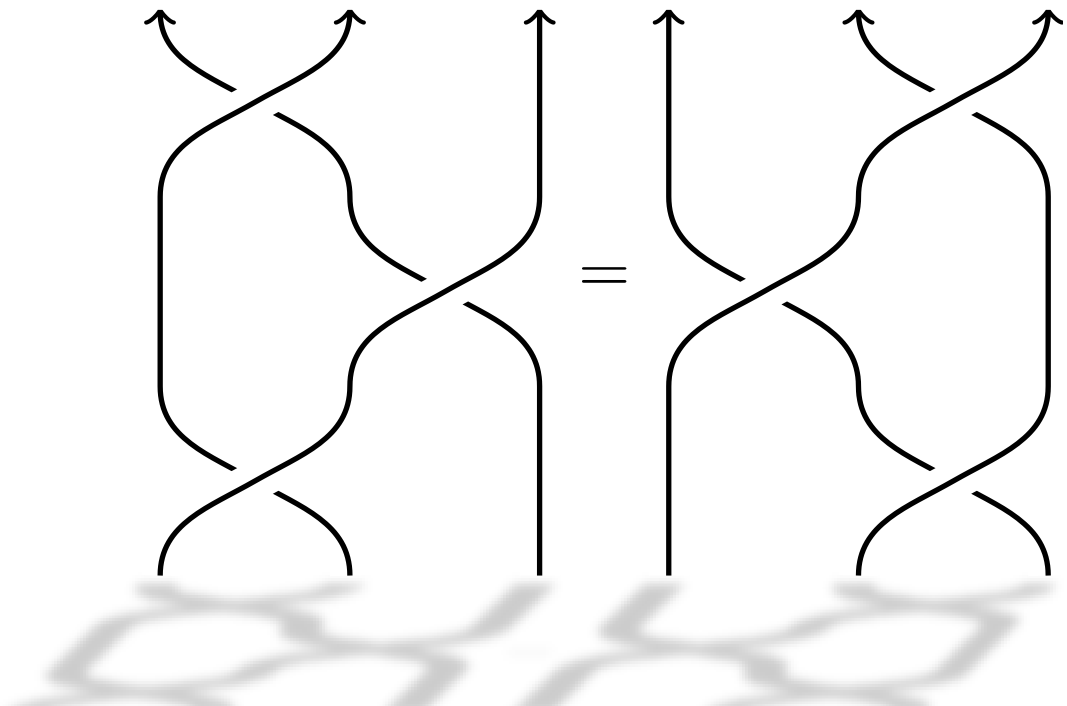
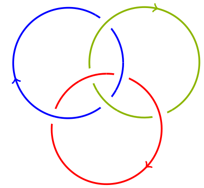
The Beauty of Braids



Catharina Stroppel (Bonn, Germany)

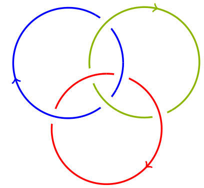
ICM 2022

Representation theory



Topology

Representation theory



Topology

Fundamental problem: find (interesting) topological invariants

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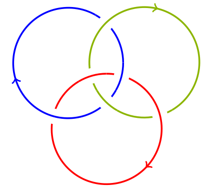

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Fundamental problem: find (interesting) topological invariants

of (oriented, smooth, compact)

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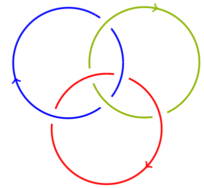

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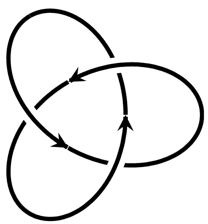

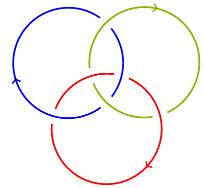

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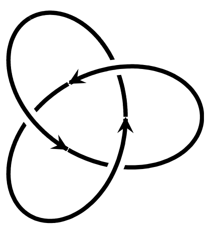
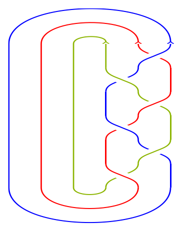


LINKS



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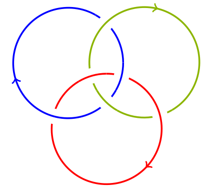

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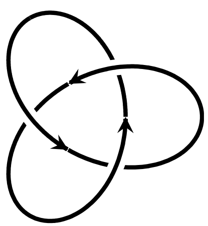

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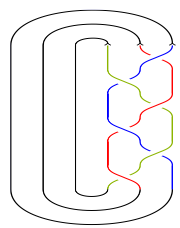
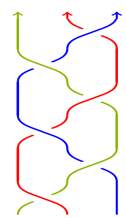
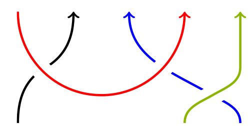


BRAIDS



TANGLES



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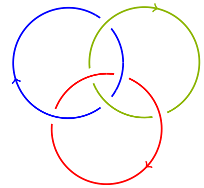

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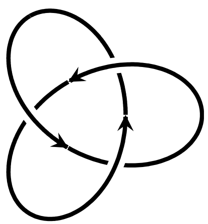

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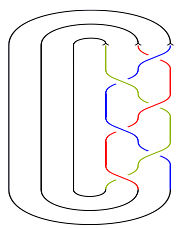

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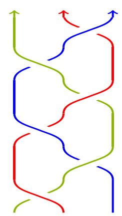

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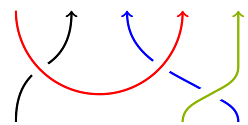

Representation theory = study of algebraic objects (groups, algebras, Lie algebras, …)

by **representing** their elements as **linear transformations** (of a vector space)

Selected success stories (1970s–1990s)

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Knot and link invariants via polynomials

Jones polynomial

HOMFLY–PT polynomial

Skein theory and Hecke algebras

Tangle and 3-manifold invariants via Quantum groups

Quantum groups

Reshetikhin–Turaev invariants

Crane–Yetter invariants

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Topological invariants via higher homotopies

- Operads
- E_n-algebras
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stronger invariants ?

CATEGORIFICATION

Tangle and 3-manifold invariants via Quantum groups

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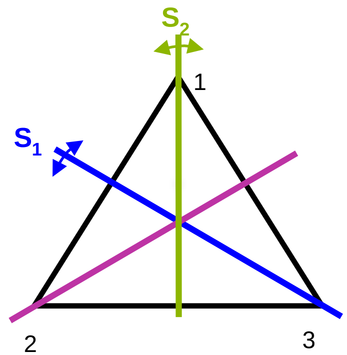
A representation: the symmetric group S_3 as group of symmetries

- Define representation on generators

$$S_3 \longrightarrow \mathrm{Sym}(\triangle) \subset \mathrm{End}(\mathbb{R}^2)$$

$$s_1 = (1,2) \quad \longmapsto$$
$$s_2 = (2,3) \quad \longmapsto$$



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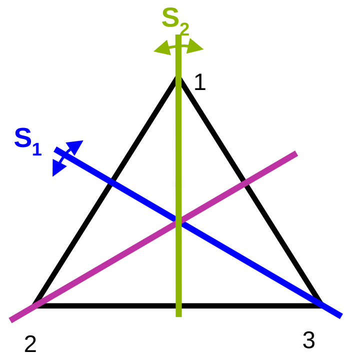

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dim 3	S_3	$\curvearrowright$	$\{(x_1,x_2,x_3) \in \mathbb{R}^3\}$
dim 2	$\mathrm{Sym}(\triangle)$	$\curvearrowright$	plane $= \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3=0\}$

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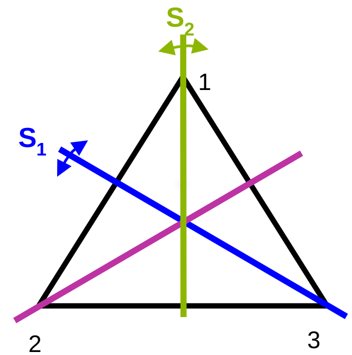

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dim 3

dim 2

$$\begin{array}{ccc} S_3 & \curvearrowright & \{(x_1,x_2,x_3) \in \mathbb{R}^3\} \\ \downarrow & & \downarrow \text{projection} \\ \mathrm{Sym}(\triangle) & \curvearrowright & \text{plane} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 = 0\} \end{array}$$

"projection […] which is an excellent tool for intuitive investigations, is a very clumsy one for rigorous proofs**.** This has lead me to abandon projections altogether."

E. Artin (1935)

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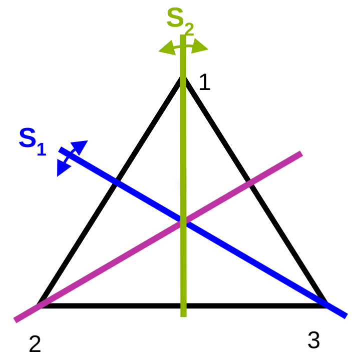

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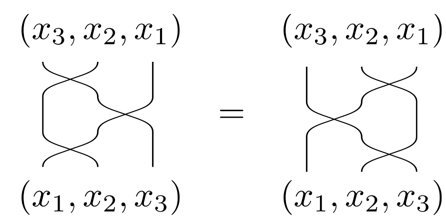
$$s_i^2 = \mathrm{id} \qquad {\color{magenta}s_1 s_2 s_1 = s_2 s_1 s_2}$$



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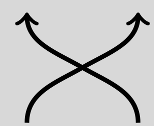
From permutations to braids

$$S_3 = \langle s_1, s_2 \rangle / (s_1 s_2 s_1 = s_2 s_1 s_2, s_i^2 = \mathrm{id})$$

symmetric group



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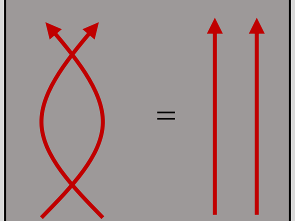

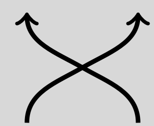

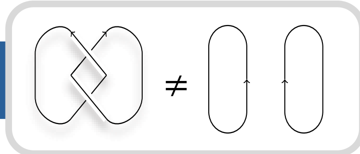

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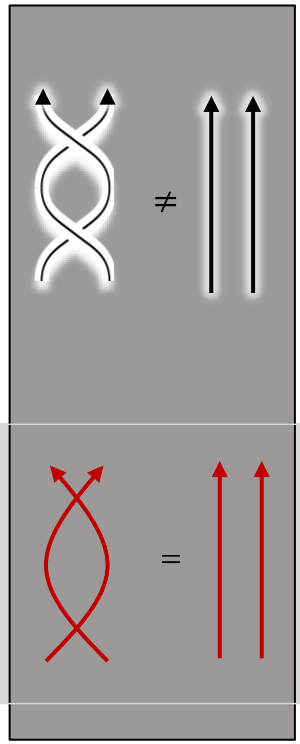

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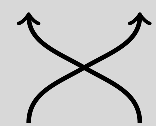

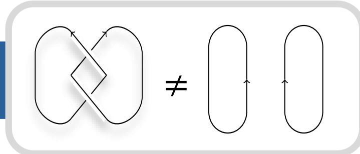

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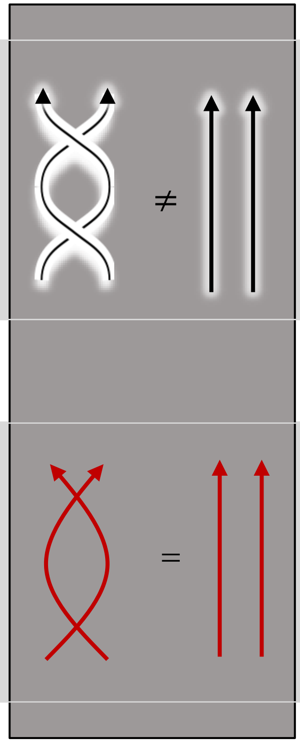
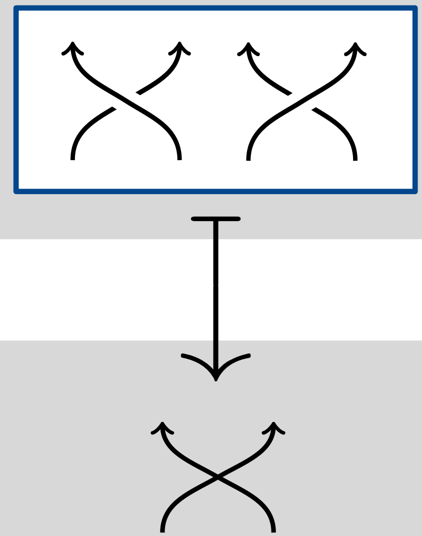
$$\mathrm{Br}_3 = \langle \beta_1, \beta_2 \rangle / (\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2)$$





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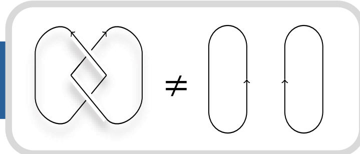

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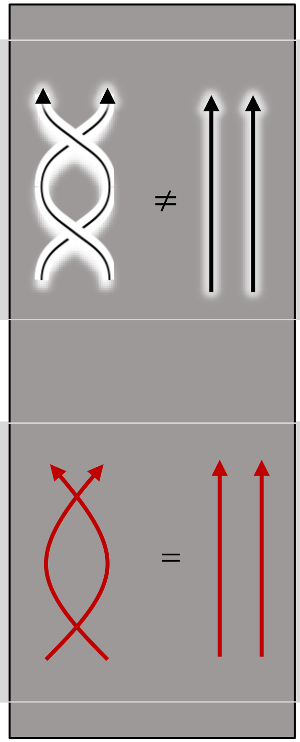

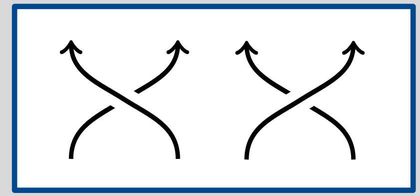

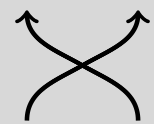

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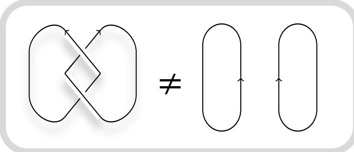

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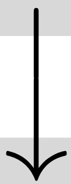

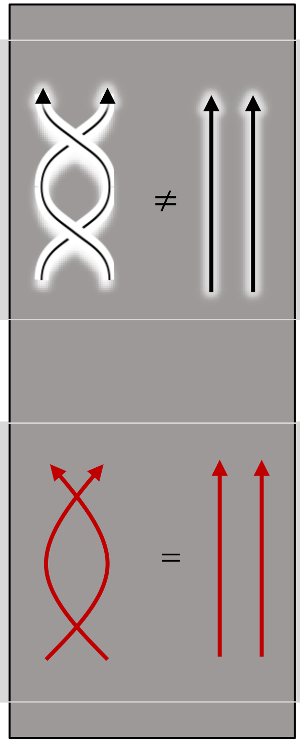

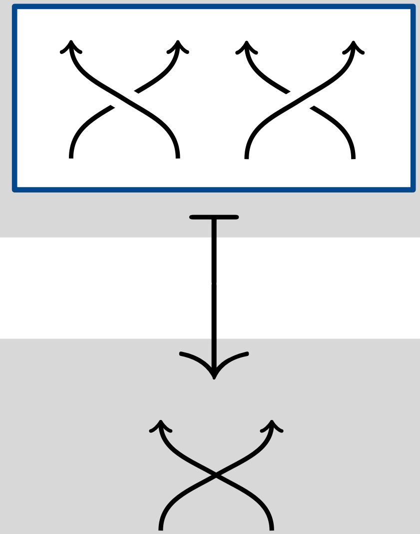

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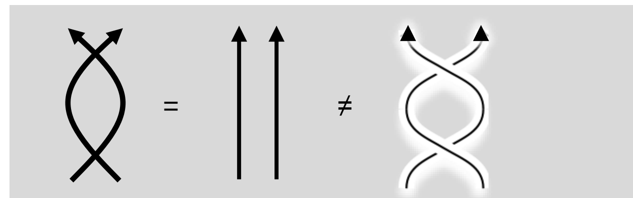
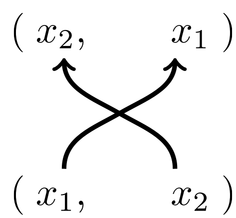
Upshot: braids turn into elements of braid groups

Representations of braid groups ?





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braids

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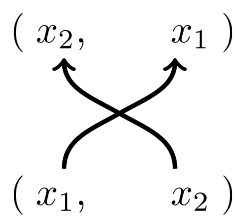

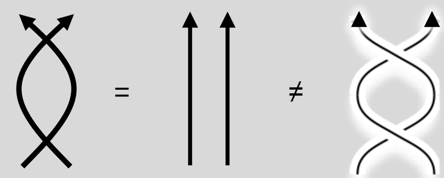

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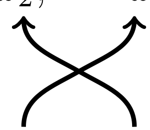
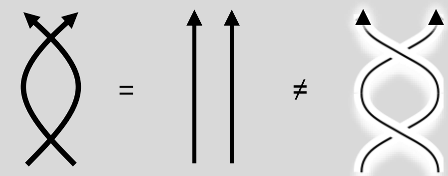
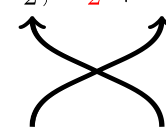


$(\; x_1, \qquad x_2 \;)$



$(\; x_2, \; x_2 + x_1 + (-x_2) \;)$



$(\; x_1, \qquad x_2 \qquad)$

Quantum trick

quantum principle:

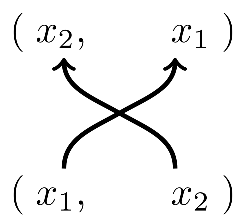
variables don't commute

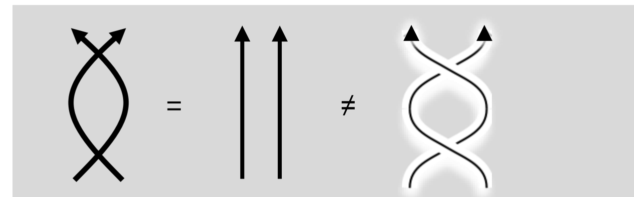
$q_i = q^{x_i}$

braids

quantum  trick

permutations of coordinates





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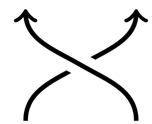
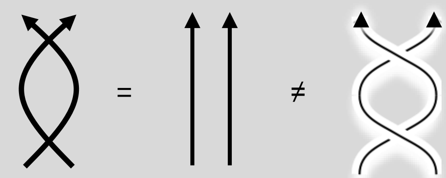
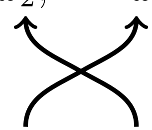


$(q_1, \quad q_2 \quad)$



$(\ x_2, \quad x_1\ )$



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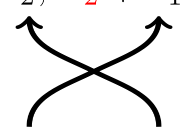
$(\ x_2, \ x_2 + x_1 + (-x_2)\ )$



$(\ x_1, \quad x_2 \quad)$

Quantum trick

quantum principle:

variables don't commute

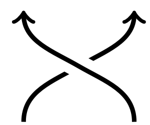
$q_i = q^{x_i}$

braids

quantum trick

permutations of coordinates

$(q_2, \quad q_2 \cdot q_1 \cdot q_2^{-1})$



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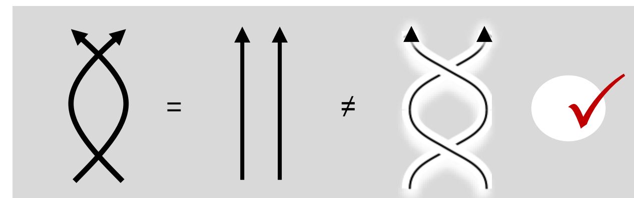

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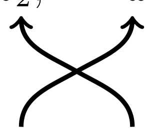

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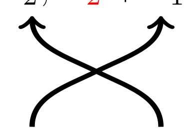

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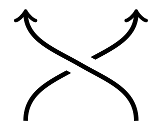

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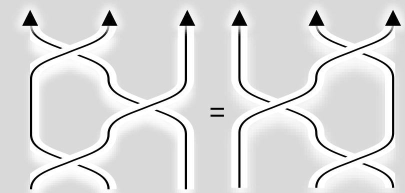

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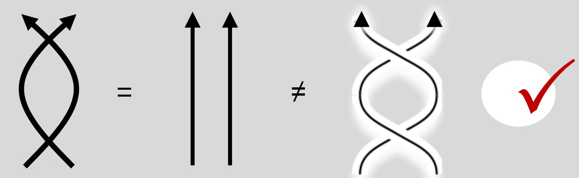

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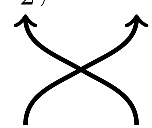

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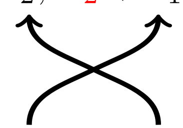

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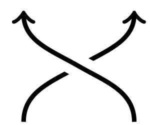

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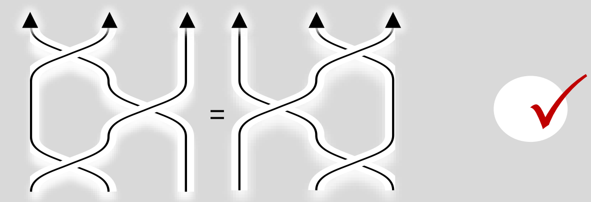

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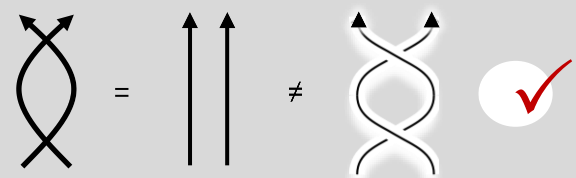

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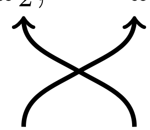

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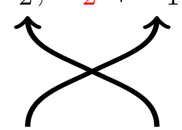
$(\ x_2, \ \textcolor{red}{x_2} + x_1 + (\textcolor{red}{-x_2}) \ )$



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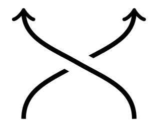

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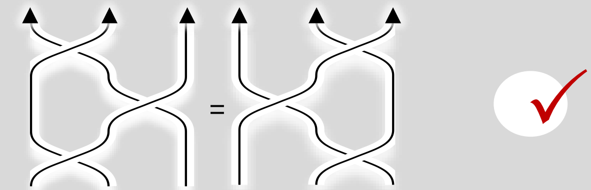

