorname{Spec}\mathbb{Z}[x, x^{-1}]$. In this group scheme, the unit map is given by $x \mapsto 1$, multiplication by $x \mapsto x \otimes x$ and inverse by $x \mapsto x^{-1}$.

Example 3.17. The antipode in Sym is determined by the primitive elements $S(p_n) = -p_n$ for all $n \in \mathbb{Z}_{>0}$, for the same reason that we gave in Example 3.4.

The antipode on elementary symmetric functions is given by $S(e_n) = (-1)^n h_n$ for every $n \in \mathbb{Z}_{>0}$, and we have $S(h_n) = (-1)^n e_n$ for every $n \in \mathbb{Z}_{>0}$.

To see this, first recall the coproduct formula from Example 3.6. For every $n \in \mathbb{Z}_{>0}$ we have

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j. \tag{3.65}$$

Thus, by the defining relations for the antipode (in Definition 3.3), it follows that

$$\sum_{i+j=n} S(e_i)e_j = u\epsilon(e_n) = \delta_{0,n} = \sum_{i+j=n} e_i\, S(e_j), \tag{3.66}$$

and analogously for complete homogeneous symmetric functions. Since we have the following identity from Remark 1.24

$$\sum_{i+j=n} (-1)e_i h_j = \delta_{0,n}, \tag{3.67}$$

we obtain the desired formulas $S(e_n) = (-1)^n h_n$ and $S(h_n) = (-1)^n e_n$ by induction. ■

Lemma 3.18. *We have the following relation between the fundamental involution and the antipode in the Hopf algebra of symmetric functions. For any $f \in \mathrm{Sym}_n$,*

$$S(f) = (-1)^n \omega(f). \tag{3.68}$$

Proof. Recall from Remark 3.14 that Sym is cocommuative, so all $\mathbb{C}$-algebra anti-endomorphisms $\mathrm{Sym} \to \mathrm{Sym}$, such as the antipode, are in fact $\mathbb{C}$-algebra endomorphisms. Consider the $\mathbb{C}$-bialgebra homomorphism $a_{-1}\colon \mathrm{Sym} \to \mathrm{Sym}$ given by $f \mapsto (-1)^n f$ for any $f \in \mathrm{Sym}_n$, and precompose it with ω to get

$$\omega \circ a_{-1} = \omega((-1)^n e_n) = (-1)^n h_n. \tag{3.69}$$

Since we have $S(e_n) = (-1)^n h_n$, the above identity yields $(\omega \circ a_{-1})(e_n) = S(e_n)$ for each $n \in \mathbb{Z}_{>0}$. Hence, we get that $\omega \circ a_{-1} = S$ as $\mathbb{C}$-algebra homomorphisms on Sym. ■

Lemma 3.19. *The fundamental involution ω and the antipode S are Hopf algebra automorphisms of* Sym.

Proof. It is expedient to consider complete homogeneous symmetric functions for checking these relations,

$$((\omega \otimes \omega) \circ \Delta)(h_n) = (\omega \otimes \omega)(\Delta(h_n)) \tag{3.70}$$

$$= (\omega \otimes \omega)\left(\sum_{i+j=n} h_i \otimes h_j \right) \tag{3.71}$$

$$= \sum_{i+j=n} \omega(h_i) \otimes \omega(h_j) \tag{3.72}$$

$$= \sum_{i+j=n} e_i \otimes e_j \tag{3.73}$$

$$= \Delta(e_n) \tag{3.74}$$

$$= \Delta(\omega(h_n)) \tag{3.75}$$

$$= (\Delta \circ \omega)(h_n). \tag{3.76}$$

Thus, for all $n \in \mathbb{Z}_{>0}$ we have

$$((\omega \otimes \omega) \circ \Delta)(h_n) = (\Delta \circ \omega)(h_n), \tag{3.77}$$

and consequently we have $(\omega \otimes \omega) \circ \Delta = \Delta \circ \omega$. Similarly, we can also see that $\epsilon = \epsilon \circ \omega$. Therefore, ω is a morphism of bialgebras from a Hopf algebra to itself, and as such, it is a Hopf algebra morphism. Being an involution, ω is invertible, so ω is indeed a Hopf algebra automorphism.

From Lemma 3.18 and Example 3.17, we know that

$$S(h_n) = (-1)^n e_n = (-1)^n \omega(h_n), \tag{3.78}$$

and we have seen that ω is a Hopf algebra automorphism. Hence the antipode must also be a Hopf algebra automorphism (since the antipode in Sym is a $\mathbb{C}$-algebra automorphism,). ■

Lemma 3.20. *The fundamental involution ω is a graded map.*

Proof. Since the structure of a graded connected bialgebra gives rise to a unique graded algebra homomorphism that satisfies the diagram of the antipode, S must be graded, i.e. $S(\mathrm{Sym}_n) \subseteq \mathrm{Sym}_n$ for any $n \in \mathbb{Z}_{>0}$. From Example 3.17, we know that for any $f \in \mathrm{Sym}_n$, we have $S(f) = (-1)^n \omega(f)$, so

$$(-1)^n S(f) = (-1)^n (-1)^n \omega(f) = \omega(f). \tag{3.79}$$

Consequently, we end up with

$$\omega(f) = (-1)^n S(f) \in (-1)^n S(\mathrm{Sym}_n) \subset \mathrm{Sym}_n . \tag{3.80}$$

■

Lemma 3.21. *The Frobenius character map (from Definition 1.46) is a Hopf algebra automorphism.*

Proof. We have seen that the character map is an algebra automorphism in Lemma 1.49, and over the course of this subsection we have endowed the ring of symmetric function Sym with a Hopf algebra structure. Recall from Definition 1.46 that the $\mathbb{C}$-algebra R_n of class of functions of the symmetric group S_n extends to a graded $\mathbb{C}$-algebra $R = \bigoplus_{n \geq 0} R_n$, which is the domain of the character map ch. Irreducible characters form an orthonormal basis of R, and the suitable product on R, so that the character map is an algebra morphism, is given by $m = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}} : R_i \otimes R_j \to R_{i+j}$. Then it is natural to define a coproduct on R by

$$\Delta := \bigoplus_{i+j=n} \mathrm{Res}_{S_i\times S_j}^{S_n} : R_n \to \bigoplus_{i+j=n} R_i \otimes R_j, \tag{3.81}$$

and one has to show the compatibility condition

$$\mathrm{ch}\left(\bigoplus_{k=0}^{n} \mathrm{Res}_{S_k\times S_{n-k}}^{S_n}(\chi)\right) = \Delta(\mathrm{ch}(\chi)). \tag{3.82}$$

The proof demonstrating that these operations give rise to a bialgebra structure on R is quite cumbersome, and it can be found in [GR], Corollary 4.3.10, in the broader context of PSHs. We will provide a proof in Lemma 3.44 that pertains specifically to the context we require. ■

3.2 Hopf Algebra of Quasi-Symmetric Functions

We are now prepared to study quasi-symmetric functions from an algebraic perspective. The Hopf algebra of symmetric functions lies within a larger Hopf algebra QSym of quasi-symmetric functions, which arises as a terminal object in the category of graded connected Hopf algebras. This fact will be demonstrated in Theorem 3.35, and we will subsequently use this universal property to present chromatic symmetric and quasi-symmetric functions of graphs within an algebraic framework.

Construction 3.22. The ring of *quasi-symmetric functions* consists of power series in infinitely many variables $\{x_i\}_{i\in\mathbb{Z}_{>0}}$ with complex coefficients and of bounded degree, such that the coefficient of any two monomials $x_{i_1}^{\alpha_1}\cdots x_{i_r}^{\alpha_r}$ and $x_{j_1}^{\alpha_1}\cdots x_{j_r}^{\alpha_r}$ coincides whenever $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$. Similarly to Sym, the $\mathbb{C}$-algebra QSym is graded by degree.

As before, everything in this section works analogously over any algebraically closed field k of characteristic zero. In Subsection 3.3, we will need $\mathbb{C}(t)$-coefficients as well as $\mathbb{C}$-coefficients.

For any composition $\alpha = (\alpha_1, \ldots, \alpha_r)$, we can define, analogously to monomial symmetric functions, *monomial quasi-symmetric functions*,

$$M_\alpha := \sum_{0<i_1<\cdots<i_r} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2}\cdots x_{i_r}^{\alpha_r}. \tag{3.83}$$

Let us list the first few examples of monomial quasi-symmetric functions,

$$\begin{aligned}
M_{(1)} &= x_1 + x_2 + x_3 + \cdots = m_{(1)} = s_{(1)} = e_1 = h_1 = p_1,\\
M_{(2)} &= x_1^2 + x_2^2 + x_3^2 + \cdots = m_{(2)} = p_2,\\
M_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \cdots = m_{(1,1)} = e_2\\
M_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \cdots = m_{(3)} = p_3,\\
M_{(2,1)} &= x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots,\\
M_{(1,2)} &= x_1x_2^2 + x_1x_3^3 + x_2x_3^2 + \cdots,\\
M_{(1,1,1)} &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3 + x_4 + \cdots = m_{(1,1,1)} = e_3.
\end{aligned}$$

Analogously to the proof of Lemma 1.16, we can show that $\{M_\alpha\}$ forms a basis of QSym, and if we restrict to compositions $\alpha \models n$, then monomial quasi-symmetric functions form a basis of the degree n part QSym_n.

Monomial quasi-symmetric functions give the means to describe monomial symmetric functions in terms of compositions, not just weak compositions, like did in Lemma 1.16. For any partition λ,

$$m_\lambda = \sum_\alpha M_\alpha, \tag{3.84}$$

where the sum runs over all compositions $\alpha = (\alpha_1, \ldots, \alpha_r)$ of shape λ.

In particular, the $\mathbb{C}$-dimension of the degree n part QSym_n is the number of compositions of n, which is given by 2^{n-1}.

Construction 3.23. Analogously to Sym (as in Construction 3.5), the product $\mathrm{QSym}(x) \otimes \mathrm{QSym}(y)$ can be embedded in the k-algebra $R(x, y)$ by identifying each $f \otimes g$ with $fg \in R(x, y)$. Thus, we have a string of inclusion $\mathrm{QSym}(x, y) \subset \mathrm{QSym}(x) \otimes \mathrm{QSym}(y) \subset R(x, y)$, and the *comultiplication* $\Delta\colon\ \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ can be defined as the composition

$$\mathrm{QSym} \cong \mathrm{QSym}(x, y) \hookrightarrow \mathrm{QSym}(x) \otimes \mathrm{QSym}(y) \cong \mathrm{QSym} \otimes \mathrm{QSym}, \tag{3.85}$$

where the isomorphism $\mathrm{QSym} \cong \mathrm{QSym}(x, y)$ is given by $f \mapsto f(x, y) = f(x_1, x_2, \ldots, y_1, y_2, \ldots)$.

