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A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products

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Abstract. The purpose of this paper is to study categorifications of tensor products of finite-dimensional modules for the quantum group for $\mathfrak{sl}_2$. The main categorification is obtained using certain Harish-Chandra bimodules for the complex Lie algebra $\mathfrak{gl}_n$. For the special case of simple modules we naturally deduce a categorification via modules over the cohomology ring of certain flag varieties. Further geometric categorifications and the relation to Steinberg varieties are discussed. We also give a categorical version of the quantised Schur–Weyl duality and an interpretation of the (dual) canonical bases and the (dual) standard bases in terms of projective, tilting, standard and simple Harish-Chandra bimodules.

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Contents

Introduction

A categorification program of the simplest quantum group, $U_q(\mathfrak{sl}_2)$, has been formulated in [BFK99]. Its ultimate goal is to construct a certain tensor 2-category with a Grothendieck ring equivalent to the representation category of $U_q(\mathfrak{sl}_2)$. In [BFK99], the categorification problem was studied for modules over the classical algebra $\mathcal{U}(\mathfrak{sl}_2)$. There, two versions of categorification of the $\mathcal{U}(\mathfrak{sl}_2)$-action on $V_1^{\otimes n}$, the n-th power of the two-dimensional fundamental representation for $\mathfrak{sl}_2$, were obtained: one using certain singular blocks of the category $\mathcal{O}(\mathfrak{gl}_n)$ of highest weight $\mathfrak{gl}_n$-modules, another one via certain parabolic subcategories of the regular block of $\mathcal{O}(\mathfrak{gl}_n)$. The quantum versions of these two categorifications were conjectured in [BFK99] and established for the parabolic case in [Str05] using the graded version of the category $\mathcal{O}(\mathfrak{gl}_n)$ from [Soe90] and [BGS96] and graded lifts of translation functors. In the present paper, among other results, we develop a categorification of the $U_q(\mathfrak{sl}_2)$-action on $V_1^{\otimes n}$ using (a graded version of) certain singular blocks of $\mathcal{O}(\mathfrak{gl}_n)$ (see Section 3). It was also conjectured in [BFK99] that the two categorifications, the one using parabolic subcategories of the regular block of $\mathcal{O}(\mathfrak{gl}_n)$ and the one using singular blocks of $\mathcal{O}(\mathfrak{gl}_n)$, are related by the Koszul duality functor of [BGS96]. In fact, it was shown in [RH04] (see also [MOS05]) that the key functors used in both categorifications, namely graded versions of translation functors and Zuckerman functors, are Koszul dual to each other. This completes the general picture of categorifications of the $U_q(\mathfrak{sl}_2)$-action in $V_1^{\otimes n}$ initiated in [BFK99] and opens a way for further steps in the general program.

The main purpose of the present paper is to study the categorification of the tensor products of arbitrary finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules (as recalled in Section 1), i.e. of modules of the form

$$V_{\mathbf{d}} = V_{d_1} \otimes \cdots \otimes V_{d_r}, \tag{1}$$

where V_{d_i} denotes an irreducible representation of dimension $d_i + 1$. Our basic observation is that $V_{\mathbf{d}}$ admits a categorification via blocks of the category of Harish-Chandra bimodules $\mathcal{H}$ for $\mathfrak{gl}_n$, where $n = \sum_{i=1}^r d_i$. To make this more precise, we go back and recall, from [BFK99], that the $\mathcal{U}(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$ has a categorification via the category

$$\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{O}(\mathfrak{gl}_n), \tag{2}$$

where ${}_{\omega_i}\mathcal{O}(\mathfrak{gl}_n)$ is the block corresponding to an integral weight ω_i with stabiliser $S_i \times S_{n-i}$. We establish a categorification of the corresponding $U_q(\mathfrak{sl}_2)$-module using a graded version of the category (2). The categorification of an arbitrary tensor product $V_{\mathbf{d}}$ is then given by a graded version of

$$\mathcal{H}_\mu := \bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{H}_\mu,$$

where ω_i, $i = 0, \ldots, n$, are as above, and μ is a dominant (with respect to the dot-action) integral weight with stabiliser isomorphic to

$$S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_r}.$$

The connection of our categorification with the one in [BFK99] is given by a functor introduced in [BG80], namely the functor of tensoring with some Verma module, $M(\mu)$:

$$ {}_\lambda F_\mu : {}_\lambda\mathcal{H}^1_\mu \to {}_\lambda\mathcal{O}(\mathfrak{gl}_n), \qquad X \mapsto X \otimes_{\mathcal{U}(\mathfrak{gl}_n)} M(\mu), \tag{3}$$

for dominant integral λ and μ. This functor defines an equivalence of categories for regular μ. If μ is not regular, then ${}_\lambda F_\mu$ yields an equivalence with a certain full subcategory of ${}_\lambda\mathcal{O}$. (Note that, although ${}_\lambda\mathcal{H}^1_\mu$ is in fact a full subcategory of ${}_\lambda\mathcal{H}_\mu$, they have the same Grothendieck group.)

In the present paper we study a categorification based on singular blocks of Harish-Chandra bimodules. By analogy with [BFK99] one expects the existence of a second categorification that uses parabolic subcategories, related to the first by (a certain generalisation of) the Koszul duality functor. (The first steps in this direction can be found in [MOS05].)

To achieve our main goal, we first construct a categorification of tensor products of the form (1) for the classical algebra $\mathcal{U}(\mathfrak{sl}_2)$ in Section 2. After recalling and further developing the graded version of the category $\mathcal{O}(\mathfrak{gl}_n)$ in Section 3 we obtain the categorification of the tensor products (1) for the quantum algebra $U_q(\mathfrak{sl}_2)$ in Section 4.

Our approach to the grading of the category $\mathcal{O}$ and the category of Harish-Chandra modules is based on the Soergel functor ([Soe90])

$$_\lambda\mathbb{V} : {_\lambda\mathcal{O}} \to C^\lambda,$$

where C^λ is the algebra of endomorphisms of the unique indecomposable projective tilting module in $_\lambda\mathcal{O}$. One of the main results of [Soe90] is an explicit description of C^λ as a ring of invariants in coinvariants. The latter is known to be isomorphic to the cohomology ring of the partial flag variety $\mathcal{F}_\lambda$ corresponding to λ. In particular, there is a natural $\mathbb{Z}$-grading on C^λ. Moreover, it was shown in [Soe90] that the functor $_\lambda\mathbb{V}$ intertwines the translation functors $\mathbb{T}_\lambda^\mu : {_\lambda\mathcal{O}} \to {_\mu\mathcal{O}}$ in the category $\mathcal{O}$ with the induction $C^\mu \otimes_{C^\lambda} \bullet : C^\mu \to C^\lambda$ or with the restriction functor Res_μ^λ : C^λ-mod $\to C^\mu$-mod, depending on whether the stabiliser of μ is contained in the stabiliser of λ or vice versa. This fact is employed for constructing the functors that provide a categorification of the $\mathcal{U}(\mathfrak{sl}_2)$-action. The natural grading of the algebra C^λ is used to define graded lifts of the previous functors which yield the quantum counterpart.

The categorification of tensor products (1) of $U_q(\mathfrak{sl}_2)$-modules via Harish-Chandra bimodules gives rise to special bases resulting from indecomposable tilting, simple, standard and dual standard modules. In Section 5 we prove in the special case of $V_1^{\otimes n}$ (i.e. $d_1 = \cdots = d_n = 1$) that, at the Grothendieck level, these bases can be identified as canonical, dual canonical, standard and dual standard bases in $V_1^{\otimes n}$. This is established using the Kazhdan–Lusztig theory, Schur–Weyl duality and the graphical calculus for tensor products. A generalisation of these results to arbitrary tensor products requires a more extensive study of the category of Harish-Chandra bimodules, which will be postponed to a future paper.

Our main theorem about the categorification of the tensor products of the form (1) in the special case of a single factor (i.e. $r = 1$), combined with the properties of Soergel's functor, leads to a "geometric" categorification of irreducible representations of $U_q(\mathfrak{sl}_2)$ which we explain in Section 6. We conclude that section with a discussion and a conjecture about a possible geometric categorification of the general tensor product, based on the Borel–Moore homology of the generalised Steinberg varieties $X^{\lambda,\mu}$ of triples (see [DR04]). The proposed geometric approach is connected with our geometric categorification of simple $U_q(\mathfrak{sl}_2)$-modules, since in the special case $r = 1$, the generalised Steinberg variety degenerates into the partial flag variety $\mathcal{F}_\lambda$. It is important to note that the generalised Steinberg varieties $X^{\lambda,\mu}$ are precisely the tensor product varieties of Malkin ([Mal03]) and Nakajima ([Nak01]) in the special case of tensor products of finite-dimensional irreducible

$U_q(\mathfrak{sl}_2)$-modules (see [Sav03]). The geometric categorification that we propose is a natural next step in the geometric description of tensor products of the form (1) of $U_q(\mathfrak{sl}_2)$-modules via Steinberg varieties, which was started in [Sav03]. We also remark that the generalised Steinberg varieties naturally appear as characteristic varieties of Harish-Chandra bimodules ([BB85]). These facts strongly indicate that the geometric categorification (via Borel–Moore homology), its relation to the algebraic categorification (by means of Harish-Chandra bimodules), and to the theory of characteristic varieties is a very rich area for future research.

Finally, we would like to mention that the categorification of the representation theory of $U_q(\mathfrak{sl}_2)$ has powerful applications to different areas in mathematics and physics. In particular, the general notion of an $\mathfrak{sl}_2$-categorification via abelian categories has been introduced in [CR07] and effectively used to solve various outstanding problems in the representation theory of finite groups. Also, in the same way as the representation theory of $U_q(\mathfrak{sl}_2)$, especially in the framework of the tensor products (1), is applied to invariants of knots in three dimensions, the categorification of all these structures is believed to produce invariants of 2-knots in four dimensions. So far, invariants of link cobordisms were obtained via a categorification of the Jones polynomial in [Jac04] and [Kho06], [Str05], [Str06]. Their representation-theoretic interpretation requires further steps in the "categorification program" of the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$.

1. Quantum $\mathfrak{sl}_2$ and its finite-dimensional representations

1.1. Definitions and preliminaries

We start by recalling some basics about the quantised enveloping algebra of $\mathfrak{sl}_2$. For details we refer for example to [CP94], [Jan98]. Let $\mathbb{C}(q)$ be the field of rational functions with complex coefficients in an indeterminate q.

Definition 1.1. The quantum group $U_q(\mathfrak{sl}_2)$ is an associative algebra over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad KK^{-1} = 1 = K^{-1}K,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We will also denote $U_q(\mathfrak{sl}_2)$ by U. For $a \in \mathbb{Z}$ define $E^{(a)} = E^a/[a]!$ and $F^{(a)} = F^{(a)}/[a]!$ where

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} \quad \text{and} \quad [a]! = [a][a-1]\dots[1].$$

Let $[a, b] = [a]!/[b]![a - b]!$ for $0 \leq b \leq a$. Let $^-$ be the $\mathbb{C}$-linear involution of $\mathbb{C}(q)$ which changes q into q^{-1}. A $\mathbb{C}$-linear (anti-)automorphism ϕ of U is called $\mathbb{C}(q)$-*antilinear* if $\phi(fx) = \bar{f}\phi(x)$ for $f \in \mathbb{C}(q)$, $x \in U$.

The algebra U has an antilinear anti-automorphism τ given by

$$\tau(E) = qFK^{-1}, \quad \tau(F) = qEK, \quad \tau(K) = K^{-1}. \tag{4}$$

Let $\sigma : U \to U$ be the $\mathbb{C}(q)$-linear algebra involution defined by

$$\sigma(E) = F, \quad \sigma(F) = E, \quad \sigma(K) = K^{-1}. \tag{5}$$

Let $\psi : U \to U$ be the $\mathbb{C}(q)$-antilinear algebra involution

$$\psi(E) = E, \quad \psi(F) = F, \quad \psi(K) = K^{-1}.$$

The algebra U is also a Hopf algebra with the comultiplication $\triangle$ given by

$$\begin{gathered}\triangle(E) = 1 \otimes E + E \otimes K^{-1}, \quad \triangle(F) = K \otimes F + F \otimes 1, \\ \triangle(K^{\mp 1}) = K^{\mp 1} \otimes K^{\mp 1}.\end{gathered} \tag{6}$$

(The antipode S is defined as $S(K) = K^{-1}$, $S(E) = -EK$ and $S(F) = -K^{-1}F$.)

1.2. The $n+1$-dimensional representation V_n

For any positive integer n, the algebra U has a unique (up to isomorphism) irreducible $n+1$-dimensional representation V_n on which K acts semisimply with powers of q as eigenvalues. The V_n constitute a complete set of representatives for the iso-classes of simple U-modules of type I. We can choose in V_n a basis of weight vectors $\{v_0, v_1, \dots, v_n\}$ such that U acts as follows:

$$K^{\pm 1} v_k = q^{\pm(2k-n)} v_k, \quad E v_k = [k+1] v_{k+1}, \quad F v_k = [n-k+1] v_{k-1}. \tag{7}$$

We call this basis the *canonical basis* of V_n, since it is a special, though rather trivial, case of the Lusztig–Kashiwara canonical bases in finite-dimensional irreducible modules over quantum groups (see [Lus90], [Kas91]). Let $\mathcal{U}(\mathfrak{sl}_2)$ denote the universal enveloping algebra of the semisimple complex Lie algebra $\mathfrak{sl}_2$. We denote by $\overline{V}_n$ the $n+1$-dimensional irreducible representation of $\mathfrak{sl}_2$.

Given a $\mathbb{C}(q)$-vector space V, a $\mathbb{C}$-bilinear form $V \times V \to \mathbb{C}(q)$ is called *semilinear* if it is $\mathbb{C}(q)$-antilinear in the first variable and $\mathbb{C}(q)$-linear in the second, i.e.

$$<fx, y> = \bar{f}<x, y>, \quad <x, fy> = f<x, y>, \quad f \in \mathbb{C}(q), x, y \in U. \tag{8}$$

On V_n, there is a (unique up to scaling) nondegenerate semilinear form

$$< \, , \, > : V_n \times V_n \to \mathbb{C}(q) \tag{9}$$

which satisfies $<xu, v> = <u, \tau(x)v>$ for any $x \in U_q(\mathfrak{sl}_2)$ and $u, v \in V_n$. In the basis $\{v_k\}_{0 \le k \le n}$ the form is given by

$$<v_k, v_l> = \delta_{k,l} q^{k(n-k)} [n, k]. \tag{10}$$

Define the *dual canonical basis* $\{v^k\}_{0 \le k \le n}$ of V_n by $<v_l, v^k> = \delta_{k,l} q^{k(n-k)}$. Then $v_k = [n,k] v^k$ and the action of E, F and K in the dual canonical basis is

$$K^{\pm 1} v^k = q^{\pm(2k-n)} v^k, \quad E v^k = [n-k] v^{k+1}, \quad F v^k = [k] v^{k-1}. \tag{11}$$

The involutions ψ and σ of U give rise to endomorphisms of V_n as follows: Let $\sigma_n : V_n \to V_n$ be the $\mathbb{C}(q)$-linear map defined by $\sigma_n(v_n) = v_0$ and $\sigma_n(xa) = \sigma(x)\sigma_n(a)$ for $x \in U$ and $a \in V_n$. Then

$$\sigma_n(v_k) = v_{n-k} \quad \text{for any } k. \tag{12}$$

Let $\psi_n : V_n \to V_n$ be the $\mathbb{C}$-linear map defined by $\psi_n(v_n) = v_n$ and $\psi_n(xa) = \psi(x)\psi_n(a)$ for $x \in U$ and $a \in V_n$. Then ψ_n is $\mathbb{C}(q)$-antilinear and

$$\psi_n(v_k) = v_k \quad \text{for any } k. \tag{13}$$

1.3. Tensor products of finite-dimensional representations

Given a positive integer n and a composition $\mathbf{d} = (d_1, \dots, d_r)$ of n we use the comultiplication (6) to define the U-module

$$V_{\mathbf{d}} = V_{d_1} \otimes \cdots \otimes V_{d_r}.$$

Let $\overline{V_{\mathbf{d}}} = \overline{V}_{d_1} \otimes \cdots \otimes \overline{V}_{d_r}$ be the corresponding $\mathfrak{sl}_2$-module. The *standard basis* (and *dual standard basis*) of $V_{\mathbf{d}}$ is given by $\{v_{\mathbf{a}} = v_{a_1} \otimes \cdots \otimes v_{a_r}\}$ ($\{v^{\mathbf{a}} = v^{a_1} \otimes \cdots \otimes v^{a_r}\}$ respectively), where $\mathbf{a}$ runs through all sequences $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ such that $0 \le a_j \le d_j$ for $1 \le j \le r$. Both bases are $\mathbb{C}(q)$-bases for $V_{\mathbf{d}}$. Likewise, we get the standard and dual standard $\mathbb{C}$-bases for $\overline{V_{\mathbf{d}}}$. We denote by $S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_r}$ the Young subgroup, corresponding to $\mathbf{d}$, of the symmetric group S_n.

2. The Grothendieck group of $\mathcal{O}$ and of the category of Harish-Chandra bimodules

2.1. Preliminaries

For an abelian category $\mathcal{B}$ let $\mathbf{G}(\mathcal{B}) = \mathbb{C} \otimes_{\mathbb{Z}} [\mathcal{B}]$, where $[\mathcal{B}]$ denotes the Grothendieck group of $\mathcal{B}$. The latter is the abelian group generated by symbols $[M]$ where M ranges over all objects of $\mathcal{B}$, subject to relations $[M_2] = [M_1] + [M_3]$ for all short exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{14}$$

in $\mathcal{B}$. An exact functor $F : \mathcal{B} \to \mathcal{B}'$ between abelian categories $\mathcal{B}$ and $\mathcal{B}'$ induces a $\mathbb{C}$-linear map $F^{\mathbf{G}} : \mathbf{G}(\mathcal{B}) \to \mathbf{G}(\mathcal{B}')$.

For a complex Lie algebra $\mathfrak{g}$ we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra with centre $\mathcal{Z}(\mathfrak{g})$. We fix an integer $n \ge 2$ and set $\mathfrak{g} = \mathfrak{gl}_n$. We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. Let $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$ denote the corresponding Bernstein–Gelfand–Gelfand category for $\mathfrak{g} = \mathfrak{gl}_n$ (see [BGG76]), i.e. the category of finitely generated $\mathfrak{g}$-modules which are $\mathfrak{h}$-diagonalisable and locally $\mathcal{U}(\mathfrak{n}_+)$-nilpotent. Let $W = S_n$ denote the Weyl group generated by the simple reflections s_i, $1 \le i \le n-1$, where $s_i s_j = s_j s_i$ if $|i - j| > 1$. Let ρ be the half-sum of the positive roots. For $w \in W$ and $\lambda \in \mathfrak{h}^*$ let $w \cdot \lambda = w(\lambda + \rho) - \rho$. Let $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$ be the stabiliser of λ with respect to this action. We denote by W^λ the set of shortest (with respect to the length function) coset representatives in W/W_λ. We denote by w_0 the longest element in W and by w_0^λ the longest element in W_λ. A weight $\lambda \in \mathfrak{h}^*$ is *integral* if $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ for any coroot $\check{\alpha}$. We call a weight $\lambda \in \mathfrak{h}^*$ *(strictly) dominant* if $\langle \lambda + \rho, \check{\alpha} \rangle \ge 0$ (or $\langle \lambda, \check{\alpha} \rangle \ge 0$ resp.) for any coroot $\check{\alpha}$ corresponding to a positive root α such that $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$. Note that this terminology is not commonly used in the literature, usually a weight is called dominant and integral if it is strictly dominant and integral in our terminology.

With this notion, the action of the centre of $\mathcal{U}(\mathfrak{gl}_n)$ gives a block decomposition $\mathcal{O} = \bigoplus {}_\lambda\mathcal{O}$, where the sum is taken over the set of dominant weights λ. The finite-dimensional simple objects in $\mathcal{O}$, however, are naturally indexed by strictly dominant and integral weights. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ denote the Verma module with highest weight λ. Let $P(\lambda)$ be its projective cover with simple head $L(\lambda)$. If $\lambda \in \mathfrak{h}^*$ is dominant and integral, then ${}_\lambda\mathcal{O}$ denotes the block containing all $M(x \cdot \lambda)$ with $x \in W$. (Based on the functor (3) we prefer the notation ${}_\lambda\mathcal{O}$ to the more common notation $\mathcal{O}_\lambda$ in order to be consistent with the fact that the objects of $\mathcal{O}$ are *left* $\mathfrak{g}$-modules.) We denote by d the usual contravariant duality on $\mathcal{O}$ preserving the simple objects. Recall that a module in ${}_\lambda\mathcal{O}$ having a Verma flag and a dual Verma flag is called a *tilting module*. Let $T(x \cdot \lambda) \in {}_\lambda\mathcal{O}$ be the indecomposable tilting module with $M(x \cdot \lambda)$ occurring as a submodule in any Verma flag. (For the classification we refer to [CI89], and for the general theory to [DR89], for example.)

2.2. Tensor products of finite-dimensional $\mathcal{U}(\mathfrak{sl}_2)$-modules and Harish-Chandra bimodules

The purpose of this section is to associate with any finite tensor product $\overline{V_{\mathbf{d}}}$ of finite-dimensional $\mathcal{U}(\mathfrak{sl}_2)$-modules an abelian category $\mathcal{B}$ of Harish-Chandra bimodules together with an isomorphism $\overline{\Phi} : \mathbf{G}(\mathcal{B}) \cong \overline{V_{\mathbf{d}}}$ of $\mathbb{C}$-vector spaces. In the following section we will define exact endofunctors on $\mathcal{B}$ which give rise to a $\mathcal{U}(\mathfrak{sl}_2)$-module structure on $\mathbf{G}(\mathcal{B})$ and show that the morphisms $\overline{\Phi}$ become isomorphisms of $\mathcal{U}(\mathfrak{sl}_2)$-modules.

We choose an ONB $\{e_i\}_{1\leq i\leq n}$ of $\mathbb{R}^n$ and identify $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n$ with $\mathfrak{h}^*$ such that $R_+ = \{e_i - e_j \mid i < j\}$ is the set of positive roots. The simple reflection $s_i \in W$ acts by permuting e_i and e_{i+1}. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ let $M(\mathbf{a})$ also denote the Verma module with highest weight $\sum_{i=1}^n a_i e_i - \rho$. If $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$ then we set $W_I = \langle s_i \mid i \notin I\rangle$. Corresponding to I we fix an integral block ${}_{I,n}\mathcal{O} = {}_{\lambda_I}\mathcal{O}$ such that $\{w \in W \mid w \cdot \lambda_I = \lambda_I\} = W_I$ and λ_I is minimal with this property. In the following we will also just write ${}_{i_1,\dots,i_r;n}\mathcal{O} = {}_{I;n}\mathcal{O}$, $W_I = W_{i_1,\dots,i_r}$, $W^I = W^{i_1,\dots,i_r}$ etc. In particular, we have the "maximal singular" blocks ${}_{\omega_i}\mathcal{O} = {}_{i;n}\mathcal{O}$ where $0 \leq i \leq n$, and ω_i is the i-th fundamental weight. (Note that by definition $W_0 = W_n = W$. We also remark that ${}_{1;2}\mathcal{O}$ is in fact regular.) Let ${}_{i;n}\mathcal{O}$ be the category containing only the zero module for $n < i$ or $i < 0$. Note that $M(\mathbf{a}) \in {}_{i;n}\mathcal{O}$ if and only if $a_j \in \{0, 1\}$ for all j and $\mathbf{a}$ contains exactly i ones. For $x \in W/W_I$ we denote by $\mathbf{a}(x) = \mathbf{a}_I(x) \in \mathbb{Z}^n$ the sequence such that

$$M(x \cdot \lambda_I) = M(\mathbf{a}(x)). \tag{15}$$

By definition, the vector space $\overline{V}_1^{\otimes n}$ has a basis of the form $\{v_{\mathbf{a}}\}$, where $\mathbf{a}$ runs through all $\{0,1\}$-sequences of length n. On the other hand, $\mathbf{G}({}_{i;n}\mathcal{O})$ has a basis of the form $\{[M(\mathbf{a})]\}$, where $\mathbf{a}$ runs through all $\{0,1\}$-sequences of length n, containing exactly i ones. Therefore (see also [BFK99, (34)]), there is an isomorphism of vector spaces

$$\mathbf{G}\Big(\bigoplus_{i=0}^{n} {}_{i;n}\mathcal{O}\Big) \cong \overline{V}_1^{\otimes n}, \qquad 1 \otimes [M(\mathbf{a})] \mapsto v_{a_1} \otimes \cdots \otimes v_{a_n}. \tag{16}$$

Before generalising this to arbitrary tensor products we give

Example 2.1. If $n = 2$ then ${}_{0;2}\mathcal{O}$ and ${}_{2;2}\mathcal{O}$ are semisimple with one simple object $L((0,0)) = M((0,0)) = P((0,0)) = T((0,0))$ (or $L((1,1)) = M((1,1)) = P((1,1)) = T((1,1))$ respectively), whereas ${}_{1;2}\mathcal{O}$ has the two simple objects $L((1,0))$ and $L((0,1)) = M((0,1)) = T((0,1))$ and $[M((1,0))] = [P((1,0))] = [L((1,0))] + [L((0,1))]$ and $[P((0,1))] = [M((1,0))] + [M((0,1))] = [L((0,1))] + [L((1,0))] + [L((0,1))]$. Hence $\mathbf{G}(\bigoplus_{i=0}^{2} {}_{i;2}\mathcal{O}) \cong \overline{V}_1^{\otimes 2}$ as vector spaces.