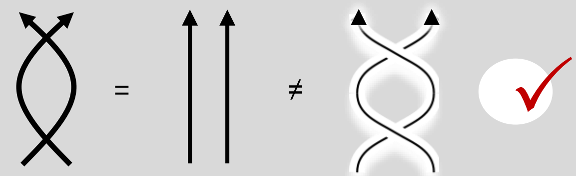

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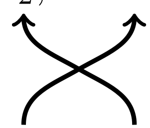

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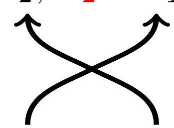


$(\, x_1, \quad x_2 \quad)$

Upshot: representations of S_n turn into representations of Br_n via quantum trick

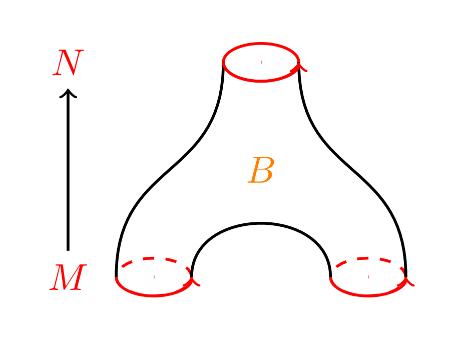
Atiyah–Segal Axioms of TQFT (1980s)

“Representation theory of manifolds”

Atiyah–Segal Axioms of TQFT (1980s)

"Representation theory of manifolds"

A Topological Quantum Field Theory of dimension $d = (d-1)+1$ is an assignment Z



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any closed $(d-1)$-manifold M $\longmapsto$ (complex) vector space $Z(M)$

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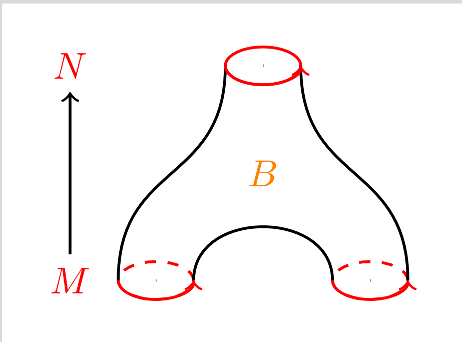

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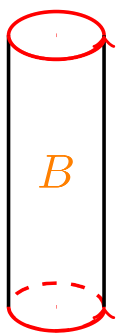
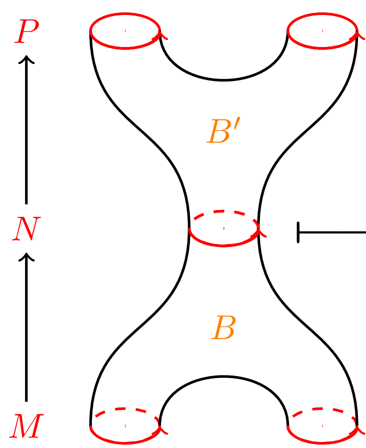


$\longmapsto \mathrm{id}_{Z(M)}$



$\longmapsto$ $Z(B') \circ Z(B)$

invariant under diffeomorphisms

$$Z\left(\sqcup M_i\right) = \bigotimes Z(M_i)$$

$$Z(\ \varnothing\ ) = \mathbb{C}$$

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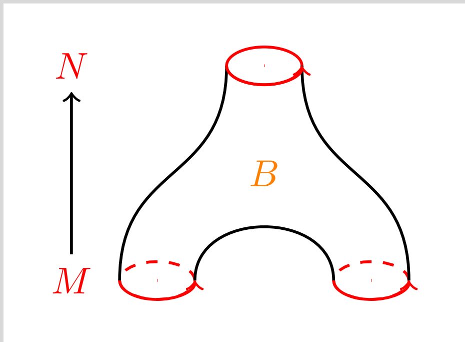

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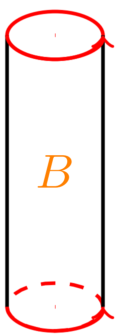

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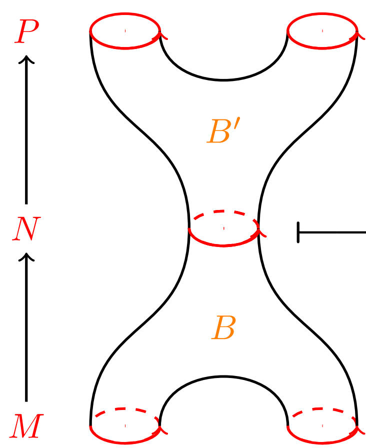

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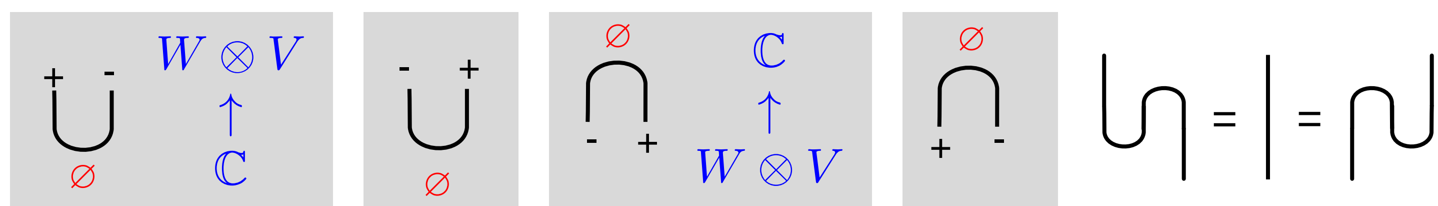
Categorical formulation: symmetric $\otimes$-functor $Z : \ \mathrm{Cob}(2) \longrightarrow \mathrm{Vect}$

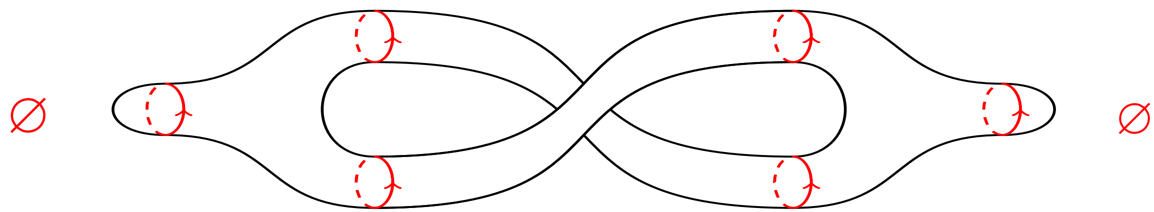
d = 1

(co)evaluation maps

1-bordisms:



closed 0-mfds: $Z(\overset{+}{\bullet}) = V$ $\quad Z(\overset{-}{\bullet}) = W$ $\quad (\cong V^*$ dualizability) d = 2



Z () $= A$

$$\mathbb{C} \xrightarrow{\text{unit}} A \xrightarrow{\text{m}^*} A \otimes A \xrightarrow{\text{swap}} A \otimes A \xrightarrow{\text{m}} A \xrightarrow{\text{tr}} \mathbb{C}$$

---

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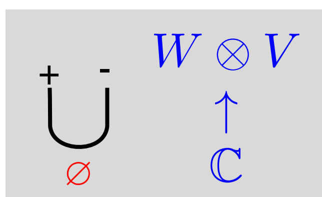

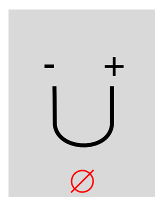

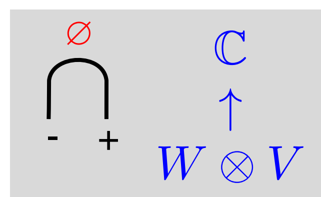

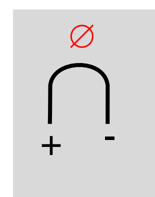

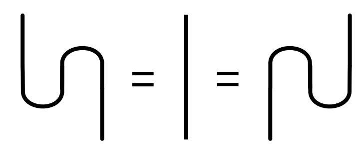

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Given a TQFT : Get invariants by CUTTING-ASSIGNING-COMPOSING

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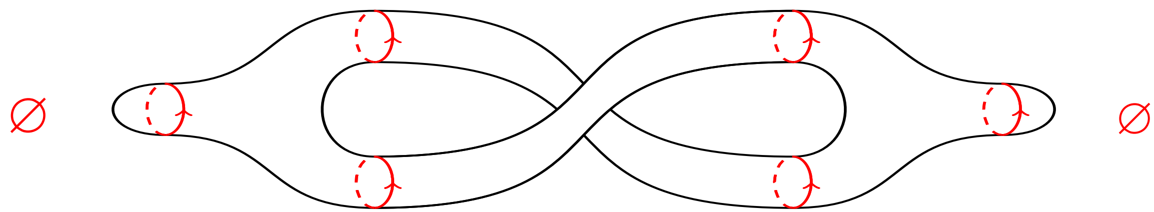

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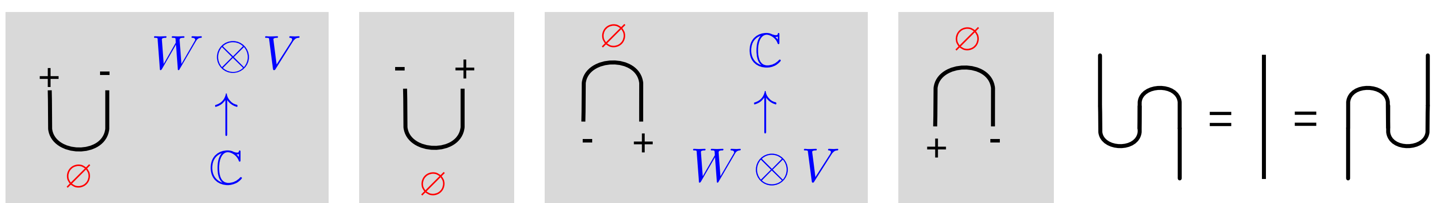

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> “quantum field theories have, because of the difficulties involved in constructing them, often been described axiomatically. This identifies their essential structural feature and postpones the question of their existence”
>
> Sir M. Atiyah (1988)

Given a TQFT : Get invariants by CUTTING-ASSIGNING-COMPOSING

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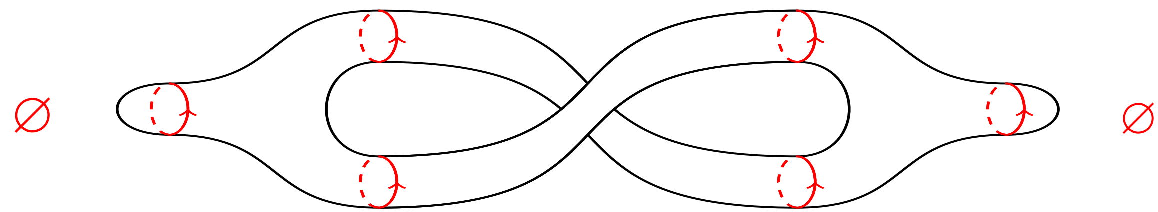

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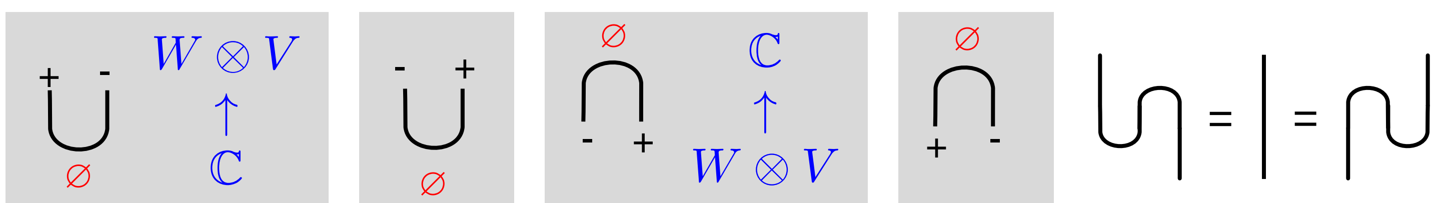

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Reshetikhin–Turaev $\mathfrak{gl}_k$ - invariants of tangles

$$\mathfrak{gl}_k = \mathrm{Lie}(\mathrm{GL}_k(\mathbb{C}))$$

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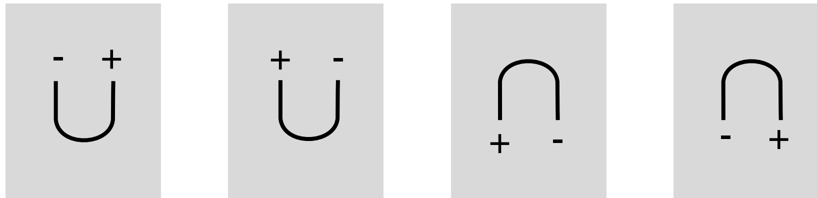

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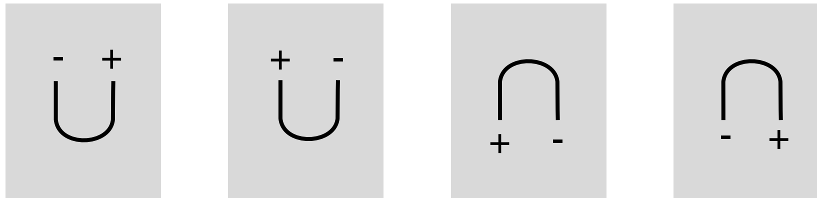

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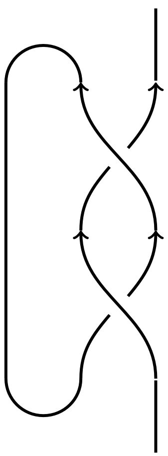

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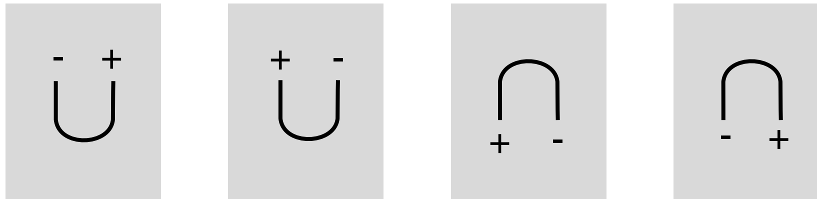

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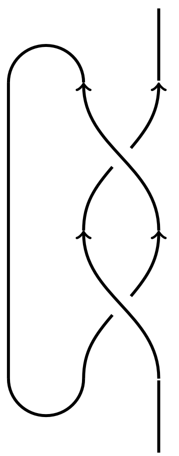

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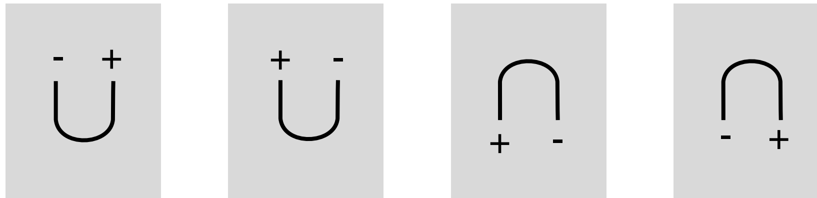

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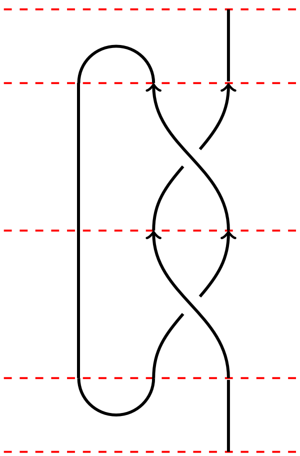

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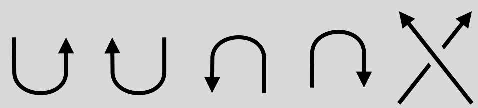 $\longmapsto$

Procedure: invariants by cutting-assigning-composing + checking relations

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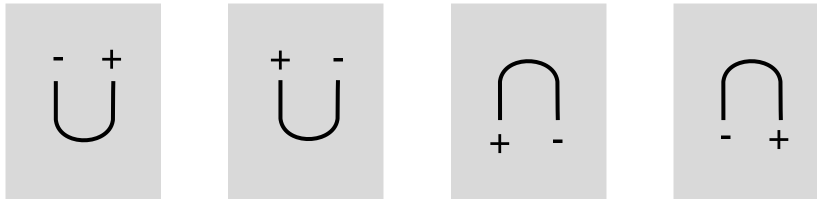

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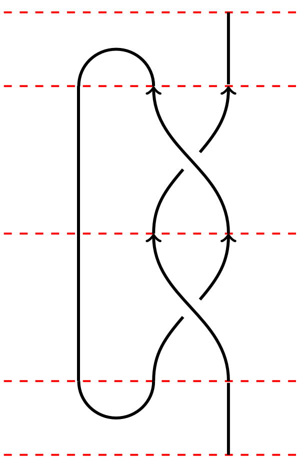

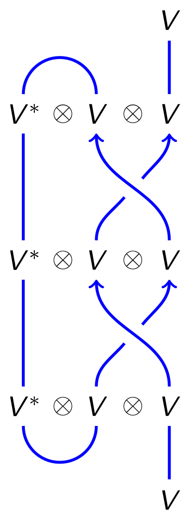

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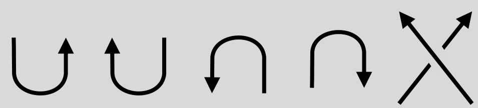 $\longmapsto$

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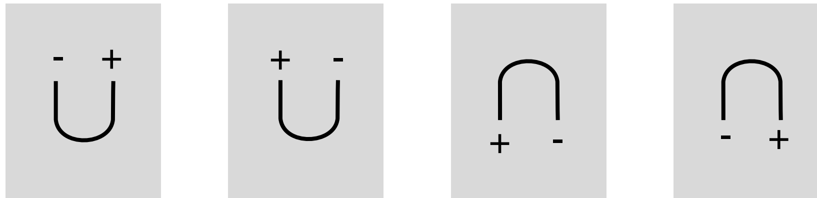

(co) evaluation maps

$V = \mathbb{C}^k$

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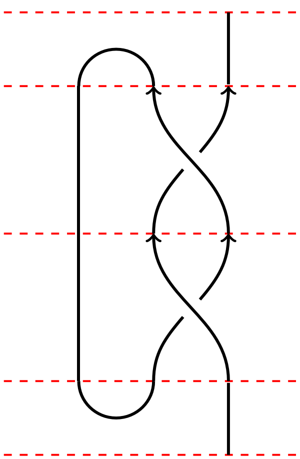

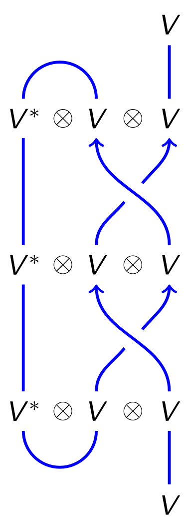

Reshetikhin–Turaev:

⊗ - functor

{tangles} $\longrightarrow$ {linear maps}

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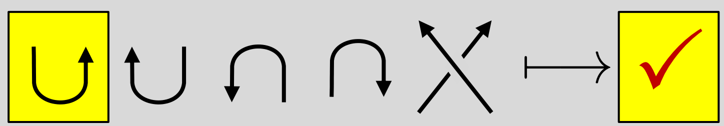

Procedure: invariants by cutting-assigning-composing + checking relations

Reshetikhin–Turaev $\mathfrak{gl}_k$ - invariants of tangles

$$\mathfrak{gl}_k = \mathrm{Lie}(\mathrm{GL}_k(\mathbb{C}))$$

d = 1

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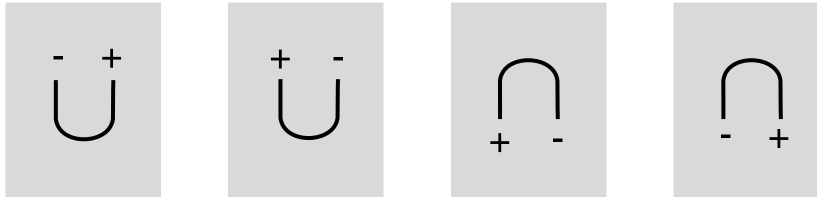

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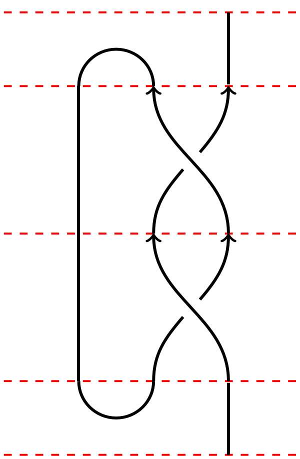

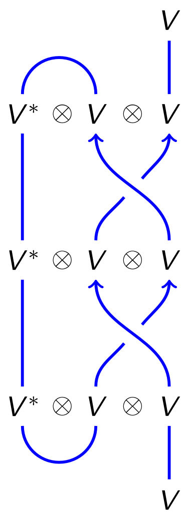

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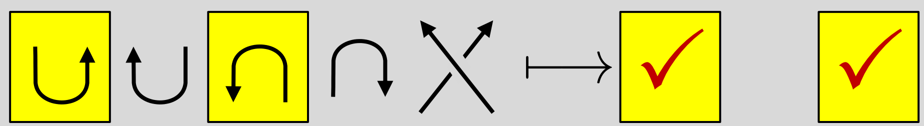

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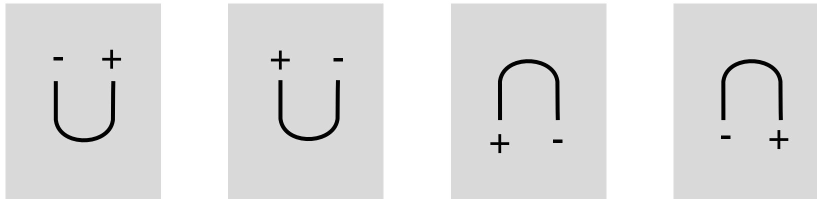