Example 3.24. The comultiplication on monomial quasi-symmetric functions is given by

$$\Delta M_\alpha = \sum_{(\beta,\gamma)\,:\,\beta\cdot\gamma=\alpha} M_\beta \otimes M_\gamma, \tag{3.86}$$

where $\beta \cdot \gamma$ is the concatenation of the two compositions.

Similarly to comultiplication of monomial symmetric functions in Example 3.6 part (a), one can compute

$$\begin{aligned}
\Delta M_{(2,3^2)} &= M_{(2,3^2)}(x_1, x_2, \ldots, y_1, y_2, \ldots) \\
&= x_1^2 x_2^3 x_3^3 + x_1^2 x_2^3 x_4^3 + \cdots + x_1^2 x_2^3 y_1^3 + x_1^2 x_2^3 y_2^3 + \cdots \\
&\quad + x_1^2 y_1^3 y_2^3 + x_1^2 y_1^3 y_3^3 + \cdots + y_1^2 y_2^3 y_3^3 + y_1^2 y_2^3 y_4^3 + \cdots \\
&= M_{(2,3^2)}(x) + M_{2,3}(x) M_{(3)}(y) + M_{(2)}(x) M_{(3^2)}(y) + M_{(2,3^2)}(y) \\
&= M_{(2,3^2)} \otimes 1 + M_{(2,3)} \otimes M_{(3)} + M_{(2)} \otimes M_{(3^2)} + 1 \otimes M_{(2,3^2)}.
\end{aligned}$$

In particular, if $\alpha = (n)$, then M_α is a primitive element.

On the other hand, multiplication can be expressed of a kind of *shuffle* on the monomial basis, for example

$$M_{(2,3^2)} M_{(5)} = M_{(2,3,3,5)} + M_{(2,3,5,3)} + M_{(2,3,8)} + M_{(2,5,3,3)} + M_{(2,8,3)} + M_{(5,2,3,3)} + M_{(7,3,3)}.$$

In particular, the primitive monomial quasi-symmetric function $M_{(n)}$ appears in a product $M_\alpha M_\beta$ only if $\alpha = (i)$ and $\beta = (j)$ with $i + j = n$. ■

Lemma 3.25. *The ring of quasi-symmetric functions* QSym *is a graded connected Hopf algebra, containing the ring of symmetric functions* Sym *as a Hopf subalgebra.*

Proof. To show coassociativity $(\Delta \circ \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, consider the monomial basis, and compute

$$((\Delta \circ \mathrm{id}) \circ \Delta) M_\alpha = \sum_{k=0}^{\ell} \Delta(M_{(\alpha_1,\ldots,\alpha_k)}) \otimes M_{(\alpha_{k+1},\ldots,\alpha_\ell)} \tag{3.87}$$

$$= \sum_{k=0}^{\ell} \sum_{i=0}^{k} M_{(\alpha_1,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\ldots,\alpha_k)} \otimes M_{\alpha_{k+1},\ldots,\alpha_\ell)} \tag{3.88}$$

$$= \sum_{k=0}^{\ell} (M_{(\alpha_1,\ldots,\alpha_k)}) \otimes \Delta M_{(\alpha_{k+1},\ldots,\alpha_\ell)} \tag{3.89}$$

$$= ((\mathrm{id} \otimes \Delta) \circ \Delta) M_\alpha. \tag{3.90}$$

The coproduct in QSym is an algebra morphism, being the composition of two algebra morphisms. The counit in both Sym and QSym will be the evaluation map that sends $f(x)$ to $f(0, 0, \ldots)$, which is an algebra morphism. Consequently, QSym is a graded connected bialgebra, and hence a Hopf algebra.

For the second claim, that Sym arises as a Hopf subalgebra of QSym, note that the generators e_n of Sym can be written as $e_n = M_{(1,\ldots,1)}$, so the formula for comultiplication on the monomial basis implies that the comultiplication on QSym restricts to

$$\Delta e_n = \sum_{i=1}^{n} e_i \otimes e_{n-i}, \tag{3.91}$$

which coincides with our comultiplication on the Hopf algebra Sym. Moreover, the antipode in a graded connected Hopf algebra is uniquely determined. ∎

Example 3.26. We are not going to use any explicit formula for the antipode in QSym, but let us briefly explain how to express it in the monomial basis.

For any $\alpha, \beta \in \mathrm{Comp}_n$, α is said to *refine* β (or β coarsens α) if one can obtain β from α by combining some of its adjacent parts. For any composition α, its *reverse composition* is $\mathrm{rev}(\alpha) = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_2, \alpha_1)$, which induces a poset automorphism of Comp_n, given by $\alpha \mapsto \mathrm{rev}(\alpha)$. Then the antipode of QSym in the monomial basis is given by

$$S(M_\alpha) = (-1)^{\ell(\alpha)} \sum_{\substack{\gamma \in \mathrm{Comp}: \\ \gamma \text{ coarsens } \mathrm{rev}(\alpha)}} M_\gamma. \tag{3.92}$$

The proof of this formula (which can be found in [GR], Theorem 5.1.11) relies on a sign reversing involution and the fact that $M_{(n)}$ is a primitive element. ∎

Remark 3.27. There is another important basis of QSym, called the basis of *fundamental quasi-symmetric functions.* For a composition $\alpha \in \mathrm{Comp}_n$, the corresponding fundamental quasi-symmetric function is defined as

$$F_\alpha := \sum_{\substack{\beta \in \mathrm{Comp}_n \\ \beta \text{ refines } \alpha}} M_\beta. \tag{3.93}$$

For instance, for the eight compositions of 4, we have

$$\begin{aligned}
F_{(1,1,1,1)} &= M_{(1,1,1,1)}.\\
F_{(2,1,1)} &= M_{(1,1,1,1)} + M_{(2,1,1)},\\
F_{(1,2,1)} &= M_{(1,1,1,1)} + M_{(1,2,1)},\\
F_{(1,1,2)} &= M_{(1,1,1,1)} + M_{(1,1,2)},\\
F_{(2,2)} &= M_{(1,1,1,1)} + M_{(1,1,2)} + M_{(2,1,1)} + M_{(2,2)},\\
F_{(3,1)} &= M_{(1,1,1,1)} + M_{(1,2,1)} + M_{(2,1,1)} + M_{(3,1)},\\
F_{(1,3)} &= M_{(1,1,1,1)} + M_{(1,2,1)} + M_{(1,1,2)} + M_{(1,3)},\\
F_{(4)} &= M_{(1,1,1,1)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(1,3)} + M_{(2,1,1)} + M_{(2,2)} + M_{(3,1)} + M_{(4)}.
\end{aligned}$$

Similarly to Schur functions, F_α is also a bridge between elementary symmetric functions and complete homogeneous symmetric functions, since we have

$$F_{(1^n)} = M_{(1^n)} = e_n, \qquad \text{and} \qquad F_{(n)} = \sum_{\alpha \in \mathrm{Comp}_n} M_\alpha = h_n. \tag{3.94}$$

There is an operation on compositions, called *near-concatenation*, which helps describe comultiplication on the fundamental basis. For two compositions $\alpha = (\alpha_1, \ldots, \alpha_i)$ and $\beta = (\beta_1, \ldots, \beta_j)$, their near-concatenation is defined by $\alpha \odot \beta \coloneqq (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \beta_1, \ldots, \beta_j)$. Then the comultiplication on the fundamental basis is given by

$$\Delta F_\alpha = \sum_{\substack{(\beta,\gamma)\\ \beta\cdot\gamma=\alpha \text{ or } \beta\odot\gamma=\alpha}} F_\beta \otimes F_\gamma. \tag{3.95}$$

The antipode on the fundamental basis is expressed by $S(F_\alpha) = (-1)^{|\alpha|} F_{\omega(\alpha)}$, where $\omega(\alpha)$ is the (uniquely defined) composition, whose partial sums form a complementary set within $[n-1]$ to the partial sums of the reverse-composition $\mathrm{rev}(\alpha)$. For example, if $\alpha = (4,1,3)$, a composition of 8, then $\mathrm{rev}(\alpha) = (3,1,4)$, and the set of partial sums of $\mathrm{rev}(\alpha)$ is $\{3,4\} \subset [7]$. The complementary set is $\{1,2,5,6,7\}$, and there is a unique composition $\omega(\alpha) = (1,1,3,1,1)$, whose set of partial sums is this set. We will not use these explicit formulas later on, and proofs can be found in [GR], Proposition 5.2.15.

Example 3.28. Recall from Theorem 1.69 that the chromatic symmetric function of the path P_3 with labels $1,2,3$, in this linear order, lies in $\mathrm{Sym}[t]$, since this path is the indifference graph of the Dyck path $(2,3,3)$. In Example 1.61, we computed e-expansion of this chromatic quasi-symmetric function $e_{(3)} + (e_3 + e_{(2,1)})t + e_3 t^2$. We have also computed the chromatic quasi-symmetric function of the path P_3 with labels $2,1,3$, in Example 1.61, and realised that this quasi-symmetric functions does not lie in the ring $\mathrm{Sym}[t]$. It can be written as

$$(2 + 2t + 2t^2) m_{(1^3)} + \sum_{i<j} x_i^2 x_j + t^2 \sum_{i<j} x_i x_j^2, \tag{3.96}$$

which can be expressed in terms of fundamental quasi-symmetric functions as

$$(e_{(3)} + F_{(2,1)}) + 2e_{(3)} t + (e_{(3)} + F_{(1,2)}) t^2. \tag{3.97}$$

Similarly, if we calculate the chromatic quasi-symmetric functions of the path P_3 with labels $1,3,2$, then we obtain the expansion

$$(e_{(3)} + F_{(1,2)}) + 2e_{(3)} t + (e_{(3)} + F_{(2,1)}) t^2. \tag{3.98}$$

■

Remark 3.29. Since $\mathrm{Sym} \subset \mathrm{QSym}$, we may consider the expansion of the symmetric functions we encountered earlier in terms of distinguished bases of QSym. Example 3.28 provided a small illustration of this phenomenon. The monomial quasi-symmetric function expansion of symmetric functions can also be helpful, as we will see in Lemma 3.46.

Let V be a finite dimensional S_n-representation. Then the coefficient of the primitive monomial quasi-symmetric function $M_{(n)}$ in the Frobenius character $\mathrm{ch}(V) \in \mathrm{Sym}$ is equal to the dimension of the space of S_n-invariants in V. In other words, it is the dimension of the subspace of V on which S_n acts trivially. This result is known as the Murnaghan-Nakayama rule, whose proof can be found in [Sa1], Section 4.10.

On the other hand, the coefficient of $M_{(n)}$ in $\omega(\mathrm{ch}(V))$ is the dimension of subspace V' of V on which S_n acts by the sign representation, i.e. for any $x \in V'$, we have $w \cdot x = (-1)^{\ell(w)}x$.

Recall from Lemma 3.11 that the Hopf algebra of symmetric functions Sym is self-dual, and that ω and the Hall inner product plays an important role in describing this duality on the distinguished bases. We will shortly see that QSym is no longer self-dual.

