CATEGORIFICATION OF THE TEMPERLEY-LIEB CATEGORY, TANGLES, AND COBORDISMS VIA PROJECTIVE FUNCTORS

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Abstract

To each generic tangle projection from the three-dimensional real vector space onto the plane, we associate a derived endofunctor on a graded parabolic version of the Bernstein-Gel′fand category $\mathcal{O}$. We show that this assignment is (up to shifts) invariant under tangle isotopies and Reidemeister moves and defines therefore invariants of tangles. The occurring functors are defined via so-called projective functors. The first part of the paper deals with the indecomposability of such functors and their connection with generalised Temperley-Lieb algebras which are known to have a realisation via decorated tangles. The second part of the paper describes a categorification of the Temperley-Lieb category and proves the main conjectures of [BFK]. Moreover, we describe a functor from the category of 2-cobordisms into a category of projective functors.

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0. Introduction

On the way to finding topological invariants for knots and links, some new ideas concerning a connection to representation theory recently appeared (see, e.g., [Kh1], [FK]). Our paper was mainly motivated by [BFK] and contains a proof of the main conjectures therein. Bernstein, Frenkel, and Khovanov constructed a realisation of the Temperley-Lieb algebra via projective functors on parabolic versions of the Bernstein-Gel′fand-Gel′fand category $\mathcal{O}$. The category $\mathcal{O}$ is given by representations (with certain finiteness conditions) of a complex semisimple Lie algebra $\mathfrak{g}$. It is stable under tensoring with a finite-dimensional $\mathfrak{g}$-module E. A direct summand of $\bullet \otimes E$ is called a *projective functor* since it preserves projectivity. Such functors play a crucial role in representation theory. The indecomposable projective functors on $\mathcal{O}$ were classified by Bernstein and Gel′fand [BG]. When restricted to the main block $\mathcal{O}_0$, their isomorphism classes are in bijection to the Weyl group. The famous Kazhdan-Lusztig theory is based on the fact that the Grothendieck ring of projective functors is described by the corresponding (specialised) (Iwahori-)Hecke algebra. In other words, this algebra has a "functorial realisation"; that is, there is a ring homomorphism from the specialised (Iwahori-)Hecke algebra into the Grothendieck ring of projective functors on a regular integral block of $\mathcal{O}$. In type A, there is a well-known quotient of the Iwahori-Hecke algebra called the *Temperley-Lieb algebra*. Because of its diagrammatical description, it is directly linked with knot theory and has several applications in physics and science (see, e.g., [K]). In [BFK], the authors considered the action of the specialised Iwahori-Hecke algebra induced via projective functors on the direct sum over all maximal parabolic subcategories of $\mathcal{O}_0$. They proved that it factors through the specialised Temperley-Lieb algebra. On the level of the Grothendieck group, the resulting representation coincides with the natural representation on the n-fold tensor product of $\mathbb{C}^2$ given by place permutations.

The following questions appeared in this context (and are the content of our paper).

(I) Is there a "functorial realisation" of the Temperley-Lieb algebra where the deformation variable comes into the picture (see [BFK])?

(II) Is there a classification of indecomposable projective functors in the parabolic setup (see [B2])?

(III) Is it possible to generalise the results of [BFK] to other types?

(IV) Is there a "functorial realisation" of the Temperley-Lieb 2-category and of arbitrary tangles (see [BFK])?

The first problem can be solved using the graded version of category $\mathcal{O}$ introduced in [BGS]. In [St], a graded version of translation functors is defined such that one can easily get the required "functorial realisation" (Theorem 4.1). In this context we also obtain a "functorial realisation" of the Temperley-Lieb algebras of Types B, C, and

D. This might be interesting since these algebras can be realised via *decorated* tangles (see, e.g., [G]).

The classification problem, however, seems to be much more complicated. We are far away from a reasonable answer. Nevertheless, we give a combinatorial formula (Proposition 3.6, Theorem 5.7) for the number of isomorphism classes of indecomposable functors. This formula was motivated by discussions with W. Soergel, who conjectured that it should determine the number of indecomposable projective functors. However, in non–simply laced cases we do not have equality in general (see Examples 3.7). In Proposition 3.8 we prove that it is sufficient to study the case of simple Lie algebras.

A very nice (and helpful!) result is given by Theorem 5.1, where we prove that an indecomposable projective functor on $\mathscr{O}_0(\mathfrak{sl}_n)$ either stays indecomposable or becomes zero after restricting to a maximal parabolic subcategory. (This was conjectured in [BFK]. Note that it is *not* true for other types; see Examples 3.7.) Moreover, the indecomposable functors corresponding to non-braid-avoiding Weyl group elements are always trivial after restriction (see Lemma 5.2).

Our main results are the proofs of [BFK, Conjectures 1 – 4]. Let $\mathscr{O}_n^{\max}$ be the direct sum of all parabolic subcategories of the main block of $\mathscr{O}(\mathfrak{sl}_n)$ given by parabolic subgroups of the form $S_k \times S_{n-k}$. We associate to each morphism f of the Temperley-Lieb 2-category, that is, to each (m,n)-tangle projection without crossings, a projective functor $F(f) : \mathscr{O}_m^{\max} \to \mathscr{O}_n^{\max}$ and prove the following in Theorem 6.2.

If $f \simeq g$ via planar isotopies, then $F(f) \cong F(g)$ as functors.

We extend this "functorial realisation" to tangles with crossings as follows. Let $\mathscr{D}^b\big(\mathscr{O}_n^{\max}\big)$ denote the bounded derived category of $\mathscr{O}_n^{\max}$. To each (m,n)-tangle projection t we associate a functor $\mathscr{T}(t) : \mathscr{D}^b\big(\mathscr{O}_m^{\max}\big) \to \mathscr{D}^b\big(\mathscr{O}_n^{\max}\big)$. The functors assigned to a right or left basic braid are given as mapping cones of the adjunction morphisms between the identity functor and translation functors through the wall. That means that they coincide with the derived functors of Irving's shuffling functors (see [I2]). We prove the following in Theorem 7.1.

If $t \simeq t'$ via ambient isotopies, then $\mathscr{T}(t) \cong \mathscr{T}(t')$

up to a grading shift and a shift in the derived category. These results prove [BFK, Conjectures 3, 4]. The dependency on the chosen representation of the tangle in the form of shifts disappears if one works with oriented tangles (see Remark 7.2 and cf. [Kh2]). Using the fact that projective functors are Koszul dual to Zuckerman's functors (as proved in [Ry]), a "functorial realisation" of tangles via singular blocks of category $\mathcal{O}$ follows (see [BFK, Conjectures 1, 2]).

Therefore, we get functor invariants for tangles. In particular, we can assign to a disjoint union of closed oriented 1-manifolds a certain endofunctor on a parabolic version of category $\mathcal{O}(\mathfrak{sl}_n)$. Our final result (Theorem 8.1) is a "functorial realisation" of the category of 2-cobordisms. In other words, we assign to each cobordism a natural transformation between the corresponding functors and prove that this assignment is invariant under isomorphisms of cobordisms. Since all the occurring functors can be lifted to a $\mathbb{Z}$-graded version (as explained in [St]), the natural transformations corresponding to cobordisms can be interpreted as (homogeneous) transformations between $\mathbb{Z}$-functors. It turns out that the Euler characteristic of the cobordism surface coincides with the degree of the assigned natural transformation. A way to realise the 2-category of tangle cobordisms in terms of projective functors will be explained in a subsequent paper.

The paper is organised as follows. In the first section we recall the main results on Category $\mathcal{O}$, its parabolic version, and its combinatorics. In Section 2 we explain how the deformation variable of the (Iwahori-)Hecke algebra can be interpreted as grading shifts. In Section 3 we define the categories of projective functors and prove some basic and general results. The problem about indecomposability of projective functors is worked out, including a description of how to define graded lifts of projective functors. In Section 4 we describe "functorial realisations" of generalised Temperley-Lieb algebras. Section 5 considers the maximal parabolic situation of type A. It includes the theorem on indecomposability of indecomposable projective functors after restriction to the parabolic category. Since some proofs rely on explicit calculations, we have attached an appendix containing the description of distinguished coset representatives for maximal parabolic subgroups. Sections 6 and 7 contain (the proof of) the two "functorial realisation" theorems for tangles. In Section 8 we finally describe "functorial realisations" of the 2-cobordisms category and mention how the Euler characteristic of cobordism surfaces can be realised as degrees of natural transformations between $\mathbb{Z}$-functors.

1. Category $\mathcal{O}$ and its combinatorics

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with fixed Borel and Cartan subalgebras. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. The universal enveloping algebras are denoted by $\mathscr{U} = \mathscr{U}(\mathfrak{g})$, $\mathscr{U}(\mathfrak{b})$, and so on. Let $\mathscr{Z} \subset \mathscr{U}$ be the centre.

We consider the category $\mathcal{O}$ of Bernstein, Gel′fand, and Gel′fand [BGG], which is the full subcategory of the category of all $\mathscr{U}$-modules given by the set of objects

$$\mathrm{Ob}(\mathscr{O}) := \left\{ M \in \mathfrak{g} - \mathrm{mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated as a } \mathscr{U}(\mathfrak{g})\text{-module,} \\ M \text{ is locally finite for } \mathfrak{b}, \\ \mathfrak{h} \text{ acts diagonally on } M \end{array} \right\},$$

where the second condition means that $\dim_{\mathbb{C}} \mathscr{U}(\mathfrak{n}) \cdot m < \infty$ for all $m \in M$ and the last says that $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$ is the μ-weight space of M. Many results about this category can be found, for example, in [BGG], [J1], and [J2].