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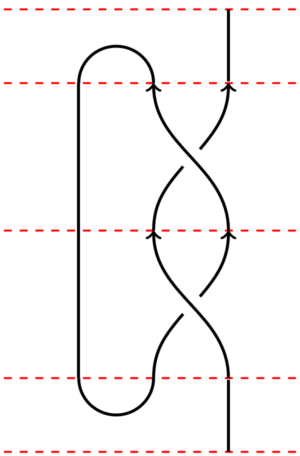

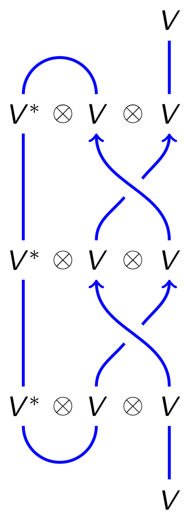

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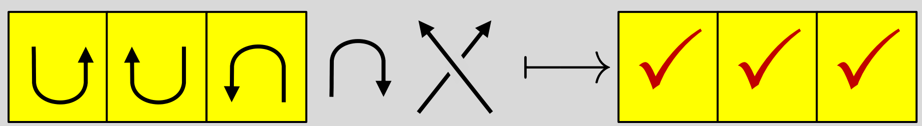

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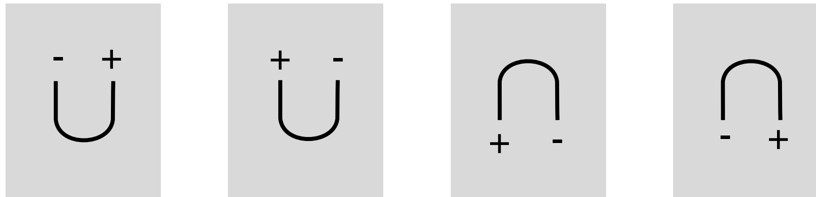

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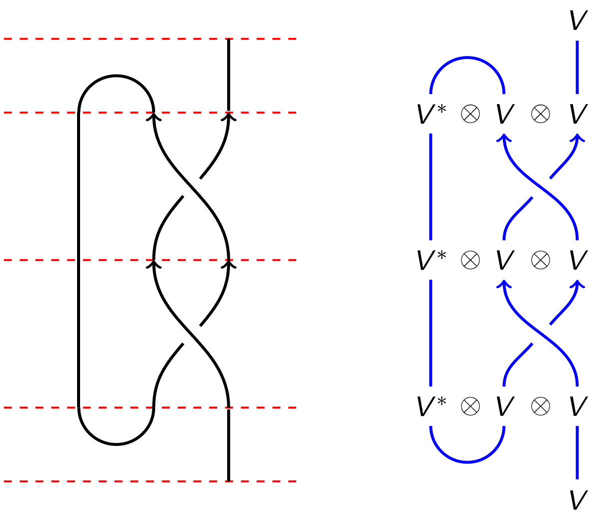

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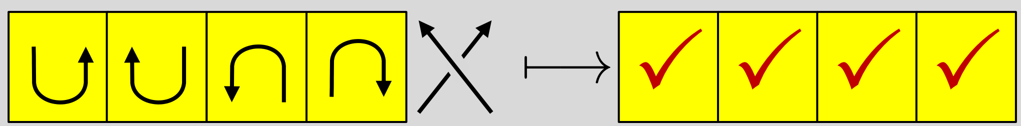

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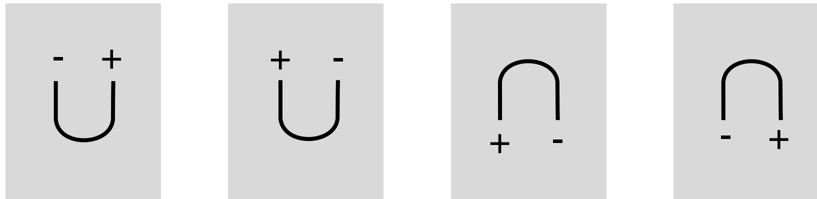

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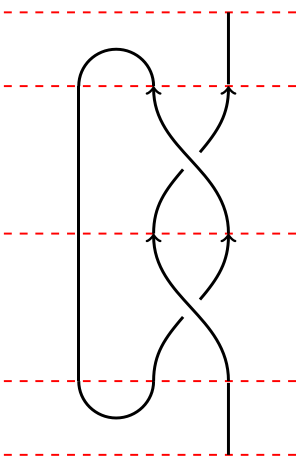

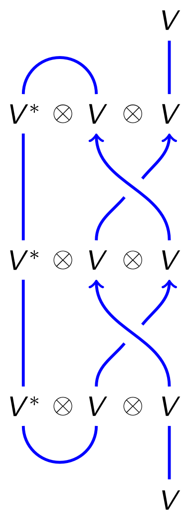

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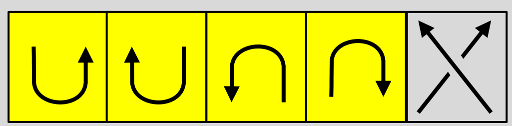
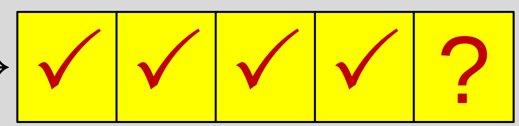
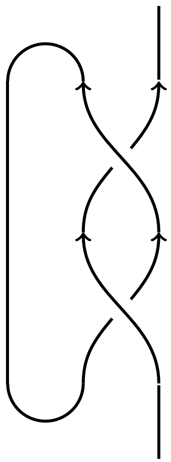
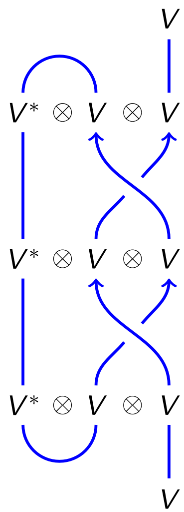
 $\longmapsto$ 

Procedure: invariants by cutting-assigning-composing + checking relations

What to do with braids ?





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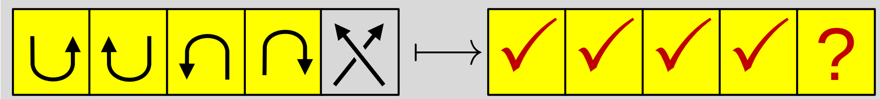

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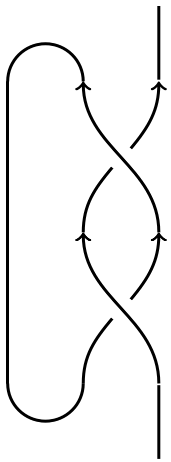

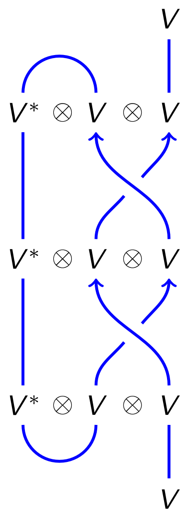

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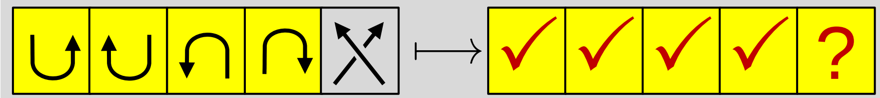

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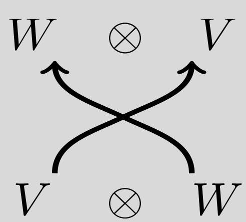
obvious map

symmetric
$\otimes$-category



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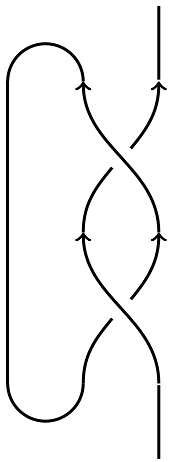

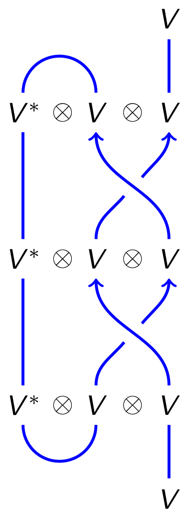

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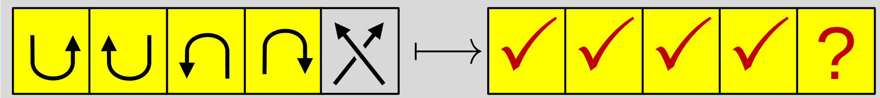

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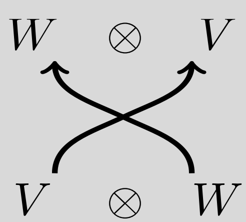

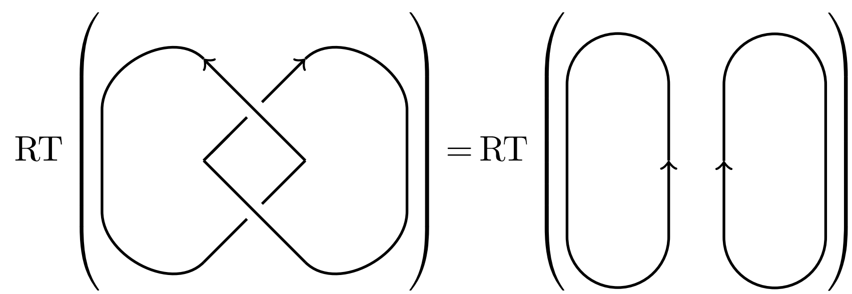

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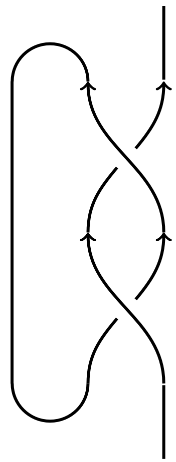

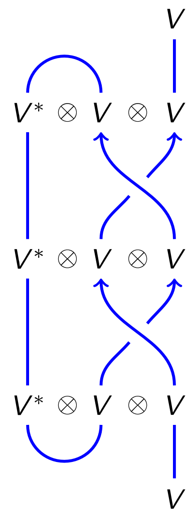

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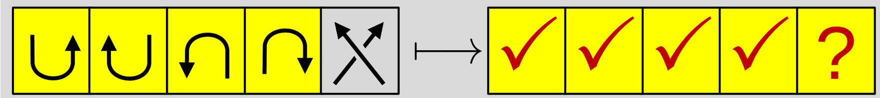

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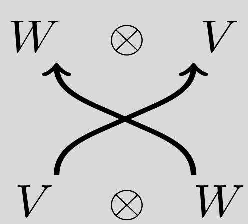

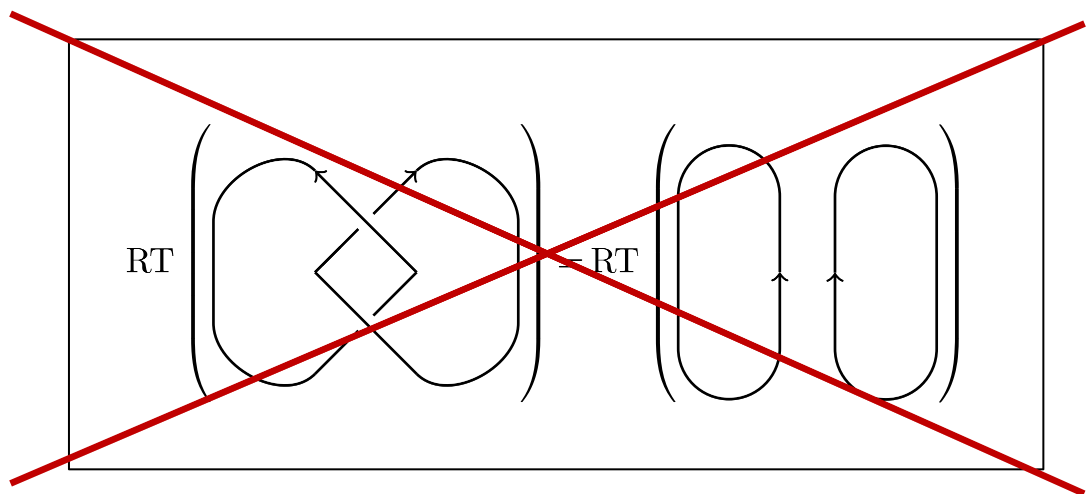

Quantum trick via Quantum groups (Faddeev school, Drinfeld, Jimbo 1980s)

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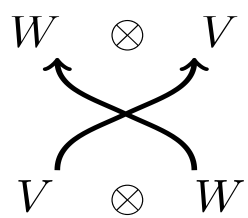

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braids

quantum trick

permutations of coordinates

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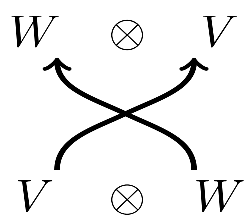

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$\mathfrak{gl}_k$-actions
representations of Lie algebras

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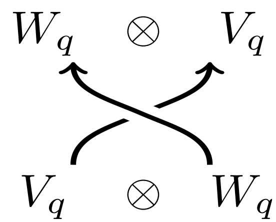

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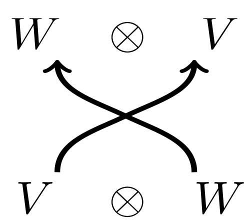

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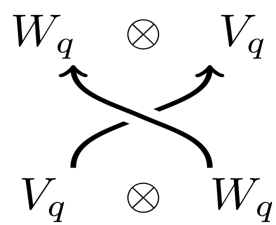

interesting

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braids

quantum

trick

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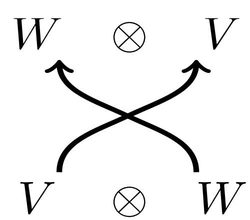

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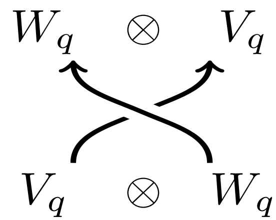

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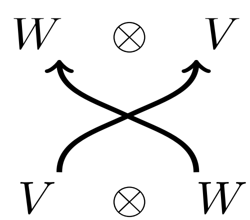

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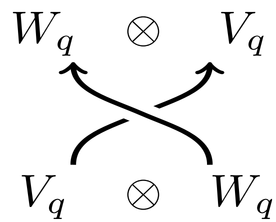

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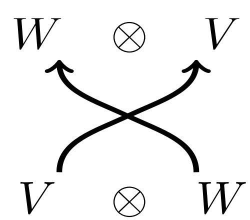

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Upshot: braids, knots and tangles turn into morphisms of representations of Quantum groups

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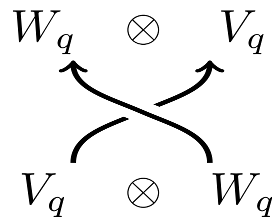

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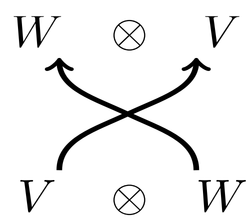

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for knots / links get $f : \mathbb{C}_q \to \mathbb{C}_q$ with $f(1) =$ Jones polynomial $\in \mathbb{Z}[q, q^{-1}]$ in case $k = 2$

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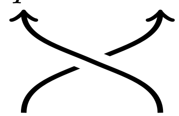
braided
$\otimes$-category

$W_q \quad \otimes \quad V_q$



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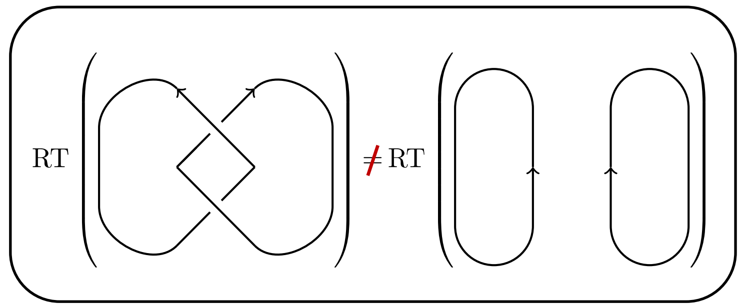

$$q^6 + q^4 + q^2 + 1 \quad \neq \quad q^2 + 2 + q^{-2}$$

symmetric
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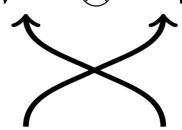

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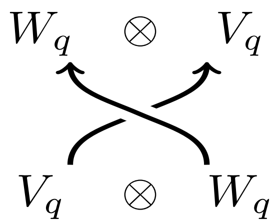

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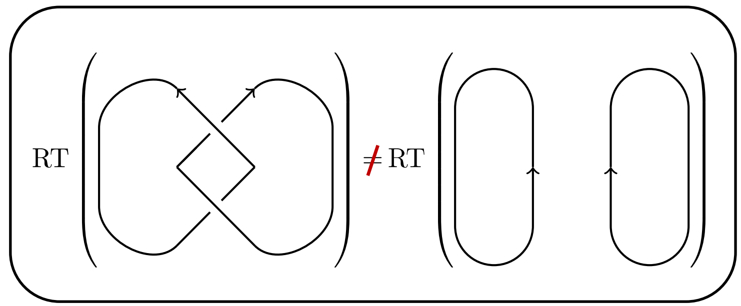

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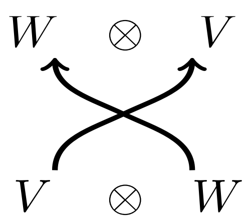

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More general $\mathfrak{gl}_k$ - representations ?

Used so far: $V = \mathbb{C}^k$

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What about $V = \wedge^d V$ for arbitrary d ?

?

More general $\mathfrak{gl}_k$ - representations ?

Used so far: $V = \mathbb{C}^k$

What about $V = \wedge^d V$ for arbitrary d ? **?**

Need to consider: $V = \wedge^{d_1} V \otimes V \wedge^{d_2} \otimes \cdots \otimes \wedge^{d_r} V$ **!**

More general $\mathfrak{gl}_k$ - representations ?

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What about $V = \wedge^d V$ for arbitrary d ?

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Need to consider: $V = \wedge^{d_1} V \otimes V \wedge^{d_2} \otimes \cdots \otimes \wedge^{d_r} V$

!

Invariants of knots, braid, tangles

Understand complicated representations !

More general $\mathfrak{gl}_k$ - representations ?

Used so far: $V = \mathbb{C}^k$ | What about $V = \wedge^d V$ for arbitrary d ? **?**

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classical

Decomposition? Constituents?

- combinatorics, partitions
- Clebsch–Gordan coefficients
- Kronecker coefficients (in #P)
- fusion rules, Pieri rules
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categorical

classical

- Maps / morphisms between the

$$\wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V$$

- Complete set of relations?

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- fusion rules, Pieri rules
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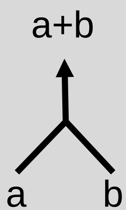
Upshot: understanding ALL morphisms = understanding the (category of) representations

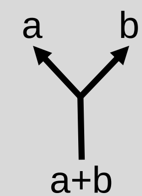
Constructing morphisms

generalised (co)evaluation morphisms

obvious morphisms: wedging and splitting



$$\textstyle\bigwedge^a V \otimes \bigwedge^b V \underset{\text{split}}{\overset{x\otimes y\mapsto x\wedge y}{\rightleftharpoons}} \bigwedge^{a+b} V$$



Constructing morphisms

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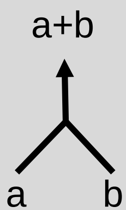

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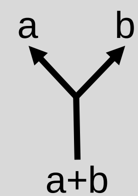

Provide morphisms between

$$\bigwedge^{d_1} V \otimes \bigwedge^{d_2} V \otimes \cdots \otimes \bigwedge^{d_r} V$$

$$d_1 + d_2 + \cdots + d_r = n$$

Constructing morphisms

generalised (co)evaluation morphisms

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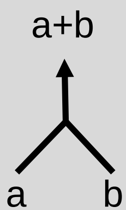

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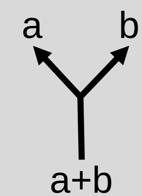

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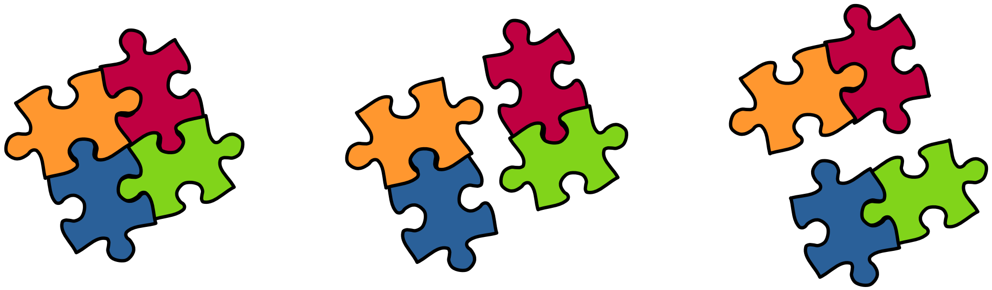
$$d_1 + d_2 + \cdots + d_r = n$$

Can we find ALL morphisms

?

