Gradings on the Brauer algebras and double affine BMW algebras

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Anna Mkrtchyan)

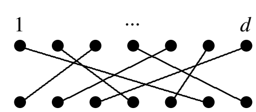
To my family

Lay Summary

In the pure mathematical field of representation theory, we study an abstract algebraic structure (e.g. finite groups) by describing its elements through matrices and its algebraic operations through multiplication of matrices. Since the theory of matrices is well-understood, it allows to obtain a better understanding of complicated objects.

In general, there can be many ways to represent an algebraic structure through matrices. Moreover, we can combine representations or, in some representations, we can find special subsets of the matrices that define representations on their own. The representations that do not have such subsets are called irreducible representations. Finding the full set of irreducible representations can give a fair understanding of the algebraic structure itself.

Furthermore, it is possible to find correspondences between different representations of algebraic structures. One of the most fundamental results of Representation theory, known as Schur-Weyl duality, connects the representations of the symmetric group with those of the group of invertible matrices. The symmetric group S_d is the group that consists of the permutation operations that can be performed on d symbols, furthermore it has an insightful diagrammatic description. Elements of the group can be presented as diagrams with d points on the top and d points on the bottom. Every point on the top is connected to exactly one point on the bottom (see an illustration below). Multiplication of diagrams is given by the concatenation fixing the top d points and the bottom d points..



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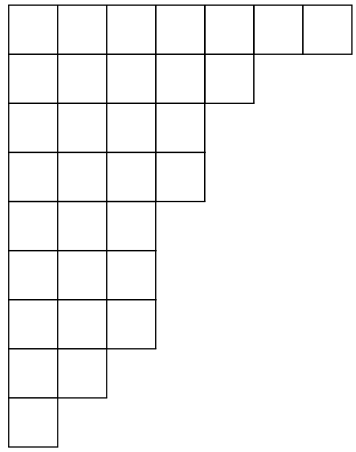
The Schur-Weyl duality gives a mysterious connection between irreducible representations of the Symmetric group and group of invertible matrices with special diagrams called *Young diagrams*. Intuitively, a Young diagram is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order (see an example below). Such diagrammatic descriptions significantly simplify the study of algebraic structures.



The Schur-Weyl duality was generalised in many different directions, where the group of invertible matrices were replaced by other fundamental structures of the Representation theory and the Symmetric group by the structures which similarly have insightful diagrammatic descriptions. Furthermore, all these generalisations similarly have deep connections to the Young diagrams.

Another way to study abstract algebraic structures, is to group their elements in certain ways, and then work with each group separately. Such a procedure is called a grading. One can define different gradings on objects. Identifying whether two gradings are the same can help to combine different approaches and obtain much deeper understanding of the object.

In the first part of the thesis, we recall the algebras that first appear in the context of the Schur-Weyl duality, called Brauer algebras, and describe two different gradings on them. The main goal of that part is to prove that the two given gradings on the Brauer algebras are the same, which gives a much richer understanding of the Brauer algebras.

In the second part of the thesis, we study another generalisation of the Schur-Weyl duality. Namely, we construct new algebras and connect their representations with representations of an abstract object that has a particular interest in Algebraic Geometry.

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Last but not least, I am indebted to my beloved family, my parents, Arevik, Hinnerk and my new-born niece Mia, for everything they have done for me. Without your love and support I would never made it through.

Abstract

In this thesis we study algebras that appear in different generalisations of the well-known Schur-Weyl duality. This classical result gives a remarkable connection between the irreducible finite-dimensional representations of the general linear and symmetric groups. In this work we are interested in the algebras appearing in similar correspondences for the orthogonal/symplectic groups and their quantum analogues.

In the first part of the thesis, we work with Brauer algebras. These algebras were first introduced by Brauer to obtain a better understanding of the irreducible finite-dimensional representations of the orthogonal/symplectic groups. Afterwards, it was shown that Brauer algebras are also important examples of graded cellular algebras. Moreover, two completely different approaches were considered to give a grading on the Brauer algebras. Our goal is to show that the two constructions give the same grading on the Brauer algebras. Namely, we give an explicit graded isomorphism between two constructions.

In the second part of thesis, we discuss generalizations of the quantised version of the previous construction, where the Brauer algebras are replaced by BMW algebras and symplectic/orthogonal groups are replaced by its quantum groups. In particular, to study specific representations of D-modules on the quantum group corresponding to the sympletic groups, we introduce *double affine BMW algebras*. Furthermore, we give some representation of these algebras and give its combinatorial description.

Contents

Chapter 1

Introduction

In this thesis we study algebras that appear in different generalisations of the well-known Schur-Weyl duality. This classical result gives a remarkable connection between the irreducible finite-dimensional representations of the general linear and symmetric groups. In this work we are interested in the algebras appearing in similar correspondences for the orthogonal/symplectic groups and their quantum analogues.

In the first part of the thesis, we work with Brauer algebras. These algebras were first introduced by Brauer to obtain a better understanding of the irreducible finite-dimensional representations of the orthogonal and symplectic groups. Later, it was shown that Brauer algebras are also important examples of graded cellular algebras. Moreover, two completely different approaches were considered to give a grading on the Brauer algebras. Our goal is to show that the two constructions give the same grading on the Brauer algebras. Namely, we give an explicit graded isomorphism between these two constructions.

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1.1 Classical Schur-Weyl duality

Let us first briefly recall a classical setting, which served as motivation to study the algebras discussed later.

Theorem 1.1.1 (Double centraliser property, [21,42])**.** *Let A, B be two subalgebras of the algebra* $\mathrm{End}E$ *of endomorphisms of a finite dimensional vector space E, such that A is semisimple and $B = \mathrm{End}_A E$. Then*

- *$A = \mathrm{End}_B E$ (i.e., the centralizer of the centralizer of A is A).*
- *B is semisimple.*
- *As a representation of $A \otimes B$, E decomposes as*

$$E = \bigoplus_{i \in I} V_i \otimes W_i,$$

where V_i are all the irreducible representations of A and W_i are all the irreducible representations of B. Furthermore, $W_i \cong \mathrm{Hom}_A(V_i, E)$ and, in particular, we have a natural bijection between irreducible representations of A and B.

One of the (non-trivial) special cases of this theorem is the well-known Schur-Weyl duality, which gives a remarkable correspondence between the irreducible finite-dimensional representations of the general linear and symmetric groups. Let us recall the setup.

Let us fix $d, n \in \mathbb{N}$ and let $V = \mathbb{C}^n$ be the fundamental representation of $\mathrm{GL_n}$. On $V^{\otimes d}$ we have two natural actions: the left action of the general linear group $\mathrm{GL_n}$ which acts diagonally and the right action of the symmetric group S_d which permutes the factors. Let ϕ, ψ be the corresponding natural representations

$$\phi : (S_d)^{\mathrm{op}} \to \mathrm{End}(V^{\otimes d}), \qquad \psi : \mathrm{GL_n} \to \mathrm{End}(V^{\otimes d}).$$

It is immediate to see that these two actions commute. Moreover, the following holds.

Theorem 1.1.2 (Classical Schur-Weyl duality, [9, 13, 40, 45, 49])**.** *We have*

a) $\phi((S_d)^{\mathrm{op}}) = \mathrm{End}_{\mathrm{GL_n}}(V^{\otimes d})$*, and if $n \geq d$ then ϕ is injective, and hence an isomorphism onto* $\mathrm{End}_{\mathrm{GL_n}}(V^{\otimes d})$*,*

b) $\psi(\mathrm{GL_n}) = \mathrm{End}_{S_d}(V^{\otimes d})$*,*

c) as a $(S_d, \mathrm{GL_n})$*-bimodule, we have the decomposition*

$$V^{\otimes d} = \bigoplus_{\lambda} S_\lambda \otimes L_\lambda,$$

where the summation is taken over partitions of d, S_λ are the Specht modules for S_d, and L_λ are some distinct irreducible representations of $\mathrm{GL_n}$ *or 0.*

Recall that Specht modules form a complete set of irreducible representations of S_d. The basis in each S_λ can be labeled by the set of standard tableaux of shape λ (see Definition 2.1.9). Moreover, the vectors of this basis are eigenvectors for the subalgebra generated by the Jucys-Murphy elements x_k of S_d, where

$$x_k = \sum_{l=1}^{k-1} (l, k), \qquad k = 1, \ldots, d,$$

and $(l, k) \in S_d$ is a transposition.

Let $\mathcal{S}$ be the category of finite-dimensional representations of the symmetric group S_d and $\mathcal{G}$ be the category of finite-dimensional representations of the general linear group $\mathrm{GL_n}$. Define a functor

$$F : \mathcal{G} \longrightarrow \mathcal{S} \tag{1.1.1}$$

$$M \mapsto \mathrm{Hom}_{\mathrm{GL_n}}(\mathbf{1}, M \otimes V^{\otimes d}). \tag{1.1.2}$$

It can be shown (see e.g [2]) that the functor is exact and takes irreducible $\mathrm{GL_n}$-modules to irreducible S_d-modules. In the later sections, we will discuss different generalizations of this functor.

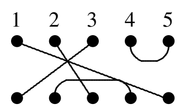
1.2 Gradings on Brauer algebras

The main focus of the first part of this thesis is laid on the Brauer algebras. These algebras were introduced by Brauer in order to study the d-th tensor power of the natural representations of orthogonal and symplectic groups, and, in particular, to generalise the Schur-Weyl duality for these groups. In particular, these algebras are centraliser algebras for the action of orthogonal and symplectic groups.

Brauer algebras have an insightful diagrammatic description, which we present here and in Section 2.4, we give more detailed description.

Let us fix δ. The Brauer algebra $B_d(\delta)$ is a diagrammatically defined algebra with a basis given by *Brauer diagrams on* $2d$ *points*, which is a matching of $2d$ points with d strands such that each point is the endpoint of exactly one strand.

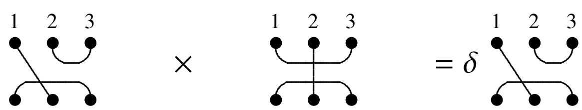
Example 1.2.1. Let $d = 5$. The following is an example of a Brauer diagram.



(1.2.1)

Diagrams are considered up to isotopy[1] in the plane fixing the top d and bottom d points. Multiplication of diagrams is given by the concatenation fixing the top d points and the bottom d points and replacing each interior loop by the multiplication by δ.

Example 1.2.2.



(1.2.2)

For $n \in \mathbb{Z}$, there exist a natural action on V^d.[2] Combining this action with a natural action of orthogonal/symplectic groups on V^d, we can obtain a completely analogous correspondence for Brauer algebras and orthogonal/symplectic groups in case $n > d$ (see Section 2.4, [5–7], also Theorem 2.4.7)[3]. In particular, we obtain that in that case $B_d(\delta)$ is semisimple. The basis of each irreducible module can be labeled by the so-called *up-down tableaux*, which are generalisations of standard tableaux (see Definition 2.1.12). Furthermore, it was shown [43] that $B_d(\delta)$ is semisimple for $\delta \notin \mathbb{Z}$, therefore here we will only consider the case $\delta \in \mathbb{Z}$.

Brauer algebras are also important examples of cellular algebras (see [22] and Chapter 2.3). A cellular algebra A is a finite-dimensional associative algebra with a distinguished cellular basis which is particularly well-adapted to studying the representation theory of A. Each cellular algebra A has a distinguished set of A-modules, called *cell modules*, which in semisimple case represent the set of irreducible modules.

Later it was shown that Brauer algebras are also graded cellular algebras (see Chapters 4, 5, [8, 16–18]). Graded cellular algebras are graded algebras with a cellular basis which is

[1] Or simply saying, the diagrams are considered to be the same or equivalent if they link the same d pairs of points.

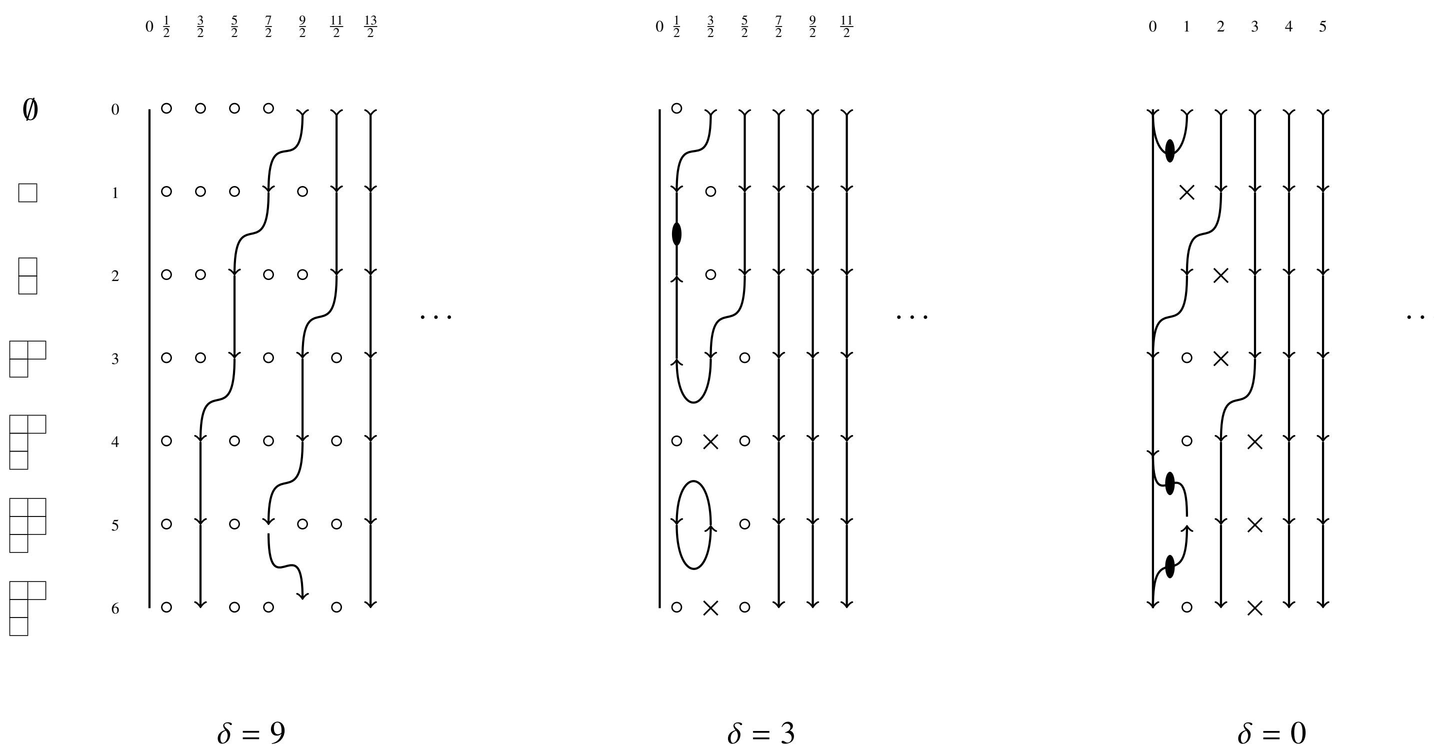
[2] In this natural action, it is crucial that $|\delta| = n = \dim(V)$.

[3] Notice that here $|n| > d$, unlike the classical case. Intuitively, it is clear that since $B_d(\delta)$ is bigger we need more generators to distinguish endomorphisms.

"compatible" with the grading (see Definition 2.3.5). The notion of graded cellular algebras was introduced by Hu and Mathas in [23], where they also showed that $\mathbb{C}S_d$ is a graded cellular algebra. Intuitively, grading on $B_d(\delta)$ shows how "far" the algebra is from being semisimple. In particular, the grading in the semisimple case is trivial.

Ehrig-Stroppel and Li independently proved that Brauer algebras are graded cellular algebras. Although their approaches are completely different, behind both of them is the usage of Jucys-Murphy elements and their actions on representations of Brauer algebras, encoded in residue sequences (see Definition 2.1.17). However, note that in contrast to the symmetric group case, representations of Brauer algebras are in general not diagonalisable under the actions of Jucys-Murrphy elements. Let us briefly discuss these two constructions.

The first graded cellular structure was given in [17] by Ehrig and Stroppel who were motivated by geometry of Springer fibers and isotropic Grassmanians. In [18] they constructed "arc algebras" $C_d(\delta)$, naturally graded algebras with an explicit basis described topologically via diagrams, called *Verma paths*, which are in one-to-one correspondence with the up-down tableaux but additionally depend on δ (please see Section 2.2 for a detailed description). To see what these diagrams look like, let us give an example of three Verma paths corresponding to the same up-down tableau but for different δ.



Since $C_d(\delta)$ is described diagrammatically, it is very intuitive. Let us name some of features of this algebra. First of all, all primitive orthogonal idempotents of the algebra are in the basis. Moreover, it is almost immediate to determine whether the product of two basis elements is zero or not. The degrees of basis elements are defined topologically and are easily read off.

In particular, it is easy to determine for which δ, d the algebra is semisimple. Furthermore, the cellular structure is clear from the construction. One main of the main challenges of this algebra is to determine the relations between elements, and in general to determine the generators and relations of the algebra. Ehrig and Stroppel [17] showed that $C_d(\delta) \cong B_d(\delta)$, which naturally gives a grading on Brauer algebras. It is worth to mention that the construction is very complicated and consists of few steps (see [17, Example 7.1], for the illustration of one of the steps for $d = 2$ in a semisimple case). Furthermore, to give explicit formulas even for rather small d is a non-trivial task. The main reason is hidden in fact that in $B_d(\delta)$ formulas for its primitive orthogonal idempotents are extremely complicated, which can be seen even in the case $d = 2$.

The second grading was given by Li in [33] and was motivated by the algebras, called *the quiver Hecke algebras* or KLR algebras, which were defined by Khovanov and Lauda in [28,29] and, independently, Rouquier in [41]. KLR algebras, R_d, are defined for each oriented quiver and categorify the positive part of the enveloping algebras of the corresponding quantum groups. These algebras are naturally $\mathbb{Z}$-graded. In [8], Brundan and Kleshchev showed that every degenerate and non-degenerate cyclotomic Hecke algebra H_d^Λ of type $G(l,1,d)$ (in particular S_d) over a field is isomorphic to a cyclotomic quiver Hecke algebra R_d^Λ of type A. KLR algebras are given by generators, which include orthogonal idempotents, and rather difficult relations (see [33, Definition 2.20]). In [33], Li constructed algebras $G_d(\delta)$ with a similar but significantly more involved presentation. Namely, he defined naturally graded algebras $G_d(\delta)$ with the set of generators $e(\mathbf{i}), y_1, \ldots, y_d, \psi_1, \ldots, \psi_{d-1}, \epsilon_1, \ldots, \epsilon_{d-1} \in G_d(\delta)$ with the idempotents of the algebra $e(\mathbf{i})$ labeled by the *residue sequences* of length d (see Definition 5.2.4). Furthermore, he showed that $G_d(\delta)$ is a graded cellular algebra and proved that $G_d(\delta) \cong B_d(\delta)$ and hence, gave another grading for Brauer algebras. The advantage of these algebras is that it gives relations between the elements of the algebra, and in particular, the elements of the cellular basis. But since the formulas are very complicated, the algebra is not very intuitive. It is difficult to determine when the product of elements is 0 and determining the degree of the elements requires computations. Furthermore, as in the previous case, since the explicit isomorphism requires formulas for idempotents of the $B_d(\delta)$, it is very complicated to write it even for rather small d.

The natural question that appears is whether these two constructions define the same grading on $B_d(\delta)$, moreover, whether there is an *explicit* graded isomorphism between algebras $C_d(\delta)$ and $G_d(\delta)$? Although the constructions are very different, the structures of the algebras are very similar. In particular, since idempotents are given in both algebras, we can expect to have a direct isomorphism between $C_d(\delta)$ and $G_d(\delta)$. In Section 6.1, we define elements

$$\bar{e}(\mathbf{i}), \bar{y}_1, \ldots, \bar{y}_d, \bar{\psi}_1, \ldots, \bar{\psi}_{d-1}, \bar{\epsilon}_1, \ldots, \bar{\epsilon}_{d-1} \in C_d(\delta)$$

as a non-trivial linear combination of basis elements and prove the following main result of the first part of the thesis.

Theorem 1.2.3. *The map* $\Phi : G_d(\delta) \longrightarrow C_d(\delta)$ *determined by*

$$e(\mathbf{i}) \mapsto \bar{e}(\mathbf{i}), \qquad y_r \mapsto \bar{y}_r, \qquad \psi_k \mapsto \bar{\psi}_k, \qquad \epsilon_k \mapsto \bar{\epsilon}_k,$$

where $1 \le r \le d, 1 \le k \le d-1$*, is an isomorphism. Moreover, for any homogeneous element* $u \in G_d(\delta)$*, we have* $\deg u = \deg \Phi(u)$*.*

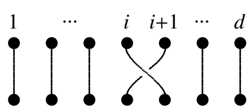
This theorem shows that we can use both approaches in parallel to combine advantages of each algebras listed before to obtain a clear and intuitive understanding of $B_d(\delta)$ and its representations even in cases when the Brauer algebras are not semisimple. We would like to stress that the isomorphism is not constructed through the isomorphisms to $B_d(\delta)$ and, in particular, does not use the previous constructions.

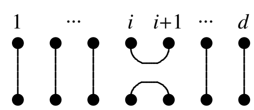
1.3 Double affine BMW algebras

The main focus of the second part of the thesis is about the newly defined algebras, called (reduced) double affine BMW algebras (which for simplicity will be called double affine BMW algebras). These algebras generalise some of the properties of double affine Hecke algebras discussed below in the Schur-Weyl analogue in type C.

Let us fix $q, z \in \mathbb{C}^*$. There are also completely analogous correspondences (as in the classical case) in quantum settings, where $\mathrm{GL_n}$ (or $\mathrm{SL_n}$) is replaced by its quantum analogue and S_d is replaced by the Hecke algebra (see [25]). Similarly, orthogonal/symplectic groups can be replaced by its quantum groups and BMW algebras play the role of $B_d(\delta)$ (see [11]). We will give more detailed explanation in Chapter 7, here we are only focusing on the intuitive description of BMW algebras and its generalisations. Furthermore, for simplicity we only discuss the generalisations for the symplectic group. [4]

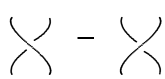
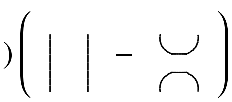
Birman-Murakami-Wenzl (BMW) algebras $\hat{W}_d$ have an insightful diagrammatic presentation similar to the presentation of Brauer algebras. The main difference is that in this presentation we distinguish over/under crossing of the lines in the diagrams. Namely, the elements of the BMW algebra can be viewed as linear combinations of *tangles*, where a tangle consists of d strands such that each strand can connect any two vertexes, but each vertex belongs to only one strand. Then the $\hat{W}_d$ is generated by $T_1, \ldots, T_{d-1}, E_1, \ldots, E_{d-1}$ where

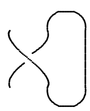
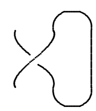
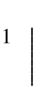
$T_i =$ 

and $E_i =$ 

. (1.3.1)

and with the relations expressed in the form

 $= (q - q^{-1})$  (1.3.2)

 $= z$ and  $= z^{-1}$  (1.3.3)

 $= \dfrac{z - z^{-1}}{q - q^{-1}} + 1.$ (1.3.4)

Similarly, multiplication is given by the concatenation of the diagrams fixing the top d points and the bottom d points.

Notice that the first relation allows us to swap over-crossing to under-crossing by adding some additional terms. Furthermore, loops are also replaced by multiplication by some parameter (see [50] for more details).

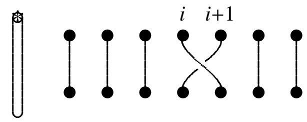
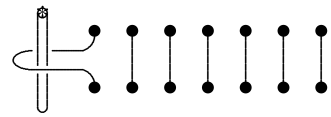
Then we can define Hecke algebra $H_d(q)$ as the quotient of $\hat{W}_d$ by the additional relations

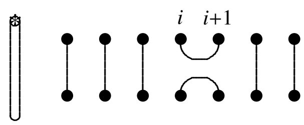
$$E_i = 0 \qquad \text{for } i = 1, \ldots, d-1.$$

[4] Notice that many of the classical results hold for generations of the orthogonal group too.

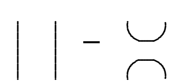
One of the important features of the quantum case is that both algebras can be seen as quotients of the group algebra of the well-known *Braid group* B_d, which is generated by $T_1^{\pm1}, \ldots, T_{d-1}^{\pm1}$ (see Chapter 7 for more details). This gives a unified way of constructing representations for H_d and $\hat{W}_d$.

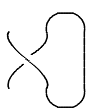
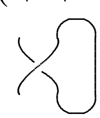
There is a natural question whether we can generalize the classical functor to a bigger category, which also includes infinite dimensional modules of the corresponding quantum group. In [39], Orellana and Ram constructed a series of functors F_λ from the category O (which also includes infinite dimensional modules, see Definition 7.0.11) to the category of finite-dimensional modules of the *affine BMW algebras*, which are infinite-dimensional algebras with the following diagrammatic description. Let the generators be given by the following tangles with a flagpole

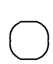
$T_i =$  and $Y^1 =$  . (1.3.5)

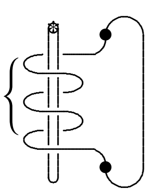
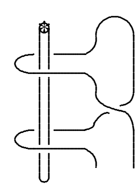
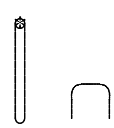
$E_i =$  (1.3.6)

Then $\hat{W}_d$ is the algebra of linear combinations of tangles generated by $T_1, \ldots, T_{d-1}, E_1, \ldots, E_{d-1}$ with the relations expressed in the form

 $-$  $= (q - q^{-1})\Big($  $\Big)$ (1.3.7)

 $= z \,|$ and  $= z^{-1} \,|$ (1.3.8)

 $= \dfrac{z - z^{-1}}{q - q^{-1}} + 1.$ (1.3.9)

ℓ loops  $= \omega_\ell$  and  $= z^{-1} \cdot$  (1.3.10)

Notice that the algebras are infinite dimensional since Y_1 can go around the flagpole any number of times. Furthermore, the subalgebra generated by T_i and E_i is isomorphic to BMW algebra. Now let us carefully look at the left relation in the last row, which seems to be the least "natural" relation. First of all, it is easy to see that the loops in that relation are central. Moreover, notice that the loops themselves are not in the algebra, the loops only can be obtained as "part" of the elements $E_1 Y_1^l E_1$. Therefore, to indicate that they are central and to simplify the calculations, these loops are set to some constants (although there are dependencies between these constants [15, Remark 2.7]).

Similarly, the affine Hecke algebra $H_d(q)$ can be defined as the quotient of $\hat{W}_d$ by the additional relations

$$E_i = 0 \qquad \text{for } i = 1, \dots, d-1.$$

Further generalisations were motivated by the double affine Hecke algebras (DAHA), which were introduced by Cherednik in [12] to prove the Macdonald's constant term conjecture for Macdonald polynomials. Simply saying, DAHA contains two copies of affine Hecke algebras, namely they are generated by elements

$$T_1, \dots, T_{d-1}, Y_1, \dots, Y_d, X_1, \dots, X_d$$

such that $T_1, \dots, T_{d-1}, Y_1, \dots, Y_d$ and $T_1, \dots, T_{d-1}, X_1, \dots, X_d$ satisfy the relations of the affine Hecke algebra with a special choice of a parameter (see Definition 7.2.6 for more details).

In [26], Jordan generalises previous constructions and defines a functor F^D from the category of D-modules on the quantum group corresponding to $U_q(\mathfrak{g})$ (see [26, Definition 10]) to the category of representations of the *elliptic braid group* (see Definition 7.2.1). The elliptic braid group can be realised as braids on the torus, or on a cube with some opposite side glued. Furthermore, Jordan showed that in case $\mathfrak{g} = \mathfrak{sl}_n$, this functor lands in the category of the representations of DAHA. Moreover, in [27], Jordan and Vazirani consider the special case of the DAHA module $F^D(\bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda)$, where the sum runs over all dominant weight of $\mathfrak{sl}_n$. They show that it is an irreducible representation of DAHA and give a combinatoral description for its basis.

Before describing our result of the second part of the thesis, let us give an informal explanation of our motivation. The construction described above gave some hints how to define double affine analogues for the quantum groups of type C. But it is worth to mention that there were numerous problems on the way. Let us name few. Recall that in the affine BMW case, we saw special central loops that were set to constants. If we think of double affine BMW algebras as tangles on a torus with "corresponding" relations, we will see that we will have loops coming from the elements $E_1 X_1^y Y_1^j E_1$, but unlike the previous case they are not central. Using the relation that allows to change over-crossing to under-crossing, we can predict the relations between loops for small numbers but the general formulas seem to be extremely hard. Moreover, since these elements are not central, we cannot set them to constants and present the predicted algebra as a quotient of the elliptic braid group. Following the same idea as in [37], we could add them as generators as long as we find their relations with other generators. Another problem that we encountered is that in this "predicted" algebra we have additional relations on affine BMW algebra which clearly suggests that we cannot expect the isomorphism between double affine BMW, which could generalise the Schur-Weyl duality with this diagrammatic algebra.

In the second part of the thesis we define a "smaller" algebra than the one we expect to have for a Schur-Weyl duality purposes. Namely, we define algebra and give few equivalent presentations of it. Further, we show that it can be presented as a quotient of the group algebra of the *extended* elliptic group, which consists of the elliptic group with additional generators. We consider $F^D(\bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda)$, where the sum runs over all dominant weight of $\mathfrak{sp}_{2n}$.[5] and extend this representation to the "extended" elliptic braid group representation and show the following.

Theorem 1.3.1. *The extended module gives an irreducible representation of the Double affine BMW algebras.*

[5] Note that we take $\mathfrak{sp}_{2n}$ not $\mathfrak{sl}_n$.

Furthermore, we give a combinatorial description for its basis. Moreover, for the case $d = 2$ we develop an alternative combinatorial description for the basis.

Chapter 2

Preliminaries

In this section we recall some background and set up some notation. In the following, we work over the field of complex numbers $\mathbb{C}$.

2.1 Classical combinatorics

Let $m \in \mathbb{N}_0$. Recall that a *partition* of m, denoted by $\lambda \vdash m$, is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \ldots = m$. If $m = 0$, we simply write $\lambda = \emptyset$. Otherwise, since $\lambda_1 \geq \lambda_2 \geq \ldots$, there are only finitely many nonzero λ_i for $i \geq 1$ and there exists k minimal such that $\lambda_k > 0$ and $\lambda_{k+1} = \lambda_{k+2} = \ldots = 0$. Therefore, we can write $\lambda = (\lambda_1, \ldots, \lambda_k)$ instead of an infinite sequence and denote by $l(\lambda) = k$.

The conjugate (transpose) partition of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is $\lambda^t = (\lambda_1^t, \ldots, \lambda_{\lambda_1}^t)$, where λ_i^t is the number of j's such that $\lambda_j \geq i$.

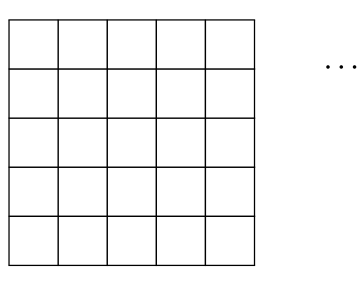
Let Par_m be the set of all partitions of m. On Par_m, we can define a partial ordering $\trianglelefteq$, called the *dominance ordering* and a total ordering $\leq$, called the *lexicographic ordering* as follows: given $\lambda, \mu \in \mathrm{Par}_m$, define

- $\lambda \trianglelefteq \mu$ if $|\sum_{i=1}^k \lambda_i| \leq |\sum_{i=1}^k \mu_i|$ for all $k \geq 1$, and $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.
- $\lambda < \mu$ if there exist k such that $\lambda_i = \mu_i$ for $i < k$ and $\lambda_k < \mu_k$; further $\lambda \leq \mu$ if $\lambda < \mu$ or $\lambda = \mu$.

Remark 2.1.1. Notice that $\lambda \trianglelefteq \mu$ implies $\lambda \leq \mu$.

Example 2.1.2. Let $\mu = (m)$. Then $\lambda \trianglelefteq \mu$ for any $\lambda \in \mathrm{Par}_m$.

Definition 2.1.3. The *box lattice* is the set $B = \{ (r, c) \mid r, c \in \mathbb{N} \}$. We call elements of B boxes and display the box lattice in the following form

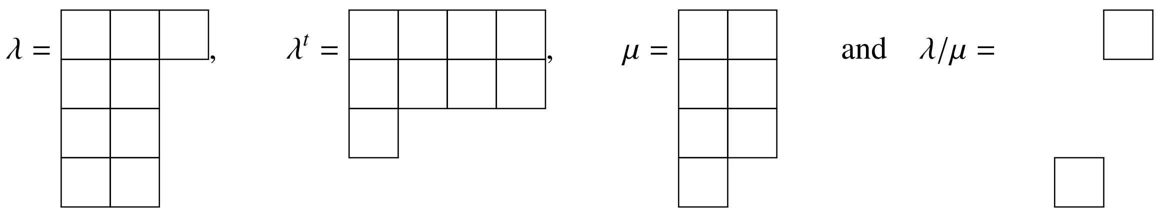


where the box (r, c) is in the row r and column c.

Definition 2.1.4. The *Young diagram* of a partition λ is the set $\{ (r, c) \mid 1 \leq r \leq k, 1 \leq c \leq \lambda_r \}$, considered as a subset of the box lattice. In the following, we abuse notation and write λ for both a partition and its Young diagram.

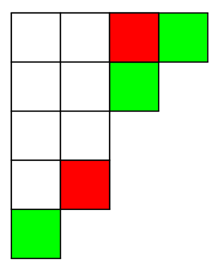
Definition 2.1.5. Let λ and μ be partitions such that $\mu \subseteq \lambda$. Then the *skew diagram* λ/μ is the configuration of boxes of λ which are not in μ.

Example 2.1.6. Let $\lambda = (3, 2, 2, 2)$ and $\mu = (2, 2, 2, 1)$. Then the Young diagrams of λ, λ^t, μ and the skew diagram λ/μ are the following



Definition 2.1.7. Suppose λ is a partition. A box α is *addable* (respectively, *removable*) if $\lambda \cup \{\alpha\}$ (respectively, $\lambda \backslash \{\alpha\}$) is again a partition. Let $\mathsf{A}(\lambda)$ and $\mathsf{R}(\lambda)$ be the sets of addable and removable boxes of λ, respectively, and set $\mathrm{AR}(\lambda) = \mathsf{A}(\lambda) \cup \mathsf{R}(\lambda)$. Furthermore, denote by λ^α the partition $\lambda \cup \{\alpha\}$ $(\lambda \backslash \{\alpha\})$ if $\alpha \in A(\lambda)$ (respectively, $\alpha \in R(\lambda)$).

Example 2.1.8. Let $\lambda = (3, 2, 2, 2)$, then if we color the set of addable boxes in green and the set of removable boxes in red, we obtain



Definition 2.1.9. Let λ be a partition of m. *A standard tableau* t of a shape λ is a filling of the Young diagram λ with the numbers $1, 2, \ldots, m$ such that the entries increase along each row and each column of t. Denote by $\mathrm{Std}(\lambda)$ the set of all standard tableau of the shape λ.

Let us fix $d \in \mathbb{N}_0$.

Definition 2.1.10. Let λ/μ be a skew diagram. An *up-down tableau of shape λ/μ and length d* is a sequence of partitions $\mathsf{t} = (\mu = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}, \lambda^{(d)} = \lambda)$ such that

$$\text{(a)} \ \ \lambda^{(i)} \supseteq \lambda^{(i-1)} \text{ and } \lambda^{(i)}/\lambda^{(i-1)} = \square, \qquad \text{or} \qquad \text{(b)} \ \ \lambda^{(i-1)} \supseteq \lambda^{(i)} \text{ and } \lambda^{(i-1)}/\lambda^{(i)} = \square.$$

An up-down tableau $\mathsf{t} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d)})$ of the shape $\lambda/\emptyset$ can be identified with a d-tuple of nodes:

$$\mathsf{t} = (\alpha_1, \alpha_2, \ldots, \alpha_d), \tag{2.1.1}$$

where $\alpha_k = (r, c)$ if $\lambda^{(k)} = \lambda^{(k-1)} \cup \{(r, c)\}$ and $\alpha_k = -(r, c)$ if $\lambda^{(k)} = \lambda^{(k-1)} \backslash \{(r, c)\}$ for $1 \leq k \leq d$. In the first case we say a node is *added in step k*, and in the second case – *removed on the step k*.

The following is immediate from the Definition.

Lemma 2.1.11. *For any partition λ, we have a bijection of sets*

$$\{\text{standard tableaux of the shape } \lambda\} \overset{1:1}{\longleftrightarrow} \{\text{up-down tableaux of the shape } \lambda/\emptyset \text{ and length } |\lambda|\}$$

For our convenience, we use the notations from [33] and for $k = 0, 1, \dots, d$, denote by $\mathsf{t}_k = \lambda^{(k)}$ and define the truncation of t to the level k to be the up-down tableau

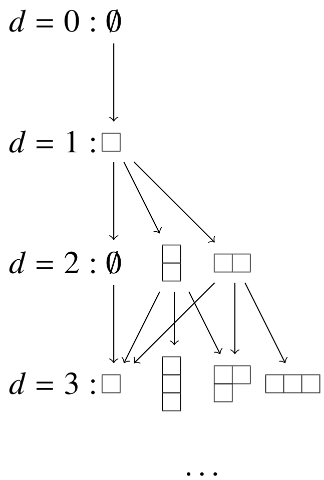
$$\mathsf{t}|_k = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)}).$$

Now we construct the branching graph for the Brauer algebras.

Define $\widehat{B}_d := \{ (\lambda, f) \mid \lambda \in \mathrm{Par}_{d-2f} \text{ and } 0 \le f \le \lfloor \frac{d}{2} \rfloor \}$ and let $\widehat{B}$ be the oriented graph with

(1) vertices $\cup_{d\in\mathbb{N}_0} \widehat{B}_d$ and

(2) an arrow $(\lambda, f) \to (\mu, g)$ for $(\lambda, f) \in \widehat{B}_{d-1}$ and $(\mu, g) \in \widehat{B}_d$, if $\mu \in \mathrm{AR}(\lambda)$.



Let us extend the orderings defined before to $\widehat{B}_d$. For $(\lambda, f), (\mu, g) \in \widehat{B}_d$, we define

- $(\lambda, f) \trianglelefteq (\mu, g)$ if $f < g$, or ($f = g$ and $\lambda \trianglelefteq \mu$),
- $(\lambda, f) \le (\mu, g)$ if $f < g$, or ($f = g$ and $\lambda \le \mu$).

Similarly, we can define $\triangleleft$ and $<$.

Definition 2.1.12. Let $(\lambda, f) \in \widehat{B}_d$. Consider an oriented path in $\widehat{B}_d$ starting at $(\emptyset, 0)$ and ending at (λ, f). An *up-down tableau* of shape (λ, f) is a sequence

$$\mathsf{t} = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \dots, (\lambda^{(d)}, f_d)), \tag{2.1.2}$$

where $(\lambda^{(0)}, f_0) = (\emptyset, 0)$, $(\lambda^{(d)}, f_d) = (\lambda, f)$ and $(\lambda^{(k-1)}, f_{k-1}) \to (\lambda^{(k)}, f_k)$ is an arrow in the path, for $k = 1, \dots, d$. We set $\mathrm{Shape}(\mathsf{t}) = (\lambda, f)$.

Remark 2.1.13. Any up-down tableau of the shape (λ, f) is also an up-down tableau in the sense of Definition 2.1.10 with the shape $\lambda/\emptyset$, which we simply denote by λ.

For any $0 \le f \le \lfloor \frac{d}{2} \rfloor$ and $\lambda \vdash (d-2f)$, define

$$\mathrm{T}_d^{\mathrm{ud}}(\lambda) := \{\, \mathsf{t} \mid \mathsf{t} \text{ is an up-down tableau of shape } \lambda \in \widehat{B}_d \,\}\,.$$

Suppose $\mathsf{s}, \mathsf{t} \in \mathrm{T}_d^{\mathrm{ud}}(\lambda)$. We define the *dominance ordering* $\mathsf{s} \trianglelefteq \mathsf{t}$ if $\mathsf{s}_k \trianglelefteq \mathsf{t}_k$ for any k with $1 \le k \le d$ and $\mathsf{s} \triangleleft \mathsf{t}$ if $\mathsf{s} \trianglelefteq \mathsf{t}$ and $\mathsf{s} \neq \mathsf{t}$.

Example 2.1.14. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathrm{Par}_d$. Set t be the following standard tableau

$$\mathsf{t} = (\emptyset, (1), (2), \ldots, (\lambda_1), (\lambda_1, 1), \ldots, (\lambda_1, \lambda_2), (\lambda_1, \lambda_2, 1), \ldots, \lambda).$$

Then for any $\mathsf{s} \in \mathrm{Std}(\lambda)$, we have $\mathsf{s} \trianglelefteq \mathsf{t}$.

Let us fix $\delta \in \mathbb{C}$.

Definition 2.1.15. The *residue* of a box $\alpha = (r, c)$ is defined by $\mathrm{res}(\alpha) = \frac{\delta-1}{2} + c - r$.

Example 2.1.16. Let $\lambda = (3, 2, 2, 2)$ and $\delta = 1$. Then, if we fill-in each box with its residue, we obtain

0	1	2
−1	0	
−2	−1	
−3	−2	

Definition 2.1.17. Let $I = \frac{\delta-1}{2} + \mathbb{Z}$. Suppose $\mathsf{t} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d)})$ is an up-down tableau. Define the *residue sequence* of t to be $\mathbf{i}_{\mathsf{t}} = (i_1, i_2, \ldots, i_d)$ where

$$i_k = \begin{cases} \mathrm{res}(\alpha), & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}, \\ -\mathrm{res}(\alpha), & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}. \end{cases}$$

Example 2.1.18. Let $\delta = 1$ and $\mathsf{t} = \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{3,1}, \ydiagram{3}\right)$. Then the residue sequence of t is $(1, -1, 1, 1, 2, -1, 1)$.

Finally, we define $T^d \subset I^d$ to be the set containing all residue sequences of up-down tableaux and $\mathrm{T}_d^{\mathrm{ud}}(\mathbf{i})$ to be the set containing all the up-down tableaux with residue sequence $\mathbf{i} \in T^d$.

2.2 Verma paths

In this subsection, following [16–18] we recall an alternative presentation of up-down tableaux which depends on δ and comes from the Lie theory using the Verma modules.

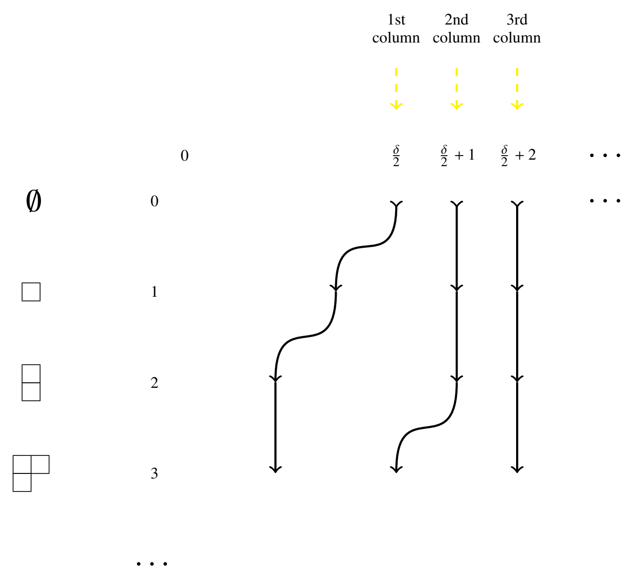
Let us fix δ. To each partition we want to assign a unique infinite sequence consisting of $\{\vee, \wedge, \circ, \times\}$, called a *diagrammatic partition* and to each up-down tableau - a unique diagram, called *Verma path*. But before giving explicit definitions, let us give an informal explanation of this construction. The key points that we emphasise here are very important for the results that come later.

First let us assume $\delta \ge 0$ and let us fix an up-down tableau t with a residue sequence $\mathbf{i}$ (see Definition 2.1.17). To construct a Verma path corresponding to t, we can follow the following steps.

Step 1.

First, to the up-down t we assign a *simple walk* and to each partition in t we assign a *simple diagrammatic partition* defined as follows.

Imagine that we have infinitely many directed lines moving down, such that initially their ends have coordinates $(\frac{\delta}{2}, 0), (\frac{\delta}{2}+1, 0), (\frac{\delta}{2}+2, 0), \dots$ [1]. This "position" corresponds to an empty partition. Every time we add/remove a box, all lines move to the next level[2] according to the following rule. If we add (respectively, remove) a box to the i-th column then the i-th[3] line moves to the left (respectively, right) and all other lines simply go down. More precisely, if we add (respectively, remove) a box the the i-th column and the end of the i-th line has a coordinate (a, b), then the i-th lines moves to the coordinate $(a-1, b+1)$ ($(a+1, b+1)$, respectively). Notice, that the first coordinate can become *negative*. Below we illustrate a simple path for $\mathsf{t} = (\emptyset, \square, \dots)$.



The following remarks give *very important* properties of this construction.

Remark 2.2.1. Since at each level we obtain a partition, lines can never intersect or cross. Namely, if at level k, i-th line points to (a, k) and $i+1$-th line points to (c, k), then $a < c$.

Remark 2.2.2. At each level k, given the tuple containing the first coordinates of the lines

$$(a_1, a_2, \dots) \quad \text{such that} \quad a_r < a_l \quad \text{for} \quad r > l,$$

we can *uniquely* determine the partition $\lambda := \mathsf{t}_k$. Indeed, since the first line points to the coordinate a_1,, λ must have $\frac{\delta}{2} - a_1$ boxes in the first column; since the second lines points to the coordinate a_2, λ must have $\frac{\delta}{2} + 1 - a_2$ boxes in the second column; etc. We denote this tuple by $S(\lambda)$.

Remark 2.2.3. Notice that if at step k, the line moves from the position $(a, k-1)$ to the position

[1]Notice that the first coordinates increase along the horizontal line moving right, and the second coordinates increase along the vertical line moving down.

[2]We indicate levels and second coordinates along the vertical line, next to the partitions.

[3]From the left.

$(a-1, k)$, then

$$i_k = a - 1/2.$$[4]

And if the line moves from the position $(a-1, k-1)$ to the position (a, k), then

$$i_k = -(a - 1/2).$$

Step 2.

Let us now slightly modify this construction. Imagine that there is a vertical wall at level 0 (see the *left* diagrams of Examples 2.2.5, 2.2.6, 2.2.7 below). Every time when a line "attempts" to cross the wall, it changes its direction from $\vee$ to $\wedge$, or other way around. Furthermore, if the line directed $\wedge$, then when we add (remove, respectively) the lines goes to the right (left, respectively). Alternatively, we can think of this modification as folding the diagram along its mirror axis. For our convenience, we draw lines in different colours. Now let us look more carefully at the *left* diagrams in Examples below.

Let us start with Example 2.2.5 (with $\delta = 1$). We can see that when the first (red) line goes from the level 1 to the level 2, it hits the wall and, hence, changes its direction. On the next step, it "crashes" with the second (blue) line. Therefore, we can see that after the lines pass the level k, each node (a, k) can either

- be empty (denoted as $\circ$),
- contain a single arrow $\vee$,
- contain a single arrow $\wedge$,
- contain both $\vee$ and $\wedge$.

We call a node with both $\vee$ and $\wedge$ *crowded*. Notice that if the node (a, k) has a line pointed $\wedge$, then in the previous presentation this line was in the position $(-a, k)$. Therefore, the rightmost $\wedge$ (if there is one) indicates the position of the first line, the second rightmost $\wedge$ (if there is one) indicates the position of the second line, etc.

Now let us look at the Example 2.2.7 (with $\delta = 2$). Here, we have a slightly more complicated situation since the line can "stand" at the wall (see levels 1,3,5). We would like to emphasise that in *this* construction (*unlike the following construction*) if the arrow touches the wall but does not attempt to cross it, it *does not* change its direction (see levels 1, **5**, 10, **12**). [5]

Similarly to the previous case, we can see that after the lines pass the level k, we can "recover" the partition that corresponds to the position of the arrows on the level k. We will give a slightly more precise explanation of this statement in the next Step.

We have the following *important* property.

Remark 2.2.4. Assume $a \geq 1$. If at step k, *independently of the orientation of the line*, the line moves from the position $(a, k-1)$ to the position $(a-1, k)$, then

$$i_k = a - 1/2.$$

And if the line moves from the position $(a-1, k-1)$ to the position (a, k), then

$$i_k = -(a - 1/2).$$

[4]Hence it is the mean of $a-1$ and a.

[5]Please pay attention to the levels 5,12, since the convention will be different.

If $a = 1/2$, and the lines moves from the position $(a, k-1)$ to the position (a, k) by hitting the wall, then $i_k = 0$.

To see this, compare in Example 2.2.6, levels 14 –>15 with 10–>11, and levels 12–>13 with 13–>14.

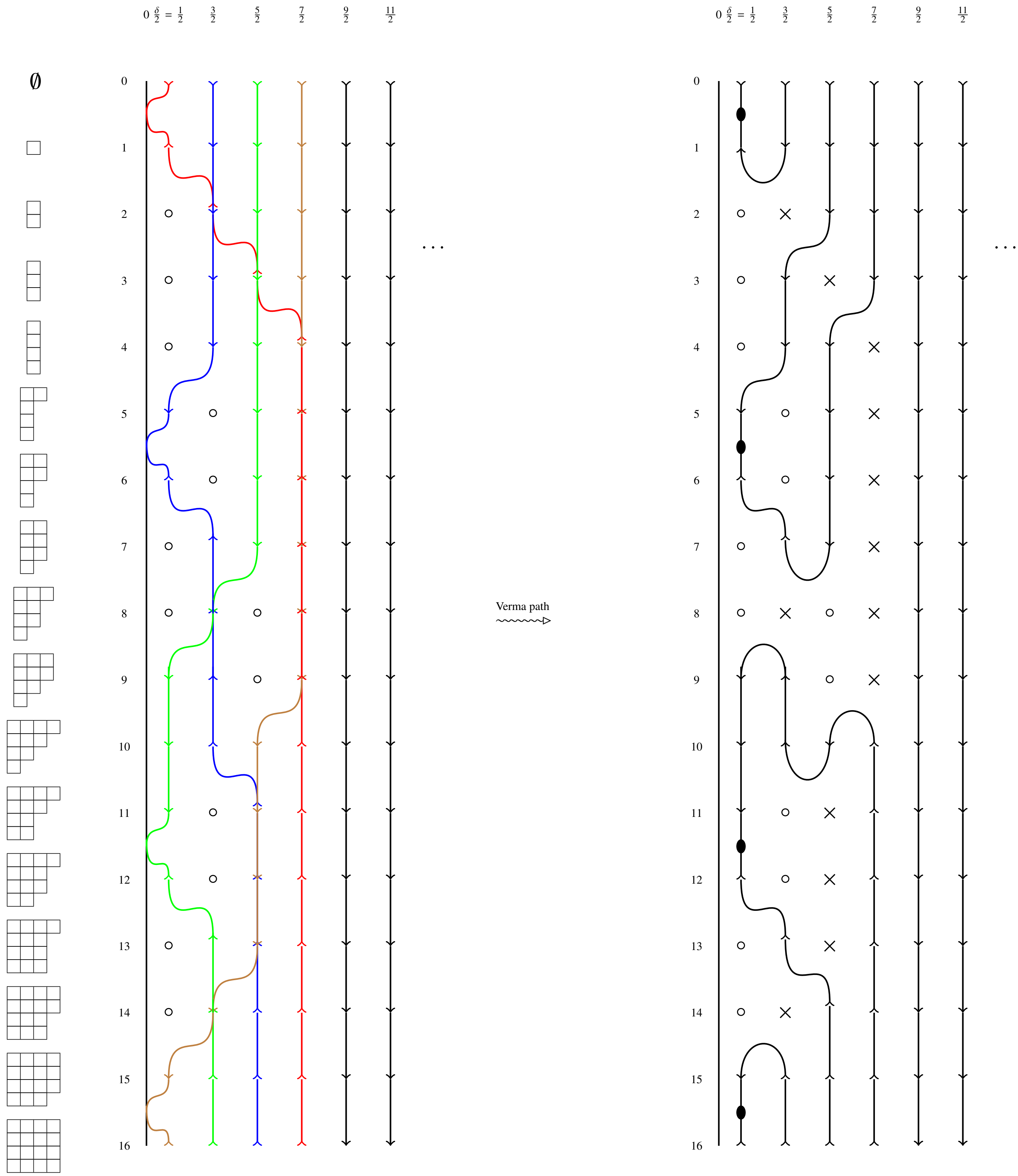
Step 3

Now we are ready to explain the construction of *diagrammatic partitions* and (uncoloured) *Verma paths*. To construct them, we make the following changes with the previous coloured diagram:

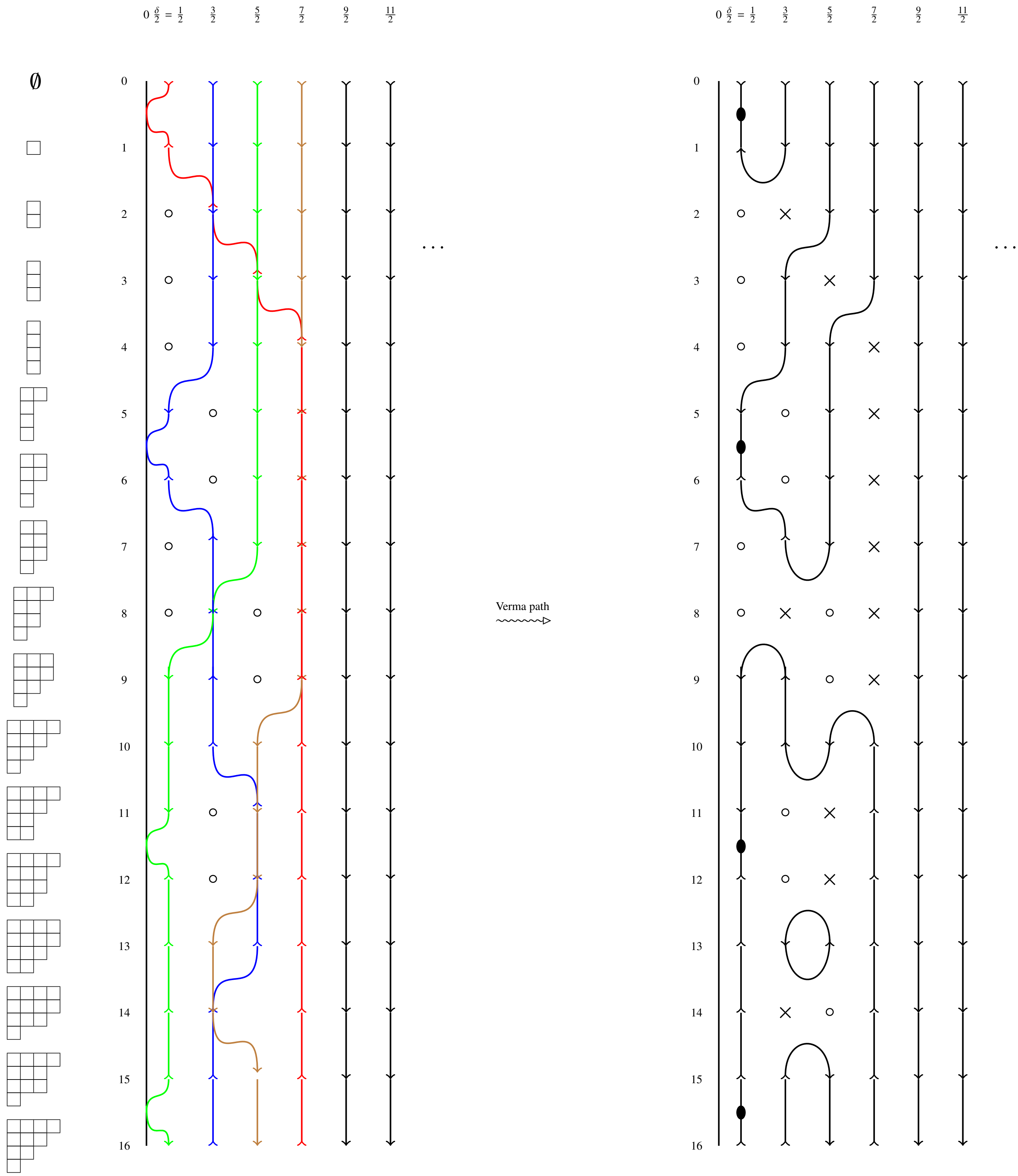
- remove all colours,
- if δ is even, change all ups *on the wall* to $\vee$ but keep the rest of the directions (see Example 2.2.7 level 4–>5),
- if δ is odd and we hit the wall, we draw a dotted straight line instead of "curved line"(see Example 2.2.6 levels 0–>1, 5–>6, **15–>16**),
- if δ is even and the line moves from/to the wall from the single occupied node to the empty node but the directions do not match, we put a dot on the line (see Example 2.2.7 levels 3–>4, 4–>5, 5–>6)
- if a line moves from a single occupied node to a crowded node, we put cross on that node and connect the two lines with a cup (see Example 2.2.7 levels 6–>7, 8–>9, 14–>15). If the directions do not match, we put a dot on a cup (see Example 2.2.7 level 12–>13).
- if a line moves from a crowded node to a single node, we connect two lines with a cap (see Example 2.2.7 levels 7–>8, 9–>10, 13–>14). If the directions do not match, we put a dot on a cup (see Example 2.2.7 level 11–>12).
- if a node moves from a crowded node to a crowded node, then replace crowded nodes with crosses and connect the other two lines (see Example 2.2.5 levels 2–>3, 3–>4, 13–>14).

See Examples 2.2.5, 2.2.6, 2.2.7 for more details.

Example 2.2.5.

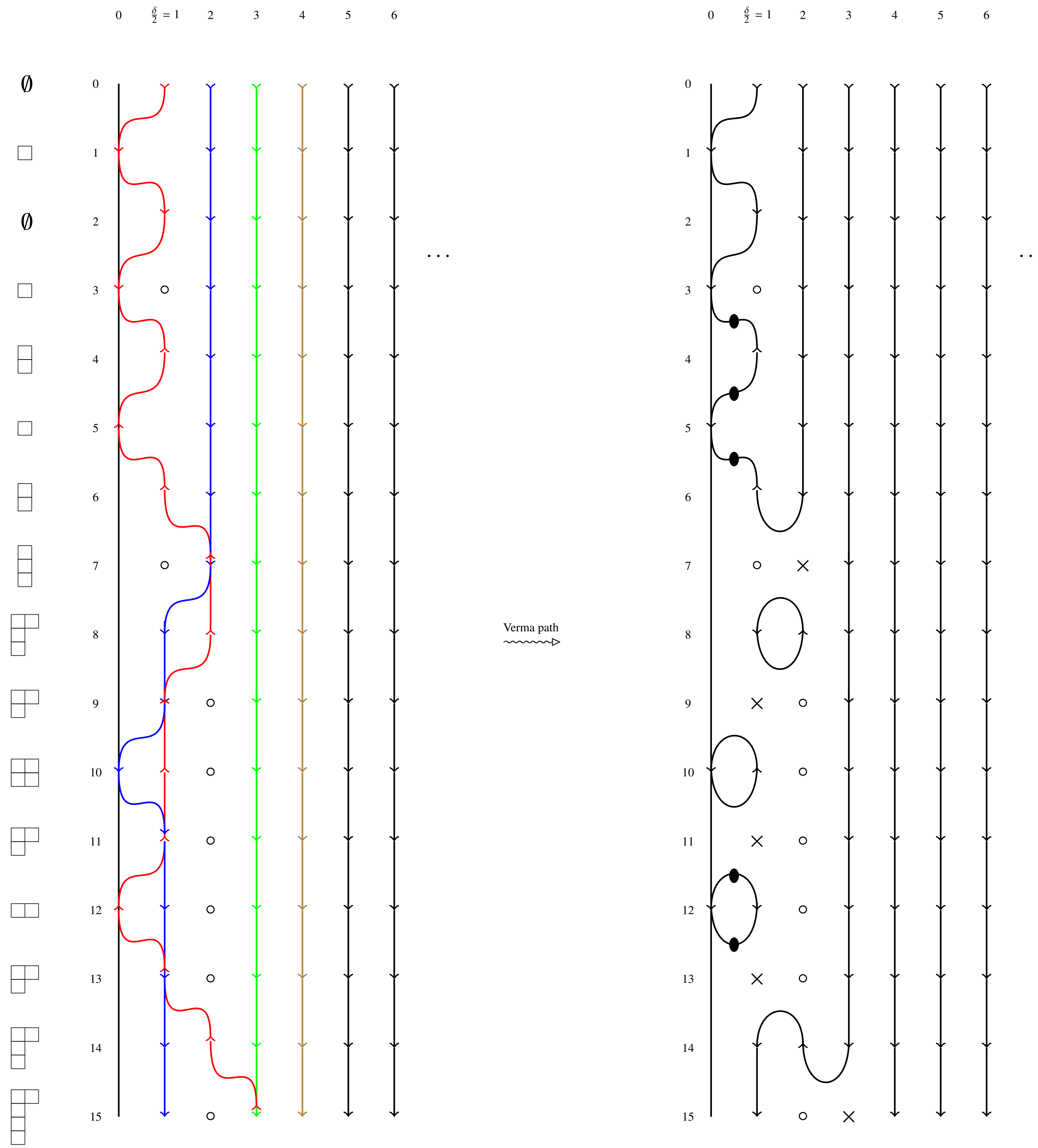

0 $\frac{\delta}{2} = \frac{1}{2}$
$\frac{3}{2}$
$\frac{5}{2}$
$\frac{7}{2}$
$\frac{9}{2}$
$\frac{11}{2}$
∅
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
...
Verma path
0 $\frac{\delta}{2} = \frac{1}{2}$
$\frac{3}{2}$
$\frac{5}{2}$
$\frac{7}{2}$
$\frac{9}{2}$
$\frac{11}{2}$
...

Example 2.2.6.



$0\ \frac{\delta}{2} = \frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$
$\emptyset$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Verma path
. . .

Example 2.2.7.



The obtained diagram is called a *Verma path*. Assume δ is odd. We call the sequence of $\circ, \vee, \wedge, \times$ starting from the coordinate $1/2$ a *diagrammatic partition* corresponding to a partition (on the left).

Example 2.2.8. Let $\delta = 1$

- $\emptyset$ corresponds to $\vee \vee \vee \dots$
- $\square$ corresponds to $\wedge \vee \vee \dots$
- $\ydiagram{1,1}$ corresponds to $\circ \times \vee \dots$

Now assume δ is even. We call the sequence of $\circ, \vee, \wedge, \times$ starting from the coordinate 0 a *diagrammatic partition* corresponding to a partition (on the left). Notice that by definition, we can only have $\circ$ and $\vee$ on the first place.

Example 2.2.9. Let $\delta = 2$

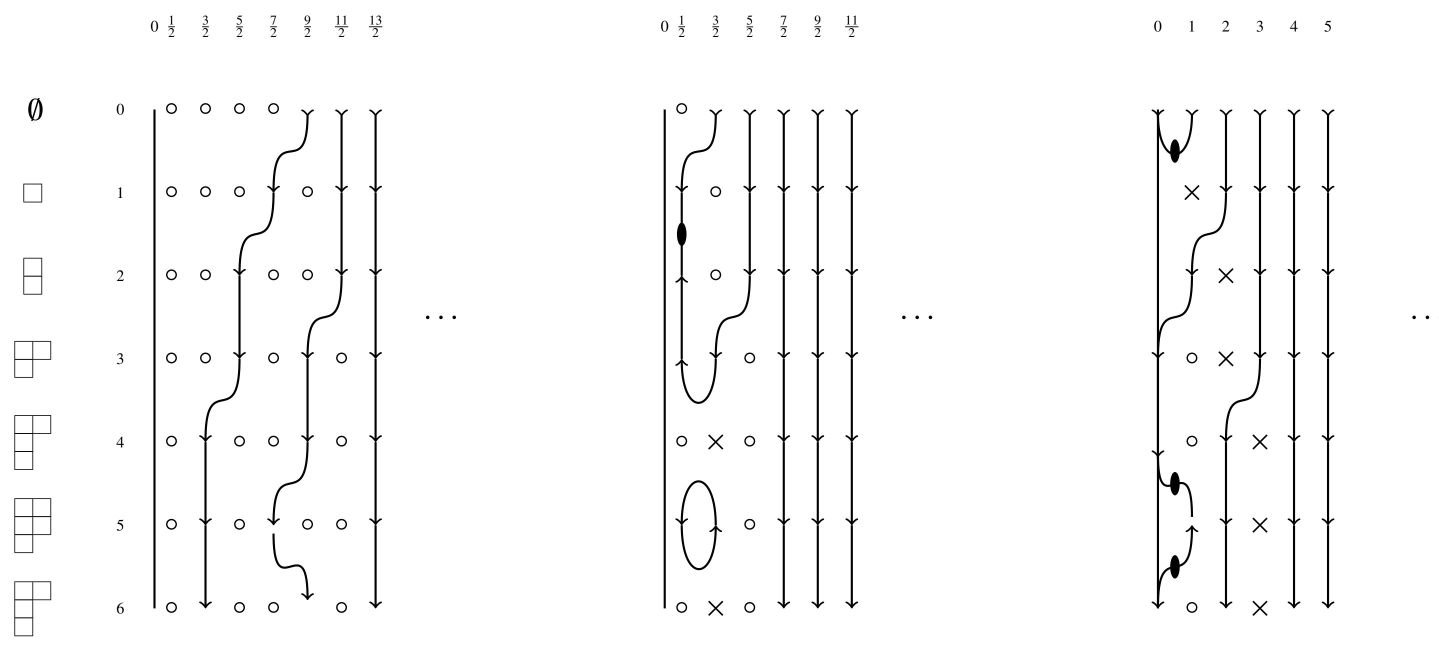
- $\emptyset$ corresponds to $\circ \vee \vee \dots$
- $\square$ corresponds to $\vee \circ \vee \vee \dots$
- $\ydiagram{1,1}$ corresponds to $\circ \wedge \vee \dots$

The following follows from the previous remarks.