Definition 3.30. The Hopf algebra of *noncommutative symmetric functions* NSym is the dual of QSym. Consider the dual pairing $(\cdot,\cdot)\colon \mathrm{NSym} \otimes \mathrm{QSym} \to k$, and let $\{H_\alpha\}$ be the k-basis of NSym dual to the k-basis $\{M_\alpha\}$ of QSym, so that $(H_\alpha, M_\beta) = \delta_{\alpha,\beta}$ for all compositions α and β.

Lemma 3.31. *For any $n \in \mathbb{Z}_{>0}$, we will abbreviate $H_n := H_{(n)}$ and use the convention $H_0 = 1$. Then we have $\mathrm{NSym} \cong k\langle H_1, H_2, \ldots\rangle$, i.e. it is the free associative (but not commutative) algebra with generators $\{H_1, H_2, \ldots\}$. Moreover, the coproduct is determined by*

$$\Delta H_n = \sum_{i+j=n} H_i \otimes H_j. \tag{3.99}$$

Proof. Recall, from Example 3.24, what comultiplication in QSym does to the monomial basis,

$$\Delta M_\alpha = \sum_{k=0}^{\ell} M_{(\alpha_1,\ldots,\alpha_k)} \otimes M_{(\alpha_{k+1},\ldots,\alpha_\ell)} = \sum_{(\beta,\gamma)\colon \beta\cdot\gamma=\alpha} M_\beta \otimes M_\gamma, \tag{3.100}$$

where $\beta \cdot \gamma$ is the concatenation of the two compositions. Consequently, multiplication on the dual $\{H_\alpha\}$ is given by $H_\beta H_\gamma = H_{\beta\cdot\gamma}$, and inductively, we have $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_\ell}$, where $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Hence, there is an algebra isomorphism $\mathrm{NSym} \cong k\langle H_1, H_2, \ldots\rangle$.

For the second assertion, recall from Example 3.24 that the primitive monomial quasi-symmetric function $M_{(n)}$ appears in a product $M_\alpha M_\beta$ only if $\alpha = (i)$ and $\beta = (j)$ with $i + j = n$. In this case, we have $M_{(i)}M_{(j)} = M_{(i+j)} + M_{(i,j)} + M_{(j,i)}$. Since H_n is dual to $M_{(n)}$, this yields the desired equality (3.99). ■

Lemma 3.32. *The homomorphism $\pi\colon \mathrm{NSym} \to \mathrm{Sym}$, $H_n \mapsto h_n$ is a surjective Hopf algebra morphism, which is adjoint to the inclusion $i\colon \mathrm{Sym} \hookrightarrow \mathrm{QSym}$ with respect to the dual pairing.*

Proof. Let V be the graded free $\mathbb{C}$-module on the set $\{H_i\}$. Recall from Lemma 1.25 and Lemma 1.25 that $\mathrm{Sym} \cong k[h_1, h_2, \ldots]$ and $\mathrm{NSym} \cong k\langle H_1, H_2, \ldots\rangle$. As π is an algebra morphism, it corresponds to a surjective map from the free module V to $\mathrm{Sym}(V)$. Furthermore, recall from Example 3.6 part (b) and Lemma 3.31, that the expressions for comultiplication also match up,

$$\Delta h_n = \sum_{i+j=n} h_i \otimes h_j \qquad \text{and} \qquad \Delta H_n = \sum_{i+j=n} H_i \otimes H_j. \tag{3.101}$$

Hence, π is a bialgebra morphism, and as a result, being a bialgebra morphism between graded connected Hopf algebras, a Hopf algebra morphism.

For the second assertion, let α be a composition and let $\lambda(\alpha)$ be the corresponding partition, i.e. a weakly decreasing rearrangement of α. Note that $\langle \pi(H_\alpha), m_\mu \rangle = \langle h_{\lambda(\alpha)}, m_\mu \rangle$ by Construction 1.44, which is 1 if $\lambda(\alpha) = m_\mu$ and 0 otherwise. This coincides with the pairing

$$\left\langle H_\alpha, \sum_{\beta : \lambda(\beta)=\lambda} M_\beta \right\rangle = \langle H_\alpha, i(m_\lambda) \rangle, \tag{3.102}$$

which shows adjointness. ■

Definition 3.33. Let k be a field of characteristic zero (for us, this will be $\mathbb{C}$ or $\mathbb{C}(t)$). A *character* of a Hopf algebra A over k is an algebra morphism $\zeta\colon A \to k$ satisfying $\zeta(1_A) = 1_k$, k-linearity and $\zeta(ab) = \zeta(a)\zeta(b)$ for any $a, b \in A$.

Definition 3.34. The *distinguished character* $\zeta_Q\colon \mathrm{QSym} \to k$ is given by $f(x) \mapsto f(1, 0, 0, \ldots)$. Note that

$$\zeta_Q(M_\alpha) = \zeta_Q(F_\alpha) = \begin{cases} 1 & \text{if } \alpha = (n) \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.103}$$

Consequently, the restriction $\zeta_Q|_{\mathrm{QSym}_n}$ is the functional $H_n \in \mathrm{NSym}_n = \mathrm{Hom}_k(\mathrm{QSym}_n, k)$, i.e. for any $f \in \mathrm{QSym}_n$, we have $\zeta_Q(f) = (H_n, f)$.

Theorem 3.35. *Let A be a connected graded Hopf algebra, and $\zeta\colon A \to k$ a character. Then there is a unique graded Hopf algebra homomorphism $\psi\colon A \to \mathrm{QSym}$ making the following diagram commute*

$$\begin{tikzcd} & \mathrm{QSym} \arrow[d, "\zeta_Q"] \\ A \arrow[ur, dashed, "\psi"] \arrow[r, "\zeta"'] & k \end{tikzcd} \tag{3.104}$$

For any homogeneous element $a \in A_n$, we have

$$\psi(a) = \sum_{\alpha \in \mathrm{Comp}_n} \zeta_\alpha(a) M_\alpha, \tag{3.105}$$

where Comp_n denotes the set of compositions of n and ζ_α for $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathrm{Comp}_n$ is given by

$$A_n \xrightarrow{\Delta^{(\ell-1)}} A^{\otimes \ell} \xrightarrow{\pi_\alpha} A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_\ell} \xrightarrow{\zeta^{\otimes \ell}} k, \tag{3.106}$$

and where $\pi_\alpha : A^{\otimes \ell} \to A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_\ell}$ denotes the canonical projection onto the graded parts determined by α. Furthermore, when A is cocommutative, $\mathrm{im}\,\psi \subset \mathrm{Sym}$.

Proof. There is a bijection between graded k-linear maps $A \to \mathrm{QSym}$ and graded k-linear maps $\mathrm{QSym}^o \to A^o$, given by taking the dual $f \mapsto f^*$. A graded map $f\colon A \to \mathrm{QSym}$ is a k-coalgebra morphism if and only if f^* is a k-algebra morphism. Regard ψ as a graded k-coalgebra map instead of a Hopf algebra map, and consider the corresponding k-algebra map under the above correspondence $\mathrm{NSym} = \mathrm{QSym}^o \xrightarrow{\psi^*} A^o$. By commutativity of diagram (3.104), for any $a \in A_n$, we have $(\psi^*(H_n), a) = (H_n, \psi(a)) = \zeta_Q(\psi(a)) = \zeta(a)$. Since ψ^* is graded, we have $(\psi^*(H_m), a) = 0$ if $a \in A_n$ and $m \neq n$. Hence, $\psi^*(H_n) \in A^o$ is given by

$$\psi^*(H_n)(a) = \begin{cases} \zeta(a) & \text{if } a \in A_n, \\ 0 & \text{if } a \in A_m \text{ for some } m \neq n. \end{cases} \tag{3.107}$$

Since $\mathrm{NSym} \cong k\langle H_1, H_2, \ldots \rangle$, by the universal property of free associative k-algebras, for any k-linear map $\zeta \colon A \to k$, we have a unique k-algebra morphism $\psi^* \colon \mathrm{QSym}^o \to A^o$ satisfying equation (3.107) for any $n \in \mathbb{Z}_{>0}$. By definition, ψ^* also satisfies equation (3.107) for $n = 0$, and we have $\psi^*(1) = 1$, since $\zeta(1) = 1$ and A is connected. Therefore, for any $\zeta \colon A \to k$ with $\zeta(1) = 1$, there exists a unique k-algebra morphism $\psi^* \colon \mathrm{QSym} \to A$ satisfying equation (3.107) for all $n \in \mathbb{Z}_{>0}$. Taking the dual of ϕ^* yields

$$\psi(a) = \sum_{\alpha \in \mathrm{Comp}} (H_\alpha, \psi(a)) M_\alpha. \tag{3.108}$$

For any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, the projection to M_α in equation (3.108) can be expressed as

$$\begin{aligned}
(H_\alpha, \psi(a)) &= (\psi^*(H_\alpha), a) && (3.109)\\
&= (\psi^*(H_{\alpha_1}) \cdots \psi^*(H_{\alpha_\ell}), a) && (3.110)\\
&= (\psi^*(H_{\alpha_1}) \otimes \cdots \otimes \psi^*(H_{\alpha_\ell}), \Delta^{(\ell-1)}(a)) && (3.111)\\
&= (\zeta^{\otimes \ell} \circ \pi_\alpha)(\Delta^{(\ell-1)}(a)) && (3.112)\\
&= \zeta_\alpha(a), && (3.113)
\end{aligned}$$

and hence formula (3.105) follows.

When $\zeta \colon A \to k$ is a character and A is a Hopf algebra, we need to show that $\psi \colon A \to \mathrm{QSym}$ is an algebra morphism, i.e. the maps $\psi \circ m$ and $m \circ (\psi \otimes \psi) \colon A \otimes A \to \mathrm{QSym}$ coincide. This follows from the universal property applied to the above compositions,

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, "\zeta"] & \mathrm{QSym} \arrow[d, "\zeta_Q"] \\ A \otimes A \arrow[u, "m"] \arrow[r, "\zeta\otimes\zeta"'] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{QSym}^{\otimes 2} \arrow[r, "m"] \arrow[dr, "\zeta_Q\otimes\zeta_Q"] & \mathrm{QSym} \arrow[d, "\zeta_Q"] \\ A \otimes A \arrow[u, "\psi\otimes\psi"] \arrow[r, "\zeta\otimes\zeta"'] & k \end{tikzcd} \tag{3.114}$$

where the two diagrams commute since ζ and ζ_Q are algebra morphisms and the uniqueness part above, applied to the character $\zeta \otimes \zeta \colon A \otimes A \to k$, ensures that $\psi \circ m = m \circ (\psi \otimes \psi)$.