Let $\mathcal{H} = \mathcal{H}(\mathfrak{g})$ denote the category of *Harish-Chandra bimodules* for $\mathfrak{g}$. That is the full subcategory, inside the category of finitely generated $\mathcal{U}(\mathfrak{g})$-bimodules of finite length, given by all objects which are locally finite with respect to the adjoint action of $\mathfrak{g}$ (see e.g. [BG80], or for an overview [Jan83, Kapitel 6]). As for the category $\mathcal{O}$, the action of $\mathcal{Z}(\mathfrak{g})$ gives a block decomposition $\mathcal{H} = \bigoplus {}_{\lambda}\mathcal{H}_{\mu}$, where λ, $\mu \in \mathfrak{h}^*$ are dominant weights. More precisely, it is given as follows: Let $\lambda \in \mathfrak{h}^*$ be dominant. We denote by $\ker \chi_\lambda$ the $\mathcal{Z}(\mathfrak{g})$-annihilator of the Verma module $M(\lambda)$. Note that $\ker \chi_\lambda$ is a maximal ideal in $\mathcal{Z}(\mathfrak{g})$. A Harish-Chandra bimodule X is an object of ${}_{\lambda}\mathcal{H}_{\mu}$ if and only if $(\ker \chi_\lambda)^m X = 0 = X(\ker \chi_\mu)^m$ for large enough $m \in \mathbb{Z}_{>0}$.

For any two $\mathfrak{g}$-modules M and N, the space $\operatorname{Hom}_{\mathbb{C}}(M,N)$ is naturally a $\mathcal{U}(\mathfrak{g})$-bimodule. Let $\mathcal{L}(M,N)$ denote its maximal submodule which is locally finite with respect to the adjoint action of $\mathfrak{g}$. Then the simple objects in ${}_{\lambda}\mathcal{H}_{\mu}$ are of the form $\mathcal{L}(M(\mu), L(x\cdot\lambda))$ where x is a longest coset representative in $W_\mu \backslash W / W_\lambda$ (see e.g. [Jan83, 6.26]). However, ${}_{\lambda}\mathcal{H}_{\mu}$ does not have enough projective objects. Therefore, we consider the category ${}_{\lambda}\mathcal{H}_{\mu}^1$ which is by definition the full subcategory of ${}_{\lambda}\mathcal{H}_{\mu}$ given by all objects such that $X \ker \chi_\mu = 0$. (Note that the simple objects stay the same.) Let $\lambda \in \mathfrak{h}^*$ be integral and dominant. If μ is integral, regular and dominant then the functor $F_\mu = \bullet \otimes_{\mathcal{U}(\mathfrak{g})} M(\mu)$ defines an equivalence of categories

$${}_{\lambda}F_{\mu} : {}_{\lambda}\mathcal{H}_{\mu}^1 \cong {}_{\lambda}\mathcal{O}. \tag{17}$$

The inverse functor is given by $\mathcal{L}(M(\mu), \bullet)$. If μ is singular, then the functor ${}_{\lambda}F_{\mu}$ defines only an embedding. (All this is proved in [BG80], for an overview see also [Jan83]).

In this setup, one of the main ideas of this paper is that formula (16) is in fact only a very special case of the following more general fact:

Proposition 2.2. *Let $\mathbf{d} = (d_1, \ldots, d_r)$ be a composition of n. Let $\mu \in \mathfrak{h}^*$ be dominant and integral such that $W_\mu = S_{\mathbf{d}}$. There is an isomorphism of vector spaces*

$$\overline{\Phi} : \mathbf{G}\Big(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{H}_{\mu}^1(\mathfrak{gl}_n)\Big) \cong \overline{V}_{\mathbf{d}}, \quad 1 \otimes [\mathcal{L}(M(\mu), M(\mathbf{a}))] \mapsto v^{\mathbf{a}(\mu)} = v^{a(\mu)_1} \otimes \cdots \otimes v^{a(\mu)_r},$$

where $a(\mu)_j = |\{a_k = 1 \mid d_{j-1} < k \leq d_j\}|$ with $d_0 = 0$, $d_{r+1} = n$.

Note that Proposition 2.2 is in fact a generalisation of (16), because if $\mathbf{d} = (1,\dots,1)$, then ${}_{\omega_i}\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n) \cong {}_{i;n}\mathcal{O}$ via the equivalence (17) and the isomorphism $\overline{\Phi}$ gives rise to the one from (16), because $v_k = v^k$ in V_1. The "asymmetry" with respect to the central characters associated to $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)$ (i.e. we consider bimodules with a fixed central character from the right hand side, but with a fixed *generalised* character from the left hand side) appears to be unnatural. In fact, we could also work with the category $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}_\mu(\mathfrak{gl}_n)$ instead, since its Grothendieck group coincides with the one of the previous category and the functors $\mathcal{E}$ and $\mathcal{F}$ (which will be introduced later) can be extended naturally. However, proofs become much simpler if we make use of the functor F_μ. Moreover, the interpretation of the canonical basis in terms of tilting objects (see Theorem 5.3 and Remark 5.6) also militate in favour of using the category $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)$. Before we prove the Proposition 2.2 we give

Example 2.3. We consider the Lie algebra $\mathfrak{gl}_3$. The block ${}_{0;3}\mathcal{O}$ is semisimple with the only (simple and projective) Verma module $M((0,0,0))$. Likewise, the block ${}_{3;3}\mathcal{O}$ is semisimple with the only (simple and projective) Verma module $M((1,1,1))$. Each of the blocks ${}_{1;3}\mathcal{O}$ and ${}_{2;3}\mathcal{O}$ contains exactly three Verma modules, namely $M((1,0,0))$, $M((0,1,0))$ and $M((0,0,1))$ (or the Verma modules $M((1,1,0))$, $M((1,0,1))$ and $M((0,1,1))$ respectively). We have the following possibilities:

- W_μ is trivial, i.e. μ is regular: the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ are exactly the $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $L(\mathbf{a})$ occurs as the head of one out of these eight Verma modules. This "models" the 8-dimensional vector space $V_1^{\otimes 3}$.
- $W_\mu \cong S_1 \times S_2$: the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ are the $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $\mathbf{a} \in \{(0,0,0),(1,0,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$. This "models" the 6-dimensional vector space $V_1 \otimes V_2$.
- $W_\mu \cong S_2 \times S_1$: the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ are the $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $\mathbf{a} \in \{(0,0,0),(0,1,0),(0,0,1),(0,1,1),(1,1,0),(1,1,1)\}$. This "models" the 6-dimensional vector space $V_2 \otimes V_1$.
- $W_\mu \cong S_3$: the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ are the $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $\mathbf{a} \in \{(0,0,0),(0,0,1),(0,1,1),(1,1,1)\}$. This "models" the 4-dimensional vector space V_3.

Note that the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ are of the form $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $\mathbf{a} = (a_1,a_2,a_3)$ and the a_i's are (weakly) increasing within the parts of the composition given by μ.

Proof of Proposition 2.2. Recall that the simple objects in ${}_{\omega_i}\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n)$ are of the form $\mathcal{L}(M(\mu), L(x\cdot\omega_i))$, where x runs through the set of longest coset representatives D for the double cosets $W_\mu\backslash W/W_i$. On the other hand, $x \in D$ if and only if $x\cdot\omega_i = \sum_{j=1}^n a_j e_j - \rho$, where $a_k \leq a_j$ if $d_{l-1} < k < j \leq d_l$ for some $l \in \{1,\dots,r\}$ and the number of j's such that $a_j = 1$ is i. That is, the simple objects in $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)$ are exactly the bimodules of the form $\mathcal{L}(M(\mu), L(\mathbf{a}))$, where $\mathbf{a} = (a_1,\dots,a_n)$ is a $\{0,1\}$-sequence with exactly i ones such that the a_i's are (weakly) increasing within the parts of the composition given by μ. Let now $y \in W_\mu$ and $x \in W_\mu \backslash W / W_i$ be a longest coset representative. The following formula holds (see [Jos82, Lemma 2.5]):

$$\mathcal{L}(M(\mu), M(x \cdot \omega_i)) = \mathcal{L}(M(y \cdot \mu), M(x \cdot \omega_i)) \cong \mathcal{L}(M(\mu), M(y^{-1}x \cdot \omega_i)). \quad (18)$$

Therefore, the isomorphism classes $[\mathcal{L}(M(\mu), M(x \cdot \omega_i))]$ $(x \in D)$ give rise to a basis of $\mathbf{G}({}_{\omega_i}\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n))$. The statement of the proposition follows. □

In the situation of Proposition 2.2 let $x \in W_\mu \backslash W / W_i$ be a longest coset representative. Write $[\Delta(\mathbf{a}(x))] = \sum [M]$, where $[M] \in \{[\mathcal{L}(M(\mu), M(yx \cdot \lambda)] \mid y \in W_\mu\}$. (Note the difference between the symbols Δ and $\triangle$, the latter denoting the comultiplication.) From the formula (18) it follows that the $1 \otimes [\Delta(\mathbf{a}(x))]$, where x runs through all longest coset representatives in $W_\mu \backslash W / W_i$, form a basis of $\mathbf{G}({}_{\omega_i}\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n))$ (although the $[\Delta(\mathbf{a}(x))]$ do not form a basis of $[{}_{\omega_i}\mathcal{H}_{\mathbf{d}}(\mathfrak{gl}_n)]$ in general, see e.g. Remark 2.5(2)).

The following holds:

Corollary 2.4. *With the assumption of Proposition* 2.2 *we have*

$$\overline{\Phi}(1 \otimes [\Delta(\mathbf{a})]) = v_{\mathbf{a}(\mu)} := v_{a(\mu)_1} \otimes \cdots \otimes v_{a(\mu)_r},$$

where $a(\mu)_j = |\{a_k = 1 \mid d_{j-1} < k \leq d_j\}|$ *with* $d_0 = 0$, $d_{r+1} = n$.

Proof. The statement follows immediately from the definitions of $\overline{\Phi}$ and $[\Delta(\mathbf{a})]$ together with the formula (18) and the equality $v_k = [n,k]v^k$ in V_n. □

Remark 2.5.
1. In general, the category $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}^1_\mu$ is not necessarily a highest weight category, in the sense of [CPS88]. However (see [KM02]), it is equivalent to a module category over a *properly stratified algebra* (as introduced in [Dla00], generalising the notion of quasi-hereditary algebras and highest weight categories). In particular, this algebra might have infinite global dimension. If A is a properly stratified algebra then, by definition, the projective A-modules have a filtration such that subquotients are so-called *standard modules* $\Delta(\mathbf{a})$; any standard module $\Delta(\mathbf{a})$ has a filtration with subquotients, each isomorphic to the (same) *proper costandard module* $\overline{\Delta}(\mathbf{a})$. If the standard modules coincide with proper standard modules, then A is quasi-hereditary. For example, for any block of category $\mathcal{O}$, the (proper) standard modules are given by the Verma modules. In ${}_\lambda\mathcal{H}^1_\mu$, the proper standard modules are given by bimodules of the form $\mathcal{L}(M(\mu), M(\mathbf{a}))$, whereas the standard modules are certain modules $\Delta(\mathbf{a})$, where its image in the Grothendieck group $[\Delta(\mathbf{a})]$ is as above (for definition and general theory see e.g. [Dla00], for the special situation see e.g. [MS05]).
2. As an example consider the case $\mathfrak{g} = \mathfrak{gl}_2$. The category ${}_0\mathcal{H}^1_{-\rho}$ has one indecomposable projective object, $P = \mathcal{L}(M(-\rho), P(-2\rho))$, and one simple object, $S = \mathcal{L}(M(-\rho), M(-2\rho))$. Then P is the unique standard module, whereas S is the unique proper standard module and P is a self-extension

of S. The category ${}_0\mathcal{H}^1_{-\rho}$ is equivalent to $\mathbb{C}[x]/(x^2)$-mod (by [Soe90, Endomorphismensatz]), in particular, it has infinite global dimension. Note that the functor ${}_0F_{-\rho}$ maps S to the *dual* Verma module $\mathrm{d}\,M(0)$ with highest weight zero.

2.3. The submodule V_n inside $V_1^{\otimes n}$

There is a unique direct summand isomorphic to V_n inside $V_1^{\otimes n}$. The inclusion is given by the map

$$i_n : V_n \hookrightarrow V_1^{\otimes n}, \qquad v_k \mapsto \sum_{|\mathbf{a}|=k} q^{\mathbf{a}^-} v_{\mathbf{a}}, \tag{19}$$

where $|\mathbf{a}| = \sum_{i=n}^{r} a_i$ and $\mathbf{a}^-$ is the cardinality of the set $\{(i,j) \mid 1 \leq i < j \leq r, a_i > a_j\}$. A split of this inclusion is given by the projection

$$\pi_n : V_1^{\otimes n} \to V_n, \qquad v^{\mathbf{a}} \mapsto q^{-\mathbf{a}^+} v^{|\mathbf{a}|}, \tag{20}$$

where $\mathbf{a}^+$ is the cardinality of the set $\{(i,j) \mid 1 \leq i < j \leq n, a_i < a_j\}$ (see e.g. [FK97, Proposition 1.3]). The composition $i_n \circ \pi_n$ is the *Jones–Wenzl projector*. We define a linear map $F : \mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)) \to \mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{O})$ as follows: For $x \in W_\mu \backslash W / W_i$ a longest coset representative let $F(1 \otimes [\Delta(\mathbf{a}(x))]) = \sum 1 \otimes [M]$, where the sum runs over all $[M] \in \{[M(yx \cdot \omega_i)] \mid y \in W_\mu\}$. Then the following holds:

Proposition 2.6. *Let* $\mathbf{d} = (d_1, \ldots, d_r)$ *be a composition of* n. *Let* $\mu \in \mathfrak{h}^*$ *be dominant and integral such that* $W_\mu = S_{\mathbf{d}}$. *The following diagrams commute:*

$$\begin{array}{ccc}
\mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)) & \xrightarrow{\overline{\Phi}} & \overline{V}_{\mathbf{d}} \\
\downarrow F & & \downarrow i_{d_1}\otimes\cdots\otimes i_{d_r} \\
\mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{O}) & \xrightarrow{\overline{\Phi}} & \overline{V}_1^{\otimes d_1} \otimes \cdots \otimes \overline{V}_1^{\otimes d_r} = \overline{V}_1^{\otimes n}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{H}^1_\mu(\mathfrak{gl}_n)) & \xrightarrow{\overline{\Phi}} & \overline{V}_{\mathbf{d}} \\
\downarrow F & & \uparrow \pi_{d_1}\otimes\cdots\otimes \pi_{d_r} \\
\mathbf{G}(\bigoplus_{i=0}^{n} {}_{\omega_i}\mathcal{O}) & \xrightarrow{\overline{\Phi}} & \overline{V}_1^{\otimes d_1} \otimes \cdots \otimes \overline{V}_1^{\otimes d_r} = \overline{V}_1^{\otimes n}
\end{array}$$

Proof. Let $x \in W_\mu \backslash W / W_i$ be a longest coset representative. By Corollary 2.4 we have $i \circ \overline{\Phi}([\Delta(\mathbf{a}(x))]) = i(v_{\mathbf{a}(\mu)})$. Now $i_{d_j}(v_{a(\mu)_j}) = \sum_{|\mathbf{b}(j)|=a(\mu)_j} v_{\mathbf{b}(j)}$. This means

$$i \circ \overline{\Phi}([\Delta(\mathbf{a}(x))]) = \bigotimes_{j=1}^{r} \Big(\sum_{|\mathbf{b}(j)|=a(\mu)_j} v_{\mathbf{b}(j)} \Big).$$

On the other hand, we get $\overline{\Phi} \circ F([\Delta(\mathbf{a}(x))]) = \sum \overline{\Phi}([M])$, where the sum runs over all $[M] \in \{[M(yx \cdot \omega_i)] \mid y \in W_\mu\}$. This means

$$\overline{\Phi} \circ F([\Delta(\mathbf{a}(x))]) = \bigotimes_{j=1}^{r} \Big(\sum_{|\mathbf{b}(j)|=a(\mu)_j} v_{\mathbf{b}(j)} \Big).$$

Hence, the first diagram commutes. Let now $\pi = \pi_{d_1} \otimes \cdots \otimes \pi_{d_r}$. The first diagram says that $i \circ \overline{\Phi} = \overline{\Phi} \circ F$. We get $\overline{\Phi} = \pi \circ i \circ \overline{\Phi} = \pi \circ \overline{\Phi} \circ F$, hence the second diagram commutes as well. □

We remark that the map F has in fact a categorical interpretation. To explain this we have to recall some facts on the properly stratified structure of ${}_\lambda\mathcal{H}_\mu^1$ for integral dominant weights λ and μ. The indecomposable projective objects in ${}_\lambda\mathcal{H}_\mu^1$ are exactly the bimodules X such that ${}_\lambda F_\mu(X) \cong P(x \cdot \lambda) \in {}_\lambda\mathcal{O}$ for some longest coset representative $x \in W_\mu \backslash W / W_\lambda$ (see [BG80] or [Jan83, 6.17, 6.18]). For a longest coset representative $x \in W_\mu \backslash W / W_\lambda$ let $P(x)$ be the indecomposable projective object ${}_\lambda\mathcal{H}_\mu^1$ such that ${}_\lambda F_\mu(P(x)) = P(x \cdot \lambda)$. Let $P^{<x} = \bigoplus P(y)$, where the sum runs over all longest coset representatives $y \in W_\mu \backslash W / W_\lambda$ with $y < x$ in the Bruhat ordering. Then the standard module corresponding to x is defined as $\Delta(x) = P(x)/M$, where M is the trace of $P^{<x}$ in $P(x)$ (see [Dla96, Definition 3]). One can show that in fact $[\Delta(x)] = [\Delta(\mathbf{a}(x))]$ as defined above, and, moreover, the images of the standard modules from ${}_\lambda\mathcal{H}_\mu^1$ under the functor ${}_\lambda F_\mu$ are objects in ${}_\lambda\mathcal{O}$ having a Verma flag (i.e. a filtration with subquotients isomorphic to Verma modules). A proof of this fact can be found for example in [MS05]. More precisely $[{}_\lambda F_\mu \Delta(x)] = \sum [M]$, where the sum runs over all $[M] \in \{[M(yx \cdot \lambda)] \mid y \in W_\mu\}$ (see [MS05, Proposition 2.18]; note that the proof there works also for singular λ). So, F should be considered as a replacement for ${}_\lambda F_\mu^{\mathbf{G}}$ (which is not defined in general, since ${}_\lambda F_\mu$ is not necessarily exact). The properties of the derived functor of ${}_\lambda F_\mu$ were studied by J. Sussan, who constructed a categorification of the Jones–Wenzl projector in the setting of derived categories ([Sus05]).

2.4. Categorification of finite tensor products of finite-dimensional $\mathcal{U}(\mathfrak{sl}_2)$-modules via representations of $\mathfrak{gl}_n$

The purpose of this section is to lift the isomorphisms $\overline{\Phi}$ from Proposition 2.2 to isomorphisms of $\mathcal{U}(\mathfrak{sl}_2)$-modules. In other words, we *categorify* tensor products of finite-dimensional $\mathcal{U}(\mathfrak{sl}_2)$-modules. Our main results (Theorems 2.7 and 4.1) are *categorifications* of the modules $\overline{V_\mathbf{d}}$ and $V_\mathbf{d}$. Let us make this more precise: By a *categorification of* $\overline{V_\mathbf{d}}$ we mean an abelian category $\mathcal{C}$ together with exact endofunctors $\mathcal{E}$ and $\mathcal{F}$ and an isomorphism (of vector spaces) $\Psi : \mathbf{G}(\mathcal{C}) \cong \overline{V_\mathbf{d}}$ such that

$$\Psi([\mathcal{E}M]) = E\Psi([M]) \quad \text{and} \quad \Psi([\mathcal{F}M]) = F\Psi([M]) \tag{21}$$

for any object $M \in \mathcal{C}$. That is, if $\mathcal{C}$ is a categorification of $\overline{V_\mathbf{d}}$ then the functors $\mathcal{E}$ and $\mathcal{F}$ define an $\mathfrak{sl}_2$-action on $\mathbf{G}(\mathcal{C})$ such that Ψ becomes an isomorphism of $\mathfrak{sl}_2$-modules.

For $\lambda, \mu \in \mathfrak{h}^*$ dominant and integral let $\mathbb{T}_\lambda^\mu : {}_\lambda\mathcal{O} \to {}_\mu\mathcal{O}$ denote the translation functor (see [BG80], [Jan83]). With the notations above we also define $\mathbb{T}_I^J = \mathbb{T}_{i_1,\dots,i_r}^{j_1,\dots,j_s} : {}_I\mathcal{O} \to {}_J\mathcal{O}$ for $J = \{j_1, \dots, j_s\} \subseteq \{1, \dots, n\}$. Let $k \in \mathbb{Z}_{>0}$. Recall from [BFK99] the projective functors $\mathcal{E}_i^{(k)} : {}_{i;n}\mathcal{O}(\mathfrak{gl}_n) \to {}_{i+k;n}\mathcal{O}(\mathfrak{gl}_n)$ given by tensoring with the k-th exterior power of the natural representation for $\mathfrak{gl}_n$ composed with the projection onto $\mathcal{O}(\mathfrak{gl}_n)_{i+k;n}$. In particular, $\mathcal{E}_i = \mathcal{E}_i^{(1)}$ is given by tensoring with the natural representation composed with the projection onto $\mathcal{O}(\mathfrak{gl}_n)_{i+1;n}$. Let $\mathcal{F}_i^{(k)} : {}_{i;n}\mathcal{O} \to {}_{i-k;n}\mathcal{O}$ be the adjoint of $\mathcal{E}_{i-k}^{(k)} : {}_{i-k;n}\mathcal{O} \to {}_{i;n}\mathcal{O}$. Set $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{E}_i$, $\mathcal{F} = \bigoplus_{i=0}^n \mathcal{F}_i$. Since the functor F_μ commutes with tensoring with finite-dimensional (left) $\mathfrak{g}$-modules, the functors $\mathcal{E}$ and $\mathcal{F}$ give rise to endofunctors of $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}_\mu^1$ via restriction. We will denote these functors with the same letters $\mathcal{E}$ and $\mathcal{F}$ respectively.

We get the following result (which can also be viewed as a specialisation of Theorem 4.1):

Theorem 2.7. *Let $\mathbf{d} = (d_1, \dots, d_r)$ be a composition of n. Let $\mu \in \mathfrak{h}^*$ be dominant and integral such that $W_\mu = S_{\mathbf{d}}$. Then the isomorphism*

$$\overline{\Phi} : \mathbf{G}\Big(\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}_\mu^1(\mathfrak{gl}_n)\Big) \cong \overline{V_{\mathbf{d}}}, \qquad [\mathcal{L}(M(\mu), M(\mathbf{a}))] \mapsto v^{\mathbf{b}} = v^{b(\mu)_1} \otimes \cdots \otimes v^{b(\mu)_r},$$

where $b(\mu)_j = |\{a_k = 1 \mid d_{j-1} < k \leq d_j\}|$ with $d_0 = 0$, $d_{r+1} = n$, together with the endofunctors $\mathcal{E}$, $\mathcal{F}$ of $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}_\mu^1(\mathfrak{gl}_n)$ give rise to a categorification of $\overline{V_{\mathbf{d}}}$.

Proof. By Proposition 2.2 and [BFK99, Theorem 1], we only have to check the formulas (21) for $\Psi = \overline{\Phi}$, that is, $\overline{\Phi}([\mathcal{E}\bullet]) = E\overline{\Phi}([\bullet])$, and $\overline{\Phi}([\mathcal{F}\bullet]) = F\overline{\Phi}([\bullet])$.