Corollary 2.2.10. *Given a diagrammatic partition, we can uniquely "recover" the partition that corresponds to this sequence* [6].

Let us now compare Verma paths corresponding to the same up-down tableau but with different δ.

Example 2.2.11. Let $d = 6$ and $\mathtt{t} = \left(\emptyset, \square, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,1,1}, \ydiagram{2,2,1}, \ydiagram{2,1,1}\right)$. Then the Verma paths corresponding this up-down tableau for $\delta = 9, \delta = 3, \delta = 0$ are the following



$\delta = 9$ $\qquad$ $\delta = 3$ $\qquad$ $\delta = 0$

[6]The one that is on the left side on each level in the Examples below.

Remark 2.2.12. Following the same argument as before, we can show that in Example 2.2.6 the residue sequences on the levels 7–>8 and 13->14 are the same and equal $(3/2 + 5/2)/2$. Moreover, the order of the $\circ$ and $\times$ is important. We can see that the residue sequence on the level 10–>11 equals $-(3/2 + 5/2)/2$. Similarly for caps.

Definition 2.2.13. A *graph* of the Verma path t, denoted by g_{t}, is the diagram of t obtained by forgetting all orientations and removing all dots.

We already showed that if Verma paths t_1 and t_1 have the same graphs, then they have the same residue sequences, i.e $\mathbf{i}_{\mathsf{t}_1} = \mathbf{i}_{\mathsf{t}_2}$. But the following statement also holds: if Verma paths t_1 and t_1 have the same residue sequences, i.e $\mathbf{i}_{\mathsf{t}_1} = \mathbf{i}_{\mathsf{t}_2}$, then they have the same graphs. Indeed, to construct a graph we can follow the same procedure as for constructing a Verma path, and since the residue unique determines the direction of a move (in particular, the order of $\circ$ and $\times$), we can uniquely construct a graph. Alternatively, we can see it using the diagrams from the Step 2 (by removing colours and directions). We have

Corollary 2.2.14. *Suppose $\mathbf{i} \in I^d$. There exists a unique unoriented graph $g_{\mathbf{i}}$ with residue sequence $\mathbf{i}$ if and only if $\mathbf{i} \in T^d$.*

Let us consider the following problem. Assume $\lambda \in \widehat{B}_d$ and $\mathsf{t} \in \mathrm{T}^{\mathrm{ud}}_d(\lambda)$. Given a graph of Verma path t, what is the minimal information we should give to determine the Verma path?

Definition 2.2.15. We call a cup or cap anticlockwise (respectively, clockwise) if its rightmost vertex is directed $\wedge$ (respectively, $\vee$). Similarly, we call a loop anticlockwise (respectively, clockwise) if its rightmost vertex is directed $\wedge$ (respectively, $\vee$).

Example 2.2.16. In 2.2.7, the loop on the levels 7-9 is anticlockwise, the loop on the levels 11-13 is clockwise.

For a simplicity, let us first assume that δ is odd. Then it is easy to see that it is enough to give all directions of the loops to determine the Verma path t completely. To show this, first notice that in case of δ is odd, we can uniquely "recover" all dots of t (they appear on the levels when "nothing" happens, i.e all strands go down). Furthermore, since t moves from $\emptyset$ to λ, we can determine all directions of the strands and cups/caps when they are not in the loops,[7] therefore we can determine t completely.

Example 2.2.17. Let us look at Example 2.2.5. We can see that given the directions on levels 0 and 16, we can easily recover the Verma path from its graph.

Now let us assume δ is even. If we do the same in this case, we will immediately see some problems. To see this, let us look at the levels 0-6 in 2.2.7. Given a graph and the direction on the level 0, we cannot uniquely determine the positions of the dots. Therefore, we need to give more information.

Let us consider the Verma path t and additionally mark levels, when the lines "cross" the wall, e.g. (only) level 3 in 2.2.7 (see also the diagram on the left). We define the *extended graph* g^e_{t} of the Verma path t be the its graph with a marked levels (where t "crosses" the wall). [8]

Remark 2.2.18. Notice that in case δ is odd, we can obtain an extended graph from the graph itself. Therefore, in this case, we set $g^e_{\mathsf{t}} := g_{\mathsf{t}}$.

[7] Its directions determined by the directions on the levels 0 and d.

[8] We will use the extended graphs in the definition of the surgery procedure.

Hence we have the following

Corollary 2.2.19. *Given the extended graph of* $\mathfrak{t}$ *and orientations of all circles, we can uniquely recover the Verma path* $\mathfrak{t}$*.*

Finally, if $\delta < 0$, then we start with an infinite number of lines directed $\wedge$, such that the first line is at the position $\delta/2$ (which is a negative number), the second line at the position $\delta/2 - 1$ etc. We follow the convention that in this case the line on the wall should be always directed $\wedge$. The rest of the construction is the same.

Now we can give precise definitions. Let

$$I_\delta = \begin{cases} \mathbb{Z}_{\geq 0} + \frac{1}{2}, & \text{if } \delta \text{ is odd,} \\ \mathbb{Z}_{\geq 0}, & \text{if } \delta \text{ is even,} \end{cases} \qquad c_\delta = \begin{cases} \frac{\delta-1}{2}, & \text{if } \delta \text{ is odd,} \\ \frac{\delta}{2}, & \text{if } \delta \text{ is even,} \end{cases} \qquad \text{and} \qquad o_\delta = \begin{cases} \vee, & \text{if } \delta \text{ is negative,} \\ \wedge, & \text{if } \delta \text{ is non-negative.} \end{cases}$$

Definition 2.2.20. We define the set D_δ of *(Brauer) diagrammatic partitions* as the set of sequences $a = (a_i)_{i \in I_\delta}$ such that $a_i \in \{\wedge, \vee, \times, \circ, \diamond\}$, $a_i \neq \diamond$ for $i \neq 0$, $a_0 \in \{\circ, \diamond\}$,

$$\#\{a_i \mid a_i \in \{\circ\}\} - \#\{a_i \mid a_i \in \{\times\}\} = c_\delta \quad \text{and} \quad \#\{a_i \mid a_i = o_\delta\}\} < \infty$$

Obviously a diagrammatic partition a is uniquely determined by the sets

$$\mathrm{P}_*(a) = \{i \mid a_i = *\},$$

for $* \in \{\wedge, \vee, \times, \circ, \diamond\}$.

To a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we assign a diagrammatic partition λ^δ in the following way. Let $\lambda' = (\lambda'_1, \lambda'_2, \ldots) = (\frac{\delta}{2} - \lambda^t_1, \frac{\delta}{2} + 1 - \lambda^t_2, \ldots, \frac{\delta}{2} + \lambda_1 - 1 - \lambda^t_{\lambda_1}, \frac{\delta}{2} + \lambda_1, , \frac{\delta}{2} + \lambda_1 + 1, \ldots)$,[9] where λ^t is the conjugate partition. Then λ^δ is defined by

$$\begin{aligned} \mathrm{P}_\vee(\lambda^\delta) &= \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\}, \\ \mathrm{P}_\wedge(\lambda^\delta) &= \{-\lambda'_i \mid \lambda'_i < 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\}, \\ \mathrm{P}_\times(\lambda^\delta) &= \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ appears in } \lambda'\}, \\ \mathrm{P}_\diamond(\lambda^\delta) &= \{\lambda'_i \mid \lambda'_i = 0\}, \\ \mathrm{P}_\circ(\lambda^\delta) &= I_\delta \setminus (\mathrm{P}_\vee \cup \mathrm{P}_\wedge \cup \mathrm{P}_\times \cup \mathrm{P}_\diamond). \end{aligned}$$

Remark 2.2.21. We can see that in the pictures above, $\diamond$ is replaced by

- $\vee$ if $\delta \geq 0$ (and even),
- $\wedge$ if $\delta < 0$ (and even).

We have the following easy result.

Lemma 2.2.22. *The assignment $\lambda \mapsto \lambda^\delta$ defines a bijection between partitions and diagrammatic partitions.* [10]

Proof. First notice that since $\lambda_1^\delta \neq \lambda_2^\delta$ for $\lambda_1 \neq \lambda_2$, injectivity follows immediately.[11] For surjectivity, we show that to each diagrammatic partition a we can assign a partition λ such that $a = \lambda^\delta$.

[9] Please compare λ' with $S(\lambda)$. Notice that λ'_i is the position of the ith line in the diagrams defined in Step 1.

[10] Please compare this statement with the one discussed before.

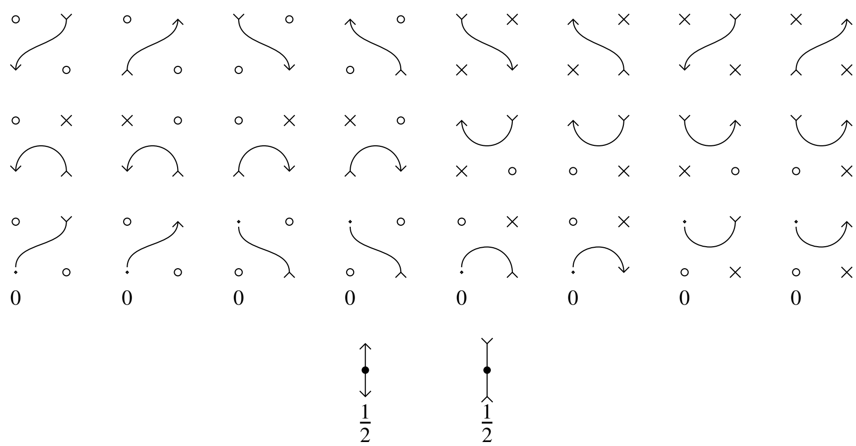
[11] Indeed, if $\lambda_1 \neq \lambda_2$, then $\lambda_1^t \neq \lambda_2^t$. Hence, by the definition $\lambda_1^\delta \neq \lambda_2^\delta$.

First, to each $a = (a_i) \in D_\delta$ assign a sequence $b = (b_i)$,[12] where $i \in \mathbb{Z}$ if δ is even and $i \in \mathbb{Z} + \frac{1}{2}$ if δ is odd, as:

$$b_i = \begin{cases} \vee & \text{if } i \leq 0 \text{ and } (a_{-i} = \wedge \text{ or } a_{-i} = \times), \\ \vee & \text{if } i > 0 \text{ and } (a_i = \times \text{ or } a_{-i} = \vee), \\ \circ & \text{otherwise .} \end{cases}$$

Now define a sequence $c = (c_j)_{j\in\mathbb{N}}$ such that c_i is the coordinate of the i-th leftmost (respectfully, rightmost) $\vee$ in b for $\delta \geq 0$ (respectfully, $\wedge$ for $\delta < 0$), which is well defined since $\#\{a_i \mid a_i \in \{\wedge\}\} < \infty$ ($\#\{a_i \mid a_i \in \{\vee\}\} < \infty$ respectively). Then we set $\lambda_i^t = \frac{\delta}{2} + i - 1 - c_i$. The first conditions in each case in the Definition 2.2.20 guarantee that λ is indeed a partition and $a = \lambda^\delta$.

Definition 2.2.23. Let $\mathsf{t} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d)})$ be an up-down tableau of shape λ with residue sequence $i_1, \ldots, i_d$. The *Verma path* associated with an up-down tableau t is given as follows: first draw the corresponding sequence of partition diagrams from top to bottom. Then for each $k = 1, \ldots, d-1$, we insert vertical line segments connecting all coordinates strictly smaller than $|i_k| - \frac{1}{2}$ and strictly larger than $|i_k| + \frac{1}{2}$ that are labelled $\vee$ or $\wedge$ in λ^i and λ^{i+1} and connect the remaining coordinates of λ^i and λ^{i+1} as in the appropriate one from the following list of moves, called *elementary moves*.



(2.2.1)

Remark 2.2.24. From now on we abuse notation and write t for both an up-down tableau t and its Verma path t.

From now on we will display Verma paths in the following way. In the elementary moves in the third row of (2.2.1) any strand might need to be decorated with a $\bullet$ as in the fourth row. To determine this, replace $\diamond$ by $\wedge$ if $\delta < 0$ or by $\vee$ if $\delta \geq 0$. If the strand is not oriented as in the first or second row, then a decoration must be added.

2.3 Graded cellular algebras

Cellular algebras were introduced by Graham and Lehrer in [22]. The definition was motivated by matrix algebras and easy diagram algebras, e.g. Temperley-Lieb algebras, and provides a

[12] Hence we are recovering the original positions of the lines given in Step 1.

tool to classify its irreducible representations. As we will see later, Brauer algebras are an example of (graded) cellular algebras.

For our convenience, we use the notations from [33]. Let A be a unital finite-dimensional $\mathbb{C}$-algebra.

Definition 2.3.1. A *cell datum* for A is a tuple $(\Lambda, *, T, C)$ where

- $\Lambda = (\Lambda, >)$ is a poset, either finite or infinite;
- $T(\lambda)$ is a finite set for each $\lambda \in \Lambda$;
- $* : A \longrightarrow A$ is a $\mathbb{C}$-linear anti-automorphism with $i^2 = id_A$;
- $C : \prod_{\lambda\in\Lambda} T(\lambda) \times T(\lambda) \longrightarrow A$ is an injective map which sends (s, t) to a^{λ}_{st} such that:.

  (1) $\{ a^{\lambda}_{\mathsf{st}} \mid \lambda \in \Lambda, s, t \in T(\lambda) \}$ is a basis of A;

  (2) for any $r \in A$ and $\mathsf{t} \in T(\lambda)$, there exists scalars $c^{v}_{\mathsf{t}}(r)$ such that, for any $\mathsf{s} \in T(\lambda)$,
  $$a^{\lambda}_{\mathsf{st}} \cdot r \equiv \sum_{\mathsf{v}\in T(\lambda)} c^{\mathsf{v}}_{\mathsf{t}}(r) a^{\lambda}_{\mathsf{sv}} \mod A^{>\lambda} \tag{2.3.1}$$
  where $A^{>\lambda}$ is the subspace of A spanned by $\{ a^{\mu}_{\mathsf{xy}} \mid \mu > \lambda, \mathsf{x}, \mathsf{y} \in T(\mu) \}$;

  (3) $*(a^{\lambda}_{\mathsf{st}}) = a^{\lambda}_{\mathsf{ts}}$, for all $\lambda \in \Lambda$ and $\mathsf{s}, \mathsf{t} \in T(\lambda)$.

The algebra A is a *cellular* if there exist a cell datum for A.

Assume A is a cellular algebra with a cell datum $(\Lambda, *, T, C)$. For any $\lambda \in \Lambda$, define $A^{\geq\lambda}$ to be the subspace of A spanned by
$$\{ a^{\mu}_{\mathsf{st}} \mid \mu \geq \lambda, \mathsf{s}, \mathsf{t} \in T(\mu) \}.$$
Then, from (2.3.1) we obtain that $A^{>\lambda}$ is an ideal of $A^{\geq\lambda}$ and hence $A^{\geq\lambda}/A^{>\lambda}$ is an A-module. For any $\mathsf{s} \in T(\lambda)$ we define C^{λ}_{s} to be the A-submodule of $A^{\geq\lambda}/A^{>\lambda}$ with basis $\{ a^{\lambda}_{\mathsf{st}} + A^{>\lambda} \mid \mathsf{t} \in T(\lambda) \}$. Again from (2.3.1) we have $C^{\lambda}_{\mathsf{s}} \cong C^{\lambda}_{\mathsf{t}}$ for any $\mathsf{s}, \mathsf{t} \in T(\lambda)$.

Definition 2.3.2. Let $\lambda \in \Lambda$. We define the *cell module* C^{λ} of A to be $C^{\lambda} = C^{\lambda}_{\mathsf{s}}$ for any $\mathsf{s} \in T(\lambda)$, which has basis $\{ a^{\lambda}_{\mathsf{t}} \mid \mathsf{t} \in T(\lambda) \}$ and for any $r \in A$,
$$a^{\lambda}_{\mathsf{t}} \cdot r = \sum_{\mathsf{u}\in T(\lambda)} c^{r}_{\mathsf{u}} a^{\lambda}_{\mathsf{u}}$$
where c^{r}_{u} are determined by
$$a^{\lambda}_{\mathsf{st}} \cdot r = \sum_{\mathsf{u}\in T(\lambda)} c^{r}_{\mathsf{u}} a^{\lambda}_{\mathsf{su}} + A^{>\lambda}.$$

Example 2.3.3. Let us consider the algebra of $d \times d$-matrices over $\mathbb{C}$. This algebra has one irreducible module isomorphic to $\mathbb{C}^d$ with an action given by right multiplication, and its standard basis is given by matrix units. It is easy to see that the algebra of $d \times d$-matrices is cellular and the cell datum is given by $*(A) = A^T$, $\Lambda := \{1\}$, $T(1) := \{1, \cdots, d\}$ and $a^{\lambda}_{st} := E_{st}$, the standard matrix units. Moreover, it is also a cell module. Indeed, $A = A^{\geq\lambda}/A^{>\lambda} = A^{\geq 1}/A^{>1}$ and $C^1 = \mathbb{C}^d$ with basis $a_1, \ldots, a_d$, where a_i is a i-th vector unit in $\mathbb{C}^d$. Then the action of A is the right matrix multiplication on $\mathbb{C}^d$.

Example 2.3.4. Let F be any filed. The group algebra of the symmetric group FS_d is an example of a cellular algebra. The cell basis $a^\lambda_{\mathsf{s},\mathsf{t}}$ is labelled by Young diagrams λ of size d and standard tableaux s, t.

The following definition is due Hu and Mathas (see [23]). Assume A is a graded algebra.

Definition 2.3.5. A *graded cell datum* for a graded algebra A is a tuple $(\Lambda, *, T, C, \deg)$ such that the tuple $(\Lambda, *, T, C,)$ is a cell datum for A and

$$\deg : \coprod_\lambda T(\lambda) \longrightarrow \mathbb{Z}$$

is a function such that each basis element $a^\lambda_{\mathsf{st}} \in A$ is homogeneous of degree $\deg a^\lambda_{\mathsf{st}} = \deg \mathsf{s} + \deg \mathsf{t}$.

If a (graded) cell datum exists for A then A is called a *(graded) cellular algebra*.

Notice that Definition 2.3.2 holds in the graded settings as well. Furthermore, using the cell modules, the one can construct a *complete set of pairwise non-isomorphic* (graded) simple A-modules (see [22, Theorem 3.4] and [23, Theorem 2.10]).

Example 2.3.6. Brauer algebras are examples of graded cellular algebras. See Theorems 4.1.14 and 5.2.10 for details.

2.4 Brauer algebras

Let us now give a more precise description of the Brauer algebras $B_d(\delta)$ discussed in Introduction. Recall $d \in \mathbb{N}$ and $\delta \in \mathbb{C}$.

Definition 2.4.1. *The Brauer algebra $B_d(\delta)$* is a unital associative algebra with generators

$$\{s_1, s_2, \ldots, s_{d-1}\} \cup \{e_1, e_2, \ldots, e_{d-1}\},$$

subject to the relations (whenever the expressions make sense)

(1) (Inverses) $s_i^2 = 1$.

(2) (Essential idempotent relation) $e_i^2 = \delta e_i$.

(3) (Braid relations) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|i - j| > 1$.

(4) (Commutation relations) $s_i e_j = e_j s_i$ and $e_i e_j = e_j e_i$ for $|i - j| > 1$.

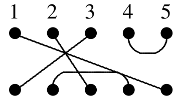
(5) (Tangle relations) $e_i e_{i+1} e_i = e_i$, $e_{i+1} e_i e_{i+1} = e_{i+1}$, $s_i e_{i+1} e_i = s_{i+1} e_i$ and $e_i e_{i+1} s_i = e_i s_{i+1}$.

(6) (Untwisting relations) $s_i e_i = e_i s_i = e_i$.

Recall the diagrammatic description given in Introduction. It can be easily checked that the relations of the Brauer algebra in this diagrammatic presentation are satisfied.

Definition 2.4.2. A *Brauer diagram* consists of two rows of d dots with each dot joined to one other dot. Denote the set of Brauer diagrams by $\mathbf{D}_d$.

Example 2.4.3. Let $d = 5$. The following is an example of a Brauer diagram.



(2.4.1)

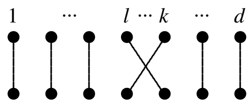
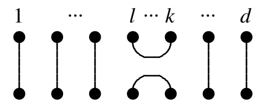
Moreover, it is possible to show that the set of Brauer diagrams $\mathbf{D}_d$, consitute the $\mathbb{C}$-basis of $B_d(\delta)$. Furthermore, it is easy to see that there are $(2d-1)!! = (2d-1)\cdot(2d-3)\cdot \ldots 3\cdot 1$ Brauer diagrams, which implies that the dimension of $B_d(\delta)$ is $(2d-1)!!$.

Remark 2.4.4. The group algebra of the symmetric group $\mathbb{C}S_d$ is a subalgebra of $B_d(\delta)$ generated by s_i.

Definition 2.4.5. For every $k = 1, \ldots, d$, we define the following elements of $B_d(\delta)$, called *Jucys-Murphy elements*,

$$x_k = \frac{\delta - 1}{2} + \sum_{l=1}^{k-1} (l,k) - \overline{(l,k)},$$

where

$(l,k) =$  and $\overline{(l,k)} =$  . (2.4.2)

The Jucys-Murphy elements play an important role in representation theory, which we will see below.

Definition 2.4.6. Define *Gelfand-Zetlin subalgebra* of $B_d(\delta)$, denoted by $\mathscr{L}_d$, to be the subalgebra of $B_d(\delta)$ generated by its Jucys-Murphy elements.

One can show that Jucys-Murphy elements pairwise commute and hence $\mathscr{L}_d$ is commutative.

Let us fix $n \in \mathbb{N}$ and let $V = \mathbb{C}^n$. Furthermore, let G denote O_n or Sp_n. On $V^{\otimes d}$ we have the natural G action. Brauer algebras $B_d(\pm n)$ also naturally act on the tensor space $V^{\otimes d}$[13]. Namely, each s_i acts naturally and permutes i-th and $i+1$-th tensor factors of $V^{\otimes d}$. Similarly, each generator e_i acts only on the i-th and $i+1$-th tensor factor of $V^{\otimes d}$. In the other words,

$$e_i|_{V^{\otimes d}} = \mathrm{id}^{\otimes i-1} \otimes e \otimes \mathrm{id}^{\otimes d-i-1},$$

where $e : V \otimes V \mapsto V \otimes V$ is defined as follows: for $v_1, \ldots, v_n$ the standard basis of $V = \mathbb{C}^n$, $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{C}^n$ and $v, w \in V$ arbitrary vectors, we have

$$v \otimes w \mapsto \pm \langle v, w \rangle \sum_j v_j \otimes v_j.$$

Let ϕ, ψ be the corresponding natural representations

$$\phi : (B_d(\pm n))^{\mathrm{op}} \to \mathrm{End}(V^{\otimes d}), \qquad \psi : G \to \mathrm{End}(V^{\otimes d}),$$

respectively.

Theorem 2.4.7 ([5–7]). *We have the following:*

[13] In this natural action, it is crucial that $|\delta| = n = \dim(V)$, so that the relations $e_i^2 = \delta e_i$ are satisfied.

a) $\phi((B_d(n))^{\mathrm{op}}) = \mathrm{End}_G(V^{\otimes d})$ *and if* $n > d$[14] *then* ϕ *is injective, and hence an isomorphism onto* $\mathrm{End}_G(V^{\otimes d})$,

b) $\psi(G) = \mathrm{End}_{B_d(n)}(V^{\otimes d})$,

c) as a $(B_d(n), G)$*-bimodule, we have the decomposition*

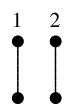
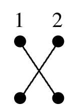
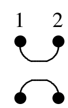
$$V^{\otimes d} = \bigoplus_{f=0}^{[d/2]} \bigoplus_{\lambda \vdash d-2f,\ \lambda_1^t + \lambda_2^t \le n} V_\lambda \otimes L_\lambda = \bigoplus_{\lambda \in \widehat{B}_d,\ \lambda_1^t + \lambda_2^t \le n} V_\lambda \otimes L_\lambda,$$

where the summation is taken over partitions $\lambda \in \widehat{B}_d$ *such that* $\lambda_1^t + \lambda_2^t \le n$[15]*,* V_λ *and* L_λ *are the respective irreducible representations of* $B_d(n)$ *and* G *associated to* λ*.*

Remark 2.4.8. The irreducible representations V_λ of $B_d(n)$ are labeled by $\lambda \in \widehat{B}_d$. In fact, this is also the case for $\delta \notin \mathbb{Z}$.

It is clear from Schur-Weyl duality (and the double centraliser property) that $B_d(\delta)$ is semisimple for $\delta \in \mathbb{N}$ and $\delta > d$, but in fact it is not true for all pairs d, δ (see [17] and references therein).

Example 2.4.9. Assume $d = 2$. Then the basis for $B_2(\delta)$ consists of the following three diagrams

$1 =$  $s =$  $e =$  (2.4.3)

- If $\delta \neq 0$, the algebra $B_2(\delta)$ has the following three primitive orthogonal idempotents.
$$e_1 = \frac{1}{\delta}e, \quad e_2 = \frac{1+s}{2} - \frac{1}{\delta}e, \quad e_3 = \frac{1-s}{2},$$
hence it is semisimple. Furthermore, we have
$$1 = e_1 + e_2 + e_3, \quad s = e_1 + e_2 - e_3, \quad e = \delta * e_1.$$
Therefore, the primitive orthogonal idempotents also give a basis for $B_2(\delta)$.

- If $\delta = 0$, one can show that $B_2(0)$ has two primitive orthogonal idempotents
$$e_2 = \frac{1+s}{2}, \quad e_3 = \frac{1-s}{2},$$
hence it cannot be semisimple. Furthermore, we have
$$1 = e_2 + e_3 \quad s = e_2 - e_3 \quad e_1 := e.$$
Hence, e_1, e_2, e_3 give a basis for $B_2(\delta)$. [16]

[14] Notice that here $n > d$, unlike the classical case. Intuitively, one can predict that since $B_d(\delta)$ is bigger we need more basis elements to distinguish endomorphisms.

[15] Recall that λ^t denotes the conjugate partition of λ.

[16] We will need this statement later.

Chapter 3

Degrees of up-down tableax and Verma paths

3.1 The degrees of up-down tableaux

In this subsection we follow the paper by Li [33].

We now introduce the degree function on the set of up-down tableaux. Ultimately this degree function will describe the grading on $B_d(\delta)$. For any integer k with $1 \le k \le d$ and up-down tableau $\mathsf{t} = (\lambda^{(0)}, \ldots, \lambda^{(d)})$, define

$$\mathsf{A}_{\mathsf{t}}(k) = \begin{cases} \emptyset, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}, \\ \{\, \beta \in \mathsf{A}(\lambda^{(k)}) \mid \operatorname{res}(\beta) = -\operatorname{res}(\alpha) \text{ and } \beta \neq \alpha \,\}, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}; \end{cases}$$

$$\mathsf{R}_{\mathsf{t}}(k) = \begin{cases} \emptyset, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}, \\ \{\, \beta \in \mathsf{R}(\lambda^{(k)}) \mid \operatorname{res}(\beta) = -\operatorname{res}(\alpha) \,\}, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}. \end{cases}$$

Definition 3.1.1 ([33], Definition 2.7)**.** Suppose $\mathsf{t} = (\lambda^{(0)}, \ldots, \lambda^{(d)})$ is an up-down tableau of size d. For integer k with $1 \le k \le d$, define

$$\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) := \begin{cases} 0, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}, \\ |\mathsf{A}_{\mathsf{t}}(k)| - |\mathsf{R}_{\mathsf{t}}(k)| + \delta_{\operatorname{res}(\alpha), -\frac{1}{2}}, & \text{if } \lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}; \end{cases} \tag{3.1.1}$$

and the *degree* of t is

$$\deg \mathsf{t} := \sum_{k=1}^{d} \deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k).$$

Example 3.1.2 ([33], Examples 2.9-2.10)**.** Let $d = 6$, $\lambda = (1, 1)$ and $\mathsf{t} = \left(\emptyset, \yng(1), \yng(2), \yng(2,1), \yng(2,2), \yng(2,1), \yng(1,1)\right) \in \mathrm{T}^{\mathrm{ud}}_d(\lambda)$. Then we have

$$\deg \mathsf{t} = \begin{cases} -1, & \text{if } \delta = 1, \\ 0, & \text{if } \delta = 0. \end{cases}$$

Indeed, if $\delta = 1$ by (3.1.1), we have $\deg \mathsf{t} = \sum_k \deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k)$, where k takes values such that $\mathsf{t}|_k$ is obtained by removing a node from $\mathsf{t}|_{k-1}$. Hence, we have $\deg \mathsf{t} = \deg(\mathsf{t}|_4 \Rightarrow \mathsf{t}|_5) + \deg(\mathsf{t}|_5 \Rightarrow \mathsf{t}|_6)$. We have $A_{\mathsf{t}}(5) = R_{\mathsf{t}}(5) = A_{\mathsf{t}}(6) = \emptyset, R_{\mathsf{t}}(6) = \{(2, 1)\}$. Since $\delta = 1$, we have $\delta_{\operatorname{res}(\alpha), -\frac{1}{2}} = 0$. Hence, the degree of t is

$$\deg \mathsf{t} = \deg(\mathsf{t}|_4 \Rightarrow \mathsf{t}|_5) + \deg(\mathsf{t}|_5 \Rightarrow \mathsf{t}|_6) = 0 - 1 = -1.$$

Now assume $\delta = 0$. Following the same argument, we have $\deg \mathrm{t} = \deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5) + \deg(\mathrm{t}|_5 \Rightarrow \mathrm{t}|_6)$. We have $A_{\mathrm{t}}(5) = \emptyset, R_{\mathrm{t}}(5) = \{(1,2)\}$. Notice $\mathrm{t}_5 = \mathrm{t}_4 \backslash \{\alpha\}$ where $\alpha = (2,2)$. Since $\mathrm{res}(\alpha) = -\frac{1}{2}$, we have

$$\deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5) = |A_{\mathrm{t}}(5)| - |R_{\mathrm{t}}(5)| + \delta_{\mathrm{res}(\alpha), -\frac{1}{2}} = 0 - 1 + 1 = 0.$$

Similarly, we have $A_{\mathrm{t}}(6) = R_{\mathrm{t}}(6) = \emptyset$. Notice $\mathrm{t}_6 = \mathrm{t}_5 \backslash \{\alpha\}$ where $\alpha = (1,2)$. Since $\mathrm{res}(\alpha) = \frac{1}{2}$, we obtain

$$\deg(\mathrm{t}|_5 \Rightarrow \mathrm{t}|_6) = |A_{\mathrm{t}}(6)| - |R_{\mathrm{t}}(6)| + \delta_{\mathrm{res}(\alpha), -\frac{1}{2}} = 0.$$

Hence, the degree of t is $\deg \mathrm{t} = \deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5) + \deg(\mathrm{t}|_5 \Rightarrow \mathrm{t}|_6) = 0 + 0 = 0$.

Example 3.1.3. Let $d = 5$, $\delta = 0$, $\lambda = (1)$ and $\mathrm{t} = (\emptyset, \square, \emptyset, \square, \square\square, \square) \in \mathrm{T}^{\mathrm{ud}}_d(\lambda)$. Then $\deg \mathrm{t} = 0$. By definition, $\deg \mathrm{t} = \deg(\mathrm{t}|_1 \Rightarrow \mathrm{t}|_2) + \deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5)$. We have $A_{\mathrm{t}}(2) = R_{\mathrm{t}}(2) = \emptyset$. Notice $\mathrm{t}_2 = \mathrm{t}_1 \backslash \{\alpha\}$ where $\alpha = (1,1)$. Since $\mathrm{res}(\alpha) = \frac{1}{2}$, we obtain

$$\deg(\mathrm{t}|_1 \Rightarrow \mathrm{t}|_2) = |A_{\mathrm{t}}(2)| - |R_{\mathrm{t}}(2)| + \delta_{\mathrm{res}(\alpha), -\frac{1}{2}} = 1.$$

Similarly, $A_{\mathrm{t}}(5) = \emptyset, R_{\mathrm{t}}(5) = (1,1)$. Notice $\mathrm{t}_5 = \mathrm{t}_4 \backslash \{\alpha\}$ where $\alpha = (1,2)$. Since $\mathrm{res}(\alpha) = \frac{1}{2}$, we obtain

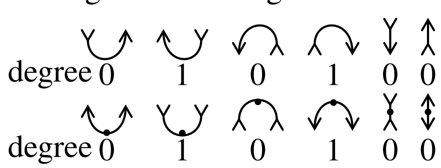
$$\deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5) = |A_{\mathrm{t}}(6)| - |R_{\mathrm{t}}(6)| + \delta_{\mathrm{res}(\alpha), -\frac{1}{2}} = -1.$$

Hence, the degree of t is $\deg \mathrm{t} = \deg(\mathrm{t}|_1 \Rightarrow \mathrm{t}|_2) + \deg(\mathrm{t}|_4 \Rightarrow \mathrm{t}|_5) = 1 - 1 = 0$.

3.2 The degrees of Verma paths

In this subsection we follow the paper by Ehrig and Stroppel [16].

Definition 3.2.1 ([16], Definition 3.14)**.** We define the *degree of a Verma path* associated to the up-down tableau t as the sum of degrees of all figures included in the Verma path minus the total number of cups, where the degree of each figure defined as follows [1]



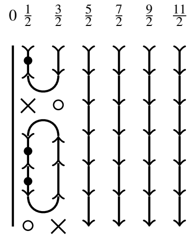
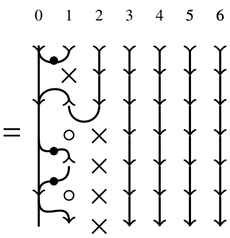
(3.2.1)

Remark 3.2.2. Alternative, we can say that the degree of the Verma path is equal

$$\#\text{ clockwise caps } - \#\text{ anticlockwise cups.}$$

Example 3.2.3. Let us consider the up-down tableaux from the Example 3.1.2. We assign to these up-down tableau via Definition 2.2.23 the following diagrams u_4 and u_5 for $\delta = 1$ and $\delta = 0$ respectively.

[1] Notice that the last two figures in each row are simply directed lines.

$u_4 =$  $\cdots$ $\quad u_5 =$  $\cdots$

$\delta = 1$ $\qquad \delta = 0$

It is easy to see that in these cases the degrees agree with the ones in Example 3.1.2.

In the next subsection we show that the two definitions of degrees agree. But first, let us state an easy lemma which is proven in [16, Proposition 4.9].

Lemma 3.2.4. *The degree of any circle contained in a Verma path* $\mathfrak{t}$ *equals*

- *1 if the rightmost label is* $\vee$*,*
- *-1 if the rightmost label is* $\wedge$ *.*

Remark 3.2.5. Notice that the definition of the degree of a figure given here is shifted by the total number of cups in the figure in comparison to the definition given in [16].

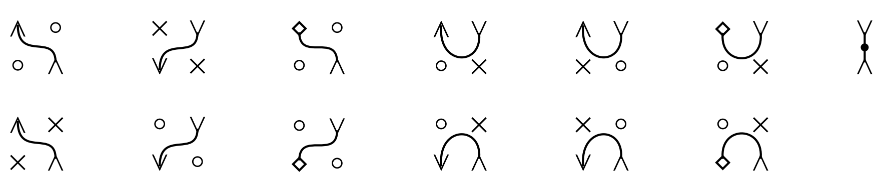
3.3 Equivalence of the two definitions of degrees

We now present the first new result of this thesis – obtained in joint work with Li and Stroppel – which shows that the notion of degree of up-down tableaux and the notion of degree of a Verma path, coincides. In Section 6, we will build on this to obtain an isomorphism between the graded cellular algebras structures each one defines on the Brauer algebra.

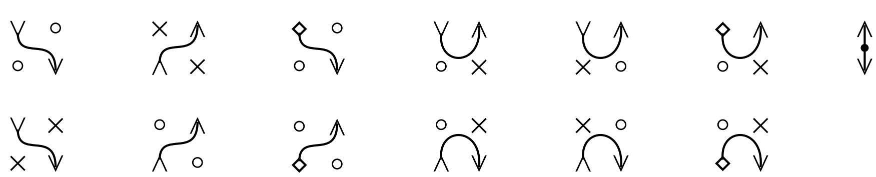
In the rest of this section we fix an up-down tableau $\mathfrak{t} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d)})$ with residue sequence $\mathbf{i} \in T^d$.

Lemma 3.3.1. *We have the following results:*

1) *Suppose* $\lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}$ *for some node* α*. Then the k-th elementary move of the Verma path* $\mathfrak{t}$ *is one of the followings:*



2) *Suppose* $\lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}$ *for some node* α*. Then k-th elementary move of the Verma path* $\mathfrak{t}$ *is one of the followings:*



Proof. This immediately follows from the proof of Lemma 2.2.22 and Definition 2.2.23, since the Young diagrams corresponding to the bottom sequence has one box added (one box removed) compared to the the Young diagram corresponding to the top lines in case (1) (in case (2), respectively). Furthermore, since the diagrams in (1) and (2) cover all possible cases from Definition 2.2.23, we can see that those give all diagrams with one boxed added or removed.

Lemma 3.3.2. *Suppose* $\lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}$ *for some node* α. *Then*

$$\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = \begin{cases} -1 + \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}, & \textit{if there exists } \beta \in \mathsf{R}(\lambda^{(k)}) \textit{ such that } \mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0, \\ 1, & \textit{if there exists } \beta \in \mathsf{A}(\lambda^{(k)}) \textit{ distinct to } \alpha \textit{ such that } \mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0, \\ \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}, & \textit{if for any } \beta \in \mathrm{AR}(\lambda^{(k)}) \textit{ distinct to } \alpha, \textit{ we have } \mathrm{res}(\beta) + \mathrm{res}(\alpha) \neq 0. \end{cases}$$

Proof. We consider each case separately.

Case 1: There exists $\beta \in \mathsf{R}(\lambda^{(k)})$ such that $\mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0$. Therefore there is no $\beta \in \mathsf{A}(\lambda^{(k)})$ with $\mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0$ (it follows from the definition of a Young tableaux that we cannot add two boxes in a row with the same residues) and, thus, we obtain $\mathsf{A}_\mathsf{t}(k) = \emptyset$ and $\mathsf{R}_\mathsf{t}(k) = \{\beta\}$. Hence, $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = -1 + \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}$.

Case 2: There exists $\beta \in \mathsf{A}(\lambda^{(k)})$ distinct to α such that $\mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0$.

In this case we have $\mathsf{A}_\mathsf{t}(k) = \{\beta\}$ and $\mathsf{R}_\mathsf{t}(k) = \emptyset$. Hence we have $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = 1 + \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}$. Notice that by the construction of up-down tableaux, if $\mathrm{res}(\alpha) = -\frac{1}{2}$, this case would never happen. Indeed, in this case we would have that $res(\beta) - res(\alpha) = 1$. Therefore, β would have to be added either above or on the right hand side, which is not possible since α is removed. Hence $\delta_{\mathrm{res}(\alpha),-\frac{1}{2}} = 0$ and we have $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = 1$.

Case 3: For any $\beta \in \mathrm{AR}(\lambda^{(k)})$ distinct to α, we have $\mathrm{res}(\beta) + \mathrm{res}(\alpha) \neq 0$.

In this case we have $\mathsf{A}_\mathsf{t}(k) = \emptyset$ and $\mathsf{R}_\mathsf{t}(k) = \emptyset$. Hence we have $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}$.

Using Lemma 3.3.1 and Lemma 3.3.2, it is easy to obtain the following:

Proposition 3.3.3. *Suppose a is the k-th elementary move of the Verma path* t. *Then we have*

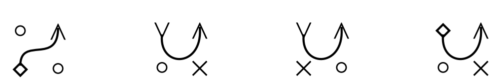
$$\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = \deg(a).$$

Proof. Suppose $\lambda^{(k)} = \lambda^{(k-1)} \cup \{\alpha\}$. We have $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = 0$. By Lemma 3.3.1 (1), we always have $\deg(a) = 0$, which proves the statement in the first case.

Now suppose $\lambda^{(k)} = \lambda^{(k-1)} \backslash \{\alpha\}$. We consider again the three cases as in Lemma 3.3.2.

Case 1: There exists $\beta \in \mathsf{R}(\lambda^{(k)})$ such that $\mathrm{res}(\beta) + \mathrm{res}(\alpha) = 0$.

Using the intuitive description of Verma paths, it is immediate to see that the only graphs that will allow to remove a box with the required residue, are of one of the following forms:



In the first graph, $\mathrm{res}(\alpha) = -i_k = -\frac{1}{2}$ and in the remaining graphs, $\mathrm{res}(\alpha) \neq -\frac{1}{2}$. Hence it is easy to see that $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = -1 + \delta_{\mathrm{res}(\alpha),-\frac{1}{2}}$.

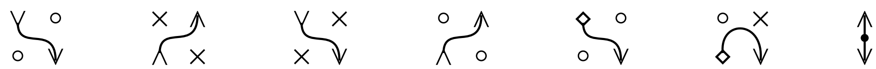
Case 2: There exists $\beta \in \mathsf{A}(\lambda^{(k)})$ distinct to α such that $\operatorname{res}(\beta) + \operatorname{res}(\alpha) = 0$.

In this case the graph a is of one of the following forms:

Both of the graphs have degree 1, which implies $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = 1$.

Case 3: For any $\beta \in \mathrm{AR}(\lambda^{(k)})$ distinct to α we have $\operatorname{res}(\beta) + \operatorname{res}(\alpha) \neq 0$.

In this case the graph a is of one of the following forms:



In the sixth graph, $\operatorname{res}(\alpha) = -i_k = -\frac{1}{2}$ and in the rest of graphs, $\operatorname{res}(\alpha) \neq -\frac{1}{2}$. Hence it is easy to see that $\deg(\mathsf{t}|_{k-1} \Rightarrow \mathsf{t}|_k) = \delta_{\operatorname{res}(\alpha), -\frac{1}{2}}$.

Corollary 3.3.4. *For an up-down tableau* t*, the degrees from Definition 3.1.1 and Definition 3.2.1 coincide.*

In the following sections, we will use the discussed degrees to describe gradings on the $B_d(\delta)$.

Chapter 4

Koszul grading on $B_d(\delta)$

In this section we discuss a framework that gives a tool to study non-semisimple Brauer algebras. We mainly follow the papers by Ehrig, Stroppel [16–18].

4.1 The algebra $C_d(\delta)$

In this subsection we define the algebra $C_d(\delta)$ and discuss some of its properties.

Definition 4.1.1. Let u be a Verma path corresponding to the up-down tableau $((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \ldots, (\lambda^{(d)}, f_d))$. We denote by u^* the Verma path corresponding to the up-down tableaux $((\lambda^{(d)}, f_d), \ldots, (\lambda^{(1)}, f_1), (\lambda^{(0)}, f_0))$ and call it the *reverse Verma path of* u.

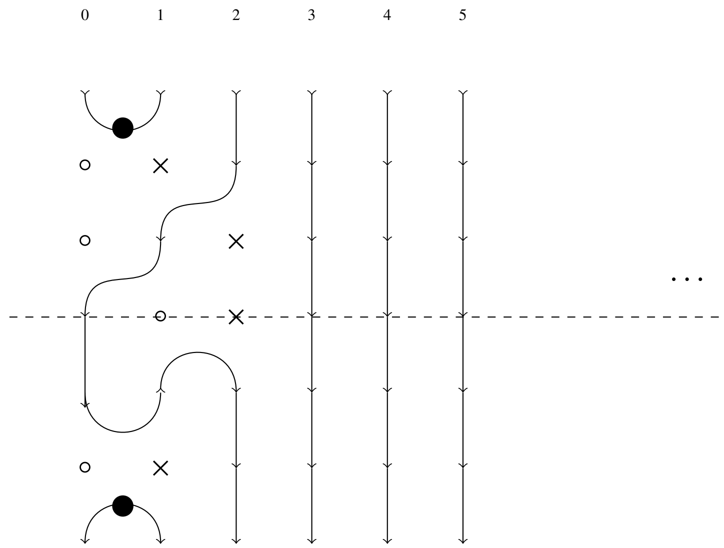
Recall the Definition 3.2.1. As a graded vector space the algebra $C_d(\delta)$ is of the following form.

Definition 4.1.2. Let $C_d(\delta)$ be a vector space with a basis given by $C_d = \{\mathsf{u}\lambda\mathsf{s}^* |\text{ for all } \lambda \in \widehat{B}_d \text{ and } \mathsf{u}, \mathsf{s} \in \mathrm{T}^{\mathrm{ud}}_d(\lambda)\}$. We put a grading on $C_d(\delta)$ by defining $\deg(\mathsf{u}\lambda\mathsf{s}^*) = \deg(\mathsf{u}) + \deg(\mathsf{s})$.

Remark 4.1.3. It is easy to see that C_d can be also identified with the set of Verma paths of shape $(\emptyset, d) \in \widehat{B}_{2d}$.

Definition 4.1.4. Let $\mathsf{t} = \mathsf{u}\lambda\mathsf{s}^* \in C_d(\delta)$, we call u the upper part of a Verma path t, and s^* - the lower part.

Example 4.1.5. Consider the following Verma path. The dashed line separates the upper part from the lower part.



To give $C_d(\delta)$ the structure of an associative unital algebra, we first have to define a multiplication on C_d which heavily depends on a topological structure of the diagrams in $C_d(\delta)$.

Definition 4.1.6. To a given Verma path, let us apply the following

(1) remove all $\circ$ and $\times$'s,

(2) put one dot on the strand if and only if the number of dots on the strand is odd,

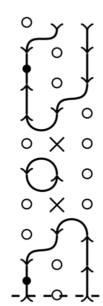
then we call the resulting topological diagram the *topological equivalent* of the original Verma path.

If in addition we apply the following

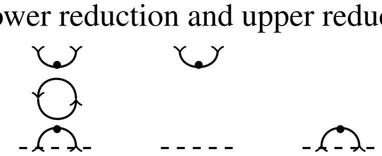
(3) remove all internal circles and cups (respectively, caps),

then the resulting diagram is called the *upper reduction* (respectively, *lower reduction*) of a given Verma path.

Example 4.1.7. Consider the Verma path:



Its topological equivalent, lower reduction and upper reduction are respectively



Recall the notions of the graph and extended graph of a Verma path given before Remark 2.2.18.

Remark 4.1.8. Recall that if two up-down tableaux t_1 and t_2 have the same residue sequence $\mathbf{i}$ then $g_{\mathsf{t}_1} = g_{\mathsf{t}_2}$. Therefore, for an up-down tableau t with a residue sequence $\mathbf{i}$, we denote the graph of t by $g_{\mathbf{i}}$.

Now we define a multiplication making the graded vector space $C_d(\delta)$ into a positively graded associative algebra.

Definition 4.1.9. Let $\mathsf{u}_1\lambda_1\mathsf{s}_1^*, \mathsf{u}_2\lambda_2\mathsf{s}_2^* \in C_d(\delta)$. We define the product $\mathsf{u}_1\lambda_1\mathsf{s}_1^* * \mathsf{u}_2\lambda_2\mathsf{s}_2^*$ in the following way

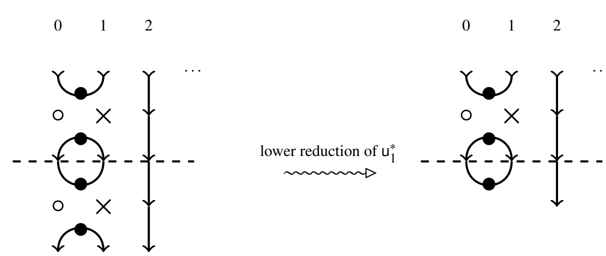
- if $g^e_{s_1^*} = g^e_{u_2^*}$ and and all mirror image pairs of internal circles in s_1^* and u_2 are oriented so that one is clockwise, the other anti-clockwise, then proceed as follows
  (1) glue s_2^* under b to obtain a new oriented diagram $b\lambda_2\mathsf{s}_2^*$, where b is the graph of upper reduction of u_2;
  (2) glue b^* under u_1 to obtain a new oriented diagram $\mathsf{u}_1\lambda_1 b^*$;
  (3) draw $b\lambda_2\mathsf{s}_2$ under the $\mathsf{u}_1\lambda_1 b^*$;
  (4) iterate the generalised surgery procedure described in Section 4.2 below to eliminate all cap-cup pairs in the symmetric middle section of the diagram;
  (5) collapse the middle section to obtained the desired linear combination of basis vectors of $C_d(\delta)$ (see Section 4.2).
- otherwise the product is defined to be zero.

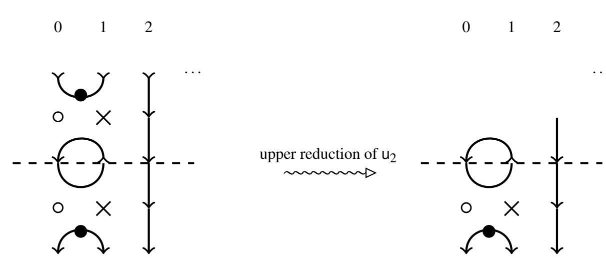
Remark 4.1.10. Note that the orientations of b and b^* in $b\lambda_2\mathsf{s}_2$ and $\mathsf{u}_1\lambda_1 b^*$ are uniquely determined by λ_1 and λ_2.

Example 4.1.11. Assume $d = 2, \delta = 0$. Denote by $\lambda_1 = \emptyset, \lambda_2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ and

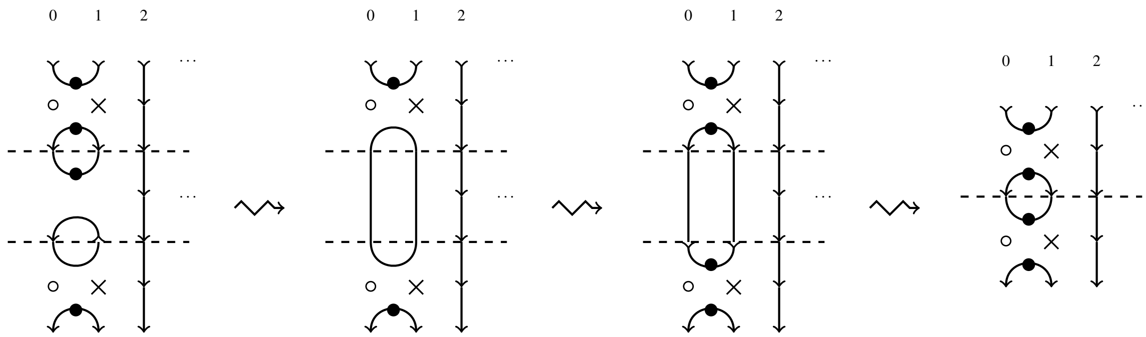
$$\mathsf{u}_1 = (\emptyset, \square, \emptyset), \qquad \mathsf{u}_1 = (\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}).$$

Define $\mathsf{t}_1 := \mathsf{u}_1\lambda_1\mathsf{u}_1^*, \mathsf{t}_2 := \mathsf{u}_2\lambda_2\mathsf{u}_2^*$, then we have

$\mathsf{t}_1 =$ 

$\mathsf{t}_2 =$ 

We can see that the extended graphs of u_1 and u_2 are equal, hence we can apply the surgery procedure. We have



First we perform a single a cut (see (4.2.1)). Notice that in this case two circles are *merged* to one, and since one of the circles is oriented clockwise and another one - anticlockwise, we orient the circle in the third diagram clockwise. Furthermore, since the rightmost vertices of the circles coincide, we multiply the diagram by $(-1)^0$. Finally, we cut the middle part and glue the horizontal dotted line of the upper diagram with the horizontal dotted line of the lower diagram, i.e *collapse the middle part*.

Therefore, we obtain $\mathsf{t}_1 * \mathsf{t}_2 = \mathsf{t}_1$. Similarly, we can show that $\mathsf{t}_2 * \mathsf{t}_1 = \mathsf{t}_1$, $\mathsf{t}_1 * \mathsf{t}_1 = 0$, $\mathsf{t}_2 * \mathsf{t}_2 = \mathsf{t}_2$.

For a residue sequence $\mathbf{i}$, consider the unoriented graph $g_{\mathbf{i}} g_{\mathbf{i}}^*$. It is symmetric around the horizontal middle line. All the loops in this graph are either crossing the middle line, or come in pairs as floating loops (i.e not crossing the middle line). In particular, every line segment involved in an elementary move from 2.2.1 at each level $l \in \{1, \dots, d\}$ has its mirror image at the level $2d + 1 - l$.

Definition 4.1.12. Let $\mathsf{u}\lambda\mathsf{s}^* \in C_d(\delta)$ with a graph g. Assume a line segment in g, involved in an elementary move from 2.2.1 at a level $l \in \{1, \dots, d\}$, has its mirror image at the level $2d + 1 - l$. Then we call such a pair of segments *partners*.

We have the following proposition.

Proposition 4.1.13. *$C_d(\delta)$ is a an unital associative algebra with multiplication described in Definition 4.1.9 and unit*

$$1 = \sum_{i \in T^d} \sum_{\lambda} g_{\mathbf{i}} \lambda g_{\mathbf{i}}^*,$$

where the sum takes over all the oriented graphs $g_{\mathbf{i}} \lambda g_{\mathbf{i}}^*$ such that

- *the loops crossing the middle line are oriented anti-clockwise;*
- *partners contained in strands or loops crossing the middle have the same orientation;*
- *partners contained in floating loops are oriented oppositely, and hence the pairs of floating loops are oriented oppositely.* [1]

The main theorem of this section is the following.

[1] The condition on partners reassures that all Verma paths that appear in the sum have the following property: the extended graphs of the upper part and the reversed lower part are the same. Therefore, all elements in the sum are idempotents.

Theorem 4.1.14 ([17], Proposition 5.3)**.** *The algebra $C_d(\delta)$ is a graded cellular algebra with a cell datum given by*

- $\Lambda := \widehat{B}_d$,
- $T(\lambda) := \mathsf{T}^{ud}_d(\lambda)$,
- $a^{\lambda}_{\mathsf{st}} := \mathsf{s}\lambda\mathsf{t}$,
- $*(\mathsf{s}\lambda\mathsf{t}^*) = \mathsf{t}\lambda\mathsf{s}^*$,
- $\deg(\mathsf{s}\lambda\mathsf{t}^*) = \deg(\mathsf{s}) + \deg(\mathsf{t})$.

for $\mathsf{s}, \mathsf{t} \in \widehat{B}_d, \lambda \in \Lambda$.

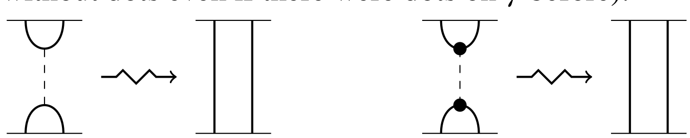
4.2 Surgery procedure

In this subsection, we describe the generalized surgery procedure for cap-cup pairs given in [16]. We would like to stress that the surgery procedure is quite involved, but in this thesis, we will only need rather simple cases.

Definition 4.2.1. Let λ be a partition and i be an integer or half-interger. Define $\mathrm{pos}_\lambda(i)$ to be the number of $\wedge'$s and $\vee'$s on the left of i in the diagrammatic partition corresponding to λ including the one at i.

Let $\mathsf{u}_1\lambda_1 b^*$ and $b\lambda_2\mathsf{s}_2$ be as defined in Definition 4.1.9. Pick a symmetric pair γ of a cup and a cap (possibly dotted) in the middle section of $(\mathsf{u}_1\lambda_1 b^*)(b\lambda_2\mathsf{s}_2)$ that can be connected without crossing rays or arcs and such that to the right of γ there are no dotted arcs.

Now perform this replacement, called a *single cut*, forgetting orientations for a while, cut open the cup and the cap in γ and stitch the loose ends together to form a pair of vertical line segments (always without dots even if there were dots on γ before):



(4.2.1)

Definition 4.2.2. All possible *unoriented* diagrams obtained after applying a finite number of single cuts are called *double diagrams* corresponding to the pair $(\mathsf{u}_1\lambda_1 b^*, b\lambda_2\mathsf{s}_2)$.

Since $\mathsf{u}_1\lambda_1 b^*$ and $b\lambda_2\mathsf{s}_2$ are fixed, we refer to these diagrams simply as double diagrams. Furthermore for any double diagram D let V_D denote the vector space with basis of all orientations of the diagram D. To describe a surgery procedure, we need a linear map $\mathrm{surg}_{D,D'} : V_D \to V_{D'}$, called a *surgery map*, where D is a double diagram and D' is some single cut of D. But before let us outline the surgery procedure.

We start from $(\mathsf{u}_1\lambda_1 b^*)(b\lambda_2\mathsf{s}_2)$ and each time we perform a singe cut, we apply to it a surgery map corresponding to its diagram and the single cut. We compose these maps until we reach a situation without any cup-cap-pairs left in the middle part. It is very important, that at each step, the obtained diagram is *admissible*, i.e each dot on the oriented diagram can be connected to a wall [2] without crossing a diagram. That is the result of the multiplication of $(\mathsf{u}_1\lambda_1 b^*)$ and

[2]Recal the notion of the wall defined in Section 2.2

$(b\lambda_2\mathsf{s}_2)$ will be a *linear combination* of oriented diagrams where the underlying shape is just obtained by applying (4.2.1) to all cup-cap pairs (see Example below).

Now let us define the surgery maps. The definition of the map $\mathrm{surg}_{D,D'}$ depends on whether the number of components increases, decreases or stays the same when passing from D to D'. There are three possible scenarios:

(1) *Merge:* two components are replaced by one component, or

(2) *Split:* one component is replaced by two components, or

(3) *Reconnect:* two components are replaced by two new components.

To define the surgery map $\mathrm{surg}_{D,D'}$ attached to γ denote by i the left and by j the right vertex of the cup (or equivalently the cap) in γ. Depending on these numbers $i < j$ we define $\mathrm{surg}_{D,D'}$ on the basis of orientations of D:

(1) *Merge:* Assume components C_1 and C_2 are merged into component C.

  (a) If both are clockwise circles or one is a clockwise circle and the other a line, the result is zero.

  (b) If both circles are anticlockwise, then apply (4.2.1) and orient the result anticlockwise.

  (c) If one component is anticlockwise and one is clockwise, apply (4.2.1), orient the result clockwise and also multiply with $(-1)^a$, where a is defined in (4.2.2).

(2) *Split:* Assume component C splits into C_i and C_j (containing the vertices at i respectively j). If, after applying (4.2.1), the diagram is not orientable, the map is just zero. Hence assume that the result is orientable, then we have:

  (a) f C is clockwise, then apply (4.2.1) and orient C_i and C_j clockwise. Finally multiply with $(-1)^{\mathrm{pos}(i)}(-1)^{a_r}$, where $\mathrm{pos}(i) = \mathrm{pos}_{\lambda_1}(i)$ (which also equals $\mathrm{pos}_{\lambda_2}(i)$) and a_r is defined in (4.2.3).

  (b) If C is anticlockwise, then apply (4.2.1) and take two copies of the result. In one copy orient C_j anticlockwise and C_i clockwise and moreover multiply with $(-1)^{\mathrm{pos}(i)}(-1)^{a_i}$; in the other copy orient C_i anticlockwise and C_j clockwise and multiply with $(-1)^{\mathrm{pos}(i)}(-1)^{a_j}$, where a_i and a_j are defined in (4.2.4).

(3) *Reconnect:* In this case two lines are transformed into two lines by applying (4.2.1). If the new diagram is orientable, necessarily with the same orientation as before, do so, otherwise the result is zero.

For a component C in an orientable diagram D obtained by the above procedure we denote by $\mathrm{t}(C)$ the rightmost vertex on C. For two vertices r, s in D connected by a sequence of arcs let $\mathrm{a}_D(r, s)$ be the total number of undotted cups plus undotted caps on such a connection. (In fact, the pairity of this number is independent of the choice of the connection, see [16, Remark 5.8].)

Then the signs are defined as follows: in the merge case in question

$$a = \mathrm{a}_D(\mathrm{t}(C_r), \mathrm{t}(C)), \tag{4.2.2}$$

where C_r ($r \in \{1,2\}$) is the clockwise component (necessarily a circle); and for the split cases

$$a_r = \mathrm{a}_{D'}(r, \mathrm{t}(C_r)) + u, \tag{4.2.3}$$

where $r \in \{i, j\}$ such that C_r does not contain $\mathrm{t}(C)$ and $u = 1$ if γ is undotted and $u = 0$ if it is dotted. Finally

$$a_j = \mathrm{a}_{D'}(j, \mathrm{t}(C_j)) \text{ and } a_i = \mathrm{a}_{D'}(i, \mathrm{t}(C_i)) + u, \tag{4.2.4}$$

where $u = 1$ if γ is undotted and $u = 0$ if γ is dotted.

Remark 4.2.3. Let us recall Example 4.1.11, and discuss some of the important features of the surgery procedure. Notice, when we apply the surgery procedure to $\mathsf{u}_1\lambda_1 b^*$ and $b\lambda_2\mathsf{s}_2$, *only* the part that contains the upper reduction of u_1, denoted by a_1, can be changed, the rest of the diagram stays the same in all steps of the procedure. Similarly, *only* the part that contains the lower reduction of s_2, denoted by a_2^*, can be changed, the rest of the diagram stays the same. For example, in Example 4.1.11, the cup of t_1 and the cap of t_2^* were not affected during the procedure. Furthermore, if we denote by $\mathsf{u}\lambda\mathsf{s}^* := \mathsf{u}_1\lambda_1\mathsf{s}_1^* * \mathsf{u}_2\lambda_2\mathsf{s}_2^*$, then u has the same extended graph as u_1, and s has the same extended graph as s_2^*. Therefore, the determine $\mathsf{u}\lambda\mathsf{s}^*$, it is enough to determine new orientations for the part of the graph of $\mathsf{u}\lambda\mathsf{s}^*$ that contains a_1 and a_2^*. To this end, we can only apply the surgery procedure to $a_1\lambda b^*$ and $b\lambda a_2$. In the following, we will only illustrate the *"essential"* part of the surgery procedure, and work with $a_1\lambda b^*$ and $b\lambda a_2$, instead of illustrating the whole Verma paths.

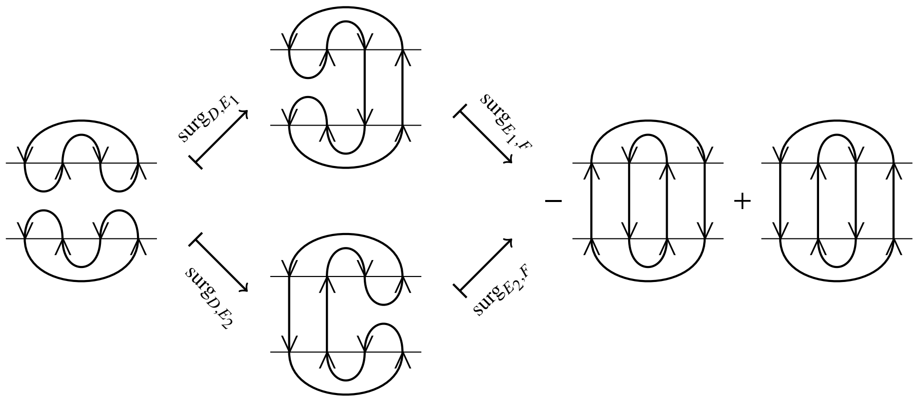
Remark 4.2.4. Please find numerous examples of the surgery procedure in Appendixes 9.2-9.5, where we also only illustrate the "essential" part for the calculations (we will give more details later).

The following examples are taken from [16]. For simplicity, we assume that the first coordinate of the leftmost vertex is $1/2$[3] (and hence δ is odd). Recall Examples 2.2.5-2.2.7, to see how these type of shapes can appear in $a_1\lambda b^*$ and $b\lambda a_2$.

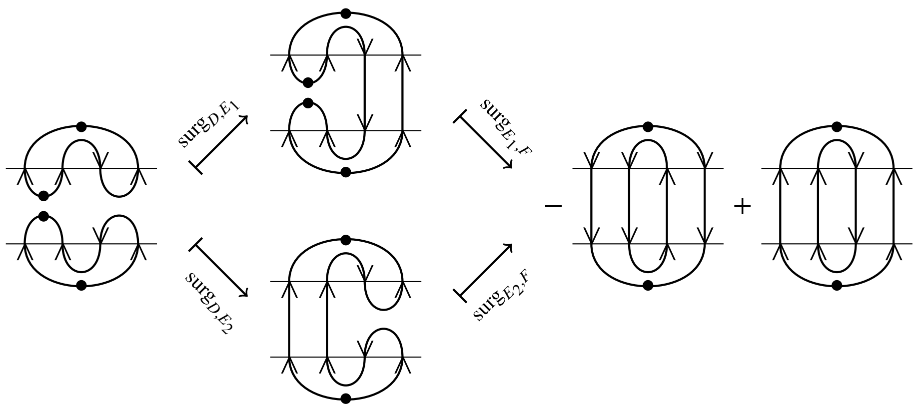
Example 4.2.5. The following example illustrates that the result does not depend on the order we choose for the cup-cap-pairs. Indeed, in the first step, we are in the case (1.b), i.e. merge two anti-clockwise circles and obtain one anti-clockwise circle, denoted by C. In the second step, we are in the case (2.b), i.e. we split the diagram. Let us assume we follow the "upper case", i.e apply $\mathrm{surg}_{E_1,F}$. Then, we have $i = 1/2$, $j = 3/2$. Furthermore, $\mathrm{pos}(i) = 1$, $a_i = 1 + 1 = 2$, $a_j = 1$. Hence, in the copy where C_i is clockwise, with multiply with (-1), in the copy where C_i is anti-clockwise with multiply with $(-1)^0 = 1$.

Now assume we follow the lower case, i.e apply $\mathrm{surg}_{E_2,F}$. Then, we have $i = 5/2$, $j = 7/2$. Furthermore, $\mathrm{pos}(i) = 3$, $a_i = 0 + 1$, $a_j = 0$. Hence, in the copy where C_i is clockwise, with multiply with $(-1)^0 = 1$, in the copy where C_i is anti-clockwise with multiply with $(-1)^1$.