For the last assertion, consider the explicit description of $\psi(a)$ in equation (3.106) when A is also cocommutative, and note that for any rearrangement β of α, we have $\zeta_\alpha = \zeta_\beta$. ■

3.3 Hopf Algebra of Graphs

Definition 3.36. An *isomorphism* of simple graphs $G = (V, E)$ and $G' = (V', E')$ is a bijective map $f \colon V \to V'$ such that $v, w \in V$ are adjacent in G if and only if $f(v)$ and $f(w)$ are adjacent in G'. Consider the free k-module $\mathcal{G}$ on isomorphism classes of finite simple graphs, where again, k denotes an algebraically closed field of characteristic zero, primarily $\mathbb{C}$ or $\mathbb{C}(t)$. We define *multiplication* of two classes of simple graphs to be the isomorphism class of their disjoint union. The *comultiplication* $\Delta \colon \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ is defined by

$$\Delta[\mathcal{G}] \coloneqq \sum_{(V_1, V_2)} [G|_{V_1}] \otimes [G|_{V_1}], \tag{3.115}$$

where the sum runs over all partitions $V_1 \cup V_2 = V$, and the restriction $G|_{V_i}$ is the induced subgraphs by the vertex set V_i. For example, we have

$$\Delta[\bullet\!-\!\bullet\!-\!\bullet] = 1 \otimes [\bullet\!-\!\bullet\!-\!\bullet] + 2[\bullet] \otimes [\bullet\!-\!\bullet] + 2[\bullet\!-\!\bullet] \otimes [\bullet] \tag{3.116}$$

$$+2[\bullet] \otimes [\bullet \quad \bullet] + 2[\bullet \quad \bullet] \otimes [\bullet] + [\bullet\!-\!\bullet\!-\!\bullet] \otimes 1. \tag{3.117}$$

Furthermore, the counit $\epsilon\colon \mathcal{G} \to k$ will be $\epsilon[G] = 1$ if G is the empty graph and 0 otherwise.

Remark 3.37. To see that the above operations give rise to a bialgebra structure on $\mathcal{G}$, we need to verify the identities in Definition 3.1 directly by definition of the operations. For instance, note that if V_1 and V_2 are disjoint vertex sets and $G|_{V_1} = G_1$, $G|_{V_2} = G_2$, then we have

$$\Delta \circ m([G_1] \otimes [G_2]) = \sum_{(V_{11}, V_{12}, V_{21}, V_{22})} [G_1|_{V_{12}} \cup G_2|_{V_{21}}] \otimes [G_1|_{V_{11}} \cup G_2|_{V_{22}}] \tag{3.118}$$

$$= (m \otimes m) \circ (\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)([G_1] \otimes [G_2]), \tag{3.119}$$

where the sum runs over all pairs of partitions $(V_1 = V_{11} \cup V_{12}, V_2 = V_{21} \cup V_{22})$.

Note that $\mathcal{G}$ is commutative and cocommutative, since $G_1 \cup G_2 = G_2 \cup G_1$, and $\mathcal{G}$ is finite type, since for each graded part $\mathcal{G}_n$, there are only finitely many isomorphism classes of simple graphs of order n.

Although not necessary for what follows, it is worth noting that the multiset of graphs obtained by deleting vertices (as opposed to edges, as discussed in Remark 1.10) can be recovered from the comultiplication operation.

Remark 3.38. Since $\mathcal{G}$ is a graded connected bialgebra, there is a unique (graded) antipode. Moreover, Takeuchi's formula from Remark 3.15 takes the form

$$S[G] = \sum_F (-1)^{|V| - \mathrm{rank}(F)} \mathrm{acyc}(G/F)[G_{V,F}], \tag{3.120}$$

where the sum runs over all subsets F of the edge set E that form the *graphic matroid* for G, i.e. if $e = \{v, v'\} \in E$ such that F has a path from v to v', then $e \in F$. The graph G/F is the *quotient graph*, i.e. all edges of F are contracted, $\mathrm{acyc(G/F)}$ denotes the number of acyclic orientations, $G_{V,F}$ denotes the subgraph of G spanned by F, and *rank* of F is the maximum cardinality of a subset F' of F such that $G_{V,F'}$ is acyclic. We will not use this combinatorial description of the antipode in what follows; its proof can be found in [HM], Theorem 3.1. However, the statement prompts an interesting observation.

Recall from Remark 1.55 that the quantity $\mathrm{acyc(G/F)}$ is also featured in an explicit description of the e-expansion of chromatic symmetric functions. The above description of the antipode motivates the introduction of a new basis for $\mathcal{G}$, which we describe briefly, in the following remark.

Mimicking the Möbius inversion formula for posets, as in Section 3.7 of [St1], given a finite graph $G = (V, E)$, we can define the class

$$[G]^{\#} := \sum_{\substack{H=(V,E') \\ E' \supset E^c}} (-1)^{|E' \setminus E^c|} [H] \in \mathcal{G}, \tag{3.121}$$

where E^c denotes the complement of E in the edge set of the corresponding complete graph. Note that $[G]^{\#}$ depends only on the isomorphism class $[G]$. Every finite graph $G = (V, E)$ satisfies

$$[G] = \sum_{\substack{H=(V,E') \\ E' \cap E = \emptyset}} [H]^{\#}. \tag{3.122}$$

The elements $[G]^{\#}$, where $[G]$ ranges over all isomorphism classes of finite graphs, also form a basis of $\mathcal{G}$ as a free k-module. For any graph $H = (V, E)$, the comultiplication can be described on the new basis as

$$\Delta[H]^{\#} = \sum_{\substack{(V_1,V_2) \\ V=V_1 \sqcup V_2 \\ H=H|_{V_1} \sqcup H|_{V_2}}} [H|_{V_1}]^{\#} \otimes [H|_{V_2}]^{\#}. \tag{3.123}$$

Furthermore, for any two finite graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, the structure constants of multiplication under this new basis can be described as

$$[H_1]^{\#}[H_2]^{\#} = \sum_{\substack{H=(V_1 \sqcup V_2, E) \\ H|_{V_1}=H_1 \\ H|_{V_2}=H_2}} [H]^{\#}. \tag{3.124}$$

These formulas are reminiscent of the Littlewood-Richardson rule (from Remark 3.9), and comultiplication of Schur functions (from Example 3.8). But they will not appear later on, so we refer the reader to [GR], Proposition 7.3.9 for further details.

Construction 3.39. There is a natural generalization of the distinguished character, which appears in the universal property of graded connected Hopf algebras, in Theorem 3.35. Let $f \in \mathrm{QSym}$, then the *principal specialization* $\mathrm{ps}^1(f)(m)$ of f at m is defined by evaluating the first m variables at 1 and the rest at 0.

Note that this map produces the chromatic polynomial, when f is the chromatic quasi-symmetric function of a graph.

For any $f \in \mathrm{QSym}$, there is a unique element in $\mathbb{C}[x]$ which takes the values $\mathrm{ps}^1(f)(m)$ on positive integers. One can see this by calculating $\mathrm{ps}^1(M_\alpha)(m)$ on the monomial basis,

$$\mathrm{ps}^1(M_\alpha)(m) = M_\alpha(1, \ldots, 1, 0, 0, \ldots) = \sum_{1 \le i_1 < \cdots < i_\ell \le m} [x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}]_{x_j=1}, \tag{3.125}$$

where we evaluated all variables at 1, in the last sum. This expression is $\binom{m}{\ell}$, which is a polynomial in m of degree ℓ, the length of α. Hence, it is determined by the prescribed (infinitely many) values.

Principal specialization is a Hopf algebra map from QSym to $\mathbb{C}[x]$ endowed with the Hopf algebra structure described in Example 3.4. To see this, first note that any evaluation map is an algebra morphism. One can show that it is also a coalgebra morphism by computing both sides of the following identity on the monomial basis. On the one hand, we have

$$(\Delta \circ \mathrm{ps}^1)(M_\alpha) = \Delta\binom{m}{\ell} = \binom{m \otimes 1 + 1 \otimes m}{\ell}. \tag{3.126}$$

By Vandermonde summation, this can be expressed as

$$\sum_{k=1}^{\ell} \binom{m \otimes 1}{k}\binom{1 \otimes m}{\ell - k} = \sum_{k=0}^{\ell} \binom{m}{k} \otimes \binom{m}{\ell - k}. \tag{3.127}$$

On the other hand, we have

$$((\mathrm{ps}^1 \otimes \mathrm{ps}^1) \circ \Delta)(M_\alpha) = \sum_{k=0}^{\ell} \mathrm{ps}^1(M_{(\alpha_1,\ldots,\alpha_k)}) \otimes \mathrm{ps}^1(M_{(\alpha_{k+1},\ldots,\alpha_\ell)}) \tag{3.128}$$

$$= \sum_{k=1}^{\ell} \binom{m}{k} \otimes \binom{m}{\ell - k}. \tag{3.129}$$

Hence, principal specialization is a bialgebra morphism between two Hopf algebras, and as a result, a Hopf algebra morphism.

Construction 3.40. Analogously, one can consider the Hopf algebra of ordered graphs $\tilde{\mathcal{G}}$, where the vertices of a graph of order n are labelled by the set $[n]$. Multiplication, comultiplication, unit and counit can be defined in the same way as the operations on isomorphism classes of graphs. The order on the disjoint union obtained by multiplication $m(G \otimes H)$ for two ordered graphs G and H with vertex sets $\{g_1 < \cdots < g_a\}$ and $\{h_1 < \cdots < h_b\}$, respectively, is given by $\{g_1 < \cdots < g_a < h_1 < \cdots < h_b\}$.

However, since we would like to emulate the construction of chromatic quasi-symmetric functions, it is expedient to introduce a *twist* in the comultiplication formula. We define the r-fold comultiplication by

$$\Delta_r(G) = \sum_{\kappa: V \to [r]} t^{\mathrm{asc}(\kappa)} G|_\kappa. \tag{3.130}$$

One can check that this new operation also endows ordered graphs with a graded connected bialgebra structure, and hence a Hopf algebra structure by comparing the suitable formulas, as in Remark 3.37. The only difference is that one has to keep track of the twist in the comultiplication. The explicit computation can be found in [GP], Proposition 11. Now, we can leverage the universal property of graded connected Hopf algebras in the Hopf algebra of ordered graphs, and compute the induced map for any particular character.

