Braid group actions on derived categories

Fabian Lenzen

Born 18. Sept. 1991 in Wuppertal, Germany

10. 10. 2017

Master's Thesis Mathematics

Advisor: Prof. Dr. C. Stroppel

Second Advisor: Prof. Dr. J. Schröer

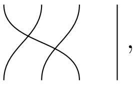
MATHEMATICAL INSTITUTE

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT DER

RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT BONN

Introduction; outline

By the name *braid group* one usually refers to the *Artin braid group*, first introduced and studied by Artin in [Art25]. Similarly to the symmetric group, the Artin braid group consists of permutations (called *braids*) of a fixed finite set which—in contrast to the symmetric group—keep track how elements pass "behind each other". Whereas it is standard to depict a permutation on say four elements by images like



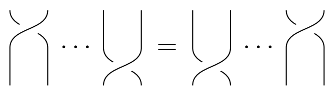
a braid in four strands is best thought of as a diagram like

or

which are in fact two different braids. The symmetric group in particular is a quotient of the Artin braid group, and both of the above braids have the same image under the quotient map. To make things rigorous consider the following definition:

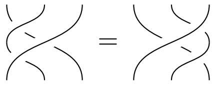
Definition. The Artin braid group B_{S_n} is the group with generators $s_1, \ldots, s_{n-1}$, depicted by a crossing $s_i = |\cdots|$ ⤫ $|\cdots|$ of the i-th strand over the $(i+1)$-st one (whose multiplication is depicted by stacking pictures), subject to the following (pictorial) relations:

$$s_i s_j = s_j s_i$$



for $|i-j| > 1$

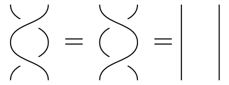
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



for all $1 \leq i < n-1$.

In other words, B_{S_n} has the same presentation as S_n, except that it misses the relation $s_i = s_i^{-1}$ and instead only satisfies the automatic relation

$$s_i s_i^{-1} = s_i^{-1} s_i = e$$



for all $1 \leq i < n$.

Thus B_{S_n} admits the quotient map $B_{S_n} \twoheadrightarrow S_n, s_i \mapsto s_i$.

Braid groups can also be defined for arbitrary *Coxeter groups*; these are a certain type of (finite or infinite) groups introduced by Coxeter in [Cox34] as a generalisation of reflection groups.

Definition. A *Coxeter system* (W, S) with Coxeter matrix $(m_{st})_{s,t \in S}$ consists of a group W, called *Coxeter group*, with generators $S \subset W$ and presentation

$$W = \langle s \in S \mid \forall s, t \in S : \overbrace{sts\cdots}^{m_{st}\text{ factors}} = \underbrace{tst\cdots}_{m_{st}\text{ factors}}, s^2 = e \rangle.$$

The *braid group* B_W associated to the Coxeter group W is the group with presentation

$$B_W \coloneqq \langle s \in S \mid \forall s, t \in S : \overbrace{sts\cdots}^{m_{st}\text{ factors}} = \underbrace{tst\cdots}_{m_{st}\text{ factors}} \rangle$$

such that the Coxeter group W is a quotient of its braid group B_W.

Coxeter groups have been classified in the finite case by Coxeter in [Cox35]. The symmetric group S_n is a particular Coxeter group with generating set $S = \{s_1, \ldots, s_{n-1}\}$ the simple transpositions and associated braid group B_{S_n}. According to the nomenclature for Coxeter groups, S_n is the Coxeter group of type A_{n-1}.

Part I. Braid group actions on the bounded derived BGG category $D^{\mathrm{b}}(\mathcal{O})$

Coxeter groups occur naturally as Weyl groups of root systems; they are thus strongly related to the representation theory of Lie algebras and are naturally realised as reflection groups. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Its Weyl group W is a Coxeter group and acts naturally on the set of integral weights by reflections. It is known that the classes $[M(w\,.\,\lambda)]$ of Verma modules for $w \in W$ form a basis of the Grothendieck group $K_0(\mathcal{O}_\lambda(\mathfrak{g}))$ of the block of the BGG category $\mathcal{O}(\mathfrak{g})$ associated to a regular integral weight λ. The map $s_i\colon [M(w\,.\,\lambda)] \mapsto [M(ws_i\,.\,\lambda)]$ thus induces a right action of W on $K_0(\mathcal{O}_\lambda)$.

Associated with the (self-biadjoint) *translation through the s-wall* functor Θ_s there are the *shuffling* and *coshuffling functors* $\mathrm{Sh}_s := \mathrm{coker}(\mathrm{adj}\colon \Theta_s \Rightarrow \mathrm{id}_{\mathcal{O}})$ and $\mathrm{Csh}_s := \ker(\mathrm{adj}'\colon \mathrm{id}_{\mathcal{O}} \Rightarrow \Theta_s)$ [see MS07, introduction, and references therein] whose derived functors induce an action of B_W on the bounded derived category $D^{\mathrm{b}}(\mathcal{O}_\lambda)$, yielding the above action of W on $K_0(\mathcal{O}_\lambda)$ [MS05, §5; Rou06, §10]. This action plays an important role in representation theory. It even lifts to a higher action in the sense that the relations of the braid group can be lifted to relations between functors [Rou06].

This thesis studies several aspects of these functors and tries to put them into a general framework. We shall explain in more detail how to calculate the images of Verma modules and indecomposable projectives under Sh_s. We shall exploit the fact that $\mathcal{O}_\lambda$ is equivalent as a category to Mod-A for a quasi-hereditary basic algebra A [Maz12, §4.4] where the distinguished modules of the quasi-hereditary structure correspond to Verma-, simple- and indecomposable projective modules. In small ranks, the algebra A can be described as a path algebra of a certain quiver associated to $\mathfrak{g}$ [Str03b; Mar06]. For instance, for $\mathfrak{g} = \mathfrak{sl}_2$ the category $\mathcal{O}_0(\mathfrak{sl}_2)$ is equivalent to modules over the path algebra $\mathbf{C}[\overset{e}{\bullet} \underset{b}{\overset{a}{\rightleftarrows}} \overset{s}{\bullet}]/(ba)$; see Section 2.1.1.

By a result of Beilinson, Ginzburg and Soergel the algebra A (and hence the category $\mathcal{O}_\lambda$) can be endowed with a positive grading [BGS96] which allows to define a graded category $\mathcal{O}^{\mathbf{Z}}_\lambda$; see Section 2.3. Moreover, the translation and shuffling functors Θ_s and Sh_s can be taken to *graded* functors [Str03a]. The grading coincides with the natural grading of A as a path algebra under the above equivalence of categories. We shall see that when gradings are taken into account, the functors Sh induce an action of the Iwahori-Hecke algebra $\mathsf{H}_q(W)$ and for $\mathfrak{g} = \mathfrak{sl}_n$ we shall see the following in Theorem 2.23:

Theorem. There is an isomorphism of $\mathbf{Z}[q^{\pm 1}]$-modules

$$K_0\big(\mathcal{O}^{\mathbf{Z}}_0(\mathfrak{sl}_n)\big) \to \mathsf{H}_q(S_n), \quad [M(w)\langle q\rangle] \mapsto qH_w, \quad [P(w)\langle q\rangle] \mapsto qC_w.$$

The shuffling functor Sh_s then acts by $\cdot H_s$.

The algebra $\mathsf{H}_q(W)$ is related to W and B_W by algebra homomorphisms $\mathbf{C}[q^{\pm 1}][B_W] \twoheadrightarrow \mathsf{H}_q(W) \twoheadrightarrow \mathbf{C}[W]$. The Iwahori-Hecke algebra $\mathsf{H}_q(W)$ is of particular interest as it is employed in the refinement and categorification of knot invariants [cf. Kho06]. The algebra $\mathsf{H}_q(W)$ has two distinguished bases, namely the standard- and the Kazhdan-Lusztig basis. It is a deep result of representation theory that the base change coefficients encode the composition series of modules in $\mathcal{O}_0$; see Section 2.3.1.

Spherical objects and spherical twist functors

We shall also interpret the action of B_W on $D^{\mathrm{b}}(\mathcal{O}_\lambda)$ by derived (co)shuffling functors as an extension of another action of braid groups on derived categories: in [ST01] Seidel and Thomas have constructed an action of B_{S_n} on the derived category $D^{\mathrm{b}}(\mathcal{C})$ of an arbitrary abelian linear category $\mathcal{C}$ of finite global dimension by the so-called *spherical twist functors*[1]

$$T_{E_k}\colon F \mapsto \mathrm{cone}(\hom^\bullet_{\mathcal{C}}(E_k, F) \otimes E_k \xrightarrow{ev} F)$$

associated to an A_{n-1}-configuration $(E_1, \ldots, E_{n-1})$ of *spherical objects*. Since $\mathcal{O}_\lambda$ satisfies the conditions of [ST01] on the category $\mathcal{C}$, it is natural to ask whether one can find spherical objects in $D^{\mathrm{b}}(\mathcal{O}(\mathfrak{sl}_n)_\lambda)$ such that the associated action of B_{S_n} is isomorphic to the action induced by the (co)shuffling functors.

After having explained the set-up and constructions of [ST01] we shall deal with that question both for the principal block $\mathcal{O}_0(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_n$ as well as their parabolic counterparts $\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_n)$. We shall show the following in Theorem 4.23:

---

[1] The *spherical twist functors* in the sense of [ST01] are *not* to be confused with Arkhipov's twisting functors [AS03]. We shall only deal with the former family of functors in this thesis.

Theorem. For a maximal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{sl}_n$ corresponding to the parabolic subgroup $S_{n-1} \times S_1 < S_n$, there is an A_{n-2}-configuration of spherical objects such that the associated spherical twist functor (in the sense of [ST01]) and the restriction $\mathbf{L}\,\mathrm{Sh}_{s_i}|_{D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})}$ are naturally isomorphic auto-equivalences of $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$. This set-up corresponds to the quiver

$$\overset{1}{\bullet} \rightleftarrows \overset{2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{n}{\bullet}$$

of type A_{n-2}, subject to certain relations.

The first part of the thesis is devoted to proving this statement and explaining the necessary background. We shall recall briefly the definitions of the BGG category $\mathcal{O}$ as well as the construction of translation- and shuffling functors in Section 1. In Sections 2.1 and 2.2 we shall explain how to calculate images of the shuffling functor, how to obtain the quiver describing $\mathcal{O}_0$ for the Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ and how to make use of this description. The construction of spherical twist functors from [ST01] will be recalled in Section 3 and applied to $\mathcal{O}_0$; first by showing that the spherical twist functor and the derived shuffling functor have isomorphic images and then by showing that both are indeed naturally isomorphic functors in the case of the theorem.

Part II. Towards super Soergel bimodules

We begin the second part of this thesis with presenting in Section 5 Rouquier's proof from [Rou06] for the that the derived (co)shuffling functors induce an action of B_W on $D^{\mathrm{b}}(\mathcal{O}_\lambda)$. In fact, Rouquier's proof shows more: namely, there is an action of B_W on the homotopy category of *Soergel bimodules* and hence on the homotopy category $K^{\mathrm{b}}(C_\lambda)$ of modules over the *coinvariant algebra* of W [see Soe90; BKM01, §2] which yields the desired action on $D^{\mathrm{b}}(\mathcal{O}_\lambda)$ by Soergel's functor $\mathbf{V}$ [Soe90]. We devote Section 5 to retracing Rouquier's proof and elaborating on some details.

An important tool both for Rouquier's proof and the study of Soergel bimodules are *symmetric polynomials.* The symmetric polynomials in n indeterminates are the invariants $\Lambda\mathrm{Pol}_n \coloneqq \mathrm{Pol}_n^{S_n}$ of the polynomial ring $\mathrm{Pol}_n \coloneqq k[x_1, \dots, x_n]$ under the action of S_n by permutation of indeterminates (see e. g. [Man98]). It is classical that $\Lambda\mathrm{Pol}_n \cong k[\varepsilon_1, \dots, \varepsilon_n]$ again is a polynomial ring, generated by the *elementary symmetric polynomials* ε_m. It is known that Pol_n is a free $\Lambda\mathrm{Pol}_n$-module of rank $n!$ [Che55; Dem73, thm. 6.2].

There is another way to define symmetric polynomials. Namely, the invariants $\mathrm{Pol}_n^{s_i}$ under a single simple transposition coincide with the kernel $\ker \partial_i$ of the *Demazure operator* $\partial_i \colon f \mapsto \frac{f - s_i(f)}{x_{i+1} - x_i}$ [Dem73]. These operators allow to define certain subalgebras of $\mathrm{End}_k(\mathrm{Pol}\mathfrak{C}_n)$ such as the *NilHecke Algebra* $\mathsf{NH}_n \coloneqq \langle x_1, \dots, x_n, \partial_1, \dots, \partial_{n-1} \rangle \subseteq \mathrm{End}_k(\mathrm{Pol}_n)$. The algebra NH_n has been generalised to the so-called *KLR algebra* in [KL09; KL11; KL10; Rou08].

Super-KLR algebra and its polynomial representation

Our aim is to investigate super-algebraic analogues for these invariants. A *superalgebra* is a $\mathbf{Z}/2\mathbf{Z}$-graded algebra whose degrees usually are referred to as *parities.* We recall the relevant definitions in Section 6. We shall encounter a particular superalgebra, namely the so-called *Clifford algebra* (see Example 6.5), which plays an important role in our considerations.

We shall present in Section 7 two superalgebras $\tilde{\mathsf{H}}_n(C)$ and $\mathsf{H}\mathfrak{C}_n(C)$ as super-analogues of the KLR-algebra. Their definitions are mostly inspired from [KKT16]. To that end we shall employ—besides a usual definition in terms of generators and relations—a diagrammatic description of $\mathsf{H}\mathfrak{C}(C)$ and $\tilde{\mathsf{H}}(C)$, which is adapted for the super-set-up from [KL09; Bru16; BE17b].

The ordinary KLR algebra has a faithful polynomial representation $\mathrm{Pol}(C)$ [KL09]. We shall define a Clifford analogue $\mathrm{Pol}\mathfrak{C}(C)$ (see Section 7.4) and show in Proposition 7.16:

Proposition. The superalgebras $\mathsf{H}\mathfrak{C}(C)$ and $\tilde{\mathsf{H}}(C)$ have a faithful polynomial representation $\mathrm{Pol}\mathfrak{C}(C)$.

The proof is mostly an adaptation of [KL09] for the super-algebraic set-up.

Clifford-symmetric polynomials

The construction of $\mathsf{H}\mathfrak{C}_n(C)$ involves the definition of a super-algebraic analogue $\mathfrak{d}_i$ for the *Demazure operator* or *divided difference operator*, also motivated by [KKT16]. We call this the *Clifford Demazure operator.*

In Section 8 we shall develop a short theory of these operators and try to understand which properties from the ordinary set-up can also be shown in the super-algebraic world. In a (non-commutative) polynomial superalgebra $\mathrm{Pol}\mathfrak{C}_n$, we define $\mathfrak{d}$-*symmetric polynomials* to be the common kernel $\bigcap_{k=1}^{n-1} \ker \mathfrak{d}_n$ of the Clifford Demazure operators. In Lemma 8.13 we recursively define *elementary* $\mathfrak{d}$-*symmetric polynomials* $\mathfrak{e}_m^{(n)}$, analogously to the recursion relation of the ordinary elementary symmetric polynomials. Let $\Lambda\mathrm{Pol}\mathfrak{C}_n$ be the sub-superalgebra of $\mathrm{Pol}\mathfrak{C}_n$ generated by the $\mathfrak{e}_m^{(n)}$'s. In the most of Section 8 we work on proving the following (see Theorem 8.23):

Theorem. The $\mathfrak{d}$-symmetric polynomials are generated by the elementary $\mathfrak{d}$-symmetric polynomials. Furthermore, $\mathrm{Pol}\mathfrak{C}_n$ is a free $\Lambda\mathrm{Pol}\mathfrak{C}_n$-algebra.

To that end, we also define analogues for the Schubert polynomials (see Section 8.4) and adapt the classical proof of the theorem to the super-set-up.

Complete symmetric polynomials, Grassmannians and partial flag varieties

We also define a super-analogue for for the complete homogeneous symmetric polynomials, see Section 8.5. We relate these to a NilHecke Clifford algebra $\mathsf{NH}\mathfrak{C}_n$ (see Definition 7.1) and show the following in Theorem 8.36:

Theorem. The yet to be defined *cyclotomic quotient* $\mathsf{NH}\mathfrak{C}_n^m$ of $\mathsf{NH}\mathfrak{C}_n$ and a super-algebraic analogue $H\mathfrak{C}_{(m,n)}$ for the cohomology ring of Grassmannians (see Definition 8.34) are Morita equivalent.

Finally, we define polynomial rings $\Lambda\mathrm{Pol}\mathfrak{C}_{\boldsymbol{k}}$ in Section 8.8 which play the super-analogous role of invariants $\mathrm{Pol}_n^{S'}$ for $S' \subseteq S$ a subset of the simple transpositions. In the classical case, such rings occur as cohomology rings of partial flag varieties $\mathrm{Fl}(\boldsymbol{k})$. We shall thus define in Section 8.8.2 a super-analogue $H\mathfrak{C}_{\boldsymbol{k}}$ for these cohomology rings which generalise the Clifford cohomology ring $H\mathfrak{C}_{(m,n)}$ of Grassmannians; see Proposition 8.48.

These steps can be be considered as first steps towards super-algebraic Clifford-Soergel bimodules. Future research has to show whether the algebras defined in this thesis turn out to be beneficial for this purpose.

Acknowledgements

I want to thank my advisor Prof. Catharina Stroppel for her continuous support, helpful advice and in particular her patient explanations during this project. I had the great opportunity to dive into many interesting topics I had barely worked on before.

Contents

Notation and conventions

We shall make use of the following notational conventions throughout the thesis:

k	An arbitrary field of characteristic distinct from 2.
$\mathbf{N}$	Natural numbers are the numbers $\{0, 1, 2, \dots\}$.
π, ι	As maps out of (into) a direct sum, these denote the projection (inclusion) of one direct summand.
$\langle - \rangle$	A group (algebra, module, vector space etc.) generated by the given elements. It should be clear from the context which structure is meant.

—*Graded algebras*: By a graded algebra A we always mean a $\mathbf{Z}$-graded algebra.

deg, $\lvert - \rvert$	The degree of an element of a graded ring (module, vector space etc.) is denoted by deg. In particular, polynomial degrees are denoted deg. We reserve $\lvert - \rvert$ for the parity when working with superalgebras.
$\langle - \rangle$	The grading shift of graded rings (modules, vector spaces etc.). We define $\langle - \rangle$ shifting upwards, i. e. for a graded module $M = \bigoplus_k M_k$ with degree k-part M_k, the shifted module has degree k-part $M\langle i \rangle_k := M_{k-i}$.
$\mathfrak{a}_+$	For $\mathfrak{a}$ a homogeneous ideal of a graded algebra $A = \bigoplus_{k \geq 0}$, we set $\mathfrak{a}_+ := \bigoplus_{k>0} \mathfrak{a}_k$.

—*Homological algebra*: For us, the differential of chain complexes *increases* the homological degree.

$F \dashv G$	F is a left adjoint functor to G.
$\mathrm{Ch}(\mathcal{C})$	The category of chain complexes in $\mathcal{C}$. We shall not always indicate chain complexes by a $\bullet$.
$D^{\mathrm{b}}(\mathcal{C})$	The bounded derived category of $\mathcal{C}$.
$[-]$	For us, the homological degree shift shifts to the left, i. e. for a chain complex X, we set $X[i]^k := X^{k+i}$.
$\cong$	An isomorphism.
$\simeq$	A quasi-isomorphism, or an equivalence of categories.
$\approxeq$	A homotopy equivalence.

Hom and hom: The symbol hom always denotes a chain complex of maps whereas Hom is a space of chain maps.

$\{\cdots\}$	We use this to denote both the total complex of a double (triple) complex and the mapping cone. This should not cause confusion: if we consider a chain map $X \to Y$ as a double complex with Y in column 0, then its total complex coincides with cone($X \to Y$).
$\lvert - \rvert$	The homological degree of a chain complex (used in Sections 3 and 5).
f^*, f_*	For $f \colon X \to Y$ a morphism, we denote the induced natural transformations of the Hom-functor by $f^* \colon \mathrm{Hom}(Y, -) \Rightarrow \mathrm{Hom}(X, -)$ and $f_* \colon \mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$.

Part I.
Braid group actions on the bounded derived BGG category $D^{\mathrm{b}}(\mathcal{O})$

1. Shuffling functors on the BGG category $\mathcal{O}$

Given a semisimple complex Lie algebra $\mathfrak{g}$, we consider the so-called *BGG-category* $\mathcal{O}$, first defined in [BGG76], which is defined as follows: Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a fixed Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a corresponding Cartan decomposition, fixing a choice of simple roots.

Definition 1.1. The *BGG-category* $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-Mod consisting of modules M that ($\mathcal{O}1$) are finitely generated, ($\mathcal{O}2$) have a weight space decomposition $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$ and ($\mathcal{O}3$) are locally $\mathfrak{n}$-finite, i. e. $U(\mathfrak{n}^+)\cdot v$ is finite dimensional for every $v \in M$.

This category contains in particular the Verma modules $M(\lambda) = U(\mathfrak{n}^+) \otimes_{U(\mathfrak{h})} \mathbf{C}_\lambda$ of highest weight $\lambda \in \mathfrak{h}^*$ and their simple quotients $L(\lambda)$, which exhaust all simple objects of $\mathcal{O}$. The category $\mathcal{O}$ is a *Krull-Schmidt category* by Weyl's theorem [Hum72, thm. 6.3; Hum08, §1.11], i. e. any $M \in \mathcal{O}$ decomposes uniquely (up to up to isomorphism and permutation [Kra15, thm. 4.2]) into a finite direct sum of indecomposable modules. The category $\mathcal{O}$ has enough projectives. Every Verma module $M(\lambda)$ has a projective cover $P(\lambda)$, and the modules $P(\lambda)$ are the indecomposable projective objects of $\mathcal{O}$. There is a duality functor $(-)^\vee \colon \mathcal{O} \to \mathcal{O}$ which is an exact auto-equivalence of $\mathcal{O}$.

Definition 1.2. A *composition series* for a module $M \in \mathcal{O}$ is a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\ell = M \tag{1.1}$$

such that for all $i \leq \ell$ we have $M_i/M_{i-1} \cong L(\lambda_i)$ for some weights $\lambda_i \in \mathfrak{h}^*$. The quotients M_i/M_{i-1} are called *composition factors* of M. By the Jordan-Hölder theorem, the composition factors of such a series are unique up to isomorphism and order of appearance. Every module $M \in \mathcal{O}$ admits such a composition series [Jan79, Satz 1.13; Hum08, §1.11]. Write $[M : L(\lambda)]$ for the multiplicity of $L(\lambda)$ in a filtration for M. We depict composition series by stacking the quotients from the simple *socle* M_0 at the bottom to the simple *head* $M/M_{\ell-1}$ at the top.

The category $\mathcal{O}$ admits a direct sum decomposition into subcategories associated to linkage classes of weights: Denote by $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with simple roots Δ, half sum of positive roots ρ and associated Weyl group W; see [Hum72, part III] for the notions. W acts on Φ and on $\mathfrak{h}^*$ by reflections. Recall the *dot action* of W on $\mathfrak{h}^*$ given by $w \cdot \lambda := w(\lambda + \rho) - \rho$. We denote the dot action of W by "$W\cdot$".

Two weights $\lambda, \mu \in \mathfrak{h}^*$ are said to be *linked* if they belong to the same orbit of $W\cdot$. Let $\mathcal{O}_\lambda$ denote the full subcategory of $\mathcal{O}$ containing those modules that have composition factors of highest weights linked to λ. Then the category $\mathcal{O}$ decomposes as the direct sum

$$\mathcal{O} = \bigoplus_{\lambda\in\mathfrak{h}^*/W\cdot} \mathcal{O}_\lambda. \tag{1.2}$$

In particular, $\mathcal{O}_\lambda$ is a Serre subcategory of $\mathcal{O}$ and contains the Verma modules $M(w\cdot\lambda)$ for $w \in W$ as well as the respective simple- and indecomposable projective modules. If a dominant regular integral weight λ is fixed, we often write $M(w)$ for $M(w\cdot\lambda)$. The category $\mathcal{O}_0$ is called *principal block* of $\mathcal{O}$. It has finite global dimension [Maz07, thm. 2]. It contains in particular the trivial representation $\mathbf{C}$ of $\mathfrak{g}$.

Apart from the composition series defined in Definition 1.2, another kind of filtration is of particular interest:

Definition 1.3. A module M is said to admit a *standard filtration* if it admits a finite filtration by submodules of the form (1.1) whith quotients $M_i/M_{i+1} \cong M(\lambda)$. If it exists, the length of such a filtration and the occurring Verma modules are again unique (up to isomorphism and order) by the Jordan-Hölder theorem. We write $(M : M(\lambda))$ for the multiplicity of the Verma module $M(\lambda)$ in a standard filtration of M.

Fact 1.4 [Hum08, §§3.9-3.11]. For multiplicities in composition series and standard filtrations, the following useful properties hold:

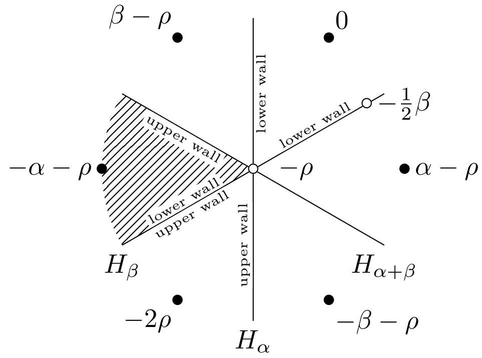


Figure 1.1: Roots, the associated reflection hyperplanes (called *walls*) and the enclosed chambers for $\mathfrak{sl}_3$. The hatched chamber lies below the α-wall, below the $(\alpha+\beta)$-wall and above the β-wall; it is thus uniquely identified by $\Phi_C^+ = \{\beta\}$ and $\Phi_C^- = \{\alpha, \alpha+\beta\}$. Black bullets • are the weights in the orbit $W \cdot 0$ (which all are regular). The weights $-\rho$ and $-\frac{1}{2}\beta$ (white bullets ∘) are examples for non-regular weights. In particular, $-\rho$ always lies on *every* wall for any $\mathfrak{g}$.

(i) Let $M \in \mathcal{O}$ be a module with standard filtration. Its composition factors then have multiplicities $[M : L(\lambda)] = \sum_{\mu \in W\cdot\lambda} (M : M(\mu))[M(\mu) : L(\lambda)]$.

(ii) The indecomposable projectives $P(\lambda)$ admit a standard filtration with $(P(\lambda) : M(\mu)) = \begin{cases} 0 \text{ if } \mu \not\leq \lambda, \\ 1 \text{ if } \mu = \lambda \end{cases}$. The case $\mu < \lambda$ will be dealt with in Section 2.3.1.

(iii) Any projective module $P \in \mathcal{O}$ can be written as a direct sum $P \cong \bigoplus_\lambda P(\lambda)^{\oplus n_\lambda}$ of indecomposable projectives with multiplicities $n_\lambda = [P, L(\lambda)]$.

(iv) For every $M \in \mathcal{O}$, $\dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$

(v) BGG Reciprocity theorem: $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$. Together with the other statements, this in particular implies $\dim_{\mathbf{C}} \operatorname{End}_{\mathcal{O}}(P(\lambda)) = \sum_{\mu \in W\cdot\lambda} (P(\lambda) : M(\mu))^2$ and $[M(\mu) : L(\nu)] = \begin{cases} 0 \text{ if } \mu \not\leq \lambda, \\ 1 \text{ if } \mu = \lambda \end{cases}$.

1.1. Translation functors

We shall recall a certain collection of functors connecting the different blocks $\mathcal{O}_\lambda$ of $\mathcal{O}$ from [Hum08, §7], originally defined in [Jan79, §2.10]. For simplicity of notation we stick to integral weights in our considerations. Consider [Hum08, §7.4] for a treatment of non-integral weights.

Definition 1.5. Given integral weights $\lambda, \mu \in \mathfrak{h}^*$ let ν be the unique dominant weight $\nu \in W(\mu - \lambda)$ (w. r. t. the normal action). The functor $T_\lambda^\mu \colon \mathcal{O}_\lambda \to \mathcal{O}_\mu, M \mapsto \pi_\mu(L(\nu) \otimes_{\mathbf{C}} M)$ is called *translation functor*, where π_μ denotes the projection $\mathcal{O} \twoheadrightarrow \mathcal{O}_\mu$ onto the respective summand of the decomposition (1.2).

Since ν is dominant, $L(\nu)$ is finite dimensional, and T_λ^μ is an example of a *projective functor* [Maz12, def. 5.1], i. e. a direct summand of the functor $V \otimes_{\mathbf{C}} -$ for some finite dimensional $V \in \mathcal{O}$. Projective functors preserve projective modules [Hum08, prop. 3.8].

Lemma 1.6. The functors T_λ^μ and T_μ^λ are biadjoint; in particular they are exact.

Proof. Since $L(\nu)$ is finite dimensional. there are adjunctions $(L(\nu) \otimes -) \dashv (L(\nu)^* \otimes -)$ and $(L(\nu)^* \otimes -) \dashv (L(\nu) \otimes -)$ as endofunctors of the category of vector spaces. For instance, the first adjunction is established by the isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbf{C}}(L(\nu) \otimes M, N) &\to \operatorname{Hom}_{\mathbf{C}}(M, L(\nu)^* \otimes N) \\ f &\mapsto \Big(m \mapsto \sum_{L(\nu)=\langle e\rangle} e^* \otimes f(e,m)\Big) \\ \big(l \otimes m \mapsto (ev_l \otimes \mathrm{id}_N)(g(m))\big) &\mapsfrom g, \end{aligned}$$

which (as well as the isomorphism for the other adjunction) is compatible with the action of $U(\mathfrak{g})$. Furthermore, there is an adjunction $\pi_\lambda : \mathcal{O} \underset{\longleftarrow}{\overset{\dashv}{\twoheadrightarrow}} \mathcal{O}_\lambda : \iota_\lambda$ of the projection π_λ and the inclusion ι_λ of the direct summand $\mathcal{O}_\lambda$ of the decomposition (1.2). This proves the claim. ☐

Chambers and walls Consider the space $\mathfrak{h}^*$ of all weights and the reflection hyperplanes

$$H_\lambda := \{\mu \in \mathfrak{h}^* \mid \langle \mu + \rho, \alpha\lambda^\vee \rangle = 0\} \tag{1.3}$$

orthogonal to a root $\lambda \in \Phi$, called the λ-*wall* in $\mathfrak{h}^*$. It is the fixed point set of the reflection s_λ orthogonal to λ. The connected components of the complement $\mathfrak{h}^* \setminus \bigcup_{\alpha\in\Phi} H_\alpha$ are called *chambers*. A chamber thus is a cone with cone point $-\rho$ enclosed by walls and is uniquely identified by a partitioning $\Phi_C^+ \cup \Phi_C^- = \Phi^+$

of the set Φ^+ of positive roots such that

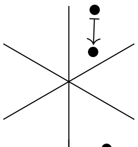
$$C = \big\{\lambda \in \mathfrak{h}^* \mid \langle \lambda + \rho, \alpha \rangle \gtrless 0 \text{ if } \alpha \in \Phi_C^{\pm}\big\}. \tag{1.4}$$

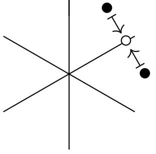
The chamber C is said to lie *above/below the* λ*-wall* for $\lambda \in \Phi_C^{\pm}$ and its *upper closure* is defined as

$$\hat{C} \coloneqq C \cup \big\{\lambda \in \mathfrak{h}^* | \langle \lambda + \rho, \alpha \rangle \geq 0 \text{ if } \lambda \in \Delta_C^{+}\big\}. \tag{1.5}$$

See Figure 1.1 for a depiction of these notions for the root system of $\mathfrak{sl}_3$. Note that the fundamental chamber (i. e. the one containing all dominant weights, in particular the weight 0) lies above all of its walls and the chamber containing all antidominant weights (in particular -2ρ) lies below all of its walls. A weight $\lambda \in \mathfrak{h}^*$ is said to be a *regular weight* if it is contained in none of the walls H_α. For instance, 0 (and every weight in the orbit $W \cdot 0$) always is a regular integral weight for any semisimple complex Lie algebra $\mathfrak{g}$.

Recall 1.7 [Hum08, §7 and references therein]. Let $\lambda \in C$ be a regular integral weight[2] contained in a chamber C.

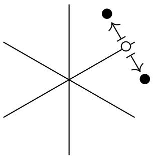




(i) If $\mu \in C$ is another weight contained in the same chamber, then $T_\lambda^\mu : \mathcal{O}_\lambda \overset{\simeq}{\longleftrightarrow} \mathcal{O}_\mu : T_\mu^\lambda$ are mutually inverse equivalences of categories.

(ii) Translation to a wall: If μ lies in the closure of C, then translation preserves Verma modules, dual Verma modules, and, if μ lies in the upper closure, also simple modules:

$$\begin{aligned} T_\lambda^\mu M(w\cdot\lambda) &\cong M(w\cdot\mu), \\ T_\lambda^\mu M(w\cdot\lambda)^\vee &\cong M(w\cdot\mu)^\vee, \\ T_\lambda^\mu L(w\cdot\lambda) &\cong \begin{cases} L(w\cdot\mu) & \text{if } w\cdot\mu \in \widehat{w\cdot C}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



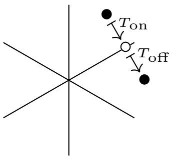
(iii) Translation from a wall: If $w \cdot \mu \in \widehat{w \cdot C}$, then $T_\mu^\lambda P(w\cdot\mu) \cong P(w\cdot\lambda)$. If furthermore μ lies solely on the s_α-wall, then

$$T_\lambda^\mu T_\mu^\lambda L(w\cdot\mu) = L(w\cdot\mu) \oplus L(w\cdot\mu).$$

If furthermore $\ell(ws_\alpha) > \ell(w)$, then there is a nonsplit short exact sequence

$$0 \to M(w\cdot\lambda) \to \underbrace{T_\mu^\lambda M(w\cdot\mu)}_{\overset{(*)}{\cong} T_\mu^\lambda M(ws_\alpha\cdot\mu)} \to M(ws_\alpha\cdot\lambda) \to 0, \tag{1.6}$$

where $(*)$ holds since μ lies in the s_α-wall of the chamber and thus is fixed by s_α.



Definition 1.8. Let λ be contained in a chamber and μ precisely in its s-wall. Denote the translation functors by $T_{\text{on}} \coloneqq T_\lambda^\mu$ and $T_{\text{off}} \coloneqq T_\mu^\lambda$. Their composite $\Theta_s = T_{\text{off}} T_{\text{on}}$ is called *translation through the wall*. By biadjointness of T_{on} and T_{off}, the functor Θ_s is an exact self-adjoint functor. Note that given two Verma modules of the same block which lie inside two chambers separated by a wall, translating the Verma modules through this wall gives isomorphic modules.

1.2. Derived kernel and cokernel functors

Given a natural transformation $\eta\colon F \Rightarrow G$ of two exact functors $F, G\colon \mathcal{C} \to \mathcal{D}$ of abelian categories $\mathcal{C}$ and $\mathcal{D}$, one can define functors

$$\operatorname{coker}\eta\colon \mathcal{C} \to \mathcal{D}; \quad X \mapsto \operatorname{coker}(\eta_X\colon FX \to GX), \tag{1.7}$$

$$\ker\eta\colon \mathcal{C} \to \mathcal{D}; \quad X \mapsto \ker(\eta_X\colon FX \to GX). \tag{1.8}$$

By the snake lemma, $\operatorname{coker}\eta$ is right exact and $\ker\eta$ is left exact.

---

[2] It is not necessary to assume regularity nor integrity for all of the listed properties, but simplifies the notation considerably and suffices for our purposes; see [Hum08, §7] for a more general treatment.

Definition 1.9. Given a category $\mathcal{C}$, its *arrow category* $\mathcal{C}^{[1]}$ is the category with objects given by morphisms $f\colon X \to Y$ in $\mathcal{C}$, and morphisms from $f\colon X \to Y$ to $g\colon X' \to Y'$ given by commutative squares. If $\mathcal{C}$ is an abelian category, then so is $\mathcal{C}^{[1]}$.

The adjunction map $\eta\colon F \Rightarrow G$ may be regarded as a functor $\mathcal{C} \to \mathcal{D}^{[1]}$ which inherits exactness from F and G. The cokernel can be regarded as a right exact functor $\mathcal{D}^{[1]} \to \mathcal{D}$ [Gro16, lem 3.1.4], and $\operatorname{coker}\eta$ is the composition $\mathcal{C} \xrightarrow{\eta} \mathcal{D}^{[1]} \xrightarrow{\text{coker}} \mathcal{D}$.

Now consider the bounded derived categories $D^{\mathrm{b}}(\mathcal{C})$ and $D^{\mathrm{b}}(\mathcal{D})$. The exact functor η induces $\eta\colon D^{\mathrm{b}}(\mathcal{C}) \to D^{\mathrm{b}}(\mathcal{D}^{[1]})$. Since coker is right exact, one can form its left derived functor $\mathbf{L}\operatorname{coker}\colon D^{\mathrm{b}}(\mathcal{D}^{[1]}) \to D^{\mathrm{b}}(\mathcal{D})$.

Lemma 1.10 [Gro16, thm. 3.5.6]. The left derived functor $\mathbf{L}\operatorname{coker}\colon D^{\mathrm{b}}(\mathcal{D}^{[1]}) \to D^{\mathrm{b}}(\mathcal{D})$ is naturally isomorphic to the mapping cone $(f\colon X \to Y) \mapsto \{X \xrightarrow{f} Y\}$. Similarly, $\mathbf{R}\ker$ is naturally isomorphic to the mapping cocone.

Caveat 1.11. It is important to keep in mind that the obvious functor $D^{\mathrm{b}}(\mathcal{D}^{[1]}) \hookrightarrow D^{\mathrm{b}}(\mathcal{D})^{[1]}$, is *not* an equivalence of categories. We consider the derived category as subcategory of the homotopy category. Objects in $D^{\mathrm{b}}(\mathcal{D}^{[1]})$ are certain objects of $\operatorname{Ch}(\mathcal{D}^{[1]}) \cong \operatorname{Ch}(\mathcal{D})^{[1]}$, i. e. (strictly) commutative diagrams

$$\begin{tikzcd}
X^\bullet \arrow[d] & \cdots \arrow[r] \arrow[dr, phantom, "/\!/"] & X^k \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & X^{k+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & \cdots \\
Y^\bullet & \cdots \arrow[r] & Y^k \arrow[r] & Y^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

in $\mathcal{D}$. Morphisms in $D^{\mathrm{b}}(\mathcal{D}^{[1]})$ are homotopy classes of commutative squares in $\operatorname{Ch}\mathcal{D}$, called *coherent diagrams* [Gro16, def. 4.1.1]. This is to say, a morphism in $D^{\mathrm{b}}(\mathcal{D}^{[1]})$ is represented by a *strictly* commutative diagram

$$\begin{tikzcd}
 & X'^\bullet \arrow[dd] & & \cdots \arrow[rr] & & X'^k \arrow[rr] \arrow[dd] & & X'^{k+1} \arrow[rr] \arrow[dd] & & \cdots & \\
X^\bullet \arrow[ur, dashed] \arrow[dd] \arrow[dr, phantom, "/\!/"] & & \cdots \arrow[rr] & & X^k \arrow[ur, dashed] \arrow[rr] \arrow[dd] & & X^{k+1} \arrow[ur, dashed] \arrow[rr] \arrow[dd] & & \cdots & & \Bigg\} \in \operatorname{Ch}\left(\mathcal{D}^{[1]}\right) \\
 & Y'^\bullet & & \cdots \arrow[rr] & & Y'^k \arrow[rr] & & Y'^{k+1} \arrow[rr] & & \cdots & \\
Y^\bullet \arrow[ur, dashed] & & \cdots \arrow[rr] & & \underbrace{Y^k}_{\in\mathcal{D}^{[1]}} \arrow[ur, dashed] \arrow[rr] & & Y^{k+1} \arrow[ur, dashed] \arrow[rr] & & \cdots & &
\end{tikzcd}$$

in $\mathcal{D}$, up to homotopy for the dashed arrows. In contrast, an object in $D^{\mathrm{b}}(\mathcal{D})^{[1]}$ is a morphism $X^\bullet \to Y^\bullet$ in $D^{\mathrm{b}}(\mathcal{D})$, which is only defined up to homotopy. A morphism in $D^{\mathrm{b}}(\mathcal{D})^{[1]}$ is a *homotopy*-commutative square

$$\begin{tikzcd}
X^\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\sim"] & X'^\bullet \arrow[d] \\
Y^\bullet \arrow[r] & Y'^\bullet
\end{tikzcd}$$

in $D^{\mathrm{b}}(\mathcal{D})$, i. e. $X^\bullet \to Y^\bullet \to Y'^\bullet$ and $X^\bullet \to X'^\bullet \to Y'^\bullet$ are homotopic chain maps. A homotopy-commutative square need not be homotopy equivalent to a strictly commutative square. These are called *incoherent diagrams*. The mapping cone is functorial only as a functor $D^{\mathrm{b}}(\mathcal{D}^{[1]}) \to D^{\mathrm{b}}(\mathcal{D})$ but does not factor through $D^{\mathrm{b}}(\mathcal{D})^{[1]}$ [Gro16, warn. 4.1.8].

1.3. Shuffling functors

Since the functors T_{on} and T_{off} from Definition 1.8 are biadjoint, there are the corresponding adjunction unit and counit maps $\eta\colon \mathrm{id} \Rightarrow T_{\text{off}}T_{\text{on}} = \Theta_s$ from the adjunction $T_{\text{off}} \dashv T_{\text{on}}$ and $\varepsilon\colon \Theta_s = T_{\text{off}}T_{\text{on}} \Rightarrow \mathrm{id}$ from the adjunction $T_{\text{on}} \dashv T_{\text{off}}$.

Definition 1.12. Define the right (resp. left) exact endofunctors

$$\mathrm{Sh}_s := \operatorname{coker}(\mathrm{id} \overset{\varepsilon}{\Rightarrow} \Theta_s), \qquad \mathrm{Csh}_s := \ker(\Theta_s \overset{\eta}{\Rightarrow} \mathrm{id})$$

of $\mathcal{O}_0$, called *shuffling* and *coshuffling functor* respectively. They have been firstly defined in [Irv93, §3].

Lemma 1.13. Let $w \in W$ and s be a simple reflection such that $\ell(ws) > \ell(w)$. There are isomorphisms

$$\begin{aligned} \operatorname{Sh}_s M(w \cdot \lambda) &\cong M(ws \cdot \lambda), \\ \operatorname{Csh}_s M(ws \cdot \lambda) &\cong M(w \cdot \lambda), \\ \Theta_s M(w) &\cong \Theta_s M(ws). \end{aligned} \tag{1.9}$$

Furthermore, translation through the wall and the shuffling functor fit into short exact sequences

$$\begin{tikzcd} M(w) \arrow[dr, hook] & & \\ & \Theta_s M(w) \arrow[dl, two heads] \arrow[r, two heads, bend right] & M(ws). \arrow[l, hook, bend right] \\ \operatorname{Sh}_s M(ws) & & \end{tikzcd} \tag{1.10}$$

The modules $M(w)$ and $\operatorname{Sh}_s^2 M(w)$ in particular have the same composition factors.

Proof. Given an adjunction $F \colon \mathcal{C} \overset{\dashv}{\rightleftarrows} \mathcal{D} \colon G$ of functors between arbitrary categories, the adjunction unit $\eta_X \colon X \to GFX$ is given by the image of $\mathrm{id}_{\mathcal{D}}$ under the natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(FX, FX) \cong \operatorname{Hom}_{\mathcal{D}}(X, GFX)$. Recall that for all Verma modules the endomorphism ring $\operatorname{End}_{\mathcal{O}}(M(w \cdot \lambda))$ is one-dimensional [Hum08, thm. 4.2]. Thus the adjunction map $\eta \colon \mathrm{id} \Rightarrow \Theta_s$, given by

$$\begin{aligned} \mathbf{C} \cong \operatorname{Hom}\big(\underbrace{T_{\mathrm{on}} M(w \cdot \lambda)}_{M(w \cdot \mu)}, T_{\mathrm{on}} M(w \cdot \lambda)\big) &\xrightarrow{\cong} \operatorname{Hom}\big(M(w \cdot \lambda), \Theta_s M(w \cdot \lambda)\big), \\ \mathrm{id} &\longmapsto \varepsilon_{M(w \cdot \lambda)}, \end{aligned}$$

is the unique (up to scalars) non-zero map $M(w \cdot \lambda) \to \Theta_s M(w \cdot \lambda)$. If $\ell(ws) > \ell(w)$, the map ε thus fits into a short exact sequence of the form (1.6), exhibiting $M(ws \cdot \lambda)$ as the cokernel of $\eta_{M(w \cdot \lambda)}$. A similar argument holds for the coshuffling functor. ☐

Lemma 1.14. Translating indecomposable projectives through the wall twice yields

$$\Theta_s^2 P(w) \cong \Theta_s P(w) \oplus \Theta_s P(w).$$

Proof. Since Θ_s is exact, $\Theta_s^2 P(w \cdot \lambda)$ is a projective module and thus can be written as a direct sum $\Theta_s^2 P(w \cdot \lambda) \cong \bigoplus_{w' \in W} P(w' \cdot \lambda)^{n_{w'}}$ of indecomposable projectives. The module $P(w' \cdot \lambda)$ occurs with multiplicity

$$\begin{aligned} n_{w'} &= \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(\Theta_s^2 P(w \cdot \lambda), L(w' \cdot \lambda)) && \textbf{??}\ 1.4.\text{(iii)} \\ &= \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(T_{\mathrm{on}} P(w \cdot \lambda), T_{\mathrm{on}} T_{\mathrm{off}} T_{\mathrm{on}} L(w' \cdot \lambda)) && \text{Lemma 1.6} \\ &= \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(T_{\mathrm{on}} P(w \cdot \lambda), T_{\mathrm{on}} T_{\mathrm{off}} L(w' \cdot \mu)) && \textbf{??}\ 1.7.\text{(ii)} \\ &= \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(T_{\mathrm{on}} P(w \cdot \lambda), L(w' \cdot \mu) \oplus L(w' \cdot \mu)) && \textbf{??}\ 1.7.\text{(iii)} \\ &= 2 \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(T_{\mathrm{on}} P(w \cdot \lambda), L(w' \cdot \mu)) && \\ &= 2 \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(T_{\mathrm{on}} P(w \cdot \lambda), T_{\mathrm{on}} L(w' \cdot \lambda)) && \textbf{??}\ 1.7.\text{(ii)} \\ &= 2 \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(\Theta_s P(w \cdot \lambda), L(w' \cdot \lambda)) && \text{Lemma 1.6.} \end{aligned}$$

☐

Since Sh_s and Csh_s are not exact in general, we pass to the bounded derived category $D^{\mathrm{b}}(\mathcal{O}_0)$ and consider the derived functors $\mathbf{L} \operatorname{Sh}_s$ and $\mathbf{R} \operatorname{Csh}_s$.

Lemma 1.15. The derived shuffling functors $\mathbf{L} \operatorname{Sh}_s$ satisfy the following properties:

(i) $\mathbf{L}_i \operatorname{Sh}_s = \begin{cases} \ker \eta & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$

(ii) For modules M with a standard filtration, the adjunction map $\eta_M \colon M \to \Theta_s M$ is an inclusion.

(iii) Modules having a standard filtration are Sh_s-acyclic, i. e. $\mathbf{L}_i \operatorname{Sh}_s M = 0$ for $i > 0$ if M admits a standard filtration.

Proof. (i) The long exact sequence of $\mathbf{L} \operatorname{Sh}_s$ coincides with the long exact sequence of the snake lemma for the exact sequence $\mathrm{id} \Rightarrow \Theta_s \Rightarrow \operatorname{Sh}_s \Rightarrow 0$ of functors. This shows that $\mathbf{L}_i \operatorname{Sh}_s = 0$ for $i > 1$ and $\mathbf{L}_1 \operatorname{Sh}_s = \ker \eta$.

(ii) By Lemma 1.13, the adjunction map η is injective for Verma modules. Hence by the four lemma, η_M is also injective if M is an extension of a Verma module by another Verma module. The statement follows by induction of the length of the filtration.

(iii) This follows immediately from the first two statements. ☐

1.4. W acts on the Grothendieck group, B_W acts on $D^{\mathrm{b}}(\mathcal{O})$

Definition 1.16. Given an abelian (resp. triangulated) category $\mathcal{C}$, its *Grothendieck group* $K_0(\mathcal{C})$ is the abelian group generated by isoclasses $[X]$ of objects $X \in \mathcal{C}$, with the relation $[X] = [Y] + [Z]$ whenever there is a short exact sequence $Y \hookrightarrow X \twoheadrightarrow Z$ (triangle $Y \to X \to Z \to Y[1]$) in $\mathcal{C}$. If $\mathcal{C}$ is a (braided) monoidal category, then $K_0(\mathcal{C})$ becomes a (commutative) ring by virtue of the multiplication induced by the tensor product.

Before letting B_W act on $D^{\mathrm{b}}(\mathcal{O}_0)$, we consider an action of W on the Grothendieck group.

Lemma 1.17 [Gro77, §§1-4]. The Grothdendieck group has the following properties:

(i) If $F\colon \mathcal{C} \to \mathcal{D}$ is an exact (triangulated) functor of abelian (triangulated) categories, it induces a homomorphism $[F]\colon K_0(\mathcal{C}) \to K_0(\mathcal{D}), [X] \mapsto [FX]$ of abelian groups.
(ii) If $\mathcal{T}$ is a triangulated category, then $[X[1]] = -[X]$ in $K_0(\mathcal{T})$.
(iii) If $\mathcal{C}$ is an abelian category, there is an isomorphism of abelian groups

$$\begin{aligned} [\mathrm{can}] : K_0(\mathcal{C}) &\overset{\cong}{\rightleftarrows} K_0(D^{\mathrm{b}}(\mathcal{C})) : \chi \\ [X] &\longmapsto [X] \\ \sum_k (-1)^k [H^k(X^\bullet)] &\longmapsfrom [X^\bullet] \end{aligned}$$

with the canonical embedding $\mathrm{can} : \mathcal{C} \hookrightarrow D^{\mathrm{b}}(\mathcal{C})$ and the *Euler characteristic* χ.

Corollary 1.18 of Lemma 1.13. Shuffling functors induce a right action of W on the Grothendieck group $K_0(\mathcal{O}_0)$ by the assignment $s \mapsto ([\mathbf{L}\,\mathrm{Sh}_s]\colon [M(w)] \mapsto [M(ws)])$ for simple reflections $s \in S$.

Proof. Recall that every module of $\mathcal{O}_0$ admits a composition series of the form (1.1) with simple factors $L(w)$. Hence, the isoclasses $[L(w)]$ form a generating set of $K_0(\mathcal{O}_0)$. Since the composition factors are unique (up to isomorphism and order of occurrence) by the Jordan-Hölder theorem, the $[L(w)]$'s are even a $\mathbf{Z}$-basis of $K_0(\mathcal{O}_0)$ [Hum08, §§1.11f].

Furthermore, recall that $L(w)$ is the unique simple quotient of $M(w)$ w. r. t. a submodule $M \subseteq M(w)$, where M only has composition factors $L(w')$ with $w' < w$ w. r. t. the Bruhat order (**??** 1.4.(v)). By induction on w this shows that also the isoclasses $[M(w)]$ of Verma modules form a basis of $K_0(\mathcal{O}_0)$ [Hum08, ex. 1.12] which is isomorphic to $K_0(D^{\mathrm{b}}(\mathcal{O}_0))$. It thus suffices to show the assertion for Verma modules. The short exact sequences in (1.10) show that Sh_s (and by Lemma 1.15 also $\mathbf{L}\,\mathrm{Sh}_s$) acts by s on Verma modules. The argument for Csh_s is similar. ☐

Definition 1.19. Given a reduced expression $w = s_{i_1} \cdots s_{i_k} \in W$ for an element of the Weyl group W, let $\mathrm{Sh}_w \coloneqq \mathbf{L}\,\mathrm{Sh}_{s_{i_1}} \cdots \mathbf{L}\,\mathrm{Sh}_{s_{i_k}}$ and $\mathrm{Csh}_w \coloneqq \mathbf{R}\,\mathrm{Csh}_{s_{i_1}} \cdots \mathbf{R}\,\mathrm{Csh}_{s_{i_k}}$.

Theorem 1.20 [MS05, thm. 5.7, lem. 5.10; Rou06, thm. 10.4]. Up to natural isomorphism, the functors Sh_w and Csh_w are independent of the choice of a reduced expression for w. The functors $\mathbf{L}\,\mathrm{Sh}_s$ and $\mathbf{R}\,\mathrm{Csh}_s$ are mutually inverse auto-equivalences of $D^{\mathrm{b}}(\mathcal{O}_0)$. Hence the assignment

$$B_W \to \mathrm{Aut}(D^{\mathrm{b}}(\mathcal{O}_0)), \quad s \mapsto \mathbf{L}\,\mathrm{Sh}_s, \quad s^{-1} \mapsto \mathbf{R}\,\mathrm{Csh}_s$$

defines an action of the braid group B_W on the category $\mathcal{O}_\lambda$. This action induces the action from Corollary 1.18 on $K_0(\mathcal{O}_\lambda)$.

Whereas Mazorchuk and Stroppel's proof works directly in the set-up of category $\mathcal{O}$, Rouquier's proof in fact shows the statement for the homotopy category of Soergel bimodules, then passes to the coinvariant algebra and obtains the theorem by virtue of Soergel's functor $\mathbf{V}$. We shall recall this proof in Section 5.

Note that this action of B_{S_n} on $D^{\mathrm{b}}(\mathcal{O}_0)$ does not come from an action of B_{S_n} on $\mathcal{O}$. In algebra and geometry, there are plenty of other examples for braid group actions on derived or other triangulated categories, all of which do not seem to exist in the abelian category [Kho02, §6.5].

2. Explicitly computing shuffling functors

After the formulation of the general theory of shuffling and translation functors in Section 1, we like to consider the involved functors more explicitly in the cases of $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_2$. We shall do so by dealing with modules in $\mathcal{O}$ in terms of their composition series and descriptions of them in the terminolgy of quiver representations. We firstly introduce our toolbox for $\mathfrak{g} = \mathfrak{sl}_2$ before applying it to the (more involved) example $\mathfrak{g} = \mathfrak{sl}_2$.

2.1. Shuffling functors on $\mathcal{O}_0(\mathfrak{sl}_2)$

After the formulation of the general theory of shuffling and translation functors in Section 1, we like to consider the involved functors more explicitly in the case $\mathfrak{g} = \mathfrak{sl}_2$. Recall the root system of $\mathfrak{sl}_2$ (see Figure 2.1) with Weyl group $S_2 = \{e, s\}$. We want to explicitly provide the images of Verma modules, simple modules and their projective covers under the shuffling functor Sh_s. There is only one wall which corresponds to the dominant and antidominant weight -1, hence the Verma module $M(-1)$ is both projective and simple [Hum08, prop. 3.8, thm. 4.8]. Since we know that T_{on} preserves Verma modules $M(w)$ and T_{off} preserves indecomposable projectives $P(w)$ (**??** 1.7), we know that $P(e) = M(e)$ and $M(s) = L(s)$ are mapped to $P(s)$ by Θ_s. The module $L(e)$ is annihilated, since it is translated to a lower wall (**??** 1.7.(iii)). For determining the image under the shuffling functor, we argue as follows:

$M(e)$: We know that $\Theta_s M(e) = P(s)$ and $\mathrm{Sh}_s M(e) = M(s)$ (Lemma 1.13), The corresponding short exact sequence $M(e) \overset{\eta}{\hookrightarrow} P(s) \twoheadrightarrow M(s)$ is the standard filtration of $P(s) = \Theta_s M(e)$, see Table 2.1.

$M(s)$: The module $P(s)$ has standard filtration $M(e) \hookrightarrow P(s) \twoheadrightarrow M(s)$, see Table 2.1. In fact, this is (up to scalars) the only extension of $M(s)$ by $M(e)$ [Car86, thm 3.8]. Thus, there is a unique non-zero map (up to scalars) $M(s) \hookrightarrow M(e) \hookrightarrow P(s)$, which can be written in terms of composition factors:

$$\underbrace{L(s)}_{M(s)} \hookrightarrow \underbrace{\begin{matrix} L(s) \\ L(e) \\ L(s) \end{matrix}}_{P(s)} \twoheadrightarrow \underbrace{\begin{matrix} L(s) \\ L(e) \end{matrix}}_{M(e)^\vee} \tag{2.1}$$

The picture is to be read as follows: It states for instance that $P(s)$ has simple quotient $L(s)$ (top) and also simple submodule $L(s)$ (bottom). Se Notation 2.5 for more details on the notation. Comparing composition sequences shows that $M(s) \hookrightarrow P(s)$ has $M(e)^\vee$ as its cokernel.

$P(s)$: We know that $P(s)$ occurs as the image $P(s) = \Theta_s M(e) = \Theta_s M(s)$. Since Θ_s is exact and the shuffling functor is right exact (see Definitions 1.8 and 1.12), we obtain a commutative diagram

$$\begin{tikzcd}
\Big\{ \arrow[d, bend right, "\Theta_s"'] \arrow[dd, bend right=60, "\mathrm{Sh}_s"'] & 0 \arrow[r] & M(e) \arrow[r] \arrow[d] & P(s) \arrow[r] \arrow[d] & M(s) \arrow[r] \arrow[d] & 0 \Big\} \\
\Big\{ & 0 \arrow[r] & P(s) \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & P(s) \arrow[r] \arrow[d] & 0 \Big\} \\
\Big\{ & & M(s) \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & M(e)^\vee \arrow[r] \arrow[d] & 0 \Big\} \\
& & 0 & 0 & 0 &
\end{tikzcd}$$

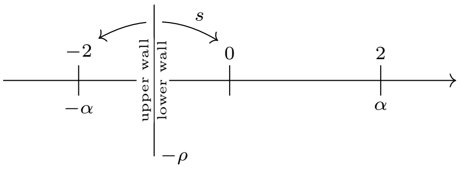


Figure 2.1: The root system of $\mathfrak{sl}_2$ is given by $\Phi = \{\pm\alpha\}$ with simple root α and reflection $s = s_\alpha$. The integral weight lattice is identified with $2\mathbf{Z}$.

Table 2.1: Composition series of the Verma modules, dual Verma modules and standard filtrations of indecomposable projectives in the principal block $\mathcal{O}_0$ for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$.

$M(e)$	$M(s)$	$M(e)^\vee$	$P(e)$	$P(s)$
$L(e)$ $L(s)$	$L(s)$	$L(s)$ $L(e)$	$M(e)$	$M(s)$ $M(e)$

module M	$\Theta_s M$	$\mathrm{Sh}_s\, M$
$M(e)$	$P(s)$	$M(s)$
$M(s)$	$P(s)$	$M(e)^\vee$
$P(s)$	$P(s)^{\oplus 2}$	$P(s)$
$L(e)$	0	0

Table 2.2: Images of Verma modules and indecomposable projectives in the principal $\mathcal{O}_0(\mathfrak{sl}_2)$ under the translation Θ_s through the s-wall and under shuffling Sh_s.

with split exact middle row. This gives $\Theta_s P(s) = P(s) \oplus P(s)$ and $\mathrm{Sh}_s\, P(s) \cong P(s)$ by commutativity. We thus obtain the images under Θ_s and Sh_s listed in Table 2.2.

2.1.1. $\mathcal{O}_0(\mathfrak{sl}_2)$ as module category over a path algebra

We want to find an approach to describe the structure of modules in $\mathcal{O}_0$ other than stacking composition factors as in (2.1). This is of particular importance if the extensions of simple modules in the composition series are more involved and are not just linearly ordered. We will see that this will already happen for $\mathfrak{sl}_3$, which necessitates a more rigid description of the modules in $\mathcal{O}_0$. Recall the following statement:

Theorem 2.1 (Gabriel, Mitchell, Morita) [Bas68, thm. II.1.3]. Let $\mathcal{C}$ be an abelian category and $P \in \mathcal{C}$. There is a functor $\mathcal{C} \to \text{Mod-}(\mathrm{End}_{\mathcal{C}}(P))$; $M \mapsto \mathrm{Hom}_{\mathcal{C}}(P, M)$, where $\mathrm{Hom}_{\mathcal{C}}(P, M)$ obtains a right $\mathrm{End}_{\mathcal{C}}(P)$-module by precomposition with endomorphisms of P, i. e. $f \,.\, g := f \circ g$ for $f \in \mathrm{Hom}_{\mathcal{C}}(P, M)$ and $g \in \mathrm{End}_{\mathcal{C}}(P)$. If P is a compact projective generator in $\mathcal{C}$, this functor is an exact equivalence of categories.

Corollary 2.2. For $\mathfrak{g} = \mathfrak{sl}_2$, there is an equivalence of categories $\mathcal{O}_0 \simeq \text{Mod-}A$, where A is the path algebra of the quiver with relations

$$Q := \overset{e}{\bullet} \overset{a}{\underset{b}{\rightleftarrows}} \overset{s}{\bullet} / (ba = 0). \tag{2.2}$$

Proof. The category $\mathcal{O}_0$ admits a projective generator $P = \bigoplus_{\lambda \in W \cdot 0} P(\lambda)$, see [Hum08, §3.13]. For $\mathfrak{sl}_2$ the endomorphism ring of $P = P(e) \oplus P(s)$ is generated as an algebra by the identity morphisms e_e of $P(e)$ and e_s of $P(s)$, the inclusion $a : P(e) \hookrightarrow P(s)$ and the quotient $b : P(s) \twoheadrightarrow M(s) \hookrightarrow P(e)$. By the standard filtration of $P(s)$, these maps admit the relation $b \circ a = 0$. The endomorphism ring $\mathrm{End}\big(P(e) \oplus P(s)\big)$ thus is isomorphic to the path algebra A. $\square$

Notation 2.3. We point to the convention we use for composing arrows of quivers, namely by writing them down like morphisms. Explicitly, this means that ba is the path $e \to as \to be$ (and not $s \to be \to as$). We denote the trivial path attached to a vertex v of a quiver by e_v. Let A be the path algebra of Q. Then $e_v A \leq A$ is the right ideal of all paths *ending* in the vertex v.