[3] This will allow us to calculate $pos(i)$ in the formulas.



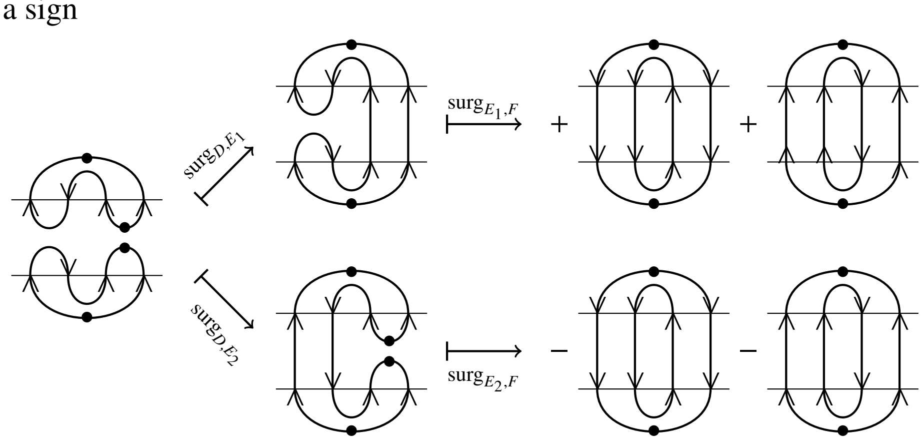
The next example is similar, but with a different number of dotted arcs.

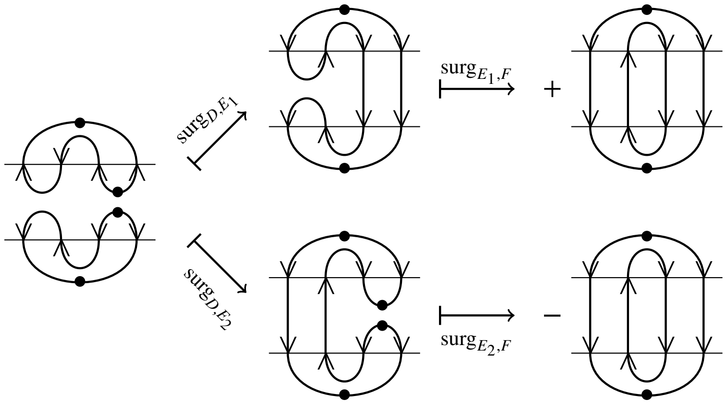


Notice that we multiply two degree 1 elements resulting in a linear combination of degree 2 elements.

Recall that we only defined the surgery maps for admissible diagrams, not all surgery maps in the next example are formally defined. Nevertheless it illustrates what goes wrong if one would ignore the admissibility assumptions and define the surgery in the "obvious" way.

Example 4.2.6. In these examples we illustrate that the admissibility assumptions are really necessary to get a well-defined multiplication. Namely, we apply the surgery maps formally even if the obtained diagram is not admissible. One observes that the two maps differ exactly by a sign





The one can find more examples in [16, Section 6.3]

4.3 The isomorphism between $B_d(\delta)$ and $C_d(\delta)$

The main theorem of this section is the following.

Theorem 4.3.1. *[17, Theorem 4.3] There exist an isomorphism $\Phi_C : C_d(\delta) \longrightarrow B_d(\delta)$ of (ungraded) algebras $C_d(\delta)$ and $B_d(\delta)$. Therefore, the Brauer algebras are graded cellular algebras.*

As we mentioned before, the proof of the theorem above is very complicated even in case $d = 2$ and consists of few steps. The idea is to connect a certain level 2-cyclotomic quotient of VW-algebra with a bigger algebra containing Verma paths and then to realize the Brauer algebras as idempotent truncation of that cyclotomic quotient.

Let us also state the following result, which will be used later.

Proposition 4.3.2 ([17], Proposition 4.6)**.** *The isomorphic image of Gelfand-Zetlin algebra $\mathscr{L}_d$ is generated by Verma paths with graphs $g_{\mathbf{i}}g^*_{\mathbf{i}}$, where $\mathbf{i} \in T^d$.*

Now let us consider the case $d = 2$.

Example 4.3.3. Assume $d = 2$. Then we have $\widehat{B_2} = \{\, \lambda_1 := \emptyset, \lambda_2 := \square\square, \lambda_3 := \begin{smallmatrix}\square\\\square\end{smallmatrix} \,\}$. Denote by

$$\mathsf{u}_1 = (\emptyset, \square, \emptyset) \quad \mathsf{u}_2 = (\emptyset, \square, \square\square) \quad \mathsf{u}_3 = (\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix})$$

Then, the basis of $C_2(\delta)$ is given by

$$\{\, \mathsf{t}_1 := \mathsf{u}_1\lambda_1\mathsf{u}_1^*, \mathsf{t}_2 := \mathsf{u}_2\lambda_2\mathsf{u}_2^*, \mathsf{t}_3 := \mathsf{u}_3\lambda_3\mathsf{u}_3^* \,\}$$

Furthermore, we have

$$\deg t_1 = 2\delta_{0,\delta}, \quad \deg \mathsf{t}_2 = \deg t_3 = 0.$$

Recall the notations from Example 2.4.9. The following is the consequence of the Theorem above.

Theorem 4.3.4. *There exist an isomorphism Φ_C of (ungraded) algebras $B_2(\delta)$ and $C_2(\delta)$ given by* [4]

$$\begin{array}{rccl} \Phi_C : & B_2(\delta) & \longrightarrow & C_2(\delta) \\ & s & \longmapsto & \mathsf{t}_1 + \mathsf{t}_2 - \mathsf{t}_3, \\ & e & \longmapsto & \delta\mathsf{t}_1 + \delta_{0,\delta}\mathsf{t}_1. \end{array} \tag{4.3.1}$$

[4] Here $\delta_{*,*}$ is the Kronecker-Symbol.

Proof. Using calculations given in Example 4.1.11, it is easy to check that Φ_C is a well-defined algebra homomorphism. Furthermore, since both algebras have dimension 3, we obtain that Φ_C is an isomorphism.

Remark 4.3.5. Using the formulas from Example 2.4.9, we can easily check that we have the following

$$\begin{array}{rccc} \Phi_C: & B_2(\delta) & \longrightarrow & C_2(\delta) \\ & e_1 & \longmapsto & \mathsf{t}_1 \\ & e_2 & \longmapsto & \mathsf{t}_2 \\ & e_3 & \longmapsto & \mathsf{t}_3 \end{array} \tag{4.3.2}$$

Chapter 5

KLR grading on $B_d(\delta)$

In this section we follow the paper by Li [33], and recall his construction which gives another tool to study non-semisimple Brauer algebras.

5.1 *KLR*-grading on S_d

Before defining $G_d(\delta)$, we recall the *KLR* presentation of the group algebra of symmetric group S_d given in [8].

Definition 5.1.1. The *cyclotomic Khovanov-Lauda–Rouquier algebra*, R_d is the unital associative $\mathbb{C}$-algebra with generators

$$\{\psi_1, \dots, \psi_{d-1}\} \cup \{y_1, \dots, y_d\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in \mathbb{Z}^d \,\}$$

and relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in\mathbb{Z}^d} e(\mathbf{i}) = 1, \qquad y_1^{\delta_{i_1,0}}e(\mathbf{i}) = 0,$$

$$y_r e(\mathbf{i}) = e(\mathbf{i})y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$

$$\begin{aligned} \psi_r y_s &= y_s \psi_r, && \text{if } s \neq r, r+1, \\ \psi_r \psi_s &= \psi_s \psi_r, && \text{if } |r-s| > 1, \end{aligned}$$

$$\psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r\psi_r + 1)e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ y_r\psi_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$y_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r + 1)e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \pm 1, \\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_{r+1} = i_r - 1 \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r = i_{r+1} + 1, \\ \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{i}), & \text{otherwise.} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ and all admissible r and s.

Moreover, R_d is naturally $\mathbb{Z}$-graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg y_r = 2 \qquad \text{and} \qquad \deg \psi_s e(\mathbf{i}) = \begin{cases} -2, & \text{if } i_s = i_{s+1}, \\ 0, & \text{if } i_s \neq i_{s+1} \pm 1, \\ 1, & \text{if } i_s = i_{s+1} \pm 1, \end{cases}$$

for $1 \leq r \leq d$, $1 \leq s < d$ and $\mathbf{i} \in \mathbb{Z}^d$.

Theorem 5.1.2 ([8])**.** *We have $R_d \cong \mathbb{C}S_d$.*

Remark 5.1.3. Similar presentation holds RS_d, where R is a field of characteristic 0 (see [8]).

The first cellular basis for $\mathbb{C}S_d$ was constructed by Murphy in [36]. Later in [23], Hu-Mathas constructed a graded cellular basis for $\mathbb{C}S_d$, and hence showed that $\mathbb{C}S_d$ is a graded cellular algebra. Let us recall this basis.

Suppose λ is a partition and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We consider t as an up-down tableau for $\delta = 1$. Let $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$ be its residue sequence.

Let $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda)$ be such that $\mathsf{t}^\lambda \trianglerighteq \mathsf{t}$ for all $\mathsf{t} \in \mathrm{Std}(\lambda)$ (see Example 2.1.14) and let $\mathbf{i}_\lambda = (i_1, \dots, i_n)$ be the residue sequence of t^λ. Furthermore, for any $\mathsf{t} \in \mathrm{Std}(\lambda)$, let $d(\mathsf{t}) \in S_d$ be such that $\mathsf{t} = \mathsf{t}^\lambda d(\mathsf{t})$.

For $w \in S_d$ define

$$\psi_w = \psi_{i_1}\psi_{i_2}\dots\psi_{i_\ell} \in R_d \quad \text{and} \quad \psi_w^* = \psi_{i_\ell}\psi_{i_{\ell-1}}\dots\psi_{i_2}\psi_{i_1} \in R_d,$$

where $w = s_{i_1}s_{i_2}\dots s_{i_m}$ is the reduced expression of w.

Definition 5.1.4. Suppose λ is a partition and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$. Define

$$\psi_{\mathsf{st}}^{R_d} = \psi_{d(\mathsf{s})}^* e(\mathbf{i}_\lambda)\psi_{d(\mathsf{t})} \in R_d.$$

Theorem 5.1.5 ([23], Theorem 5.14)**.** *We have*

$$\{\, \psi_{\mathsf{st}}^{R_d} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \textit{ for } \lambda \in \mathrm{Par}_d \,\}$$

is a graded cellular basis of R_d.

5.2 The algebra $G_d(\delta)$

The definition of the algebra $G_d(\delta)$ is similar to Definition 5.1.1 but uses more relations/cases and requires more notation. One could skip this part and have just a brief glance on Definition 5.2.4. Please see also Section 5.4 for the detailed description of $G_2(\delta)$.

Recall $I = \frac{\delta-1}{2} + \mathbb{Z}$. Now we define a mapping $h_k : I^d \longrightarrow \mathbb{Z}$ which divides I^d into three mutually exclusive subsets. We will use this division to determine the relations and the degree of generators of the graded algebra $G_d(\delta)$.

Suppose $\mathbf{i} = (i_1, i_2, \dots, i_d) \in I^d$ and k is an integer with $1 \leq k \leq d$. We define

$$\begin{aligned} h_k(\mathbf{i}) \;:=\; & \delta_{i_k, -\frac{\delta-1}{2}} + \#\{\, 1 \leq r \leq k-1 \mid i_r = -i_k \pm 1 \,\} + 2\#\{\, 1 \leq r \leq k-1 \mid i_r = i_k \,\} \\ & -\delta_{i_k, \frac{\delta-1}{2}} - \#\{\, 1 \leq r \leq k-1 \mid i_r = i_k \pm 1 \,\} - 2\#\{\, 1 \leq r \leq k-1 \mid i_r = -i_k \,\}. \end{aligned}$$

Definition 5.2.1 ([33], p. 10)**.** Let λ be a partition. We define the λ-signature and λ-residue of a box α in the following way

$$\mathrm{sign}_\lambda(\alpha) = \begin{cases} 0, & \text{if } \alpha \in \mathsf{A}(\lambda), \\ 1, & \text{if } \alpha \in \mathsf{R}(\lambda), \end{cases} \qquad \text{and} \qquad \mathrm{res}_\lambda(\alpha) = \begin{cases} \mathrm{res}(\alpha), & \text{if } \alpha \in \mathsf{A}(\lambda), \\ -\mathrm{res}(\alpha), & \text{if } \alpha \in \mathsf{R}(\lambda), \end{cases}$$

and for any $i \in I$, denote $\mathrm{AR}_\lambda(i) = \{\, \alpha \in \mathrm{AR}(\lambda) \mid \mathrm{res}_\lambda(\alpha) = i \,\}$.

Recall $T^d \subset I^d$ is the set containing all the residue sequences of up-down tableaux. The key point of $h_k(\mathbf{i})$ is the following lemma proven in [33, 3.4 - 3.6], which explicitly expresses the connection of $\mathbf{i}$ and the construction of λ.

Lemma 5.2.2 ([33], Lemmas 3.4 - 3.6)**.** *Suppose $\mathbf{i} \in I^d$ such that $\mathbf{i}|_{k-1}$ is the residue sequence of some up-down tableaux with shape (λ, f). Then we have the following properties:*

(1) $h_k(\mathbf{i}) = |\,\mathrm{AR}_\lambda(-i_k)| - |\,\mathrm{AR}_\lambda(i_k)|$,

(2) if $\mathbf{i} \in T^d$ *then* $h_k(\mathbf{i}) \in \{-2, -1, 0\}$,

(3) if $\mathbf{i} \in T^d$ *and* $i_k = 0$ *then* $h_k(\mathbf{i}) = 0$,

(4) if $\mathbf{i} \in T^d$ *and* $i_k = \pm\frac{1}{2}$ *then* $h_k(\mathbf{i}) \in \{-1, -2\}$.

Remark 5.2.3. For given $\mathfrak{t} \in \mathscr{T}^{\mathrm{ud}}_d(\mathbf{i})$ and $\lambda = \mathfrak{t}_{k-1}$ where $1 \le k \le d$, the value of $h_k(\mathbf{i})$ uniquely determines $|\,\mathrm{AR}_\lambda(\pm i_k)|$. Since $|\,\mathrm{AR}_\lambda(i_k)| \ge 1$ as $\mathbf{i}$ is the residue sequence of $\mathfrak{t}$ and $0 \le |\,\mathrm{AR}_\lambda(-i_k)| + |\,\mathrm{AR}_\lambda(i_k)| \le 1$ by the construction of up-down tableaux, we have

$$(|\,\mathrm{AR}_\lambda(-i_k)|, |\,\mathrm{AR}_\lambda(i_k)|) = \begin{cases} (0,2), & \text{if } h_k(\mathbf{i}) = -2, \\ (0,1), & \text{if } h_k(\mathbf{i}) = -1, \\ (1,1), & \text{if } h_k(\mathbf{i}) = 0. \end{cases}$$

Define $I^d_{k,+}$, $I^d_{k,-}$ and $I^d_{k,0}$ as subsets of I^d by

$$I^d_{k,+} := \{\, \mathbf{i} \in I^d \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -1 \,\}. \tag{5.2.1}$$

$$I^d_{k,-} := \{\, \mathbf{i} \in I^d \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -2, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -3 \,\}, \tag{5.2.2}$$

$$I^d_{k,0} := I^d \backslash (I^d_{k,+} \cup I^d_{k,-}). \tag{5.2.3}$$

Hence we split I^d into three mutually exclusive subsets. These subsets will be used in defining the degree of $e(\mathbf{i})\epsilon_k e(\mathbf{j})$ in the graded algebra.

For $\mathbf{i} \in I^d$ and $1 \le k \le d-1$, define $\alpha_k(\mathbf{i}) \in \mathbb{Z}$ and $A^{\mathbf{i}}_{k,1}, A^{\mathbf{i}}_{k,2}, A^{\mathbf{i}}_{k,3}, A^{\mathbf{i}}_{k,4} \subset \{1, 2, \ldots, k-1\}$ by

$$\begin{aligned}\alpha_k(\mathbf{i}) = \#\{\, 1 \le r \le k-1 \mid i_r = i_k \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -i_k \,\} \\ + \#\{\, 1 \le r \le k-1 \mid i_r = i_k - 1 \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -(i_k - 1) \,\} + \delta_{i_k,0} + \delta_{i_k,\frac{\delta-1}{2}},\end{aligned}$$

and

$$\begin{aligned} A^{\mathbf{i}}_{k,1} &:= \{\, 1 \le r \le k-1 \mid i_r = -i_k \pm 1 \,\}, & A^{\mathbf{i}}_{k,2} &:= \{\, 1 \le r \le k-1 \mid i_r = i_k \,\}, \\ A^{\mathbf{i}}_{k,3} &:= \{\, 1 \le r \le k-1 \mid i_r = i_k \pm 1 \,\}, & A^{\mathbf{i}}_{k,4} &:= \{\, 1 \le r \le k-1 \mid i_r = -i_k \,\}; \end{aligned}$$

and for $\mathbf{i} \in I^d_{k,0}$ and $1 \le k \le d-1$, define $z_k(\mathbf{i}) \in \mathbb{Z}$ by

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if } h_k(\mathbf{i}) < -2, \text{ or } h_k(\mathbf{i}) \ge 0 \text{ and } i_k \neq 0, \\ (-1)^{\alpha_k(\mathbf{i})}(1 + \delta_{i_k,-\frac{1}{2}}), & \text{if } -2 \le h_k(\mathbf{i}) < 0, \\ \frac{1+(-1)^{\alpha_k(\mathbf{i})}}{2}, & \text{if } i_k = 0. \end{cases}$$

Now we can give the main definition of this section.

Definition 5.2.4 ([33])**.** Let $G_d(\delta)$ be an unital associate $\mathbb{C}$-algebra with generators

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^d \,\} \cup \{\, y_k \mid 1 \le k \le d \,\} \cup \{\, \psi_k \mid 1 \le k \le d-1 \,\} \cup \{\, \epsilon_k \mid 1 \le k \le d-1 \,\}$$

subject to the following relations:

(1) Idempotent relations: Let $\mathbf{i}, \mathbf{j} \in I^d$ and $1 \le k \le d-1$. Then

$$y_1^{\delta_{i_1, \frac{\delta-1}{2}}} e(\mathbf{i}) = 0, \qquad \sum_{\mathbf{i} \in I^d} e(\mathbf{i}) = 1, \qquad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i}), \qquad e(\mathbf{i})\epsilon_k = 0 \text{ if } i_k + i_{k+1} \neq 0; \tag{5.2.4}$$

(2) Commutation relations: Let $\mathbf{i} \in I^d$, $1 \le k, r \le d-1$, $1 \le l, m \le d$, such that $|k-r| > 1$ and $|k-l| > 1$. Then

$$y_l e(\mathbf{i}) = e(\mathbf{i}) y_l, \qquad \psi_k e(\mathbf{i}) = e(\mathbf{i} \cdot s_k)\psi_k \qquad y_l y_m = y_m y_l, \qquad y_l \psi_k = \psi_k y_l, \tag{5.2.5}$$

$$y_l \epsilon_k = \epsilon_k y_l, \qquad \psi_k \psi_r = \psi_r \psi_k, \qquad \psi_k \epsilon_r = \epsilon_r \psi_k, \qquad \epsilon_k \epsilon_r = \epsilon_r \epsilon_k. \tag{5.2.6}$$

(3) Essential commutation relations: Let $\mathbf{i} \in I^d$ and $1 \le k \le d-1$. Then

$$e(\mathbf{i}) y_k \psi_k = e(\mathbf{i}) \psi_k y_{k+1} + e(\mathbf{i}) \epsilon_k e(\mathbf{i} \cdot s_k) - \delta_{i_k, i_{k+1}} e(\mathbf{i}), \tag{5.2.7}$$

$$e(\mathbf{i}) \psi_k y_k = e(\mathbf{i}) y_{k+1} \psi_k + e(\mathbf{i}) \epsilon_k e(\mathbf{i} \cdot s_k) - \delta_{i_k, i_{k+1}} e(\mathbf{i}). \tag{5.2.8}$$

(4) Inverse relations: Let $\mathbf{i} \in I^d$ and $1 \le k \le d-1$. Then

$$e(\mathbf{i}) \psi_k^2 = \begin{cases} (y_{k+1} - y_k) e(\mathbf{i}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0, \\ (y_k - y_{k+1}) e(\mathbf{i}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ 0, & \text{if } i_k = i_{k+1} \text{ or } i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0, \\ e(\mathbf{i}), & \text{otherwise;} \end{cases} \tag{5.2.9}$$

(5) Essential idempotent relations: Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^d$ and $1 \le k \le d-1$. Then

$$e(\mathbf{i}) \epsilon_k e(\mathbf{i}) = \begin{cases} (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i}), & \text{if } \mathbf{i} \in I^d_{k,0} \text{ and } i_k = -i_{k+1} \neq \pm\frac{1}{2}, \\ (-1)^{\alpha_k(\mathbf{i})+1} (y_{k+1} - y_k) e(\mathbf{i}), & \text{if } \mathbf{i} \in I^d_{k,+}; \end{cases} \tag{5.2.10}$$

$$\begin{aligned} y_{k+1} e(\mathbf{i}) &= y_k e(\mathbf{i}) - 2(-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i}) \epsilon_k e(\mathbf{i}) y_k \\ &= y_k e(\mathbf{i}) - 2(-1)^{\alpha_k(\mathbf{i})} y_k e(\mathbf{i}) \epsilon_k e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in I^d_{k,0} \text{ and } i_k = -i_{k+1} = \frac{1}{2}, \end{aligned} \tag{5.2.11}$$

$$\begin{aligned} e(\mathbf{i}) &= (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i}) \epsilon_k e(\mathbf{i}) + 2(-1)^{\alpha_{k-1}(\mathbf{i})} e(\mathbf{i}) \epsilon_{k-1} e(\mathbf{i}) \\ &\quad - e(\mathbf{i}) \epsilon_{k-1} \epsilon_k e(\mathbf{i}) - e(\mathbf{i}) \epsilon_k \epsilon_{k-1} e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in I^d_{k,0} \text{ and } -i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2}, \end{aligned} \tag{5.2.12}$$

$$e(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i}) (\epsilon_k y_k + y_k \epsilon_k) e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in I^d_{k,-} \text{ and } i_k = -i_{k+1}, \tag{5.2.13}$$

$$e(\mathbf{j}) \epsilon_k e(\mathbf{i}) \epsilon_k e(\mathbf{k}) = \begin{cases} z_k(\mathbf{i}) e(\mathbf{j}) \epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in I^d_{k,0}, \\ (-1)^{\alpha_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}) (\sum_{r \in A^{\mathbf{i}}_{k,1}} y_r - 2 \sum_{r \in A^{\mathbf{i}}_{k,2}} y_r, & \\ \qquad + \sum_{r \in A^{\mathbf{i}}_{k,3}} y_r - 2 \sum_{r \in A^{\mathbf{i}}_{k,4}} y_r) e(\mathbf{j}) \epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in I^d_{k,+}; \\ 0, & \text{if } \mathbf{i} \in I^d_{k,-}, \end{cases} \tag{5.2.14}$$

(6) Untwist relations: Let $\mathbf{i}, \mathbf{j} \in I^d$ and $1 \le k \le d-1$. Then

$$e(\mathbf{i}) \psi_k \epsilon_k e(\mathbf{j}) = \begin{cases} (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i}) \epsilon_k e(\mathbf{j}), & \text{if } \mathbf{i} \in I^d_{k,+} \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases} \tag{5.2.15}$$

$$e(\mathbf{j})\epsilon_k\psi_k e(\mathbf{i}) = \begin{cases} (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{j})\epsilon_k e(\mathbf{i}), & \text{if } \mathbf{i} \in I^d_{k,+} \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise}; \end{cases} \tag{5.2.16}$$

(7) Tangle relations: Let $\mathbf{i}, \mathbf{j} \in I^d$ and $1 < k < d$. Then

$$e(\mathbf{j})\epsilon_k\epsilon_{k-1}\psi_k e(\mathbf{i}) = e(\mathbf{j})\epsilon_k\psi_{k-1}e(\mathbf{i}), \quad e(\mathbf{i})\psi_k\epsilon_{k-1}\epsilon_k e(\mathbf{j}) = e(\mathbf{i})\psi_{k-1}\epsilon_k e(\mathbf{j}), \tag{5.2.17}$$

$$e(\mathbf{i})\epsilon_k\epsilon_{k-1}\epsilon_k e(\mathbf{j}) = e(\mathbf{i})\epsilon_k e(\mathbf{j}); \quad e(\mathbf{i})\epsilon_{k-1}\epsilon_k\epsilon_{k-1}e(\mathbf{j}) = e(\mathbf{i})\epsilon_{k-1}e(\mathbf{j}); \quad e(\mathbf{i})\epsilon_k e(\mathbf{j})(y_k + y_{k+1}) = 0; \tag{5.2.18}$$

(8) Braid relations: Let $\mathcal{B}_k = \psi_k\psi_{k-1}\psi_k - \psi_{k-1}\psi_k\psi_{k-1}$, $\mathbf{i} \in I^d$ and $1 < k < d$. Then

$$e(\mathbf{i})\mathcal{B}_k = \begin{cases} e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1), \quad (5.2.19) \\ -e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1), \quad (5.2.20) \\ e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1), \quad (5.2.21) \\ -e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1), \quad (5.2.22) \\ -(-1)^{\alpha_{k-1}(\mathbf{i})} e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \quad (5.2.23) \\ (-1)^{\alpha_k(\mathbf{i})} e(\mathbf{i})\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2} \text{ and } h_{k-1}(\mathbf{i}) = 0, \quad (5.2.24) \\ e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k - 1, \quad (5.2.25) \\ -e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k + 1, \quad (5.2.26) \\ 0, & \text{otherwise.} \quad (5.2.27) \end{cases}$$

The algebra has a natural grading with $\deg e(\mathbf{i}) = 0$, $\deg y_k = 2$, $\deg e(\mathbf{i})\epsilon_k e(\mathbf{j}) = \deg_k(\mathbf{i}) + \deg_k(\mathbf{j})$, where

$$\deg_k(\mathbf{i}) = \begin{cases} 1, & \text{if } \mathbf{i} \in I^d_{k,+}, \\ -1, & \text{if } \mathbf{i} \in I^d_{k,-}, \\ 0, & \text{if } \mathbf{i} \in I^d_{k,0}; \end{cases} \quad \text{and} \quad \deg e(\mathbf{i})\psi_k = \begin{cases} 1, & \text{if } i_k = i_{k+1} \pm 1, \\ -2, & \text{if } i_k = i_{k+1}, \\ 0, & \text{otherwise}; \end{cases} \tag{5.2.28}$$

Remark 5.2.5 ([33])**.** There exists an involution $*$ on $G_d(\delta)$ such that $e(\mathbf{i})^* = e(\mathbf{i})$, $y_k^* = y_k$, $\psi_r^* = \psi_r$ and $\epsilon_r^* = \epsilon_r$ for $1 \leq k \leq d$ and $1 \leq r \leq d-1$.

Lemma 5.2.6 ([33], Corollary 5.29, Lemma 7.13)**.** *We have the following*

- $e(\mathbf{i}) = 0$ *if* $\mathbf{i} \notin T^d$,
- $e(\mathbf{i})\psi_k e(\mathbf{i}) = 0$ *if* $i_k = i_{k+1} = 0$.

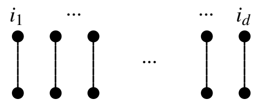
Remark 5.2.7. Notice that since, $e(\mathbf{i}) = 0$ for $\mathbf{i} \notin T^d$ we have only finitely many generators. Moreover, since for any $\mathbf{i} \in T^d$, $e(\mathbf{i})$ is an *orthogonal idempotent* and $\sum_{\mathbf{i} \in I^d} e(\mathbf{i}) = 1$, it is enough to work with the following generators

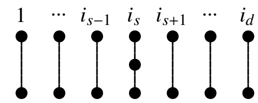
- $e(\mathbf{i})$ for $\mathbf{i} \notin T^d$,
- $e(\mathbf{i})y_s = e(\mathbf{i})y_s e(\mathbf{i})$ for $\mathbf{i} \notin T^d$, $s = 1, \ldots, d$ (since $y_s e(\mathbf{i}) = e(\mathbf{i})y_s$),
- $e(\mathbf{i})\psi_r = e(\mathbf{i})\psi_r e(\mathbf{i}{\cdot}s_r)$ for $\mathbf{i} \notin T^d$, $r = 1, \ldots, d-1$ (since $\psi_r e(\mathbf{i}) = e(\mathbf{i}{\cdot}s_r)\psi_r$), [1]

[1] Where $\mathbf{i}{\cdot}s_r = (i_1, \ldots, i_{r-1}, i_{r+1}, i_r, i_{r+2})$.

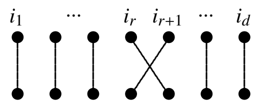
- $e(\mathbf{i})\epsilon_r e(\mathbf{j})$ for $\mathbf{i}, \mathbf{j} \in T^d$ such that $i_r + i_{r+1} = j_r + j_{r+1}$ (since $e(\mathbf{i})\epsilon_r = 0$ if $i_r + i_{r+1} \neq 0$).

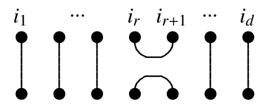
Remark 5.2.8. There is also a diagrammatic presentation of $G_d(\delta)$. We associate to each generator of $G_d(\delta)$ the following diagrams on $2d$ dots:

$e(\mathbf{i}) =$ 

and $y_s e(\mathbf{i}) =$ 

,

$e(\mathbf{i})\psi_r =$ 

and $e(\mathbf{i})\epsilon_r e(\mathbf{j}) =$ 

.

for $\mathbf{i} = (i_1, \ldots, i_d) \in I^d$, $\mathbf{j} = (j_1, \ldots, j_n) \in I^d$, $1 \leq s \leq n$ and $1 \leq r \leq n-1$. The labels connected by a verticle string have to be the same, and the sum of labels connected by a horizontal string equals 0 (see relation 5.2.4).

Diagrams are considered up to isotopy, and multiplication of diagrams is given by concatenation, subject to the relations (5.2.4) - (5.2.27). This diagrammatic presentation will not be used in this thesis.

In [33], Li generalised the construction of the set $\{ \psi^{R_d}_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathrm{Par}_d \}$ and showed the following

Lemma 5.2.9 ([33], Definition 4.9, Lemma 4.10, Remark 5.31)**.** *Let $\lambda \in \widehat{B}_d, \mathsf{s}, \mathsf{t} \in \mathrm{T}^{ud}_d(\lambda)$. Then there exist a homogeneous element $\psi_{\mathsf{st}} \in G_d(\delta)$, such that for any $\mathbf{i}, \mathbf{j} \in I^d$, we have*

$$e(\mathbf{i})\psi_{\mathsf{st}}e(\mathbf{j}) = \begin{cases} \psi_{\mathsf{st}}, & \text{if } \mathbf{i} = \mathbf{i}_{\mathsf{s}} \text{ and } \mathbf{j} = \mathbf{i}_{\mathsf{t}}, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the set $\{ \psi^{R_d}_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathrm{Par}_d \}$ spans $G_d(\delta)$.

Moreover, he have the following.

Theorem 5.2.10. *[33, Theorem 7.77] The algebra $G_d(\delta)$ is graded cellular algebra with a cell datum is given by*

- $\Lambda := \widehat{B}_d$,
- $T(\lambda) := \mathrm{T}^{ud}_d(\lambda)$,
- $a^\lambda_{\mathsf{st}} := \psi^\lambda_{\mathsf{st}}$,
- $\deg \psi^\lambda_{\mathsf{st}} = \deg(\mathsf{s}) + \deg(\mathsf{t})$
- $*(\psi^\lambda_{\mathsf{st}}) = \psi^\lambda_{\mathsf{ts}}$,

for $\mathsf{s}, \mathsf{t} \in \widehat{B}_d, \lambda \in \Lambda$.

5.3 The isomorphism between $B_d(\delta)$ and $G_d(\delta)$

We have the following, which was proved by using the theory of semi-orthogonal forms to define a new presentation of $B_d(\delta)$.

Theorem 5.3.1. *[33, Theorem 7.76] There exist an isomorphism $\Phi_G : B_d(\delta) \longrightarrow G_d(\delta)$ of (ungraded) algebras $G_d(\delta)$ and $B_d(\delta)$. Therefore, the Brauer algebras are graded cellular algebras.*

Furthermore, we have

Proposition 5.3.2 ([33], Section 6.1)**.** *The isomorphic image of Gelfand-Zetlin algebra $\mathscr{L}_d$ is generated by $e(\mathbf{i})$ and $y_1, \dots, y_d$ where $\mathbf{i} \in T^d$.*

5.4 $G_2(\delta)$

In this section we give a rather informal explanation of the structure of $G_2(\delta)$. All results in this Section are consequences of the previous Sections.

Let $d = 2$, recall $I = \frac{\delta-1}{2} + \mathbb{Z}$ and $\mathbf{i} = (i_1, i_2) \in I^2$. We have

$$h_1(\mathbf{i}) := \delta_{i_1,-\frac{\delta-1}{2}} - \delta_{i_1,\frac{\delta-1}{2}}$$
$$h_2(\mathbf{i}) := \delta_{i_2,-\frac{\delta-1}{2}} + \delta_{i_2,(-i_1\pm1)} + 2\delta_{i_2,i_1} - \delta_{i_2,\frac{\delta-1}{2}} - \delta_{i_2,(i_1\pm1)} - 2\delta_{i_2,-i_1}$$

$$\alpha_1(\mathbf{i}) = \delta_{i_1,0} + \delta_{i_1,\frac{\delta-1}{2}},$$

We have

$$I^2_{1,+} := \{\, \mathbf{i} \in I^2 \mid i_1 \neq 0, -\frac{1}{2} \text{ and } h_1(\mathbf{i}) = 0, \text{ or } i_1 = -\frac{1}{2} \text{ and } h_1(\mathbf{i}) = -1 \,\}. \tag{5.4.1}$$
$$I^2_{1,-} := \{\, \mathbf{i} \in I^2 \mid i_1 \neq 0, -\frac{1}{2} \text{ and } h_1(\mathbf{i}) = -2, \text{ or } i_1 = -\frac{1}{2} \text{ and } h_1(\mathbf{i}) = -3 \,\}, \tag{5.4.2}$$
$$I^2_{1,0} := I^2 \backslash (I^2_{1,+} \cup I^2_{1,-}). \tag{5.4.3}$$

Define $z_1(\mathbf{i}) \in \mathbb{Z}$ by

$$z_1(\mathbf{i}) = \begin{cases} 0, & \text{if } h_1(\mathbf{i}) < -2, \text{ or } h_1(\mathbf{i}) \geq 0 \text{ and } i_1 \neq 0, \\ (-1)^{\alpha_1(\mathbf{i})}(1 + \delta_{i_1,-\frac{1}{2}}), & \text{if } -2 \leq h_1(\mathbf{i}) < 0, \\ \frac{1+(-1)^{\alpha_1(\mathbf{i})}}{2}, & \text{if } i_1 = 0. \end{cases}$$

Definition 5.4.1. Let $B_2(\delta)$ be an unital associate $\mathbb{C}$-algebra with generators

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^2 \,\} \cup \{\, y_1, y_2, \psi, \epsilon \,\}$$

subject to the following relations:

(1) Idempotent relations: Let $\mathbf{i}, \mathbf{j} \in I^2$. Then
$$y_1^{\delta_{i_1,\frac{\delta-1}{2}}} e(\mathbf{i}) = 0, \qquad \sum_{\mathbf{i} \in I^2} e(\mathbf{i}) = 1, \qquad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i}), \qquad e(\mathbf{i})\epsilon = 0 \text{ if } i_1 + i_2 \neq 0;$$

(2) Commutation relations: Let $\mathbf{i} \in I^2$, $l = 1, 2$. Then
$$y_l e(\mathbf{i}) = e(\mathbf{i}) y_l, \qquad \psi e(\mathbf{i}) = e(\mathbf{i} \cdot s_1)\psi \qquad y_1 y_2 = y_2 y_1$$

(3) Essential commutation relations: Let $\mathbf{i} \in I^d$. Then
$$\begin{aligned} e(\mathbf{i})y_1\psi &= e(\mathbf{i})\psi y_2 + e(\mathbf{i})\epsilon e(\mathbf{i}{\cdot}s_1) - \delta_{i_1,i_2}e(\mathbf{i}), \\ e(\mathbf{i})\psi y_1 &= e(\mathbf{i})y_2\psi + e(\mathbf{i})\epsilon e(\mathbf{i}{\cdot}s_1) - \delta_{i_1,i_2}e(\mathbf{i}). \end{aligned}$$

(4) Inverse relations: Let $\mathbf{i} \in I^d$. Then
$$e(\mathbf{i})\psi^2 = \begin{cases} (y_2 - y_1)e(\mathbf{i}), & \text{if } i_1 = i_2 + 1 \text{ and } i_1 + i_2 \neq 0, \\ (y_1 - y_2)e(\mathbf{i}), & \text{if } i_1 = i_2 - 1 \text{ and } i_1 + i_2 \neq 0, \\ 0, & \text{if } i_1 = i_2 \text{ or } i_1 + i_2 = 0 \text{ and } h_1(\mathbf{i}) \neq 0, \\ e(\mathbf{i}), & \text{otherwise;} \end{cases}$$

(5) Essential idempotent relations: Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^d$. Then
$$e(\mathbf{i})\epsilon e(\mathbf{i}) = \begin{cases} (-1)^{\alpha_1(\mathbf{i})}e(\mathbf{i}), & \text{if } \mathbf{i} \in I^2_{1,0} \text{ and } i_1 = -i_2 \neq \pm\frac{1}{2}, \\ (-1)^{\alpha_1(\mathbf{i})+1}(y_2 - y_1)e(\mathbf{i}), & \text{if } \mathbf{i} \in I^2_{1,+}; \end{cases}$$
$$\begin{aligned} y_2e(\mathbf{i}) &= y_1e(\mathbf{i}) - 2(-1)^{\alpha_1(\mathbf{i})}e(\mathbf{i})\epsilon e(\mathbf{i})y_1 \\ &= y_1e(\mathbf{i}) - 2(-1)^{\alpha_1(\mathbf{i})}y_1e(\mathbf{i})\epsilon e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in I^2_{1,0} \text{ and } i_1 = -i_2 = \frac{1}{2}, \end{aligned}$$
$$e(\mathbf{i}) = (-1)^{\alpha_1(\mathbf{i})}e(\mathbf{i})(\epsilon y_1 + y_1\epsilon)e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in I^d_{1,-} \text{ and } i_1 = -i_2,$$
$$e(\mathbf{j})\epsilon e(\mathbf{i})\epsilon e(\mathbf{k}) = \begin{cases} z_1(\mathbf{i})e(\mathbf{j})\epsilon e(\mathbf{k}), & \text{if } \mathbf{i} \in I^d_{1,0}, \\ 0, & \text{if } \mathbf{i} \in I^2_{1,-} \text{ or } \mathbf{i} \in I^2_{1,+} \end{cases}$$

(6) Untwist relations: Let $\mathbf{i}, \mathbf{j} \in I^2$. Then
$$e(\mathbf{i})\psi\epsilon e(\mathbf{j}) = \begin{cases} (-1)^{\alpha_1(\mathbf{i})}e(\mathbf{i})\epsilon e(\mathbf{j}), & \text{if } \mathbf{i} \in I^d_{1,+} \text{ and } i_1 \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases}$$
$$e(\mathbf{j})\epsilon\psi e(\mathbf{i}) = \begin{cases} (-1)^{\alpha_1(\mathbf{i})}e(\mathbf{j})\epsilon_1 e(\mathbf{i}), & \text{if } \mathbf{i} \in I^d_{1,+} \text{ and } i_1 \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases}$$

(7) Tangle relations:
$$e(\mathbf{i})\epsilon e(\mathbf{j})(y_1 + y_2) = 0.$$

Corollary 5.4.2. *The following immediately follows from the definition*

(1) $e(\mathbf{i}) = 0$ *if* $i_1 \neq \frac{\delta-1}{2}$ *(rel. 1.1)*

(2) $y_1e(\mathbf{i}) = 0$ *if* $i_1 = \frac{\delta-1}{2}$*, and hence* $y_1 = y_1 * 1 = \sum_{\mathbf{i}\in I^2} y_1e(\mathbf{i}) = \sum_{(\frac{\delta-1}{2},i_2)} y_1e(\frac{\delta-1}{2}, i_2) = 0$, *(rel. 1.1)*

(3) $e(\mathbf{i})\epsilon = 0$ *unless* $i_1 = -i_2 = \frac{\delta-1}{2}$, *(rel. 1.4)*

(4) $\psi e(\mathbf{i}) = 0$ *unless* $i_1 = i_2 = \frac{\delta-1}{2}$ *(rel. 2.2)*

(5) $0 = e(\frac{\delta-1}{2}, \frac{\delta-1}{2})\psi y_2 + e(\frac{\delta-1}{2}, \frac{\delta-1}{2})\epsilon e(\frac{\delta-1}{2}, \frac{\delta-1}{2}) - e(\frac{\delta-1}{2}, \frac{\delta-1}{2})$, *(rel 3)*

- $e(\frac{\delta-1}{2}, \frac{\delta-1}{2})\psi y_2 = e(\frac{\delta-1}{2}, \frac{\delta-1}{2})$, *if* $\delta \neq 1$

(6) $e(\frac{\delta-1}{2}, \frac{\delta-1}{2})\psi y_2 = e(\frac{\delta-1}{2}, \frac{\delta-1}{2})y_2\psi$ *(rel 3)*

(7) $e(\mathbf{i})\psi^2 = 0$ *(rel. 4)*

(8) $e(\mathbf{i}) = 0$ *unless* $\mathbf{i}$ *is one of the following (rel. 4.4)*[2]

- $(\frac{\delta-1}{2}, \frac{\delta-1}{2} + 1)$
- $(\frac{\delta-1}{2}, \frac{\delta-1}{2} - 1)$
- $(\frac{\delta-1}{2}, \frac{\delta-1}{2})$
- $(\frac{\delta-1}{2}, -\frac{\delta-1}{2})$

(9) *we have*

- $h_1(\mathbf{i}) = 0$ *if* $\delta = 1$, *otherwise* $h_1(\mathbf{i}) = -1$
- $I^2_{1,-} \cap \{ (\frac{\delta-1}{2}, a) \mid a \in I \} = \emptyset$
- $I^2_{1,+} \cap \{ (\frac{\delta-1}{2}, a) \mid a \in I \} = \emptyset$ *if* $\delta \neq 0$
- $I^2_{1,+} \cap \{ (\frac{\delta-1}{2}, a) \mid a \in I \} = \{ (\frac{\delta-1}{2}, a) \mid a \in I \}$ *if* $\delta = 0$

(10) $e(\frac{\delta-1}{2}, \frac{\delta-1}{2}) = 0$ *unless* $\delta = 1$. *Indeed, if* $\delta \neq 1$, *then using (rel. 2.1), (rel. 2.3), (rel. 4) and (5)*

$$0 = (e(\frac{\delta-1}{2}, \frac{\delta-1}{2})\psi y_2)^2 = e(\frac{\delta-1}{2}, \frac{\delta-1}{2})$$

(11) $\psi = 0$.

It follows from (10) that we only need to show that $\psi e(0,0) = 0$. *Indeed, we have* $\alpha_1(0,0) = 0$. *From (rel. 5.1), we obtain* $e(0,0)\epsilon e(0,0) = e(0,0)$. *Hence, from (rel. 6), we have* $e(0,0)\psi = 0$

Lemma 5.4.3. *Assume* $\delta = 0$, *then we have*

- $h_1(\mathbf{i}) = -1$,
- $\alpha_1(\mathbf{i}) = 1$,
- $I^2_{1,+} \cap \{ (\frac{\delta-1}{2}, a) \} = \{ (-1/2, 1/2), (-1/2, -3, 2) \}$
- $z_1(-1/2, 1/2) = -2$ *and* $z_1(\mathbf{i}) = -1$ *otherwise*

Lemma 5.4.4. *Assume* $\delta = 0$. *Then* $B_2(0)$ *is generated by*

$$e_2 := e(-\frac{1}{2}, \frac{1}{2}), e_3 := e(-\frac{1}{2}, -\frac{3}{2}), y_2, \epsilon,$$

with relations

(1) $e_2 + e_3 = 1$,

(2) $e_3 e_2 = e_2 e_3 = 0$, $e_i e_i = e_i$ *for* $i = 2, 3$,

(3) $e_3 \epsilon = 0$,

(4) $y_2 e_i = e_i y_2$,

(5) $y_2 e_3 = 0$,

[2] Very important relation which shows that there are only finitely many nonzero $e(\mathbf{i})$.

(6) $e_2\epsilon e_2 = y_2 e_2$,

(7) $e_2\epsilon e_2\epsilon e_2 = 0$.

Moreover, $\deg e_2 = \deg e_3 = 0, \deg y_2 = 2, \deg e_2\epsilon e_2 = 1 + 1 = 2$.

Theorem 5.4.5. *Assume $\delta = 0$. There exist an isomorphism Φ_G of (ungraded) algebras $B_2(\delta)$ and $G_2(\delta)$ given by* [3]

$$\begin{array}{rccc} \Phi_G: & G_2(\delta) & \longrightarrow & B_2(\delta) \\ & \epsilon & \longmapsto & e_1 \\ & y_2 & \longmapsto & e_1 \\ & e_2 & \longmapsto & e_2 \\ & e_3 & \longmapsto & e_3 \end{array} \tag{5.4.4}$$

Therefore, we can see the following

Theorem 5.4.6. *Assume $\delta = 0$. There exist an isomorphism Φ of graded algebras $G_2(\delta)$ and $C_2(\delta)$ given by*

$$\begin{array}{rccc} \Phi_G: & G_2(\delta) & \longrightarrow & C_2(\delta) \\ & e_2 & \longmapsto & \mathrm{t}_2 \\ & e_3 & \longmapsto & \mathrm{t}_2 \\ & \epsilon & \longmapsto & \mathrm{t}_1 \\ & y_2 & \longmapsto & \mathrm{t}_1 \end{array} \tag{5.4.5}$$

Let us carefully look at the Verma paths in the image. We can notice that e_2 and t_2 (respectfully, e_3 and t_3) are both idempotents and the both correspond to the same residue sequence[4] $(-\frac{1}{2}, \frac{1}{2})$ (respectfully, $(-\frac{1}{2}, -\frac{3}{2})$). Furthermore, if the one considers also other values for δ[5] in case $d = 2$, the one can notice that ϵ corresponds to the Verma path[6], where we add and then remove a box (for $d = 2$). Moreover, the one also can guess that $y_2 e(\mathbf{i})$ either is zero or an idempotent with a circle with a reverse orientation.

The one can guess from this map and some other cases (for small d), that there might exist an explicit isomorphism between $G_2(\delta)$ and $C_2(\delta)$. In the following Chapter we prove this result.

[3] Recall the notation from Example 2.4.9

[4] The graph of the upper part and reversed lower part corresponds to this residue sequence.

[5] Another interesting case is $\delta = 2$.

[6] Up to a sign.

Chapter 6

Graded isomorphism between $C_d(\delta)$ and $G_d(\delta)$

In this section we define elements $\bar{e}(\mathbf{i}), \bar{y}_r, \bar{\psi}_k, \bar{\epsilon}_k \in C_d(\delta)$, where $\mathbf{i} \in T^d, 1 \leq r \leq d, 1 \leq k \leq d-1$, as a non-trivial linear combination of basis elements and prove the following main theorem.

Theorem 6.0.1. *The map $\Phi : G_d(\delta) \longrightarrow C_d(\delta)$ determined by*

$$e(\mathbf{i}) \mapsto \bar{e}(\mathbf{i}), \qquad y_r \mapsto \bar{y}_r, \qquad \psi_k \mapsto \bar{\psi}_k, \qquad \epsilon_k \mapsto \bar{\epsilon}_k,$$

where $\mathbf{i} \in T^d, 1 \leq r \leq d, 1 \leq k \leq d-1$, is an isomorphism. Moreover, for any homogeneous element $u \in G_d(\delta)$, we have $\deg u = \deg \phi(u)$.

The results obtained jointly with Li and Stroppel.

6.1 Generators of $C_d(\delta)$

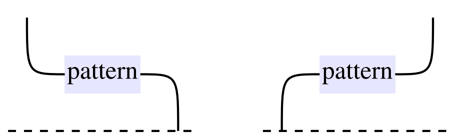
In this subsection we construct elements $\bar{e}(\mathbf{i}), \bar{y}_r, \bar{\psi}_k, \bar{\epsilon}_k \in C_d(\delta)$, where $\mathbf{i} \in T^d, 1 \leq r \leq d, 1 \leq k \leq d-1$.

6.1.1 Preliminaries

As we mentioned above, the elements $\bar{e}(\mathbf{i}), \bar{y}_r, \bar{\psi}_k, \bar{\epsilon}_k$ are linear combinations of the basis elements, but the signs in the combinations are quite complicated and depend on the graphs of the basis elements. In this subsection we define few parameters which will help to determine these signs.

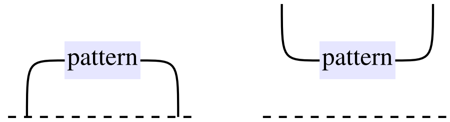
Let $\mathbf{i} \in T^d$ and $1 \leq k \leq d$. We call the pattern of graph $\mathbf{i}$ at level k as *k-pattern*. [1] One can see that a pattern is in one of the following forms (in the topological equivalent of the graph):

(1) as a part of a vertical strand:



[1] More precisely, k-pattern is the elementary move from Definition 2.2.23 that appears between levels $k-1$ and k.

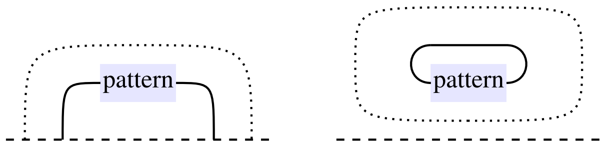
(2) as a part of a horizontal strand:



(3) as a part of a loop:

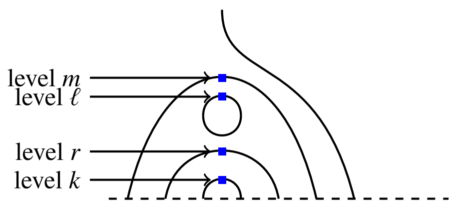
Definition 6.1.1. We say a pattern is *bounded* if the strand or loop containing the pattern is bounded by a horizontal strand, or a loop.

For example, in the following diagrams, the pattern is bounded by the dashed strand or loop:



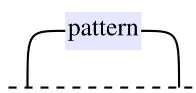
Definition 6.1.2. Define $b_k(\mathbf{i})$ to be the number of strands or loops bound the k-pattern of $\mathbf{i}$.

Example 6.1.3. Suppose the graph of $\mathbf{i}$ is

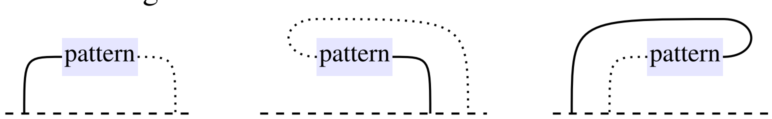


Then the numbers of strands or loops bounding the patterns at level k, r, ℓ and m, respectively are 2, 1, 1 and 0. Hence we have $b_k(\mathbf{i}) = 2$, $b_r(\mathbf{i}) = 1$, $b_\ell(\mathbf{i}) = 1$ and $b_m(\mathbf{i}) = 0$.

Definition 6.1.4. Let the k-pattern belong to a horizontal strand which connects to the bottom line on both sides, i.e.



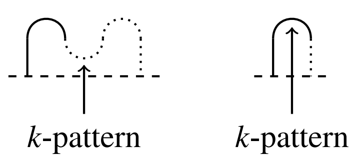
The *right strand* is the part of the strand on the right side of the pattern. In the following graph the dashed strands are the right strands:[2]



The k-pattern itself is a part of the right strand if and only if the k pattern is of the form


or


Example 6.1.5. In the following graph the dashed strands are the right strands:



[2]See Examples 2.2.5, 2.2.6, 2.2.7 to see how can these typo of shapes appear.

Definition 6.1.6. Suppose $\mathbf{i}$ is a graph. For $1 \leq k \leq n$, define $d_k(\mathbf{i})$ such that

(1) If the k-pattern belongs to a horizontal strand which connects to the bottom line on both sides, then define $d_k(\mathbf{i})$ to be the number of dots[3] on the right strand attached to the k-pattern of $\mathbf{i}$.

(2) In the rest of the cases, define $d_k(\mathbf{i}) = 0$.

Remark 6.1.7. Here we are using (an extreme) abuse of notations to *significantly* simplify the explanations and proofs. Notice that $d_k(\mathbf{i})$ does not *only* depend on $\mathbf{i}$ (i.e. graph, see Corollary 2.2.14), but rather on the dotted graph of a Verma path. Recall from the discussion before Corollary 2.2.19 that if δ is odd, then the dotted graph can be "recovered" from the graph. But if δ is even, it might not be possible to do. Therefore, technically it would be more correct to write $d_k(\mathsf{t})$ or $d_k(g_{\mathsf{t}}^{dot})$, where t is a Verma path with a residue sequence $\mathbf{i}$, and g_{t}^{dot} is a dotted graph of the t. But instead let us clarify our notation and show how it will be used.

Assume $\mathsf{u}\lambda\mathsf{s}^* \in C_d(\delta)$ is a *basis* element of $C_d(\delta)$, such that u has a residue sequence $\mathbf{i}$. Then

$$(-1)^{d_k(\mathbf{i})}\mathsf{u}\lambda\mathsf{s}^* := (-1)^{d_k(\mathsf{u})}\mathsf{u}\lambda\mathsf{s}^*.$$

Furthermore, if $\mathsf{u}_1\lambda_1\mathsf{s}_1^*, \mathsf{u}_2\lambda_2\mathsf{s}_2^* \in C_d(\delta)$, are two *basis* elements of $C_d(\delta)$, such that $\mathsf{u}_1, \mathsf{u}_2$ have a residue sequence $\mathbf{i}$, then

$$(-1)^{d_k(\mathbf{i})}(\mathsf{u}_1\lambda_1\mathsf{s}_1^* + \mathsf{u}_2\lambda_2\mathsf{s}_2^*) := (-1)^{d_k(\mathsf{u}_1)}\mathsf{u}_1\lambda_1\mathsf{s}_1^* + (-1)^{d_k(\mathsf{u}_2)}\mathsf{u}_2\lambda_2\mathsf{s}_2^*.$$

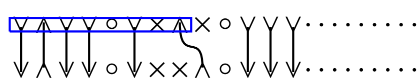
Definition 6.1.8. Suppose $\mathbf{i}$ is a graph and $1 \leq k \leq d$. Define $m_k(\mathbf{i}) = b_k(\mathbf{i}) + d_k(\mathbf{i})$.

Remark 6.1.9. Continuing the previous remark, we set

$$(-1)^{m_k(\mathbf{i})}(\mathsf{u}_1\lambda_1\mathsf{s}_1^* + \mathsf{u}_2\lambda_2\mathsf{s}_2^*) := (-1)^{(d_k(\mathsf{u}_1)+b_k(\mathbf{i}))}\mathsf{u}_1\lambda_1\mathsf{s}_1^* + (-1)^{(d_k(\mathsf{u}_2)+b_k(\mathbf{i}))}\mathsf{u}_2\lambda_2\mathsf{s}_2^*.$$

Definition 6.1.10. Suppose $\mathbf{i}$ is a graph and $1 \leq k \leq d$. Define $\mathrm{pos}_k(\mathbf{i})$ to be the number of $\wedge$'s or $\vee$'s on the left between level $k-1$ and level k if $i_k \neq 0$, and $\mathrm{pos}_k(\mathbf{i}) = 0$ if $i_k = 0$.

Example 6.1.11. Suppose the graph at level $k-1$ and k of graph $\mathbf{i}$ is



The number of $\wedge$ and $\vee$'s on the left between level $k-1$ and level k is 6, and hence $\mathrm{pos}_k(\mathbf{i}) = 6$.

Example 6.1.12. In Example 2.2.5, we have $\mathrm{pos}_k(\mathbf{i}) = 6$

- $\mathrm{pos}_2(\mathbf{i}) = 1$,
- $\mathrm{pos}_3(\mathbf{i}) = 0$,
- $\mathrm{pos}_7(\mathbf{i}) = 1$,
- $\mathrm{pos}_9(\mathbf{i}) = 0$.

[3]Please compare this with the number of (additional) marks on the extended graph

6.1.2 The elements $\hat{e}(\mathbf{i})$'s and $\hat{y}_k$'s

Before defining elements $\bar{e}(\mathbf{i}), \bar{y}_r, \bar{\psi}_k, \bar{\epsilon}_k \in C_d(\delta)$, in this and the following two subsections, we construct auxiliary elements of $C_d(\delta)$ to simplify the construction. Up to some signs, these elements will give us desired generators.

Assume $\mathbf{i} \in I^d$. Recall Corollary 2.2.14. Consider the unoriented graph $g_{\mathbf{i}}g_{\mathbf{i}}^*$. It is symmetric around the horizontal middle line. All the loops in this graph are either crossing the middle line, or come in pairs as floating loops (i.e not crossing the middle line). In particular, every line segment involved in an elementary move from (2.2.1) at each level $l \in \{1, \ldots, d\}$ has its mirror image at the level $2d + 1 - l$; we call such a pair of segments *partners*.

Let us define elements $\hat{e}(\mathbf{i}) \in C_d(\delta)$ as follows

Definition 6.1.13. Let

$$\hat{e}(\mathbf{i}) = \begin{cases} \sum g_{\mathbf{i}}\lambda g_{\mathbf{i}}^*, & \text{if } \mathbf{i} \in T^d, \\ 0, & \text{if } \mathbf{i} \notin T^d, \end{cases} \tag{6.1.1}$$

where the sum runs over all the oriented graphs $g_{\mathbf{i}}\lambda g_{\mathbf{i}}^*$ such that

- the loops crossing the middle line are oriented anti-clockwise;
- partners contained in strands or loops crossing the middle have the same orientation;
- partners contained in floating loops are oriented oppositely, and hence the pairs of floating loops are oriented oppositely.

In the rest of this paper we only consider $\mathbf{i} \in T^n$, since otherwise $\hat{e}(\mathbf{i}) = 0$.

Lemma 6.1.14. *The elements $\hat{e}(\mathbf{i})$ are homogeneous of degree zero.*

Proof. First, notice that by definition all cups in the upper part and caps in the lower part do not give any contribution to the degree. Furthermore the floating loops appear in pairs whose degree cancel each other. Finally the loops crossing the middle line are of degree zero by assumption.

We define analogues of the degree two elements $y_k e(\mathbf{i}) \in G_d(\delta)$ from [33] as follows:

Definition 6.1.15. Let

$$\hat{y}_k\hat{e}(\mathbf{i}) = \begin{cases} \sum_\lambda g_{\mathbf{i}}g_{\mathbf{i}}^* \in C_d(\delta), & \text{if the pattern at level } k \text{ of the graph } g_{\mathbf{i}}g_{\mathbf{i}}^* \text{ is contained in a loop}, \\ 0, & \text{if the pattern at level } k \text{ of the graph } g_{\mathbf{i}}g_{\mathbf{i}}^* \text{ is not contained in a loop}, \end{cases} \tag{6.1.2}$$

where the sum runs over all the oriented graphs $g_{\mathbf{i}}\lambda g_{\mathbf{i}}^*$ such that

- the loop (or the pair of floating loops) containing the pattern at level k is oriented clockwise;
- all other loops crossing the middle line are oriented anti-clockwise;
- partners contained in strands, loops crossing the middle or pair of floating loops, containing the pattern at level k, have the same orientation;

- partners contained in all other floating loops are oriented oppositely, and hence such pairs of floating loops are oriented oppositely.

Lemma 6.1.16. *Assume that* $\hat{y}_k\hat{e}(\mathbf{i}) \neq 0$, *then* $\deg \hat{y}_k\hat{e}(\mathbf{i}) = 2$.

Proof. Immediately follows from Lemma 3.2.4 since $\hat{y}_k$ changes one anticlockwise circle to one anti-clockwise circle which increases the degree by two.

Definition 6.1.17. Let $\mathsf{u}, \mathsf{s} \in \widehat{B}_d$ of the shape λ and $\mathsf{t} = \mathsf{u}\lambda\mathsf{s}^*$ with a graph g_{t}. Let c be some connected part of the upper part of g_{t} and assume it has a mirror image c' in the lower part. If the orientations of c and c' satisfy the conditions of the Definition 6.1.9, then the pair (c, c') is called *inessential*.

Furthermore, we call *an essential part of a Verma path* $\mathsf{t} = \mathsf{u}\lambda\mathsf{s}^*$ *(respectively, graph* g_{t}*)*, the topological equivalent of the diagram obtained by removing all inessential pairs of t (respectively, g_{t}).

Remark 6.1.18. The definition of the elements defined below and calculations connected to them heavily depend on the essential part of a Verma path.

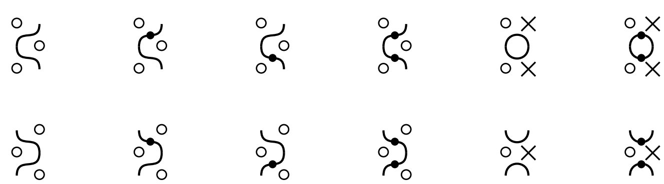
6.1.3 The elements $\hat{\epsilon}_k$'s

In this subsection we define elements ϵ_k whose definition, in particular, is motivated by the degree $\hat{\epsilon}_k$ defined in (5.2.28). The construction will use a concatenation of two graphs $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ of the expected degree.

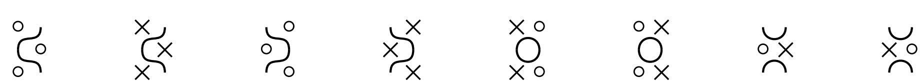
Suppose $\mathbf{i} \in T^d$ and $i_k + i_{k+1} = 0$. At level k and $k+1$ the graph of $\mathbf{i}$ is of one of the following forms:

(1) If $i_k = 0$

(2) If $i_k = \pm\frac{1}{2}$



(3) If $i_k \neq 0$ or $\pm\frac{1}{2}$



Lemma 6.1.19. *Suppose* $\mathbf{i} \in T^d$. *Then*

(1) If $i_k = 0$, *the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following forms:*

(2) If $i_k = \pm\frac{1}{2}$

(a) *if* $\deg_k(\mathbf{i}) = 1$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following forms:*

(b) *if* $\deg_k(\mathbf{i}) = -1$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following*

(c) *if* $\deg_k(\mathbf{i}) = 0$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following*

(3) *if* $i_k \neq 0$ *or* $\pm\frac{1}{2}$

(a) *if* $\deg_k(\mathbf{i}) = 1$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following forms:*

(b) *if* $\deg_k(\mathbf{i}) = -1$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following*

(c) *if* $\deg_k(\mathbf{i}) = 0$*, the k-pattern and* $k+1$*-pattern of* $\mathbf{i}$ *is of one of the following*

Proof. This Lemma can be proved following the similar argument as the proof of Proposition 3.3.3.

For a residue sequence $\mathbf{i}$ let us denote by $\mathbf{i}'$ the following sequence

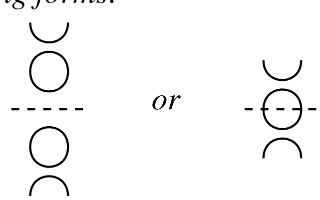
$$\mathbf{i}' = \left(\frac{\delta-1}{2}, -\frac{\delta-1}{2}, i_1, i_2, \ldots, i_{k-1}, i_{k+2}, i_{k+3}, \ldots, i_d\right). \tag{6.1.3}$$

In the following we assume that $\mathbf{i}' \in T^d$.

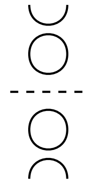
Lemma 6.1.20. *Suppose* $\mathbf{i} \in T^d$ *and* $\mathbf{i}'$ *is defined as above. By ignoring the dots, we have:*

Case 1: *Suppose* $k = 1$ *or* 2 *and* $\delta \neq 0$*. The graph of* $g_{\mathbf{i}} g^*_{\mathbf{i}'}$ *is symmetric and there are no loops involving patterns of level* 1 *and* 2*.*

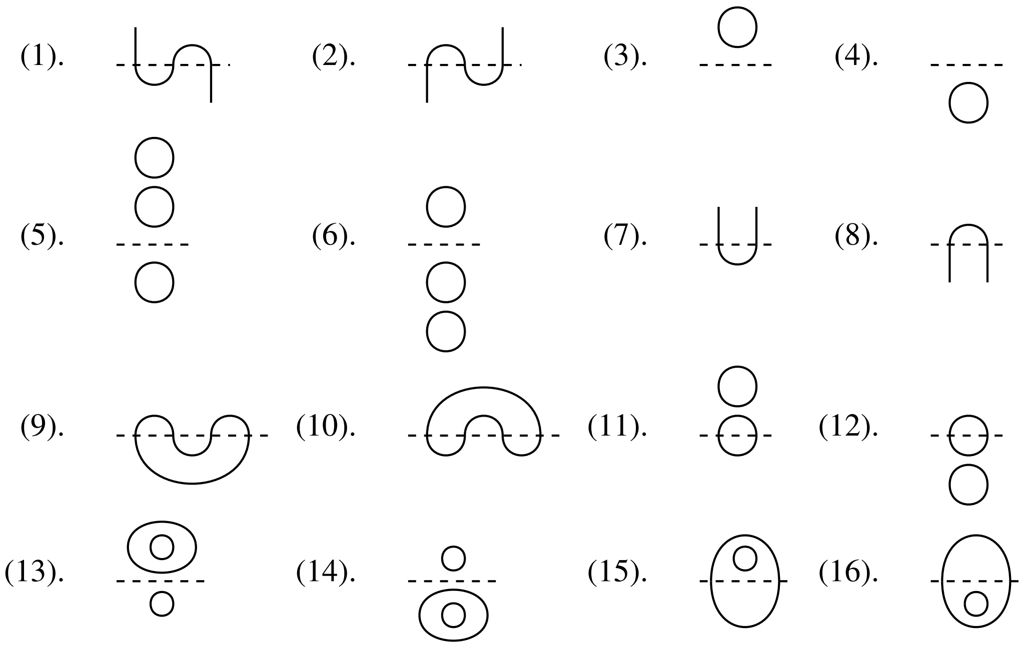
Case 2: *Suppose* $k = 1$ *and* $\delta = 0$*. The graph of* $g_{\mathbf{i}} g^*_{\mathbf{i}'}$ *is symmetric and are topologically equivalent to one of the following forms:*



Case 3: *Suppose $k = 2$ and $\delta = 0$. The graph of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is symmetric and topologically equivalent to*



Case 4: *Suppose $k > 2$ and $\delta \neq 0$. When* $\deg_k(\mathbf{i}) = 1$*, the graph of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is topologically equivalent to one of the following forms:*



(6.1.4)

When $\deg_k(\mathbf{i}) = 0$*, the graph of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is symmetric; and when* $\deg_k(\mathbf{i}) = -1$*, the graph of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is symmetric, except at level k and $k+1$ of $g_{\mathbf{i}}$ there is an extra floating loop.*

Case 5: *Suppose $k > 2$ and $\delta = 0$. The graph of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is essentially the same as in Case 4, except at level 2 and 3 of $g^*_{\mathbf{i}'}$ there will be an extra floating loop.*

Proof. Cases 1-3 are trivial. Before considering the Case 4, let us clarify our conventions. The main goal of the following calculations is to check that the special elements defined below satisfy the relations of $G_d(\delta)$. Since in the algebra $C_d(\delta)$, in the most cases it is immediate to determine whether the product of two basis elements is 0, we will define these special elements in such a way that we will only need to multiply Verma paths which differ only in "certain" way. Therefore, we will only indicate the topological equivalent of these parts, and only those ones which are essential for the surgery procedure and for determining the signs. We will make it clearer later.