Lemma 3.41. *Let us consider the character* $\zeta_0 \colon \tilde{\mathcal{G}} \to \mathbb{C}(t)$ *given by*

$$\zeta_0(G) := \begin{cases} 1 & \text{if } G \text{ is edgeless,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.131}$$

Then, for every ordered graph G, the induced map from Theorem 3.35 is the chromatic quasi-symmetric function,

$$\psi_0(G) = X(G; x, t).$$

Proof. To see this, we need to unravel the explicit construction of $\psi_0(G)$. By Theorem 3.35, for any composition $\alpha = (\alpha_1, \ldots, \alpha_r)$, the coefficient of the monomial quasi-symmetric function M_α in $\psi_0(G)$ is given by

$$\underbrace{(\zeta_0 \otimes \cdots \otimes \zeta_0)}_{r\text{-fold}} \circ (\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \Delta_r(G). \tag{3.132}$$

Using the defining formula of comultiplication in $\tilde{\mathcal{G}}$, this expression turns into

$$\sum_{\kappa: V \to [r]} t^{\mathrm{asc}(\kappa)} (\zeta_0 \circ \pi_{\alpha_1}(G|_{V_1})) \otimes \cdots \otimes (\zeta_0 \circ \pi_{\alpha_r}(G|_{V_r})), \tag{3.133}$$

where $V_i = \kappa^{-1}(i)$ and π_k is the projection onto the kth homogeneous component. By definition of ζ_0, a summand in (3.133) is nonzero if and only if the order of each V_i is α_i and κ is a proper colouring, i.e. there are no monochromatic edges. Thus, the coefficient of M_α in $\psi_0(G)$ is given by

$$\sum_{\substack{\kappa: V \to [r] \text{ proper} \\ |V_i| = \alpha_i}} t^{\mathrm{asc}(\kappa)}. \tag{3.134}$$

Let us write the monomial quasi-symmetric function M_α as

$$M_\alpha = \sum_{j_1 < \cdots < j_r} x_{j_1}^{\alpha_1} \cdots x_{j_r}^{\alpha_r}, \tag{3.135}$$

and let $\kappa' : V \longrightarrow \mathbb{Z}_{>0}$ be a proper colouring using colours j_i, each of them α_i times, respectively. Each such colouring can be factored uniquely as $V \xrightarrow{\kappa} [r] \xhookrightarrow{\iota} \mathbb{Z}_{>0}$, where κ is a proper colouring that uses α_i times each colour i, and ι is an order preserving inclusion. Hence, we have expressed $\psi_0(G)$ as

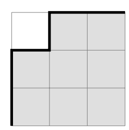
$$\psi_0(G) = \sum_{\substack{\kappa: V \to \mathbb{Z}_{>0} \\ \text{proper}}} t^{\operatorname{asc}(\kappa)} x^{\kappa}, \tag{3.136}$$

which is the chromatic quasi-symmetric function $X(G; x, t)$. ∎

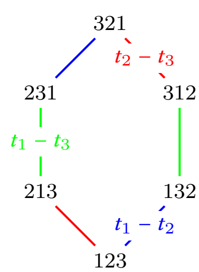
Remark 3.42. Recall from Construction 1.68, part (c), that an ordered graph $G = ([n], E, <)$ is the incomparability graph of a Dyck path if and only if $ij \in E$ implies that $i'j' \in E$ for any $i \leq i' < j' \leq j$. Note that this closure condition ensures that the set $\mathcal{D}$ of such graphs is closed under multiplication and comultiplication operations of $\tilde{\mathcal{G}}$. Hence, $\mathcal{D}$ is a graded connected Hopf subalgebra of $\tilde{\mathcal{G}}$.

Construction 3.43. Let us revisit flow-up classes, the dot action and the Frobenius character of this action from a Hopf algebraic point of view.

Recall from Construction 2.20 that the moment graph of a Hessenberg variety takes into account both the left and the right regular action of the symmetric group. For instance, the moment graph of the permutohedral variety of rank 3, which is determined by the Dyck path $(2, 3, 3)$,



is given by

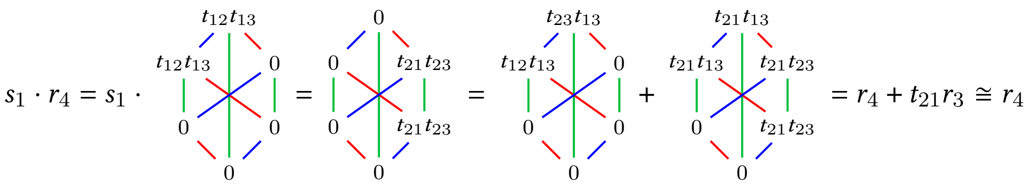


where the vertices correspond to permutations in S_3, the edges correspond to right-multiplication by the transpositions (12) and (23), and the edge labels correspond to left-multiplication by the transpositions (12) and (23) and (13). Recall from Construction 2.20 that we only take into account right-multiplication by transporitions (ij), where the cell with coordinates (i, j) lies below the Dyck path and above the diagonal.

Therefore, it is expedient to introduce two ordered sets of variables $L = (L_1, \ldots, L_n)$ and $R = (R_1, \ldots, R_n)$. Let us identify $\mathbb{C}[L]$ with $H_T^*(\mathrm{pt})$, and recall from Construction 2.9 that we have a tower of graded connected $\mathbb{C}$-algebras $\mathbb{C} \subset \mathbb{C}[L] \subset H_T^*(X(h))$, where $H_T^*(X(h))$ is a free module over $\mathbb{C}[L]$. By dint of GKM theory, we can express the ordinary cohomology as the quotient

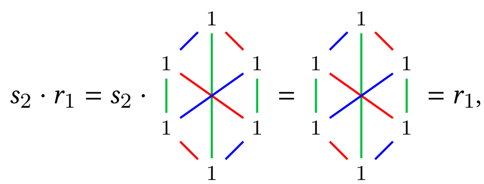
$$H^*(X(h)) \cong H_T^*(X(h))/\langle L \rangle, \tag{3.137}$$

which we also pointed out in Construction 2.9, and utilised in Example 2.26 and 2.28. Here, $\langle L \rangle$ denotes the $\mathbb{C}$-submodule of $H_T^*(X(h))$ generated by the labelled moment graphs, where all vertices are assigned the label L_i. For examples, we saw that

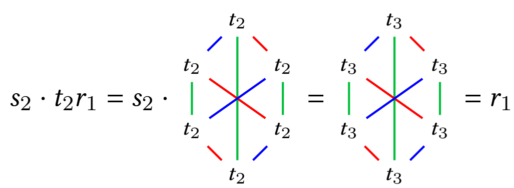


(3.138)

Recall from Construction 2.9 that the dot action fixes $\mathbb{C}$ pointwise, while it preserves $\mathbb{C}[L]$ with a twisted $\mathbb{C}$-algebra automorphism, given by $w \cdot L_i = L_{w(i)}$, e.g., in Example 2.28 we computed



(3.139)



(3.140)

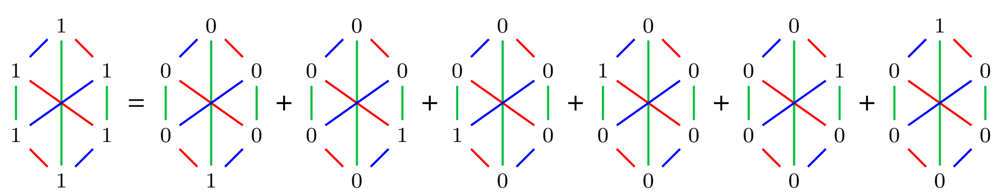
As a result, the dot action is twisted $\mathbb{C}[L]$-linear.

Let us denote the permutation group of L by $S(L)$ and the permutation group of R by $S(R)$. Furthermore, we denote the set of bijections from R to L by $S(L \leftarrow R)$. Guay-Paquet used this notation in [GP] to emphasize that $S(L)$ acts on $S(L \leftarrow R)$ by composition on the left.

It will be useful to think of the tuples of polynomials associated to the vertices in the moment graph of $X(h)$ as elements of the product

$$\mathcal{T} \cong \prod_{\alpha \in S(L \to R)} \mathbb{C}[L]. \tag{3.141}$$

Let 1_α denote the element of $\mathcal{T}$, that has label 1 at the vertex corresponding to α and 0 elsewhere. Then the unit element of $\mathcal{T}$ can be written as $1 = \sum_\alpha 1_\alpha$. For example,



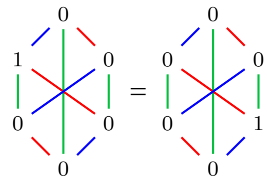
(3.142)

We shall identify $\mathbb{C}[L]$ with the subring $\mathbb{C}[L] \cdot 1$ of $\mathcal{T}$. Any element $f \in \mathcal{T}$ can be written as

$$f = \sum_{\alpha \in S(L \leftarrow R)} f_\alpha(L_1, \ldots, L_n)\, 1_\alpha. \tag{3.143}$$

The element $f \in \mathcal{T}$ is *homogeneous* if each f_α is homogeneous. This grading turns $\mathcal{T}$ into a graded connected $\mathbb{C}$-algebra. Note that $S(R)$ acts $\mathbb{C}[L]$-linearly on $\mathcal{T}$ by $1_\alpha \cdot w = 1_{\alpha \circ w}$ for any $w \in S(R)$. For example, we have

$s_1 \cdot 1_{231} = s_1 \cdot$



$= 1_{132} = 1_{s_1(231)}.$ (3.144)

One can also regard the ring $\mathcal{T}$ as a product of $\mathbb{C}[R]$'s, given by

$$\mathcal{T} \cong \prod_{\alpha \in S(L \leftarrow R)} \mathbb{C}[R], \tag{3.145}$$

but the two products (3.141) and (3.145) are not naturally isomorphic. We pick the isomorphism given by $1_\alpha\, R_i = \alpha(R_i)\, 1_\alpha$ for every $\alpha \in S(L \leftarrow R)$ and every variable R_i. If we express an element $f \in \mathcal{T}$ as

$$\mathcal{T} = \prod_{\alpha \in S(L \leftarrow R)} 1_\alpha\, g_\alpha(R_1, \dots, R_n), \tag{3.146}$$

then the above isomorphism can be expressed as $f_\alpha(L_1, \dots, L_n) = g_\alpha(\alpha(R_1), \dots, \alpha(R_n))$. This way, $\mathcal{T}$ can also be endowed with a right $\mathbb{C}[R]$-module structure, via the action $w \cdot 1_\alpha = 1_{w \circ \alpha}$ for any $w \in S(L)$. Note that the left action by $S(L)$ commutes with the right action by $S(R)$, and we have twisted $\mathbb{C}[L]$ and $\mathbb{C}[R]$ linearity,

$$w \cdot (L_i 1_\alpha) = w(L_i) 1_{w \cdot \alpha} \qquad \text{and} \qquad (1_\alpha R_i) \cdot w = 1_{\alpha \circ w} w(R_i). \tag{3.147}$$

For a Dyck path h, let $\mathcal{T}_h$ denote the subring of $\mathcal{T}$ that satisfy the edge conditions of the moment graph $M(h)$ of $X(h)$, for a regular semisimple Hessenberg variety corresponding to h. Recall from Construction 2.22 that $\mathcal{T}_h$ has a homogeneous basis given by flow-up classes, so that $\mathcal{T}_h$ is a free graded module of rank $n!$ over both $\mathbb{C}[L]$ and $\mathbb{C}[R]$.

Let us denote by the graded Frobenius character (Definition 1.51) of these two actions by $\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L])$ and $\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])$, respectively.

Lemma 3.44. *The maps $\mathcal{D} \to \mathrm{Sym}[t]$, given by*

$$\mathrm{Frob}_R \colon G(h) \to \mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R]) \quad \textit{and} \quad \mathrm{Frob}_L \colon G(h) \to \mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L]) \tag{3.148}$$

are maps of graded connected Hopf algebras.