For a given weight $\lambda \in \mathfrak{h}^*$, let $M(\lambda) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{b})} \mathbb{C}_\lambda$ denote the Verma module with highest weight λ and simple head $L(\lambda)$. Let $P(\lambda) \in \mathscr{O}$ be the projective cover of $L(\lambda)$.

Let $\pi \subset R$ be the set of simple roots inside the set of all roots. For $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the α-weight space of $\mathfrak{g}$ under the adjoint action. The coroot of α is denoted by $\check{\alpha}$. We use W for the Weyl group with unit element e and denote by $\mathscr{S}$ the set of simple reflections. The length of $w \in W$ is denoted by $l(w)$. For $w, a_1, \ldots, a_r \in W$, we call an expression $w = a_1 a_2 \cdots a_r$ *minimal* if $\sum_{i=1}^r l(a_i) = l(w)$. In particular, any reduced expression is minimal. The Weyl group acts in a natural way on $\mathfrak{h}^*$ (with fix-point zero); for any $\lambda \in \mathfrak{h}^*$, we denote by $w \cdot \lambda = w(\lambda + \rho) - \rho$ the image of λ under the "translated" action of W with fix-point $-\rho$, where ρ is the half-sum of positive roots.

Let W_λ denote the stabiliser of λ under this action. We denote by $\mathscr{O}_\lambda$ the full subcategory of $\mathscr{O}$ having as objects all modules annihilated by a large enough power of the maximal ideal $\ker \chi_\lambda = \mathrm{Ann}_{\mathscr{Z}} M(\lambda)$ in the centre of $\mathscr{U}$. We call $\lambda \in \mathfrak{h}^*$ *dominant* (with respect to $-\rho$) if $\langle \lambda + \rho, \check{\alpha} \rangle \geq 0$, and we always label the subcategories $\mathscr{O}_\lambda$ with dominant weights.

1.1. The parabolic category $\mathscr{O}^{\mathfrak{p}}$

Let $S \subseteq \pi$ be a subset of the simple roots with corresponding root system $R_S = R \cap \mathbb{Z}S$. We define the Lie algebra $\mathfrak{g}_S \subseteq \mathfrak{g}$ as

$$\mathfrak{g}_S = \mathfrak{n}_S^- \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S^+,$$

where $\mathfrak{n}_S^\mp = \bigoplus_{\alpha \in \mp R \cap R_S} \mathfrak{g}_\alpha$. Then $\mathfrak{g}_S$ is semisimple with Cartan subalgebra $\mathfrak{h}_S = \bigoplus_{\alpha \in S} \mathbb{C}\check{\alpha}$ and root system R_S. Let us denote the corresponding Weyl group by W_S, and let W^S be the set of minimal-length coset representatives for $W_S \backslash W$, that is,

$$W^S = \left\{ w \in W \,\middle|\, \forall s \in \mathscr{S} \cap W_S : l(sw) > l(w) \right\}.$$

The parabolic subalgebras (containing $\mathfrak{b}$) of $\mathfrak{g}$ are parametrised by the elements of the power set of π in such a way that $S \subseteq \pi$ corresponds to

$$\mathfrak{p}_S = (\mathfrak{g}_S \oplus \mathfrak{h}^S) + \mathfrak{n},$$

where $\mathfrak{h}^S = \bigcap_{\alpha \in S} \ker \alpha$. Using this bijection, we identify $W_{\mathfrak{p}_S} = W_S$, $W^S = W^{\mathfrak{p}_S}$, and so on.

Let $\mathfrak{p} = \mathfrak{p}_S$ be a parabolic subalgebra (containing $\mathfrak{b}$) of $\mathfrak{g}$ with universal enveloping algebra $\mathscr{U}(\mathfrak{p})$. The category $\mathscr{O}^S = \mathscr{O}^{\mathfrak{p}}$ is the full subcategory of $\mathscr{O}$ whose objects are exactly the locally $\mathfrak{p}$-finite modules of $\mathscr{O}$; that is, $M \in \mathscr{O}^{\mathfrak{p}}$ if and only if $\dim_{\mathbb{C}} \mathscr{U}(\mathfrak{p})m < \infty$ for all $m \in M$. This category is called the *parabolic category* $\mathscr{O}$ (with respect to $\mathfrak{p}$ or S, resp.).

Let $P_{\mathfrak{p}}^{+} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{N},\ \forall \alpha \in S\}$ be the strictly dominant integral weights with respect to S. The map that sends a simple $\mathscr{U}(\mathfrak{p})$-module to its highest weight gives (see [Ro]) a bijection

$$\{\text{iso-classes of finite-dimensional simple } \mathscr{U}(\mathfrak{p})\text{-modules}\} \overset{1:1}{\longleftrightarrow} P_{\mathfrak{p}}^{+}. \tag{1.1}$$

We denote the (unique up to isomorphism) simple $\mathfrak{p}$-module of highest weight λ by $E(\lambda)$. The *parabolic Verma module* (with respect to S or $\mathfrak{p}$, resp.) of highest weight λ is defined as

$$M^{\mathfrak{p}}(\lambda) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} E(\lambda).$$

It has a unique simple quotient $L^{\mathfrak{p}}(\lambda) \cong L(\lambda)$ (see [Ro, Proposition 3.3]). Note that if $S = \emptyset$, then $\mathfrak{p}_S = \mathfrak{b}$ and $M^{\mathfrak{p}}(\lambda) = M(\lambda)$ is the "ordinary" Verma module. In the other extreme case, where $S = \pi$, we have $M^{\mathfrak{g}}(\lambda) \cong L(\lambda)$.

There is a bijection between the isomorphism classes of simple modules in $\mathscr{O}^{\mathfrak{p}}$ and the elements of $P_{\mathfrak{p}}^{+}$ by mapping a module to its highest weight. The category $\mathscr{O}^{\mathfrak{p}}$ has enough projectives. We denote the projective cover of the simple module $L(\lambda)$ corresponding to $\lambda \in P_{\mathfrak{p}}^{+}$ by $P^{\mathfrak{p}}(\lambda)$ (for details, see [Ro, Proposition 3.3, Corollaries 4.2, 4.4]). A categorical characterisation of the parabolic Verma modules is given by the following fact.

LEMMA 1.1 (Parabolic Verma modules as projective objects)
Let $\lambda \in P_{\mathfrak{p}}^{+}$. The module $M^{\mathfrak{p}}(\lambda)$ is projective in the full subcategory $\mathscr{O}_{\lambda\geq}^{\mathfrak{p}}$ of $\mathscr{O}^{\mathfrak{p}}$, objects of which have only composition factors of the form $L(\mu)$ with $\mu \not> \lambda$.

Proof
For $M \in \mathscr{O}_{\lambda\geq}^{\mathfrak{p}}$, we have by Frobenius's reciprocity

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{g}}\left(M^{\mathfrak{p}}(\lambda), M\right) &= \operatorname{Hom}_{\mathfrak{g}}\left(\mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} E(\lambda), M\right) \\ &\cong \operatorname{Hom}_{\mathfrak{p}}\left(E(\lambda), M\right) \cong M_{\lambda}.\end{aligned}$$

Therefore, $\operatorname{Hom}_{\mathfrak{g}}(M^{\mathfrak{p}}(\lambda), \bullet)$ is exact and $M^{\mathfrak{p}}(\lambda)$ is projective. (Note that the last isomorphism follows from (1.1) and the fact that λ is by assumption a maximal possible weight.) □

The following proposition describes how to construct the projective covers in $\mathscr{O}^{\mathfrak{p}}$ given a projective cover in $\mathscr{O}$.

PROPOSITION 1.2 (Projective covers in $\mathscr{O}^{\mathfrak{p}}$)
Let $Q \in \mathscr{O}^{\mathfrak{p}}$ with projective cover $P \in \mathscr{O}$. Then the projective cover of Q in $\mathscr{O}^{\mathfrak{p}}$ is (up to isomorphism) the quotient P/M, where M is the smallest submodule of P containing all composition factors of P not contained in $\mathscr{O}^{\mathfrak{p}}$.

Proof
First of all, it is clear from the definition of $\mathscr{O}^{\mathfrak{p}}$ that $P/M \in \mathscr{O}^{\mathfrak{p}}$. Since $\mathrm{Hom}_{\mathscr{O}}(P, \bullet) = \mathrm{Hom}_{\mathscr{O}}(P/M, \bullet) = \mathrm{Hom}_{\mathscr{O}^{\mathfrak{p}}}(P/M, \bullet)$ on $\mathscr{O}^{\mathfrak{p}}$, the projectivity of P/M follows. If a submodule of P/M surjects onto Q, then its preimage under the canonical map $P \twoheadrightarrow P/M$ maps surjectively onto Q as well. Hence, P/M is a projective cover by the minimality of P. □

Restriction to the subcategory $\mathscr{O}^{\mathfrak{p}}_{\lambda\geq}$ gives the following.

COROLLARY 1.3
For $\lambda \in \mathfrak{h}^*$, there is an isomorphism $M^{\mathfrak{p}}(\lambda) \cong M(\lambda)/M$, where M denotes the smallest submodule containing all composition factors not contained in $\mathscr{O}^{\mathfrak{p}}$.