Given a module $M \in \mathcal{O}_0$, we can interpret $M \in \text{Mod-}A$ as representation of Q via

$$M = \big(\mathrm{Hom}(P(e), M) \overset{\circ a}{\underset{\circ b}{\rightleftarrows}} \mathrm{Hom}(P(s), M)\big), \tag{2.3}$$

where the structure maps are given by composition with the endomorphisms a and b of P. We shall freely identify modules $M \in \mathcal{O}_0$ with their images under the equivalence $\mathcal{O}_0 \simeq \text{Mod-}A$. We want to find bases of the Verma modules and indecomposable projectives as representations of Q as in (2.3). Since by Theorem 2.1 the equivalence $\mathcal{O}_0 \simeq \text{Mod-}A$ is exact, it preserves indecomposable projectives, whose images can be characterised explicitly:

Proposition 2.4 [Bar15, cor. 4.18, rmk. 4.20]. Let A be a finite dimensional basic $\mathbf{C}$-algebra, with a complete set $E \subseteq A$ of pairwise orthogonal primitive idempotents, e. g. the path algebra of a bounded quiver Q with the collection of trivial paths $\{e_v\}$ attached to all vertices $v \in Q_0$. Then the indecomposable projective A-modules are precisely the right ideals eA for $e \in E$.

For the path algebra A of Q, the indecomposable projective modules thus have bases

$$P(e) = e_e A = \Big(\langle e_e \rangle \overset{1}{\underset{0}{\rightleftarrows}} \langle b \rangle\Big), \qquad P(s) = e_s A = \Big(\langle a \rangle \xrightleftarrows[(\,1\ 0\,)]{\binom{0}{1}} \langle e_s, ab \rangle\Big). \tag{2.4}$$

Recall that any $P(\lambda)$ admits a standard filtration by Verma modules (see **??** 1.4). By exactness of the equivalence $\mathcal{O}_0 \simeq \text{Mod-}A$ we can carry over these standard filtrations from $\mathcal{O}_0$ to Mod-A. Explicitly, as a Q-representation, $M(s)$ fits into the short exact sequence

$$\underbrace{\langle e_e\rangle \overset{1}{\underset{0}{\rightleftarrows}} \langle b\rangle}_{P(e)=M(e)} \overset{a\circ}{\hookrightarrow} \underbrace{\langle a\rangle \overset{\binom{0}{1}}{\underset{(1\ 0)}{\rightleftarrows}} \langle e_s, ab\rangle}_{\Theta_s M(e)=P(s)} \twoheadrightarrow \underbrace{0 \rightleftarrows \langle e_s\rangle}_{M(s)} \tag{2.5}$$

corresponding to $\operatorname{Sh}_s M(e) = M(s)$. The action of Sh_s on the other Verma module $M(s)$ is exhibited by the sequence

$$\underbrace{0 \rightleftarrows \langle e_s\rangle}_{M(s)} \overset{ab\circ}{\hookrightarrow} \underbrace{\langle a\rangle \overset{\binom{0}{1}}{\underset{(1\ 0)}{\rightleftarrows}} \langle e_s, ab\rangle}_{\Theta_s M(s)=P(s)} \twoheadrightarrow \underbrace{\langle a\rangle \overset{0}{\underset{1}{\rightleftarrows}} \langle e_s\rangle}_{\operatorname{Sh}_s M(s)=M(e)^\vee}. \tag{2.6}$$

At this point, we want to emphasize the difference between $M(e)$ and $M(e)^\vee$: Although both have the same simple composition factors[3] $L(e) \cong (\mathbf{C} \rightleftarrows 0)$ and $L(s) \cong (0 \rightleftarrows \mathbf{C})$, the structure maps linking them are interchanged; namely in $M(e)^\vee$ socle and head are exchanged in comparison to $M(e)$.

2.2. Shuffling functors on $\mathcal{O}_0(\mathfrak{sl}_3)$

As a more involved example, we now want to apply the strategies developed so far to $\mathfrak{g} = \mathfrak{sl}_3$. Recall the root system of $\mathfrak{sl}_3$ from Figure 1.1 with Weyl group $W = S_3 = \langle s, t\rangle$, whose longest element $sts = tst$ we denote w_0. Verma modules in $\mathcal{O}_0$ have the composition series listed in Table 2.3, where we employ the following notation:

Notation 2.5. We depict composition factors as follows: consider the entry for $M(e)$ in Table 2.3. The depiction states that $M(e)$ has $L(w_0)$ as its unique simple submodule ($L(w_0)$ is the only entry in the bottom layer) and $L(e)$ as its unique simple quotient (the only entry in the top layer). The "intermediate layer" for $M(e)$ for instance depict the fact $M(e)/L(w_0)$ in turn has simple submodules $L(st)$ and $L(ts)$, or that $M(e)$ has e. g. a (non-simple) quotient $\genfrac{}{}{0pt}{}{L(e)}{L(s)}$. We also write down morphisms in terms of composition factors by drawing arrows which indicate where each composition factor is mapped: for example in $\mathfrak{g} = \mathfrak{sl}_2$, the nontrivial endomorphism ab of $P(s)$ is depicted

$$ab\colon P(s) = \left\{\begin{tikzcd}[row sep=tiny] L(s) \arrow[rdd] & L(s) \\ L(e) & L(e) \\ L(s) & L(s) \end{tikzcd}\right\} = P(s).$$

Any factor from which no arrow starts is in the kernel of the morphism, and any factor at which no arrow ends is in the cokernel. We use a similar notation for standard filtrations.

Again, our first step is to deduce the composition series of the images $\operatorname{Sh}_s M(ws)$ for $\ell(ws) > \ell(w)$ from the already known images $\operatorname{Sh}_s M(w) \cong M(ws)$, using the short exact sequences (1.10). The image of $\operatorname{Sh}_s M(s)$ under Sh_s is exhibited by the following short exact sequence:

$$\begin{tikzcd} M(s) \arrow[r, hook] \arrow[d, equal] & \Theta_s M(s) = P(s) \arrow[r, two heads] \arrow[d, equal] & \operatorname{Sh}_s M(s) \arrow[d, equal] \\ \begin{array}{c} L(s) \\ L(st)\ \ L(ts) \\ L(w_0) \end{array} & \begin{array}{c} L(s) \\ L(e)\ \ L(st)\ \ L(ts) \\ L(s)\ \ L(t)\ \ L(w_0) \\ L(st)\ \ L(ts) \\ L(w_0) \end{array} & \begin{array}{c} L(s) \\ L(e)\ \ L(st)\ \ L(ts) \\ L(t)\ \ L(w_0) \end{array} \end{tikzcd} \tag{2.7}$$

Remark 2.6. We see from (2.7) that $\operatorname{Sh}_s M(s)$ and $M(e)$ have the same composition factors (see Lemma 1.13), but arranged in a different order. For instance, $M(e)$ has simple quotient $L(e)$ and simple submodule $L(w_0)$ (see Table 2.3), whereas (2.7) shows that $\operatorname{Sh}_s M(s)$ has simple quotient $L(s)$ and simple submodules $L(w_0)$ and $L(t)$.

---

[3] To put it another way, both have the same dimension vector.

Table 2.3: Composition series of the Verma modules and standard filtrations of indecomposable projectives in the principal block $\mathcal{O}_0$ for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. See Notation 2.5 for an explanation of the depiction of the composition series. Note that (by definition) every Verma module $M(w)$ has unique simple quotient $L(w)$ and unique simple submodule $L(w_0)$. $M(w)$ is a submodule of $M(w')$ if and only if $w \geq w'$ in the Bruhat order. Likewise, $P(w)$ by definition has quotient $M(w)$, and $P(w')$ is a submodule of $P(w)$ if and only if $w \leq w'$.

$M(e)$	$M(s)$	$M(t)$	$M(st)$	$M(ts)$	$M(w_0)$
$L(e)$ $L(s)$ $L(t)$ $L(st)$ $L(ts)$ $L(w_0)$	$L(s)$ $L(st)$ $L(ts)$ $L(w_0)$	$L(t)$ $L(st)$ $L(ts)$ $L(w_0)$	$L(st)$ $L(w_0)$	$L(ts)$ $L(w_0)$	$L(w_0)$
$P(e)$	$P(s)$	$P(t)$	$P(st)$	$P(ts)$	$P(w_0)$
$M(e)$	$M(s)$ $M(e)$	$M(t)$ $M(e)$	$M(st)$ $M(s)$ $M(t)$ $M(e)$	$M(ts)$ $M(s)$ $M(t)$ $M(e)$	$M(w_0)$ $M(st)$ $M(ts)$ $M(s)$ $M(t)$ $M(e)$

For understanding the structure of $\operatorname{Sh}_s M(ts)$, we first need to deduce the structure of $\Theta_s M(ts)$ from the extension

$$\begin{tikzcd} M(t) \arrow[r, hook] \arrow[d, equal] & \Theta_s M(t) \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & M(ts) \arrow[d, phantom, "\shortparallel"] \\ \begin{smallmatrix} & L(t) & \\ L(st) & & L(ts) \\ & L(w_0) & \end{smallmatrix} & \begin{smallmatrix} & & L(ts) \\ & L(t) & L(w_0) \\ L(st) & L(ts) & \\ & L(w_0) & \end{smallmatrix} & \begin{smallmatrix} L(ts) \\ L(w_0) \end{smallmatrix} \end{tikzcd}$$

of $\operatorname{Sh}_s M(t) = M(ts)$ by $M(t)$. Since Lemma 1.13 asserts that $\Theta_s M(t) = \Theta_s M(ts)$, we obtain the quotient

$$\begin{tikzcd} M(ts) \arrow[r, hook] & \Theta_s M(ts) \arrow[r, two heads] & \operatorname{Sh}_s M(ts) \arrow[r, equal] & \begin{smallmatrix} & L(ts) & \\ L(t) & & L(w_0) \\ & L(st) & \end{smallmatrix}. \end{tikzcd}$$

By the same means, we obtain $\operatorname{Sh}_s M(w_0) = \begin{smallmatrix} L(w_0) \\ L(st) \end{smallmatrix}$. One observes the following:

Lemma 2.7. Let $w \in W$ and $s \in S$ be a simple reflection such that $\ell(ws) > \ell(w)$. Then $\operatorname{Sh}_s^2 M(w) = \operatorname{Sh}_s M(ws)$ has simple quotient $L(ws)$.

Proof. Since $\operatorname{Sh}_s M(w) = M(ws)$ is a quotient $\Theta_s M(w) \twoheadrightarrow M(ws)$, the module $\Theta_s M(w)$ has simple quotient $L(ws)$. Recall that $\Theta_s M(ws) = \Theta_s M(w)$ (Lemma 1.13). Since $M(ws) \hookrightarrow \Theta_s M(ws)$ is a proper inclusion, the quotient $\operatorname{Sh}_s = \Theta_s M(ws)/M(ws)$ has simple quotient $L(ws)$. ☐

However, studying the module structure solely relying on composition series is not satisfactory: Which submodule is generated by a certain simple composition factor? Does any of the extensions involved in the composition series split? Considering the composition series for the indecomposable projectives $P(w)$, one notes that the inclusions $P(w) \hookrightarrow \Theta_s P(w)$ are not uniquely determined by the composition series; how to determine $\operatorname{Sh}_s P(w)$ in this case?

These questions are difficult to answer in this framework; especially the last one is crucial.W We saw already in the $\mathfrak{sl}_2$-case that we can describe the modules in $\mathcal{O}_0$ as representations of a certain quiver, which will turn out to be an advantageous description for $\mathfrak{sl}_3$ as well.

2.2.1. $\mathcal{O}_0(\mathfrak{sl}_3)$ as module category over a path algebra

Unfortunately, we find ourselves trapped in a vicious circle: as we cannot really understand the homomorphisms between indecomposable projectives, we want to consider representations of quivers. However, in the case of $\mathfrak{g} = \mathfrak{sl}_2$, we obtained the quiver from exactly these maps (see Section 2.1.1).

Nevertheless, we know indecomposable projective quiver representations to be of the form $e_v A$ for A the path algebra and e_v the trivial path at the vertex v (see Proposition 2.4). Hence we can rely on [Str03b] providing the desired quiver in order to reconstruct all the other modules.

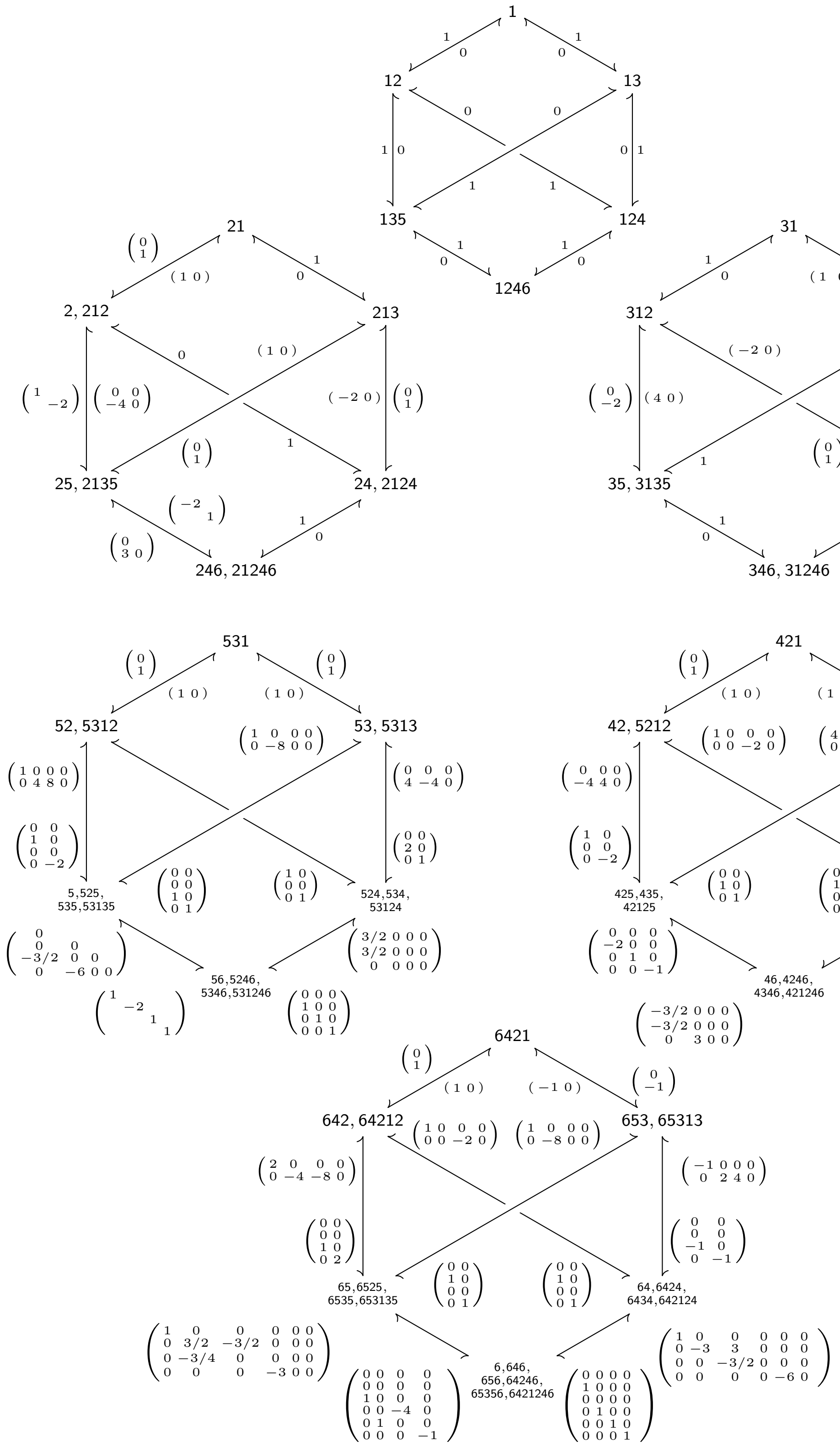


Figure 2.2: Indecomposable projective modules $P(w)$ of highest weight w for $\mathfrak{sl}_3$ as representations of the quiver Q from (2.9). Each hexagon denotes one module $P(w)$ for some Weyl group element $w \in W$ as in (2.10). The meaning of the hexagonal shape and the numbers is explained in Notation 2.8. For the vector space attached to one vertex we have chosen the denoted paths as basis vectors; see the explanation to (2.10). We provide the structure maps in terms of these bases, writing just 1 and 0 for the identity and the zero map. Letting w run through W, we arrange the modules $P(w)$ hexagonally as in (2.10).

Before discussing the relevant modules and the action of the shuffling functor on them we introduce some notation:

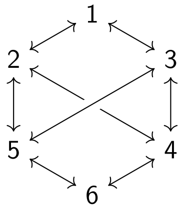
Notation 2.8. We arrange the weights in the following shape[4] and number them throughout this section as indicated in the second diagram:

$$\begin{tikzcd}
 & e \arrow[dl, dash] \arrow[dr, dash] & & & & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
s \arrow[d, dash] & & t \arrow[d, dash] & =: & 2 \arrow[d, dash] & & 3 \arrow[d, dash] \\
st \arrow[dr, dash] & & ts \arrow[dl, dash] & & 5 \arrow[dr, dash] & & 4, \arrow[dl, dash] \\
 & w_0 & & & & 6 &
\end{tikzcd} \tag{2.8}$$

where we denote by w_0 the longest element of $W = S_3$. We abbreviate paths by the numbers of the corners they pass, for instance we write 421 for the path $1 \to 2 \to 4$.

Recall from the proof of Corollary 2.2 that $\mathcal{O}_0 \simeq$ Mod-A for $A = \operatorname{End}_{\mathcal{O}} P$ the endomorphism ring of the projective generator $P = \bigoplus_{w \in W} P(w)$ (Theorem 2.1, [Hum08, §3.13]). It has been shown in [Str03b, §5.1.2; Mar06, thm. 4.1] that A is the path algebra $\mathbf{C}[Q]$ of the quiver with relations

$Q \quad =$



$$\begin{aligned}
121 &= 0 & 131 &= 0\\
242 &= 0 & 353 &= 0\\
431 &= 421 & 521 &= -2 \cdot 531\\
352 &= -2 \cdot 342 & 243 &= -2 \cdot 213\\
&= 4 \cdot 312 & &= -\frac{1}{2} \cdot 253\\
252 &= -4 \cdot 212 & 353 &= -313\\
652 &= 2 \cdot 642 & 643 &= -653\\
124 &= 134 & 135 &= -\frac{1}{2} \cdot 125\\
464 &= -3 \cdot 424 & 565 &= -\frac{3}{2} \cdot 535\\
256 &= -2 \cdot 246 & 346 &= 356\\
564 &= \frac{3}{2} \cdot (524 + 534) & 465 &= -\frac{3}{2} \cdot (425 + 435).
\end{aligned} \tag{2.9}$$

We denote an element of A by a path representing it up to the relations of Q.

2.2.2. Bases of representations of Q by paths

Indecomposable projectives By the equivalence $\mathcal{O}_0 \simeq$ Mod-A an arbitrary module $M \in \mathcal{O}_0$ is mapped to $\operatorname{Hom}(P, M)$, which becomes a representation of Q via

$$\operatorname{Hom}(P, M) = \begin{tikzcd}
 & \operatorname{Hom}\big(P(e), M\big) \arrow[dl, shift left, "\circ 21"] \arrow[dr, shift left, "\circ 31"] & \\
\operatorname{Hom}\big(P(s), M\big) \arrow[ur, shift left, "\circ 12"] \arrow[d, leftrightarrow] \arrow[drr, leftrightarrow] & & \operatorname{Hom}\big(P(t), M\big) \arrow[ul, shift left, "\circ 13"] \arrow[d, leftrightarrow] \arrow[dll, leftrightarrow] \\
\operatorname{Hom}\big(P(st), M\big) \arrow[dr, leftrightarrow] & & \operatorname{Hom}\big(P(ts), M\big). \arrow[dl, leftrightarrow] \\
 & \operatorname{Hom}\big(P(w_0), M\big) &
\end{tikzcd} \tag{2.10}$$

In particular, the indecomposable projectives $P(w) \in \mathcal{O}_0$ are (by abuse of notation) mapped to $P(w) = e_w A$. This becomes a representation of Q by attaching to the vertex v the vector space $e_w A e_v$ spanned by paths from v to w. We then can describe the structure maps of $P(w)$ as matrices after having chosen some paths in Q as basis vectors for each corner $e_w A e_v$.

Example 2.9. Let $M = P(e)$. The vector space $\operatorname{Hom}(P(e), P(e))$ in the top corner of (2.10) is one dimensional with basis vector $\mathrm{id}_{P(e)}$, which corresponds to the path $1 = e_e$. Similarly, $\operatorname{Hom}(P(s), P(e))$ (top left corner) contains only one homomorphism (up to scalars), denoted by 12. The entire representation

---

[4] Note that our schematic intentionally deviates from the arrangement in [Str03b]. Although Stroppel's hexagon resembles the familiar picture for the root system of $\mathfrak{sl}_3$, it turns out that the corners 4 and 5 stand for ts and ts resp.

$\mathrm{Hom}(P, P(e))$ has the following basis by paths with structure maps:

$$P(e) = Ae_e = \begin{tikzcd}
 & \mathsf{1} \arrow[dl, leftrightarrow, "1"', "0"] \arrow[dr, leftrightarrow, "1", "0"'] & \\
\mathsf{12} \arrow[d, leftrightarrow, "1"', "0"] \arrow[drr, leftrightarrow, "0", "1"'] & & \mathsf{13} \arrow[d, leftrightarrow, "0"', "1"] \arrow[dll, leftrightarrow, "0"', "1"] \\
\mathsf{135} \arrow[dr, leftrightarrow, "1", "0"'] & & \mathsf{124} \arrow[dl, leftrightarrow, "1"', "0"] \\
 & \mathsf{1246} &
\end{tikzcd} \tag{2.11}$$

The coefficients for the structure maps are obtained from the relations listed in (2.9); for instance,

$$\circ\mathsf{12} \colon \mathrm{Hom}\big(P(e), P(e)\big) \to \mathrm{Hom}\big(P(s), P(t)\big). \quad \mathsf{1} \mapsto \mathsf{12};$$
$$\circ\mathsf{21} \colon \mathrm{Hom}\big(P(s), P(t)\big) \to \mathrm{Hom}\big(P(e), P(e)\big), \ \mathsf{12} \mapsto \mathsf{121} = 0.$$

Note that we can see immediately from this structure that $P(e)$ has

- socle $L(w_0)$: all outgoing maps from the the bottom corner are zero (i. e. from the vector space $\mathrm{Hom}(P(w_0), P(e)) = \langle \mathsf{6421} \rangle$). The representation of Q with $\mathrm{Hom}(P(w_0), P(e))$ at corner $\mathsf{6}$ (and zero in all other corners) thus is a (simple) subrepresentation of $P(e)$;
- head $L(e)$: all maps into the top corner (i. e. into the vector space $\mathrm{Hom}(P(e), P(e)) = \langle \mathsf{1} \rangle$ whose only basis vector is the trivial path at vertex e) are zero; and
- each simple module $L(-)$ exactly once as composition factor since at every corner there is precisely one basis vector.

In fact, we can infer the entire composition series of $P(e)$ from this picture immediately. The procedure for the other indecomposable projectives is similar. A basis for every $P(w)$ together with matrices for the structure maps is given in Figure 2.2.

Verma modules Recall that the indecomposable projectives have standard filtrations by Verma modules, listed in Table 2.3. We can thereupon recover the Verma modules $M(-)$ (to be precise: their images as representations of the quiver Q) as quotients of the modules $P(-)$, beginning with $M(e) = P(e)$ and proceeding to $M(w_0)$. A basis for each Verma module is given in Figure 2.3. In particular, there are the following inclusions

$$\begin{tikzcd}
 & P(e) \arrow[dl, hook, "\mathsf{21}\circ"'] \arrow[dr, hook, "\mathsf{31}\circ"] \arrow[rrrr, equal, bend left] & & & & M(e) & \\
P(s) \arrow[d, hook, "\mathsf{52}\circ"'] \arrow[drr, hook, "\mathsf{42}\circ"] & & P(t) \arrow[d, hook, "\mathsf{43}\circ"] \arrow[dll, hook, "\mathsf{53}\circ"'] & & M(s) \arrow[ur, hook, "\mathsf{12}\circ"] & & M(t) \arrow[ul, hook, "\mathsf{13}\circ"'] \\
P(st) \arrow[dr, hook, "\mathsf{65}\circ"'] & & P(ts), \arrow[dl, hook, "\mathsf{64}\circ"] & & M(st) \arrow[u, hook, "\mathsf{25}\circ"] \arrow[urr, hook, "\mathsf{35}\circ"'] & & M(ts) \arrow[u, hook, "\mathsf{34}\circ"'] \arrow[ull, hook, "\mathsf{24}\circ"] \\
 & P(w_0) & & & & M(w_0) \arrow[ul, hook, "\mathsf{56}\circ"] \arrow[ur, hook, "\mathsf{46}\circ"'] &
\end{tikzcd} \tag{2.12}$$

of representations of Q.

2.2.3. Sh_s in terms of representations of Q

Verma modules We can use the description of indecomposable projectives and Verma modules from Section 2.2.2 to compute their images under the shuffling functor Sh_s as representations of Q. Recall that we already know the images of $M(e)$, $M(t)$ and $M(st)$ under Sh_s and that $\Theta_s M(w) = \Theta_s M(ws)$ by Lemma 1.13.

The adjunction map $\eta_{M(s)} \colon M(s) \to \Theta_s M(s) = P(s)$ is the unique inclusion $M(s) \hookrightarrow P(s)$ (Lemma 1.13 and comparison of composition factors from Table 2.3). Since we have chosen bases for $M(s)$ and $P(s) = \Theta_s M(s)$ (for instance the ones in Figures 2.2 and 2.3), $\eta_{M(s)}$ can explicitly be formulated as a collection of matrices attached to every vertex v of Q, representing the maps $M(s)_v \hookrightarrow P(s)_v$. We can compute a basis of its cokernel, which by Definition 1.12 is the image under Sh_s:

$$\mathrm{Sh}_s M(s) := \operatorname{coker} \eta_{M(s)} = \begin{tikzcd}
 & \mathsf{21} \arrow[dl, leftrightarrow, dashed, "0"', "1"] \arrow[dr, leftrightarrow, "1", "0"'] & \\
\mathsf{2} \arrow[d, leftrightarrow, "1"', "0"] \arrow[drr, leftrightarrow, "0", "1"'] & & \mathsf{213} \arrow[d, leftrightarrow, dashed, "-4"', "0"] \arrow[dll, leftrightarrow, dashed, "-2"', "0"] \\
\mathsf{25} \arrow[dr, leftrightarrow, "-2", "0"'] & & \mathsf{24}. \arrow[dl, leftrightarrow, "1"', "0"] \\
 & \mathsf{246} &
\end{tikzcd} \tag{2.13}$$

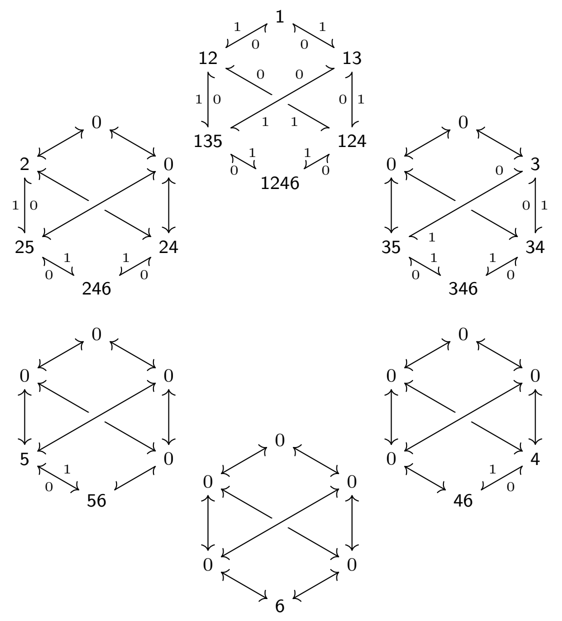
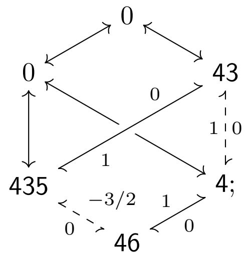


Figure 2.3: Bases for Verma modules $M(w)$ of highest weight w for $\mathfrak{sl}_3$ as representations of the quiver (2.9). Each hexagon denotes one Verma module. The arrangement of weights into the hexagon is as in (2.8). Although we realise Verma modules as quotients of the projective modules from Figure 2.2 by Table 2.3, we do *not* distinguish in our notation between paths (as basis vectors of some $P(w)$) and their image under $P(w) \twoheadrightarrow M(w)$. The depiction thus provides basis vectors (in terms of representing paths) attached to each corner of Q. The structure maps are either the identity 1 or the zero map 0.

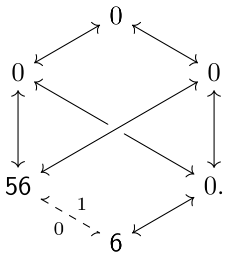
Remark 2.10. As representation of Q, The image $\mathrm{Sh}_s\, M(s)$ has the same dimension vector as $M(e)$, i. e. both have the same simple composition factors with multiplicities. The dashed structure maps indicate extensions in the composition series whose head and socle exchanged w. r. t. the series of $M(s)$. This observation is analogous to Remark 2.6.

In the same manner, we obtain the images of the other Verma modules under Sh_s:

$\mathrm{Sh}_s\, M(ts) =$



$\mathrm{Sh}_s\, M(w_0) =$



Projective covers We already know $P(e) = M(e)$ and hence $\mathrm{Sh}_s\, P(e) = P(s)$. To obtain $\mathrm{Sh}_s\, P(t)$ and $\mathrm{Sh}_s\, P(st)$ we apply the following strategy. Recall that any $P(w)$ has a standard filtration $0 = M_0 \subset \cdots \subset M_\ell = P(w)$ with subquotients isomorphic to Verma modules. Applying the exact functor Θ_s yields a filtration of $\Theta_s P(w)$ with subquotients isomorphic to translated Verma modules. By naturality, the adjunction map η gives a commutative diagram

$$\begin{tikzcd}
M_1 \arrow[r, hook] \arrow[d, hook] & M_2 \arrow[r, hook] \arrow[d, hook] & \dots \arrow[r, hook] & M_{\ell-1} \arrow[r, hook] \arrow[d, hook] & P(w) \arrow[d, hook] \\
\Theta_s M_1 \arrow[r, hook] & \Theta_s M_2 \arrow[r, hook] & \dots \arrow[r, hook] & \Theta_s M_{\ell-1} \arrow[r, hook] & \Theta_s P(w)
\end{tikzcd} \tag{2.14}$$

of inclusions (see Lemma 1.15), which allows us to infer the adjunction map $P(w) \hookrightarrow \Theta_s P(w)$. Making things explicit, we compute the following images of indecomposable projectives:

$P(t)$: we compute $\mathrm{Sh}_s\, P(t)$ by applying Θ_s and Sh_s to the standard filtration of $P(s)$:

$$\begin{tikzcd}
\Big\{ \arrow[d, bend right, "\Theta_s"'] \arrow[dd, bend right=60, "\mathrm{Sh}_s"'] & M(e) \arrow[r, hook] \arrow[d] & P(t) \arrow[r, two heads] \arrow[d] & M(t) \arrow[d] & \Big\} \\
\Big\{ & P(s) \arrow[r, hook] \arrow[d, two heads] & \Theta_s P(t) \arrow[r, two heads] \arrow[d, two heads] & \Theta_s M(t) \arrow[d, two heads] & \Big\} \\
\Big\{ & M(s) \arrow[r] & \mathrm{Sh}_s\, P(t) \arrow[r, two heads] & M(ts) & \Big\}.
\end{tikzcd} \tag{2.15}$$

Since we already know bases of $P(s)$ and $\Theta_s M(t)$ as representations of Q,[5] we can compute a basis for their extension $\Theta_s P(ts)$, which by comparison shows $\Theta_s P(t) \cong P(ts)$. Commutativity of the diagram allows us to find the vertical adjunction map $P(t) \to \Theta_s P(t)$ and its cokernel $\operatorname{Sh}_s P(t)$, which in turn fits into the (exact, see Lemma 1.15) bottom row of (2.15):

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & & & 0 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & & & 0 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
2 \arrow[dd, leftrightarrow] \arrow[ddrr, leftrightarrow] & & 0 \arrow[dd, leftrightarrow] \arrow[ddll, leftrightarrow] & & 42 \arrow[dd, leftrightarrow] \arrow[ddrr, leftrightarrow] & & 0 \arrow[dd, leftrightarrow] \arrow[ddll, leftrightarrow] & & 0 \arrow[dd, leftrightarrow] \arrow[ddrr, leftrightarrow] & & 0 \arrow[dd, leftrightarrow] \arrow[ddll, leftrightarrow] \\
 & & {} \arrow[rr, hook, "42\circ"] & & {} & & {} \arrow[rr, two heads] & & {} & & \\
25 \arrow[dr, leftrightarrow] & & 24 \arrow[dl, leftrightarrow] & & 425 \arrow[dr, leftrightarrow] & & {4,524} \arrow[dl, leftrightarrow] & & 0 \arrow[dr, leftrightarrow] & & 4. \arrow[dl, leftrightarrow] \\
 & 246 & & & & {46,4246} & & & & 46 & \\
 & \underbrace{\hphantom{xxxxxxxx}}_{\operatorname{Sh}_s M(e)=M(s)} & & & & \underbrace{\hphantom{xxxxxxxx}}_{\operatorname{Sh}_s P(t)} & & & & \underbrace{\hphantom{xxxxxxxx}}_{\operatorname{Sh}_s M(t)=M(ts)} &
\end{tikzcd} \tag{2.16}$$

$P(st)$: There are similar diagrams for the standard filtration of $P(st)$. We want to make this explicit once again, primarily since the images of $P(st)$ under translation and shuffling turn out to have an interesting appearance, and secondly in order to illustrate an extensive argumentation based on filtration factors.

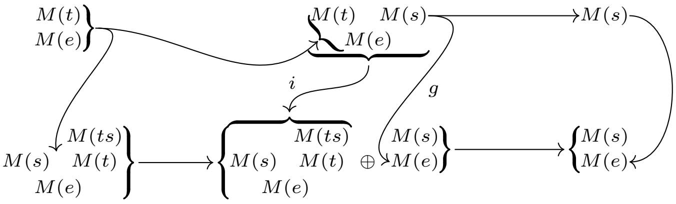
Recall from Table 2.3 that $P(st)$ has a standard filtration consisting of the two short exact sequences

$$K \hookrightarrow P(st) \twoheadrightarrow M(st), \qquad P(t) \hookrightarrow K \twoheadrightarrow M(s). \tag{2.17}$$

Apply Θ_s to the second sequence:

$$\begin{tikzcd}
{} \arrow[dd, bend right=60, "\Theta_s"'] & \Big\{ & P(t) \arrow[r, hook] \arrow[dd] & K \arrow[r, two heads] \arrow[dd, "\eta_K"', "\begin{pmatrix} i \\ g \end{pmatrix}"] & M(s) \arrow[dd] & \Big\} \\
 & & & & & \\
{} & \Big\{ & P(ts) \arrow[r, hook] & P(ts) \oplus P(s) \arrow[r, two heads] & P(s) & \Big\},
\end{tikzcd} \tag{2.18}$$

where the second row splits by exactness. The adjunction map η_K is obtained from composition factors through the diagram. Namely, (2.18) looks as follows in terms of standard filtrations (see Notation 2.5 for an explanation of the notation):



(2.19)

which yields $i\colon K \hookrightarrow P(ts)$ and $g\colon K \twoheadrightarrow M(s) \hookrightarrow M(e) \hookrightarrow P(s)$.

Now that we have a standard filtration of K and know its adjunction maps, we may obtain $\Theta_s P(st)$ by applying Θ_s to the first sequence from (2.17). Explicitly, we see that $\Theta_s P(st) = P(w_0) \oplus P(s)$ fits into the bottom sequence of

$$\begin{tikzcd}
{} \arrow[dd, bend right=60, "\Theta_s"'] & \Big\{ & K \arrow[r, hook] \arrow[dd, "\eta_K"] & P(st) \arrow[r, two heads] \arrow[dd, "\eta_{P(st)}"', "\begin{pmatrix} j \\ h \end{pmatrix}"] & M(st) \arrow[dd] & \Big\} \\
 & & & & & \\
{} & \Big\{ & P(ts) \oplus P(s) \arrow[r, hook] & P(w_0) \oplus P(s) \arrow[r, two heads] & \Theta_s M(st) & \Big\}
\end{tikzcd} \tag{2.20}$$

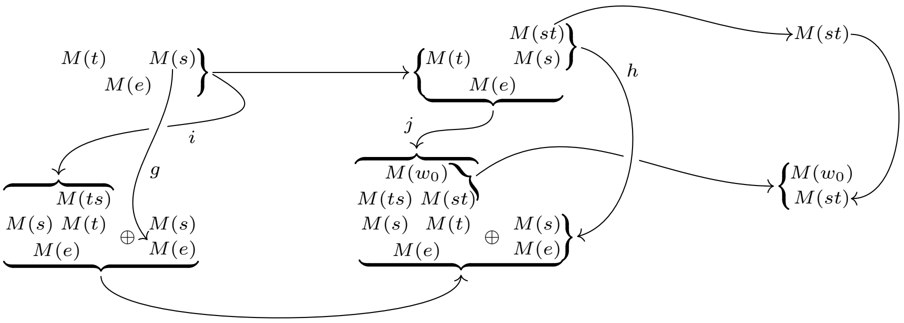
---

[5] The latter is an extension of $M(ts)$ by $M(t)$, and this extension is unique up to scalars [Car86, thm. 3.8].

Table 2.4: Translated and shuffled indecomposable projective modules in $\mathcal{O}_0$ for $\mathfrak{sl}_3$.

$P(-)$	$\Theta_s P(-)$	$\operatorname{Sh}_s P(-)$
$P(e)$	$P(s)$	$M(s)$
$P(s)$	$P(s)^{\oplus 2}$	$P(s)$
$P(st)$	$P(w_0) \oplus P(s)$	$\Theta_t M(ts) \oplus P(s)$
$P(t)$	$P(ts)$	see (2.16)
$P(ts)$	$P(ts)^{\oplus 2}$	$P(ts)$
$P(w_0)$	$P(w_0)^{\oplus 2}$	$P(w_0)$

by considering the diagram in terms of standard filtrations:



(2.21)

The maps i, g are taken from (2.19); the other two are the inclusion $j : P(st) \hookrightarrow P(w_0)$ and the inclusion $h : \begin{smallmatrix} M(st) \\ M(s) \end{smallmatrix} \hookrightarrow \begin{smallmatrix} M(s) \\ M(e) \end{smallmatrix}$. The image $\operatorname{Sh}_s P(st)$ thus is

$$\operatorname{Sh}_s P(st) = \operatorname{coker}\bigl(\begin{smallmatrix} j \\ h \end{smallmatrix}\bigr) \cong \Bigl(\begin{smallmatrix} M(w_0) \\ M(ts) \end{smallmatrix}\Bigr) \oplus P(s) \tag{2.22}$$

We point to the circumstance that the right summand $P(s)$ in (2.22) is *not* identical to the respective summand $P(s)$ of (2.21) but intersects both $P(w_0)$ and $P(s)$ non-trivially. To see this, one chases composition factors through the diagram. Note that the first summand is just $\Theta_t M(ts)$, so that $\operatorname{Sh}_s P(st) \cong \Theta_t M(ts) \oplus P(s)$.

$P(ts)$, $P(w_0)$: Recall from Lemma 1.14 that $\Theta_s^2 P(w) = \Theta_s P(w) \oplus \Theta_s P(w)$. Since we just saw that $\Theta_s P(t) = P(ts)$, we thereupon obtain $\Theta_s P(ts) = P(ts) \oplus P(ts)$.

The last module $P(w_0)$ occurs in $P(w_0) \oplus P(s) = \Theta_s P(st)$; and we have $\Theta_s(P(w_0) \oplus P(s)) = (P(w_0) \oplus P(s)) \oplus (P(w_0) \oplus P(s))$. Recall from Section 1 that $\mathcal{O}_0$ is a Krull-Schmidt category; in particular every projective object decomposes uniquely (up to permutation and isomorphism) into a direct sum of finitely many indecomposable projective modules $P(w)$. Uniqueness allows to split off $\Theta_s P(s) = P(s) \oplus P(s)$ from $\Theta_s(P(w_0) \oplus P(s))$ [Kra15, cor. 4.3]; we thus obtain $\Theta_s P(w_0) = P(w_0) \oplus P(w_0)$.

To understand the images $\operatorname{Sh}_s P(ts)$ and $\operatorname{Sh}_s P(w_0)$, we use that Sh_s is exact on modules with standard filtrations.

Altogether, we obtain the images of indecomposable projectives under translation and shuffling listed in Table 2.4.

Remark 2.11. The argumentation for $P(st)$ can be formulated more rigorously in the same fashion as for $P(s)$. It is indeed enlightening to track how the maps involved are formed as module homomorphisms over the path algebra A.

2.3. Gradings on $\mathcal{O}_0$; Iwahori-Hecke algebra

The path algebra of a quiver comes with a natural grading having the trivial paths in degree 0 and all arrows in degree 1. Since all relations of the quiver Q in (2.9) are homogeneous, we can endow Mod-A (and by Theorem 2.1 also $\mathcal{O}_0$) with a grading which was not visible from the Lie-theoretic point of view.

Let $\mathcal{O}_0^{\mathbf{Z}} := \text{grMod-}A$ be the thus graded category $\mathcal{O}_0$. See [Str03a; RS15] for more information on $\mathcal{O}_0^{\mathbf{Z}}$. We denote by $\langle - \rangle \colon \text{grMod-}A \xrightarrow{\simeq} \text{grMod-}A$ the upward shift of the grading. We shall freely identify modules in $\mathcal{O}_0$ and their graded lifts in $\mathcal{O}_0^{\mathbf{Z}}$.

Such a graded category $\mathcal{O}_0^{\mathbf{Z}}$ can be defined for an arbitrary semisimple complex Lie algebra [Str03b, sec. 5.1.2]. The mutual inclusions among projective covers and among Verma modules are of order one, which can be seen in (2.12) for $\mathfrak{g} = \mathfrak{sl}_3$. Note that, since A is positively graded, any inclusion of a proper submodule into an A-module is a map of positive order. We can thus recover the order of the composition factors of a module from the grading [Str03a].

If we look at the inclusions $M(w) \hookrightarrow \Theta_s M(w)$ in terms of the chosen homogeneous basis, we see that they are of order one if w lies above an s-wall, and of order two if w lies below the s-wall. If we let $\Theta_s M(ws) \cong \Theta_s M(w)\langle -1 \rangle$ for $\ell(ws) > \ell(w)$ and shift $\operatorname{Sh}_s M(ws)$ accordingly, we can turn (1.10) into short exact sequences of graded modules

$$M(w)\langle 1 \rangle \hookrightarrow \Theta_s M(w) \twoheadrightarrow M(ws); \qquad M(ws)\langle 1 \rangle \hookrightarrow \Theta_s M(ws) \twoheadrightarrow \operatorname{Sh}_s M(ws). \tag{2.23}$$

Definition 2.12. Given a graded algebra A, the Grothendieck group $K_0(\text{grMod-}A)$ becomes a $\mathbf{Z}[q^{\pm 1}]$-module by letting the grading shift $\langle - \rangle$ of grMod-A act by q, i. e. $[M\langle 1 \rangle] := q[M]$ for any graded A-module M.

With the two short exact sequences above, we thus obtain

$$\begin{aligned} [\Theta_s M(w)] &= q[M(w)] + [M(ws)] & [\operatorname{Sh}_s M(w)] &= [M(ws)] \\ [\Theta_s M(ws)] &= [M(w)] + q^{-1}[M(ws)] & [\operatorname{Sh}_s M(ws)] &= [\Theta_s M(ws)] - q[M(ws)] \\ & & &= [M(w)] + (q^{-1} - q)[M(ws)] \end{aligned} \tag{2.24}$$

whenever $\ell(ws) > \ell(w)$. To spell it out more concisely, Sh_s acts by s if the word gets longer and by $s + (q^{-1} - q)$ if it gets shorter when multiplied with s. These are precisely the relations of the following algebra:

Definition 2.13 [Lus03]. The *Iwahori-Hecke algebra* $\mathsf{H}_q(W)$ of a Coxeter system (W, S) is the $\mathbf{Z}[q^{\pm 1}]$-algebra with generators H_s for $s \in S$, and relations

$$H_s^2 = 1 + (q^{-1} - q)H_s, \qquad \underbrace{H_s H_t H_s \cdots}_{m_{st} \text{ factors}} = \underbrace{H_t H_s H_t \cdots}_{m_{st} \text{ factors}}$$

for two simple reflections $s, t \in S$ with $s \neq t$. $\mathsf{H}_q(W)$ thus is a quotient of the group algebra $\mathbf{Z}[q^{\pm 1}][B_W]$. It is a deformation of the group algebra $\mathbf{Z}[W]$ since $\mathsf{H}_q(W) \otimes_{\mathbf{Z}[q^{\pm 1}]} \mathbf{Z} \cong \mathbf{Z}[W]$ where we let q act on $\mathbf{Z}$ by 1.

The element H_s is a unit with inverse $H_s^{-1} = H_s - (q^{-1} - q)$. Given a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ of $w \in W$, let $H_w := H_{s_{i_1}} \cdots H_{s_{i_1}}$. This is independent of the reduced expression [Mat64, thm. 4]. The algebra $\mathsf{H}_q(W)$ is a free $\mathbf{Z}[q^{\pm 1}]$ module with basis $\{H_w\}_{w \in W}$ [Bou07, exc. 22; Jac12, thm. 2.3; Bum04, thm. 46.3], called *standard basis*. In terms of the standard basis, multiplication is given by

$$H_w \cdot H_s = \begin{cases} H_{ws} & \text{if } \ell(ws) = \ell(w) + \ell(s) \\ H_w + (q^{-1} - q)H_{ws} & \text{otherwise} \end{cases} \tag{2.25}$$

by the defining relations.

Remark 2.14. The seemingly arbitrary relations of $\mathsf{H}_q(W)$ arise, for $W = S_n$, by realising $\mathsf{H}_q(S_n)$ as follows: Let $B \subseteq GL_n(\mathbf{F}_q)$ be the (Borel) subgroup of upper triangular matrices. Then $\mathsf{H}_q(S_n)$ is realised as the convolution algebra of complex-valued functions on $GL_n(\mathbf{F}_q)$ which are B-biinvariant (i. e. constant on double cosets $B \backslash GL_n(\mathbf{F}_q)/B$). The parameter q hence arises as the cardinality of the finite field $\mathbf{F}_q$ [Bum04, thm. 46.4].

2.3.1. Kazhdan-Lusztig theory

Theorem 2.15 (Kazhdan-Lusztig) [KL79, thm. 1.1]. The algebra $\mathsf{H}_q(W)$ has an involution $\overline{(-)}$ defined by $\bar{q} := q^{-1}$ and $\bar{H}_w := H_{w^{-1}}^{-1}$. There is a unique basis of $\mathsf{H}_q(W)$ by elements C_w satisfying

$$\bar{C}'_w = C'_w; \qquad C'_w = \sum_{v \leq w} p_{vw} H_v$$

with coefficients $p_{ww} = 1$ and, for $v < w$, $p_{vw} \in \mathbf{Z}[q]_+$ is a polynomial without constant term. We set $p_{vw} = 0$ for $v \nless w$.

Definition 2.16. The basis $\{C'_w\}_{w\in W}$ of $\mathsf{H}_q(W)$ is called *Kazhdan-Lusztig basis*. The base change coefficients p_{wv} are called *Kazhdan-Lusztig polynomials*.

For the computation of Kazhdan-Lusztig polynomials there is a recursive algorithm, see [KL79, thm. 1.1]. A self-contained expository article on the matter is given in [Soe97]. Kazhdan-Lusztig polynomials occur as the multiplicities of simple factors in composition series of Verma modules, and hence by the BGG reciprocity theorem (see **??** 1.4.(v)) as multiplicities of Verma modules in the standard filtrations of indecomposable projectives:

Theorem 2.17 [BB81, §4; BK81, §8]. For any two elements $v, w \in W$,

$$[M(w) : L(v)] = (P(v) : M(w)) = p_{v,w}(1). \tag{2.26}$$

Remark 2.18. The formula in (2.26) differs from the usual form $(P(v) : M(w)) = n_{w_0v,w_0w}(1)$ found in the literature. This is owing to the fact that we take $M(w) := M(w\,.\,0)$ with the *dominant* weight 0 whereas many other sources use $M(w) := M(-2\rho)$ for the *antidominant* weight -2ρ. Secondly, the polynomials n_{vw} are defined with respect to another presentation for $\mathsf{H}_q(W)$ which seems to be slightly more widespread than the one from Definition 2.13. Namely, the Iwahori-Hecke algebra may also be defined as the $\mathbf{Z}[t^{\pm 1/2}]$-algebra with presentation

$$\mathsf{H}'_t(W) = \langle T_s \text{ for } s \in S \mid \text{distant and braid relations, and for all } i\text{: } T_i^2 = t + (t-1)T_i\rangle. \tag{2.27}$$

In contrast, convention from Definition 2.13 is used e. g. in [Lus03; Soe97].

For $q = t^{-1/2}$ there is an isomorphism $\mathsf{H}_q(W) \xrightarrow{\cong} \mathsf{H}'_t(W)$; $H_s \mapsto t^{-1/2}T_s$. The adapted version of Theorem 2.15 in terms of the presentation (2.27) asserts that the Kazhdan-Lusztig basis vectors C_w relate to the generators T_s by

$$C'_w = t^{-\ell(w)/2} \sum_{v\in W} n_{vw} T_v$$

for the appropriate (and more common) definition Kazhdan-Lusztig polynomials n_{vw}. By $T_v = t^{\ell(v)/2}H_v$ this shows

$$C'_w = \sum_{v\in W} t^{\ell(v)/2+\ell(w)/2} n_{vw} H_v$$

and therefore by comparison with Theorem 2.15 we get $p_{vw} = q^{\ell(v)-\ell(w)} n_{vw}$. For the investigation of composition series of modules in $\mathcal{O}$ we consider the presentation in Definition 2.13 by H_s's to be the more useful one.

In fact, even more than stated in Theorem 2.17 is true. Namely, the exponents of q in the Kazhdan-Lusztig polynomial also encode the "layer" in which a composition factor occurs.

Definition 2.19 [Hum08, §8.14]. Let $v \in W$ and $M(v) = M^0 \supset M^1 \supset \cdots M^{\ell\ell} \supset 0$ a filtration for $M(v)$ of *minimal* length $\ell\ell$ whose subquotients $M_k := M^k/M^{k+1}$ are semisimple.[6] Such a filtration is called *Loewy filtration* and the subquotients M_k are called *Loewy layers*.

Example 2.20. The extremal cases of a Loewy filtration are the following:

(i) Recall that a proper submodule $N \subset M$ is a *maximal submodule* if there is no submodule $N \subset S \subset M$ properly lying "between" N and M. The *radical* $\operatorname{rad} M$ of M is intersection of all maximal submodules. It is the smallest submodule such that $M/\operatorname{rad} M$ is semisimple. The *radical filtration* $M^\bullet_{\text{rad}}$ of M is the Loewy filtration $M^0_{\text{rad}} = M, M^1_{\text{rad}} = \operatorname{rad} M, M^2_{\text{rad}} = \operatorname{rad} M^1_{\text{rad}}, \ldots, M^{\ell\ell}_{\text{rad}} = \operatorname{rad} M^{\ell\ell-1}_{\text{rad}}$, where $M^{\ell\ell}$ is semisimple.

(ii) Recall that the *socle* $\operatorname{soc} M$ of a module M is its largest semisimple submodule. The *socle filtration* $M^\bullet_{\text{soc}}$ of M is the Loewy filtration defined by the Loewy layers $M^{\text{soc}}_1 = \operatorname{soc} M, M^{\text{soc}}_2 = \operatorname{soc}(M/M^{\text{soc}}_1), \ldots, M^{\text{soc}}_{\ell\ell} = \operatorname{soc}(M/M^{\text{soc}}_{\ell\ell-1})$ with $M_{\ell\ell}$ semisimple.

---

[6] A filtration of *maximal* length is just a composition series of $M(v)$. In particular, such a filtration of finite minimal length exists.

Table 2.5: Kazhdan-Lusztig polynomials for $\mathfrak{sl}_3$. See the explanations on our conventions for the polynomials in Remark 2.18. By Theorem 2.21, one can infer the composition series listed in Table 2.3 from these polynomials. By **??** 1.4.(iv) these can be used to compute dimensions of Hom-spaces between projectives. For $\mathfrak{sl}_3$, see Table 4.9 for these dimensions.

$p_{\downarrow,\rightarrow}$	e	s	t	st	ts	w_0
e	1	q	q	q^2	q^2	q^3
s	0	1	0	q	q	q^2
s	0	0	1	q	q	q^2
st	0	0	0	1	0	q
ts	0	0	0	0	1	q
w_0	0	0	0	0	0	1

A module M is called *rigid* if both coincide. That is, a rigid module has a unique (up to isomorphism) Loewy filtration.

Theorem 2.21 (generalised Kazhdan-Lusztig theorem) [Irv88, thm. 1, 2, cor. 7; see also Str03a, thm. 7.6].

(i) Verma modules are rigid.

(ii) The composition factor multiplicities of their Loewy layers relate to the Kazhdan-Lusztig polynomials by

$$p_{vw} = \sum_{k\geq 0}[M_{\ell(v)-\ell(w)+2k} : L(w)]q^k.$$

To state it differently, the coefficient of the q^k-term of p_{vw} is the multiplicity of $L(w)$ in the k-th Loewy layer of $M(w)$.

(iii) Consider an indecomposable projective module $P(w)$. Let $P \supset P^1 \supset \cdots \supset P^\ell \supset 0$ be a filtration of maximal length whose subquotients $P_k = P^k/P^{k+1}$ are isomorphic to Verma modules.[7] The standard filtration multiplicities of the subquotients of this filtration relate to the Kazhdan-Lusztig polynomials by

$$p_{vw} = \sum_{k\geq 0}(P_{\ell(w)-\ell v+2k} : M(v))q^k.$$

The last statement is a q-analogue of the BGG reciprocity theorem, see **??** 1.4.(v). Note that Irving states his theorem for antidominant weights instead dominant ones. Later we shall also use a parabolic analogue of these statements, see Theorem 4.10.

Example 2.22. For $\mathfrak{sl}_3$, recall the composition series of Verma modules from Table 2.3. Each line in the depictions constitutes a Loewy layer. Compare this to the Kazhdan-Lusztig polynomials for $\mathfrak{sl}_3$ listed in Table 2.5.

2.3.2. $K_0\big(\mathcal{O}_0^{\mathbf{Z}}(\mathfrak{sl}_n)\big)$ is isomorphic to the Iwahori-Hecke algebra

Using Kazhdan-Lusztig theory we obtain the following connection of graded category $\mathcal{O}^{\mathbf{Z}}$ and the Hecke algebra:

Theorem 2.23. There is an isomorphism of $\mathbf{Z}[q^{\pm 1}]$-modules

$$K_0\big(\mathcal{O}_0^{\mathbf{Z}}(\mathfrak{sl}_n)\big) \to \mathsf{H}_q(S_n), \quad [M(w)\langle q\rangle] \mapsto qH_w, \quad [P(w)\langle q\rangle] \mapsto qC_w.$$

The shuffling functor Sh_s then acts by $\cdot H_s$.

Proof. Recall that $K_0(\mathcal{O}_0(\mathfrak{sl}_n))$ admits a basis $\{[M(w)]\}_{w\in W}$ as free abelian group; see the proof of Corollary 1.18. Therefore, taking gradings into account, $\{[M(w)]\}_{w\in W}$ is a basis of $K_0(\mathcal{O}_0^{\mathbf{Z}}(\mathfrak{sl}_n))$ as free $\mathbf{Z}[q^{\pm 1}]$-module, and $\mathsf{H}_q(W)$ and $K_0(\mathcal{O}_0^{\mathbf{Z}}(\mathfrak{sl}_n))$ are isomorphic free $\mathbf{Z}[q^{\pm 1}]$-modules by sending $[M(w)]$ to H_w. By Theorem 2.17, one can express the class $[P(w)] = \sum_{v\leq w} p_{v,w}[M(v)]$ in $K_0(\mathcal{O}_0^{\mathbf{Z}}(\mathfrak{sl}_n))$. This shows that the isomorphism assigns $[P(w)] \mapsto C_w$. The action of Sh_s is taken from (2.24). ☐

To express it more concisely, the theorem shows that by virtue of Verma modules, indecomposable projectives and the shuffling functor, the category $\mathcal{O}_0(\mathfrak{sl}_n)$ categoryfies the Iwahori-Hecke algebra; see [KMS09, and references; Maz12] for an overview of algebraic catorification.

---

[7] Albeit lacking an appropriate name for such a series, we should consider it as the analogue of a Loewy filtration corresponding to standard filtrations instead of composition series.

3. Spherical twist functors

There is a general construction of an action of the braid group B_{S_n} of the symmetric group on the derived category $D^{\mathrm{b}}(\mathfrak{S})$ of some suitable category $\mathfrak{S}$ [ST01, sec. 2], given in terms of *twist functors* that are parametrised by *spherical objects*. Seidel and Thomas aim at letting $\mathfrak{S}$ be some category of quasi-coherent sheaves, which does not have enough projectives, necessitating some technicalities for the category that is acted on.

However, we will not have such difficulties with category $\mathcal{O}_0$ which has enough projectives. This section is dedicated to recapitulating the construction and proof given in [ST01] in a manner sufficient for our case with some details added sporadically. We later shall examine whether the braid group action given by shuffling functors can be understood as twisting.

Seidel and Thomas' spherical twist functors Let $\mathcal{C}$ be an abelian $\mathbf{C}$-linear category of finite global dimension. Given a finite dimensional vector space V and an object $X \in \mathcal{C}$, we can always form their tensor product $V \otimes_{\mathbf{C}} X \in \mathcal{C}$ which lies in $\mathcal{C}$ and is just a fancy way to express a direct sum of copies of X indexed by a basis of V. Likewise, we can construct the space $\mathrm{Lin}_{\mathbf{C}}(V, X)$ of $\mathbf{C}$-linear maps from V to X as a product of copies of X indexed by a basis of V. If $\mathcal{C}$ contains also infinite direct sums and products, both constructions can be extended to infinite dimensional vector spaces; however, we do not need this for our purposes. The functors $V \otimes_{\mathbf{C}} -$ and $\mathrm{Lin}_{\mathbf{C}}(V, -)$ are exact since these are just finite direct sums (products).[8] Given two objects X, Y such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite dimensional, we define the *evaluation* and the *coevaluation* map

$$\begin{aligned} ev\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathbf{C}} X &\to Y & \qquad ev'\colon X &\to \mathrm{Lin}_{\mathbf{C}}(\mathrm{Hom}_{\mathcal{C}}(X,Y), Y), \\ f \otimes x &\mapsto f(x) & x &\mapsto (f \mapsto f(x)), \end{aligned} \tag{3.1}$$

which are natural transformations $ev\colon (V \otimes_{\mathbf{C}} -) \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $ev'\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{Lin}_{\mathbf{C}}(V, -)$.

3.1. Passage to Chain Complexes

Both constructions pass down to functors of chain complexes of objects in $\mathcal{C}$. Recall that, given two chain complexes $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathcal{C})$, we can form a chain complex $\hom^\bullet_{\mathcal{C}}(X, Y)$ of vector spaces given by

$$\hom^k_{\mathcal{C}}(X^\bullet, Y^\bullet) := \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^{i+k})$$

with differential

$$\mathrm{d}^k(f) := \mathrm{d}_Y \circ f - (-1)^k \circ \mathrm{d}_X$$

for $f \in \hom^k_{\mathcal{C}}(X, Y)$ a degree k-map. Note the notational difference between the following objects:

- $\hom^\bullet_{\mathcal{C}}(X^\bullet, Y^\bullet)$: The chain complex of $\mathbf{C}$-vector spaces, with degree k-maps of chain complexes in $\mathcal{C}$ in homological degree k.
- $\mathrm{Hom}_{\mathrm{Ch}\,\mathcal{C}}(X^\bullet, Y^\bullet)$: The $\mathbf{C}$-vector space of chain maps from $X^\bullet$ to $Y^\bullet$. It relates to $\hom^\bullet_{\mathcal{C}}(X^\bullet, Y^\bullet)$ by $\mathrm{Hom}_{\mathrm{Ch}\,\mathcal{C}}(X^\bullet, Y^\bullet) = Z^0\big(\hom^\bullet_{\mathcal{C}}(X^\bullet, Y^\bullet)\big)$.
- $\mathrm{Hom}_{\mathcal{C}}(X, Y)$: The $\mathbf{C}$-vector space of maps $X \to Y$ in $\mathcal{C}$.
- $\mathrm{Hom}^*_{K^{\mathrm{b}}(\mathcal{C})}(X, Y)$, $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(X, Y)$: The graded $\mathbf{C}$-vector space of degree k-maps in $K^{\mathrm{b}}(\mathcal{C})$ (resp. $D^{\mathrm{b}}(\mathcal{C})$. Note that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathcal{C})}(X, Y) = H^*(\hom_{\mathcal{C}}(X, Y))$ [Wei94, §2.7.5].

Notation 3.1. From now on, we shall drop the "$(-)^\bullet$" from chain complexes. Most of the time, X, Y will denote chain complexes in $\mathrm{Ch}(\mathcal{C})$ and V a chain complex of $\mathbf{C}$-vector spaces. For an element $f \in X$ of a chain complex X, we denote its homological degree by $|f|$, i. e. $f \in X^{|f|}$. All tensor products are understood to be taken over $\mathbf{C}$.