Let now $M = \mathcal{L}(M(\mu), M(\mathbf{a}))$ for some $\{0,1\}$-sequence $\mathbf{a}$ of length n. Then we get

$$\overline{\Phi}([\mathcal{E}M]) = \overline{\Phi}([\mathcal{L}(M(\mu), \mathcal{E}M(\mathbf{a}))]) = \overline{\Phi}\Big(\sum_{\mathbf{a}'}[\mathcal{L}(M(\mu), M(\mathbf{a}'))]\Big),$$

where the sum runs over all sequences $\mathbf{a}'$ built up from $\mathbf{a}$ by replacing one zero by a one (see [BFK99, (38)]). From the definition of Φ and formula (18) we get $\overline{\Phi}([\mathcal{E}M]) = \sum_{\mathbf{a}'} \alpha_{\mathbf{a},\mathbf{a}'} v^{\mathbf{a}'(\mu)}$, where the sum runs over all $\mathbf{a}'$ where $a'_{j_0} = b(\mu)_{j_0} + 1$ for some j_0 and $a'_j = b(\mu)_j$ for $j \neq j_0$ and $\alpha_{\mathbf{a},\mathbf{a}'}$ is the number of zeros occurring in $\{a_k \mid d_{j_0} - 1 < k \leq d_{j_0}\}$. On the other hand, $Ev^{\mathbf{b}} = \sum_{\mathbf{a}'} \beta_{\mathbf{a},\mathbf{a}'} v^{\mathbf{a}'}$, where the sum runs over all $\mathbf{a}'$ as in the previous sum and $\beta_{\mathbf{a},\mathbf{a}'}$ is defined by the equation $Ev^{b(\mu)_{j_0}} = \beta_{\mathbf{a},\mathbf{a}'} v^{b(\mu)_{j_0}+1}$ in $V_{d_{j_0} - d_{j_0-1}+1}$. By formula (11) we actually have $\beta_{\mathbf{a},\mathbf{a}'} = d_{j_0} - d_{j_0-1} + 1 - b(\mu)_{j_0}$. The latter is, by definition of $b(\mu)_{j_0}$, exactly the number of zeros occurring in $\{a_k \mid d_{j_0} - 1 < k \leq d_{j_0}\}$. We therefore get $\overline{\Phi}([\mathcal{E}M]) = E\overline{\Phi}([M])$, even for all objects M. The similar arguments to show $\overline{\Phi}([\mathcal{F}M]) = F\overline{\Phi}([M])$ are omitted. □

3. The graded version of $\mathcal{O}$ and the action of $U_q(\mathfrak{sl}_2)$

Our main goal is to "categorify" the U-modules $V_{\mathbf{d}}$. In Theorem 2.7, we already obtained a categorification of the $\mathcal{U}(\mathfrak{sl}_2)$-modules $\overline{V}_{\mathbf{d}}$ via certain Harish-Chandra bimodules. The main idea is to introduce a graded version, say $\mathcal{H}_\mu^{\mathrm{gr}}$, of $\bigoplus_{i=0}^n {}_{\omega_i}\mathcal{H}_\mu^1(\mathfrak{gl}_n)$. Then the complexified Grothendieck group $\mathbf{G}(\mathcal{H}_\mu^{\mathrm{gr}})$ becomes a $\mathbb{C}(q)$-module, where q acts by shifting the grading. The second step will be to introduce exact functors $\mathbf{E}$, $\mathbf{F}$, $\mathbf{K}$, $\mathbf{K}^{-1}$ which induce a U-action on $\mathbf{G}(\mathcal{H}_\mu^{\mathrm{gr}})$ such that $\mathbf{G}(\mathcal{H}_\mu^{\mathrm{gr}}) \cong V_{\mathbf{d}}$ as U-modules. In the present section we consider first the special case $V_1^{\otimes n}$ and work with the category $\mathcal{O}$ instead of Harish-Chandra bimodules. The general case will easily be deduced afterwards.

3.1. Graded algebras and modules

For a ring or algebra A we denote by A-mod (resp. mod-A) the category of finitely generated left (resp. right) A-modules. If A is $\mathbb{Z}$-graded then we denote the corresponding categories of graded modules by A-gmod and gmod-A respectively. For $k \in \mathbb{Z}$ we denote by $\langle k \rangle$: gmod-$A \to$ gmod-A the functor of shifting the grading by k, i.e. $(\langle k \rangle M)_i = M_{i-k}$ for $M = \bigoplus_{i\in\mathbb{Z}} M_i \in$ gmod-A. We will normally write $M\langle k \rangle$ instead of $\langle k \rangle M$. For a finite-dimensional nonnegatively graded algebra A such that A_0 is semisimple, the Grothendieck group [gmod-A] is the free $\mathbb{Z}[q, q^{-1}]$-module, freely generated by a set of isomorphism classes of simple objects in gmod-A which give rise to a basis of [mod-A] after forgetting the grading. The $\mathbb{Z}[q, q^{-1}]$-module structure comes from the grading, namely, for an object M of gmod-A we define $q^i[M] = [M\langle i \rangle]$. We set $\mathbf{G}(\text{gmod-}A) = \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\text{gmod-}A]$.

3.2. The graded version of $\mathcal{O}$

We first recall (from [BGS96]) the graded version of each integral block of the category $\mathcal{O}$ which is defined by introducing a (Koszul) grading on the endomorphism ring of a minimal projective generator in this block. Fixing $P_\lambda = \bigoplus_{x\in W^\lambda} P(x\cdot\lambda) \in {}_\lambda\mathcal{O}$ a minimal projective generator defines an equivalence of categories $\epsilon_\lambda : {}_\lambda\mathcal{O} \cong \text{mod-}\operatorname{End}_{\mathfrak{g}}(P_\lambda)$ via the functor $M \mapsto \operatorname{Hom}_{\mathfrak{g}}(P_\lambda, M)$ (see e.g. [Bas68]). We fix such pairs $(P_\lambda, \epsilon_\lambda)$ for any dominant integral λ. Now we have to introduce a grading on $\operatorname{End}_{\mathfrak{g}}(P_\lambda)$. This is done in [BGS96] using the connection between the category $\mathcal{O}$ and modules over the cohomology ring of certain partial flag varieties (as described in [Soe90]). Which partial flag we have to take depends on the block of $\mathcal{O}$ we consider. More precisely, we have to do the following: In each integral block ${}_\lambda\mathcal{O}$, there is a unique indecomposable self-dual projective-injective (tilting) module T_λ. We consider Soergel's functor $\mathbb{V}_\lambda : {}_\lambda\mathcal{O} \to \text{mod-}\operatorname{End}_{\mathfrak{g}}(T_\lambda)$, $M \mapsto \operatorname{Hom}_{\mathcal{O}}(T_\lambda, M)$, which has been introduced in [Soe90]. By [Soe90, Endomorphismensatz], the algebra $\operatorname{End}_{\mathcal{O}}(T_\lambda)$ is canonically isomorphic to the invariants C^{W_λ} inside the coinvariants $C = S(\mathfrak{h})/(S(\mathfrak{h})_+^W)$, i.e. it is isomorphic to the cohomology ring of the partial flag variety corresponding to λ inside the cohomology ring of the full flag variety (see e.g. [Hil82]).

With the above notation we also denote by $T_{i_1,\dots,i_r} \in {}_I\mathcal{O}$ the unique indecomposable projective-injective (tilting) module with corresponding functor $\mathbb{V}_I : {}_I\mathcal{O} \to \mathrm{mod} - \mathrm{End}_{\mathcal{O}}(T_I)$ and set $C^I = \mathrm{End}_{\mathcal{O}}(T_I)$. We will abbreviate C^{W_λ} as C^λ.

If we fix a $\mathbb{Z}$-grading on the algebra $S(\mathfrak{h})$ by putting $\mathfrak{h}$ in degree two, then C, and hence also any C^λ, inherits a grading (which is known to coincide with the cohomology grading; see e.g. [Hil82] and references therein). For any $x \in W^\lambda$, the module $\mathbb{V}P(x \cdot \lambda)$ has a graded lift (see [BGS96]); i.e. there exists a module $M \in C^\lambda$-gmod which is isomorphic to $\mathbb{V}P(x\cdot\lambda)$ after forgetting the grading. Since $\mathrm{End}_{C^\lambda}(\mathbb{V}P(x\cdot\lambda)) \cong \mathrm{End}_{\mathcal{O}}(P(x\cdot\lambda))$ ([Soe90, Struktursatz]), the module $\mathbb{V}P(x\cdot\lambda)$ is indecomposable. Hence a graded lift is unique up to isomorphism and grading shift (see e.g. [BGS96, Lemma 2.5.3]). We fix for any $x \in W^\lambda$ a graded lift of $\mathbb{V}P(x\cdot\lambda)$ such that its lowest degree is $-l(x)$. By abuse of notation we denote this graded lift also $\mathbb{V}P(x\cdot\lambda)$. This defines a grading on $\mathbb{V}P_\lambda = \bigoplus_{x\in W^\lambda} \mathbb{V}P(x\cdot\lambda)$ and induces a grading on $\mathrm{End}_{\mathcal{O}}(P_\lambda) = \mathrm{End}_{C^\lambda}(\mathbb{V}P_\lambda)$ by Soergel's structure theorem ([Soe90]). In fact, it turns $\mathrm{End}_{\mathcal{O}}(P_\lambda)$ into a (nonnegatively graded) Koszul algebra (see [BGS96]) which we denote by ${}_\lambda A$. As usual we also write ${}_iA$ instead of ${}_{\omega_i}A$.

3.3. Graded lifts of modules and functors

Now we have a graded version for any integral block of $\mathcal{O}$. We need graded lifts of modules in ${}_\lambda\mathcal{O}$ which are defined as follows:

Definition 3.1. Let λ, μ be dominant and integral weights. Let f_μ : gmod-${}_\mu A \to$ mod-${}_\mu A$ denote the functor which forgets the grading.

1. Let $\overline{M} \in {}_\mu\mathcal{O}$. A *graded lift of* $\overline{M}$ is a module $M \in$ gmod-${}_\mu A$ such that $f_\lambda(M) \cong \epsilon_\mu(\overline{M})$.
2. Let $\overline{F} : {}_\lambda\mathcal{O} \to {}_\mu\mathcal{O}$ be a functor. A *graded lift of* $\overline{F}$ is a functor F : gmod-${}_\lambda A \to$ gmod-${}_\mu A$ such that
   (i) $F\langle k\rangle \cong \langle k\rangle F$,
   (ii) $f_\mu F \cong \epsilon_\mu \overline{F} \epsilon_\lambda^{-1} f_\lambda$.

If $\overline{M} \in {}_\lambda\mathcal{O}$ is indecomposable, then a graded lift of $\overline{M}$, if it exists, is unique up to isomorphism and grading shift (see e.g. [BGS96, Lemma 2.5.3]). In general, for an arbitrary module $\overline{M} \in {}_\lambda\mathcal{O}$, a graded lift does not have to exist (see e.g. [Str03, Section 4]). However, it is known that indecomposable projective modules, (dual) Verma modules and simple modules have graded lifts: see [BGS96, Section 3.11], [Str03, Section 3]. (For a more general setup for quasi-hereditary algebras we refer to [Zhu04].) We denote by $\tilde{P}(x\cdot\lambda)$, $\tilde{M}(x\cdot\lambda)$, $\tilde{L}(x\cdot\lambda)$ the (uniquely defined up to isomorphism) graded lifts of the modules $P(x\cdot\lambda)$, $M(x\cdot\lambda)$, $L(x\cdot\lambda)$ with the property that their heads are concentrated in degree zero. Let $\tilde{\nabla}(x\cdot\lambda)$ be the graded lift of the dual Verma module $\nabla(x\cdot\lambda) = \mathrm{d}\,\Delta(x\cdot\lambda)$ such that the socle is concentrated in degree zero. Let $\tilde{I}(x\cdot\lambda)$ be the injective hull of $\tilde{L}(x\cdot\lambda)$. Let $\mathrm{d} = \mathrm{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$ denote a graded lift of the duality functor such that $\mathrm{d}(L) = L$ for any simple module L concentrated in degree zero (for properties see e.g. [Str03]).

3.4. Projective functors and the cohomology ring of the flag variety

The purpose of this section is to give the tools for a construction of graded lifts of the functors $\mathcal{E}$ and $\mathcal{F}$. The following result gives first of all a categorical interpretation of the divided powers of E and F and secondly provides an alternative description of the functors $\mathcal{E}$ and $\mathcal{F}$ which makes it possible to relate them to the cohomology ring of the flag variety in Proposition 3.3 afterwards. From these results, the desired graded lifts of functors can be constructed easily.

Proposition 3.2. *Let $0 \leq i \leq n$ and $k \in \mathbb{Z}_{>0}$.*

(a) *There are isomorphisms of projective functors*

$$\bigoplus_{j=1}^{k!} \mathcal{E}_i^{(k)} \cong \mathcal{E}_{i+k-1} \cdots \mathcal{E}_{i+1}\mathcal{E}_i, \tag{22}$$

$$\bigoplus_{j=1}^{k!} \mathcal{F}_i^{(k)} \cong \mathcal{F}_{i+k+1} \cdots \mathcal{F}_{i-1}\mathcal{F}_i. \tag{23}$$

(b) *Let $k \in \mathbb{Z}_{>0}$. There are isomorphisms of indecomposable projective functors*

$$\mathcal{E}_i^{(k)} \cong \mathbb{T}_{i,i+k}^{i+k}\, \mathbb{T}_i^{i,i+k}, \tag{24}$$

$$\mathcal{F}_i^{(k)} \cong \mathbb{T}_{i,i-k}^{i-k}\, \mathbb{T}_i^{i,i-k}. \tag{25}$$

Proof. By adjointness properties it is enough to prove the formulas (22) and (24). Recall (see e.g. [Jan83, 4.6(1)]) that for any $\lambda \in \mathfrak{h}^*$ and any finite-dimensional $\mathfrak{g}$-module E we have $[M(\lambda)\otimes E] = [\bigoplus_\nu M(\lambda+\nu)]$, where ν runs through the multiset of weights of E. From the definition of $\mathcal{E}_i^{(k)}$ we get in particular $[\mathcal{E}_i^{(k)}(M(\mathbf{a}))] = [\bigoplus_{\mathbf{a}'} M(\mathbf{a}')]$, where $\mathbf{a}' = (a_1', a_n')$ runs through the set of $\{0,1\}$-sequences containing exactly $i+k$ ones and where $a_j = 1$ implies $a_j' = 1$ for $1 \leq j \leq n$. In particular $[\bigoplus_{j=1}^{k!} \mathcal{E}_i^{(k)}(M(\lambda_i))] = [\mathcal{E}_{i+k-1} \cdots \mathcal{E}_{i+1}\mathcal{E}_i(M(\lambda_i))]$ (see [BFK99, Proposition 6]). Hence, (22) follows from the classification of projective functors ([BG80]).

To prove the second part let first $k = 1$. The formula [BFK99, Proposition 6] shows that $\mathcal{E}_i(M(\omega_i))$ has a Verma flag with subquotients isomorphic to $M(x \cdot \omega_{i+1})$, where $x \in W_i/W_{i,i+1}$ is a shortest coset representative. The same is true for the module $\mathbb{T}_{i,i+1}^{i+1}\, \mathbb{T}_i^{i,i+1}\, M(\omega_i)$ (using [Jan83, 4.13(1), 4.12 (2)]). Since both modules are projective, they are isomorphic. The classification theorem of projective functors ([BG80, Section 3]) provides the required isomorphism of functors for $k = 1$. Using again [Jan83, formulas 4.13(1), 4.12(2)] we easily get $[\mathcal{E}_i^{(k)}(M(\lambda_i))] = [\mathbb{T}_{i,i+k}^{i+k}\, \mathbb{T}_i^{i,i+k}\, M(\lambda_i)]$. The classification theorem for projective functors ([BG80]) implies $\mathcal{E}_i^{(k)} \cong \mathbb{T}_{i,i+k}^{i+k}\, \mathbb{T}_i^{i,i+k}$. Note that $M(\lambda_{i+k})$ occurs with multiplicity one in $\mathcal{E}_i^{(k)}(M(\lambda_i))$, hence $\mathcal{E}_i^{(k)}(M(\lambda_i))$ is indecomposable, and therefore so is $\mathcal{E}_i^{(k)}$ (again by the classification of projective functors [BG80]). □

We have the restriction functors $\mathrm{Res}_J^I : C^I\text{-mod} \to C^J\text{-mod}$ if $J \subseteq I$ and $\mathrm{Res}_\mu^\lambda : C^\lambda\text{-mod} \to C^\mu\text{-mod}$ if $W_\lambda \subseteq W_\mu$.

Proposition 3.3. *Let λ, μ be dominant and integral weights such that $W_\lambda \subseteq W_\mu$. There are isomorphisms of functors*

$$\mathbb{V}_\mu \, \mathbb{T}_\lambda^\mu \cong \mathrm{Res}_\mu^\lambda \, \mathbb{V}_\mu, \qquad \mathbb{V}_\lambda \, \mathbb{T}_\mu^\lambda \cong C^\lambda \otimes_{C^\mu} \mathbb{V}_\mu(\bullet).$$

In particular,

$$\mathbb{V}_{i+k} \, \mathcal{E}_i^{(k)} \cong \mathrm{Res}_{i+k}^{i,i+k} \; C^{i,i+k} \otimes_{C^i} \mathbb{V}_i(\bullet), \qquad \mathbb{V}_{i-k} \, \mathcal{F}_i^{(k)} \cong \mathrm{Res}_{i-k}^{i,i-k} \; C^{i,i-k} \otimes_{C^i} \mathbb{V}_i(\bullet),$$

for any $k \in \mathbb{Z} > 0$.

Proof. This is [Soe92, Theorem 12, Proposition 6] together with Proposition 3.2. □

To keep track of the grading, we first need the following well-known, but crucial fact that C^I is a free C^J-module of finite rank whenever $J \subseteq I$. More precisely, we need the following statement:

Lemma 3.4. *Let $1 \leq i \leq n-1$. There are isomorphisms of graded C^i-modules*

$$C^{i,i+1} \cong \bigoplus_{r=0}^{n-i-1} C^i \langle 2r \rangle, \qquad C^{i,i-1} \cong \bigoplus_{r=0}^{i-1} C^i \langle 2r \rangle.$$

Proof. By classical invariant theory ([Hil82, II.3]), $C^{i,i\pm1}$ is a free C^i-module of rank $|W_i/W_{i,i\pm1}|$, and a basis can be chosen homogeneous in the degrees length of x, where x runs through the set of shortest coset representatives from $W_i/W_{i,i\pm1}$. □

The following adjoint pairs of functors will be used later:

Proposition 3.5. *Let λ, μ be dominant and integral. Assume $W_\lambda \subseteq W_\mu$. Then there are pairs of adjoint functors*

$$(C^\lambda \otimes_{C^\mu} \bullet, \mathrm{Res}_\mu^\lambda) \quad \textit{and} \quad (\mathrm{Res}_\mu^\lambda, C^\lambda \otimes_{C^\mu} \bullet)$$

between C^λ-mod *and C^μ*-mod*; and*

$$(C^\lambda \otimes_{C^\mu} \bullet, \mathrm{Res}_\mu^\lambda) \quad \textit{and} \quad (\mathrm{Res}_\mu^\lambda, C^\lambda \otimes_{C^\mu} \bullet \langle -\max \rangle) \tag{26}$$

considered as functors between C^λ-gmod *and C^μ*-gmod*. Here* max $\in I$ *denotes the maximal element in I where $C^\lambda \cong \bigoplus_{i \in I} C^\mu \langle i \rangle$ as graded C^μ-module.*

Proof. By Proposition 3.3, the pairs of adjoint functors $(\mathbb{T}_\mu^\lambda, \mathbb{T}_\lambda^\mu)$ and $(\mathbb{T}_\lambda^\mu, \mathbb{T}_\mu^\lambda)$ give rise to the pairs $(C^\lambda \otimes_{C^\mu} \bullet, \mathrm{Res}_\mu^\lambda)$ and $(\mathrm{Res}_\mu^\lambda, C^\lambda \otimes_{C^\mu} \bullet)$, since $\mathbb{V}$ is a quotient functor. However, for the graded version we have to be more explicit. For $M \in C^\mu$-mod we consider the inclusion $i_M : M \hookrightarrow C^\lambda \otimes_{C^\mu} M$, $m \mapsto 1 \otimes m$. This is a C^μ-morphism and defines a map

$$\Phi_{M,N} : \mathrm{Hom}_{C^\lambda}(C^\lambda \otimes_{C^\mu} M, N) \to \mathrm{Hom}_{C^\mu}(M, \mathrm{Res}_\mu^\lambda N), \qquad f \mapsto f \circ i_M,$$

for any $N \in C^\lambda$-mod. The map is functorial in M and N, and it is injective, since $f \circ i_M = 0$ implies $f(c \otimes m) = cf(1 \otimes m) = c(f \circ i_M(m)) = 0$ for any $m \in M$, $c \in C^\lambda$. To show the surjectivity it is enough to compare the dimensions.

Moreover, the functors are exact and additive. Therefore, it is sufficient to consider the case $M = C^\mu$ (for general M choose a free 2-step resolution and use the Five-Lemma). In this case we have $\dim \mathrm{Hom}_{C^\lambda}(C^\lambda \otimes_{C^\mu} C^\mu, N) = \dim \mathrm{Hom}_{C^\lambda}(C^\lambda, N) = \dim N = \dim \mathrm{Hom}_{C^\mu}(C^\mu, \mathrm{Res}^\lambda_\mu N)$. This proves the first adjunction. Moreover, the adjunction is compatible with the grading, since i_M is homogeneous of degree zero for graded C^μ-modules M.

Let $p_N : C^\lambda \otimes_{C^\mu} N \cong \bigoplus_{i\in I} N\langle i\rangle \to N\langle\max\rangle$ be the projection for $N \in C^\mu$-gmod. It defines a natural morphism

$$\Phi'_{M,N} : \mathrm{Hom}_{C^\lambda}(M, C^\lambda \otimes_{C^\mu} N) \to \mathrm{Hom}_{C^\mu}(\mathrm{Res}^\lambda_\mu M, N), \qquad f \mapsto p_N \circ f.$$

We show that Φ' is injective: Assume $p_N \circ f = 0$. Then $f(M) \subseteq \bigoplus_{i\in I-\max} N\langle i\rangle$. On the other hand, for any $n \in C^\lambda \otimes_{C^\mu} N$ there exists a $c \in C$ such that $cn \notin \bigoplus_{i\in I-\{\max\}} N\langle i\rangle$. Hence $f = 0$ and the injectivity of $\Phi'_{M,N}$ follows. The map $\Phi'_{M,N}$ is surjective for $M = C^\lambda$, because $\dim \mathrm{Hom}_{C^\lambda}(C^\lambda, C^\lambda \otimes_{C^\mu} N) = \dim(C^\lambda \otimes_{C^\mu} N) = i \cdot \dim N = \dim \mathrm{Hom}_{C^\mu}(C^\lambda, N)$, where $i = |W_\mu/W_\lambda|$ is the rank of C^λ as C^μ-module. The surjectivity in general then follows from the Five-Lemma. By construction, the isomorphism Φ' is homogeneous of degree max. The existence of the second adjunction from (26) follows. $\square$

3.5. A functorial action of $U_q(\mathfrak{sl}_2)$

We are prepared to introduce the graded lifts of our functors $\mathcal{E}$ and $\mathcal{F}$. We define

$$\mathbf{E}_i^{(k)} = \mathrm{Hom}_{C^{i+k}}(\mathbb{V}P_{i+k}, \mathrm{Res}^{i,i+k}_{i+k} C^{i,i+k} \otimes_{C^i} \mathbb{V}P_i\langle -r_{i,k}\rangle), \tag{27}$$

$$r_{i,k} = \sum_{r=i}^{i+k-1} (n-r-1);$$

$$\mathbf{F}_i^{(k)} = \mathrm{Hom}_{C^{i-1}}(\mathbb{V}P_{i-k}, \mathrm{Res}^{i,i-k}_{i-k} C^{i,i-k} \otimes_{C^i} \mathbb{V}P_i\langle -r'_{i,k}\rangle), \tag{28}$$

$$r'_{i,k} = \sum_{r=i}^{i+k-1} r-1.$$

We have $\mathbf{E}_i^{(k)} \in \mathrm{End}_{C^i}(\mathbb{V}P_i)$-gmod-$\mathrm{End}_{C^{i+k}}(\mathbb{V}P_{i+k})$ via $g.f.h = (\mathrm{Id}\otimes g)\circ f \circ h$ for $g \in \mathrm{End}_{C^i}(\mathbb{V}P_i)$, $h \in \mathrm{End}_{C^{i+k}}$ and $f \in \mathbf{E}_i^{(k)}$. From the definitions we may also consider $\mathbf{E}_i^{(k)}$ as an object in ${}_iA$-gmod-${}_{i+k}A$. Tensoring with $\mathbf{E}_i^{(k)}$ defines a functor $\mathbf{E}_i^{(k)}$: gmod-${}_iA \to$ gmod-${}_{i+k}A$ (which we denote, abusing notation, by the same symbol). Analogously, $\mathbf{F}_i^{(k)}$ defines a functor $\mathbf{F}_i^{(k)}$: gmod-${}_iA \to$ gmod-${}_{i-k}A$. Set $\mathbf{E}^{(k)} = \bigoplus_{i=0}^n \mathbf{E}_i^{(k)}$, considered as an endofunctor of $\bigoplus_{i=0}^n$ gmod-${}_iA$. Similarly, $\mathbf{F}^{(k)} = \bigoplus_{i=0}^n \mathbf{F}_i^{(k)}$. Let $\mathbf{K}_i = \langle 2i-n\rangle$ be the endofunctor of gmod-${}_iA$ which shifts the degree by $2i-n$, and $\mathbf{K} = \bigoplus_{i=0}^n \mathbf{K}_i$.

The following result is the crucial step towards our main categorification theorem (Theorem 4.1):

Theorem 3.6. (a) *The functors* $\mathbf{E}_i^{(k)}$ *and* $\mathbf{F}_i^{(k)}$ *are graded lifts of* $\mathcal{E}_i^{(k)}$ *and* $\mathcal{F}_i^{(k)}$ *respectively.*

(b) *The functors* $\mathbf{E}$, $\mathbf{F}$, *and* $\mathbf{K}$ *satisfy the relations*

$$\mathbf{KE} \cong \mathbf{EK}\langle 2\rangle,$$
$$\mathbf{KF} \cong \mathbf{FK}\langle -2\rangle,$$
$$\mathbf{KK}^{-1} \cong \mathrm{Id} \cong \mathbf{K}^{-1}\mathbf{K},$$
$$\mathbf{E}_{i-1}\mathbf{F}_i \oplus \bigoplus_{r=0}^{n-i-1} \mathrm{Id}\langle n-1-2r-2i\rangle \cong \mathbf{F}_{i+1}\mathbf{E}_i \oplus \bigoplus_{r=0}^{i-1} \mathrm{Id}\langle 2i-n-2r-1\rangle.$$

(c) *In the Grothendieck group we have the equality*

$$(q-q^{-1})(\mathbf{E}_{i-1}^{\mathbf{G}}\mathbf{F}_i^{\mathbf{G}} - \mathbf{F}_{i+1}^{\mathbf{G}}\mathbf{E}_i^{\mathbf{G}}) = \mathbf{K}_i^{\mathbf{G}} - (\mathbf{K}_i^{-1})^{\mathbf{G}}.$$

Moreover, $\mathbf{E}_{i-1}\mathbf{F}_i$ *is a summand of* $\mathbf{F}_{i+1}\mathbf{E}_i$ *if* $n-2i>0$. *Likewise,* $\mathbf{F}_{i+1}\mathbf{E}_i$ *is a summand of* $\mathbf{E}_{i-1}\mathbf{F}_i$ *if* $n-2i<0$.