The puzzle game

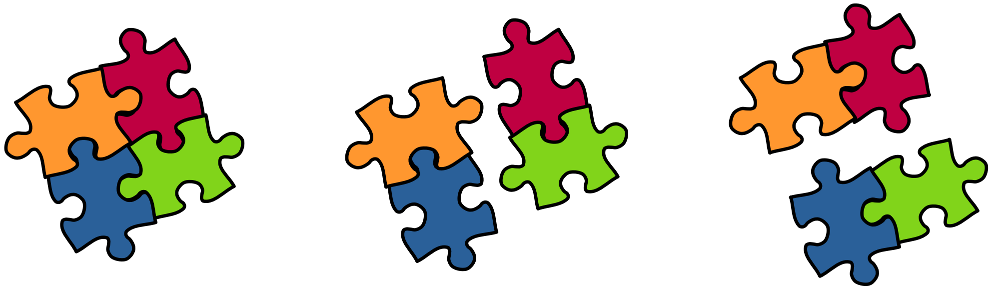
The puzzle game



The puzzle game for tensor products of maps

$$\boxed{\begin{array}{c} f \\ \circ \\ g \end{array}} \otimes \boxed{\begin{array}{c} h \\ \circ \\ k \end{array}} = \begin{array}{c} \boxed{f \otimes h} \\ \circ \\ \boxed{g \otimes k} \end{array}$$

The puzzle game



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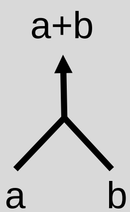
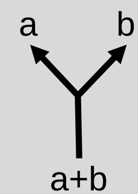


$$\textstyle\bigwedge^a V \otimes \bigwedge^b V \underset{\mathrm{split}}{\overset{x\otimes y \mapsto x\wedge y}{\rightleftharpoons}} \textstyle\bigwedge^{a+b} V$$



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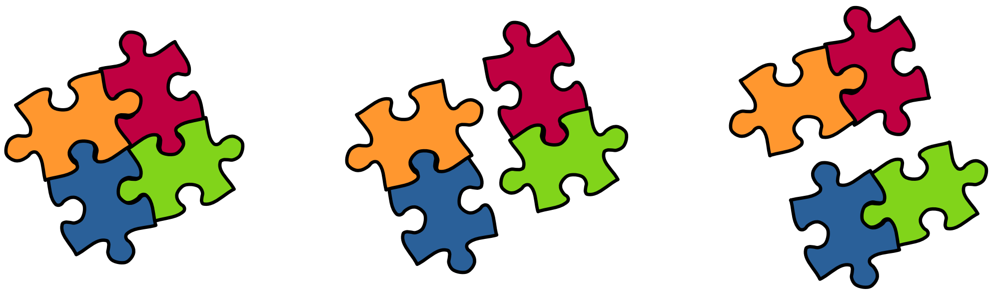
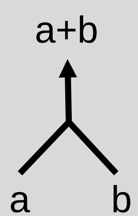
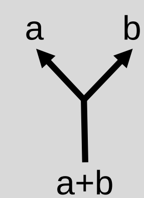


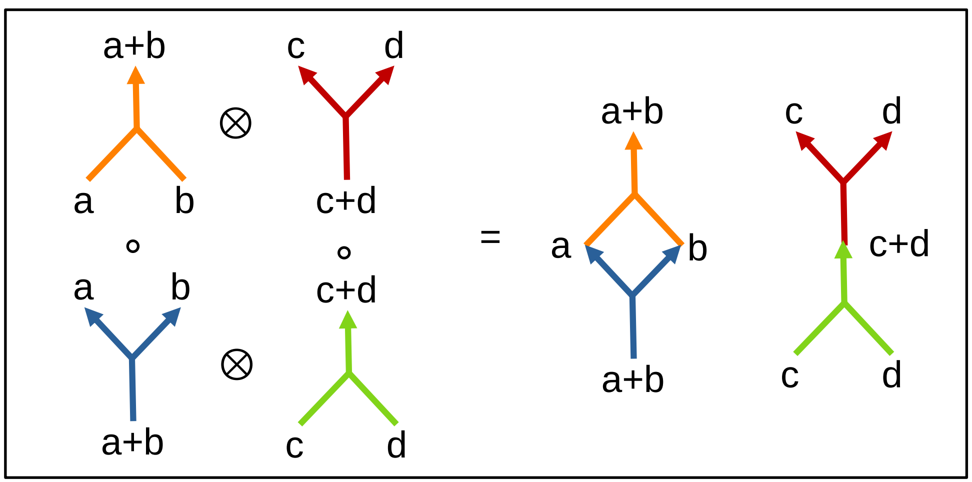
The puzzle game for tensor products of maps

$$(f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k)$$



$$\Lambda^a V \otimes \Lambda^b V \underset{\text{split}}{\overset{x\otimes y \mapsto x\wedge y}{\rightleftarrows}} \Lambda^{a+b} V$$





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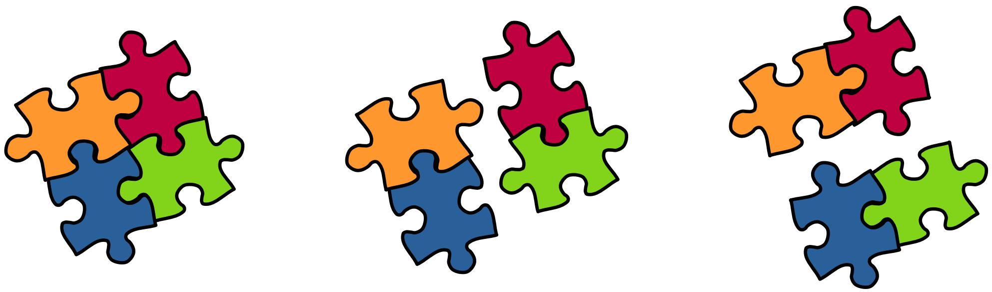

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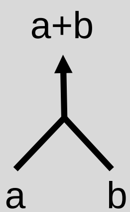

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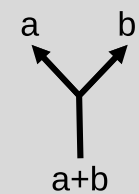
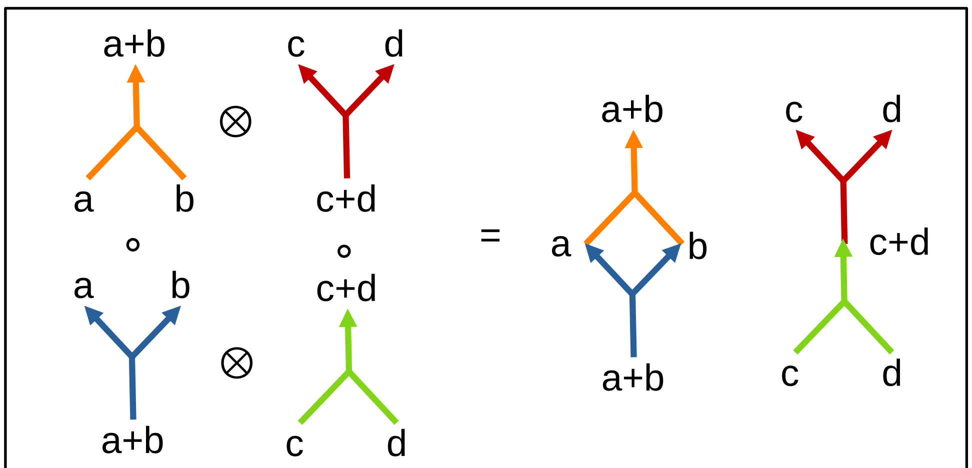
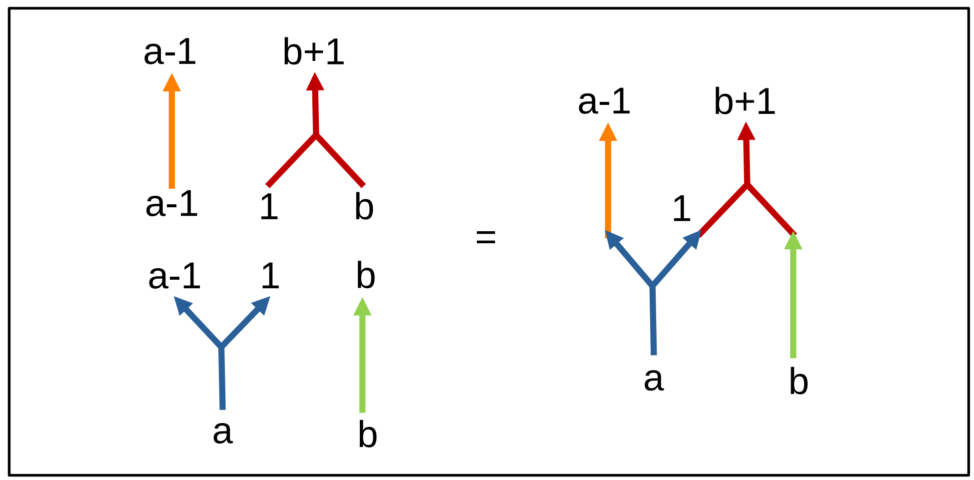
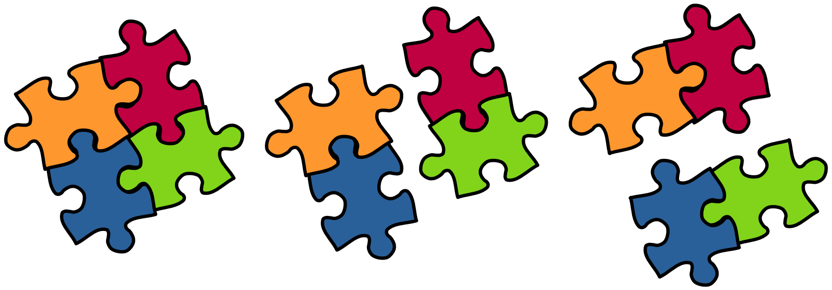




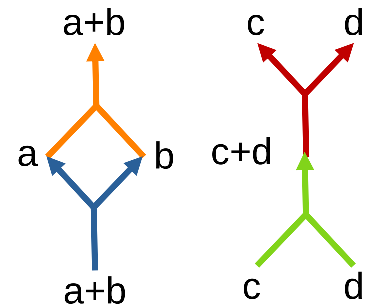


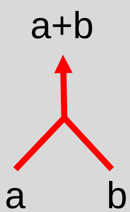
The puzzle game



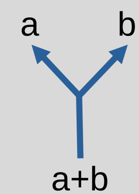
The puzzle game for tensor products of linear maps

$$(f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k)$$



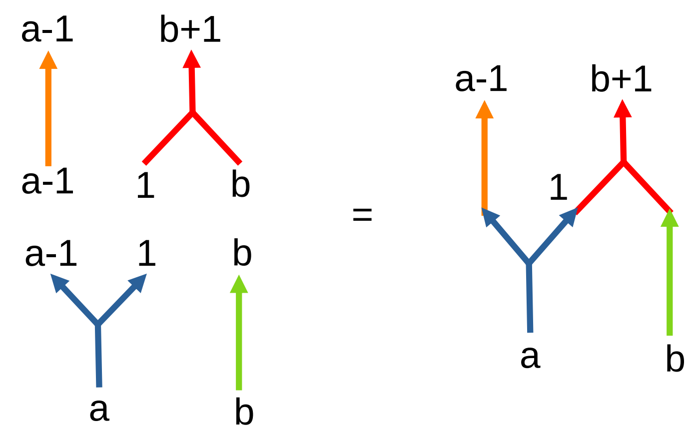


$$\bigwedge^a V \otimes \bigwedge^b V \underset{\text{split}}{\overset{x\otimes y\mapsto x\wedge y}{\rightleftarrows}} \bigwedge^{a+b} V$$



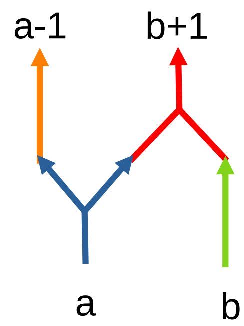
Can compose them vertically and horizontally and take linear combinations

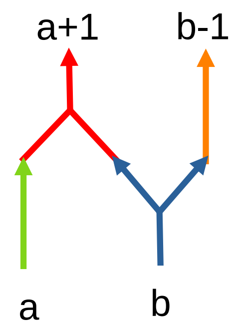
(Linear) **GLUING**



(Linear) gluing

$$\begin{array}{c} \bigwedge^{a-1} V \otimes \bigwedge^{b+1} V \\ \uparrow \\ \bigwedge^{a} V \otimes \bigwedge^{b} V \end{array}$$





$$\begin{array}{c} \bigwedge^{a+1} V \otimes \bigwedge^{b-1} V \\ \uparrow \\ \bigwedge^{a} V \otimes \bigwedge^{b} V \end{array}$$

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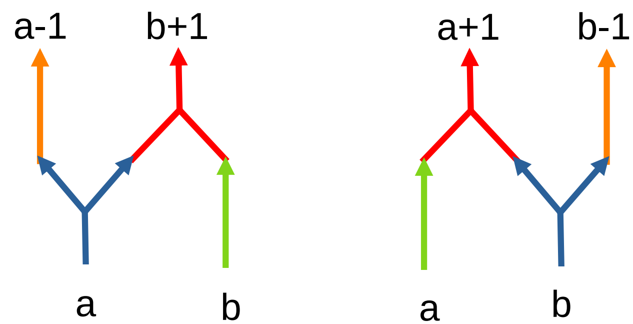
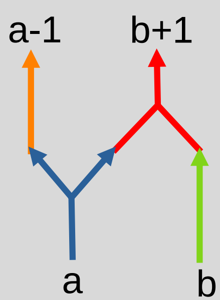
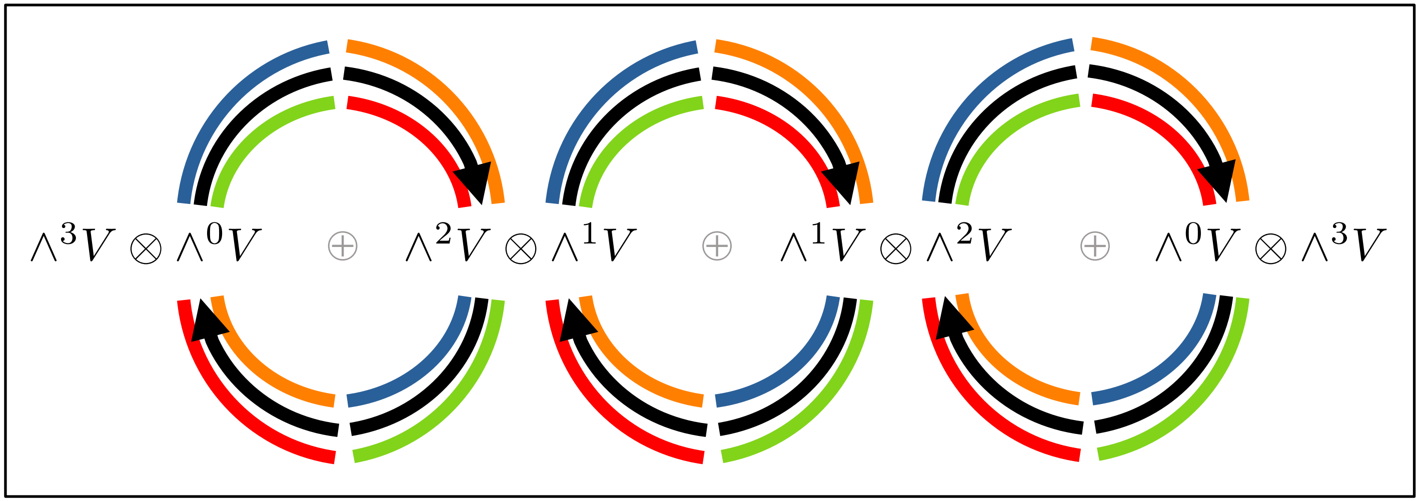


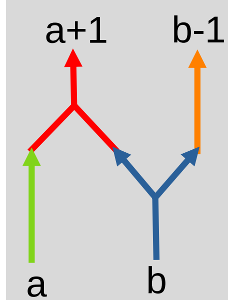
$$\begin{array}{c}\wedge^{a+1}V \otimes \wedge^{b-1}V \\ \uparrow \\ \wedge^{a}V \otimes \wedge^{b}V\end{array}$$

Give morphisms:





$$\wedge^3 V \otimes \wedge^0 V \quad \oplus \quad \wedge^2 V \otimes \wedge^1 V \quad \oplus \quad \wedge^1 V \otimes \wedge^2 V \quad \oplus \quad \wedge^0 V \otimes \wedge^3 V$$



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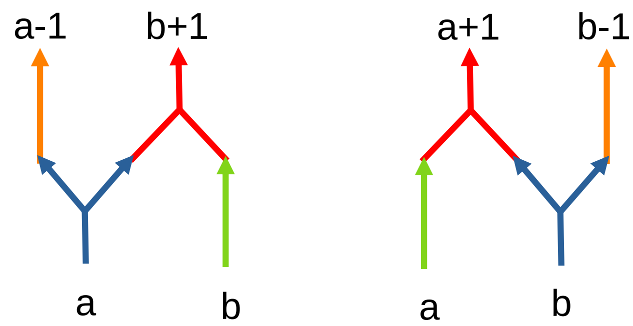

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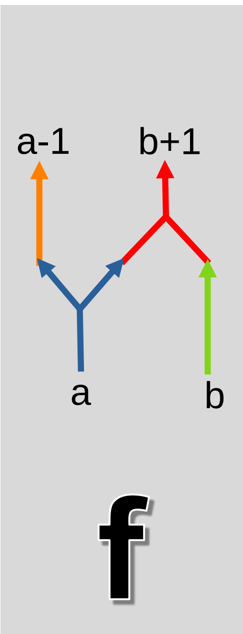
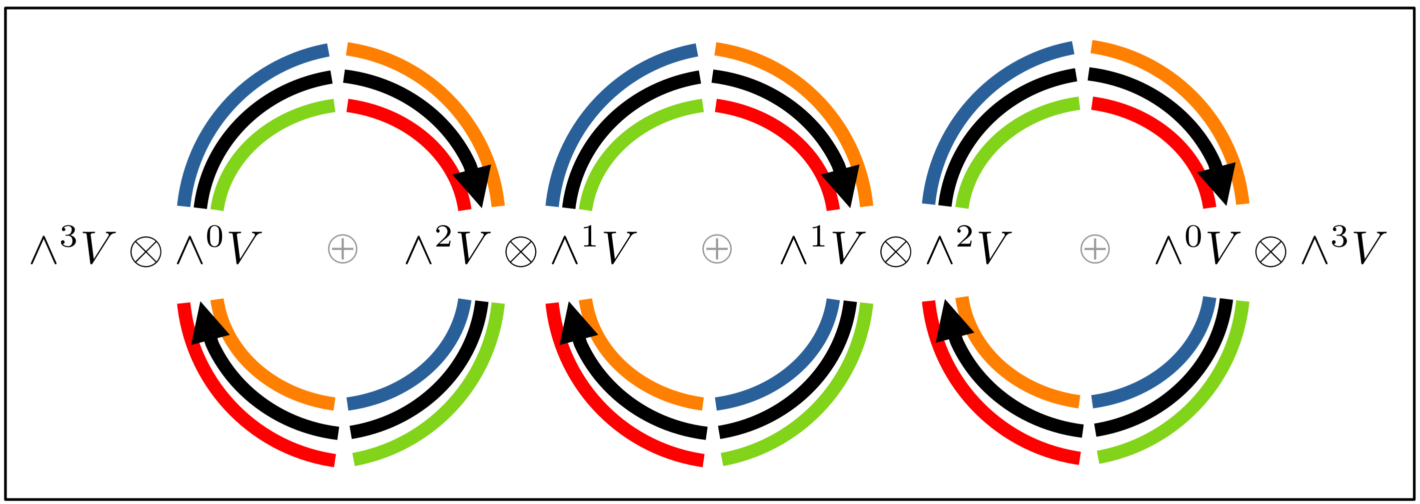
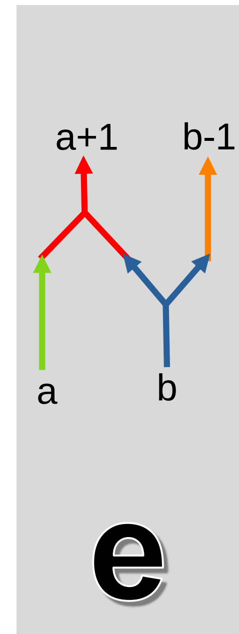
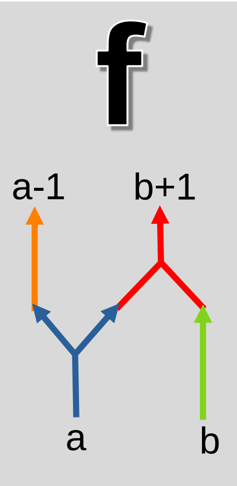


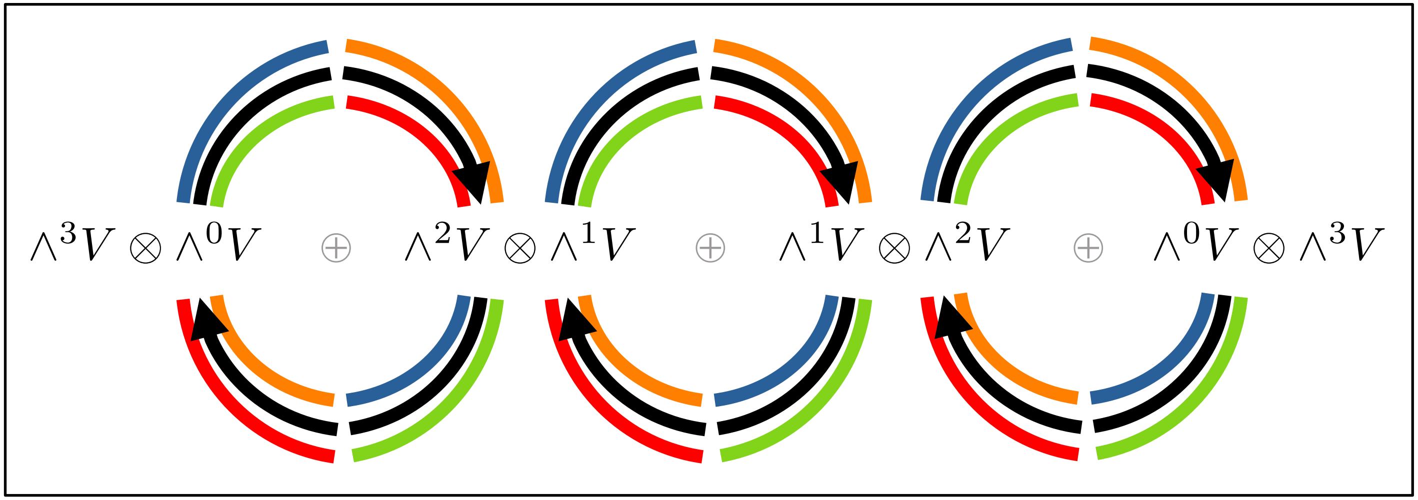


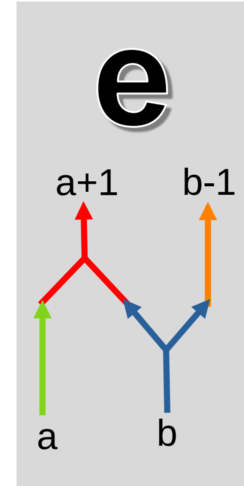


$\mathfrak{gl}_2$-action

Lie algebra actions everywhere !