Proof. Since both $\mathcal{D}$ and Sym are graded connected Hopf algebras, it suffices to show that Frob_R and Frob_L respect multiplication and comultiplication, as well as the grading of $\mathcal{D}$ and Sym.

Recall from Example 1.71 that the indifference graph of a Dyck path is disconnected if and only if the Dyck path touches the diagonal. Assume that we have a decomposition of $G(h)$ of the form $G(h_1) \uplus \cdots \uplus G(h_r)$, where the $G(h_i)$ themselves are not necessarily connected and r is some natural number. Let us denote the Dyck path corresponding to $G(h_i)$ by $h_i \colon n_i \to n_i$, so that $n = n_1 + \cdots + n_r$, where n is the rank of the Dyck path $h \colon [n] \to [n]$. Then the r-fold multiplication is given by $m_r(G(h_1) \otimes \cdots \otimes G(h_r)) = G(h)$.

If we introduce new sets of variables $L_i := (L_{i,1}, \dots, L_{i,n_i})$ and $R_i = (R_{i,1}, \dots, R_{i,n_i})$ for each $i \in [r]$, then $\mathcal{T}_{h_1} \otimes \cdots \otimes \mathcal{T}_{h_r}$ is a module over both $\mathbb{C}[L]$ and $\mathbb{C}[R]$, where the action is induced by the Young subgroup $Y_L := S(L_1) \times \cdots \times S(L_r) \subset S(L)$ corresponding to the composition of n given by L_i. Analogously, we denote by $Y_R := S(R_1) \times \cdots \times S(R_r) \subset S(R)$ the Young subgroup corresponding to the composition of n given by R_i. Later we will consider orbits of Young subgroups corresponding to other compositions too.

If we can show that
$$\prod_{i\in[r]} \mathrm{Frob}(S(L_i), \mathcal{T}_{h_i}, \mathbb{C}[L_i]) = \mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L]), \tag{3.149}$$
then we see that multiplication is respected by Frob_L (and analogously, by Frob_R).

In other words, we need to show that
$$\mathcal{T}_h \cong \mathrm{Ind}_{S(L_1)\times\cdots\times S(L_r)}^{S(L)}(\mathcal{T}_{h_1}\otimes\cdots\otimes\mathcal{T}_{h_r}). \tag{3.150}$$

First, note that the moment graph $M(h)$ has at least r connected components, by Construction 2.20, since the Dyck path h touches the diagonal at least $r-1$ times. Let $\alpha_0 \in S(L \leftarrow R)$ denote the element given by $\alpha_0(R_i) = L_i$ for all $i \in [n]$. Recall from Construction 3.43 that α_0 can be regarded as a vertex of $M(h)$. Let us denote its Y_R-orbit by $\mathcal{O}$. Then the restriction of $M(h)$ to $\mathcal{O}$ is the Cartesian product $M(h_1)\times\cdots\times M(h_r)$. Consequently, the $\mathbb{C}$-linear map $\mathcal{T}_{h_1}\otimes\cdots\otimes\mathcal{T}_{h_r} \to \mathcal{T}_h|_{\mathcal{O}}$, where $\mathcal{T}_h|_{\mathcal{O}}$ denotes the subspace whose elements have zeros for coordinates outside $\mathcal{O}$, takes flow-up classes to flow-up classes. Hence, it is an isomorphism.

The subspace $\mathcal{T}_h|_{\mathcal{O}}$ is preserved by the Y_L-action, since $\mathcal{O}$ is the orbit of α_0 under the Y_L-action. Let $w_1,\ldots,w_k$ be coset representatives for Y_L in $S(L)$, then we can decompose $\mathcal{T}_h$ as
$$\mathcal{T}_h = \bigoplus_{i\in[k]} w_i \cdot \mathcal{T}_h|_{\mathcal{O}}, \tag{3.151}$$
which is the desired induced representation. Hence, the maps Frob_R and Frob_L are compatible with multiplication, as required.

To show that Frob_R and Frob_L respect comultiplication, let us compare the formulas for comultiplication in $\mathcal{D}$ and Sym. In the Hopf algebra $\mathcal{D}$ of ordered graphs arising from Dyck paths, we have
$$\Delta_r(G(h)) = \sum_{\kappa:[n]\to[r]} t^{\mathrm{asc}(\kappa)} G(h)|_\kappa, \tag{3.152}$$
and in the Hopf algebra Sym, we have
$$\Delta_r(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L]) = \sum_\alpha \mathrm{Frob}(Y_{L,\alpha}, \mathcal{T}_h, \mathbb{C}[L]), \tag{3.153}$$
where the sum runs over all compositions $\alpha = (\alpha_1,\ldots,\alpha_r)$ of n, of length r, and $Y_{L,\alpha}$ denotes the Young subgroup corresponding to α. One can partition the set of colourings according to the compositions determined by the number of times each colour is being used, and rewrite the comultiplication formula (3.152) by splitting the sum into (not necessarily proper) colourings $\kappa\colon [n]\to[r]$,
$$\Delta_r(G(h)) = \sum_\alpha \sum_{\substack{\kappa\colon [n]\to[r] \\ \text{of type } \alpha}} t^{\mathrm{asc}(\kappa)} G(h)|_\kappa. \tag{3.154}$$

We need to show the two formulas in (3.153) and (3.154) coincide, and thereby show that comultiplication is respected by the map Frob_L. The computation for Frob_R is analogous.

For any $\beta \in S(L \leftarrow R)$, the orbit $\mathcal{O}_\beta$ under the action of the Young subgroup $Y_{L,\alpha}$ on the vertices of the moment graph $M(h)$ consists of those $\beta' \in S(L \leftarrow R)$ with $\beta^{-1}(L_i) = \beta'^{-1}(L_i)$ for all $i \in [r]$. Let us denote the set $\beta^{-1}(L_i)$ by $R_{i,\beta}$ for any $\beta \in S(L\leftarrow R)$ and $i \in [r]$.

By the combinatorial description of the moment graph $M(h)$ in Construction 2.20, if the directed edge ij is an ascent of the colouring κ, then every vertex of the orbit $\mathcal{O}_\beta$ has an incoming edge labelled by (R_i, R_j).

Similarly, if ij is a descent of κ, then every vertex of O_β has an outgoing vertex labelled by (R_i, R_j). On the other hand, if ij is a monochromatic edge for κ, then the edges labelled by (R_i, R_j) form a perfect matching of O_β. In other words, the subgraph of $M(h)$ induced by O_β can be written as the Cartesian product $M(h)|_{O_\beta} \cong M(h_1) \times \cdots \times M(h_r)$, where the h_i are the restricted Dyck paths determined by the restriction $G(h)|_\kappa = (G(h_1), \ldots, G(h_r))$. Moreover, if there is a directed edge in $M(h)$ from β to β', then there is also a directed edge from any vertex of O_β to a distinct vertex of $O_{\beta'}$, and consequently, the quotient $M(h)/Y_{L,\alpha}$ is a directed acyclic graph.

Let us denote the orbits of the $Y_{L,\alpha}$-action on $M(h)$ by $O_{\beta_1}, \ldots, O_{\beta_k}$, listed in a way that there are no directed edges from O_{β_i} to O_{β_j} if $i < j$, which is feasible, since $M(h)/Y_{L,\alpha}$ is acyclic. Then we have a chain of nested ideals,

$$\mathcal{I}_1 := \mathcal{T}_h|_{O_{\beta_1}} \subseteq \mathcal{I}_2 := \mathcal{T}_h|_{O_{\beta_1} \cup O_{\beta_2}} \subseteq \cdots \subseteq \mathcal{I}_k := \mathcal{T}_h|_{O_{\beta_1} \cup \cdots \cup O_{\beta_k}}, \tag{3.155}$$

where the restrictions again denotes those tuples of polynomials in $\mathcal{T}_h$ that are zero for all vertices of the moment graph $M(h)$ outside the restriction. The projections

$$\mathcal{T}_h \to \mathcal{T}_h/\mathcal{I}_1 \to \mathcal{T}_h/\mathcal{I}_2 \to \cdots \to \mathcal{T}_h/\mathcal{I}_k = 0 \tag{3.156}$$

are equivariant with respect to the $\mathbb{C}[L]$ and $\mathbb{C}[R]$-actions, as well as the $Y_{L,\alpha} \subseteq S(L)$-action. Denote the kernel of the ith projection by K_i, and the ith orbit by O_i with induced subgraph

$$M(h)|_{O_i} \cong M(h_1) \times \cdots \times M(h_r) \tag{3.157}$$

determined by the colouring $G(h)|_\kappa = (G(h_1), \ldots, G(h_r))$. Note that $\mathcal{T}_{h_1} \otimes \cdots \otimes \mathcal{T}_{h_r}$ are r-tuples of polynomials in $\mathbb{C}[R_1] \otimes \cdots \otimes \mathbb{C}[R_r] \cong \mathbb{C}[R]$. So we can construct a $\mathbb{C}$-linear map,

$$\varphi \colon \mathcal{T}_{h_1} \otimes \cdots \otimes \mathcal{T}_{h_r} \to K_i, \tag{3.158}$$

sending a tuple of vertices in $M(h)|_{O_i} = M(h_1) \times \cdots \times M(h_r)$ to the corresponding vertex of O_i, and multiplying each coordinate by

$$\prod_{ij \in \mathrm{Asc}(\kappa)} (R_i - R_j), \tag{3.159}$$

where the product runs over all ascents ij of κ on $G(h)$. Indeed, φ maps to the kernel K_i since the edge conditions of the restriction $M(h)|_{O_i} = M(h_1) \times \cdots \times M(h_r)$ are satisfied, and since multiplication by the product (3.159) ensures that all edge condition are satisfied on directed edges from β to β', where $\beta' \in O_i$ and $\beta \notin O_i$.

In fact, φ maps flow-up vectors to flow-up vectors, which implies that φ is an isomorphism. This shows that Formulas (3.153) and (3.154) coincide, and the degree shift comes from the product (3.159). ∎

Remark 3.45. Recall from Formula 3.105 in Theorem 3.35 that the Hopf algebra maps Frob_L and Frob_R are determined by the values of their multiplicative characters, which are given by postcomposing Frob_L and Frob_R with the canonical character $\zeta_Q \colon \mathrm{QSym} \to \mathbb{C}(t)$, where QSym denotes the Hopf algebra of quasi-symmetric functions over the field $\mathbb{C}(t)$. Recall from Construction 3.43 that $\mathcal{T}_h$ is a twisted $\mathbb{C}[L]$-linear $S(L)$-representation, so let us compute the multiplicative character of $\omega(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R]))$.