Proof. The first statement follows from the definitions, as well as $\mathbf{K}_{i+1}\mathbf{E}_i \cong \mathbf{E}_i\mathbf{K}_i\langle 2\rangle$, and $\mathbf{K}_{i-1}\mathbf{F}_i \cong \mathbf{F}_i\mathbf{K}_i\langle -2\rangle$. Obviously $\mathbf{K}$ is an auto-equivalence with inverse $\mathbf{K}^{-1}$. To prove the remaining isomorphisms we first claim that

$$\mathcal{E}_{i-1}\,\mathcal{F}_i \cong G \oplus \bigoplus_{r=1}^{i-1}\mathrm{Id} \quad \text{and} \quad \mathcal{F}_{i+1}\,\mathcal{E}_i \cong G \oplus \bigoplus_{r=1}^{n-i-1}\mathrm{Id} \tag{29}$$

for some indecomposable endofunctor G of ${}_{i;n}\mathcal{O}$. Let $M(\mathbf{a})$ be the projective Verma module in ${}_{i;n}\mathcal{O}$, i.e. $\mathbf{a}=(1,\dots,1,0,\dots,0)$ with i ones. Then

$$\begin{aligned}[]
[\mathcal{E}_{i-1}\mathcal{F}_i M(\mathbf{a})] &= \Big[\mathcal{E}_{i-1}\bigoplus_{k=1}^{i} M(\mathbf{a}(k))\Big] \\
&= \Big[\bigoplus_{k=1}^{i}\bigoplus_{l=i+1}^{n} M(\mathbf{a}(k,l))\Big] + \Big[\bigoplus_{k=1}^{i} M(\mathbf{a})\Big],
\end{aligned}$$

where $\mathbf{a}(k), \mathbf{a}(k,l) \in \mathbb{Z}^n$ are such that

$$\mathbf{a}(k)_j = \begin{cases} 0 & \text{if } i<j\le n \text{ or } j=k, \\ 1 & \text{otherwise,}\end{cases}$$
$$\mathbf{a}(k,l)_j = \begin{cases} 0 & \text{if } i<j\ne l\le n \text{ or } j=k, \\ 1 & \text{otherwise.}\end{cases}$$

On the other hand,

$$\begin{aligned}[]
[\mathcal{F}_{i+1}\mathcal{E}_i M(\mathbf{a})] &= \Big[\mathcal{F}_{i+1}\bigoplus_{l=i+1}^{n} M(\mathbf{b}(l))\Big] \\
&= \Big[\bigoplus_{k=1}^{i}\bigoplus_{l=i+1}^{n} M(\mathbf{b}(l,k))\Big] + \Big[\bigoplus_{k=i+1}^{n} M(\mathbf{a})\Big],
\end{aligned}$$

where $\mathbf{b}(l), \mathbf{b}(l,k) \in \mathbb{Z}^n$ are such that

$$\mathbf{b}(l)_j = \begin{cases} 1 & \text{if } 1 \leq j \leq i \text{ or } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{b}(l,k)_j = \begin{cases} 1 & \text{if } 1 \leq j \neq k \leq i \text{ or } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 0 & \text{if } i < j \neq l \leq n \text{ or } j = k, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $[\mathcal{F}_{i+1}\mathcal{E}_i M(\mathbf{a})] = [\bigoplus_{k=1}^{i} \bigoplus_{l=i+1}^{n} M(\mathbf{a}(k,l))] + [\bigoplus_{k=i+1}^{n} M(\mathbf{a})]$.

Let P be the projective cover of $M((0,1,\dots,1,0,\dots,0,1)) \in {}_{i;n}\mathcal{O}$. An easy calculation (in the Hecke algebra) shows that $[P] = [\bigoplus_{k=1}^{i} \bigoplus_{l=i+1}^{n} M(\mathbf{a}(k,l))] + [M(\mathbf{a})]$. The decompositions (29) then follow from the classification theorem ([BG80]) of projective functors. The functor G is the indecomposable projective functor which sends $M(\mathbf{a})$ to P.

Now we have to consider the graded picture. By Lemma 3.4 we get isomorphisms of graded C^i-modules

$$\begin{aligned} C^{i,i+1} \otimes_{C^{i+1}} C^{i,i+1} \otimes_{C^i} \mathbb{C} &\cong \bigoplus_{l=0}^{i} \bigoplus_{k=0}^{n-i-1} \mathbb{C}\langle 2k+2l \rangle, \\ C^{i,i-1} \otimes_{C^{i-1}} C^{i,i-1} \otimes_{C^i} \mathbb{C} &\cong \bigoplus_{l=0}^{n-i} \bigoplus_{k=0}^{i-1} \mathbb{C}\langle 2l+2k \rangle. \end{aligned} \tag{30}$$

Since their lowest degrees coincide and G is indecomposable, the decompositions (29) give rise to isomorphisms of endofunctors of gmod-${}_iA$,

$$\mathbf{E}_{i-1}\,\mathbf{F}_i \cong \mathbf{G} \oplus \bigoplus_{r=0}^{i-1} \mathrm{Id}\langle m_r \rangle \quad \text{and} \quad \mathbf{F}_{i+1}\,\mathbf{E}_i \cong \mathbf{G} \oplus \bigoplus_{r=0}^{n-i-1} \mathrm{Id}\langle n_r \rangle, \tag{31}$$

where $\mathbf{G}$ is a certain graded lift of G and m_r, $n_r \in \mathbb{Z}$. The formulas (30), together with the definition of the $\mathbf{E}_i$, $\mathbf{F}_i$ and the formula $\mathrm{Hom}(M, N\langle i \rangle) = \mathrm{Hom}(M,N)\langle -i \rangle$ for graded morphisms between graded modules M, N, imply that, if $n - 2i \geq 0$, then

$$\begin{aligned} \mathbf{F}_{i+1}^{\mathbf{G}}\mathbf{E}_i^{\mathbf{G}} - \mathbf{E}_{i-1}^{\mathbf{G}}\mathbf{F}_i^{\mathbf{G}} &= \Big(\bigoplus_{k=0}^{n-i-1} \langle -(2k+2i) \rangle \langle -(-n+1) \rangle \Big)^{\mathbf{G}} \\ &\quad - \Big(\bigoplus_{k=0}^{i-1} \langle -(2(n-i)+2k) \rangle \langle n-1 \rangle \Big)^{\mathbf{G}} \\ &= \begin{cases} 0 & \text{if } n-2i = 0, \\ (\bigoplus_{k=0}^{n-2i-1} \langle -(2i+2k) \rangle \langle n-1 \rangle)^{\mathbf{G}} & \text{if } n-2i > 0, \end{cases} \\ &= [n-2i]\,\mathrm{Id}\,. \end{aligned}$$

If $n-2i<0$, then

$$\begin{aligned}\mathbf{E}^{\mathbf{G}}_{i-1}\mathbf{F}^{\mathbf{G}}_{i}-\mathbf{F}^{\mathbf{G}}_{i+1}\mathbf{E}^{\mathbf{G}}_{i} &\cong \Big(\bigoplus_{k=0}^{i-1}\langle -(2(n-i)+2k)\rangle\langle -(-n+1)\rangle\Big)^{\mathbf{G}}\\ &\quad -\Big(\bigoplus_{k=0}^{n-i-1}\langle -(2k+2i)\rangle\langle n-1\rangle\Big)^{\mathbf{G}}\\ &=\Big(\bigoplus_{k=0}^{2i-n-1}\langle -(n-2i+1+2k)\rangle\Big)^{\mathbf{G}}\\ &=[2i-n]\,\mathrm{Id}\,.\end{aligned}$$

In particular,

$$(q-q^{-1})(\mathbf{E}^{\mathbf{G}}_{i-1}\mathbf{F}^{\mathbf{G}}_{i}-\mathbf{F}^{\mathbf{G}}_{i+1}\mathbf{E}^{\mathbf{G}}_{i})=\mathbf{K}^{\mathbf{G}}_{i}-(\mathbf{K}^{-1}_{i})^{\mathbf{G}},$$

and the formula of part (c) holds.

Since for $n-2i\geq 0$ the element $\mathbf{F}^{\mathbf{G}}_{i+1}\mathbf{E}^{\mathbf{G}}_{i}-\mathbf{E}^{\mathbf{G}}_{i-1}\mathbf{F}^{\mathbf{G}}_{i}$ is a (positive) sum of certain $(\langle k\rangle)^{\mathbf{G}}$ with $k\in\mathbb{Z}$, the decomposition (31) implies that $\mathbf{E}_{i-1}\mathbf{F}_i$ is a summand of $\mathbf{F}_{i+1}\mathbf{E}_i$. Likewise, $\mathbf{F}_{i+1}\mathbf{E}_i$ is a summand of $\mathbf{E}_{i-1}\mathbf{F}_i$ if $n-2i<0$. If now $n-2i>0$ then the formula from (c) implies

$$\mathbf{E}_{i-1}\mathbf{F}_i\oplus\bigoplus_{k=0}^{n-2i-1}\mathrm{Id}\langle n-2i-1-2k\rangle\cong\mathbf{F}_{i+1}\mathbf{E}_i.$$

If $n-2i<0$ then we get

$$\mathbf{F}_{i+1}\mathbf{E}_i\oplus\bigoplus_{k=0}^{2i-n-1}\mathrm{Id}\langle 2i-n-1-2k\rangle\cong\mathbf{E}_{i-1}\mathbf{F}_i.$$

From this we finally deduce the last formula in part (b) by adding the missing parts on both sides. □

4. Harish-Chandra bimodules and the graded category $\mathcal{O}$

4.1. The categorification theorem

In this section we deduce the main result which provides a categorification of arbitrary finite tensor products $V_{\mathbf{d}}$ of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules.

This will be in fact a fairly easy consequence from Theorem 3.6 using the embedding of categories ${}_\lambda\mathcal{H}^1_\mu\hookrightarrow{}_\lambda\mathcal{O}$ for any reductive Lie algebra and dominant integral weights λ, μ from [BG80]. The image of this functor is a full subcategory of ${}_\lambda\mathcal{O}$ given by $\mathcal{P}_\mu$-presentable objects; an object M is $\mathcal{P}_\mu$*-presentable* if there is an exact sequence of the form $P_2\to P_1\to M\to 0$ such that P_2, P_1 are direct sums of projective modules, each indecomposable summand isomorphic to some $P(x\cdot\lambda)$ such that x is a longest double coset representative in $W_\mu\backslash W/W_\lambda$. Let $\mathcal{A}^\mu_i=\mathcal{A}^\mu_i(\mathfrak{gl}_n)$ be the full subcategory of gmod-${}_iA$ given by $\mathcal{P}_\mu$-presentable objects, i.e. by objects M such that there is an exact sequence (in gmod-${}_iA$) of the form $P_2 \to P_1 \to M \to 0$, where P_2, P_1 are direct sums of projective modules, each indecomposable summand after forgetting the grading isomorphic to some $\epsilon_\lambda(P(x \cdot \lambda))$ where x is a longest double coset representative in $W_\mu \backslash W / W_i$. The category $\mathcal{A}_i^\mu$ is abelian, since the categories ${}_\lambda\mathcal{H}_\mu^1$ are. (It is not completely trivial to describe this abelian structure of $\mathcal{A}_i^\mu$ when realized as a subcategory of gmod-${}_iA$; see e.g. [MS05]).

Theorem 3.6 implies the following main result

Theorem 4.1 (Categorification Theorem). *Let* $\mathbf{d}$ *be a composition of* n *and let* $\mu \in \mathfrak{h}^*$ *be dominant and integral such that* $W_\mu \cong S_{\mathbf{d}}$. *There is an isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules*

$$\mathbf{G}\Big(\bigoplus_{i=0}^{n} \mathcal{A}_i^\mu(\mathfrak{gl}_n)\Big) \cong V_{\mathbf{d}},$$

where the left hand side becomes a $U_q(\mathfrak{sl}_2)$*-module structure via the induced action of the exact functors* $\mathbf{E}$, $\mathbf{F}$, *and* $\mathbf{K}$.

Proof. We only have to show that the functors in question preserve the category $\bigoplus_{i=0}^n \mathcal{A}_i^\mu$; then the statement follows from Theorem 3.6, Proposition 2.2 and Theorem 2.7. Recall that the embedding ${}_\lambda\mathcal{H}_\mu^1 \hookrightarrow {}_\lambda\mathcal{O}$ is given by $X \mapsto X \otimes_{\mathcal{U}(\mathfrak{g})} M(\mu)$. In particular, it commutes with translation functors. Hence $\bigoplus_{i=0}^n \mathcal{A}_i^\mu$ is stable under $\mathbf{E}$ and $\mathbf{F}$. That it is also stable under grading shifts, in particular under $\mathbf{K}$, follows directly from the definitions. □

Additionally, we are able to give a categorical interpretation of the involutions introduced in Section 1. This will be the topic of the following subsections.

4.2. The anti-automorphism τ as taking left adjoints

The anti-automorphism τ (and its inverse) can be considered as the operation of taking left (respectively right) adjoints:

Proposition 4.2. *There are pairs of adjoint functors*

$$(\mathbf{E}, \mathbf{F}\mathbf{K}^{-1}\langle 1\rangle), \quad (\mathbf{F}, \mathbf{E}\mathbf{K}\langle 1\rangle), \quad (\mathbf{K}, \mathbf{K}^{-1}), \quad (\mathbf{K}^{-1}, \mathbf{K})$$

and $(\langle -k\rangle, \langle k\rangle)$ *for* $k \in \mathbb{Z}$.

Proof. This follows directly from the definitions (27) and (28) using Proposition 3.5 and Lemma 3.4. □

4.3. The Cartan involution σ as an equivalence of categories

The involution σ (see (5)) has the following functorial interpretation:

Proposition 4.3. *There is an equivalence of categories*

$$\hat{\sigma} : \bigoplus_{i=0}^{n} \text{gmod-}{}_iA \to \bigoplus_{i=0}^{n} \text{gmod-}{}_iA$$

such that

$$\Sigma(\mathbf{E}) \cong \mathbf{F}, \quad \Sigma(\mathbf{F}) \cong \mathbf{E}, \quad \Sigma(\mathbf{K}) \cong \mathbf{K}^{-1}, \quad \Sigma(\langle k\rangle) \cong \langle k\rangle$$

for all $k \in \mathbb{Z}$, *where* $\Sigma(F) = \hat{\sigma} F \hat{\sigma}^{-1}$ *for any endofunctor* F *of* $\bigoplus_{i=0}^n \text{gmod-}{}_iA$. *Moreover* $\Sigma(G_1G_2) \cong \Sigma(G_1)\Sigma(G_2)$ *and* $\Sigma^2(G_1) \cong G_1$ *for any endofunctors* G_1, G_2 *of* $\bigoplus_{i=0}^n \text{gmod-}{}_iA$.

Proof. Let $t : W \to W$ be the isomorphism given by $s_i \mapsto s_{n-i}$. By [Soe90, Theorem 11], it induces an equivalence of categories $\hat{\sigma}_i : \text{mod-}{}_iA \to \text{mod-}{}_{n-i}A$ such that $\hat{\sigma}_i M(x \cdot \omega_i) \cong M(t(x) \cdot \omega_{n-1})$ which lifts even to an equivalence of categories $\hat{\sigma}_i : \text{gmod-}{}_iA \to \text{gmod-}{}_{n-i}A$. In particular $\hat{\sigma}_i \mathbf{K}_i \cong \mathbf{K}_{n-i}^{-1}\hat{\sigma}_i$. Set $\omega = \bigoplus_{i=0}^n \hat{\sigma}_i$. From the definitions we get $\Sigma(\langle k\rangle) \cong \langle k\rangle$. We also have $\hat{\sigma}_{i+1}\,\mathbf{E}_i \cong \mathbf{F}_{n-i}\,\hat{\sigma}_i$ after forgetting the grading, hence $\hat{\sigma}_{i+1}\,\mathbf{E}_i \cong \mathbf{F}_{n-i}\,\hat{\sigma}_i\langle j\rangle$ for some j, because the functors involved are indecomposable (see Proposition 3.2). A direct calculation in the Grothendieck group shows that in fact $\hat{\sigma}_{i+1}\,\mathbf{E}_i \cong \mathbf{F}_{n-i}\,\hat{\sigma}_i$ and $\hat{\sigma}_{i-1}\,\mathbf{F}_i \cong \mathbf{E}_{n-i}\,\hat{\sigma}_i$. Since t is an involution, so are $\hat{\sigma}$ and so is Σ by definition. The formula $\Sigma(G_1G_2) \cong \Sigma(G_1)\Sigma(G_2)$ is then also clear. □

4.4. The involution ψ as a duality functor

For $\lambda \in \mathfrak{h}^*$ dominant and integral let $\mathrm{d} = \mathrm{Hom}_{\mathbb{C}}(\cdot, \mathbb{C}) : C^\lambda\text{-gmod} \to C^\lambda\text{-gmod}$ be the graded duality, i.e. $\mathrm{d}(M)_i = \mathrm{Hom}_{\mathbb{C}}(M_{-i}, \mathbb{C})$. With the conventions on the graded lifts of $\mathbb{V}_\lambda P(x\cdot\lambda)$ we find in particular that $\mathbb{V}P(x\cdot\lambda) \cong \mathrm{d}\,\mathbb{V}P(x\cdot\lambda)$ is self-dual ([Soe90, Lemma 9]). Hence, d defines an isomorphism of graded algebras ${}_\lambda A \cong {}_\lambda A^{\mathrm{opp}}$. We get a contravariant duality $\mathrm{d}_\lambda : \text{gmod-}{}_\lambda A \to {}_\lambda A\text{-gmod} \cong \text{gmod-}{}_\lambda A$, $M \mapsto \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$, where $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ has the dual grading, that is, $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})_i = \mathrm{Hom}_{{}_\lambda A}(M_{-i}, \mathbb{C})$. Let $\mathrm{d} = \bigoplus_{i=0}^n \mathrm{d}_{\omega_i}$ be the duality on $\bigoplus_{i=0}^n \text{gmod-}{}_iA$. Put $\mathrm{d}'_i = \langle 2i(n-i)\rangle\,\mathrm{d}_{\omega_i}$. Of course, d and d′ are involutions. We first mention an important fact:

Lemma 4.4. *There are isomorphisms of functors*

$$\mathbf{E}_i\,\mathrm{d}'_i \cong \mathrm{d}'_{i+1}\,\mathbf{E}_i, \quad \mathbf{F}_i\,\mathrm{d}'_i \cong \mathrm{d}'_{i-1}\,\mathbf{F}_i, \quad \mathbf{K}_i\,\mathrm{d}'_i \cong \mathrm{d}'_i\,\mathbf{K}_i^{-1},$$

for any $0 \leq i \leq n$.

Proof. Set $\mathrm{d}_i = \mathrm{d}_{\omega_i}$. After forgetting the grading we have $\mathrm{d}_{i+1}\,\mathbf{E}_i \cong \mathbf{E}_i\,\mathrm{d}_i$ ([Jan83, 4.12(9)]). Since, considered as a functor from ${}_{i;n}\mathcal{O}$ to ${}_{i+1;n}\mathcal{O}$, the functor $\mathrm{d}_{i+1}\,\mathcal{E}_i\,\mathrm{d}_i \cong \mathcal{E}_i$ is indecomposable (Proposition 3.2), we get $\mathrm{d}_{i+1}\,\mathbf{E}_i \cong \mathbf{E}_i\,\mathrm{d}_i\langle k\rangle$ for some $k \in \mathbb{Z}$ ([BGS96, Lemma 2.5.3] applied to ${}_{i+1}A \otimes {}_iA^{\mathrm{opp}}$). Hence, it is enough to prove that there exists some $M \in \text{gmod-}{}_iA$ which is not annihilated by $\mathrm{d}_{i+1}\,\mathbf{E}_i$ and satisfies $\mathrm{d}'_{i+1}\,\mathbf{E}_i(M) \cong \mathbf{E}_i\,\mathrm{d}'_i(M)$. Let $M \in \text{gmod-}{}_iA$ be the graded lift of the projective module $P(w_0 \cdot \omega_i)$ with head concentrated in degree zero. From the definition of d'_i it follows that $\mathrm{d}'_i\,M \cong M$, since $P(w_0\cdot\omega_i) \in \mathcal{O}$ is self-dual and the grading filtration of M is of length $2i(n-i)$ ([BGS96, Theorem 3.11(ii)]). By definition we have

$$\begin{aligned}\mathbf{E}_iM &= M \otimes_{{}_iA} \mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}P_{i+1}, C^{i,i+1} \otimes_{C^i} \mathbb{V}_iP_i\langle -(n-i-1)\rangle) \\ &= M \otimes_{{}_iA} \mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}P_{i+1}, C^{i,i+1} \otimes_{C^i} \mathbb{V}_iT_i\langle -(n-i-1)\rangle).\end{aligned}$$

By definition of the ${}_iA$-action we only have to consider

$$M \otimes_{{}_iA} \mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}P_{i+1}, C^{i,i+1} \otimes_{C^i} \mathbb{V}_iT_i\langle -(n-i-1)\rangle).$$

From Lemma 3.4 we have isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}P_{i+1}, C^{i,i+1} \otimes_{C^i} \mathbb{V}_iT_i\langle -(n-i-1)\rangle)\\ &\qquad \cong \mathrm{Hom}_{C^{i+1}}\Big(\mathbb{V}_{i+1}P_{i+1}, \bigoplus_{k=0}^{i} C^{i+1}\langle -(n-i-1)\rangle\langle -i(n-i)\rangle\langle 2k\rangle\Big)\end{aligned}$$

Since we are in fact only interested in knowing the head and socle of $\mathbf{E}_iM$, it is enough to consider

$$\begin{aligned}&\mathrm{Hom}_{C^{i+1}}\Big(\mathbb{V}_{i+1}T_{i+1}, \bigoplus_{k=0}^{i} C^{i+1}\langle -(n-i-1)\rangle\langle -i(n-i)\rangle\langle 2k\rangle\Big)\\ &\quad = \mathrm{Hom}_{C^{i+1}}\Big(C^{i+1}, \bigoplus_{k=0}^{i} C^{i+1}\langle -(n-i-1)\rangle\langle -i(n-i)\rangle\langle 2k\rangle\langle (i+1)(n-i-1)\rangle\Big).\end{aligned}$$

There are (up to scalars) unique morphisms of minimal (resp. maximal) degree, namely of degree $s = -(n-i-1) - i(n-i) + (i+1)(n-i-1) = -i$ (and $t = -i + 2i + 2(i+1)(n-i-1) = i + 2(i+1)(n-i-1)$ respectively). Hence, the module $\mathbf{E}_iM = \mathbf{E}_i\, \mathrm{d}'_i\, M$ has minimal degree $-i$ and maximal degree t. Therefore, $\mathrm{d}_{i+1}\mathbf{E}_iM$ has maximal degree i and minimal degree $-t$. By definition of d'_{i+1}, we see that $\mathrm{d}'_{i+1}\, \mathbf{E}_iM$ has maximal degree $i + 2(i+1)(n-i-1) = t$ and minimal degree $-t+2(i+1)(n-i-1) = -i$. This proves the first formula. The second then follows by the adjointness properties from Proposition 4.2 as follows: The functor $\mathbf{E}_i$ has right adjoint $\mathbf{F}_{i+1}(\mathbf{K}_{i+1})^{-1}\langle 1\rangle$ and the functor $\mathrm{d}'_{i+1}\, \mathbf{E}_i\, \mathrm{d}'_i$ has right adjoint

$$\begin{aligned}\mathrm{d}'_i\, \mathbf{K}_i\langle 1\rangle \mathbf{F}_{i+1}\, \mathrm{d}'_{i+1} &\cong \mathrm{d}'_i\, \mathbf{F}_{i+1}\langle 2i-n+1\rangle\, \mathrm{d}'_{i+1} \cong \mathrm{d}'_i\, \mathbf{F}_{i+1}\, \mathrm{d}'_{i+1}\langle n-2i-1\rangle\\ &\cong \mathrm{d}'_i\, \mathbf{F}_{i+1}\, \mathrm{d}'_{i+1}(\mathbf{K}_{i+1})^{-1}\langle 1\rangle.\end{aligned}$$

Hence $\mathbf{F}_{i+1} \cong \mathrm{d}'_i\, \mathbf{F}_{i+1}\, \mathrm{d}'_{i+1}$, or $\mathrm{d}'_i\, \mathbf{F}_{i+1} \cong \mathbf{F}_{i+1}\, \mathrm{d}'_{i+1}$. The last isomorphism of the lemma follows from $\mathbf{K}_i\, \mathrm{d}'_i = \mathbf{K}_i\langle 2i(n-i)\rangle\, \mathrm{d}_i \cong \langle 2i(n-i)\rangle\, \mathrm{d}_i\, \mathbf{K}_i^{-1} \cong \mathrm{d}'_i\, \mathbf{K}_i^{-1}$. □

Let $\mathrm{d}' = \bigoplus_{i=0}^n \mathrm{d}'_i : \bigoplus_{i=0}^n \text{gmod-}{}_iA$ be the duality from above. For an endofunctor F of $\bigoplus_{i=0}^n \text{gmod-}{}_iA$ let $\Psi(F)$ denote the functor $\mathrm{d}'\, F\, \mathrm{d}'$. The involution ψ has the following functorial interpretation:

Proposition 4.5. *The functor Ψ is an involution satisfying $\Psi(\mathbf{E}) \cong \mathbf{E}$, $\Psi(\mathbf{F}) \cong \mathbf{F}$, $\Psi(\mathbf{K}) \cong \mathbf{K}^{-1}$, and $\Psi(\langle k\rangle) \cong \langle -k\rangle$ for any $k \in \mathbb{Z}$.*

Proof. By definition, Ψ is an involution satisfying $\Psi(\langle k\rangle) \cong \langle -k\rangle$. The rest follows from Lemma 4.4. □

5. Schur–Weyl duality and special bases

Permuting the factors of the $\mathfrak{sl}_2$-module $\overline{V}_1^{\otimes n}$ gives rise to an additional S_n-module structure which commutes with the action of the Lie algebra. In the quantised version we get an action of the Hecke algebra corresponding to the symmetric group S_n. We would like to give a categorical version of this bimodule $V_1^{\otimes n}$.