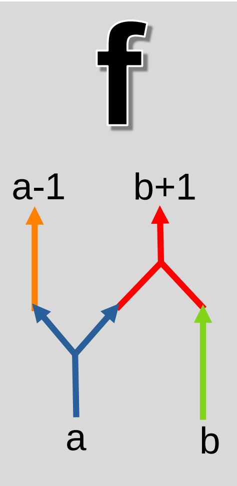


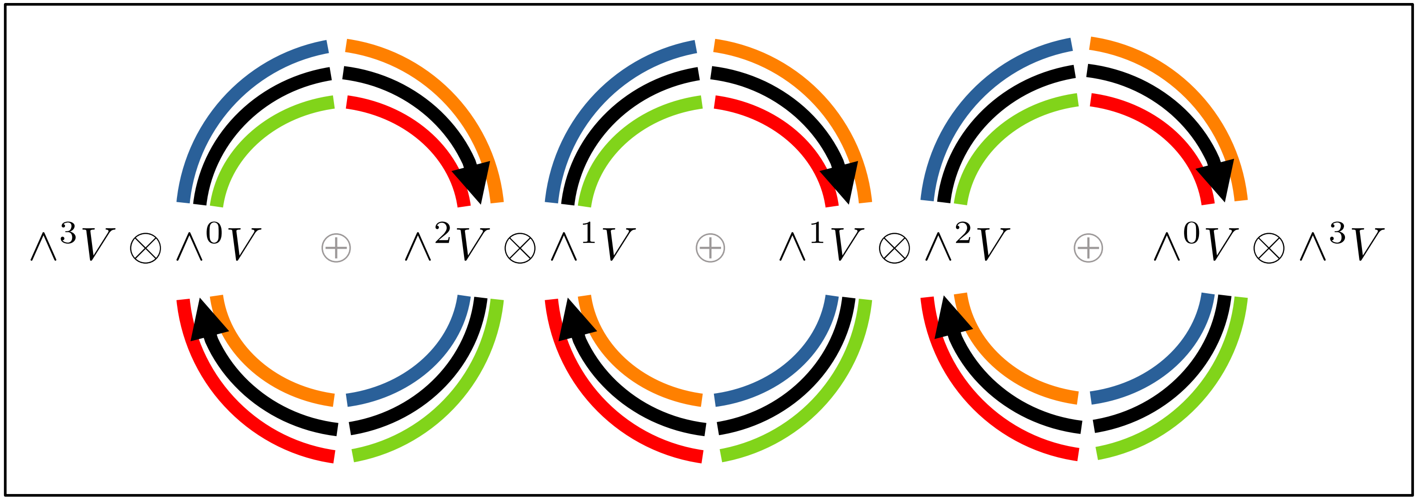


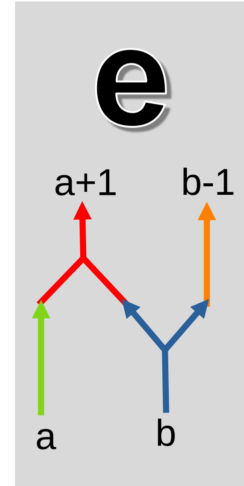
Theorem (Howe 1990s)

- $\mathfrak{gl}_k \curvearrowright \displaystyle\bigoplus_{d_1+d_2+\cdots+d_r=n} \wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V \curvearrowleft \mathfrak{gl}_r$
- Gluing gives all morphisms between the $\wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V$

Lie algebra actions everywhere !







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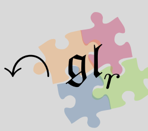
- $\mathfrak{gl}_k \curvearrowright \bigoplus_{d_1+d_2+\cdots+d_r=n} \wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V \curvearrowleft \mathfrak{gl}_r$ 
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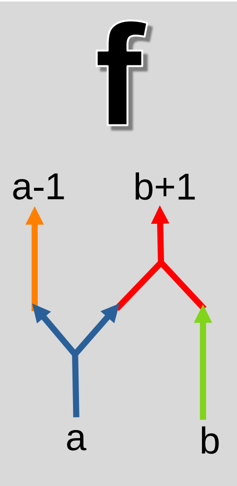
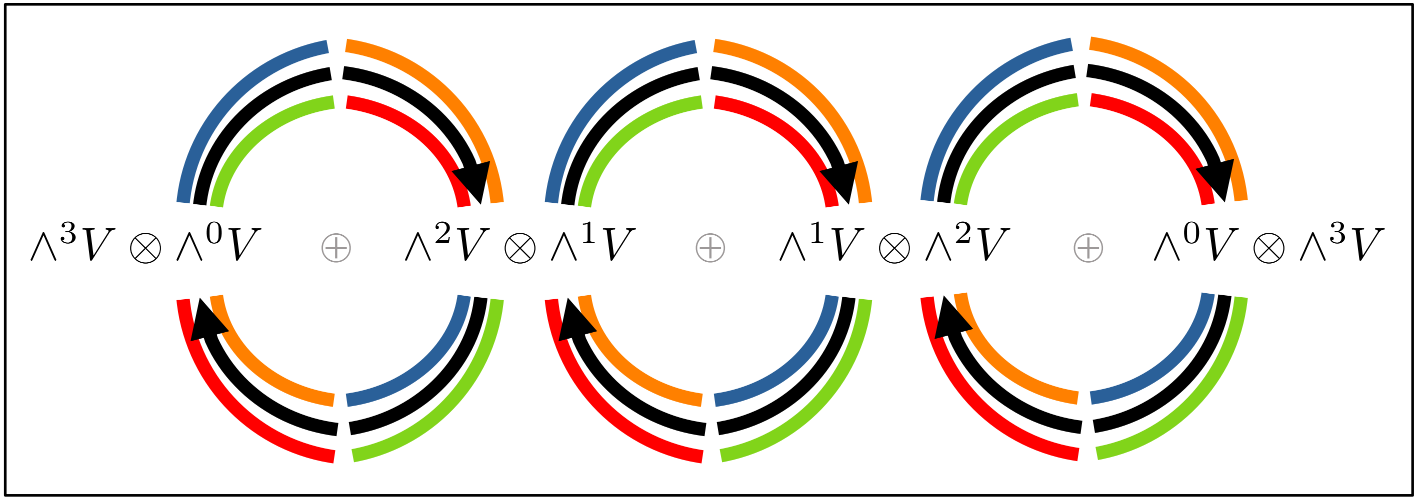
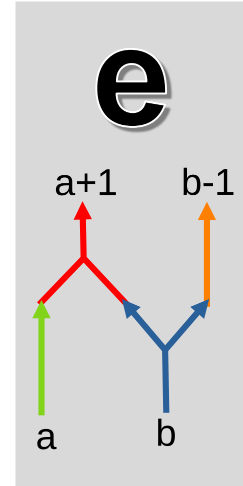
Upshot: (another) Lie algebra action controls ALL ✓ morphisms

Lie algebra actions everywhere !







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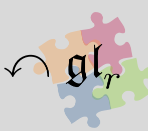
- $\mathfrak{gl}_k \curvearrowright \quad \bigoplus_{d_1+d_2+\cdots+d_r=n} \wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V \quad \curvearrowleft \mathfrak{gl}_r$ 
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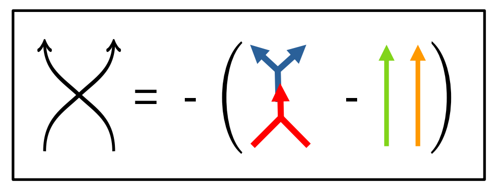


Upshot: (another) Lie algebra action controls ALL ✓ morphisms

Quantum group actions everywhere !

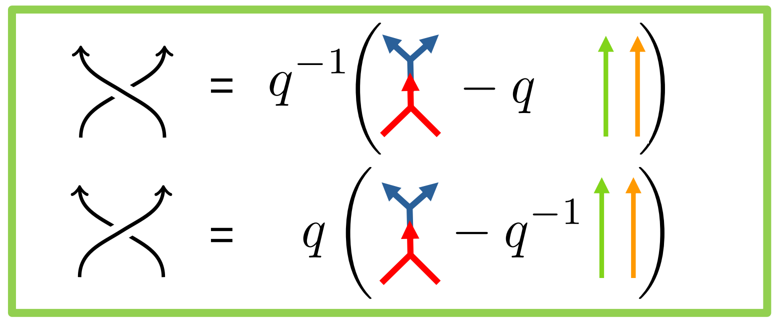
Theorem (Quantum Howe duality)

- $U_q(\mathfrak{gl}_k) \curvearrowright \bigoplus_{d_1+d_2+\cdots+d_r=n} \wedge^{d_1} V_q \otimes \cdots \otimes \wedge^{d_r} V_q \curvearrowleft U_q(\mathfrak{gl}_r)$
- Gluing gives all morphisms between the $\wedge^{d_1} V_q \otimes \cdots \otimes \wedge^{d_r} V_q$

braids

quantum trick

permutations



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braids

quantum trick

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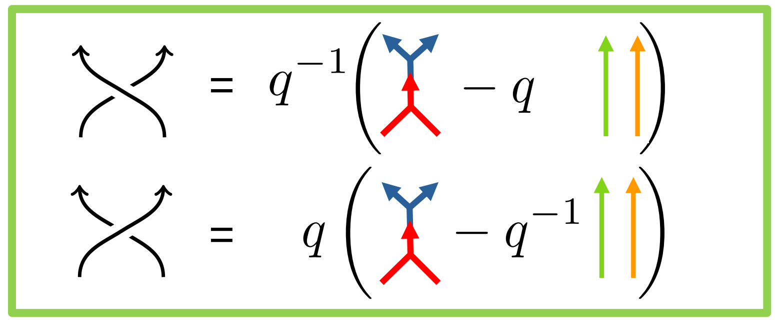

- Complete set of relations
  (Cautis–Kamnitzer–Morrison 2014)

Upshot: ALL morphisms thus links / braids / tangles are controlled by Quantum group actions

Quantum group actions everywhere !

Theorem (Quantum Howe duality)

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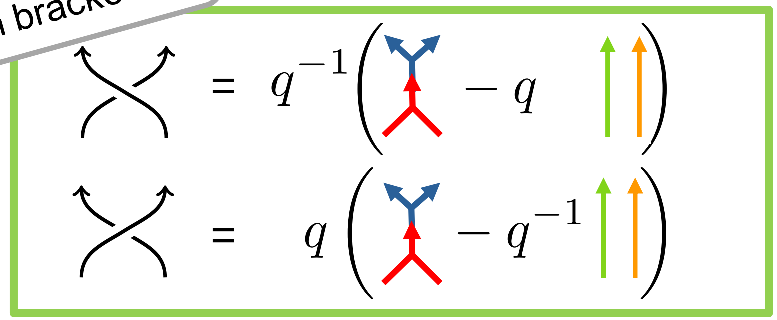
braids

quantum trick

permutations

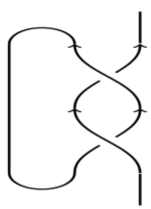
skein theory, generalized Kauffman bracket



- Complete set of relations (Cautis–Kamnitzer–Morrison 2014)

Upshot: ALL morphisms ✓ thus links / braids / tangles ✓ are controlled by Quantum group actions

Invariants of tangle bordisms ?



tangles

oriented points

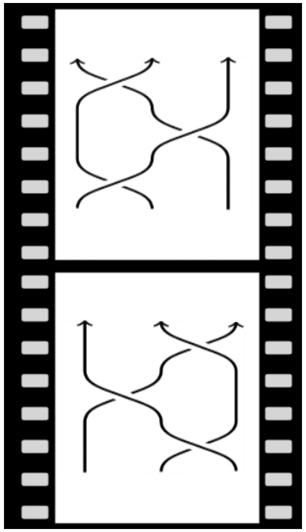
dim 1

+ -

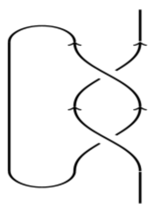
dim 0

added braids

Invariants of tangle bordisms ?



tangle bordisms



tangles

oriented points

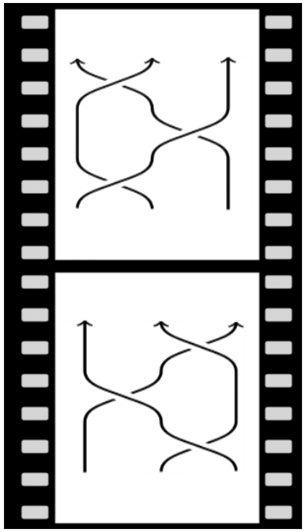
added braids

dim 1

\+ -

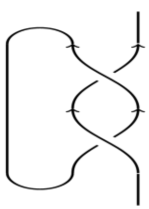
dim 0

Invariants of tangle bordisms ?



invariants ?

tangle bordisms



tangles

oriented points

?

$\cup$ $\cap$ dim 1

$\overset{+}{\bullet}$ $\overset{-}{\bullet}$ dim 0

added braids

Higher dimensional TQFTs? (1990–2015)

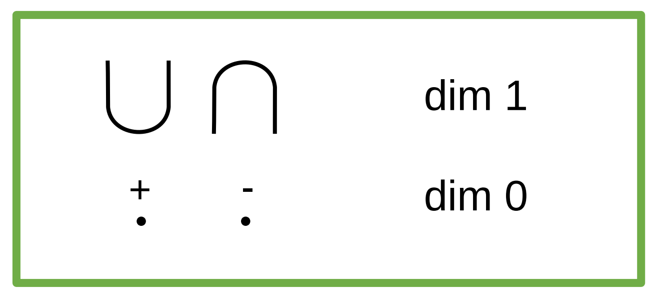
Observation:
Cutting along codim=1 submanifolds might not suffice

dim d
dim d-1
dim d-2

dim 1
dim 0

want to include

?



Higher dimensional TQFTs? (1990–2015)

Observation:
Cutting along codim=1 submanifolds might not suffice

d
d-1
⋮
0

\- TQFT

(Baez–Dolan, Lurie, …)

dim d
dim d-1
dim d-2

dim 1
dim 0

want to include

?

dim 1

\+ •   - •

dim 0

Higher dimensional TQFTs? (1990–2015)

Observation:
Cutting along codim=1 submanifolds might not suffice

d
d-1
⋮
0
- TQFT

(Baez–Dolan, Lurie, …)

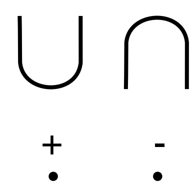
dim d
dim d-1
dim d-2
dim 1
dim 0

want to include

d = 2

2
1
0
- TQFT

?



dim 1

dim 0

Higher dimensional TQFTs? (1990–2015)

Observation:
Cutting along codim=1 submanifolds might not suffice

dim d
dim d-1
dim d-2
dim 1
dim 0

$\Big\}$ want to include

$\begin{pmatrix} d \\ d\text{-}1 \\ \vdots \\ 0 \end{pmatrix}$ - TQFT

(Baez–Dolan, Lurie, …)

---

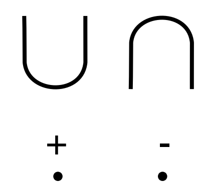
d = 2

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ - TQFT

$$\text{symmetric } \otimes\text{-functor: } \mathrm{Cob}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \mathrm{Vect}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

(Schommer-Pries,… 1990-2014)

?



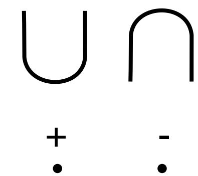
dim 1

dim 0

$\mathrm{Vect}\begin{pmatrix}2\\1\\0\end{pmatrix}$

Higher dimensional TQFTs ?

natural transformation	$\longleftarrow$	dim 2	bordisms
linear functor		dim 1	
$\mathbb{C}$-linear category		dim 0	



$\mathrm{Vect}\begin{pmatrix}2\\1\\0\end{pmatrix}$

Higher dimensional TQFTs ?

natural transformation
linear functor
$\mathbb{C}$-linear category

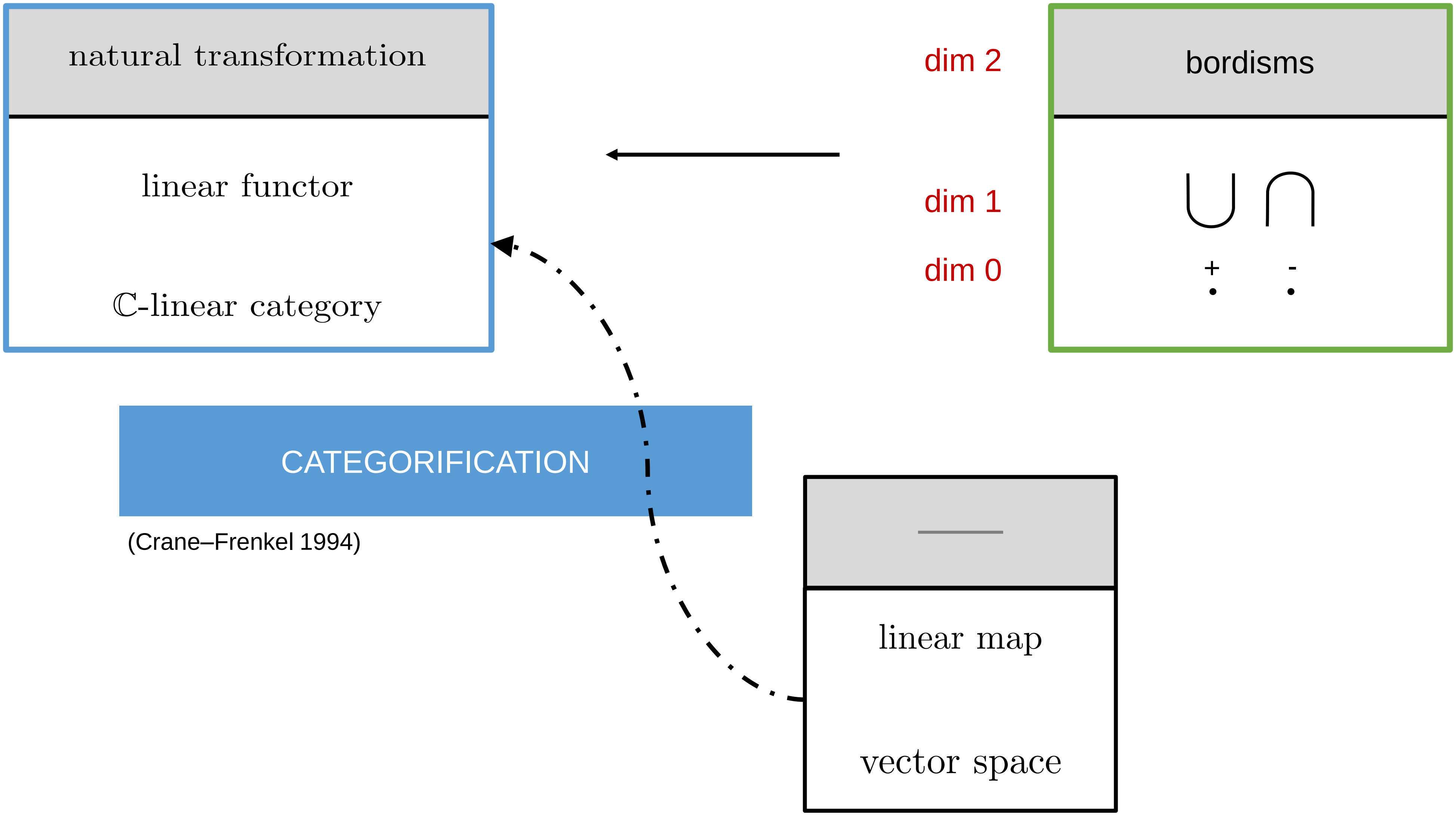
$\longleftarrow$

	bordisms
dim 2	
dim 1	$\cup$ $\cap$
dim 0	$\overset{+}{\bullet}$ $\overset{-}{\bullet}$

—
linear map
vector space

Vect $\begin{pmatrix}2\\1\\0\end{pmatrix}$

Higher dimensional TQFTs ?



Vect $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Higher dimensional TQFTs ?

natural transformation

linear functor

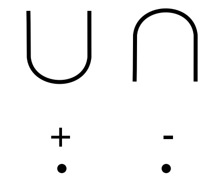
$\mathbb{C}$-linear category

dim 2

dim 1

dim 0

bordisms



CATEGORIFICATION

(Crane–Frenkel 1994)

rather an art than a theory

———

linear map

vector space

Categorification ?

Categorification ?

Vector spaces, linear maps, vectors

$$\begin{tikzcd} \Lambda^3 V \otimes \Lambda^0 V \arrow[r, shift left] & \Lambda^2 V \otimes \Lambda^1 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^1 V \otimes \Lambda^2 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^0 V \otimes \Lambda^3 V \arrow[l, shift left] \end{tikzcd}$$

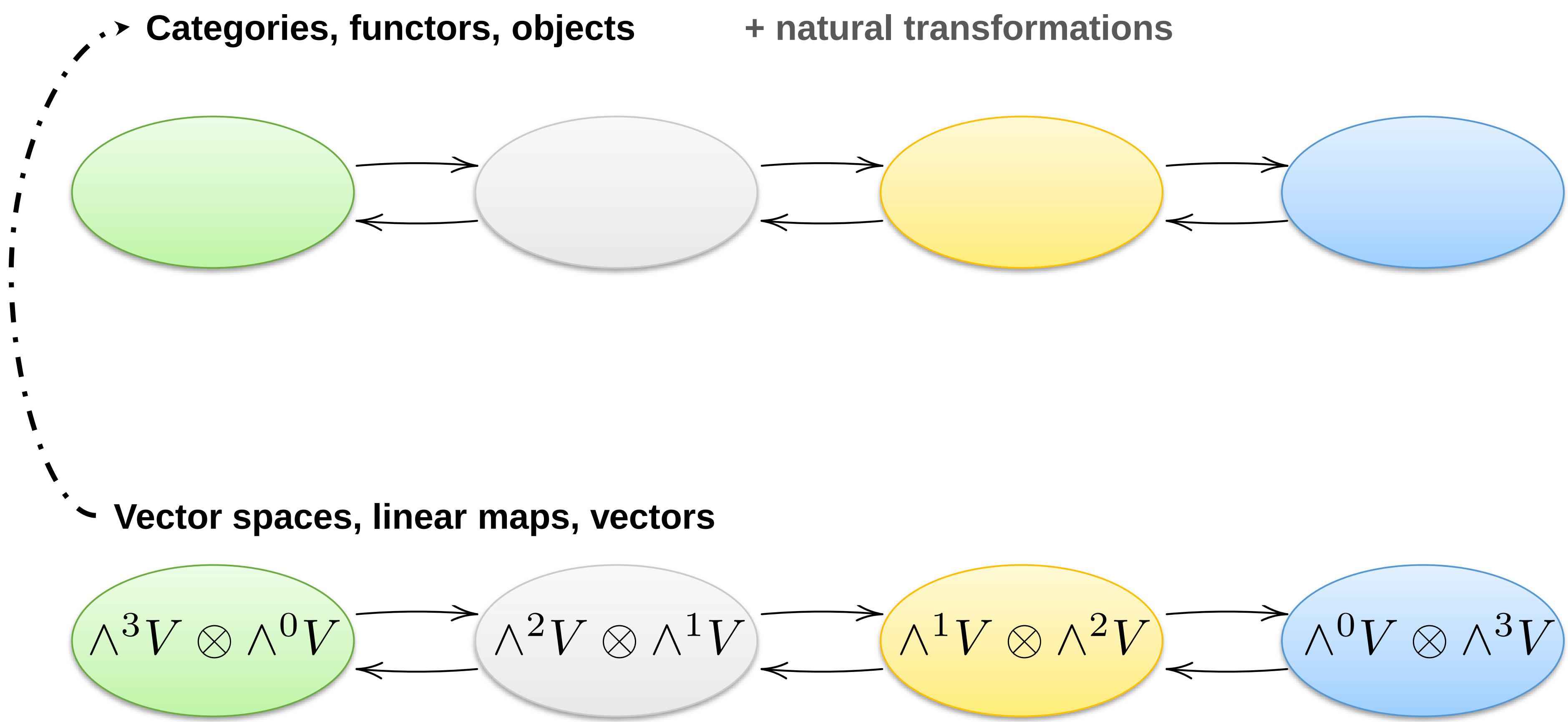
Categorification ?