---

[8] For infinite dimensional V one would need in addition that $\mathcal{C}$ satisfies the axioms (Ab4) and (Ab4*) from Grothendieck's Tôhoku-article [Gro57].,

Definition 3.2. Given a chain complex V of vector spaces and a chain complex X of objects in $\mathcal{C}$, define chain complexes $V \otimes X$ and $\mathrm{lin}_{\mathbf{C}}(V, X)$ by

$$\begin{aligned} (V \otimes X)^k &:= \bigoplus_{k=i+j} V^i \otimes X^k & \qquad \mathrm{lin}^k_{\mathbf{C}}(V, X) &:= \prod_{i \in \mathbf{Z}} \mathrm{Lin}_{\mathbf{C}}(V^i, X^{i+k}) \\ \mathrm{d}(v \otimes x) &:= \mathrm{d}v \otimes x + (-1)^{|v|} v \otimes \mathrm{d}x & \mathrm{d}(f)(v) &:= (-1)^{|v|}\big[\mathrm{d}(f(v)) - f(\mathrm{d}v)\big]. \end{aligned}$$

It is a standard fact that $V \otimes X$ is a chain complex; for $\mathrm{lin}_{\mathbf{C}}(V, X)$ we verify

$$\begin{aligned} &\mathrm{d}(\mathrm{d}f)(v) \\ &= (-1)^{|v|}\Big[\mathrm{d}\big((\mathrm{d}f)(v)\big) - (\mathrm{d}f)(\mathrm{d}v)\Big] \\ &= (-1)^{|v|}\Big[\mathrm{d}\Big((-1)^{|v|}(\mathrm{d}f(v) - f(\mathrm{d}v))\Big) - (-1)^{|v|+1}\big(\mathrm{d}f\mathrm{d}(v) + f(\mathrm{d}\mathrm{d}v)\big)\Big] \\ &= -(-1)^{|v|}\mathrm{d}f\mathrm{d}(v) - (-1)^{|v|+1}\mathrm{d}f\mathrm{d}(v) \\ &= 0. \end{aligned}$$

Remark 3.3. We could have defined $\mathrm{lin}_{\mathbf{C}}(V, X)$ with the same sign convention for the differential as for $\mathrm{hom}_{\mathcal{C}}(X, Y)$. However, this would necessitate introducing signs in the following maps.

Lemma 3.4. Let $X, Y \in \mathrm{Ch}(\mathcal{C})$ and $V \in \mathrm{Ch}(\mathbf{C}\text{-Vect})$. The maps ev and ev' from (3.1) induce chain maps

$$ev\colon \mathrm{hom}_{\mathcal{C}}(X, Y) \otimes X \to Y, \qquad ev'\colon X \to \mathrm{lin}_{\mathbf{C}}(\mathrm{hom}_{\mathcal{C}}(X, Y), Y).$$

Furthermore, there are monomorphic maps of chain complexes

$$\begin{aligned} \alpha\colon V \otimes \mathrm{hom}_{\mathcal{C}}(X, Y) &\to \mathrm{hom}_{\mathcal{C}}(X, V \otimes Y), \\ v \otimes f &\mapsto (x \mapsto v \otimes f(x)), \\ \beta\colon \mathrm{hom}_{\mathcal{C}}(X, Y) \otimes V &\to \mathrm{hom}_{\mathcal{C}}(\mathrm{lin}_{\mathbf{C}}(V, X), Y), \\ f \otimes v &\mapsto (\phi \mapsto (f \circ \phi)(v)), \\ \gamma\colon \mathrm{hom}_{\mathcal{C}}(X, \mathrm{lin}_{\mathbf{C}}(V, Y)) \otimes Z &\to \mathrm{lin}_{\mathbf{C}}(V, \mathrm{hom}_{\mathcal{C}}(X, Y) \otimes Z), \\ f \otimes z &\mapsto (v \mapsto [y \mapsto f(y)(v)] \otimes z), \end{aligned}$$

induced by the respective monomorphisms of vector spaces. These maps are isomorphisms (resp. quasi-isomorphisms) if V (resp. $H^*(V)$) has finite total dimension.

Proof. We only have to show that the above maps are compatible with the sign convention we have chosen for the differential. Throughout this proof x, y, z and v denote elements of the respective chain complexes. For an element $f \otimes x \in \mathrm{hom}_{\mathcal{C}}(X, Y) \otimes X$,

$$\begin{aligned} ev(\mathrm{d}(f \otimes x)) &= ev\Big\{\big[\mathrm{d} \circ f - (-1)^{|f|} f \circ \mathrm{d}\big] \otimes x + (-1)^{|f|} f \otimes \mathrm{d}x\Big\} \\ &= \mathrm{d}f(x) - (-1)^{|f|} f\mathrm{d}(x) + (-1)^{|f|} f\mathrm{d}(x) \\ &= \mathrm{d}(ev(f \otimes x)). \end{aligned}$$

For $x \in X$, denote $ev'(x)\colon f \mapsto f(x)$. Then

$$\begin{aligned} \mathrm{d}(ev'(x)(f)) &= (-1)^{|f|}\left[\mathrm{d}ev'(x)(f) - ev'_x(\mathrm{d}(f))\right] \\ &= (-1)^{|f|}\Big[\mathrm{d}f(x) - \mathrm{d}f(x) + (-1)^{|f|} f\mathrm{d}(x)\Big] \\ &= f\mathrm{d}(x) \\ &= ev'(\mathrm{d}x)(f). \end{aligned}$$

For the map $\alpha\colon V \otimes \mathrm{hom}_{\mathcal{C}}(X, Y) \to \mathrm{hom}_{\mathcal{C}}(X, V \otimes Y)$:

$$\begin{aligned} \mathrm{d}\big[\alpha(v \otimes f)\big](x) &= \mathrm{d}v \otimes f(x) + (-1)^{|v|} v \otimes (\mathrm{d}f)(x) \\ &= \mathrm{d}v \otimes f(x) + (-1)^{|v|} v \otimes \mathrm{d}f(x) - (-1)^{|v|+|f|} v \otimes f(\mathrm{d}x) \end{aligned}$$

$$\begin{aligned}
&= \alpha\big[\mathrm{d}v \otimes f + (-1)^{|v|} v \otimes (\mathrm{d} \circ f - (-1)^{|f|} f \circ \mathrm{d}\big](x) \\
&= \alpha\big[\mathrm{d}(v \otimes f)\big](x).
\end{aligned}$$

For the map $\beta\colon \hom_{\mathcal{C}}(X, Y) \otimes V \to \hom_{\mathcal{C}}(\mathrm{lin}_{\mathbf{C}}(V, X), Y)$:

$$\begin{aligned}
\mathrm{d}\big[\beta(f \otimes v)\big](\phi) &= \mathrm{d}f\phi(v) - (-1)^{|f|+|v|} f(\mathrm{d}\phi)(v) \\
&= \mathrm{d}f\phi(v) - (-1)^{|f|+|v|+|v|}\big[f\mathrm{d}\phi(v) - f\phi\mathrm{d}(v)\big] \\
&= \mathrm{d}f\phi(v) - (-1)^{|f|} f\mathrm{d}\phi(v) + (-1)^{|f|} f\phi\mathrm{d}(v) \\
&= \beta\big[(d \circ f - (-1)^{|f|} f \circ \mathrm{d}) \otimes v) + (-1)^{|f|} f \otimes \mathrm{d}v\big](\phi) \\
&= \beta\mathrm{d}(f \otimes \phi).
\end{aligned}$$

For $\gamma\colon \hom_{\mathcal{C}}(X, \mathrm{lin}_{\mathbf{C}}(V, Y)) \otimes Z \to \mathrm{lin}_{\mathbf{C}}(V, \hom_{\mathcal{C}}(X, Y) \otimes Z)$, we abbreviate the map $y \mapsto f(y)(v)$ by $f(-)(v)$. Then $\mathrm{d}f(-)(v)$ denotes the differential of this map, whereas $\mathrm{d} \circ f(-)(v)$ denotes the map followed by the differential. Then $\mathrm{d} \circ \gamma$ is given by

$$\begin{aligned}
\mathrm{d}\big[\gamma(f \otimes z)\big](v) &= (-1)^{|v|}\left\{\mathrm{d}\big[(f(-)(v)) \otimes d\big] - \big[f(-)(\mathrm{d}v)\big] \otimes z\right\} \\
&= (-1)^{|v|}\Big\{\Big[\mathrm{d} \circ \big[f(-)(v)\big] - (-1)^{|f|+|v|} f(\mathrm{d}-)(v)\Big] \otimes z + \\
&\qquad + (-1)^{|f|+|v|} f(-)(v) \otimes \mathrm{d}z - f(-)(\mathrm{d}v) \otimes z\Big\} \\
&= (-1)^{|v|}\mathrm{d} \circ [f(-)(v)] \otimes z - (-1)^{|f|} f(\mathrm{d}-)(v) \otimes z + \\
&\qquad + (-1)^{|f|} f(-)(v) \otimes \mathrm{d}z - (-1)^{|v|} f(-)(\mathrm{d}v) \otimes z \\
&= (\mathrm{d} \circ f)(-)(v) \otimes z - (-1)^{|f|} f(\mathrm{d}-)(v) \otimes z + (-1)^{|f|} f(-)(v) \otimes \mathrm{d}z \\
&= \gamma\Big\{\big[\mathrm{d} \circ f - (-1)^{|f|} f \circ \mathrm{d}\big] \otimes z + (-1)^{|f|} f \otimes z\Big\}(v) \\
&= \gamma\big[\mathrm{d}(f \otimes z)\big])(v),
\end{aligned}$$

so these maps are indeed maps of chain complexes. ☐

Remark 3.5. Alternatively, we can describe the chain complex $V \otimes X$ (resp. $\mathrm{lin}_{\mathbf{C}}(V, X)$) as a direct sum (product) of shifted copies of X: Given a homogeneous basis (x_i) of V, let $\mathrm{d}(x_i) = \sum_{ij} z_{ji} x_j$. Both chain complexes $V \otimes X$ and $\mathrm{lin}_{\mathbf{C}}(V, X)$ then have degree-k-part

$$\begin{aligned}
(V \otimes X)^k &= \bigoplus_{x} X^{k+|x_i|}_{x_i} \\
\mathrm{lin}^k_{\mathbf{C}}(V, X) &= \prod_{x_i} X^{k+|x_i|}_{x_i}
\end{aligned}$$

where the the subscript x_i is only used for indexing. The differential of the latter (see Definition 3.2) has components

$$\mathrm{d}^k_{ij}\colon X^{k+|x_i|}_{x_i} \to X^{k+1+|x_j|}_{x_j}; \qquad \mathrm{d}^k_{ij} = \begin{cases} (-1)^{|x_i|}\mathrm{d}_X & \text{if } i = j, \\ (-1)^{|x_i|} z_{ij} \cdot \mathrm{id} & \text{if } |x_i| = |x_j| + 1. \end{cases}$$

3.2. Spherical objects, spherical twist functors

The following terminology is from [ST01]:

Definition 3.6. A chain complex $E \in D^{\mathrm{b}}(\mathcal{C})$ is called *d-spherelike*[9] for an integer $d \geq 0$ if

- (S1) for any chain complex $F \in D^{\mathrm{b}}(\mathcal{C})$, the graded vector spaces $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(E, F)$ and $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(F, E)$ have finite total dimension, and
- (S2) there is an isomorphism $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(E, E) \cong \mathbf{C}[x]/(x^2)$ of graded algebras, with the identity in degree zero and a non-trivial degree d-morphism x.

---

[9] There is no name for this property in [ST01]. The term *spherelike object* has been coined in [HKP16], where property (S3) is called *Calabi-Yau property.*

It is called *d-spherical*[9] if furthermore

(S3) the composition map $\operatorname{Hom}^i_{D^{\mathrm{b}}}(F,E) \otimes \operatorname{Hom}^{d-i}_{D^{\mathrm{b}}}(F,E) \to \operatorname{Hom}^d_{D^{\mathrm{b}}}(E,E) \cong \langle x \rangle_{\mathbf{C}}$ is non-degenerate for all F and i.

Remark 3.7. We point out that there might also be 0-spherical objects. We will encounter some when applying the theory. The object $\mathbf{C}[x]/(x^2) \in D^{\mathrm{b}}\big(\mathbf{C}[x]/(x^2)\text{-Mod}\big)$, considered as concentrated in degree zero, is an obvious example for a 0-spherical object.

Remark 3.8. An object X of a linear category $\mathcal{C}$ is said to have a *Serre dual* if there is an object $\mathrm{S}X$ such that there is an isomorphism of functor $\operatorname{Hom}_{\mathcal{C}}(X,-)^* \cong \operatorname{Hom}_{\mathcal{C}}(-,\mathrm{S}X)$. If $\mathrm{S}X$ is functorial in X and an auto-equivalence of $\mathcal{C}$, then S is said to be a *Serre functor* on $\mathcal{C}$; see [MS08, §3] for an overview of the notion.

In fancy language, (S3) thus says that $E[d]$ is a Serre dual of E. Assuming that E is d-spherelike and has *some* Serre dual $\mathrm{S}E$, one can find a triangulated subcategory of $D^{\mathrm{b}}(\mathcal{C})$ containing both E and $\mathrm{S}E$, called the *spherical subcategory* of E in $D^{\mathrm{b}}(\mathcal{C})$ in which E becomes spherical [HKP16, thm. 4.4].

Notation 3.9. We denote the homological degree of chain complexes (resp. double) by writing a 0 below the entry in homological degree zero. Curly braces around a double complex (resp. triple complex) denote its total complex. Explicitly, given a double complex X, its total complex

$$\{\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots\} \coloneqq \left\{\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & X^{-1,0} \arrow[r, "\mathrm{d_h}"] \arrow[d, "\mathrm{d_v}"'] & X^{0,0} \arrow[r] \arrow[d, "\mathrm{d_v}"] & \cdots \\ \cdots \arrow[r] & X^{-1,1} \arrow[r, "\mathrm{d_h}"'] \arrow[d] & X^{0,1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd}\right\}$$

has degree k-part $\bigoplus_{i+j=k} X^{ij}$ with differential given by components

$$\mathrm{d_{tot}} \coloneqq \mathrm{d_h} + (-1)^i \mathrm{d_v} \colon X^{ij} \to X^{i-1,j} \oplus X^{i,j-1}.$$

In particular, let $f\colon X \to Y$ be a map of chain complexes. We can consider f as a double complex concentrated in horizontal degrees $i = -1, 0$. Its total complex

$$\left\{X \xrightarrow{f} \underset{0}{Y}\right\} \coloneqq \left\{\cdots \to X^1 \oplus Y^0 \xrightarrow{\begin{pmatrix} -\mathrm{d}_X & 0 \\ f & \mathrm{d}_Y \end{pmatrix}} X^2 \oplus Y^1 \to \cdots\right\}$$

then is just the mapping cone of f. We will henceforth use the notation $\{X \to Y\}$ rather than $\operatorname{cone}(X \to Y)$.

Recall 3.10. Recall from Caveat 1.11 the distinction between the coherent arrow category $D^{\mathrm{b}}(\mathcal{C}^{[1]})$ and the incoherent arrow category $D^{\mathrm{b}}(\mathcal{C})^{[1]}$. Let $X \in D^{\mathrm{b}}(\mathcal{C})$ be a fixed chain complex. Since $ev\colon \hom_{\mathcal{C}}(X,-) \otimes X \Rightarrow \mathrm{id}_{\mathrm{Ch}\,\mathcal{C}}$ is a natural transformation in $\mathrm{Ch}(\mathcal{C})$ (i. e. its naturality square strictly commutes, not just up to homotopy), it induces a functor $ev\colon D^{\mathrm{b}}(\mathcal{C}) \to D^{\mathrm{b}}(\mathcal{C}^{[1]})$ into the coherent arrow category. Taking its mapping cone thus gives an endofunctor of $D^{\mathrm{b}}(\mathcal{C})$.

Definition 3.11. Given an object $E \in D^{\mathrm{b}}(\mathcal{C})$ satisfying (S1) we define the associated *(spherical) twist functor* as the mapping cone

$$\begin{aligned} T_E\colon D^{\mathrm{b}}(\mathcal{C}) &\to D^{\mathrm{b}}(\mathcal{C}) \\ F &\mapsto \left\{\hom_{\mathcal{C}}(E,F) \otimes E \xrightarrow{ev} \underset{0}{F}\right\} \end{aligned} \tag{3.2}$$

of the evaluation map, and the *cotwist functor* as the cocone

$$T'_E\colon F \mapsto \left\{\underset{0}{F} \xrightarrow{ev'} \operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F,E),E\big)\right\} \tag{3.3}$$

of the coevaluation. Since the Hom-complexes are assumed to have finite total dimension, the resulting chain complex is bounded and hence belongs to $D^{\mathrm{b}}(\mathcal{C})$.

Recall that $\operatorname{Hom}(E,-)$ is a homological functor for any chain complex $E \in D^{\mathrm{b}}(\mathcal{C})$ [Wei94, ex. 10.2.8]. Applying it to an exact triangle $F \to G \to H \to F[1]$ yields an exact sequence

$$\cdots \to \operatorname{Hom}(E,F) \to \operatorname{Hom}(E,G) \to \operatorname{Hom}(E,H) \to \operatorname{Hom}(E,F[1]) = \operatorname{Hom}^1(E,F) \to \cdots$$

which shows that $\hom_{\mathcal{C}}(E,-)$ and hence T_E and T'_E are triangulated functors.

Remark 3.12. The concept of spherical twist functors T_E and $T_{E'}$ for a spherical object E has been taken to a more general notion of twist functors on triangulated categories: A functor $F\colon \mathcal{C} \to \mathcal{D}$ of triangulated categories is said to be a *spherical functor* if it admits both adjoints $L \dashv F \dashv R$ such that the twist functor $T_F \coloneqq \{FR[1] \to \mathrm{id}_{\mathcal{D}}\}$ and the cotwist functor $C_F \coloneqq \{\mathrm{id}_{\mathcal{C}} \to RF[-1]\}$ are mutual inverse auto-equivalences [Seg16, def. 2.1].

The shuffling functor Sh_s thus is an example for a spherical twist functor $\mathrm{Sh}_s = T_{\Theta_s}$ on $D^{\mathrm{b}}(\mathcal{O})$, corresponding to the spherical functor Θ_s. It seems remarkable that *every* auto-equivalence of a triangulated category arises in this way [Seg16, thm. 2.10].

Recall the definition of the Grothendieck group $K_0(D^{\mathrm{b}}\mathcal{C}))$ from Definition 1.16. The following is not from [ST01] but an easy to see consequence of Definition 3.11.

Proposition 3.13. For a $2d$-spherelike object $E \in D^{\mathrm{b}}(\mathcal{C})$, the functors T_E and T'_E induce an $[T_E] = [T'_E]$ of K_0.

Proof. Consider the composition $T_E T_E$, which assigns to $F \in D^{\mathrm{b}}(\mathcal{C})$ the total complex

$$T_E T_E\colon F \mapsto \left\{ \begin{tikzcd} \hom_{\mathcal{C}}\big(E, \hom_{\mathcal{C}}(E,F) \otimes E\big) \otimes E \arrow[r, "ev"] \arrow[d, "\hom_{\mathcal{C}}(E,ev)\otimes E"] & \hom_{\mathcal{C}}(E,F) \otimes E \arrow[d, "ev"] \\ \hom_{\mathcal{C}}(E,F) \otimes E \arrow[r, "ev"] & F \end{tikzcd} \right\}.$$

Note that shifting the homological degree by ±1 induces a sign in the Grothendieck group[10]. Given a complex V of $\mathbf{C}$-vector spaces, recall from Remark 3.5 that $V \otimes F$ is just a direct sum of copies of F indexed by a basis of V, each copy shifted by the homological degree of the respective basis vector of V. We thus have $[V \otimes F] = [F]^V \coloneqq \sum_k (-1)^k \dim V^k [F]$ in $K_0(\mathcal{C})$. The functor $T_E T_E$ therefore induces a homomorphism

$$\begin{aligned} [T_E T_E]\colon K_0(\mathcal{C}) &\to K_0(\mathcal{C}), \\ [F] &\mapsto [F] - 2[E]^{\hom_{\mathcal{C}}(E,F)} + [E]^{\hom_{\mathcal{C}}(E, \hom_{\mathcal{C}}(E,F) \otimes E)}; \end{aligned}$$

We apply Lemma 3.4 to obtain

$$\begin{aligned} &= [F] - 2[E]^{\hom_{\mathcal{C}}(E,F)} + [E]^{\overbrace{\hom_{\mathcal{C}}(E,F) \otimes \hom_{\mathcal{C}}(E,E)}^{(\mathrm{id} \oplus [2d])\, \hom_{\mathcal{C}}(E,F)}} \\ &= [F] - 2[E]^{\hom_{\mathcal{C}}(E,F)} + \big(1 + (-1)^{2d}\big)[E]^{\hom_{\mathcal{C}}(E,F)} \end{aligned}$$

since $\hom_{\mathcal{C}}(E,E)$ is concentrated in degrees 0 and $2d$. Since $2d$ is even, the second two summands cancel such that we get $[T_E T_E F] = [F]$. For the other compositions, the argument is similar. ☐

Lemma 3.14. The twist and cotwist functors have the following immediate properties:

(i) Shifted objects induce naturally isomorphic (co)twist functors.
(ii) There is an adjunction $T'_E \dashv T_E$.

Proof. (i) Recall that the shift $[-]$ of chain complexes acts on objects by $E[k]^i \coloneqq E^{i+k}$. Putting this into the definition of the chain complex $\hom_{\mathcal{C}}(E,F)$ (see Definition 3.2) gives

$$\begin{aligned} \hom_{\mathcal{C}}(E,F)[k]^l = \prod_i \operatorname{Hom}_{\mathcal{C}}(E^i, F^{i+k+l}) &= \prod_i \operatorname{Hom}_{\mathcal{C}}(E^{i-k}, F^{i+l}) \\ = \hom^l_{\mathcal{C}}(E, F[k]) \qquad\quad &= \hom^l_{\mathcal{C}}(E[-k], F). \end{aligned}$$

---

[10] This can be seen from rotating the triangle $X \to X \to 0 \to X[1]$.

Similarly, $(E \otimes F)[k] = E[k] \otimes F = E \otimes F[k]$. Putting this into the definition of T_E and T'_E (see Definition 3.11) shows

$$\begin{aligned} T_{E[i]}F &= \big\{\hom_{\mathcal{C}}(E[i], F) \otimes E[i] \to F\big\} \\ &= \big\{\hom_{\mathcal{C}}(E, F)[-i] \otimes E[i] \to F\big\} \\ &= T_E F; \end{aligned} \qquad \begin{aligned} T'_{E[i]}F &= \big\{F \to \operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F, E[i]), E[i]\big)\big\} \\ &= \big\{F \to \operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F, E)[i], E[i]\big)\big\} \\ &= T'_E F. \end{aligned}$$

(ii) Let $G \in D^{\mathrm{b}}(\mathcal{C})$. Recall that all hom-complexes $\hom_{\mathcal{C}}(E, -)$ are finite dimensional. The map α from Lemma 3.4 for $V = \hom_{\mathcal{C}}(E, G)$ thus gives a quasi-isomorphism

$$\begin{aligned} &\hom_{\mathcal{C}}(F, \hom_{\mathcal{C}}(E, G) \otimes E) \\ \simeq\ &\hom_{\mathcal{C}}(E, G) \otimes \hom_{\mathcal{C}}(E, F), \end{aligned}$$

and the map β for $V = \hom_{\mathcal{C}}(E, F)$ gives an quasi-isomorphism

$$\simeq \hom_{\mathcal{C}}\big\{\operatorname{lin}_{\mathbf{C}}[\hom_{\mathcal{C}}(F, E), E], G\big\}.$$

Since $H^0 \hom_{\mathcal{C}}(-,-) = \operatorname{Hom}_{K^{\mathrm{b}}(\mathcal{C})}(-,-)$ and since D^{b} is a localisation of K^{b} [see Wei94, §§10.3, 10.7], these induce isomorphisms of Hom-spaces in $D^{\mathrm{b}}(\mathcal{C})$. The adjunction is revealed by

$$\begin{aligned} &\operatorname{Hom}_{D^{\mathrm{b}}}(F, T_E(G)) \\ =\ &\big\{\operatorname{Hom}_{D^{\mathrm{b}}}\big(F, \hom_{\mathcal{C}}(E, G) \otimes E\big) \to \operatorname{Hom}_{D^{\mathrm{b}}}(F, G)\big\} \\ \simeq\ &\big\{\operatorname{Hom}_{D^{\mathrm{b}}}\big(\operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F, E), E\big), G\big) \to \operatorname{Hom}_{D^{\mathrm{b}}}(F, G)\big\} \\ =\ &\operatorname{Hom}_{D^{\mathrm{b}}}(T'_E F, G). \end{aligned}$$

Note that, although this is not necessary for what we do, this shows that both functors are even dg-adjoint. ☐

3.3. Braid relations

Our goal is to find a criterion for a collection of objects sufficient for the associated (co)twist functors to satisfy the relations of the braid group.

Definition 3.15. A collection of d-spherical objects $E_1, \dots, E_n$ is said to be an A_n*-configuration* if the total dimensions of Hom-complexes are $\dim \operatorname{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(E_i, E_j) = \begin{cases} 1 \text{ if } |i-j| = 1 \\ 0 \text{ if } |i-j| = 0. \end{cases}$

Theorem 3.16 [ST01, thm. 2.17]. Given an A_n-configuration of d-spherical objects, the (co)twist functors associated to such a configuration induce an action of the braid group $B_{S_{n+1}}$ of type A_n on $D^{\mathrm{b}}(\mathcal{C})$, i. e. there are natural isomorphisms

$$\begin{aligned} T_{E_i} &\cong (T'_{E_i})^{-1}, \\ T_{E_i}T_{E_j} &\cong T_{E_j}T_{E_i} && \text{if } |i-j| > 1, \\ T_{E_i}T_{E_{i+1}}T_{E_i} &\cong T_{E_{i+1}}T_{E_i}T_{E_{i+1}} && \text{for } 1 \le i < n. \end{aligned} \tag{3.4}$$

The rest of this section is devoted to presenting the proof of this theorem.

Proposition 3.17. The (co)twist functors T_E and T'_E are mutually inverse auto-equivalences of $D^{\mathrm{b}}(\mathcal{C})$ if and only if E is d-spherical for some integer $d \ge 0$.

Proof. —Showing $T_E T'_E F \simeq F$: The first composition $T_E T'_E F$ is the total complex

$$T_E T'_E F = \left\{ \begin{tikzcd} \hom_{\mathcal{C}}(E, F) \otimes E \arrow[r, "ev'_* \otimes \mathrm{id}_E"] \arrow[d, "ev"'] & \hom_{\mathcal{C}}\big[E, \operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F, E), E\big)\big] \otimes E \arrow[d, "ev"] \\ F \arrow[r, "ev'"'] & \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F, E), E\big] \end{tikzcd} \right\} \tag{3.5}$$

with F in degree $(0,0)$. Our goal is to show that this complex is quasi-isomorphic to F. First we use the quasi-isomorphism γ from Lemma 3.4 on $V = \hom_{\mathcal{C}}(F,E)$ to replace the top right corner. We obtain the quasi-isomorphic complex

$$T_E T'_E F \simeq \left\{ \begin{tikzcd} \hom_{\mathcal{C}}(E,F) \otimes E \arrow[r, "\psi"] \arrow[d, "ev"'] & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), \hom_{\mathcal{C}}(E,E) \otimes E\big] \arrow[d, "ev_*"] \\ F \arrow[r, "ev'"'] & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \end{tikzcd} \right\} \tag{3.6}$$

with $\psi := \gamma \circ (ev'_* \otimes \mathrm{id}_E)$. The map $ev_*\colon f \mapsto ev \circ f$ has a right inverse $\phi\colon g \mapsto \big(h \mapsto \mathrm{id}_E \otimes g(h)\big)$, i.e. $ev_* \circ \phi$ is the identity of $\mathrm{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(F,E), E\big)$. We can use the map ϕ to construct the dashed inclusion

$$\left\{ \begin{tikzcd} & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \arrow[dd, dashed, equal, "\mathrm{id}"'] \arrow[dr, dashed, hook, "\phi"] & \\ \hom_{\mathcal{C}}(E,F) \otimes E \arrow[rr, "\psi"] \arrow[dd, "ev"'] & & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), \hom_{\mathcal{C}}(E,E) \otimes E\big] \arrow[dd, "ev_*"] \\ & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \arrow[dr, dashed, hook, "\mathrm{id}"] & \\ F \arrow[rr, "ev'"'] & & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \end{tikzcd} \right\} \tag{3.7}$$

of a null-homotopic double complex into (3.6). This is indeed an inclusion of complexes since $ev_* \circ \phi = \mathrm{id}_{\mathrm{lin}_{\mathbf{C}}[\hom_{\mathcal{C}}(F,E),E]}$. Hence the original chain complex $T_E T'_E F$ is quasi-isomorphic to the cokernel

$$\left\{ \begin{tikzcd} \hom_{\mathcal{C}}(E,F) \otimes E \arrow[r, "\psi"] \arrow[d, "ev"'] & \mathrm{lin}_{\mathbf{C}}\Big[\hom_{\mathcal{C}}(F,E), \frac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle} \otimes E\Big] \arrow[d] \\ F \arrow[r] & 0 \end{tikzcd} \right\} \tag{3.8}$$

of this inclusion. To show that the obvious inclusion of F into (3.8) is a quasi-isomorphism is the same as to show that its cokernel is acyclic.

—*ψ is a quasi-isomorphism*: To do so, it suffices to show that the map

$$\begin{aligned} \psi\colon \hom_{\mathcal{C}}(E,F) \otimes E &\to \mathrm{lin}_{\mathbf{C}}\left[\hom_{\mathcal{C}}(F,E), \frac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle} \otimes E\right] \\ f \otimes e &\mapsto \big[g \mapsto g \circ f \otimes e\big] \end{aligned}$$

is a quasi-isomorphism. Since all vector spaces involved are finite dimensional, we can rewrite Hom-complexes as tensoring with dual complexes. The map ψ thus becomes

$$\begin{aligned} \psi\colon \hom_{\mathcal{C}}(E,F) \otimes E &\to \hom_{\mathcal{C}}(F,E)^\vee \otimes \frac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle} \otimes E \\ f \otimes e &\mapsto \sum_{\hom_{\mathcal{C}}(F,E) = \langle h \rangle} h^* \otimes (h \circ f) \otimes e, \end{aligned}$$

where the h's form a basis of $\hom_{\mathcal{C}}(F,E)$ with dual basis vectors the h^*'s. Now ψ is just the adjoint map of the composition

$$\circ\colon \hom_{\mathcal{C}}(E,F) \otimes \hom_{\mathcal{C}}(F,E) \to \hom_{\mathcal{C}}(E,E)/\langle \mathrm{id}_E \rangle, \tag{3.9}$$

tensored with the identity on E. Recall that taking homology commutes with $V \otimes = -$ and $\mathrm{lin}_{\mathbf{C}}(V,-)$ for finite dimensional V (finite direct sum resp. product). There is a natural map $H^*(\hom_{\mathcal{C}}(-,-)) = \mathrm{Hom}^*_{K^{\mathrm{b}}(\mathcal{C})}(-,-) \twoheadrightarrow \mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(-,-)$ [Wei94, §10.7; Ver96, def. I.1.2.2]. Applying H^* to (3.9) thus gives a map

$$\mathrm{Hom}^*_{D^{\mathrm{b}}}(E,F) \otimes \mathrm{Hom}^*_{D^{\mathrm{b}}}(F,E) \to \mathrm{Hom}^*_{D^{\mathrm{b}}}(E,E)/\langle \mathrm{id}_E \rangle,$$

which by (S3) is a perfect pairing if and only if E is spherical. In this case, its adjoint ψ is a quasi-isomorphism. Therefore, the complex (3.8) is quasi-isomorphic to the cokernel of the inclusion (3.7), which already has been shown to be quasi-isomorphic to (3.5). This shows $T_E T'_E F \simeq F$.

—*Showing* $T'_E T_E F \simeq F$: The other composition $T'_E T_E F$ is given by the total complex

$$\left\{\begin{tikzcd} \hom_{\mathcal{C}}(E,F) \otimes E \arrow[r, "ev"] \arrow[d, "ev'_{\hom_{\mathcal{C}}(E,F)}"'] & F \arrow[d, "ev'_F"] \\ \operatorname{lin}_{\mathbf{C}}\big[\hom\big(\hom_{\mathcal{C}}(E,F) \otimes E, E\big), E\big] \arrow[r, "ev_*"'] & \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \end{tikzcd}\right\}. \tag{3.10}$$

The bottom map ev_* and the left map ev' are given by

$$\begin{aligned} ev_*\colon\ \operatorname{lin}_{\mathbf{C}}\big[\hom\big(\hom_{\mathcal{C}}(E,F) \otimes E, E\big), E\big] &\to \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \\ g &\mapsto \big[f \mapsto g(f \circ ev)\big]; \\ ev'\colon\ \hom_{\mathcal{C}}(E,F) \otimes E &\to \hom\big[E, \operatorname{lin}_{\mathbf{C}}\big(\hom_{\mathcal{C}}(E,E), F\big)\big] \\ f \otimes e &\mapsto g(f \otimes e). \end{aligned}$$

The square (3.10) indeed commutes with these maps: one verifies

$$\begin{tikzcd} f \otimes e \arrow[r, mapsto, "ev"] \arrow[dd, mapsto, "ev'"'] & f(e) \arrow[d, mapsto, "ev'"] \\ & \big[h \mapsto h(f(e))\big] \\ \big(g \mapsto g(f \otimes e)\big) \arrow[r, mapsto, "ev_*"'] & = \big[h \mapsto (h \circ ev)(f \otimes e) = h(f(e))\big]. \end{tikzcd} \tag{3.11}$$

—*quasi-isomorphic replacements*: We replace the object in the bottom left corner of (3.10) by

$$\begin{aligned} &\operatorname{lin}_{\mathbf{C}}\big[\hom\big(\hom_{\mathcal{C}}(E,F) \otimes E, E\big), E\big] \\ \cong\ &\operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E,F)^\vee \otimes \hom_{\mathcal{C}}(E,E), E\big] \\ \cong\ &\operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E,E), \hom_{\mathcal{C}}(E,F) \otimes E\big]. \end{aligned}$$

These are indeed isomorphisms by finite dimensionality of all Hom-complexes involved. The original total complex (3.10) is quasi-isomorphic to

$$\left\{\begin{tikzcd} \hom_{\mathcal{C}}(E,F) \otimes E \arrow[r, "ev"] \arrow[d, "\widetilde{ev}'"'] & F \arrow[d, "ev'_F"] \\ \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E,E), \hom_{\mathcal{C}}(E,F) \otimes E\big] \arrow[r, "\widetilde{ev}_*"'] & \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \end{tikzcd}\right\} \tag{3.12}$$

with the new bottom map

$$\begin{aligned} \widetilde{ev}_*\colon\ \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E,E), \hom_{\mathcal{C}}(E,F) \otimes E\big] &\to \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F,E), E\big] \\ g &\mapsto \big[f \mapsto (g \circ ev \circ f)(\mathrm{id}_E)\big]. \end{aligned} \tag{3.13}$$

The new vertical map $\widetilde{ev}'$ is induced by $\mathbf{C} \to \hom_{\mathcal{C}}(E,E), 1 \mapsto \mathrm{id}_E$, i. e.

$$\begin{aligned} \widetilde{ev}'\colon\ \hom_{\mathcal{C}}(E,F) \otimes E &\to \operatorname{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E,E), \hom_{\mathcal{C}}(E,F) \otimes E\big], \\ f \otimes e &\mapsto g \mapsto \begin{cases} f \otimes e & \text{if } g = \mathrm{id}_E, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.14}$$

It has a left inverse

$$h(\mathrm{id}_E) \overset{\phi}{\mapsfrom} h, \tag{3.15}$$

i. e. $\phi \circ \widetilde{ev}'$ equals the identity of $\hom_{\mathcal{C}}(E, F)$. The map ψ is induced by $(\mathrm{id}_E)^* \colon \hom_{\mathcal{C}}(E, E) \to \mathbf{C}$. To verify that the square (3.12) still commutes one calculates

$$\begin{tikzcd}
f \otimes e \arrow[r, mapsto, "ev"] \arrow[dd, mapsto, "\widetilde{ev}'"'] & f(e) \arrow[d, mapsto, "ev'_F"] \\
& \big[h \mapsto h(f(e))\big] \\
\Big[\underbrace{g \mapsto \left\{ \begin{smallmatrix} f\otimes e & \text{if } g = \mathrm{id}, \\ 0 & \text{otherwise} \end{smallmatrix} \right.}_{=:a}\Big] \otimes e \arrow[r, mapsto, "\widetilde{ev}_*"'] & = \big[h \mapsto h(ev(a(\mathrm{id}_E))) = h(f(e))\big].
\end{tikzcd} \tag{3.16}$$

The fact that $\widetilde{ev}'$ has a left inverse ϕ allows us to consider the dashed surjection

$$\left\{ \begin{tikzcd}
\hom_{\mathcal{C}}(E, F) \otimes E \arrow[rr, "ev"] \arrow[dd, "\widetilde{ev}'"'] \arrow[dr, dashed, two heads, "\mathrm{id}"'] & & F \arrow[dd, "ev'"] \\
& \hom_{\mathcal{C}}(E, F) \otimes E \arrow[dd, dashed, equal, "\mathrm{id}"] & \\
\mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E, E), \hom_{\mathcal{C}}(E, F) \otimes E\big] \arrow[rr, crossing over, "\widetilde{ev}_*"] \arrow[dr, dashed, two heads, "\phi"'] & & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F, E), E\big] \\
& \hom_{\mathcal{C}}(E, F) \otimes E &
\end{tikzcd} \right\}$$

onto a null-homotopic chain complex. This is indeed a surjection of chain complexes since $(\phi) \circ (\widetilde{ev}') = \mathrm{id}_{\hom_{\mathcal{C}}(E,F) \otimes E}$. This surjection has kernel

$$\left\{ \begin{tikzcd}
0 \arrow[r] \arrow[d] & F \arrow[d, "ev'"] \\
\ker \phi \arrow[r, "\widetilde{ev}_*"'] & \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(F, E), E\big]
\end{tikzcd} \right\}. \tag{3.17}$$

It follows from the definition of ϕ in (3.15) that one sees

$$\begin{aligned}
\ker \phi &= \big\{ g \in \mathrm{lin}_{\mathbf{C}}\big[\hom_{\mathcal{C}}(E, E), \hom_{\mathcal{C}}(E, F) \otimes E\big] \bigm| g(\mathrm{id}_E) = 0 \big\} \\
&= \mathrm{lin}_{\mathbf{C}}\Big[\tfrac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle}, \hom_{\mathcal{C}}(E, F) \otimes E\Big].
\end{aligned}$$

There is an obvious surjection from (3.17) onto F. Since every complex involved has finite total dimension, we can write hom-complexes as tensoring with duals. The surjection onto F thus has kernel

$$\left\{ \begin{tikzcd}
\Big(\tfrac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle}\Big)^{\vee} \otimes \hom_{\mathcal{C}}(E, F) \otimes E \arrow[r, "\widetilde{ev}_*"] & \hom_{\mathcal{C}}(F, E)^{\vee} \otimes E \\
\eta \otimes f \otimes e \arrow[r, mapsto] & \big[f \mapsto \eta(f \circ g)\big] \otimes e
\end{tikzcd} \right\}. \tag{3.18}$$

We are done if we can show that this complex is acyclic. The map ev_* in (3.18) is just the adjoint of the dual composition

$$\circ^{\vee} \colon \Big(\tfrac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle}\Big)^{\vee} \to \hom_{\mathcal{C}}(E, F)^{\vee} \otimes \hom_{\mathcal{C}}(F, E)^{\vee}, \tag{3.19}$$

tensored with id_E. The map $\circ \colon \hom_{\mathcal{C}}(E, F) \otimes \hom_{\mathcal{C}}(F, E) \to \frac{\hom_{\mathcal{C}}(E,E)}{\langle \mathrm{id}_E \rangle}$ and hence $\circ^{\vee}$ is a quasi-isomorphism if and only if E is spherical. This proves the statement. ☐

The following statement is the crucial step for establishing the braid- and commutativity relations for the (co)twist functors.

Proposition 3.18. Given two d-spherical objects $E_1, E_2 \in D^{\mathrm{b}}(\mathcal{C})$,

(i) the object $T_{E_2} E_1$ is d-spherelike, and
(ii) there is a natural isomorphism of functors $T_{T_{E_2} E_1} T_{E_2} \cong T_{E_2} T_{E_1}$.

Proof. (i) We first show that T_{E_2} is d-spherelike.

(S1) Let $F \in D^{\mathrm{b}}(\mathcal{C})$. There is an isomorphism $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(F, T_{E_2}E_1) \cong \mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(T'_{E_2}F, E_1)$ by the adjunction $T'_{E_2} \dashv T_{E_2}$. The latter has finite total dimension since by assumption E_1 satisfies (S1). Since E_2 is assumed to be d-spherical, the equivalence from Proposition 3.17 gives $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(T_{E_2}E_1, F) \cong \mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(E_1, T'_{E_2}F)$ the latter of which is again finite dimensional.

(S2) Since E_2 is assumed to be spherical, the equivalence from Proposition 3.17 and the assumption on E_1 to satisfy (S2) gives

$$\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(T_{E_2}E_1, T_{E_2}E_1) \cong \mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{C})}(T'_{E_2}T_{E_2}E_1, E_1) \cong \mathbf{C}[x]/(x^2).$$

(ii) We now show that there is natural isomorphism $T_{T_{E_2}E_1}T_{E_2}F \cong T_{E_2}T_{E_1}F$ as endofunctors of $D^{\mathrm{b}}(\mathcal{C})$. Explicitly, we want to show that there is a quasi-isomorphism between the total complex

$$T_{T_{E_2}E_1}T_{E_2}F = \left\{\begin{tikzcd} \mathrm{hom}_{\mathcal{C}}\big[T_{E_2}E_1, \mathrm{hom}_{\mathcal{C}}(E_2,F)\otimes E_2\big] \otimes T_{E_2}E_1 \arrow[r, "ev_*"] \arrow[d, "ev"'] & \mathrm{hom}_{\mathcal{C}}\big[T_{E_2}E_1, F\big] \otimes T_{E_2}E_1 \arrow[d, "ev"] \\ \mathrm{hom}_{\mathcal{C}}(E_2,F)\otimes E_2 \arrow[r, "ev"'] & F \end{tikzcd}\right\}$$

and the total complex

$$T_{E_2}T_{E_1}F = \left\{\begin{tikzcd} \mathrm{hom}_{\mathcal{C}}\big[E_2, \mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes E_1\big]\otimes E_2 \arrow[r, "ev_*"] \arrow[d, "ev"'] & \mathrm{hom}_{\mathcal{C}}(E_2,F)\otimes E_2 \arrow[d, "ev"] \\ \mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes E_1 \arrow[r, "ev"'] & F \end{tikzcd}\right\}.$$

We take the quasi-isomorphism α from Lemma 3.4 for $V = \mathrm{hom}_{\mathcal{C}}(E_1, F)$ and apply $\alpha \otimes E_2$ on the top left corner of the diagram; this allows us to replace this double complex quasi-isomorphically by

$$T_{E_2}T_{E_1}F \simeq \left\{\begin{tikzcd} \overbrace{\mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes \mathrm{hom}_{\mathcal{C}}(E_2,E_1)\otimes E_2}^{=:(*)} \arrow[r, "\eta\otimes\mathrm{id}_{E_2}"] \arrow[d, "ev_*"'] & \mathrm{hom}_{\mathcal{C}}(E_2,F)\otimes E_2 \arrow[d, "ev"] \\ \underbrace{\mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes E_1}_{=:(**)} \arrow[r, "ev"'] & F \end{tikzcd}\right\}$$

with the composition $\eta\colon f \otimes g \mapsto g \circ f$. This double complex is just

$$T_{E_2}T_{E_1}F \simeq \Big\{\mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes T_{E_2}(E_1) \xrightarrow{\phi} T_{E_2}F\Big\} \tag{3.20}$$

for some map ϕ.

Claim. This map ϕ factors as

$$\phi : \mathrm{hom}_{\mathcal{C}}(E_1,F)\otimes T_{E_2}E_1 \xrightarrow{T_{E_2}\otimes\mathrm{id}_{T_{E_2}E_1}} \underbrace{\mathrm{hom}_{\mathcal{C}}(T_{E_2}E_1, T_{E_2}F)\otimes T_{E_2}E_1}_{=:U} \xrightarrow{ev} T_{E_2}F,$$

where T_{E_2} denotes the map $\mathrm{hom}_{\mathcal{C}}(E_1,F)i \to \mathrm{hom}_{\mathcal{C}}(T_{E_2}E_1, T_{E_2}F)$ induced by the functor T_{E_2}.

The first map is an isomorphism since T_{E_2} is spherical by assumption; hence, by Proposition 3.17 it is an equivalence. Writing down U as triple complex

$$\begin{tikzcd}
& \begin{array}{c}\hom_{\mathcal{C}}\left[E_1, \hom_{\mathcal{C}}(E_2,F)\otimes E_2\right]\\ \otimes \hom_{\mathcal{C}}(E_2,E_1)\otimes E_2\end{array} \arrow[rr] \arrow[dl, "ev^*"'] \arrow[dd] & & \left.\begin{array}{c}\hom_{\mathcal{C}}(E_1,F)\\ \otimes \hom_{\mathcal{C}}(E_2,E_1)\otimes E_2\end{array}\right\} = (*) \arrow[dl] \arrow[dd] \\
\begin{array}{c}\hom_{\mathcal{C}}\left[\hom_{\mathcal{C}}(E_2,E_1)\otimes E_2, \hom_{\mathcal{C}}(E_2,F)\otimes E_2\right]\\ \otimes \hom_{\mathcal{C}}(E_2,E_1)\otimes E_2\end{array} \arrow[rr, "ev_*"] \arrow[dd, "ev*"'] & & \begin{array}{c}\hom_{\mathcal{C}}\left[\hom_{\mathcal{C}}(E_2,E_1)\otimes E_2, F\right]\\ \otimes \hom_{\mathcal{C}}(E_2,E_1)\otimes E_2\end{array} \arrow[dd] & \\
& \begin{array}{c}\hom_{\mathcal{C}}\left[E_1, \hom_{\mathcal{C}}(E_2,F)\otimes E_2\right]\\ \otimes E_1\end{array} \arrow[rr, crossing over] \arrow[dl] & & \hom_{\mathcal{C}}(E_1,F)\otimes E_1 = (**) \arrow[dl] \\
\begin{array}{c}\hom_{\mathcal{C}}\left[\hom_{\mathcal{C}}(E_2,E_1)\otimes E_2, \hom_{\mathcal{C}}(E_2,F)\otimes E_2\right]\\ \otimes E_1\end{array} \arrow[rr, "ev"'] & & \hom_{\mathcal{C}}\left[\hom_{\mathcal{C}}(E_2,E_1)\otimes E_2, F\right]\otimes E_1 &
\end{tikzcd}$$

shows that ϕ factors as required. Hence

$$T_{E_2}T_{E_1}F \simeq \{U \xrightarrow{ev} T_{E_2}F\} = T_{T_{E_2}E_1}T_{E_2}F.$$

This proves the assertion. ☐

Proposition 3.19. The spherical twist functors satisfy the braid relations: Let E_1, E_2 be d-spherical for some $d \geq 0$.

(i) If $\mathrm{Hom}^*_{D^{\mathrm{b}}}(E_2, E_1) = 0$, then T_{E_1} and T_{E_2} naturally commute, i. e. there is a natural isomorphism of functors $T_{E_1}T_{E_2} \cong T_{E_2}T_{E_1}$.

(ii) If $\mathrm{Hom}^*_{D^{\mathrm{b}}}(E_2, E_1)$ has total dimension one, then there is a natural isomorphism of functors $T_{E_1}T_{E_2}T_{E_1} \cong T_{E_2}T_{E_1}T_{E_2}$.

Proof. (i) Twisting E_1 with E_2 yields

$$T_{E_2}E_1 = \big\{\underbrace{\hom_{\mathcal{C}}(E_2,E_1)}_{\text{acyclic}} \otimes E_2 \to E_1\big\} \simeq E_1.$$

Proposition 3.18 then yields

$$T_{E_1}T_{E_2} \cong T_{T_{E_2}E_1}E_2 \cong T_{E_2}T_{E_1}.$$

(ii) Let f be the unique (up to scalars) non-trivial map $f\colon E_2 \to E_1$. Let k be the degree of f, i. e. $f\colon E_2[k] \to E_1$ is a homogeneous chain map and is the only (up to scalars) non-trivial map $E_2[k] \to E_1$. Since the spherical twist functors are not affected by shifting the spherical object we may w. l. o. g. assume that $k = 0$. Writing down the (co)twisted objects exhibits an isomorphism

$$(T_{E_2}T_1) = \{E_2 \xrightarrow{f} \underset{0}{E_1}\} \cong \{\underset{0}{E_2} \xrightarrow{f} E_1\}[1] = T'_{E_1}E_2[1]. \tag{3.21}$$

Again we can apply the previous statements to obtain

$$\begin{aligned}
T_{E_1}T_{E_2}T_{E_1} &\cong T_{E_1}T_{T_{E_2}E_1}T_{E_2} && T_{E_1} \circ \text{ Proposition 3.18}\\
&\cong T_{E_1}T_{T'_{E_1}E_2[1]}T_{E_2} && (3.21)\\
&\cong T_{E_1}T_{T'_{E_1}E_2}T_{E_2} && \text{Lemma 3.14}\\
&\cong T_{\underbrace{T_{E_1}T'_{E_1}}_{\mathrm{id}}E_2}T_{E_1}T_{E_2} && \text{Proposition 3.18 } \circ T_{E_2}\\
&\cong T_{E_2}T_{E_1}T_{E_2} && \text{Proposition 3.17.}
\end{aligned}$$

☐

This completes the proof of Theorem 3.16.

4. Relating shuffling functors and spherical twist functors on $D^{\mathrm{b}}(\mathcal{O}_0)$

The category $\mathcal{O}$ as well as each of its blocks has the properties required in the set-up of Section 3. In the remainder of the first part of this thesis, we thereupon want to pursue the following question:

Question 4.1. Can we find spherical objects in $D^{\mathrm{b}}(\mathcal{O}_0)$ such that the associated spherical twist functors are isomorphic to the shuffling functor?

We shall first show that for certain Lie algebras there are spherical objects (namely some indecomposable projective modules) such that the associated twist functor and the shuffling functor have isomorphic images (see Observations 4.2 and 4.14). For certain other Lie algebras, none of the indecomposable projectives are spherical (see Observations 4.9 and 4.15). We finally show that we can answer **??** 4.1 positively for certain Lie algebras (see Theorems 4.21 to 4.22).

4.1. Spherical objects in $D^{\mathrm{b}}\big(\mathcal{O}_0(\mathfrak{sl}_2)\big)$

We see from (2.4) that the endomorphism ring of $P(s)$ is isomorphic to $\mathbf{C}[x]/(x^2)$ with the morphism $x = \big(P(s) \xrightarrow{b} P(e) \xrightarrow{a} P(s)\big)$, hence $P(s)$ is a 0-spherelike object in the sense of Definition 3.6 if considered as a complex concentrated in degree zero.

Recall that $\mathcal{O}_0$ has finite global dimension [Maz07, thm. 2]. Since we are working in the derived category of $\mathcal{O}_0$, we may replace every module $M \in D^{\mathrm{b}}(\mathcal{O}_0)$ by a projective resolution $P^\bullet \xrightarrow{\simeq} M$ of finite length. It is thus enough to check (S3) for $F \in \{P(e), P(s)\}$ ranging over the indecomposable projective modules. But apart from $P(s)$ the only indecomposable projective is $P(e)$, and the corresponding composition

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}}(P(e), P(s)) \times \mathrm{Hom}_{\mathcal{O}}(P(s), P(e)) &\to \mathrm{Hom}_{\mathcal{O}}(P(s), P(s))/\langle \mathrm{id}_{P(s)} \rangle, \\ (a, b) &\mapsto ab = x \end{aligned}$$

is a perfect pairing since the Hom-spaces are all one-dimensional. Hence $P(s)$ is a spherical object whose cotwisting functor yields

$$\begin{aligned} T'_{P(s)}P(s) &= \{\underset{0}{P(s)} \to P(s) \oplus P(s)\} \simeq \{0 \to P(s)\} \ = P(s)[-1] \ = \mathrm{Sh}_s\, P(s)[-1], \\ T'_{P(s)}P(e) &= \{\underset{0}{P(e)} \to P(s)\} \qquad\qquad \simeq \{0 \to M(s)\} = M(s)[-1] = \mathrm{Sh}_s\, P(e)[-1] \end{aligned} \tag{4.1}$$

on indecomposable projectives. We observe:

Observation 4.2. For $\mathfrak{sl}_2$, the indecomposable projective object $P(s)$ is spherical, and for all $M \in D^{\mathrm{b}}(\mathcal{O}_0)$ there is an isomorphism of images $T'_{P(s)}M \simeq \mathbf{L}\,\mathrm{Sh}\, M[-1]$ under the spherical cotwist and the derived shuffling functor.

Can we find further d-spherical objects, maybe for $d > 0$? Consider the simple module $L(e)$ with projective resolution $P(e) \to P(s) \to P(e) \xrightarrow{\simeq} L(e)$. This can be seen to be 2-spherelike and in fact is a spherical object. Its associated twist functor has images

$$\begin{aligned} T'_{L(e)}P(s) &= \big\{\underbrace{\mathrm{hom}_{\mathcal{O}}(L(e), P(s))}_{\overset{(*)}{=}0} \to P(s)\big\} \simeq P(s)[-1], \\ T'_{L(e)}P(e) &= \{\underset{0}{P(e)} \to P(s)\} \qquad\qquad\quad \simeq M(s)[-1]; \end{aligned} \tag{4.2}$$

where $(*)$ can be seen from the composition series in Table 2.1. Therefore, also cotwisting with $L(e)$ has images $T'_{L(e)}M$ quasi-isomorphic to the images $\mathrm{Sh}_s\, M$ of the shuffling functor Sh_s for all modules $M \in D^{\mathrm{b}}(\mathcal{O}_0)$. We in particular observe:

Observation 4.3. There may be non-isomorphic spherical objects whose associated spherical twist functors have isomorphic images.

4.2. Spherical objects in $\mathcal{O}_0(\mathfrak{sl}_3)$

Can we transfer what we have learnt for $\mathfrak{sl}_2$ to $D^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_3))$? Unfortunately the object $P(s)$, albeit still 0-spherelike in $D^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_3))$, is not spherical anymore because the composite $P(s) \to P(t) \to P(s)$ is the zero map; this can be seen from Table 2.3 or Figure 2.2. Since according to Definition 3.11 we only need the object to be spherelike to define its associated spherical twist functor, we may still wonder whether at least one of the two functors $\mathbf{L}\,\mathrm{Sh}_s$ or $\mathbf{R}\,\mathrm{Csh}_s$ might be isomorphic to the (co)twist functors

$$T_{P(s)}P(t) = \{P(s) \to \underset{0}{P(t)}\}; \qquad T'_{P(s)}P(t) = \{\underset{0}{P(t)} \to P(s)\}. \tag{4.3}$$

However, we already saw that $\mathrm{Sh}_s\,P(t)$ has composition factors $M(ts), M(s)$, which do not occur as homologies of any of the two (co)twisted objects. Using the Csh_s-acyclic resolution $P(t) \xrightarrow{\simeq} P(ts) \to \begin{smallmatrix} M(ts) \\ M(s) \end{smallmatrix}$ we see that $\mathbf{R}\,\mathrm{Csh}_s\,P(t) = \{P(ts) \to P(t)\}$, which is also not quasi-isomorphic to $T_{P(s)}P(t)$.

Remark 4.4. We notice that $T_{P(s)}$ and $T'_{P(s)}$ still induce mutual inverse automorphisms in the Grothendieck group by Proposition 3.13 since $P(s)$ is 0-spherelike.

4.3. Spherical objects in the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$

We have seen in Section 4.2 that the sphericity of $P(s)$ is not preserved under the inclusion $\mathfrak{sl}_2 \subset \mathfrak{sl}_3$, owing to the existence of objects $F \in \mathcal{O}_0(\mathfrak{sl}_3) \setminus \mathcal{O}_0(\mathfrak{sl}_2)$ on which the condition (S3) fails. We thus want to find a suitable intermediate category $\mathcal{O}_0(\mathfrak{sl}_2) \subseteq \mathcal{I} \subseteq \mathcal{O}_0(\mathfrak{sl}_3)$, ideally with a structure relating to that of $\mathcal{O}$, such that the sphericity of $P(s)$ is preserved under $\mathcal{O}_0(\mathfrak{sl}_2) \subseteq \mathcal{I}$. Wee shall see in this section that the parabolic category $\mathcal{O}^{\mathfrak{p}}$ has this property for an appropriate choice of the parabolic subalgebra $\mathfrak{p}$. In this set-up, we will eventually prove that $T_{P(s)}$ and $\mathbf{L}\,\mathrm{Sh}_s$ are isomorphic as functors.

4.3.1. The parabolic category $\mathcal{O}^{\mathfrak{p}}$

We recall the definition and the most important properties of parabolic category $\mathcal{O}^{\mathfrak{p}}$, mostly taken from [Hum08, §9; Maz12, §4.6; KM16, §2.4].

Definition 4.5. Given a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ (i. e. a Lie subalgebra containing a Borel subalgebra), $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $\mathcal{O}$ containing those $U(\mathfrak{g})$-modules for which the action of $U(\mathfrak{p})$ is locally finite, i. e. for any element m of a $U(\mathfrak{g})$-module, $U(\mathfrak{p}) \cdot m$ is finite dimensional.

The category $\mathcal{O}^{\mathfrak{p}}$ shares the direct sum decomposition of $\mathcal{O}$, and we consider its principal block $\mathcal{O}_0^{\mathfrak{p}}$. Let $\Phi^{\mathfrak{p}} \subseteq \Phi$ be the root system of $\mathfrak{p}$ with simple roots $\Delta^{\mathfrak{p}} \subseteq \Delta$. We obtain a parabolic subgroup $W_{\mathfrak{p}} := \langle s_\alpha \mid \alpha \in \Delta^{\mathfrak{p}} \rangle \leq W$ of the Weyl group of $\mathfrak{g}$. Conversely, a parabolic subgroup $W_{\mathfrak{p}} \leq W$ determines a set $\Delta^{\mathfrak{p}} \subseteq \Delta$ of simple roots, which in turn fixes a parabolic subalgebra $\mathfrak{p}$. Each left coset in $W_{\mathfrak{p}} \backslash W$ has a unique representative of minimal length [Hum90, §1.10]; denote by $W^{\mathfrak{p}}$ the set of such representatives.

Example 4.6. Let $W = S_3$ with generators s, t, and consider the parabolic subalgebra $W_{\mathfrak{p}} := \langle s \rangle \leq W$. The left quotient $W_{\mathfrak{p}} \backslash W$ comprises cosets $W_{\mathfrak{p}} \backslash W = \big\{\{e, s\}, \{t, st\}, \{ts, sts\}\big\}$ which have minimal length representatives $W^{\mathfrak{p}} = \{e, t, ts\}$.

The category $\mathcal{O}_0^{\mathfrak{p}}$ consists precisely of those modules in $\mathcal{O}_0$ whose composition series only contains simple factors $L(w)$ for $w \in W^{\mathfrak{p}}$ [KM16, §2.4]. This makes $\mathcal{O}_0^{\mathfrak{p}}$ a Serre subcategory of $\mathcal{O}_0$.

Example 4.7. (i) In the situation of Example 4.6, $\mathcal{O}_0^{\mathfrak{p}}$ is the subcategory of $\mathcal{O}_0(\mathfrak{sl}_3)$ generated under extensions by the simple modules $L(e)$, $L(t)$ and $L(ts)$.

(ii) Consider the extremal case $\mathfrak{p} = \mathfrak{g}$. This corresponds to $\Delta^{\mathfrak{p}} = \Delta$, hence $W_{\mathfrak{p}} = W$ and $W^{\mathfrak{p}} = \langle e \rangle \leq W$. The category $\mathcal{O}_0^{\mathfrak{g}}$ therefore is the abelian category generated by $L(e)$ under direct sums. The other extremal case $\mathfrak{p} = \mathfrak{b}$ the Borel subalgebra of upper triangular matrices corresponds to $\Delta^{\mathfrak{p}} = \emptyset$, hence $W_{\mathfrak{p}} = \langle e \rangle$ and $W^{\mathfrak{p}} = W$. Therefore, $\mathcal{O}_0^{\mathfrak{b}} = \mathcal{O}_0$.

The inclusion $\iota : \mathcal{O}_0^{\mathfrak{p}} \subseteq \mathcal{O}_0$ has both left and right adjoint functors: its left adjoint is the *Zuckerman functor* $Z^{\mathfrak{p}} \dashv \iota$ which maps a module to its largest quotient in $\mathcal{O}^{\mathfrak{p}}$, and its right adjoint is the *dual Zuckerman functor* $\iota \dashv Z_{\mathfrak{p}}$ which assigns to a module its largest submodule in $\mathcal{O}^{\mathfrak{p}}$ [Maz12, §4.6; KM16, §2.4].

Table 4.1: Composition series and standard filtrations of Verma modules and indecomposable projective modules in $\mathcal{O}_0^{\mathfrak{p}}$ for the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{sl}_3$ corresponding to $W_{\mathfrak{p}} = \langle t \rangle$. The modules are obtained from the composition series from Table 2.3 by $M^{\mathfrak{p}}(w) = Z^{\mathfrak{p}} M(w)$ and $P^{\mathfrak{p}}(w) = Z^{\mathfrak{p}} P(w)$, where $Z^{\mathfrak{p}}$ discards the smallest submodule generated by all composition factors $L(w)$ for $w \in W_{\mathfrak{p}}$; see Section 4.3.1.