1.2. The parabolic Hecke module $\mathscr{N}$

We recall some facts on the Kazhdan-Lusztig combinatorics developed in [KL] and [D]. We use the notation of [S2].

Let $\mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in one variable v. Let $\mathscr{H} = \mathscr{H}(W, \mathscr{S})$ denote the Hecke algebra of $(W, \mathscr{S})$, that is, the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_x \mid x \in W\}$ and relations

$$H_s^2 = H_e + (v^{-1} - v)H_s \quad \text{for } s \in \mathscr{S}, \tag{1.2}$$

$$H_x H_y = H_{xy} \quad \text{if } l(x) + l(y) = l(xy). \tag{1.3}$$

We denote by $H \mapsto \overline{H}$ the *duality* on $\mathscr{H}$, that is, the ring homomorphism given by $H_x \mapsto (H_{x^{-1}})^{-1}$ and $v \mapsto v^{-1}$. The *Kazhdan-Lusztig basis* is given by elements $\underline{H}_x$ (for $x \in W$) such that $\underline{H}_x$ is self-dual (i.e., $\overline{\underline{H}_x} = \underline{H}_x$) and $\underline{H}_x \in H_x + \sum_{y \in W} v\mathbb{Z}[v]H_y$. In particular, $C_s := \underline{H}_s = H_s + v$ is a Kazhdan-Lusztig basis element for each simple reflection $s \in W$. For $S \subseteq \pi$, a subset of the simple roots, let $\mathscr{H}_S = \mathscr{H}(W_S, W_S \cap \mathscr{S})$ be the corresponding Hecke algebra. We consider $\mathbb{Z}[v, v^{-1}]$ as a right $\mathscr{H}_S$-module, where H_s for $s \in \mathscr{S}$ acts by multiplication with $-v$. On the other hand, the Hecke algebra $\mathscr{H}$ is in a natural way, via restriction, a left $\mathscr{H}_S$-module. Therefore, the following definition of the *parabolic Hecke module* (with respect to S or $\mathfrak{p}$) makes sense:

$$\mathcal{N}^{\mathfrak{p}} := \mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}_S} \mathcal{H}.$$

Hence, the parabolic Hecke module $\mathcal{N}^{\mathfrak{p}}$ is a right $\mathcal{H}$-module and a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{N_x^{\mathfrak{p}} := 1 \otimes H_x \mid x \in W^{\mathfrak{p}}\}$. The structure as a right $\mathcal{H}$-module is given by the following.

LEMMA 1.4 (see [S2])
We have

$$N_x^{\mathfrak{p}} C_s = \begin{cases} N_{xs}^{\mathfrak{p}} + v N_x^{\mathfrak{p}} & \text{if } xs > x \text{ and } xs \in W^{\mathfrak{p}}, \\ N_{xs}^{\mathfrak{p}} + v^{-1} N_x^{\mathfrak{p}} & \text{if } xs < x \text{ and } xs \in W^{\mathfrak{p}}, \\ 0 & \text{if } xs \notin W^{\mathfrak{p}}. \end{cases}$$

1.3. Translation through the wall and the parabolic Hecke module
For $\lambda \in \mathfrak{h}^*$ dominant and integral, let $\theta_0^\lambda : \mathcal{O}_0 \to \mathcal{O}_\lambda$ (resp., $\theta_\lambda^0 : \mathcal{O}_\lambda \to \mathcal{O}_0$) be the corresponding translation functors. For $W_\lambda = \{e, s\}$, $s \in \mathcal{S}$, we denote by $\theta_s = \theta_\lambda^0 \theta_0^\lambda$ the translation functor through the wall (for more details, see, e.g., [J1], [J2]).

Note that if M is $\mathfrak{p}$-locally finite, then so is the tensor product $M \otimes E$ for any finite-dimensional $\mathfrak{g}$-module E. Hence, the functor θ_s restricts to a functor $\mathcal{O}_0^{\mathfrak{p}} \longrightarrow \mathcal{O}_0^{\mathfrak{p}}$ for any parabolic subalgebra containing $\mathfrak{b}$.

Let $[\mathcal{O}_0^{\mathfrak{p}}]$ denote the Grothendieck group of $\mathcal{O}_0^{\mathfrak{p}}$. Since θ_s is exact, it induces a group homomorphism on $[\mathcal{O}_0^{\mathfrak{p}}]$ which is denoted by $[[\theta_s]]$. Each of the following sets is a basis of $[\mathcal{O}_0^{\mathfrak{p}}]$: $\{[P^{\mathfrak{p}}(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$, $\{[M^{\mathfrak{p}}(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$, and $\{[L(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$. In the following we use the abbreviations $P^{\mathfrak{p}}(x) = P^{\mathfrak{p}}(x \cdot 0)$, $M^{\mathfrak{p}}(x) = M^{\mathfrak{p}}(x \cdot 0)$, and $L(x) = L(x \cdot 0)$ for any $x \in W$.

We state the following well-known results.

PROPOSITION 1.5
Let $x \in W^{\mathfrak{p}}$. Let s be a simple reflection.

(1) *If $xs \in W^{\mathfrak{p}}$ and $x < xs$, then $\theta_s M^{\mathfrak{p}}(x) \cong \theta_s M^{\mathfrak{p}}(xs)$ and there is a short exact sequence of the form*

$$0 \to M^{\mathfrak{p}}(x) \longrightarrow \theta_s M^{\mathfrak{p}}(x) \longrightarrow M^{\mathfrak{p}}(xs) \to 0.$$

(2) *If $xs \notin W^{\mathfrak{p}}$, then $\theta_s M^{\mathfrak{p}}(x) = 0$.*

(3) *The following diagram commutes:*

$$\begin{tikzcd} \mathscr{N}^{\mathfrak{p}} \arrow[r, "N_x^{\mathfrak{p}} \mapsto [M^{\mathfrak{p}}(x)]", "v=1"'] \arrow[d, "\cdot C_s"'] & {[\mathscr{O}_0^{\mathfrak{p}}]} \arrow[d, "{\big[[\theta_s]\big]}"] \\ \mathscr{N}^{\mathfrak{p}} \arrow[r, "N_x^{\mathfrak{p}} \mapsto [M^{\mathfrak{p}}(x)]", "v=1"'] & {[\mathscr{O}_0^{\mathfrak{p}}]} \end{tikzcd}$$

Proof

The first part of the theorem is [I1, Proposition v]. For the second part, we assume $xs \notin W^{\mathfrak{p}}$; hence, $xs > x$. (Otherwise, choose $t \in W_{\mathfrak{p}} \cap \mathscr{S}$ such that $txs < xs$. Then $l(txs) = l(xs) - 1 = l(x) - 2 = l(tx) - 3$. This is a contradiction.) Any nonzero quotient of $\theta_s M(x)$ contains $L(xs)$ as a composition factor; hence, there is no nontrivial quotient that is $\mathfrak{p}$-locally finite. In particular, $\theta_s M^{\mathfrak{p}}(x) = 0$. The commutativity of the diagram is then clear by Lemma 1.4. □

2. Gradable modules and graded translation

In the following we consider an integral regular block (say, $\mathscr{O}_0$) of the category $\mathscr{O}$ with its parabolic subcategory. Let $P = \bigoplus_{x\in W} P(x)$ be the sum over all indecomposable projectives in this block. This is a minimal projective generator. How its endomorphism ring becomes a $\mathbb{Z}$-graded ring is explained in [BGS] and [St]. In the following, let $A = \mathrm{End}_{\mathfrak{g}}(P)$ be equipped with this ($\mathbb{Z}$-)grading. By Morita equivalence, we can consider $\mathscr{O}_0$ as a category of finitely generated (nongraded!) right modules over a graded ring A. If we denote by $\mathrm{mof}-A$ the category of finitely generated right A-modules, this means

$$\mathscr{O}_0 \cong \mathrm{mof}-A.$$

We denote by (g) $\mathrm{mof}-A$ the category of finitely generated *graded* right A-modules. As in [St], we call a module $M \in \mathscr{O}_0$ *gradable* if $\mathrm{Hom}_{\mathfrak{g}}(P, M)$ is gradable, that is, if there exists a graded right A-module $\tilde{M}$ such that $\mathrm{f}(\tilde{M}) \cong \mathrm{Hom}_{\mathfrak{g}}(P, M)$. (Here f denotes the grading forgetting functor $\mathrm{gmof}-A \longrightarrow \mathrm{mof}-A$.) In this case, $\tilde{M}$ is called *a lift* of M. We call $M \in \mathscr{O}_0^{\mathfrak{p}}$ *gradable* if it is gradable considered as an object of $\mathscr{O}_0$.

In [St] and [BGS] it is shown that all "important" objects of $\mathscr{O}_0$, such as projective modules, simple modules, and Verma modules, are gradable. We generalise this result to the parabolic situation.

THEOREM 2.1

Let $M \in \mathscr{O}_0^{\mathfrak{p}}$ be a simple object, a projective object, or a parabolic Verma module. Then M is gradable (considered as an object in $\mathscr{O}_0$).