Categories, functors, objects **+ natural transformations**

Vector spaces, linear maps, vectors

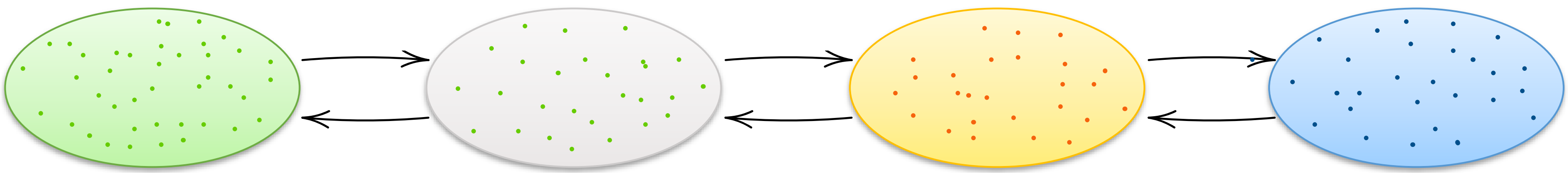
$$\begin{tikzcd} \bigwedge^3 V \otimes \bigwedge^0 V \arrow[r, shift left] & \bigwedge^2 V \otimes \bigwedge^1 V \arrow[l, shift left] \arrow[r, shift left] & \bigwedge^1 V \otimes \bigwedge^2 V \arrow[l, shift left] \arrow[r, shift left] & \bigwedge^0 V \otimes \bigwedge^3 V \arrow[l, shift left] \end{tikzcd}$$

Categorification ?



Categorification ?

Categories, functors, objects + natural transformations

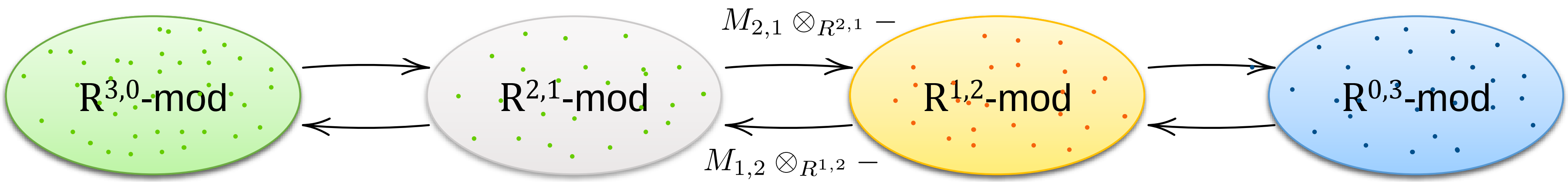


Vector spaces, linear maps, vectors

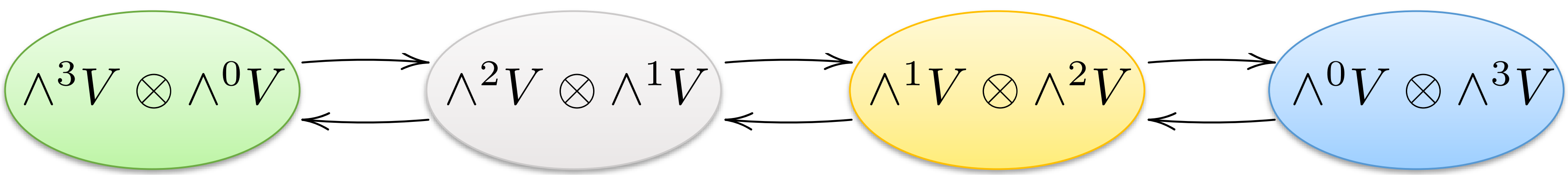
$$\begin{tikzcd} \Lambda^3 V \otimes \Lambda^0 V \arrow[r, shift left] & \Lambda^2 V \otimes \Lambda^1 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^1 V \otimes \Lambda^2 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^0 V \otimes \Lambda^3 V \arrow[l, shift left] \end{tikzcd}$$

Categorification ?

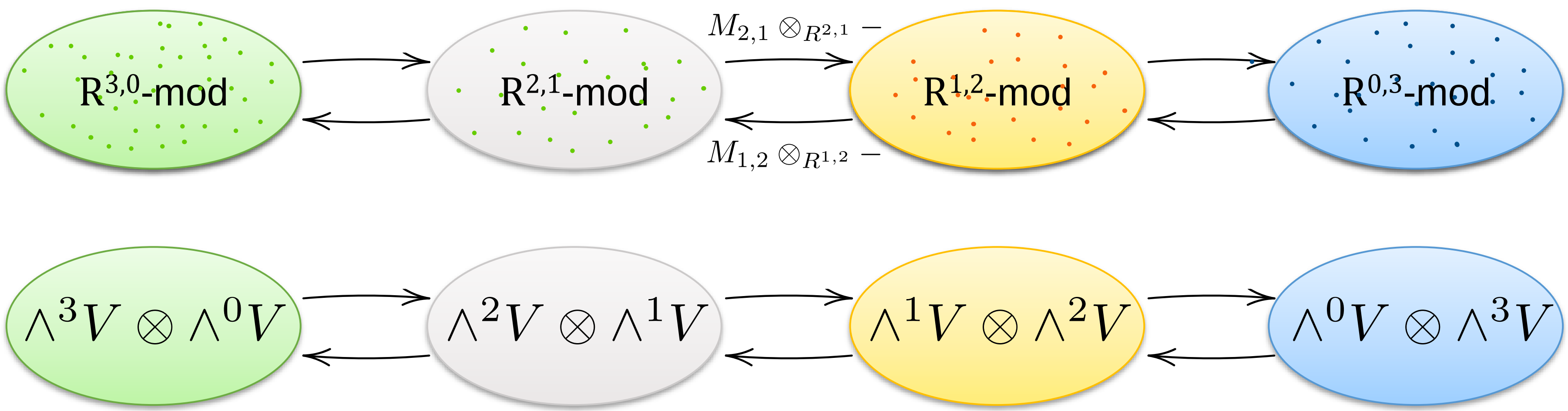
Categories, functors, objects **+ natural transformations**



Vector spaces, linear maps, vectors



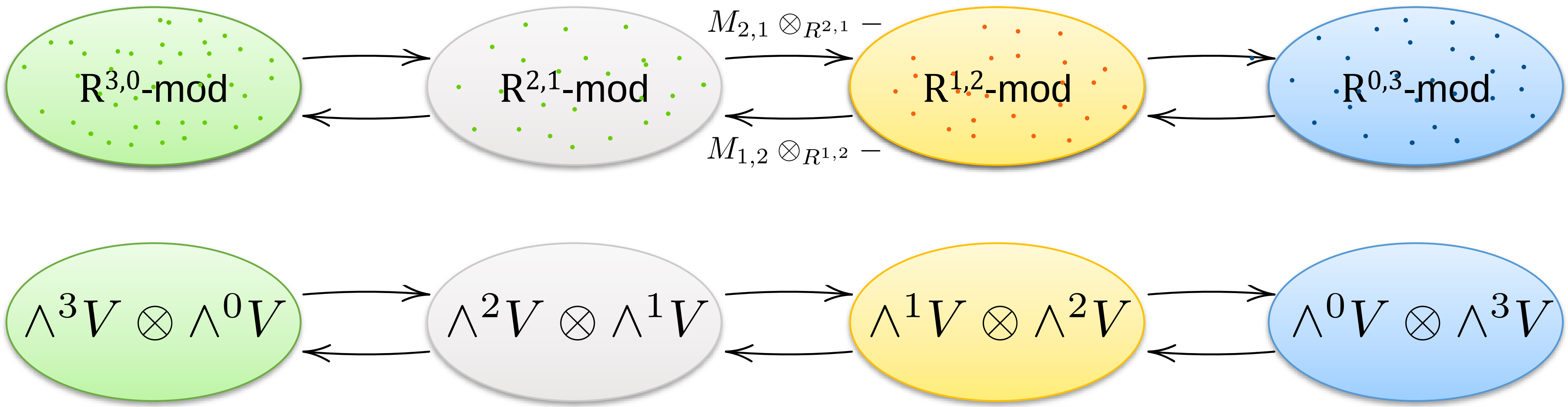
Trying to find a solution …



Trying to find a solution …

$S_a \times S_b \subseteq S_3 \quad \curvearrowright \quad R^{a,b} = \mathbb{C}[x_1, x_2, x_3]^{S_a \times S_b} \quad \supseteq \quad \mathbb{C}[x_1, x_2, x_3]^{S_{a+b}} = R^{a+b}$

$a + b = 3$ (partial) invariants



Trying to find a solution …

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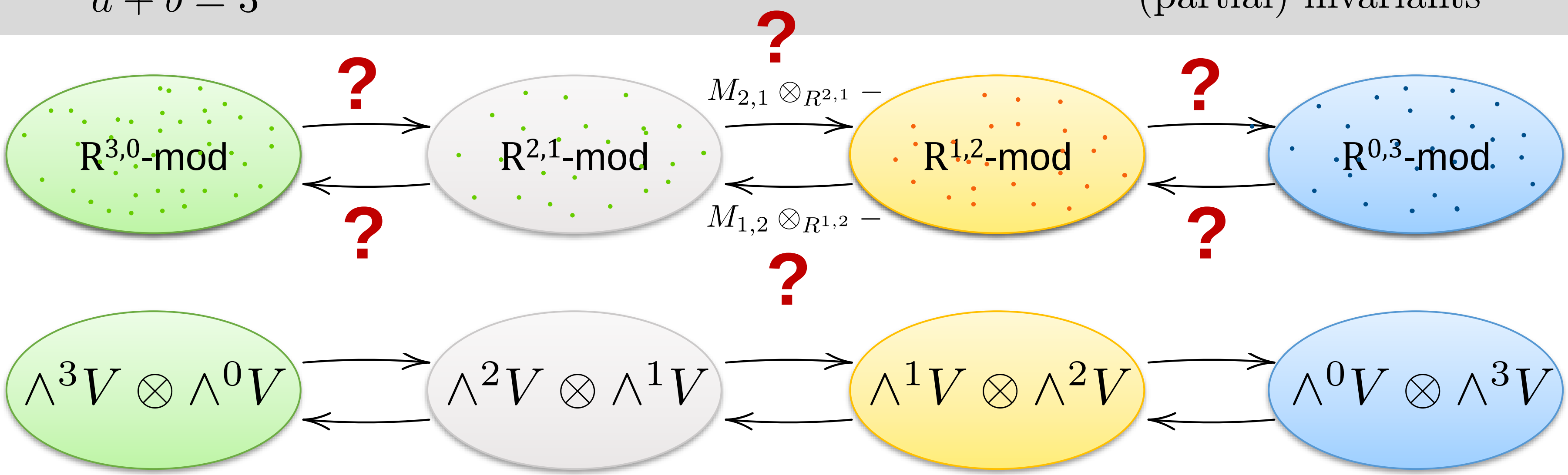
$a + b = 3$ (partial) invariants

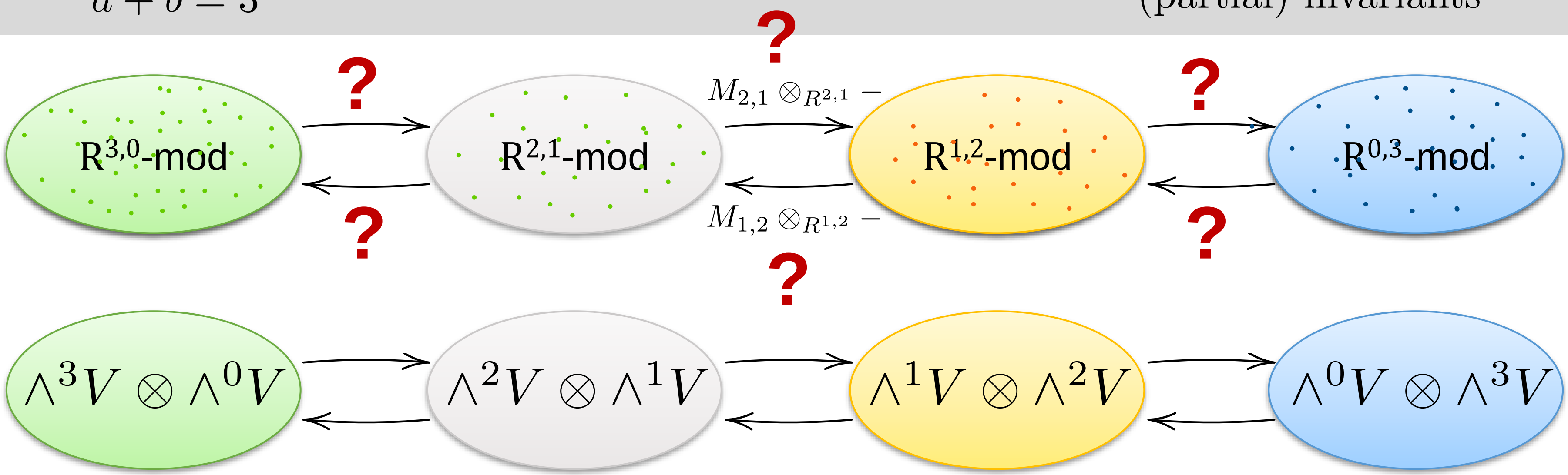


What to assign to linear maps ?

Trying to find a solution …

$$S_a \times S_b \subseteq S_3 \quad \curvearrowright \quad R^{a,b} = \mathbb{C}[x_1, x_2, x_3]^{S_a \times S_b} \quad \supseteq \quad \mathbb{C}[x_1, x_2, x_3]^{S_{a+b}} = R^{a+b}$$

$a + b = 3$ (partial) invariants



What to assign to linear maps **?**

Observation: $M = R^{a,b}$ has an action of both $R^{a,b}$ and R^{a+b} **!**

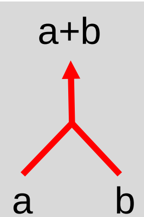
Linear maps turn into linear functors

$$S_a \times S_b \subseteq S_3 \quad \curvearrowright \quad R^{a,b} = \mathbb{C}[x_1, x_2, x_3]^{S_a \times S_b} \quad \supseteq \quad \mathbb{C}[x_1, x_2, x_3]^{S_{a+b}} = R^{a+b}$$

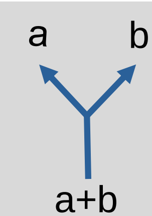
$a + b = 3$ (partial) invariants

$$\begin{tikzcd} R^{3,0}\text{-mod} \arrow[r, shift left] & R^{2,1}\text{-mod} \arrow[l, shift left] \arrow[r, shift left] & R^{1,2}\text{-mod} \arrow[l, shift left] \arrow[r, shift left] & R^{0,3}\text{-mod} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^3 V \otimes \Lambda^0 V \arrow[r, shift left] & \Lambda^2 V \otimes \Lambda^1 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^1 V \otimes \Lambda^2 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^0 V \otimes \Lambda^3 V \arrow[l, shift left] \end{tikzcd}$$



$$\underset{\text{ind} = R^{a,b} \otimes_{R^{a+b}} -}{\overset{\text{res} = R^{a,b} \otimes_{R^{a,b}} -}{\rightleftharpoons}}$$



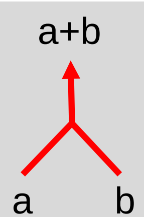
Linear maps turn into linear functors

$$S_a \times S_b \subseteq S_3 \quad \curvearrowright \quad R^{a,b} = \mathbb{C}[x_1, x_2, x_3]^{S_a \times S_b} \quad \supseteq \quad \mathbb{C}[x_1, x_2, x_3]^{S_{a+b}} = R^{a+b}$$

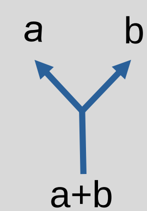
$a + b = 3$ (partial) invariants

$$\begin{tikzcd} R^{3,0}\text{-mod} \arrow[r, shift left] & R^{2,1}\text{-mod} \arrow[l, shift left] \arrow[r, shift left] & R^{1,2}\text{-mod} \arrow[l, shift left] \arrow[r, shift left] & R^{0,3}\text{-mod} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \wedge^3 V \otimes \wedge^0 V \arrow[r, shift left] & \wedge^2 V \otimes \wedge^1 V \arrow[l, shift left] \arrow[r, shift left] & \wedge^1 V \otimes \wedge^2 V \arrow[l, shift left] \arrow[r, shift left] & \wedge^0 V \otimes \wedge^3 V \arrow[l, shift left] \end{tikzcd}$$



$$\underset{\text{ind} = R^{a,b} \otimes_{R^{a+b}} -}{\overset{\text{res} = R^{a,b} \otimes_{R^{a,b}} -}{\rightleftharpoons}}$$



Outcome: categorified puzzle game produces enough functors / bimodules

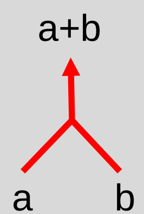
Linear maps turn into linear functors

$$S_a \times S_b \subseteq S_3 \quad \curvearrowright \quad R^{a,b} = \mathbb{C}[x_1, x_2, x_3]^{S_a \times S_b} \quad \supseteq \quad \mathbb{C}[x_1, x_2, x_3]^{S_{a+b}} = R^{a+b}$$

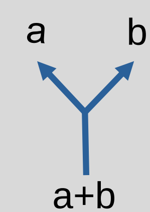
$a + b = 3$ (partial) invariants

$$\begin{tikzcd}
\text{R}^{3,0}\text{-mod} \arrow[r, shift left, "\checkmark"] & \text{R}^{2,1}\text{-mod} \arrow[l, shift left, "\checkmark"] \arrow[r, shift left, "\checkmark"] & \text{R}^{1,2}\text{-mod} \arrow[l, shift left, "\checkmark"] \arrow[r, shift left, "\checkmark"] & \text{R}^{0,3}\text{-mod} \arrow[l, shift left, "\checkmark"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^3 V \otimes \Lambda^0 V \arrow[r, shift left] & \Lambda^2 V \otimes \Lambda^1 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^1 V \otimes \Lambda^2 V \arrow[l, shift left] \arrow[r, shift left] & \Lambda^0 V \otimes \Lambda^3 V \arrow[l, shift left]
\end{tikzcd}$$



$$\begin{array}{c} \text{res} = R^{a,b} \otimes_{R^{a,b}} - \\ \rightleftharpoons \\ \text{ind} = R^{a,b} \otimes_{R^{a+b}} - \end{array}$$



Outcome: categorified puzzle game produces enough functors / bimodules

The categorified puzzle game

The puzzle game for functors / bimodules

$$\boxed{\begin{array}{c} F \\ \circ \\ G \end{array}} \otimes \boxed{\begin{array}{c} H \\ \circ \\ K \end{array}} = \begin{array}{c} \boxed{F \otimes H} \\ \circ \\ \boxed{G \otimes K} \end{array}$$

The categorified puzzle game

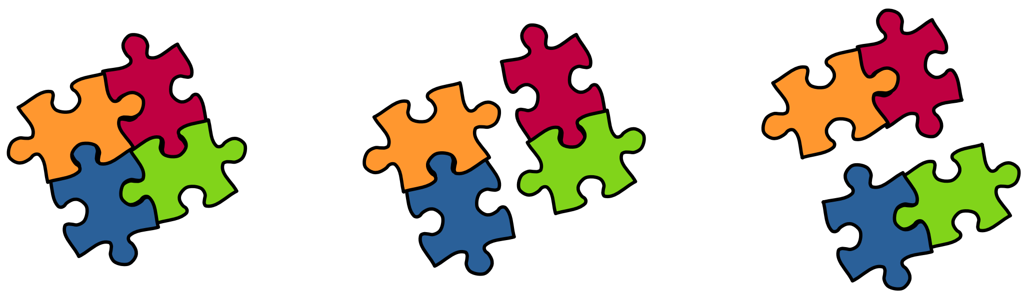
The puzzle game for functors / bimodules

$$\begin{matrix} F \\ \circ \\ G \end{matrix} \otimes \begin{matrix} H \\ \circ \\ K \end{matrix} = \begin{matrix} F \otimes H \\ \circ \\ G \otimes K \end{matrix}$$