$M^{\mathfrak{p}}(e)$	$M^{\mathfrak{p}}(s)$	$M^{\mathfrak{p}}(st)$	$P^{\mathfrak{p}}(e)$	$P^{\mathfrak{p}}(s)$	$P^{\mathfrak{p}}(st)$
$\begin{matrix} L(e) \\ L(s) \end{matrix}$	$\begin{matrix} L(s) \\ L(st) \end{matrix}$	$L(st)$	$M(e)$	$\begin{matrix} M(s) \\ M(e) \end{matrix} = \begin{matrix} & L(s) & \\ L(st) & & L(e) \\ & L(s) & \end{matrix}$	$\begin{matrix} M(st) \\ M(s) \end{matrix} = \begin{matrix} L(st) \\ L(s) \\ L(st) \end{matrix}$

The category $\mathcal{O}_0^{\mathfrak{p}}$ shares many of the properties used so far with $\mathcal{O}_0$; in particular, it has enough projectives, and the indecomposable projective modules are precisely the images $P^{\mathfrak{p}}(w) := Z^{\mathfrak{p}} P(w)$ of the indecomposable projectives in $\mathcal{O}_0$ [Maz12, §4.6]. The category $\mathcal{O}_0^{\mathfrak{p}}$ has a projective generator

$$P^{\mathfrak{p}} = \bigoplus_{w \in W^{\mathfrak{p}}} P^{\mathfrak{p}}(w) = Z^{\mathfrak{p}} P \tag{4.4}$$

for P the projective generator of $\mathcal{O}_0$ from Corollary 2.2.

Caveat 4.8. The indecomposable projective objects $P^{\mathfrak{p}}(w)$ in $\mathcal{O}_0^{\mathfrak{p}}$ are in general not projective if considered as objects in $\mathcal{O}_0$.

The $P^{\mathfrak{p}}(w)$'s admit standard filtrations with quotients isomorphic to *parabolic Verma* modules $M^{\mathfrak{p}}(w) := Z^{\mathfrak{p}} M(w)$. There is a parabolic analogue of BGG reciprocity theorem (see **??** 1.4.(v)) and of **??** 1.4.(iv); namely, for all $M \in \mathcal{O}_0^{\mathfrak{p}}$ and $v, w \in W^{\mathfrak{p}}$ we have

$$[M^{\mathfrak{p}}(w) : L^{\mathfrak{p}}(v)] = (P^{\mathfrak{p}}(v) : M^{\mathfrak{p}}(w)), \tag{4.5}$$

$$\dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}_0^{\mathfrak{p}}}(P^{\mathfrak{p}}(w), M) = [M : L(w)]; \tag{4.6}$$

see [Roc80, prop. 4.5, thm. 6.1]. There are parabolic analogues $p_{wv}^{\mathfrak{p}} \in \mathbf{Z}[q^{\pm 1}]$ of Kazhdan-Lusztig polynomials, defined in [Deo87, §3], allowing to compute the composition multiplicities by

$$(P^{\mathfrak{p}}(v) : M^{\mathfrak{p}}(w)) = [M^{\mathfrak{p}}(w) : L^{\mathfrak{p}}(v)] = p_{v,w}^{\mathfrak{p}}(1) \tag{4.7}$$

just as in the non-parabolic case in Theorem 2.17; see [CC87, thm. 3.2.8]. A recursive algorithm for computing the $p_{v,w}^{\mathfrak{p}}$ is concisely described in [Soe97, thm. 3.1] There is an equivalence of categories

$$\mathcal{O}_0^{\mathfrak{p}} \simeq \text{Mod-}A_{\mathfrak{p}} \ \text{ for } A_{\mathfrak{p}} := A / A e_{\mathfrak{p}} A \tag{4.8}$$

for a basic quasi-hereditary algebra A with $\mathcal{O}_0 \simeq \text{Mod-}A$. Here, $A e_{\mathfrak{p}} A \trianglelefteq A$ denotes the two sided ideal generated by a maximal idempotent $e_{\mathfrak{p}}$ of A annihilating $\mathfrak{p}$. The quotient $A_{\mathfrak{p}}$ itself is a basic quasi-hereditary algebra by virtue of the family $\{M^{\mathfrak{p}}(w)\}$ and their duals. We shall obtain $A_{\mathfrak{p}}$ as a path algebra of a quiver with relations. For more information on $\mathcal{O}_0^{\mathfrak{p}}$, see [Hum08, §9; Maz12, §4.6; KM16, §2.4].

The Zuckerman functors $Z^{\mathfrak{p}}$ and $Z_{\mathfrak{p}}$ naturally commute with projective functors [Maz12, prop. 6.1], in particular with translation Θ_s through the s-wall. Since $Z^{\mathfrak{p}}$ is right exact, it commutes with the cokernel Sh_s, and likewise $Z_{\mathfrak{p}}$ commutes with Csh_s. We want to decide whether the shuffling functor and the spherical twist with an appropriate object have isomorphic images when restricted to $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$.

4.3.2. Spherical objects in $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$: $\mathfrak{sl}_3$

We start with the parabolic subalgebra[11] $\mathfrak{p} = \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right) \subseteq \mathfrak{sl}_3$ corresponding to the parabolic subgroup $W_{\mathfrak{p}} = \langle t \rangle \leq W$. The left cosets have representatives $W^{\mathfrak{p}} = \{e, s, st\}$ see Example 4.6. The category $\mathcal{O}_0^{\mathfrak{p}}$ contains Verma modules and indecomposable projectives with composition series listed in Table 4.1.

Note that both $P^{\mathfrak{p}}(s)$ and $P^{\mathfrak{p}}(st)$ can be seen from Table 4.1 to have precisely one non-trivial endomorphism (up to scalars) which is nilpotent and sends the head to the socle. Hence, both have endomorphism algebra isomorphic to $\mathbf{C}[x]/(x^2)$ and thus are 0-spherelike. In fact, $P^{\mathfrak{p}}(s)$ and $P^{\mathfrak{p}}(st)$ are even self-dual. We obtain images of the Vermas and projectives as follows:

---

[11] This notation means the subalgebra of $\mathfrak{sl}_3$ of matrices with arbitrary entries for $*$ and a zero for 0.

Module M	$\Theta_s M$	$\mathbf{L}_1 \operatorname{Sh}_s M$	$\operatorname{Sh}_s M$
$M^{\mathfrak{p}}(e)$	$P^{\mathfrak{p}}(s)$	0	$M^{\mathfrak{p}}(s)$
$M^{\mathfrak{p}}(s)$	$P^{\mathfrak{p}}(s)$	$L(st)$	$\begin{smallmatrix} & L(s) & \\ L(e) & & L(st) \end{smallmatrix}$
$M^{\mathfrak{p}}(st)$	0	$L(st)$	0
$P^{\mathfrak{p}}(s)$	$P^{\mathfrak{p}}(s)^{\oplus 2}$	0	$P^{\mathfrak{p}}(s)$
$P^{\mathfrak{p}}(st)$	$P^{\mathfrak{p}}(s)$	$L(st)$	$M^{\mathfrak{p}}(e)^{\vee}$

Table 4.2: Images of Verma modules and indecomposable projectives in $D^{\mathrm{b}}\big(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_2)\big)$ under the translation Θ_s through the s-wall and the derived shuffling $\mathbf{L}_i \operatorname{Sh}_s$ for $i = 0, 1$.

$M^{\mathfrak{p}}(e)$: The Zuckerman functor $Z^{\mathfrak{p}}$ naturally commutes with Θ_s. The short exact sequence (1.10) associated to $\operatorname{Sh}_s M(e)$ therefore passes via $Z^{\mathfrak{p}}$ to the respective right exact sequence

$$M^{\mathfrak{p}}(e) \to P^{\mathfrak{p}}(s) \twoheadrightarrow M^{\mathfrak{p}}(s) \to 0,$$

which in fact is the standard filtration of $P^{\mathfrak{p}}(s)$ and has an injection on its left. This yields $\operatorname{Sh}_s M^{\mathfrak{p}}(e) = M^{\mathfrak{p}}(s)$.

$M^{\mathfrak{p}}(s)$: The adjunction map $M^{\mathfrak{p}}(s) \to P^{\mathfrak{p}}(s)$ has non-trivial kernel and cokernel; hence by Lemma 1.15.(i):

$$\begin{aligned} \operatorname{Sh}_s M^{\mathfrak{p}}(st) &= \operatorname{coker}\big(M^{\mathfrak{p}}(s) \to P^{\mathfrak{p}}(s)\big) = \left(\begin{smallmatrix} & L(s) & \\ L(e) & & L(st) \end{smallmatrix}\right); \\ \mathbf{L}_1 \operatorname{Sh}_s M^{\mathfrak{p}}(st) &= \quad \ker\big(M^{\mathfrak{p}}(s) \to P^{\mathfrak{p}}(s)\big) = L(st). \end{aligned}$$

In particular, $M^{\mathfrak{p}}(s)$ is not Sh_s-acyclic.

$M(st)$: Since $\Theta_s M(st)$ has head $L(w_0)$ and $w_0 \notin W^{\mathfrak{p}}$, we obtain $\Theta_s M^{\mathfrak{p}}(st) = 0$ for the translate in $\mathcal{O}_0^{\mathfrak{p}}$.

$P^{\mathfrak{p}}(st)$: The module $P^{\mathfrak{p}}(st)$ fits into the short exact sequence

$$\begin{tikzcd}
\{0 \arrow[r] \arrow[d, bend right=60, "\Theta_s"'] & M^{\mathfrak{p}}(s) \arrow[r] \arrow[d] & P^{\mathfrak{p}}(st) \arrow[r] \arrow[d] & M^{\mathfrak{p}}(st) \arrow[r] \arrow[d] & 0\} \\
\{0 \arrow[r] & P^{\mathfrak{p}}(s) \arrow[r, "\cong"] & \Theta_s P^{\mathfrak{p}}(st) \arrow[r] & 0 \arrow[r] & 0\}.
\end{tikzcd}$$

This gives $\Theta_s P^{\mathfrak{p}}(st) \cong P^{\mathfrak{p}}(s)$ and allows us to infer $\mathbf{L} \operatorname{Sh}_s P^{\mathfrak{p}}(st) = \big\{M^{\mathfrak{p}}(e) \to L(st)\big\}$, e. g. by the long exact sequence of the snake lemma.

Altogether, we obtain the translated and shuffled modules listed in Table 4.2. Note that $M^{\mathfrak{p}}(s)$, $M^{\mathfrak{p}}(st)$ and $P^{\mathfrak{p}}(st)$ are not Sh_s-acyclic.[12] This in particular shows that for a module $M \in \mathcal{O}_0$ with a standard filtration Lemma 1.15.(iii) need not hold for $Z^{\mathfrak{p}} M$.

We now want to continue our search for spherical objects. We are lucky: our candidate $P^{\mathfrak{p}}(s)$ turns out to be spherical this time. Its endomorphism space

$$\operatorname{End}(P^{\mathfrak{p}}(s)) = \left\{ \mathrm{id}, \begin{tikzcd}[row sep=tiny, column sep=tiny] & L(s) \arrow[ddrrr, bend left] & & & L(s) & \\ L(e) & & L(st) & L(e) & & L(st) \\ & L(s) & & & L(s) & \end{tikzcd} \right\} \cong \mathbf{C}[x]/(x^2) \tag{4.9}$$

shows that $P^{\mathfrak{p}}(s)$ indeed is a 0-spherelike object (recall the arrow notation from Notation 2.8). Computing the "composition pairings"

$$\circ : \operatorname{Hom}(P^{\mathfrak{p}}(e), P^{\mathfrak{p}}(s)) \times \operatorname{Hom}(P^{\mathfrak{p}}(s), P(e))$$

$$\longrightarrow \left\{ \begin{tikzcd}[row sep=tiny, column sep=tiny] & L(s) & \big\} \arrow[dr, bend left] & & & L(s) & \\ L(st) & & L(e) & \left\{\begin{smallmatrix} L(s) \\ L(e) \end{smallmatrix}\right. \arrow[drr, bend left] & L(e) & & L(st) \\ & L(s) & & & & L(s) & \end{tikzcd} \right\} = \{x\} \tag{4.10}$$

$$\circ : \operatorname{Hom}(P^{\mathfrak{p}}(st), P^{\mathfrak{p}}(s)) \times \operatorname{Hom}(P^{\mathfrak{p}}(s), P(st))$$

$$\longrightarrow \left\{ \begin{tikzcd}[row sep=tiny, column sep=tiny] & L(s) & \big\} \arrow[dr, bend left] & & & L(s) & \\ L(e) & & L(st) & \begin{smallmatrix} L(st) \\ L(s) \\ L(st) \end{smallmatrix} \arrow[r, bend left] & \big\{ L(st) & & L(e) \\ & L(s) & & & & L(s) & \end{tikzcd} \right\} = \{x\} \tag{4.11}$$

---

[12] We should not be surprised by $P^{\mathfrak{p}}(st)$ not being Sh_s-acyclic since we are considering $P^{\mathfrak{p}}(st)$ as an object of $\mathcal{O}_0$ where it is not projective; see Caveat 4.8.

Table 4.3: Parabolic Kazhdan-Lusztig polynomials for the parabolic subgroup $W_\mathfrak{p} = \langle s_2, s_3, \dots, s_{n-1}\rangle = S_1 \times S_{n-1} < S_n$. The dimensions $\dim_{\downarrow,\to} := \dim \operatorname{Hom}(P^\mathfrak{p}(\downarrow), P^\mathfrak{p}(\to))$ of Hom-spaces in the right half of the table are computed by (4.13).

$p^\mathfrak{p}_{\downarrow,\to}$	e	s_1	s_1s_2	$s_1s_2s_3$	$\cdots$	$\dim_{\downarrow,\to}$	e	s_1	s_1s_2	$s_1s_2s_3$	$\cdots$
e	1	q					1	1	0	0	
s_1		1	q				1	2	1	0	
s_1s_2			1	q			0	1	2	1	
$s_1s_2s_3$				1	$\ddots$		0	0	1	2	$\ddots$

with the other indecomposable projectives $P^\mathfrak{p}(e)$ and $P^\mathfrak{p}(st)$ shows that $P^\mathfrak{p}(s)$ satisfies (S3) and therefore indeed is a spherical object. For all modules M listed in Table 4.2 which have $P^\mathfrak{p}(s)$ as their translate $\Theta_s M$ it is immediate that there is an isomorphism

$$T'_{P^\mathfrak{p}(s)} M = \{\underset{0}{M} \to P^\mathfrak{p}(s)\} \simeq \mathbf{L}\operatorname{Sh}_s M$$

of images. Since for the remaining module $M^\mathfrak{p}(st)$ there is no non-zero morphism $M^\mathfrak{p}(st) \to P^\mathfrak{p}(s)$, we also have $T'_{P^\mathfrak{p}(s)} M^\mathfrak{p}(st) \simeq \mathbf{L}\operatorname{Sh}_s M^\mathfrak{p}(st)$. Hence we obtain the following:

Observation 4.9. In the subcategory $D^\mathrm{b}(\mathcal{O}_0^\mathfrak{p}) \subset D^\mathrm{b}(\mathcal{O}_0)$ for $\mathfrak{sl}_3$, the object $P^\mathfrak{p}(s)$ is spherical, and for all $M \in D^\mathrm{b}(\mathcal{O}_0^\mathfrak{p})$ there is an isomorphism $T'_{P^\mathfrak{p}(s)} M \simeq \mathbf{L}\operatorname{Sh}_s M$ in $D^\mathrm{b}(\mathcal{O}_0^\mathfrak{p})$.

We shall see in Theorem 4.22 that this is in fact a natural isomorphism of functors.

4.3.3. Spherical objects in $D^\mathrm{b}(\mathcal{O}_0^\mathfrak{p})$: from $\mathfrak{sl}_3$ to $\mathfrak{sl}_n$

Now consider a (maximal) parabolic subalgebra

$$\mathfrak{p} = \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \subseteq \mathfrak{sl}_n \tag{4.12}$$

corresponding to the parabolic subgroup $W_\mathfrak{p} = \langle s_2, s_3, \dots, s_{n-1}\rangle = S_1 \times S_{n-1} < S_n$ and minimal length coset representatives $W^\mathfrak{p} = \{e, s_1, s_1s_2, s_1s_2s_3, \dots, s_1 \cdots s_{n-1}\}$. This is a somehow extremal case of a maximal parabolic subalgebra, i. e. one corresponding to a subgroup of the form $S_m \times S_{n-m} \subseteq S_n$.

Parabolic Kazhdan-Lusztig theory To find spherical objects we need to understand the composition series of the relevant objects and the dimensions of Hom-spaces between indecomposable projectives. For maximal parabolic subalgebras there is a handy graphical calculus for computing the parabolic Kazhdan-Lusztig polynomials $p^\mathfrak{p}_{vw}$ [BS11a, §5; LS13], which yields the polynomials listed in Table 4.3. We may compute the dimensions

$$\begin{aligned} \dim \operatorname{Hom}(P^\mathfrak{p}(v), P^\mathfrak{p}(w)) &= [P^\mathfrak{p}(w) : L(v)] \\ &= \sum_{u \in W^\mathfrak{p}} (P^\mathfrak{p}(w) : M^\mathfrak{p}(u))[M^\mathfrak{p}(u) : L(v)] \\ &= \sum_u p_{uw}(1) p_{uv}(1) \end{aligned} \tag{4.13}$$

of Hom-space between each two indecomposable projectives by (4.5–4.7). We thus obtain the dimensions listed in Table 4.3, which indicates that the collection $(P^\mathfrak{p}(s_1), P^\mathfrak{p}(s_1s_2), \dots)$ is a candidate for an A_{n-1}-configuration.

We shall have a deeper look into the structure of these modules to be able to check whether (S3) is satisfied in this context. By (4.7), we obtain the multiplicity of composition factors of Verma modules resp. indecomposable projective modules from Table 4.3.

In fact more is true: recall from Section 2.3 the graded version $\mathcal{O}_0^\mathbf{Z}$ of $\mathcal{O}_0$. We have seen the grading to (partially) encode the "order" of simple factors in a composition series. Recall the equivalence $\mathcal{O}_0^\mathfrak{p} \simeq \text{Mod-}A_\mathfrak{p}$ from (4.8). The algebra $A_\mathfrak{p}$ has been obtained from the (graded, see Section 2.3) algebra A by quotienting out the homogeneous ideal $Ae_\mathfrak{p}A$ and thus is graded itself. Therefore, we obtain a graded parabolic category $\mathcal{O}_0^{\mathbf{Z},\mathfrak{p}} := \text{grMod-}A_\mathfrak{p}$. We shall freely identify modules in $\mathcal{O}_0^\mathfrak{p}$ with their graded lifts.

Table 4.4: Composition series of Verma modules and indecomposable projectives in $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_n)$ for the parabolic subalgebra $\mathfrak{p}$ from (4.12). The series are obtained from Table 4.3 by virtue of Theorem 4.10.

	e	s_1	s_1s_2	$\cdots$	$s_1\cdots s_{n-1}$
$M^{\mathfrak{p}}(-)$	$L(e)$ $L(s_1)$	$L(s_1)$ $L(s_1s_2)$	$L(s_1s_2)$ $L(s_1s_2s_3)$	$\cdots$	$L(s_1\cdots s_{n-1})$
$P^{\mathfrak{p}}(-)$	″	$L(s_1)$ $L(e)\quad L(s_1s_2)$ $L(s_1)$	$L(s_1s_2)$ $L(s_1)\quad L(s_1s_2s_3)$ $L(s_1s_2)$	$\cdots$	$L(s_1\cdots s_{n-1})$ $L(s_1\cdots s_{n-2})$ $L(s_1\cdots s_{n-1})$

Furthermore, recall from Theorem 2.21 that we may obtain the order of Loewy layers[13] from Kazhdan-Lusztig polynomials. The following parabolic analogues of the generalised Kazhdan-Lusztig-Theorem 2.21 hold true in our set-up. Recall the definition of the radical filtration and of rigidity of a module from Example 2.20.(i).

Theorem 4.10 (parabolic generalised Kazhdan-Lusztig-theorem).

(i) Consider a parabolic Verma module $M^{\mathfrak{p}}(w)$. The composition factor multiplicities of layers of the radical filtration $(M^{\mathfrak{p}}(w))^{\mathrm{rad}}_{\bullet}$ of $M^{\mathfrak{p}}(w)$ relate to the parabolic Kazhdan-Lusztig polynomials by

$$p^{\mathfrak{p}}_{vw} = \sum_{k\geq 0}\big[(M^{\mathfrak{p}}(w))^{\mathrm{rad}}_{\ell(v)-\ell(w)+2k} : L(w)\big]q^k.$$

In other words, the coefficient of the q^k-term of $p^{\mathfrak{p}}_{vw}$ is the multiplicity of $L(w)$ in the k-th layer of $(M^{\mathfrak{p}}(w))^{\mathrm{rad}}_{\bullet}$ [Irv90, cor. 7.1.3].

(ii) For a maximal parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}$, parabolic Verma modules $M^{\mathfrak{p}}(w)$ remain rigid [CIS88, thm. 1.3].

Remark 4.11. As in the non-parabolic case, the filtration associated to the grading of modules in $\mathcal{O}_0^{\mathbf{Z},\mathfrak{p}}$ coincides with the radical filtration. Theorem 4.10.(i) thus states that

$$p^{\mathfrak{p}}_{vw} = \sum_{k\geq 0}[M^{\mathfrak{p}}(v) : L(w)\langle k\rangle]q^k = \sum_{k\geq 0}(P^{\mathfrak{p}}(v) : M^{\mathfrak{p}}(w)\langle k\rangle)q^k;$$

see [BGS96, thm. 3.11.4] where the graded lift of $M^{\mathfrak{p}}(w)$ is denoted by M^Q_w and P^Q_{xy} denotes the parabolic Kazhdan-Lusztig polynomials (see also [Soe97, rmk. 3.2.2] on the relation to [BGS96, thm. 3.11.4]).

Translation and shuffling functor From the parabolic Kazhdan-Lusztig polynomials in Table 4.3 we obtain the composition series of Verma modules and indecomposable projectives listed in Table 4.4. Each of the indecomposable projectives hence has $\mathbf{C}[x]/(x^2)$ as endomorphism space. Any non-trivial composition $P^{\mathfrak{p}}(w_1)\to P^{\mathfrak{p}}(w_2)\to P^{\mathfrak{p}}(w_3)$ vanishes unless it is of the form

$$x\colon P^{\mathfrak{p}}(s_1\cdots s_i)\to P^{\mathfrak{p}}(s_1\cdots s_{i\pm 1})\to P^{\mathfrak{p}}(s_1\cdots s_i)$$

for $2\leq i<n-1$ (resp. $2<i\leq n-1$), up to isomorphism. This proves the following:

Observation 4.12. The $P^{\mathfrak{p}}(s_1\cdots s_{n-1})$ indeed form an A_{n-1}-configuration.

Remark 4.13. It is interesting to note that—similar to the non-parabolic setting—we can realise the category $\mathcal{O}_0^{\mathbf{Z},\mathfrak{p}}$ as graded modules over the path algebra $A_{\mathfrak{p}}$ of the quiver with relations

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & n-1 \arrow[l, bend left] \end{tikzcd}$$

$$\begin{gathered} 1\to 2\to 1=0 \\ i\to i\pm 1\to i\pm 2=0 \text{ for } 1<i<n-1, \quad i\to(i+1)\to i = i\to(i-1)\to i. \end{gathered} \tag{4.14}$$

For more details on the case of general maximal parabolic subalgebras we refer to [BS11b].

---

[13] See Definition 2.19.

Table 4.5: Images of the translation and derived shuffling functors Θ_w and $\mathbf{L}\,\mathrm{Sh}_w$ for $w \in W^{\mathfrak{p}}$ for the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{sl}_n$ defined in (4.12). A table cell containing a single object as image $\mathbf{L}\,\mathrm{Sh}_w M$ means that M is Sh_w-acyclic and the cell contains just $\mathrm{Sh}_w M$. A table cell containing a chain complex (written vertically) means that that $\mathbf{L}\,\mathrm{Sh}_w M$ is a chain complex concentrated in degrees 0 and 1 (see also Lemma 1.15) with homological degree 0 at the bottom.

M	$M^{\mathfrak{p}}(e)$	$P^{\mathfrak{p}}(s_1)$	$M^{\mathfrak{p}}(s_1)$	$P^{\mathfrak{p}}(s_1s_2)$	$M^{\mathfrak{p}}(s_1s_2)$	$\dots$
$\Theta_{s_1}M$	$P^{\mathfrak{p}}(s_1)$	$P^{\mathfrak{p}}(s_1)^{\oplus 2}$	$P^{\mathfrak{p}}(s_1)$	0	0	
$\mathbf{L}\,\mathrm{Sh}_{s_1} M$	$M^{\mathfrak{p}}(s_1)$	$P^{\mathfrak{p}}(s_1)$	$\left\{\begin{matrix} M^{\mathfrak{p}}(s_1) \\ \downarrow \\ P^{\mathfrak{p}}(s_1) \end{matrix}\right\}$	$\left\{\begin{matrix} P^{\mathfrak{p}}(s_1s_2) \\ \downarrow \\ 0 \end{matrix}\right\}$	$\left\{\begin{matrix} M^{\mathfrak{p}}(s_1s_2) \\ \downarrow \\ 0 \end{matrix}\right\}$	
$\Theta_{s_2}M$	0	0	$P^{\mathfrak{p}}(s_1s_2)$	$P^{\mathfrak{p}}(s_1s_2)^{\oplus 2}$	$P^{\mathfrak{p}}(s_1s_2)$	
$\mathbf{L}\,\mathrm{Sh}_{s_2} M$	$\left\{\begin{matrix} M^{\mathfrak{p}}(s_1) \\ \downarrow \\ 0 \end{matrix}\right\}$	$\left\{\begin{matrix} P^{\mathfrak{p}}(s_1) \\ \downarrow \\ 0 \end{matrix}\right\}$	$M^{\mathfrak{p}}(s_1s_2)$	$P^{\mathfrak{p}}(s_1s_2)$	$\left\{\begin{matrix} M^{\mathfrak{p}}(s_1s_2) \\ \downarrow \\ P^{\mathfrak{p}}(s_1s_2) \end{matrix}\right\}$	$\ddots$

We now come back to **??** 4.1: Does this A_{n-1}-configuration have something to do with shuffling functors? If—as we might expect—$P^{\mathfrak{p}}(s)$ again induces a spherical cotwist $T'_{P^{\mathfrak{p}}(s)}$ with images are isomorphic to those of Sh_s: what functor arises from the other spherical indecomposable projectives? In contrast to the non-parabolic case, all Verma modules and projectives fit into short exact sequences

$$M^{\mathfrak{p}}(s_1\cdots s_i) \hookrightarrow P^{\mathfrak{p}}(s_1\cdots s_{i+1}) \twoheadrightarrow M^{\mathfrak{p}}(s_1\cdots s_{i+1}) \tag{4.15}$$

for $0 \le i < n-1$, hence *all* Verma modules have indecomposable projectives as their translates (and not only the ones corresponding to dominant weights). We thus obtain the translated and shuffled modules listed in Table 4.5. Note that by Lemma 1.15 the image $\mathrm{Sh}_w M$ is *always* concentrated at most in homological degrees 0 and 1 for all M and $\mathfrak{g}$.

Shuffling and spherical cotwist have isomorphic images We argue as for $\mathfrak{sl}_3$. Let M be a module with translate $\Theta_{s_i}M = P^{\mathfrak{p}}(w)$ and such that $\mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}(M, P^{\mathfrak{p}}(w))$ is one dimensional. It is then obvious that $T'_{P^{\mathfrak{p}}(w)}M \cong \mathbf{L}\,\mathrm{Sh}_{s_i} M$ since

$$\begin{aligned} T'_{P^{\mathfrak{p}}(w)}M &\overset{\text{def}}{=} \Big\{\underset{0}{M} \xrightarrow{ev'} \hom_{\mathbf{C}}\big[\hom_{\mathcal{O}^{\mathfrak{p}}}(M, P^{\mathfrak{p}}(w)), P^{\mathfrak{p}}(w)\big]\Big\} \\ &\cong \big\{M \xrightarrow{\varepsilon} \underset{0}{\Theta_s M}\big\}[1] \\ &\overset{\text{def}}{=} \mathbf{L}\,\mathrm{Sh}_{s_i} M[1]. \end{aligned}$$

Recall from Table 4.3 that for $M = P^{\mathfrak{p}}(v)$ the Hom-spaces $\mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}(P^{\mathfrak{p}}(v), P^{\mathfrak{p}}(w))$ for distinct $v, w \in W^{\mathfrak{p}}$ are zero- or one-dimensional; hence $T'_{P^{\mathfrak{p}}(w)}P^{\mathfrak{p}}(v) \simeq \mathrm{Sh}_{s_i} P^{\mathfrak{p}}(v)[1]$. For $M = P^{\mathfrak{p}}(w)$ itself, we have

$$\begin{aligned} T'_{P^{\mathfrak{p}}(w)}P^{\mathfrak{p}}(w) &\cong \big\{\underset{0}{P^{\mathfrak{p}}(w)} \xrightarrow{\binom{\mathrm{id}}{x}} P^{\mathfrak{p}}(w) \oplus P^{\mathfrak{p}}(w)\big\} \\ &\cong \{0 \xrightarrow{x} \underset{0}{P^{\mathfrak{p}}(w)}\}[1] \\ &\overset{\text{def}}{=} \mathrm{Sh}_{s_i} P^{\mathfrak{p}}(w)[1]. \end{aligned}$$

where x is (up to scalars) the non-trivial endomorphism of $P^{\mathfrak{p}}(w)$. We deduce:

Observation 4.14. Let $\mathfrak{p} \subseteq \mathfrak{sl}_n$ be the maximal parabolic subalgebra corresponding to the partition $(n-1, 1)$ of n. For all $M \in D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$ and all integers $1 \le i \le n-1$, there is an isomorphism of images $T'_{P^{\mathfrak{p}}(s_1\cdots s_i)}M \simeq \mathbf{L}\,\mathrm{Sh}_{s_i} M[1]$.

We shall show in Theorem 4.21 that this in fact is a natural isomorphism of functors.

Table 4.6: Parabolic Kazhdan-Lusztig polynomials for the parabolic subgroup $\langle s\rangle \times \langle u\rangle = S_2 \times S_2 < S_4 = \langle s,t,u\rangle$ The dimensions $\dim_{\downarrow,\rightarrow} := \dim \operatorname{Hom}(P^{\mathfrak{p}}(\downarrow), P^{\mathfrak{p}}(\rightarrow))$ of Hom-spaces in the right half of the table are computed by (4.13).

$p^{\mathfrak{p}}_{\downarrow,\rightarrow}$	e	t	ts	tu	tsu	$tsut$	$\dim_{\downarrow,\rightarrow}$	e	t	ts	tu	tsu	$tsut$
e	1	q	0	0	0	q^2		1	1	0	0	0	1
t	0	1	q	q	q^2	1		1	2	1	1	1	2
ts	0	0	1	0	q	0		0	1	2	1	2	1
tu	0	0	0	1	q	0		0	1	1	2	2	1
tsu	0	0	0	0	1	q		0	1	2	2	4	2
$tsut$	0	0	0	0	0	1		1	2	1	1	2	4

Table 4.7: Composition series of parabolic Verma modules and indecomposable projectives in $\mathcal{O}_0^{\mathfrak{p}}$ for $\mathfrak{p} \subseteq \mathfrak{sl}_4$ the parabolic subalgebra corresponding to the parabolic subgroup $W_{\mathfrak{p}} := S_2 \times S_2 \leq S_4$ (see (4.16)). The series are obtained from the parabolic Kazhdan-Lusztig polynomials listed in Table 4.6.

$w \in W^{\mathfrak{p}}$	$M^{\mathfrak{p}}(w)$	$P^{\mathfrak{p}}(w)$
e	$L(e)$ $L(t)$ $L(tsut)$	dto.
t	$L(t)$ $L(ts)$ $L(tsut)$ $L(tu)$ $L(tsu)$	$L(t)$ $L(e)$ $L(ts)$ $L(tsut)$ $L(tu)$ $L(t)$ $L(tsu)$ $L(tsut)$
ts	$L(ts)$ $L(tsu)$	$L(ts)$ $L(t)$ $L(tsu)$ $L(ts)$ $L(tsut)$ $L(tsu)$ $L(tsu)$
tu	$L(tu)$ $L(tsu)$	$L(tu)$ $L(t)$ $L(tsu)$ $L(ts)$ $L(tsut)$ $L(tsu)$ $L(tsu)$
tsu	$L(tsu)$ $L(tsut)$	$L(tsu)$ $L(ts)$ $L(tu)$ $L(tsut)$ $L(t)$ $L(tsu)$ $L(tsu)$ $L(ts)$ $L(tsut)$ $L(tsu)$ $L(tsu)$
$tsut$	$L(tsut)$	$L(tsut)$ $L(tsu)$ $L(t)$ $L(tsut)$ $L(e)$ $L(ts)$ $L(tsut)$ $L(tu)$ $L(t)$ $L(tsu)$ $L(tsut)$

4.3.4. Maximal parabolic subalgebras with different partitions

Is it necessary to choose a parabolic subalgebra $\mathfrak{p}$ which corresponds to the “extremal” partition $(n-1,1)$, i. e. to the parabolic subgroup $S_{n-1} \times S_1 < S_n$? The maximal parabolic subalgebra

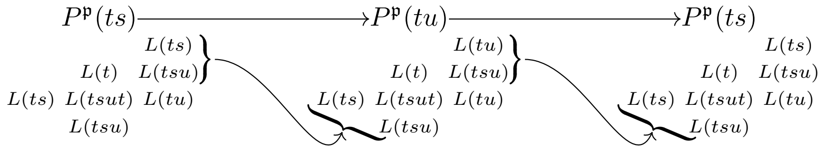
$$\mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \subset \mathfrak{sl}_4, \tag{4.16}$$

corresponding to th parabolic subgroup $W_{\mathfrak{p}} = \langle s \rangle \times \langle u \rangle = S_2 \times S_2 < S_4 = \langle s,t,u \rangle$ with minimal length coset representatives $W^{\mathfrak{p}} = \{e, t, tu, ts, tsu, tsut\}$, has the parabolic Kazhdan-Lusztig polynomials listed in Table 4.6 (see also [BS11b]).

By (4.13), these imply the dimensions of Hom-spaces listed in Table 4.6. There is no possible A$_3$ configuration consisting of indecomposable projectives. The dimensions suggest that $P^{\mathfrak{p}}(t)$, $P^{\mathfrak{p}}(ts)$ and $P^{\mathfrak{p}}(tsu)$ might be spherelike, so we wonder whether we can find an A$_2$-configuration or at least a single spherical object. Indeed, the composition series listed in Table 4.7 show that $P^{\mathfrak{p}}(t)$, $P^{\mathfrak{p}}(ts)$ and $P^{\mathfrak{p}}(tsu)$ have endomorphism rings isomorphic to $\mathbf{C}[x]/(x^2)$, i. e. are spherelike.

Are these objects spherical? Unfortunately, we encounter the very same problem of the present spherelike objects as in the non-parabolic $\mathfrak{sl}_3$-case, revealed by the composition series:

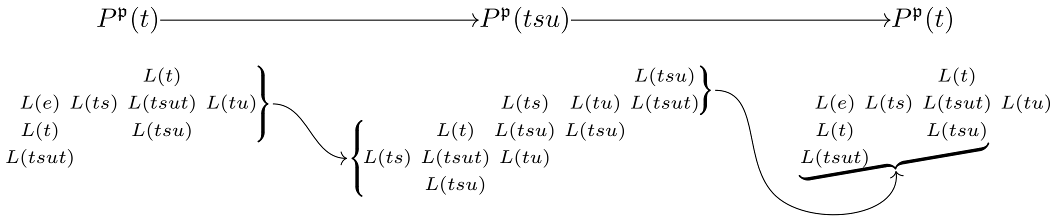
$P^{\mathfrak{p}}(ts)$, $P^{\mathfrak{p}}(tu)$: The projectives $P^{\mathfrak{p}}(ts)$, $P^{\mathfrak{p}}(tu)$ corresponding to the two incomparable weights $ts \ngtrless tu$ have non-trivial morphisms $P^{\mathfrak{p}}(ts) \to P^{\mathfrak{p}}(tu)$ and $P^{\mathfrak{p}}(tu) \to P^{\mathfrak{p}}(ts)$ whose composition



(4.17)

is the zero morphism. The same holds true for $\big(P^{\mathfrak{p}}(tu) \to P^{\mathfrak{p}}(t) \to P^{\mathfrak{p}}(tu)\big) = 0$. Hence neither $P^{\mathfrak{p}}(tu)$ nor $P^{\mathfrak{p}}(ts)$ is spherical.

$P^{\mathfrak{p}}(t)$: For $P^{\mathfrak{p}}(t)$ the composition



(4.18)

shows that $P^{\mathfrak{p}}(t)$ it is not spherical either.

We summarise:

Observation 4.15. For $\mathfrak{p} \subseteq \mathfrak{sl}_4$ the parabolic subalgebra corresponding to the parabolic subgroup $S_2 \times S_2 \leq S_4$ (see (4.16)), $P^{\mathfrak{p}}(t)$, $P^{\mathfrak{p}}(ts)$ and $P^{\mathfrak{p}}(tu)$ are the spherelike indecomposable projective modules. None of them is spherical.

4.4. Shuffling and spherical twist functors are isomorphic functors…

Until now, we have only investigated the behaviour of the twisting- and the shuffling functor on objects. It remains to be shown that in the set-up of Observation 4.14 the two functors are indeed naturally isomorphic, which we shall address in this section.

We shall make use of the following possibility to explicitly describe a right exact functor on module categories. We know that the tensor product is a right exact additive functor. The converse also holds:

Lemma 4.16. Any right exact functor $F\colon$ Mod-$A \to$ Mod-B that preserves arbitrary direct sums is isomorphic to tensoring with the A-B-bimodule FA.

Remark 4.17. The image FA of the A-module A indeed has a left action on A. Namely, any element $a \in A$ induces a bimodule endomorphism $(a\cdot)$ of A. The functor F then yields a B-module endomorphism $F(a\cdot)$ of FA.

Proof of Lemma 4.16. Let $M \in \text{Mod-}A$ be an A-module. We can identify M with $\operatorname{Hom}_A(A, M)$ via the evaluation map $\operatorname{Hom}_A(A, M) \otimes_A A \to M$. Applying F yields a map $\operatorname{Hom}_A(A, M) \otimes_A FA \to FM, \phi \otimes a \mapsto F(\phi)(a)$ which is natural in M. It is clear that this is an isomorphism for $M = A$ and hence, since F and the tensor product both commute with direct sums, for arbitrary free A-modules. For an arbitrary module M, we can choose a free presentation of M which maps to a presentation of FM under F by right exactness of F. By the five lemma applied to the natural isomorphism already found for free modules, this shows that indeed $FM \cong M \otimes_A FA$. □

Corollary 4.18. Given abelian categories $\mathcal{A}$ and $\mathcal{B}$ with projective generators $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$, any right exact functor $F\colon \mathcal{A} \to \mathcal{B}$ which commutes with arbitrary direct sums is naturally isomorphic to the tensor product functor

$$- \otimes_{\operatorname{End} P_{\mathcal{A}}} \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})\colon \mathcal{A} \to \mathcal{B}. \tag{4.19}$$

We denote the $\operatorname{End}_{\mathcal{A}}(P_{\mathcal{A}})$-$\operatorname{End}_{\mathcal{B}}(P_{\mathcal{B}})$-bimodule $\operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})$ by M_F.

Proof. Recall from Theorem 2.1 that there are equivalences $G_{\mathcal{A}} : \mathcal{A} \simeq \text{Mod-}\operatorname{End}_{\mathcal{A}}(P_{\mathcal{A}})$ and $G_{\mathcal{B}} : \mathcal{B} \simeq \text{Mod-}\operatorname{End}_{\mathcal{B}}(P_{\mathcal{B}})$ of categories. Under these equivalences, Lemma 4.16 states that $G_{\mathcal{B}}^{-1} F G_{\mathcal{A}}$ is isomorphic to

$$- \otimes_{\operatorname{End}_{\mathcal{A}} P_{\mathcal{A}}} \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})\colon \text{ Mod-}\operatorname{End}_{\mathcal{A}}(P_{\mathcal{A}}) \to \text{Mod-}\operatorname{End}_{\mathcal{B}}(P_{\mathcal{B}})\,.$$

where the bimodule structure of $\operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})$ is given by

$$\phi \,.\, f \,.\, \psi := F(\phi) \circ f \circ \psi$$

for $f \in \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})$, $\phi \in \operatorname{End}_{\mathcal{A}}(P_{\mathcal{A}})$ and $\psi \in \operatorname{End}_{\mathcal{B}}(P_{\mathcal{B}})$; see Remark 4.17. By abuse of notation, we just write

$$F \cong - \otimes_{\operatorname{End}_{\mathcal{A}}(P_{\mathcal{A}})} \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{A}}, FP_{\mathcal{A}})\colon \mathcal{A} \to \mathcal{B}.$$

where the endomorphism ring acts by precomposition on homomorphisms. □

In the context of chain complexes, a functor $\operatorname{Ch}(\mathcal{A}) \to \operatorname{Ch}(\mathcal{B})$ that is given by a complex of functors $\mathcal{A} \to \mathcal{B}$ is seen immediately to be isomorphic to tensoring with a complex of modules. This also passes to the derived category.

4.4.1. …on $D^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_2))$

We now return to answering **??** 4.1. Let A denote the endomorphism algebra of the projective generator of $\mathcal{O}_0$ (see Corollary 2.2 and (4.4)). Recall that by curly braces we denote mapping cones (see Notation 3.9). By the above considerations, the derived shuffling functor and the cotwist functor resp. are isomorphic to

$$\begin{aligned} \mathbf{L}\operatorname{Sh}_s[-1] &= \{\underset{0}{\mathrm{id}_{\mathcal{O}}} \to \Theta_s\} && \cong - \otimes_A \{A \to M_{\Theta_s}\} \\ T'_{P(s)} &= \left\{\underset{0}{\mathrm{id}_{\mathcal{O}}} \to \operatorname{lin}_{\mathbf{C}}\bigl(\hom_{\mathcal{O}}(-, P(s)), P(s)\bigr)\right\} && \cong - \otimes_A \{A \to M_{P(s)}\}. \end{aligned}$$

For proving that both functors are naturally isomorphic, it thus suffices to prove that M_{Θ_s} and $M_{P(s)}$ are isomorphic bimodules.

Complexes “representing” $\mathbf{L}\operatorname{Sh}_s$ and $T'_{P(s)}$ For writing down the latter module explicitly, recall that the Hom-spaces in question are finite dimensional by the definition of a spherical object in Definition (S1). Hence,

$$\begin{aligned} M_{P(s)} &\cong \operatorname{Hom}_A\bigl[P, \operatorname{lin}_{\mathbf{C}}\bigl(\hom_A(A, P(s)), P(s)\bigr)\bigr] \\ &\cong \operatorname{Hom}_{\mathcal{O}}\bigl[P, P \otimes_A P(s)^{\vee} \otimes_{\mathbf{C}} P(s)\bigr] \\ &\cong P(s)^{\vee} \otimes_{\mathbf{C}} P(s) \end{aligned}$$

as A-bimodules. Let us spell out this module explicitly for the case $\mathfrak{g} = \mathfrak{sl}_2$. We quickly recall the following from Section 2.1.1:

Recall 4.19. The endomorphism algebra A is the path algebra of the quiver with relations $Q = e \rightleftarrows s/(e \leftarrow s \leftarrow e)$, see Corollary 2.2. As a representation of Q, we have seen that the module $P(s) \cong e_s A$ is given by assigning to a vertex v the space $e_s A e_v$ of all paths from v to the vertex s. We have chosen the basis $\{\mathrm{id}_s, s \leftarrow e, s \leftarrow e \leftarrow s\}$ for $P(s)$. The module $P(s)$ is a right A-module by precomposing a path of A to a path in $P(s)$. See Section 2.1.1 for a detailed explanation.

Notation 4.20. Consider a $\mathbf{C}$-algebra B as well as finitely dimensional modules $M \in B\text{-Mod}$ and $N \in \text{Mod-}B$ with $\mathbf{C}$-bases $M = \langle m_i \mid i \in I \rangle_{\mathbf{C}}$ and $N = \langle n_j \mid j \in J \rangle_{\mathbf{C}}$ resp.

(i) We draw diagrams like

$$\begin{tikzcd} m_1 \arrow[r, bend left, "b_1"] \arrow[rr, bend right, "b_2"'] & m_2 \arrow[r, bend left, "b_2"] & m_3 & \cdots \end{tikzcd}$$

which are to be read as follows: the elements b_1 and $b_2 \in B$ act on m_1 as indicated. That there is no arrow labelled b_2 starting at m_2 means that b_2 acts by zero on m_2.

(ii) Consider B-B-bimodule $M \otimes_{\mathbf{C}} N$. It has a $\mathbf{C}$-basis $\{m_i \otimes n_j \mid i \in I, j \in J\}$ on which the bimodule action is given by $b \,.\, m_i \otimes n_j \,.\, b' = (bm_i) \otimes (n_j b)$. We depict a concrete action on a bimodule by diagrams like the following:

$$\left.\begin{tikzcd} m_1 \arrow[d, bend right, "b_1"'] \arrow[dd, bend right=70, "b_2"'] \\ m_2 \arrow[d, bend right, "b_2"'] \\ \vdots \end{tikzcd}\right\} \otimes \left\{\begin{tikzcd} n_1 \\ n_2 \arrow[u, bend right, "b_2"'] \arrow[d, bend left, "b_1"] \\ \vdots \end{tikzcd}\right. \tag{4.20}$$

(iii) Analogously, consider the B-B-bimodule $\mathrm{Hom}_{\mathbf{C}}(M, N)$. Let $n_j \mapsto m_i$ be the $\mathbf{C}$-linear map $M \to N, n_k \mapsto \left\{\begin{smallmatrix} m_i & k=j \\ 0 & k \neq j \end{smallmatrix}\right.$. The bimodule $\mathrm{Hom}_{\mathbf{C}}(M, N)$ then has a $\mathbf{C}$-basis $\{m_i \mapsfrom n_j \mid i \in I, j \in J\}$. In terms of this basis, the bimodule structure is given by

$$b \,.\, (m_i \mapsfrom n_j) \,.\, b' = (b\cdot) \circ (m_i \mapsfrom n_j) \circ (\cdot b')$$

which we depict similarly to (4.20).

(iv) Finally, assume that $M = \bigoplus_{i \in I} M_i$ and $N = \bigoplus_{j \in J} N_j$ as left (resp. right) B-modules. Given a homomorphism $f \in \mathrm{Hom}_B(N_j, M_i)$, let f also denote the homomorphism $N \xrightarrow{\pi_j} N_j \xrightarrow{f} M_i \xrightarrow{\iota_i} M$. The B-B bimodule $\mathrm{Hom}_B(M, N) = \bigoplus_{i,j} \mathrm{Hom}_B(M_i, N_j)$ then has a $\mathbf{C}$-basis

$$\{M_i \xleftarrow{f} N_j \mid i \in I, j \in J, f \in \mathrm{Hom}_B(N_i, M_j)\}$$

on which the action is given by

$$b \,.\, (M_i \xleftarrow{f} N_j) \,.\, b' = (b\cdot) \circ (M_i \xleftarrow{f} N_j) \circ (\cdot b').$$

Assume that all M_i are identical, i. e. $M = (M_*)^{\oplus I}$ for some module M_*. We then depict this basis of $\mathrm{Hom}_B(M, N)$ by pictures like

$$\left.\begin{tikzcd} M_1 \leftarrow \arrow[d, bend right, "b_1"'] \arrow[dd, bend right=70, "b_2"'] \\ M_2 \leftarrow \arrow[d, bend right, "b_2"'] \\ \vdots \end{tikzcd}\right\} \leftarrow \left\{\begin{tikzcd} \xleftarrow{f} N_1 \\ \xleftarrow{g} N_2 \arrow[u, bend right, "b_2"'] \arrow[d, bend left, "b_1"] \\ \vdots \end{tikzcd}\right. \tag{4.21}$$

for $f \in \mathrm{Hom}_B(N_1, M_*)$ and $f \in \mathrm{Hom}_B(N_2, M_*)$. We shall not attach the label "f" to the morphism if $\mathrm{Hom}_B(N_j, M_*)$ is one-dimensional.

Module representing $T'_{P(s)}$ We apply this notation for the left and right A-module structures on $P(s)^\vee \otimes_{\mathbf{C}} P(s)$. In terms of the basis recalled in **??** 4.19, the bimodule structure of $P(s)^\vee \otimes_{\mathbf{C}} P(s)$ is given by the following action on basis vectors:

$$P(s)^\vee \otimes_{\mathbf{C}} P(s) \quad = \quad \left.\begin{tikzcd} (s \leftarrow e \leftarrow s)^* \arrow[d, bend right, "\circ(e \leftarrow s)"'] \\ (s \leftarrow e)^* \arrow[d, bend right, "\circ(s \leftarrow e)"'] \\ e^* \end{tikzcd}\right\} \otimes \left\{\begin{tikzcd} e \arrow[d, bend left, "\circ(s \leftarrow e)"] \\ (s \leftarrow e) \arrow[d, bend left, "\circ(s \leftarrow e)."] \\ (s \leftarrow e \leftarrow s) \end{tikzcd}\right. \tag{4.22}$$

$\mathrm{End}(P) \ni f$	$\mapsto$	$\Theta_s f \in \mathrm{End}(\Theta_s P)$
$P(e) \to P(e)$		$P(s) \xrightarrow{1} P(s)$
$P(e) \to P(s)$		$P(s) \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} P(s)^{\oplus 2}$
$P(s) \to P(e)$		$P(s)^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} P(s)$
$P(s) \overset{\mathrm{id}}{\to} P(s)$		$P(s)^{\oplus 2} \xrightarrow{\mathrm{id}} P(s)^{\oplus 2}$
$P(s) \overset{x}{\to} P(s)$		$P(s)^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right)} P(s)^{\oplus 2}$

Table 4.8: Images of all endomorphisms of the projective generator $P = P(e) \oplus P(s)$ of $\mathcal{O}_0(\mathfrak{sl}_2)$ under the translation functor Θ_s.

Module representing Θ_s In order to understand the left action on the other bimodule $M_{\Theta_s} = \mathrm{Hom}\big(P(e) \oplus P(s), P(s) \oplus P(s)^{\oplus 2}\big)$, we need to know how Θ_s acts on endomorphisms of the projective generator P, since $A = \mathrm{End}_{\mathcal{O}}(P)$ acts from the left via Θ_s. This can be revealed by fitting $P(e)$ and $P(s)$ into short exact sequences: the image of the following sequence splits canonically (which yields the image of the inclusion under Θ_s) and exhibits the adjunction map for $P(s)$ to be the vertical map in the commutative diagram

$$\begin{tikzcd}
P(e) \arrow[r, hook] \arrow[d, hook] & P(s) \arrow[r, two heads] \arrow[d, hook, "\left(\begin{smallmatrix}1\\x\end{smallmatrix}\right)"] & M(s) \arrow[d, hook] \\
P(s) \arrow[r, hook, "\iota_1"] & P(s)^{\oplus 2} \arrow[r, two heads, "\pi_2"] & P(s).
\end{tikzcd} \tag{4.23}$$

A similar sequence holds for the morphism $\Theta_s(P(s) \to P(e))$. The non-trivial endomorphism x of $P(s)$ fits into the following sequence. Commutativity of the diagram allows us to obtain the image of x in the middle of the diagram

$$\begin{tikzcd}
P(e), \arrow[r, hook] \arrow[d, hook] \arrow[d, mapsto, bend right=60, "\Theta_s"] \arrow[dd, mapsto, bend right=80, "\mathrm{Sh}_s"'] & P(s) \arrow[r, "x"] \arrow[d, hook, "\left(\begin{smallmatrix}1\\x\end{smallmatrix}\right)"'] & P(s) \arrow[r, two heads] \arrow[d, hook, "\left(\begin{smallmatrix}1\\x\end{smallmatrix}\right)"] & P(e)^\vee \arrow[d] \\
P(s) \arrow[r, hook] \arrow[d, two heads] & P(s)^{\oplus 2} \arrow[r, "\left(\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right)", "=\Theta_s x"'] \arrow[d, two heads, "\left(\begin{smallmatrix}x & -1\end{smallmatrix}\right)"'] & P(s)^{\oplus 2} \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\left(\begin{smallmatrix}x & -1\end{smallmatrix}\right)"] & P(s) \arrow[d, two heads] \\
M(s) \arrow[r, hook, "\iota"'] & P(s) \arrow[r, "-x"'] & P(s) \arrow[r, two heads, "-\pi"'] & P(e)^\vee.
\end{tikzcd} \tag{4.24}$$

The functor Θ_s thus assigns the images listed in Table 4.8 to the endomorphisms of P.

We can now return to the description of the bimodule action of A on M_{Θ_s}. With the notation explained in Notation 4.20.(iv) and the left action from Table 4.8, the left and right actions of A on M_{Θ_s} on basis vectors are as follows:

$$M_{\Theta_s} \quad = \quad \begin{array}{r} \circ(e\leftarrow s) \\ \circ(s\leftarrow e) \end{array} \left.\begin{array}{l} P(s)_3 \leftarrow \\ P(s)_1 \leftarrow \\ P(s)_2 \leftarrow \end{array}\right\} \leftarrow \left\{\begin{array}{l} \xleftarrow{1} P(s) \\ \leftarrow P(e) \\ \xleftarrow{x} P(s). \end{array}\right. \begin{array}{l} \circ(s\leftarrow e) \\ \circ(s\leftarrow e) \end{array} \tag{4.25}$$

One sees that there is an isomorphism $\mathrm{Hom}(P, \Theta_s P) \cong P(s)^\vee \otimes_{\mathbb{C}} P(s)$ of A-bimodules by comparing the action on these bases; hence we have shown:

Theorem 4.21. Let $\mathfrak{g} = \mathfrak{sl}_2$. The functors $\mathbf{L}\,\mathrm{Sh}_s[-1]$ and $T'_{P(s)}$ are naturally isomorphic autoequivalences of $D^{\mathrm{b}}(\mathcal{O}_0)$.

4.4.2. …on $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_n))$ for certain $\mathfrak{p} \subseteq \mathfrak{sl}_n$

We first generalise our insights to the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{sl}_3$ considered in Section 4.3.2. We shall notice later that this contains already all information required to understand the parabolic $\mathfrak{sl}_n$-case.

Recall the equivalence $\mathcal{O}_0^{\mathfrak{p}} \simeq \text{Mod-}A_{\mathfrak{p}}$ from (4.8). and the indecomposable projectives of $\mathcal{O}_0^{\mathfrak{p}}$ with composition factors listed in Table 4.1. The algebra $A_{\mathfrak{p}}$ is the path algebra of the quiver with relations

$$Q_{\mathfrak{p}} \coloneqq \quad e \rightleftarrows s \rightleftarrows st, \qquad \begin{array}{ll} e \leftarrow s \leftarrow e = 0, & e \leftarrow s \leftarrow st = 0, \\ st \leftarrow s \leftarrow e = 0, & s \leftarrow e \leftarrow s = s \leftarrow st \leftarrow s. \end{array} \tag{4.26}$$

With Notation 4.20 the $A_{\mathfrak{p}}$-$A_{\mathfrak{p}}$-bimodule $P^{\mathfrak{p}}(s)^\vee \otimes_{\mathbf{C}} P^{\mathfrak{p}}(s)$ associated to the cotwist functor $T'_{P^{\mathfrak{p}}(s)}$ has the following basis with $A_{\mathfrak{p}}$-action:

$$\begin{tikzcd}[row sep=small]
(s \leftarrow e \leftarrow s)^* \arrow[d, bend right, "(e\leftarrow s)\circ"'] \arrow[dd, bend right=60, "(st\leftarrow s)\circ"'] & & e_s \arrow[d, bend left, "\circ(s\leftarrow e)"] \arrow[dd, bend left=60, "\circ(s\leftarrow st)"] \\
(s \leftarrow st)^* \arrow[dd, bend right, "(e\leftarrow s)\circ"'] & \otimes & s \leftarrow e \arrow[dd, bend left, "\circ(e\leftarrow s)"] \\
(s \leftarrow e)^* \arrow[d, bend right, "(s\leftarrow st)\circ"'] & & s \leftarrow st \arrow[d, bend left, "\circ(st\leftarrow s)."] \\
e_s^* & & s \leftarrow e \leftarrow s
\end{tikzcd} \tag{4.27}$$

For computing the bimodule M_{Θ_s}, we notice that the images of morphisms listed in Table 4.8 remain valid for the respective parabolic modules. Additionally, there are images

$$\begin{aligned}
\Theta_s \colon P^{\mathfrak{p}}(st) \to P^{\mathfrak{p}}(s) \quad &\longmapsto \quad P^{\mathfrak{p}}(s) \xrightarrow{\binom{1}{0}} P^{\mathfrak{p}}(s)^{\oplus 2}, \\
P^{\mathfrak{p}}(s) \to P^{\mathfrak{p}}(st) \quad &\longmapsto \quad P^{\mathfrak{p}}(s)^{\oplus 2} \xrightarrow{(0\ 1)} P^{\mathfrak{p}}(s).
\end{aligned} \tag{4.28}$$

We thus deduce that the left and right actions on the following vector space basis of $\operatorname{Hom}_{\mathcal{O}}(P, \Theta_s P)$ are

$$\begin{tikzcd}[row sep=small]
P^{\mathfrak{p}}(s)_3 \leftarrow \arrow[d, bend right, "\Theta_s(e\leftarrow s)\circ"'] \arrow[dd, bend right=60, "\Theta_s(st\leftarrow s)\circ"'] & & \xleftarrow{1} P^{\mathfrak{p}}(s) \arrow[d, bend left, "\circ(s\leftarrow e)"] \arrow[dd, bend left=60, "\circ(s\leftarrow st)"] \\
P^{\mathfrak{p}}(s)_1 \leftarrow \arrow[dd, bend right, "\Theta_s(e\leftarrow s)\circ"'] & \leftarrow & \leftarrow P^{\mathfrak{p}}(e) \arrow[dd, bend left, "\circ(e\leftarrow s)"] \\
P^{\mathfrak{p}}(s)_4 \leftarrow \arrow[d, bend right, "\Theta_s(s\leftarrow st)\circ"'] & & \leftarrow P^{\mathfrak{p}}(st) \arrow[d, bend left, "\circ(st\leftarrow s)."] \\
P^{\mathfrak{p}}(s)_2 \leftarrow & & \xleftarrow{x} P^{\mathfrak{p}}(s)
\end{tikzcd} \tag{4.29}$$

Again, the bimodules in (4.27) and (4.29) are isomorphic; hence we have shown:

Theorem 4.22. Let $\mathfrak{p} \subseteq \mathfrak{sl}_3$ be a maximal parabolic subalgebra. The functors $\mathbf{L}\operatorname{Sh}_s[-1]$ and $T'_{P(s)}$ are naturally isomorphic autoequivalences of $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$.

We eventually consider the case for the parabolic $\mathfrak{sl}_n$-case. However, there remains nothing to do: the dimensions of Hom-spaces listed in Table 4.3 show that tensoring with $P^{\mathfrak{p}}(s_1)^\vee \otimes_{\mathbf{C}} P^{\mathfrak{p}}(s_1)$ annihilates all indecomposable projectives $P^{\mathfrak{p}}(s_1 s_2 s_3 \cdots)$ not regarded so far. We already know that this is the case for Θ_s. Furthermore, the above considerations hold true for any copy of $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_3)$ in $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_n)$, hence we deduce:

Theorem 4.23. For a maximal parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{sl}_n$ corresponding to the parabolic subgroup $S_{n-1} \times S_1 \leq S_n$ there is an A_{n-2}-configuration $\big(P^{\mathfrak{p}}(s_1), P^{\mathfrak{p}}(s_1 s_2), \ldots, P^{\mathfrak{p}}(s_1 \cdots s_{n-2})\big)$. The associated cotwist functors $T'_{P^{\mathfrak{p}}(s_1 \cdots s_i)}$ and the restriction of the derived shuffling functor $\mathbf{L}\operatorname{Sh}_{s_i}[-1]$ to $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$ are naturally isomorphic auto-equivalences of $D^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}})$.

Remark 4.24. In Section 4.2 we have seen that for $\mathfrak{sl}_3$ none of the indecomposable projectives is spherical. Since $T'_{P^{\mathfrak{p}}(t)}$ therefore cannot be an auto-equivalence but $\mathbf{L}\operatorname{Sh}_t$ certainly is, both functors cannot be isomorphic. In Observation 4.15 we have encountered similar problems if we consider maximal parabolic subgroups other than $S_{n-1} \times S_1 \leq S_n$. We suggest another (easier) argument to see $\mathbf{L}\operatorname{Sh}_t[-1] \ncong T'_{P^{\mathfrak{p}}(t)}$ in these cases:

We have used Corollary 4.18 to answer **??** 4.1 by scrutinising the (easier) question whether the modules that “represent” $\mathbf{L}\operatorname{Sh}_t$ and $T'_{P^{\mathfrak{p}}(t)}$ are isomorphic. A dimension argument easily reveals $\mathbf{L}\operatorname{Sh}_t[-1] \ncong T'_{P^{\mathfrak{p}}(t)}$ based on this consideration.

Consider for instance $\mathfrak{sl}_3$. Although $P(s) \in \mathcal{O}_0$ is infinite dimensional, its *image* under the equivalence $\mathcal{O}_0 \simeq A$-Mod is finite dimensional, namely $\dim_{\mathbf{C}} P(s) = 10$, which can be taken from the basis of $P(s)$ given in Figure 2.2. Therefore, $\dim_{\mathbf{C}} M_{P(s)} = \dim_{\mathbf{C}} P(s)^\vee \otimes_{\mathbf{C}} P(s) = 100$. On the other hand, consider the dimensions of $\operatorname{Hom}_{\mathcal{O}}(P(w), \Theta_s P(v))$ for $v, w \in S_3$ listed in Table 4.9. The table yields $\dim_{\mathbf{C}} M_{\Theta_s} = \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}}(P, \Theta_s P) = 145$, hence $M_{\Theta_s} \ncong M_{P(s)}$.

Remark 4.25. Recall the notion of spherical subcategories from Remark 3.8. Since $\mathcal{O}_0(\mathfrak{sl}_n)$ is equivalent to modules over a quasi-hereditary algebra of finite global dimension [Maz12, §4.4], its derived category $D^{\mathrm{b}}(\mathcal{O}_0)$ admits a Serre functor [BK89], namely $\mathrm{S} = \mathbf{L}\operatorname{Sh}^2_{w_0}$ [MS08, prop. 4.1]. In the light of Remark 3.8 this means that the spherelike object $P(s)$ has a spherical subcategory in $D^{\mathrm{b}}(\mathcal{O}_0)$. We refrain from giving the construction of spherical subcategories; see [HKP17, §4] for the definition. We do not know yet the shape of this spherical subcategory. It would be interesting to understand how it relates to $\mathcal{O}_0^{\mathfrak{p}}$ for the parabolic subalgebra we considered in Section 4.3.3.