Proof
Since the simple modules in the parabolic subcategory are also simple in $\mathcal{O}_0$, the statement for simple objects is proved in [St]. By Lemma 1.2, for each $x \in W^{\mathfrak{p}}$ there is an isomorphism $P^{\mathfrak{p}}(x) \cong P(x)/M$, where M is the smallest submodule containing all simple composition factors of the form $L(y)$ with $y \notin W^{\mathfrak{p}}$. We consider the graded lift $\tilde{P}(x)$ of $P(x)$, which is defined in [St]. Let M be its smallest submodule that is generated by the collection of one-dimensional subspaces corresponding to simple composition factors of the form $L(y)$ of $P(x)$ with $y \notin W^{\mathfrak{p}}$. Therefore, M is by definition generated by homogeneous elements; hence the module $\tilde{P}(x)/M$ is a lift of $P^{\mathfrak{p}}(x)$. For the parabolic Verma modules, we can do (by Lemmas 1.1 and 1.2) an analogous construction. The theorem follows. □

The proof of Theorem 2.1 gives the following.

COROLLARY 2.2
Let $P^{\mathfrak{p}} := \bigoplus_{x \in W^{\mathfrak{p}}} P^{\mathfrak{p}}(x)$ *be a minimal projective generator of* $\mathcal{O}_0^{\mathfrak{p}}$. *Then* $\operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$ *is a quotient of* $\operatorname{End}_{\mathfrak{g}}(P)$ *even as a graded ring.*

Remark 2.3
(1) By construction, the graded rings $A = \operatorname{End}_{\mathfrak{g}}(P)$ and $A^{\mathfrak{p}} = \operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$ coincide with the ones introduced in [BGS].
(2) The graded lifts of $P^{\mathfrak{p}}(x)$ (with $x \in W^{\mathfrak{p}}$) are unique up to isomorphism and a shift of the grading (see [St, Lemma 1.5]). We defined the lifts in such a way that the simple head is concentrated in degree zero. The same is true for the lifts of the parabolic Verma modules and of the simple objects.
(3) It follows directly from the construction that a module $M \in \mathcal{O}_0^{\mathfrak{p}}$ is gradable if and only if it is gradable as an object of $\mathcal{O}_0^{\mathfrak{p}}$, that is, if there exists a graded right $A^{\mathfrak{p}}$-module $\tilde{M}$ such that $\mathrm{f}(\tilde{M}) \cong \operatorname{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}, M)$ (as nongraded right $A^{\mathfrak{p}}$-modules).
(4) In [B1] it is proved that $A^{\mathfrak{p}}$ (even for singular blocks) becomes a Koszul ring, generalising the results of [BGS].

2.1. Combinatorics of graded translation functors
Let us from now on denote by $\tilde{P}^{\mathfrak{p}}(x)$, $\tilde{M}^{\mathfrak{p}}(x)$, and $\tilde{L}(x)$ the graded lifts of $P^{\mathfrak{p}}(x)$, $M^{\mathfrak{p}}(x)$, and $L(x)$, respectively, as defined in the proof of Theorem 2.1, that is, with head concentrated in degree zero. We consider $A^{\mathfrak{p}} := \operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$, the endomorphism ring of the minimal projective generator $P^{\mathfrak{p}} = \bigoplus_{x \in W^{\mathfrak{p}}} P^{\mathfrak{p}}(x)$ of $\mathcal{O}_0^{\mathfrak{p}}$, as a graded ring. For $m \in \mathbb{Z}$, let $M\langle m\rangle$ be the graded module defined by $M\langle m\rangle_n := M_{n-m}$ with the same module structure as M; that is, $\mathrm{f}(M\langle m\rangle) = \mathrm{f}(M)$. Let [gmof$-A^{\mathfrak{p}}$] be the

Grothendieck group of the category of all finitely generated right $A^{\mathfrak{p}}$-modules. Each of the following three sets is a basis of $[\mathrm{gmof}-A^{\mathfrak{p}}]$: $\{[\tilde{L}(x)\langle i\rangle] \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$, $\{[\tilde{M}^{\mathfrak{p}}(x)\langle i\rangle] \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$, $\{[\tilde{P}^{\mathfrak{p}}(x)\langle i\rangle \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$. Let $\tilde{\theta}_s : \mathrm{gmof}-A \longrightarrow \mathrm{gmof}-A$ denote the graded version of θ_s with the graded adjunction morphisms $\mathrm{ID}\langle 1\rangle \to \tilde{\theta}_s$ and $\tilde{\theta}_s \to \mathrm{ID}\langle -1\rangle$ as defined in [St]. We get the following generalisation of [St, Theorems 3.6 and 5.3].

THEOREM 2.4

Let $s \in W$ be a simple reflection.

(1) *Let x, $xs \in W^{\mathfrak{p}}$ such that $x < xs$. The graded lifts of the parabolic Verma modules fit into the following short exact sequences of graded modules:*

$$0 \to \tilde{M}^{\mathfrak{p}}(x)\langle 1\rangle \to \tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(x) \to \tilde{M}^{\mathfrak{p}}(xs) \to 0,$$
$$0 \to \tilde{M}^{\mathfrak{p}}(x) \to \tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(xs) \to \tilde{M}^{\mathfrak{p}}(xs)\langle -1\rangle \to 0.$$

(2) *Let $x \in W^{\mathfrak{p}}$ such that $xs \notin W^{\mathfrak{p}}$. Then $\tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(x) = 0$.*

Proof

Note that the maps have to be (up to a scalar) the adjunction morphisms since the homomorphism spaces in question are all one-dimensional. Hence, the upper inclusion and the lower surjection are clear. On the other hand, the canonical surjection $\tilde{M}(xs) \twoheadrightarrow \tilde{M}^{\mathfrak{p}}(xs)$ is homogeneous of degree zero by definition. The surjection of graded modules (see [St, Theorem 3.6]) $\tilde{\theta}_s\tilde{M}(x) \twoheadrightarrow \tilde{M}(xs)$ has kernel $\tilde{M}(x)\langle 1\rangle$. Therefore, the surjection in the first row has to be homogeneous of degree zero. For the injection in the second row, we consider the inclusion of graded modules $\tilde{M}(x) \hookrightarrow \tilde{\theta}_s\tilde{M}(xs)$ (see [St, Theorem 5.3]). This induces the injection in the second sequence.

For the second part, we already know the statement when forgetting the grading (Proposition 1.5); hence, there is nothing to do. □

We get a combinatorial description of the graded translation functors.

COROLLARY 2.5

The following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}^{\mathfrak{p}} \arrow[r, "v^n N_x \mapsto [\tilde{M}^{\mathfrak{p}}(x)\langle n\rangle]"] \arrow[d, "\cdot C_s"'] & {[\mathrm{gmof}-A^{\mathfrak{p}}]} \arrow[d, "{[\tilde{\theta}_s]}"] \\
\mathcal{N}^{\mathfrak{p}} \arrow[r, "v^n N_x \mapsto [\tilde{M}^{\mathfrak{p}}(x)\langle n\rangle]"] & {[\mathrm{gmof}-A^{\mathfrak{p}}]}
\end{tikzcd}$$

Proof
This follows from Theorem 2.4 using Lemma 1.4. □

Remark 2.6
The horizontal maps in the corollary are in fact isomorphisms of $\mathbb{Z}[v, v^{-1}]$-modules, where the action on [gmof $-A$] is given by $v^i[M] = [M\langle i\rangle]$.

3. The categories of projective functors
In this section we define the additive categories of projective functors.

For each dominant weight $\mathfrak{h}^*$, we denote by $\mathrm{p}_\lambda^{\mathfrak{p}} : \mathscr{O}^{\mathfrak{p}} \to \mathscr{O}_\lambda^{\mathfrak{p}}$ the canonical projection. An endofunctor on $\mathscr{O}_\lambda^{\mathfrak{p}}$ is called *projective* if it is a (nonzero) direct summand of $\mathrm{p}_\lambda^{\mathfrak{p}}(\bullet \otimes E)$ for some finite-dimensional $\mathfrak{g}$-module E. Note that the direct sum of two such functors is again projective. Together with the zero functor, these functors form an additive category $\mathscr{P}_\lambda^{\mathfrak{p}}$ with the usual morphisms (i.e., natural transformation between functors) and the usual notation of (finite) direct sums.

A (projective) functor F on $\mathscr{O}_\lambda^{\mathfrak{p}}$ is *indecomposable* if $F \cong F_1 \oplus F_2$ for some endofunctors F_1, F_2 on $\mathscr{O}_\lambda^{\mathfrak{p}}$ implies $F_1 = 0$ or $F_2 = 0$. In particular, a projective functor on $\mathscr{O}_\lambda^{\mathfrak{p}}$ is indecomposable if and only if it is an indecomposable object in $\mathscr{P}_\lambda^{\mathfrak{p}}$; hence, there is no obstacle to calling such functors *indecomposable projective functors* (on $\mathscr{O}_\lambda^{\mathfrak{p}}$).

In the following we write just $\mathscr{P}^{\mathfrak{p}}$ instead of $\mathscr{P}_0^{\mathfrak{p}}$, and $\mathscr{P}_\lambda = \mathscr{P}_\lambda^{\mathfrak{b}}$. We denote by $\mathrm{IndP}(\mathfrak{g}, \mathfrak{p})$ the set of isomorphism classes of indecomposable objects in $\mathscr{P}^{\mathfrak{p}}$ and by $\#\,\mathrm{IndP}(\mathfrak{g}, \mathfrak{p})$ the order of this set. The indecomposable functors on $\mathscr{O}_\lambda$ are classified by the following.