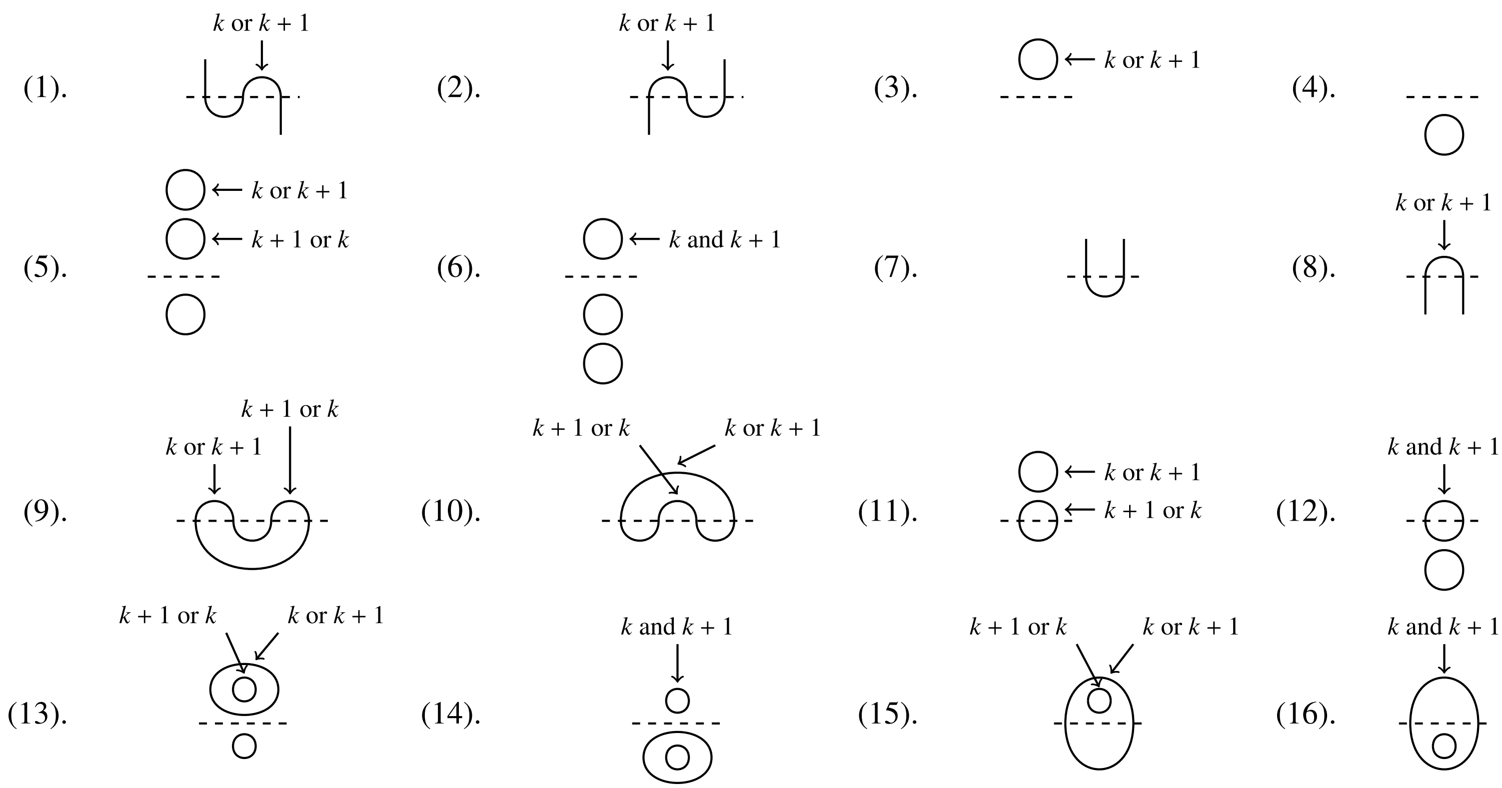
Now let us discuss Case 4 when $\deg_k(\mathbf{i}) = 1$. To obtain a graph of $\mathbf{i}'$, we cut from the graph of $\mathbf{i}$ the levels from the levels $k-1$ to $k+1$ [4] and glue the $k-1$ level with $k+1$ level[5]. In this case, we cut the part which is of the form (2b) in Lemma 6.1.19. Therefore, we should consider all possible cases of positions of the cap and cup that we cut. To understand which cases 1-16

[4] Recall our notations from Section 2.2

[5] It is easy to see by drawing the graph of $\mathbf{i}'$ and comparing with the graph of $\mathbf{i}$.

corresponds to which positions, we can notice the following. The upper part corresponds to a graph with a residue sequence $\mathbf{i}$, and the lower part - to $\mathbf{i}'$. Hence if we "reverse" the upper graph, we will see the essential parts the graph $g_{\mathbf{i}}g_{\mathbf{i}^*}$. For example in (1), we will obtain a line a circle, and if cup is on the line and the cap is on the circle, and we cut the levels $k-1$ to $k+1$, we will obtain the figure in (1). Similarly, considering all possible cases, we can obtain the 16 cases, given above.

Remark 6.1.21. We evaluate the structures of $g_{\mathbf{i}}g^*_{\mathbf{i}'}$'s in (6.1.4) by identifying the k and $k+1$-patterns in $\mathbf{i}$ contained in the loops or horizontal strands among the 16 cases. This result will be used later when we multiply elements. We note that we will only consider the loops or horizontal strands but not vertical strands. Note the k and $k+1$-patterns in $\mathbf{i}$ are involved in vertical strands in case 4 and 7. Hence we do not give any information in case 4 and 7.



Definition 6.1.22. For every $l \in \{1, \dots, d\}$ such that $l \neq k, k+1$ define

$$l^{(k)} = \begin{cases} 2d - 1 - l, & \text{if } l < k, \\ 2d + 1 - l, & \text{if } l > k+1, \end{cases} \tag{6.1.5}$$

Consider the unoriented graph $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ given by the pair $(\mathbf{i}, \mathbf{i}')$. It can be divided into pieces $g_{\mathbf{i}}g^*_{\mathbf{i}'} = abcc^*a^*e^*$, where a is the part the level 1 to the level $k-1$, b, is the part from the level k to the level $k+1$, c is the part from the level $k+2$ to the level d, e^* is the part from the level $2d-1$ to the level $2d$, a^* and c^* are reverse Verma paths of a and c respectively. In particular, $g_{\mathbf{i}}g^*_{\mathbf{i}'}$ is not symmetric but has some partial symmetry. Namely, for each $l \in \{1, \dots, d\}$ such that $l \neq k, k+1$, every segment at each level l has its mirror image at the level $l^{(k)}$, we call such a pair of segments *partners*. Furthermore, each loop in the upper half, that is not crossing the middle line and does not contain the patterns at the levels k and $k+1$, have its symmetric pair in the lower part. We call them the *pair of floating loops*.

Definition 6.1.23. Define $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$[6].

$$\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \begin{cases} \sum \beta^*_{g_\mathbf{i}\lambda g^*_{\mathbf{i}'}} g_\mathbf{i}\lambda g^*_{\mathbf{i}'}, & \text{if } i_k + i_{k+1} = 0, \\ 0, & \text{if } i_k + i_{k+1} \neq 0, \end{cases} \tag{6.1.6}$$

where the sum runs over all the oriented graphs $g_\mathbf{i}\lambda g^*_{\mathbf{i}'}$ such that

- it corresponds to an up-down tableau t with $\mathsf{t}(2d-2) = (1,1)$, $\mathsf{t}(2d-1) = -(1,1)$, $\mathsf{t}(2d) = (1,1)$;
- the loops in the middle, in case they do not contain the patterns of level k and $k+1$, are oriented anti-clockwise;
- the pair of floating loops, in case they do not contain the patterns of level k and $k+1$, are oriented oppositely;
- all other loops are oriented such that the degree of $g_\mathbf{i}\lambda g^*_{\mathbf{i}'}$ equals $\deg_k(\mathbf{i}) + \delta_{\delta,0}$;
- partners contained in strands, loops crossing the middle or in the loops, oriented in the same direction, have the same orientation;
- partners contained in the pair of floating loops or the loops, oriented oppositely, are oriented oppositely.

and coefficients β^*_t equal

- ∘ 1 if $\begin{cases} \deg_k(\mathbf{i}) = 0 \quad \text{or} \\ \deg_k(\mathbf{i}) = 1 \text{ and the essential part of the graph } g_\mathbf{i} g^*_{\mathbf{i}'} \text{ like in the cases (2),(4),(6),(8);} \end{cases}$
- ∘ $(-1)^{m_k(\mathbf{i})}$ if $\begin{cases} \deg_k(\mathbf{i}) = -1 \quad \text{or} \\ \deg_k(\mathbf{i}) = 1 \text{ and the essential part of the graph } g_\mathbf{i} g^*_{\mathbf{i}'} \text{ like in the cases (1),(3),(5),(9);} \end{cases}$
- ∘ 1 or -1 in all other cases, uniquely determined by the form of $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ from the table below.

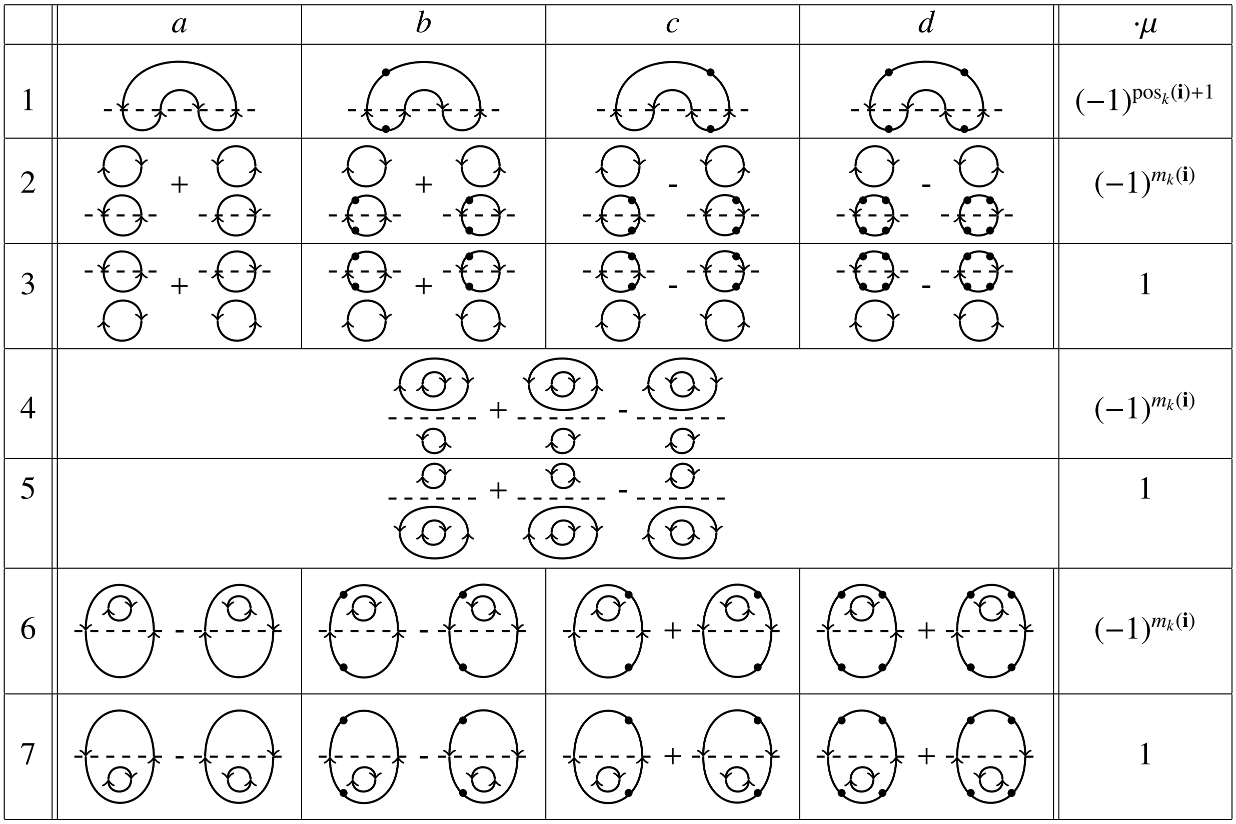
- for (10) we have $\begin{cases} 1(a) \text{ or } 1(b) \text{ or } 1(c) \text{ or } 1(d), & \text{if } k \text{ is on inside arc} \\ 1(a) \text{ or } 1(b) \text{ or } -1(c) \text{ or } -1(d), & \text{if } k+1 \text{ is on inside arc;} \end{cases}$
- for (11) we have $\begin{cases} 2(a) \text{ or } 2(b) \text{ or } 2(c) \text{ or } 2(d), & \text{if } k \text{ is in the floating loop,} \\ 2(a) \text{ or } 2(b) \text{ or } -2(c) \text{ or } -2(d), & \text{if } k+1 \text{ is in the floating loop;} \end{cases}$
- for (12) we have$3(a)$ or $3(b)$ or $3(c)$ or $3(d)$;
- for (13) we have $\begin{cases} 4, & \text{if } k \text{ is on inside loop,} \\ -4, & \text{if } k+1 \text{ is on inside loop;} \end{cases}$
- for (14) we have 5;

[6] Please find below more intuitive description. Notice that the condition on partners controls the extended graphs

- for (15) we have $\begin{cases} 6(a) \text{ or } 6(b) \text{ or } 6(c) \text{ or } 6(d), & \text{if } k \text{ is on inside loop,} \\ -6(a) \text{ or } -6(b) \text{ or } 6(c) \text{ or } 6(d), & \text{if } k+1 \text{ is on inside loop;} \end{cases}$

- for (16) we have$7(a)$ or $7(b)$ or $7(c)$ or $7(d)$;

Table 6.1: For $1 \leq k \leq 7$, we denote by k, respectively $k(x)$ with $x \in \{a, b, c, d\}$, the diagram from the k-th row and column x multiplied by the number in k-th row and the last column for $k = 4, 5$, respectively $k = 1, 2, 3, 6, 7$.



Similarly, we define $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$

Definition 6.1.24. Let

$$\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i}) = \begin{cases} \sum \beta_{g_{\mathbf{i}'} \lambda g_{\mathbf{i}}^*} g_{\mathbf{i}'} \lambda g_{\mathbf{i}}^*, & \text{if } i_k + i_{k+1} = 0, \\ 0, & \text{if } i_k + i_{k+1} \neq 0, \end{cases} \tag{6.1.7}$$

where the sum runs over all the oriented graphs $g_{\mathbf{i}'} \lambda g_{\mathbf{i}}^*$ such that

- it corresponds to an up-down tableau t with $\mathrm{t}(1) = (1, 1)$, $\mathrm{t}(2) = -(1, 1)$, $\mathrm{t}(3) = (1, 1)$;
- the loops in the middle, in case they do not contain the patterns of level $2d + 1 - k$ and $2d - k$, are oriented anti-clockwise;
- the pair of floating loops, in case they do not contain the patterns of level $2d + 1 - k$ and $2d - k$, are oriented oppositely;
- all other loops are oriented such that the degree of $g_{\mathbf{i}'} \lambda g_{\mathbf{i}}^*$ equals $\deg_k(\mathbf{i}) + \delta_{\delta,0}$;

- partners contained in strands, loops crossing the middle or in the loops, oriented in he same direction, have the same orientation;
- partners contained in the pair of floating loops or the loops, oriented oppositely, are oriented oppositely.

and the coefficients β_{t} in the sum (6.1.7) equal

- 1 if $\begin{cases} \deg_k(\mathbf{i}) = 0 & \text{or} \\ \deg_k(\mathbf{i}) = 1 \text{ and the essential part of the graph } g_{\mathbf{i}}g^*_{\mathbf{i}'} \text{ like in the cases (2),(4),(6),(8);} \end{cases}$
- $(-1)^{m_k(\mathbf{i})}$ if $\begin{cases} \deg_k(\mathbf{i}) = -1 & \text{or} \\ \deg_k(\mathbf{i}) = 1 \text{ and the essential part of the graph } g_{\mathbf{i}}g^*_{\mathbf{i}'} \text{ like in the cases (1),(3),(5),(9);} \end{cases}$
- 1 or -1 in all other cases, uniquely determined by the form of $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ from the table below.

- for (10) we have $\begin{cases} 1(a) \text{ or } 1(b) \text{ or } 1(c) \text{ or } 1(d), & \text{if } k \text{ is on left arc} \\ 1(a) \text{ or } 1(b) \text{ or } -1(c) \text{ or } -1(d), & \text{if } k+1 \text{ is on left arc;} \end{cases}$
- for (11) we have $2(a)$ or $2(b)$ or $-2(c)$ or $-2(d)$;
- for (12) we have $3(a)$ or $3(b)$ or $3(c)$ or $3(d)$;
- for (13) we have -4;
- for (14) we have 5;
- for (15) we have $-6(a)$ or $-6(b)$ or $6(c)$ or $6(d)$;
- for (16) we have $7(a)$ or $7(b)$ or $7(c)$ or $7(d)$;

Alternatively, we can give a definition in terms of topological equivalents.

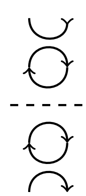
(1) The element $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is defined as:

Case 1: $k = 1$ or 2, $\mathbf{i} = \mathbf{i}'$ and $\delta \neq 0$.

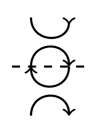
The elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

Case 2: $k = 1$ and $\delta = 0$.

The elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is define as

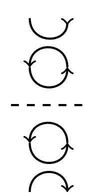


or



Case 3: $k = 2$, $\mathbf{i} = \mathbf{i}'$ and $\delta = 0$.

The elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is define as

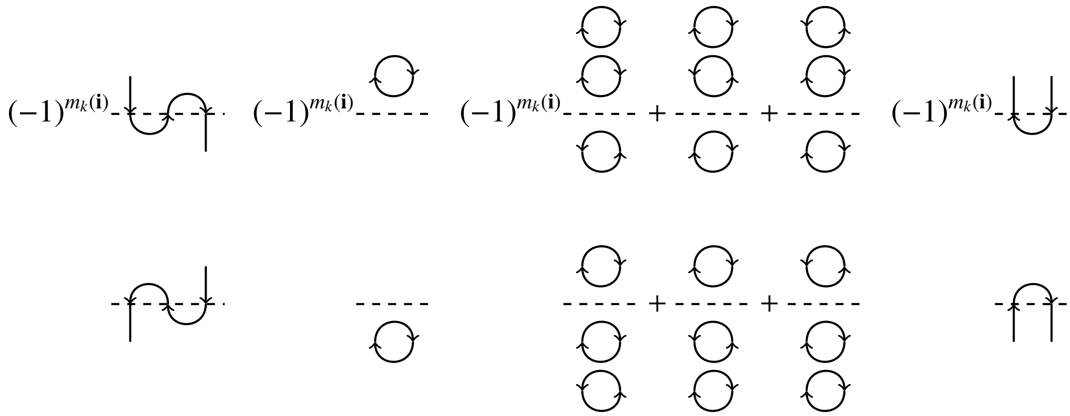


Case 4: either $k > 2$, or $k = 2$ and $\mathbf{i} \neq \mathbf{i}'$, and $\delta \neq 0$.

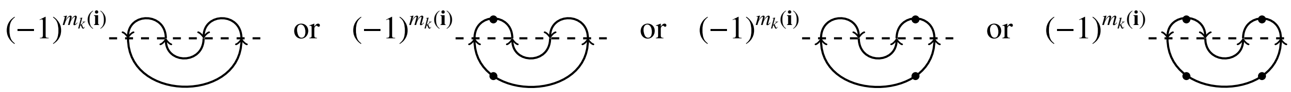
Case 4.1: $\deg_k(\mathbf{i}) = 1$.

By considering the 16 cases of (6.1.4), define $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ as

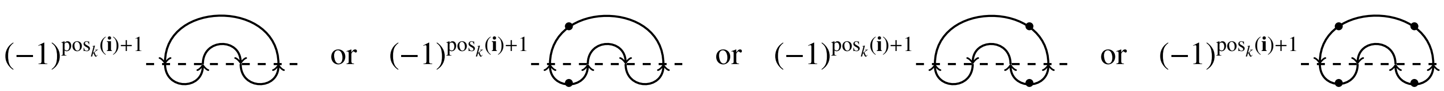
- In case (1) - (8), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:



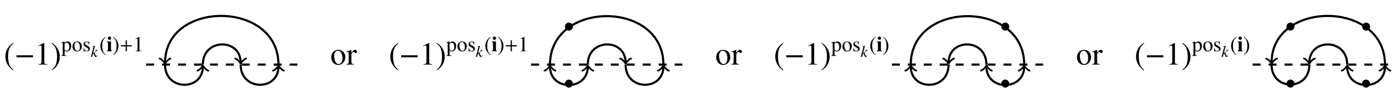
- In case (9), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:



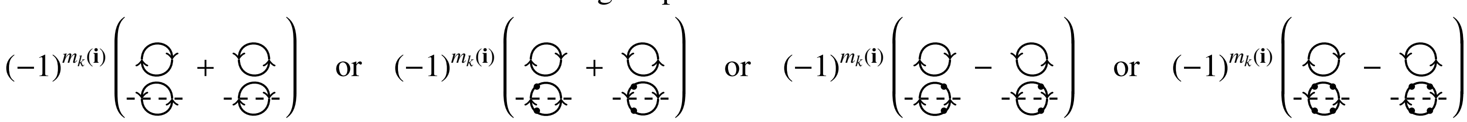
- In case (10), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:
  - when k is on the inside arc



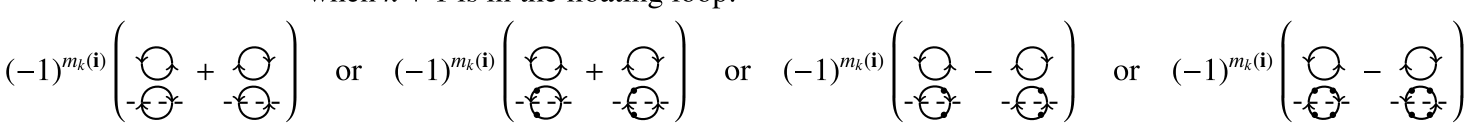
  - when $k + 1$ is on the inside arc



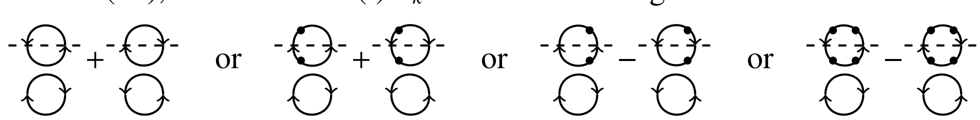
- In case (11), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:
  - when k is in the floating loop:



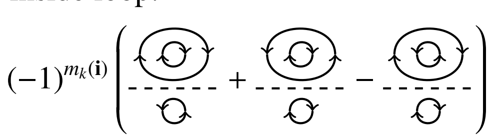
  - when $k + 1$ is in the floating loop:



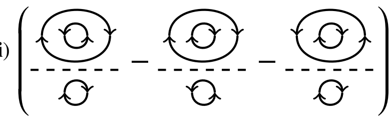
- In case (12), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:



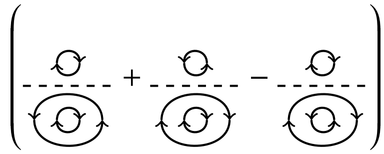
- In case (13), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:
  - when k is on the inside loop:



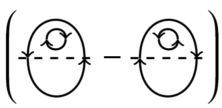
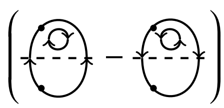
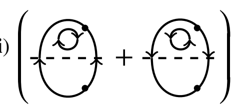
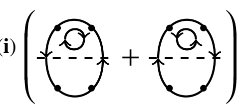
– when $k+1$ is on the inside loop:

$(-1)^{m_k(\mathbf{i})}$ ;

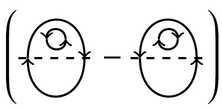
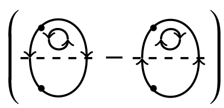
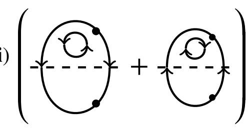
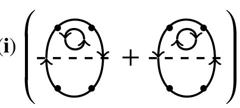
- In case (14), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:



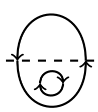
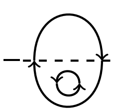
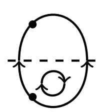
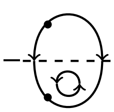
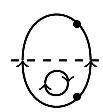
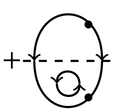
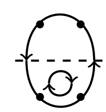
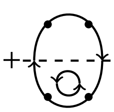
- In case (15), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:
  – when k is in the inside loop:

$(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$ 

  – when $k+1$ is in the inside loop:

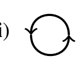
$(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$  or $(-1)^{m_k(\mathbf{i})}$ 

- In case (16), the elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is of the following forms:

 −  or  −  or  +  or  + 

Case 4.2: $\deg_k(\mathbf{i}) = -1$.

In this case there is an extra floating loop at level k and $k+1$ of $g_{\mathbf{i}}$. Then the element $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is defined as

$(-1)^{m_k(\mathbf{i})}$ 

Case 4.3: $\deg_k(\mathbf{i}) = 0$.

In this case elements $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

Case 5: either $k > 2$, or $k = 2$ and $\mathbf{i} \neq \mathbf{i}'$, and $\delta = 0$.

In this case the element $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ is defined similar to Case 4, except the extra floating loop at level 2 and 3 in $g^*_{\mathbf{i}'}$ is oriented clockwise.

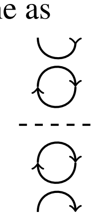
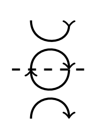
(2) The element $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is defined as:

Case 1: $k = 1$ or 2, $\mathbf{i} = \mathbf{i}'$ and $\delta \neq 0$.

The elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

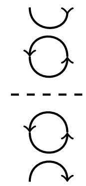
Case 2: $k = 1$ and $\delta = 0$.

The elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is define as

 or 

Case 3: $k = 2$, $\mathbf{i} = \mathbf{i}'$ and $\delta = 0$.

The elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is define as

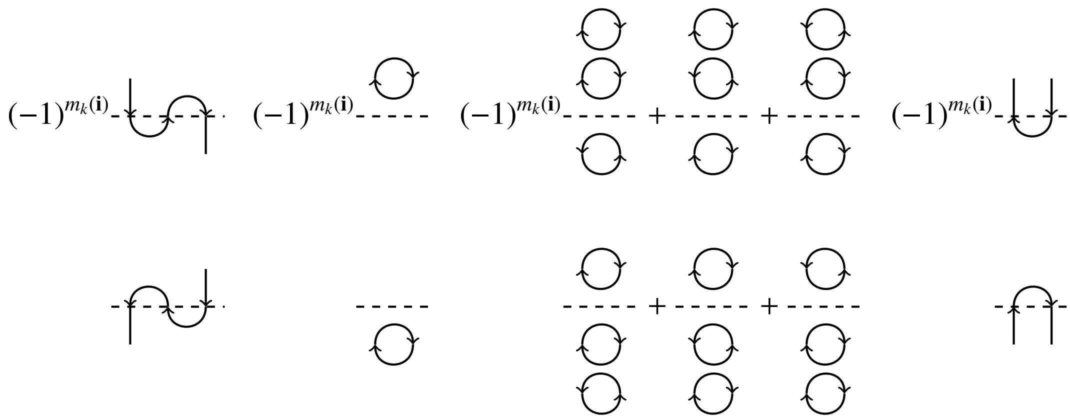


Case 4: either $k > 2$, or $k = 2$ and $\mathbf{i} \neq \mathbf{i}'$, and $\delta \neq 0$.

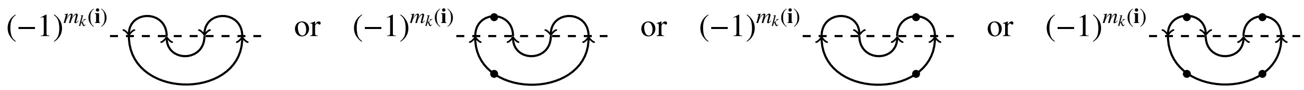
Case 4.1: $\deg_k(\mathbf{i}) = 1$.

By considering the 16 cases of (6.1.4), define $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ as

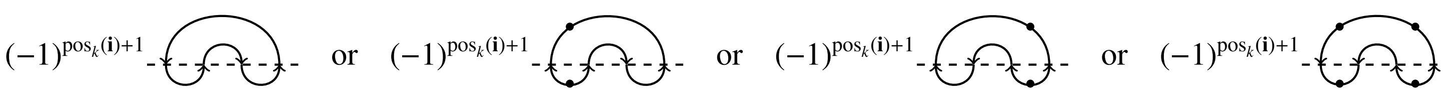
- In case (1) - (8), the elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is of the following forms:



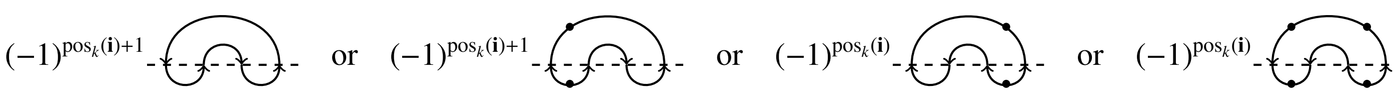
- In case (9), the elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is of the following forms:



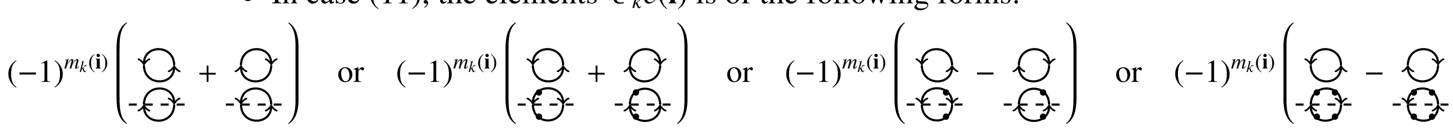
- In case (10), the elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is of the following forms:
  - when k is on the left arc



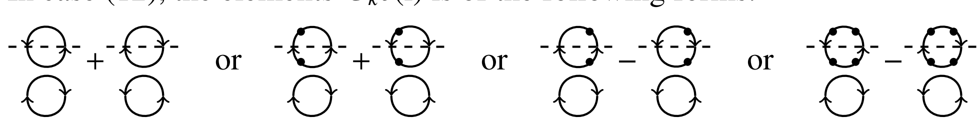
  - when $k + 1$ is on the left arc



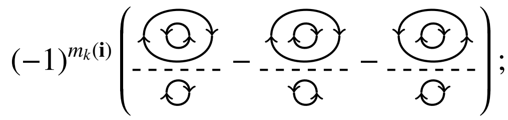
- In case (11), the elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is of the following forms:



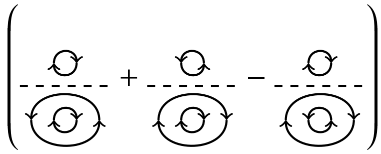
- In case (12), the elements $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ is of the following forms:



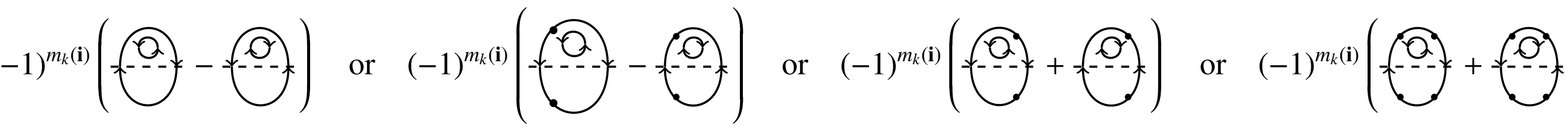
- In case (13), the elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is of the following forms:



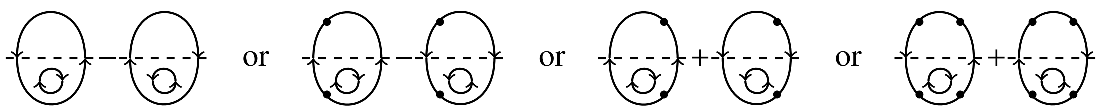
- In case (14), the elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is of the following forms:



- In case (15), the elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is of the following forms:

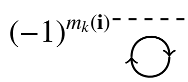


- In case (16), the elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is of the following forms:



Case 4.2: $\deg_k(\mathbf{i}) = -1$.

In this case there is an extra floating loop at level k and $k+1$ of $g_{\mathbf{i}}$. Then the element $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is defined as



Case 4.3: $\deg_k(\mathbf{i}) = 0$.

In this case elements $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

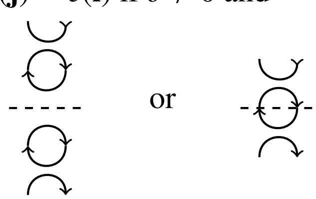
Case 5: either $k > 2$, or $k = 2$ and $\mathbf{i} \neq \mathbf{i}'$, and $\delta = 0$.

In this case the element $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ is defined similar to Case 4, except the extra floating loop at level 2 and 3 in $g^*_{\mathbf{i}'}$ is oriented clockwise.

Now we are ready to give a main definition of this subsection.

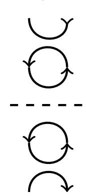
Definition 6.1.25. Suppose $\mathbf{i}, \mathbf{j} \in T^d$ with $i_k + i_{k+1} = j_k + j_{k+1} = 0$ for some $1 \leq k \leq d-1$ and $i_r = j_r$ if $r \neq k, k+1$. Then we define $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j})$ by

(1) If $k = 1$, we have $\mathbf{i} = \mathbf{j}$. Define $\hat{e}(\mathbf{i})\hat{\epsilon}_1\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})$ if $\delta \neq 0$ and

$\hat{e}(\mathbf{i})\hat{\epsilon}_1\hat{e}(\mathbf{j}) =$ 

if $\delta = 0$.

(2) If $k = 2$ and $\mathbf{i} = \mathbf{j} = \mathbf{i}'$, define $\hat{e}(\mathbf{i})\hat{\epsilon}_2\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})$ if $\delta \neq 0$ and

$\hat{e}(\mathbf{i})\hat{\epsilon}_2\hat{e}(\mathbf{j}) =$ 

if $\delta = 0$.

(3) In the rest of the cases, define

$$\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{j}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j}).$$

Notice that even for $\delta \neq 0$, we have $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{j}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$. We will use this equation in the calculations.

The following is immediate from the definition above.

Lemma 6.1.26. *We have* $\deg \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j}) = \deg_k(\mathbf{i}) + \deg_k(\mathbf{j})$, *where*

$$\deg_k(\mathbf{i}) = \begin{cases} 1, & \text{if } \mathbf{i} \in I^d_{k,+}, \\ -1, & \text{if } \mathbf{i} \in I^d_{k,-}, \\ 0, & \text{if } \mathbf{i} \in I^d_{k,0}. \end{cases}$$

6.1.4 The elements $\hat{\psi}_k$'s

In this subsection we define analogues of elements ψ_k whose definition, in particular, is motivated by the degree ψ_k defined in (5.2.28).

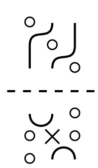
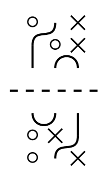
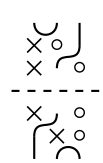
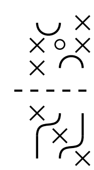
Recall that we defined an action of S_d on the Verma path and on residue sequences. Suppose $\mathbf{i} \in T^d$ such that $\mathbf{i}\cdot s_k \in T^d$ with $1 \leq k \leq d-1$.

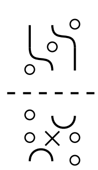
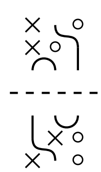
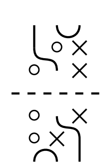
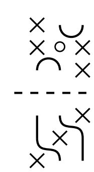
Consider the unoriented graph $g_{\mathbf{i}}g^*_{\mathbf{i}\cdot s_k}$ given by the pair $(\mathbf{i}, \mathbf{i}\cdot s_k)$. It can be divided into pieces $(\mathbf{i}, \mathbf{i}\cdot s_k) = abcc^*e^*a^*$, where a is the part the level 1 to the level $k-1$, b, is the part from the level k to the level $k+1$, c is the part from the level $k+2$ to the level d, e^* is the part from the level $2d-k$ to the level $2d-k+1$, a^* and c^* are reverse Verma paths of a and c respectively. In particular, $g_{\mathbf{i}}g^*_{\mathbf{i}\cdot s_k}$ is not symmetric but has some partial symmetry. Namely, for each $l \in \{1, \ldots, d\}$ such that $l \neq k, k+1$, every segment at each level l has its mirror image at the level $2d+1-l$, we call such a pair of segments *partners*. Furthermore, each loop in the upper half, that is not crossing the middle line and does not contain the patterns at the levels k and $k+1$, have its symmetric pair in the lower part. We call them the *pair of floating loops*.

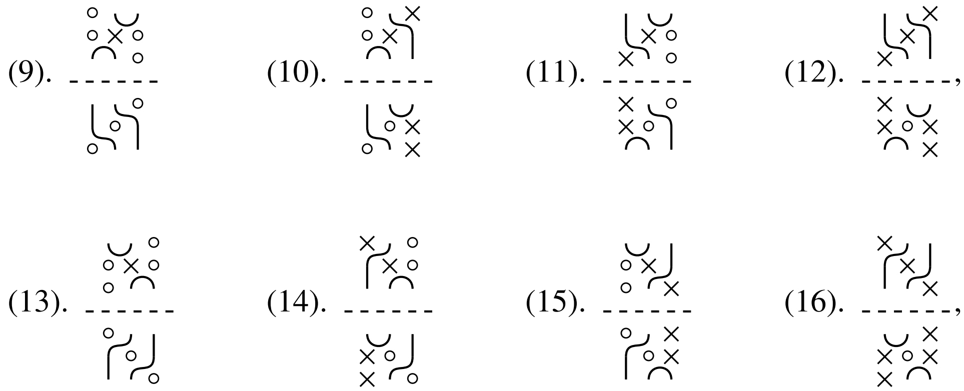
Let us compare the graphs of $\mathbf{i}$ and $\mathbf{i}\cdot s_k$.

Lemma 6.1.27. *(1) if $|i_k - i_{k+1}| > 1$ and $i_k + i_{k+1} \neq 0$, the graphs of* $\mathbf{i}$ *and* $\mathbf{i}\cdot s_k$ *are topologically the same.*

(2) If $|i_k - i_{k+1}| = 1$ and $i_k + i_{k+1} \neq 0$, the graph of $g_{\mathbf{i}}g^*_{\mathbf{i}\cdot s_k}$ between level k and $k+1$ is of one of the following forms

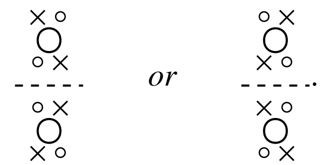
(1).  (2).  (3).  (4). ,

(5).  (6).  (7).  (8). ,

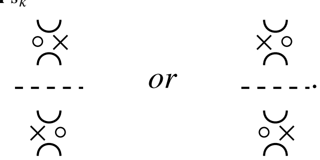


or the graphs swaping all the ∘'s and ×'s.

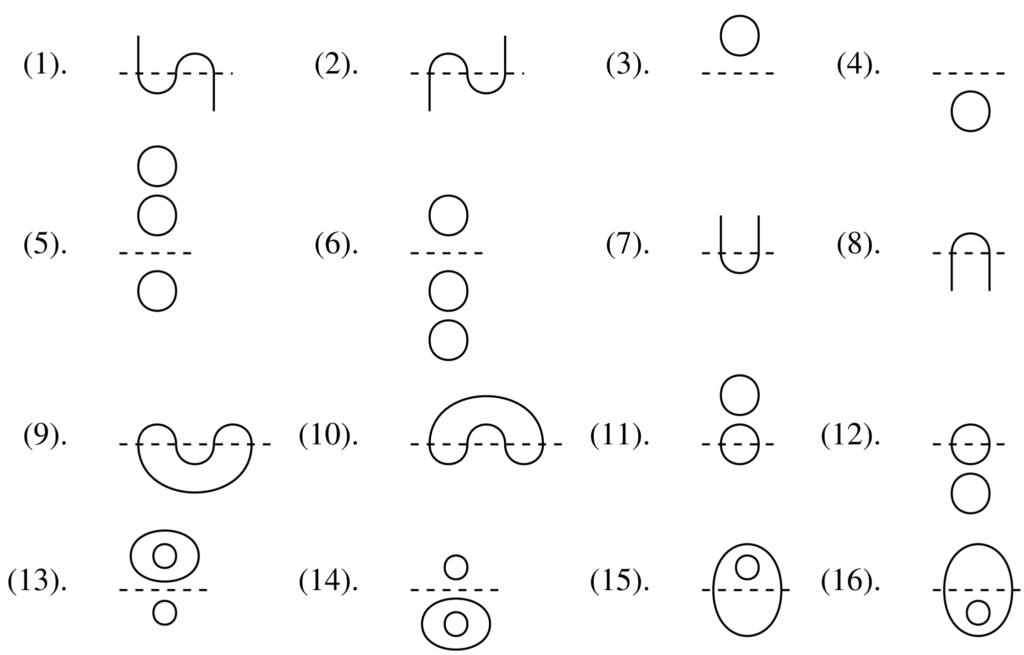
(3) *If* $|i_k - i_{k+1}| = 0$ *and* $i_k + i_{k+1} \neq 0$*, the graph of* $g_{\mathbf{i}} g^*_{\mathbf{i}\cdot s_k}$ *between level* k *and* $k+1$ *is of one of the following forms*



(4) *If* $i_k + i_{k+1} = 0$*, the graph of* $g_{\mathbf{i}} g^*_{\mathbf{i}\cdot s_k}$ *between level* k *and* $k+1$ *is of the following forms*

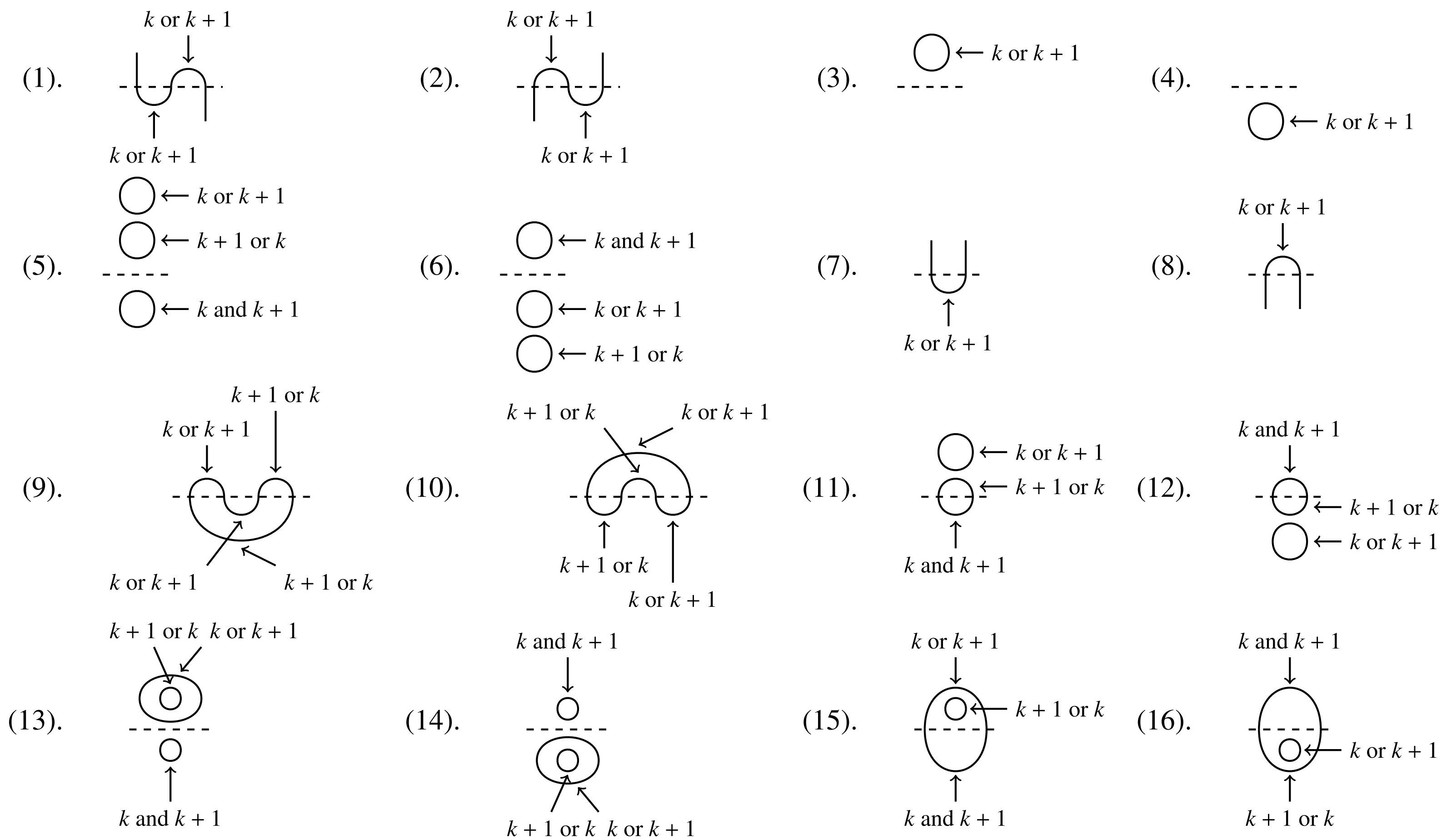


Lemma 6.1.28. *Suppose* $|i_k - i_{k+1}| = 1$ *and* $i_k + i_{k+1} \neq 0$*. By ignoring the dots, the graph of* $g_{\mathbf{i}} g^*_{\mathbf{i}\cdot s_k}$ *is topologically equivalent to one of the following cases topologically:*



(6.1.8)

Remark 6.1.29. We evaluate the structure of $g_{\mathbf{i}} g^*_{\mathbf{i}\cdot s_k}$. The next diagram expresses the part of the graph that contains pattern of level k and $k+1$ on both sides.



Definition 6.1.30. Define

$$\hat{e}(\mathbf{i})\hat{\psi}_k = \begin{cases} \sum_\lambda \beta^*_{g_\mathbf{i}\lambda g^*_{\mathbf{i}\cdot s_k}} g_\mathbf{i}\lambda g^*_{\mathbf{i}\cdot s_k}, & \text{if } \mathbf{i}\cdot s_k \in T^d \text{ and } i_k^2 + i_{k+1}^2 \neq 0, \\ 0, & \text{if } \mathbf{i}\cdot s_k \notin T^d \text{ or } i_k = i_{k+1} = 0, \end{cases} \tag{6.1.9}$$

where the sum runes over all the oriented graphs $g_\mathbf{i}\lambda g^*_{\mathbf{i}'}$ such that

- the loops in the middle, in case they do not contain the patterns of level k and $k+1$, are oriented anti-clockwise;
- the pair of floating loops, in case they do not contain the patterns of level k and $k+1$, are oriented oppositely;
- all other loops are oriented such that the degree of $g_\mathbf{i}\lambda g^*_{\mathbf{i}\cdot s_k}$ equals $e(\mathbf{i})\psi_k$;
- partners contained in strands, loops crossing the middle or in the loops, oriented in the same direction, have the same orientation;
- partners contained in the pair of floating loops or the loops, oriented oppositely, are oriented oppositely.

and coefficients $\beta^*_\mathfrak{t}$ equal

- $\circ$ 1 if $\begin{cases} |i_k - i_{k+1}| > 1 \text{ and } i_k + i_{k+1} \neq 0, & \text{or} \\ i_k + i_{k+1} = 0, & \text{or} \\ |i_k - i_{k+1}| = 1 \text{ and the essential part of the graph } g_{\mathbf{i}} g^*_{\mathbf{i}'} \text{ like in the cases (2),(4),(6),(8) ;} \end{cases}$

- $\circ$ $(-1)^{m_k(\mathbf{i})}$ if $\begin{cases} |i_k - i_{k+1}| = 0 \text{ and } i_k + i_{k+1} \neq 0 & \text{or} \\ |i_k - i_{k+1}| = 1 \text{ and the essential part of the graph } g_{\mathbf{i}} g^*_{\mathbf{i}'} \text{ like in the cases (1),(3),(5),(9);} \end{cases}$

- $\circ$ 1 or -1 in all other cases, uniquely determined by the form of $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i})$ from the table below.

- for (10) we have $\begin{cases} -1(a) \text{ or } -1(b) \text{ or } -1(c) \text{ or } -1(d), & \text{if } k \text{ of } \mathbf{i} \text{ is on ins. arc and } k \text{ of } \mathbf{i}\cdot s_k \text{ is on the left arc} \\ -1(a) \text{ or } -1(b) \text{ or } 1(c) \text{ or } 1(d), & \text{if } k \text{ of } \mathbf{i} \text{ is on ins. arc and } k+1 \text{ of } \mathbf{i}\cdot s_k \text{ is on the left arc} \\ 1(a) \text{ or } 1(b) \text{ or } 1(c) \text{ or } 1(d), & \text{if } k+1 \text{ of } \mathbf{i} \text{ is on ins. arc and } k \text{ of } \mathbf{i}\cdot s_k \text{ is on the left arc} \\ 1(a) \text{ or } 1(b) \text{ or } -1(c) \text{ or } -1(d), & \text{if } k+1 \text{ of } \mathbf{i} \text{ is on ins. arc and } k+1 \text{ of } \mathbf{i}\cdot s_k \text{ is on the left} \end{cases}$

- for (11) we have $\begin{cases} 2(a) \text{ or } 2(b) \text{ or } 2(c) \text{ or } 2(d), & \text{if } k \text{ is in the floating loop,} \\ 2(a) \text{ or } 2(b) \text{ or } -2(c) \text{ or } -2(d), & \text{if } k+1 \text{ is in the floating loop;} \end{cases}$

- for (12) we have $3(a)$ or $3(b)$ or $3(c)$ or $3(d)$;

- for (13) we have $\begin{cases} 4, & \text{if } k \text{ is on inside loop,} \\ -4, & \text{if } k+1 \text{ is on inside loop;} \end{cases}$

- for (14) we have 5;

- for (15) we have $\begin{cases} 6(a) \text{ or } 6(b) \text{ or } 6(c) \text{ or } 6(d), & \text{if } k \text{ is in the inside loop,} \\ -6(a) \text{ or } -6(b) \text{ or } 6(c) \text{ or } 6(d), & \text{if } k+1 \text{ is in the inside loop;} \end{cases}$

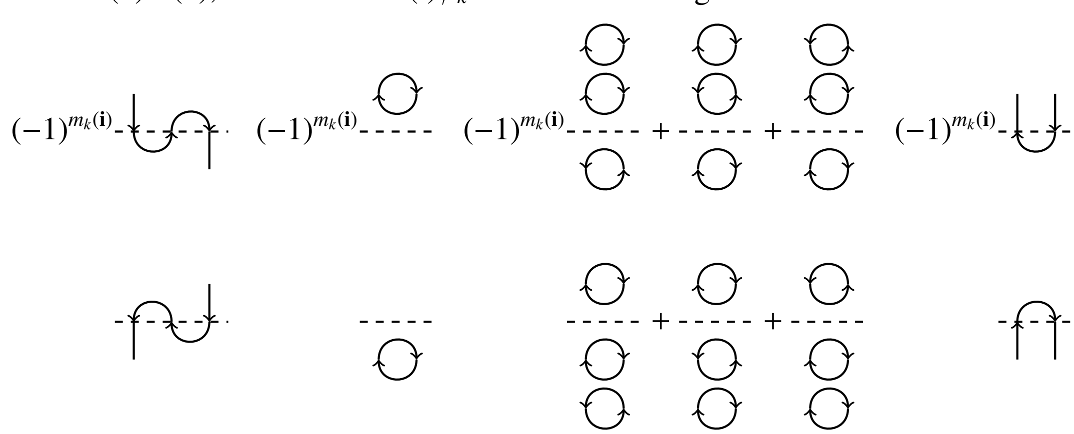
- for (16) we have $7(a)$ or $7(b)$ or $7(c)$ or $7(d)$;

Alternatively, one can define in terms of the topological equivalences.

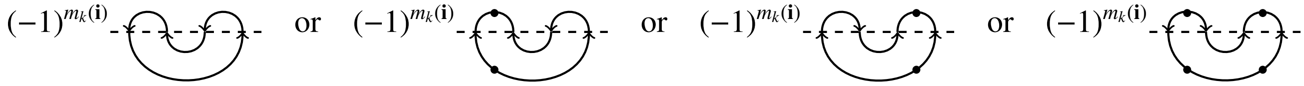
Case 1: If $|i_k - i_{k+1}| > 1$ and $i_k + i_{k+1} \neq 0$, the element $\hat{e}(\mathbf{i})\hat{\psi}_k$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

Case 2: If $|i_k - i_{k+1}| = 1$ and $i_k + i_{k+1} \neq 0$, by considering the 16 cases of (6.1.8), we define $\hat{e}(\mathbf{i})\hat{\psi}_k$ as the following cases:

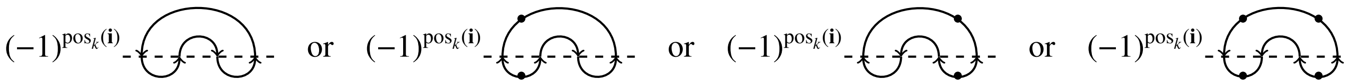
- In case (1) - (8), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:



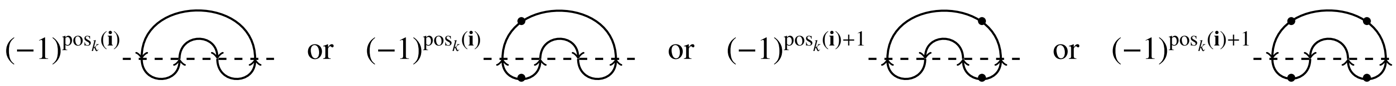
- In case (9), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:



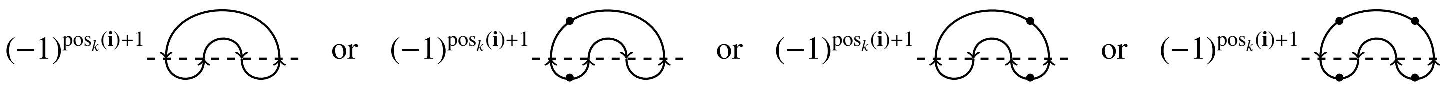
- In case (10), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:
  - when k-pattern of $\mathbf{i}$ is on the inside arc and k-pattern of $\mathbf{i}\cdot s_k$ is on the left arc



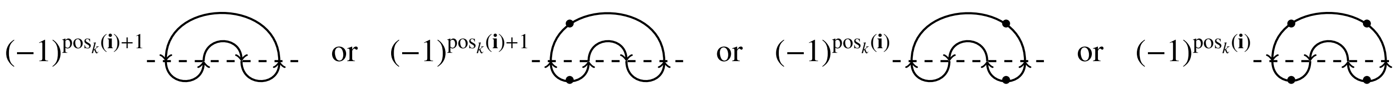
  - when k-pattern of $\mathbf{i}$ is on the inside arc and $k+1$-pattern of $\mathbf{i}\cdot s_k$ is on the left arc



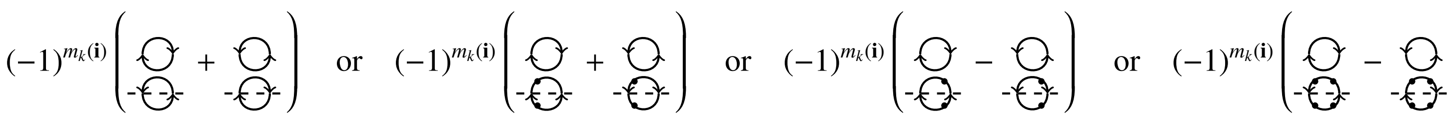
  - when $k+1$-pattern of $\mathbf{i}$ is on the inside arc and k-pattern of $\mathbf{i}\cdot s_k$ is on the left arc



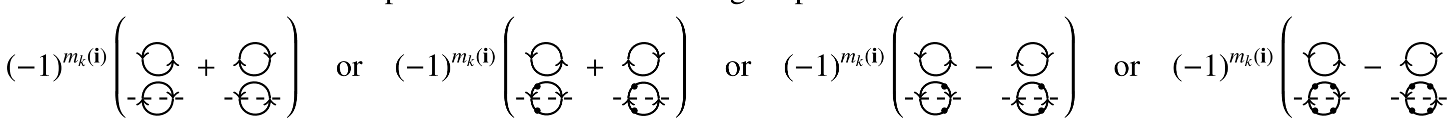
  - when $k+1$-pattern of $\mathbf{i}$ is on the inside arc and $k+1$-pattern of $\mathbf{i}\cdot s_k$ is on the left arc



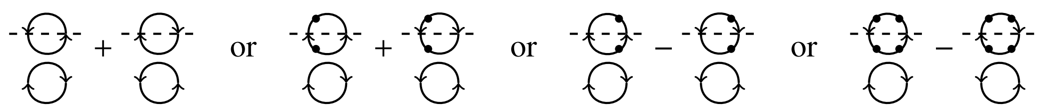
- In case (11), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:
  - when k-pattern of $\mathbf{i}$ is on the floating loop:



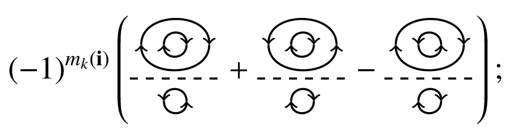
  - when $k+1$-pattern of $\mathbf{i}$ is on the floating loop:



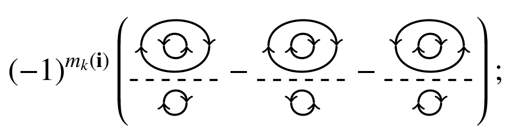
- In case (12), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:



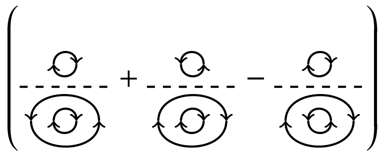
- In case (13), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:
  - when k is on the inside loop:



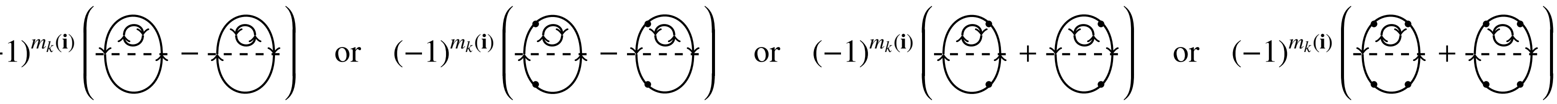
  - when $k+1$ is on the inside loop:



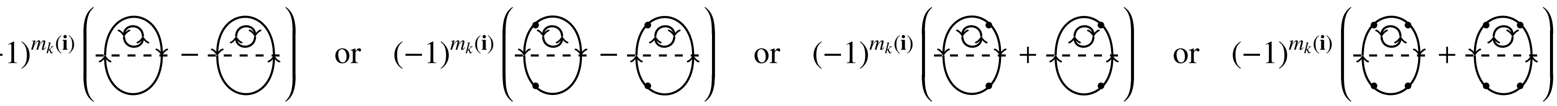
- In case (14), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:



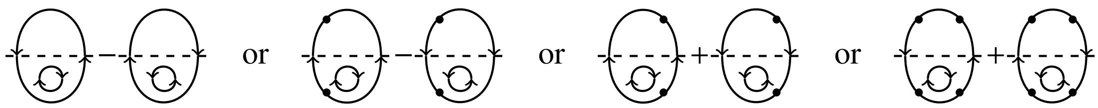
- In case (15), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:
  - when k is in the inside loop:



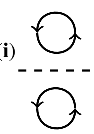
  - when $k+1$ is in the inside loop:



- In case (16), the elements $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following forms:



Case 3: If $|i_k - i_{k+1}| = 0$ and $i_k + i_{k+1} \neq 0$, the element $\hat{e}(\mathbf{i})\hat{\psi}_k$ is of the following form:

$(-1)^{m_k(\mathbf{i})}$ 

Case 4: If $i_k + i_{k+1} = 0$, the graphs of $\mathbf{i}$ and $\mathbf{i}'$ are topologically the same. We define $\hat{e}(\mathbf{i})\hat{\psi}_k$ topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$.

Since we have that $\psi_k e(\mathbf{i}) = e(\mathbf{i} \cdot s_k)\psi_k$, we give the following definition.

Definition 6.1.31. Let $\hat{\psi}_k \hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i} \cdot s_k)\hat{\psi}_k$.

The following is immediate.

Lemma 6.1.32.

$$\deg \hat{e}(\mathbf{i})\hat{\psi}_k = \begin{cases} 1, & \text{if } i_k = i_{k+1} \pm 1, \\ -2, & \text{if } i_k = i_{k+1}, \\ 0, & \text{otherwise;} \end{cases}$$

6.1.5 The twisted generators $\bar{e}(\mathbf{i}), \bar{y}_r, \bar{\psi}_k$ and $\bar{\epsilon}_k$

Finally, in this subsection we define elements of $C_d(\delta)$

$$\{\, \bar{e}(\mathbf{i}) \mid \mathbf{i} \in I^d \,\} \cup \{\, \bar{y}_k \mid 1 \le k \le d \,\} \cup \{\, \bar{\psi}_k \mid 1 \le k \le k-1 \,\} \cup \{\, \bar{\epsilon}_k \mid 1 \le k \le d-1 \,\}.$$

Definition 6.1.33. Suppose $1 \le k \le d-1$ and $\mathbf{i} \in I^d$ with $i_k + i_{k+1} = 0$. We define $\alpha_k^L(\mathbf{i}), \alpha_k^R(\mathbf{i}), \beta_k^L(\mathbf{i}), \beta_k^R(\mathbf{i}) \in \mathbb{Z}$ by

$\alpha_k^L(\mathbf{i}) = \#\{\, 1 \le r \le k-1 \mid i_r = i_k \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -i_k \,\} + \delta_{i_k,0} + \delta_{i_k,\frac{\delta-1}{2}}$,

$\alpha_k^R(\mathbf{i}) = \#\{\, 1 \le r \le k-1 \mid i_r = i_k - 1 \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -(i_k - 1) \,\}$,

$$\beta_k^L(\mathbf{i}) = \#\{\, 1 \le r \le k-1 \mid i_r = i_k \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -i_k \,\}$$
$$+ \#\{\, 1 \le r \le k-1 \mid i_r = i_k + 1 \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -(i_k+1) \,\} + \delta_{i_k,0} + \delta_{i_k,-\frac{1}{2}} + \delta_{i_k,\frac{\delta-1}{2}},$$
$$\beta_k^R(\mathbf{i}) = \#\{\, 1 \le r \le k-1 \mid i_r = i_k \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -i_k \,\}$$
$$+ \#\{\, 1 \le r \le k-1 \mid i_r = i_k - 1 \,\} + \#\{\, 1 \le r \le k-1 \mid i_r = -(i_k-1) \,\} + \delta_{i_k,-\frac{\delta-1}{2}} + 1.$$

Definition 6.1.34. Suppose $\mathbf{i} \in I^d$, $1 \le r \le d$ and $1 \le k \le d-1$, we define the elements:

(1). $\operatorname{sign}_{\mathbf{i}}(\hat{y}_r) = \operatorname{pos}_r(\mathbf{i})$;

(2). $\operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k) = \begin{cases} \operatorname{pos}_{k+1}(\mathbf{i}), & \text{if } i_k = i_{k+1} \text{ and } i_k + i_{k+1} \neq 0, \\ \operatorname{pos}_k(\mathbf{i}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ 0, & \text{if } i_k \neq i_{k+1}, i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ \operatorname{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) + \operatorname{sign}_{\mathbf{i}\cdot s_k}(\overrightarrow{\epsilon}_k) + \operatorname{pos}_k(\mathbf{i}), & \text{if } i_k + i_{k+1} = 0; \end{cases}$

(3). $\operatorname{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_k) = \alpha_k^R(\mathbf{i}) + \beta_k^R(\mathbf{i}) \operatorname{pos}_k(\mathbf{i})$;

(4). $\operatorname{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) = \alpha_k^L(\mathbf{i}) + \beta_k^L(\mathbf{i}) \operatorname{pos}_k(\mathbf{i})$.