Table 4.9: Dimensions of $\mathrm{Hom}_{\mathcal{O}}(P(w), \theta_s P(v))$ for $v, w \in W$ for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. The dimensions can either be computed by Kazhdan-Lusztig polynomials (see Table 2.5) or by using the explicit bases from Figure 2.2.

$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}}(\rightarrow, \downarrow)$	$P(e)$	$P(s)$	$P(t)$	$P(st)$	$P(ts)$	$P(w_0)$	$\bigoplus_w P(w)$
$\Theta_s P(e) = P(s)$	1	2	1	2	2	2	10
$\Theta_s P(s) = P(s)^{\oplus 2}$	2	4	2	4	4	4	20
$\Theta_s P(t) = P(ts)$	1	2	2	3	4	4	16
$\Theta_s P(st) = P(w_0) \oplus P(s)$	2	4	3	6	6	8	29
$\Theta_s P(ts) = P(ts)^{\oplus 2}$	2	4	4	6	8	8	32
$\Theta_s P(w_0) = P(w_0)^{\oplus 2}$	2	4	4	8	8	12	38
$\Theta_s P = \bigoplus_w \Theta_s P(w)$	10	20	16	29	32	38	145

Remark 4.26. Recall from Section 2.1.1 that when thinking about spherical objects we started with the path algebra of the quiver $Q = e \rightleftarrows s/(e \to s \to e)$. We generalised Q to a certain quiver, see (2.9), corresponding to the inclusion $\mathfrak{sl}_2 \subseteq \mathfrak{sl}_3$. Another quiver containing Q was given in (4.14), corresponding to $\mathfrak{sl}_2 \subseteq \mathfrak{p}$ for certain $\mathfrak{p} \subseteq \mathfrak{sl}_n$. A third possible generalisation are circular quivers. An investigation on spherelike representations of circular quivers and the associated spherical subcategories has been performed in [HKP17, §6].

Part II.

From symmetric functions towards super Soergel bimodules

5. Rouquier Complexes and the action of B_W on $D^{\mathrm{b}}(\mathcal{O}_0)$

In this section, we want to present a proof for Theorem 1.20 given by Rouquier in [Rou06]. Instead of directly proving that the braid relations are satisfied by the shuffling functor, Rouquier introduces functors F_s which act on the homotopy category of so-called *Soergel bimodules*. We shall explain in Section 5.3 how this construction links back to the category $\mathcal{O}$.

5.1. Representations of Coxeter groups

Recall the definition of a Coxeter group (W, S) and the associated braid group B_W from the introduction.

Definition 5.1. Let (W, S) be a Coxeter system. An element $w \in W$ is called a *reflection* if it is conjugate to a generator $s \in S$. We call the elements of S *simple reflections*.

Let V be a finite dimensional k-vector space. A *reflection* $\phi \in \operatorname{End}_k(V)$ is an involution whose fixed point set V^{ϕ} has codimension one. Since we assume k not to be of characteristic 2, this implies that V decomposes into eigenspaces $V = V^{\phi} \oplus V^{-\phi}$ with $\dim V^{-\phi} = 1$. The vector space V is a *reflection representation* of W if every reflection $w \in W$ acts by a reflection on V. The representation V is called *reflection faithful* if any $w \in W$ acts by a reflection on V if and only if w is a reflection in W. In this case, for any two reflections $v, w \in W$, we have that $V^{-v} \neq V^{-w}$ implies $v \neq w$.

Definition 5.2 (geometric representation) [Hum90, §5.8]. Let the k-vector space $V_{\mathrm{g}} := \langle e_s \mid s \in S\rangle_k$ be endowed with the bilinear form that is given on basis vectors by $\langle e_s, e_t\rangle := -\cos(\pi/m_{s,t})$. A generator $s \in S$ acts on $v \in V_{\mathrm{g}}$ by $s\,.\,v := v - 2\langle e_s, v\rangle e_s$, i. e. by reflecting v across the plane orthogonal to e_s w. r. t. $\langle\cdot,\cdot\rangle$. The vector space V_{g} endowed with this action is called the *geometric representation* of W. This representation is reflection-faithful [Soe07, prop. 2.1].

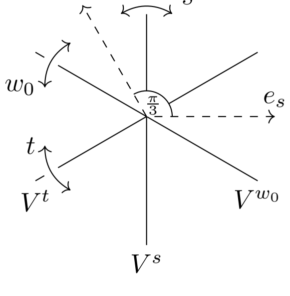
Example 5.3. For $W = S_3$ the symmetric group, the geometric representation is depicted in Figure 5.1. In general, if W is a Weyl group of a semisimple complex Lie algebra, its geometric representation is just $\mathfrak{h}^*$ with the usual action of W.

Definition 5.4 (permutation representation). For the symmetric group S_n, which is a Coxeter group of type A_{n-1}, the *permutation representation* is the vector space $V_{\mathrm{p}} := \langle e_i, \dots, e_n\rangle_k$ on which S_n acts by permuting coordinates. This representation is reflection-faithful, too. It has V_{g} as quotient by the one-dimensional subspace $\langle e_1 + \cdots + e_n\rangle_k$. A respective representation is given for other permutation groups, in particular for the Coxeter group series BC_n and D_n.

Definition 5.5. Given a finite dimensional k-vector space V, the *coordinate ring* of V or *ring of regular functions* on V is the k-algebra $k[V] := k[\alpha_i]$ for a basis $\{\alpha_i\}$ of the dual space V^*.

(i) Let $A_{\mathrm{g}} := k[V_{\mathrm{g}}] = k[\alpha_s]_{s\in S}$, where $\alpha_s \in V_{\mathrm{g}}^*$ is the functional with $\ker\alpha_s = V^s$, normalised such that $\alpha_s(e_t) = m_{s,t}$. The W-action on V_{g} induces the *contragredient action* on V_{g}^* given by $g\,.\,a(v) := a(g^{-1}\,.\,v)$ for $g \in W, a \in V_{\mathrm{g}}^*$ and $v \in V_{\mathrm{g}}$. The algebra A_{g} thus inherits a W-action such that $s\,.\,\alpha_s = -\alpha_s$.

Figure 5.1: Geometric representation of the Coxeter group $W = S_3$ with generators s, t. The basis vectors e_s and e_t are the -1-eigenvectors of s and t respectively. They enclose an angle of $\frac{\pi}{m_{st}}$ where for S_3 we have $m_{st} = 3$. This representation corresponds to the root system of $\mathfrak{sl}_3$, depicted in Figure 1.1.



(ii) For $W = S_n$ the symmetric group, we similarly define $A_\mathrm{p} \coloneqq k[V_\mathrm{p}] = k[x_1, \dots, x_n]$ with an S_n-action on A_p by permuting variables. This algebra has the quotient $A_\mathrm{g} = A_\mathrm{p}/(x_1 + \cdots + x_n)$. We set $\alpha_{s_i} \coloneqq x_{i+1} - x_i$ such that again $s \,.\, \alpha_s = -\alpha_s$.

In the following, let V denote either of the two representations V_g and V_p, and let A denote the respective ring of regular functions.

The algebra A is endowed with a natural grading by the degree of the polynomial. We follow Rouquier's convention to have the α_s in degree one rather then Soergel's convention to place them in degree two [Soe07]. We point out again that we denote by $\langle 1 \rangle$ the *upwards shift* of the grading, which is opposite to the convention in [Rou06].

Lemma 5.6. The algebra A is a free graded A^s-module of rank 2: Given a simple reflection $s \in S$ of W, the algebra A decomposes as $A \cong A^s \oplus A^s \alpha_s$ as graded A^s-A-bimodule.

Proof. The isomorphism sending a to its symmetric and anti-symmetric part is given by $a \mapsto \big(\frac{1}{2}(a + s \,.\, a), \frac{1}{2}(a - s \,.\, a)\big)$, noting that $\frac{1}{2}(a - s \,.\, a) = \langle \alpha_t, \alpha_s \rangle \alpha_s$ is divisible by α_s. ☐

Definition 5.7 [Dem73]. The *Demazure operator* is the homogeneous map

$$\partial_s \colon A\langle -1 \rangle \to A^s; \quad x \mapsto \tfrac{x - sx}{2\alpha_s}.$$

Its image lies in A^s since both $x - sx$ and α_s are antisymmetric; hence their quotient is symmetric.

Our goal is to establish an action of B_W on the category of graded A-bimodules A-gMod-A, following the construction of Rouquier. From now on, let us work with $k = \mathbf{C}$.

5.2. Braid group action

Define the *twisted diagonal* $\Delta_w = \{(wv, v)\} \subseteq V \times V$ with algebra of regular functions $\mathbf{C}[\Delta_w] = (A \otimes_\mathbf{C} A)/(wa \otimes 1 - 1 \otimes a)_{a \in A}$. Denote the ideal quotiented out by $\mathfrak{a}_w$. Furthermore, define the subset $W_{\leq w} \coloneqq \{v \in W \mid v \leq w\}$ as well as the union $\Delta_{\leq w} \coloneqq \bigcup_{v \leq w} \Delta_v$ of twisted diagonals which is a hyperplane in $V \times V$. Its algebra of regular functions is $A_{\leq w} \coloneqq A / \prod_{v \leq w} \mathfrak{a}_v$. We have

$$\operatorname{Hom}_A\big(A_{\leq w}, A_{\leq w'}\big) = \begin{cases} \mathrm{can} & \text{if } w' \leq w, \\ 0 & \text{otherwise,} \end{cases}$$

with the canonical quotient map $\mathrm{can} : A_{\leq w} \twoheadrightarrow A_{\leq w'}$, given by the restriction of functions along the inclusion $\Delta_{\leq w'} \subseteq \Delta_{\leq w}$.

Definition 5.8. The *Rouquier complexes* are defined to be the complexes

$$\begin{aligned} F_s &= \big\{0 \longrightarrow 0 \longrightarrow \underset{0}{A \otimes_{A^s} A} \xrightarrow{\varepsilon'} A \longrightarrow 0\big\} \\ F_s^{-1} &= \big\{0 \longrightarrow A \xrightarrow{\eta} \underset{0}{A \otimes_{A^s} A\langle -1 \rangle} \longrightarrow 0 \longrightarrow 0\big\} \\ &\qquad\quad 1 \longmapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s \end{aligned} \tag{5.1}$$

of graded A-bimodules, where the map ε' is the multiplication in the algebra A and η is the *comultiplication* for a yet to be defined coalgebra structure.

We explain the proof of the following theorem of Rouquier:

Theorem 5.9 [Rou06, prop. 9.4]. Tensoring with the complexes F_s and F_s^{-1} establishes an action of B_W on $K^\mathrm{b}(A\text{-gMod-}A)$, i. e. there is a map

$$B_W \to \operatorname{End}\big(K^\mathrm{b}(A\text{-gMod-}A)\big)$$

defined on simple reflections by $s^{\pm 1} \mapsto (F_s^{\pm 1} \otimes_A -)$. We occasionally write F_s for the tensor product $F_s \otimes -$.

We leave the following statement unproven:

Proposition 5.10 [Rou06, prop. 8.1]. Let $F\colon \mathcal{S} \rightleftarrows \mathcal{T} :G$ be two triangulated functors between triangulated categories and $\langle 1\rangle$ be a triangulated auto-equivalence of $\mathcal{S}$ such that there are adjunctions $F \dashv G \dashv F\langle 1\rangle$. Assume that the following adjunction (co)units fit into a distinguished triangle $\mathrm{id}_{\mathcal{S}} \to GF \to \langle -1\rangle \xrightarrow{0} \mathrm{id}_{\mathcal{S}}[1]$. Then the mapping (co)cones $C = \{\mathrm{id}_{\mathcal{T}} \xrightarrow{\varepsilon'} GF\}$ and $K \coloneqq \{\mathrm{id}_{\mathcal{T}} \xrightarrow{\eta} F\langle 1\rangle G\}[-1]$ of the adjunction (co)units are mutually inverse auto-equivalences of $\mathcal{D}$.

Corollary 5.11. The (co)shuffling functors Sh_s and Csh_s are mutually inverse auto-equivalences $\mathcal{O}_0$.

Proof. Recall from Definition 1.12 the definitions $\mathrm{Sh}_s = \{\mathrm{id}_{\mathcal{O}_0} \to T_{\mathrm{off}}T_{\mathrm{on}}\}$ and $\mathrm{Csh}_s = \{T_{\mathrm{off}}T_{\mathrm{on}} \to \mathrm{id}_{\mathcal{O}_0}\}[-1]$ for the functors $T_{\mathrm{off}}\colon \mathcal{O}_\mu \rightleftarrows \mathcal{O}_0 :T_{\mathrm{on}}$ and a weight μ contained in the s-wall of the fundamental chamber. The functors T_{off} and T_{on} are biadjoint, and Proposition 5.10 yields the statement with $\langle 1\rangle \coloneqq \mathrm{id}_{\mathcal{O}_0}$.

Recall from Section 2.3 the definition of the graded category $\mathcal{O}_0^{\mathbf{Z}}$ as well as the graded shuffling $\mathrm{Sh}_s = \{\mathrm{id}\langle 1\rangle \to \Theta_s\}$ and coshuffling $\mathrm{Csh}_s = \{\Theta_s \to \mathrm{id}_{\mathcal{O}_0}\langle -1\rangle\}[-1]$. These are adjunction maps from adjunctions $T_{\mathrm{off}} \dashv T_{\mathrm{on}}\langle 1\rangle$, $T_{\mathrm{on}} \dashv T_{\mathrm{off}}\langle -1\rangle$ of graded translation functors; see [Str03a, thm. 8.4]. Proposition 5.10 yields the statement with $\langle 1\rangle$ the grading shift. ☐

The braid relations are the harder part. In fact, the graded version of the corollary is proven by Rouquier without referring to [Str03a].

Lemma 5.12. The functors $F_s \otimes_A -$ and $F_s^{-1} \otimes_A - \in \mathrm{End}\big(K^{\mathrm{b}}(A\text{-gMod-}A)\big)$ for $s \in S$ indeed are mutual inverses.

Proof. As the notation suggests, the structure maps of F_s and F_s^{-1} are part of adjunctions $F \dashv G \dashv F\langle -1\rangle$. Namely, consider the functors

$$F = A \otimes_{A^s} -\colon\; A^s\text{-gMod} \rightleftarrows A\text{-gMod} :(A^s \oplus A^s\langle 1\rangle) \otimes_A - = G,$$

where the right A-action on $A^s \oplus A^s\langle 1\rangle$ is given by the isomorphism $A^s \oplus A^s\langle 1\rangle \cong A$ as left A^s-modules. The two adjunctions are established by tensoring with the following maps:

$$\begin{array}{lrll}
F \dashv G: & FG = A \otimes_{A^s} A & \xrightarrow{\varepsilon'} A & \text{in } A\text{-gMod}\\
 & a \otimes b & \longmapsto ab & \\
 & \mathrm{id} & \xrightarrow[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]{\eta'} \mathrm{id} \oplus \mathrm{id}\langle 1\rangle = GF & \text{in } A^s\text{-gMod}\\[2ex]
G \dashv F\langle -1\rangle: & GF\langle -1\rangle = \mathrm{id}\langle -1\rangle \oplus \mathrm{id} & \xrightarrow[(0\ 1)]{\partial_s} \mathrm{id} & \text{in } A^s\text{-gMod}\\
 & A & \xrightarrow{\eta} A \otimes_{A^s} A\langle -1\rangle & \text{in } A\text{-gMod}\\
 & & = FG\langle -1\rangle & \\
 & 1 & \longmapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s. &
\end{array}$$

The middle two maps obviously fit into a split exact sequence $\mathrm{id} \overset{\eta'}{\hookrightarrow} \mathrm{id} \oplus \mathrm{id}\langle 1\rangle \overset{\partial_s}{\twoheadrightarrow} \mathrm{id}\langle 1\rangle$ of endofunctors of A^s-gMod. When passing to the homotopy category, this sequence gives rise to natural transformations $\mathrm{id} \overset{\eta'}{\Rightarrow} \mathrm{id} \oplus \mathrm{id}\langle 1\rangle \overset{\partial_s}{\Rightarrow} \mathrm{id}\langle 1\rangle \Rightarrow \mathrm{id}[1]$ of functors which yield distinguished triangles when evaluated on objects. Proposition 5.10 now asserts that the (co)cones of η and ε' are mutually inverse auto-equivalences of $K^{\mathrm{b}}(A\text{-gMod})$. Unravelling the definition of F and G exhibits that these cones are precisely the Rouquier complexes

$$\begin{aligned}
\{\mathrm{id} \xrightarrow{\eta} FG\langle -1\rangle\} &= \{A \to \underset{0}{A \otimes_{A^s} A\langle -1\rangle} \to 0\} \otimes_A - = F_s^{-1},\\
\{FG \xrightarrow{\varepsilon'} \mathrm{id}\}[-1] &= \{0 \to \quad \underset{0}{A \otimes_{A^s} A} \quad \to A\} \otimes_A - = F_s.
\end{aligned}$$

This shows that F_s^{-1} and F_s are indeed mutually inverse auto-equivalences. ☐

Proposition 5.13 [Rou06, prop. 9.2]. Assume that $W = I_n$ is a dihedral group of order $2n$, i. e. is given by the presentation $W = \langle s_+, s_- \mid s_+^2 = s_-^2 = (s_+s_-)^m = 1\rangle$ with $m < \infty$. Then the functors $F_{s_+}, F_{s_-} \in \mathrm{End}\big(K^{\mathrm{b}}(A\text{-grMod}\,A)\big)$ satisfy the braid relations.

Proof. Our strategy is to construct a complex $F^\ell_\pm$ homotopy equivalent to the ℓ-fold tensor product $F_{s_\pm} \otimes F_{s_\mp} \otimes \cdots$ of complexes, such that $F^\ell_\pm$ only depends on the group element represented by the word $s_\pm s_\mp \cdots$ of length ℓ. This will prove the lemma.

—*Constructing the complex* $F^\ell_\pm$: We introduce some notation: Let $w^\ell_\pm := s_\pm s_\mp s_\pm \cdots$ with ℓ factors, $D^\ell_\pm := A_{\le w^\ell_\pm}$ and the respective ideal $\mathfrak{a}^\ell_\pm := \mathfrak{a}_{w^\ell_\pm}$. Note that whenever $\ell < j$, we have $w^\ell_* < w^j_*$ for any allocation of the two asterisks with signs. Consider the complex

$$F^\ell_\pm := \Big\{ 0 \to \underset{0}{[0]} D^\ell_\pm \xrightarrow{\left(\begin{smallmatrix}+\\+\end{smallmatrix}\right)} D^{\ell-1}_\pm - \oplus D^{\ell-1}_\mp \xrightarrow{\left(\begin{smallmatrix}+&-\\+&-\end{smallmatrix}\right)} \cdots \to D^+_1 \oplus D^-_1 \xrightarrow{(+\ -)} \underset{0}{[\ell]} A_e \to 0 \Big\}$$

with the indicated multiples of the quotient map can. Note that $F^m_+ = F^m_-$.

—*Homology of* $F^\ell_\pm$: Our first goal is to show that the complex $F^\ell_\pm$ has vanishing homology in all degrees except of zero by showing that it is obtained from splicing short exact sequences together. For any ring R and ideals $\mathfrak{b}, \mathfrak{c} \lhd R$ there is a short exact sequence $R/\mathfrak{b}\mathfrak{c} \hookrightarrow R/\mathfrak{b} \oplus R/\mathfrak{c} \twoheadrightarrow R/(\mathfrak{b}+\mathfrak{c})$. For the ideals $\mathfrak{a}^r_\pm$ for $0 \le r \le \ell$ we thus obtain

$$0 \to \underbrace{A/\mathfrak{a}^r_+\mathfrak{a}^r_-}_{\mathbf{C}\left[\Delta_{\le w^r_+} \cap \Delta_{\le w^r_-}\right]} \to A/\mathfrak{a}^r_+ \oplus A/\mathfrak{a}^r_- \to \underbrace{A/(\mathfrak{a}^r_+ + \mathfrak{a}^r_-)}_{\mathbf{C}\left[\Delta_{\le w^r_+} \cup \Delta_{\le w^r_-}\right]} \to 0. \tag{5.2}$$

Since $w^r_\pm$ is built by alternating s_+ and s_-, the only sub-words of $w^r_\pm$ of length $r-1$ can be taken from

$$w^r_+ = \mathrlap{\overbrace{}^{w^{r-1}_+}} s_+ \underbrace{s_- s_+ \cdots s_\mp s_\pm}_{w^{r-1}_-}, \quad w^r_- = \mathrlap{\underbrace{}_{w^{r-1}_-}} s_- \overbrace{s_+ \cdots s_\mp s_\pm s_\mp}^{w^{r-1}_+}.$$

Therefore, the set $\{v \in W \mid v \le w^r_+, w^r_-\}$ of words smaller than both of them equals the union

$$\begin{aligned} W_{\le w^r_+} \cap W_{\le w^r_-} &= W_{\le w^{r-1}_+} \cup W_{\le w^{r-1}_-}; \quad \text{hence} \\ \Delta_{\le w^r_+} \cap \Delta_{\le w^r_-} &= \Delta_{\le w^{r-1}_+} \cup \Delta_{\le w^{r-1}_-} \end{aligned}$$

with coordinate rings

$$A/(\mathfrak{a}^r_+ + \mathfrak{a}^r_-) \cong A/(\mathfrak{a}^{r-1}_+\mathfrak{a}^{r-1}_-).$$

The short exact sequences (5.2) thus can be spliced together:

$$\begin{tikzcd}[column sep=small]
 & & & & & & A/(\mathfrak{a}^{\ell-1}_\pm + \mathfrak{a}^{\ell-1}_\mp) \arrow[dr] & & & \\
F^\ell_\pm & = & 0 \arrow[r] & \underset{0}{A/I\mathfrak{a}^\ell_\pm} \arrow[rr] & & A/\mathfrak{a}^{\ell-1}_\pm \oplus A/\mathfrak{a}^{\ell-1}_\mp \arrow[rr] \arrow[ur] & & A/\mathfrak{a}^{\ell-2}_\pm \oplus A/\mathfrak{a}^{\ell-2}_\mp \arrow[rr] \arrow[dr] & & \cdots \\
 & & & & A/(\mathfrak{a}^{\ell-1}_\pm\mathfrak{a}^{\ell-1}_\mp) \arrow[ur] & & & & A/(\mathfrak{a}^{\ell-2}_\pm + \mathfrak{a}^{\ell-2}_\mp). &
\end{tikzcd}$$

This shows that the complex $F^\ell_\pm$ is exact in all degrees except zero. We now show inductively that there is a homotopy equivalence $F_{s_\pm} \otimes_A F^\ell_\mp \approxeq F^{\ell+1}_\pm$.

—*Induction base*: For the case $\ell = 1$, we first show:

Claim. The quotient map

$$\begin{aligned} A \otimes_{A^s} A &\twoheadrightarrow (A \otimes_{A^s} A)/(a \otimes 1 - 1 \otimes a)_{a \in A}(sa' \otimes 1 - 1 \otimes a')_{a' \in A} \\ &= (A \otimes_{\mathbf{C}} A)/(a \otimes 1 - 1 \otimes a)_{a \in A}(sa' \otimes 1 - 1 \otimes a')_{a' \in A} \\ &= D^1_\pm \end{aligned}$$

is an isomorphism.

By virtue of the isomorphism $A \cong A^s \oplus A^s\alpha_s$ (see Lemma 5.6) we take decompositions $a = u + v\alpha_s$ and $a' = u' + v'\alpha_s$ for $u, v, u', v' \in A^s$. We obtain that $\ker(A \otimes_{A^s} A \twoheadrightarrow D^1_\pm)$ is generated by elements

$$\begin{aligned}
&(a \otimes 1 - 1 \otimes a)(sa' \otimes 1 - 1 \otimes a') \\
&= \big[(u + v\alpha_s) \otimes 1 - 1 \otimes (u + v\alpha_s)\big]\big[(u' - v'\alpha_s) \otimes 1 - 1 \otimes (u' + v'\alpha_s)\big] \\
&= vv'(-\alpha_s^2 \otimes 1 + 1 \otimes \alpha_s^2) \\
&= 0.
\end{aligned}$$

We thus have an isomorphism of complexes

$$\begin{tikzcd}
F^1_\pm = \big\{0 \arrow[r] & D^1_\pm \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, "\cong"] & 0\big\} \\
F_{s_\pm} = \big\{0 \arrow[r] & A \otimes_{A^s} A \arrow[r] & A \arrow[r] & 0\big\}.
\end{tikzcd}$$

—*Induction step*: Assume the claim holds for all $F^r_\pm$ for $r < \ell$. The tensor product $F_{s_\pm} \otimes_A F^\ell_\mp$ is given by the total complex

$$\begin{aligned}
F_{s_\pm} \otimes_A F^\ell_\mp &= \left\{\begin{tikzcd}
D_\pm \otimes_A D^\ell_\mp \arrow[r] \arrow[d] & D_\pm \otimes_A \big(D^{\ell-1}_\pm \oplus D^{\ell-1}_\mp\big) \arrow[r] \arrow[d] & \cdots \arrow[r] & D_\pm \otimes_A A \arrow[d] \\
A \otimes_A D^\ell_\mp \arrow[r] & A \otimes_A \big(D^{\ell-1}_\pm \oplus D^{\ell-1}_\mp\big) \arrow[r] & \cdots \arrow[r] & A \otimes_A A
\end{tikzcd}\right\} \\
&= \left\{\begin{tikzcd}
D_\pm D^\ell_\mp \arrow[r] \arrow[d] & D_\pm D^{\ell-1}_\pm \oplus D_\pm D^{\ell-1}_\mp \arrow[r] \arrow[d] & \cdots \arrow[r] & D_\pm \arrow[d] \\
D^\ell_\mp \arrow[r] & D^{\ell-1}_\pm \oplus D^{\ell-1}_\mp \arrow[r] & \cdots \arrow[r] & A
\end{tikzcd}\right\}
\end{aligned} \tag{5.3}$$

where each of the morphisms is the quotient map.

Claim. There are homotopy equivalences

$$A \otimes_{A^{s_\pm}} \big(0 \to D^\ell_\mp \to D^{\ell-1}_\pm \to 0\big) \approxeq \big(0 \to D^{r+1}_\pm \to D^{r-1}_\mp \to 0\big), \tag{5.4}$$

$$\big(0 \to D^1_\pm \to A \to A \to 0\big) \otimes_A D^r_\pm \approxeq \big(0 \to D^r_\mp \to 0 \to 0\big). \tag{5.5}$$

See Lemma 5.15 below.

In the following, we use these two equivalences to replace (5.3) by a simpler double complex step-by-step. Consider the following diagram with explanation given afterwards:

$$\begin{tikzcd}
& \overset{(v)}{D^{\ell+1}_\pm} \arrow[r] \arrow[dr, phantom, "\approxeq"] & \overset{(iv)}{D^{\ell-1}_\pm}\oplus\overset{(v)}{D^\ell_\pm} \arrow[rr, "\begin{pmatrix} * & * \\ * & * \end{pmatrix}"] \arrow[ddr, bend left, "\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}"] \arrow[drr, phantom, "\approxeq"] & & \overset{(iv)}{D^{\ell-2}_\pm}\oplus\overset{(v)}{D^{\ell-1}_\mp} \arrow[rr] \arrow[ddr, bend left] \arrow[dr, phantom, "\approxeq\ (iii)"] & & \cdots \\
& D_\pm D^\ell_\mp \arrow[u, "(i)"] \arrow[r] \arrow[drr] \arrow[dd] & D_\pm D^{\ell-1}_\pm \oplus D_\pm D^{\ell-1}_\mp \arrow[u, "\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}", "(i)"'] \arrow[rr, "\begin{pmatrix} * & * \\ * & * \end{pmatrix}"] \arrow[dr, "(ii)"] \arrow[dd] \arrow[ur, phantom, "\approx"] & & D_\pm D^{\ell-2}_\pm \oplus D_\pm D^{\ell-2}_\mp \arrow[u, "(i)"] \arrow[rr] \arrow[dr, "(ii)"] \arrow[dd, "\begin{pmatrix} * & 0 \\ 0 & \mathrm{id} \end{pmatrix}"] \arrow[ur, phantom, "\approx"] & & \cdots \\
F_{s_\pm} \otimes_A F^\ell_\mp = & & & D^{\ell-1}_\mp \oplus D_\pm D^{\ell-1}_\mp \arrow[rr] \arrow[dd] \arrow[dl, phantom, "\approx"] & & D^{\ell-2}_\mp \oplus D_\pm D^{\ell-2}_\mp \arrow[r] \arrow[dd] \arrow[dl, phantom, "\approx"] & \cdots \\
& \underset{(v)}{D^\ell_\mp} \arrow[r] \arrow[drr] & D^{\ell-1}_\pm \oplus D^{\ell-1}_\mp \arrow[rr] \arrow[dr, "(ii)"] & & D^{\ell-2}_\pm \oplus D^{\ell-2}_\mp \arrow[rr] \arrow[dr, "(ii)", "\begin{pmatrix} 0 & \mathrm{id} \end{pmatrix}"'] & & \cdots \\
& & & \underset{(v)}{0 \oplus D^{\ell-1}_\mp} \arrow[rr] & & \underset{(v)}{0 \oplus D^{\ell-2}_\mp} \arrow[r] & \cdots
\end{tikzcd} \tag{5.6}$$

(5.3) is highlighted by the brace. We perform the following steps:

(i) Applying (5.4) to each of the morphisms in the top row of the diagram yields the first homotopy equivalence.

(ii) Similarly, use (5.5) to replace the left half of the vertical morphisms.

(iii) The composition of these two homotopy equivalences is again a homotopy equivalence. The first and the second homotopy equivalence have been obtained by replacing one summand at a time; hence they are of diagonal form and so is composition. But $\operatorname{Hom}(D_{\pm}^{\ell-1}, D_{\mp}^{\ell-1})$ is zero, hence the composite is of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix}\right)$.

(iv) This can only remain a homotopy equivalence if the complex formed by the indicated summands is nullhomotopic.

(v) Thus the original double complex is homotopy equivalent to the double complex consisting of the remaining indicated summands in the rear top and front bottom row.

This exhibits that there is a homotopy equivalence

$$F_{s_\pm} \otimes F_{\mp}^{\ell} \approxeq \big(0 \to D_{\pm}^{\ell+1} \to D_{\pm}^{\ell} \oplus D_{\mp}^{\ell} \to \cdots \to A \to 0\big) =: C.$$

We do not know yet what the differentials of C are.

—*Showing* $C \cong F_{\mp}^{\ell+1}$: Recall that $F_{s_\pm}$ and $F_{\pm}^{\ell}$ have homology $H^r(F_{s_\pm}) = 0$ and $H^r(F_{\pm}^{\ell}) = 0$ for $r > 0$. In degree 0, homology is given by

$$H^0(F_{s_\pm}) = \ker\big(D_{\pm}^1 \xrightarrow{\varepsilon'} A\big),$$

$$H^0(F_{\pm}^{\ell}) = \ker\Big(D_{\pm}^{\ell} \xrightarrow{\left(\begin{smallmatrix} + \\ + \end{smallmatrix}\right)} D_{\pm}^{\ell-1} \oplus D^{\ell-1} D_{\mp}\Big)$$

respectively. There is an isomorphism

$$D_{\pm}^1 \cong A \otimes_{A^{s_\pm}} A \cong (A^{s_\pm} \oplus A^{s_\pm}\langle 1\rangle) \otimes_{A^{s_\pm}} A \cong A^{\oplus 2}$$

of right A-modules; hence $D_{\pm}^1$ is a free right A-module of rank 2. The 0-th homology module $H^0(F_{s_\pm}) \in$ Mod-A thus is a submodule of a free right A-module and thus is free itself (A is a PID). The Künneth theorem [Wei94, thm. 3.6.3] thereupon implies that the homology of C is given by

$$\begin{aligned} H^r(C) &\cong \bigoplus_{m+n=r} H^m(F_{s_\pm}) \otimes H^n(F_{\pm}^{\ell}) \oplus \bigoplus_{m+n=r-1} \overbrace{\operatorname{Tor}_1^A\big(H^m(F_{s_\pm}), H^n(F_{\pm}^{\ell})\big)}^{0} \\ &= \begin{cases} H^0(F_{s_\pm}) \otimes H^0(F_{\pm}^{\ell}) & \text{if } r = 0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, the complex C has the same entries and the same homology as the complex $F_{\pm}^{\ell+1}$ in degrees other than zero. Since all spaces $\operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(D_*^{r+1}, D_*^{\ell})$ are one-dimensional, the complex C is completely described by the scalar multiples of the canonical projection maps. We can understand C as the complex

$$0 \to * \xrightarrow{\left(\begin{smallmatrix} a_{\ell+1} \\ c_{\ell+1} \end{smallmatrix}\right)} * \oplus * \xrightarrow{\left(\begin{smallmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{smallmatrix}\right)} \cdots \xrightarrow{\left(\begin{smallmatrix} c_1 & d_1 \end{smallmatrix}\right)} * \to 0.$$

Claim. None of the coefficients a_r, b_r, c_r and d_r, $1 \leq r \leq \ell + 1$, can be zero.

Assume that there are vanishing coefficients, and let $r \leq \ell$ be the smallest index for which a coefficient vanishes, say $c_r = 0$. Since C is a chain complex, the coefficients satisfy

$$\begin{pmatrix} a_{r-1} & b_{r-1} \\ c_{r-1} & d_{r-1} \end{pmatrix} \left(\begin{smallmatrix} a_r & b_r \\ 0 & d_r \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_r a_{r-1} & \cdots \\ \cdots & \cdots \end{smallmatrix}\right) \overset{!}{=} 0;$$

in particular $a_r a_{r-1} = 0$. Since r was chosen minimally, a_r vanishes. On the other hand, the next degree of the complex forces the maps to satisfy

$$\left(\begin{smallmatrix} 0 & b_r \\ 0 & d_r \end{smallmatrix}\right) \begin{pmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{pmatrix} = \begin{pmatrix} b_r c_{r+1} & b_r d_{r+1} \\ d_r c_{r+1} & d_r d_{r+1} \end{pmatrix} \overset{!}{=} 0$$

which requires either $b_r = d_r = 0$ or $c_{r+1} = d_{r+1} = 0$. If $b_r = d_r = 0$, C decomposes as

$$C \cong \left(\begin{array}{l} \Big\{ \cdots \longrightarrow D_{\pm}^{r+1} \oplus D_{\mp}^{r+1} \longrightarrow D_{\pm}^{r} \oplus D_{\mp}^{r} \longrightarrow 0 \longrightarrow \cdots \Big\} \\ \oplus \Big\{ \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow D_{\pm}^{r-1} \oplus D_{\mp}^{r-1} \longrightarrow \cdots \Big\} \end{array} \right).$$

If otherwise $c_{r+1} = d_{r+1} = 0$, there is a direct sum decomposition

$$C \cong \left(\begin{array}{l} \Big\{ \cdots \longrightarrow D_{\pm}^{r+1} \oplus D_{\mp}^{r+1} \xrightarrow[(\,a_{r+1}\ b_{r+1}\,)]{} D_{\pm}^{r} \longrightarrow 0 \longrightarrow \cdots \Big\} \\ \oplus \Big\{ \cdots \longrightarrow 0 \longrightarrow D_{\mp}^{r} \xrightarrow[\binom{b_r}{d_r}]{} D_{\pm}^{r-1} \oplus D_{\mp}^{r-1} \longrightarrow \cdots \Big\} \end{array} \right).$$

In both cases, the second summand has non-vanishing homology in its leftmost degree since all the structure maps of the complex are non-trivial quotient maps. This is a contradiction; hence all coefficients of the matrices are non-zero.

By imposing the condition on the coefficients that the product of every two consecutive matrices is zero, we obtain linear equations that allow us to determine the complex only by the variables a_r and c_r:

$$\begin{aligned} c_1a_2 + d_1c_2 &= 0 \quad \text{and therefore} & d_1 &= -\tfrac{c_1a_2}{c_2}, \\ c_1b_2 + d_1d_2 &= 0 & &\vdots \\ &\vdots & & \\ a_ra_{r+1} + b_rc_{r+1} &= 0 & b_r &= -\tfrac{a_ra_{r+1}}{c_{r+1}}, \\ a_rb_{r+1} + b_rd_{r+1} &= 0 & & \\ c_ra_{r+1} + d_rc_{r+1} &= 0 & d_2 &= -\tfrac{c_ra_{r+1}}{c_{r+1}}, \\ c_rb_{r+1} + d_rd_{r+1} &= 0 & &\vdots \\ &\vdots & & \\ a_\ell a_{\ell+1} + b_\ell c_{\ell+1} &= 0 & b_\ell &= -\tfrac{a_\ell a_{\ell+1}}{c_{\ell+1}}, \\ c_\ell a_{\ell+1} + d_\ell c_{\ell+1} &= 0 & d_\ell &= -\tfrac{c_\ell a_{\ell+1}}{c_{\ell+1}}. \end{aligned}$$

Lines without an entry on the right are already determined by the others. Using this we can construct an isomorphism

$$\begin{tikzcd} F_{\pm}^{\ell+1} = \big(0 \arrow[r] & * \arrow[r, "\binom{+}{+}"] \arrow[d, "1"'] & * \oplus * \arrow[r, "\left(\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right)"] \arrow[d, "f_\ell"] & \cdots \arrow[r, "(+\ -)"] & * \arrow[r] \arrow[d, "f_0"] & 0\big) \\ C = \big(0 \arrow[r] & * \arrow[r, "\binom{a_{\ell+1}}{c_{\ell+1}}"'] & * \oplus * \arrow[r, "\left(\begin{smallmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{smallmatrix}\right)"'] & \cdots \arrow[r, "(\,c_1\ d_1\,)"'] & * \arrow[r] & 0\big). \end{tikzcd}$$

To ensure that the vertical maps form a map of complexes, we construct them as follows. By fixing the leftmost map to be one, we obtain f_ℓ since it has to satisfy

$$\underbrace{\begin{pmatrix} a_{\ell+1} & \\ & c_{\ell+1} \end{pmatrix}}_{f_\ell} \binom{+}{+} = \binom{a_{\ell+1}}{c_{\ell+1}}.$$

Continuing downward induction yields

$$\begin{aligned} \underbrace{\begin{pmatrix} a_{\ell+1}a_\ell & \\ & a_{\ell+1}c_\ell \end{pmatrix}}_{f_{\ell-1}} \begin{pmatrix} + & - \\ + & - \end{pmatrix} &= \begin{pmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{pmatrix} f_\ell &&= \begin{pmatrix} a_{\ell+1}a_\ell & \overbrace{c_{\ell+1}b_\ell}^{-a_{\ell+1}a_\ell} \\ a_{\ell+1}c_\ell & \underbrace{c_{\ell+1}d_\ell}_{-a_{\ell+1}c_\ell} \end{pmatrix} \\ &\ \ \vdots \\ \underbrace{\begin{pmatrix} a_{\ell+1}\cdots a_{r+1} & \\ & a_{\ell+1}\cdots a_{r+2}c_{r+1} \end{pmatrix}}_{f_r} \begin{pmatrix} + & - \\ + & - \end{pmatrix} &= \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} f_{r+1} &&= \begin{pmatrix} a_{\ell+1}\cdots a_{r+1} & a_{\ell+1}\cdots a_{r+2}\ \overbrace{c_{r+1}b_r}^{-a_{r+1}a_r} \\ a_{\ell+1}\cdots a_{r+1}c_r & a_{\ell+1}\cdots a_{r+2}\ \underbrace{c_{r+1}d_r}_{-a_{r+1}c_r} \end{pmatrix} \\ &\ \ \vdots \\ \underbrace{(a_\ell \cdots a_2c_1)}_{f_0} \cdot (\,+\ -\,) &= (\,c_1\ d_1\,) f_1 &&= (\,a_{\ell+1}\cdots a_2c_1\ \ a_{\ell+1}\cdots a_3\ \underbrace{c_2d_1}_{-a_2c_1}\,). \end{aligned}$$

The maps f_r exhibit the complex C to be isomorphic to $F_{\pm}^{\ell+1}$ since the coefficients a_r and c_r are non-zero for all $0 \leq r \leq \ell + 1$. This finishes the proof. $\square$

Corollary 5.14. The proposition holds for an arbitrary Coxeter system (W,S) and generators $s,t \in S$ such that $m_{st} < \infty$.

Proof. The subgroup $\langle s,t\rangle \leq W$ generated by s and t is isomorphic to the dihedral group $I_{m_{st}}$. We can decompose the reflection representation $V \cong V^{s,t} \oplus V_{\perp}^{s,t}$ of W into its $\langle s,t\rangle$-invariant part and its complement. This yields a factorisation

$$A = \mathbf{C}[V] \cong \mathbf{C}\big[V^{s,t}\big] \otimes_{\mathbf{C}} \mathbf{C}\big[V_{\perp}^{s,t}\big]$$

of algebras. We denote both tensor factors by $A \cong A_1 \otimes_{\mathbf{C}} A_2$. There is an induction functor

$$\begin{aligned}\mathrm{Ind}_{A_2}^{A} \colon\ & A_2\text{-gMod-}A_2 \to A\text{-gMod-}A,\\ & M \mapsto A_1 \otimes_{\mathbf{C}} M\end{aligned}$$

where we let A_1 act on the factor A_1 from both sides and let A_2 act on M from both sides. There also is a restriction functor $\mathrm{Res}_A^{A_2}$, coming from the algebra homomorphism $1 \otimes \mathrm{id}_{A_2} \colon A_2 \to A_1 \otimes A_2$. Since $V^{s,t}$ (and hence A_1) is invariant under the action of $\langle s,t\rangle$, the left and right actions of A_1 can be slid across all tensor products $\otimes_{A^s}$ and $\otimes_{A^t}$; hence both the left and right actions of A_1 coincide on all modules considered in the proof. Thus, when restricted to these modules there are natural isomorphisms of functors

$$\begin{aligned}&\mathrm{Ind}_{A_2}^{A} \circ (A_2 \otimes_{A_2^s} -) \circ \mathrm{Res}_A^{A_2}\\ \cong\ & (A_1 \otimes_{\mathbf{C}} A_2) \otimes_{A_2^s} -\\ \cong\ & A \otimes_{\left(A_1 \otimes_{\mathbf{C}} A_2^s\right)} -\\ \cong\ & A \otimes_{A^s} -\end{aligned} \qquad \begin{aligned}&\mathrm{Ind}_{A_2}^{A} \circ (A_2 \otimes_{A_2} -) \circ \mathrm{Res}_A^{A_2}\\ \cong\ & A_1 \otimes_{\mathbf{C}} A_2 \otimes_{A_2} -\\ \cong\ & (A_1 \otimes_{\mathbf{C}} A_2) \otimes_{A_1 \otimes_{\mathbf{C}} A_2} -\\ \cong\ & A \otimes_{A} -\end{aligned}$$

of A-bimodules. The proof for the dihedral case thus passes from A_2-grMod-A_2 to A-grMod-A. ☐

The following lemma finishes the proof of Theorem 5.9:

Lemma 5.15. Let $x \in W$ such that $stx > tx > x \neq e$. Then there are homotopy equivalences

$$\begin{aligned} A \otimes_{A^s} \big(0 \to A_{\leq tx} \xrightarrow{\mathrm{can}} A_{\leq x} \to 0\big) &\approxeq \big(0 \to A_{\leq stx} \to A_{\leq x} \to 0\big)\\ \big(0 \to A \otimes_{A^s} A \to A \to 0\big) \otimes_A A_{\leq x} &\approxeq \big(0 \to A_{\leq x}\langle 1\rangle \to 0 \to 0\big)\end{aligned}$$

as claimed above.

Proof. See [Rou06, lem. 9.1]. ☐

5.3. Back to category $\mathcal{O}$

We now want to use the results we obtained for the category of graded A-modules to prove that the derived shuffling functor indeed admits the braid relations and is an auto-equivalence of $D^{\mathrm{b}}(\mathcal{O})$. First, recall the definition of the *coinvariant algebra* $C = A/\big(A_+^W\big)$, i. e. the quotient of A by its non-constant W-invariant polynomials. Note that C_W is independent of the choice for $A \in \{A_{\mathrm{g}}, A_{\mathrm{p}}\}$ [Che55]. By the induced inclusion of module categories, we can act on complexes of C-modules:

$$K^{\mathrm{b}}(C\text{-gMod}) \hookrightarrow K^{\mathrm{b}}(A\text{-gMod}) \xrightarrow{F_s} K^{\mathrm{b}}(A\text{-gMod}).$$

However, consider the A-A-bimodule $A \otimes_{A^s} A$ used in the construction of F_s. Tensoring this to the A-C-bimodule C yields

$$A \otimes_{A^s} \big(A/(A_+^W)\big) \cong \big(A/(A_+^W)\big) \otimes_{A^s} \big(A/(A_+^W)\big) \cong C \otimes_{C^s} C$$

since W-invariants can in particular be slid across the tensor product $\otimes_{A^s}$. We thus obtain also an action of the respective complexes F_s and F_s^{-1} on $K^{\mathrm{b}}(C\text{-gMod})$. The algebra C arises also in the context of category $\mathcal{O}$:

Theorem 5.16 (Soergel's Endomorphismensatz) [Soe90, thm. 3]. Let μ be a (not necessarily regular) integral weight, W_μ its stabiliser subgroup under the dot-action and $w_0 \in W_\mu$ the longest element therein. Denote $E_\mu := \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot \mu))$. Then $C^{W_\mu} \cong E_\mu$.

This isomorphism allows to define *Soergel's functor* $\mathbf{V}_\mu \coloneqq \operatorname{Hom}_{\mathcal{O}}(P(w_0 \cdot \mu), -)\colon \mathcal{O}_\mu \to \text{Mod-}E_\mu \simeq \text{Mod-}C^{W_\mu}$ which resembles the functor in Morita's theorem. Although not an equivalence, it still has the following useful property:

Theorem 5.17 (Soergel's Struktursatz) [Soe90, thm. 2]. When restricted to the subcategory $\mathcal{O}_\mu$-Proj of projective modules in $\mathcal{O}_\mu$, this functor is fully faithful.

Now recall the translation functors T_{on} and T_{off} from Section 1.1 between the regular block $\mathcal{O}_0$ and the block $\mathcal{O}_\mu$ for a weight μ on the s-wall adjacent to 0, i. e. with stabiliser subgroup $W_\mu = \langle s \rangle$. Recall that T_{off} preserves indecomposable projectives and in particular $T_{\mathrm{off}} P(w_0 \cdot \mu)$ is naturally isomorphic to $P(w_0 \cdot 0)$. The thus induced map

$$C^\mu \cong \operatorname{End}_{\mathcal{O}}\bigl(P(w_0 \cdot \mu)\bigr) \xrightarrow{T_{\mathrm{off}}} \operatorname{End}_{\mathcal{O}}\bigl(P(w_0 \cdot 0)\bigr) \cong C$$

is the inclusion $\iota : C^\mu \subseteq C$ [Soe90, rmk. ad thm. 8]. The restriction functor induced by this inclusion hence gives the top left square in

$$\begin{tikzcd}
C\text{-Mod} \arrow[r, "\operatorname{Res}_C^{C^\mu}", "\iota^*"'] \arrow[d, "\simeq"] & C^\mu\text{-Mod} \arrow[r, "\operatorname{Ind}_{C^\mu}^C"] \arrow[d, "\simeq"] & C\text{-Mod} \arrow[d, "\simeq"] \\
\text{Mod-}E_{\mathrm{o}} \arrow[r, "T_{\mathrm{off}}^*"] & \text{Mod-}E_\mu \arrow[r, "T_{\mathrm{on}}^*"] & \text{Mod-}E_{\mathrm{o}} \\
\mathcal{O}_0 \arrow[r, "T_{\mathrm{on}}"'] \arrow[u, "\mathbf{V}_0"] & \mathcal{O}_\mu \arrow[r, "T_{\mathrm{off}}"'] \arrow[u, "\mathbf{V}_\mu"] & \mathcal{O}_0. \arrow[u, "\mathbf{V}_0"]
\end{tikzcd}$$

Given a module $M \in \text{Mod-}E_{\mathrm{o}}$, its image $T_{\mathrm{off}}^* M$ denotes the right E_μ-module with action $m\,.\,\phi \coloneqq m\,.\,T_{\mathrm{off}}(\phi)$ for $m \in M$ and $\phi \in E_\mu$. The top right square is obtained from the left one via the adjunctions $\operatorname{Ind}_{C^\lambda}^C \dashv \operatorname{Res}_C^{C^\lambda}$ and $T_{\mathrm{on}} \dashv T_{\mathrm{off}}$. The commutativity of the lower half of the diagram can be seen as follows: $T_{\mathrm{off}}^* \circ \mathbf{V}_0(M) = \operatorname{Hom}(P(w_0 \cdot 0), M)$ with $\phi \in E_0$ acting from right by $f\,.\,\phi = f \circ T_{\mathrm{off}}(\phi)$. This yields an isomorphism

$$T_{\mathrm{off}}^* \circ \mathbf{V}_0 \cong \operatorname{Hom}_{\mathcal{O}}\left(T_{\mathrm{off}} P(w_0 \cdot \mu), -\right) \overset{\dashv}{\cong} \operatorname{Hom}_{\mathcal{O}}\left(P(w_0 \cdot \mu), T_{\mathrm{on}}(-)\right) = \mathbf{V}_\mu \circ T_{\mathrm{on}}$$

and dually

$$T_{\mathrm{on}}^* \circ \mathbf{V}_\mu \cong \operatorname{Hom}_{\mathcal{O}}\left(T_{\mathrm{on}} P(w_0 \cdot 0), -\right) \overset{\dashv}{\cong} \operatorname{Hom}_{\mathcal{O}}\left(P(w_0 \cdot 0), T_{\mathrm{off}}(-)\right) = \mathbf{V}_0 \circ T_{\mathrm{off}}.$$

This in particular shows that translation through the s-wall, is taken, under composition with $\mathbf{V}_0$, to

$$\mathbf{V}_0 \circ \Theta_s = \mathbf{V}_0 \circ T_{\mathrm{off}} T_{\mathrm{on}} = \operatorname{Ind}_{C^\mu}^C \circ \mathbf{V}_\mu \circ T_{\mathrm{on}} = \underbrace{\operatorname{Ind}_{C^\mu}^C \circ \operatorname{Res}_C^{C^\mu}}_{= C \otimes_{C^s} C \otimes_C -} \circ \mathbf{V}_0.$$

Comparing this to the definition of the Rouquier complexes shows

$$\mathbf{V}_0 \circ \underbrace{\{\mathrm{id} \to \Theta_s\}}_{\mathbf{L}\,\mathrm{Sh}_s} \cong \underbrace{\{C \to C \otimes_{C^s} C\}}_{F_s^{-1}} \circ \mathbf{V}_0; \qquad \mathbf{V}_0 \circ \underbrace{\{\Theta_s \to \mathrm{id}\}}_{\mathbf{R}\,\mathrm{Csh}_s} \cong \underbrace{\{C \otimes_{C^s} C \to C\}}_{F_s} \circ \mathbf{V}_0$$

as functors $K^{\mathrm{b}}(\mathcal{O}_0) \to K^{\mathrm{b}}(C)$. Finally, Soergel's Struktursatz asserts that $\mathbf{V}$ is fully faithful when restricted to $\mathcal{O}$-Proj. The natural isomorphisms which ensure the braid relations for F_s and F_s^{-1} hence can be lifted via $\mathbf{V}_0$ to the respective natural isomorphisms for $\mathbf{L}\,\mathrm{Sh}$ and $\mathbf{R}\,\mathrm{Csh}$ as endofunctors of $K^{\mathrm{b}}(\mathcal{O}_{\mathrm{o}}\text{-Proj}) \simeq D^{\mathrm{b}}(\mathcal{O}_0)$. This finally proves Theorem 1.20.

5.4. Soergel bimodules

In the proof of Theorem 5.9 we have already seen the objects of a remarkable subcategory of A-grMod-A, the so-called Soergel bimodules.

Definition 5.18. The category of *Bott-Samelson modules*, denoted by $\mathcal{BS}$, is the full subcategory of A-grMod-A generated as a monoidal category by the $A_s \coloneqq A \otimes_{A^s} A\langle -1 \rangle$ for $s \in S$. Its objects are thus modules of the form $A_w \coloneqq A_{s_1} \otimes_A \cdots \otimes_A A_{s_r}$ for a word $w \coloneqq s_1 \cdots s_r$ of simple reflections.

Its graded (resp. additive) envelope is the smallest category containing $\mathcal{BS}$ that is closed under grading shifts (resp. direct sums). Its *Karoubi envelope* is the smallest category containing $\mathcal{BS}$ in which every idempotent morphism splits. As a subcategory of an abelian category, this is a fancy way to express that the Karoubi envelope is the smallest category containing all direct summands.

Definition 5.19. The category of *Soergel bimodules*, denoted by $\mathcal{S}$, is the Karoubi envelope of the additive and graded envelopes of $\mathcal{BS}$. That is to say that Soergel bimodules are direct sums of grading shifts of direct summands of Bott-Samelson modules.

Remark 5.20. Originally, Soergel bimodules have been defined for the geometric representation A_{g}. However, one may pass between A_{p}- and A_{g}-modules by quotienting out resp. adjoining $(x_1 + \cdots + x_n)$; see [EK10a, rmk. 2.2 and §4.6] for a discussion about the difference between the construction for A_{g} and A_{p}.

Theorem 5.9 then in fact shows that the braid group acts on $D^{\mathrm{b}}(\mathcal{S})$. The indecomposable objects of $\mathcal{S}$ are parametrised by the elements of W [Soe07, §6]. Recall the definition of the Hecke algebra from Definition 2.13. Soergel bimodules (called “spezielle Bimoduln” in [Soe07, def. 5.11]) have been proven to categorify the Hecke algebra of $\mathsf{H}_q(W)$ in [Soe07, thm. 1.10; Soe92, thm. 1]:

Theorem 5.21. For A the ring of regular functors of a reflection faithful representation (e. g. A as above), there is a ring isomorphism $\mathsf{H}_q(W) \to K_0(\mathcal{S})$ such that $q \mapsto [A\langle 1\rangle]$, $H_s \mapsto \langle A_s\rangle$ and, for $k = \mathbf{C}$, $C_w \mapsto \langle A_w\rangle$.

Theorem 5.9 then in fact shows that the braid group acts on $D^{\mathrm{b}}(\mathcal{S})$. The category $\mathcal{S}$ admits a well-established graphical description [Lib10; EK10a; EW16] A comprehensive introduction to Soergel bimodules and their diagrammatics is given in [Lib17].

Remark 5.22. The Rouquier complexes have been subject to research on there own right. For instance, [EK10b] uses the graphical description for Soergel bimodules from [EK10a] to show that Rouquier complexes are functorial over braid cobordisms. In fancy language, this means that $K^{\mathrm{b}}(\mathcal{S})$ is a 2-representation of the braid 2-category, which is the 2-category with objects—natural numbers, 1-morphisms from n to m—braids in B_{S_n} and 2-morphisms—braid cobordisms.

6. Superalgebras, supermodules and supercategories

If S names some algebraic structure, then a *super-S* is a $\mathbf{Z}/2\mathbf{Z}$-graded S. The $\mathbf{Z}/2\mathbf{Z}$-grading will be called *super grading* if it is necessary to distinguish it from other gradings such as polynomial gradings. To tell apart S and its super-analogue, we refer to S as the *classical* or *ordinary* structure. The super-degree is called *parity* and will be denoted by $|-|$, whereas we reserve "deg" for polynomial gradings. Homogeneous elements of parity 0 are called *even* and such of parity 1 are called *odd.* Explicitly:

Definition 6.1. A *super vector space* V is a $\mathbf{Z}/2\mathbf{Z}$-graded vector space $V = V_0 \oplus V_1$. The super-dimension of V is written $\dim_k V = \dim_k V_0 | \dim_k V_1$, and we write $k^{m|n}$ for the canonical $m|n$-dimensional k-super vector space. A morphism of super vector spaces is a graded vector space homomorphism. The thus defined category is denoted by k-sVect.

It admits a parity shift automorphism Π which exchanges the odd and even parts of a super vector space. The category k-sVect is a closed symmetric monoidal category with the usual tensor product of graded vector spaces $(V \otimes W)_p = \bigoplus_{q+r=p} V_q \otimes W_r$ and the braiding $v \otimes w \mapsto (-1)^{|v||w|}$. Its internal Hom is the super vector space whose degree p-part consists of degree p-morphisms, i. e. $\hom_k(V,W)_0$ consists of grading-preserving morphisms whereas $\hom_k(V,W)_1$ consists of grading-reversing morphisms from V to W. Given finite dimensional super vector spaces $V = k^{n|m}$ and $W = k^{p|q}$, we may chose homogeneous bases, such that a homomorphism in $\hom_k(V,W)$ can be written as an $(m|n) \times (p|q)$-block matrix

$$\begin{array}{c} \\ p \\ q \end{array}\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} T_{0,0} & T_{0,1} \\ T_{1,0} & T_{1,1} \end{pmatrix} \end{array}$$

where the diagonal blocks constitute the even part and the off-diagonal blocks constitute the odd part of $\hom_k(V,W)$. The tensor product of two homomorphisms $f \in \hom_k(V,W)$ and $g \in \hom_k(V',W')$ satisfies $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$.

We now have to admit that we cheated when we said that k-sVect is a monoidal category: unlike in ordinary monoidal category, we have to replace the interchange law by a *super interchange law*: The composition of tensor products of morphisms on $V \otimes V'$ obeys the rule $(f \otimes g)(f' \otimes g') = (-1)^{|g||f'|} ff' \otimes gg'$.

Remark 6.2. The name *interchange law* originates from the theory of 2-categories where it says that the composition of 1-morphisms commutes with the composition of 2-morphisms. A monoidal category can be regarded as a 2-category by defining the 1-morphisms to be generated by tensor products of two factors. Then both notions of the interchange law coincide.

6.1. Superalgebras

Definition 6.3. A k-superalgebra A is a super vector space A with a graded k-algebra structure. It is called *supercommutative*, or (possibly a source of confusion) just commutative, if the multiplication commutes with the braiding, that is $ab = (-1)^{|a||b|} ba$ for all $a, b \in A$. This means in particular that in a commutative superalgebra every odd element squares to zero. The tensor product (of super vector spaces) of two superalgebras A,B carries a superalgebra structure by $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$.

Example 6.4. The exterior algebra on n generators is the k-algebra

$$\Lambda[\omega_1, \dots, \omega_n] \coloneqq k\langle \omega_1, \dots, \omega_n \rangle / (\omega_i \omega_j + \omega_j \omega_i)_{1 \le i,j \le n}.$$

It is a commutative superalgebra concentrated in its odd part (apart from k in the even part). The *polynomial superalgebra* on $m|n$ indeterminates is the commutative superalgebra $k[y_1, \dots, y_m | \omega_1, \dots, \omega_n] \coloneqq k[y_i]_{1 \le i \le m} \otimes_k \Lambda[\omega_j]_{1 \le j \le n}$. There is an obvious action of $S_m \times S_n$ on this algebra.

Example 6.5. The *Clifford algebra* on n generators is the k-superalgebra

$$\mathfrak{C}_n \coloneqq k\langle \mathfrak{c}_1, \dots, \mathfrak{c}_n \rangle / (\mathfrak{c}_i^2 = 1, \mathfrak{c}_i \mathfrak{c}_j = -\mathfrak{c}_j \mathfrak{c}_i | i \neq j) \tag{6.1}$$

with all generators odd. It is not (super-) commutative. There is an obvious action of S_n on $\mathfrak{C}_n$ by permuting the generators.

Definition 6.6. Given two k-superalgebras A, B, an *A-B-super bimodule* is a graded algebra bimodule; that means in particular that the algebra action is an even map of super vector spaces. By the super interchange law for vector space homomorphisms, a morphism f of bimodules has to satisfy $f(a\,.\,m\,.\,b) = (-1)^{|f||a|}a\,.\,f(m)\,.\,b$. We denote the category of A-B-super bimodules by A-sMod-B. One-sided modules are defined in the obvious way.

Over a commutative superalgebra, a left module obtains right module structure by $a\,.\,m = (-1)^{|a||m|}m\,.\,a$. Super bimodules admit the obvious tensor product over their superalgebras.

6.2. Supercategories

The category A-sMod of modules over a super k-algebra A should be the archetypical k-supercategory. It is equipped with the auto-equivalence Π which interchanges the even and odd parts. This motivates the following definition:

Definition 6.7 [KKT16, §2]. A *supercategory* is a category $\mathcal{C}$ equipped with an endofunctor Π and a natural isomorphism $\xi\colon \Pi^2 \Rightarrow \mathrm{id}_{\mathcal{C}}$ which turns Π into an auto-equivalence, such that $\xi\Pi = \Pi\xi$. A *superfunctor* consists of an ordinary functor $F\colon \mathcal{C} \to \mathcal{D}$ of the underlying categories and natural isomorphisms $F\Pi_{\mathcal{C}} \cong \Pi_{\mathcal{D}}F$ such that the chain of natural isomorphisms $F \cong F\Pi_{\mathcal{C}}^2 \cong \Pi_{\mathcal{D}}F\Pi_{\mathcal{C}} \cong \Pi_{\mathcal{D}}^2F \cong F$ equals the identity of F. A *supernatural transformation* is a natural transformation $\eta\colon F \Rightarrow G$ such that

$$\begin{tikzcd}
F\Pi_{\mathcal{C}} \arrow[r, Rightarrow, "\eta\circ\Pi_{\mathcal{D}}"] \arrow[d, Rightarrow] & G\Pi_{\mathcal{C}} \arrow[d, Rightarrow] \\
\Pi_{\mathcal{D}}F \arrow[r, Rightarrow, "\Pi_{\mathcal{D}}\circ\eta"'] & \Pi_{\mathcal{D}}G
\end{tikzcd}$$

commutes.