$\circ$ = Tensor product of bimodules

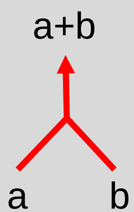
$\otimes = \otimes_{\mathbb{C}}$

The categorified puzzle game

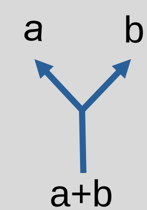


The puzzle game for functors / bimodules

$$\boxed{\begin{matrix}F\\ \circ\\ G\end{matrix}} \otimes \boxed{\begin{matrix}H\\ \circ\\ K\end{matrix}} = \begin{matrix}\boxed{F \otimes H}\\ \circ\\ \boxed{G \otimes K}\end{matrix}$$



$$\mathrm{res} = R^{a,b} \otimes_{R^{a,b}} - \quad \rightleftharpoons \quad \mathrm{ind} = R^{a,b} \otimes_{R^{a+b}} -$$

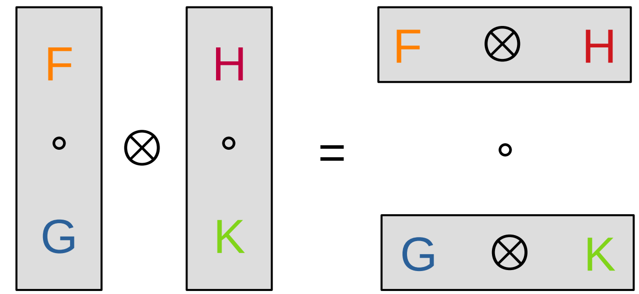


$\circ$ = Tensor product of bimodules

$\otimes = \otimes_{\mathbb{C}}$

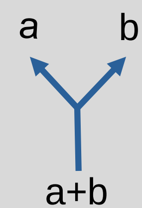
The categorified puzzle game

The puzzle game for functors / bimodules



$$\text{res} = R^{a,b} \otimes_{R^{a,b}} -$$

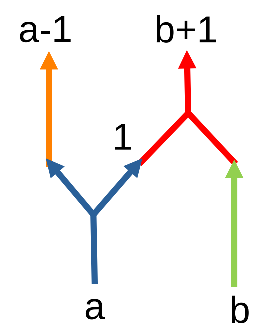
$$\text{ind} = R^{a,b} \otimes_{R^{a+b}} -$$



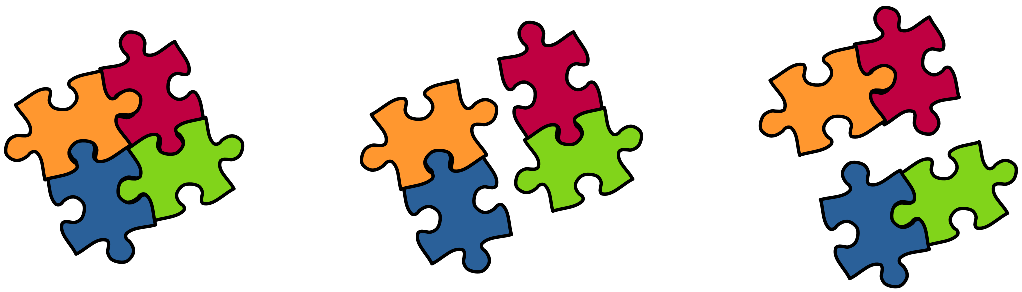
$\circ$ $=$ Tensor product of bimodules

$\otimes = \otimes_{\mathbb{C}}$

$$\begin{array}{c} F \otimes H \\ \circ \\ G \otimes K \end{array}$$



Categorical gluing



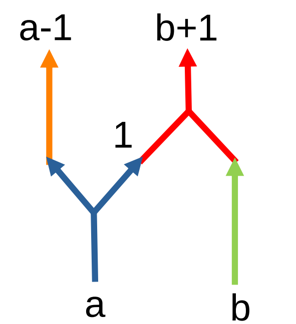
The puzzle game for functors / bimodules

$$\begin{array}{|c|}\hline F \\ \circ \\ G \\ \hline\end{array} \otimes \begin{array}{|c|}\hline H \\ \circ \\ K \\ \hline\end{array} = \begin{array}{c}\boxed{F \otimes H} \\ \circ \\ \boxed{G \otimes K}\end{array}$$

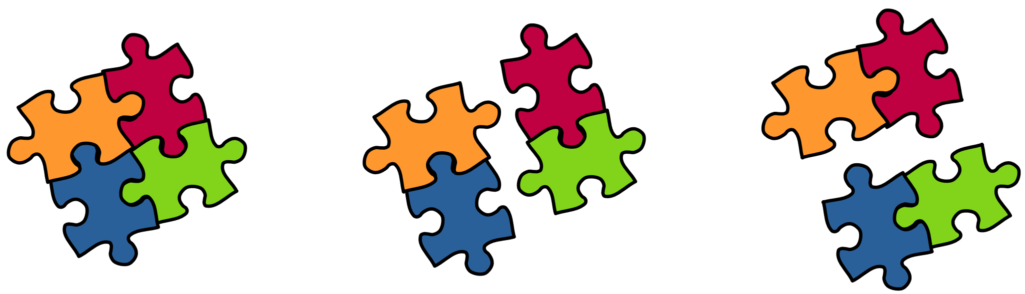
Compose vertically and horizontally
and take
direct sums and direct summands

(Categorical) **GLUING**

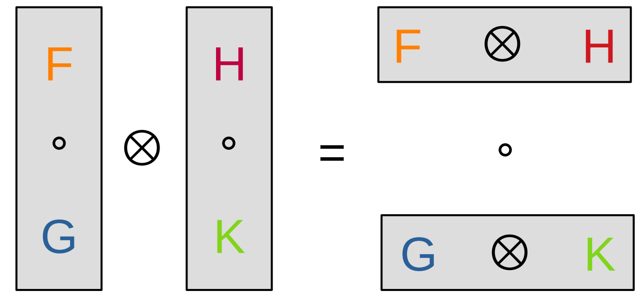
$$\begin{array}{ccc} F & \otimes & H \\ & \circ & \\ G & \otimes & K \end{array}$$



Categorical gluing



The puzzle game for functors / bimodules



Compose vertically and horizontally
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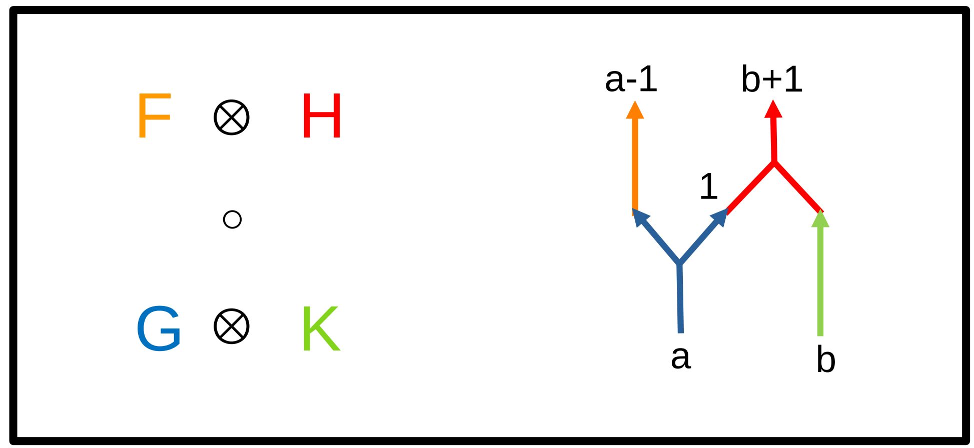
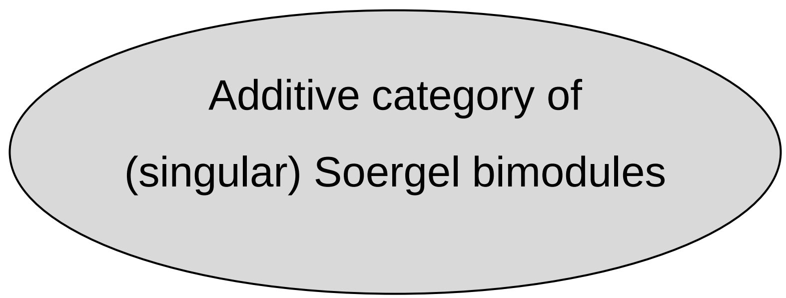
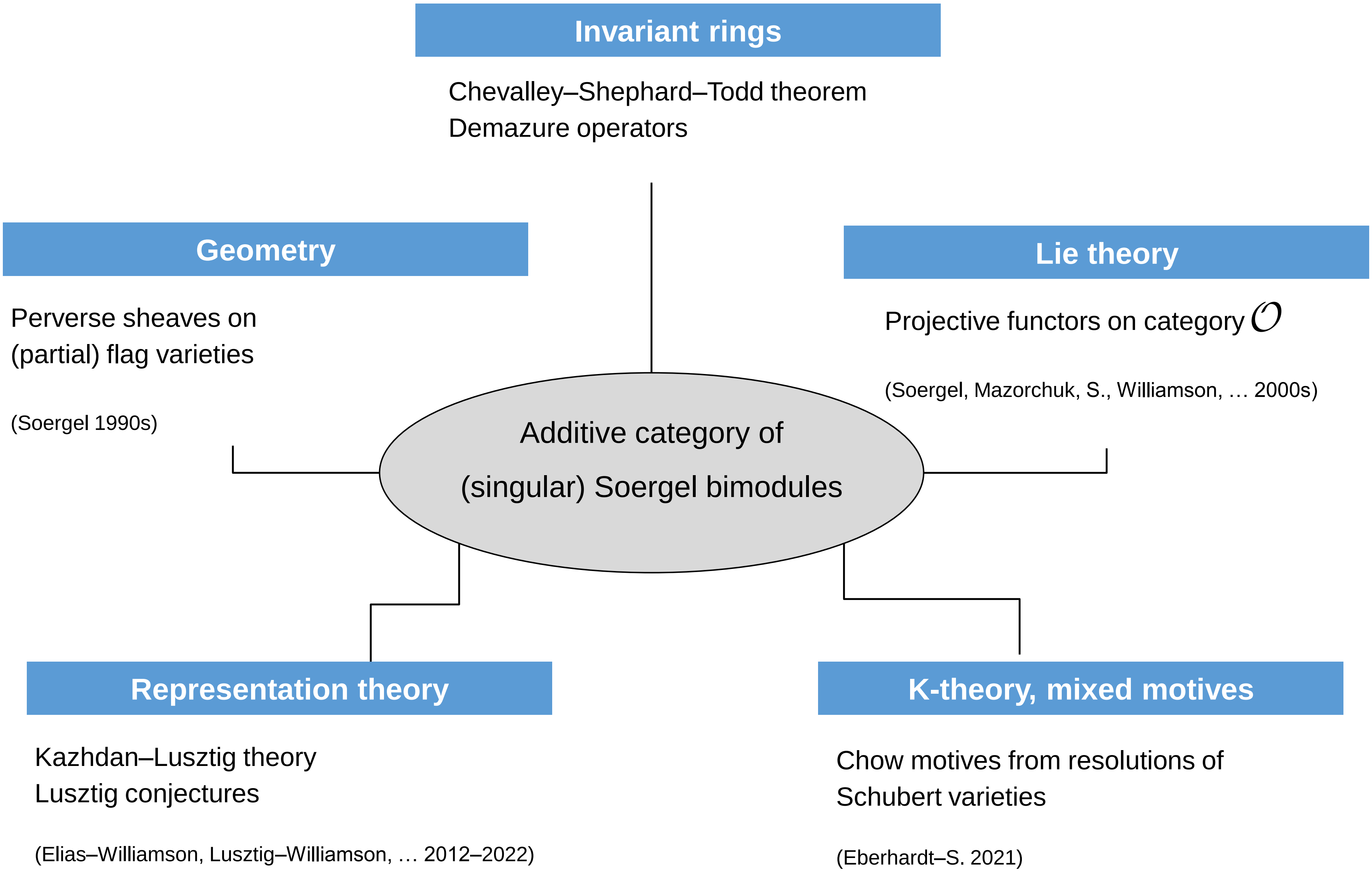
(Categorical) **GLUING**

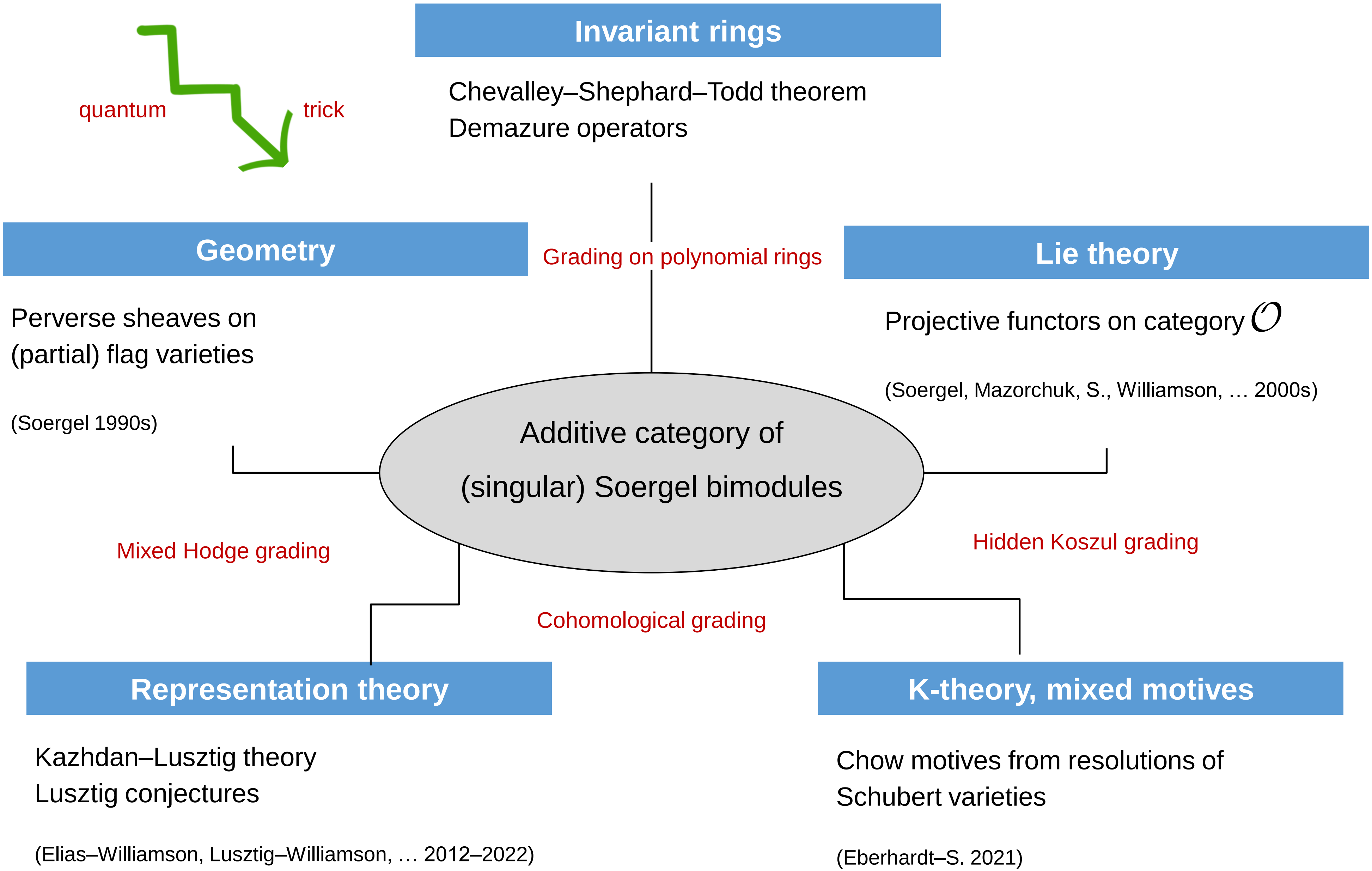


Outcome: (singular) Soergel bimodules

(Soergel 1990s)


Additive category of
(singular) Soergel bimodules


Invariant rings
Chevalley–Shephard–Todd theorem
Demazure operators
Geometry
Perverse sheaves on
(partial) flag varieties
(Soergel 1990s)
Lie theory
Projective functors on category $\mathcal{O}$
(Soergel, Mazorchuk, S., Williamson, … 2000s)
Additive category of
(singular) Soergel bimodules
Representation theory
Kazhdan–Lusztig theory
Lusztig conjectures
(Elias–Williamson, Lusztig–Williamson, … 2012–2022)
K-theory, mixed motives
Chow motives from resolutions of
Schubert varieties
(Eberhardt–S. 2021)


Invariant rings
Chevalley–Shephard–Todd theorem
Demazure operators
quantum
trick
Geometry
Grading on polynomial rings
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Additive category of
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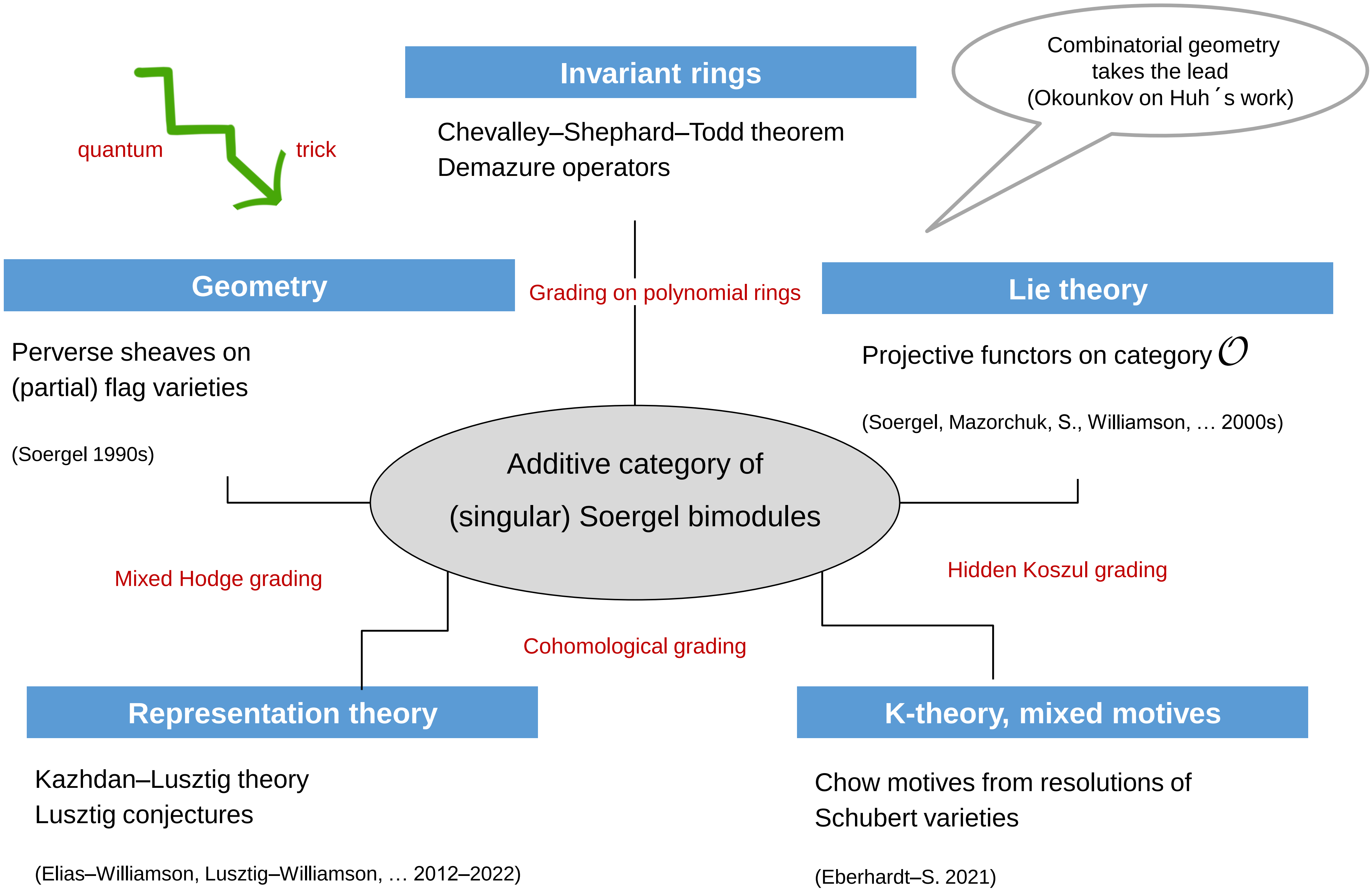
Mixed Hodge grading
Hidden Koszul grading
Cohomological grading
Representation theory
K-theory, mixed motives
Kazhdan–Lusztig theory
Lusztig conjectures
(Elias–Williamson, Lusztig–Williamson, … 2012–2022)
Chow motives from resolutions of
Schubert varieties
(Eberhardt–S. 2021)


quantum
trick
Invariant rings
Chevalley–Shephard–Todd theorem
Demazure operators
Combinatorial geometry
takes the lead
(Okounkov on Huh´s work)
Geometry
Grading on polynomial rings
Lie theory
Perverse sheaves on
(partial) flag varieties
(Soergel 1990s)
Projective functors on category $\mathcal{O}$
(Soergel, Mazorchuk, S., Williamson, … 2000s)
Additive category of
(singular) Soergel bimodules
Mixed Hodge grading
Hidden Koszul grading
Cohomological grading
Representation theory
K-theory, mixed motives
Kazhdan–Lusztig theory
Lusztig conjectures
(Elias–Williamson, Lusztig–Williamson, … 2012–2022)
Chow motives from resolutions of
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A categorification theorem

Theorem (S. 2002, Mazorchuk–S. 2009)

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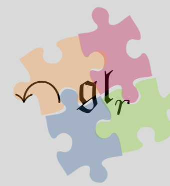
$$\mathfrak{gl}_k \curvearrowright \bigoplus_{d_1+d_2+\cdots+d_r=n} \wedge^{d_1} V \otimes \wedge^{d_2} V \otimes \cdots \otimes \wedge^{d_r} V \curvearrowleft \mathfrak{gl}_r$$



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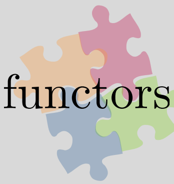
functors $\curvearrowright$ $\bigoplus$ some abelian categories $\curvearrowleft$ functors



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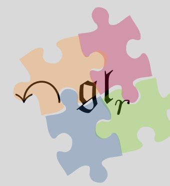

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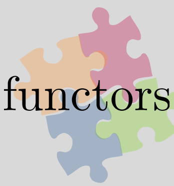