Definition 6.1.35. Suppose $\mathbf{i} \in T^d$, $1 \le r \le d$ and $1 \le k \le d-1$, we define a twisted generators $C_d(\delta)$

$$\{\, \bar{e}(\mathbf{i}) \mid \mathbf{i} \in T^d \,\} \cup \{\, \bar{y}_r \mid 1 \le r \le d \,\} \cup \{\, \bar{\psi}_k \mid 1 \le k \le d-1 \,\} \cup \{\, \bar{\epsilon}_k \mid 1 \le k \le d-1 \,\}$$

as follows

(1) $\bar{e}(\mathbf{i}) = \hat{e}(\mathbf{i})$,

(2) $\bar{y}_r \bar{e}(\mathbf{i}) = (-1)^{\operatorname{sign}_{\mathbf{i}}(\hat{y}_r)} \hat{y}_k \hat{e}(\mathbf{i})$ for $\mathbf{i} \in T^d$, and $\bar{y}_r = \sum_{\mathbf{i}\in T^d} \bar{y}_r \bar{e}(\mathbf{i})$,

(3) $\bar{e}(\mathbf{i})\bar{\psi}_k = (-1)^{\operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k)} \hat{e}(\mathbf{i})\hat{\psi}_k$ for $\mathbf{i} \in T^d$, and $\bar{\psi}_k = \sum_{\mathbf{i}\in T^d} \bar{e}(\mathbf{i})\bar{\psi}_k$,

(4) $\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{j}) = (-1)^{\operatorname{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) + \operatorname{sign}_{\mathbf{j}}(\overrightarrow{\epsilon}_k)} \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j})$ for $\mathbf{i}, \mathbf{j} \in T^d$, and $\bar{\epsilon}_k = \sum_{\mathbf{i},\mathbf{j}\in T^d} \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{j})$.

6.2 Well-definedness of Φ

To prove the main Theorem 6.3.1, we first need to show that the map Φ is well-defined. Therefore the goal of this section is to show that the elements

$$\{\, \bar{e}(\mathbf{i}) \mid \mathbf{i} \in T^d \,\} \cup \{\, \bar{y}_r \mid 1 \le r \le d \,\} \cup \{\, \bar{\psi}_k \mid 1 \le k \le d-1 \,\} \cup \{\, \bar{\epsilon}_k \mid 1 \le k \le d-1 \,\}$$

of $C_d(\delta)$ satisfy the relations from Definition 5.2.4.

6.2.1 Preliminary Lemmas

In this subsection we prove some technical results which will be used frequently in the rest of this section for checking the relations from Definition 5.2.4.

Here many calculations are in $\mathbb{Z}_2$, which we indicate by the symbol " $\equiv$ ". Namely, whenever we say $a \equiv b$ for some $a, b \in \mathbb{Z}$, it means $a \equiv b \pmod 2$.

Let us fix $\mathbf{i} \in I^d$. Recall the definition of $\alpha_k(\mathbf{i})$ and $h_k(\mathbf{i})$ given in the Section 5.2.

Lemma 6.2.1. *We have the following equations*

(1) $\alpha_k(\mathbf{i}) = \alpha_k^L(\mathbf{i}) + \alpha_k^R(\mathbf{i})$.

(2) if $i_k + i_{k+1} = 0$ *then* $\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv \deg_k(\mathbf{i})$.

Proof. (1) Follows immediately from the definitions of $\alpha_k(\mathbf{i}), \alpha_k^L(\mathbf{i})$ and $\alpha_k^R(\mathbf{i})$.

(2) Assume that $i_k + i_{k+1} = 0$, then we have

$$\begin{aligned}
&\beta_k^L(\mathbf{i}) - \delta_{i_k,0} - \delta_{i_k,-\frac{1}{2}} - \delta_{i_k,\frac{\delta-1}{2}} + \beta_k^R(\mathbf{i}) - \delta_{i_k,-\frac{\delta-1}{2}} - 1\\
= \ & \#\{\, r \mid i_r = i_k \,\} + \#\{\, r \mid i_r = -i_k \,\} + \ \#\{\, r \mid i_r = i_k + 1 \,\} + \#\{\, r \mid i_r = -(i_k + 1) \,\}\\
& + \#\{\, r \mid i_r = i_k \,\} + \#\{\, r \mid i_r = -i_k \,\} + \ \#\{\, r \mid i_r = i_k - 1 \,\} + \#\{\, r \mid i_r = -(i_k - 1) \,\}\\
= \ & \#\{\, r \mid i_r = -i_k \pm 1 \,\} + 2\#\{\, r \mid i_r = i_k \,\} + \ \#\{\, r \mid i_r = i_k \pm 1 \,\} + 2\#\{\, r \mid i_r = -i_k \,\}\\
\equiv \ & \#\{\, r \mid i_r = -i_k \pm 1 \,\} + 2\#\{\, r \mid i_r = i_k \,\} + \ \#\{\, r \mid i_r = i_k \pm 1 \,\} + 2\#\{\, r \mid i_r = -i_k \,\}\\
& -2\,(\#\{\, r \mid i_r = i_k \pm 1 \,\} + 2\#\{\, r \mid i_r = -i_k \,\})\\
= \ & h_k(\mathbf{i}) - \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}},
\end{aligned}$$

where in all equations above r satisfies $1 \le r \le k-1$. Hence we have $\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv h_k(\mathbf{i}) + \delta_{i_k,0} + \delta_{i_k,-\frac{1}{2}} + 1$. Then from (5.2.1-5.2.3) and (5.2.28) we have

$$\deg_k(\mathbf{i}) = \begin{cases} h_k(\mathbf{i}), & \text{if } i_k = 0 \text{ or } -\frac{1}{2};\\ h_k(\mathbf{i}) + 1, & \text{otherwise ,}\end{cases}$$

and hence $\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv \deg_k(\mathbf{i})$.

Lemma 6.2.2. *Assume* $\mathbf{i}{\cdot}s_k \in I^d$. *We have the following properties:*

(1) If $|i_k - i_{k+1}| > 1$ *and* $i_k + i_{k+1} \neq 0$*, we have*

$$\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}{\cdot}s_k), \qquad \mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}{\cdot}s_k);$$

(2) If $|i_k - i_{k+1}| = 1$ *and* $i_k + i_{k+1} \neq 0$*, we have*

$$\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}{\cdot}s_k) + 1, \qquad \mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}{\cdot}s_k) + 1, \qquad \mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1;$$

(3) If $i_k = i_{k+1}$ *and* $i_k + i_{k+1} \neq 0$*, we have*

$$\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}{\cdot}s_k), \qquad \mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}{\cdot}s_k), \qquad \mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1;$$

(4) If $i_k + i_{k+1} = 0$*, we have*

$$\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}{\cdot}s_k), \qquad \mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}{\cdot}s_k), \qquad \mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1.$$

Proof. This is again a case by case argument and can be proven similarly as Lemma 6.2.1 using Lemma 6.1.27.

Lemma 6.2.3. *Suppose* $\mathbf{i} \in T^d$. *Then*

(1) $\mathbf{i}{\cdot}s_k \in T^d$ *if and only if* $h_k(\mathbf{i}) = 0$;

(2) if $i_k = -i_{k+1} = \pm\frac{1}{2}$ *we have* $\mathbf{i}{\cdot}s_k \notin T^d$.

Proof. See [33, Lemma 3.20] and [33, Lemma 3.7].

6.2.2 Idempotent relations

Lemma 6.2.4. *Let* $\mathbf{i}, \mathbf{j} \in I^d$ *and* $1 \leq k \leq d-1$. *Then*

(1) $\bar{y}_1^{\delta_{i_1, \frac{\delta-1}{2}}} \bar{e}(\mathbf{i}) = 0$,

(2) $\sum_{\mathbf{i} \in I^d} \bar{e}(\mathbf{i}) = 1$,

(3) $\bar{e}(\mathbf{i})\bar{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}\bar{e}(\mathbf{i})$,

(4) $\bar{e}(\mathbf{i})\hat{\epsilon}_k = 0$ *if* $i_k + i_{k+1} \neq 0$.

Proof. The statements (2), (3), (4) follow immediately from the definition.

(1) if $i_1 \neq \frac{\delta-1}{2}$ then by the definition $\hat{e}(\mathbf{i}) = \bar{e}(\mathbf{i}) = 0$. Furthermore, if $i_1 = \frac{\delta-1}{2}$, then we have $\bar{y}_1\bar{e}(\mathbf{i}) = 0$ since the pattern at level 1 can never be contained in a loop.

6.2.3 Commutation relations

Lemma 6.2.5. *Let* $\mathbf{i} \in I^d$, $1 \leq k, r \leq d-1$, $1 \leq l, m \leq d$. *Moreover, assume that* $|k - r| > 1$ *and* $|k - l| > 1$. *Then*

$$\bar{y}_l\bar{e}(\mathbf{i}) = \bar{e}(\mathbf{i})\bar{y}_l, \qquad \bar{\psi}_k e(\mathbf{i}) = \bar{e}(\mathbf{i}\cdot s_k)\bar{\psi}_k, \qquad \bar{y}_l\bar{y}_m = \bar{y}_m\bar{y}_l, \qquad \bar{y}_l\bar{\psi}_k = \bar{\psi}_k\bar{y}_l,$$
$$\bar{y}_l\bar{\epsilon}_k = \bar{\epsilon}_k y_l, \qquad \bar{\psi}_k\bar{\psi}_r = \bar{\psi}_r\bar{\psi}_k, \qquad \bar{\psi}_k\bar{\epsilon}_r = \bar{\epsilon}_r\bar{\psi}_k, \qquad \bar{\epsilon}_k\bar{\epsilon}_r = \bar{\epsilon}_r\bar{\epsilon}_k.$$

Proof. All equations follow immediately from the definition.

6.2.4 Essential commutation relations

In this subsection we prove essential commutation relations. Note that in order to proof equities in this and the following subsections, we have to check that two elements have the same graph, topologically equivalent and have the same sign. Since to check that two elements have the same graph is trivial, we will only check their the topological equivalence and the signs.

Lemma 6.2.6. *Suppose* $\mathbf{i} \in T^d$ *and* $1 \leq k \leq d-1$. *Then*

(1) if $i_k = i_{k+1}$ *and* $i_k + i_{k+1} \neq 0$, *we have*
$$\hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} + \hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k = \hat{e}(\mathbf{i}), \qquad \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k = \hat{e}(\mathbf{i});$$

(2) if $|i_k - i_{k+1}| = 1$ *and* $i_k + i_{k+1} \neq 0$, *we have*
$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k - \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} = 0, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k - \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k = 0;$$

(3) if $|i_k - i_{k+1}| > 1$ *and* $i_k + i_{k+1} \neq 0$, *we have*
$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1}, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k = \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k;$$

(4) if $i_k + i_{k+1} = 0$ *and* $i_k \neq 0, \pm\frac{1}{2}$, *we have*
$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k), \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k + \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k);$$

(5) if $i_k = -i_{k+1} = \pm\frac{1}{2}$, we have

$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k) = 0, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k + \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k) = 0;$$

(6) if $i_k = -i_{k+1} = 0$, we have

$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k+\hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k)-\hat{e}(\mathbf{i}) = 0, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k+\hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k)-\hat{e}(\mathbf{i}) = 0.$$

Proof. We consider all 6 cases separately.

(1) if $i_k = i_{k+1}$ and $i_k + i_{k+1}, \neq 0$ it is easy to see that $\hat{e}(\mathbf{i})\hat{\psi}_k y_{k+1}$ is the sum of all oriented graphs with

- the pair of floating loops between levels k and $k+1$ are oriented oppositely, such that the top one is oriented anti-clockwise and the bottom one is oriented clockwise.
- the remaining pairs of floating loops are oriented oppositely,
- the loops in the middle are oriented anti-clockwise.

Analogously, we obtain that $\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k$ is the sum of all oriented graphs with

- the pair of floating loops between levels k and $k+1$ are oriented oppositely, such that the top one is oriented clockwise and the bottom one is oriented anti-clockwise.
- the remaining pairs of floating loops are oriented oppositely,
- the loops in the middle are oriented anti-clockwise.

Hence $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1} + \hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k = \hat{e}(\mathbf{i})$. Following the same argument we obtain $\hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k = \hat{e}(\mathbf{i})$.

(2) the equations

$$\hat{y}_k\hat{e}(\mathbf{i})\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1}, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k = \hat{y}_{k+1}\hat{e}(\mathbf{i})\hat{\psi}_k.$$

for $|i_k - i_{k+1}| = 1$ and $i_k + i_{k+1} \neq 0$ can be proven by checking Appendixes 9.1, 9.3, 9.4 for each case separately.

(3) if $|i_k - i_{k+1}| > 1$ and $i_k + i_{k+1} \neq 0$ we have that $\hat{e}(\mathbf{i})\hat{y}_k = \hat{y}_k\hat{e}(\mathbf{i})$ and $\psi_r e(\mathbf{i}) = e(s_r\cdot\mathbf{i})\psi_r$ and hence we obtain

$$\hat{e}(\mathbf{i})\hat{y}_k\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1}, \qquad \hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k = \hat{e}(\mathbf{i})\hat{y}_{k+1}\hat{\psi}_k.$$

(4) if $i_k + i_{k+1} = 0$ and $i_k \neq 0, \pm\frac{1}{2}$, we have that $\hat{e}(\mathbf{i})\hat{\psi}_k$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$. Hence this case is equivalent to $\hat{e}(\mathbf{i})(\hat{y}_k + \hat{y}_{k+1}) = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i})$, which can be proven by checking Appendixes 9.1, 9.2.

(5) if $i_k = -i_{k+1} = \pm\frac{1}{2}$ then by Lemma 6.2.3 we have $\mathbf{i}\cdot s_k \notin T^d$. Hence $\hat{e}(\mathbf{i}\cdot s_k) = 0$, which means all the terms involving in the equalities are 0.

(6) if $i_k = -i_{k+1} = 0$ then by definitions we have $\hat{e}(\mathbf{i})\hat{\psi}_k = 0$ and $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})$.

Proposition 6.2.7. *We have the following*

$$\bar{e}(\mathbf{i})\bar{y}_k\bar{\psi}_k = \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_{k+1} + \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}}\bar{e}(\mathbf{i}),$$
$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_k = \bar{e}(\mathbf{i})\bar{y}_{k+1}\bar{\psi}_k + \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}}\bar{e}(\mathbf{i}).$$

Proof. We consider different cases.

(1) if $i_k = i_{k+1}$ and $i_k + i_{k+1} \neq 0$, then by Lemma 6.2.2 (3), we have $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}\cdot s_k) + 1$ and $\mathrm{pos}_k(\mathbf{i}\cdot s_k) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1$. Hence by Lemma 6.2.6(1),

$$\bar{e}(\mathbf{i})\bar{y}_k\bar{\psi}_k = \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_{k+1} - \bar{e}(\mathbf{i}), \qquad \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_k = \bar{e}(\mathbf{i})\bar{y}_{k+1}\bar{\psi}_k - \bar{e}(\mathbf{i}).$$

(2) if $|i_k - i_{k+1}| \geq 1$ and $i_k + i_{k+1} \neq 0$, then by Lemma 6.2.2 (1) and (2), we have $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}\cdot s_k)$ and $\mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}\cdot s_k)$. Hence by Lemma 6.2.6(2) and (3),

$$\bar{e}(\mathbf{i})\bar{y}_k\bar{\psi}_k = \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_{k+1}, \qquad \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_k = \bar{e}(\mathbf{i})\bar{y}_{k+1}\bar{\psi}_k.$$

(3) if $i_k + i_{k+1} = 0$ and $i_k \neq 0, \pm\frac{1}{2}$, then by Lemma 6.2.2 (4), $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}\cdot s_k) + 1$ and $\mathrm{pos}_k(\mathbf{i}\cdot s_k) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1$. Hence by Lemma 6.2.6(4),

$$\bar{e}(\mathbf{i})\bar{y}_k\bar{\psi}_k = \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_{k+1} + \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}}\bar{e}(\mathbf{i}), \qquad \bar{e}(\mathbf{i})\bar{\psi}_k\bar{y}_k = \bar{e}(\mathbf{i})\bar{y}_{k+1}\bar{\psi}_k + \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}}\bar{e}(\mathbf{i}).$$

(4) if $i_k = -i_{k+1} = \pm\frac{1}{2}$, then all the terms in equation are 0, hence the statement is obvious.

(5) if $i_k = -i_{k+1} = 0$, then by Lemma 6.2.1, we have $\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv \deg_k(\mathbf{i}) = 0$. Hence

$$\alpha_k^L(\mathbf{i}) + \beta_k^L(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \alpha_k^R(\mathbf{i}) + \beta_k^R(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}) \equiv 0.$$

Therefore the proposition follows from Lemma 6.2.6(6), .

6.2.5 Inverse relations

Lemma 6.2.8. *Suppose* $\mathbf{i} \in T^d$. *Then we have*

$$\hat{e}(\mathbf{i})\hat{\psi}_k^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1} \text{ or } (i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0); \\ (\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}), & \text{if } |i_k - i_{k+1}| = 1 \text{ and } i_k + i_{k+1} \neq 0; \\ \hat{e}(\mathbf{i}), & \text{otherwise.} \end{cases}$$

Proof. We consider different cases:

(1) Assume $i_k = i_{k+1}$. Then if $i_k = 0$, we have $\hat{e}(\mathbf{i})\hat{\psi}_k = 0$. If $i_k \neq 0$, then the floating loops in $\hat{e}(\mathbf{i})\hat{\psi}_k$ at level k of $g_{\mathbf{i}}g_{\mathbf{i}}^*$ are both anti-clockwise, hence the product is 0.

(2) Assume $|i_k - i_{k+1}| = 1$ and $i_k + i_{k+1} \neq 0$. Then the statement can be proven by checking Appendix A and B for each case separately.

(3) Assume $|i_k - i_{k+1}| > 1$ and $i_k + i_{k+1} \neq 0$. Since $\hat{e}(\mathbf{i})\hat{\psi}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{\psi}_k$ are topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$, we have $\hat{e}(\mathbf{i})\hat{\psi}_k^2 = \hat{e}(\mathbf{i})$.

(4) Assume $i_k + i_{k+1} = 0$. If $h_k(\mathbf{i}) \neq 0$, then by Lemma 6.2.3 we have $\mathbf{i}\cdot s_k = 0$ and hence $\hat{e}(\mathbf{i})\hat{\psi}_k^2 = 0$. If $h_k(\mathbf{i}) = 0$, we have that $\hat{e}(\mathbf{i})\hat{\psi}_k$ and $\hat{\psi}_k\hat{e}(\mathbf{i})$ are topologically equivalent to the idempotents, therefore $\hat{e}(\mathbf{i})\hat{\psi}_k^2 = \hat{e}(\mathbf{i})$.

Proposition 6.2.9. *Let* $\mathbf{i} \in T^d$. *We have*

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1} \text{ or } (i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0); \\ (\bar{y}_{k+1} - \bar{y}_k)\bar{e}(\mathbf{i}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0; \\ (\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0; \\ \bar{e}(\mathbf{i}), & \text{otherwise.} \end{cases}$$

Proof. We consider different cases:

(1) Assume $i_k = i_{k+1}$ or ($i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) \neq 0$). Then the statement follows from Lemma 6.2.8.

(2) Assume $i_k = i_{k+1} + 1$ and $i_k + i_{k+1} \neq 0$.

From Lemma 6.2.8, we have

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}\hat{e}(\mathbf{i})\hat{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}(\hat{y}_k + \hat{y}_{k+1})\bar{e}(\mathbf{i}).$$

From Lemma 6.2.2, we have

$$\mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}\cdot s_k) \equiv \mathrm{pos}_k(\mathbf{i}) + 1,$$

which implies

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = ((-1)^{\mathrm{pos}_{k+1}(\mathbf{i})}\hat{y}_{k+1} - (-1)^{\mathrm{pos}_k(\mathbf{i})}\hat{y}_k)\hat{e}(\mathbf{i}) = (\bar{y}_{k+1} - \bar{y}_k)\bar{e}(\mathbf{i}).$$

(3) Assume $i_k = i_{k+1} - 1$ and $i_k + i_{k+1} \neq 0$.

Then from Lemma 6.2.8, we have

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i})}\hat{e}(\mathbf{i})\hat{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}).$$

From Lemma 6.2.2, we have

$$\mathrm{pos}_{k+1}(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}) + 1,$$

which implies

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = ((-1)^{\mathrm{pos}_k(\mathbf{i})}\hat{y}_k - (-1)^{\mathrm{pos}_{k+1}(\mathbf{i})}\hat{y}_{k+1})\hat{e}(\mathbf{i}) = (\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}).$$

(4) Assume $|i_k - i_{k=1}| > 1$ and $i_k + i_{k+1} \neq 0$. Then it follows from Lemma 6.2.8 that $e(\mathbf{i})\psi_k^2 = e(\mathbf{i})$.

(5) Assume $i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) = 0$.

Then from Lemma 6.2.8 we have $\hat{e}(\mathbf{i})\hat{\psi}_k^2 = \hat{e}(\mathbf{i})$. Hence

$$\bar{e}(\mathbf{i})\bar{\psi}_k^2 = (-1)^{\alpha_k^L(\mathbf{i})+\alpha_k^R(\mathbf{i}\cdot s_k)+\beta_k^L(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i})+\beta_k^R(\mathbf{i}\cdot s_k)\,\mathrm{pos}_k(\mathbf{i}\cdot s_k)+\mathrm{pos}_k(\mathbf{i}\cdot s_k)+\alpha_k^L(\mathbf{i}\cdot s_k)+\alpha_k^R(\mathbf{i})+\beta_k^L(\mathbf{i}\cdot s_k)\,\mathrm{pos}_k(\mathbf{i}\cdot s_k)+\beta_k^R(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i})+\mathrm{pos}_k(\mathbf{i})}\hat{e}(\mathbf{i}).$$

By definition, we have

$$\begin{aligned}
&\alpha_k^L(\mathbf{i}) + \alpha_k^R(\mathbf{i}\cdot s_k) + \alpha_k^L(\mathbf{i}\cdot s_k) + \alpha_k^R(\mathbf{i})\\
=\ & 2\#\{\,1 \le r \le k-1 \mid i_r = \pm i_k\,\} + \#\{\,1 \le r \le k-1 \mid i_r = \pm(i_k - 1), \pm(i_k + 1)\,\} + \delta_{i_k,\frac{\delta-1}{2}} + \delta_{i_k,-\frac{\delta-1}{2}}\\
\equiv\ & h_k(\mathbf{i}) \equiv 0.
\end{aligned}$$

Moreover, from Lemma 6.2.1 we have

$$\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv h_k(\mathbf{i}) \equiv \beta_k^L(\mathbf{i}\cdot s_k) + \beta_k^R(\mathbf{i}\cdot s_k).$$

Then by Lemma 6.2.2 (4) we have $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}\cdot s_k)$, wich implies

$$\begin{aligned}
&\alpha_k^L(\mathbf{i}) + \alpha_k^R(\mathbf{i}\cdot s_k) + \beta_k^L(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \beta_k^R(\mathbf{i}\cdot s_k)\,\mathrm{pos}_k(\mathbf{i}\cdot s_k) + \mathrm{pos}_k(\mathbf{i}\cdot s_k)\\
&+\alpha_k^L(\mathbf{i}\cdot s_k) + \alpha_k^R(\mathbf{i}) + \beta_k^L(\mathbf{i}\cdot s_k)\,\mathrm{pos}_k(\mathbf{i}\cdot s_k) + \beta_k^R(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \mathrm{pos}_k(\mathbf{i})\\
\equiv\ & \left(\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i})\right)\mathrm{pos}_k(\mathbf{i}) + \left(\beta_k^L(\mathbf{i}\cdot s_k) + \beta_k^R(\mathbf{i}\cdot s_k)\right)\mathrm{pos}_k(\mathbf{i})\\
\equiv\ & 0.
\end{aligned}$$

Hence, we have $\bar{e}(\mathbf{i})\bar{\psi}_k^2 = \bar{e}(\mathbf{i})$.

6.2.6 Essential idempotent relations

Lemma 6.2.10. *Suppose* $\mathbf{i} \in T^d_{k,0}$ *and* $i_k = -i_{k+1} \neq \pm\frac{1}{2}$. *Then we have* $\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})$.

Proof. Let us first assume that $\delta \neq 0$. By definition, $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j})$. Furthermore, since $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ are topologically equivalent to the idempotents $\hat{e}(\mathbf{i})$, we have $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})$ (notice that, although, the graphs of $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ are different from the graph of $\hat{e}(\mathbf{i})$, the graph of their product is the same as the graph of $\hat{e}(\mathbf{i})$).

Now assume $\delta = 0$ (and, hence, $k \geq 2$). By definition, $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j}) = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{j}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j})$, where $\mathbf{i}'$ defined in (6.1.3). Each $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ has one additional floating loop oriented clockwise, which cancels out in a product $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{j}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j})$ since additional floating loops in $\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{j}')$ are oriented anti-clockwise. Therefore, we have $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})$.

Furthermore,
$$\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = (-1)^{\alpha^L_k(\mathbf{i})+\alpha^R_k(\mathbf{i})+\beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i})}\bar{e}(\mathbf{i}).$$

From Lemma 6.2.1, we have $\beta^L_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \equiv \deg_k(\mathbf{i}) = 0$. Hence
$$\alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) + \beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}),$$
which implies $\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})$.

Lemma 6.2.11. *Suppose* $\mathbf{i} \in T^d_{k,+}$. *Then we have* $\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}(\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i})$.

Proof. For $\delta \neq 0$, one can show that $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = (\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i})$ by checking Appendix A and B.

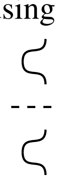
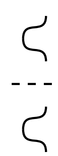
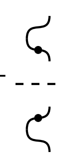
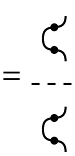
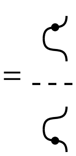
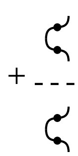
If $\delta = 0$, notice that again additional floating loops cancel out in a product $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{j})$.

Hence we have
$$\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = (-1)^{\alpha^L_k(\mathbf{i})+\alpha^R_k(\mathbf{i})+\beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i})}(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}).$$

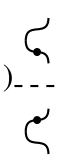
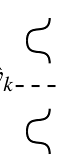
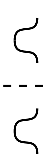
From Lemma 6.2.1, we have $\beta^L_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \equiv \deg_k(\mathbf{i}) = 1$. Therefore, we obtain
$$\alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) + \beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i}),$$
which implies
$$\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})+\operatorname{pos}_k(\mathbf{i})}(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}(\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}).$$

Lemma 6.2.12. *Suppose* $\mathbf{i} \in T^d_{k,0}$ *and* $i_k = -i_{k+1} = \frac{1}{2}$, *then*
$$(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = 2\hat{y}_k\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = 2\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i})\hat{y}_k.$$

Proof. By focusing on k and $k+1$-patterns of $\mathbf{i}$, we have

$\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) =$  $\hat{e}(\mathbf{i}) =$  $+$  or $\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) =$  $\hat{e}(\mathbf{i}) =$  $+$ 

Without loss of generality, we only consider the first situation. It is easy to see that

$(\hat{y}_k + \hat{y}_{k+1})$  $= 0$ and $\hat{y}_k$  $= \hat{y}_{k+1}$ .

Therefore we have
$$(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = (\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) = 2\hat{y}_k\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}).$$

Following the same argument, we have $(\hat{y}_k + \hat{y}_{k+1})\hat{e}(\mathbf{i}) = 2\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i})\hat{y}_k$.

Lemma 6.2.13. *Suppose* $\mathbf{i} \in T^d_{k,0}$ *and* $i_k = -i_{k+1} = \frac{1}{2}$, *then*

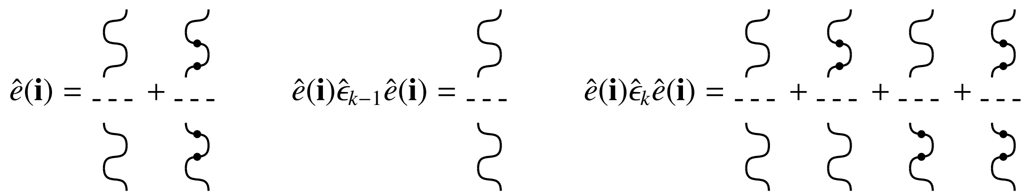
$$(\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}) = 2(-1)^{\alpha_k(\mathbf{i})}\bar{y}_k\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = 2(-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i})\bar{y}_k.$$

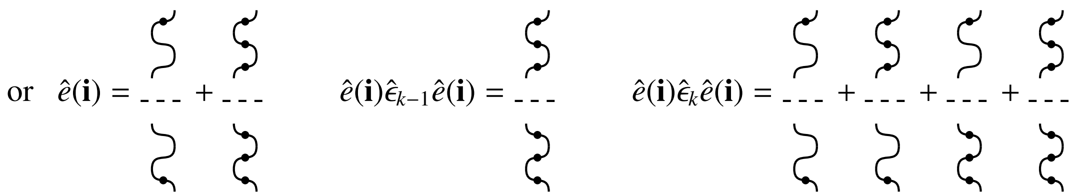
Proof. By Lemma 6.2.1, we have $\beta^L_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \equiv \deg_k(\mathbf{i}) = 0$. Hence we have $\mathrm{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) + \mathrm{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_k) \equiv \alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) = \alpha_k(\mathbf{i})$; and by Lemma 6.2.2, we have $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}) + 1$. Therefore the statement follows from Lemma 6.2.12.

Lemma 6.2.14. *Suppose* $\mathbf{i} \in T^d_{k,0}$ *and* $-i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2}$, *then*

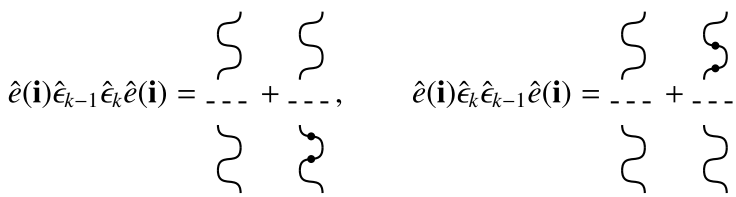
$$\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) + 2\hat{e}(\mathbf{i})\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i}) - \hat{e}(\mathbf{i})\hat{\epsilon}_{k-1}\hat{\epsilon}_k\hat{e}(\mathbf{i}) - \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i}).$$

Proof. By focusing on $k-1$, k and $k+1$-patterns of $\mathbf{i}$, we have





Without loss of generality, we only consider the first situation. It is easy to see that



Hence we have

$$\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}) + 2\hat{e}(\mathbf{i})\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i}) - \hat{e}(\mathbf{i})\hat{\epsilon}_{k-1}\hat{\epsilon}_k\hat{e}(\mathbf{i}) - \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i}).$$

Lemma 6.2.15. *Suppose* $\mathbf{i} \in T^d_{k,0}$ *and* $-i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2}$, *then*

$$\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) + 2(-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}) - \bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{i}) - \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}).$$

Proof. From Lemma 6.2.1, we have $\beta^L_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \equiv \deg_k(\mathbf{i}) = 0$. Hence $\mathrm{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) + \mathrm{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_k) \equiv \alpha^L_k(\mathbf{i})+\alpha^R_k(\mathbf{i}) = \alpha_k(\mathbf{i})$. Similarly, we have $\mathrm{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_{k-1})+\mathrm{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_{k-1}) \equiv \alpha_{k-1}(\mathbf{i})$. From Lemma 6.2.34 we can see that $\mathrm{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_k) + \mathrm{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_{k-1}) \equiv \mathrm{sign}_{\mathbf{i}}(\overrightarrow{\epsilon}_k) + \mathrm{sign}_{\mathbf{i}}(\overleftarrow{\epsilon}_{k-1}) \equiv 0$. Therefore the statement follows from Lemma 6.2.14.

Lemma 6.2.16. *Suppose* $\mathbf{i} \in T^d_{k,-}$ *and* $i_k = -i_{k+1}$. *Then* $\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}(\bar{\epsilon}_k\bar{y}_k + \bar{y}_k\bar{\epsilon}_k)\bar{e}(\mathbf{i})$.

Proof. If $\delta \neq 0$, we can show that $\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})(\hat{\epsilon}_k\hat{y}_k + \hat{y}_k\hat{\epsilon}_k)\hat{e}(\mathbf{i})$ by checking Appendix E.

If $\delta = 0$ (and hence, $k \geq 2$) and $k = 2$, then we can see that $\hat{e}(\mathbf{i})(\hat{\epsilon}_k\hat{y}_k + \hat{y}_k\hat{\epsilon}_k)\hat{e}(\mathbf{i})$ is the sum of elements with clockwise floating loop on the top and anti-clockwise floating loop on the bottom at the level two, which is exactly $\hat{e}(\mathbf{i})$.

If $\delta = 0$ and $k > 2$, then we use can similar arguments as for the case $\delta \neq 0$, noticing that additional floating loops will cancel out.

Hence we have

$$\bar{e}(\mathbf{i})(\bar{\epsilon}_k\bar{y}_k + \bar{y}_k\bar{\epsilon}_k)\bar{e}(\mathbf{i}) = (-1)^{\alpha_k^L(\mathbf{i})+\alpha_k^R(\mathbf{i})+\beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\beta_k^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\operatorname{pos}_k(\mathbf{i})}\hat{e}(\mathbf{i}).$$

From Lemma 6.2.1, we have $\beta_k^L(\mathbf{i}) + \beta_k^R(\mathbf{i}) \equiv \deg_k(\mathbf{i}) \equiv 1$. Hence

$$\alpha_k^L(\mathbf{i}) + \alpha_k^R(\mathbf{i}) + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta_k^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}),$$

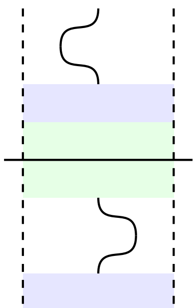
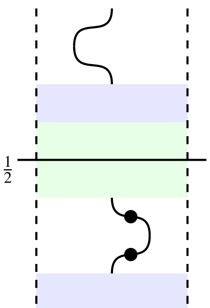
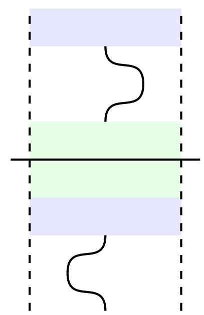
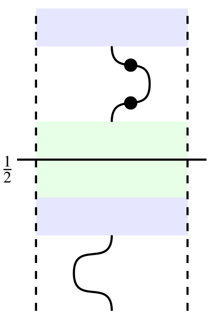
which implies $\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}(\bar{\epsilon}_k\bar{y}_k + \bar{y}_k\bar{\epsilon}_k)\bar{e}(\mathbf{i})$.

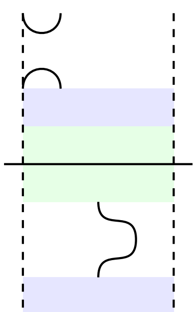
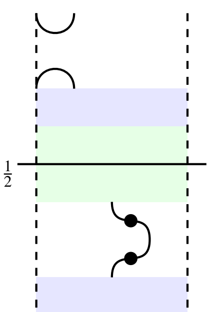
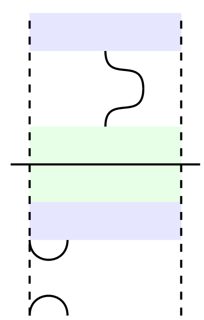
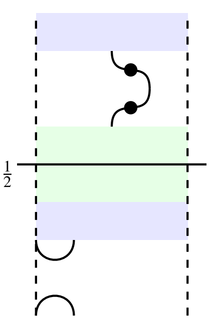
Lemma 6.2.17. *Suppose* $\mathbf{i} \in T^d$. *Then*

$$\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \begin{cases} \left(1 + \delta_{i_k,-\frac{1}{2}}\right)\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}'), & \text{if } \mathbf{i} \in T^d_{k,0}, \\ 0, & \text{if } \mathbf{i} \in T^d_{k,-}, \end{cases}$$

where $\mathbf{i}'$ *defined in (6.1.3).*

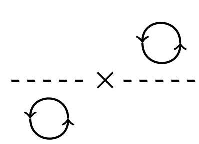
Proof. Suppose $\mathbf{i} \in T^d_{k,0}$. Then we have

$\delta \neq 0:$ $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) =$  $+\delta_{i_k,-\frac{1}{2}}$  $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k =$  $+\delta_{i_k,-\frac{1}{2}}$ 

$\delta = 0:$ $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) =$  $+\delta_{i_k,-\frac{1}{2}}$  $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k =$  $+\delta_{i_k,-\frac{1}{2}}$ 

Since the shaded areas have the same graph and degree 0, it is evident that $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = (1 + \delta_{i_k,-\frac{1}{2}})\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$.

Suppose $\mathbf{i} \in T^d_{k,-}$. Then

$\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k =$  $= 0.$

Lemma 6.2.18. *Suppose* $\mathbf{i} \in T^d_{k,0}, \mathbf{j}, \mathbf{k} \in T^d$ *and* $1 \leq k \leq d-1$.*Then*

$$\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{k}) = (-1)^{\alpha_k(\mathbf{i})}(1 + \delta_{i_k,-\frac{1}{2}})\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k}). \tag{6.2.1}$$

Proof. We have $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i}) \overleftarrow{\epsilon}_k = \left(1 + \delta_{i_k, -\frac{1}{2}}\right) \hat{e}(\mathbf{i}')\hat{\epsilon}_1 \hat{e}(\mathbf{i}')$

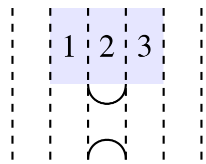
Now let us show the second equation in (5.2.14) and suppose $\mathbf{i} \in T^d_{k,+}$. Recall that

$$A^{\mathbf{i}}_{k,1} = \{\, 1 \le r \le k-1 \mid i_r = -i_k \pm 1 \,\}, \qquad A^{\mathbf{i}}_{k,2} = \{\, 1 \le r \le k-1 \mid i_r = i_k \,\},$$
$$A^{\mathbf{i}}_{k,3} = \{\, 1 \le r \le k-1 \mid i_r = i_k \pm 1 \,\}, \qquad A^{\mathbf{i}}_{k,4} = \{\, 1 \le r \le k-1 \mid i_r = -i_k \,\}.$$

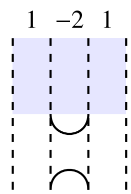
Let $x_1, x_2, \dots, x_{k-1}$ be invariant. Define the polynomial

$$f_{\mathbf{i}}(x_1, x_2, \dots, x_{k-1}) = \sum_{r \in A^{\mathbf{i}}_{k,1}} x_r + \sum_{r \in A^{\mathbf{i}}_{k,3}} x_r - 2 \sum_{r \in A^{\mathbf{i}}_{k,2}} x_r - 2 \sum_{r \in A^{\mathbf{i}}_{k,4}} x_r.$$

Let us fix $\mathbf{i} \in T^d_{k,+}$ and for simplicity write f for $f_{\mathbf{i}}$. Furthermore, let us assume that $i_k = -i_{k+1} \neq -\frac{1}{2}, \pm 1$. Consider the graph of $\mathbf{i}$ between level 1 and $k+1$:

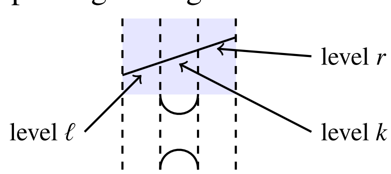


We focus on the shaded area. It can be seen that the shaded area is separated into three regions. A pattern in region 1 has residue $\pm(i_k - 1)$, which belongs $A^{\mathbf{i}}_{k,1}$ or $A^{\mathbf{i}}_{k,3}$; a pattern in region 2 has residue $\pm i_k$, which belongs $A^{\mathbf{i}}_{k,2}$ or $A^{\mathbf{i}}_{k,4}$; and a pattern in region 3 has residue $\pm(i_k + 1)$, which belongs $A^{\mathbf{i}}_{k,1}$ or $A^{\mathbf{i}}_{k,3}$. More precisely, for $1 \le r \le k-1$, if the r-pattern is in region 1 or 3, then $+\bar{y}_r$ is involved in $f(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{k-1})$; and if the r-pattern is in region 2, then $-2\bar{y}_r$ is involved in $f(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{k-1})$:



(6.2.2)

For instance, suppose a strand passing the region as below:

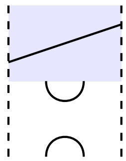


Then it can be seen $r, \ell \in A^{\mathbf{i}}_{k,1}$ or $A^{\mathbf{i}}_{k,3}$ and $k \in A^{\mathbf{i}}_{k,2}$ or $A^{\mathbf{i}}_{k,4}$. Hence $\bar{y}_r + \bar{y}_\ell - 2\bar{y}_k$ is involved in $f(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{k-1})$.

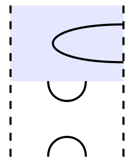
Lemma 6.2.19. *Suppose* $\mathbf{i} \in T^d_{k,+}$ *and* $i_k = -i_{k+1} \neq -\frac{1}{2}, \pm 1$. *For any strand passing through the shaded area (as long as the strand touches the left or right boundaries of the area, it leaves the area), all the patterns on the strand involved in* $f(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{k-1})$ *cancel with each other.*

Proof. Without loss of generality, we assume that the strand enters the area from the right.

There are two ways for the strand to exit the area:

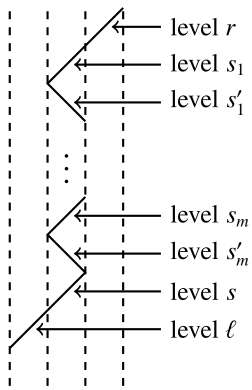


or



Case 1: The strand leaves the area from the left.

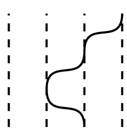
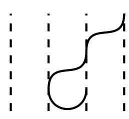
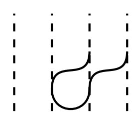
In this case the strand is of the following form:

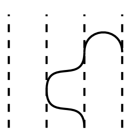
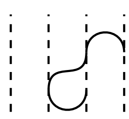
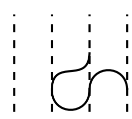


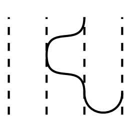
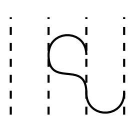
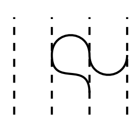
and we need to prove that

$$\bar{y}_r - 2(\bar{y}_{s_1} + \bar{y}_{s'_1}) - 2(\bar{y}_{s_2} + \bar{y}_{s'_2}) - \ldots - 2(\bar{y}_{s_m} + \bar{y}_{s'_m}) - 2\bar{y}_s + \bar{y}_\ell = 0. \tag{6.2.3}$$

First we evaluate the part of the strand that contains $\bar{y}_r - 2(\bar{y}_{s_1} + \bar{y}_{s'_1}) - 2(\bar{y}_{s_2} + \bar{y}_{s'_2}) - \ldots - 2(\bar{y}_{s_m} + \bar{y}_{s'_m})$. It is of one of the following forms:

(1.1).  (1.2).  (1.3). 

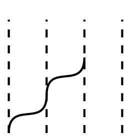
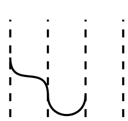
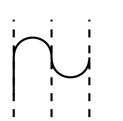
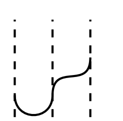
(1.4).  (1.5).  (1.6). 

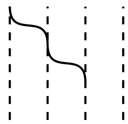
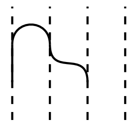
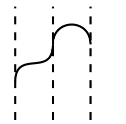
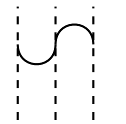
(1.7).  (1.8).  (1.9). 

It can be seen from the above diagrams that $\mathrm{pos}_{s_i}(\mathbf{i}) = \mathrm{pos}_{s'_i}(\mathbf{i}) + 1$ and $\hat{y}_{s_i} = \hat{y}_{s'_i}$ for any $1 \leq i \leq m$, and $\bar{y}_{s_i} + \bar{y}_{s'_i} = 0$. Hence

$$\bar{y}_r - 2(\bar{y}_{s_1} + \bar{y}_{s'_1}) - 2(\bar{y}_{s_2} + \bar{y}_{s'_2}) - \ldots - 2(\bar{y}_{s_m} + \bar{y}_{s'_m}) = \bar{y}_r = (-1)^{\mathrm{pos}_r(\mathbf{i})}\hat{y}_r. \tag{6.2.4}$$

Then we evaluate the part of the strand that contains $-2\bar{y}_s + \bar{y}_l$. It is of one of the following forms:

(2.1).  (2.2).  (2.3).  (2.4). 

(2.5).  (2.6).  (2.7).  (2.8). 

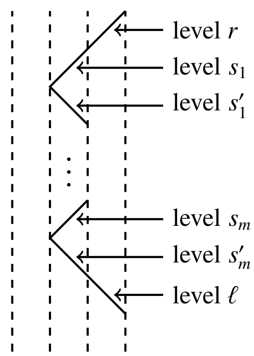
It can be seen from the above diagrams that $\mathrm{pos}_s(\mathbf{i}) = \mathrm{pos}_\ell(\mathbf{i})$ and $\hat{y}_s = \hat{y}_\ell$. Hence

$$-2\bar{y}_s + \bar{y}_\ell = -\bar{y}_s = -(-1)^{\mathrm{pos}_k(\mathbf{i})}\hat{y}_s. \tag{6.2.5}$$

Since (2.1) − (2.4) can be connected with (1.1), (1.4) and (1.9); and (2.5) − (2.8) can be connected with (1.3), (1.6) and (1.7), in all these cases we have $\mathrm{pos}_r(\mathbf{i}) = \mathrm{pos}_s(\mathbf{i})$ and $\hat{y}_r = \hat{y}_s$. Hence (6.2.3) follows by (6.2.4) and (6.2.5).

Case 2: The strand leaves the area from the right.

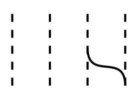
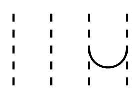
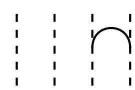
In this case the strand is of the following form:



and we need to prove that

$$\bar{y}_r - 2(\bar{y}_{s_1} + \bar{y}_{s_1'}) - 2(\bar{y}_{s_2} + \bar{y}_{s_2'}) - \ldots - 2(\bar{y}_{s_m} + \bar{y}_{s_m'}) + \bar{y}_\ell = 0. \tag{6.2.6}$$

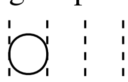
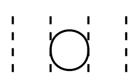
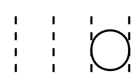
Following the same argument as in Case 1, we can obtain (6.2.4). The pattern of the strand that contains $\bar{y}_\ell$ is of one of the following forms:

(3.1).  (3.2).  (3.3). 

Since (3.1) and (3.2) can be connected with (1.1), (1.4), (1.8) and (1.9); and (3.3) can be connected with (1.2), (1.3), (1.5), (1.6) and (1.7), in all these cases, we have $\mathrm{pos}_r(\mathbf{i}) = \mathrm{pos}_\ell(\mathbf{i})$ and $\hat{y}_r = \hat{y}_\ell$. Hence by (6.2.6) the equality (6.2.3) holds.

Lemma 6.2.20. *Suppose $\mathbf{i} \in T^d_{k,+}$ and $i_k = -i_{k+1} \neq -\frac{1}{2}, \pm 1$. When a floating loop is completely inside the region, all terms on the loop involved in $f(\bar{y}_1, \ldots, \bar{y}_{k-1})$ cancel each other.*

Proof. The floating loop is of the forms

(1).  (2).  (3). 

Suppose the upper half of the loop is in level r and the lower half of the loop is in level ℓ. In case (1) we have that terms on the loop involved in $f(\bar{y}_1, \ldots, \bar{y}_{k-1})$ are

$$\bar{y}_r + \bar{y}_\ell = ((-1)^{\mathrm{pos}_r(\mathbf{i})} + (-1)^{\mathrm{pos}_\ell(\mathbf{i})})\hat{y}_r = 0;$$

and in case (2) we have

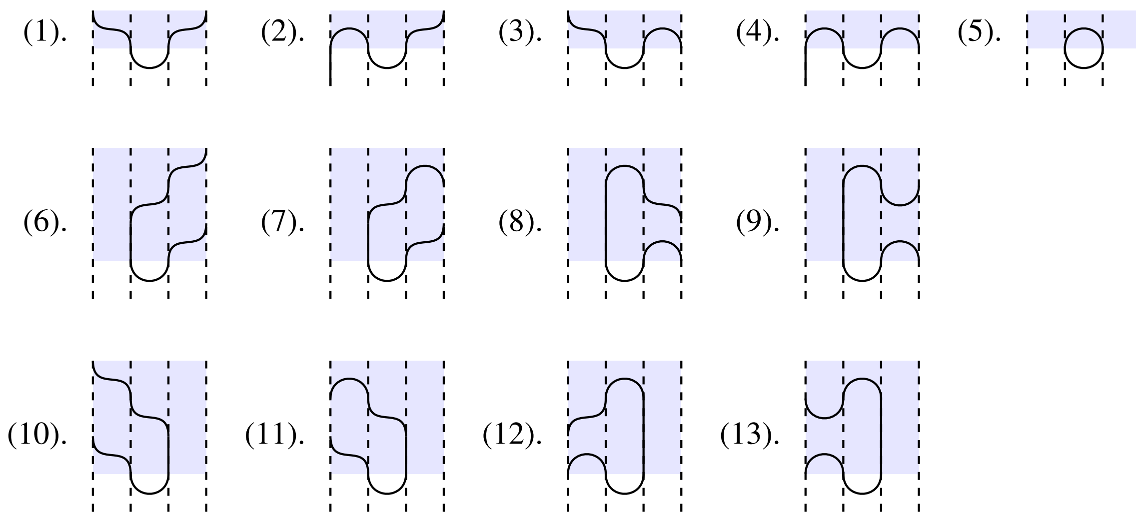
$$-2\bar{y}_r - 2\bar{y}_\ell = -2((-1)^{\mathrm{pos}_r(\mathbf{i})} + (-1)^{\mathrm{pos}_\ell(\mathbf{i})})\hat{y}_r = 0;$$

and in case (3) we have

$$\bar{y}_r + \bar{y}_\ell = ((-1)^{\mathrm{pos}_r(\mathbf{i})} + (-1)^{\mathrm{pos}_\ell(\mathbf{i})})\hat{y}_r = 0.$$

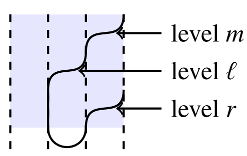
Lemma 6.2.21. *Suppose* $\mathbf{i} \in T^d_{k,+}$ *and* $i_k = -i_{k+1} \neq -\frac{1}{2}, \pm 1$. *Let the pattern be connected with k-pattern directly from right on the level r, and directly from left on the level* ℓ. *Then the term involved in* $f(\bar{y}_1, \ldots, \bar{y}_{k-1})$ *directly connected with pattern at level k in the region is* $(-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)$.

Proof. The pattern directly connected with the pattern at level k is of the form:



In case (1)−(4), we have $\mathrm{pos}_r(\mathbf{i}) = \mathrm{pos}_l(\mathbf{i}) = \mathrm{pos}_k(\mathbf{i})$, which implies $\bar{y}_r + \bar{y}_\ell = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)$. In case (5), we have $r = l$ and $\mathrm{pos}_r(\mathbf{i}) = \mathrm{pos}_k(\mathbf{i}) + 1$. Hence we have $-2\bar{y}_r = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)$.

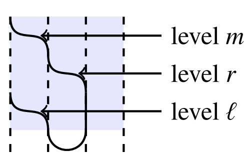
In case (6) – (9), suppose



Since $\mathrm{pos}_m(\mathbf{i}) = \mathrm{pos}_\ell(\mathbf{i}) = \mathrm{pos}_r(\mathbf{i}) + 1 = \mathrm{pos}_k(\mathbf{i}) + 1$, we have

$$\bar{y}_m - 2\bar{y}_\ell + \bar{y}_r = \bar{y}_r - \bar{y}_\ell = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell).$$

Similarly, for case (10) - (13), we have



Since $\mathrm{pos}_m(\mathbf{i}) = \mathrm{pos}_r(\mathbf{i}) = \mathrm{pos}_\ell(\mathbf{i}) + 1 = \mathrm{pos}_k(\mathbf{i}) + 1$, we have

$$\bar{y}_m - 2\bar{y}_r + \bar{y}_\ell = \bar{y}_\ell - \bar{y}_r = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell).$$

Combining Lemma 6.2.19 - 6.2.21, we have the following result:

Lemma 6.2.22. *Suppose* $\mathbf{i} \in T^d_{k,+}$ *and* $i_k = -i_{k+1} \neq -\frac{1}{2}, \pm 1$. *let the pattern be connected with k-pattern directly from right on level r, and directly from left on level* ℓ. *Then we have*

$$f(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{k-1})\bar{e}(\mathbf{i}) = (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{i}).$$

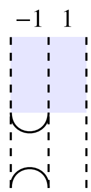
It is only left to consider the cases when $i_k = -i_{k+1} = -\frac{1}{2}$ or ± 1. Since

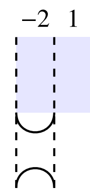
$$i_k = -i_{k+1} = -\frac{1}{2}: \quad \begin{array}{llll} A^{\mathbf{i}}_{k,1} & = \{ 1 \leq r \leq k-1 \mid i_r = -\frac{1}{2}, \frac{3}{2} \}, & A^{\mathbf{i}}_{k,2} & = \{ 1 \leq r \leq k-1 \mid i_r = -\frac{1}{2} \}, \\ A^{\mathbf{i}}_{k,3} & = \{ 1 \leq r \leq k-1 \mid i_r = \frac{1}{2}, -\frac{3}{2} \}, & A^{\mathbf{i}}_{k,4} & = \{ 1 \leq r \leq k-1 \mid i_r = \frac{1}{2} \}; \end{array}$$

$$i_k = -i_{k+1} = 1: \quad \begin{array}{llll} A^{\mathbf{i}}_{k,1} & = \{ 1 \leq r \leq k-1 \mid i_r = -2, 0 \}, & A^{\mathbf{i}}_{k,2} & = \{ 1 \leq r \leq k-1 \mid i_r = 1 \}, \\ A^{\mathbf{i}}_{k,3} & = \{ 1 \leq r \leq k-1 \mid i_r = 0, 2 \}, & A^{\mathbf{i}}_{k,4} & = \{ 1 \leq r \leq k-1 \mid i_r = -1 \}; \end{array}$$

$$i_k = -i_{k+1} = -1: \quad \begin{array}{llll} A^{\mathbf{i}}_{k,1} & = \{ 1 \leq r \leq k-1 \mid i_r = 0, 2 \}, & A^{\mathbf{i}}_{k,2} & = \{ 1 \leq r \leq k-1 \mid i_r = -1 \}, \\ A^{\mathbf{i}}_{k,3} & = \{ 1 \leq r \leq k-1 \mid i_r = -2, 0 \}, & A^{\mathbf{i}}_{k,4} & = \{ 1 \leq r \leq k-1 \mid i_r = 1 \}. \end{array}$$

Following the same argument as before, we have analogues graphs as in (6.2.2):

$i_k = -i_{k+1} = -\frac{1}{2}$: 

$i_k = -i_{k+1} = \pm 1$: 

By the similar arguments, we have an analogue of Lemma 6.2.22.

Lemma 6.2.23. *Suppose* $\mathbf{i} \in T^d_{k,+}$ *and* $i_k = -i_{k+1} = -\frac{1}{2}$ *or* ± 1. *Let the pattern be connected with k-pattern directly from right on level r, and directly from left on level* ℓ. *Then we have*

$$f(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{k-1})\bar{e}(\mathbf{i}) = \begin{cases} \frac{(-1)^{\mathrm{pos}_k(\mathbf{i})}}{2}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{i}), & \text{if } i_k = -i_{k+1} = -\frac{1}{2}, \\ (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{i}), & \text{if } i_k = -i_{k+1} = \pm 1. \end{cases}$$

Lemma 6.2.24. *Suppose* $\mathbf{i} \in T^d_{k,+}$. *Then we have*

$$\bar{e}(\mathbf{j})\bar{\epsilon}_k\hat{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{k}) = (-1)^{\alpha_k(\mathbf{i})}\left(1 + \delta_{i_k, -\frac{1}{2}}\right)\left(\sum_{i \in A^{\mathbf{i}}_{k,1}} \bar{y}_i + \sum_{i \in A^{\mathbf{i}}_{k,3}} \bar{y}_i - 2\sum_{i \in A^{\mathbf{i}}_{k,2}} \bar{y}_i - 2\sum_{i \in A^{\mathbf{i}}_{k,4}} \bar{y}_i\right)\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k}).$$

Proof. By the definition it is easy to see that for $1 \leq i \leq 4$, $r \in A^{\mathbf{i}}_{k,i}$ if and only if $r + 2 \in A^{\mathbf{i}'}_{k,i}$. Therefore from Lemma 6.2.22 and Lemma 6.2.23, we have

$$f(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{k-1})\bar{e}(\mathbf{i}) = \begin{cases} \frac{(-1)^{\mathrm{pos}_k(\mathbf{i})}}{2}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{i}), & \text{if } i_k = -i_{k+1} = -\frac{1}{2}, \\ (-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{i}), & \text{if } i_k = -i_{k+1} \neq -\frac{1}{2}, \end{cases}$$

where in the diagram $\mathbf{i}$ the pattern is connected with k-pattern directly from right on the level r, and directly from left on the level ℓ.

By checking Appendixes A and B, we can obtain $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = (\hat{y}_{r+2} + \hat{y}_{\ell+2})\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$. Hence by Proposition 6.2.41, we have

$$\begin{aligned} \hat{e}(\mathbf{j})\hat{\epsilon}_k\hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{k}) &= \hat{e}(\mathbf{j})(\overleftarrow{\epsilon}_k\hat{e}_2\overrightarrow{\epsilon}_k)\hat{e}(\mathbf{i})(\overleftarrow{\epsilon}_k\hat{e}_2\overrightarrow{\epsilon}_k)\hat{e}(\mathbf{k}) \\ &= \hat{e}(\mathbf{j})\overleftarrow{\epsilon}_k\hat{\epsilon}_2(\hat{y}_{r+2} + \hat{y}_{\ell+2})\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')\hat{\epsilon}_2\overrightarrow{\epsilon}_k\hat{e}(\mathbf{k}) \\ &= (\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{j})\overleftarrow{\epsilon}_k\hat{\epsilon}_2\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')\hat{\epsilon}_2\overrightarrow{\epsilon}_k\hat{e}(\mathbf{k}) \\ &= (\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{j})\overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k\hat{e}(\mathbf{k}) \end{aligned}$$

$$= (\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{j})\hat{\epsilon}_k\hat{e}(\mathbf{k}).$$

Finally, since $\mathrm{sign}_{\mathbf{i}}\overrightarrow{\epsilon}_k + \mathrm{sign}_{\mathbf{i}}\overleftarrow{\epsilon}_k = \alpha_k(\mathbf{i}) + \mathrm{pos}_k(\mathbf{i})$ from Lemma 6.2.11, we have

$$\begin{aligned}
\bar{e}(\mathbf{j})\bar{\epsilon}_k\hat{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{k}) &= (-1)^{\mathrm{sign}_{\mathbf{j}}(\overleftarrow{\epsilon}_k)+\mathrm{sign}_{\mathbf{k}}(\overrightarrow{\epsilon}_k)}(-1)^{\alpha_k(\mathbf{i})}(-1)^{\mathrm{pos}_k(\mathbf{i})}(\hat{y}_r + \hat{y}_\ell)\hat{e}(\mathbf{j})\hat{\epsilon}_k\hat{e}(\mathbf{k})\\
&= (-1)^{\alpha_k(\mathbf{i})}\left(1+\delta_{i_k,-\frac{1}{2}}\right) f(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_{k-1})\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k})\\
&= (-1)^{\alpha_k(\mathbf{i})}\left(1+\delta_{i_k,-\frac{1}{2}}\right)\left(\sum_{i\in A^{\mathbf{i}}_{k,1}}\bar{y}_i + \sum_{i\in A^{\mathbf{i}}_{k,3}}\bar{y}_i - 2\sum_{i\in A^{\mathbf{i}}_{k,2}}\bar{y}_i - 2\sum_{i\in A^{\mathbf{i}}_{k,4}}\bar{y}_i\right)\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k}).
\end{aligned}$$

.

Combining the results of this subsection, we obtain the following Proposition.

Proposition 6.2.25. *Let* $\mathbf{i},\mathbf{j},\mathbf{k} \in I^d$ *and* $1 \le k \le d-1$. *Then*

$$\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) = \begin{cases}(-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i}), & \text{if } \mathbf{i}\in T^d_{k,0} \text{ and } i_k = -i_{k+1} \neq \pm\frac{1}{2},\\ (-1)^{\alpha_k(\mathbf{i})}(\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}), & \text{if } \mathbf{i}\in T^d_{k,+};\end{cases}$$

$$(\bar{y}_k - \bar{y}_{k+1})\bar{e}(\mathbf{i}) = 2(-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i})\bar{y}_k = 2(-1)^{\alpha_k(\mathbf{i})}\bar{y}_k\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}), \qquad \text{if } \mathbf{i}\in T^d_{k,0} \text{ and } i_k = -i_{k+1} = \frac{1}{2},$$

$$\begin{aligned}\bar{e}(\mathbf{i}) &= (-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}) + 2(-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i})\\ &\quad - \bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{i}) - \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}), \qquad \text{if } \mathbf{i}\in T^d_{k,0} \text{ and } -i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2},\end{aligned}$$

$$\bar{e}(\mathbf{i}) = (-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})(\bar{\epsilon}_k\bar{y}_k + \bar{y}_k\bar{\epsilon}_k)\bar{e}(\mathbf{i}), \qquad \text{if } \mathbf{i}\in T^d_{k,-} \text{ and } i_k = -i_{k+1}.$$

$$\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{k}) = \begin{cases}(-1)^{\alpha_k(\mathbf{i})}(1+\delta_{i_k,-\frac{1}{2}})\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k}), & \text{if } \mathbf{i}\in T^d_{k,0},\\ 0, & \text{if } \mathbf{i}\in T^d_{k,-},\\ (-1)^{\alpha_k(\mathbf{i})}(1+\delta_{i_k,-\frac{1}{2}})(\sum_{r\in A^{\mathbf{i}}_{k,1}}\bar{y}_r - 2\sum_{r\in A^{\mathbf{i}}_{k,2}}\bar{y}_r, & \\ \qquad + \sum_{r\in A^{\mathbf{i}}_{k,3}}\bar{y}_r - 2\sum_{r\in A^{\mathbf{i}}_{k,4}}\bar{y}_r)\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{e}(\mathbf{k}), & \text{if } \mathbf{i}\in T^d_{k,+}.\end{cases}$$

6.2.7 Untwist relations

In this subsection we prove the untwist relations.

Lemma 6.2.26. *Suppose* $\mathbf{i}\in I^d$ *such that* $i_k + i_{k+1} = 0$. *Then we have*

$$\overrightarrow{\epsilon}_k\hat{\psi}_k\hat{e}(\mathbf{i}) = \begin{cases}\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}), & \text{if } \mathbf{i}\in T^d_{k,+} \text{ and } i_k \neq 0,\pm\frac{1}{2},\\ 0, & \text{otherwise;}\end{cases}$$

and

$$\hat{e}(\mathbf{i})\hat{\psi}_k\overleftarrow{\epsilon}_k = \begin{cases}\hat{e}(\mathbf{i})\overrightarrow{\epsilon}_k, & \text{if } \mathbf{i}\in T^d_{k,+} \text{ and } i_k \neq 0,\pm\frac{1}{2},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Assume $i_k + i_{k+1} = 0$. Then, unless $\mathbf{i}\in T^d_{k,+}$ and $i_k \neq 0, \pm\frac{1}{2}$, we have $\hat{\psi}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})\hat{\psi}_k = 0$ (since in this case $\mathbf{i}\cdot s_k$ is not a residue sequence). We only show the first equality $\overrightarrow{\epsilon}_k\hat{\psi}_k\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ since the other one follows by the same argument.

Assume $\mathbf{i}\in T^d_{k,+}$, then the graph of $\hat{\psi}_k\hat{e}(\mathbf{i})$ is of the following form

or

which implies that $\hat{\psi}_k\hat{e}(\mathbf{i})$ is topologically equivalent to the idempotent. Hence it is easy to see that $\overrightarrow{\epsilon}_k\hat{\psi}_k\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$.

Proposition 6.2.27. *Suppose* $\mathbf{i} \in I^d$ *such that* $i_k + i_{k+1} = 0$. *Then we have*

$$\bar{\epsilon}_k\bar{\psi}_k\bar{e}(\mathbf{i}) = \begin{cases} (-1)^{a_k(\mathbf{i}\cdot s_k)}\bar{\epsilon}_k\bar{e}(\mathbf{i}), & \text{if } \mathbf{i} \in T^n_{k,+}, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\epsilon}_k = \begin{cases} (-1)^{a_k(\mathbf{i}\cdot s_k)}\bar{e}(\mathbf{i})\bar{\epsilon}_k, & \text{if } \mathbf{i} \in T^n_{k,+}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From Lemma 6.2.26, it follows that it sufficies to prove that

$$\alpha^R_k(\mathbf{i}\cdot s_k) + \beta^R_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \alpha^L_k(\mathbf{i}\cdot s_k) + \alpha^R_k(\mathbf{i}) + \beta^L_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}\cdot s_k),$$
$$\alpha^L_k(\mathbf{i}\cdot s_k) + \beta^L_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}\cdot s_k) + \beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \operatorname{pos}_k(\mathbf{i}\cdot s_k) \equiv \alpha_k(\mathbf{i}\cdot s_k).$$

From Lemma 6.2.2 and Lemma 6.2.1 we have,

$$\alpha^R_k(\mathbf{i}\cdot s_k)+\alpha^R_k(\mathbf{i}\cdot s_k) = \alpha_k(\mathbf{i}\cdot s_k), \quad \operatorname{pos}_k(\mathbf{i}\cdot s_k) \equiv \operatorname{pos}_k(\mathbf{i}), \quad \text{and} \quad \beta^L_k(\mathbf{i}\cdot s_k)+\beta^R_k(\mathbf{i}\cdot s_k) \equiv h_k(\mathbf{i}\cdot s_k)+1 \equiv 1,$$

and hence

$$\begin{aligned} &\alpha^R_k(\mathbf{i}\cdot s_k) + \beta^R_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \alpha^L_k(\mathbf{i}\cdot s_k) + \alpha^R_k(\mathbf{i}) + \beta^L_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \beta^R_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i}) \\ &\equiv \alpha_k(\mathbf{i}\cdot s_k) + \operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i}) \equiv \alpha_k(\mathbf{i}\cdot s_k), \\ &\alpha^L_k(\mathbf{i}\cdot s_k) + \beta^L_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}\cdot s_k) + \beta^L_k(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i}\cdot s_k)\operatorname{pos}_k(\mathbf{i}\cdot s_k) + \operatorname{pos}_k(\mathbf{i}\cdot s_k) \\ &\equiv \alpha_k(\mathbf{i}\cdot s_k) + \operatorname{pos}_k(\mathbf{i}\cdot s_k) + \operatorname{pos}_k(\mathbf{i}\cdot s_k) \equiv \alpha_k(\mathbf{i}\cdot s_k). \end{aligned}$$

6.2.8 Tangle relations

In this subsection we prove the tangle relations. To do this, we first show the relation

$$\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{\epsilon}_{k-1} = \bar{\epsilon}_{k-1}, \qquad \bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k = \bar{\epsilon}_k.$$

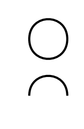
The key idea is to show these relations is to obtain the following

$$\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = \overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i}')\bar{\epsilon}_1\hat{e}(\mathbf{i}').$$

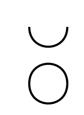
Suppose $\mathbf{i} \in T^d$ such that $i_{k-1} = -i_k = i_{k+1}$. Let us list the possible local structures of the graph $\mathbf{i}$.