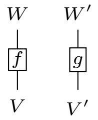
Since we are working with k-linear categories all the time, the following less opaque definition is sufficient for our needs:

Definition 6.8 [BE17a]. A k-linear *supercategory* (resp. *superfunctor*, *supernatural transformation*) is a category (functor, natural transformation) enriched in k-super vector spaces. Explicitly, in a *supercategory* $\mathcal{C}$ all Hom-spaces lie in k-sVect and composition is an even map of k-super vector spaces. A *superfunctor* induces even maps of super vector spaces on Hom-spaces. A *supernatural transformation* is a natural transformation $\eta\colon F \Rightarrow G$ whose naturality square supercommutes. This means that for each object $x \in \mathcal{C}$ the map η_x decomposes as a direct sum $\eta_{x,0} \oplus \eta_{x,1} \in \mathrm{Hom}_{\mathcal{D}}(Fx, Gx)$ of homogeneous maps, such that $\eta_{x,p} \circ Ff = (-1)^{p|f|}Gf \circ \eta_{x,p}$.

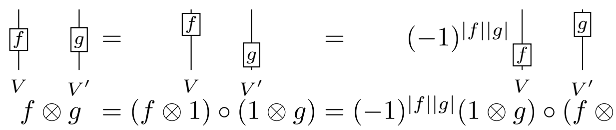
A *monoidal supercategory* is defined like a monoidal structure on the underlying category whose structure maps are the respective super-analogues, and whose interchange law is replaced by the above super interchange law. see [BE17a] for a comprehensive treatment.

6.3. Super-Diagrams

For ordinary monoidal k-linear categories it is common depict the morphism $f \otimes g\colon V \otimes V' \to W \otimes W'$ graphically by



The super interchange law thus states that



$$f \otimes g \;= (f \otimes 1) \circ (1 \otimes g) = (-1)^{|f||g|}(1 \otimes g) \circ (f \otimes 1).$$

We emphasize that such diagrams are to be understood as first vertical and then horizontal composition of morphisms [cf. BE17a]. We shall keep this in mind when working with super-diagrams.

7. Super-KLR algebras

We now recall the notion of KLR-superalgebras following [KKT16]. This is a generalisation of the ordinary KLR-algebra from [KL09; Rou08]. We start with an index set I which is assumed to admit a decomposition $I = I_{\mathrm{ev}} \sqcup I_{\mathrm{odd}}$ which turns it into a superset. Our goal is to extend the classical theory to the super-setting, of course in such a way that the new constructions yield the classical case when restricted to an index set with $I_{\mathrm{odd}} = \emptyset$.

7.1. NilHecke Superalgebra

Recall the definition of the NilHecke algebra NH_n of type $\mathrm{A_n}$ [Dem73, §2] and the definition of the Clifford algebra $\mathfrak{C}_n$ from Example 6.5. The following definitions are motivated from the definition of the quiver Hecke Clifford superalgebra in [KKT16, §3.3], which we shall consider later on.

Definition 7.1. The *NilHecke Clifford superalgebra* $\mathsf{NH}\mathfrak{C}_n$ of type A_n is the k-superalgebra with even generators y_i and $\mathfrak{d}_i$, and odd generators $\mathfrak{c}_i$, subject to the relations

$$\begin{aligned} y_i y_j &= y_j y_i && \forall i,j, \\ \mathfrak{c}_i\mathfrak{c}_j &= -\mathfrak{c}_j\mathfrak{c}_i && \forall i \neq j, \\ \mathfrak{c}_i^2 &= 1 && \forall i, \\ y_i\mathfrak{c}_j &= (-1)^{\delta_{i,j}}\mathfrak{c}_j y_i && \forall i,j, \end{aligned} \qquad \begin{aligned} \mathfrak{d}_i y_j - y_{s_i(j)}\mathfrak{d}_i &= \begin{cases} -1 - \mathfrak{c}_i\mathfrak{c}_{i+1} & \text{if } j = i \\ 1 - \mathfrak{c}_i\mathfrak{c}_{i+1} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \\ \mathfrak{d}_i\mathfrak{c}_j &= \quad \mathfrak{c}_{s_i(j)}\mathfrak{d}_i \qquad\qquad \forall i,j. \end{aligned} \tag{7.1}$$

and the NilCoxeter-relations

$$\begin{aligned} \mathfrak{d}_i^2 &= 0 \\ \mathfrak{d}_i\mathfrak{d}_j &= \quad \mathfrak{d}_j\mathfrak{d}_i \qquad \text{for } |i-j| > 1 \\ \mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i &= \mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}. \end{aligned} \tag{7.2}$$

The *NilCoxeter Clifford algebra* $\mathsf{NC}\mathfrak{C}_n$ is the subalgebra generated by the $\mathfrak{d}_i$'s and the $\mathfrak{c}_j$'s.

Definition 7.2. An *algebra* B over a (not necessarily commutative) ring A is defined to be an A-bimodule equipped with a multiplication map $B \otimes_A B \to B$ that is a bimodule homomorphism. The algebra B is said to be generated by some of its elements if they generate B under the A-bimodule structure and multiplication.

By this definition, we can consider $\mathsf{NH}\mathfrak{C}_n$ as a $\mathfrak{C}_n$ algebra generated by the y_i's and the $\mathfrak{d}_i$'s. We introduce the following analogues to the polynomial representation of the classical NilHecke algebra:

Definition 7.3. The *polynomial Clifford superalgebra* is the $\mathfrak{C}_n$-algebra with generators and relations

$$\mathrm{Pol}\mathfrak{C}_n \coloneqq \left\langle y_1, \dots, y_n \;\middle|\; y_i y_j = y_j y_i, \quad y_i\mathfrak{c}_j = (-1)^{\delta_{ij}}\mathfrak{c}_j y_i \quad \forall i,j \right\rangle_{\mathfrak{C}_n}.$$

As a Clifford-analogue of the Demazure operator (see Definition 5.7), we define an even k-linear homomorphism $\mathfrak{d}_i$ on $\mathrm{Pol}\mathfrak{C}_n$ by

$$\begin{aligned} \mathfrak{d}_i(y_i) &\coloneqq -1 - \mathfrak{c}_i\mathfrak{c}_{i+1}, & \mathfrak{d}_i(y_j) &\coloneqq 0 && \forall j \neq i, i+1, \\ \mathfrak{d}_i(y_{i+1}) &\coloneqq 1 - \mathfrak{c}_i\mathfrak{c}_{i+1}, & \mathfrak{d}_i(\mathfrak{c}_j) &\coloneqq 0 && \forall j, \end{aligned}$$

such that $\mathfrak{d}_i$ is a s_i-derivation. This means that

$$\mathfrak{d}_i(fg) \coloneqq \mathfrak{d}_i(f)g + s_i(f)\mathfrak{d}_i(g) \qquad\qquad \forall j \neq i, i+1,$$

where the symmetric group S_n acts on $\mathrm{Pol}\mathfrak{C}_n$ by permuting the y_i's and $\mathfrak{c}_j$'s independently.

Lemma 7.4. $\mathrm{Pol}\mathfrak{C}_n$ is a free $\mathfrak{C}_n$-left module and a free $\mathfrak{C}_n$-right module. For both module structures, a basis is given by $\{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \in \mathbf{N}\}$. In particular, $\mathrm{Pol}\mathfrak{C}_n$ has the same graded rank as left and as right $\mathfrak{C}_n$-module.

Proof. Take any monomial in y_i's and $\mathfrak{c}_j$'s from $\mathrm{Pol}\mathfrak{C}_n$. By the defining relation $y_i\mathfrak{c}_j = (-1)^{\delta_{ij}}\mathfrak{c}_j y_i$ of $\mathrm{Pol}\mathfrak{C}_n$, we may slide all $\mathfrak{c}_i$'s to the left (right), possible at the cost of introducing a sign when sliding them past y_i's with the same index. Furthermore, since the y_i's commute, we may sort powers of y_i's by their index. As a left (right) $\mathfrak{C}_n$-module, $\mathrm{Pol}\mathfrak{C}_n$ thus is isomorphic to $\mathfrak{C}_n \otimes_k \mathrm{Pol}_n$ ($\mathfrak{C}_n \otimes_k \mathrm{Pol}_n$). ☐

Lemma 7.5. The Clifford Demazure operator $\mathfrak{d}_i$ is indeed a well-defined operator. Furthermore, it is a right $\mathfrak{C}_n$-module homomorphism.

Proof. Well-definedness is shown by induction. Assume for element $f \in \mathrm{Pol}\mathfrak{C}_n$ that $\mathfrak{d}_i(f)$ is independent of a presentation of minimal length of f in the generators y_i and $\mathfrak{c}_j$. The assertion than follows from showing $\mathfrak{d}_i$ is compatible with the commutator relations involving y_j and $\mathfrak{c}_j$. Let w. l. o. g. $i = 1$. For the commutativity of the y's,

$$\mathfrak{d}_1(y_1y_2f) = y_1y_2\mathfrak{d}_1(f) + \underbrace{\mathfrak{d}_1(y_1y_2)}_{\substack{=(-1-\mathfrak{c}_1\mathfrak{c}_2)y_2+y_2(1-\mathfrak{c}_1\mathfrak{c}_2)\\ =0}} f, \qquad \mathfrak{d}_1(y_2y_1f) = y_2y_1\mathfrak{d}_1(f) + \underbrace{\mathfrak{d}_1(y_2y_1)}_{\substack{=(1-\mathfrak{c}_1\mathfrak{c}_2)y_1+y_1(-1-\mathfrak{c}_1\mathfrak{c}_2)\\ =0}} f;$$

hence $\mathfrak{d}_1(y_1y_2f) = \mathfrak{d}_1(y_2y_1f)$. For the relations involving $\mathfrak{c}$'s and y's, one calculates for instance

$$\begin{aligned} \mathfrak{d}_1(\mathfrak{c}_1y_1f) &= \underbrace{\mathfrak{d}_1(\mathfrak{c}_1y_1)}_{\substack{=\mathfrak{c}_2(-1-\mathfrak{c}_1\mathfrak{c}_2)\\ =\mathfrak{c}_1-\mathfrak{c}_2}} f + \mathfrak{c}_2y_2\mathfrak{d}_1(f) \qquad & \mathfrak{d}_1(\mathfrak{c}_2y_1f) &= \underbrace{\mathfrak{d}_1(\mathfrak{c}_2y_1)}_{\substack{=\mathfrak{c}_1(-1-\mathfrak{c}_1\mathfrak{c}_2)\\ =-\mathfrak{c}_1-\mathfrak{c}_2}} f + \mathfrak{c}_2y_2\mathfrak{d}_1(f)\\ -\mathfrak{d}_1(y_1\mathfrak{c}_1f) &= \overbrace{-\mathfrak{d}_1(y_1\mathfrak{c}_1)}^{=-(-1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{c}_1} f - y_2\mathfrak{c}_2\mathfrak{d}_1(f); & \mathfrak{d}_1(y_1\mathfrak{c}_2f) &= \overbrace{\mathfrak{d}_1(y_1\mathfrak{c}_2)}^{=(-1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{c}_2} f - y_2\mathfrak{c}_2\mathfrak{d}_1(f); \end{aligned}$$

hence $(\mathfrak{c}_1y_1f) = -\mathfrak{d}_1(y_1\mathfrak{c}_1f)$ and $(\mathfrak{c}_2y_1f) = -\mathfrak{d}_1(y_1\mathfrak{c}_2f)$. The calculations for $\mathfrak{d}_1(y_2\mathfrak{c}1) = \mathfrak{d}_1(\mathfrak{c}_2y_1)$ and $\mathfrak{d}_1(\mathfrak{c}_2y_2) = -\mathfrak{d}_1(y_2\mathfrak{c}_2)$ are similar. $\mathfrak{d}_i$ is a right $\mathfrak{C}_n$-module homomorphism since $\mathfrak{d}_i(f\mathfrak{c}_j) = \mathfrak{d}_i(f)\mathfrak{c}_j + s_i(f)\mathfrak{d}_i(\mathfrak{c}_j) = \mathfrak{d}_i(f)\mathfrak{c}_j$ for all $f \in \mathrm{Pol}\mathfrak{C}_n$ and $1 \leq j \leq n$. ☐

Lemma 7.6. The kernel $\ker \mathfrak{d}_i$ is a unitary $\mathfrak{C}_n$-subalgebra of $\mathrm{Pol}\mathfrak{C}_n$ in the sense of Definition 7.2.

Proof. A priori $\ker \mathfrak{d}_i$ is just a $\mathfrak{C}_n$-submodule of $\mathrm{Pol}\mathfrak{C}_n$. We have $\mathfrak{C}_n \subseteq \ker \mathfrak{d}_i$ by definition of $\mathfrak{d}_i$. Let $f, g \in \ker \mathfrak{d}_i$. Since $\mathfrak{d}_i$ is a s_i-derivation by definition, we have $\mathfrak{d}_i(fg) = \mathfrak{d}(f)g + s_i(f)\mathfrak{d}_i(g) = 0$. Hence $\ker \mathfrak{d}_i$ is multiplicatively closed. ☐

Lemma 7.7. (i) $\mathrm{Pol}\mathfrak{C}_n$ is a representation of the NilHecke Clifford superalgebra.

(ii) There is a vector space isomorphism $\ker \mathfrak{d}_i \cong \operatorname{im} \mathfrak{d}_i$. This endows $\operatorname{im} \mathfrak{d}_i$ with a $\mathfrak{C}_n$-algebra structure. Since $\operatorname{im} \mathfrak{d}_i \subseteq \ker \mathfrak{d}_i$, this is an equality.

Proof. (i) We have to check that the Clifford Demazure operators satisfy the defining relations from Definition 7.1. It is immediate from the definition of $\mathfrak{d}_i$ that the relations (7.1) are satisfied, so we show that the $\mathfrak{d}_i$'s satisfy (7.2):

$\mathfrak{d}_i^2(f) = 0$: Assume that the statement has been proven for $f \in \mathrm{Pol}\mathfrak{C}_n$. it follows from

$$\begin{aligned} \mathfrak{d}_1^2(fg) &= \mathfrak{d}_i\big(\mathfrak{d}_i(f)g + s_i(f)\mathfrak{d}_i(g)\big)\\ &= s_i(\mathfrak{d}_i(f))\,\mathfrak{d}_i(g) + \mathfrak{d}_i(s_if)\,\mathfrak{d}_i(g) \end{aligned}$$

that it suffices to show that $s_i\mathfrak{d}_i = -\mathfrak{d}_is_i$. Assume that $s_i\mathfrak{d}_i(f) = -\mathfrak{d}_i(s_if)$. We need to show that $s_i\mathfrak{d}_i(y_jf) = -\mathfrak{d}_i\big(s_i(y_jf)\big)$, which is obvious for $j \neq i$. For $j = i, i+1$ one calculates

$$\begin{aligned} s_i\mathfrak{d}_i(y_if) &= s_i\big[(-1-\mathfrak{c}_i\mathfrak{c}_{i+1})f + y_{i+1}\,\mathfrak{d}_if\big]\\ &\overset{*}{=} (-1+\mathfrak{c}_i\mathfrak{c}_{i+1})\,s_if - y_{i+1}\,\mathfrak{d}_is_if\\ &= -\mathfrak{d}_i(y_{i+1}\,s_if), \end{aligned}$$

with the induction hypothesis applied in $*$. The calculation for $j = i+1$ is similar.

$\mathfrak{d}_i\mathfrak{d}_j(f) = \mathfrak{d}_j\mathfrak{d}_i(f)$ *for* $|i-j| > 1$: Clear from the definition of $\mathfrak{d}_i$.

$\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(f) = \mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}(f)$: Assume that the statement has been proven for $f \in \mathrm{Pol}\mathfrak{C}_n$. It is clear that the statement then also holds true for $\mathfrak{c}_jf\ \forall j$ and for $y_jf\ \forall j \neq i, i+1, i+2$. For these cases, let w. l. o. g. $i = 1$. We compute:

$$\begin{aligned} \mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1(y_1f) &= \mathfrak{d}_1\mathfrak{d}_2(-1-\mathfrak{c}_1\mathfrak{c}_2)f + y_2\,\mathfrak{d}_2f)\\ &= \mathfrak{d}_1\big((-1-\mathfrak{c}_1\mathfrak{c}_3)\mathfrak{d}_2f + (-1-\mathfrak{c}_2\mathfrak{c}_3)\mathfrak{d}_1f + y_3\,\mathfrak{d}_2\mathfrak{d}_1f\big)\\ &= (-1-\mathfrak{c}_2\mathfrak{c}_3)\mathfrak{d}_1\mathfrak{d}_2f + y_3\,\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1f \end{aligned}$$

$$
\begin{aligned}
&\overset{*}{=} \mathfrak{d}_2\big((-1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{d}_2 f + y_2\,\mathfrak{d}_1\mathfrak{d}_2 f\big)\\
&= \mathfrak{d}_2\mathfrak{d}_1(y_1\,\mathfrak{d}_2 f)\\
&= \mathfrak{d}_2\mathfrak{d}_1\mathfrak{d}_2(y_1 f);\\
\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1(y_2 f) &= \mathfrak{d}_1\mathfrak{d}_2\big((-1-\mathfrak{c}_1\mathfrak{c}_2)f + y_1\,\mathfrak{d}_1 f\big)\\
&= \mathfrak{d}_1\big((1-\mathfrak{c}_1\mathfrak{c}_3)\mathfrak{d}_2 f + y_1\,\mathfrak{d}_2\mathfrak{d}_1 f\big)\\
&= (-1-\mathfrak{c}_2\mathfrak{c}_3)\mathfrak{d}_1\mathfrak{d}_2 f + (-1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{d}_2\mathfrak{d}_1 f + y_2\,\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1 f\\
&\overset{*}{=} \mathfrak{d}_2\big((-1-\mathfrak{c}_1\mathfrak{c}_3)\mathfrak{d}_1 + y_3\,\mathfrak{d}_1\mathfrak{d}_2 f\big)\\
&= \mathfrak{d}_2\mathfrak{d}_1\big((-1-\mathfrak{c}_2\mathfrak{c}_3)f + y_3\,\mathfrak{d}_2 f\big)\\
&= \mathfrak{d}_2\mathfrak{d}_1\mathfrak{d}_2(y_2 f);\\
\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1(y_3 f) &= \mathfrak{d}_1\mathfrak{d}_2(y_3\,\mathfrak{d}_1 f)\\
&= \mathfrak{d}_1\big((1-\mathfrak{c}_2\mathfrak{c}_3)\mathfrak{d}_1 f + y_2\,\mathfrak{d}_2\mathfrak{d}_1 f\big)\\
&= (1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{d}_2\mathfrak{d}_1 f + y_1\,\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_1 f\\
&\overset{*}{=} \mathfrak{d}_2\big((1-\mathfrak{c}_1\mathfrak{c}_3)f + (1-\mathfrak{c}_1\mathfrak{c}_2)\mathfrak{d}_2 f + y_1\,\mathfrak{d}_1\mathfrak{d}_2 f\big)\\
&= \mathfrak{d}_2\mathfrak{d}_1\big((1-\mathfrak{c}_2\mathfrak{c}_3)f + y_2\,\mathfrak{d}_2 f\big)\\
&= \mathfrak{d}_2\mathfrak{d}_1\mathfrak{d}_2(y_3\, f);
\end{aligned}
$$

with the induction hypothesis applied in $*$.

(ii) Since $\mathfrak{d}_i^2 = 0$ we may view $(\mathrm{Pol}\mathfrak{C}_n, \mathfrak{d}_i)$ as a chain complex of $\mathfrak{C}_n$-right modules by its polynomial grading in the y_i's. We show that $\mathrm{Pol}\mathfrak{C}_n$ then is contractible by the chain homotopy

$$
h_k\colon (\mathrm{Pol}\mathfrak{C}_n)_k \to (\mathrm{Pol}\mathfrak{C}_n)_{k+1}, f \mapsto \begin{cases} \frac{1}{2}(-1-\mathfrak{c}_i\mathfrak{c}_{i+1})y_i f & \text{if } 2 \mid k,\\ \frac{1}{2}(+1-\mathfrak{c}_i\mathfrak{c}_{i+1})y_{i+1} f & \text{if } 2 \nmid k.\end{cases}
$$

Assume $2 \mid \deg f$.

$$
\begin{aligned}
&h_{k-1}(\mathfrak{d}_i f) + \mathfrak{d}_i h_k(f)\\
&= \tfrac{1}{2}(1-\mathfrak{c}_i\mathfrak{c}_{i+1})y_{i+1}(\mathfrak{d}_i f) + \tfrac{1}{2}\mathfrak{d}_i\big((-1-\mathfrak{c}_i\mathfrak{c}_{i+1})y_i f\big)\\
&= \tfrac{1}{2}(1-\mathfrak{c}_i\mathfrak{c}_{i+1})y_{i+1}(\mathfrak{d}_i f) + \tfrac{1}{2}(-1+\mathfrak{c}_i\mathfrak{c}_{i+1})\big[(-1-\mathfrak{c}_i\mathfrak{c}_{i+1})f + y_{i+1}\mathfrak{d}_i(f)\big]\\
&= \tfrac{1}{2}\big[\underbrace{(1-\mathfrak{c}_i\mathfrak{c}_{i+1}) + (-1+\mathfrak{c}_1\mathfrak{c}_{i+1})}_{0}\big]y_{i+1}\mathfrak{d}_i(f) + \underbrace{\tfrac{1}{2}(-1+\mathfrak{c}_1\mathfrak{c}_{i+1})(-1-\mathfrak{c}_i\mathfrak{c}_{i+1})}_{1}\, f.
\end{aligned}
$$

The computation is similar if $2 \nmid \deg f$. This shows that h_* indeed is a contracting chain homotopy $\mathrm{id} \sim 0$. In particular, the chain complex $(\mathrm{Pol}\mathfrak{C}_n)_*$ is acyclic, i. e. $\ker\big(\mathfrak{d}_i|_{(\mathrm{Pol}\mathfrak{C}_n)_k}\big) / \operatorname{im}\big(\mathfrak{d}_i|_{(\mathrm{Pol}\mathfrak{C}_n)_{k+1}}\big) = 0$. This shows the assertion. ☐

We shall show in Lemma 8.22 independently from the following results that the representation $\mathrm{Pol}\mathfrak{C}_n$ of $\mathsf{NH}\mathfrak{C}_n$ is faithful.

7.2. Quiver Hecke superalgebra

We now want to recall from [KKT16, §3] the definition of a super-algebraic analogue of the KLR-algebra.

Definition 7.8. A *generalised Cartan matrix* on I is a matrix $C = (-d_{ij}) \in \mathbf{Z}^{I \times I}$ such that

(i) $-d_{ii} = 2$ for all $i \in I$,
(ii) $-d_{ij} \leq 0$ for distinct $i, j \in I$,
(iii) $-d_{ij} = 0$ if and only if $-d_{ji} = 0$, and
(iv) $2 \mid d_{i,j}$ if $i \in I_{\mathrm{odd}}$.

We define a $\mathbf{Z}$-valued bilinear form on $\mathbf{N}[I]$ by $\langle i, j\rangle := -d_{ij}$. Equivalently, we may describe the same datum by a graph Γ with vertex set I and d_{ij} directed edges from i to j for $i \neq j$, called the *Coxeter graph* of the Cartan matrix. For distinct indices i, j we thus write $i - j$ if $d_{ij} \neq 0$ and $i \not- j$ if $-d_{ij} = 0$.

For the remainder of this section let $C \in \mathbf{Z}^{I \times I}$ be a symmetrisable generalised Cartan matrix. For each edge $i - j$ of the Coxeter graph Γ fix an orientation. We write $i \to j$ if the edge is oriented from i to j.

Definition 7.9 ($\tilde{\mathsf{H}}_n(C)$, algebraically) [KKT16, §3]. The *quiver Hecke superalgebra* $\tilde{\mathsf{H}}_n(C)$ is the k-linear supercategory with objects $\nu \in I^n$, their identities 1_ν, as well as generating morphisms $x_{k,\nu} \colon \nu \to \nu$ for $1 \leq k \leq n$ and $\tau_{k,\nu} \colon \nu \to s_k(\nu)$ for $1 \leq kn$. Morphisms are equipped with the parities $|x_{k,\nu}| := |\nu_k|$ and $|\tau_{k,\nu}| := |\nu_k||\nu_{k+1}|$. Composition is subject to the following relations:

$$x_{k,\nu}x_{l,\nu} = \begin{cases} (-1)^{|\nu_k||\nu_l|}x_{k,\nu}x_{l,\nu} & \text{if } \nu_k \neq \nu_l \\ x_{k,\nu}x_{l,\nu} & \text{if } \nu_k = \nu_l \end{cases} \tag{7.3}$$

$$\tau_{k,s_i\nu}\tau_{l,\nu} = \quad (-1)^{|\nu_k||\nu_l|}\tau_{l,\nu}\tau_{k,\nu} \qquad \text{for } i - j > 1 \tag{7.4}$$

$$\tau_{k,s_k\nu}\tau_{k,\nu} = \begin{cases} 0 & \text{if } \nu_i = \nu_j \\ 1_\nu & \text{if } \nu_k \not- \nu_{k+1} \\ t_{\nu_k,\nu_{k+1}}x_{k,\nu}^{d_{\nu_k,\nu_{k+1}}} + t_{\nu_{k+1},\nu_k}x_{k+1,\nu}^{d_{\nu_{k+1},\nu_k}} & \text{if } \nu_k - \nu_{k+1} \end{cases} \tag{7.5}$$

$$\tau_{k,\nu}x_{l,\nu} - (-1)^{|\nu_k||\nu_{k+1}||\nu_l|} \cdot x_{s_k(l),\nu}\tau_{k,\nu} = \begin{cases} 0 & \text{if } l \notin \{k, k+1\} \text{ or } \nu_k \neq \nu_{k+1} \\ 1_\nu & \text{if } l = k \\ -1_\nu & \text{if } l = k+1 \end{cases} \tag{7.6}$$

$$\tau_{k,\nu}\tau_{k+1,\nu}\tau_{k,\nu} - \tau_{k+1,\nu}\tau_{k,\nu}\tau_{k+1,\nu} = \begin{cases} t_{\nu_k,\nu_{k+1}}1_\nu & \text{if } \nu_k = \nu_{k+2} \text{ and } \nu_k - \nu_{k+1} \\ 0 & \text{otherwise} \end{cases} \tag{7.7}$$

If $I_{\text{odd}} = \emptyset$, this definition just gives the ordinary quiver Hecke algebra from [KL09; Rou08].

In the non-super case, there is a handy graphical calculus for the NilHecke algebra, developed in [KL09; KL11]. We suggest the following altered diagrammatic definition for the super-setting:

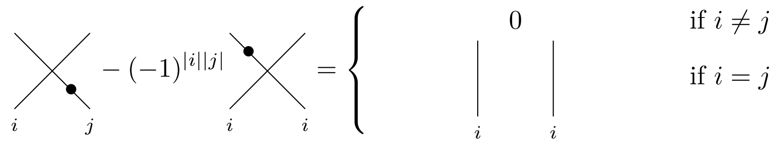
Definition 7.10. By a *string diagram* with n strands we understand n continuous paths $\phi_k : [0,1] \hookrightarrow \mathbf{R} \times [0,1]$ in a strip, $1 \leq k \leq n$, subject to the following conditions:

(i) Each path starts in $\mathbf{N} \times \{0\}$ and ends in $\mathbf{N} \times \{1\}$.
(ii) The projection of any path to the second coordinate is strictly monotonically increasing.
(iii) By their endpoints, the strands constitute a self-bijection of the natural numbers $\{1, \ldots, n\}$.
(iv) Strands may intersect; however, there must not be any intersections of more than two strands in one point.

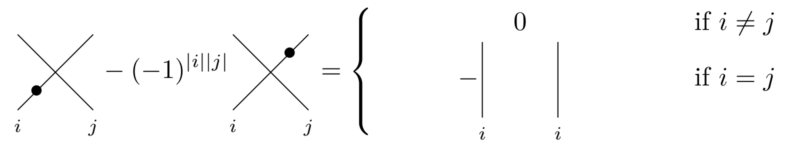
A string diagram is defined up to isotopy. Strings in a diagram may be endowed with certain point-like decorations; these are required not to lie on the crossings, and their positions are also specified only up to isotopy. In the super-setting, we endow critical points (such as crossings) and decorations with parities. The remarks in Section 6.3 apply. We count strands at the bottom from the left: the k-th strand is the one that starts at $k \times \{0\}$.

Definition 7.11 ($\tilde{\mathsf{H}}_n(C)$, diagrammatically). With the same set-up as in Definition 7.9, let $\tilde{\mathsf{H}}_n(C)$ be the k-linear supercategory with

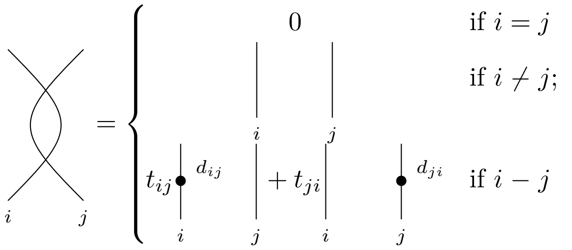
- objects: sequences $\nu \in I^n$
- morphisms: if ν' is not permutation of ν, $\mathrm{Hom}(\nu, \nu')$ is the zero space. Otherwise, it consists of formal linear combinations of string diagrams on n strands which connect identical entries of ν and ν'. We say that the k-th strand is labelled by ν_k and usually write its label below the strand. Strings may be decorated by an arbitrary non-negative number of dots, distant from the crossings.
- parities: The dots $x_{k,\nu} := \overset{\bullet}{|}_{\nu_k}$ and crossings $\tau_{k,\nu} := {}_{\nu_k}\times_{\nu_{k+1}}$ have parities $|\overset{\bullet}{|}_{\nu_k}| := |\nu_k|$ and $|{}_{\nu_k}\times_{\nu_{k+1}}| := |\nu_k||\nu_{k+1}|$. The explanations from Section 6.3 about vertical positioning apply here.
- composition: given by vertically stacking diagrams, subject to the following local relations:



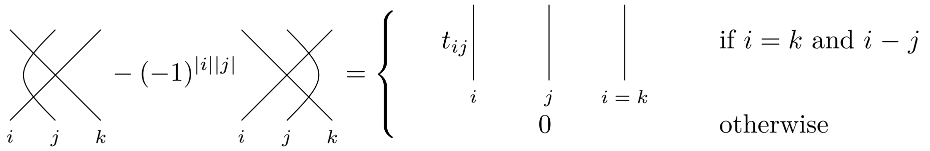
(7.8)



(7.9)



(7.10)



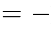
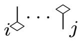
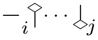
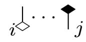
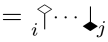
(7.11)

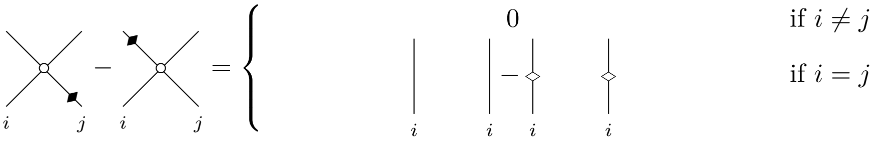
It is clear that both definitions of $\tilde{\mathsf{H}}_n(C)$ define the same category.

Remark 7.12. The diagrammatic definition of $\tilde{\mathsf{H}}_n(C)$ makes it clear that there is an embedding of the quiver Hecke superalgebra into Brundan's diagrammatic super Kac-Moody 2-category [BE17b] by adding an arrow tip pointing upwards to every string.

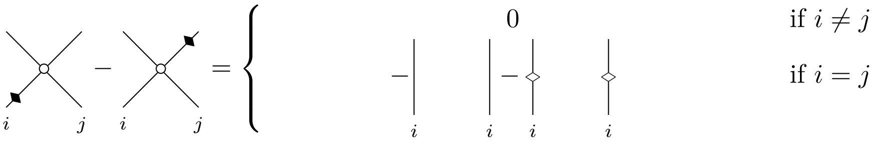
7.3. Quiver Hecke Clifford superalgebra

It turns out to be advantageous for calculations to *adjoin* odd generators for all odd indices $i \in I_{\text{odd}}$ rather than to endow the x_i's with a super grading themselves. This is captured by the following definition from [KKT16], where we propose a diagrammatic calculus based on the one for $\tilde{\mathsf{H}}_n(C)$.

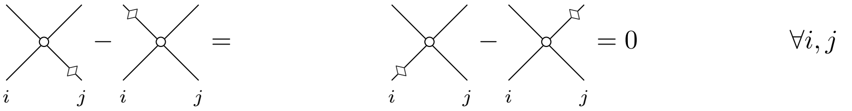
Definition 7.13 ($\mathsf{H}\mathfrak{C}_n(C)$, diagrammatically). With the same data as above, the *quiver Hecke Clifford superalgebra* $\mathsf{H}\mathfrak{C}_n(C)$ is the supercategory with the same objects $\nu \in I^n$ as $\tilde{\mathsf{H}}_n(C)$ and $\mathsf{H}_n(C)$, the even generators $y_{ik,\nu} =$ $_{\nu_k}$, $\sigma_{k,\nu} = {}_{\nu_k}$  $_{\nu_{k+1}}$ the odd generator $\mathfrak{c}_{k,\nu} =$ $_{\nu_k}$, subject to the relations $\mathfrak{c}_{k,\nu} = 0$ if $|\nu_k| = 0$, $_i = |_i$, the commutativity relations  $= -$$_i$,  $= -$ ,  $=$  and the following relations for the interaction with the crossing:



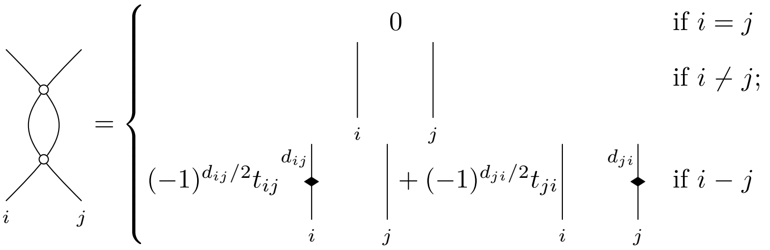
(7.12)



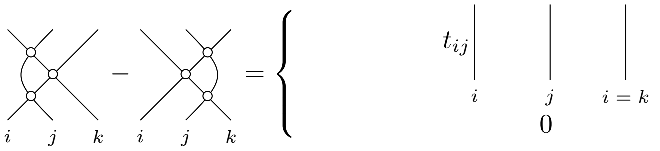
(7.13)



(7.14)



(7.15)



if $i = k$
and $i - j$

otherwise. (7.16)

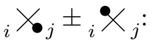
For every odd index $\mathsf{H}\mathfrak{C}_n(C)$ contains a copy of $\mathsf{NH}\mathfrak{C}_n$.

Lemma 7.14. There is an embedding

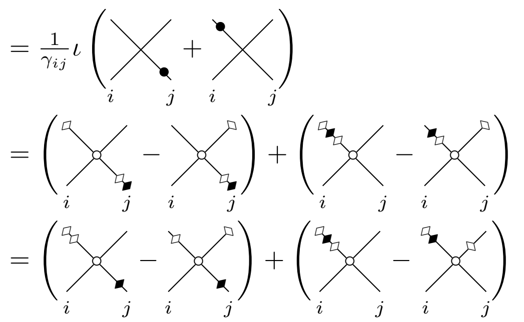
$$\begin{aligned} \iota : \tilde{\mathsf{H}}_n(C) &\hookrightarrow \mathsf{H}\mathfrak{C}_n(C), \\ x_{k,\nu} &\mapsto (\mathfrak{c}_{k,\nu})^{|\nu_k|} y_{k,\nu} \\ \tau_{k,\nu} &\mapsto \gamma_{\nu_k,\nu_{k+1}} (\mathfrak{c}_{k,\nu} - \mathfrak{c}_{k+1,\nu})^{|\nu_k||\nu_{k+1}|} \sigma_{k,\nu} \end{aligned}$$

with factors $\gamma_{i,j}$ such that $\gamma_{i,j} = 1$ if at least one index is even, $\gamma_{i,i} = \frac{1}{2}$ if i is odd and $\gamma_{i,j}\gamma_{j,i} = -\frac{1}{2}$ otherwise.

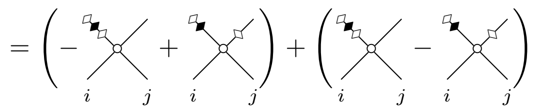
Proof. We verify the well-definedness of this map and refer to [KKT16, thm. 3.3] for injectivity. We have to check:

: If both indices i and j are even, domain and codomain locally reduce to the ordinary KLR-algebra; hence nothing remains to prove. Let us thus assume that both i and j are odd indices.

$$\tfrac{1}{\gamma_{ij}} \iota(\tau_{i,\nu} x_{k+1,\nu} + \tau_{i,\nu} x_{k+1,\nu})$$

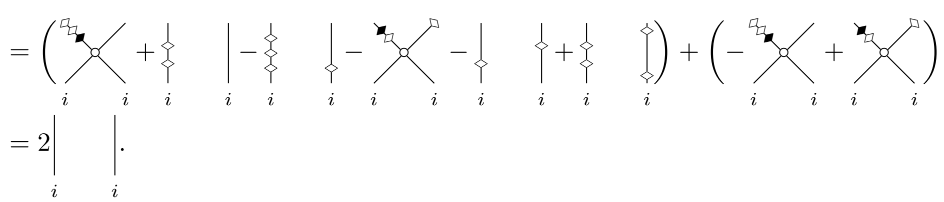


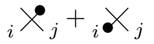
by (7.14). If $i \neq j$, then (7.12) asserts that we can slide all diamonds across the crossing. By the commutativity relations for ✦'s and ✧'s we obtain for $i \neq j$:

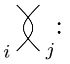


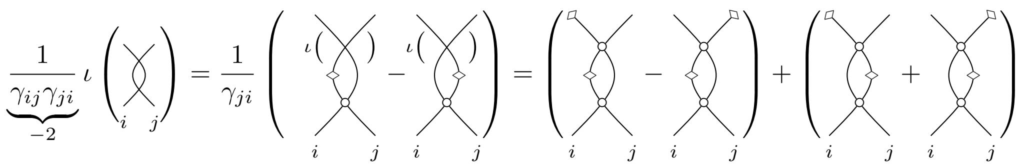
$= 0.$

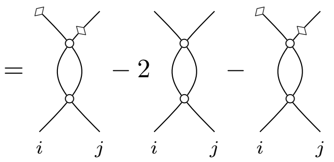
If instead $i = j$, we obtain from (7.12):



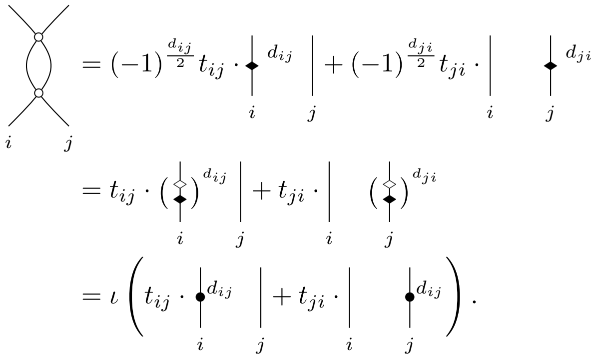
The computation for  is similar.

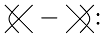
: If $i = j$ this is immediate. For $i \neq j$ nothing remains to prove if one index is even, so assume $|i||j| = 1$.





since we may drag white diamons across crossings by (7.14). If $i \not- j$, this equals $-2{}_i|\ \ |_j$ since we may resolve the double crossings by (7.15). If $i - j$, we calculate



: For the braid relation, we resort to an "algebraic" rather than diagrammatic calculation. We assume w. l. o. g. that $k = 1$. The braid relation then follows from the computation

$$\begin{aligned}
&\tau_1\tau_2\tau_{1,\nu}\\
&= (\mathfrak{c}_1 - \mathfrak{c}_2)^{|s_2s_1(\nu)_1||s_2s_1(\nu)_2|}\sigma_1\,(\mathfrak{c}_2 - \mathfrak{c}_3)^{|s_1(\nu)_2||s_1(\nu)_3|}\sigma_2\,(\mathfrak{c}_1 - \mathfrak{c}_2)^{|\nu_1||\nu_2|}\sigma_{1,\nu}\\
&= (\mathfrak{c}_1 - \mathfrak{c}_2)^{|\nu_1||\nu_2|}(\mathfrak{c}_1 - \mathfrak{c}_3)^{|\nu_1||\nu_3|}(\mathfrak{c}_2 - \mathfrak{c}_3)^{|\nu_2||\nu_3|}\sigma_1\sigma_2\sigma_{1,\nu}\\
&\overset{*}{=} (\mathfrak{c}_2 - \mathfrak{c}_3)^{|\nu_2||\nu_3|}(\mathfrak{c}_1 - \mathfrak{c}_3)^{|\nu_1||\nu_3|}(\mathfrak{c}_1 - \mathfrak{c}_2)^{|\nu_1||\nu_2|}\sigma_2\sigma_1\sigma_{2,\nu}\\
&= (\mathfrak{c}_2 - \mathfrak{c}_3)^{|s_1s_2(\nu)_2||s_1s_2(\nu)_3|}\sigma_2\,(\mathfrak{c}_1 - \mathfrak{c}_2)^{|s_2(\nu)_1||s_2(\nu)_2|}\sigma_1\,(\mathfrak{c}_2 - \mathfrak{c}_3)^{|\nu_2||\nu_3|}\sigma_{2,\nu}\\
&= \tau_2\tau_1\tau_{2,\nu}.
\end{aligned}$$

The equation $*$ is clear if at least one index is even and is verified by multiplying out the $\mathfrak{c}_k$'s if every index is odd.

For injectivity see [KKT16, thm. 3.3]. ☐

7.4. Faithful polynomial representation

In the vein of the representation $\mathrm{Pol}\mathfrak{C}_n$ of the NilHecke Clifford algebra $\mathsf{NH}\mathfrak{C}_n$ defined in Lemma 7.14, we define polynomial representation of the quiver Hecke Clifford algebra $\mathsf{NH}\mathfrak{C}_n(C)$.

Definition 7.15. Let $\mathrm{Pol}\mathfrak{C}_n(C)$ be the k-linear supercategory with

- objects: the free k-super vector spaces $\mathrm{Pol}\mathfrak{C}_\nu \coloneqq k[y_{1,\nu}, \dots, y_{n,\nu}, \mathfrak{c}_{1,\nu}, \dots, \mathfrak{c}_{n,\nu}]$, indexed by sequences $\nu \in I^n$. The $y_{k,\nu}$'s and $\mathfrak{c}_{k,\nu}$ have the same parities and satisfy the same relations as in Definition 7.13.
- morphisms: super vector space homomorphisms.

$\mathrm{Pol}\mathfrak{C}_n(C)$ has a full subcategory $\mathrm{Pol}\mathfrak{C}(\nu)$ with objects $\{\mathrm{Pol}\mathfrak{C}_{\nu'} \mid \nu' \in S_n \,.\, \nu\}$. We may regard $\mathrm{Pol}\mathfrak{C}_n(C)$ as a free k-super vector space by taking the direct sum of its objects.

The quiver Hecke Clifford superalgebra $\mathsf{H}\mathfrak{C}_n(C)$ then can also be understood in terms of linear algebra by realising it as the following algebra of endomorphisms:

Proposition 7.16. $\mathsf{H}\mathfrak{C}(\nu)$ acts faithfully on $\mathrm{Pol}\mathfrak{C}(\nu)$, where $y_{k,\nu}$ and $\mathfrak{c}_{k,\nu}$ act by multiplication and $\sigma_{k,\nu}$ acts by

$$\begin{cases}
s_k & \text{if } \nu_k \not- \nu_{k+1} \text{ or } \nu_k \leftarrow \nu_{k+1},\\
\mathfrak{d}_k & \text{if } \nu_k = \nu_{k+1},\\
\Big(t_{\nu_k,\nu_{k+1}}(\mathfrak{c}_{k,\nu}y_{k,\nu})^{d_{\nu_k,\nu_{k+1}}} + \\
\quad + t_{\nu_{k+1},\nu_k}(\mathfrak{c}_{k+1,\nu}y_{k+1,\nu})^{d_{\nu_{k+1},\nu_k}}\Big)s_k & \text{if } \nu_k \to \nu_{k+1}.
\end{cases} \tag{7.17}$$

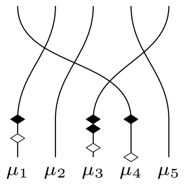
Proof. In the non-super case, a similar faithful representation of the quiver Hecke algebra was proven in [KL09, §2.3]. Our Clifford-analogue only necessitates minor alterations.

—*Spanning set*: As a vector space, $\mathsf{H}\mathfrak{C}_n(\nu)$ is isomorphic to a direct sum $\bigoplus_{\mu,\mu' \in S_n.\nu}({}_{\mu'}\,\mathsf{H}\mathfrak{C}_\mu)$ of subspaces, where ${}_{\mu'}\,\mathsf{H}\mathfrak{C}_\mu$ is given by diagrams connecting the bottom sequence μ with the top sequence ν. Let ${}_{\mu'}S_\mu \leq S_n$ be the subgroup of permutations that map μ to μ':

$$ {}_{\mu'}S_\mu \leq S_n = \{w \in S_n \mid \mu_{w(k)} = \mu'_k\}. $$

Given a diagram of ${}_{\mu'}\,\mathsf{H}\mathfrak{C}_\mu$, assume there are two strands intersecting more than once. By the relation (7.15) it can be replaced by a linear combination of diagrams with fewer crossings, possibly decorated with more diamonds. Diamonds can be slid across crossings by (7.12–7.14), possibly at the cost of introducing summands with less crossings, and past other diamonds, possibly introducing signs.

The vector space ${}_{\mu'}\,\mathsf{H}\mathfrak{C}_\mu$ thus is spanned by diagrams with any two strands intersecting at most once, all black diamonds below all crossings and all white diamonds below the black ones, with at most one white diamond per strand. The white ones may be arranged descending to the right. A typical diagram in this spanning set looks like the following:



This spanning set can be written as

$$ \begin{aligned} {}_{\mu'}\,\mathsf{B}\mathfrak{C}_\mu &= \{\tau_{\pi,\mu} y_\mu^{\boldsymbol{\alpha}} \mathfrak{c}_\mu^{\boldsymbol{\beta}}\} \\ &= \{\tau_{\pi,\mu'} y_{1,\mu}^{\alpha_1} \cdots y_{n,\mu}^{\alpha_n} \mathfrak{c}_{1,\mu}^{\beta_1} \cdots \mathfrak{c}_{n,\mu}^{\beta_n}\} \end{aligned} \tag{7.18} $$

where π is a reduced expression for an element of ${}_{\mu'}S_\mu$, μ and μ' are permutations of ν, $\boldsymbol{\alpha} \in \mathbf{N}^n$, $\boldsymbol{\beta} \in \{0,1\}^n$ are multiindices and $\mathfrak{c}_\mu^{\boldsymbol{\beta}}$ is an ordered monomial. Choose a complete order $\leq$ on I such that $i < j$ whenever there is an edge $i \to j$. This order induces a lexicographic order on I^n. We show by induction on μ w. r. t. this order on I^n that ${}_{\mu'}\,\mathsf{B}\mathfrak{C}_\mu$ is a k-basis on which $\mathsf{H}\mathfrak{C}(\nu)$ acts faithfully.

—*Base of induction*: Let n_i be the number of entries i in ν. Let

$$ \mu = (\underbrace{i_1, \ldots, i_1}_{n_1}, \underbrace{i_2, \ldots,}_{n_2} \ldots) $$

such that $i_1 < i_2 < \cdots$. The tuple μ then is the lowest element in the orbit $S_n \,.\, \nu$ w. r. t. the order $\leq$. We may write a permutation $w \in {}_{\nu'}S_\nu$ as $w = w_2 w_1$ where $w_2 \in S_{n_1} \times S_{n_2} \times \cdots$ only permutes the i_1, i_2 etc. independently, and w_2 is of minimal length, i. e. does not interchange identical labels and interchanges distinct labels at most once. The spanning set thus can be written as

$$ {}_{\nu'}\,\mathsf{B}\mathfrak{C}_\nu = \{\mathfrak{d}_{w_2,\mu} \mathfrak{d}_{w_1,\mu} y_\mu^{\boldsymbol{\alpha}} \mathfrak{c}_\mu^{\boldsymbol{\beta}}\}. $$

We want to check that these elements act linearly independently on $\mathrm{Pol}\mathfrak{C}(\nu)$.

- The terms $y_\mu^{\boldsymbol{\alpha}} \mathfrak{c}_\mu^{\boldsymbol{\beta}}$ take $y_\mu^{\boldsymbol{\alpha}'} \mathfrak{c}_\mu^{\boldsymbol{\beta}'}$ to $\pm y_\mu^{\boldsymbol{\alpha}+\boldsymbol{\alpha}'} \mathfrak{c}_\mu^{\boldsymbol{\beta}+\boldsymbol{\beta}'}$.
- Since w_1 permutes strands with distinct labels ordered increasingly, $\mathfrak{d}_{w_1,\mu}$ acts by a permutation.
- Since $w_2 \in S_{n_1} \times \cdots \times S_{n_m}$ is a permutation which only permutes identically labelled strands, $\mathfrak{d}_{w_2,\nu}$ is contained in the product $\mathsf{NC}\mathfrak{C}_{n_1} \times \mathsf{NC}\mathfrak{C}_{n_2} \times \cdots \times \mathsf{NC}\mathfrak{C}_{n_m}$. We shall prove in Lemma 8.22 that the action of $\mathsf{NC}\mathfrak{C}_n$ on polynomials is faithful.

—*Induction step*: It suffices to show that if ${}_{\mu'}\,\mathsf{B}\mathfrak{C}_\mu$ is linearly independent, then ${}_{s_k(\mu')}\,\mathsf{B}\mathfrak{C}_\mu$ is linearly independent for μ'_k, μ'_{k+1} distinct and connected, for otherwise ${}_{s_k(\mu')}\,\mathsf{B}\mathfrak{C}_\mu$ maps bijectively to ${}_{\mu'}\,\mathsf{B}\mathfrak{C}_\mu$ by $\mathfrak{d}_{k,\mu'}$. The multiplication map $(\sigma_k\cdot) : {}_{s_k(\mu')}\,\mathsf{B}\mathfrak{C}_\nu \hookrightarrow {}_{\mu'}\,\mathsf{B}\mathfrak{C}_\mu$ is seen to be injective by the same argumentation as in [KL09]:

Let ${}_{\mu'}\,\mathsf{B}\mathfrak{C}_{\mu}$ be endowed with a partial order $\prec$ such that if we assign to an element $\mathfrak{d}_{w,\mu}y_{\mu}^{\boldsymbol{\alpha}}\mathfrak{c}_{\mu}^{\boldsymbol{\beta}}$ of the spanning set the tuple $(\ell(w),\boldsymbol{\alpha},\boldsymbol{\beta})$, the order $\prec$ coincides with the lexicographic ordering on these tuples. Define a map

$$\begin{aligned}\varsigma\colon {}_{s_k(\mu')}\,\mathsf{B}\mathfrak{C}_{\mu} &\to {}_{\mu'}\,\mathsf{B}\mathfrak{C}_{\mu}\\ D &\mapsto \begin{cases}\mathfrak{d}_k D & \text{if the } k\text{-th and } (k+1)\text{-st} \\ & \text{strand do not intersect} \\ t_{\mu_k,\mu_{k+1}}D'(\mathfrak{c}_{k,\mu}y_{k,\mu})^{d_{\mu_k,\mu_{k+1}}} & \text{otherwise}\end{cases}\end{aligned} \tag{7.19}$$

where D' is obtained from D by removing the crossing. The map ς is injective, and multiplication by $\sigma_{k,\mu'}$ satisfies

$$\sigma_{k,\mu'}\cdot D \in \{\pm\varsigma(D)\} + \sum_{d\prec\varsigma(D)} \mathbf{Z}d \subseteq {}_{\mu'}\,\mathsf{B}\mathfrak{C}_{\mu}\,.$$

By the induction hypothesis on ${}_{\mu'}\,\mathsf{B}\mathfrak{C}_{\mu}$, we obtain that the multiplication map $(\sigma_{k,\mu'}\cdot)$ must be injective. ☐

Definition 7.17. We equip $\mathrm{Pol}\mathfrak{C}(\nu)$ with a polynomial grading by setting $\deg(y_{k,\mu})$ and $\deg(\mathfrak{c}_{k,\mu}) = 0$. $\mathsf{H}\mathfrak{C}(\nu)$ admits a similar polynomial grading by additionally setting $\deg(\sigma_{k,\mu}) = -1$. Then $\mathrm{Pol}\mathfrak{C}(\nu)$ is a faithful graded $\mathsf{H}\mathfrak{C}(\nu)$-module.

Corollary 7.18. With this grading, $\mathrm{Pol}\mathfrak{C}(\nu)$ is a faithful graded $\mathsf{H}\mathfrak{C}(\nu)$-module.

8. Clifford symmetric Polynomials

Recall from Definition 7.1 the definition of the Clifford Demazure operator $\mathfrak{d}_i$ and how it acts on the polynomial ring $\mathrm{Pol}\mathfrak{C}_n$, see Definition 7.3. In the ordinary case, the symmetric polynomials, i. e. the ones which are invariant under the action of S_n, are precisely the common kernel of the Demazure operators.

Let the index set $I = \{1, \dots, n\} = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$ be endowed with parities. From now on, we consider the algebra $\mathfrak{C}_n/(\mathfrak{c}_i = 0 \text{ if } |i| = 0)$. We shall denote this algebra also by $\mathfrak{C}_n$ for the remainder of the thesis. $\mathrm{Pol}\mathfrak{C}_n$ and $\mathfrak{d}_i$ are defined as in Definition 7.3 with the additional relation that $\mathfrak{c}_i = 0$ if $|i| = 0$.

Definition 8.1. The $\mathfrak{C}_n$-algebra of $\mathfrak{d}$-*symmetric polynomials*[14] is the intersection $\bigcap_{k=1}^{n-1} \ker \mathfrak{d}_k = \bigcap_{k=1}^{n-1} \operatorname{im} \mathfrak{d}_i$.

Remark 8.2. Since each of the kernels $\ker \mathfrak{d}_i$ is a $\mathfrak{C}_n$-algebra by Lemma 7.6, so is their intersection. The equality $\ker \mathfrak{d}_i = \operatorname{im} \mathfrak{d}_i$ has been shown in Lemma 7.7.

In this section, we want to investigate some of their properties. To this end, we shall introduce the notion of elementary $\mathfrak{d}$-symmetric polynomials and eventually show that $\bigcap_{i=1}^{n-1} \ker \mathfrak{d}_i$ is a polynomial ring over $\mathfrak{C}_n$ generated by the $\mathfrak{d}$-symmetric polynomials.

Remark 8.3. Let $\tilde{\partial}_k := \left(\frac{1}{2}(\mathfrak{c}_k - \mathfrak{c}_{k+1})\right)^{|k||k+1|} \mathfrak{d}_k$ and let $\widetilde{\mathrm{Pol}}_n$ be the k-superalgebra with generators $x_1, \dots, x_n$ of parities $|x_k| = |k|$ and commutativity relations $x_k x_l = (-1)^{1-\delta_{kl}|k||l|} x_l x_k$. There is an inclusion $\widetilde{\mathrm{Pol}}_n \hookrightarrow \mathrm{Pol}\mathfrak{C}_n, x_k \mapsto (\mathfrak{c}_k)^{|k|} y_k$. If all indices are even, $\widetilde{\mathrm{Pol}}_n = \mathrm{Pol}\mathfrak{C}_n = \mathrm{Pol}_n$. If k and $k+1$ are odd indices, we have under this inclusion

$$\begin{aligned}
\tilde{\partial}_k(x_k) &= \tfrac{1}{2}(\mathfrak{c}_k - \mathfrak{c}_{k+1})\mathfrak{c}_{k+1}(-1 - \mathfrak{c}_k\mathfrak{c}_{k+1}) = 1, \\
\tilde{\partial}_k(x_{k+1} &= \tfrac{1}{2}(\mathfrak{c}_k - \mathfrak{c}_{k+1})\mathfrak{c}_k(1 - \mathfrak{c}_k\mathfrak{c}_{k+1}) \qquad = 1;
\end{aligned}$$

and if at least one of k, $k+1$ is an even index, then

$$\begin{aligned}
\tilde{\partial}_k(x_k) &= (\mathfrak{c}_{k+1})^{|k|}(-1 - \mathfrak{c}_k\mathfrak{c}_{k+1}) &&= -(\mathfrak{c}_{k+1})^{|k|}, \\
\tilde{\partial}_k(x_{k+1}) &= (\mathfrak{c}_k)^{|k+1|}(1 - \mathfrak{c}_k\mathfrak{c}_{k+1}) &&= +(\mathfrak{c}_k)^{|k+1|}.
\end{aligned}$$

$\tilde{\partial}_k$ indeed is a well-defined operator by Lemma 7.14. We prefer to work with $\mathfrak{d}_k$ though. Note that $\left(\frac{1}{2}(\mathfrak{c}_k - \mathfrak{c}_{k+1})\right)^{|k||k+1|}$ is a unipotent element in $\mathfrak{C}_n$ and therefore in particular is a unit. This implies that $\ker \tilde{\partial}_k = \ker \mathfrak{d}_k$ as $\mathfrak{C}_n$-bimodules (and $\mathfrak{C}_n$-algebras). Hence all our considerations about $\mathfrak{d}$-symmetric polynomials also hold true for $\tilde{\partial}$, which justifies considering them instead of $\tilde{\partial}$-symmetric polynomials.

8.1. Interlude: counting graded ranks

Before we come to the definition of elementary $\mathfrak{d}$-symmetric polynomials, we recall a useful tool for working with polynomial rings. Note that we are dealing with ordinary (commutative) polynomial rings in this section.

Let Pol_n, the NilCoxeter algebra NC_n and the NilHecke algebra NH_n be endowed with a grading such that y_i is of degree 1 and $\mathfrak{d}_i$ is of degree -1. We shall call these gradings the *polynomial grading*, denoted by "deg".

Definition 8.4. Given a (graded) ring R and a free $\mathbf{Z}$-graded R-module $M = \bigoplus_{k \in \mathbf{Z}} M_k$, its *graded rank* or *Poincaré series* is the formal power series $\mathrm{rk}_{q,R}(M) := \sum_{k \in \mathbf{Z}} \mathrm{rk}_R(M_k) q^k \in \mathbf{Z}[[q]]$. We write $\mathrm{rk}_{q,\mathbf{Z}}$ for the graded rank of free abelian groups. With respect to a field k, we denote the graded dimension by $\dim_{q,k}$.

For expressing graded ranks, the notion of *q-integers* $(n)_q := 1 + \cdots + q^{n-1}$ is notably useful. One usually assumes $1-q$ to be a unit such that $(n)_q = \frac{1-q^n}{1-q}$. For $q \to 1$ the q-analogues then converge to the respective ordinary notions. For an introduction to q-numbers and a conversion between the notations $(n)_q$ and $[n]_q$ see [LQ15].

In analogy with rk_q we define a q-analogue ord_q of the order of a Coxeter group where powers of q encode the length ℓ of words w. r. t. a chosen presentation.

Definition 8.5. Let (W, S) be a Coxeter system with a fixed generating set S. Recall that S determines a length function ℓ on W. We define $\mathrm{ord}_q(W) := \sum_{w \in W} q^{\ell(w)}$.

---

[14] One may be tempted to call these supersymmetric polynomials; this term has already been coined for another notion though [Ste85].

Lemma 8.6 [cf. Win01, §1]. The Coxeter group series A_n, BC_n and D_n have the following q-orders:

(i) $\operatorname{ord}_q(S_n) = (n)_q!$ with the *q-factorial* $(n)_q! := (n)_q(n-1)_q \cdots (1)_q$.
(ii) $\operatorname{ord}_q(\mathrm{BC}_n) = (2n)_q!!$ with the *q-double factorial* $(2n)_q!! := (2n)_q(n-2)_q \cdots (2)_q$.
(iii) $\operatorname{ord}_q(\mathrm{D}_n) = (2n-2)_q!!(n)_q$.

We included $\operatorname{ord}_q(\mathrm{BC}_n)$ and $\operatorname{ord}_q(\mathrm{D}_n)$ for the sake of completeness because we could not find a concise explanation for these two cases in the literature. We shall not need them in the following though.

Proof. (i) The length $\ell(w)$ of a permutation $w \in S_n$ equals the number of *inversions* [BB05, prop. 1.5.2], i.e. the cardinality of $\operatorname{inv} w := \{(i,j) \mid 1 \le i < j \le n, w(i) > w(j)\}$. Assume that $\sum_{w \in S_n} p(w) = (n)_q!$. Consider the permutations

$$\begin{aligned} \pi_0 &\colon (1,\dots,n+1) \mapsto (1,\dots,n,n+1), \\ \pi_1 &\colon (1,\dots,n+1) \mapsto (1,\dots,n+1,n), \\ &\vdots \\ \pi_n &\colon (1,\dots,n+1) \mapsto (n+1,1,\dots,n) \end{aligned}$$

contained in S_{n+1}, where $\pi_0 = e$ is the trivial permutation. Moving the rightmost entry in π_0 to the left will subsequently create new inversions such that π has the k inversions

$$\operatorname{inv} \pi_k = \big\{(n-k+1, n-k+2), \dots, (n-k+1, n+1)\big\}.$$

and $\sum_k q^{\ell(\pi_k)} = (1 + q + \cdots q^n) = (n+1)_q$. Since any $w \in S_n$ only interchanges the first n slots, the inversions of w and π_k do not interfere and we have that $\operatorname{inv}(\pi_k w) = w^{-1}(\operatorname{inv} \pi_k) \sqcup \operatorname{inv}(w)$. Therefore, $\sum_k q^{\ell(\pi_k w)} = (n+1)_q q^{\ell(w)}$. By the induction hypothesis, letting w traverse all of S_n proves the claim.