THEOREM 3.1 ([BG, Theorems 3.3, 3.5])
(1) *Let λ be an integral dominant weight. Let $F, G \in \mathscr{P}_\lambda$. Then*

$$F \cong G \iff F\big(M(\lambda)\big) \cong G\big(M(\lambda)\big).$$

(2) *The assignment $F \mapsto F(M(\lambda))$ defines a bijection between* $\mathrm{IndP}(\mathfrak{g}, \mathfrak{b})$ *and the set of isomorphism classes of indecomposable projective objects in $\mathscr{O}_\lambda$.*

For $\lambda = 0$, let $F_w \in \mathscr{P}$ such that $F_w M(e) \cong P(w)$. For any $F \in \mathscr{P}^{\mathfrak{p}}$, we denote by $\big[[F]\big]$ the induced homomorphism on the Grothendieck group.

Remark 3.2
Since for $F \in \mathscr{P}$ the module $F(M(e))$ is projective, we can reformulate the first part of the theorem as

$$F \cong G \iff \big[F(M(e))\big] = \big[G(M(e))\big] \iff \big[[F]\big] = \big[[G]\big].$$

Unfortunately, the obvious generalisation of Theorem 3.1 to the parabolic situation is no longer true. Let s be a simple reflection. Choose $\mathfrak{p}$ such that $L(s) \notin \mathcal{O}_0^{\mathfrak{p}}$. We have $\theta_s \neq 0$ in general, but $\theta_s(M^{\mathfrak{p}}(e)) = 0$. Nevertheless, we conjecture a generalisation of Remark 3.2.

CONJECTURE 3.3
Let $F, G \in \mathscr{P}^{\mathfrak{p}}$. Then

$$F \cong G \iff \big[[F]\big] = \big[[G]\big].$$

3.1. Indecomposable projective functors

In this subsection, we state some characterisations of indecomposable (projective) functors in a general setup and define graded lifts.

PROPOSITION 3.4
Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Let $F \in \mathscr{P}_\lambda^{\mathfrak{p}}$. The following are equivalent.

(i) *F is indecomposable.*

(ii) *The only idempotents in* $\operatorname{End}(F)$ *are* 0 *and* 1.

(iii) $\operatorname{End}(F)$ *is a local ring.*

Proof

(ii) $\Rightarrow$ (i). Assume that F is decomposable and $F \cong F_1 \oplus F_2$. The natural transformation given by projection onto the first factor obviously defines a nontrivial idempotent.

(i) $\Rightarrow$ (ii). Let $\pi \in \operatorname{End}(F)$ be a nontrivial idempotent. This defines an endofunctor F_π on $\mathcal{O}_\lambda^{\mathfrak{p}}$ by $F_\pi(M) = \pi(F(M))$ on objects and $F_\pi(f) = \pi_N \circ F(f)|_{\pi_M(F(M))}$ on morphisms $f \in \operatorname{Hom}(M, N)$. Since π is idempotent, it is $F_\pi(\mathrm{Id}_M) = \mathrm{Id}_{\pi(M)}$ for any object M. Let $f \in \operatorname{Hom}(M, N)$ and $g \in \operatorname{Hom}(Q, M)$. We get $\pi_N \circ F(f) \circ \pi_M \circ F(g)|_{\pi_Q(F(Q))} = \pi_N \circ F(f) \circ F(g) \circ \pi_Q|_{\pi_Q(F(Q))} = \pi_N \circ F(f \circ g)|_{\pi_Q(F(Q))}$ since π is idempotent. Hence, F_π is indeed a functor. On the other hand, $(\mathrm{Id} - \pi) \in \operatorname{End}(F)$ is also idempotent, and we have $F \cong F_\pi \oplus F_{\mathrm{Id} - \pi}$.

(ii) $\Leftrightarrow$ (iii). Note that $\dim \operatorname{End}(F) \leq \dim \operatorname{End}_{\mathfrak{g}}(F(P))$ for some (minimal) projective generator P of $\mathcal{O}_0^{\mathfrak{p}}$; hence, $\dim \operatorname{End}(F) < \infty$. Then the equivalence is well known (see, e.g., [L]). □

We next get a generalisation of the classical Krull-Remak-Azumaya-Schmidt theorem (see, e.g., [L]).

COROLLARY 3.5
Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Let $F \in \mathscr{P}_\lambda^{\mathfrak{p}}$. Then F is isomorphic to a finite direct sum of indecomposable projective functors on $\mathcal{O}_\lambda^{\mathfrak{p}}$. Moreover, this decomposition

is unique up to isomorphism and order of the summands.

Proof

Let $\mathrm{ll}(F)$ denote the length of a Jordan-Hölder series of $F(P^{\mathfrak{p}})$. Of course, $\mathrm{ll}(F_1) < \mathrm{ll}(F)$ when F_1 is a direct summand of F. This shows that the desired decomposition exists. The uniqueness follows then by standard arguments (see, e.g., [L, Corollary 19.23]) using Proposition 3.4. □

3.2. The image of the Hecke algebra

The action of the Hecke algebra on the parabolic Hecke module (see Lemma 1.4) induces a homomorphism

$$\Phi^{\mathfrak{p}} : \mathscr{H} \longrightarrow \mathrm{End}_{\mathbb{Z}}(\mathscr{N}^{\mathfrak{p}}_{v=1}) = \mathrm{End}_{\mathbb{Z}}([\mathscr{O}^{\mathfrak{p}}_0]), \tag{3.1}$$

where $\mathscr{N}^{\mathfrak{p}}_{v=1}$ denotes the specialisation $v \rightsquigarrow 1$ of $\mathscr{N}^{\mathfrak{p}}$. The $\mathbb{Z}$-rank of the image of $\Phi^{\mathfrak{p}}$ is denoted by $\mathrm{R}(\mathfrak{g},\mathfrak{p})$. The following lemma gives a lower bound for the number of indecomposable projective functors.

PROPOSITION 3.6

For any parabolic subalgebra $\mathfrak{p}$ *containing* $\mathfrak{b}$*, the following holds:*

$$\#\,\mathrm{IndP}(\mathfrak{g},\mathfrak{p}) \geq \mathrm{R}(\mathfrak{g},\mathfrak{p}). \tag{3.2}$$

Proof

The image of $\Phi^{\mathfrak{p}}$ is generated by the multiplications $\cdot\underline{H}_x$ with $x \in W$. Let $\{\cdot\underline{H}_x \mid x \in I\}$ be a maximal linear independent subset. Let $\{F_x \mid x \in I\}$ be the corresponding projective functors on $\mathscr{O}_0$. Let $\{G_i \mid 1 \leq i \leq m\}$ be a system of representatives for $\mathrm{IndP}(\mathfrak{g},\mathfrak{p})$. For $x \in I$, we therefore have $F_{x\,|\mathscr{O}^{\mathfrak{p}}_0} \cong \bigoplus_{i=1}^m G_i^{\alpha_i}$ for some nonnegative integers α_i. (Here, $G_i^{\alpha_i}$ denotes the direct sum of α_i copies of G_i.) Hence $\big[[F_{x\,|\mathscr{O}^{\mathfrak{p}}_0}]\big] = \sum_{i=1}^m \alpha_i \big[[G_i]\big]$. Thus, the $\big[[G_i]\big]$ generate the image of $\Phi^{\mathfrak{p}}$ and the claim follows. □

We list a few examples, including some where we have strict inequality in formula (3.2).

Examples 3.7

(a) Let $\mathfrak{g}$ be arbitrary semisimple. For $\mathfrak{p} = \mathfrak{b}$, both sides of formula (3.2) are equal to the order of the Weyl group: the left-hand side by Theorem 3.1, and the right-hand side because the self-dual elements $\underline{H}_x = H_e\underline{H}_x$ with $x \in W$ constitute a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathscr{H}$ giving rise to a $\mathbb{Z}$-basis after specialisation.

(b) In the other extremal case, we have $\mathrm{IndP}(\mathfrak{g},\mathfrak{g}) = 1$. The isomorphism $\mathrm{End}_{\mathbb{Z}}(\mathscr{N}^{\mathfrak{p}}_{v=1}) \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$ implies $\mathrm{R}(\mathfrak{g},\mathfrak{g}) = 1$.

(c) Let $\mathfrak{g}$ be of type B_2 or G_2, with $\mathfrak{p}$ a maximal parabolic subalgebra. Then

$$10 = \#\,\mathrm{IndP}(\mathfrak{so}_3, \mathfrak{p}) > \mathrm{R}(\mathfrak{g}, \mathfrak{p}) = 6,$$
$$26 = \#\,\mathrm{IndP}(\mathfrak{g} \text{ of type } G_2, \mathfrak{p}) > \mathrm{R}(\mathfrak{g}, \mathfrak{p}) = 10.$$

Consider $\mathfrak{g} = \mathfrak{so}_3$. Let $W_{\mathfrak{p}} = \langle t\rangle \subseteq \langle s, t\rangle$; hence, $W^{\mathfrak{p}} = \langle e, s, st, sts\rangle$. By Theorem 3.9 below, θ_t is indecomposable (with $X = \{st\}$). For $\lambda \in \mathfrak{h}^*$ dominant and integral such that $W_\lambda = \{e, s\}$, the category $\mathscr{O}_\lambda^{\mathfrak{p}}$ is semisimple. (Note that $\theta_0^\lambda P^{\mathfrak{p}}(e) \cong M^{\mathfrak{p}}(\lambda)$ and $[\theta_0^\lambda P^{\mathfrak{p}}(st)] = [\theta_0^\lambda(M^{\mathfrak{p}}(st) \oplus M^{\mathfrak{p}}(sts))] = [M^{\mathfrak{p}}(st\cdot\lambda) \oplus M^{\mathfrak{p}}(st\cdot\lambda)]$. Therefore, $P^{\mathfrak{p}}(x\cdot\lambda) = M^{\mathfrak{p}}(x\cdot\lambda) = L(x\cdot\lambda)$ for $x \in \{e, st\}$.)