Lemma 6.2.28. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = -i_k = i_{k+1}$. *Then*

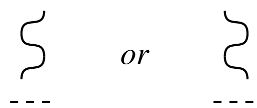
(1) if $\deg_k(\mathbf{i}) = 1$ *and* $\deg_{k-1}(\mathbf{i}) = -1$*, then the graph of* $\mathbf{i}$ *at level* $k-1$*,* k *and* $k+1$ *is of the form*



(2) if $\deg_k(\mathbf{i}) = -1$ *and* $\deg_{k-1}(\mathbf{i}) = 1$*, then the graph of* $\mathbf{i}$ *at level* $k-1$*,* k *and* $k+1$ *is of the form*

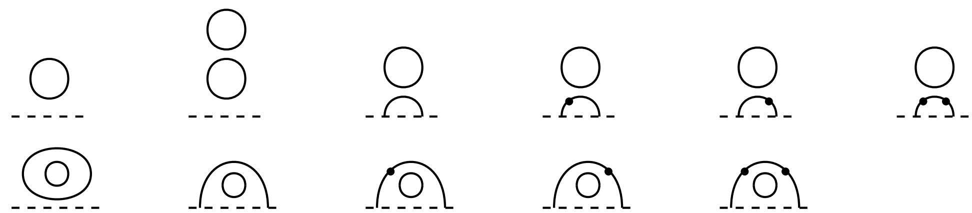


(3) *if* $\deg_k(\mathbf{i}) = \deg_{k-1}(\mathbf{i}) = 0$*, the graph of* $\mathbf{i}$ *at level* $k-1$*,* k *and* $k+1$ *is of the form*

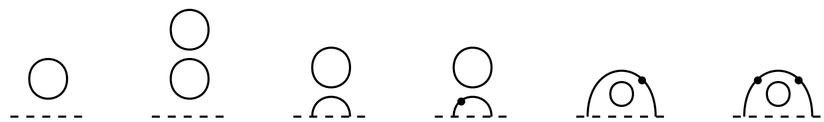


Corollary 6.2.29. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = -i_k = i_{k+1}$*. Then in case* $\deg_k(\mathbf{i}) = \pm 1$*, we have the following properties:*

(1) *the graph of* $\mathbf{i}$ *is topologically equivalent to one of the following graphs:*

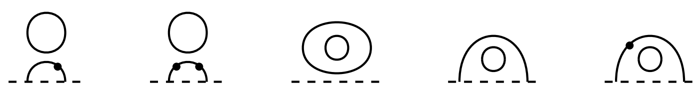


(2) *assume* $\mathbf{i}$ *is topologically equivalent to the following graphs:*



then $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$*.*

(3) *assume* $\mathbf{i}$ *is topologically equivalent to the following graphs:*



we have the following

(a) *if* $\deg_k(\mathbf{i}) = 1$*, then* k *is on the floating loop (or inside loop) and* $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$*.*

(b) *if* $\deg_k(\mathbf{i}) = -1$*, then* k *is on the floating loop (or inside loop) and* $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i}) + 1$*.*

Proof. (1) follows from Lemma 6.2.28. Let us prove (2). if $\deg_{k-1}(\mathbf{i}) = -1$, the $k-1$-pattern and k-pattern of $\mathbf{i}$ are on the same floating loop, which implies $b_{k-1}(\mathbf{i}) = b_k(\mathbf{i})$ and $d_{k-1}(\mathbf{i}) = d_k(\mathbf{i}) = 0$, and hence $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$; whereas if $\deg_{k-1}(\mathbf{i}) = 1$, by checking (3), (5), (11.a.i), (11.a.ii), (11.b.i), (11.b.ii), (15.a.iii), (15.a.iv), (15.b.iii) and (15.b.iv) of Appendix A, we always have $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$.

Now let us prove prove (3). If $\deg_k(\mathbf{i}) = 1$, then it follows from Lemma 6.2.28(1) that the k-pattern is on the floating loop (or inside loop), and $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$ since $k-1$-pattern and k-pattern of $\mathbf{i}$ are on the same floating loop; and if $\deg_k(\mathbf{i}) = -1$, then from Lemma 6.2.28(2) we have that the k-pattern is on the floating loop (or inside loop), and $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i}) + 1$ by checking (11.a.iii), (11.a.iv), (15.a.i) and (15.a.ii) of Appendix A.

Lemma 6.2.30. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = -i_k = i_{k+1}$*. Then we have*

$$\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i}) \overleftarrow{\epsilon}_{k-1} = \overrightarrow{\epsilon}_{k-1} \hat{e}(\mathbf{i}) \overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}').$$

Proof. We only show $\overrightarrow{\epsilon}_k \hat{e}(\mathbf{i}) \overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$, since the relation $\overrightarrow{\epsilon}_{k-1} \hat{e}(\mathbf{i}) \overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$ follows by the same calculations.

We consider the following cases:

Case 1: $\deg_{k-1}(\mathbf{i}) = \deg_k(\mathbf{i}) = 0$ and $\delta \neq 0$.

In this case, the graph of $\mathbf{i}$ and $\mathbf{i}'$ are topologically equivalent. Hence $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ and $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1}$ are topologically equivalent to the idempotents $\hat{e}(\mathbf{i}')$. Therefore we have

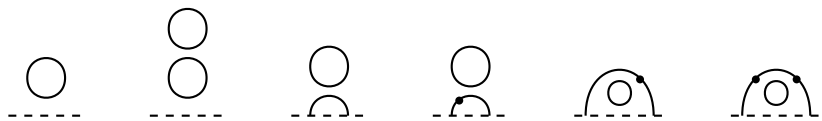
$$\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}').$$

Case 2: $\deg_{k-1}(\mathbf{i}) = -1$, $\deg_k(\mathbf{i}) = 1$ and $\delta \neq 0$.

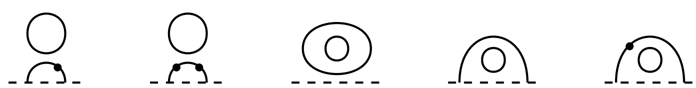
In this case, we have $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{i}')$ by checking all 11 cases of Corollary 6.2.29 using the calculations of Appendix E. By the definition of $\hat{\epsilon}_k$'s, we have $\hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$ if $\delta \neq 0$. Hence we have $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$.

Case 3: $\deg_{k-1}(\mathbf{i}) = 1$, $\deg_k(\mathbf{i}) = -1$ and $\delta \neq 0$.

In this case, the calculations are essentially the involution of Case 2. From Corollary 6.2.29 we have that if $\mathbf{i}$ is topologically equivalent to the following graphs:



then $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i})$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = (-1)^{m_{k-1}(\mathbf{i})+m_k(\mathbf{i})}\hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i}')$; and if $\mathbf{i}$ is topologically equivalent to the following graphs:



then $m_{k-1}(\mathbf{i}) = m_k(\mathbf{i}) + 1$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = (-1)^{m_{k-1}(\mathbf{i})+m_k(\mathbf{i})+1}\hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i}')$. Therefore in all 11 cases we have $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}')$.

Case 4: $\delta = 0$.

in this case there will be a floating circle at levels 2 and 3 of the graph of $\mathbf{i}'$. Following the same calculations as in Cases 1 - 3, the product $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1}$ is the sum of all graphs $g_{\mathbf{i}'}g^*_{\mathbf{i}'}$ with the following orientations:

- the pair of floating loops at level 2 and 3 are oriented clockwise
- the rest pairs of floating loops are oriented oppositely
- the loop in the middle is oriented anti-clockwise

Hence, the product equals $\hat{e}(\mathbf{i})\hat{\epsilon}_1\hat{e}(\mathbf{i})$.

Lemma 6.2.31. *We have*

$$\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}').$$

Proof. The Lemma can be verified by direct calculations by considering both cases $\delta = 0$ and $\delta \neq 0$.

Lemma 6.2.32. *For any* $\mathbf{i} \in I^d$ *such that* $i_k + i_{k+1} = 0$, *we have*

$$\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{\epsilon}_{k-1}\ldots\hat{\epsilon}_1\hat{e}(\mathbf{i}'), \qquad \overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i}')\hat{\epsilon}_1\ldots\hat{\epsilon}_{k-1}\hat{\epsilon}_k\hat{e}(\mathbf{i}).$$

Proof. We only prove the first equality. The second equality can be verified by the same arguments. We proceed by induction on k. If $k = 1$, it is trivial. Assume

$$\hat{e}(\mathbf{j})\overleftarrow{\epsilon}_{k-1} = \hat{e}(\mathbf{j})\hat{\epsilon}_{k-1}\hat{\epsilon}_{k-2}\ldots\hat{\epsilon}_1\hat{e}(\mathbf{j}')$$

for any $\mathbf{j} \in T^d$.

For $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, let

$$\mathbf{j} = (i_1, \ldots, i_{k-2}, i_{k-1}, -i_{k-1}, i_{k-1}, i_{k+2}, i_{k+3}, \ldots, i_n).$$

It can be seen that $\mathbf{i}' = \mathbf{j}'$. Then by Lemma 6.2.30, Lemma 6.2.31 and induction, we have

$$\begin{aligned}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k &= \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{i}')\hat{\epsilon}_1\hat{e}(\mathbf{i}') = \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\hat{\epsilon}_2\hat{e}(\mathbf{i}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j})\overleftarrow{\epsilon}_{k-1}\\ &= \hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k\hat{e}(\mathbf{j})\hat{\epsilon}_{k-1}\hat{\epsilon}_{k-2}\ldots\hat{\epsilon}_1\hat{e}(\mathbf{i}')\\ &= \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{\epsilon}_{k-2}\ldots\hat{\epsilon}_1\hat{e}(\mathbf{i}'),\end{aligned}$$

.

Now we start to prove the tangle relations. First we prove that

$$\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k = \bar{\epsilon}_k, \qquad \bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{\epsilon}_{k-1} = \bar{\epsilon}_{k-1}.$$

Lemma 6.2.33. *We have $\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{\epsilon}_k = \hat{\epsilon}_k$ and $\hat{\epsilon}_{k-1}\hat{\epsilon}_k\hat{\epsilon}_{k-1} = \hat{\epsilon}_{k-1}$.*

Proof. For convenience we omit $\hat{e}(\mathbf{i})$'s in the following calculations. By Lemma 6.2.30 and Lemma 6.2.31 we have

$$\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{\epsilon}_k = (\overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k)(\overleftarrow{\epsilon}_{k-1}\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1})(\overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k) = \overleftarrow{\epsilon}_k\hat{\epsilon}_2(\overrightarrow{\epsilon}_k\overleftarrow{\epsilon}_{k-1})\hat{\epsilon}_2(\overrightarrow{\epsilon}_{k-1}\overleftarrow{\epsilon}_k)\hat{\epsilon}_2\overrightarrow{\epsilon}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_2\overrightarrow{\epsilon}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k = \hat{\epsilon}_k,$$

and

$$\hat{\epsilon}_{k-1}\hat{\epsilon}_k\hat{\epsilon}_{k-1} = \overleftarrow{\epsilon}_{k-1}\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1}\overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k\overleftarrow{\epsilon}_{k-1}\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1} = \overleftarrow{\epsilon}_{k-1}\hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1} = \hat{\epsilon}_{k-1}.$$

Lemma 6.2.34. *Suppose $\mathbf{i} \in T^d$ such that $i_{k-1} = -i_k = i_{k+1}$. Then we have*
$\alpha_k^R(\mathbf{i})+\beta_k^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\alpha_{k-1}^L(\mathbf{i})+\beta_{k-1}^L(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) = \alpha_{k-1}^R(\mathbf{i})+\beta_{k-1}^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\alpha_k^L(\mathbf{i})+\beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) = 0.$

Proof. First we show $\alpha_k^L(\mathbf{i})+\beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i})+\alpha_{k-1}^R(\mathbf{i})+\beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) = 0$. Note that by Lemma 6.2.2 we have $\operatorname{pos}_k(\mathbf{i}) = \operatorname{pos}_{k-1}(\mathbf{i}) + 1$. We consider the following three cases:

Case 1: $i_k = 0$.

By the direct calculation one can see that $\beta_k^L(\mathbf{i}) = \beta_{k-1}^R(\mathbf{i})$, $\alpha_k^L(\mathbf{i}) = \alpha_{k-1}^L(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha_{k-1}^L(\mathbf{i}) + \alpha_{k-1}^R(\mathbf{i}) = \beta_{k-1}^R(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}&\alpha_k^L(\mathbf{i}) + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \alpha_{k-1}^R(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i})\\ =\ &\alpha_{k-1}^L(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \alpha_{k-1}^R(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i})\\ =\ &\beta_{k-1}^R(\mathbf{i}) + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i})\\ =\ &\beta_{k-1}^R(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i})\\ =\ &\beta_{k-1}^R(\mathbf{i})\,(1 + \operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i})) = 0.\end{aligned}$$

Case 2: $i_k = -\frac{1}{2}$.

By the direct calculation one can see that $\beta_k^L(\mathbf{i}) = \beta_{k-1}^R(\mathbf{i})$, $\alpha_k^L(\mathbf{i}) = \alpha_{k-1}^L(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha_{k-1}^L(\mathbf{i}) + \alpha_{k-1}^R(\mathbf{i}) + 1 = \beta_{k-1}^R(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}
& \alpha^L_k(\mathbf{i}) + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \alpha^L_{k-1}(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) \left(1 + \operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i})\right) = 0.
\end{aligned}$$

Case 3: $i_k \neq 0$ or $-\frac{1}{2}$.

By the direct calculation one can see that $\beta^L_k(\mathbf{i}) = \beta^R_{k-1}(\mathbf{i})$, $\alpha^L_k(\mathbf{i}) = \alpha^L_{k-1}(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha^L_{k-1}(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + 1 = \beta^R_{k-1}(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} - \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}
& \alpha^L_k(\mathbf{i}) + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \alpha^L_{k-1}(\mathbf{i}) + 1 - \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_{k-1}(\mathbf{i}) \left(1 + \operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i})\right),
\end{aligned}$$

Therefore $\alpha^L_k(\mathbf{i}) + \beta^L_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) = 0$. Now let us show $\alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) = 0$. Note by Lemma 6.2.2 we have $\operatorname{pos}_k(\mathbf{i}) = \operatorname{pos}_{k-1}(\mathbf{i}) + 1$. Similarly consider the following three cases.

Case 1: $i_k = 0$.

By the direct calculation one can see that $\beta^L_{k-1}(\mathbf{i}) = \beta^R_k(\mathbf{i})$, $\alpha^L_{k-1}(\mathbf{i}) = \alpha^L_k(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) = \beta^R_k(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}
& \alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^L_k(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i})(\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{i}) + 1) = 0.
\end{aligned}$$

Case 2: $i_k = \frac{1}{2}$.

By the direct calculation one can see that $\beta^L_{k-1}(\mathbf{i}) = \beta^R_k(\mathbf{i})$, $\alpha^L_{k-1}(\mathbf{i}) = \alpha^L_k(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) + 1 = \beta^R_k(\mathbf{i}) + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}
& \alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \alpha^L_k(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i}) \operatorname{pos}_{k-1}(\mathbf{i}) \\
= \;& \beta^R_k(\mathbf{i})(\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{i}) + 1) = 0.
\end{aligned}$$

Case 3: $i_k \neq 0, \frac{1}{2}$.

By the direct calculation one can see that $\beta^L_{k-1}(\mathbf{i}) = \beta^R_k(\mathbf{i})$, $\alpha^L_{k-1}(\mathbf{i}) = \alpha^L_k(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$, $\alpha^L_k(\mathbf{i}) + \alpha^R_k(\mathbf{i}) + 1 = \beta^R_k(\mathbf{i}) - \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}}$.

Hence we have

$$\begin{aligned}
&\alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i})\,\mathrm{pos}_{k-1}(\mathbf{i}) \\
= \; &\alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \alpha^L_k(\mathbf{i}) + 1 + \delta_{i_k,-\frac{\delta-1}{2}} + \delta_{i_k,\frac{\delta-1}{2}} + \beta^L_{k-1}(\mathbf{i})\,\mathrm{pos}_{k-1}(\mathbf{i}) \\
= \; &\beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i})\,\mathrm{pos}_{k-1}(\mathbf{i}) \\
= \; &\beta^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_{k-1}(\mathbf{i}) \\
= \; &\beta^R_k(\mathbf{i})(\mathrm{pos}_k(\mathbf{i}) + \mathrm{pos}_{k-1}(\mathbf{i}) + 1) = 0.
\end{aligned}$$

Therefore $\alpha^R_k(\mathbf{i}) + \beta^R_k(\mathbf{i})\,\mathrm{pos}_k(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i})\,\mathrm{pos}_{k-1}(\mathbf{i}) = 0$.

Lemma 6.2.35. *We have* $\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k = \bar{\epsilon}_k$ *and* $\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{\epsilon}_{k-1} = \bar{\epsilon}_{k-1}$.

Proof. The statement is obviously true by Lemma 6.2.33 and Lemma 6.2.34.

Lemma 6.2.36. *Suppose* $\mathbf{i} \in T^d$. *Then we have*

$$\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k, \qquad \hat{y}_k\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{y}_{k+1}\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}).$$

Proof. We first show $\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$. By checking Appendix C, we have

- $(-1)^{m_k(\mathbf{i})}\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = (-1)^{m_{k+1}(\mathbf{i})}\hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ in case
  $(1\!-\!8), (9.a\!-\!b), (10.c\!-\!d), (11.a.i\!-\!ii), (11.b.i\!-\!ii), (12), (14), (15.a.iii\!-\!iv), (15.b.iii\!-\!iv), (16)$.
  Furthermore, in these cases we have $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i})$ (see Appendix A), which implies $\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$.
- $(-1)^{m_k(\mathbf{i})}\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = -(-1)^{m_{k+1}(\mathbf{i})}\hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ in case
  $(9.c - d), (10.a - b), (11.a.iii - iv), (11.b.iii - iv), (13), (15.a.i - ii), (15.b.i - ii)$.
  Moreover, we have $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1$ (see Appendix A,), which implies $\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k = \hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$.

Following the same argument we can show $\hat{y}_k\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}) = \hat{y}_{k+1}\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$ by checking Appendix D and A.

Lemma 6.2.37. *Suppose* $\mathbf{i} \in T^d$. *Then we have*

$$(\bar{y}_k + \bar{y}_{k+1})\bar{\epsilon}_k = \bar{\epsilon}_k(\bar{y}_k + \bar{y}_{k+1}) = 0.$$

Proof. By Lemma 6.2.2 we have $\mathrm{pos}_k(\mathbf{i}) = \mathrm{pos}_{k+1}(\mathbf{i}) + 1$, and from Lemma 6.2.36 we obtain $(\bar{y}_k + \bar{y}_{k+1})\bar{\epsilon}_k = \bar{\epsilon}_k(\bar{y}_k + \bar{y}_{k+1}) = 0$.

Lemma 6.2.38. *Suppose* $\mathbf{i} \in I^d$ *such that* $i_{k-1} + i_{k+1} = 0$. *Then we have*

$$\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i}), \qquad \hat{e}(\mathbf{i})\hat{\psi}_{k-1}\overleftarrow{\epsilon}_k = \hat{e}(\mathbf{i})\hat{\psi}_k\overleftarrow{\epsilon}_{k-1}.$$

Proof. We only prove the first equality. The second equality is basically the involution of the first equality. We consider the following cases.

Case 1: $|i_{k-1} - i_k| > 1$ and $|i_{k+1} - i_k| > 1$.

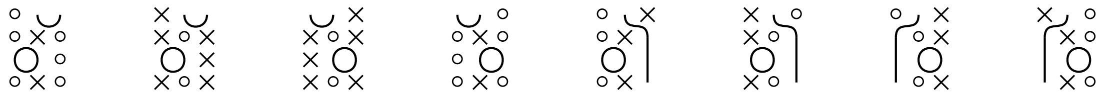
This part is obvious.

Case 2: $|i_{k-1} - i_k| = 1$ and $|i_{k+1} - i_k| > 1$.

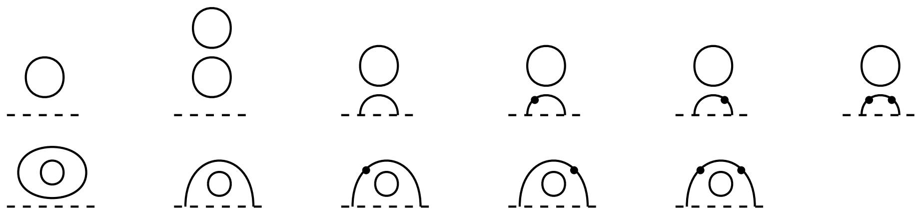
In this case, we have $\deg \hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = 1$ and $\deg \hat{\psi}_k\hat{e}(\mathbf{i}) = 0$. This forces that $\deg_k(\mathbf{i}\cdot s_{k-1}) = -1$ or $\deg_k(\mathbf{i}\cdot s_{k-1}) = 0$, if $\deg_{k-1}(\mathbf{i}\cdot s_k) = 0$ or $\deg_{k-1}(\mathbf{i}\cdot s_k) = 1$, respectively. Hence we consider these cases separately.

Case 2.1: $\deg_k(\mathbf{i}\cdot s_{k-1}) = -1$ and $\deg_{k-1}(\mathbf{i}\cdot s_k) = 0$.

In this case, the graph $\mathbf{i}\cdot s_{k-1}$ is of one of the following forms:



Similarly to Corollary 6.2.29, the graph of $\mathbf{i}\cdot s_{k-1}$ is topologically the same as one of the following graphs:



Since $\overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i}\cdot s_k)$ and $\psi_k\hat{e}(\mathbf{i})$ have degree 0, the product $\overrightarrow{\epsilon}_{k-1}\psi_k\hat{e}(\mathbf{i})$ is topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$. Hence following the same argument as in Case 2 of the proof of Lemma 6.2.30, we can show $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$.

Case 2.2: $\deg_k(\mathbf{i}\cdot s_{k-1}) = 0$ and $\deg_{k-1}(\mathbf{i}\cdot s_k) = 1$.

In this case, since $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_{k-1})$ and $\hat{\psi}_k\hat{e}(\mathbf{i})$ have degree 0, they are topologically equivalent to idempotents. Hence we have $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) \sim \hat{\psi}_{k-1}\hat{e}(\mathbf{i})$ and $\overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i}) \sim \overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i}\cdot s_k)$. As $|i_k - i_{k+1}| > 1$, the graphs of $\mathbf{i}$ and $\mathbf{i}\cdot s_k$ are topologically the same. Hence by the definition we have $\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) \sim \overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i}\cdot s_k)$, which implies $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) \sim \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$. Moreover, since these two elements have the same graph, it forces $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$.

Case 3: $|i_{k-1} - i_k| > 1$ and $|i_{k+1} - i_k| = 1$.

This case is essentially the same as Case 2.

Case 4: $|i_{k-1} - i_k| = |i_{k+1} - i_k| = 1$.

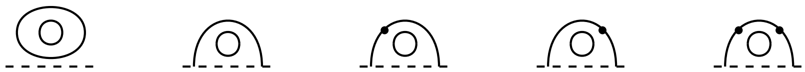
In this case we have $i_{k-1} = i_{k+1} = \pm 1$ and $i_k = 0$. Since $\mathbf{i}$ is a residue sequence, it implies that $\mathbf{i}\cdot s_k$ and $\mathbf{i}\cdot s_{k+1}$ are not residue sequences. Hence we have $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = 0 = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$.

Case 5: $i_{k-1} = i_k$.

In this case we can assume that $i_{k-1} = i_k = -i_{k+1}$, since otherwise $i_{k-1} + i_{k+1} \neq 0$, and we would have $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = 0 = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$. Under this assumption we have,

$$\deg \overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_{k-1}) = 1, \qquad \deg \hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = -2, \qquad \deg \overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i}\cdot s_k) = -1, \qquad \deg \hat{\psi}_k\hat{e}(\mathbf{i}) = 0.$$

Similarly to Corollary 6.2.29, the graph of $\mathbf{i}$, $\mathbf{i}\cdot s_{k-1}$ and $\mathbf{i}\cdot s_k$ are all topologically the same as one of the following graphs:



Following the similar argument as in Case 2 of the proof of Lemma 6.2.30, we can show $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$.

Case 6: $i_{k+1} = i_k$.

This case is essentially the same as Case 5.

By combining all the 6 cases above, we obtain $\overrightarrow{\epsilon}_k\hat{\psi}_{k-1}\hat{e}(\mathbf{i}) = \overrightarrow{\epsilon}_{k-1}\hat{\psi}_k\hat{e}(\mathbf{i})$.

Corollary 6.2.39. *Suppose* $\mathbf{i} \in I^d$. *Then we have*

$$\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{\psi}_k = \hat{\epsilon}_k\hat{\psi}_{k-1}, \qquad \hat{\psi}_k\hat{\epsilon}_{k-1}\hat{\epsilon}_k = \hat{\psi}_{k-1}\hat{\epsilon}_k.$$

Proof. We only prove the first equality. The second equality follows from the same arguments. Here by Lemma 6.2.31 and Lemma 6.2.38 we have

$$\hat{\epsilon}_k\hat{\epsilon}_{k-1}\hat{\psi}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k\overleftarrow{\epsilon}_{k-1}\overrightarrow{\epsilon}_{k-1}\hat{\epsilon}_2\hat{\psi}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1}\hat{\psi}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_{k-1}\hat{\psi}_k = \overleftarrow{\epsilon}_k\hat{\epsilon}_2\overrightarrow{\epsilon}_k\hat{\psi}_{k-1} = \hat{\epsilon}_k\hat{\psi}_{k-1}.$$

Proposition 6.2.40. *Suppose* $\mathbf{i} \in I^d$ *such that* $i_{k-1} + i_{k+1} = 0$. *Then we have*

$$\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\psi}_k = \bar{\epsilon}_k\bar{\psi}_{k-1}, \qquad \bar{\psi}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k = \bar{\psi}_{k-1}\hat{\epsilon}_k.$$

Proof. By Corollary 6.2.39, it suffices to prove that

$$\operatorname{sign}_{\mathbf{i}\cdot s_{k-1}}(\overrightarrow{\epsilon}_k) + \operatorname{sign}_{\mathbf{i}\cdot s_{k-1}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}\cdot s_k}(\overrightarrow{\epsilon}_{k-1}) + \operatorname{sign}_{\mathbf{i}\cdot s_k}(\hat{\psi}_k),$$
$$\operatorname{sign}_{\mathbf{i}\cdot s_{k-1}}(\overleftarrow{\epsilon}_k) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}\cdot s_k}(\overleftarrow{\epsilon}_{k-1}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Denote $\mathbf{j} = \mathbf{i}\cdot s_{k-1}$ and $\mathbf{k} = \mathbf{i}\cdot s_k$. By definition we need to show that

$$\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),$$
$$\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Similar to Lemma 6.2.38, we separate the Proposition into several cases.

Case 1: $|i_{k-1} - i_k| > 1$ and $|i_{k+1} - i_k| > 1$.

Let $i_{k-1} = -i_{k+1} = i$ and $i_k = j$. In this case we have

$$\begin{array}{llllll} \mathbf{j}: & \cdots & j & i & -i & \cdots \\ \mathbf{i}: & \cdots & i & j & -i & \cdots \end{array} \qquad \begin{array}{llllll} \mathbf{k}: & \cdots & i & -i & j & \cdots \\ \mathbf{i}: & \cdots & i & j & -i & \cdots \end{array}$$

Hence we have

$$\alpha_k^R(\mathbf{j}) = \alpha_{k-1}^R(\mathbf{k}), \quad \beta_k^R(\mathbf{j}) = \beta_{k-1}^R(\mathbf{k}), \quad \operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k}), \quad \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k);$$
$$\alpha_k^L(\mathbf{j}) = \alpha_{k-1}^L(\mathbf{k}), \quad \beta_k^L(\mathbf{j}) = \beta_{k-1}^L(\mathbf{k}), \quad \operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k}), \quad \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k),$$

which implies

$$\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),$$
$$\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Case 2: $i_{k-1} - i_k = 1$ and $|i_{k+1} - i_k| > 1$.

Let $i_{k-1} = -i_{k+1} = i$ and $i_k = i + 1$. In this case we have

$$\begin{array}{lccccc} \mathbf{j}: & \cdots & i+1 & i & -i & \cdots \\ \mathbf{i}: & \cdots & i & i+1 & -i & \cdots \end{array} \qquad \begin{array}{lccccc} \mathbf{k}: & \cdots & i & -i & i+1 & \cdots \\ \mathbf{i}: & \cdots & i & i+1 & -i & \cdots \end{array}$$

Hence we have

$\alpha_k^R(\mathbf{j}) = \alpha_{k-1}^R(\mathbf{k})$, $\beta_k^R(\mathbf{j}) = \beta_{k-1}^R(\mathbf{k})$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k)$;
$\alpha_k^L(\mathbf{j}) = \alpha_{k-1}^L(\mathbf{k})$, $\beta_k^L(\mathbf{j}) = \beta_{k-1}^L(\mathbf{k}) + 1$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k) + \operatorname{pos}_{k-1}(\mathbf{i})$,

which implies

$$\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),$$
$$\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Case 3: $i_{k-1} - i_k = -1$ and $|i_{k+1} - i_k| > 1$.

Let $i_{k-1} = -i_{k+1} = i$ and $i_k = i - 1$. In this case we have

$$\begin{array}{lccccc} \mathbf{j}: & \cdots & i-1 & i & -i & \cdots \\ \mathbf{i}: & \cdots & i & i-1 & -i & \cdots \end{array} \qquad \begin{array}{lccccc} \mathbf{k}: & \cdots & i & -i & i-1 & \cdots \\ \mathbf{i}: & \cdots & i & i-1 & -i & \cdots \end{array}$$

Hence we have

$\alpha_k^R(\mathbf{j}) = \alpha_{k-1}^R(\mathbf{k}) + 1$, $\beta_k^R(\mathbf{j}) = \beta_{k-1}^R(\mathbf{k}) + 1$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) + \operatorname{pos}_{k-1}(\mathbf{j}) = \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k)$;
$\alpha_k^L(\mathbf{j}) = \alpha_{k-1}^L(\mathbf{k})$, $\beta_k^L(\mathbf{j}) = \beta_{k-1}^L(\mathbf{k})$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k)$,

which implies

$$\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),$$
$$\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Case 4: $|i_{k-1} - i_k| > 1$ and $i_{k+1} - i_k = 1$.

Let $i_{k-1} = -i_{k+1} = i$ and $i_k = -i + 1$. In this case we have

$$\begin{array}{lccccc} \mathbf{j}: & \cdots & -i+1 & i & -i & \cdots \\ \mathbf{i}: & \cdots & i & -i+1 & -i & \cdots \end{array} \qquad \begin{array}{lccccc} \mathbf{k}: & \cdots & i & -i & -i+1 & \cdots \\ \mathbf{i}: & \cdots & i & -i+1 & -i & \cdots \end{array}$$

Hence we have

$\alpha_k^R(\mathbf{j}) = \alpha_{k-1}^R(\mathbf{k}) + 1$, $\beta_k^R(\mathbf{j}) = \beta_{k-1}^R(\mathbf{k}) + 1$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k) + \operatorname{pos}_k(\mathbf{k})$;
$\alpha_k^L(\mathbf{j}) = \alpha_{k-1}^L(\mathbf{k})$, $\beta_k^L(\mathbf{j}) = \beta_{k-1}^L(\mathbf{k})$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k)$,

which implies

$$\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),$$
$$\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1}) = \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).$$

Case 5: $|i_{k-1} - i_k| > 1$ and $i_{k+1} - i_k = -1$.

Let $i_{k-1} = -i_{k+1} = i$ and $i_k = -i-1$. In this case we have

$$
\begin{array}{lccccccccl}
\mathbf{j}: & \cdots & -i-1 & & i & & -i & \cdots \\
 & & & \times & & & & \\
\mathbf{i}: & \cdots & i & & -i-1 & & -i & \cdots
\end{array}
\qquad
\begin{array}{lccccccccl}
\mathbf{k}: & \cdots & i & & -i & & -i-1 & \cdots \\
 & & & & & \times & & \\
\mathbf{i}: & \cdots & i & & -i-1 & & -i & \cdots
\end{array}
$$

Hence we have

$\alpha_k^R(\mathbf{j}) = \alpha_{k-1}^R(\mathbf{k})$, $\beta_k^R(\mathbf{j}) = \beta_{k-1}^R(\mathbf{k})$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{j}}(\hat\psi_{k-1}) = \operatorname{sign}_{\mathbf{k}}(\hat\psi_k)$;

$\alpha_k^L(\mathbf{j}) = \alpha_{k-1}^L(\mathbf{k})$, $\beta_k^L(\mathbf{j}) = \beta_{k-1}^L(\mathbf{k}) + 1$, $\operatorname{pos}_k(\mathbf{j}) = \operatorname{pos}_{k-1}(\mathbf{k})$, $\operatorname{sign}_{\mathbf{i}}(\hat\psi_{k-1}) = \operatorname{sign}_{\mathbf{i}}(\hat\psi_k) + \operatorname{pos}_k(\mathbf{i})$,

which implies

$$
\begin{aligned}
\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat\psi_{k-1}) &= \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat\psi_k),\\
\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat\psi_{k-1}) &= \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat\psi_k).
\end{aligned}
$$

Case 6: $|i_{k-1} - i_k| = |i_{k+1} - i_k| = 1$.

In this case we have $\overrightarrow{\epsilon}_k\hat\psi_{k-1}\hat e(\mathbf{i}) = 0 = \overrightarrow{\epsilon}_{k-1}\hat\psi_k\hat e(\mathbf{i})$. Hence we always have $\bar\epsilon_k\bar\epsilon_{k-1}\bar\psi_k\bar e(\mathbf{i}) = \bar\epsilon_k\bar\psi_{k-1}\bar e(\mathbf{i})$. Similarly, we have $\bar e(\mathbf{i})\bar\psi_k\bar\epsilon_{k-1}\bar\epsilon_k = \bar e(\mathbf{i})\bar\psi_{k-1}\bar\epsilon_k$.

Case 7: $i_{k-1} = i_k$.

Let $i_{k-1} = i_k = -i_{k+1} = i$. In this case we have

$$
\begin{array}{lccccccccl}
\mathbf{j}: & \cdots & i & & i & & -i & \cdots \\
 & & & \times & & & & \\
\mathbf{i}: & \cdots & i & & i & & -i & \cdots
\end{array}
\qquad
\begin{array}{lccccccccl}
\mathbf{k}: & \cdots & i & & -i & & i & \cdots \\
 & & & & & \times & & \\
\mathbf{i}: & \cdots & i & & i & & -i & \cdots
\end{array}
$$

Hence by Lemma 6.2.34 we have

$$
\begin{aligned}
&\alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_k(\hat\psi_k)\\
&= \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \alpha_k^L(\mathbf{k}) + \alpha_k^R(\mathbf{i}) + \beta_k^L(\mathbf{k})\operatorname{pos}_k(\mathbf{k}) + \beta_k^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{k})\\
&= \alpha_k^R(\mathbf{i}) + \beta_k^R(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{k}) + 1\\
&= \alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{pos}_{k-1}(\mathbf{j}) + 1\\
&= \alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{pos}_k(\mathbf{j})\\
&= \alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat\psi_{k-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat\psi_k)\\
&= \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \alpha_k^L(\mathbf{i}) + \alpha_k^R(\mathbf{k}) + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \beta_k^R(\mathbf{k})\operatorname{pos}_k(\mathbf{k}) + \operatorname{pos}_k(\mathbf{i})\\
&= \alpha_k^L(\mathbf{i}) + \beta_k^L(\mathbf{i})\operatorname{pos}_k(\mathbf{i}) + \operatorname{pos}_k(\mathbf{i})\\
&= \alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{pos}_k(\mathbf{j})\\
&= \alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat\psi_{k-1}).
\end{aligned}
$$

Case 8: $i_{k+1} = i_k$.

Let $i_{k-1} = -i_k = -i_{k+1} = i$. In this case we have

$$
\begin{array}{lccccccccl}
\mathbf{j}: & \cdots & -i & & i & & -i & \cdots \\
 & & & \times & & & & \\
\mathbf{i}: & \cdots & i & & -i & & -i & \cdots
\end{array}
\qquad
\begin{array}{lccccccccl}
\mathbf{k}: & \cdots & i & & -i & & -i & \cdots \\
 & & & & & \times & & \\
\mathbf{i}: & \cdots & i & & -i & & -i & \cdots
\end{array}
$$

Hence by Lemma 6.2.34 we have

$$\begin{aligned}
&\alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{j}}(\hat{\psi}_{k-1})\\
&= \alpha_k^R(\mathbf{j}) + \beta_k^R(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \alpha_{k-1}^L(\mathbf{j}) + \alpha_{k-1}^R(\mathbf{i}) + \beta_{k-1}^L(\mathbf{j})\operatorname{pos}_{k-1}(\mathbf{j}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{j})\\
&= \alpha_{k-1}^R(\mathbf{i}) + \beta_{k-1}^R(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{k})\\
&= \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{pos}_{k+1}(\mathbf{k})\\
&= \alpha_{k-1}^R(\mathbf{k}) + \beta_{k-1}^R(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{k}}(\hat{\psi}_k),
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_{k-1})\\
&= \alpha_k^L(\mathbf{j}) + \beta_k^L(\mathbf{j})\operatorname{pos}_k(\mathbf{j}) + \alpha_{k-1}^L(\mathbf{i}) + \alpha_{k-1}^R(\mathbf{j}) + \beta_{k-1}^L(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + \beta_{k-1}^R(\mathbf{j})\operatorname{pos}_{k-1}^R(\mathbf{j}) + \operatorname{pos}_{k-1}(\mathbf{i})\\
&= \alpha_{k-1}^L(\mathbf{i}) + \beta_{k-1}^L(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{i})\\
&= \alpha_{k-1}^L(\mathbf{j}) + \beta_{k-1}^L(\mathbf{j})\operatorname{pos}_{k-1}(\mathbf{j}) + \operatorname{pos}_{k+1}(\mathbf{i})\\
&= \alpha_{k-1}^L(\mathbf{k}) + \beta_{k-1}^L(\mathbf{k})\operatorname{pos}_{k-1}(\mathbf{k}) + \operatorname{sign}_{\mathbf{i}}(\hat{\psi}_k).
\end{aligned}$$

Combining the results of this subsection, we have the following Proposition:

Proposition 6.2.41. *Let* $\mathbf{i}, \mathbf{j} \in T^d$ *and* $1 < k < d$. *Then*

$$\bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\psi}_k\bar{e}(\mathbf{i}) = \bar{e}(\mathbf{j})\bar{\epsilon}_k\bar{\psi}_{k-1}\bar{e}(\mathbf{i}), \quad \bar{e}(\mathbf{i})\bar{\psi}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{j}) = \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{j}),$$
$$\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{j}) = \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{j}); \qquad \bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{j}) = \bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{j}); \qquad \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{j})(\bar{y}_k + \bar{y}_{k+1}) = 0.$$

6.2.9 Braid Relations

In this subsection we prove the Braid relations. We first assume that either $i_k + i_{k+1} = 0$ or $i_k + i_{k-1} = 0$.

Lemma 6.2.42. *Suppose* $\mathbf{i} \in T^d$. *Then we have*

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \begin{cases} \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1),\\ -\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1),\\ \bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1),\\ -\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{\epsilon}_k\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1).\end{cases}$$

Proof. We only prove the first equality. The rest of the equalities can be verified using the same arguments. So we assume that $i_k + i_{k+1} = 0$ and $i_{k-1} = \pm(i_k - 1)$. By [33, Lemma 7.60] we have either $\mathbf{i}\cdot s_k \in T^d$ or $\mathbf{i}\cdot s_{k-1}s_k \in T^d$, but not both, which implies

Case 1: $\mathbf{i}\cdot s_k \in T^d$

In this case we have $\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = 0$. Moreover, since $i_k + i_{k+1} = 0$ and $\mathbf{i}\cdot s_k \in T^d$ we have $\mathbf{i} \in T^d_{k,+}$, and by the definition of h_k we have $\mathbf{i}\cdot s_k s_{k-1} s_k = (i_1, \dots, -i_k, i_k, \pm(i_k - 1), \dots) \in T^d_{k,0}$. Hence, using the relations proven before, we have

$$\begin{aligned}
\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} &= \bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)\\
&= (-1)^{\alpha_{k-1}(\mathbf{i}\cdot s_k s_{k-1} s_k)}\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)\\
&= (-1)^{\alpha_{k-1}(\mathbf{i}\cdot s_k s_{k-1} s_k)}\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}^2\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)\\
&= (-1)^{\alpha_{k-1}(\mathbf{i}\cdot s_k s_{k-1} s_k)}\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)\\
&= (-1)^{\alpha_{k-1}(\mathbf{i}\cdot s_k s_{k-1} s_k) + \alpha_k(\mathbf{i}\cdot s_k)}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k).
\end{aligned}$$

Hence we have $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)$ since $(-1)^{\alpha_{k-1}(\mathbf{i}\cdot s_k s_{k-1} s_k)+\alpha_k(\mathbf{i}\cdot s_k)} = 1$ by the definition of α_k.

Case 2: $\mathbf{i}\cdot s_{k-1}s_k \in T^d$

In this case we have $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k = 0$. Moreover, since $\mathbf{i}\cdot s_{k-1}s_k \in T^d$ we have $\mathbf{i} \in T^n_{k,0}$. Hence

$$\begin{aligned}
\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} &= -\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} \\
&= -(-1)^{\alpha_k(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1}\bar{e}(\mathbf{i}\cdot s_{k-1}s_k s_{k-1}) \\
&= (-1)^{\alpha_k(\mathbf{i})+1}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\psi}_k^2\bar{\psi}_{k-1}\bar{e}(\mathbf{i}\cdot s_{k-1}s_k s_{k-1}) \\
&= (-1)^{\alpha_k(\mathbf{i})+1}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{\psi}_{k-1}\bar{e}(\mathbf{i}\cdot s_{k-1}s_k s_{k-1}) \\
&= (-1)^{\alpha_k(\mathbf{i})+\alpha_k(\mathbf{i}\cdot s_{k-1}s_k)+1}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_{k-1}s_k s_{k-1}).
\end{aligned}$$

Hence we have $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}\cdot s_k s_{k-1} s_k)$ since $(-1)^{\alpha_k(\mathbf{i})+\alpha_k(\mathbf{i}\cdot s_{k-1}s_k)+1} = 1$ by [33, Lemma 6.15].

The next result can be verified by the same arguments as Lemma 6.2.42.

Lemma 6.2.43. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = -i_k = i_{k+1}$. *Then*

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \begin{cases} -(-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}), & \textit{if } h_k(\mathbf{i}) = 0, \\ (-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_k\bar{e}(\mathbf{i}), & \textit{if } h_{k-1}(\mathbf{i}) = 0. \end{cases}$$

Proof. We only prove the first equality, since the second equality follows from the similar argument. We assume $h_k(\mathbf{i}) = 0$. By the definition of h_k, we have $h_{k-1}(\mathbf{i}) = -2$, then Lemma 6.2.3 implies $\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = 0$. Hence it suffices to prove that

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k = -(-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i}).$$

Since $\hat{e}(\mathbf{i})\hat{\psi}_k$ and $\hat{\psi}_k\hat{e}(\mathbf{i})$ are topologically equivalent to the idempotent $\hat{e}(\mathbf{i})$, and $\hat{e}(\mathbf{i}\cdot s_k)\hat{\psi}_{k-1}$ is topologically equivalent to $\hat{e}(\mathbf{i})\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i})$, we have

$$\hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\epsilon}_k\hat{e}(\mathbf{i}).$$

By Lemma 6.2.1, $\beta^R_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i}) \equiv \deg_{k-1}(\mathbf{i}) \equiv 1$; and by Lemma 6.2.2, we have $\operatorname{pos}_{k-1}(\mathbf{i}) + 1 \equiv \operatorname{pos}_k(\mathbf{i}) \equiv \operatorname{pos}_k(\mathbf{i}\cdot s_k)$. Therefore

$$\begin{aligned}
&\alpha_{k-1}(\mathbf{i}) + \alpha^L_{k-1}(\mathbf{i}) + \alpha^R_{k-1}(\mathbf{i}) + \beta^L_{k-1}(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + \beta^R_{k-1}(\mathbf{i})\operatorname{pos}_{k-1}(\mathbf{i}) + 1 \\
\equiv\ &\alpha_{k-1}(\mathbf{i}) + \alpha_{k-1}(\mathbf{i}) + \operatorname{pos}_{k-1}(\mathbf{i}) + 1 \equiv \operatorname{pos}_{k-1}(\mathbf{i}) + 1 \equiv \operatorname{pos}_k(\mathbf{i}\cdot s_k),
\end{aligned}$$

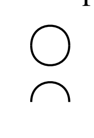
which proves $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = -(-1)^{\alpha_{k-1}(\mathbf{i})}\bar{e}(\mathbf{i})\bar{\epsilon}_{k-1}\bar{e}(\mathbf{i})$.

Lemma 6.2.44. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = i_{k+1} = i_k \pm 1$. *Then we have*

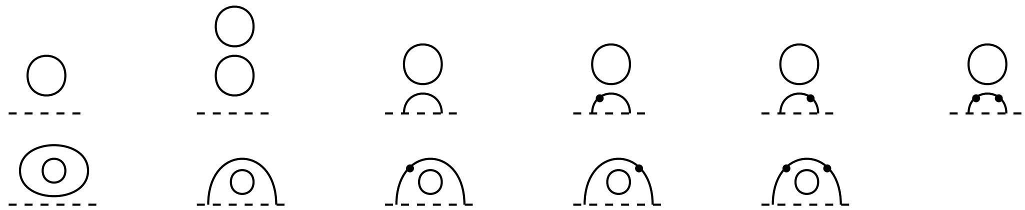
$$\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\hat{\psi}_k\hat{\psi}_{k-1} + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = \hat{e}(\mathbf{i}).$$

Proof. From the construction of up-down tableaux, one can see that for $\mathbf{i} \in T^d$, we have either $\mathbf{i}\cdot s_k \in T^d$ or $\mathbf{i}\cdot s_{k-1} \in T^d$, but not both. This implies that either $\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\hat{\psi}_k\hat{\psi}_{k-1} = 0$ or $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = 0$. Without loss of generality, we assume that $\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\hat{\psi}_k\hat{\psi}_{k-1} = 0$. If $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = 0$, the calculations are the same.

Since $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k \neq 0$, the $k-1$, k and $k+1$-patterns of $\mathbf{i}\cdot s_k$ is



which implies the diagram of $\mathbf{i}\cdot s_k$ is topologically equivalent to one of the following diagrams:



One can see that $\hat{e}(\mathbf{i})\hat{\psi}_k$ is topologically equivalent to $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i}\cdot s_k)$; $\hat{\psi}_k\hat{e}(\mathbf{i})$ is topologically equivalent to $\hat{e}(\mathbf{i}\cdot s_k)\overleftarrow{\epsilon}_k$; and $\hat{e}(\mathbf{i}\cdot s_k)\hat{\psi}_{k-1}\hat{e}(\mathbf{i}\cdot s_k)$ is topologically equivalent to $\hat{e}(\mathbf{i}\cdot s_k)\hat{\epsilon}_{k-1}\hat{e}(\mathbf{i}\cdot s_k)$. Hence following the calculations of Appendix E, we have

$$\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\hat{\psi}_k\hat{\epsilon}_{k-1} + \hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = \hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k = \hat{e}(\mathbf{i}),$$

.

Now we check the signs.

Lemma 6.2.45. *Suppose* $\mathbf{i} \in T^d$ *such that* $i_{k-1} = i_{k+1} = i_k \pm 1$. *Then we have*

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \begin{cases} \bar{e}(\mathbf{i}), & \text{if } i_{k-1} = i_{k+1} = i_k - 1, \\ -\bar{e}(\mathbf{i}), & \text{if } i_{k-1} = i_{k+1} = i_k + 1. \end{cases}$$

Proof. We consider the following cases:

Case 1: $i_{k-1} = i_{k+1} = i_k - 1$

Case 1.1: $\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = 0$

In this case we have

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \hat{e}(\mathbf{i})\hat{\psi}_k\left((-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}\hat{\psi}_{k-1}\right)\left((-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}\hat{\psi}_k\right) = \hat{e}(\mathbf{i}) = \bar{e}(\mathbf{i}).$$

Case 1.2: $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k+1}\bar{\psi}_k = 0$

By Lemma 6.2.2 we have $\mathrm{pos}_{k-1}(\mathbf{i}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}\cdot s_{k-1}) + 1$, which implies

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = -\hat{e}(\mathbf{i})\left((-1)^{\mathrm{pos}_{k-1}(\mathbf{i})}\hat{\psi}_{k-1}\right)\left((-1)^{\mathrm{pos}_{k+1}(\mathbf{i}\cdot s_k)}\hat{\psi}_k\right)\hat{\psi}_{k-1} = \hat{e}(\mathbf{i}) = \bar{e}(\mathbf{i}).$$

Case 2: $i_{k-1} = i_{k+1} = i_k + 1$

Case 1.1: $\bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = 0$.

By Lemma 6.2.2 we have $\mathrm{pos}_k(\mathbf{i}) \equiv \mathrm{pos}_k(\mathbf{i}\cdot s_k) + 1$, which implies

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = \hat{e}(\mathbf{i})\left((-1)^{\mathrm{pos}_k(\mathbf{i})}\hat{\psi}_k\right)\left((-1)^{\mathrm{pos}_k(\mathbf{i}\cdot s_k)}\hat{\psi}_{k-1}\right)\hat{\psi}_k = -\hat{e}(\mathbf{i}) = -\bar{e}(\mathbf{i}).$$

Case 1.2: $\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k+1}\bar{\psi}_k = 0$

From Lemma 6.2.2 we have $\mathrm{pos}_{k-1}(\mathbf{i}\cdot s_{k-1}) \equiv \mathrm{pos}_{k+1}(\mathbf{i}\cdot s_{k-1})$, which implies

$$\bar{e}(\mathbf{i})\bar{\psi}_k\bar{\psi}_{k-1}\bar{\psi}_k - \bar{e}(\mathbf{i})\bar{\psi}_{k-1}\bar{\psi}_k\bar{\psi}_{k-1} = -\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\left((-1)^{\mathrm{pos}_{k+1}(\mathbf{i}\cdot s_{k-1})}\hat{\psi}_k\right)\left((-1)^{\mathrm{pos}_{k-1}(\mathbf{i}\cdot s_{k-1})}\hat{\psi}_{k-1}\right) = -\hat{e}(\mathbf{i}) = -\bar{e}(\mathbf{i}).$$

.

Proposition 6.2.46. *Let $\bar{\mathcal{B}}_k = \bar\psi_k\bar\psi_{k-1}\bar\psi_k - \bar\psi_{k-1}\bar\psi_k\bar\psi_{k-1}$, $\mathbf{i} \in T^d$ and $1 < k < d$. Then*

$$\bar e(\mathbf{i})\bar{\mathcal{B}}_k = \begin{cases} \bar e(\mathbf{i})\bar\epsilon_k\bar\epsilon_{k-1}\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1), \quad (6.2.7)\\ -\bar e(\mathbf{i})\bar\epsilon_k\bar\epsilon_{k-1}\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1), \quad (6.2.8)\\ \bar e(\mathbf{i})\bar\epsilon_{k-1}\bar\epsilon_k\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1), \quad (6.2.9)\\ -\bar e(\mathbf{i})\bar\epsilon_{k-1}\bar\epsilon_k\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1), \quad (6.2.10)\\ -(-1)^{\alpha_{k-1}(\mathbf{i})}\bar e(\mathbf{i})\bar\epsilon_{k-1}\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\tfrac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \quad (6.2.11)\\ (-1)^{\alpha_k(\mathbf{i})}\bar e(\mathbf{i})\bar\epsilon_k\bar e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\tfrac{1}{2} \text{ and } h_{k-1}(\mathbf{i}) = 0, \quad (6.2.12)\\ \bar e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k - 1, \quad (6.2.13)\\ -\bar e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k + 1, \quad (6.2.14)\\ 0, & \text{otherwise.} \quad (6.2.15) \end{cases}$$

Proof. We can prove (6.2.7 - 6.2.10) by Lemma 6.2.42, (6.2.11 - 6.2.12) by Lemma 6.2.43, and (6.2.13 - 6.2.14) by Lemma 6.2.45. For the rest of the cases, we have

(1) $i_{k-1} + i_k = 0$

   (a) $i_{k+1} = i_{k-1} = 0$ or $\pm\frac{1}{2}$

   (b) $i_{k+1} = i_{k-1}$ and $h_{k-1}(\mathbf{i}) \neq 0$, and $h_k(\mathbf{i}) \neq 0$.

   (c) $|i_{k+1} - i_{k-1}| > 1$ and $|i_{k+1} - i_k| > 1$.

   In the first two cases above we have that $\bar\psi_k\bar\psi_{k-1}\bar\psi_k = \bar\psi_{k-1}\bar\psi_k\bar\psi_{k-1} = 0$, hence $\bar e(\mathbf{i})\bar{\mathcal{B}}_k = 0$. In the third case, we have that $\bar\psi_k\bar\psi_{k-1}\bar\psi_k\bar e(\mathbf{i})$ and $\bar\psi_{k-1}\bar\psi_k\bar\psi_{k-1}\bar e(\mathbf{i})$ are topologically equivalent to $\bar e(\mathbf{i})$.

(2) $i_{k-1} + i_k \neq 0$.

   (a) $|i_{k-1} - i_k| > 1$

      i. $|i_{k-1} + i_k| > 1$

      ii. $|i_{k-1} + i_k| = 1$ and $i_k + i_{k+1} \neq 0$

   (b) $|i_{k-1} - i_k| = 1$ and $i_k + i_{k+1} \neq 0$

   (c) $i_{k-1} = i_k$

      i. $|i_{k+1} - i_k| > 1$

      ii. $|i_{k+1} - i_k| = 1$

      iii. $i_{k+1} = i_k$

By considering all the above case, we can show that $\bar e(\mathbf{i})\bar\psi_k\bar\psi_{k-1}\bar\psi_k = \bar e(\mathbf{i})\bar\psi_{k-1}\bar\psi_k\bar\psi_{k-1}$. We omit the proof here.

6.3 Proof of the main theorem

Now we are ready to prove the main result of this section.

Theorem 6.3.1. *The map $\Phi : G_d(\delta) \longrightarrow C_d(\delta)$ determined by*

$$e(\mathbf{i}) \mapsto \bar{e}(\mathbf{i}), \qquad y_r \mapsto \bar{y}_r, \qquad \psi_k \mapsto \bar{\psi}_k, \qquad \epsilon_k \mapsto \bar{\epsilon}_k,$$

where $\mathbf{i} \in T^d, 1 \leq r \leq d, 1 \leq k \leq d-1$, is an isomorphism. Moreover, for any homogeneous element $u \in G_d(\delta)$, we have $\deg u = \deg \phi(u)$.

Proof. Since the elements

$$\{\, \bar{e}(\mathbf{i}) \mid \mathbf{i} \in T^d \,\} \cup \{\, \bar{y}_r \mid 1 \leq r \leq d \,\} \cup \{\, \bar{\psi}_k \mid 1 \leq k \leq d-1 \,\} \cup \{\, \bar{\epsilon}_k \mid 1 \leq k \leq d-1 \,\}$$

satisfy the defining relations (5.2.4 - 5.2.27) of $G_d(\delta)$, proven in Section 6.2, we have that Φ is a well-defined algebra homomorphism. It is graded by Lemmas 6.1.14, 6.1.16, 6.1.26,6.1.32. Furthermore, since $B_d(\delta)$ and $G_d(\delta)$ have the same dimension (Theorems 4.3.1, 5.3.1), it suffices to show that Φ is surjective.

Let us choose arbitrary $\mathbf{i}, \mathbf{j} \in I^d$ such that $g_{\mathbf{i}} g^*_{\mathbf{j}}$ is a graph of some Verma path of length $2d$, and let $\bar{a}_{\mathbf{i},\mathbf{j}}$ be the element in $C_d(\delta)$ with the graph $g_{\mathbf{i}} g^*_{\mathbf{j}}$, such that all the loops in $\bar{a}_{\mathbf{i},\mathbf{j}}$ are oriented anti-clockwise.

Claim 6.3.2. It is enough to show that for any $\mathbf{i}, \mathbf{j} \in I^d$, such that $g_{\mathbf{i}} g^*_{\mathbf{j}}$ is a graph of some Verma path of length $2d$, $\bar{a}_{\mathbf{i},\mathbf{j}}$ is in the image of the map Φ.

Proof. Assume that $\bar{c} \in C_d(\delta)$ and has the graph $g_{\mathbf{i}} g^*_{\mathbf{j}}$ but not all circles in the Verma path $\bar{c}$ are oriented anti-clockwise. Then there exist $l_1, \ldots, l_k, r_1, \ldots, r_s$ such that $\hat{y}_{l_1} \ldots \hat{y}_{l_k} \bar{a}_{\mathbf{i},\mathbf{j}} \hat{y}_{r_1} \ldots \hat{y}_{r_s} = \bar{c}$. Therefore, since $\Phi(y_k) = \hat{y}_k$ for any $1 \leq k \leq d$, we obtain that $\bar{c}$ is also in the image of the map ϕ.

Let $\bar{a} := \bar{a}_{\mathbf{i},\mathbf{j}}$ and $\bar{b}$ be the Verma path with graph $g_{\mathbf{j}} g^*_{\mathbf{i}}$ such that all the loops in b are oriented anti-clockwise. Furthermore, let $a := \Phi_G^{-1} \circ \Phi_C^{-1}(\bar{a})$ and $b := \Phi_G^{-1} \circ \Phi_C^{-1}(\bar{b})$. Recall that $\mathscr{L}_d$ is the Gelfand-Zetlin algebra of $B_d(\delta)$. Let us abuse notation and use the same notation for its isomorphic image in $C_d(\delta)$ and $G_d(\delta)$. Recall from Proposition 4.3.2 that in $B_d(\delta)$, $\mathscr{L}_d$ is generated by all Verma paths with the graphs $g_{\mathbf{i}} g^*_{\mathbf{i}}$ and from Proposition 5.3.2 that in $G_d(\delta)$, it is generated by $e(\mathbf{i}), \mathbf{i} \in T^d$ and $y_1, \ldots, y_d$. Hence, we have $\bar{e}(\mathbf{i}) \in \mathscr{L}_d \bar{a} \mathscr{L}_d \bar{b} \mathscr{L}_d \subseteq C_d(\delta)$, which implies that $e(\mathbf{i}) \in \mathscr{L}_d a \mathscr{L}_d b \mathscr{L}_d \subseteq G_d(\delta)$ (it follows from [17] and [33] that idempotents with a fixed residue sequence are preserved by Φ_C and Φ_G).

Now, consider the elements $\Phi(a), \Phi(b) \in C_d(\delta)$. Clearly, $\Phi(a), \Phi(b) \neq 0$, otherwise $\Phi(e(\mathbf{i})) \in \Phi(\mathscr{L}_d a \mathscr{L}_d b \mathscr{L}_d) = 0$, which contradicts to $\Phi(e(\mathbf{i})) \neq 0$. Moreover, we have that $\Phi(a) = k \cdot \bar{a} \in C_d(\delta)$ and $\Phi(b) = k' \cdot \bar{b} \in C_d(\delta)$ for some scalars k and k' because Φ preserves the degrees (note that $\Phi(a), \Phi(b)$ are the (up to scalars) unique elements of minimal possible degree among elements with graphs $g_{\mathbf{i}} g^*_{\mathbf{j}}$ and $g_{\mathbf{j}} g^*_{\mathbf{i}}$, respectively). Hence, the map Φ is surjective.

Chapter 7

Quantisation

In this and the following sections, we fix $q, z \in \mathbb{C}^*$ and $n, d \in \mathbb{N}$. Let $V = \mathbb{C}^{2n}$. The Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}$ is given by

$$\mathfrak{sp}_{2n} = \{x \in \mathrm{End}(V) \mid (xv_1, v_2) + (v_1, xv_2) = 0 \text{ for all } v_1, v_2 \in V\},$$

where $(,) : V \otimes V \to \mathbb{C}$ is a nondegenerate bilinear form satisfying $(v_1, v_2) = -(v_2, v_1)$.

We choose a basis $\{v_i \mid i \in \hat{V}\}$ of V, where $\hat{V} = \{-n, \ldots, -1, 1, \ldots, n\}$ so that the matrix of the bilinear form $(,) : V \otimes V \to \mathbb{C}$ is

$$J = \mathrm{antidiag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{n}), \quad \text{and} \quad \mathfrak{sp}_{2n} = \{x \in \mathrm{End}(V) \mid x^t J + Jx = 0\},$$

where x^t is the transpose of x.

Proposition 7.0.1 ([35], (7.9); [4], Ch. 8 §13 2.I, 3.I, 4.I)**.** *We have*

$$\mathfrak{sp}_{2n} = \mathrm{span}\{\, F_{ij} \mid i, j \in \hat{V}\} \text{ where } F_{ij} = E_{ij} - \mathrm{sign}(i)\,\mathrm{sign}(i) E_{-j,-i},$$

where E_{ij} is the matrix with 1 in the (i, j)-entry and 0 elsewhere. A Cartan subalgebra of $\mathfrak{sp}_{2n}$ is given by

$$\mathfrak{h} = span\{F_{ii} \mid i \in \hat{V}\} \qquad \text{with basis} \qquad \{F_{11}, F_{22}, \ldots, F_{nn}\}. \tag{7.0.1}$$

And the dual basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\mathfrak{h}^*$ is given by

$$\varepsilon_i \colon \mathfrak{h} \to \mathbb{C} \qquad \text{with} \qquad \varepsilon_i(F_{jj}) = \delta_{ij}. \tag{7.0.2}$$

The form

$$\langle , \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C} \qquad \text{given by} \qquad \langle x, y\rangle = \frac{1}{2}\mathrm{tr}_V(xy) \tag{7.0.3}$$

is a nondegenerate ad-invariant symmetric bilinear form on $\mathfrak{g}$ such that the restriction to $\mathfrak{h}$ is a nondegenerate form $\langle , \rangle \colon \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ on $\mathfrak{h}$. Notice that since $\langle , \rangle$ is nondegenerate, the map $\nu \colon \mathfrak{h} \to \mathfrak{h}^*$ given by $\nu(h) = \langle h, \cdot\rangle$ is a vector space isomorphism which induces a nondegenerate form $\langle , \rangle$ on $\mathfrak{h}^*$. Let $\langle , \rangle \colon \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ be the form on $\mathfrak{h}^*$ induced by the form on $\mathfrak{h}$ and the vector space isomorphism $\nu \colon \mathfrak{h} \to \mathfrak{h}^*$ given by $\nu(h) = \langle h, \cdot\rangle$. Furthermore, we have that

$$\{F_{11}, \ldots, F_{nn}\} \quad \text{and} \quad \{\varepsilon_1, \ldots, \varepsilon_n\} \text{ are orthonormal bases of } \mathfrak{h} \text{ and } \mathfrak{h}^*.$$

Let Δ denote the set of all roots α and we choose $\Delta_+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \mid 1 \leq i \leq n\}$ be the set of positive roots. Let $\rho = 1/2 \sum_{\alpha>0} \alpha$, where the sum runs over all positive roots. Further, we fix an ordered sequence of simple roots $\alpha_1, \ldots, \alpha_n$.

We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \qquad \mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}_\alpha$ is the eigenspace corresponding to the root α under the adjoint action on $\mathfrak{h}$.

The $n \times n$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \frac{2 < \alpha_i, \alpha_j >}{< \alpha_i, \alpha_i >}$$

is called the *Cartan matrix* of $\mathfrak{g}$. Furthermore, let $D = \mathrm{diag}(d_1, \dots, d_n)$ be the diagonal matrix with entries

$$d_{i,i} =< \alpha_i, \alpha_i > /2.$$

Denote $q_i = q^{d_i}$ and suppose that $q_i^2 \neq 1$.