(ii) The Coxeter group BC_n, called the *hyperoctahedral group* or *signed permutation group*, is the group of permutations π of the set $\{\pm 1, \dots, \pm n\}$ such that $\pi(-k) = -\pi(k)$. It has as generators the simple transpositions $s_k \colon \pm k \leftrightarrows \pm k+1$ for $1 \le k \le n-1$ and the additional generator $s_0 \colon 1 \leftrightarrows -1$. A signed permutation w has set of inversions

$$\operatorname{inv}(w) := \big\{(i,j) \bigm| 1 \le i < j \le n, w(i) > w(j)\big\} \cup \big\{(-i,j) \bigm| 1 \le i \le j \le n, w(-i) > w(j)\big\}.$$

Then for a permutation w $\ell(w) = |\operatorname{inv}(w)|$, see [BB05, prop. 8.1.1]. We count the number of inversions as in part (i). Assume that $\operatorname{ord}_q \mathrm{BC}_n = (n)_q!!$ and consider the permutations

$$\begin{aligned} \pi_0 &\colon (1,\dots,n+1) \mapsto (1,\dots,n,n+1), \\ &\vdots \\ \pi_n &\colon (1,\dots,n+1) \mapsto (n+1,1,\dots,n), \\ \pi_{n+1} &\colon (1,\dots,n+1) \mapsto (-n-1,1,\dots,n), \\ &\vdots \\ \pi_{2n+1} &\colon (1,\dots,n+1) \mapsto (1,\dots,w(n),-n-1). \end{aligned}$$

They have inversions

$$\operatorname{inv} \pi_k = \begin{cases} \Big\{(n-k+1, n-k+2), \dots, (n-k+1, n+1)\Big\} & \text{if } 0 \le k \le n, \\ \Big\{\underbrace{(1,l), \dots, (l-1,l)}_{l-1 \text{ many}}, \underbrace{(-1,l), \dots, (l+1,l)}_{l-1 \text{ many}} \\ \qquad \underbrace{(-l,l), \dots, (l,n+1)}_{n-l+2 \text{ many}}\Big\} & \text{if } 1 < l \le n+1 \text{ for } l := k-n. \end{cases}$$

Hence, $|\operatorname{inv} \pi_k| = k$ and $\sum_k q^{\ell(\pi_k)} = (2n+2)_q$. The rest of the argument is as in part (i).

(iii) The Coxeter group D_n is the subgroup of BC_n generated by $s_1, \dots, s_n$ and the additional generator $\tilde{s}_0 := s_0 s_1 s_0 \colon (1,2,3,\dots) \leftrightarrows (-2,-1,3,\dots)$. It is the subgroup of signed permutations that flip an even number of signs, called *demihypercube group*. A permutation $w \in \mathrm{D}_n$ has inversions

$$\operatorname{inv}(w) := \big\{(i,j) \bigm| 1 \le i < j \le n, \quad w(i) > w(j)\big\} \cup \big\{(-i,j) \mid 1 \le i \overset{*}{<} j \le n, w(-i) > w(j)\big\};$$

(note the difference with BC_n w. r. t. the inequality $\overset{*}{<}$). Then $\ell(w) = |\operatorname{inv}(w)|$, see [BB05, prop. 8.2.1]. Assume by induction that $\operatorname{ord}_q \mathrm{D}_n = (2n-2)_q!!(n)_q$. Consider the permutations

$$\begin{aligned}
\pi_0 \colon (1,\dots,n+1) &\mapsto (w(1),\dots,w(n),n+1),\\
&\vdots\\
\pi_{n-1} \colon (1,\dots,n+1) &\mapsto (1,n+1,\dots,w(n)),\\
\pi_n \colon (1,\dots,n+1) &\mapsto (n+1,1,2,\dots,w(n)),\\
\pi'_n \colon (1,\dots,n+1) &\mapsto (-n-1,-1,2,\dots,w(n)),\\
&\vdots\\
\pi_{2n} \colon (1,\dots,n+1) &\mapsto (1,\dots,n,-n-1).
\end{aligned}$$

Their set of inversions

$$\operatorname{inv} \pi_k^{(\prime)} = \begin{cases} \big\{(n-k+1,n-k+2),\dots,(n-k+1,n+1)\big\} & \text{if } 0 \le k \le n,\\[1ex] \big\{\underbrace{(1,l),\dots,(l-1,l)}_{l-1 \text{ many}}, \underbrace{(-1,l),\dots,(l+1,l)}_{l-1 \text{ many}} \\ \qquad\qquad \underbrace{(-l,l+1),\dots,(l,n+1)}_{n-l+1 \text{ many}}\big\} & \text{if } 1 < l \le n+1 \text{ for } l \coloneqq k-n \end{cases}$$

has cardinality $|\operatorname{inv} \pi_k^{(\prime)}| = k$ and therefore

$$\sum_k q^{\ell(\pi_k^{(\prime)})} = 1 + \dots + q^n + q^n + \dots + q^{2n} = (1+q^n)(n+1)_q.$$

By the same argumentation as in part (i) and the induction hypothesis we see that

$$\operatorname{ord}_q \mathrm{D}_{n+1} = (2n-2)_q!! \underbrace{(1+q^n)(n)_q}_{=(2n)_q}(n+1)_q = (2n)_q!!(n+1)_q. \qquad \square$$

Lemma 8.7. There are the following graded ranks of abelian groups we are particularly interested in:

$$\operatorname{rk}_{q,\mathbf{Z}} \mathrm{Pol}_n = \frac{1}{(1-q)^n}, \qquad \operatorname{rk}_{q,\mathbf{Z}} \Lambda\mathrm{Pol}_n = \frac{1}{(n)_q!(1-q)^n}, \tag{8.1}$$

$$\operatorname{rk}_{q,\mathbf{Z}} \mathsf{NC}_n = (n)_q!, \qquad \operatorname{rk}_{q,\mathbf{Z}} \mathsf{NH}_n = \frac{(n)_q!}{(1-q)^n}. \tag{8.2}$$

Where NC_n denotes the *NilCoxeter-algebra* $\mathsf{NC}_n = \mathbf{Z}[\partial_1,\dots,\partial_n]$ and NH_n denotes the *NilHecke-algebra* $\mathsf{NH}_n = \mathrm{Pol}_n \otimes_{\mathbf{Z}} \mathsf{NC}_n$.

Proof. Pol_n: Every indeterminate y_i independently generates a free abelian group $\langle 1, y_i, y_i^2, \dots\rangle_{\mathbf{Z}}$ of graded rank $1 + q + q^2 + \dots = \frac{1}{1-q}$. Since $\mathrm{Pol}_n \cong \mathbf{Z}[y]^{\otimes n}$ as abelian groups, Pol_n has graded rank $\operatorname{rk}_{q,\mathbf{Z}} \mathrm{Pol}_n = (1-q)^{-n}$.

$\Lambda\mathrm{Pol}_n$: Recall that the symmetric polynomials $\Lambda\mathrm{Pol}_n = \mathrm{Pol}_n^{S_n}$ are isomorphic to the polynomial ring $\Lambda\mathrm{Pol}_n \cong \mathbf{Z}\big[\varepsilon_1^{(n)},\dots,\varepsilon_n^{(n)}\big]$ in the elementary symmetric polynomials $\varepsilon_m^{(n)}$. The polynomial $\varepsilon_m^{(n)}$ is of degree m and thus generates a subgroup of graded rank $\frac{1}{1-q^m}$. $\Lambda\mathrm{Pol}_n$ therefore has graded rank $\operatorname{rk}_{q,\mathbf{Z}} \Lambda\mathrm{Pol}_n = \prod_{m=1}^n \frac{1}{1-q^m}$. Since $(1-q^m) = (m)_q(1-q)$ (telescoping sum), one obtains inductively that $\operatorname{rk}_{q,\mathbf{Z}} \Lambda\mathrm{Pol}_n = \frac{1}{(n)_q!(1-q)^n}$.

NC_n: By the relations of the NilCoxeter-algebra NC_n, there is a bijection between the symmetric group S_n and a $\mathbf{Z}$-basis of NC_n which maps Coxeter length to polynomial degree. By the Lemma 8.6 this shows $\operatorname{rk}_{q,\mathbf{Z}} \mathsf{NC}_n = (n)_q!$.

NH_n: By using $\mathsf{NH}_n = \mathrm{Pol}_n \otimes_{\mathbf{Z}} \mathsf{NC}_n$, we obtain that $\operatorname{rk}_{q,\mathbf{Z}}(\mathsf{NH}_n) = \frac{(n)_q!}{(1-q)^n}$. $\square$

Remark 8.8. Similar statements hold for invariant polynomials for the series BC_n and D_n of Coxeter groups.

(i) The hyperoctahedral group BC_n acts on the polynomial ring Pol_n by permuting the indeterminates and flipping their signs. It thus has invariants $\mathrm{Pol}_n^{\mathrm{BC}_n} = \mathbf{Z}[y_1^2, \dots, y_n^2]^{S_n}$, which yields

$$\mathrm{rk}_{q,\mathbf{Z}} \mathrm{Pol}_n^{\mathrm{B}_n} = \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \frac{1}{(2n)_q!!(1-q)^n}.$$

(ii) The demihypercube group D_n acts on Pol_n by permuting the indeterminates and flipping two signs at once. The invariants thus are

$$\mathrm{Pol}_n^{\mathrm{D}_n} = \mathbf{Z}\big[\varepsilon_1(y_1^2, \dots, y_n^2), \dots, \varepsilon_{n-1}(y_1^2, \dots, y_n^2), \varepsilon_n(y_1, \dots, y_n)\big]$$

with graded rank

$$\mathrm{rk}_{q,\mathbf{Z}}(\mathrm{Pol}_n) = \frac{1}{(1-q^2)\cdots(1-q^{2n-2})(1-q^n)} = \frac{1}{(2n-2)_q!!(n)_q(1-q)^n}.$$

We notice in particular that for $W \in \{S_n, \mathrm{BC}_n, \mathrm{D}_n\}$ we have that $\mathrm{rk}_{q,\mathbf{Z}} \mathrm{Pol}_n \,/\, \mathrm{rk}_{q,\mathbf{Z}} \mathrm{Pol}_n^W = \mathrm{ord}_q W$. In fact, one has even $\mathrm{rk}_{q,\mathbf{Z}}\big(\mathrm{Pol}_n \,/\, \mathrm{Pol}_n^W\big) = \mathrm{ord}_q W$ for general finite reflection groups, see [Che55]. Even more is true: namely, Pol_n is a free $\Lambda\mathrm{Pol}_n$-module of graded rank $(q)_q!$ (resp. $\mathrm{ord}_q W$ for other finite reflection groups W) [Dem73, thm. 6.2]. We shall show a respective statement in the Clifford set-up for $W = S_n$ in Theorem 8.23.

Remark 8.9. It is clear that the same statements hold true when replacing $\mathbf{Z}$ by a field k by applying $k \otimes_{\mathbf{Z}} -$ to each of the abelian groups considered so far.

Corollary 8.10. We have seen in Lemma 7.4 that $\mathrm{Pol}\mathfrak{C}_n \cong \mathfrak{C}_n \otimes_k \mathrm{Pol}_n$ as left $\mathfrak{C}_n$-module and $\mathrm{Pol}\mathfrak{C}_n \cong \mathrm{Pol}_n \otimes_k \mathfrak{C}_n$ as right $\mathfrak{C}_n$-module. We endow $\mathrm{Pol}\mathfrak{C}_n$ with a (polynomial) grading such that the odd generators $\mathfrak{c}_i$ are of degree zero. Lemma 8.7 then shows that

$$\mathrm{rk}_{q,\mathfrak{C}_n} \mathrm{Pol}\mathfrak{C}_n = \frac{1}{(1-q)^n}, \quad \mathrm{rk}_{q,\mathfrak{C}_n} \Lambda\mathrm{Pol}\mathfrak{C}_n = \frac{1}{(n)_q!(1-q)^n}, \quad \mathrm{rk}_{q,\mathfrak{C}_n} \mathsf{NH}\mathfrak{C}_n = \frac{(n)_q!}{(1-q)^n}$$

both as left and right $\mathfrak{C}_n$-modules.

8.2. Elementary $\mathfrak{d}$-symmetric polynomials

We want to find an analogue of the elementary symmetric polynomials for $\Lambda\mathrm{Pol}\mathfrak{C}_n$. These are expected to coincide with the ordinary elementary symmetric polynomials if all indices are even.

Lemma 8.11. The operator $\mathfrak{d}_i$ on Clifford-polynomials has as kernel the $\mathfrak{C}_n$-subalgebra

$$\ker \mathfrak{d}_i = \langle \gamma_{i,i+1} y_i + \gamma_{i+1,i} y_{i+1},\ y_i y_{i+1},\ y_j \mid j \neq i, i+1 \rangle \tag{8.3}$$

of $\mathrm{Pol}\mathfrak{C}_n$, where we define

$$\gamma_{i,i\pm 1} := \begin{cases} \pm\mathfrak{c}_i & \text{if } i, i\pm 1 \text{ both are of odd parity,} \\ 1 & \text{otherwise.} \end{cases} \tag{8.4}$$

In the following, we stick to the convention that $0^0 := 1$ so that we can simply write $\gamma_{i,i\pm 1} := (\pm\mathfrak{c}_i)^{|i||i\pm 1|}$. Note that always $\gamma_{i,i\pm 1}^2 = 1$.

Proof. Recall that in the purely even case the kernel of the Demazure operator Definition 5.7 is the subalgebra $\ker \partial_i = \langle y_i + y_{i+1}, y_i y_{i+1}, y_j \mid j \neq i, i+1 \rangle \subseteq \mathrm{Pol}_n$. In the super-setting, we have already seen that $\ker \mathfrak{d}_i = \mathrm{im}\, \mathfrak{d}_i \subseteq \mathrm{Pol}\mathfrak{C}_n$ is a $\mathfrak{C}_n$-subalgebra in Lemma 7.7. Assume w. l. o. g. that $i = 1$ and $n = 2$, so the only indeterminates are y_1, y_2 and $\mathfrak{c}_1, \mathfrak{c}_2$. Let $\Lambda := \langle y_1 y_2,\ \gamma_{1,2} y_1 + \gamma_{2,1} y_2 \rangle$ as $\mathfrak{C}_2$-algebra. We have to show that $\Lambda = \ker \mathfrak{d}_1$ as subalgebras of $\mathrm{Pol}\mathfrak{C}_2$.

($\subseteq$) In general, $y_1 y_2 \in \ker \mathfrak{d}_1$ since

$$\mathfrak{d}_1(y_1 y_2) = (-1 - \mathfrak{c}_1\mathfrak{c}_2) y_2 + y_2(1 - \mathfrak{c}_1\mathfrak{c}_2) = 0.$$

Furthermore, we have $\mathfrak{d}_1(y_1) y_1 - \mathfrak{d}_1(y_2) y_2 \in \ker \mathfrak{d}_1$ since

$$\mathfrak{d}_1\big(\mathfrak{d}_1(y_1) y_1 - \mathfrak{d}_1(y_2) y_2\big) = -\mathfrak{d}_1(y_2)\mathfrak{d}_1(y_1) + \mathfrak{d}_1(y_1)\mathfrak{d}_1(y_2) = 0.$$

Thus $\ker \mathfrak{d}_1$ also contains the polynomial[15]

$$\begin{aligned}
&\big(-\tfrac{1}{2}(\mathfrak{c}_1-\mathfrak{c}_2)\big)^{|1||2|}\,(\mathfrak{d}_1(y_1)y_1-\mathfrak{d}_1(y_2)y_2)\\
&=\big(-\tfrac{1}{2}(\mathfrak{c}_1-\mathfrak{c}_2)\big)^{|1||2|}\big((-1-\mathfrak{c}_1\mathfrak{c}_2)y_1-(1-\mathfrak{c}_1\mathfrak{c}_2)y_1\big)\\
&=\begin{cases}\mathfrak{c}_1y_1-\mathfrak{c}_2y_2 & \text{if both 1, 2 are odd,}\\ y_1+y_2 & \text{otherwise}\end{cases}\\
&=\gamma_{1,2}y_1+\gamma_{2,1}y_2.
\end{aligned}$$

($\supseteq$) The algebra $\mathrm{Pol}\mathfrak{C}_2$ contains an element $\alpha_1 \coloneqq y_1-y_2$ which satisfies $\mathfrak{d}_1(\alpha_1)=-2$. Recall that $\mathrm{Pol}\mathfrak{C}_2$ is a free left and right $\mathfrak{C}_2$-module of graded rank $\mathrm{rk}_{q,\mathfrak{C}_2}\mathrm{Pol}\mathfrak{C}=\frac{1}{(1-q)^2}$. Its subalgebra Λ is also $\mathfrak{C}_2$-free of graded rank $\mathrm{rk}_{q,\mathfrak{C}_2}\Lambda=\frac{1}{1-q}\cdot\frac{1}{1-q^2}$. The $\mathfrak{C}_2$-submodule $\Lambda\alpha_1\subseteq\mathrm{Pol}\mathfrak{C}$ does not intersect $\ker\mathfrak{d}_1$ since for any $\lambda\in\Lambda$ non-zero, we have $\mathfrak{d}_1(\lambda\alpha_1)=-2s_1(\lambda)\neq 0$. It has graded rank $\mathrm{rk}_{q,\mathfrak{C}_2}\Lambda\alpha_1=q\,\mathrm{rk}_{q,\mathfrak{C}_2}\Lambda$. Since

$$\mathrm{rk}_{q,\mathfrak{C}}\Lambda+\mathrm{rk}_{q,\mathfrak{C}_2}\Lambda\alpha_1=\frac{1+q}{(1+q)(1-q^2)}=\frac{1}{(1-q)^2}=\mathrm{rk}_{q,\mathfrak{C}_2}\mathrm{Pol}\mathfrak{C}_2,$$

we obtain that $\mathrm{Pol}\mathfrak{C}_2=\Lambda\oplus\Lambda\alpha_s$ as $\mathfrak{C}_2$-module and in particular that the inclusion $\Lambda\subseteq\ker\mathfrak{d}_1$ in fact is an equality. □

Let $\gamma_{n,n+1}\coloneqq 1$ if $n+1$ exceeds the number of indeterminates. Since

$$y_iy_{i+1}=\gamma_{i+1,i+2}\gamma_{i,i+1}\big(\gamma_{i+1}\gamma_{i+1,i+2}\gamma_{i+1,i+2}\,y_iy_{i+1}\big) \tag{8.5}$$

$$=(-1)^{|i||i+1|+|i+1||i+2|}\big(y_iy_{i+1}\,\gamma_{i+1,i+2}\gamma_{i+1,i+2}\big)\gamma_{i+1,i+2}\gamma_{i,i+1} \tag{8.6}$$

we may replace the second generator by $\gamma_{i,i+1}y_1\,\gamma_{i+1,i+2}y_{i+1}$ without changing the $\mathfrak{C}_n$-span.

Definition 8.12. We denote the generators of $\ker\mathfrak{d}_i$ by

$$\mathfrak{e}_1^{(i-1,i+1)}\coloneqq\gamma_{i,i+1}y_i+\gamma_{i+1,i}y_{i+1};\qquad \mathfrak{e}_2^{(i-1,i+1)}\coloneqq\gamma_{i,i+1}y_1\,\gamma_{i+1,i+2}y_{i+1}.$$

If $i=1$ we just write $\mathfrak{e}_m^{(2)}\coloneqq\mathfrak{e}_m^{(i-1,i+1)}$ so that we can write $\ker\mathfrak{d}_1=\langle\mathfrak{e}_1^{(2)},\mathfrak{e}_2^{(2)}\rangle\subseteq\mathrm{Pol}\mathfrak{C}_2$ as $\mathfrak{C}_2$-algebra.

Our goal is to show that there are polynomials $\mathfrak{e}_m^{(n)}$, $1\le m\le n$ of polynomial degree m which generate the intersection $\bigcap_{k=1}^{n-1}\ker\mathfrak{d}_n$ of $\mathfrak{C}_n$-algebras. These are supposed to serve as a Clifford-replacement for the ordinary elementary symmetric polynomials.

Our first task is to find a recursive formula for polynomials lying in $\bigcap_{k=1}^{n-1}\ker\mathfrak{d}_n$. Proving that they are indeed generators necessitates some more work. We shall prove this in Theorem 8.23.

Before coming to the n-fold intersection $\bigcap_{k=1}^{n-1}\ker\mathfrak{d}_k$, we shall for a moment think about the intersection $\ker\mathfrak{d}_1\cap\ker\mathfrak{d}_2\subseteq\mathrm{Pol}\mathfrak{C}_3$ of just the first two kernels. We can multiply the first generator of $\ker\mathfrak{d}_2$ with the unit $\gamma_{2,1}\gamma_{2,3}$ from the left without changing its span. Thus

$$\begin{aligned}
\ker\mathfrak{d}_1\cap\ker\mathfrak{d}_2=\Big\langle\, &\overbrace{\gamma_{1,2}y_1+\rlap{$\underbrace{\phantom{\gamma_{2,1}y_2+\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\;y_0}}_{\gamma_{2,1}\gamma_{2,3}\,\mathfrak{e}_1^{(1,3)}}$}\gamma_{2,1}y_2}^{\mathfrak{e}_1^{(0,2)}}+\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\;y_0,\\
&\overbrace{\gamma_{1,2}\,y_1\;\;\gamma_{2,1}\,y_2+\rlap{$\underbrace{\phantom{\gamma_{1,2}\,y_1\;\;\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\,y_3+\gamma_{2,1}\,y_2\;\;\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\,y_3}}_{\mathfrak{e}_1^{(0,2)}\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\,y_3}$}\gamma_{1,2}\,y_1\;\;\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\,y_3}^{\gamma_{1,2}\,y_1\,\gamma_{2,1}\gamma_{2,3}\,\mathfrak{e}_1^{(1,3)}}+\gamma_{2,1}\,y_2\;\;\gamma_{2,1}\gamma_{2,3}\,\gamma_{3,2}\,y_3,\\
&\overbrace{\gamma_{1,2}\,y_1\;\;\gamma_{2,3}\,y_2}^{\mathfrak{e}_2^{(0,2)}}\,\gamma_{3,4}\,y_3\,\Big\rangle
\end{aligned} \tag{8.7}$$

as a $\mathfrak{C}_n$-algebra. This depiction serves as a template for the following lemma:

---

[15] It seems more natural to put all Clifford-generators on the right since $\mathfrak{d}_1$ is $\mathfrak{C}_n$-right linear. We shall stick to putting them on the left though in order to preserve compatibility with the notation employed in [KKT16].

Table 8.1: Elementary $\mathfrak{d}$-symmetric polynomials $\mathfrak{e}_m^{(n)}$ as defined in Lemma 8.13, spelled out explicitly for $m \leq n \leq 4$.

$m\backslash n$	1	2	3	4
1	$\gamma_{1,2}y_1$	$\gamma_{1,2}\, y_1$ $+\, \gamma_{2,1}\, y_2$	$\gamma_{1,2}\, y_1$ $+\, \gamma_{2,1}\, y_2$ $+\, \gamma_{2,1}\gamma_{2,3}\gamma_{3,2}\ y_3$	$\gamma_{1,2}\, y_1$ $+\, \gamma_{2,1}\, y_2$ $+\, \gamma_{2,1}\gamma_{2,3}\gamma_{3,2}\ y_3$ $+\, \gamma_{2,1}\gamma_{2,3}\ \gamma_{3,2}\gamma_{3,4}\gamma_{4,3}\ y_4$
2		$\gamma_{1,2}, y_1\ \gamma_{2,3}\, y_2$	$\gamma_{1,2}\, y_1\ \gamma_{2,3}\, y_2$ $+\, \gamma_{1,2}\, y_1\ \gamma_{3,2}\, y_3$ $+\, \gamma_{2,1}\, y_2\ \gamma_{3,2}\, y_3$	$\gamma_{1,2}\, y_1\ \gamma_{2,3}\, y_2$ $+\, \gamma_{1,2}\, y_1\ \gamma_{3,2}\, y_3$ $+\, \gamma_{2,1}\, y_2\ \gamma_{3,2}\, y_3$ $+\, \gamma_{1,2}\, y_1\ \gamma_{3,2}\gamma_{3,4}\gamma_{4,3}\, y_4$ $+\, \gamma_{2,1}\, y_2\ \gamma_{3,2}\gamma_{3,4}\gamma_{4,3}\ y_4$ $+\, \gamma_{2,1}\gamma_{2,3}\gamma_{3,2}\, y_3\ \gamma_{3,2}\gamma_{3,4}\ \gamma_{4,3}\ y_4$
3			$\gamma_{1,2}\, y_1\, \gamma_{2,3}\, y_3\, \gamma_{3,4}\, y_3$	$\gamma_{1,2}\, y_1\ \gamma_{2,3}\, y_2\ \gamma_{3,4}\, y_3$ $+\, \gamma_{1,2}\, y_1\ \gamma_{2,3}\, y_2\ \gamma_{4,3}\, y_4$ $+\, \gamma_{1,2}\, y_1\ \gamma_{3,2}\, y_3\ \gamma_{4,3}\, y_4$ $+\, \gamma_{2,1}\, y_2\ \gamma_{3,2}\, y_3\ \gamma_{4,3}\, y_4$
4				$\gamma_{1,2}\, y_1\ \gamma_{2,3}\, y_2\ \gamma_{3,4}\, y_3\ \gamma_{4,5}\, y_4$

Table 8.2: Ordinary elementary symmetric polynomials $\varepsilon_m^{(n)}$ for $m \leq n \leq 4$. They are a specialisation of the ones listed in Table 8.1 at $\gamma_{i,i\pm1} = 1$ for all i, obtained when all i have *even* parity.

$m\backslash n$	1	2	3	4
1	y_1	$y_1 + y_2$	$y_1 + y_2 + y_3$	$y_1 + y_2 + y_3 + y_4$
2		y_1y_2	$y_1y_2 + y_1y_3 + y_2y_3$	$y_1y_2 + y_1y_3$ $+\, y_1y_4 + y_2y_3$ $+\, y_2y_4 + y_3y_4$
3			$y_1y_2y_3$	$y_1y_2y_3 + y_1y_2y_4$ $+\, y_1y_3y_4 + y_2y_3y_4$
4				$y_1y_2y_3y_4$

Lemma 8.13. The recursively defined polynomials

$$\begin{aligned}
\mathfrak{e}_1^{(1)} &= \gamma_{1,2}y_1 \\
&\vdots \\
\mathfrak{e}_1^{(n)} &= \mathfrak{e}_1^{(n-1)} + (\gamma_{2,1}\gamma_{2,3}\ \gamma_{3,2}\gamma_{3,4}\cdots\gamma_{n-1,n-2}\gamma_{n-1}n)\gamma_{n,n-1}\ y_n, \\
\mathfrak{e}_1^{(n)}2 &= \mathfrak{e}_1^{(n-1)} + \mathfrak{e}_1^{(n)}(\gamma_{3,2}\gamma_{3,4}\cdots\gamma_{n-1,n-2}\gamma_{n-1}n)\gamma_{n,n-1}\ y_n, \\
&\vdots \\
\mathfrak{e}_m^{(n)} &= \mathfrak{e}_m^{(n-1)} + \mathfrak{e}_{m-1}^{(n-1)}(\gamma_{m+1,m}\gamma_{m+1,m+2}\cdots\gamma_{n-1,n-2}\gamma_{n-1}n)\gamma_{n,n-1}\ y_n, \\
\mathfrak{e}_n^{(n)} &= \mathfrak{e}_n^{(n-1)}\ \gamma_{n+1,n+2}\ y_n.
\end{aligned} \tag{8.8}$$

are contained in the common kernel $\bigcap_{k=1}^{n-1} \ker \mathfrak{d}_k \subseteq \mathrm{Pol}\mathfrak{C}_n$ of the Clifford Demazure operators. We shall refer to these polynomials as *elementary* $\mathfrak{d}$*-symmetric polynomials* of degree m in n indeterminates. We define the $\mathfrak{C}_n$-algebra $\Lambda\mathrm{Pol}\mathfrak{C}_n := \mathfrak{C}_n\langle \mathfrak{e}_1^{(n)}, \ldots, \mathfrak{e}_n^{(n)}\rangle$. By the statement, this is a subalgebra of $\bigcap_{k=1}^{n-1} \ker \mathfrak{d}_n$.

Example 8.14. Before giving the proof of the lemma we make the definition of the $\mathfrak{e}$'s explicit for small numbers of indeterminates in particular for purely even and purely odd indices.

(i) For $n = 1, \ldots, 4$ variables, the elementary $\mathfrak{d}$-symmetric polynomials are listed explicitly in Table 8.1.
(ii) If all indices are even, then all $\gamma_{i,i\pm1} = 1$. In this case the $\mathfrak{d}$-elementary symmetric polynomials from Table 8.1 specialise to those from Table 8.2 at $\gamma_{i,i\pm1} = 1$, which we recognise at the ordinary elementary symmetric polynomials $\varepsilon_m^{(n)}$. The induction formula from Lemma 8.13 indeed gives

$$\begin{aligned}
\varepsilon_1^{(1)} &= x_1 \\
\varepsilon_m^{(n)} &= \varepsilon_m^{(n-1)} + \varepsilon_{m-1}^{(n-1)}x_n
\end{aligned} \tag{8.9}$$

Table 8.3: Odd elementary symmetric polynomials $\varepsilon_m^{(n)}$ for $m \leq n \leq 4$. They are a specialisation of the ones listed in Table 8.1 at $\gamma_{i,i\pm 1} = \pm 1$ for all i, obtained when all i have *odd* parity.

$m\backslash n$	1	2	3	4
1	x_1	$x_1 - x_2$	$x_1 - x_2 + x_3$	$x_1 - x_2 + x_3 - x_4$
2		x_1x_2	$x_1x_2 - x_1x_3$ $+ x_2x_3$	$x_1x_2 - x_1x_3 + x_2x_3$ $+ x_1x_4 - x_2x_4 - x_3x_4$
3			$x_1x_3x_3$	$x_1x_3x_3 - x_1x_2x_3$ $+ x_1x_3x_4 - x_2x_3x_3$
4				$x_1x_3x_3x_4$

by setting each of the γ's in (8.8) to 1. This is a well-known recursive formula for the ordinary elementary symmetric polynomials

$$\varepsilon_m^{(n)} = \sum_{1\leq k_1 < \cdots < k_m \leq n} x_{k_1} \cdots x_{k_m}. \tag{8.10}$$

(iii) If all indices are odd, we have $\gamma_{i,i\pm 1} = \pm \mathfrak{c}_i$. We set $x_i := (\mathfrak{c}_i)^{|i|} y_i$ (cf. Remark 8.3 and Lemma 7.14), thus in this case $\gamma_{i,i+1} y_i = x_i$. The polynomials from Table 8.1 then specialise at $\gamma_{i,i\pm 1} = \pm \mathfrak{c}_i$ to ones listed in Table 8.2. We call these the *odd elementary symmetric* polynomials, denoted $o\varepsilon_m^{(n)}$. The recursion formula reduces to

$$\begin{aligned} o\varepsilon_1^{(1)} &= x_1 \\ o\varepsilon_m^{(n)} &= o\varepsilon_m^{(n-1)} + (-1)^{n-m} o\varepsilon_{m-1}^{(n-1)}\, x_n \\ o\varepsilon_n^{(n)} &= o\varepsilon_{m-1}^{(n-1)}\, x_n \end{aligned} \tag{8.11}$$

Consider also the odd elementary symmetric polynomials

$$_{\mathrm{EKL}}o\varepsilon_m^{(n)} := \sum_{1\leq k_1 < \cdots < k_m \leq n} (-1)^{k_m - 1} x_{k_1} \cdots (-1)^{k_1 - 1} x_{k_m} \tag{8.12}$$

as defined in [EKL14, (2.21–2.23)] which admit the recursion formula

$$_{\mathrm{EKL}}o\varepsilon_m^{(n)} = {}_{\mathrm{EKL}}o\varepsilon_m^{(n-1)} + (-1)^{n-1}\, {}_{\mathrm{EKL}}o\varepsilon_m^{(n-1)} x_n.$$

Our odd elementary symmetric polynomials indeed coincide with those from [EKL14] up to an overall sign $(-1)^{\frac{m(m-1)}{2}}$:

$$\begin{aligned} o\varepsilon_m^{(n)} &= o\varepsilon_m^{(n-1)} + (-1)^{n-m} o\varepsilon_{m-1}^{(n-1)}\, x_n \\ &= (-1)^{\frac{m(m-1)}{2}} \cdot {}_{\mathrm{EKL}}o\varepsilon_m^{(n-1)} + (-1)^{\left(\frac{(m-1)(m-2)}{2}\right) + (n-m)} \cdot {}_{\mathrm{EKL}}o\varepsilon_m^{(n-1)}\, x_n \\ &= (-1)^{\frac{m(m-1)}{2}} \cdot {}_{\mathrm{EKL}}o\varepsilon_m^{(n-1)} + (-1)^{n-1} x_n \cdot {}_{\mathrm{EKL}}o\varepsilon(n-1)_m \\ &= (-1)^{\frac{m(m-1)}{2}} \cdot {}_{\mathrm{EKL}}o\varepsilon_m^{(n)}. \end{aligned} \tag{8.13}$$

Note that in our sign convention the first monomial $x_1 \cdots x_m$ of the odd polynomials always has +1 as coefficient. Odd symmetric functions have already been treated extensively in [EKL14; KE12; EQ16].

Proof of Lemma 8.13. The statement is proven by induction on n and m. It is convenient to set $\mathfrak{e}_0^{(n)} = 1$ for all $n \geq 0$. The claim then trivially holds for $n = 1$ and $m = 0, 1$.

—*Induction step on n for m* = 1: Assume that $\mathfrak{e}_1^{(k)} \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_{n-1}$ for all $k \leq n$. We want to show that $\ker \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_n$ contains the polynomial

$$\mathfrak{e}_1^{(n+1)} = \mathfrak{e}_1^{(n)} + (\gamma_{2,1}\gamma_{2,3} \cdots \gamma_{n,n-1}\gamma_{n-1,n})\gamma_{n+1,n}\, y_{n+1}.$$

The first summand $\mathfrak{e}_1^{(n)}$ is contained in $\ker \mathfrak{d}_1 \cap \cdots \cap \ker \mathfrak{d}_{n-1}$ by the induction hypothesis. The second summand contains no y_k for $k \leq n$ and thus lies in $\ker \mathfrak{d}_1 \cap \cdots \cap \ker \mathfrak{d}_{n-1}$ as well (cf. the definition of $\mathfrak{d}$ in (7.1)). To show that $\mathfrak{e}_1^{(n+1)} \in \ker \mathfrak{d}_n$, expand the first summand once more:

$$\mathfrak{e}_1^{(n+1)} = \mathfrak{e}_1^{(n-1)} + \big(\gamma_{2,1} \cdots \gamma_{n-1,n-2}\gamma_{n-1,n}\big)\gamma_{n,n-1}\, y_n + \big(\gamma_{2,1} \cdots \gamma_{n,n-1}\gamma_{n,n+1}\big)\gamma_{n+1,n}\, y_{n+1};$$

and regroup:

$$= \mathfrak{e}_1^{(n-1)} + \underbrace{\gamma_{2,1} \cdots \gamma_{n,n-1}\gamma_{n-1,n}\ \gamma_{n,n-1}}_{\in \mathfrak{C}_n^*} \underbrace{\big(y_n + \gamma_{n,n+1}\gamma_{n+1,n}\ y_{n+1}\big)}_{=\gamma_{n,n+1}\mathfrak{e}_1^{(n-1,n+1)}}.$$

The first summand $\mathfrak{e}_1^{(n-1)}$ clearly is contained in $\ker \mathfrak{d}_n$ since it contains no y_n, y_{n+1}. The second summand lies in the $\mathfrak{C}_n$-algebra $\big\langle \mathfrak{e}_1^{(n-1,n+1)} \big\rangle \subseteq \ker \mathfrak{d}_n$ (see Lemma 8.11 and Definition 8.12). Hence we have $\mathfrak{e}_1^{(n+1)} \in \ker \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_n$.

—*Induction step for* $m > 1$: Assume $\mathfrak{e}_m^{(k)} \in \ker \mathfrak{d}_1 \cap \cdots \cap \ker \mathfrak{d}_{n-1}$ for all $k \leq n$ and $m \leq k$. We want to show that $\ker \mathfrak{d}_1 \cap \cdots \cap \ker \mathfrak{d}_n$ contains

$$\mathfrak{e}_m^{(n+1)} = \mathfrak{e}_m^{(n)} + \mathfrak{e}_{m-1}^{(n)}\big(\gamma_{m+1,m}\gamma_{m+1,m+2} \cdots \gamma_{n,n-1}\gamma_{n-1,n}\big)\gamma_{n+1,n}\, y_{n+1}. \tag{8.14}$$

$\mathfrak{e}_m^{(n+1)} \in \mathfrak{d}_k$ *for* $k \leq n-1$: By the induction hypothesis, $\mathfrak{e}_m^{(n)} \in \ker \mathfrak{d}_k$. For the second summand,

$$\begin{aligned} &\mathfrak{d}_k\big(\underbrace{\mathfrak{e}_{m-1}^{(n)}}_{\in \ker \mathfrak{d}_k}\ \underbrace{\gamma_{m+1,m}\gamma_{m+1,m+2} \cdots \gamma_{n,n-1}\gamma_{n-1,n}}_{\in \mathfrak{C}_n^*}\gamma_{n+1,n}y_{n+1}\big) \\ &= s_i\big(\mathfrak{e}_{m-1}^{(n)} \cdot \gamma_{m+1,m}\gamma_{m+1,m+2} \cdots \gamma_{n,n-1}\gamma_{n-1,n}\gamma_{n+1,n}\big)\, \mathfrak{d}_k(y_{n+1}) \\ &= 0 \end{aligned}$$

by the induction hypothesis on $\mathfrak{e}_{m-1}^{(n)}$.

$\mathfrak{e}_m^{(n+1)} \in \mathfrak{d}_n$: For $\mathfrak{d}_n$, first expand both $\mathfrak{e}$'s in (8.14) once more by the recursion formula:

$$\begin{aligned} \mathfrak{e}_m^{(n+1)} = \Big[&\overbrace{\mathfrak{e}_m^{(n-1)} + \mathfrak{e}_{m-1}^{(n-1)}\big(\gamma_{m+1,m} \cdots \gamma_{n-1,n-2}\gamma_{n-1,n}\big)\gamma_{n,n-1}\, y_n}^{\mathfrak{e}_m^{(n)}}\Big] + \\ &+ \Big[\underbrace{\mathfrak{e}_{m-1}^{(n-1)} + \mathfrak{e}_{m-2}^{(n-1)}\big(\gamma_{m,m-1} \cdots \gamma_{n-1,n}\big)\gamma_{n,n-1}\, y_n\big)}_{\mathfrak{e}_{m-1}^{(n)}}\Big] \cdot \\ &\cdot \big(\gamma_{m+1,m} \cdots \gamma_{n,n-1}\gamma_{n,n+1}\big)\gamma_{n+1,n}\, y_{n+1} \end{aligned}$$

and regroup both terms containing $\mathfrak{e}_{m-1}^{(n-1)}$:

$$\begin{aligned} &= \mathfrak{e}_m^{(n-1)} + \mathfrak{e}_{m-2}^{(n-1)} \overbrace{\big(\gamma_{m,m-1} \cdots \gamma_{n-1,n}\big)\gamma_{n,n-1}\, y_n \big(\gamma_{m+1,m} \cdots \gamma_{n,n+1}\big)\gamma_{n+1,n-1}\, y_{n+1}}^{=\pm(\gamma_{m,m-1}\cdots\gamma_{n,n-1})(\gamma_{m+1,m}\cdots\gamma_{n+1,n})\overbrace{y_n y_{n+1}}^{=\gamma_{n+1,n+2}\gamma_{n,n+1}\,\mathfrak{e}_2^{(n-1,n+1)}}} + \\ &+ \mathfrak{e}_{m-1}^{(n-1)} \underbrace{\big(\cdots \gamma_{n-1,n-2}\gamma_{n-1,n}\ \gamma_{n,n-1}\big)}_{\in \mathfrak{C}_n^*} \underbrace{\big(\gamma_{n,n-1}y_n + \gamma_{n+1,n}y_{n+1}\big)}_{=\mathfrak{e}_1^{(n-1,n+1)}}. \end{aligned}$$

When applying the operator $\mathfrak{d}_n$, the terms $\mathfrak{e}_m^{(n-1)}$, $\mathfrak{e}_{m-2}^{(n-1)}$ and $\mathfrak{e}_{m-1}^{(n-1)}$ vanish since they contain no y_n, y_{n+1}. The second summand vanishes when applying $\mathfrak{d}_n$ to it since it is contained in the $\mathfrak{C}_n$-bimodule $\mathfrak{e}_{m-1}^{(n-1)} \cdot \langle \mathfrak{e}_2^{(n-1,n+1)} \rangle \subseteq \ker \mathfrak{d}_n$. The last summand vanishes when applying $\mathfrak{d}_n$ since it is contained in $\mathfrak{e}_{m-1}^{(n-1)} \cdot \langle \mathfrak{e}_2^{(n-1,n+1)} \rangle \subseteq \ker \mathfrak{d}_n$.

Thus $\mathfrak{e}_m^{(n+1)} \in \ker \mathfrak{d}_n$, and we already saw $\mathfrak{e}_m^{(n+1)} \in \ker \mathfrak{d}_1 \cap \cdots \cap \ker \mathfrak{d}_{n-1}$. This proves the assertion. ☐

Remark 8.15. The $\mathfrak{C}_n$-algebra $\Lambda\mathrm{Pol}\mathfrak{C}_n$ has a basis $\big\{(\mathfrak{e}_1^{(n)})^{\alpha_i}, \ldots, (\mathfrak{e}_n^{(n)})^{\alpha_n}\big\}$, $\alpha_i \in \mathbf{N}$, as left and as right $\mathfrak{C}_n$-module (cf. the proof of Lemma 7.4). With the same argumentation as in Lemma 8.7 we see that $\mathrm{rk}_{q,\mathfrak{C}_n} \Lambda\mathrm{Pol}\mathfrak{C}_n = \frac{1}{(n)_q!(1-q)^n}$ both as left and right $\mathfrak{C}_n$-module.

8.3. Invariance under Hecke-algebra action

Instead of knowing the (classical) elementary symmetric polynomials as the generators of the common kernel $\ker \partial_1 \cap \cdots \cap \ker \partial_n$, the reader is most likely to know the symmetric polynomials to be defined as the invariants $\mathrm{Pol}_n^{S_n}$, where the symmetric group acts by exchanging indeterminates. One thus may ask which action is "detected" by the $\mathfrak{d}_i$'s.

Another way to see the classical action of the symmetric group on polynomials is by letting s_i act by the commutator $[x_i, \partial_i]$ on Pol_n. Indeed, one verifies that this yields the correct action. Recall from Definition 2.13 the Iwahori-Hecke algebra $\mathsf{H}_q(S_n)$, which is a deformation of the group algebra $k[S_n]$. We shall employ henceforth the presentation from Remark 2.18, namely

$$\mathsf{H}_t(W) \cong \langle T_s \text{ for } s \in S \mid \text{distant and braid relations; } \forall i : T_i^2 = t + (t-1)T_i \rangle \tag{2.27}$$

as a $k[t^{\pm 1}]$ algebra.

Lemma 8.16 [APR00, §3]. The Hecke algebra $\mathsf{H}_t(S_n)$ acts on Pol_n by virtue of the *t-commutator* $[x_i, \partial_i]_t \coloneqq x_i\partial_i - t\partial_i x_i$. For the specialisation at $t = 1$, this yields the usual action of $k[S_n]$ by permutations.

Caveat 8.17. Unless for the special case $t = 1$, this is *not* an action by algebra homomorphisms but only by vector space homomorphisms.

Proof. We check that the t-commutators satisfy the relations of $\mathsf{H}_t(S_n)$.

distant relation: Clear since the ∂_i also satisfy the distant relation.

quadratic relation:

$$\begin{aligned}
[x_1, \partial_2]_t^2 &= (x_1\partial_1 - t\partial_1 x_1)^2 \\
&= x_1\partial_1 x_1\partial_1 - \underbrace{t x_1\partial_1\partial_1 x_1}_{0} - \partial_1 x_1^2 \partial_1 + t^2 \partial_1 x_1 \partial_1 x_1 \\
&= -x_1\partial_1 + t \underbrace{(x_1 + x_2)\partial_1}_{x_1\partial_1 + \partial_1 x_1 + 1} - t^2 \partial_1 x \\
&= t + (t-1)(x_1\partial_1 - t\partial_1 x_1).
\end{aligned}$$

braid relation: See [APR00, §3].

Let $t = 1$. It is clear that $[x_i, \partial_i]_t(x_j f) = x_j [x_i, \partial_i]_t(f)$ for $j \neq i, i+1$. Assume w.l.o.g. that $i = 1$. One then computes

$$\begin{aligned}
&(x_1\partial_1 - \partial_1 x_1)(x_1 f) \\
&= -x_1 f + x_1 x_2 \partial_1 f + x_1 f - x_2 \partial_1 x_1 f \\
&= x_2(x_1\partial_1 - \partial_1 x_1)(f),
\end{aligned}
\qquad
\begin{aligned}
&(x_1\partial_1 - \partial_1 x_1)(x_2 f) \\
&= x_1 f + x_1^2 \partial_1 f - x_1 f - x_1 \partial_1 x_1 f \\
&= x_1(x_1\partial_1 - \partial_1 x_1)(f).
\end{aligned}$$

Hence $[x_i, \partial_i]_t$ indeed acts on polynomials by the simple transposition s_i. ☐

The odd symmetric polynomials, albeit not invariant any more under the action of the symmetric group, are the invariants under the action of the Iwahori-Hecke algebra $\mathsf{H}_{-1}(S_n)$ at $t = -1$ [LR14, §2.4].

Question 8.18. Is there a (possibly multi-parameter) Iwahori-Hecke algebra acting on the $\mathfrak{C}_n$-algebra $\mathrm{Pol}\mathfrak{C}_n$ such that the invariants under this action are precisely the polynomials in $\Lambda\mathrm{Pol}\mathfrak{C}_n$?

We have not yet managed to answer this question. One might attempt to construct an algebra

$$\tilde{\mathsf{H}}_t(S_n) \cong \left\langle T_s \text{ for } s \in S \;\middle|\; \begin{matrix} \text{distant and braid relations;} \\ \forall i : T_i^2 = \left\{ \begin{smallmatrix} 1 & \text{if } i \text{ even} \\ -2T_i - 1 & \text{if } i \text{ odd} \end{smallmatrix} \right. \end{matrix} \right\rangle, \tag{8.15}$$

which would lie somewhere between $\mathsf{H}_1(S_n)$ and $\mathsf{H}_{-1}(S_n)$, depending on the partitioning of I, such that the $\mathfrak{d}$-symmetric polynomials are the invariants of an appropriate action of $\tilde{}_t(S_n)$. However, this necessitates further investigations which we shall not cover in this thesis.

Remark 8.19. It is interesting to note that the action by t-commutators allows to define new algebras such as the algebra of t-symmetric polynomials, which are the $\mathsf{H}_t(S_n)$-invariants in the polynomial ring with the commutativity relation $x_i x_j = t x_j x_i$ for $i > j$ [Rag17, §§5sq.]. We shall not pursue this further.

8.4. Schubert Polynomials; freeness of Pol$\mathfrak{C}_n$

Definition 8.20. For a reduced word $w \in S_n$ let the $\mathfrak{d}$-*Schubert polynomial* $\mathfrak{s}_w \in \mathrm{Pol}\mathfrak{C}_n$ be

$$\mathfrak{s}_w := \mathfrak{d}_{w^{-1}w_0}(y_1^{n-1} \cdots y_{n-1}^1), \tag{8.16}$$

which makes sense since according to Lemma 7.7 the $\mathfrak{d}_i$ satisfy the relations of the symmetric group. This is the definition of the ordinary Schubert polynomials [Man98, §2.3], with $\mathfrak{d}$ instead of ∂. Hence in the purely even case, the $\mathfrak{d}$-Schubert polynomials coincide with the ordinary ones.

Lemma 8.21. The $\mathfrak{d}$-Schubert polynomials have the following properties:

(i) $\mathfrak{d}_v \mathfrak{s}_w = \mathfrak{s}_{wv^{-1}}$ if wv^{-1} is a reduced expression.
(ii) For any tuple $\boldsymbol{\alpha} \in \mathbf{N}^n$, the following holds:

$$y^{\boldsymbol{\alpha}} \mathfrak{d}_v \mathfrak{s}_w \begin{cases} \in \mathfrak{C}_n^* y^{\boldsymbol{\alpha}} & \text{if } v = w^{-1}, \\ = 0 & \text{if } \ell(u) = \ell(w) \text{ but } v \neq w^{-1}, \\ = 0 & \text{if } \ell(v) > \ell(w). \end{cases} \tag{8.17}$$

In particular, $\mathfrak{s}_e$ is non-zero.

Proof. The first statement is clear from the definition of $\mathfrak{s}_w$. For the second one, we argue as follows:

$\ell(v) > \ell(w)$: Every $\mathfrak{d}_i$ reduces the polynomial degree by -1. Recall that the longest element w_0 of the symmetric group has $\ell(w_0) = \frac{n(n-1)}{2} = \deg(y_1^{n-1} \cdots y_{n-1}^1)$. Thus, $\mathfrak{s}_w$ has polynomial degree $\ell(w)$. Since $\mathfrak{d}_i$ acts on constants by zero, the assertion follows.

$\ell(v) = \ell(w), v \neq w^{-1}$: In this case $v(w^{-1}w_0)$ is not a reduced expression, which implies $\mathfrak{d}_v \mathfrak{d}_{w^{-1}w_0} = 0$.

$v = w^{-1}$: Take the reduced expression $w_0 = s_{n-1} \cdots s_2 s_1 s_{n-2} \cdots s_3 s_2 s_1 s_2 s_1$. We have

$$\begin{aligned} &\mathfrak{d}_{w_0}(y_1^{n-1} \cdots y_{n-1}^1) \\ &= \mathfrak{d}_{w_0 s_1} \mathfrak{d}_1 (y_1\, y_2 y_1\, y_3 y_2 y_1\, y_4 y_3 y_2 y_1 \cdots y_{n-1} y_{n-2} \cdots y_1) \\ &= \mathfrak{d}_{w_0 s_1} \left[\mathfrak{d}_1(y_1)\, (y_1\, y_2 y_1 y_3 y_2 y_1 \cdots) + y_2 \mathfrak{d}(y_1\, y_2 y_1\, y_3 y_2 y_1 \cdots) \right]. \end{aligned}$$

The second summand vanishes since in the argument of $\mathfrak{d}_1$ the indeterminates y_1, y_2 only occur in products $y_1 y_2$, which lie in $\ker \mathfrak{d}_1$ (see Lemma 8.11). We continue with the next operators. It turns out that in every step we can apply $\mathfrak{d}_i$ to precisely one factor y_i since the remaining y_i, y_{i+1}'s occur pairwise:

$$\begin{aligned} &= \mathfrak{d}_{w_0 s_1 s_2} s_2 (\mathfrak{d}_1 y_1) \left[(\mathfrak{d}_2 y_2)\, (y_1\, y_3 y_2 y_1 \cdots) + \overbrace{\mathfrak{d}_2 (y_1\, y_3 y_2 y_1 \cdots)}^{0} \right] \\ &= \mathfrak{d}_{w_0 s_1 s_2 s_1} s_1 s_2 (\mathfrak{d}_1 y_1)\, s_1 (\mathfrak{d}_2 y_2)\, \left[\mathfrak{d}_1 (y_1)\, y_3 y_2 y_1 \cdots \right] \\ &\vdots \\ &= (w_0 s_1) \underbrace{(\mathfrak{d}_1 y_1)}_{\in \mathfrak{C}_n^*} (w_0 s_1 s_2) \underbrace{(\mathfrak{d}_2 y_2)}_{\in \mathfrak{C}_n^*} (w_0 s_1 s_2 s_1) \underbrace{(\mathfrak{d}_1 y_1)}_{\cdots} \cdots s_1 (\mathfrak{d}_2 y_2) \cdot \mathfrak{d}_1 (y_1). \end{aligned}$$

Since the permutation action of S_n preserves $\mathfrak{C}_n^*$, we obtain that $\mathfrak{s}_e \in \mathfrak{C}_n^*$ is non-zero. ☐

Lemma 8.22. The action of $\mathsf{NH}\mathfrak{C}_n$ (and hence the action of $\mathsf{NC}\mathfrak{C}_n$) on Pol$\mathfrak{C}_n$ is faithful.

Proof. We recall briefly the proof of [EKL14, prop. 2.11] which can be applied nearly literally. By the defining relations, it is clear that $\{y^{\boldsymbol{\alpha}} \mathfrak{d}_v\}_{\boldsymbol{\alpha}, v \in S_n}$ is a generating set of $\mathsf{NH}\mathfrak{C}_n$ as $\mathfrak{C}_n$-algebra. We show by induction on $\ell(v)$ that $y^{\boldsymbol{\alpha}} \mathfrak{d}_v \mathfrak{s}_w$ are linearly independent elements of the $\mathfrak{C}_n$-bimodule Pol$\mathfrak{C}_n$. This proves that the generating set $\{y^{\boldsymbol{\alpha}} \mathfrak{d}_v\}$ acts linearly independently in Pol$\mathfrak{C}_n$.

—*Induction base*: The only element of length 1 is e, and by (8.17) $y^{\boldsymbol{\alpha}} \mathfrak{d}_v \mathfrak{s}_e \neq 0$ only if $v = e$, and in this case $y^{\boldsymbol{\alpha}} \mathfrak{d}_v \in \mathfrak{C}_n y^{\boldsymbol{\alpha}}$. Thus $y^{\boldsymbol{\alpha}} \mathfrak{d}_e$ cannot be a linear combination of $\{y^{\boldsymbol{\alpha}'} \mathfrak{d}_v\}_{v > e}$.

—*Induction step*: Assume that $\{y^{\boldsymbol{\alpha}'} \mathfrak{d}_{v'}\}_{v' < v}$ is linearly independent for a fixed $v \in S_n$. Assume that $y^{\boldsymbol{\alpha}} \mathfrak{d}_v$ were a linear combination of $\{y^{\boldsymbol{\alpha}'} \mathfrak{d}_{v'}\}_{\ell(v') < \ell(v)}$. But $y^{\boldsymbol{\alpha}} \mathfrak{d}_v \mathfrak{s}_{v^{-1}} \in \mathfrak{C}_n^* y^{\boldsymbol{\alpha}}$ by the first case in (8.17), whereas a linear combination of $\{y^{\boldsymbol{\alpha}'} \mathfrak{d}_{v'}\}$ maps $\mathfrak{s}_{v^{-1}}$ either to zero or to a polynomial of degree strictly larger than 0. This is a contradiction, so there cannot be any linear relations between $y^{\boldsymbol{\alpha}} \mathfrak{d}_v$ and elements $y^{\boldsymbol{\alpha}'} \mathfrak{d}_{v'}$ for $\ell(v') < \ell(v)$. Assume now $y^{\boldsymbol{\alpha}} \mathfrak{d}_v$ were a non-trivial linear combination of $\{y^{\boldsymbol{\alpha}'} \mathfrak{d}_{v'}\}_{\ell(v') \geq \ell(v)}$. But by the second two cases in (8.17), this is also a contradiction. ☐

By virtue of Schubert polynomials, we may prove the following analogue of the respective theorem for ordinary symmetric polynomials. Recall that the respective Clifford-analogues $\Lambda\mathrm{Pol}\mathfrak{C}_n$ and $\Lambda\mathrm{Pol}\mathfrak{C}_n$ of Pol_n and $\Lambda\mathrm{Pol}_n$ have graded ranks $\mathrm{rk}_{q,\mathfrak{C}_n}\Lambda\mathrm{Pol}\mathfrak{C}_n = \frac{1}{(n)_q!(1-q)^n}$ and $\mathrm{rk}_{q,\mathfrak{C}_n}\mathrm{Pol}\mathfrak{C}_n = \frac{1}{(1-q)^n}$; see Remark 8.15 and Corollary 8.10.

Theorem 8.23. The following facts, which are well-known for the purely even and the purely odd case, remain true for the Clifford-setting:

(i) $\mathrm{Pol}\mathfrak{C}_n$ is a free $\Lambda\mathrm{Pol}\mathfrak{C}_n$-module of graded rank $(n)_q!$.
(ii) The inclusion $\Lambda\mathrm{Pol}\mathfrak{C}_n \subseteq \bigcap_{k=1}^{n-1}\ker\mathfrak{d}_n$ from Lemma 8.13 is in fact an equality.
(iii) The faithful action of $\mathsf{NH}\mathfrak{C}_n$ on $\mathrm{Pol}\mathfrak{C}_n$ gives an isomorphism $\mathsf{NH}\mathfrak{C}_n \xrightarrow{\cong} \mathrm{End}_{\Lambda\mathrm{Pol}\mathfrak{C}_n}(\mathrm{Pol}\mathfrak{C}_n)$.

Proof. Since Schubert polynomials behave completely analogously to the classical case (apart from $\mathfrak{C}_n$), the classical proof applies entirely. We nevertheless explain briefly the arguments to see this. For details, consider [Lau10, §3.2] for the even and [EKL14, prop. 2.15] for the odd case.

(i) Define the $\mathfrak{C}_n$-bimodule $\mathfrak{H}_n := \langle y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} \mid a_i \leq n-i\rangle_{\mathfrak{C}_n}$. This module has a basis given by the Schubert polynomials $\mathfrak{s}_w$; see the proof of [EKL14, prop. 2.12]. For every index i one may chose an exponent $0 \leq a_i \leq n-i$ which yields a generator $y_i^{a_i}$ of (polynomial) degree a_i. The submodule generated by y_i^* thus has graded rank $(n-i)_q$ and hence $\mathrm{rk}_{q,\mathfrak{C}_n}(\mathfrak{H}_n) = (n)_q!$. The very same proof as in the even (resp. odd) case shows that the multiplication

$$\Lambda\mathrm{Pol}\mathfrak{C}_n \otimes_{\mathfrak{C}_n} \mathfrak{H}_n \to \mathrm{Pol}\mathfrak{C}_n$$

is injective, and since both sides have the same graded $\mathfrak{C}_n$-rank $\frac{1}{(1-q)^n}$ (see (8.1) in Lemma 8.7), it is an isomorphism of $\mathfrak{C}_n$-modules, which exhibits $\mathrm{Pol}\mathfrak{C}_n$ to be a free $\Lambda\mathrm{Pol}\mathfrak{C}_n$-module of graded rank $(n)_q!$.

(ii) In (i) we have shown that $\mathrm{Pol}\mathfrak{C}_n$ has a $\Lambda\mathrm{Pol}\mathfrak{C}_n$-basis by Schubert polynomials $\mathfrak{s}_w$. Given a polynomial $p \in \mathrm{Pol}\mathfrak{C}_n$, we take a linear combination $\sum_{w\in S_n} q_w\mathfrak{s}_w$ for $q_w \in \Lambda\mathrm{Pol}\mathfrak{C}_n$ with q_w non-zero for some $w > e$. Assume the image

$$\mathfrak{d}_i(p) = \mathfrak{d}_i\Big(\sum_w \mathfrak{d}_i(q_w)\mathfrak{s}_w\Big) = \sum_w s_i(q_w)\mathfrak{s}_{s_iw}$$

under $\mathfrak{d}_i$ were zero. Since by assumption there was some non-zero q_w for $w > e$, there still is a non-zero term $s_i(p_w)\mathfrak{s}_{s_iw}$ in $\mathfrak{d}_i(p)$. But since $\mathfrak{d}_i\mathfrak{s}_v = \mathfrak{s}_{s_iv}$ if s_iv is a reduced expression by Lemma 8.21.(i), $\mathfrak{d}_i$ maps no other Schubert polynomial to $\mathfrak{s}_{s_iv}$ Therefore, $\mathfrak{d}_i(p)$ cannot be zero. Hence there are no $\Lambda\mathrm{Pol}\mathfrak{C}_n$-linear combinations of Schubert polynomials in $\mathrm{Pol}\mathfrak{C}_n$ not already contained in $\ker\mathfrak{d}_i$, which are annihilated by $\mathfrak{d}_i$. This proves the statement.

(iii) We have already seen in Lemma 8.22 that $\mathsf{NH}\mathfrak{C}_n$ acts faithfully on the free $\mathfrak{C}_n$-module $\mathrm{Pol}\mathfrak{C}_n$. This action is compatible with the $\Lambda\mathrm{Pol}\mathfrak{C}_n$-module structure from (i): as $\mathsf{NH}\mathfrak{C}_n = \mathsf{NC}\mathfrak{C}_n \otimes_{\mathfrak{C}_n} \mathrm{Pol}\mathfrak{C}_n$, we see that $\mathrm{Pol}\mathfrak{C}_n$ acts on itself, $\mathsf{NC}\mathfrak{C}_n$ acts trivially on $\Lambda\mathrm{Pol}\mathfrak{C}_n$ by the very definition of the latter and $\mathsf{NC}\mathfrak{C}_n$ indeed acts on $\mathfrak{H}_n$ since it acts by decreasing exponents. A graded rank computation

$$\begin{aligned}\mathrm{rk}_{q,\mathfrak{C}_n}\mathrm{End}_{\Lambda\mathrm{Pol}\mathfrak{C}_n}(\mathrm{Pol}\mathfrak{C}_n) &= \mathrm{rk}_{q,\mathfrak{C}_n}(\Lambda\mathrm{Pol}\mathfrak{C}_n)\cdot\mathrm{rk}_{q,\Lambda\mathrm{Pol}\mathfrak{C}_n}(\mathrm{Pol}\mathfrak{C}_n)^2\\ &= \frac{((n)_q!)^2}{(n)_q!(1-q)^n}\\ &= \mathrm{rk}_{q,\mathfrak{C}_n}\mathsf{NH}\mathfrak{C}_n\end{aligned}$$

shows the asserted isomorphism.

Note that [Lau10, prop. 3.5] gives a more explicit proof for surjectivity in the classical case. □

8.5. Complete symmetric polynomials

Just as in the classical case, we may give another collection of generators for $\Lambda\mathrm{Pol}\mathfrak{C}_n$, namely a Clifford-analogue for the complete symmetric polynomials. We beg the reader's pardon for the advent of a confusing amount of new notation.