5.1. A categorical version of the Schur–Weyl duality

Let (W, S) be a Coxeter system. The corresponding Hecke algebra $\mathscr{H}(W, S)$ is the associative algebra (with 1) over $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in one variable, with generators H_s for $s \in S$ and relations

$$\begin{aligned}(H_s + q)(H_s - q^{-1}) = 0,&\\ H_sH_tH_s\cdots H_t = H_tH_sH_t\cdots H_s &\quad \text{if } sts\cdots t = tst\cdots s,\\ H_sH_tH_s\cdots H_s = H_tH_sH_t\cdots H_t &\quad \text{if } sts\cdots s = tst\cdots t.\end{aligned}$$

In particular, if $x \in W$ with reduced expression $x = s_{i_1}\cdots s_{i_r}$ then $H_x = H_{s_{i_1}}\cdots H_{s_{i_r}}$ does not depend on the reduced expression and $\{H_x \mid x \in W\}$ is a $\mathbb{Z}[q, q^{-1}]$-basis of $\mathscr{H}(W, S)$. For any subset S' of S we get $W_{S'} \subset W$ and define $M^{S'} = \mathscr{H}(W, S) \otimes_{\mathscr{H}(W_{S'}, S')} \mathbb{Z}$, the corresponding permutation module. (Here, $H_s \in (W_{S'}, S')$ acts on $\mathbb{Z}$ by multiplication with q^{-1}.) We denote by $\mathscr{H}$ the complexified Hecke algebra corresponding to the symmetric group S_n, and by $\mathcal{M}^i = \mathbb{C} \otimes_{\mathbb{Z}} M^i$ the complexified permutation module corresponding to $W = S_n$ and $W_{S'} = W_i$. Then the $\mathcal{M}^i$ has a basis $\{M_x^i = 1 \otimes (H_x \otimes 1) \mid x \in W^i\}$. For more details we refer for example to [Soe97]. (Note that, beside the complexification, we work with left $\mathscr{H}$-modules, whereas the modules in [Soe97] are right $\mathscr{H}$-modules).

In [FKK98], the authors describe explicitly a well-known isomorphism of $\mathscr{H}$-modules

$$\alpha : \bigoplus_{i=0}^{n} \mathcal{M}^i \cong V_1^{\otimes n}, \qquad M_x^i \mapsto v_{\mathbf{a}(x)}, \tag{32}$$

where $\mathbf{a}(x)$ is the $\{0, 1\}$-sequence such that $M(x \cdot \omega_i) = M(\mathbf{a}(x))$. The algebra $\mathscr{H}$ acts on the right via the so-called R-matrix. We refer to [FKK98, Proposition 2.1′] for details. (Note that our q is the v there and our H_i is vT_i there.)

On the other hand, there is an isomorphism of $\mathbb{C}(q)$-modules

$$\beta : \bigoplus_{i=0}^{n} \mathcal{M}^i \cong \mathbf{G}\Big(\bigoplus_{i=0}^{n} \text{gmod-}_i A\Big), \qquad M_x^i \mapsto 1 \otimes [\tilde{M}(x \cdot \omega_i)]. \tag{33}$$

It induces a right $\mathscr{H}$-action on the space.

Our next task will be to "categorify" this action. For any simple reflection $s \in W$, there is a *twisting functor* $T_s : \mathcal{O} \to \mathcal{O}$ which preserves blocks, in particular induces $T_s : {}_\lambda\mathcal{O} \to {}_\lambda\mathcal{O}$ for any integral dominant weight λ. These functors were studied for example in [AL03], [AS03], [KM05]. The most convenient description (for our purposes) of these functors is given in [KM05] in terms of partial coapproximation: Let $M \in {}_\lambda\mathcal{O}$ be projective. Let $M' \subset M$ be the smallest submodule such that M/M' has only composition factors of the form $L(x \cdot \lambda)$, where $sx > x$. Then $M \mapsto M'$ defines a functor T_s from the additive category of projective modules in ${}_\lambda\mathcal{O}$ to ${}_\lambda\mathcal{O}$. This functor extends in a unique way to a right exact functor $T_s : {}_\lambda\mathcal{O} \to {}_\lambda\mathcal{O}$ (for details see [KM05]). This definition of T_s has the advantage that it is immediately clear that this functor is gradable. Moreover, we have the following:

Proposition 5.1. *For any simple reflection $s \in W$ and integral dominant weight λ, the twisting functor $T_s : {}_\lambda\mathcal{O} \to {}_\lambda\mathcal{O}$ is gradable. A graded lift is unique up to isomorphism and shift in the grading.*

Proof. We only have to show the uniqueness of a graded lift. Since T_s is right exact, it is given by tensoring with some bimodule (see e.g. [Bas68, 2.2]). By [BGS96, Lemma 2.5.3] it is enough to show that T_s is indecomposable. Let G_s be the right adjoint functor of T_s. If λ is regular then $G_sT_s \cong \mathrm{Id}$ on the additive category given by all projective modules in ${}_\lambda\mathcal{O}$ ([AS03, Corollary 4.2]). Since T_s and G_s commute with translation functors ([AS03, Section 3]), the adjunction morphism defines an isomorphism $G_sT_s \cong \mathrm{Id}$ on the additive category given by all projective modules in ${}_\lambda\mathcal{O}$ even for singular integral λ. Hence $\mathrm{End}_\mathcal{O}(T_sP) \cong \mathrm{End}_\mathcal{O}(P)$ for any projective module $P \in {}_\lambda\mathcal{O}$. In particular, T_sP is indecomposable, if so is P. Assume now $T_s \cong F_1 \oplus F_2$. For any indecomposable projective $P \in {}_\lambda\mathcal{O}$ there exists $i(P) \in \{1, 2\}$ such that $F_{i(P)}(P) = 0$. Since $M(\lambda)$ is a submodule of any projective module P, we have $i(P) = i(M(\lambda))$ for any P. Hence T_s is indecomposable when restricted to projective modules, hence also when considered as a functor on ${}_\lambda\mathcal{O}$. □

The next result lifts the action of the Hecke algebra to a functorial action (this should be compared with [FKK98, Proposition 1.1] or [Soe97, p. 86]):

Proposition 5.2. *Fix $i \in \{0, 1, \dots, n\}$. There are right exact functors $\overline{H}_j : {}_{i;n}\mathcal{O} \to {}_{i;n}\mathcal{O}$, $1 \le j < n$, satisfying the following properties:*

(a) *They are exact on the subcategory of modules having a filtration with subquotients isomorphic to Verma modules.*

(b) *They have graded lifts $\mathbf{H}_j$ satisfying*

$$[\mathbf{H}_j(\tilde{M}(x \cdot \omega_i))] = \begin{cases} [\tilde{M}(s_jx \cdot \omega_i)] + (q^{-1} - q)[\tilde{M}(x \cdot \omega_i)] & \text{if } s_jx < x,\ s_jx \in W^i, \\ [\tilde{M}(s_jx \cdot \omega_i)] & \text{if } s_jx > x,\ s_jx \in W^i, \\ q^{-1}[\tilde{M}(x \cdot \omega_i)] & \text{if } s_jx \notin W^i. \end{cases} \tag{34}$$

(c) *Let $w_0 = s_{i_1} \cdots s_{i_r}$ be a reduced expression for the longest element in S_n and $\mathbf{H}_{w_0} = \mathbf{H}_{i_1} \cdots \mathbf{H}_{i_r}$ the corresponding composition of functors. Then $\mathbf{H}_{w_0}$ is exact on modules with Verma flag and*

$$\mathbf{H}_{w_0}\tilde{M}(x \cdot \lambda) \cong (\mathrm{d}\,\tilde{M}(w_0x \cdot \lambda))\,\langle -l(w_0^i)\rangle, \tag{35}$$

$$\mathbf{H}_{w_0}\tilde{P}(x \cdot \lambda) \cong \tilde{T}(w_0x \cdot \lambda)\,\langle -l(w_0^i)\rangle, \tag{36}$$

$$\mathbf{H}_{w_0}\tilde{T}(x \cdot \lambda) \cong \tilde{I}(w_0x \cdot \lambda)\,\langle -l(w_0^i)\rangle. \tag{37}$$

where $\tilde{T}(x \cdot \lambda)$ *denotes the graded lift of the tilting module* $T(x \cdot \lambda)$ *such that* $\tilde{M}(x \cdot \lambda)$ *occurs as a submodule in a Verma flag and* $\tilde{I}(x \cdot \lambda) = \mathrm{d}\, \tilde{P}(x \cdot \lambda)$ *is the injective hull of the simple module* $\tilde{L}(x \cdot \lambda)$*.*

Proof. We claim that if we forget the grading, these functors are the twisting functors T_{s_i}. They are right exact by definition, are exact when restricted to the subcategory of modules having a Verma flag ([AS03, Theorem 2.2]), and satisfy the relations (34) if we forget the grading ([AL03, 6.5 and 6.6 or Lemma 2.1]).

We know that these functors are gradable and indecomposable when restricted to an integral block (Proposition 5.1). Therefore, we just need a "correct" lift of these functors. We choose a graded lift $\tilde{T}_s$ of $T_s : {}_\lambda\mathcal{O} \to {}_\lambda\mathcal{O}$ such that

$$\tilde{T}_s \tilde{M}(\lambda) \cong \begin{cases} \tilde{M}(s \cdot \lambda) & \text{if } s \in W^\lambda, \\ \tilde{M}(s \cdot \lambda)\langle -1 \rangle & \text{otherwise.} \end{cases} \tag{38}$$

Let $sx > x$. Then $T_s M(x \cdot \lambda) \cong M(sx \cdot \lambda)$ ([AL03, Lemma 2.1]). Hence $\tilde{T}_s \tilde{M}(x \cdot \lambda) \cong \tilde{M}(sx \cdot \lambda)\langle k \rangle$ for some $k \in \mathbb{Z}$. On the other hand, we have an inclusion

$$\tilde{M}(x \cdot \lambda)\langle l(x) \rangle \hookrightarrow \tilde{M}(\lambda) \tag{39}$$

for any $x \in W^\lambda$ (for example by [BGS96, Proposition 3.11.6]).

- Assume $sx > x$, $sx \in W^\lambda$. Then $\tilde{M}(sx \cdot \lambda)\langle l(x) + k \rangle \cong \tilde{T}_s \tilde{M}(x \cdot \lambda)\langle l(x) \rangle \hookrightarrow \tilde{T}_s \tilde{M}(\lambda)$. From (38) it follows that $\tilde{M}(sx \cdot \lambda)\langle l(x) + k \rangle \hookrightarrow \tilde{M}(\lambda)\langle -1 \rangle$, hence $l(x) + k + 1 = l(sx) = l(x) + 1$. That means $k = 0$. We get $\tilde{T}_s \tilde{M}(x \cdot \lambda) \cong \tilde{M}(sx \cdot \lambda)$.
- Assume $sx \notin W^\lambda$ (in particular $sx > x$), hence $\tilde{T}_s \tilde{M}(x \cdot \lambda) \cong \tilde{M}(sx \cdot \lambda)\langle k \rangle$ for some $k \in \mathbb{Z}$ ([AL03, 6.5 and 6.6 or Lemma 2.1]). Then
$$\tilde{M}(x \cdot \lambda)\langle l(x) + k \rangle \cong \tilde{T}_s \tilde{M}(x \cdot \lambda)\langle l(x) \rangle \hookrightarrow \tilde{T}_s \tilde{M}(\lambda).$$
From (38) it follows that $\tilde{M}(x \cdot \lambda)\langle l(x) + k \rangle \hookrightarrow \tilde{M}(\lambda)\langle -1 \rangle$, hence $l(x) + k = l(x) - 1$. That means $k = -1$. We get $\tilde{T}_s \tilde{M}(x \cdot \lambda) \cong \tilde{M}(x \cdot \lambda)\langle -1 \rangle$.
- Assume $sx < x$, $sx \in W^\lambda$. From the translation principle it follows that there is a unique nonsplit extension
$$0 \to M(y \cdot \lambda) \to M \to M(sy \cdot \lambda) \to 0$$
whenever $y, sy \in W^\lambda$ and $sy > y$. (To see this one could first consider the case where λ is regular. If we write $sy = yt$ for some simple reflection t then the statement becomes familiar. The general statement then follows by translation.) If λ is regular, the main result of [KM05] says that T_s is adjoint to Joseph's completion functor (see [Jos82]), in particular $T_s M(z \cdot \lambda)$ is the cokernel of the inclusion $M(z \cdot \lambda) \to M$ for $z \in \{y, sy\}$. In the graded picture we have a unique nonsplit extension
$$0 \to M(y \cdot \lambda)\langle 1 \rangle \to M \to M(sy \cdot \lambda) \to 0$$
whenever $y \in W$, $sy > x$, and then $\tilde{T}_s \tilde{M}(y \cdot \lambda)$ is the cokernel of the inclusion $\tilde{M}(y \cdot \lambda)\langle 1 \rangle \to M$ if $y < sy$, and $\tilde{T}_s \tilde{M}(sy \cdot \lambda)$ is the cokernel of the inclusion

$\tilde{M}(sy \cdot \lambda)\langle 1\rangle \to M\langle -1\rangle$ if $y > sy$ (compare [Str03, Theorem 5.3, Theorem 3.6]). We get the following formula for regular integral λ:

$$[\tilde{T}_s\tilde{M}(x \cdot \lambda)] = [\tilde{M}(sx \cdot \lambda)] + (q^{-1} - q)[\tilde{M}(x \cdot \lambda)]. \tag{40}$$

To see this we just calculate

$$[M\langle -1\rangle] - [\tilde{M}(x \cdot \lambda)\langle 1\rangle] = [\tilde{M}(sx \cdot \lambda)] + [\tilde{M}(x \cdot \lambda)\langle -1\rangle] - [\tilde{M}(x \cdot \lambda)\langle 1\rangle],$$

and the formula follows.

To get the result for singular blocks we use translation functors. Let λ, μ be dominant integral weights with μ regular. Then the translation functors $\mathbb{T}_\lambda^\mu$ and $\mathbb{T}_\mu^\lambda$ are gradable. This follows from Proposition 3.3 as follows. Since the functor $\mathbb{V}_\mu$ induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(P_\mu, \mathbb{T}_\lambda^\mu P_\lambda) \cong \mathrm{Hom}_{C^\mu}(\mathbb{V}_\mu P_\mu, \mathbb{V}_\mu \mathbb{T}_\lambda^\mu P_\lambda)$$

(by the Struktursatz of [Soe90]), a graded lift $\tilde{\mathbb{T}}_\mu^\lambda$ of $\mathbb{T}_\mu^\lambda$ is given by tensoring with the $(\mathrm{End}_{C^\mu}(\mathbb{V}_\mu P_\mu), \mathrm{End}_{C^\lambda}(\mathbb{V}_\lambda P_\lambda))$-bimodule

$$\mathrm{Hom}_{C^\lambda\text{-gmod}}(\mathbb{V}_\lambda P_\lambda, \mathrm{Res}_\lambda^\mu \mathbb{V} P_\mu).$$

We have $\tilde{\mathbb{T}}_\mu^\lambda M(\mu) \cong M(\lambda)\langle k\rangle$ for some $k \in \mathbb{Z}$. The inclusion (39) implies $\tilde{\mathbb{T}}_\mu^\lambda M(x \cdot \mu) \cong M(x \cdot \lambda)\langle k\rangle$ for any $x \in W^\lambda$. Without loss of generality we may assume $k = 0$. We have the following equalities:

$$\begin{aligned}
[\tilde{T}_s\tilde{M}(x \cdot \lambda)] &= [\tilde{T}_s\tilde{\mathbb{T}}_\mu^\lambda\tilde{M}(x \cdot \mu)] \\
&= [\tilde{\mathbb{T}}_\mu^\lambda\tilde{T}_s\tilde{M}(x \cdot \mu)] && (41)\\
&= [\tilde{\mathbb{T}}_\mu^\lambda\tilde{M}(sx \cdot \mu)] + (q + q^{-1})[\tilde{\mathbb{T}}_\mu^\lambda\tilde{M}(x \cdot \mu)] && (42)\\
&= [\tilde{M}(sx \cdot \lambda)] + (q + q^{-1})[\tilde{M}(x \cdot \lambda)].
\end{aligned}$$

The first and the last equality follow from the definitions. To see the equality (41) observe that twisting functors and translation functors commute (see [AS03, Section 3]). With standard arguments one can check that $T_s\mathbb{T}_\mu^\lambda$ is indecomposable, hence $\tilde{T}_s\tilde{\mathbb{T}}_\mu^\lambda$ is isomorphic to $\tilde{\mathbb{T}}_\mu^\lambda\tilde{T}_s$ up to a shift in the grading. However, they agree on $\tilde{M}(\mu)$, hence they are isomorphic. Finally, (42) follows from (40).

This finishes the proof of part (b) of the proposition. After forgetting the grading we have $\mathbf{H}_{w_0}M(x \cdot \lambda) \cong \mathrm{d}\,M(w_0x \cdot \lambda)$ (this is [AS03, (2.3) and Theorem 2.3] for the regular case, the general case follows easily by translation). Since the left derived functor of $\mathbf{H}_{w_0}$ defines an equivalence of derived categories ([AS03, Corollary 4.2]), by general arguments, we get isomorphisms of modules

$$\begin{aligned}
\mathbf{H}_{w_0}M(x \cdot \lambda) &\cong \mathrm{d}\,M(w_0x \cdot \lambda),\\
\mathbf{H}_{w_0}P(x \cdot \lambda) &\cong T(w_0x \cdot \lambda),\\
\mathbf{H}_{w_0}T(x \cdot \lambda) &\cong I(w_0x \cdot \lambda).
\end{aligned}$$

For details we refer for example to [GGOR03, Proposition 4.2]. In particular, tilting modules and injective modules are gradable. We are left with checking the graded version. Since all modules involved are indecomposable, we have

$$\mathbf{H}_{w_0}\tilde{M}(x\cdot\lambda) \cong (\mathrm{d}\,\tilde{M}(w_0x\cdot\lambda))\langle k_x^1\rangle,$$
$$\mathbf{H}_{w_0}\tilde{P}(x\cdot\lambda) \cong \tilde{T}(w_0x\cdot\lambda)\langle k_x^2\rangle,$$
$$\mathbf{H}_{w_0}\tilde{T}(x\cdot\lambda) \cong \tilde{I}(w_0x\cdot\lambda)\langle k_x^3\rangle,$$

for some $k_x^i \in \mathbb{Z}$. Set $s = -l(w_0^i)$. We claim that $\tilde{L}(w_0x\cdot\lambda)\langle s\rangle$ occurs as a composition factor in $\mathbf{H}_{w_0}\tilde{M}(x\cdot\lambda)$. This is clear from the formulas (34), hence $k_x^1 = s$. The inclusion $\tilde{M}(x\cdot\lambda) \hookrightarrow \tilde{T}(x\cdot\lambda)$ gives rise to an inclusion $(\mathrm{d}\,\tilde{M}(w_0x\cdot\lambda))\langle s\rangle \hookrightarrow \mathbf{H}_{w_0}T(x\cdot\lambda)$. This implies $k_x^3 = s$. The surjection $\tilde{P}(x\cdot\lambda) \to \tilde{M}(x\cdot\lambda)$ gives rise to a surjection $\mathbf{H}_{w_0}\tilde{P}(x\cdot\lambda) \to (\mathrm{d}\,\tilde{M}(w_0x\cdot\lambda))\langle s\rangle$. Hence $\tilde{L}(w_0x\cdot\lambda)\langle s\rangle$ occurs as a composition factor in $\mathbf{H}_{w_0}\tilde{P}(x\cdot\lambda)$. This implies $k_x^2 = s$. The proposition follows. □

In Proposition 5.2, to categorify the action of the Hecke algebra we restricted the (graded lifts) of the twisting functors $\overline{H}_j$ to the category of modules with Verma flag, to force them to be exact. J. Sussan studied a categorification of the Hecke algebra action by considering the derived functors associated to the twisting functors ([Sus05]).

5.2. The canonical, standard and dual canonical bases

We will now combine the three pictures: the Hecke module, the Grothendieck group of the graded version of certain blocks of $\mathcal{O}$ and the $U_q(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$. We consider $V_1^{\otimes n}$ as a U-module via the comultiplication $\triangle$.

We first have to introduce a bilinear form on $V_1^{\otimes n}$ for $n \geq 2$. There is a nondegenerate bilinear form $<\ ,\ >$ on $V_1^{\otimes 2}$ defined by $<v_i \otimes v_j, v^k \otimes v^l> = \delta_i^l \delta_j^k$. It satisfies

$$<\triangle(x)(v_i \otimes v_j), v^k \otimes v^l> = <v_i \otimes v_j, \triangle'(\sigma(x))(v^k \otimes v^l)>,$$

where $\triangle' = \psi \otimes \psi \circ \triangle \circ \psi$, explicitly

$$\triangle'(E) = 1 \otimes E + E \otimes K, \quad \triangle'(F) = K^{-1} \otimes F + F \otimes 1, \quad \triangle'(K^{\mp 1}) = K^{\mp 1} \otimes K^{\mp 1}.$$

Let $(V_1^{\otimes n})'$ denote the $U_q(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$ but with comultiplication $\triangle'$. The form above can be extended to a bilinear form $<\ ,\ > : V^{\otimes n} \times (V^{\otimes n})' \to \mathbb{C}$ by putting

$$<v_{\mathbf{a}}, v^{\mathbf{b}}> = \prod_{i=1}^{n} \delta_{a_i, b_{n-i+1}}. \tag{43}$$

Then it satisfies $<xu, v> = <u, \sigma(x)v>$ for any $x \in U_q(\mathfrak{sl}_2)$ and $u, v \in V_1^{\otimes n}$.

The module $V_1^{\otimes n}$ has two distinguished $\mathbb{C}(q)$-bases, namely

- the *standard basis* $\{v_{\mathbf{a}} = v_{a_1} \otimes \cdots \otimes v_{a_n} \mid a_j \in \{0,1\}\}$,
- the *canonical basis* $\{v_{\mathbf{a}}^{\diamondsuit} = v_{a_1} \diamondsuit \cdots \diamondsuit v_{a_n} \mid a_j \in \{0,1\}\}$.

There are also two distinguished bases in the space $(V_1^{\otimes n})'$, namely

- the *dual standard basis* $\{v^{\mathbf{a}} = v^{a_1} \otimes \cdots \otimes v^{a_n} \mid a_j \in \{0,1\}\}$,
- the *dual canonical basis* $\{v^{\mathbf{a}}_{\heartsuit} = v^{a_1} \heartsuit \cdots \heartsuit v^{a_n} \mid a_j \in \{0,1\}\}$.

The canonical and dual canonical bases were defined by Lusztig and Kashiwara ([Lus90], [Lus92], [Kas91]). Lusztig ([Lus93, Chapter 27]) defined a certain semi-linear involution Ψ on $V_1^{\otimes n}$ which determines the canonical basis uniquely by the following two properties:

(i) $\Psi(v_{\mathbf{a}}^{\diamondsuit}) = v_{\mathbf{a}}^{\diamondsuit}$.
(ii) $v_{\mathbf{a}}^{\diamondsuit} - v_{\mathbf{a}} \in \sum_{\mathbf{b}\neq\mathbf{a}} q^{-1}\mathbb{Z}[q^{-1}]v_{\mathbf{b}}$.

Given the canonical basis, the dual canonical basis is defined by

$$\langle v_{\mathbf{a}}^{\diamondsuit}, v_{\heartsuit}^{\mathbf{b}} \rangle = \prod_{i=1}^{n} \delta_{a_i, b_{n-i+1}}. \tag{44}$$

(For an explicit graphical description of these bases we refer to [FK97].)

On the other hand, the permutation module $\mathcal{M}^i$ also has several distinguished $\mathbb{C}[q,q^{-1}]$-bases, namely

- the *standard basis* $\{M_x^i = 1 \otimes H_x \otimes 1 \mid x \in W^i\}$,
- the (positive) *self-dual basis* $\{\underline{M}_x^i \mid x \in W^i\}$,
- the (negative) *self-dual basis* $\{\underline{\tilde{M}}_x^i \mid x \in W^i\}$,
- the *"twisted" standard basis* $\{(M_x^i)^{\text{Twist}} := q^{l(w_0^i)} H_{w_0} M_x^i \mid x \in W^i\}$,
- the *"twisted" positive self-dual basis*
$$\{(\underline{M}_x^i)^{\text{Twist}} := q^{l(w_0^i)} H_{w_0} \underline{M}_x^i \mid x \in W^i\},$$
- the *"twisted" negative self-dual basis*
$$\{(\underline{\tilde{M}}_x^i)^{\text{Twist}} := q^{l(w_0^i)} H_{w_0} \underline{\tilde{M}}_x^i \mid x \in W^i\}.$$

These bases were defined by Kazhdan, Lusztig and Deodhar (see [KL79], [Deo87]). We use here the notation from [Soe97], except that we have the upper index i to indicate that $M_x^i \in \mathcal{M}_x^i$ and we also use q instead of v. The bases can be characterised as follows ([KL79] in the notation of [Soe97]): Let $\Psi : \mathscr{H} \to \mathscr{H}$ be the $\mathbb{Z}$-linear involution given by $H_x \mapsto (H_{x^{-1}})^{-1}$, $q \mapsto q^{-1}$. It induces an involution on any $\mathcal{M}^i$. Then the $\underline{M}_x^i$ are uniquely defined by

(i) $\Psi(\underline{M}_x^i) = \underline{M}_x^i$,
(ii) $\underline{M}_x^i - M_x^i \in \sum_{y\neq x} v\mathbb{Z}[v]M_y^i$.