Definition 7.0.2. Let $U_q(\mathfrak{g})$, called *the Drinfel'd-Jimbo quantum group corresponding to* $\mathfrak{g}$, be the associative unitive complex algebra with $4n$ generators E_i, F_i, K_i, K_i^{-1}, $1 \leq i \leq n$, and defining relations

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1 \tag{7.0.4}$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \qquad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j, \tag{7.0.5}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{7.0.6}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\!\!\left[\begin{matrix} 1 - a_{ij} \\ r \end{matrix}\right]\!\!\right]_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0, \qquad i \neq j, \tag{7.0.7}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\!\!\left[\begin{matrix} 1 - a_{ij} \\ r \end{matrix}\right]\!\!\right]_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0, \qquad i \neq j, \tag{7.0.8}$$

where

$$\left[\!\!\left[\begin{matrix} n \\ r \end{matrix}\right]\!\!\right]_q = \frac{[n]_q!}{[r]_q![n-r]_q!} = q^{(r-n)r}\begin{bmatrix} n \\ r \end{bmatrix}_{q^2}, \qquad \text{and} \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We will write $U = U_q(\mathfrak{sp}_{2n})$.

Proposition 7.0.3 ([30], Proposition 5, Theorem 17; [32], Corollary (2.15))**.** *U is a ribbon Hopf algebra, i.e there exist a universal R-matrix $\mathcal{R} = \sum_k r_k^+ \otimes r_k^-$ in (a suitable completion of) $U \otimes U$ and an invertible central element v, called a ribbon element such that for all $u \in U$*

- *$\mathcal{R}\Delta(u)\mathcal{R}^{-1} = (T \circ \Delta)(u)$, where Δ is the coproduct on U and T is the linear map $T : U \otimes U \to U \otimes U$ given by $T(v \otimes w) = w \otimes v$,*

- *$(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$*

- *$(1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$,*

where $\mathcal{R}_{12} :: U \otimes U \to U \otimes U \otimes U$, is algebra homomorphism $\mathcal{R}_{12}(v \otimes w \otimes 1)$, similarly we define $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$;

- *$v^2 = uS(u)$, $S(v) = v$, $\epsilon(v) = 1$, $\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(v \otimes v)$,*

where $u = m(S \otimes 1)(\mathcal{R}_{21})$, and m is a multiplication in U, Δ is a co-product, u is the unit, ϵ is a co-unit and S is the antipode of U.

Denote by $\mathcal{R}_{21} = \sum_k r_k^- \otimes r_k^+$

Definition 7.0.4. We call $\lambda \in \mathfrak{h}^*$ a *dominant weight* if $\lambda = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. We denote the set of dominant weights Λ.

The following is immediate from the Definition.

Lemma 7.0.5. *We have a bijection of sets*

$$\{\text{Young diagrams } \lambda \text{ with } l(\lambda) \leq n\} \overset{1:1}{\longleftrightarrow} \{\text{Dominant weights}\}$$
$$\lambda \mapsto \lambda$$

Remark 7.0.6. From now on, we will only consider partitions λ with $l(\lambda) \leq n$. Furthermore, by abuse of notation, we will use the same symbols for dominant weight and partitions.

Theorem 7.0.7 ([30], Proposition 28)**.** *For any partition λ with $l(\lambda) \leq n$, there exist a unique irreducible U-module V_λ with the highest weight λ. Furthermore, V_λ is finite-dimensional.*

Proposition 7.0.8 ([39])**.** *We have the following decomposition.*

$$V_\lambda \otimes V_{\alpha_1} = \bigoplus_{\bar{\lambda}} V_{\bar{\lambda}}, \tag{7.0.9}$$

where the sum over $\bar{\lambda}$ denotes a sum over all partitions (with $l(\bar{\lambda}) \leq n$) obtained by adding or removing a box from λ.

Corollary 7.0.9. *We have the following decompositions for each* $1 \leq i \leq d$.

$$V_\mu \otimes V^{\otimes i} = \bigoplus_{\lambda} V_\lambda, \tag{7.0.10}$$

where the sum runs over all dominant weights λ such that there is an up-down tableau of length i from μ to λ.

The quantum group has a triangular decomposition.

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \qquad \text{and} \qquad U_q(\mathfrak{b}_+) = U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+)$$

Definition 7.0.10. The *Verma module M_λ* of the highest weight $\lambda \in \Lambda$ defined as

$$M_\lambda = U \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}v_\lambda,$$

where $\mathbb{C}v_\lambda$ is the one dimensional $U_q(\mathfrak{b}_+)$-module spanned by a vector v_λ such that $av_\lambda = \lambda(a)v_\lambda$ for $a \in \mathfrak{h}$ and $U_q(\mathfrak{n}_+)v_\lambda = 0$.

If M is a U-module and $\lambda \in \mathfrak{h}^*$ the λ weight space of M is

$$M_\lambda = \{m \in M \mid am = q^{\lambda(a)}m, \text{ for all } a \in \mathfrak{h}\}.$$

Definition 7.0.11. The *category $\mathcal{O}$* is the category of U-modules M such that

- $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$,
- For all $m \in M$, $\dim(U_q(\mathfrak{n}_+)m)$ is finite,
- M is finitely generated as a U module.

7.1 Affine Braid Group

We follow the paper by Orellana and Ram [39].

Definition 7.1.1. The *affine braid group* $\mathcal{B}_d$ is given by the braid group generators $T_1, \dots, T_{d-1}$ and commuting elements $Y_1, \dots, Y_d$ with relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad \text{for } i = 1, 2, \dots, d-2 \tag{7.1.1}$$
$$T_iT_j = T_jT_i, \qquad \text{if } j \neq i \pm 1 \tag{7.1.2}$$
$$Y_{i+1} = T_iY_iT_i, \qquad \text{for } i = 1, 2, \dots, d-1. \tag{7.1.3}$$
$$Y_1T_i = T_iY_1, \qquad \text{for } i = 2, 3, \dots, d-1. \tag{7.1.4}$$

Remark 7.1.2. Notice we have a commutative sub-algebra $\mathcal{Y} := C[Y_1^{\pm}, \dots, Y_d^{\pm}]$.

Let us fix $\omega_1, \omega_2, \dots \in \mathbb{C}$.

Definition 7.1.3. The *affine Birman–Murakami–Wenzl (BMW) algebra* $\mathcal{W}_d$ is the quotient of the group algebra $\mathbb{C}\mathcal{B}_d$ by the relations

$$(T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0 \tag{7.1.5}$$
$$E_iT_{i-1}^{\pm 1}E_i = z^{\pm 1}E_i \quad \text{and} \quad E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i \tag{7.1.6}$$
$$E_iY_iY_{i+1} = z^{-2}E_i = Y_iY_{i+1}E_i \tag{7.1.7}$$
$$E_1(Y_1)^rE_1 = \omega_rE_1, \tag{7.1.8}$$

where the E_i, are defined by the equations

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i \qquad 1 \leq i \leq d-1. \tag{7.1.9}$$

Recall the diagrammatic description of the affine BMW algebra given in Introduction. The one can find the comparison of the relations as well as some other relations of the algebra, (e.g. relations involving E_i's, similar to the ones we saw in Brauer algebras) in [15, Section 2.4], see also [14, 39].

It will be convenient to consider a different presentation for the affine BMW algebra; towards this end we have the following simple and *well-known* lemmas.

Lemma 7.1.4. *The generators T_i, $1 \leq i \leq d-1$, are invertible.*

Proof. From (7.1.5) we have

$$T_i(T_i + q^{-1})(T_i - q) - z^{-1}(T_i + q^{-1})(T_i - q) = 0 \tag{7.1.10}$$
$$T_i(T_i + q^{-1})(T_i - q) - z^{-1}T_i^2 + z^{-1}(q - q^{-1})T_i + z^{-1} = 0 \tag{7.1.11}$$
$$-zT_i(T_i + q^{-1})(T_i - q) + T_i^2 - (q - q^{-1})T_i = 1 \tag{7.1.12}$$
$$T_i^{-1} = -z(T_i + q^{-1})(T_i - q) + T_i - q + q^{-1}. \tag{7.1.13}$$

Lemma 7.1.5. *The relation (7.1.5) is equivalent to each of the relations*

$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\pm 1}E_i, \tag{7.1.14}$$

where $\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i$.

Proof. We have

$$(T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0 \tag{7.1.15}$$

$$(T_i - z^{-1})(T_i^2 + (q^{-1} - q)T_i - 1) = 0 \tag{7.1.16}$$

$$T_i^3 + (q^{-1} - q)T_i^2 - T_i - z^{-1}T_i^2 - z^{-1}(q^{-1} - q)T_i + z^{-1} = 0, \tag{7.1.17}$$

since $q - q^{-1} \neq 0$ we can divide

$$\frac{T_i^2}{q - q^{-1}}(T_i - (q - q^{-1}) - T_i^{-1}) = \frac{z^{-1}T_i}{q - q^{-1}}(T_i - (q - q^{-1}) - T_i^{-1}) \tag{7.1.18}$$

$$T_i^2\left(\frac{T_i - T_i^{-1}}{q - q^{-1}} - 1\right) = z^{-1}T_i\left(\frac{T_i - T_i^{-1}}{q - q^{-1}} - 1\right) \tag{7.1.19}$$

$$T_i^2 E_i = z^{-1}T_iE_i \tag{7.1.20}$$

$$T_iE_i = z^{-1}E_i \qquad \text{since } T_i \text{ is invertible} \tag{7.1.21}$$

Analogously, we obtain

$$E_iT_i = z^{-1}E_i.$$

To obtain the other relations, multiply by z, T_i^{-1}

$$zE_i = T_i^{-1}E_i \qquad \text{and} \qquad zE_i = E_iT_i^{-1}$$

Lemma 7.1.6. *We have*

$$E_i^2 = \left(1 + \frac{z - z^{-1}}{q - q^{-1}}\right)E_i. \tag{7.1.22}$$

Proof.

$$E_i^2 = \left(1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}\right)E_i = E_i - \frac{T_iE_i - T_i^{-1}E_i}{q - q^{-1}} = \left(1 + \frac{z - z^{-1}}{q - q^{-1}}\right)E_i.$$

We denote by $x := 1 + \frac{z-z^{-1}}{q-q^{-1}}$ the coefficient in equation 7.1.22 above.

Lemma 7.1.7. *We have*

$$T_i^2 = \tilde{q}T_i - z^{-1}\tilde{q}E_i + 1 \qquad \textit{and} \qquad T_i^{-2} = 1 - \tilde{q}T_i^{-1} + z^1\tilde{q}E_i$$

Proof. Follows from the definition

Let us give an alternative definition of the algebra $\mathcal{W}_d$.

Definition 7.1.8. The algebra $\hat{\mathcal{W}}_d$ is generated by $T_1, \ldots, T_{d-1}$, $E_1, \ldots, E_{d-1}$ and commuting elelments $Y_1, \ldots, Y_d$ with relations (7.1.1-7.1.4), (7.1.6-7.1.9) and (7.1.14), namely

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad \text{for } i = 1, 2, \ldots, d-2 \tag{7.1.23}$$

$$T_iT_j = T_jT_i, \qquad \text{if } j \neq i \pm 1 \tag{7.1.24}$$

$$Y_{i+1} = T_iY_iT_i, \qquad \text{for } i = 1, 2, \ldots, d-1. \tag{7.1.25}$$

$$Y_1T_i = T_iY_1, \qquad \text{for } i = 2, 3, \ldots, d-1. \tag{7.1.26}$$

$$E_iT_{i-1}^{\pm 1}E_i = z^{\pm 1}E_i \quad \text{and} \quad E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i \tag{7.1.27}$$

$$E_iY_iY_{i+1} = z^{-2}E_i = Y_iY_{i+1}E_i \tag{7.1.28}$$

$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\pm 1}E_i, \qquad \text{for } i = 1, 2, \ldots, d-1. \tag{7.1.29}$$

$$E_1(Y_1)^r E_1 = \omega_r E_1. \tag{7.1.30}$$

where the E_i, are defined by the equations

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i \qquad 1 \le i \le d-1. \tag{7.1.31}$$

Definition 7.1.9. Define an algebra homomorphism ϕ as follows

$$\phi : \mathcal{B}_d \to \hat{\mathcal{W}}_d \tag{7.1.32}$$

$$Y_i \mapsto Y_i \qquad i = 1, \dots, d, \tag{7.1.33}$$

$$T_i \mapsto T_i \qquad i = 1, \dots, d-1. \tag{7.1.34}$$

Lemma 7.1.10. *The homomorphism $\phi : \mathcal{B}_d \to \hat{\mathcal{W}}_d$ descends to an isomoprhism $\mathcal{W}_d \to \hat{\mathcal{W}}_d$.*

Proof. It immediately follows from Lemma 7.1.5 and Isomorphism theorem, that $\mathcal{W}_d \cong \hat{\mathcal{W}}_d$.

Furthermore, we can generalise the classical Schur-Weyl duality to the affine BMW algebras and quantum groups. But before let us define few maps. For U-modules M and N, we define a map

$$\begin{array}{rccc} \check{R}_{MN}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum_{\mathcal{R}} R_2 n \otimes R_1 m, \end{array} \tag{7.1.35}$$

where $\mathcal{R} = \sum_k r_k^+ \otimes r_k^- \in U \otimes U$ is the R-matrix of U, defined in Proposition 7.0.3.

The quasitriangularity of R gives us the following Braid relation

$$(\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP})(\check{R}_{NP} \otimes \mathrm{id}_M) = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N)(\mathrm{id}_P \otimes \check{R}_{MN}), \tag{7.1.36}$$

where P is U-module.

Let h, u be such that $q = e^{h/2}$ and $v = e^{-h\rho}u$. Let M be a U-module, V be a finite-dimensional U-module and let $\psi \in \mathrm{End}_U(M \otimes V)$. Define the *quantum trace* $(\mathrm{id} \otimes \mathrm{qtr}_V)(\psi)\colon M \to M$ by

$$(\mathrm{id} \otimes \mathrm{qtr}_V)(\psi) = (\mathrm{id} \otimes \mathrm{tr}_V)((1 \otimes uv^{-1})\psi). \tag{7.1.37}$$

Definition 7.1.11. Let M be a highest weight U-module, $V = V_{\alpha_1}$, the unique simple modules of highest weight α_1. Define a map

$$\begin{array}{rccc} \Phi\colon & \mathcal{B}_d & \longrightarrow & \mathrm{End}_U(M \otimes V^{\otimes d}) \\ & T_i & \longmapsto & \check{R}_i, \qquad\qquad 1 \le i \le d-1, \\ & Y_1 & \longmapsto & \check{R}_0^2, \end{array} \tag{7.1.38}$$

where

$$\check{R}_i = \mathrm{id}_M \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}_V^{\otimes(d-i-1)} \qquad \text{and} \qquad \check{R}_0^2 = (\check{R}_{VM}\check{R}_{MV}) \otimes \mathrm{id}_V^{\otimes(d-1)}. \tag{7.1.39}$$

Theorem 7.1.12 ([39], Theorem 6.17-6.18)**.** *We have Φ makes $M \otimes V^{\otimes d}$ into a $\mathcal{B}_d$ module. Furthermore, if $M = V_\mu$, then Φ is surjective, where V_μ is the unique simple modules of highest weights μ and μ is a partition.*

Moreover, for any k $\omega_k^\mu := -(\mathrm{id} \otimes \mathrm{qtr}_V)((z\mathcal{R}_{21}\mathcal{R})^k) \in \mathbb{C}$ and depend only on the central character of V_μ. Furthermore, Φ is a representation of the affine BMW algebra $\mathcal{W}_d$ with parameters $\omega_1, \omega_2, \dots$ and $z = -q^{2n+1}$.

Definition 7.1.13. A $\mathcal{B}_d$-module M is *calibrated* if M has a basis of simultaneous eigenvectors for the action of Y_i's.

A tuple $\bar{u} = (u_1, \dots, u_d) \in (C^*)^{\otimes d}$ is called a weight for $\mathcal{Y}$. Each tuple defines a homomorphism $Y_i \mapsto u_i$ and hence a one-dimension representation of $\mathcal{Y}$.

Definition 7.1.14. Let M be a $\mathcal{B}_d$-module and $\bar{u}$ be a weight for $\mathcal{Y}$. We define

$$M_{\bar{u}} := \{v \in M \mid hv = \bar{u}(h)v, \text{ for all } h \in \mathcal{Y}\}.$$

Recall the decompositions (7.0.9) and (7.0.10). Let us state state one property of the representation above, which will be used later.

Proposition 7.1.15 ([39], Proposition 3.6, Theorem 6.18)**.** *$V_\mu \otimes V^{\otimes d}$ is calibrated. Furthermore, the eigenvalues of Y_i separate the components of the decomposition (7.0.10) of $V_\mu \otimes V^{\otimes i}$. Hence for each weight $\bar{u}$ for $\mathcal{Y}$, the space $M_{\bar{u}}$ is either trivial or one-dimensional.*

Fix a dominant weight $\lambda \in \Lambda$ and let $\tilde{O}_d$ be the category of finite dimensional $\mathcal{B}_d$-modules. Define a functor

$$\begin{array}{rccc} F_\lambda\colon & O & \longrightarrow & \tilde{O}_d \\ & M & \longmapsto & \mathrm{Hom}_U(M(\lambda), M \otimes V^{\otimes d}), \end{array} \tag{7.1.40}$$

We have the following

Proposition 7.1.16 ([39], Proposition 3.9)**.** *Let $\lambda \in \Lambda$, then the functor F_λ is exact.*

Let μ be a partition. Consider $H^{\lambda/\mu} = F_\lambda(V_\mu)$. It is a naturally $\mathcal{W}_d$-module. Moreover, it was shown in [39] that for any λ and μ, $H^{\lambda/\mu}$ is a simple $\mathcal{W}_d$-module. Hence from the Double Centralizer theorem and Theorem 7.1.12 we obtain the following.

Corollary 7.1.17. *As $(U, \mathcal{W}_d)$-module, we have the following decomposition*

$$V_\mu \otimes V^{\otimes d} \cong \bigoplus_\lambda V_\lambda \otimes H^{\lambda/\mu},$$

where V_λ is the irreducible U-module of highest weight λ and $H^{\lambda/\mu}$ is the irreducible $\mathcal{W}_d$ module.

The irreducible representations $H^{\lambda/\mu}$ can be constructed explicitly. Let us recall this construction.

Definition 7.1.18. Let μ, λ be dominant weights and

$$M^{\lambda/\mu} = \mathrm{span}\left\{ v_T \;\middle|\; \begin{array}{c} T = (\mu = \tau^{(0)}, \ldots, \tau^{(d)} = \lambda) \text{ an} \\ \text{up down tableau of length } d \text{ from } \mu \text{ to } \lambda \end{array} \right\}$$

(so that the symbols v_T are a $\mathbb{C}$-basis of $M^{\lambda/\mu}$). Define operators Y_i, E_j, T_j, for $1 \le i \le d, 1 \le j \le d-1$, by

$$Y_i v_T = \tilde{c}(\tau^{(i)}, \tau^{(i-1)}) v_T, \qquad 1 \le i \le d,$$

$$E_j v_T = \delta_{\tau^{(j+1)}, \tau^{(j-1)}} \cdot \sum_S (E_j)_{ST} v_S, \qquad \text{and} \qquad T_j v_T = \sum_S (T_j)_{ST} v_S, \qquad 1 \le j \le d-1,$$

where both sums are over up-down tableaux $S = (\mu = \tau^{(0)}, \ldots, \tau^{(i-1)}, \sigma^{(i)}, \tau^{(i+1)}, \ldots, \tau^{(d)} = \lambda)$ that are the same as T except possibly at the ith step and

$$(E_i)_{ST} = -1 \cdot \frac{\sqrt{D_{\tau^{(i)}} D_{\sigma^{(i)}}}}{D_{\tau^{(i-1)}}},$$

$$(T_i)_{ST} = \begin{cases} \sqrt{(q^{-1} + (T_j)_{TT})(q^{-1} + (T_j)_{SS})}, & \text{if } \tau^{(i-1)} \neq \tau^{(i+1)} \text{ and } S \neq T, \\ \left(\dfrac{q - q^{-1}}{1 - \tilde{c}(\tau^{(i+1)}, \sigma^{(i)})\tilde{c}(\tau^{(i)}, \tau^{(i-1)})^{-1}}\right)(\delta_{ST} - (E_i)_{ST}), & \text{otherwise,} \end{cases}$$

$$\tilde{c}(\tau^{(i)}, \tau^{(i-1)}) = \begin{cases} q^{2c(\tau^{(i)}/\tau^{(i-1)})}, & \text{if} \tau^{(i)} \supset \tau^{(i-1)}, \\ z^{-2} q^{-2c(\tau^{(i-1)}/\tau^{(i)})}, & \text{if} \tau^{(i)} \subset \tau^{(i-1)}. \end{cases}$$

$$D_\tau = \prod_{\alpha \in \Delta} \frac{[\langle \mu + \rho, \alpha^\vee]_q}{[\langle \rho, \alpha^\vee \rangle]_q}. \tag{7.1.41}$$

Theorem 7.1.19 ([39], Theorem 6.20)**.** *Let λ/μ be a pair of partitions. Operators Y_i, E_j, T_j, for $1 \leq i \leq d, 1 \leq j \leq d-1$ turn $M^{\lambda/\mu}$ into an irreducible $\mathcal{W}_d$-module. Furthermore, $H^{\lambda/\mu}$ is isomorphic to $M^{\lambda/\mu}$ as $\mathcal{W}_d$-modules, and hence for $v_T \in M^{\lambda/\mu}$ we have*

$$E_1 (Y_1)^k E_1 . v_T = \omega_k^\mu E_1 v_T,$$

where ω_k^μ defined in Theorem 7.1.12.

7.2 Elliptic Braid Group

In this subsection we follow the papers [26] and [27].

Definition 7.2.1. The *elliptic braid group* $\mathscr{B}_d$ is given by the affine braid group generators and commuting elements $X_1, \dots, X_d$ with relations (7.1.1-7.1.2), (7.1.3-7.1.4) and

$$X_{i+1} = T_i X_i T_i, \qquad \text{for } i = 1, 2, \dots, d-1. \tag{7.2.1}$$

$$X_1 T_i = T_i X_1, \qquad \text{for } i = 2, 3, \dots, d-1. \tag{7.2.2}$$

$$X_1 \dots X_d Y_i = Y_i X_1 \dots X_d,\ Y_1 \dots Y_d X_i = X_i Y_1 \dots Y_d, \qquad \text{for } i = 1, \dots, d, \tag{7.2.3}$$

$$X_1 Y_2 = Y_2 X_1 T^2. \tag{7.2.4}$$

Definition 7.2.2. We define a U-module $O_q(\mathfrak{g}) = \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda$.

Theorem 7.2.3 ([26], Theorem 22, [27])**.** *Let V a f.d. U-module. On the vector space*

$$W = \mathrm{Hom}_U(\mathbf{1}, O_q(\mathfrak{g}) \otimes V^{\otimes d})$$

of invariants we have an action of the elliptic braid group $\mathscr{B}_d$.

Let $W^\lambda = \mathrm{Hom}_U(1, V_\lambda^* \otimes V_\lambda \otimes V^{\otimes d})$. We have a vector space decomposition

$$\mathrm{Hom}_U(\mathbf{1}, O_q(\mathfrak{g}) \otimes V^{\otimes n}) \cong \bigoplus_{\lambda \in \Lambda} W^\lambda.$$

Lemma 7.2.4 ([26,27])**.** *We have $X_1(W^\lambda) \subset \oplus_{\bar{\lambda}} W^{\bar{\lambda}}$, where $\bar{\lambda} \in AR(\lambda)$.*

Theorem 7.2.5 ([26,27])**.** *Each subspace W^λ is a finite-dimensional submodule for the affine BMW algebra isomorphic to $H^{\lambda/\lambda}$.*

Proof. Let us consider the space

$$\mathrm{Hom}_U(\mathbf{1}, O_q(\mathfrak{g}) \otimes V^{\otimes n}) = \mathrm{Hom}_U(\mathbf{1}, \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda \otimes V^{\otimes d})$$

of invariants in the U-module $O_q(G) \otimes V^{\otimes d}$.

It is enough to notice that for any $\lambda \in \Lambda$ we have

$$W^\lambda = \mathrm{Hom}_U(1, V_\lambda^* \otimes V_\lambda \otimes V^{\otimes d}) \cong \mathrm{Hom}_U(V_\lambda, V_\lambda \otimes V^{\otimes d}) = H^{\lambda/\lambda}. \tag{7.2.5}$$

All construction described in this section can also be done for $U_q(\mathfrak{sl}_n)$. Let us briefly recall some analogue of the Schur-Weyl duality for $U_q(\mathfrak{sl}_n)$ and double affine Hecke algebra.

Definition 7.2.6. Double affine Hecke algebra (DAHA) $\mathcal{HH}_d(q,t)$ is the quotient of the group algebra $\mathbb{C}\mathscr{B}_d$ n of the elliptic braid group by the additional relations

$$(T_i - q^{-1}t)(T_i + q^{-1}t^{-1}),$$

where $q, t \in \mathbb{C}$.

Chapter 8

Double Affine BMW algebras

8.1 Combinatorics

In this section we define combinatorics used to descibe representations of double affine BMW algebras, introduced in the following section.

Recall $\mathsf{A}(\lambda)$ and $\mathsf{R}(\lambda)$ from Definition 2.1.7. We have the following.

Lemma 8.1.1. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have the following*

$$(r, c) \in A(\lambda) \iff (\lambda_r < \lambda_{r-1} \text{ or } r = 1) \text{ and } \lambda_r = c - 1, \tag{8.1.1}$$

$$(r, c) \in R(\lambda) \iff \lambda_r > \lambda_{r+1} \text{ and } \lambda_r = c. \tag{8.1.2}$$

Proof. First, notice that if $r = 1$ then the box $(1, \lambda_1 + 1)$ is addable and there are no other addable boxes in the first row. Moreover, there are no addable boxes in row r if $\lambda_r = \lambda_{r-1}$ and there is one addable box $(r, \lambda_r + 1)$ if $\lambda_r < \lambda_{r-1}$ (here we use the convention $\lambda_{k+1} = 0$).

Similarly, if $\lambda_r > \lambda_{r+1}$, then the box (r, λ_r) is removable. Furthermore, there are no removable boxes in the row r if $\lambda_r = \lambda_{r+1}$.

Definition 8.1.2. A skew diagram A is called *reduced* if

(1) there exist $r, c, \in \mathbb{N}$ such that $(1, c)$ and $(r, 1)$ are in A,

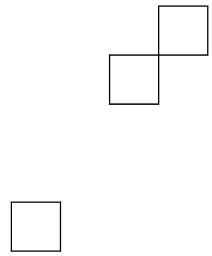
(2) for any two boxes (r_1, c_1) and (r_2, c_2), we have either

- $r_1 < r_2$ and $c_1 > c_2$ or
- $r_1 > r_2$ and $c_1 < c_2$.

Therefore, we can uniquely write $A = ((1, c_1), (r_2, c_2), \ldots, (r_l, 1))$ such that $r_i < r_{i+1}$ and $c_i > c_{i+1}$. We denote $l(A) = l$ and $\lambda(A) = (\underbrace{c_1 - 1, \ldots, c_1 - 1}_{r_2 - 1}, \underbrace{c_2 - 1, \ldots, c_2 - 1}_{r_3 - r_2}, \ldots, \underbrace{c_{l-1} - 1, \ldots, c_{l-1} - 1}_{r_l - r_{l-1}})$, the length and the shape of A respectively.

Remark 8.1.3. Notice that the second condition is equivalent to having at most one box in each row and each column.

Example 8.1.4. The tuple $A = ((1,4),(2,3),(5,1))$ is a reduced skew diagram (RSD)



Lemma 8.1.5. *Let A be RSD, then $\lambda(A)$ is a partition.*

Proof. Indeed, since A is RSD, we have $c_i > c_{i+1}$, therefore $\lambda(A)$ is a partition.

Lemma 8.1.6. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have that the set $A(\lambda)$ is RSD*

Proof. First, from Lemma 8.1.1 we have that we add boxes only to the rows r such that $\lambda_r < \lambda_{r-1}$, therefore $\lambda \cup A(\lambda)$ is a partition. Moreover, since $A(\lambda) = \lambda'/\lambda$, where $\lambda' = \lambda \cup A(\lambda)$, we have that $A(\lambda)$ is a skew diagram. Furthermore, to each row/column we can add not more than one box and we can always add boxes to the first column and first row. Therefore, $A(\lambda)$ is reduced.

Moreover, we have the following.

Lemma 8.1.7. *We have a bijection of sets*

$$\{\textit{Young diagrams}\} \overset{1:1}{\longleftrightarrow} \{\textit{RSDs}\}$$
$$\lambda \mapsto A(\lambda)$$

Proof. The map is well-defined, since, by Lemma 8.1.6, $A(\lambda)$ is a RSD for any partition λ.

Furthermore, notice that if $\lambda \neq \mu$ then $A(\lambda) \neq A(\mu)$. Indeed, let $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_s)$ and i be the minimal index such that $\lambda_i \neq \mu_i$. WLOG we can assume that $\lambda_i < \mu_i$. Then by minimality assumption, we have $\lambda_i < \mu_i \leq \mu_{i-1} = \lambda_{i-1}$, and hence the box $(i, \lambda_i + 1) \in A(\lambda)$, but not in $A(\mu)$.

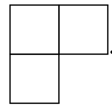
Now let us show that the map is surjective. Assume that $A = ((1, c_1), \ldots, (r_l, 1))$ is RSD. Let $\lambda = \lambda(A)$ be the shape of A. By Lemma 8.1.5 we have that λ is a partition. Finally, notice that it follows immediately from the condition in Lemma 8.1.6 that $A(\lambda) = A$, since $r_i < r_{i+1}$ and $c_i > c_{i+1}$.

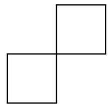
Definition 8.1.8. An extension of a RSD $A = ((1, c_1), \ldots, (r_l, 1))$, denoted by A^e, is the following subset of the box lattice

$$A^e = A \cup \{(r_2 - 1, c_1 - 1), \ldots, (r_l - 1, c_{l-1} - 1)\}.$$

Since $r_i < r_{i+1}$ and $c_i > c_{i+1}$, we can uniquely write $A^e = ((1, c_1), (r_2 - 1, c_1 - 1), (r_2, c_2), (r_3 - 1, c_2 - 1), \ldots, (r_l, 1))$.

Example 8.1.9. Let A be an RSD as in Example 8.1.4. Then $A^e = ((1,4),(1,3),(2,3),(4,2),(5,1))$ and has the following form





Proposition 8.1.10. *Let A be an RSD and A^e its extension. Then for any two boxes $(r_1, c_1), (r_2, c_2) \in A^e$ we have either*

- $r_1 \leq r_2$ *and* $c_1 \geq c_2$ *or*
- $r_1 \geq r_2$ *and* $c_1 \leq c_2$.

Lemma 8.1.11. *We have*

$$\{RSDs\} \overset{1:1}{\longleftrightarrow} \{extensions\ of\ RSD\}$$

Proof. First, notice that the map is well-defined and surjective since we define an extension for each RSS. Therefore, we only need to check injectivity.

Assume $A = ((1, c_1), \ldots, (r_l, 1))$ and $B = ((1, \bar{c}_1), \ldots, (\bar{r}_{\bar{l}}, 1))$ are RSDs such that $A \neq B$. Let i be minimal such that $(r_i, c_i) \neq (\bar{r}_i, \bar{c}_i)$. First assume that $r_i \neq \bar{r}_i$, then $(r_i - 1, c_{i-1} - 1) \in A$, but not in B, since $c_{i-1} - 1 = \bar{c}_{i-1} - 1$ by minimality of i. Now assume that $r_i = \bar{r}_i$ but $c_i \neq \bar{c}_i$. Then $(r_i, c_i) \in A$, but not in B. Therefore, $A^e \neq B^e$.

Definition 8.1.12. *A shape $\lambda(A^e)$* of the extended RSD A^e is the shape $\lambda(A)$ of the corresponding *RSD* A.

Lemma 8.1.13. *Let $A = ((1, c_1), (r_2, c_2)), \ldots, (r_l, 1))$ be an RSD with the shape $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then we have $A^e = AR(\lambda)$.*

Proof. In Lemma 8.1.7 we showed that $A = A(\lambda)$. From Lemma 8.1.1, we have that $R(\lambda) = \{(r_2 - 1, c_1 - 1), \ldots, (r_l - 1, c_{l-1} - 1)\}$. Indeed, assume that r is such that $\lambda_r > \lambda_{r+1}$, and hence the box (r, λ_r) is removable. Furthermore, we have that the box $(r + 1, \lambda_{r+1} + 1)$ is addable, and there is an addable box in the column $\lambda_r + 1$, and these boxes are consecutive in the tuple A. Hence, there exist i and a such that $(r_i, c_i) = (a, \lambda_r + 1)$ and $(r_{i+1}, c_{i+1}) = (r + 1, \lambda_{r+1} + 1)$. Therefore, we have $r = r_{i+1} - 1$ and $\lambda_r = c_i - 1$. Since all boxes with coordinates $(r_{j+1} - 1, c_j - 1)$, where $1 \leq j \leq l - 1$, are removable, we obtain $A^e/A = R(\lambda)$.

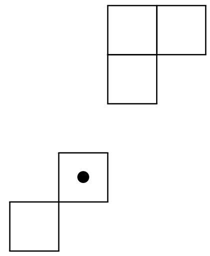
Proposition 8.1.14. *Let λ be a partition and A^e be its corresponding extended RSD. Then there are no two boxes on the same diagonal of $AR(\lambda) = A^e$. Namely, if $(r_1, c_1), (r_2, c_2)$ belong to A^e and $r_1 - c_1 = r_2 - c_2$, then $r_1 = r_2$ and $c_1 = c_2$.*

Proof. Immediately follows from the Proposition 8.1.10.

Definition 8.1.15. Let λ be a partition. *A simple λ-up-down tableau* is an up-down tableau of shape λ/λ and length 2.

Definition 8.1.16. A marked extended RSD is a tuple A_α consisting of an extended RSD A and a box $\alpha \in A$. We will display A_α as A with one box marked.

Example 8.1.17. Let $A = \{(1,4),(1,3),(2,3),(3,2),(5,1)\}$ be an extended RSD, then $A_{(3,2)}$ has the following form.



Lemma 8.1.18. *For any partition λ, we have a bijection of sets*

$$\{\textit{simple } \lambda\textit{-up-down tableaux}\} \overset{1:1}{\longleftrightarrow} \{\textit{ marked extended RSDs of shape } \lambda\}$$
$$(\lambda, \lambda^\alpha, \lambda) \mapsto A_\alpha$$

Proof. By definition the map is well-defined and injective. Moreover, the sets have the same cardinality. Therefore, the map is a bijection of sets.

Example 8.1.19. Let $A_{(3,2)}$ be the marked extended RSD from Example 8.1.17, then the corresponding up-down tableaux is $((3,2,2),(3,2,1),(3,2,2))$.

Now let us see what will happen with an extended RSS if we add/remove one box from its corresponding Young diagram.

Lemma 8.1.20. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition, $A(\lambda) = ((1,c_1),\dots,(r_l,1))$ and $\alpha = (r,c) \in AR(\lambda)$. Then*

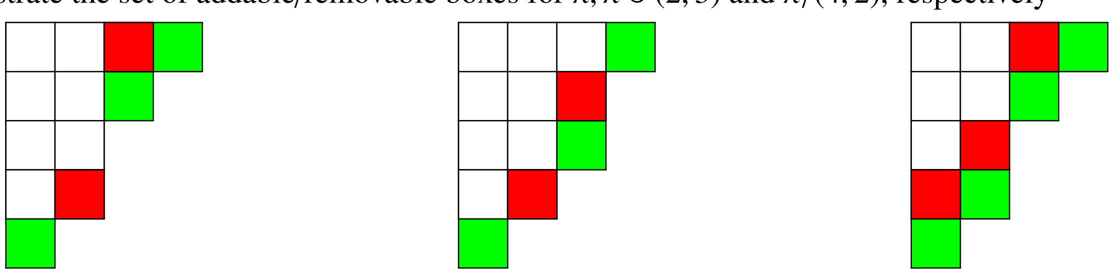
(1) if $\alpha \in A(\lambda)$ we have
$$A(\lambda^\alpha) = A(\lambda)/\alpha \cup \tilde{\delta}^{\lambda_{r-1}}_{\lambda_r+1}(r,c+1) \cup \delta^{\lambda_r}_{\lambda_{r+1}}(r+1,c),$$

(2) if $\alpha \in R(\lambda)$ we have
$$A(\lambda^\alpha) = A(\lambda) \cup \alpha/(\delta^{\lambda_r+1}_{\lambda_{r-1}}(r,c+1) \cup \delta^{\lambda_{r+1}+1}_{\lambda_r}(r+1,c)),$$

where $1 \le i \le c,\ 2 \le j \le c,\ \tilde{\delta}^b_a(r,c) = \begin{cases} (r,c) & \textit{if} \quad b > a, \\ \emptyset & \textit{otherwise.} \end{cases}$ *and* $\delta^b_a(r,c) = \begin{cases} (r,c) & \textit{if} \quad b = a, \\ \emptyset & \textit{otherwise.} \end{cases}$

Proof. 1) Consider $\lambda \cup (r,c)$. Notice that only if $\lambda_{r-1} > \lambda_r + 1$, then we can a box on the row r. Clearly, the box should have coordinates $(r, c+1)$. Furthermore, only if $\lambda_{r+1} = \lambda_r$ we can add a box on the column c, and the box should have coordinates $(r+1, c)$.
2) Notice that the box (r,c) is addable in $\lambda/(r,c)$. Furthermore, if $\lambda_r = \lambda_{r-1} - 1$ (respectively, if $\lambda_{r+1} = \lambda_r - 1$) then the box $(r, c+1)$ (respectively, $(r+1, c)$) is not addable.

Example 8.1.21. Let $\lambda = (3,2,2,2)$ as in Example 2.1.6. Following the same conventions, we illustrate the set of addable/removable boxes for $\lambda, \lambda \cup (2,3)$ and $\lambda/(4,2)$, respectively



8.2 Double affine BMW algebras

Definition 8.2.1. *Extended elliptic group* $\mathscr{B}_d^e$ is generated by

- the commuting elements $W_1, W_2, \ldots$
- the elliptic Braid group $\mathscr{B}_d$,

with cross relations

- $Y_i W_j = W_j Y_i \qquad 1 \le i \le d, j \ge 1,$
- $T_i W_j = W_j T_i \qquad 1 \le i \le d-1, j \ge 1,$

Definition 8.2.2. *Double affine BMW algebra* $\mathcal{WW}_d(q,z)$ is the quotient of the group algebra $\mathbb{C}\mathscr{B}_d^e$ of the extended elliptic Braid group by the relations

$$(T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0, \tag{8.2.1}$$

$$E_i T_{i-1}^{\pm 1} E_i = z^{\pm 1} E_i \qquad \text{and} \qquad E_i T_{i+1}^{\pm 1} E_i = z^{\pm 1} E_i, \tag{8.2.2}$$

$$E_i Y_i Y_{i+1} = z^{-2} E_i = Y_i Y_{i+1} E_i \tag{8.2.3}$$

$$E_1 (Y_1)^r E_1 = W_r E_1. \tag{8.2.4}$$

where

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i, \qquad 1 \le i \le d-1.$$

Lemma 8.2.3. *Representations of the affine BMW algebra* $\mathcal{W}_d$ *are in one-to-one correspondence with the representations of the subalgebra of* $\mathcal{WW}_d(q,z)$ *generated by* T_i, Y_j, W_k *for* $1 \le i \le d-1, 1 \le j \le d, k \ge 1$.

Proof. The statement immediately follows from the fact that W_k are central in the subalgebra generated by T_i, Y_j, W_k for $1 \le i \le d-1, 1 \le j \le d, k \ge 1$.

Similarly, we have the following properties.

Lemma 8.2.4. *We have*

- T_i *is invertible,*
- *(8.2.1) is equivealent to* $E_i T_i^{\pm 1} = T_i^{\pm 1} E_i = z^{\pm 1} E_i$,
- $E_i^2 = \left(1 + \frac{z - z^{-1}}{q - q^{-1}}\right) E_i =: x E_i$
- $T_i^2 = \tilde{q} T_i - z^{-1} \tilde{q} E_i + 1 \qquad$ *and* $\qquad T_i^{-2} = 1 - \tilde{q} T_i^{-1} + z^1 \tilde{q} E_i$

Below we give few equivalent Definitions of double affine BMW algebras.

Definition 8.2.5. The algebra $\hat{\mathcal{WW}}_d(q,z)$ is generated by $T_1^{\pm 1}, \ldots, T_{d-1}^{\pm 1}, E_1^{\pm 1}, \ldots, E_{d-1}^{\pm 1}$, pairwise commuting generators $W_1, W_2, \ldots$, pairwise commuting generators $X_1^{\pm 1}, \ldots, X_d^{\pm 1}$ and pairwise commuting generators $Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}$ with relations (7.1.1-7.1.2) and

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i, \tag{8.2.5}$$

$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\pm 1}E_i, \tag{8.2.6}$$
$$X_1Y_2 = Y_2X_1T^2, \tag{8.2.7}$$
$$T_iX_iT_i = X_{i+1}, \qquad T_iY_iT_i = Y_{i+1}, \qquad i = 1, \dots, d-1, \tag{8.2.8}$$
$$X_1T_i = T_iX_1, \qquad Y_1T_i = T_iY_1, \qquad \text{for } i = 2, 3, \dots, d-1, \tag{8.2.9}$$
$$X_1 \dots X_dY_i = Y_iX_1 \dots X_d, \qquad Y_1 \dots Y_dX_i = X_iY_1 \dots Y_d \qquad i = 1, \dots, d, \tag{8.2.10}$$
$$E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i, \tag{8.2.11}$$
$$E_iY_iY_{i+1} = z^{-2}E_i = Y_iY_{i+1}E_i \tag{8.2.12}$$
$$E_1(Y_1)^rE_1 = W_rE_1, \tag{8.2.13}$$
$$Y_iW_j = W_jY_i \qquad 1 \le i \le d, j \ge 1, \tag{8.2.14}$$
$$T_iW_j = W_jT_i \qquad 1 \le i \le d-1, j \ge 1, \tag{8.2.15}$$

Lemma 8.2.6. *The algebra $\mathcal{WW}_d(q,z)$ is isomorphic to the algebra $\hat{\mathcal{WW}}_d(q,z)$.*

Proof. Define an algebra homomorphism ϕ

$$\phi : \mathcal{B}_d \to \hat{\mathcal{WW}}_d(q,z) \tag{8.2.16}$$
$$X_i \mapsto X_i \qquad i = 1, \dots, d, \tag{8.2.17}$$
$$Y_i \mapsto Y_i \qquad i = 1, \dots, d, \tag{8.2.18}$$
$$T_j \mapsto T_j \qquad i = 1, \dots, d-1, \tag{8.2.19}$$
$$W_k \mapsto W_k \qquad k \ge 1. \tag{8.2.20}$$

Then by Lemma 8.2.4 and Isomorphism theorem, we have that $\mathcal{WW}_d(q,z) \cong \hat{\mathcal{WW}}_d(q,z)$.

Definition 8.2.7. The algebra $\tilde{\mathcal{WW}}_d(q,z)$ is generated by $T_1^{\pm 1}, \dots, T_{d-1}^{\pm 1}, E_1, \dots, E_{d-1}$, pairwise commuting generators $W_1, W_2, \dots$, pairwise commuting generators $X_1^{\pm 1}, \dots, X_{d-1}$, pairwise commuting generators $Y_1^{\pm 1}, \dots, Y_d^{\pm 1}$ and central element $X^{\pm 1}$ with relations (7.1.1-7.1.2) and

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i, \tag{8.2.21}$$
$$E_iT_i^{\pm 1} = T_i^{\pm 1}E_i = z^{\pm 1}E_i, \tag{8.2.22}$$
$$X_1Y_2 = Y_2X_1T^2, \tag{8.2.23}$$
$$T_iX_iT_i = X_{i+1}, \qquad i = 1, \dots, d-2, \tag{8.2.24}$$
$$T_jY_jT_j = Y_{j+1}, \qquad j = 1, \dots, d-1 \tag{8.2.25}$$
$$X_1T_i = T_iX_1, \qquad Y_1T_i = T_iY_1, \qquad \text{for } i = 2, 3, \dots, d-1, \tag{8.2.26}$$
$$T_{d-1}X_{d-1}T_{d-1} = XX_1^{-1} \dots X_{d-1}^{-1}, \tag{8.2.27}$$
$$E_iT_{i+1}^{\pm 1}E_i = z^{\pm 1}E_i, \tag{8.2.28}$$
$$E_iY_iY_{i+1} = z^{-2}E_i = Y_iY_{i+1}E_i \tag{8.2.29}$$
$$E_1(Y_1)^rE_1 = W_rE_1, \tag{8.2.30}$$
$$Y_iW_j = W_jY_i \qquad 1 \le i \le d, j \ge 1, \tag{8.2.31}$$
$$T_iW_j = W_jT_i \qquad 1 \le i \le d-1, j \ge 1. \tag{8.2.32}$$

Lemma 8.2.8. *The algebra $\hat{\mathcal{WW}}_d(q,z)$ is isomorphic to the algebra $\tilde{\mathcal{WW}}_d(q,z)$.*

Proof. Define two maps ϕ and ψ in the following way

$$\phi : \hat{\mathcal{WW}}_d(q,z) \to \tilde{\mathcal{WW}}_d(q,z) \tag{8.2.33}$$

$$X_i \mapsto X_i, \qquad Y_i \mapsto Y_i, \tag{8.2.34}$$
$$T_i \mapsto T_i, \tag{8.2.35}$$
$$E_i \mapsto E_i, \qquad i = 1, \ldots, d-1 \tag{8.2.36}$$
$$X_d \mapsto XX_1^{-1} \ldots X_{d-1}^{-1}, \qquad Y_d \mapsto Y_d,, \tag{8.2.37}$$
$$W_k \mapsto W_k \qquad k \geq 1. \tag{8.2.38}$$

$$\psi : \mathcal{W}\tilde{W}_d(q,z) \to \mathcal{W}\hat{W}_d(q,z) \tag{8.2.39}$$
$$X_i \mapsto X_i, \qquad Y_i \mapsto Y_i, \tag{8.2.40}$$
$$T_i \mapsto T_i, \tag{8.2.41}$$
$$E_i \mapsto E_i, \qquad i = 1, \ldots, d-1 \tag{8.2.42}$$
$$X \mapsto X_1 \ldots X_d \qquad Y_d \mapsto Y_d, \tag{8.2.43}$$
$$W_k \mapsto W_k \qquad k \geq 1. \tag{8.2.44}$$

It is easy to see that ϕ and ψ are algebra homomorphisms. Furthermore, since $\phi\psi = \psi\phi = \mathrm{id}$, we obtain that $\mathcal{W}\hat{W}_d(q,z) \cong \mathcal{W}\tilde{W}_d(q,z)$.

Definition 8.2.9. The algebra $\mathcal{W}\bar{W}_d(q,z)$ is generated by $T_1^{\pm 1}, \ldots, T_d^{\pm 1}, E_1^{\pm 1}, \ldots, E_d^{\pm 1}$, commuting generators $W_1, W_2, \ldots$, commuting generators $Y_1^{\pm 1}, \ldots, Y_d^{\pm}$ and a generator $\Pi^{\pm 1}$ such that Π^d is central and with relations (7.1.1-7.1.2) and

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i \tag{8.2.45}$$
$$E_i T_i^{\pm 1} = T_i^{\pm 1} E_i = z^{\pm 1} E_i \qquad i = 1, \ldots, d, \tag{8.2.46}$$
$$T_i Y_i T_i = Y_{i+1}, \qquad i = 1, \ldots, d-1, \tag{8.2.47}$$
$$T_n Y_n T_n = Y_1 \tag{8.2.48}$$
$$Y_1 T_i = T_i Y_1, \qquad \text{for } i = 2, 3, \ldots, d-1, \tag{8.2.49}$$
$$\Pi Y_1 = Y_2 \Pi. \tag{8.2.50}$$
$$\Pi T_i \Pi^{-1} = T_{i+1} \qquad i = 1, \ldots, d-1, \tag{8.2.51}$$
$$E_1 (Y_1)^r E_1 = W_r E_1, \tag{8.2.52}$$
$$Y_i W_j = W_j Y_i \qquad 1 \leq i \leq d, j \geq 1, \tag{8.2.53}$$
$$T_i W_j = W_j T_i \qquad 1 \leq i \leq d-1, j \geq 1. \tag{8.2.54}$$

Proposition 8.2.10. *We have* $\Pi Y_i = Y_{i+1}\Pi \qquad i = 1, \ldots, d-1$. *Furthermore,* $\Pi Y_n = Y_1 \Pi$.

Proof. We proceed by induction. The case $i = 1$ follows from the definition. Assume $\Pi Y_j = Y_{j+1}\Pi$ for all $j \leq i$, then we have

$$\begin{aligned}
\Pi Y_i &= Y_{i+1}\Pi, \qquad (\cdot T_{i+1} T_i) \\
T_{i+1}\Pi Y_i T_i &= T_{i+1} Y_{i+1} \Pi T_i, \\
\Pi T_i Y_i T_i &= T_{i+1} Y_{i+1} T_{i+1} \Pi \\
\Pi Y_{i+1} &= Y_{i+2} \Pi.
\end{aligned}$$

Similarly, we can obtain $\Pi Y_n = Y_1 \Pi$.

Lemma 8.2.11. *The algebra* $\mathcal{W}\tilde{W}_d(q,z)$ *is isomorphic to the algebra* $\mathcal{W}\bar{W}_d(q,z)$.

Proof. Define two maps ϕ and ψ in the following way

$$\phi : \mathcal{W}\tilde{\mathcal{W}}_d(q,z) \to \mathcal{W}\bar{\mathcal{W}}_d(q,z) \tag{8.2.55}$$

$$X_i \mapsto T_{i-1} \dots T_1 \Pi T_{d-1}^{-1} \dots T_i^{-1} \tag{8.2.56}$$

$$X \mapsto \Pi^d, \tag{8.2.57}$$

$$Y_i \mapsto Y_i \tag{8.2.58}$$

$$T_i \mapsto T_i, \tag{8.2.59}$$

$$E_i \mapsto E_i, \tag{8.2.60}$$

$$W_k \mapsto W_k \qquad k \geq 1. \tag{8.2.61}$$

$$\psi : \mathcal{W}\bar{\mathcal{W}}_d(q,z) \to \mathcal{W}\tilde{\mathcal{W}}_d(q,z) \tag{8.2.62}$$

$$\Pi \mapsto X_1 T_{d-1} \dots T_1, \tag{8.2.63}$$

$$Y_i \mapsto Y_i \tag{8.2.64}$$

$$T_i \mapsto T_i, \tag{8.2.65}$$

$$T_n \mapsto X_1 T_{d-1} \dots T_1 T_{d-1} (X_1 T_{d-1} \dots T_1)^{-1} \tag{8.2.66}$$

$$E_n \mapsto X_1 T_{d-1} \dots T_1 E_{d-1} (X_1 T_{d-1} \dots T_1)^{-1} \tag{8.2.67}$$

$$E_i \mapsto E_i, \tag{8.2.68}$$

$$W_k \mapsto W_k \qquad k \geq 1. \tag{8.2.69}$$

Similarly, check that that ϕ and ψ are algebra homomorphisms. Moreover, it is easy to see that $\phi\psi = \psi\phi = \text{id}$. Therefore, $\mathcal{W}\tilde{\mathcal{W}}_d(q,z) \cong \mathcal{W}\bar{\mathcal{W}}_d(q,z)$.

8.3 Representations of $\mathcal{WW}_d(q,z)$

Let $z := -q^{2n+1}$ and d be even.

Let $M = \oplus M^\lambda$ where M^λ is a vector space given by

$$M^\lambda = \operatorname{span}\left\{ v_T \;\middle|\; \begin{array}{c} T = (\lambda = \tau^{(0)}, \tau^{(1)}(= \tau^{(d+1)}), \tau^{(2)}, \ldots, \tau^{(d)} = \lambda) \text{ an} \\ \text{up down tableau of length } d \text{ from } \lambda \text{ to } \lambda \text{ with } l(\tau^{(i)}) \le n \end{array} \right\}$$

(so that the symbols v_T are a $\mathbb{C}$-basis of M^λ).

Theorem 8.3.1. *The action defined in the following way gives an irreducible representation of $\bar{\mathcal{WW}}_d(q,z)$ on the space M*

$$Y_i v_T = \tilde{c}(\tau^{(i)}, \tau^{(i-1)}) v_T, \qquad 1 \le i \le d,$$

$$E_j v_T = \delta_{\tau^{(j+1)},\tau^{(j-1)}} \cdot \sum_S (E_j)_{ST} v_S, \qquad \textit{and} \qquad T_j v_T = \sum_S (T_j)_{ST} v_S, \qquad 1 \le j \le d,$$

$$\Pi.v_T = v_{T'},$$

$$W_k.v_T = \omega_k^\lambda v_T,$$

where ω_k^λ defined in Theorem 7.1.12, both sums are over up-down tableaux $S = (\lambda = \tau^{(0)}, \ldots, \tau^{(i-1)}, \sigma^{(i)}, \tau^{(i+1)}, \ldots, \tau^{(d)} = \lambda)$ that are the same as T except possibly at the ith step, $v_{T'} = (\tau^{(d-1)}, \tau^{(d)} = \tau^{(0)}, \tau^{(1)}, \ldots, \tau^{(d-1)})$ and

$$(E_i)_{ST} = -1 \cdot \frac{\sqrt{D_{\tau^{(i)}} D_{\sigma^{(i)}}}}{D_{\tau^{(i-1)}}},$$

$$(T_i)_{ST} = \begin{cases} \sqrt{(q^{-1} + (T_j)_{TT})(q^{-1} + (T_j)_{SS})}, & \text{if } \tau^{(i-1)} \ne \tau^{(i+1)} \text{ and } S \ne T, \\ \left(\dfrac{q - q^{-1}}{1 - \tilde{c}(\tau^{(i+1)}, \sigma^{(i)}) \tilde{c}(\tau^{(i)}, \tau^{(i-1)})^{-1}} \right) (\delta_{ST} - (E_i)_{ST}), & \text{otherwise,} \end{cases}$$

$$\tilde{c}(\tau^{(i)}, \tau^{(i-1)}) = \begin{cases} z q^{2c(\tau^{(i)}/\tau^{(i-1)})}, & \text{if} \tau^{(i)} \supset \tau^{(i-1)}, \\ z^{-1} q^{-2c(\tau^{(i-1)}/\tau^{(i)})}, & \text{if} \tau^{(i)} \subset \tau^{(i-1)}. \end{cases} \tag{8.3.1}$$

Proof. Consider the subalgebra of $\bar{\mathcal{WW}}_d(q,z)$ generated by T_i, Y_j, W_k for $1 \le i \le d-1, 1 \le j \le d, k \ge 1$. Then it follows from Lemma 8.2.3 and Theorem 7.1.19, that it is enough to check only the relations $\bar{\mathcal{WW}}_d(q,z)$ involving T_n, E_n and Π.

Let us check the relation (8.2.50), hence need to show that $\Pi Y_1.v_T = Y_2 \Pi.v_T$. Indeed,

$$\Pi.Y_1.v_T = c(\tau^{(1)}, \tau^{(0)}) \Pi.v_T = c(\tau^{(1)}, \tau^{(0)}).v_{T'} = Y_2 \Pi.v_T.$$

Similarly, we check the other relations.

Now we have to show that it defines an irreducible representation. Since for any λ, the subalgebra generated by T_i, Y_j, W_k acts irreducibly on M^λ, it is enough to show that for any λ_1, λ_2 there exist (non-unique) elements $v^{\lambda_1} \in M^{\lambda_1}, v^{\lambda_2} \in M^{\lambda_2}$ and $a_{\lambda_1,\lambda_2}, a_{\lambda_2,\lambda_1} \in \bar{\mathcal{WW}}_2(q,z)$ such that $a_{\lambda_1,\lambda_2}.v^{\lambda_1} = v^{\lambda_2}$ and $a_{\lambda_2,\lambda_1}.v^{\lambda_2} = v^{\lambda_1}$. Without loss of generality, we can assume that $\lambda_2 = \lambda_1 \cup \alpha$, where $\alpha \in A(\lambda_1)$. Set $v_{\lambda_1} := v_{(\lambda_1,\lambda_2,\lambda_1,\lambda_2,\ldots)} \in M_{\lambda_1}$ and $v_{\lambda_2} := v_{(\lambda_2,\lambda_1,\lambda_2,\lambda_1,\ldots)} \in M_{\lambda_2}$. Then we have $\Pi.v_{\lambda_1} = v_{\lambda_2}$ and $\Pi^{-1}.v_{\lambda_2} = v_{\lambda_1}$. Hence we can set $a_{\lambda_1,\lambda_2} := \Pi$ and $a_{\lambda_2,\lambda_1} := \Pi^{-1}$.

Corollary 8.3.2. *Assume $d = 2$. Let $N = \oplus N^\lambda$ where*

$$N^\lambda = \text{span}\{\, v_S \mid S \textit{ is an marked extended RSD of shape } \lambda\}.$$

Using Lemma 8.1.18 we can identify this vector space with a vector space $M = \oplus M^\lambda$, and hence we can construct a representation of $\mathcal{W}\bar{\mathcal{W}}_2(q,z)$ on the space N^λ.

Definition 8.3.3. Let $T = (\lambda = \tau^{(0)}, \tau^{(1)}(= \tau^{(d+1)}), \tau^{(2)}, \ldots, \tau^{(d)} = \lambda)$ be an up-down tableau. Denote by $\bar{u}_T$ a weight $(u_1, \ldots, u_d)$ for $\mathcal{Y}$ such that $u_i = \tilde{c}(\tau^{(i)}, \tau^{(i-1)})$.

Theorem 8.3.4. *Let V be the natural representation for UThen the operators X_i, Y_i and T_i defined in Section 7.2 give a representation of $\mathcal{WW}_d(q,z)$ with parameters q and z on the space $W = \mathrm{Hom}_U(\mathbf{1}, O_q\mathfrak{g} \otimes V^{\otimes d})$. Furthermore,this representation isomorphic to M^λ.*

Proof. From Theorem 7.2.5 we have $\mathrm{Hom}_U(\mathbf{1}, O_q(\mathfrak{g}) \otimes V^{\otimes n}) \cong \bigoplus_{\lambda\in\Lambda} W^\lambda \cong \bigoplus_{\lambda\in\Lambda} M^{\lambda/\lambda} \cong \bigoplus_{\lambda\in\Lambda} H^{\lambda/\lambda}$. Hence, up to a constant we can choose a unique basis in each W^λ, which can labeled by up-down tableaux of the shape λ/λ. Moreover, it follows from Proposition 7.1.15 that for each up-down tableau S of the shape λ/λ, the space $W^\lambda_{\bar{u}_s}$ is one-dimensional. From the relations of $\mathcal{W}\bar{\mathcal{W}}_d(q,z)$, we have that for any up-down tableau S of the shape λ/λ $\Pi(W^\lambda_{\bar{u}_s}) \subset M_{\Pi(\bar{u}_s)}$ (see Definition 7.1.14).

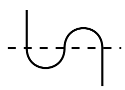
Furthermore, for any partition λ we have $T_i(W^\lambda) = W^\lambda$ and from Lemma 7.2.4 we obtain $\Pi(W^\lambda) \subset \oplus_{\bar{\lambda}} W^{\bar{\lambda}}$. Therefore, from Proposition 8.1.14 we conclude $\Pi(W^\lambda_{\bar{u}_s}) = W^\lambda_{\Pi(\bar{u})_s}$.

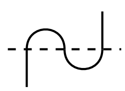
Chapter 9

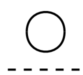
Appendix

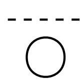
9.1 Appendix A

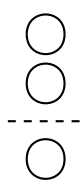
In this appendix we study the relations between $m_k(\mathbf{i})$, $m_{k+1}(\mathbf{i})$, $m_k(\mathbf{i}\cdot s_k)$ and $m_{k+1}(\mathbf{i}\cdot s_k)$ in case $|i_k - i_{k+1}| = 1$. We remind the readers that the result of this appendix can be also used to illustrate the relations between $m_k(\mathbf{i})$ and $m_{k+1}(\mathbf{i})$ in case $i_k + i_{k+1} = 0$ and $\deg_k(\mathbf{i}) = 1$.

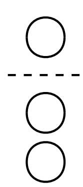
(1)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

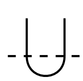
(2)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

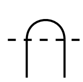
(3)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(4)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

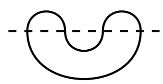
(5)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

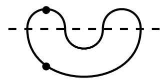
(6)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

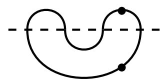
(7)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

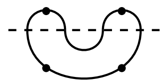
(8)  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(9) (a) when k-pattern of $\mathbf{i}$ is on the left arc, and k-pattern of $\mathbf{i}\cdot s_k$ is on the inside arc:

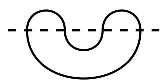
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

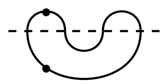
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

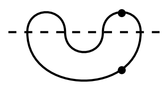
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

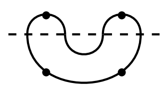
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

(b) when k-pattern of $\mathbf{i}$ is on the left arc, and $k+1$-pattern of $\mathbf{i}\cdot s_k$ is on the inside arc:

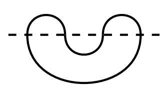
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

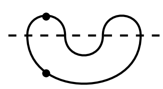
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

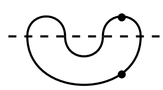
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

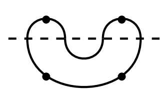
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

(c) when $k+1$-pattern of $\mathbf{i}$ is on the left arc, and k-pattern of $\mathbf{i}\cdot s_k$ is on the inside arc:

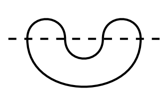
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

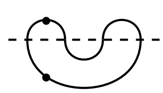
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

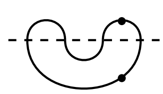
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

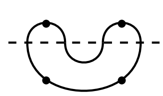
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1.$

(d) when $k+1$-pattern of $\mathbf{i}$ is on the left arc, and $k+1$-pattern of $\mathbf{i}\cdot s_k$ is on the inside arc:

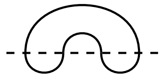
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

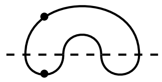
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

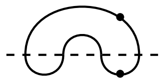
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

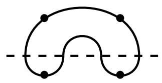
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k).$

(10) (a) when k-pattern of $\mathbf{i}$ is on the inside arc, and k-pattern of $\mathbf{i}\cdot s_k$ is on the left arc:

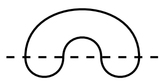
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

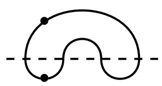
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

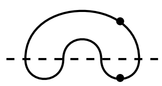
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

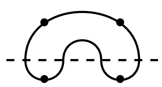
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

(b) when k-pattern of $\mathbf{i}$ is on the inside arc, and $k+1$-pattern of $\mathbf{i}{\cdot}s_k$ is on the left arc:

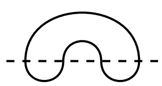
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

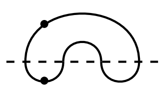
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

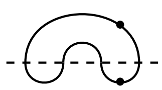
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

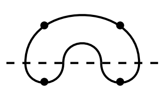
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

(c) when $k+1$-pattern of $\mathbf{i}$ is on the inside arc, and k-pattern of $\mathbf{i}{\cdot}s_k$ is on the left arc:

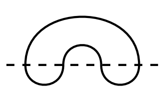
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

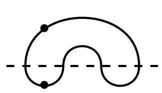
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

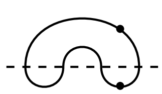
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

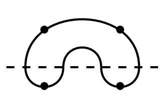
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k) + 1.$

(d) when $k+1$-pattern of $\mathbf{i}$ is on the inside arc, and $k+1$-pattern of $\mathbf{i}{\cdot}s_k$ is on the left arc:

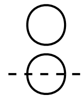
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

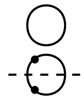
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

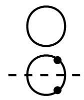
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

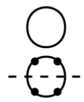
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}{\cdot}s_k) = m_{k+1}(\mathbf{i}{\cdot}s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}{\cdot}s_k).$

(11) (a) when k is on the floating loop:

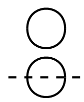
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

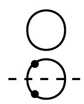
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

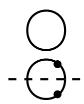
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

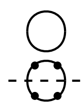
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

(b) when $k + 1$ is on the floating loop:

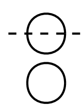
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

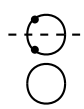
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

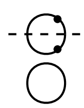
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

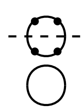
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

(12) (a) when k is on the floating loop:

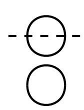
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

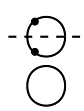
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

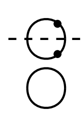
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

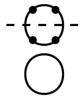
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

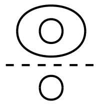
(b) when $k + 1$ is on the floating loop:

i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

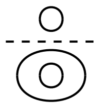
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

(13) 

(a) when k is on the inside loop: $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$.

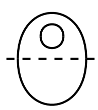
(b) when $k + 1$ is on the inside loop: $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

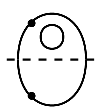
(14) 

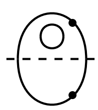
(a) when k is on the inside loop: $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$.

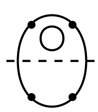
(b) when $k + 1$ is on the inside loop: $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(15) (a) when k is on the inside loop:

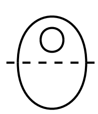
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

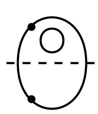
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

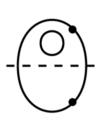
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

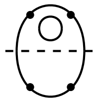
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(b) when $k + 1$ is on the inside loop:

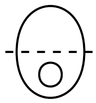
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

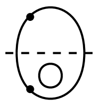
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}) + 1, m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

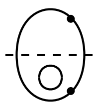
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

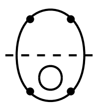
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(16) (a) when k is on the inside loop:

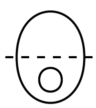
i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

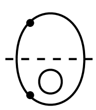
ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k) + 1$

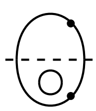
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

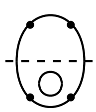
iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

(b) when $k + 1$ is on the inside loop:

i.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

ii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k) + 1, m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$

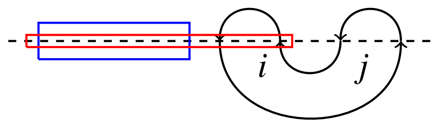
iii.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

iv.  : $m_k(\mathbf{i}) = m_{k+1}(\mathbf{i}), m_k(\mathbf{i}\cdot s_k) = m_{k+1}(\mathbf{i}\cdot s_k), m_k(\mathbf{i}) = m_k(\mathbf{i}\cdot s_k)$.