Definition 8.24. Let $M_{(n)}$ be the $n \times n$-matrix

$$M_{(n)} \coloneqq \begin{pmatrix} \mathfrak{e}_1^{(n)} & \tilde{\kappa}_{1,2} & & & \\ -\mathfrak{e}_2^{(n)} & & \tilde{\kappa}_{1,3} & & \\ \vdots & & & \ddots & \\ (-1)^{n-2}\mathfrak{e}_{n-1}^{(n)} & & & & \tilde{\kappa}_{1,n} \\ (-1)^{n-1}\mathfrak{e}_n^{(n)} & & & & 0 \end{pmatrix}$$

where we set

$$\kappa_{k,l} \coloneqq \gamma_{k+1,k}\gamma_{k+1,k+2} \cdots, \gamma_{l-1,l-2}\gamma_{l-1,l}\,\gamma_{l,l-1},$$

for $k < l$, and $\kappa_{l-1,l} \coloneqq \gamma_{l,l-1}$, $\kappa_{l,l} \coloneqq \gamma_{l,l+1}$. Furthermore, let

$$\tilde{\kappa}_{k,l} \coloneqq \kappa_{k,l}\kappa_{l,l} = \gamma_{k+1,k}\gamma_{k+1,k+2} \cdots \gamma_{l-1,l-2}\gamma_{l-1,l}\,\gamma_{l,l-1}\gamma_{l,l+1}.$$

Note that we can "split" the κ's by

$$\kappa_{k,l} = \underbrace{\gamma_{k+1,k} \cdots \gamma_{m,m-1}}_{\kappa_{k,m}}\gamma_{m,m+1}\underbrace{\gamma_{m+1,m} \cdots \gamma_{l,l-1}}_{\kappa_{m,l}} = \tilde{\kappa}_{k,m}\kappa_{m,l}. \tag{8.18}$$

The *complete symmetric polynomial* $h_m^{(n)}$ of (polynomial) degree m on n indeterminates is the top left entry of the matrix power $M_{(n)}^m$ In particular, $h_1^{(n)} = \mathfrak{e}_1^{(n)}$ and $h_0^{(n)} = \mathfrak{e}_0^{(n)} = 1$.

Remark 8.25. Note that the top row of $M_{(n)}^m$ has entries

$$\big(M_{(n)}^m\big)_{1,*} = \big(h_m^{(n)},\ h_{m-1}^{(n)}\tilde{\kappa}_{1,2},\ h_{m-2}^{(n)}\tilde{\kappa}_{1,2}\tilde{\kappa}_{1,3}, \ldots, \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,m+1}, 0, \ldots, 0\big).$$

Example 8.26. (i) If all indices are even and thus $\mathfrak{e}_m^{(n)} = \varepsilon_m^{(n)}$ and $\kappa_{k,l} = 1$ for all k, l, these polynomials coincide with the ordinary complete symmetric polynomials

$$h_m^{(n)} = \sum_{1 \le k_1 \le \cdots \le k_m \le n} x_{k_1} \cdots x_{k_m}.$$

(ii) If all indices are odd, the resulting complete homogeneous symmetric polynomials coincide with those of [EKL14] up to a renormalisation involving the κ's.

Proposition 8.27. There are identities

$$\sum_{l=0}^{m}(-1)^l\, h_{m-l}^{(n)}\, \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,l}\mathfrak{e}_l^{(n)} = \delta_{m,0} = \sum_{l=0}^{m}(-1)^l\, \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,l}\mathfrak{e}_l^{(n)}\, h_{n-l}^{(n)}.$$

Proof. The complete homogeneous symmetric polynomial $h_m^{(n)}$ is the top left entry of the matrix power $M_{(n)}^m = M_{(n)}^{m-1} \cdot M_{(n)}$ and therefore equals

$$h_m^{(n)} = \Big(h_{m-1}^{(n)},\ h_{m-2}^{(n)}\tilde{\kappa}_{1,2}, \ldots, \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,m+1}, 0, \ldots, 0\Big) \cdot \Big(\mathfrak{e}_1^{(n)}, -\mathfrak{e}_2^{(n)}, \ldots, (-1)^n\mathfrak{e}_n^{(n)}\Big)^T;$$

hence one obtains a recursive description $h_m^{(n)} = \sum_{l=1}^m (-1)^{l-1} h_{m-l}^{(n)}\tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,l}\mathfrak{e}_l^{(n)}$ of the complete homogeneous symmetric polynomials for $m \geq 1$. Putting $h_m^{(n)}$ to the other side proves the statement.

For the second equality, denote the other entries of $M_{(n)}^m$ by

$$\begin{pmatrix} h_m^{(n)} & h_{m-1}^{(n)}\tilde{\kappa}_{1,2} & h_{m-2}^{(n)}\tilde{\kappa}_{1,2}\tilde{\kappa}_{1,3} & \cdots \\ h_{(1),m+1}^{(n)} & h_{(1),m}^{(n)}\tilde{\kappa}_{1,2} & \ddots & \\ h_{(2),m+2}^{(n)} & \ddots & & \end{pmatrix} \coloneqq M_{(n)}^m,$$

such that $h_{(k),l}^{(n)}$ always is a polynomial of degree l. The polynomial $h_m^{(n)}$ is the top left entry of $M_{(n)}^m = M_{(n)} \cdot M_{(n)}^{m-1}$ and thus is given by

$$h_m^{(n)} = \mathfrak{e}_1 h_{m-1}^{(n)} + \tilde{\kappa}_{1,2}h_{(1),m}^{(n)}$$

$$\begin{aligned} h^{(n)}_{(1),m} &= -\mathfrak{e}_2 h^{(n)}_{m-2} + \tilde{\kappa}_{1,3} h^{(n)}_{(2),m} \\ &\vdots \\ h^{(n)}_{(k-1),m} &= (-1)^k \mathfrak{e}_k h^{(n)}_{m-k} + \tilde{\kappa}_{1,k+1} h^{(n)}_{(k),m} \end{aligned}$$

such that we obtain a recursion formula

$$\begin{aligned} h^{(n)}_m &= \mathfrak{e}^{(n)}_1 h^{(n)}_{m-1} + \tilde{\kappa}_{1,2} \left(-\mathfrak{e}^{(n)}_2 h^{(n)}_{m-2} + \tilde{\kappa}_{1,3} \big(\cdots (\pm \mathfrak{e}^{(n)}_{m-1} h^{(n)}_1 \mp \tilde{\kappa}_{1,m} \mathfrak{e}^{(n)}_m) \cdots \big) \right) \\ &= \sum_{l=1}^{m} (-1)^{l-1} \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,l} \mathfrak{e}^{(n)}_l \, h^{(n)}_{m-l}. \end{aligned}$$

Again, putting $h^{(n)}_m$ to the other side proves the claim. ☐

8.6. Interlude: cohomology of Grassmannians and partial flag varieties

We recall some classical (non-super) theory on the cohomology rings of Grassmannians and partial flag varieties. Our aim is to build super-analogues for these rings.

Definition 8.28. Fix natural numbers $k \leq n$. The set of all k-dimensional subspaces $F \subseteq k^n$ of the n-dimensional k-vector space forms an algebraic variety, called *Grassmann variety* (or just *Grassmannian*), denoted $\mathrm{Gr}(k, n)$.

Fact 8.29. The Grassmannian $\mathrm{Gr}(k, n)$ has cohomology ring

$$H_{(k,n)} \coloneqq H^*(\mathrm{Gr}(k,n)) \cong \frac{k\big[\varepsilon^{(n)}_1, \ldots, \varepsilon^{(n)}_m\big]}{\big(h^{(n)}_{n-k+1}, \ldots, h^{(n)}_n\big)}. \tag{8.19}$$

with coefficients in k.

This is a corollary of **??** 8.31. We need to give some more definitions beforehand:

Definition 8.30. Fix natural number $\ell \leq n$. A *partial flag* $\boldsymbol{F}_\bullet$ of length ℓ is a chain $\boldsymbol{F}_\bullet$ of k-vector subspaces $0 \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_\ell = k^n$. The set of all partial flags of length ℓ is called *(partial) flag variety* $\mathrm{Fl}(\ell; n)$. If $\ell = n$, then $\mathrm{Fl}(n) \coloneqq \mathrm{Fl}(n; n)$ is called *full flag variety.* To a flag $\boldsymbol{F}_\bullet$ one associates its *dimension vector* $\boldsymbol{k} = (\dim_k F_0, \ldots, \dim_k F_\ell)$. The connected components of $\mathrm{Fl}(\ell)$ are the varieties $\mathrm{Fl}(\boldsymbol{k})$ of partial flags with dimension vector $\boldsymbol{k}$. In particular $\mathrm{Gr}(k, n) = \mathrm{Fl}(0, k, n)$.

Fact 8.31 [Man98, prop. 3.6.15, rmk. 3.6.16; or Ful97, prop. 3]. The full flag variety $\mathrm{Fl}(n)$ has cohomology ring $H^* \mathrm{Fl}(n) \cong \mathrm{Pol}_n / (\Lambda\mathrm{Pol}_n)_+$. The partial flag variety $\mathrm{Fl}(\boldsymbol{k})$ has cohomology ring

$$H_{\boldsymbol{k}} \coloneqq H^* \mathrm{Fl}(\boldsymbol{k}) \cong \frac{\Lambda\mathrm{Pol}_{k_1 - k_0} \otimes_k \cdots \otimes_k \Lambda\mathrm{Pol}_{k_\ell - k_{\ell-1}}}{(\Lambda\mathrm{Pol}_n)_+}. \tag{8.20}$$

We denote the nominator by $\Lambda\mathrm{Pol}_{\boldsymbol{k}}$.

Recall the notion of graded ranks from Section 8.1.

Lemma 8.32. The cohomology ring $H(k, n) \coloneqq H^*(\mathrm{Gr}(k, n))$ of the Grassmannian has graded dimension $\dim_{q,k} H(k, n) = \binom{n}{k}_q$ with the *q-binomial coefficient* $\binom{n}{k}_q \coloneqq \frac{(n)_q!}{(k)_q!(n-k)_q!}$.

Proof. Without quotienting out the ideal, the polynomial ring $\Lambda\mathrm{Pol}_k \coloneqq k[\varepsilon_1, \ldots, \varepsilon_k]$ in the symmetric polynomials has graded dimension $\dim_{q,k} \Lambda\mathrm{Pol}_k = (1-q)^{-1} \cdots (1-q^k)^{-1}$. To count the graded dimension of the ideal $\mathfrak{a} = (h_{n-k+1}, \ldots, h_n)$ to be quotiented out, we note that each generator h_m contributes a principal ideal with graded dimension $\dim_{q,k}(h_m) = q^m \dim_{q,k} \Lambda\mathrm{Pol}_k$. We then proceed by the inclusion-exclusion principle to obtain

$$\begin{aligned} &\dim_{q,k}(h_{n-k+1}, \ldots, h_n) \\ &= \Big(\big(q^{n-k+1} + \cdots q^n\big) - \big(q^{2(n-k)+3} + \cdots + q^{2n-1}\big) + \cdots \pm q^{(n-k+1)+\cdots+n} \Big) \dim_{q,k} \Lambda\mathrm{Pol}_k \\ &= \big(1 - (1-q^{n-k+1}) \cdots (1-q^n)\big) \dim_{q,k} \Lambda\mathrm{Pol}_k \,. \end{aligned}$$

The quotient thus has graded dimension

$$\dim_{q,k}(\Lambda\mathrm{Pol}_k / \mathfrak{a}) = \frac{(1-q^{n-k+1}) \cdots (1-q^n)}{(1-q) \cdots (1-q^k)} = \frac{(1-q)^{n-(n-k)} (n)_q! / (n-k)_q!}{(1-q)^k (k)_q!} = \binom{n}{k}_q. \qquad ☐$$

For the cohomology ring of partial flag varieties (see **??** 8.31) this allows to deduce:

Corollary 8.33. Let $\boldsymbol{k} \coloneqq (0 = k_0 \leq \ldots \leq k_r = n)$ be a dimension vector. The cohomology ring $H(\boldsymbol{k})$ of the partial flag variety $\mathrm{Fl}(\boldsymbol{k})$ has as graded dimension the *q-multinomial coefficient*

$$\dim_{q,k} H(\boldsymbol{k}) = \binom{n}{k_1, k_2 - k_1, \ldots, k_r - k_{r-1}}_q \coloneqq \frac{(n)_q!}{(k_1)_q!(k_2 - k_1)_q! \cdots (k_r - k_{r-1})_q!}.$$

Proof. The cohomology ring is the quotient

$$\frac{\Lambda\mathrm{Pol}_{k_1} \otimes \cdots \otimes \Lambda\mathrm{Pol}_{k_l - k_{l-1}}}{\Big(1 - \big(1 + \varepsilon_1^{(k_1)} + \cdots + \varepsilon^{(k_1)_{k_1}}\big) \cdots \big(1 + \varepsilon_1^{(k_l - k_{l-1})} + \cdots + \varepsilon_{(k_l - k_{l-1})}^{k_l - k_{l-1}}\big)\Big)}$$

of $\Lambda\mathrm{Pol}_{\boldsymbol{k}} \coloneqq \Lambda\mathrm{Pol}_{k_1} \otimes \cdots \otimes \Lambda\mathrm{Pol}_{k_l - k_{l-1}}$. Denote the ideal quotiented out by $\mathfrak{a}$ and proceed as in the proof of Lemma 8.32. We see that the nominator has graded dimension

$$\dim_{q,k} \Lambda = \prod_{a=1}^{l} \frac{1}{1-q} \cdots \frac{1}{1 - q^{k_a - k_{a-1}}} = \frac{1}{(1-q)^n} \prod_{a=1}^{l} \frac{1}{(k_a - k_{a-1})_q!},$$

and the ideal $\mathfrak{a}$ quotiented out has graded dimension

$$\dim_{q,k} \mathfrak{a} = \big((1-q)^n (n)_q! - 1\big) \dim_{q,k} \Lambda.$$

This shows the assertion. □

8.7. Cyclotomic quotients

It is known that in the classical case the cohomology ring $H_{(k,n)}$ of Grassmann varieties (see **??** 8.29) is Morita equivalent to the so-called cyclotomic quotient of the NilHecke algebra [Lau12, §5]. For the odd setting, the respective Morita equivalence has already been established in [EKL14, §5]. We introduce analogues of both rings in the Clifford setting and prove that they are Morita equivalent.

Definition 8.34. The *m-th cyclotomic quotient* of the Hecke Clifford algebra $\mathsf{NH}\mathfrak{C}_n$ is the quotient $\mathsf{NH}\mathfrak{C}_n^m \coloneqq \mathsf{NH}\mathfrak{C}_n \,/\big((\kappa_{1,n} y_n)^m\big)$. The quotient $H\mathfrak{C}_{(m,n)} \coloneqq \Lambda\mathrm{Pol}\mathfrak{C}_n \,/\big(h_k^{(n)}\big)_{k > n-m}$ is called the *Clifford cohomology ring* of the Grassmannian $\mathrm{Gr}(m,n)$.

Remark 8.35. In contrast to the standard definition of the cyclotomic quotient, we mod out some power of y_n instead of y_1. However, using y_1 instead would necessitate transposing the matrix $M_{(n)}$ in Definition 8.24. We want to avoid this.

Theorem 8.36. The rings $\mathsf{NH}\mathfrak{C}_n^m$ and $H\mathfrak{C}_{(m,n)}$ are Morita-equivalent.

Proof. The proof in [Lau12, §5] does not rely on commutativity of the rings involved and thus applies immediately in our setting. We quickly recall how to show the statement. The reader is strongly encouraged to verify the following calculation for $n = 3$. Let us start with

$$\begin{aligned}
\mathfrak{e}_n^{(n)} &= \gamma_{1,2} y_1 \, \gamma_{2,3} y_2 \cdots \gamma_{n,n+1} y_n \\
&= \big(\gamma_{1,2} y_1 \, \gamma_{2,3} y_2 \cdots \gamma_{n-1,n} y_{n-1}\big) \gamma_{n,n+1} y_n \\
&= \big(\mathfrak{e}_{n-1}^{(n)} - \tilde{\mathfrak{e}}_{n-1}^{(n)}\big) \underbrace{\gamma_{n,n+1}}_{\kappa_{n,n}} y_n
\end{aligned}$$

where we set

$$\begin{aligned}
\tilde{\mathfrak{e}}_{n-1}^{(n)} &= \gamma_{1,2} y_1 \cdots \gamma_{n-2,n-1} y_{n-2} \, \gamma_{n,n-1} y_n + \cdots + \gamma_{2,1} y_2 \, \gamma_{3,2} y_3 \cdots \gamma_{n,n-1} y_n \\
&= \big(\gamma_{1,2} y_1 \cdots \gamma_{n-2,n-1} y_{n-2} + \cdots + \gamma_{2,1} y_2 \cdots \gamma_{n-1,n-2} y_{n-1}\big) \gamma_{n,n-1} y_n \\
&= \big(\mathfrak{e}_{n-2}^{(n)} - \tilde{\mathfrak{e}}_{n-2}^{(n)}\big) \underbrace{\gamma_{n,n-1}}_{\kappa_{n-1,n}} y_n
\end{aligned}$$

where we set

$$\begin{aligned}
\tilde{\mathfrak{e}}_{n-2}^{(n)} &= \big(\gamma_{1,2} y_1 \cdots \gamma_{n-3,n-2} y_{n-3} + \cdots + \gamma_{3,2} y_3 \cdots \gamma_{n-1,n-2} y_{n-1}\big) \gamma_{n-1,n-2} \gamma_{n-1,n} \gamma_{n,n-1} y_n \\
&= \big(\mathfrak{e}_{n-3}^{(n)} - \tilde{\mathfrak{e}}_{n-3}^{(n)}\big) \underbrace{\gamma_{n-1,n-2} \gamma_{n-1,n} \gamma_{n,n-1}}_{\kappa_{n-2,n}} y_n
\end{aligned}$$

$$\begin{aligned}&\vdots\\ \tilde{\mathfrak{e}}_2^{(n)} &= \left(\mathfrak{e}_1^{(n)} - \tilde{\mathfrak{e}}_1^{(n)}\right)\underbrace{\gamma_{3,2}\gamma_{3,4}\cdots\gamma_{n-1,n-2}\gamma_{n-1,n}\gamma_{n,n-1}}_{\kappa_{2,n}}y_n\end{aligned}$$

where

$$\tilde{\mathfrak{e}}_1^{(n)} = \underbrace{\gamma_{2,1}\gamma_{1,2}\cdots\gamma_{n-1,n-2}\gamma_{n-1,n}\gamma_{n,n-1}}_{\kappa_{1,n}}\,y_n.$$

To state the above calculation differently, we have that

$$\begin{aligned}0 &= \mathfrak{e}_n^{(n)} - \left(\mathfrak{e}_{n-1}^{(n)} - \left(\cdots\left(\mathfrak{e}_1^{(n)} - \kappa_{1,n}y_n\right)\cdots\right)\kappa_{n-1,n}y_n\right)\kappa_{n,n}y_n\\ &= \sum_{k=0}^{n-1}(-1)^k\mathfrak{e}_{n-k}^{(n)}\kappa_{n-k+1,n}y_n\cdots\kappa_{n,n}y_n\\ &= \sum_{l=1}^{n}(-1)^l\mathfrak{e}_l^{(n)}\prod_{k=l+1}^{n}\kappa_{k,n}y_n.\end{aligned}$$

Set $b_l := \prod_{k=l+1}^{n}\kappa_{k,n}y_n$ for $1 \le l \le n-1$; in particular $b_n := 1$. Recall now the $\mathfrak{C}_n$-bimodule $\mathfrak{H}_n$ from the proof of Theorem 8.23.(i). We define a similar free $\mathfrak{C}_n$-bimodule of graded rank $(n)_q!$ with direct sum decomposition

$$\tilde{\mathfrak{H}}_n := \langle y_1^{\alpha_1}\cdots y_{n-1}^{\alpha_{n-1}} \mid \alpha_i < i\rangle_{\mathfrak{C}_n} = \bigoplus_{\boldsymbol{\alpha}}\langle y^{\boldsymbol{\alpha}}b_1,\ldots,y^{\boldsymbol{\alpha}}y_n^{n-1}\cdot b_n\rangle_{\mathfrak{C}_n} \tag{8.21}$$

for multiindices $\boldsymbol{\alpha}$. We denote the direct summands by $B_{\boldsymbol{\alpha}}$. It is analogous to the proof of Theorem 8.23.(i) that multiplication gives an isomorphism $\tilde{\mathfrak{H}}_n \otimes_{\mathfrak{C}_n} \Lambda\mathrm{Pol}\mathfrak{C}_n \xrightarrow{\cong} \mathrm{Pol}\mathfrak{C}_n$.

Recall from (8.18) how to split the κ's. Multiplication from the left by $\kappa_{1,n}y_n$ acts on this basis of $B_{\boldsymbol{\alpha}}$ by

$$\begin{aligned}&(\kappa_{1,n}y_n\cdot)\colon B_{\boldsymbol{\alpha}} \to B_{\boldsymbol{\alpha}},\\ 1 = b_n &\mapsto \kappa_{1,n}y_n &&\overset{(8.18)}{=} \kappa_{1,n}\kappa_{n,n}\kappa_{n,n}y_n &&= \overbrace{\kappa_{1,n}\kappa_{n,n}}^{=\tilde{\kappa}_{1,n}}b_{n-1}\\ b_{n-1} &\mapsto \kappa_{1,n}y_n\kappa_{n,n}y_nb_{n-1} &&= \kappa_{1,n-1}\kappa_{n-1,n-1}\kappa_{n-1,n}y_n &&= \underbrace{\kappa_{1,n-1}\kappa_{n-1,n-1}}_{=\tilde{\kappa}_{1,n-1}}b_{n-2}\\ &\vdots\\ b_2 &\mapsto \kappa_{1,n}y_nb_2 &&= \kappa_{1,2}\kappa_{2,2}y_nb_2 &&= \underbrace{\kappa_{1,2}\kappa_{2,2}}_{\substack{=\tilde{\kappa}_{1,2}\\ =\gamma_{2,1}\gamma_{2,3}}}b_1\\ b_1 &\mapsto \textstyle\sum_{l=1}^{n}(-1)^l\mathfrak{e}_l^{(n)}\underbrace{\textstyle\prod_{k=l}^{n-1}\kappa_{k,n}y_n}_{=b_l}.\end{aligned}$$

This shows that multiplication with $\kappa_{1,n}y_n$ from the left acts on the basis from (8.21) by the matrix

$$\begin{pmatrix}\mathfrak{e}_1^{(n)} & \tilde{\kappa}_{1,2} & & & \\ -\mathfrak{e}_2^{(n)} & & \tilde{\kappa}_{1,3} & & \\ \vdots & & & \ddots & \\ (-1)^{n-2}\mathfrak{e}_{n-1}^{(n)} & & & & \tilde{\kappa}_{1,n}\\ (-1)^{n-1}\mathfrak{e}_n^{(n)} & & & & 0\end{pmatrix} = M_{(n)}.$$

Quotienting out the two-sided ideal $(\kappa_{1,n}y_n)^m \trianglelefteq \mathsf{NH}\mathfrak{C}_n$ is the same as requiring that $M_{(n)}^m = 0$.

Claim. The ideal $(h_{n-m+1},\ldots,h_n) \trianglelefteq \mathrm{Pol}\mathfrak{C}_n$ is also generated by the first column of $M_{(n)}^{m+1}$.

By definition, h_{k+1} is the top left entry of $M_{(n)}\cdot M_{(n)}^k$ for any k. Recall that we denoted the entries of the first column $\left(M_{(n)}^{k+1}\right)_{*,1}$ of $M_{(n)}^{k+1}$ by

$$\left(M_{(n)}^{k+1}\right)_{*,1} =: \left(h_{k+1}, h_{(1),k+2}, h_{(2),k+3},\ldots,h_{(n-k-1),n}\right)^T.$$

We thus have

$$h_{n-m+2} = \mathfrak{e}_1^{(n)}h_{n-m+1} + \tilde{\kappa}_{1,2}h_{(1),n-m+2}$$

Table 8.4: Elementary $\mathfrak{d}$-symmetric polynomials $\mathfrak{e}_m^{(k,n)}$ for $n = 3$, as defined in Definition 8.37. The leftmost column is the same as in Table 8.1. The braces illustrates how one can construct the polynomials recursively as stated in Corollary 8.38, starting with the rightmost polynomial $\mathfrak{e}_3^{(2,3)} = \gamma_{3,4}y_3$.

$m\backslash k+1$	1	2	3
1	$\gamma_{1,2}\, y_1 + \underbrace{\gamma_{2,1}\, y_2 + \gamma_{2,1}\gamma_{2,3}\gamma_{3,2}\; y_3}_{\gamma_{2,1}\gamma_{2,3}\mathfrak{e}_1^{(1,3)}}$	$\gamma_{2,3}\, y_2 + \underbrace{\gamma_{3,2}\, y_3}_{\gamma_{3,2}\gamma_{3,4}\; \mathfrak{e}_1^{(1,3)}}$	$\gamma_{3,4}\, y_3$
2	$\underbrace{\gamma_{1,2}\, y_1\gamma_{2,3}\, y_2 + \gamma_{1,2}\, y_1\gamma_{3,2}\, y_3}_{\gamma_{1,2}\, y_1 \mathfrak{e}(1,3)_1} + \gamma_{2,1}\, y_2\;\; \gamma_{3,2}\, y_3$	$\gamma_{2,3}\, y_2\, \gamma_{3,2}\, y_3$	
3	$\gamma_{1,2}\, y_1\;\; \underbrace{\gamma_{2,3}\, y_3\, \gamma_{3,4}\, y_3}_{\mathfrak{e}_2^{(1,3)}}$		

$$
\begin{aligned}
&\equiv \tilde{\kappa}_{1,2} h_{(1),n-m+2} \pmod{h_{n-m+1}} \\
h_{n-m+3} &= \mathfrak{e}_1^{(n)} h_{n-m+2} + \tilde{\kappa}_{1,2} h_{(1),n-m+3} \\
&= \mathfrak{e}_2^{(n)} \left(\mathfrak{e}_1^{(n)} h_{n-m+1} + \tilde{\kappa}_{1,2} h_{(1),n-m+2}\right) + \tilde{\kappa}_{1,2}\left(-\mathfrak{e}_2^{(n)} h_{n-m+1} + \tilde{\kappa}_{1,3} h_{(2),n-m+2}\right) \\
&\equiv \tilde{\kappa}_{1,2}\tilde{\kappa}_{1,3} h_{(2),n-m+3} \pmod{h_{n-m+1}, h_{n-m+2}} \\
&\vdots \\
h_n &\equiv \tilde{\kappa}_{1,2}\cdots\tilde{\kappa}_{1,m} h_{(m-1),n} \pmod{h_{n-m+1}, \ldots, h_{n-1}}.
\end{aligned}
$$

This proves the claim since all $\tilde{\kappa}$'s are units. Taking the product $M_{(n)}^m = M_{(n)}^{m-n+1} \cdot M_{(n)}^{n-1}$ shows that the last column of $M_{(n)}^m$ has entries

$$
\left(M_{(n)}^m\right)_{*,n} = \left(h_{n-m+1}, h_{(1),n-m+2}, \ldots, h_{(m-1,n)}\right)^T \tilde{\kappa}_{1,2}\cdots\tilde{\kappa}_{1,n}.
$$

Thus the entries of $\left(M_{(n)}^m\right)_{*,n}$ also generate $(h_{n-m+1}, \ldots, h_n) \trianglelefteq \mathrm{Pol}\mathfrak{C}_n$.

Claim. All entries of $M_{(n)}^m$ are $\Lambda\mathrm{Pol}\mathfrak{C}_n$-linear combinations of entries of the last column $\left(M_{(n)}^m\right)_{*,n}$.

Since $M_{(n)}^{k+1} = M_{(n)}^1 \cdot M_{(n)}^k$ for any k, the entries of $\left(M_{(n)}^{k+1}\right)_{*,1}$ are $\Lambda\mathrm{Pol}\mathfrak{C}_n$-linear combinations of entries of $\left(M_{(n)}^k\right)_{*,1}$. Since also $M_{(n)}^{k+1} = M_{(n)}^k \cdot M_{(n)}^1$, we have $\left(M_{(n)}^{k+1}\right)_{*,2} = \tilde{\kappa}_{1,2}\left(M_{(n)}^k\right)_{*,1}$. Therefore, the entries of $\left(M_{(n)}^{k+1}\right)_{*,1}$ are linear combinations of entries of the second column $\left(M_{(n)}^{k+1}\right)_{*,2}$. The claim follows by iterating this argument.

Altogether, we have shown that requiring $M_{(n)}^m = 0$ is the same as quotienting out $(h_{n-m+1}, \ldots, h_n) \trianglelefteq \mathrm{Pol}\mathfrak{C}_n$. Applying this to each summand $B_{\boldsymbol{\alpha}}$ of (8.21) yields

$$
\mathsf{NH}\mathfrak{C}_n^m \cong \mathrm{Mat}_{n!}(\Lambda\mathrm{Pol}\mathfrak{C}_n)/\left(M_{(n)}^m\right) \cong \mathrm{Mat}_{n!}(\mathsf{H}\mathfrak{C}_{(m,n)}).
$$

This establishes the asserted Morita equivalence. ☐

8.8. Clifford algebras associated to partial flag varieties

We now come back to the consideration of $\mathfrak{d}$-symmetric polynomials.

Definition 8.37. Let $\mathfrak{e}_m^{(k,n)}$ be the polynomial of degree m in the indeterminates $y_k, \ldots, y_n$ that is obtained from $\mathfrak{e}_m^{(n-k)}$ by replacing every index i by $i + k$.

Recall from Lemma 8.13 the recursion formula for the elementary $\mathfrak{d}$-symmetric polynomials. We can easily derive the following corollary by regrouping the terms of (8.8):

Corollary 8.38. There is another recursion formula

$$
\mathfrak{e}_m^{(n)} = \gamma_{1,2}\, y_1\, \mathfrak{e}_{m-1}^{(1,n)} + \gamma_{2,1}\gamma_{2,3}\, \mathfrak{e}_m^{(1,n)} \tag{8.22}
$$

for the elementary $\mathfrak{d}$-symmetric polynomials, where $\gamma_{i,i\pm1}$ is as defined in Lemma 8.11.

Example 8.39. The "regrouping" of terms is best made explicit by considering the first $\mathfrak{d}$-symmetric polynomials listed explicitly in Table 8.1. We can start with the polynomial $\mathfrak{e}_1^{(2,3)} := \gamma_{3,4} y_3$ as defined in Definition 8.37 and construct the polynomials $\mathfrak{e}_m^{(k,3)}$ for $0 \leq k < 3$ and $m \leq n - k$ as described in the corollary. The resulting polynomials are listed in Table 8.4 with the grouping indicated by braces.

The two recursive expansions (8.8) and (8.22) from Lemma 8.13 and Corollary 8.38 are subsumed in the following definition:

Definition 8.40. For $1 < k < n$, let $\lambda_{m,l}^{(0,k,n)} \in \mathfrak{C}_n$ be the coefficients defined by

$$\mathfrak{e}_m^{(n)} = \sum_{0 \leq l \leq m} \mathfrak{e}_l^{(k)} \lambda_{m,l}^{(0,k,n)} \, \mathfrak{e}_{m-l}^{(k,n)}. \tag{8.23}$$

Example 8.41. By the expansions (8.8) and (8.22) we know the coefficients $\lambda_{m,l}^{(0,k,n)}$ for two particular values for k:

(i) For $k = n - 1$, $\lambda_{m,l}^{(0,k,n)}$ is just coefficient of the expansion $\mathfrak{e}_m^{(n)} = \sum_l \mathfrak{e}_l^{(n-1)} \lambda_l \gamma_{n,n+1} y_n$. The recursion formula from Lemma 8.13 gives

$$\lambda_{m,l}^{(0,n-1,n)} = \begin{cases} 1 & \text{if } l = m \\ \gamma_{m+1,m} \gamma_{m+1,m+2} \cdots \gamma_{n-1,n-2} \gamma_{n-1} n & \text{if } l = m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $k = 1$, $\lambda_{m,l}^{(0,k,n)}$ is the coefficient of the expansion $\mathfrak{e}_m^{(n)} = \sum \gamma_{1,2} y_1 \lambda_l \mathfrak{e}_l^{(1,n-1)}$. By Corollary 8.38 these are given by

$$\lambda_{m,l}^{(0,1,n)} = \begin{cases} \gamma_{2,1} \gamma_{2,3} & \text{if } l = 0 \\ 1 & \text{if } l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In general however, it seems to be difficult to give an explicit formula for $\lambda_{m,l}^{(0,k,n)}$ for arbitrary k.

Definition 8.42. Recall from Lemma 8.13 that we defined $\Lambda\mathrm{Pol}\mathfrak{C}_n := \langle \mathfrak{e}_1^{(n)}, \ldots, \mathfrak{e}_n^{(n)} \rangle$ as the $\mathfrak{C}_n$-subalgebra of $\mathrm{Pol}\mathfrak{C}_n$ generated by the $\mathfrak{d}$-symmetric polynomials. We have shown $\Lambda\mathrm{Pol}\mathfrak{C}_n = \bigcap_{i=1}^{n-1} \ker \mathfrak{d}_i$ in Theorem 8.23.(ii). We additionally define the $\mathfrak{C}_n$-algebras $\Lambda\mathrm{Pol}\mathfrak{C}_{(k,n)} := \langle \mathfrak{e}_m^{(k,n)} \mid 1 \leq m \leq m - k \rangle \subseteq \bigcap_{i=k+1}^{n-1} \ker \mathfrak{d}_i$ and more generally

$$\begin{aligned} \Lambda\mathrm{Pol}\mathfrak{C}_{(k_0,\ldots,k_l)} &:= \mathfrak{C}_n \left\langle \mathfrak{e}_{m_0}^{(k_0,k_1)}, \mathfrak{e}_{m_1}^{(k_1,k_2)}, \ldots, \mathfrak{e}_{m_{l-1}}^{(k_{l-1},k_l)} \;\middle|\; 1 \leq m_j \leq k_{j+1} - k_j \right\rangle \\ &\overset{*}{\subseteq} \ker \mathfrak{d}_{k_0+1} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_1}} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_j}} \cap \cdots \cap \mathfrak{d}_{k_l - 1} \end{aligned}$$

where all behatted factors $\widehat{\ker \mathfrak{d}_{k_j}}$ for $1 \leq j \leq l$ are to be omitted. In particular, the $\mathfrak{C}_n$-algebra of $\mathfrak{d}$-symmetric polynomials defined in Lemma 8.13 is $\Lambda\mathrm{Pol}\mathfrak{C}_n = \Lambda\mathrm{Pol}\mathfrak{C}_{(0,n)}$.

Remark 8.43. Note that Theorem 8.23.(ii) applies to each $\left\langle \mathfrak{e}_{m_j}^{(k_j,k_{j+1})} \mid 1 \leq m_j \leq k_{j+1} - k_j \right\rangle$. Therefore, $*$ is in fact an equality.

8.8.1. One step flag varieties

For every j there are the following inclusions of $\mathfrak{C}_n$-algebras: consider

$$\begin{aligned} &\Lambda\mathrm{Pol}\mathfrak{C}_{(k_0,\ldots,\widehat{k_j},\ldots,k_l)} \\ &= \ker \mathfrak{d}_{k_0+1} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_{j-1}}} \cap \cdots \cap \ker \mathfrak{d}_{k_j} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_{j+1}}} \cap \cdots \cap \mathfrak{d}_{k_l - 1}, \end{aligned}$$

the intersection in which the k_j-th kernel is *not* omitted. Clearly there is an inclusion into the intersection of kernels where the k_j-th kernel *is* omitted:

$$\subseteq \ker \mathfrak{d}_{k_0+1} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_{j-1}}} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_j}} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_{j+1}}} \cap \cdots \cap \mathfrak{d}_{k_l - 1}$$

$$= \Lambda\mathrm{Pol}\mathfrak{C}_{(k_1,\dots,k_l)}\,. \tag{8.24}$$

Since we know from Lemma 7.6 that $\ker \mathfrak{d}_i$ is a $\mathfrak{C}_n$-algebra and since the intersection of two subalgebras is again a subalgebra, these are inclusions of $\mathfrak{C}_n$-algebras.

Lemma 8.44. Let $\boldsymbol{k} = (0, k, k+1, \dots, n)$ for some k, so that the there are inclusion

$$\Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k,\dots,n)} \subseteq \Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k,k+1,\dots,n)} \supseteq \Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k+1,\dots,n)}\,. \tag{8.25}$$

by (8.24). The two inclusions can be written explicitly in terms of $\mathfrak{d}$-symmetric polynomials, namely

$$\begin{aligned} \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)} &\hookrightarrow \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)} \\ \mathfrak{e}_m^{(0,k)} &\longmapsto \mathfrak{e}_m^{(0,k)} \\ \mathfrak{e}_m^{(k,n)} &\longmapsto \gamma_{k+1,k+2}\, y_{k+1}\, \mathfrak{e}_{m-1}^{(k+1,n)} + \gamma_{k+1,k}\gamma_{k+1,k+2}\, \mathfrak{e}_m^{(k+1,n)} \end{aligned} \tag{8.26}$$

$$\begin{aligned} \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k+1,n)} &\hookrightarrow \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)} \\ \mathfrak{e}_m^{(0,k+1)} &\longmapsto \mathfrak{e}_m^{(0,k)} + \mathfrak{e}_{m-1}^{(0,k)} \cdot \gamma_{m+1,m} \cdots \gamma_{k+1,k}\, y_{k+1} \\ \mathfrak{e}_m^{(k+1,n)} &\longmapsto \mathfrak{e}_m^{(k+1,n)}. \end{aligned} \tag{8.27}$$

Proof. A polynomial from $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)}$ is contained in all $\ker \mathfrak{d}_i$ for $i \in \{1, \dots, k-1, k+1, k+2, \dots, n\}$ by Lemma 8.13. It is a fortiori contained in all $\ker \mathfrak{d}_i$ for $i \in \{1, \dots, k-1, k+2 \dots, n\}$, i. e. in $\ker \mathfrak{d}_1 \cap \cdots \ker \mathfrak{d}_{k-1} \cap \ker \mathfrak{d}_{k+2} \cap \cdots \ker \mathfrak{d}_n$.

We know from Theorem 8.23.(ii) and Remark 8.43 that this intersection equals $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)}$, which has generators $\mathfrak{e}_m^{(0,k,k+1,n)}$. The coefficients for writing elementary $\mathfrak{d}$-symmetric polynomials from $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)}$ in terms of those polynomials from $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)}$ are given in Example 8.41. This proves the statement. □

Remark 8.45. These inclusions turn $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)}$ into a $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)}$-$\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k+1,n)}$-bimodule. In the even case, this bimodule structure corresponds to the one described in [KL10, (5.17)]. In contrast to the notation in [KL10], we let $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)}$ act from the left and $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k+1,n)}$ from the right.

8.8.2. Clifford Cohomology of partial flag varieties

Recall the Clifford Grassmann cohomology ring $H\mathfrak{C}_{(m,n)}$ from Definition 8.34 and the ordinary cohomology ring of partial flag varieties from **??** 8.31. The following definition generalises both:

Definition 8.46. For any sequence $\boldsymbol{k} = (0 = k_0 \leq k_1 \leq \cdots \leq k_\ell = n)$ let the *Clifford cohomology ring* of the flag variety $\mathrm{Fl}(\boldsymbol{k})$ be the quotient

$$H\mathfrak{C}_{(0,k_1,\dots,k_\ell=n)} \coloneqq \Lambda\mathrm{Pol}\mathfrak{C}_{(k_1,\dots,k_l)} \Big/ (\Lambda\mathrm{Pol}\mathfrak{C}_{(0,n)})_+ \tag{8.28}$$

by the two-sided ideal generated by non-constant $\mathfrak{d}$-symmetric polynomials.

Remark 8.47. The graded dimension computations carried out in Lemma 8.32 and Corollary 8.33 remain valid for $H\mathfrak{C}_{\boldsymbol{k}}$ if one replaces $\dim_{q,k}$ by $\mathrm{rk}_{q,\mathfrak{C}_n}$.

Proposition 8.48 (Grassmannians). The quotient $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)} / (\Lambda\mathrm{Pol}\mathfrak{C}_{(0,n)})_+$ is isomorphic to the Clifford cohomology ring $H\mathfrak{C}_{(k,n)}$ of Grassmannians defined in Definition 8.34.

Proof. We may expand the generators $\mathfrak{e}_k^{(0,n)}$ as in (8.23). In the quotient we therefore obtain relations

$$\sum_{l=0}^{m} \mathfrak{e}_l^{(0,k)}\, \lambda_{m,l}^{(0,k,n)}\, \mathfrak{e}_{m-l}^{(k,n)} = \delta_{m,0}.$$

Comparing coefficients with those of the identity

$$\sum_{l=0}^{m} (-1)^l\, h_{m-l}^{(k,n)}\, \tilde{\kappa}_{k,k+1} \cdots \tilde{\kappa}_{k,k+l} \mathfrak{e}_l^{(k,n)} = \delta_{m,0}$$

from Proposition 8.27 implies that

$$(-1)^l\, h_{m-l}^{(k,n)}\, \tilde{\kappa}_{k,k+1} \cdots \tilde{\kappa}_{k,k+l} = \mathfrak{e}_{m-l}^{(0,k)}\, \lambda_{m,l}^{(0,k,n)}; \tag{8.29}$$

so we indeed have an isomorphism $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)} / (\Lambda\mathrm{Pol}\mathfrak{C}_{(0,n)})_+ \cong H\mathfrak{C}_{(k,n)}$. □

We thus have arrived at a super-generalisation for the cohomology rings of Flag varieties. In the ordinary set-up, it has been proven in [KL10] that these rings admit an action by the Kac-Moody 2-category from [Bru16]. The latter has a super-analogue constructed in [BE17b]. This motivates:

Question 8.49. The Clifford cohomology rings $H\mathfrak{C}_{\boldsymbol{k}}$ from Definition 8.46 bear a bimodule structure by Remark 8.45. Can one construct an action of the Kac-Moody 2-supercategory from [BE17b] on these Clifford cohomology rings?

Answering this question has to be deferred to future work.

References

[APR00] R. M. Adin, A. Postnikov and Y. Roichman. 'Hecke algebra actions on the coinvariant algebra'. In: *J. Algebra* 233.2 (2000), pp. 594–613. DOI: `10.1006/jabr.2000.8441`. arXiv: `math/9908090`.

[Art25] E. Artin. 'Theorie der Zöpfe'. In: *Abh. Math. Sem. Univ. Hamburg* 4.1 (1925), pp. 47–72. DOI: `10.1007/BF02950718`.

[AS03] H. H. Andersen and C. Stroppel. 'Twisting functors on $\mathcal{O}$'. In: *Represent. Theory* 7 (2003), pp. 681–699. DOI: `10.1090/S1088-4165-03-00189-4`.

[Bar15] M. Barot. *Introduction to the Representation Theory of Algebras*. Springer, 2015. ISBN: 978-3-319-11474-3. DOI: `10.1007/978-3-319-11475-0`.

[Bas68] H. Bass. *Algebraic K-Theory*. 1st ed. New York, Amsterdam: W. A. Benjamin, 1968.

[BB05] A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups*. Graduate Texts in Mathematics 231. New York: Springer, 2005. ISBN: 3-540-44238-3.

[BB81] A. Beilinson and J. Bernstein. 'Localisation de $\mathfrak{g}$-modules'. In: *C. R. Acad. Sci. Paris.* Série I : Sciences mathémathique 292.1 (1981), pp. 15–18. URL: `http://gallica.bnf.fr/ark:/12148/bpt6k6226873r/f32.image`.

[BE17a] J. Brundan and A. P. Ellis. 'Monoidal supercategories'. In: *Comm. Math. Phys.* 351.3 (2017), pp. 1045–1089. DOI: `10.1007/s00220-017-2850-9`. arXiv: `1603.05928`.

[BE17b] J. Brundan and A. P. Ellis. 'Super Kac-Moody 2-Categories'. In: *P. Lond. Math. Soc.* (2017). DOI: `10.1112/plms.12055`. arXiv: `1701.04133`.

[BGG76] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand. 'A certain category of $\mathfrak{g}$-modules'. In: *Funkcional. Anal. i Prilozen.* 10.2 (1976), pp. 1–8.

[BGS96] A. Beilinson, V. Ginzburg and W. Soergel. 'Koszul duality patterns in representation theory'. In: *J. Amer. Math. Soc.* 9.2 (1996), pp. 473–527. DOI: `10.1090/S0894-0347-96-00192-0`.

[BK81] J.-L. Brylinski and M. Kashiwara. 'Kazhdan-Lusztig conjecture and holonomic systems'. In: *Invent. Math.* 64.3 (1981), pp. 387–410. DOI: `10.1007/BF01389272`.

[BK89] A. Bondal and M. M. Kapranov. 'Representable functors, Serre functors, and reconstructions'. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 53 (1989), pp. 1183–1205. DOI: `10.1070/im1990v035n03abeh000716`.

[BKM01] T. Brüstle, S. König and V. Mazorchuk. 'The Coinvariant Algebra and Representation Types of Blocks of Category $\mathcal{O}$'. In: *Bull. London Math. Soc* 33 (2001), pp. 669–681. DOI: `10.1112/S0024609301008529`.

[Bou07] N. Bourbaki. *Élements de mathémathique. Algèbres de Lie. chapitres 4 à 6.* Berlin: Springer, 2007. ISBN: 978-3-540-35337-9.

[Bru16] J. Brundan. 'On the Definition of Kac-Moody 2-Category'. In: *Math. Ann.* 364.1 (2016), pp. 353–372. DOI: `10.1007/s00208-015-1207-y`. arXiv: `1501.00350`.

[BS11a] J. Brundan and C. Stroppel. 'Highest Weight Categories Arising from Khovanov's Diagram Algebra I: Cellularity'. In: *Mosc. Math. J.* 11.4 (2011), pp. 685–722, 821–822. arXiv: `0806.1532`.

[BS11b] J. Brundan and C. Stroppel. 'Highest Weight Categories Arising from Khovanov's Diagram Algebra III: Category $\mathcal{O}$'. In: *Represent. Theory* 15 (2011), pp. 70–243. DOI: `10.1090/S1088-4165-2011-00389-7`.

[Bum04] D. Bump. *Lie groups*. 1st ed. Graduate texts in mathematics 225. New York, 2004. ISBN: 978-1-4614-8023-5.

[Car86] K. J. Carlin. 'Extensions of Verma modules'. In: *Trans. Amer. Math. Soc.* 294.1 (1986), pp. 29–43. DOI: `10.2307/2000116`.

[CC87] D. H. Collingwood and L. G. Casian. 'The Kazhdan-Lusztig Conjecture for Generalized Verma Modules'. In: *Math. Z.* 195 (1987), pp. 581–600. DOI: `10.1007/bf01166705`. EuDML: 183714.

[Che55] C. Chevalley. 'Invariants of finite groups generated by reflections'. In: *Am. J. Math.* 77.4 (1955), pp. 778–782. DOI: `10.2307/2372597`. JSTOR: `2372597`.

[CIS88] D. H. Collingwoo, R. S. Irving and B. Shelton. 'Filtrations on generalized Verma modules for Hermitian symmetric pairs'. In: *J. reine angew. Math.* 383 (1988), pp. 54–86. DOI: `10.1515/crll.1988.383.54`. EuDML: 152992.

[Cox34] H. S. M. Coxeter. 'Discrete groups generated by reflections'. In: *Ann. of Math.* 2nd ser. 35.3 (1934), pp. 588–621. DOI: `10.2307/1968753`.

[Cox35] H. S. M. Coxeter. 'The Complete Enumeration of Finite Groups of the Form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$'. In: *J. London Math. Soc.* 1st ser. 10.37 (1935), pp. 21–25. DOI: `10.1112/jlms/s1-10.37.21`.

[Dem73] M. Demazure. 'Invariants symétriques entiers des groupes de Weyl et torsion'. In: *Invent. Math.* 21 (1973), pp. 287–301. ISSN: 0020-9910. DOI: `10.1007/BF01418790`.

[Deo87] V. Deodhar. 'On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials'. In: *J. Algebra* 111.2 (1987), pp. 483–506. DOI: `10.1016/0021-8693(87)90232-8`.

[EK10a] B. Elias and M. Khovanov. 'Diagrammatics for Soergel categories'. In: *Int. J. Math. Math. Sci.* (2010), Art. ID 978635. DOI: `10.1155/2010/978635`. arXiv: `0902.4700`.

[EK10b] B. Elias and D. Krasner. 'Rouquier Complexes are Functorial over Braid Cobordisms'. In: *Homology Homotopy Appl.* 12.2 (2010), pp. 109–146. DOI: `10.4310/hha.2010.v12.n2.a4`. arXiv: `0906.4761`.

[EKL14] A. P. Ellis, M. Khovanov and A. D. Lauda. 'The odd nilHecke algebra and its diagrammatics'. In: *Int. Math. Res. Not. IMRN* 4 (2014), pp. 991–1062. DOI: `10.1093/imrn/rns240`. arXiv: `1111.1320`.

[EQ16] A. P. Ellis and Y. Qi. 'The differential graded odd nilHecke algebra'. In: *Comm. Math.* 344.1 (2016), pp. 275–331. DOI: `10.1007/s00220-015-2569-4`. arXiv: `1504.01712`.

[EW16] B. Elias and G. Williamson. 'Soergel Calculus'. In: *Represent. Theory* 20 (2016), pp. 295–374. DOI: `10.1090/ert/481`. arXiv: `1309.0865`.

[Ful97] W. Fulton. *Young Tableaux. With Applications to Representation Theory and Geometry.* London Mathematical Society Student Texts 35. Cambridge University Press, 1997. ISBN: 0-521-56144-2.

[Gro16] M. Groth. 'Book Project on Derivators'. 2016. URL: `http://www.math.uni-bonn.de/~mgroth/monos/intro-to-der-1.pdf`.

[Gro57] A. Grothendieck. 'Sur quelques points d'algèbre homologique'. In: *Tôhoku Mathematical Journal.* 2nd ser. 9 (1957), pp. 119–221. DOI: `10.2748/tmj/1178244839`.

[Gro77] A. Grothendieck. 'Groupes de classes des categories abeliennes et triangulees. Complexes parfaits.' In: *Séminaire de Géométrie Algébrique du Bois-Marie.* Vol. 5. Lecture Notes in Mathematics 589. Berlin: Springer, 1977. ISBN: 978-3-540-08248-4.

[HKP16] A. Hochenegger, M. Kalck and D. Ploog. 'Spherical subcategories in algebraic geometry'. In: *Math. Nachr.* 289.11-12 (2016), pp. 1450–1465. DOI: `10.1002/mana.201400232`. arXiv: `1208.4046`.

[HKP17] A. Hochenegger, M. Kalck and D. Ploog. 'Spherical subcategories in representation theory'. 2017. arXiv: `1502.06838`.

[Hum08] J. E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$.* Graduate Studies in Mathematics 94. American Mathematical Society, 2008. ISBN: 978-0-8218-4678-0.

[Hum72] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate Texts in Mathematics 9. New York: Springer, 1972. ISBN: 978-1-4612-6398-2.

[Hum90] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics 29. Cambridge University Press, 1990.

[Irv88] R. S. Irving. 'The socle filtration of a Verma module'. In: *Ann. Sci. ENS.* 4th ser. 21 (1988), pp. 47–65. DOI: `10.24033/asens.1550`.

[Irv90] R. S. Irving. *A filtered category $\mathcal{O}_S$ and applications.* Vol. 1. 6 vols. Memoirs of the American Mathematical Society 419. Providence, RI: Amer. Math. Soc., 1990. ISBN: 978-0-8218-2482-5.

[Irv93] R. S. Irving. 'Shuffled Verma modules and principal series modules over complex semisimple Lie algebras'. In: *J. London Math. Soc.* 2nd ser. 48.2 (1993), pp. 263–277. DOI: `10.1112/jlms/s2-48.2.263`.

[Jac12] N. Jacon. 'Representations of the symmetric group and its Hecke algebra'. 2012. HAL: hal-00731102.

[Jan79] J. C. Jantzen. *Moduln mit einem höchsten Gewicht.* Lecture Notes in Mathematics 750. Springer, 1979. ISBN: 978-3-540-34854-2.

[KE12] M. Khovanov and A. P. Ellis. 'The Hopf algebra of odd symmetric functions'. In: *Adv. Math.* 2 (2012), pp. 965–999. DOI: `10.1016/j.aim.2012.04.031`. arXiv: `1107.5610`.

[Kho02] M. Khovanov. 'A functor-valued invariant of tangles'. In: *Algebr. Geom. Topol.* 2 (2002), pp. 665–741. DOI: `10.2140/agt.2002.2.665`. arXiv: `math/0103190`.

[Kho06] M. Khovanov. 'Link homology and categorification'. In: *Invited lectures.* International Congress of Mathematicians (Madrid, 2006). Ed. by M. Sanz-Solé et al. Vol. II. Zurich: Eur. Math. Soc., 2006, pp. 989–999. ISBN: 978-3-03719-022-7. arXiv: `math/0605339`.

[KKT16] S.-J. Kang, M. Kashiwara and S. Tsuchioka. 'Quiver Hecke superalgebras'. In: *J. Reine Angew. Math.* 711 (2016), pp. 1–54. DOI: `10.1515/crelle-2013-0089`. arXiv: `1107.1039`.

[KL09] M. Khovanov and A. D. Lauda. 'A diagrammatic approach to categorification of quantum groups I'. In: *Represent. Theory* 13 (2009), pp. 309–347. DOI: `10.1090/s1088-4165-09-00346-x`. arXiv: `0803.4121`.

[KL10] M. Khovanov and A. D. Lauda. 'A categorification of quantum $\mathfrak{sl}(n)$'. In: *Quantum Topology* 1.1 (2010), pp. 1–92. DOI: `10.4171/QT/10`. arXiv: `0807.3250`.

[KL11] M. Khovanov and A. D. Lauda. 'A diagrammatic approach to categorification of quantum groups II'. In: *Trans. Amer. Math. Soc.* 363.5 (2011), pp. 2685–2700. DOI: `10.1090/s0002-9947-2010-05210-9`.

[KL79] D. Kazhdan and G. Lusztig. 'Representations of Coxeter Groups and Hecke Algebras'. In: *Inv. Math.* 53 (1979), pp. 165–184. DOI: `10.1007/bf01390031`.

[KM16] T. Kildetoft and V. Mazorchuk. 'Parabolic Projective Functors in Type A'. In: *Adv. Math.* 301 (2016), pp. 785–803. DOI: `10.1016/j.aim.2016.06.026`. arXiv: `1506.07008`.

[KMS09] M. Khovanov, V. Mazorchuk and C. Stroppel. 'A brief review of abelian categorifications'. 2009. arXiv: `0702746`.

[Kra15] H. Krause. 'Krull-Schmidt categories and projective covers'. In: *Expo. Math.* 33.4 (2015), pp. 535–549. DOI: `10.1016/j.exmath.2015.10.001`. arXiv: `1410.2822`.

[Lau10] A. D. Lauda. 'A categorification of quantum $\mathfrak{sl}_2$'. In: *Adv. Math.* 225.6 (2010), pp. 3327–3424. DOI: `10.1016/j.aim.2010.06.003`. arXiv: `0803.3652`.

[Lau12] A. D. Lauda. 'An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}_2$'. In: *Bull. Inst. Math. Acad. Sin.* new series 7.2 (2012), pp. 165–270. arXiv: `1106.2128`.

[Lib10] N. Libedinsky. 'Presentation of right-angled Soergel categories by generators and relations'. In: *J. Pure Appl. Algebra* 214.12 (2010), pp. 2265–2278. DOI: `10.1016/j.jpaa.2010.02.026`. arXiv: `0810.2395`.

[Lib17] N. Libedinsky. 'Gentle introduction to Soergel bimodules I: The basics'. 2017. arXiv: `1702.00039`.

[LQ15] B. Le Stum and A. Quirós. 'On Quantum Integers and Rationals'. In: *Trends in Number Theory.* Fifth Spanish Meeting on Number Theory (Sevilla, July 2013). Ed. by F. Chamizo et al. Contemporary Mathematics 649. Providence, RI: AMS, 2015, pp. 107–130. ISBN: 978-0-8218-9858-1. DOI: `10.1090/conm/649/13022`.

[LR14] A. D. Lauda and H. M. Russell. 'Oddification of the cohomology of type A Springer varieties'. In: *Int. Math. Res. Not. IMRN* 17 (2014), pp. 4822–4854. DOI: `10.1093/imrn/rnt098`. arXiv: `1203.0797`.

[LS13] T. Lejczyk and C. Stroppel. 'A graphical description of $(\mathrm{D}_n, \mathrm{A}_{n-1})$ Kazhdan-Lusztig polynomials'. In: *Glasgow Math. J.* 55.2 (2013), pp. 313–340. DOI: `10.1017/S0017089512000547`.

[Lus03] G. Lusztig. *Hecke algebras with unequal parameters.* Vol. 18. CRM Monograph Series. AMS, CRM, 2003. ISBN: 978-0-8218-3356-8. arXiv: `math/0208154`.

[Man98] L. Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci.* Vol. 6. SMF/AMS Texts and Monographs 3. Providence, RI: AMS, 1998. ISBN: 978-0-8218-2154-1.

[Mar06] F. Marko. 'Algebra associated with the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_3(\mathbf{C})$'. In: *Algebras, Rings and Their Representations.* International Conference on Algebras, Modules and Rings (Lisbon, 14th–18th July 2003). Ed. by A. Facchini et al. 2006, pp. 201–2014. ISBN: 978-981-256-598-3.

[Mat64] H. Matsumoto. 'Générateurs et relations des groupes de Weyl généralisés'. In: *C. R. Acad. Sci. Paris.* Série A et B : sciences mathématiques. Sciences physiques 258.4 (1964), pp. 3419–3422. URL: `http://gallica.bnf.fr/ark:/12148/bpt6k4011c/f1029.item`.

[Maz07] V. Mazorchuk. 'Some homological properties of the category $\mathcal{O}$'. In: *Pac. J. Math.* 232.2 (2007), pp. 313–341. DOI: `10.2140/pjm.2007.232.313`.

[Maz12] V. Mazorchuk. *Lectures on Algebraic Categorification.* Zurich: Eur. Math. Soc., 2012. ISBN: 978-3-03719-108-8.

[MS05] V. Mazorchuk and C. Stroppel. 'Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module'. In: *Trans. Amer. Math. Soc.* 357 (2005), pp. 2939–2973. DOI: `10.1090/S0002-9947-04-03650-5`.

[MS07] V. Mazorchuk and C. Stroppel. 'On functors associated to a simple root'. In: *314* 1 (2007), pp. 97–128. DOI: `10.1016/j.jalgebra.2007.03.015`. arXiv: `0410339`.

[MS08] V. Mazorchuk and C. Stroppel. 'Projective-injective modules, Serre functors and symmetric algebras'. In: *J. reine angew. Math.* 616 (2008), pp. 131–165. DOI: `10.1515/CRELLE.2008.020`.

[Rag17] R. Ragavender. 'Odd Dunkl Operators and nilHecke Algebras'. 2017. arXiv: `1707.04667`.

[Roc80] A. Rocha-Caridi. 'Splitting Criteria for $\mathfrak{g}$-Modules Induced from a Parabolic and the Bernstein-Gelfand-Gelfand Resolution of a Finite Dimensional, Irreducible $\mathfrak{g}$-Module'. In: *Trans. Amer. Math. Soc.* 2 (1980), pp. 335–366. DOI: `10.2307/1999832`. JSTOR: `1999832`.

[Rou06] R. Rouquier. 'Categorification of $\mathfrak{sl}_2$ and braid groups'. In: *Trends in representation theory of algebras and related topics*. Contemp. Math. 406. Providence, RI: Amer. Math. Soc., 2006, pp. 137–167. DOI: `10.1090/conm/406/07657`. arXiv: `math/0409593`.

[Rou08] R. Rouquier. '2-Kac-Moody algebras'. 2008. arXiv: `0812.5023`.

[RS15] M. Rottmaier and W. Soergel. 'Unicity of graded covers of the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand'. In: *Ark. Mat.* 53.2 (2015), pp. 383–400. DOI: `10.1007/s11512-014-0211-x`.

[Seg16] E. Segal. 'All Autoequivalences Are Spherical Twists'. 2016. arXiv: `1603.06717`.

[Soe07] W. Soergel. 'Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen'. In: *J. Inst. Math. Jussieu* 6.3 (2007), pp. 501–525. DOI: `10.1017/S1474748007000023`. arXiv: `math/0403496`.

[Soe90] W. Soergel. 'Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe'. In: *J. Amer. Math. Soc.* 3.2 (1990), pp. 421–445. DOI: `10.1090/S0894-0347-1990-1029692-5`.

[Soe92] W. Soergel. 'The combinatorics of Harish-Chandra bimodules'. In: *J. Reine Angew. Math.* 429 (1992), pp. 49–74. DOI: `10.1515/crll.1992.429.49`.

[Soe97] W. Soergel. 'Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules'. In: *Represent. Theory* 1 (1997), pp. 83–114. DOI: `10.1090/S1088-4165-97-00021-6`.

[ST01] P. Seidel and R. Thomas. 'Braid Group Actions on Derived Categories of Coherent Sheaves'. In: *Duke Math. Jour.* 108 (2001), pp. 37–108. DOI: `10.1215/s0012-7094-01-10812-0`.

[Ste85] J. R. Stembridge. 'A characterization of supersymmetric polynomials'. In: *J. Algebra* 95.2 (1985), pp. 439–444. DOI: `10.1016/0021-8693(85)90115-2`.

[Str03a] C. Stroppel. 'Category $\mathcal{O}$: Gradings and translation functors'. In: *J. Algebra* 268 (2003), pp. 301–326. DOI: `10.1016/S0021-8693(03)00308-9`.

[Str03b] C. Stroppel. 'Category $\mathcal{O}$: Quivers and endomorphism rings of projectives'. In: *Representation Theory* 7 (2003). DOI: `10.1090/S1088-4165-03-00152-3`.

[Ver96] J.-L. Verdier. 'Des Catégories Triangulées et des Categories Derivées'. PhD thesis. 1996. URL: `http://webusers.imj-prg.fr/~georges.maltsiniotis/jlv.html`.

[Wei94] C. A. Weibel. *An introduction to homological algebra*. Cambridge studies in advanced mathematics 38. Cambridge: Cambridge University Press, 1994. ISBN: 0-521-43500-5.

[Win01] R. Winkel. 'A combinatorial derivation of the Poincaré polynomials of the finite irreducible Coxeter groups'. In: *Discrete Math.* 239.1 (2001), pp. 83–99. DOI: `10.1016/S0012-365X(00)00375-7`.