We define (exact) endofunctors G_e, G_{st} on $\mathscr{O}_\lambda^{\mathfrak{p}}$ by

$$G_w L(x\cdot\lambda) = \begin{cases} L(x\cdot\lambda) & \text{if } w = x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

In particular, $\mathrm{ID} \cong G_e \oplus G_{st}$. Set $\mathscr{G}_w = \theta_\lambda^0 G_w \theta_0^\lambda$. Then $\theta_s \cong \mathscr{G}_e \oplus \mathscr{G}_{st}$ and hence is decomposable. If $x \notin \{e, s\}$, then $G_e\theta_0^\lambda M^{\mathfrak{p}}(x) = 0$. Otherwise,

$$\begin{aligned}
[G_e\theta_0^\lambda M^{\mathfrak{p}}(x)] &= [G_e M^{\mathfrak{p}}(\lambda)] = [M^{\mathfrak{p}}(\lambda)] = \big[G_e(M^{\mathfrak{p}}(\lambda) \oplus M^{\mathfrak{p}}(st\cdot\lambda))\big] \\
&= \big[G_e\theta_0^\lambda\big(M^{\mathfrak{p}}(s) \oplus M^{\mathfrak{p}}(st)\big)\big] = \big[G_e\theta_0^\lambda\theta_t\big(M^{\mathfrak{p}}(e) \oplus M^{\mathfrak{p}}(s)\big)\big] \\
&= [G_e\theta_0^\lambda\theta_t\mathscr{G}_e M^{\mathfrak{p}}(x)].
\end{aligned}$$

By the semisimplicity of $\mathscr{O}_\lambda^{\mathfrak{p}}$, we get $G_e\theta_0^\lambda\theta_t\mathscr{G}_e \cong G_e\theta_0^\lambda$; hence, $\mathscr{G}_e\theta_t\mathscr{G}_e \cong \mathscr{G}_e$. Analogously, $\mathscr{G}_{st}\theta_t\mathscr{G}_{st} \cong \mathscr{G}_{st}$. One can easily check that for $w, z \in \{e, st\}$ with $w \neq z$, the functors

$$\mathrm{ID}, \theta_t, \mathscr{G}_w, \mathscr{G}_w\theta_t, \theta_t\mathscr{G}_w, \theta_w\theta_t\mathscr{G}_z$$

induce pairwise distinct morphisms on the Grothendieck group $[\mathscr{O}_0^{\mathfrak{p}}]$. The criterion of Theorem 3.9 shows that the functors are all indecomposable. Hence (by Theorem 3.1 and Corollary 3.5), they represent $\mathrm{Ind}(\mathfrak{g}, \mathfrak{p})$.

We remark that the induced representation on the Grothendieck group is isomorphic to the one obtained by taking by $\mathfrak{g} = \mathfrak{sl}_4$, where $W_{\mathfrak{p}}$ is generated by noncommuting simple reflections s_2, s_3.

The case G_2 can be done in an analogous way.

3.3. Restriction to the simple case

To get a description of indecomposable projective functors, it is enough to consider simple Lie algebras because of the following.

PROPOSITION 3.8

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two semisimple complex Lie algebras with parabolic subalgebras $\mathfrak{p}_i$

containing the fixed Borel subalgebra $\mathfrak{b}_i \subset \mathfrak{g}_i$ *for* $i = 1, 2$. *Then*

$$\#\,\mathrm{IndP}(\mathfrak{g}_1 \times \mathfrak{g}_2, \mathfrak{p}_1 \times \mathfrak{p}_2) = \#\,\mathrm{IndP}(\mathfrak{g}_1, \mathfrak{p}_1) \cdot \#\,\mathrm{IndP}(\mathfrak{g}_2, \mathfrak{p}_2).$$

Proof

There is a triangular decomposition $\mathfrak{g}_1 \times \mathfrak{g}_2 = (\mathfrak{n}_1^- \times \mathfrak{n}_2^-) \oplus (\mathfrak{h}_1 \times \mathfrak{h}_2) \oplus (\mathfrak{n}_1 \times \mathfrak{n}_2)$ arising from the corresponding triangular decompositions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. The identification $\mathscr{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) = \mathscr{U}(\mathfrak{g}_1) \boxtimes \mathscr{U}(\mathfrak{g}_2)$ induces $\mathscr{Z}(\mathscr{U}(\mathfrak{g}_1 \times \mathfrak{g}_2)) = \mathscr{Z}(\mathscr{U}(\mathfrak{g}_1)) \boxtimes \mathscr{Z}(\mathscr{U}(\mathfrak{g}_2))$ (where $\boxtimes$ denotes the outer tensor product over $\mathbb{C}$). This corresponds to an identification $\mathfrak{h}_1^* \times \mathfrak{h}_2^* = (\mathfrak{h}_1 \times \mathfrak{h}_2)^*$ and an isomorphism between the Weyl group of $\mathfrak{g}_1 \times \mathfrak{g}_2$ and the product $W_1 \times W_2$ of the single Weyl groups. Then the outer tensor product defines a functor

$$\boxtimes : \mathscr{O}_0(\mathfrak{g}_1) \times \mathscr{O}_0(\mathfrak{g}_2) \longrightarrow \mathscr{O}_{(0,0)}(\mathfrak{g}_1 \times \mathfrak{g}_2).$$

The simple objects in $\mathscr{O}_{(0,0)}(\mathfrak{g}_1 \times \mathfrak{g}_2)$ are given as tensor products of simple objects in formulas $L((x, y)) \cong L(x) \boxtimes L(y)$ for $(x, y) \in W_1 \times W_2$ with projective cover

$$P\big((x, y)\big) \cong P(x) \boxtimes P(y) \tag{3.4}$$

(for more details concerning this, see [B2, Section 2]). On the other hand, the outer tensor product defines a map between the sets of projective functors

$$\boxtimes : \mathscr{P}(\mathfrak{g}_1) \times \mathscr{P}(\mathfrak{g}_1) \longrightarrow \mathscr{P}(\mathfrak{g}_1 \times \mathfrak{g}_2).$$

The isomorphisms (3.4) together with the classification theorem (Theorem 3.1) imply that the map is in fact a bijection. This proves the proposition for $\mathfrak{p}_i = \mathfrak{b}_i$, $i = 1, 2$.

On the other hand, for any $F, G \in \mathscr{P}$, there is an isomorphism of rings

$$\Gamma : \mathrm{End}(F) \boxtimes \mathrm{End}(G) \longrightarrow \mathrm{End}(F \boxtimes G) \tag{3.5}$$

given by $\Gamma(\phi \otimes \psi)_{(P,Q)}(p, q) = \phi_P(p) \otimes \psi_Q(q)$ on projective generators $P \in \mathscr{O}_0(\mathfrak{g}_1)$, $Q \in \mathscr{O}_0(\mathfrak{g}_2)$, and $p \in P$, $q \in Q$. (It is not difficult to see that this in fact defines a homomorphism of functors. For the bijectivity, see, e.g., [B2, Lemma 2.1].)

The isomorphism (3.5) induces an isomorphism

$$\Gamma^{\mathfrak{p}} : \mathrm{End}(F_{|\mathscr{O}_0^{\mathfrak{p}_1}}) \boxtimes \mathrm{End}(G_{|\mathscr{O}_0^{\mathfrak{p}_2}}) \longrightarrow \mathrm{End}(F \boxtimes G_{|\mathscr{O}_0(\mathfrak{p}_1 \times \mathfrak{p}_2)}).$$

Note that $\Gamma^{\mathfrak{p}}(\phi \otimes \psi)$ is an idempotent if and only if ϕ, ψ are idempotents. Moreover, $\Gamma^{\mathfrak{p}}(\phi \otimes \psi)$ is a trivial idempotent if and only if ϕ and ψ are also. Hence, pairs of indecomposable projective functors correspond to indecomposable projective functors. This proves the proposition. □

3.4. A criterion for indecomposability

For $F \in \mathscr{P}^{\mathfrak{p}}$ and $X \subseteq W^{\mathfrak{p}}$, we consider the sets of *simple objects* in $\mathscr{O}_0^{\mathfrak{p}}$,

$$\begin{aligned} \operatorname{supp}(F) &= \big\{L \,\big|\, F(L) \neq 0\big\}, \\ \operatorname{Comp}(X) &= \big\{L \,\big|\, [M^{\mathfrak{p}}(x) : L] \neq 0 \text{ for some } x \in X\big\}. \end{aligned}$$

We get the following criterion for a projective functor to be indecomposable.