9.2 Appendix B

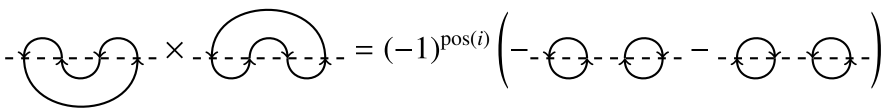
In this appendix we go through the necessary graphical calculations for $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})$. We remind the readers that these calculations can be also used to calculate $\hat{e}(\mathbf{i})\hat{\psi}_k^2$ and $\hat{e}(\mathbf{i}')\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k\hat{e}(\mathbf{i}')$ with minor modifications on the ± signs.

To explain the following calculations, let us give one example before we start. Consider the cup-cap diagram of $\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ that is topologically the same as

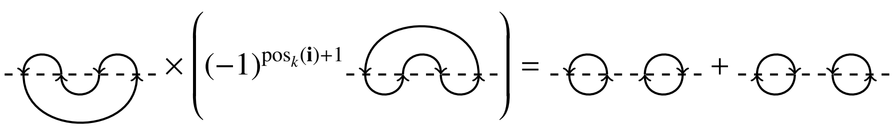


where k is on the left arc. The number of $\wedge$ and $\vee$'s in the blue box is $\mathrm{pos}_k(\mathbf{i})$ and the number of $\wedge$ and $\vee$'s in the red box is $\mathrm{pos}(i)$. Hence we have $\mathrm{pos}(i) = \mathrm{pos}_k(\mathbf{i}) + 2$.

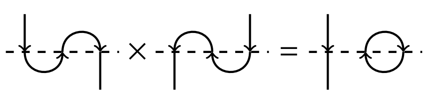
Then when we do multiplications, we have

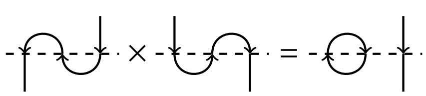


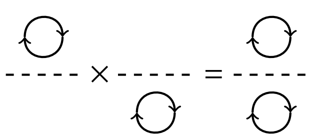
which implies

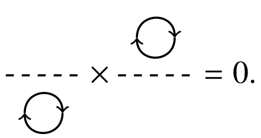


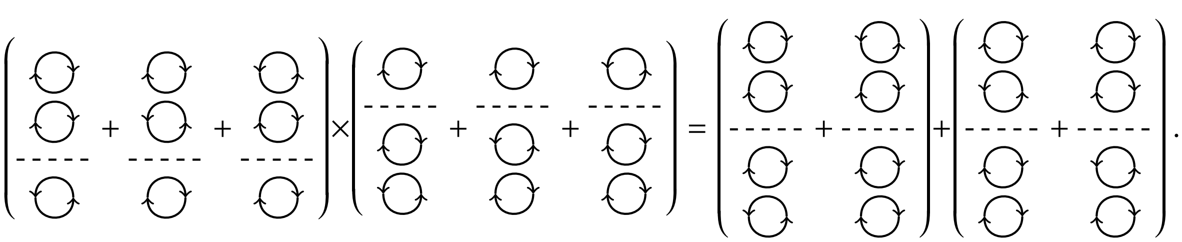
since $\mathrm{pos}(i) = \mathrm{pos}_k(\mathbf{i}) + 2$.

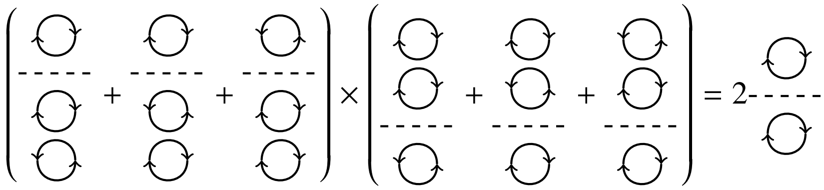
(1) 

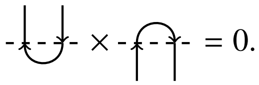
(2) 

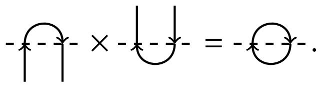
(3) 

(4) 

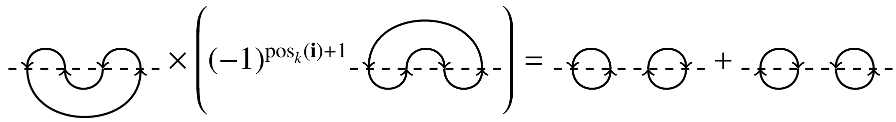
(5) 

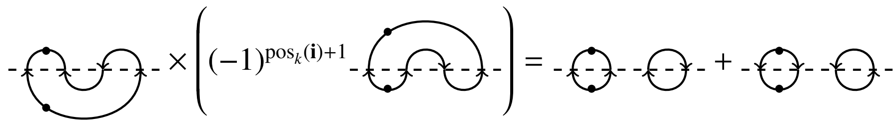
(6) 

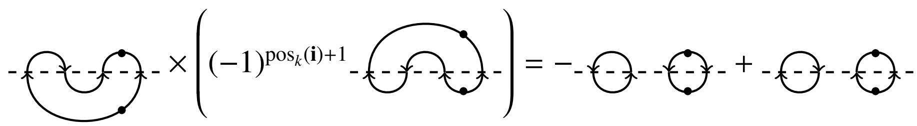
(7) 

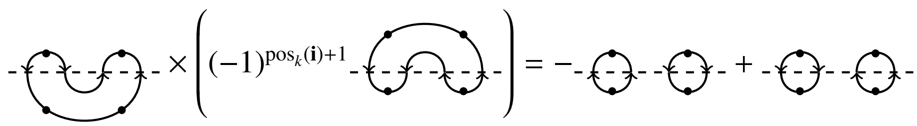
(8) 

(9) (a) when k-pattern in $\mathbf{i}$ is on the left arc:

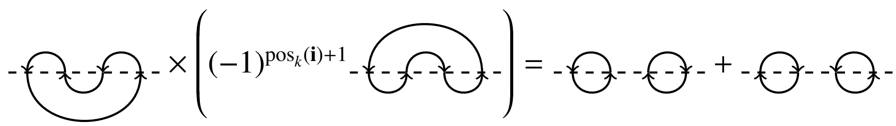
i. 

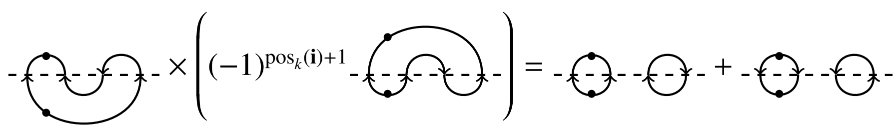
ii. 

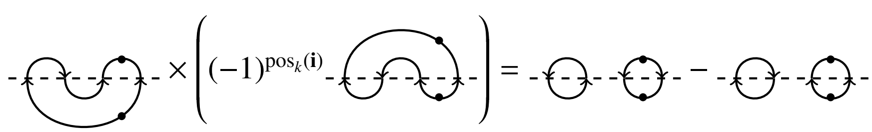
iii. 

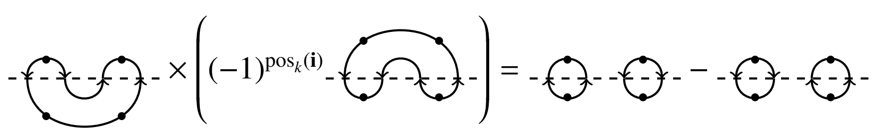
iv. 

(b) when $k+1$-pattern of $\mathbf{i}$ is on the left arc:

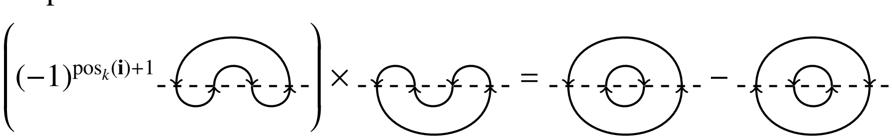
i. 

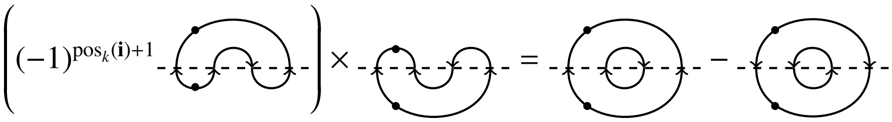
ii. 

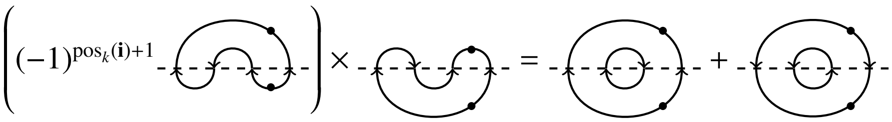
iii. 

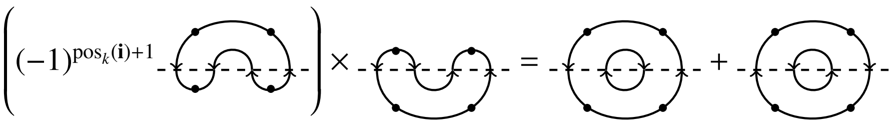
iv. 

(10) (a) when k-pattern of $\mathbf{i}$ is on the inside arc

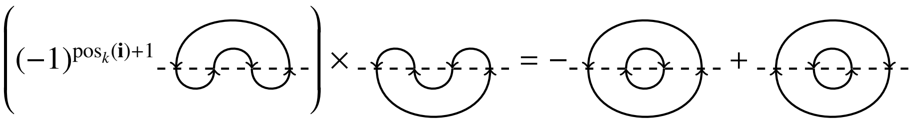
i. 

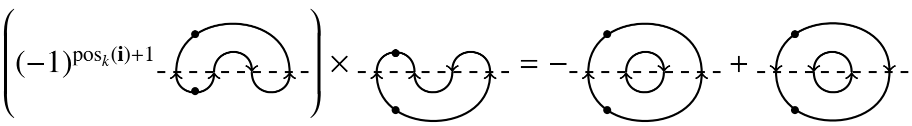
ii. 

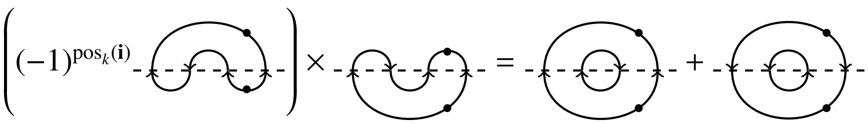
iii. 

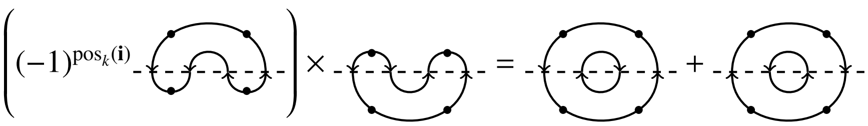
iv. 

(b) when $k+1$-pattern of $\mathbf{i}$ is on the inside arc

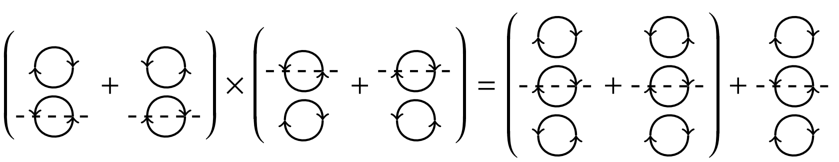
i. 

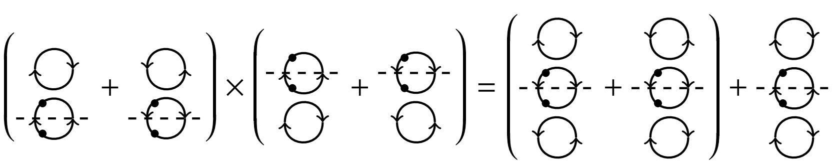
ii. 

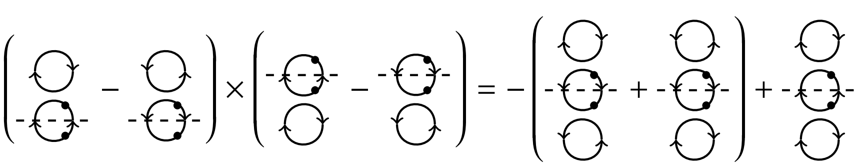
iii. 

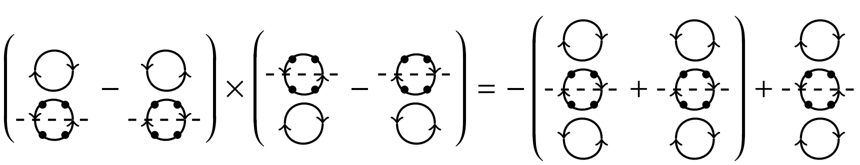
iv. 

(11) (a) when k is in the floating loop:

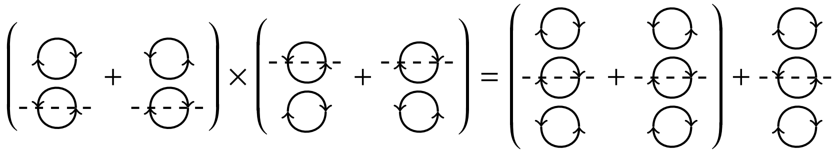
i. 

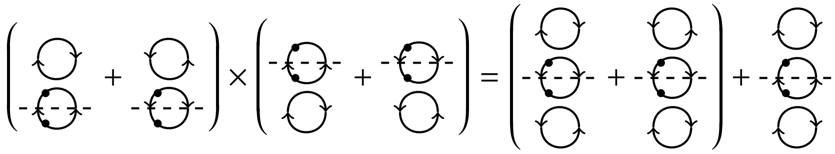
ii. 

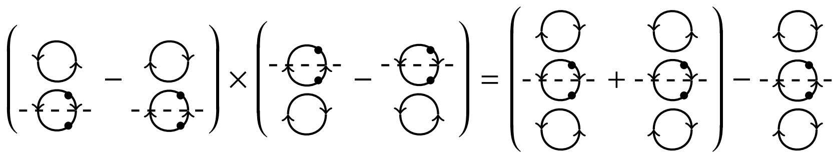
iii. 

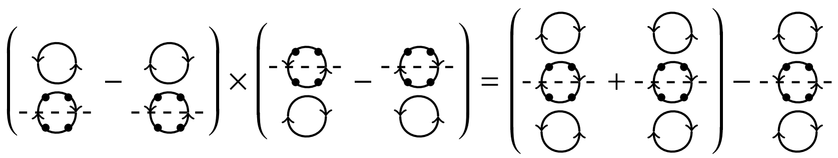
iv. 

(b) when $k+1$ is in the floating loop:

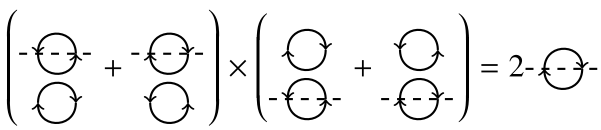
i. 

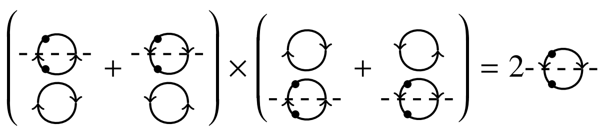
ii. 

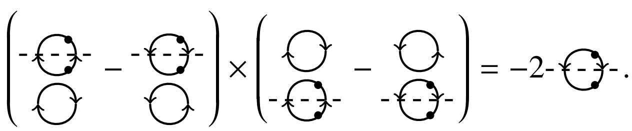
iii. 

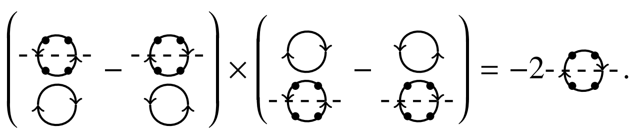
iv. 

(12) (a) when k is on the floating loop:

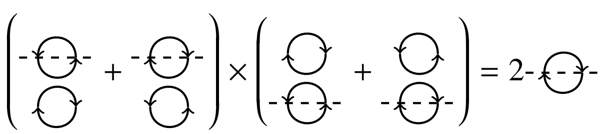
i. 

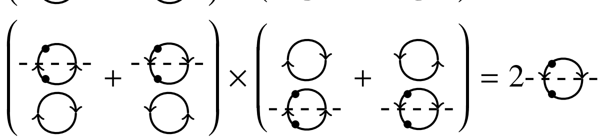
ii. 

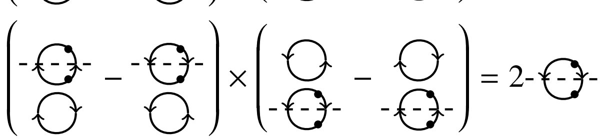
iii. 

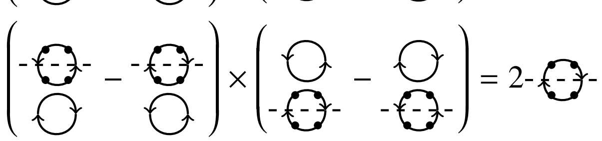
iv. 

(b) when $k+1$ is on the floating loop:

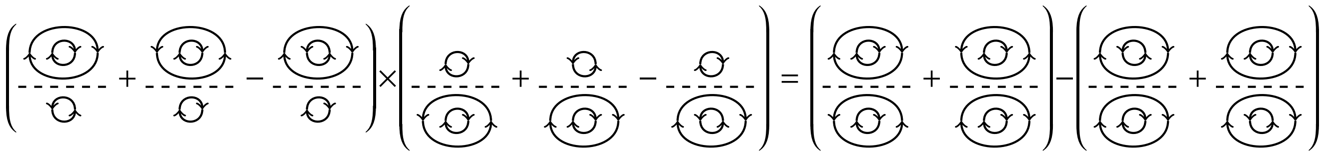
i. 

ii. 

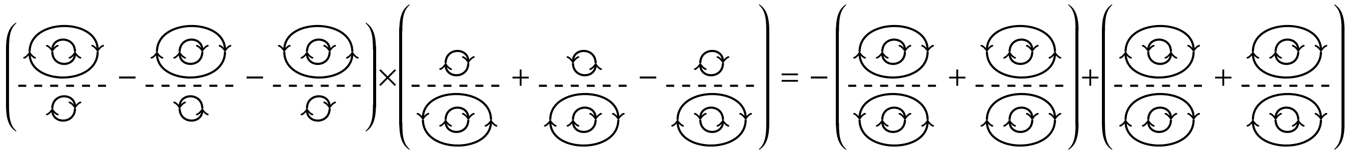
iii. 

iv. 

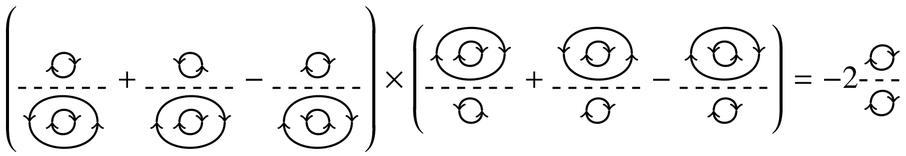
(13) (a) k is on the inside loop:



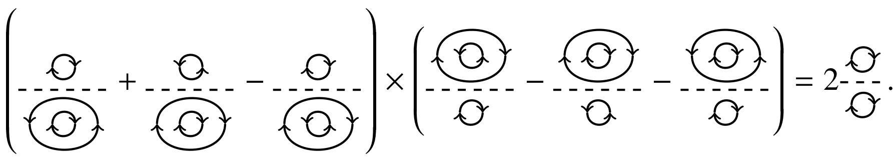
(b) $k+1$ is on the inside loop:



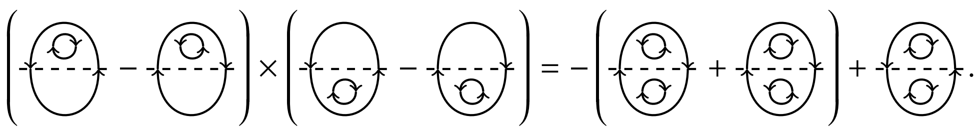
(14) (a) k is on the inside loop:

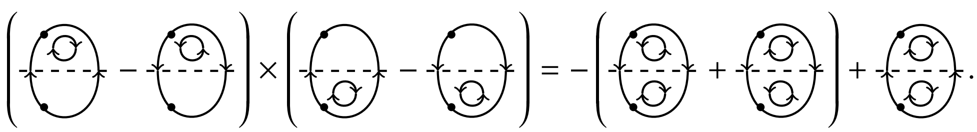


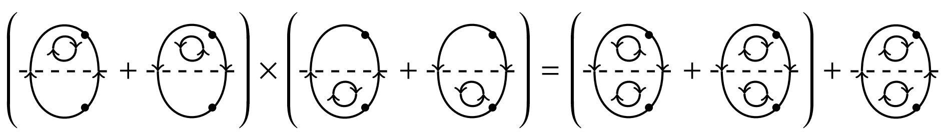
(b) $k+1$ is on the inside loop:

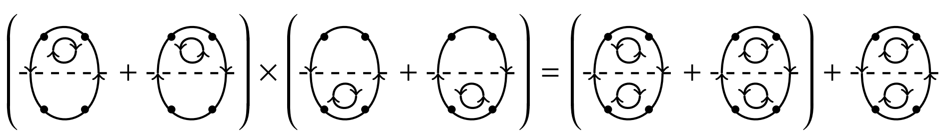


(15) (a) k is on the inside loop:

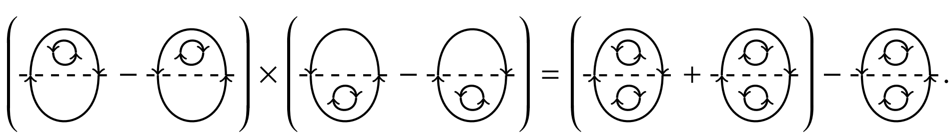
i. 

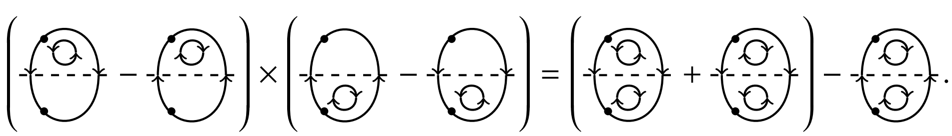
ii. 

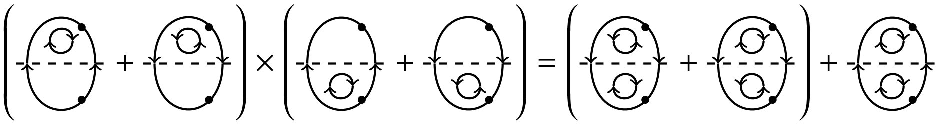
iii. 

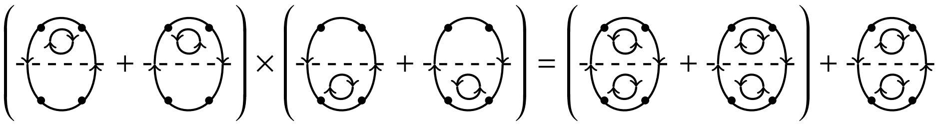
iv. 

(b) $k+1$ is on the inside loop:

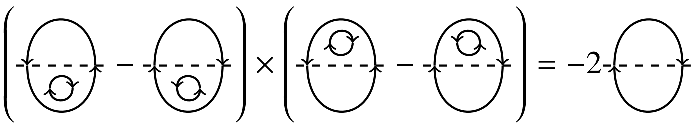
i. 

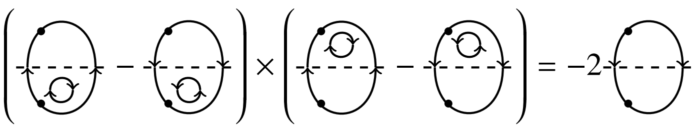
ii. 

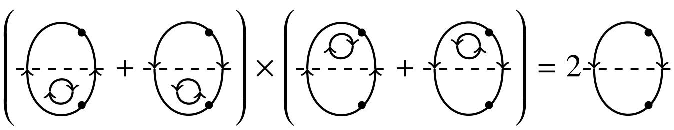
iii. 

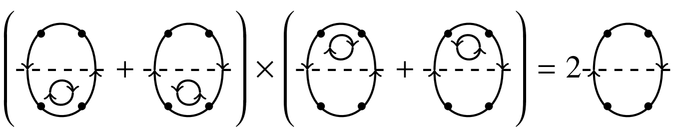
iv. 

(16) (a) k is on the inside loop:

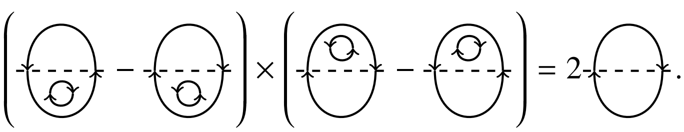
i. 

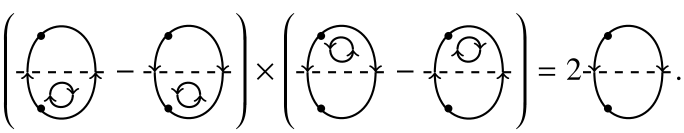
ii. 

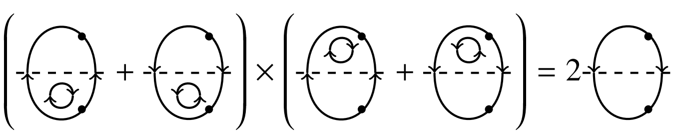
iii. 

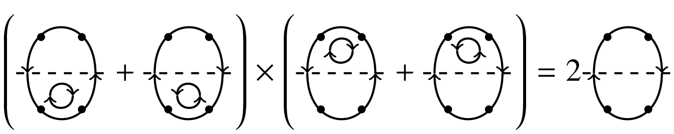
iv. 

(b) $k + 1$ is on the inside loop:

i. 

ii. 

iii. 

iv. 

9.3 Appendix C

In this appendix we go through the graphical calculations required for $\hat{y}_k\hat{e}(\mathbf{i})\hat{\psi}_k$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})\hat{\psi}_k$. These calculations can be used directly (with minor modifications on $\pm$ signs) when we calculate $\hat{y}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$. Please refer to Remark 6.1.21 and Remark 6.1.29 for the existences of $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ (i.e. when they are nonzero) and whether they are the same or different.

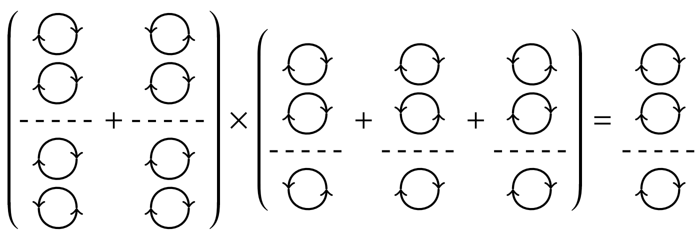
(1) the product is 0.

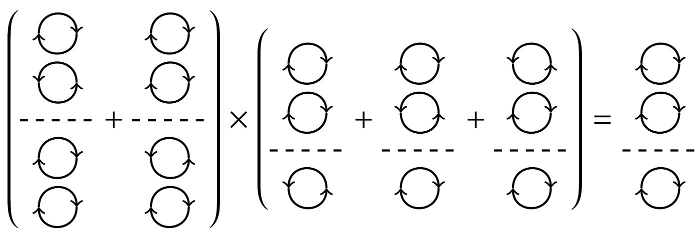
(2) the product is 0.

(3) the product is 0.

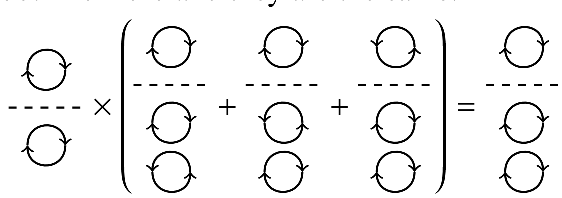
(4) the product is 0.

(5) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.





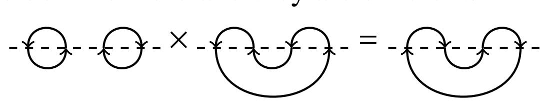
(6) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.

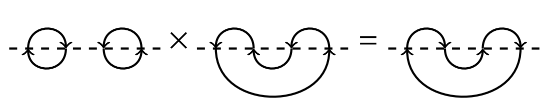


(7) the product is 0.

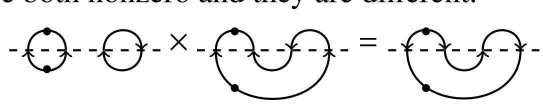
(8) the product is 0.

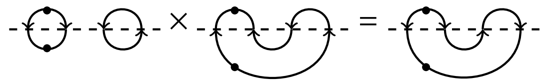
(9) (a) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



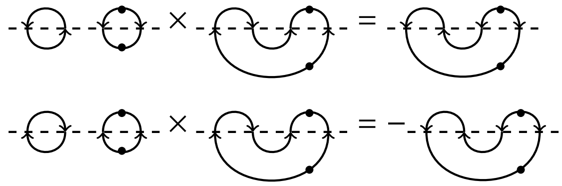


(b) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.

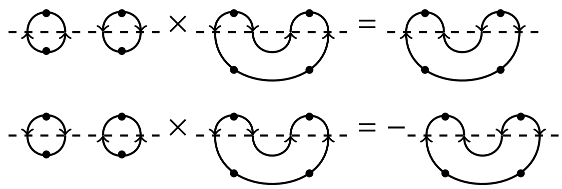




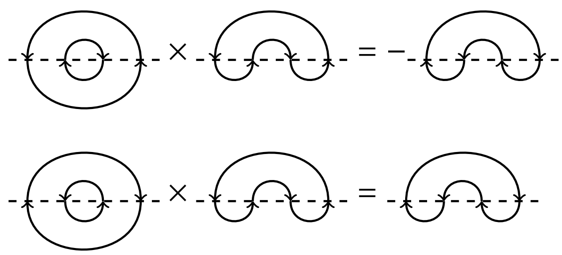
(c) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



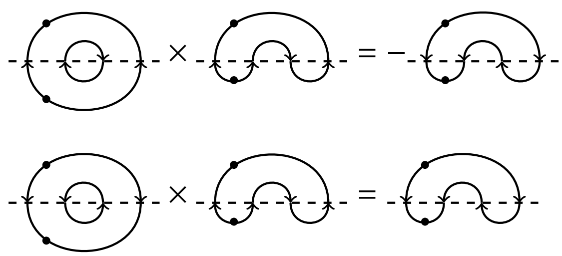
(d) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



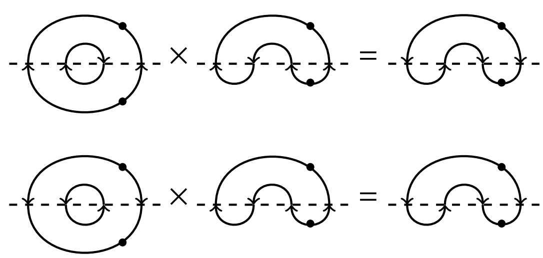
(10) (a) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



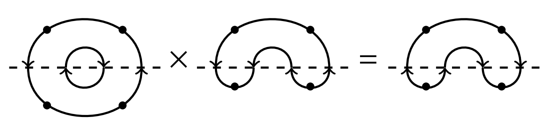
(b) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.

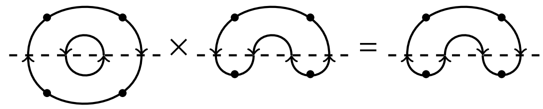


(c) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



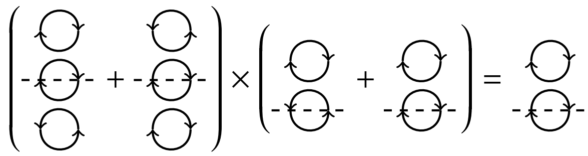
(d) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.

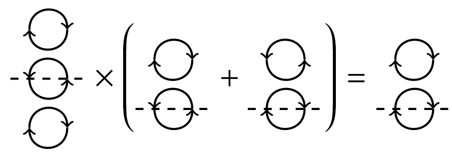




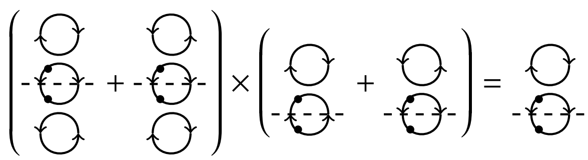
(11) (a) when k-pattern of $\mathbf{i}$ is on the floating loop:

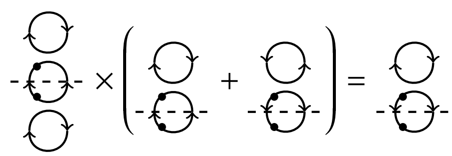
i. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



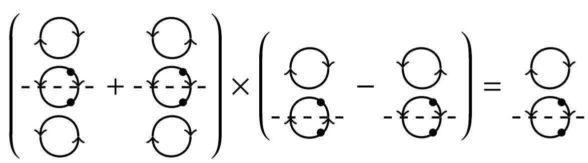


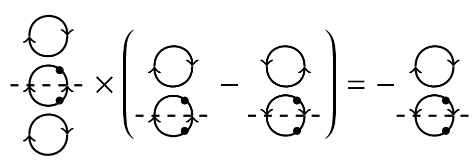
ii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



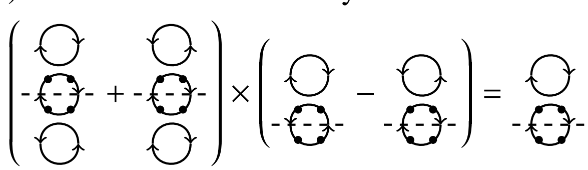


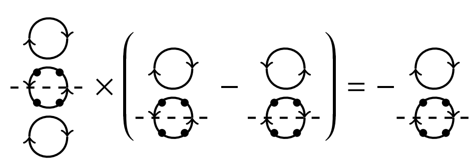
iii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.





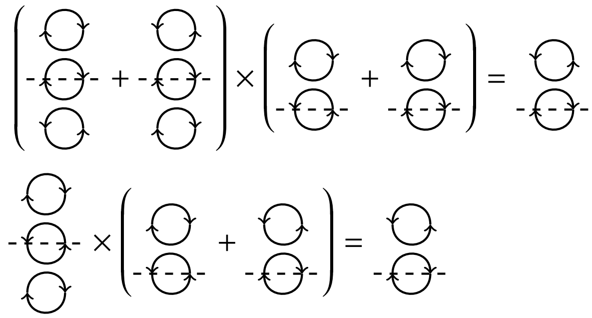
iv. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



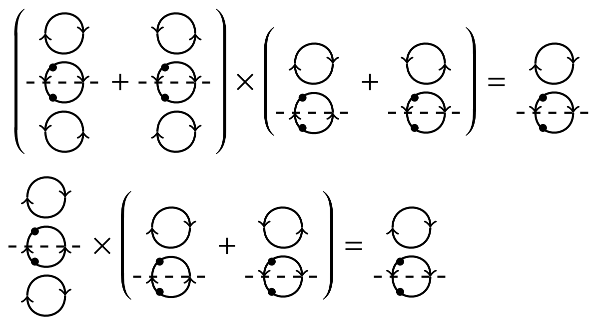


(b) when $k+1$-pattern of $\mathbf{i}$ is on the floating loop:

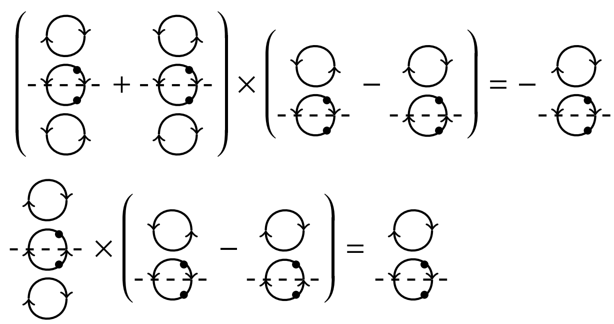
i. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



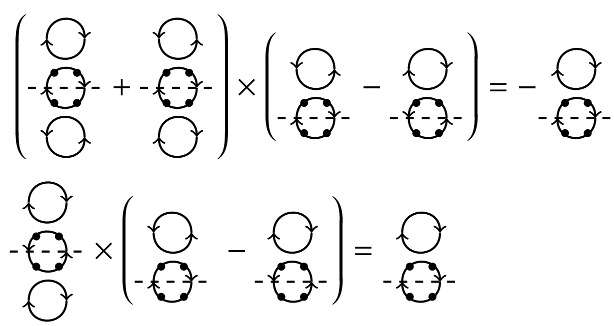
ii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



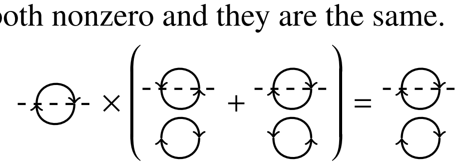
iii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



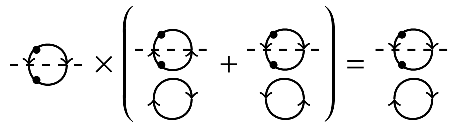
iv. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



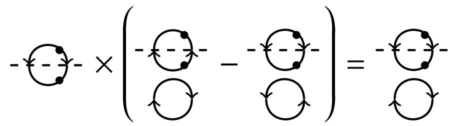
(12) (a) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



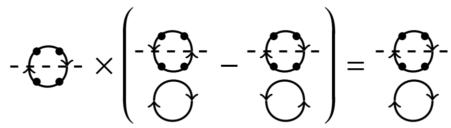
(b) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



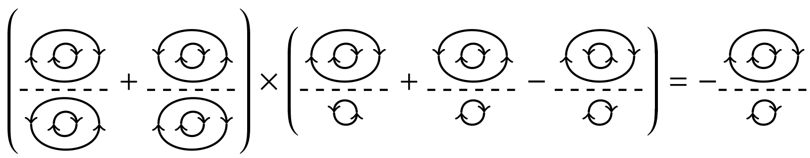
(c) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.

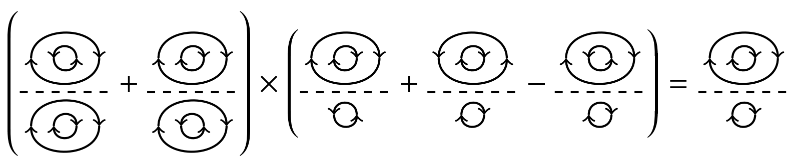


(d) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.

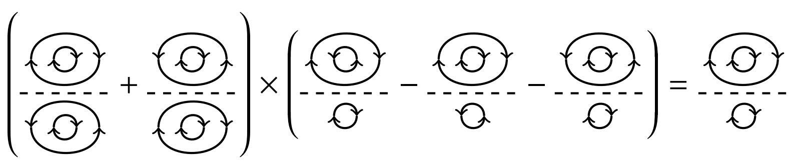


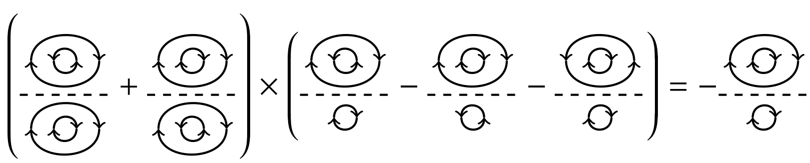
(13) (a) when k-pattern of $\mathbf{i}$ is on the inside loop: $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



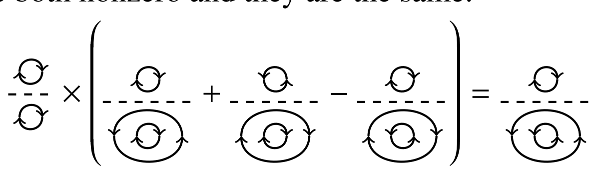


(b) when $k+1$-pattern of $\mathbf{i}$ is on the inside loop: $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



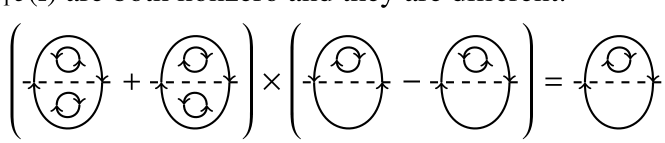


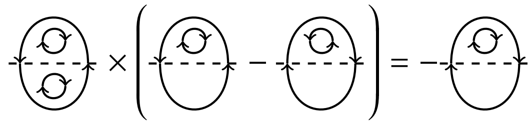
(14) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



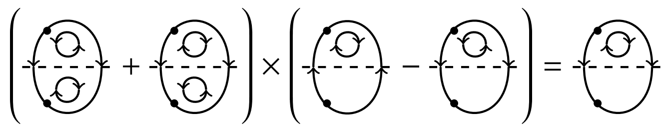
(15) (a) k-pattern of $\mathbf{i}$ is on the inside loop:

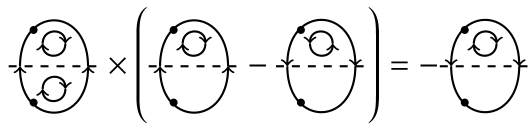
i. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



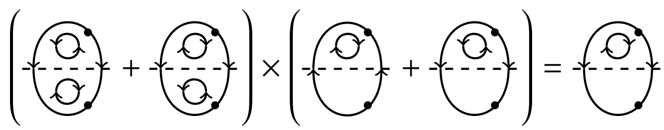


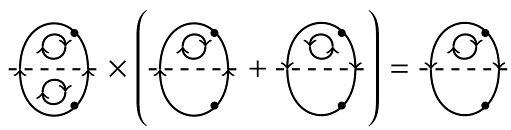
ii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



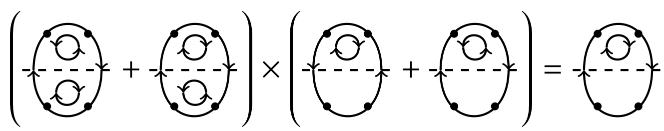


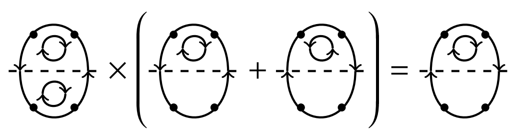
iii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.





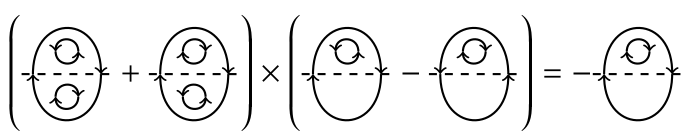
iv. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.

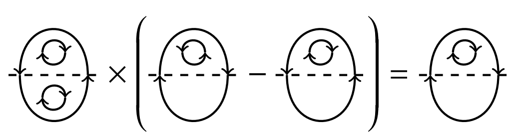




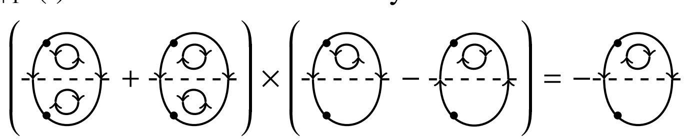
(b) $k+1$-pattern of $\mathbf{i}$ is on the inside loop:

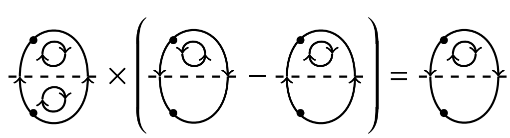
i. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



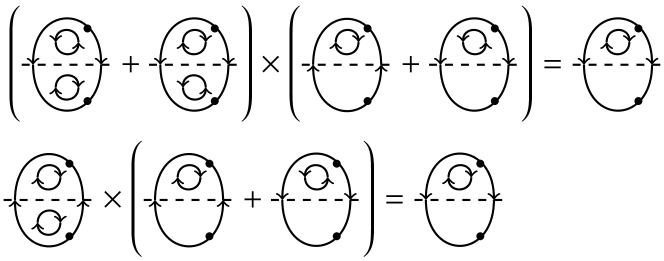


ii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.

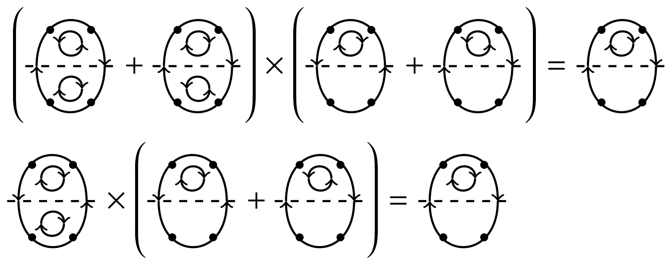




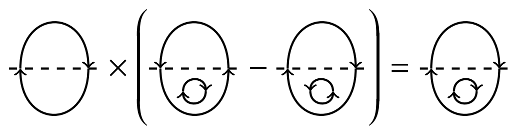
iii. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



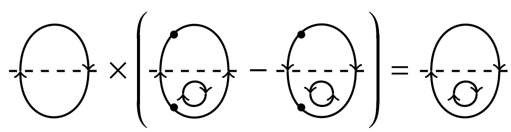
iv. $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are different.



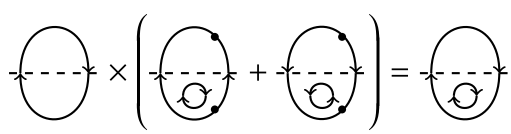
(16) (a) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



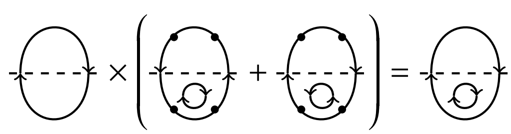
(b) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



(c) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



(d) $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ are both nonzero and they are the same.



9.4 Appendix D

In this appendix we go through the graphical calculations required for $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_k$ and $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{y}_{k+1}$. These calculations can be used directly with minor modifications on ± signs when we calculate $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\hat{y}_k$ and $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\hat{y}_{k+1}$. Please refer to Remark 6.1.21 and Remark 6.1.29 for the existences of $\hat{y}_k\hat{e}(\mathbf{i})$ and $\hat{y}_{k+1}\hat{e}(\mathbf{i})$ (i.e. when they are nonzero) and whether they are the same or different.

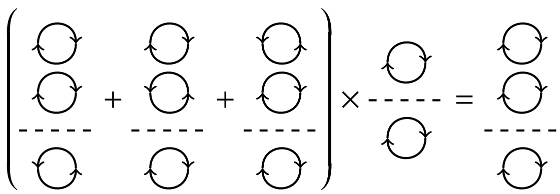
(1) the product is 0.

(2) the product is 0.

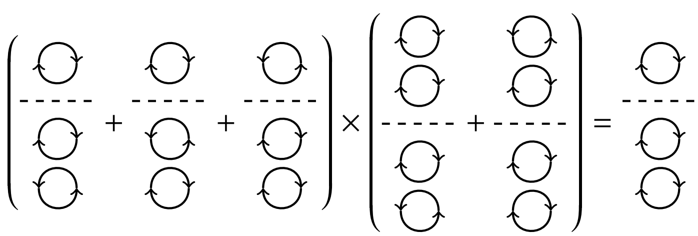
(3) the product is 0.

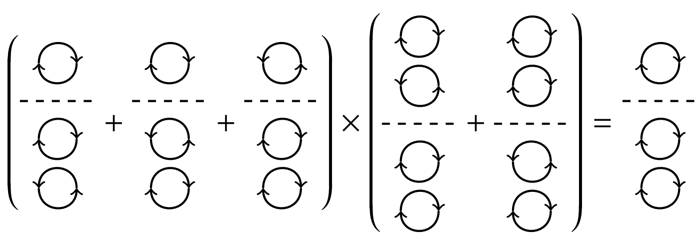
(4) the product is 0.

(5) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



(6) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

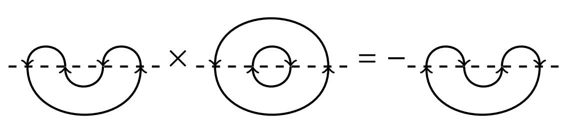


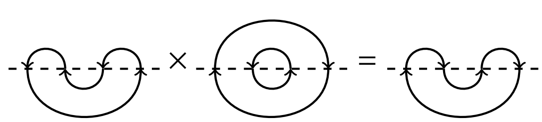


(7) the product is 0.

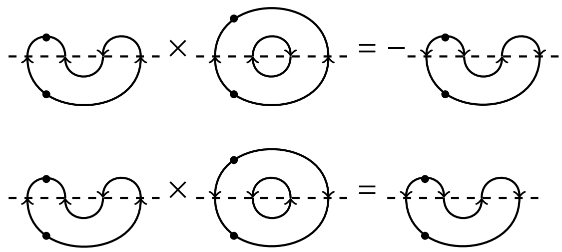
(8) the product is 0.

(9) (a) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

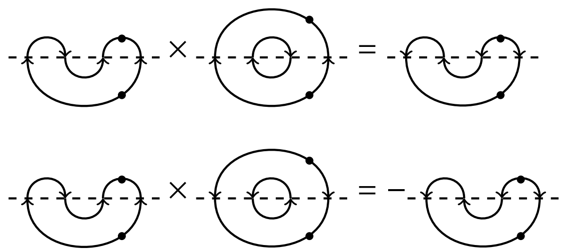




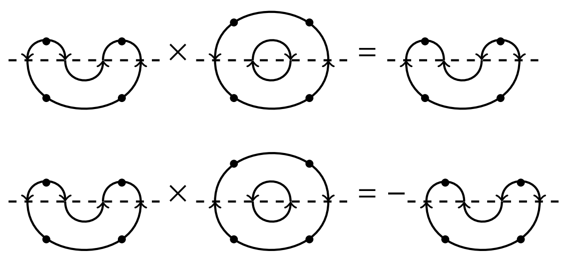
(b) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



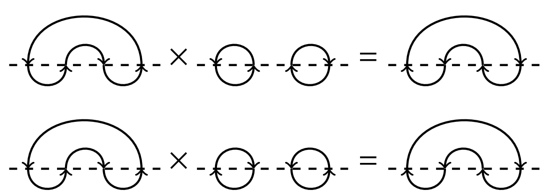
(c) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



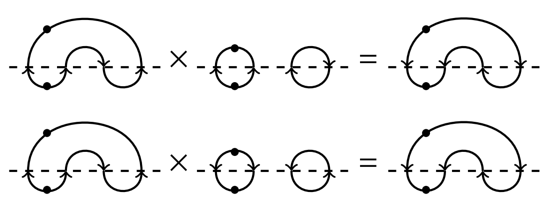
(d) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



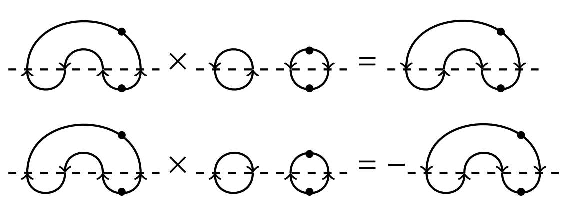
(10) (a) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



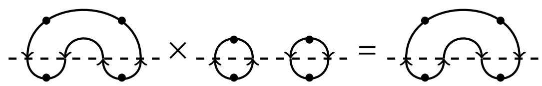
(b) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

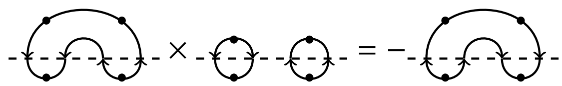


(c) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



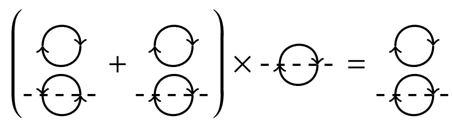
(d) $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



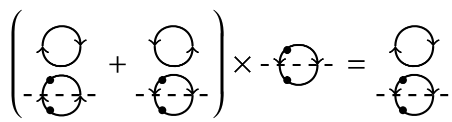


(11) (a) when k-pattern of $\mathbf{i}$ is on the floating loop:

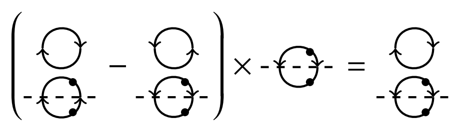
i. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



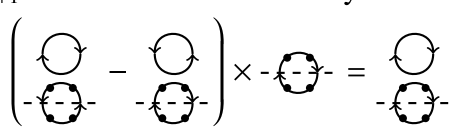
ii. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



iii. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

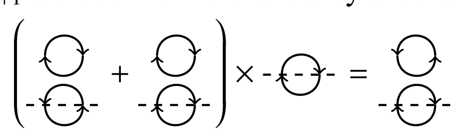


iv. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

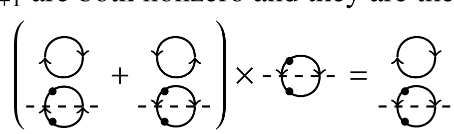


(b) when $k+1$-pattern of $\mathbf{i}$ is on the floating loop:

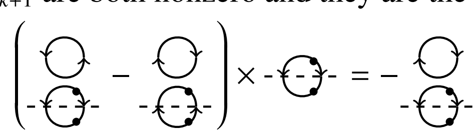
i. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



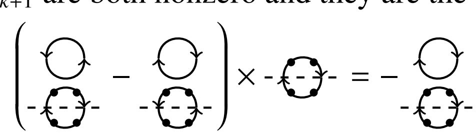
ii. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



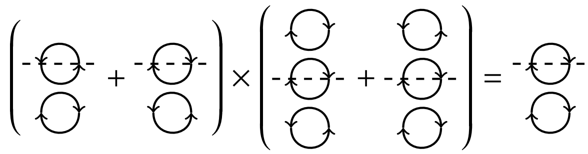
iii. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

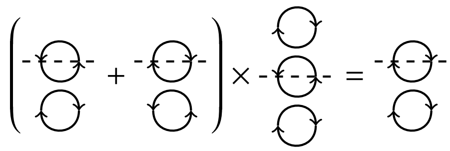


iv. $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}{\cdot}s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

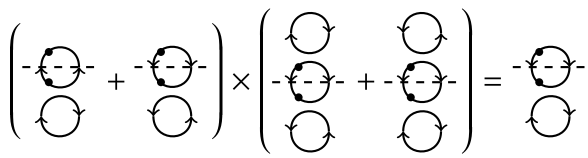


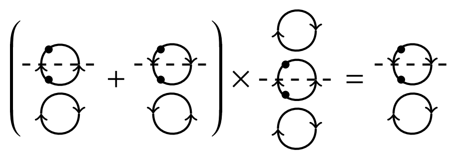
(12) (a) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



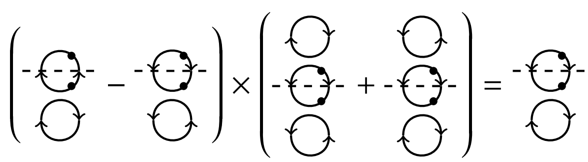


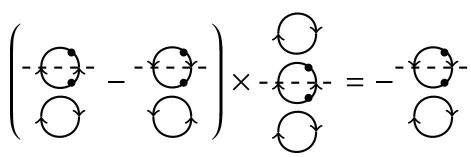
(b) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



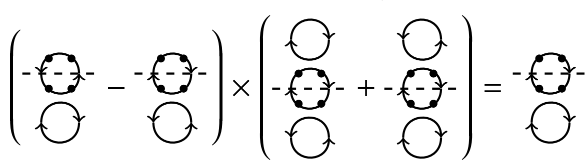


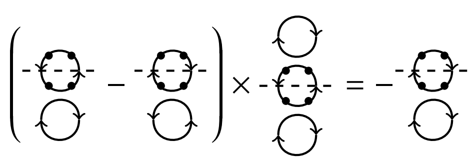
(c) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



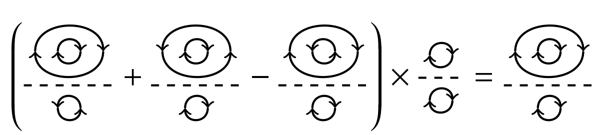


(d) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

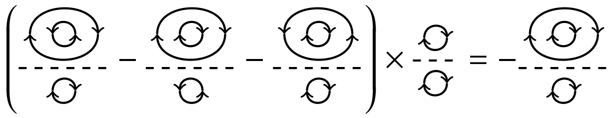




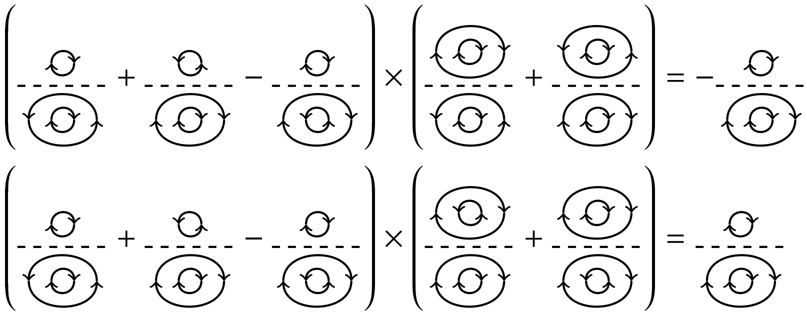
(13) (a) when k-pattern of $\mathbf{i}$ is on the inside loop: $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



(b) when $k+1$-pattern of $\mathbf{i}$ is on the inside loop: $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

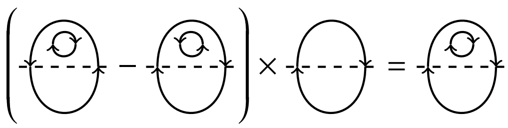


(14) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

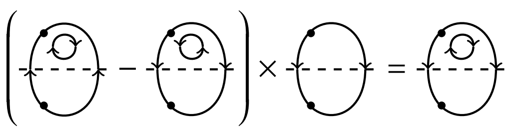


(15) (a) k-pattern of $\mathbf{i}$ is on the inside loop:

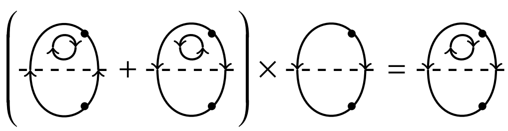
i. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



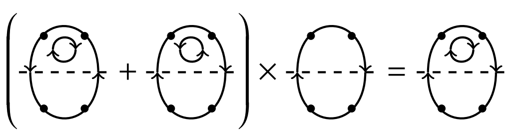
ii. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



iii. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

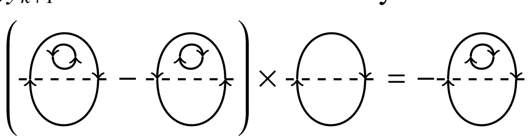


iv. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

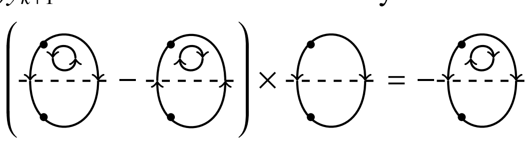


(b) $k+1$-pattern of $\mathbf{i}$ is on the inside loop:

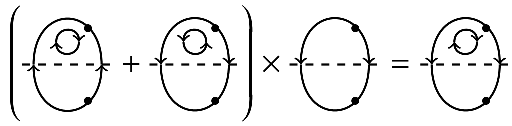
i. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



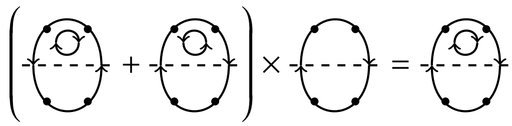
ii. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.



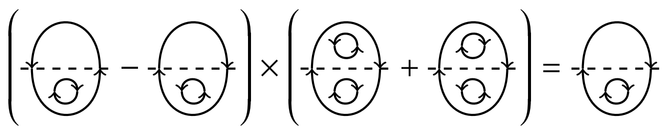
iii. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

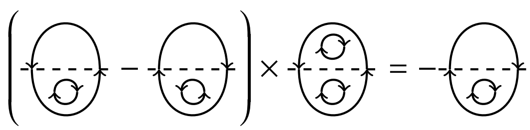


iv. $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are the same.

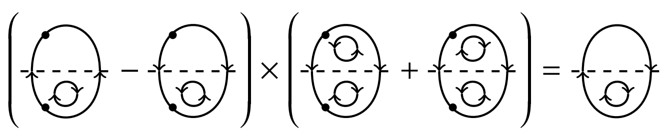


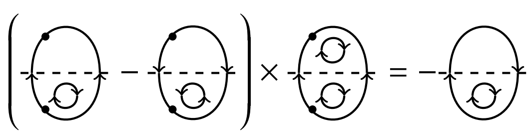
(16) (a) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



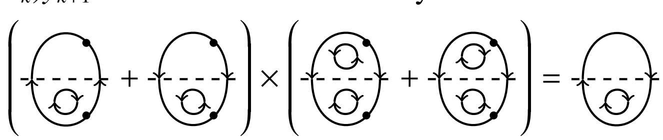


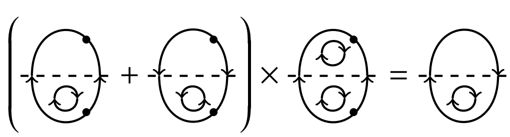
(b) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.



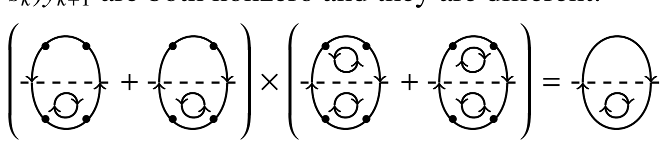


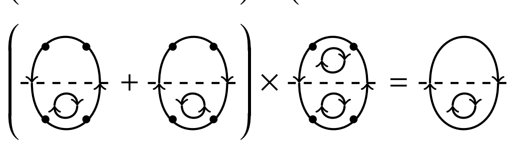
(c) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.





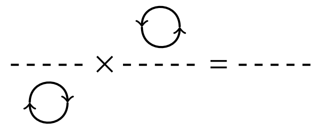
(d) $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_k$ and $\hat{e}(\mathbf{i}\cdot s_k)\hat{y}_{k+1}$ are both nonzero and they are different.

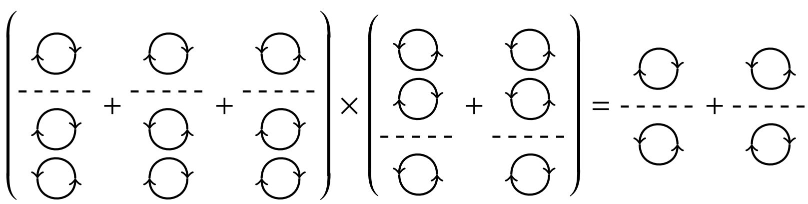


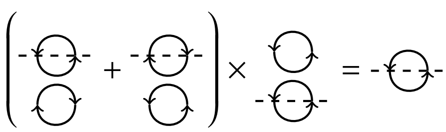


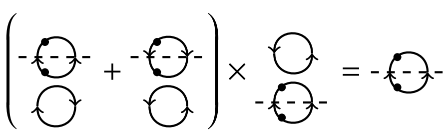
9.5 Appendix E

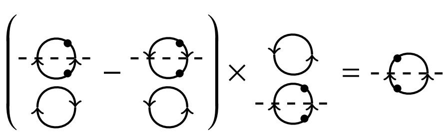
In this appendix we go through the calculations of $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1}$ when $\deg_k \mathbf{i} = 1$. The calculations in this appendix can be used to calculate $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1}$ when $\deg_k \mathbf{i} = -1$, and $\overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ when $\hat{e}(\mathbf{i}) = \pm 1$ with minor modifications with $\pm$ signs. We also want to remind the readers that the calculations in this appendix can be extended to calculate $\overrightarrow{\epsilon}_k\hat{e}(\mathbf{i})\hat{\psi}_{k-1}$ and $\hat{\psi}_{k-1}\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_k$ when $i_{k-1} = i_k$; $\overrightarrow{\epsilon}_{k-1}\hat{e}(\mathbf{i})\hat{\psi}_k$ and $\hat{\psi}_k\hat{e}(\mathbf{i})\overleftarrow{\epsilon}_{k-1}$ when $i_k = i_{k+1}$; and $\hat{e}(\mathbf{i})\hat{\psi}_{k-1}\hat{\psi}_k\hat{\psi}_{k-1}$ and $\hat{e}(\mathbf{i})\hat{\psi}_k\hat{\psi}_{k-1}\hat{\psi}_k$ when $i_{k-1} = i_{k+1} = i_k \pm 1$.

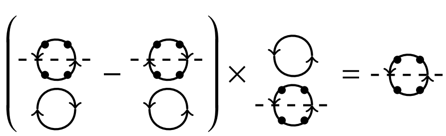
(1) 

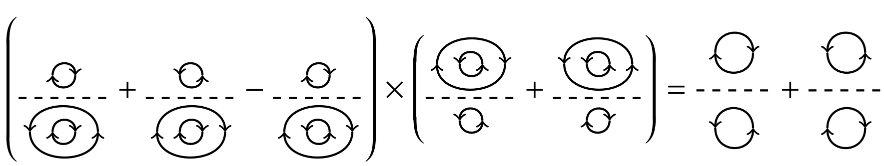
(2) 

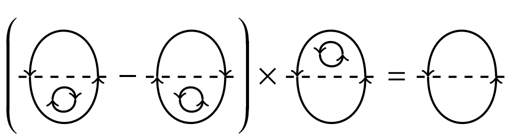
(3) 

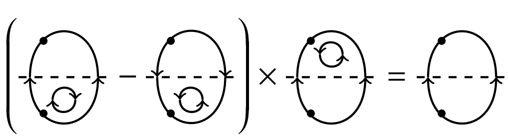
(4) 

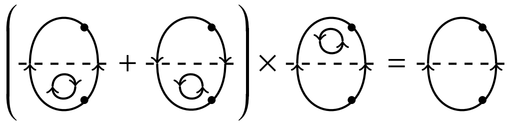
(5) 

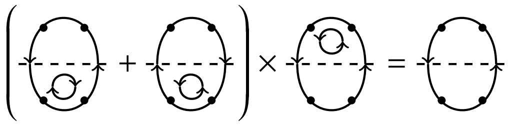
(6) 

(7) 

(8) 

(9) 

(10) 

(11) 

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