The basis elements $\underline{\tilde{M}}_x^i$ are characterised by

(i) $\Psi(\underline{\tilde{M}}_x^i) = \underline{\tilde{M}}_x^i$,
(ii) $\underline{\tilde{M}}_x^i - M_x^i \in \sum_{y\neq x} v^{-1}\mathbb{Z}[v^{-1}]M_y^i$.

Note that what we call the "twisted" bases are in fact bases, since H_{w_0} is invertible in $\mathscr{H}$.

The following theorem gives a categorical interpretation of all these bases:

Theorem 5.3. (a) *There is an isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules*

$$\Phi : \mathbf{G}\Big(\bigoplus_{i=0}^{n} \text{gmod-}{}_iA\Big) \cong V_1^{\otimes n}, \qquad 1 \otimes [\tilde{M}(\mathbf{a})\langle i\rangle] \mapsto q^i v_{\mathbf{a}} = q^i(v_{a_1} \otimes \cdots \otimes v_{a_n}),$$

where the $U_q(\mathfrak{sl}_2)$*-structure on the left hand side is induced by the functors* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{K}$.

(b) *There is an isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules*

$$\Phi' : \mathbf{G}\Big(\bigoplus_{i=0}^{n} \text{gmod-}{}_iA\Big) \cong (V_1^{\otimes n})', \qquad 1 \otimes [\tilde{\nabla}(\mathbf{a})\langle i\rangle] \mapsto q^i v^{\mathbf{a}} = q^i v^{a_1} \otimes \cdots \otimes v^{a_n},$$

where the $U_q(\mathfrak{sl}_2)$*-structure on the left hand side is induced by the functors* $\mathbf{E}' = \bigoplus_{i=0}^{n} \mathbf{E}'_i$, $\mathbf{F}' = \bigoplus_{i=0}^{n} \mathbf{F}'_i$ *and* $\mathbf{K}$, *where*

$$\mathbf{E}'_i = \langle -2(i+1)(n-i-1)\rangle \mathbf{E}_i \langle 2i(n-i)\rangle,$$
$$\mathbf{F}'_i = \langle -2(i-1)(n-i+1)\rangle \mathbf{F}_i \langle 2i(n-i)\rangle.$$

(c) *The isomorphism* β *defines bijections:*

positive self-dual basis	$\leftrightarrow$	*standard lifts of indecomposable projectives*
$\underline{M}_x^i$	$\mapsto$	$[\tilde{P}(x \cdot \omega_i)]$,
standard basis	$\leftrightarrow$	*standard lifts of Verma modules*
$\underline{M}_x^i$	$\mapsto$	$[\tilde{M}(x \cdot \omega_i)]$.

(d) *The bilinear form* $<\ ,\ >$ *can be realized as follows:*

$$<1 \otimes [M], 1 \otimes [N]> = \sum_i \sum_j (-1)^j \dim \operatorname{Hom}_{\mathcal{D}}(\mathrm{d}\, N, \mathbf{H}M\langle -i\rangle)[j])q^i,$$

where $\mathcal{D}$ *denotes the bounded derived category of* $\bigoplus$ gmod-${}_iA$ *with shift functors* $[j]$, *and* $\mathbf{H}$ *denotes the derived functor of the twisting functor* $\mathbf{H}_{w_0}$ *from Proposition* 5.2, *but shifted in the grading such that the standard lifts of Verma modules are sent to standard lifts of dual Verma modules.*

(e) *We have* $\Phi = \alpha \circ \beta^{-1}$ *(from* (32) *and* (33)*) and these isomorphisms, together with* Φ', *define bijections:*

twisted positive self-dual basis	$\leftrightarrow$	*standard lifts of indec. tilting modules*	$\leftrightarrow$	*canonical basis*
$(\underline{M}_x^i)^{\mathrm{Twist}}$	$\mapsto$	$1 \otimes [\tilde{T}(w_0 x \cdot \omega_i)]$,		
		$1 \otimes [\tilde{T}(\mathbf{a})]$	$\mapsto$	$v_{\mathbf{a}}^{\diamond}$

standard basis	$\leftrightarrow$	*standard lifts of Verma modules*	$\leftrightarrow$	*standard basis*
M_x^i	$\mapsto$	$1 \otimes [\tilde{M}(x \cdot \omega_i)]$,		
		$1 \otimes [\tilde{M}(\mathbf{a})]$	$\mapsto$	$v_{\mathbf{a}}$
twisted standard basis	$\leftrightarrow$	*standard lifts of dual Verma modules*	$\leftrightarrow$	*dual standard basis*
$(M_x^i)^{\mathrm{Twist}}$	$\mapsto$	$1 \otimes [\tilde{\nabla}(w_0 x \cdot \omega_i)]$,		
		$1 \otimes [\tilde{\nabla}(\mathbf{a})]$	$\mapsto$	$v^{\mathbf{a}}$
twisted negative self-dual basis	$\leftrightarrow$	*standard lifts of simple modules*	$\leftrightarrow$	*dual canonical basis*
$(\underline{\tilde{M}}{}_x^i)^{\mathrm{Twist}}$	$\mapsto$	$1 \otimes [\tilde{L}(w_0 x \cdot \omega_i)]$,		
		$1 \otimes [\tilde{L}(\mathbf{a})]$	$\mapsto$	$v_{\mathbf{a}}^{\heartsuit}$

(f) *There is an isomorphism of* $\mathbb{C}(q)$*-modules*

$$\gamma : \bigoplus_{i=0}^{n} \mathcal{M}^i \cong \mathbf{G}\left(\bigoplus_{i=0}^{n} \text{gmod-}{}_iA\right), \qquad M_x^i \mapsto 1 \otimes [\tilde{\nabla}(w_0 x \cdot \omega_i)].$$

Under this isomorphism, the negative self-dual basis corresponds to standard lifts of simple modules and the (twisted) positive self-dual basis corresponds to standard lifts of tilting (resp. injective) modules; more precisely,

$$\begin{aligned} \underline{\tilde{M}}{}_x^i &\mapsto 1 \otimes [\tilde{L}(w_0 x \cdot \omega_i)], \\ \underline{M}_x^i &\mapsto 1 \otimes [\tilde{T}(w_0 x \cdot \omega_i)], \\ (\underline{M}_x^i)^{\mathrm{Twist}} &\mapsto 1 \otimes [\tilde{I}(x \cdot \omega_i)]. \end{aligned}$$

Before we prove the theorem we give

Example 5.4. Consider $\mathfrak{gl}_2$ (i.e. $n = 2$). Let $i = 1$. Then gmod-${}_iA$ is equivalent to the graded version of the principal block of $\mathcal{O}(\mathfrak{gl}_2)$. Consider the Hecke module $\mathcal{M}^1$. To avoid too many indices we will omit the superscript i. The module $\mathcal{M} = \mathcal{M}^1$ has the standard basis given by the elements M_e and M_s, where s is the (only) simple reflection in the Weyl group. One can easily calculate the distinguished bases in the Hecke module $\mathcal{M}$; together with [BGS96, Theorem 3.11.4(ii)] and [FK97, Section 1.5] we get the following:

- The twisted standard basis is given by $(M_e)^{\mathrm{Twist}} = M_s$ and $(M_s)^{\mathrm{Twist}} = M_e + (q^{-1} - q)M_s$. The corresponding equations in the Grothendieck group are $[\mathrm{d}\,\tilde{\Delta}(s\cdot 0)] = [\tilde{\Delta}(s\cdot 0)]$ and $[\mathrm{d}\,\tilde{\Delta}(0)] = [\tilde{\Delta}(0)] + [\tilde{\Delta}(s\cdot 0)\langle -1\rangle] - [\tilde{\Delta}(s\cdot 0)\langle 1\rangle]$.

- The positive self-dual basis is given by $\underline{M}_e = M_e$ and $\underline{M}_s = M_s + qM_e$. These equations correspond to the equalities $[\tilde{P}(0)] = [\tilde{\Delta}(0)]$, and $[\tilde{P}(s)] = [\tilde{\Delta}(s \cdot 0)] + [\tilde{\Delta}(e)\langle 1\rangle]$ via the isomorphism β.

  Under the isomorphism γ the equations above correspond to the equalities $[\tilde{T}(s \cdot 0)] = [\tilde{\nabla}(s \cdot 0)]$ and $[\tilde{T}(0)] = [\tilde{\nabla}(0)] + q[\tilde{\nabla}(s \cdot 0)]$.
- The twisted positive self-dual basis is given by $(\underline{M}_e)^{\text{Twist}} = M_s$ and $(\underline{M}_s)^{\text{Twist}} = (M_s)^{\text{Twist}} + q(M_e)^{\text{Twist}} = M_e + q^{-1}M_s$. These equations correspond to $[\tilde{T}(s \cdot 0)] = [\tilde{\Delta}(s \cdot 0)]$ and $[\tilde{T}(0)] = [\tilde{\Delta}(0)] + [\tilde{\Delta}(s \cdot 0)\langle -1\rangle]$, or, equivalently, to the equations $[\tilde{T}((0,1))] = [\tilde{\Delta}((0,1))]$ and $[\tilde{T}((1,0))] = [\tilde{\Delta}((1,0))] + [\tilde{\Delta}((0,1))\langle -1\rangle]$. On the other hand, we have $v_0 \diamond v_1 = v_0 \otimes v_1$ and $v_1 \diamond v_0 = v_1 \otimes v_0 + q^{-1} v_0 \otimes v_1$. Under the isomorphism γ the twisted positive self-dual basis corresponds to the basis given by the standard lifts of the indecomposable injective modules; the corresponding relations are $[\tilde{I}(0)] = [\tilde{\nabla}(0)]$ and $[\tilde{I}(s \cdot 0)] = [\tilde{\nabla}(s \cdot 0)] + [\tilde{\nabla}(0)\langle -1\rangle]$.
- The negative self-dual basis is given by $\tilde{\underline{M}}_e = M_e$ and $\tilde{\underline{M}}_s = M_s - q^{-1}M_e$. Via the isomorphism γ, these equations become $[\tilde{L}(s \cdot 0)] = [\mathrm{d}\,\tilde{\Delta}(s \cdot 0)]$ and $[\tilde{L}(0)] = [\mathrm{d}\,\tilde{\Delta}(0)] - [\mathrm{d}\,\tilde{\Delta}(s \cdot 0)\langle -1\rangle]$, or equivalently, $[\tilde{L}((0,1))] = [\mathrm{d}\,\tilde{\Delta}((0,1))]$ and $[\tilde{L}((1,0))] = [\mathrm{d}\,\tilde{\Delta}((1,0))] - [\mathrm{d}\,\tilde{\Delta}((0,1))\langle -1\rangle]$. These formulas correspond to the following expressions of the dual canonical basis in terms of the dual basis: $v^0 \heartsuit v^1 = v^0 \otimes v^1$ and $v^1 \heartsuit v^0 = v^1 \otimes v^0 - q^{-1} v^0 \otimes v^1$.
- The twisted negative self-dual basis is given by $(\tilde{\underline{M}}_e)^{\text{Twist}} = M_s$ and $(\tilde{\underline{M}}_s)^{\text{Twist}} = M_e - qM_s$. The corresponding equalities are $[\tilde{L}(s \cdot 0)] = [\tilde{\Delta}(s \cdot 0)]$ and $[\tilde{L}(0)] = [\tilde{\Delta}(0)] - [\tilde{\Delta}(s \cdot 0)\langle 1\rangle]$.

We still have to prove the theorem:

Proof of Theorem 5.3. We have $\Phi = \alpha \circ \beta^{-1}$, since they agree by definition on standard modules. The existence of the isomorphism $\overline{\Phi}$ in (16) implies that Φ is an isomorphism of $\mathbb{C}(q)$-modules. We have to show that it is a $U_q(\mathfrak{sl}_2)$-morphism. Let $M = \tilde{M}(\mathbf{a}) \in \text{gmod-}_i A$. From the definitions we get $\Phi(\mathbf{K}[M]) = \Phi([M\langle 2i-n\rangle]) = q^{2i-n}\Phi([M])$. On the other hand, $Kv_{\mathbf{a}} = q^m v_{\mathbf{a}}$, where m is the number of ones minus the numbers of zeros occurring in $\mathbf{a}$, hence $m = i-(n-i) = 2i-n$. We get $\Phi(\mathbf{K}^{\pm 1}[M]) = K^{\pm 1}\Phi([M])$ for any $M \in \bigoplus_{i=0}^n \text{gmod-}_i A$.

Note that $\mathcal{E}M(1^i 0^{n-i})$ is projective. From [BFK99, Proposition 6] we see that $P(1^i 0^{n-i-1} 1)$ is a direct summand and that $M(1^i 0^{n-i})$ occurs with multiplicity one in any Verma flag. Hence we get in fact $\mathcal{E}M(1^i 0^{n-i}) \cong P(1^i 0^{n-i-1} 1)$, since $M(1^i 0^{n-i})$ would occur in a Verma flag of any other direct summand. Because of the indecomposability of $P(1^i 0^{n-i-1} 1)$ we get $\mathbf{E}\tilde{M}(1^i 0^{n-i}) \cong \tilde{P}(1^i 0^{n-i-1} 1)\langle k\rangle$ for some $k \in \mathbb{Z}$ ([BGS96, Lemma 2.5.3]). To determine k we calculate j such that

$$\mathrm{Hom}_{\text{gmod-}_{i+1}A}(\tilde{M}(1^{i+1}0^{n-i-1})\langle j\rangle, \mathbf{E}\tilde{M}(1^i 0^{n-i})) \neq 0.$$

(Since the homomorphism space in question is one-dimensional, the number j is well-defined.) We set $M = M(1^i 0^{n-i})$ and $N = M(1^{i+1} 0^{n-i-1})$. From our definitions we get

$$\mathrm{Hom}_{\mathrm{gmod}\text{-}\,_{i+1}A}(\tilde{M}(1^{i+1}0^{n-i-1})\langle j\rangle, \mathbf{E}\tilde{M}(1^i0^{n-i})) = \mathrm{Hom}_{\mathrm{gmod}\text{-}\,_{i+1}A}(X, Y),$$

where $Y = \mathrm{Hom}_C(\mathbb{V}_{i+1}P_{i+1}, \mathrm{Res}_{i+1}^{i,i+1} C^{i,i+1} \otimes_{C^i} \mathbb{V}_i M\langle -n+i+1\rangle)$ and $X = \mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}P_{i+1}, \mathbb{V}_{i+1}N)\langle j\rangle$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{gmod}\text{-}\,_{i+1}A}(X, Y)\\
&\qquad = \mathrm{Hom}_{C^{i+1}}(\mathbb{V}_{i+1}N\langle j\rangle, \mathrm{Res}_{i+1}^{i,i+1} C^{i,i+1} \otimes_{C^i} \mathbb{V}_i M)\langle -n+i+1\rangle)\\
&\qquad = \mathrm{Hom}_{C^{i+1}}(\mathbb{C}\langle j\rangle, \mathrm{Res}_{i+1}^{i,i+1} C^{i,i+1} \otimes_{C^i} \mathbb{C}\langle -n+i+1\rangle).
\end{aligned}$$

Now Lemma 3.4 tells us that j has to be the highest nonzero degree occurring in $\bigoplus_{k=0}^{n-i-1} \mathbb{C}\langle 2k\rangle\langle -n+i+1\rangle$. That is, $j = 2n-2i-2-n+i+1 = n-(i+1)$. On the other hand, the formula of [BGS96, Theorem 3.11.4(ii)] gives

$$[\tilde{P}(1^{i+1}0^{n-i})] = \sum_{k=0}^{n-i-1} [\tilde{M}(1^i0^{n-i-1-k}10^k\langle k\rangle].$$

Hence we finally get

$$\mathbf{E}\tilde{M}(1^i0^{n-i}) \cong \tilde{P}(1^i0^{n-i-1}1) \tag{45}$$

and $\Psi([\mathbf{E}\tilde{M}(1^i0^{n-i})]) = \sum_{k=0}^{n-i} \Psi([\tilde{M}(1^i0^{n-i-k}10^k\langle k\rangle])$. On the other hand, we have to calculate $\triangle(E)v_{\mathbf{a}}$, where $\mathbf{a} = 1^i0^{n-i}$. We get $\triangle(E)v_{\mathbf{a}} = \sum_{k=0}^{n-i} qv_{\mathbf{a}^k}$, where $\mathbf{a}^k = 1^i0^{n-i-1+k}10^k$. We get $\Psi([\mathbf{E}M]) = E\Psi([M])$. The relation $\Psi([\mathbf{F}M]) = F\Psi([M])$ follows from analogous calculations. The existence of the desired isomorphism Φ follows. This proves part (a).

Obviously, Φ' is an isomorphism of $\mathbb{C}(q)$-modules. We first have to verify that the functors $\mathbf{E}'$, $\mathbf{F}'$ and $\mathbf{K}$ satisfy the $U_q(\mathfrak{sl}_2)$ relations from Definition 1.1. Since we know that the functors $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{K}$ satisfy these relations (see Theorem 3.6), it is enough to verify the last equation. Since, however, $\mathbf{F}'_{i+1}\mathbf{E}'_i = \mathbf{F}_{i+1}\mathbf{E}_i$ and $\mathbf{E}'_{i-1}\mathbf{F}'_i = \mathbf{E}_{i-1}\mathbf{F}_i$, this also follows directly from Theorem 3.6. It is left to show that Φ' is in fact a $U_q(\mathfrak{sl}_2)$-morphism. We will deduce this from part (a). Recall that if $\mathrm{d} = \bigoplus_{i=0}^n \mathrm{d}_i$ denotes the duality on $\bigoplus_{i=0}^n \mathrm{gmod}\text{-}\,_iA$ which fixes simple modules concentrated in degree zero, then we put $\mathrm{d}'_i = \langle 2i(n-i)\rangle\,\mathrm{d}_i$ (see Section 4.4). Let for the moment $\nabla = \mathrm{d}\,\tilde{\Delta}(\mathbf{a})$ for some $\{0,1\}$-sequence $\mathbf{a}$ containing exactly i ones. Set $m_i = 2i(n-i)$. We have the following equalities:

$$\begin{aligned}
\Phi'(\mathbf{E}'\nabla) &= \Phi'(\langle -m_{i+1}\rangle\mathbf{E}_i\langle m_i\rangle\nabla) = \Phi'(\langle -m_{i+1}\rangle\mathbf{E}_i\,\mathrm{d}'_i\,\Delta(\mathbf{a}))\\
&= \Phi'(\langle -m_{i+1}\rangle\,\mathrm{d}'_{i+1}\,\mathbf{E}_i\Delta(\mathbf{a})) = \Phi'(\mathrm{d}_{i+1}\,\mathbf{E}_i\Delta(\mathbf{a})).
\end{aligned}$$

The first equation holds by definition of $\mathbf{E}'$, the second and the last one by definition of d'. The remaining third equation is given by Lemma 4.4. From the definitions of Φ, Φ', and the duality d we see that $\Phi'(\mathrm{d}_{i+1}\,\mathbf{E}_i\Delta(\mathbf{a}))$ is nothing else than $\Phi(\mathbf{E}_i\Delta(\mathbf{a}))$, expressed in the standard basis, but with the involution $q \mapsto q^{-1}$ applied to all coefficients. From part (a) we know that this is the same as $\mathbf{E}\Phi(\Delta(\mathbf{a}))$, expressed in the standard basis, but with the involution $q \mapsto q^{-1}$ applied to all coefficients. Since $\Phi'(\nabla) = v^{\mathbf{a}}$, whereas $\Phi(\Delta(\mathbf{a})) = v_{\mathbf{a}}$, we see directly from the definition of the comultiplication $\triangle'$ (in comparison with $\triangle$) that

$\Phi'(\mathbf{E}'\nabla) = \Phi'(\mathrm{d}_{i+1}\,\mathbf{E}_i\Delta(\mathbf{a})) = E\Phi'(\nabla)$, where E acts on the latter via the comultiplication $\triangle'$. Analogous calculations show that $\Phi'(\mathbf{F}'\nabla) = F\Phi'(\nabla)$. The equality $\Phi'(\mathbf{K}\nabla) = K\Phi'(\nabla)$ is clear. The map Φ' is now in fact a $U_q(\mathfrak{sl}_2)$-morphism because the standard lifts of the dual Verma modules give rise to a $\mathbb{C}(q)$-basis of $\mathbf{G}(\bigoplus_{i=0}^{n} \text{gmod-}{}_iA)$. Part (b) of the theorem follows.

Part (c) is well-known and follows for example from [BGS96, Theorem 3.11.4 (ii)] and [Soe97, Remark 3.2 (2)].

To prove part (d) we first verify that the form is bilinear. This follows from the equalities

$$\begin{aligned}
&<1\otimes[M\langle k\rangle], 1\otimes[N\langle l\rangle]> \\
&\qquad = \sum_i\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\mathrm{d}(N\langle l\rangle), \mathbf{H}M\langle -i+k\rangle[j])q^i \\
&\qquad = \sum_i\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\mathrm{d}(N)\langle -l\rangle, \mathbf{H}M\langle -i+k\rangle[j])q^i \\
&\qquad = \sum_r\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\mathrm{d}(N), \mathbf{H}M\langle -r\rangle[j])q^{k+l}q^r \\
&\qquad = q^{k+l}<1\otimes[M], 1\otimes[N]>.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&<1\otimes[\tilde{M}(y\cdot\omega_i)], 1\otimes[\mathrm{d}\,\tilde{M}(x\cdot\omega_i)]> \\
&\qquad = \sum_i\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\tilde{M}(x\cdot\omega_i), \mathbf{H}\tilde{M}(y\cdot\omega_i)\langle i\rangle[j])q^i \\
&\qquad = \sum_i\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\tilde{M}(x\cdot\omega_i), \mathrm{d}\,\tilde{M}(w_0y\cdot\omega_i)\langle i\rangle[j])q^i \\
&\qquad = \dim\mathrm{Hom}_{\mathcal{D}}(\tilde{M}(x\cdot\omega_i), \mathrm{d}\,\tilde{M}(w_0y\cdot\omega_i)) = \delta_{x,w_0y}.
\end{aligned}$$

From (43) the statement (d) of the theorem follows.

Let us prove part (e). By definition the standard basis of $\mathcal{M}$ is mapped to the standard lifts of Verma modules, and they are mapped to the standard basis in $V_1^{\otimes n}$. By (36) of Proposition 5.2, the twisted standard basis is mapped to the standard lifts of the dual Verma modules, and they are (by definition) mapped to the dual standard basis in $(V_1^{\otimes n})'$. From part (c) and Proposition 5.2 we find that the twisted positive basis of $\mathcal{M}$ corresponds to the standard lifts of tilting modules. The formula [FKK98, Theorem 2.6] together with [Soe97, Proposition 3.4] explicitly shows that the twisted positive basis corresponds to the canonical basis. Finally, we have

$$\begin{aligned}
<[\tilde{T}(y\cdot\lambda)], [\mathrm{d}\,\tilde{L}(x\cdot\lambda)]> &= \sum_i\sum_j(-1)^j \dim\mathrm{Hom}_{\mathcal{D}}(\tilde{L}(x\cdot\lambda), \mathbf{H}\tilde{T}(y\cdot\lambda)\langle i\rangle[j])q^i \\
&= \dim\mathrm{Hom}_{\mathcal{D}}(\tilde{L}(x\cdot\lambda), \mathrm{d}\,\tilde{I}(w_0y\cdot\lambda))q^i = \delta_{x,w_0y}.
\end{aligned}$$

From (44) it follows that the standard lifts of the simple modules correspond to the dual canonical basis. Finally, it is known that the negative self-dual basis corresponds to the dual canonical basis (see e.g. [FKK98, Theorem 2.5′].