THEOREM 3.9

Let $0 \neq F \in \mathscr{P}^{\mathfrak{p}}$. Assume that there exists $X \subseteq W$ with $\operatorname{supp}(F) \subseteq \operatorname{Comp}(X)$ such that

(a) *$F(M^{\mathfrak{p}}(x))$ is indecomposable for any $x \in X$;*

(b) *for any nontrivial decomposition $X = X_1 \cup X_2$, we have*

$$\operatorname{Comp}(X_1) \cap \operatorname{Comp}(X_2) \cap \operatorname{supp}(F) \neq \emptyset.$$

Then F is indecomposable.

Remark 3.10

• The functor $\mathrm{ID} \in \mathscr{P}^{\mathfrak{p}}$ is indecomposable; therefore, $\operatorname{IndP}(\mathfrak{g}, \mathfrak{g}) = \{\mathrm{ID}\}$.

• Let $\mathfrak{p} = \mathfrak{b}$. The theorem gives an alternative proof of the fact that the indecomposability of $F(M(e))$ implies the indecomposability of F. Let $X = \{e\}$. Since every simple module occurs as a composition factor in $M(e)$, the assumptions are satisfied if and only if $F(M(e))$ is indecomposable.

On the other hand, we could also choose $X = W$. Note that $F(M(e))$ is indecomposable if and only if $F(M(x))$ is decomposable for all $x \in W$ by [AS, Theorem 2.2, Corollary 4.2]. Since $F(L(w_o)) \neq 0$ and $L(w_o)$ occurs in the socle of each Verma module, assumption (2) is satisfied.

• Consider the situation $\mathfrak{g} = \mathfrak{so}_3$ of Example 3.7 for $F = \theta_s$. Condition (a) is always satisfied. Let us assume the existence of a set X as in the theorem. For $i \in \{1, 2\}$, set $X_i = X \cap A_i$, where $A_1 = \{e, s\}$ and $A_2 = \{st, sts\}$. Hence, condition (b) is not satisfied. It turns out that θ_s is indeed decomposable.

Proof of Theorem 3.9

We assume the existence of X. Let $\pi \in \operatorname{End}(F)$ be an idempotent. By assumption (a), it is $\pi_{M^{\mathfrak{p}}(x)} \in \{\mathrm{Id}, 0\}$ for all $x \in X$. Choose $x_1 \in X$ such that $FM^{\mathfrak{p}}(x_1) \neq 0$. Set $X_1 = \{x_1\}$ and $X_2 = X \backslash X_1$. Let L be an element of the intersection given in (b) occurring in, say, $\operatorname{Comp}(\{x_2\})$, $x_2 \in X_2$. If $\pi_{M^{\mathfrak{p}}(x_1)} = \mathrm{Id}$, then $\pi_L = \mathrm{Id}$, and therefore $\pi_{M^{\mathfrak{p}}(x_2)} = \mathrm{Id}$. Going on with $X_1 := \{x_1, x_2\}$, and so on, in the same way finally gives $\pi_{M^{\mathfrak{p}}(x)} = \mathrm{Id}$ for any $x \in X$. The same arguments work if $\pi_{M^{\mathfrak{p}}(x_1)} = 0$. Hence, π is either the identity or zero on all simple objects simultaneously.

We first consider the case where π is the identity on simple objects. We prove that $\pi_M = \mathrm{Id}$ for any M by induction on its length. Let $M_1 \hookrightarrow M \twoheadrightarrow M_2$ be a short exact sequence. Then $F(M_1) \overset{i}{\hookrightarrow} F(M) \overset{p}{\twoheadrightarrow} F(M_2)$ is exact. Let $x \in F(M)$. If $x = i(y)$ for some $y \in F(M_1)$, then $\pi_M(x) = \pi_M(i(y)) = i(y) = x$. Otherwise, $0 \neq p(x) = \pi_{M_2}(p(x)) = p(\pi_M(x))$. Hence, $x - \pi_M(x) = i(y)$ for some $y \in F(M_1)$. Since π is idempotent, we have $0 = \pi_M(\pi_M(x) - x) = \pi_M(i(y)) = i(y)$. Therefore, $y = 0$ and $\pi_M = \mathrm{Id}$. Now let $\pi_{M_i} = 0$ for $i = 1, 2$. For $x \in F(M)$, we get $p(\pi_M(x)) = \pi_{M_2}(p(x)) = 0$; hence, $\pi_M(\pi_M(x)) = \pi_M(i(y)) = i(\pi_{M_1}(y)) = 0$ for some $y \in M_1$. The theorem follows. □

3.5. Lifts of projective functors

Let F be an exact endofunctor on $\mathcal{O}_0^{\mathfrak{p}}$. We call $\tilde{F} : \mathrm{gmof}-A^{\mathfrak{p}} \longrightarrow \mathrm{gmof}-A^{\mathfrak{p}}$ a *graded lift* of F if $\tilde{F}$ is a $\mathbb{Z}$-functor (as defined in [AJS, Section E3]) and induces F after forgetting the grading and applying the equivalence $\mathrm{mof}-A^{\mathfrak{p}} \to \mathcal{O}_0^{\mathfrak{p}}$ (for details, see [St]). If such a lift exists, we call F *gradable*.

PROPOSITION 3.11

Let $F \in \mathcal{P}^{\mathfrak{p}}$ be indecomposable. A lift $\tilde{F}$ of F (if it exists) is unique up to isomorphism and grading shift.

Proof

Under the equivalence $\mathcal{O}_0^{\mathfrak{p}} \cong \mathrm{gmof}-A^{\mathfrak{p}}$, the functor F corresponds to $\bullet \otimes_{A^{\mathfrak{p}}} X$ for some $A^{\mathfrak{p}}$-bimodule X (see [Ba]). Moreover, F is indecomposable if and only if X is also (as an $A^{\mathfrak{p}}$-bimodule). A graded lift $\tilde{F}$ of F is therefore given as tensoring with some graded $A^{\mathfrak{p}}$-bimodule $\tilde{X}$ such that $\tilde{X} \cong X$ after forgetting the grading. By the indecomposability of X, a lift is unique up to isomorphism and grading shift (use [St, Lemma 1.5] for the graded ring $A^{\mathfrak{p}} \otimes (A^{\mathfrak{p}})^{\mathrm{opp}}$). □

COROLLARY 3.12

Let $F \in \mathcal{P}$ be indecomposable. Then F is gradable. A lift of F is unique up to isomorphism and grading shift.

Proof

The translation functors through a wall are gradable (see [St]); hence, so are their compositions. Theorem 3.1 shows that there is a decomposition of functors

$$\theta_{s_r}\theta_{s_{r-1}} \cdot \ldots \cdot \theta_{s_1} \cong F_x \oplus \bigoplus_{y<x} F_y^{\alpha_y} \tag{3.6}$$

for some $\alpha_y \in \mathbb{N}$ and $x = s_1 \cdot \ldots \cdot s_r$ a reduced expression of x. By the induction hypothesis, the F_y's are gradable for $y < x$. (Note that $F_e = \mathrm{ID}$ is gradable.) Therefore,

F_x is gradable (see [St, Lemma 1.4]). The statement on the uniqueness is Proposition 3.11. □

We fix a lift $\tilde{F}_w$ of F_w such that $\tilde{F}_w \tilde{M}(e) \cong \tilde{P}(w)$.

Remark 3.13

Let $[w] = s_1 s_2 \cdot \ldots \cdot s_r$ be a reduced expression of $w \in W$. With the conventions on the lift $\tilde{F}_w$ and Corollary 2.5, we get

$$\tilde{\theta}_{s_r} \tilde{\theta}_{s_{r-1}} \cdot \ldots \cdot \tilde{\theta}_{s_1} \cong \bigoplus_{y \in W} (\tilde{F}_y \langle i \rangle)^{\alpha_{[w],y,i}},$$

where the $\alpha_{[w],y,i}$ are defined as $C_{s_1} C_{s_2} \cdot \ldots \cdot C_{s_r} = \sum_{y \in W, i \in \mathbb{Z}} \alpha_{w,y,i}\, v^i \underline{H}_y$. Note that $\alpha_{w,y,i}$ does not depend on the reduced expression of w, provided w is braid-avoiding.

4. Generalised Temperley-Lieb algebras

In this section we describe “functorial realisations” of generalised Temperley-Lieb (TL) algebras. Let W be a Weyl group of type A, B, C, or D with corresponding Hecke algebra $\mathscr{H}$. We consider the (generalised) Temperley-Lieb algebra $\mathscr{H}/TL$; that is, TL is generated by all $\sum_{w \in W_{\text{adj}}} v^{-l(w)} H_w$ such that $W_{\text{adj}} \subset W$ is generated by two simple reflections, where the corresponding vertices in the Dynkin diagram are connected. These algebras were introduced by Temperley and Lieb [TL] for type A and by Dieck [Di] for other types. Alternatively, they can be defined by the following relations (with $s, t \in \mathscr{S}$):

$$C_s^2 = (v + v^{-1}) C_s, \tag{4.1}$$

$$C_s C_t = C_t C_s \quad \text{if } ts = st. \tag{4.2}$$

Additionally, for types A, B, C, and D,

$$C_s C_t C_s = C_s \quad \text{if } ts \neq st \text{ and } sts = tst, \tag{4.3}$$

and for types B and C,

$$C_t C_s C_t C_s = C_t C_s + C_t C_s \quad \text{if } ts \neq st \text{ and } sts \neq tst. \tag{4.4}$$

THEOREM 4.1 (TL algebras and projective functors)

Let $\mathfrak{g}$ be a simple Lie algebra of type A, B, C, or D. Let $\mathfrak{p} \subset \mathfrak{g}$ be maximal parabolic. With the interpretation of v^i as grading shift $\langle i \rangle$, the graded translation functors satisfy the relations (4.1)–(4.4).