Lemma 3.46. *For any Dyck path h, we have*

$$\omega(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])) = \psi_t(G(h)), \tag{3.160}$$

where $\psi_t \colon \tilde{\mathcal{G}} \to \mathrm{QSym}$ is the Hopf algebra homomorphism induced by the multiplicative character

$$\zeta_t \colon \tilde{\mathcal{G}} \to \mathbb{C}(t), \quad \zeta_t(G) := t^{|E(G)|}, \tag{3.161}$$

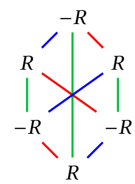
via the universal property in Theorem 3.35.

Proof. Consider the element

$$B(h) := \sum_{\beta \in S(L \leftarrow R)} (-1)^{|\mathrm{inv}(\beta)|} 1_\beta \prod_{ij \in E(G(h))} (R_i - R_j) \in \mathcal{T}_h, \tag{3.162}$$

where $\mathrm{inv}(\beta)$ denotes the set of inversions of β. For example, for $h = (3,3,3)$, we have

$$B(3,3,3) = \tag{3.163}$$



where R is the product described in (3.162).

Then $S(L)$ acts by the sign representation on the $\mathbb{C}[R]$-span of $B(h)$ in $\mathcal{T}_h$. Let us denote by $\mathcal{T}_h'$ the subspace of $\mathcal{T}_h$ on which $S(L)$ acts by the sign representation, and let

$$x = \sum_{\beta \in S(L \leftarrow R)} 1_\beta \, g_\beta(R_1, \dots, R_n) \in \mathcal{T}_h' \tag{3.164}$$

be an arbitrary element. Consider the element in the group algebra

$$\varepsilon = \sum_{w \in S(L)} (-1)^{\mathrm{sgn}(w)} w \in \mathbb{C}[S(L)] \tag{3.165}$$

that acts as the orthogonal projection onto $\mathcal{T}_h'$. Then for every transposition (ij), swapping L_i and L_j, the action can be described as $(ij) \cdot x = (ij)\varepsilon \cdot x = (-\varepsilon) \cdot x = -x$.

Consequently, the action on polynomials in the tuple x is given by $g_{(ij)\circ\beta}(R_1, \dots, R_n) = -g_\beta(R_1, \dots, R_n)$, which means that there is a unique polynomial $g(R_1, \dots, R_n)$ such that

$$x = \sum_{\beta \in S(L \to R)} (-1)^{\mathrm{inv}(\beta)} 1_\beta \, g(R_1, \dots, R_n).$$

By construction, the elements $x \in \mathcal{T}_h'$ satisfy the edge condition of the moment graph $M(h)$ if and only if $g(R_1, \dots, R_n)$ is divisible by $\prod_{ij \in E(G(h))} (R_i - R_j)$.

Recall from Remark 3.29 the representation theoretic interpretation of the coefficient of $M_{(n)}$ in the symmetric function $\omega(\mathrm{ch}(V))$, for a finite dimensional S_n-representation V: this coefficient is the dimension of the subspace V' of V on which S_n acts by the sign representation. Therefore, the corresponding the multiplicative character from Theorem 3.35 is non-trivial in the degree of V, considered a homogeneous part of a graded representation.

Note that the degree of $\mathcal{T}_h'$ within $\mathcal{T}_h$ is $|E(G(h))|$, since each vertex is assigned a polynomial, given by the product in (3.162). Consequently, by Definition 3.34, the distinguished character of this representation is

$$\zeta_Q(\omega(\mathrm{Frob}(S(L), \mathcal{T}_h', \mathbb{C}[R]))) = t^{|E(G(h))|}. \tag{3.166}$$

By the above description the sign representation is the only component of $\mathcal{T}_h$ that contributes to the term $M_{(n)}$ in the monomial expansion in the image of the graded Frobenius character $\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])$. Thus, the distinguished character of the entire graded representation $\mathcal{T}_h$ is given by the same expression,

$$\zeta_Q(\omega(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R]))) = t^{|E(G(h))|}, \tag{3.167}$$

which coincides with the multiplicative character $\zeta_t(G(h))$ from (3.161). Hence, by the universal property in Theorem 3.35, we have the desired equality,

$$\omega(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])) = \psi_t(G(h)). \tag{3.168}$$

■

Remark 3.47. In Lemma 3.41, we saw that

$$\psi_0(G) = \sum_{\substack{\kappa\colon V\to\mathbb{Z}_{>0}\\ \text{proper}}} t^{\mathrm{asc}(\kappa)} x^\kappa. \tag{3.169}$$

Similarly, we can unravel the explicit construction of ψ_t. Again, by Theorem 3.35, for any composition $\alpha = (\alpha_1, \ldots, \alpha_r)$, the coefficient of the monomial quasi-symmetric function M_α in $\psi_0(G)$ is given by

$$\underbrace{(\zeta_t \otimes \cdots \otimes \zeta_t)}_{r\text{-fold}} \circ (\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \Delta_r(G). \tag{3.170}$$

By the defining formula of comultiplication in $\tilde{\mathcal{G}}$, we obtain

$$\sum_{\kappa\colon V\to[r]} t^{\mathrm{asc}(\kappa)} (\zeta_t \circ \pi_{\alpha_1}(G|_{V_1})) \otimes \cdots \otimes (\zeta_t \circ \pi_{\alpha_r}(G|_{V_r})), \tag{3.171}$$

where $V_i = \kappa^{-1}(i)$ and π_k is the projection into the kth homogeneous component. Now we can turn to the definition of the multiplicative character,

$$\zeta_t\colon \tilde{\mathcal{G}} \to \mathbb{C}(t), \quad \zeta_t(G) := t^{|E(G)|}. \tag{3.172}$$

This time, a summand in (3.171) is nonzero if and only if the order of each V_i is α_i. The colouring κ need not be a proper colouring. Furthermore, for any colouring κ, the monomial $t^{\mathrm{asc}(\kappa)}$ in (3.171) gets multiplied by $t^{|E(G|_{V_i})|}$ for all $i \in [r]$. In other words, the induced map ψ_t can be described as

$$\psi_t(G) = \sum_{\substack{\kappa\colon V\to\mathbb{Z}_{>0}\\ \text{arbitrary}}} t^{\mathrm{weak\,asc}(\kappa)} x^\kappa, \tag{3.173}$$

where $\mathrm{weak\,asc}(\kappa)$ denotes the number of *weak ascents* of κ, i.e. the number of edges $uv \in E(G)$ with $u < v$ and $\kappa(u) \le \kappa(v)$.

To express the Frobenius character of tensor products, we will need the following operation on symmetric functions, treated extensively in [Ros].

Lemma 3.48. *For two finite dimensional S_n-representations U and V, we have*

$$\mathrm{ch}(U \otimes V) = \mathrm{ch}(U) \star \mathrm{ch}(V), \tag{3.174}$$

where $\star$ is the Kronecker product $\mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{Sym}$ *given by*

$$p_\lambda \star p_\mu = \begin{cases} z(\lambda) p_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{3.175}$$

where $z(\lambda)$ is a scaling factor.

Lemma 3.49. *The Kronecker product, from Lemma 3.48, yields the expression*

$$\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}) = \mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L]) \star \mathrm{Frob}(S(L), \mathbb{C}[L], \mathbb{C}), \tag{3.176}$$

and similarly,

$$\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}) = \mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R]) \star \mathrm{Frob}(S(L), \mathbb{C}[R], \mathbb{C}). \tag{3.177}$$

Proof. We will prove the first identity, the second one can be shown analogously. Let $\{f_i\}_{i\in[n!]}$ be a flow-up basis, which always exists by Construction 2.22. Let $\{g_i\}$ be the $\mathbb{C}$-linear basis of monomials in $\mathbb{C}[L]$. Then, by Construction 2.9, part (c), the set $\{g_j \cdot f_i\}$ is a homogenous $\mathbb{C}$-linear basis for $\mathcal{T}_h$. Let us denote the $\mathbb{C}[L]$-coefficient of f_i in $w \cdot f_i$ by a_i, for each $i \in [n!]$. Then a_i has degree 0, i.e. we have $a_i \in \mathbb{C}$. Let us denote by b_j the $\mathbb{C}$-coefficient of g_j in $w \cdot g_j$. Then the coefficient of $g_j \cdot f_i$ in $w \cdot (g_j \cdot f_i) = (w \cdot f_i)(w \cdot g_j)$, is $a_i b_j$, and we have

$$\sum_{i\in[n!]} t^{\deg(f_i g_j)} a_i b_j = \sum_{i\in[n!]} t^{\deg(f_i)} a_i \sum_{i\in[n!]} t^{\deg(g_j)} b_j, \tag{3.178}$$

and as a result, we obtain the desired formula, via the p-expansion of the graded Frobenius character, described in Remark 1.52. The proof of the second formula is analogous. ∎

Lemma 3.50. *Consider the Hopf algebra endomorphisms of* $\mathrm{Sym}[t]$ *given by* $E_t(p_{(n)}) := t^n p_{(n)}$ *for all* $n \in \mathbb{Z}_{>0}$ *and* $E_{(1-t)}(p_{(n)}) := (1-t)^n p_{(n)}$ *for all* $n \in \mathbb{Z}_{>0}$*. Then we can relate our* $\mathbb{C}[L]$ *and* $\mathbb{C}[R]$*-actions by*

$$E_{(1-t)}(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L])) = (\mathrm{id} * (S \circ E_t))(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])). \tag{3.179}$$

Proof. A monomial in $\mathbb{C}[L]$ is fixed by the action of $w \in S(L)$ if and only if for every cycle of the permutation w, the variables L_i labelled by the elements of this cycle have the same exponent.