It remains to prove part (f) of the theorem. Of course, γ defines an isomorphism of $\mathbb{C}(q)$-modules. That tilting modules correspond to the positive self-dual bases as stated follows directly from part (c) together with the isomorphisms (35) and (36). That simple modules correspond to the negative self-dual basis elements follows directly from the second half of part (e). So we are done. □

We finish with two additional remarks:

Proposition 5.5. (a) *There are isomorphisms of functors*

$$\mathbf{H}_{w_0}\,\mathbf{E} \cong \mathbf{E}\,\mathbf{H}_{w_0}, \qquad \mathbf{H}_{w_0}\,\mathbf{F} \cong \mathbf{F}\,\mathbf{H}_{w_0}, \qquad \mathbf{H}_{w_0}\,\mathbf{K} \cong \mathbf{K}\,\mathbf{H}_{w_0}.$$

(b) *The "categorical" bilinear form* $<\ ,\ >$ *from Theorem* 5.3(d) *satisfies*

$$<\triangle(x)(v_i \otimes v_j), v^k \otimes v^l> = <v_i \otimes v_j, \triangle'(\omega(x))(v^k \otimes v^l)>.$$

Proof. The first statement is clear if we forget the grading. To prove the first isomorphism it is therefore enough to show $\mathbf{H}^2_{w_0}\mathbf{E} \cong \mathbf{E}\mathbf{H}^2_{w_0}$, even $\mathbf{H}^2_{w_0}\mathbf{E}_i\tilde{M}(\mathbf{a}) \cong \mathbf{E}_i\mathbf{H}^2_{w_0}\tilde{M}(\mathbf{a})$, where $\mathbf{a} = (1,\dots,1,0,\dots,0)$ with exactly i ones. From the formula (45) we know that $\mathbf{E}_i\tilde{M}(\mathbf{a}) \cong \tilde{P}(\mathbf{b})$ for some $\{0,1\}$-sequence $\mathbf{b}$. Hence

$$\mathbf{H}^2_{w_0}\mathbf{E}_i\tilde{M}(\mathbf{a}) \cong \mathbf{H}^2_{w_0}\tilde{P}(\mathbf{b}) \cong \tilde{I}(\mathbf{b})\langle -2l(w_0^i)\rangle$$

by Proposition 5.2. On the other hand,

$$\begin{aligned}\mathbf{E}_i\mathbf{H}^2_{w_0}\tilde{M}(\mathbf{a}) &\cong \mathbf{E}_i(\mathrm{d}\,\tilde{M}(\mathbf{a}))\langle -2l(w_0^i)\rangle \cong (\mathrm{d}\,\mathbf{E}_i\tilde{M}(\mathbf{a}))\langle -2l(w_0^i)\rangle \\ &\cong (\mathrm{d}\,P(\mathbf{b}))\langle -2l(w_0^i)\rangle \cong \tilde{I}(\mathbf{b})\langle -2l(w_0^i)\rangle\end{aligned}$$

by Lemma 4.4 and, again, Proposition 5.2. The first isomorphism of the proposition follows. The second is proved analogously, and the third is clear.

To prove statement (b) set $m_i = i(n-i)$, write just $\hom(N,M)$ as shorthand for $\sum_i \sum_j (-1)^j \dim \mathrm{Hom}^j_{\mathcal{D}}(N,M)q^i$, and calculate

$$\begin{aligned}
&<\mathbf{E}_i M, N> \\
&= \hom(\mathrm{d}\,N, \mathbf{H}\mathbf{E}_i M) \\
&= \hom(\mathrm{d}\,N, \mathbf{H}_{w_0}\mathbf{E}_i M\langle l(w_0^{i+1})\rangle) && \text{(by definition of } \mathbf{H}) \\
&= \hom(\mathrm{d}\,N, \mathbf{E}_i\mathbf{H}_{w_0} M\langle l(w_0^{i+1})\rangle) && \text{(by part (a))} \\
&= \hom(\mathrm{d}\,N, \mathbf{E}_i\mathbf{K}_i\langle 1\rangle\mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) \\
&= \hom(\mathbf{F}_{i+1}(\mathrm{d}_{i+1}\,N), \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(Prop. 4.2)} \\
&= \hom(\mathbf{F}_{i+1}\,\mathrm{d}'_{i+1}\langle m_{i+1}\rangle N, \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(see Section 4.4)} \\
&= \hom(\mathrm{d}'_i\,\mathbf{F}_{i+1}\langle m_{i+1}\rangle N, \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(Lemma 4.4)} \\
&= \hom(\langle m_i\rangle\,\mathrm{d}_i\,\mathbf{F}_{i+1}\langle m_{i+1}\rangle N, \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(see Section 4.4)} \\
&= \hom(\mathrm{d}_i\langle -m_i\rangle\mathbf{F}_{i+1}\langle m_{i+1}\rangle N, \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(by definition of d)} \\
&= \hom(\mathrm{d}_i\,\mathbf{F}'_{i+1}N, \mathbf{H}_{w_0}\mathbf{K}^{-1}\langle -1\rangle M\langle l(w_0^{i+1})\rangle) && \text{(by definition of } \mathbf{F}').
\end{aligned}$$

We claim that the latter is isomorphic to

$$\begin{aligned}\hom(\mathrm{d}\,\mathbf{F}'_{i+1}(N), \mathbf{H}_{w_0}M\langle(n-2i-1+l(w_0^{i+1})\rangle\rangle) &= \hom(\mathrm{d}\,\mathbf{F}'_{i+1}(N), \mathbf{H}_{w_0}M) \\ &= <M, \mathbf{F}'_{i+1}N>.\end{aligned}$$

To verify the claim, note that $l(w_0^i) = \frac{1}{2}(i(i-1)+(n-i)(n-i-1))$. Hence

$$\begin{aligned}&l(w_0^{i+1}) - l(w_0^i) \\ &= \tfrac{1}{2}((i+1)i + (n-i-1)(n-i-2) - (i(i-1)+(n-i)(n-i-1))) \\ &= \tfrac{1}{2}(2i-2n+2i+2) = 2i-n+1.\end{aligned}$$

Together with the definition of $\mathbf{H}$, the claim follows. Similarly we get $<\mathbf{F}_iM, N> = <M, \mathbf{E}'_{i-1}N>$. The formula $<\mathbf{K}_iM, N> = <M, \mathbf{K}_iN>$ is obvious from the bilinearity of the form. The proposition follows. □

Remark 5.6. Theorem 5.3 generalises to arbitrary tensor products $V_{\mathbf{d}}$ such that the standard basis corresponds to standard modules, the dual standard basis to dual standard modules, the canonical basis to tilting modules and the dual canonical bases to simple modules. Since the proofs involve deeply the properly stratified structure of the category of Harish-Chandra bimodules, the arguments will appear in another paper, where we moreover show that Proposition 2.6 is also true in the graded setup.

5.3. Categorification dictionary

Quantum $\mathfrak{sl}_2$ and its representations	**Functors and categories**
ring $\mathbb{Z}[q, q^{-1}]$	category $\mathbb{C}$-Vect of graded vector spaces
multiplication by q	grading shift up by 1
representation V_n	category $\mathcal{C} \cong \bigoplus_{i=0}^n C^i$-gmod, where C^i is the cohomology ring of a Grassmannian
weight spaces of V_n	summands of $\mathcal{C} \cong \bigoplus_{i=0}^n C^i$-gmod
semilinear form $< \, , \, > : \quad V_n \times V_n \to \mathbb{C}(q)$	bifunctor $\mathrm{Hom}_{\mathcal{C}}(*, *)$
canonical basis of V_n	indecomposable projective modules in $\bigoplus_{i=0}^n C^i$-gmod
dual canonical basis of V_n	simple modules $\bigoplus_{i=0}^n C^i$-gmod
representation $V_1^{\otimes n}$	graded version of $\bigoplus_{i=0}^n {}_{i;n}\mathcal{O}$, certain blocks of the category $\mathcal{O}(\mathfrak{gl}_n)$
weight spaces of $V_1^{\otimes n}$	blocks in the graded version of $\bigoplus_{i=0}^n {}_{i;n}\mathcal{O}$
standard basis of $V_1^{\otimes n}$	standard (= Verma) modules

dual standard basis of $V_1^{\otimes n}$	dual standard (= dual Verma) modules
canonical basis of $V_1^{\otimes n}$	indecomposable tilting modules
dual canonical basis of $V_1^{\otimes n}$	simple modules
representation $V_{\mathbf{d}}$	graded version of $\bigoplus_{i=0}^{n} {}_{\mu}\mathcal{H}_{\omega_i}(\mathfrak{gl}_n)$, certain blocks of the category of Harish-Chandra bimodules $\mathcal{H}(\mathfrak{gl}_n)$
weight spaces of $V_{\mathbf{d}}$	blocks in the graded version of $\bigoplus_{i=0}^{n} {}_{\mu}\mathcal{H}_{\omega_i}(\mathfrak{gl}_n)$
anti-automorphism τ	taking the right adjoint functor
involutions ψ_n, ψ	duality functor
Cartan involutions σ and σ_n	equivalences $\hat{\sigma}$ and $\hat{\sigma}_n$ arising from the equivalences of categories $\mathcal{A}_i^{\mu} \cong \mathcal{A}_{n-i-1}^{\mu}$

6. Geometric categorification

Our treatment of the graded version of the category $\mathcal{O}$ (based on [BGS96]) uses substantially the cohomology rings of partial flag varieties. A more systematic study of the structure of these rings naturally leads to an alternative categorification which we call "geometric". We first consider the geometric categorification of simple finite-dimensional $U_q(\mathfrak{sl}_2)$-modules using the cohomology rings of partial flag varieties and relations between them. Then we proceed to the geometric categorification of general tensor products using certain algebras of functions which generalise the cohomology rings and point towards the Borel–Moore homology of generalised Steinberg varieties. We conclude this section by formulating open problems related to the geometric categorification.

6.1. From algebraic to geometric categorification

The categorification of simple $U_q(\mathfrak{sl}_2)$-modules we propose gives rise to a categorification of simple $\mathcal{U}(\mathfrak{sl}_2)$-modules by forgetting the grading. The categorification for simple $\mathcal{U}(\mathfrak{sl}_2)$-modules obtained in this way is exactly the one appearing in [CR07]. It does not seem to be obvious from the approach of [CR07] why exactly this categorification plays an important role. As a motivation for choosing this categorification we first show that it naturally emerges from our Theorem 4.1 as follows: Theorem 4.1 provides an isomorphism $V_n \cong \mathbf{G}(\bigoplus_{i=0}^{n} \mathcal{A}_i^{-\rho})$, since $W_{-\rho} = W$. Each of the categories $\mathcal{A}_i^{-\rho}$ contains (up to isomorphism and shift in the grading) one single indecomposable projective object ([Jan83, 6.26]), hence also (up to isomorphism and grading shift) one simple object. On the other hand, the categories C^i-gmod also have (up to isomorphism and shift in the grading) one single simple object S_i. In particular, the Grothendieck groups of the two categories coincide. However, we have the following stronger result:

Proposition 6.1. *There is an equivalence of categories*

$$F : \bigoplus_{i=0}^{n} \mathcal{A}_i^{-\rho} \to \bigoplus_{i=0}^{n} C^i\text{-gmod}$$

which intertwines the functors $\mathbf{E}_i$, $\mathbf{F}_i$, *and* $\mathbf{K}_i$ *with the functors*

$$\begin{aligned} E_i &= \mathrm{Res}_{i+1}^{i,i+1} C^{i,i+1} \otimes_{C^i} \langle -n+i+1 \rangle, \\ F_i &= \mathrm{Res}_{i-1}^{i,i-1} C^{i,i-1} \otimes_{C^i} \langle -i+1 \rangle, \\ K_i &= \langle 2i-n \rangle. \end{aligned}$$

Proof. Let $P \in \mathcal{A}_i^{-\rho}$ be the unique (up to isomorphism) indecomposable projective module such that its head is concentrated in degree zero. Then we have an isomorphism of graded algebras $\mathrm{End}_{\mathcal{A}_i^{-\rho}}(P) \cong \mathrm{End}_{\mathcal{A}_i}(P(w_0 \cdot \omega_i)) \cong C^i$. (The first isomorphism follows just from the definition of $\mathcal{A}_i^{-\rho}$, the second is [Soe90, Endomorphismensatz] together with the definition of the grading on $\mathcal{A}_i$.) Note that any module in $\mathcal{A}_i^{-\rho}$ is a quotient of some $P\langle k \rangle$, $k \in \mathbb{Z}$. In other words, P is a $\mathbb{Z}$-generator of $\mathcal{A}_i^{-\rho}$ in the sense of [AJS94, E.3]. Hence (see e.g. [AJS94, Proposition E.4]), the functor

$$\bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}_i^{-\rho}}(P\langle k \rangle, \bullet) : \mathcal{A}_i^{-\rho} \to \text{gmod-}\mathrm{End}_{\mathcal{A}_i^{-\rho}}(P) \cong C^i\text{-gmod}$$

defines an equivalence. Of course, we could fix any $l_i \in \mathbb{Z}$ and still have an equivalence of categories

$$\bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}_i^{-\rho}}(P\langle k + l_i \rangle, \bullet) : \mathcal{A}_i^{-\rho} \to \text{gmod-}\mathrm{End}_{\mathcal{A}_i^{-\rho}}(P\langle l_i \rangle) \cong C^i\text{-gmod}.$$

Up to a grading shift, these equivalences obviously intertwine the functors in question. (This follows directly from the definitions of the functors and general arguments, see e.g. [Bas68, 2.2].) It follows easily from the definitions of the graded lifts that $l_i = -i(n-i)$ gives the required equivalences. □

From Proposition 6.1 it follows in particular that $\mathbf{G}(\bigoplus_{i=0}^n C^i\text{-gmod})$ becomes a $U_q(\mathfrak{sl}_2)$-module via the functors $E = \bigoplus_{i=0}^n E_i$, $F = \bigoplus_{i=0}^n F_i$ and $K = \bigoplus_{i=0}^n K_i$. From Theorem 4.1 we know that the resulting module is isomorphic to V_n. One could, of course, ask which C^i-modules correspond to the (dual) canonical basis elements. This will be answered in the next section.

6.2. Categorification of simple $U_q(\mathfrak{sl}_2)$-modules via modules over the cohomology rings of Grassmannians

We now want to describe the geometric categorification, motivated by Proposition 6.1 alone, not as a consequence of the main categorification theorem. The reason for this is that we propose a generalisation of this construction to a "geometric" categorification of $V_{\mathbf{d}}$ using certain algebras of functions (Section 6.4). Later on we will discuss the relation of these function algebras to the Borel–Moore homology rings of Steinberg varieties and to the algebraic categorification from Section 4.

Let $n \in \mathbb{Z}_{>0}$ and let $I := \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. We consider the corresponding partial flag variety G_I given by all flags $\{0\} \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n$ where $\{\dim_{\mathbb{C}} F_j\} = I$. Let $C^I = H^\bullet(\mathrm{G}_I, \mathbb{C})$ denote its cohomology. We will be only interested in the cases $k = 1$, $k = 2$ and $I = \{i\}$, $I = \{i_1, i_2\}$. Let C^I-mod denote the category of finitely generated C^I-modules. Each of these categories has exactly one simple object, the trivial module $\mathbb{C}$. If $I = \{i\}$ then the corresponding variety is a Grassmannian and we denote its cohomology ring by C^i. The Grothendieck group of $\bigoplus_{i=0}^n C^i$-mod is free of rank $n+1$. The rings C^I have a natural (positive, even) $\mathbb{Z}$-grading and we may consider the categories C^I-gmod of finitely generated *graded* C^I-modules and the category $\bigoplus_{i=0}^n C^i$-gmod. The Grothendieck group of the latter is then a free $\mathbb{Z}[q, q^{-1}]$-module of rank $n+1$, where q^i acts by shifting the grading degree by i. Hence we have a candidate for a categorification of the simple $U_q(\mathfrak{sl}_2)$-module V_n. We also need functors giving rise to the $U_q(\mathfrak{sl}_2)$-action. If $J \subseteq I$ then there is an obvious surjection $\mathrm{G}_I \to \mathrm{G}_J$ inducing an inclusion $C^J \to C^I$ of rings. Let $\mathrm{Res}^I_J : C^I\text{-gmod} \to C^J\text{-gmod}$ denote the restriction functor. For $0 \le i \le n$ set

$$\begin{aligned} E_i &= \mathrm{Res}^{i,i+1}_{i+1} C^{i,i+1} \otimes_{C^i} \langle -n+i+1 \rangle, \\ F_i &= \mathrm{Res}^{i,i-1}_{i-1} C^{i,i-1} \otimes_{C^i} \langle -i+1 \rangle, \\ K_i &= \langle 2i - n \rangle. \end{aligned}$$

Let $S_i \in C^i$-gmod be the simple (trivial) module concentrated in degree 0. Denote by $\mathbb{V}_i T_i \in C^i$-gmod the projective cover of $S_i\langle -i(n-i)\rangle$. (The notation coincides with the one from Section 3.2.) We consider the functors $E := \bigoplus_{i=0}^n E_i$, $F := \bigoplus_{i=0}^n F_i$, $K := \bigoplus_{i=0}^n K_i$ from $\bigoplus_{i=0}^n C^i$-gmod to itself. Since they are exact, and commute with shifts in the grading, they induce $\mathbb{Z}[q, q^{-1}]$-morphisms on $[\bigoplus_{i=0}^n C^i\text{-mod}]$. Note that $\frac{q^j - q^{-j}}{q - q^{-1}} = \sum_{k=0}^{j-1} q^{j-1-2k}$ for $j \in \mathbb{Z}_{>0}$. Therefore, if $2i - n > 0$, we use the notation

$$\begin{aligned} \frac{K_i - K_i^{-1}}{q - q^{-1}} &= -\frac{K_{n-i} - {K_{n-i}}^{-1}}{q - q^{-1}} \\ &= \begin{cases} \displaystyle\bigoplus_{k=0}^{2i-n-1} \mathrm{Id}\langle 2i - n - 1 - 2k \rangle & \text{if } 2i - n > 0, \\ 0 & \text{if } 2i - n = 0, \end{cases} \end{aligned} \tag{46}$$

(as endofunctors of $\bigoplus_{i=0}^n C^i$-gmod).

The following result categorifies simple $U_q(\mathfrak{sl}_2)$-modules:

Theorem 6.2 (The categorification of V_n). *Fix $n \in \mathbb{Z}_{>0}$.*

(a) *The functors E, F and K, K^{-1} satisfy the relations*

$$KE = \langle 2 \rangle EK, \quad KF = \langle -2 \rangle FK, \quad KK^{-1} = \mathrm{Id} = K^{-1}K, \tag{47}$$

$$E_{i-1}F_i \cong F_{i+1}E_i \oplus \frac{K_i - K_i^{-1}}{q - q^{-1}} \qquad \textit{if } 2i - n \geq 0, \tag{48}$$

$$F_{i+1}E_i \cong E_{i-1}F_i \oplus -\frac{K_i - K_i^{-1}}{q - q^{-1}} \qquad \textit{if } n - 2i > 0. \tag{49}$$

In the Grothendieck group we have the equality

$$(q - q^{-1})(E^{\mathbf{G}}_{i-1}F^{\mathbf{G}}_i - F^{\mathbf{G}}_{i+1}E^{\mathbf{G}}_i) = K^{\mathbf{G}}_i - (K_i^{-1})^{\mathbf{G}}.$$

Hence they induce a $U_q(\mathfrak{sl}_2)$*-structure on* $\mathbf{G}(\bigoplus_{i=0}^n C^i\text{-mod})$.

(b) *With respect to this structure, there is an isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules*

$$V_n \cong \mathbf{G}\Big(\bigoplus_{i=0}^n C^i\text{-gmod}\Big), \qquad v^i \mapsto 1 \otimes [S_i],\ v_i \mapsto 1 \otimes [\mathbb{V}_i T_i].$$

(c) *The involution* σ_n *from* (12) *can be categorified in the following sense: There is an equivalence of categories*

$$\hat{\sigma}_n : \quad \bigoplus_{i=0}^n C^i\text{-gmod} \cong \bigoplus_{i=0}^n C^i\text{-gmod}, \qquad S_i \mapsto S_{n-i},\ \mathbb{V}_i T_i \mapsto \mathbb{V}_{n-i}T_{n-i},$$

such that

$$\hat{\sigma}_n\,\hat{\sigma}_n \cong \mathrm{Id}, \qquad \hat{\sigma}_n\,E\,\hat{\sigma}_n \cong F, \qquad \hat{\sigma}_n\,F\,\hat{\sigma}_n \cong E, \qquad \hat{\sigma}_n\,K\,\hat{\sigma}_n \cong K^{-1},$$
$$\hat{\sigma}_n\langle k\rangle\,\hat{\sigma}_n \cong \langle k\rangle \qquad \textit{for any } k \in \mathbb{Z}.$$

(d) *The involution* ψ_n *from* (13) *can be categorified in the following sense: The duality* $M \mapsto \mathrm{d}\,M := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ *defines an involution*

$$\hat{\psi}_n : \bigoplus_{i=0}^n C^i\text{-gmod} \cong \bigoplus_{i=0}^n C^i\text{-gmod}, \qquad \mathbb{V}_i P_i \mapsto \mathbb{V}_i P_i,$$

and

$$\hat{\psi}_n\,E\,\hat{\psi}_n \cong F, \qquad \hat{\psi}_n\,F\,\hat{\psi}_n \cong E, \qquad \hat{\psi}_n\,K\,\hat{\psi}_n \cong K^{-1},$$
$$\hat{\psi}_n\,\langle i\rangle\,\hat{\psi}_n \cong \langle -i\rangle \qquad \textit{for any } i \in \mathbb{Z}.$$

(e) *The semilinear form becomes*

$$<[M],[N]> = \sum_i \sum_j (-1)^j \dim(\mathrm{Ext}^j_{\mathcal{C}}(N, M\langle -i\rangle))q^i.$$

(f) *The antilinear anti-automorphism* τ *can be viewed as the operation of taking right adjoint functors, namely: There are pairs of adjoint functors*

$$(E, FK^{-1}\langle 1\rangle), \qquad (F, EK\langle 1\rangle), \quad (K, K^{-1}), \quad (K^{-1}, K)$$

and $(\langle -k\rangle, \langle k\rangle)$ *for* $k \in \mathbb{Z}$.

We would like to stress again that essentially the same categorification, in the ungraded case (without q), was previously constructed by Chuang and Rouquier [CR07]. Before we prove the theorem we state the following

Lemma 6.3. *Consider the graded duality* $\mathrm{d} : \bigoplus_{i=0}^n C^i\text{-gmod} \to \bigoplus_{i=0}^n C^i\text{-gmod}$, $M \mapsto \mathrm{d}(M)$, *where* $(\mathrm{d}\, M)_j = \mathrm{Hom}_{\mathbb{C}}(M_{-j}, \mathbb{C})$. *Then there are isomorphisms of functors* $\mathrm{d}\, E \cong E\, \mathrm{d}$ *and* $\mathrm{d}\, F \cong F\, \mathrm{d}$.

Proof. To establish $\mathrm{d}\, E \cong E\, \mathrm{d}$ it is enough to show that $\mathrm{d}\, E_i \mathbb{V}_i \cong E_i\, \mathrm{d}\, \mathbb{V}_i$. Let us forget the grading for the moment. From Proposition 3.3 we get $\mathrm{d}\, E_i \mathbb{V}_i \cong \mathrm{d}\, \mathbb{V}_i \mathcal{E}_i$. On the other hand, $\mathrm{d}\, \mathbb{V}_i \mathcal{E}_i \cong \mathbb{V}_i\, \mathrm{d}\, \mathcal{E}_i \cong \mathbb{V}_i \mathcal{E}_i\, \mathrm{d}$ by [Soe90, Lemma 8] and [Jan83, 4.12 (9)]. Finally we have $\mathbb{V}_i \mathcal{E}_i\, \mathrm{d} \cong E_i \mathbb{V}_i\, \mathrm{d} \cong E_i\, \mathrm{d}\, \mathbb{V}_i$ again by Proposition 3.3 and [Soe90, Lemma 8]. We get an isomorphism of functors $\mathrm{d}\, E_i \cong E_i\, \mathrm{d} : \bigoplus_{i=0}^n C^i\text{-mod} \to \bigoplus_{i=0}^n C^i\text{-mod}$. Since the functors $\mathcal{E}_i$ are indecomposable (see Proposition 3.3), we can find (see e.g. [BGS96, Lemma 2.5.3]) some $k_i \in \mathbb{Z}$ such that $\mathrm{d}\, E_i \cong E_i\, \mathrm{d} \langle k_i \rangle : \bigoplus_{i=0}^n C^i\text{-gmod} \to \bigoplus_{i=0}^n C^i\text{-gmod}$. On the other hand, using Lemma 3.4 we get isomorphisms of graded vector spaces

$$\begin{aligned} E_i(S_i) &\cong C^{i,i+1} \otimes_{C^i} \mathbb{C}\langle -n+i+1 \rangle \\ &\cong \bigoplus_{r=0}^{n-i+1} \mathbb{C}\langle 2r-n+i-1 \rangle \\ &\cong \mathbb{C}\langle n-i-1 \rangle \oplus \mathbb{C}\langle n-i-3 \rangle \oplus \cdots \oplus \mathbb{C}\langle -n+i+3 \rangle \oplus \mathbb{C}\langle -n+i+1 \rangle \\ &\cong \mathrm{d}\, E_i(S_i). \end{aligned}$$

Hence $k_i = 0$ and so $\mathrm{d}\, E \cong E\, \mathrm{d}$. The arguments establishing $\mathrm{d}\, F \cong F\, \mathrm{d}$ are analogous. □

Proof of Theorem 6.2. The relations (47) follow directly from the definitions. The verification of the relations (48) and (49) is much more involved. We prove them here only on the level of the Grothendieck group. For the full statement we refer to Theorem 3.6 and Proposition 6.1. By Lemma 3.4 we get isomorphisms of graded C^i-modules

$$\begin{aligned} F_{i+1}E_i(\mathbb{C}) &\cong C^{i,i+1} \otimes_{C^{i+1}} C^{i,i+1} \otimes_{C^i} \mathbb{C}\langle -n+1 \rangle \\ &\cong \bigoplus_{l=0}^{i} \bigoplus_{k=0}^{n-i-1} \mathbb{C}\langle 2k+2l \rangle \langle -n+1 \rangle, \\ E_{i-1}F_i(\mathbb{C}) &\cong C^{i,i-1} \otimes_{C^{i-1}} C^{i,i-1} \otimes_{C^i} \mathbb{C}\langle -n+1 \rangle \\ &\cong \bigoplus_{l=0}^{n-i} \bigoplus_{k=0}^{i-1} \mathbb{C}\langle 2l+2k \rangle \langle -n+1 \rangle. \end{aligned}$$

If $n - 2i \geq 0$, then

$$\begin{aligned} F_{i+1}^{\mathbf{G}} E_i^{\mathbf{G}} - E_{i-1}^{\mathbf{G}} F_i^{\mathbf{G}} &= \Big(\bigoplus_{k=0}^{n-i-1} \langle (2k+2i) \rangle \langle (1-n) \rangle \Big)^{\mathbf{G}} \\ &\quad - \Big(\bigoplus_{k=0}^{i-1} \langle (2(n-i)+2k) \rangle \langle 1-n \rangle \Big)^{\mathbf{G}} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } n-2i=0, \\ \Big(\bigoplus_{k=0}^{n-2i-1} \langle (2i+2k) \rangle \langle 1-n \rangle \Big)^{\mathbf{G}} & \text{if } n-2i>0, \end{cases}$$
$$= [n-2i] \operatorname{Id}.$$

If $n-2i<0$, then

$$\begin{aligned} E_{i-1}^{\mathbf{G}} F_i^{\mathbf{G}} - F_{i+1}^{\mathbf{G}} E_i^{\mathbf{G}} &\cong \Big(\bigoplus_{k=0}^{i-1} \langle (2(n-i)+2k) \rangle \langle (-n+1) \rangle \Big)^{\mathbf{G}} \\ &\quad - \Big(\bigoplus_{k=0}^{n-i-1} \langle (2k+2i) \rangle \langle 1-n \rangle \Big)^{\mathbf{G}} \\ &= \Big(\bigoplus_{k=0}^{2i-n-1} \langle (n-2i+1+2k) \rangle \Big)^{\mathbf{G}} \\ &= [2i-n] \operatorname{Id}. \end{aligned}$$

In particular,

$$(q-q^{-1})(E^{\mathbf{G}} F^{\mathbf{G}} - F^{\mathbf{G}} E^{\mathbf{G}}) = K^{\mathbf{G}} - (K^{-1})^{\mathbf{G}}.$$

This proves part (a) of the proposition on the level of the Grothendieck group.