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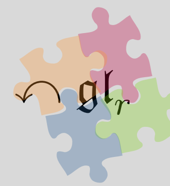

- Gluing gives all morphisms between categorified $\wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V$

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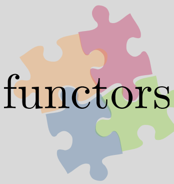

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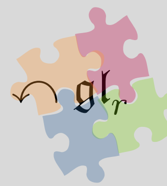

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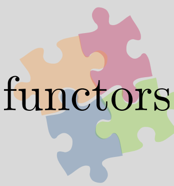

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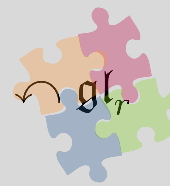

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Upshot: tangles turn into complexes of functors / bimodules

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$$\begin{array}{ccccc} \text{(derived) functors} & \curvearrowright & \bigoplus \text{some abelian categories} & \curvearrowleft & \text{(derived) functors} \end{array}$$

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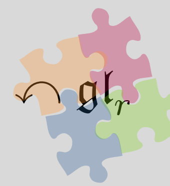

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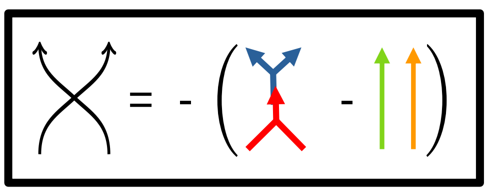

2-term complex

Upshot: tangles turn into complexes of functors / bimodules

Quantum trick again …

Theorem (S. 2002, Mazorchuk–S. 2009)

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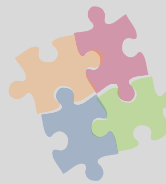
graded (**derived**) functors $\curvearrowright$	$\bigoplus$ some abelian categories	$\curvearrowleft$ **graded** (**derived**) functors



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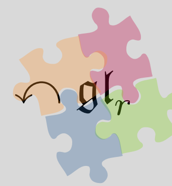
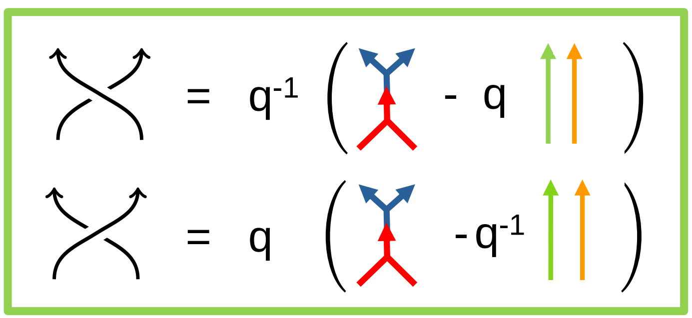


- Gluing gives all morphisms between categorified $\wedge^{d_1} V_q \otimes \cdots \otimes \wedge^{d_r} V_q$



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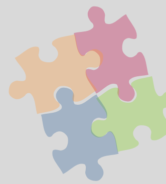

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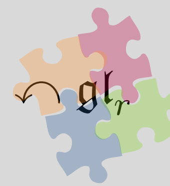

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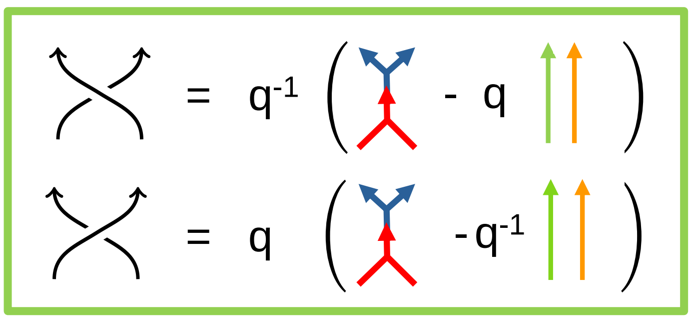

Upshot: tangles turn into complexes of graded functors / bimodules

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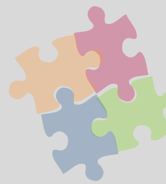

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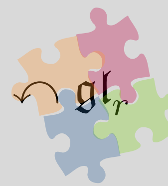

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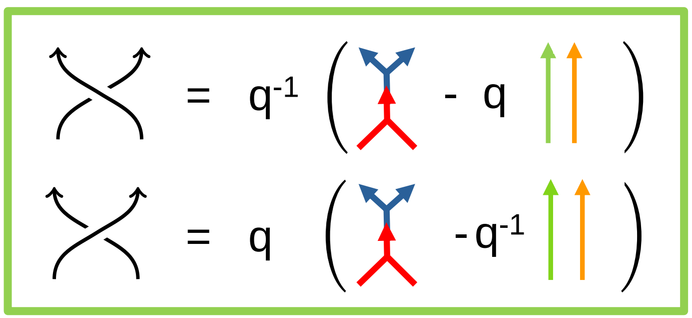

links $\longmapsto$ complexes of graded vector spaces

$$\cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M_{-1} \longrightarrow \cdots$$

$\overset{k=2}{\Longrightarrow}$ Khovanov homology (2000) $\sum_{i,j} (-1)^i \dim M_{i,j} t^i q^j$

Upshot: tangles turn into complexes of graded functors / bimodules

Tangle bordisms ?

2009–2017

(… , Blanchet,
Ehrig–Tubbenhauer–S.,
Ehrig–Tubbenhauer–Wedrich)

$$\mathfrak{g} = \mathfrak{gl}_k$$

$$\textstyle\bigwedge^{d_1} V \otimes \cdots \otimes \bigwedge^{d_r} V$$

natural transformation $\longleftarrow\!\!\shortmid$ tangle bordisms

linear functor $\longleftarrow\!\!\shortmid$

$\mathbb{C}$-linear category $\longleftarrow\!\!\shortmid$

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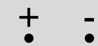
$\mathbb{C}$-linear category $\longleftarrow\!\!\dashv$ 

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2008–now

(Webster, S.–Sussan Cooper–Krushkal, Gukov, Rozansky, …)

more fundamental:

arbitrary finite-dim

$\mathfrak{gl}_2$-representations

$\rightarrow$ colored Jones polynomial

tangle bordisms

?

Tangle bordisms ?

2009–2017 (… , Blanchet, Ehrig–Tubbenhauer–S., Ehrig–Tubbenhauer–Wedrich)	$\mathfrak{g} = \mathfrak{gl}_k$ $\bigwedge^{d_1} V \otimes \cdots \otimes \bigwedge^{d_r} V$	natural transformation $\longleftarrow\!\!\mid$ tangle bordisms linear functor $\longleftarrow\!\!\mid$  $\mathbb{C}$-linear category $\longleftarrow\!\!\mid$ 	
2008–now (Webster)	arbitrary $\mathfrak{g}$	tangle bordisms ?	
2008–now (Webster, S.–Sussan Cooper–Krushkal, Gukov, Rozansky, …)	more fundamental: arbitrary finite-dim $\mathfrak{gl}_2$-representations $\rightarrow$ colored Jones polynomial	tangle bordisms ?	what is the complex attached to the unknot ?

Rigidity via natural transformations (2008–2017)

matrix factorizations string theory symplectic geometry combinatorial TQFT coherent sheaves

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Hilbert schemes diagrammatics Springer theory Khovanov homology algebraic geometry Donaldson-Thomas theory matrix factorizations string theory symplectic geometry combinatorial TQFT coherent sheaves

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Axiomatics of natural transformations

for categorified

$\mathfrak{gl}_k$ or $U_q(\mathfrak{gl}_k)$

representations W

(Chuang–Rouquier, Rouquier Khovanov–Lauda)

W irreducible

(Chuang–Rouquier)

W = tensor products of irreducibles

(Brundan–Losev–Webster)

UNIQUENESS

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Upshot: categorification of $\wedge^{d_1} V \otimes \cdots \otimes \wedge^{d_r} V$ is the unique one

Knots, braids and tangles in three-dimensional space

Mysteries of knots in three-dimensional space

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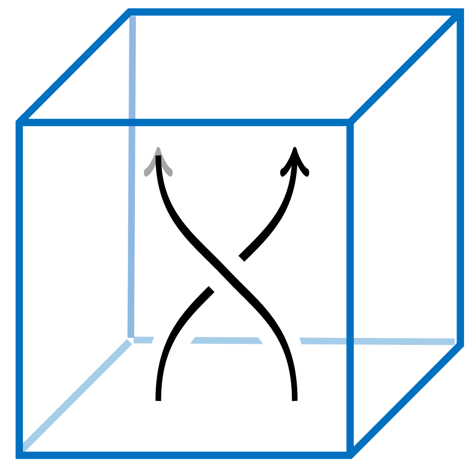
Mysteries of knots in three-dimensional space

- these objects are invariants of a three-dimensional situation



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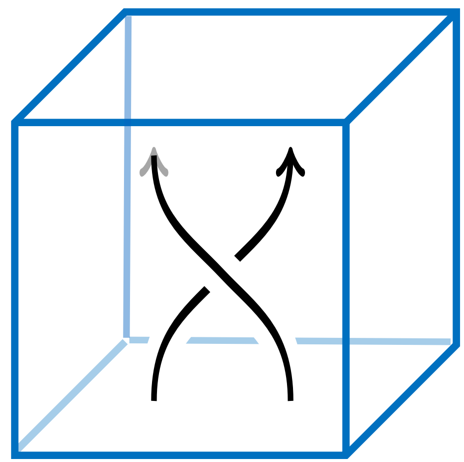

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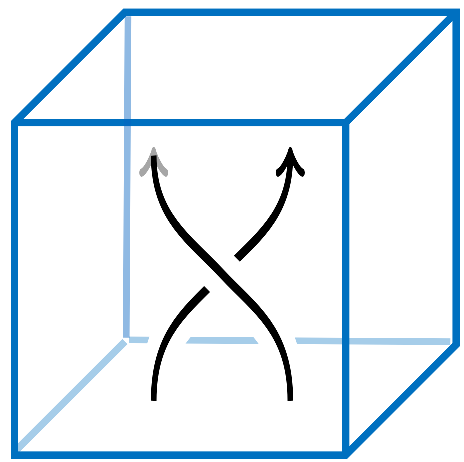

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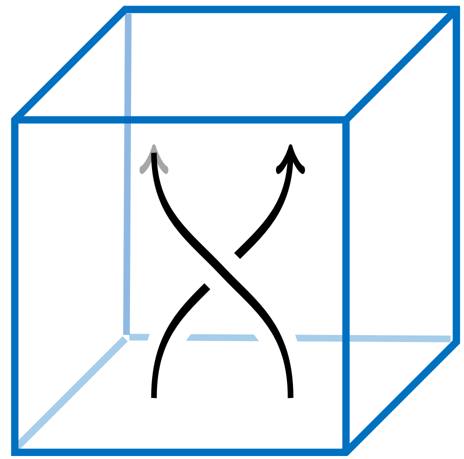
E. Witten (1989)



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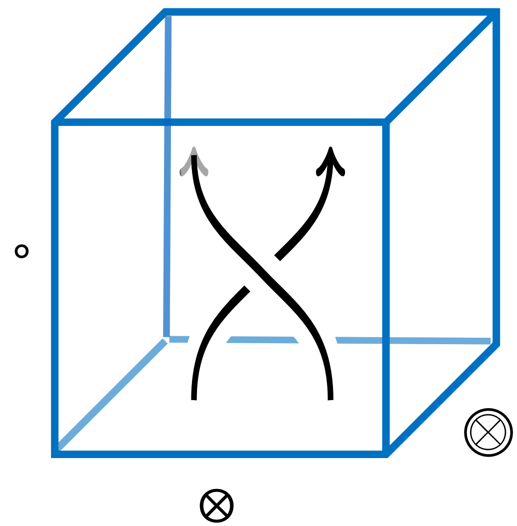

Mysteries of knots in three-dimensional space

Behind the Jones polynomial and the Reshetikhin–Turaev tangle invariants:

$\boxed{3+2+1+0}$ - TQFT

(Witten, Walker, Freed–Teleman, …)

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Breakthrough **BORDISM HYPOTHESIS** (Baez–Dolan, Lurie, …1995–2009)

$\begin{matrix} d \\ d-1 \\ \vdots \\ 1 \\ 0 \end{matrix}$ - TQFT

Z is determined by $Z(\bullet)$ **!**

- **Verification** (already for d=2), even **formulation** requires higher categories / ∞-categories

(Schommer-Pries, Pstragowski, Hopkins–Lurie, …, Lurie, Ayala–Francis, …, Haugseng, Calaque–Scheimbauer,...)

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Examples of $Z(\bullet)$ **?**

Dimension 3

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$\boxed{\begin{matrix}3\\2\\1\\0\end{matrix}}$ - TQFT $\quad Z \longleftrightarrow Z(\bullet)$ $\quad$ braided $\otimes$-category

+ dualizability conditions

(Douglas–Schommer-Pries–Snyder 2013, Freed–Teleman, Walker, …)

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Upshot: representation theory provides examples! ✓

Why braidings ?

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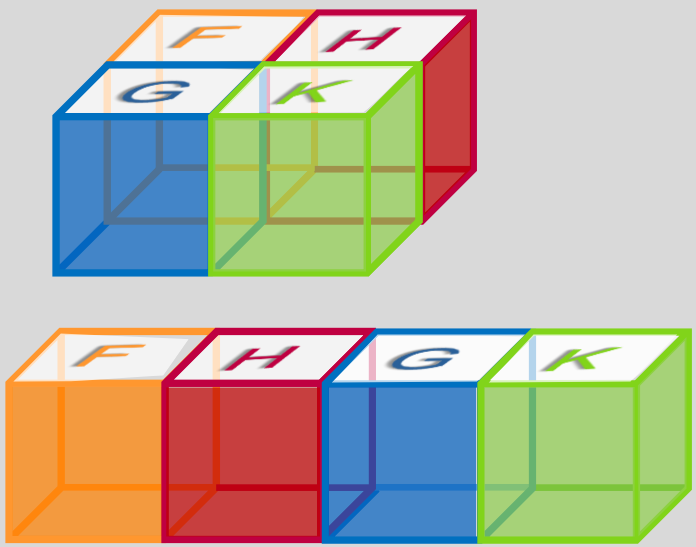
e.g. via Quantum group representations

Upshot: representation theory provides examples! ✓

Why braidings ?

Little cubes

Little cubes



Eckmann–Hilton argument $X \bigotimes Y \cong X \otimes Y$

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$$\wr|$$

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But: interesting BRAIDING isomorphisms !

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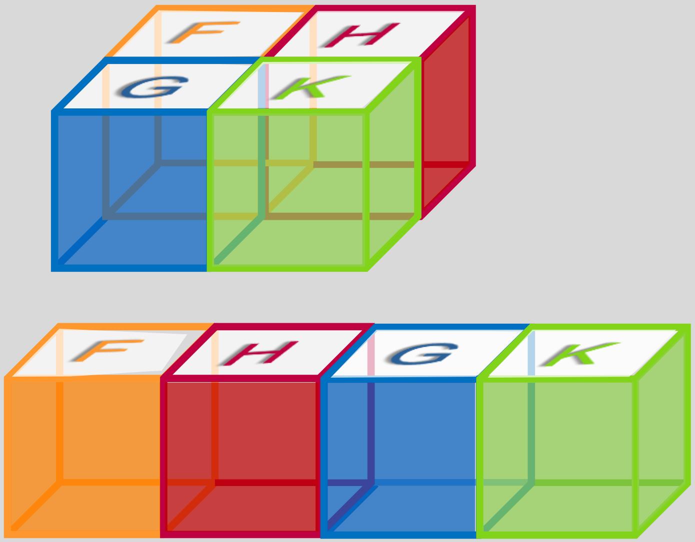

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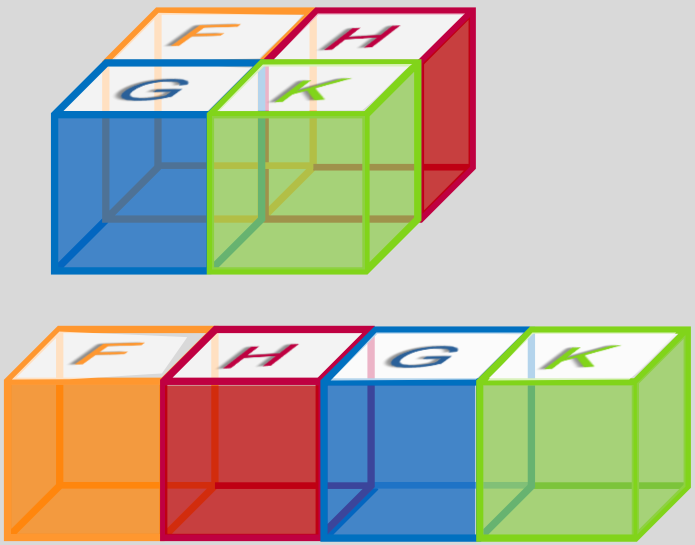
But: interesting BRAIDING isomorphisms !

Little cubes give an **operadic reformulation**:

$\mathcal{Z}(\bullet)$ = E_2-algebra in $(\infty, 1)$-categories + dualizability conditions

Little cubes



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$$\wr\|$$

$$(F \bigotimes G) \otimes (H \bigotimes K) \cong (F \otimes G) \otimes (H \otimes K)$$

But: interesting BRAIDING isomorphisms !

interaction of

$\otimes$ and $\bigotimes$

Little cubes give an **operadic reformulation**:

$\mathcal{Z}(\bullet) = E_2$-algebra in $(\infty, 1)$-categories + dualizability conditions

Challenge: TQFT for d ≥ 4

Representation theory

Topology

$\overset{+}{4}$
3
2
1
0

- TQFT s

?

braided
$\otimes$-category

$+$ dualizability conditions

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Example ? ?

Topology

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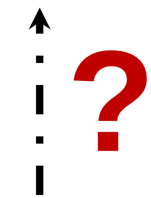

braided
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$\overset{+}{3}$, 2, 1, 0 - TQFT s

Challenge: TQFT for $d \geq 4$

Representation theory		Topology
braided $\otimes$-**2**-category + dualizability conditions	Example ??	$\begin{pmatrix} 4^+ \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ - TQFT s
		$\uparrow$?
braided $\otimes$-category + dualizability conditions		$\begin{pmatrix} 3^+ \\ 2 \\ 1 \\ 0 \end{pmatrix}$ - TQFT s

Upshot: can we construct an interesting braided $\otimes$-**2**-category?

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braided
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+ dualizability conditions

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$\overset{+}{4}$, 3, 2, 1, 0 - TQFT s

E_2 - algebra in (∞, 2) - categories

?

E_2 - algebra in (∞,1) - categories

braided
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+ dualizability conditions

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Current work:

Mazel-Gee–Liu–Reutter–S.–Wedrich

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Basic observation: Graded (singular) Soergel bimodules form a $\otimes$-$\mathbf{2}$-category :

tensor product : $\otimes = \otimes_{\mathbb{C}}$

$2-$morphisms : bimodule maps / natural transformations

$1-$morphisms : graded $(R^{\mathbf{d}'}, R^{\mathbf{d}}) -$ (singular) Soergel bimodules

objects : $\mathbf{d} = (d_1, d_2, \ldots, d_r), \quad d_i \in \mathbb{N}_0, r \in \mathbb{N}$

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Breakthrough result: generators and relations description of $\otimes$-category of Soergel bimodules
(Elias–Williamson 2012)

Precise: semi-strict monoidal 2-category (in the sense of Baez–Neuchl)

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Hecke categories

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Quantum trick again ...

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braided
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quantum $\rightsquigarrow$ trick

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braided
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(Mazel-Gee–Liu–Reutter–S.–Wedrich 2022)

∞ - version

chain complexes

of graded (singular)

Soergel bimodules

graded (singular) Soergel bimodules

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tensor product :	$\otimes = \otimes_{\mathbb{C}}$
2 – morphisms :	bimodule maps / natural transformations
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The construction of a braided $\otimes$-**2**-category

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depending on $\mathfrak{gl}_k$

Expectation: natural choice of subquotients provides enough dualizability conditions!

Combining success stories …

+
4
3
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- TQFT

Combining success stories …

Knot and link invariants via polynomials

- Jones polynomial
- HOMFLY–PT polynomial
- Skein theory and Hecke algebras

Knot and manifold invariants via TQFT

- Witten–Jones polynomial
- Chern–Simons and Axelrod–Singer theory
- Atiyah–Segal axioms

+
4
3
2
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- TQFT

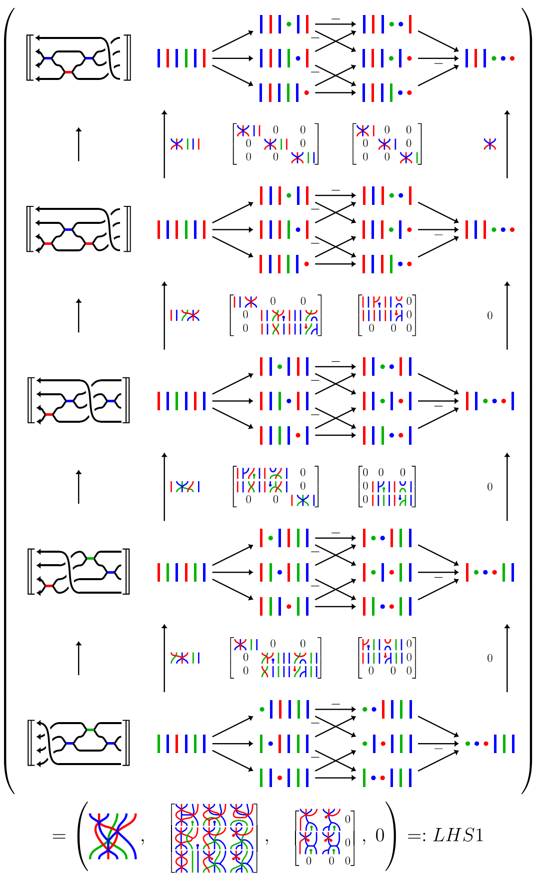
Tangle and 3-manifold invariants via Quantum groups

- Quantum groups
- Reshetikhin–Turaev invariants
- Crane–Yetter invariants

Topological invariants via higher homotopies

- Operads
- E_n-algebras
- Higher dimensional algebra

Thank you for your attention !



the joy of

explicitly
computing

higher homotopies …