Let us consider the power-sum expansion of $\mathrm{Frob}(S(L), \mathbb{C}[L], \mathbb{C})$ from Remark 1.52. If a_i denote the coefficient of the basis e_i in $w \cdot e_i$ for any $i \in I$, then we have

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{w\in S_n} \left(\sum_{i\in I} t^{\deg(e_i)} a_i \right) p_\lambda, \tag{3.180}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$, denotes the cycle type of w. Therefore, the summand corresponding to $w \in S(L)$ in the power-sum expansion of $\mathrm{Frob}(S(L), \mathbb{C}[L], \mathbb{C})$ is given by

$$\frac{1}{n!}(1 + t^{\lambda_1} + t^{2\lambda_1} + \cdots)(1 + t^{\lambda_2} + t^{2\lambda_2} + \cdots) \cdots (1 + t^{\lambda_\ell} + t^{2\lambda_\ell} + \cdots) p_\lambda = \frac{1}{n!} \frac{p_{(\lambda_1)}}{1 - t^{\lambda_1}} \frac{p_{(\lambda_2)}}{1 - t^{\lambda_2}} \frac{p_{(\lambda_\ell)}}{1 - t^{\lambda_\ell}}. \tag{3.181}$$

From Example 3.17, we can see that $(\mathrm{id} * (S \circ E_t))(p_k) = (1 - t^k) p_{(k)}$ for any $k \in \mathbb{Z}_{>0}$. The action of $S(L)$ on $\mathbb{C}[R]$ is the trivial representation, so the summand corresponding to $w \in S(L)$ in the power-sum expansion of $\mathrm{Frob}(S(L), \mathbb{C}[R], \mathbb{C})$ is given

$$\frac{1}{n!}(1 + t + t^2 + \cdots)^n p_\lambda = \frac{1}{n!} \frac{1}{(1-t)^n} p_\lambda = \frac{1}{n!} \frac{p_{(\lambda_1)}}{(1-t)^{\lambda_1}} \frac{p_{(\lambda_2)}}{(1-t)^{\lambda_2}} \cdots \frac{p_{(\lambda_\ell)}}{(1-t)^{\lambda_\ell}}. \tag{3.182}$$

By comparing these two p-expansions, Lemma 3.49 yields the desired equality. ∎

Theorem 3.51. *For any Dyck path* h*, we have*

$$\omega(\mathrm{Frob}(H^*(X_h))) = X(G(h); x, t),$$

where X_h *is a regular semisimple Hessenberg variety associated to* h*,* $H^*(X_h)$ *is the* S_n*-module given by the dot action on the cohomology ring of* X_h*,* Frob *is the graded Frobenius character map,* ω *is the fundamental involution,* $G(h)$ *is the indifference graph of* h *and* $X(G(h); x, t)$ *is its chromatic quasi-symmetric function.*

Proof. Now we are ready to present Guay-Paquet's proof of the Shareshian–Wachs conjecture, that we stated in Conjecture 2.27. We have seen in Lemma 3.46 that $\omega(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])) = \psi_t(G(h))$, where ψ_t is the induced Hopf algebra morphism by the character $\zeta_t \colon \tilde{\mathcal{G}} \to \mathbb{C}(t)$, $\zeta_t(G) \coloneqq t^{|E(G)|}$ via the universal property from Theorem 3.35. Moreover, we have seen in Lemma 3.41 that $\psi_0(G) = X(G; x, t)$ induced via the same universal property by the character $\zeta_0 \colon \tilde{\mathcal{G}} \to \mathbb{C}(t)$ given by

$$\zeta_0(G) \coloneqq \begin{cases} 1 & \text{if } G \text{ is edgeless,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.183}$$

Recall from Lemma 3.50 that the Hopf algebra endomorphism $E_{(1-t)}$ of Sym satisfies

$$E_{(1-t)}(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[L])) = (\mathrm{id} * (S \circ E_t))(\mathrm{Frob}(S(L), \mathcal{T}_h, \mathbb{C}[R])). \tag{3.184}$$

So it is enough to show that

$$E_{(1-t)}(\psi_0(G(h))) = (\mathrm{id} * (S \circ E_t))(\psi_t(G(h))). \tag{3.185}$$

Note that it suffices to verify this equality for nonempty, connected ordered graphs $G(h)$, i.e. whenever the Dyck path h does not touch the diagonal, because otherwise the chromatic quasi-symmetric function is the product of the chromatic quasi-symmetric functions of the components, by Construction 1.68, part (d). We can again appeal to the universal property in Theorem 3.35, and instead, show that the two sides have the same multiplicative characters, i.e.

$$\zeta_Q(E_{(1-t)}(\psi_0(G(h)))) = (\mathrm{id} * (S \circ E_t))(\psi_t(G(h))). \tag{3.186}$$

Since $E_{(1-t)}$ acts on Sym by multiplying by each homogeneous part by $(1-t)^k$, where k is the degree, and since ψ_0 respects the grading, we have

$$\zeta_Q(E_{(1-t)}(\psi_0(G(h)))) = \begin{cases} 1-t & \text{if } G(h) \text{ has one vertex,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.187}$$

Now we need to show that the canonical character of $(\mathrm{id} * (S \circ E_t))(\psi_t(G(h)))$ is given by the same expression. Let π_0 denote the projection onto the degree zero part of the graded Hopf algebra endomorphism $(\mathrm{id} * (S \circ E_t)) \in \mathrm{End}(\mathrm{Sym})$, and let π_+ denote the projection onto the positive degree part. By Takeuchi's formula from Remark 3.15, the antipode can be expressed as the r-fold convolution product

$$S = \sum_{r \geq 0} (-1)^r \underbrace{\pi_+ * \cdots * \pi_+}_{r\text{-fold}}. \tag{3.188}$$

Therefore, the map in question can be written as

$$\mathrm{id} * (S \circ E_t) = \sum_{r \geq 0} (-1)^r \mathrm{id} * \underbrace{(\pi_+ \circ E_t) * \cdots * (\pi_+ \circ E_t)}_{r\text{-fold}}. \tag{3.189}$$

Since the canonical character is multiplicative, we have

$$\zeta_Q \circ (\mathrm{id} * (S \circ E_t)) = \sum_{r \geq 0} (-1)^r (\zeta_Q \otimes \underbrace{(\zeta_Q \circ \pi_+ \circ E_t) \otimes \cdots \otimes (\zeta_Q \circ \pi_+ \circ E_t)}_{r\text{-fold}}) \circ \Delta_{r+1}. \tag{3.190}$$

We have seen in the proof of Remark 3.47, in equation (3.173), that the coefficient of a monomial quasi-symmetric function M_α in $\psi_t(G(h))$ is given by

$$\sum_{\kappa} t^{\mathrm{weak\,asc}(\kappa)},$$

where the sum runs over all colourings $\kappa\colon V \to \mathbb{Z}_{>0}$ corresponding to the composition α.

Then, by Example 3.24, the comultiplication can be expressed as

$$\Delta_{r+1}(M_\alpha) = \sum_{\substack{(\alpha^0,\ldots,\alpha^r) \\ \alpha^0\cdot\ldots\cdot\alpha^r=\alpha}} M_{\alpha^0} \otimes \cdots \otimes M_{\alpha^r}, \tag{3.191}$$

where the sum runs over the $(r+1)$-tuples of compositions whose concatenation is the composition α. When applying the canonical character, we get $\zeta_Q(M_{\alpha^i}) = 0$ if α^i has more than one part or if α^i is the empty composition. Consequently, after applying $\zeta_Q \circ (\mathrm{id} * (S \circ E_t))$ from equation (3.190) to the monomial quasi-symmetric function M_α, it becomes zero, unless

(a). α has $r+1$ parts α_i and each partition α^i in equation (3.191) is the singleton consisting of the $(i+1)$th part α_{i+1},

(b). or if α has r parts, α^0 is the empty composition and α^i is the singleton consisting of the ith part of α.

Hence, when we apply the map on the right side of equation (3.190) to the monomial quasi-symmetric function M_α, where $\alpha = (\alpha_1, \ldots, \alpha_r)$, we obtain the expression $(-1)^{r-1}t^{\alpha_2+\cdots+\alpha_r} + (-1)^r t^{\alpha_1+\cdots+\alpha_r}$. Therefore, the coefficients in (3.173) yield the identity

$$\zeta_Q(\mathrm{id} * (S \circ E_t))(\psi_t(G(h))) = \sum_{r\geq 0}\sum_{\kappa}(-1)^r t^{\mathrm{stat}(\kappa)}, \tag{3.192}$$

where the second sum runs over all colourings $\kappa\colon [n] \to \{0\} \cup [r]$ such that every colour in $[r]$ is used at least once (but not necessarily colour 0), and the colouring invariant $\mathrm{stat}(\kappa)$ in the exponent is given by

$$\mathrm{stat}(\kappa) := (\#\text{ vertices with colour} > 0) + (\#\text{ weak ascents of } \kappa \text{ on } G(h)). \tag{3.193}$$

If h is the trivial Dyck path (1), then $G(h)$ is the ordered graph with a single vertex, and in this case, expression (3.192) is $1 - t$, which tallies with equation (3.187). To see that (3.187) and (3.192) are equal, we need to show that for any other Dyck path h, the expression in (3.192) is 0. Again, it suffices to show this for connected graphs $G(h)$ by 1.68, part (d).

To this end, we would like to construct a suitable sign-reversing involution on the set of colourings $\kappa\colon [n] \to \{0\} \cup [r]$ such that every colour in $[r]$ is used at least once (but not necessarily colour 0), which preserves $\mathrm{inv}(\kappa)$, so that the all terms $(-1)^r t^{\mathrm{stat}(\kappa)}$ cancel out. Let us denote the set of such colourings by $C_{n,r}$ when both n and r are fixed, and $C_n := \bigcup_{r>0} C_{n,r}$.

The idea behind this construction resembles the proof of Theorem 1.69. Take a colouring $\kappa \in C_{n,r}$. First, we consider the case where vertex n has colour 0, and vertex $n-1$ has an arbitrary colour i. If $n-1$ is the only vertex with colour i, then we can construct a colouring κ' by giving vertex $n-1$ colour $i-1$, and thereby removing colour i entirely. Note that according to the expansion of ψ_t in equation (3.173) a colouring here need not be proper, whereas in the case of ψ_0, in equation (3.136) we only consider proper colourings.

If there is another vertex with colour i, other than vertex $n-1$, then we add a new colour lying between i and $i+1$ in our ordered set of available colours, and give vertex $n-1$ this new colour. Then, in both cases, we may relabel the set of available positive colours in an order-preserving way, such that these sets are $[r-1]$ and $[r+1]$, respectively. The maps described in the two cases are inverses of each other.

Now consider the case where vertex n is the only vertex with colour 1, and vertex $n-1$ has colour i. If there are no other vertices with colour i, then we give vertex $n-1$ colour $i-1$ if $i \neq 2$ and colour 0 if $i = 2$, and remove i from the list of available colours. If, on the other hand, vertex $n-1$ is not the only one with colour i, then we give vertex $n-1$ colour with a new colour that lies directly after colour i in our ordered set of available colours. Again, we may relabel the set of available positive colours in an order-preserving way, such that these sets are $[r-1]$ and $[r+1]$, respectively. The maps described inductively in the above two cases are inverses of each other.

Finally, we need to consider the case where vertex n has colour i different from 0 or 1. If vertex n is the unique vertex with vertex i, then we recolour it with colour $i-1$ and we discard colour i. If there are other vertices with colour i, then we recolour vertex n with a colour that is larger than all colours in $[i]$. Then, we may relabel the set of available positive colours in an order-preserving way, such that these sets are $[r-1]$ and $[r+1]$, respectively, and again, the maps described in the two cases are inverses of each other.

This concludes the construction of the desired sign-reversing involution. ■

Remark 3.52. Guay-Paquet's Hopf algebraic proof was written in 2016. However, it is worth noting that this is the second proof of the Shareshian–Wachs conjecture. A year earlier, Brosnan and Chow provided a proof, using advanced tools from geometry, such as perverse sheaves on flag varieties and nearby cycles, in the article [BC]. Both perspectives have their merits and limitations.

While this thesis focuses solely on the Hopf algebraic proof, I also intend to explore the geometric proof in the near future. Understanding the geometric approach may shed light on certain details that are somewhat obscured by the Hopf algebraic framework. For instance, it could provide insights into the the geometric interpretation of the fundamental involution ω or that of the sign reversing involution utilised in the proof of Theorem 3.51.

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