Obviously, the map $v^i \mapsto [S_i]$ from part (b) defines an isomorphism of vector spaces. We first have to verify that this is in fact a morphism of $U_q(\mathfrak{sl}_2)$-modules. Using Lemma 3.4 we get isomorphisms of $\mathbb{Z}[q,q^{-1}]$-modules

$$\begin{aligned} &[E_i(S_i)] = [C^{i,i+1} \otimes_{C^i} \mathbb{C}\langle -n+i+1 \rangle] \\ &\quad = \Big[\bigoplus_{r=0}^{n-i-1} \mathbb{C}\langle 2r-n+i+1 \rangle \Big] \\ &\quad = [\mathbb{C}\langle n-i-1 \rangle \oplus \mathbb{C}\langle n-i-3 \rangle \oplus \cdots \oplus \mathbb{C}\langle -n+i+3 \rangle \oplus \mathbb{C}\langle -n+i+1 \rangle] \\ &\quad = [S_{i+1}\langle n-i-1 \rangle \oplus S_{i+1}\langle n-i-3 \rangle \oplus \cdots \oplus S_{i+1}\langle -n+i+3 \rangle \oplus \mathbb{C}\langle -n+i+1 \rangle] \\ &\quad = \sum_{k=0}^{n-i-1} [S_{i+1}\langle -n+i+1+2k \rangle]. \end{aligned}$$

Similarly we have

$$\begin{aligned} [F_i(S_i)] &= [\mathbb{C}\langle -i+1 \rangle \oplus \mathbb{C}\langle -i+3 \rangle \oplus \cdots \oplus \mathbb{C}\langle i-3 \rangle \oplus \mathbb{C}\langle i-1 \rangle] \\ &= [S_{i-1}\langle -i+1 \rangle \oplus S_{i-1}\langle -i+3 \rangle \oplus \cdots \oplus S_{i-1}\langle i-3 \rangle \oplus S_{i-1}\langle i-1 \rangle] \\ &= \sum_{k=0}^{i-1} [S_{i-1}\langle 1-i+2k \rangle]. \end{aligned}$$

This fits with the formula (11). Hence, the assignment $v^i \mapsto [S_i]$ defines a $U_q(\mathfrak{sl}_2)$-morphism, where the simple modules concentrated in degree zero correspond to the dual canonical basis elements. It remains to show that v_i is mapped to $[\mathbb{V}_i T_i]$.

Recall that the algebra C^i has a basis b_y naturally indexed by elements $y \in W^i$. With our convention on the grading, b_y is homogeneous of degree $2l(y)$. Hence, from the definition of $\mathbb{V}_i T_i$ we get

$$[\mathbb{V}_i T_i] = [n,i][S_i], \tag{50}$$

and part (b) of the proposition follows directly from the formula $v_i = [n,i]v^i$.

To prove statement (c) we fix the standard basis $\{b_i\}_{1\leq i\leq n}$ of $\mathbb{C}^n$. Then there is an isomorphism of vector spaces sending b_i to b_{n-i+1}. This induces an isomorphism $C^i \cong C^{n-i}$ of graded algebras, hence an equivalence of categories

$$\hat{\sigma}_n : \bigoplus_{i=0}^{n} C^i\text{-gmod} \cong \bigoplus_{i=0}^{n} C^i\text{-gmod}, \qquad S_i \mapsto S_{n-i},\ \mathbb{V}_i T_i \mapsto \mathbb{V}_{n-i}T_{n-i}.$$

The remaining isomorphisms of functors of part (c) follow directly from the definitions.

To prove (d), recall first that $\mathbb{V}_i P_i$ is self-dual, hence $\hat{\psi}_n \mathbb{V}P_i \cong \mathbb{V}P_i$. The isomorphisms $\hat{\psi}_n \circ K^i \cong K^{-i} \circ \hat{\psi}_n$ and $\hat{\psi}_n \circ \langle i\rangle \cong \langle -i\rangle \circ \hat{\sigma}_n$ follow directly from the definitions. For the remaining isomorphisms of functors of (d) we refer to Lemma 6.3. The semilinearity of the form in part (e) follows directly from the formula $\mathrm{Hom}_{\mathcal{C}}(M\langle k\rangle, N) = \mathrm{Hom}_{\mathcal{C}}(M, N\langle -k\rangle) = \mathrm{Hom}_{\mathcal{C}}(M,N)_k$ for graded vector spaces. On the other hand, we have

$$\begin{aligned}\sum_i \sum_j (-1)^j \dim \mathrm{Ext}^j_{\mathcal{C}}(S_k, \mathbb{V}_l T_l\langle -i\rangle)q^i &= \sum_i \dim \mathrm{Hom}_{\mathcal{C}}(S_k, \mathbb{V}_l T_l\langle -i\rangle)q^i \\ &= \delta_{k,l}q^{l(n-l)} = \delta_{k,l}q^{k(n-k)},\end{aligned}$$

since $\mathbb{V}_l T_l$ is injective and its socle is concentrated in degree $l(n-l)$. From formula (50) it follows that $<[\mathbb{V}_l T_l],[\mathbb{V}_k T_k]> = \delta_{l,k}q^{l(n-l)}[n,l]$. Looking at formula (10) completes the proof. It remains to prove statement (f). From Lemmas 3.5 and 3.4 we get the following pairs of adjoint functors:

$$\begin{aligned}&(\mathrm{Res}^{i,i+1}_{i+1} C^{i,i+1} \otimes_{C^i} \langle -n+i+1\rangle, \mathrm{Res}^{i,i+1}_{i} C^{i,i+1} \otimes_{C^{i+1}} \langle -2i\rangle\langle n-i-1\rangle) \\ &\qquad = (E_i, F_{i+1}\langle n-2(i+1)\rangle\langle 1\rangle)\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{Res}^{i,i-1}_{i-1} C^{i,i-1} \otimes_{C^i} \langle -i+1\rangle, \mathrm{Res}^{i,i-1}_{i} C^{i,i-1} \otimes_{C^{i-1}} \langle -2(n-i)\rangle\langle i-1\rangle) \\ &\qquad = (F_i, E_{i-1}\langle 2(i-1)-n\rangle\langle 1\rangle).\end{aligned}$$

This gives the first two pairs of adjoint functors. The remaining ones are obvious from the definitions. □

The answer to the question raised at the end of the previous section now follows directly: Theorem 4.1 together with Theorem 6.2(b) and Proposition 6.1

provide isomorphisms of $U_q(\mathfrak{sl}_2)$-modules

$$\begin{array}{rcccc} V_n & \cong & \mathbf{G}\Big(\bigoplus_{i=0}^{n} \mathcal{A}_i^{-\rho}\Big) & \cong & \mathbf{G}\Big(\bigoplus_{i=0}^{n} C^i\text{-gmod}\Big), \\ v^k & \mapsto & 1 \otimes [L_i] & \mapsto & 1 \otimes [S_i], \\ v_k & \mapsto & 1 \otimes [P_i] & \mapsto & 1 \otimes [\mathbb{V}_i T_i], \end{array}$$

where L_i denotes a graded lift of $\mathcal{L}(M(-\rho), L(w_0 \cdot \omega_i))$ with head concentrated in degree zero and P_i denotes its projective cover.

6.3. An elementary categorification of $\overline{V}_1^{\otimes n}$ using algebras of functions

Next we would like to give a geometric categorification of the tensor products $\overline{V}_{\mathbf{d}}$. Since we do not have a general version of Soergel's theory which on the one hand side naturally leads to the algebras C^i and on the other hand generalises directly Proposition 6.1, we will start from the opposite end and propose a class of finite-dimensional algebras whose graded modules yield the desired geometric categorification.

In this subsection we will make the first step in this direction by giving a rather elementary categorification of $\overline{V}_1^{\otimes n}$ using the algebras B^i of functions on the finite set of cosets W/W_i (Proposition 6.4). We believe that this is a necessary ingredient of a more substantial and general construction which will be considered in the next subsection.

Let $W = S_n$ be the symmetric group of order $n!$ with subgroup W_i as above. For any $0 \leq i \leq n$ let $B^i = \mathrm{Func}(W/W_i)$ be the algebra of complex-valued functions on the (finite) set W/W_i. Similarly, for $0 \leq i, i+1 \leq n$ let $B^{i,i+1} = \mathrm{Func}(W/W_{i,i+1})$ be the algebra of functions on $W/W_{i,i+1}$. For any $w \in W/W_i$ we have an idempotent $e_w^{(i)} \in B^i$, namely the characteristic function of w, i.e. $e_w^{(i)}(x) = \delta_{w,x}$. In fact, the $e_w^{(i)}$, $w \in W/W_i$, form a complete set of primitive, pairwise orthogonal, idempotents. The algebra B^i is semisimple with simple (projective) modules $S_w^i = B^i e_w^{(i)}$. On the other hand, $B^{i,i+1}$ is both a B^i-module and a B^{i+1}-module as follows: Because $W_{i,i+1}$ is a subgroup of W_i and of W_{i+1}, we have surjections $\pi_i : W/W_{i,i+1} \to W/W_i$ and $\pi_{i+1} : W/W_{i,i+1} \to W/W_{i+1}$. If $g \in B^j$ for $j \in \{i, i+1\}$ and $f \in B^{i,i+1}$ we put $g.f(x) = g(\pi_j(x))f(x)$ for $x \in W/W_{i,i+1}$. The B^i's are commutative, hence we get a left and a right module structure. Clearly, $B^{i,i+1}$ becomes a free B^j-module of rank equal to the order of the group $(W/W_{i,i+1})/(W/W_i)$, hence to the order of $W_i/W_{i,i+1}$. Let

$$\mathcal{C}_{\mathrm{func}} := \bigoplus_{i=0}^{n} B^i\text{-mod}\,.$$

For technical reasons, if $i > n$ or $i < 0$, let B^i-mod denote the category consisting of the zero $\mathbb{C}$-module. We define the following endofunctors of $\mathcal{B}$:

- $E_{\mathrm{func}} = \bigoplus_{i=0}^{n} E_i$,
where $E_i : B^i\text{-mod} \to B^{i+1}\text{-mod}$ is the functor $B^{i,i+1} \otimes_{B^i} \bullet$ if $i < n$ and the zero functor otherwise.
- $F_{\mathrm{func}} = \bigoplus_{i=0}^{n} F_i$,
where $F_i : B^i\text{-mod} \to B^{i-1}\text{-mod}$ is the functor $B^{i,i-1} \otimes_{B^i} \bullet$ if $i > 0$ and the zero functor otherwise.

For any $w \in W$ we denote by $\mathbf{a}_{w,n,i}$ the $\{0,1\}$-sequence $w(1,\dots,1,0,\dots,0)$ of length n, where we used exactly i ones. We get the following elementary categorification:

Proposition 6.4. *The category $\mathcal{C}_{\mathrm{func}}$ together with the isomorphism*

$$\eta : \mathbf{G}(\mathcal{C}_{\mathrm{func}}) \to \overline{V}_1^{\otimes n}, \qquad S_w^i = B^i e_w^{(i)} \mapsto v_{\mathbf{a}_{w,n,i}},$$

and the functors E_{func} and F_{func} is a categorification (in the sense of Subsection 2.4*) of the module $\overline{V}_1^{\otimes n}$.*

Proof. Clearly, the map η is an isomorphism of vector spaces, and the functors E_{func} and F_{func} are exact. As $B^{i,i+1}$ is a free B^i-module of rank equal to $n-i$, the order of $W_i/W_{i,i+1}$, it follows that the B^{i+1}-module $B^{i,i+1} \otimes_{B^i} S_w^i$ is of dimension $n-i$. Hence it is a direct sum of $n-i$ simple B^{i+1}-modules. A basis of $B^{i,i+1} \otimes_{B^i} S_w^i$ is given by elements of the form $f_x \otimes 1$, where $x \in W_i/W_{i,i+1}$ and f_x is the characteristic function for $xw \in W_i w$. If $x \in W_i$ then $x\mathbf{a}_{w,n,i+1}$ is equal to $x\mathbf{a}_{w,n,i}$, but with exactly one zero occurring in the sequence replaced by a one. If we allow only $x \in W_i/W_{i,i+1}$ then the $x\mathbf{a}_{w,n,i}$ provide each sequence exactly once. Therefore, $\eta(E_{\mathrm{func}} S_w^i) = E\eta(S_w^i)$. Similarly, $\eta(F_{\mathrm{func}} S_w^i) = F\eta(S_w^i)$. The statement follows. □

Given any finite-dimensional algebra, say A, we could equip A with a trivial $\mathbb{Z}$-grading by putting $A = A_0$. Then a graded A-module is nothing else than an A-module M which carries the structure of a $\mathbb{Z}$-graded vector space. In particular, we could consider the function algebras B^i as trivially graded. Then $B^{i,i+1}$ becomes a $\mathbb{Z}$-graded (B^{i+1}, B^i)-bimodule by putting the characteristic function corresponding to the shortest coset representative $w \in W/W_{i,i+1}$ in degree $l(w)$. Similarly $B^{i,i-1}$ becomes a $\mathbb{Z}$-graded (B^{i-1}, B^i)-bimodule. In this way, we get graded lifts

$$\mathbf{E}_{\mathrm{func}} : \bigoplus_{i=0}^{n} B^i\text{-gmod} \to \bigoplus_{i=0}^{n} B^i\text{-gmod},$$
$$\mathbf{F}_{\mathrm{func}} : \bigoplus_{i=0}^{n} B^i\text{-gmod} \to \bigoplus_{i=0}^{n} B^i\text{-gmod}$$

of our functors E_{func} and F_{func}. We define $\mathcal{C}_{\mathrm{func}} = \bigoplus_{i=0}^{n} B^i\text{-gmod}$ with the endofunctor $\mathbf{K} = \bigoplus_{i=0}^{n} \langle 2i-n \rangle$. The following statement follows directly from Proposition 6.4 and the definition of the grading and provides a categorification of the $U_q(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$ in terms of graded modules over our function algebras:

Corollary 6.5. *The isomorphism*

$$\eta : \mathbf{G}(\mathcal{C}_{\mathrm{func}}) \to V_1^{\otimes n}, \qquad S_w^i = B^i e_w^{(i)} \mapsto v_{\mathbf{a}_{w,n,i}},$$

defines an isomorphism of $U_q(\mathfrak{sl}_2)$-modules, where the module structure on the left hand side is induced by the functors $\mathbf{E}_{\mathrm{func}}$, $\mathbf{F}_{\mathrm{func}}$ and $\mathbf{K}_{\mathrm{func}}$.

6.4. A categorification of $\overline{V}_{\mathbf{d}}$ using finite-dimensional algebras

Now we combine the categorification of irreducible modules from Section 6.2 and the elementary categorification from Proposition 6.4 to a (partly conjectural) categorification of an arbitrary tensor product $\overline{V}_{\mathbf{d}}$. The present construction is parallel to the categorification of irreducible modules as described in Section 6.2 and coincides with this categorification in the special case when $\overline{V}_{\mathbf{d}}$ has a single factor.

Let again $W = S_n$ with the Young subgroup $S_{\mathbf{d}}$ corresponding to the composition $\mathbf{d}$. Let $B^{\mathbf{d}} = \mathrm{Func}(W/S_{\mathbf{d}})$ be the algebra of (complex-valued) functions on $W/S_{\mathbf{d}}$. Recall (from Section 2.4) the subalgebras C^i, $C^{i,i+1}$, $C^{i,i-1}$ in the coinvariant algebra corresponding to W. The Weyl group W is acting on both $B^{\mathbf{d}}$ and C.

We set

$$H_{\mathbf{d}}^i = (B^{\mathbf{d}} \otimes C)^{W_i}, \qquad H_{\mathbf{d}}^{i,i+1} = (B^{\mathbf{d}} \otimes C)^{W_{i,i+1}}, \tag{51}$$

where we take the W_i-invariants with respect to the diagonal action. For any $w \in W_i \backslash W/S_{\mathbf{d}}$, there is an idempotent $f_w = e_w \otimes 1$, where $e_w(x) = e_w(yx) = \delta_{w,x}$ for any $x \in W_i \backslash W/S_{\mathbf{d}}$ and $y \in W_i$. These f_w form in fact a complete set of primitive pairwise orthogonal idempotents. In particular, the simple modules of $H_{\mathbf{d}}^i$ are naturally indexed by (longest coset representatives of) the double cosets $W_i \backslash W/S_{\mathbf{d}}$. Let $P_{\mathbf{d},x}^i = H_{\mathbf{d}}^i f_w$ be the corresponding indecomposable projective module with simple head $S_{\mathbf{d},x}^i$. We define

$$\mathcal{C}_{\mathrm{geom}} := \bigoplus_{i=0}^{n} H_{\mathbf{d}}^i\text{-mod}\,.$$

Obviously, $H_{\mathbf{d}}^j$ is a subset of $H_{\mathbf{d}}^{i,i+1}$ for $j = i, i+1$ (if they are defined). For technical reasons we denote by $H_{\mathbf{d}}^i$-mod the category containing only the zero $\mathbb{C}$-module if $i > n$ or $i < 0$.

Analogously to our elementary construction we define for $0 \leq i \leq n$ the functors

$$E_i : H_{\mathbf{d}}^i\text{-mod} \to H_{\mathbf{d}}^{i+1}\text{-mod} \quad \text{as} \quad H_{\mathbf{d}}^{i+1} \otimes_{H_{\mathbf{d}}^i} \bullet \quad \text{if } 0 \leq i < n,$$
$$F_i : H_{\mathbf{d}}^i\text{-mod} \to H_{\mathbf{d}}^{i-1}\text{-mod} \quad \text{as} \quad H_{\mathbf{d}}^{i-1} \otimes_{H_{\mathbf{d}}^i} \bullet \quad \text{if } 0 < i \leq n,$$

otherwise they should be just the zero functors. We set

$$E_{\mathrm{geom}} = \bigoplus_{i=0}^{n} E_i, \qquad F_{\mathrm{geom}} = \bigoplus_{i=0}^{n} F_i,$$

Proposition 6.6. *There are isomorphisms of vector spaces*

$$\Phi_1 : \mathbf{G}(\mathcal{C}_{\mathrm{geom}}) \to \overline{V}_{\mathbf{d}}, \qquad S^i_{\mathbf{d},w} \mapsto v^{\mathbf{a}(\mu)},$$
$$\Phi_2 : \mathbf{G}(\mathcal{C}_{\mathrm{geom}}) \to \overline{V}_{\mathbf{d}}, \qquad P^i_{\mathbf{d},w} \mapsto v_{\mathbf{a}(\mu)},$$

where $\mathbf{a}(\mu) = w(1, \dots, 1, 0, \dots, 0)$.

Proof. This follows directly from the definitions. □

Note that if the tensor product $\overline{V}_{\mathbf{d}}$ has only one factor, that is, $\mu = (n)$ and hence $S_{\mathbf{d}} = W$, we get $H^i_{\mathbf{d}} \cong C^i$ and the functors $E^{\mathbf{G}}$ and $F^{\mathbf{G}}$ become the functors E and F from Section 6.2. For the general case we formulate the following

Conjecture 6.7. *The isomorphisms Φ_1 and Φ_2 agree and are isomorphisms of $\mathfrak{sl}_2$-modules, where the action on the left hand side is induced by the functors E_{geom} and F_{geom}.*

The quantum version of this conjectural categorification should again arise from the corresponding graded version.

Remark 6.8. The conjecture implies in particular that the functors E_{geom} and F_{geom} preserve the additive category of projective modules. By direct calculations it can be shown that the conjecture is true for all cases where $n = 2, 3$. In these cases we also know that $(B^{\mathbf{d}} \otimes C)^{W'}$ is a free $(B^{\mathbf{d}} \otimes C)^{W''}$-module of rank $|W''/W'|$ for any subgroups $W \supseteq W'' \supseteq W'$.

6.5. Open problems related to a geometric categorification

The categorification of $\overline{V}_{\mathbf{d}}$ via the modules over the finite-dimensional algebras $H^i_{\mathbf{d}}$ from the previous section strongly suggests that the geometry of Grassmannians and partial flag varieties used in the case of a single factor $\overline{V}_n$ should be replaced by the geometry of generalised Steinberg varieties (as defined in [DR04]) for the general linear group $GL_n(\mathbb{C})$ to obtain a geometric categorification of arbitrary tensor products $\overline{V}_{\mathbf{d}}$. Note first that the dimension of the algebra $B \otimes C$ coincides with the dimension of the Borel–Moore homology $H_*(Z)$ of the (full) Steinberg variety for $GL_n(\mathbb{C})$ and is equal to $|W|^2$ (see e.g. [CG97, Proposition 8.1.5, Lemma 7.2.11]). Moreover, the algebras $H^i_{\mathbf{d}}$ can be viewed as the subalgebra of $W_{\mathbf{d}} \otimes W_i$-invariants in $B \otimes C$, and it was proven in [DR05, (1.1″)] that the Borel–Moore homology of the generalised Steinberg variety $Z^i_{\mathbf{d}}$ associated to the pair $(W_{\mathbf{d}}, W_i)$ is isomorphic to the subspace of $H_*(Z)$ given by $W_{\mathbf{d}} \otimes W_i$-invariants. We expect that using intersection theory (see [Ful98]), one can define a commutative algebra structure on $Z^i_{\mathbf{d}}$ which yields the algebras $H^i_{\mathbf{d}}$ introduced in Section 6.4.

We also note that the generalised Steinberg varieties $Z^i_{\mathbf{d}}$ are precisely the tensor product varieties of Malkin ([Mal03]) and Nakajima ([Nak01]) in the special case of tensor products of finite-dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules. Thus the geometric categorification we propose is also a natural next step in the geometric description of the tensor products $V_{\mathbf{d}}$ from [Sav03].

Looking from the algebraic categorification side of the picture one notices that the generalised Steinberg varieties also appear as characteristic varieties of

Harish-Chandra bimodules ([BB85]). Thus one expects that the geometry of characteristic varieties that underlies our Categorification Theorem 4.1 should provide a conceptual relation between the algebraic and geometric categorifications giving rise to isomorphisms of $\mathcal{U}(\mathfrak{sl}_2)$-modules:

$$\mathbf{G}\Big(\bigoplus_{i=0}^{n}\mathcal{A}_i^{\mu}(\mathfrak{gl}_n)\Big)\cong\mathbf{G}\Big(\bigoplus_{i=0}^{n}H_{\mathbf{d}}^{i}\text{-mod}\Big)\cong V_{\mathbf{d}}.$$

One can also relate the algebraic categorification to a geometric categorification by studying the projective functors acting on the category $\mathcal{O}(\mathfrak{gl}_n)$, extending our constructions in Section 3. In fact the complexified Grothendieck ring of projective endofunctors of the principal block of $\mathcal{O}(\mathfrak{gl}_n)$ is isomorphic to the group algebra of the Weyl group W by the classification theorem from [BG80] and the Kazhdan–Lusztig theory. On the other hand, the group algebra of W is canonically isomorphic to the top degree $H_{\mathrm{top}}(Z)$ of the Borel–Moore homology $H_*(Z)$ with respect to the convolution product (see e.g. [CG97, Theorem 3.4.1]). One can show that the whole Borel–Moore homology ring $H_*(Z)$ encodes the Grothendieck ring of these projective functors together with the action of the centre of the category. This picture can be generalised to $H_{\mathrm{top}}(Z_{\mathbf{d}}^i)$ and $H_*(Z_{\mathbf{d}}^i)$ by looking at projective functors between different singular blocks of the category $\mathcal{O}(\mathfrak{gl}_n)$. Details of this alternative approach will appear in a subsequent paper.

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