Proof

Let s, t be commuting simple reflections. By Theorem 3.1, $\theta_s \theta_t \cong \theta_t \theta_s$. The functors are indecomposable, and therefore $\tilde\theta_s\tilde\theta_t \cong \tilde\theta_t\tilde\theta_s\langle i\rangle$ for some $i \in \mathbb{Z}$ (Corollary 3.12). Since $\tilde M(e)\langle 2\rangle$ occurs in both $\tilde\theta_s\tilde\theta_t\tilde M(e)$ and $\tilde\theta_t\tilde\theta_s\tilde M(e)$ as a submodule (Theorem 2.4), it follows that $i = 0$. Therefore, relation (4.2) is satisfied. Since there is an isomorphism $\theta_s^2 \cong \theta_s \oplus \theta_s$, we get $\tilde\theta_s^2 \cong \theta_s\langle i\rangle \oplus \tilde\theta_s\langle j\rangle$ for some i, j (again using Corollary 3.12). On the other hand, Corollary 2.5 shows $[\tilde\theta_s^2\tilde M(e)] = [\tilde\theta_s\tilde M(e)\langle 1\rangle] + [\tilde\theta_s\tilde M(e)\langle -1\rangle]$. Relation (4.1) is satisfied. Now let $st \neq ts$, but let $sts = tst$. We just recall the arguments of [BFK]. We have $\theta_s\theta_t\theta_s \cong F_{sts} \oplus \theta_s$ by Theorem 3.1. Let $\eta \in \mathfrak{h}^*$ be dominant and integral such that $W_\eta = \langle s, t\rangle$. Since $F_{sts}M(e) \cong P(sts) \cong \theta_\eta^0\theta_0^\eta M(e)$ (see, e.g., [J2, Formula 14.13(1)] and [S2, Proposition 2.9]), we get $F_{sts} \cong \theta_\eta^0\theta_0^\eta$ by Theorem 3.1. In particular, $F_{sts} = 0$ when restricted to $\mathscr{O}_0^{\mathfrak{p}}$ (for $\mathfrak{p}$ maximal parabolic!). In the graded picture we have $\tilde\theta_s\tilde\theta_t\tilde\theta_s \cong \tilde F_{sts}\langle i\rangle \oplus \tilde\theta_s\langle j\rangle$ for some i, $j \in \mathbb{Z}$. Since we did not prove Remark 3.13, we determine i and j directly. By Theorem 2.4, $\tilde\theta_s\tilde\theta_t\tilde\theta_s\tilde M(e)$ surjects onto $\tilde M(sts)$, and therefore, $i = 0$. Corollary 2.5 shows that $\tilde M(e)\langle k\rangle$ occurs as a submodule in $\tilde\theta_s\tilde\theta_t\tilde\theta_s\tilde M(e)$ for $k = 3, 1$. By Theorem 2.4, $\tilde M(e)\langle 3\rangle$ is a submodule of $\tilde\theta_t\tilde\theta_s\tilde M(e)\langle 1\rangle$. The latter is contained in $\tilde\theta_s\tilde\theta_t\tilde\theta_s\tilde M(e) \cong \tilde P(sts)\oplus\tilde P(s)\langle j\rangle$; hence, it must be a submodule of $\tilde P(sts)$. Therefore, $\tilde M(e)\langle 1\rangle$ is a submodule of $\tilde\theta_s\tilde M(e)\langle j\rangle$. Theorem 2.4 implies $j = 0$. Formula (4.3) follows.

If $st \neq ts$ and $sts \neq tst$, then $\theta_s\theta_t\theta_s\theta_t \cong F_{tsts} \oplus \theta_{ts} \oplus \theta_{ts}$ by Theorem 3.1. The same arguments as above show that $F_{tsts} = 0$ when restricted to $\mathscr{O}_0^{\mathfrak{p}}$ and that $\tilde\theta_t\tilde\theta_s\tilde\theta_t\tilde\theta_s \cong \tilde F_{stst} \oplus \tilde\theta_{ts}\langle i\rangle \oplus \theta_{ts}\langle j\rangle$ for some i, $j \in \mathbb{Z}$. Corollary 2.5 implies that $\tilde M(e)\langle 4\rangle \oplus \tilde M(e)\langle 2\rangle \oplus \tilde M(e)\langle 2\rangle$ occurs as a submodule in $\tilde\theta_t\tilde\theta_s\tilde\theta_t\tilde\theta_s\tilde M(e)$. Since $\tilde P(sts)\langle 1\rangle$ is a submodule of $\tilde\theta_t P(sts)$ (hence of $\tilde P(stst)$), the module $\tilde M(e)\langle 4\rangle$ is a submodule of $P(tsts)$. Theorem 2.4 shows that $\tilde M(e)\langle 2\rangle$ is a submodule of $\tilde\theta_t\tilde\theta_s\tilde M(e)$. We get $i = j = 0$. The theorem follows. □

For W of type A and $N \in \mathbb{N}_{>0}$, the N-generalised Temperley-Lieb algebra $\mathscr{H}_N$ is defined as $\mathscr{H}/I_N$, where I_N is the $\mathbb{Z}[v, v^{-1}]$-span of all (Kazhdan-Lusztig) basis elements indexed by tableaux with more than N columns (see [H], [Li], [BK]). In particular, $\mathscr{H}_1 \cong \mathbb{Z}[v, v^{-1}]$ and $\mathscr{H}_2$ is the ordinary Temperley-Lieb algebra.

PROPOSITION 4.2

Let $n > 1$. Let $\mathfrak{p} = \mathfrak{p}_S \subset \mathfrak{sl}_{n+1}$ be a parabolic subalgebra and $N \geq n - |S| + 1$. Then the corresponding $\Phi^{\mathfrak{p}}$ from (3.1) factors through $\mathscr{H}_N$.

Proof

This is just a reformulation of, say, [H, Section 3], [M], or [BK, Theorem 3.1]. □

Remark 4.3

- Via (3.1), Proposition 4.2 provides an *injection*
$$\mathscr{H}_N \to \operatorname{End}_{\mathbb{Z}[v,v^{-1}]}\Big(\bigoplus \mathcal{N}^{\mathfrak{p}_S}\Big),$$
where the sum runs over all $S \subset \pi$ satisfying $|S| \geq n - N + 1$ (see [M] or [H]).
- The description of I_N in terms of Kazhdan-Lusztig basis elements (see [H], [Li]) provides many $F \in \mathscr{P}$ which become zero after restricting to $\mathscr{O}_0^{\mathfrak{p}}$. In particular, the case $I_2 = (\{\underline{H}_{sts}), st \neq ts\})$ and the relation $C_sC_tC_s = \underline{H}_{sts} + C_s$ imply Theorem 4.1 for type A.
- It is not clear whether the $\mathscr{H}_N$ have a diagrammatical/topological interpretation. However, the generalised Temperley-Lieb algebras of types B and D are known to have a description via *decorated* tangles (see, e.g., [G]). This might be of topological interest.

5. Type A: Maximal parabolic subalgebras

In general, an indecomposable projective functor does not stay indecomposable when restricted to parabolic subcategories (see Example 3.7(c)). However, this section is devoted to a proof of the following result (conjectured in [BFK]).

THEOREM 5.1 (Indecomposability)
Let $n > 1$. Let $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $F \in \mathscr{P}$ be indecomposable. Then its restriction to $\mathscr{O}_0^{\mathfrak{p}}$ is indecomposable or zero.

For $w \in W$ with a reduced expression $[w] = s_{i_1}s_{i_2} \cdot \ldots \cdot s_{i_r}$, let $\theta_{[w]} = \theta_{s_{i_r}}\theta_{s_{i_{r-1}}} \cdot \ldots \cdot \theta_{s_{i_1}}$. If $\mathfrak{g} = \mathfrak{sl}_n$, then $w \in W = S_n$ is *braid-avoiding* if some (resp., any) reduced expression does not contain a substring of the form sts with noncommuting simple reflections s and t. In this case, $\theta_{[w]} \in \mathscr{P}$ is indecomposable (see [BW, Theorem1]) and hence isomorphic to F_w. In particular, it is independent of the chosen reduced expression.

In the following we study the case $\mathfrak{g} = \mathfrak{sl}_n$ with corresponding category $\mathscr{O}(\mathfrak{sl}_n)$. We always consider the Weyl group of $\mathfrak{sl}_n$ as generated by $s_i = s_{\alpha_i}$, $1 \leq i \leq n$, such that $s_is_j = s_js_i$ if and only if $|i-j| > 1$. To simplify notation, set $\mathscr{O}_i(\mathfrak{sl}_n) = \mathscr{O}(\mathfrak{sl}_n)_\lambda$, where $\lambda \in \mathfrak{h}^*$ is dominant and integral such that $W_\lambda = \{e, s_i\}$. For $1 \leq k \leq n$, let $S_k = \pi\backslash\{\alpha_k\}$, and set $S_0 = S_{n+1} = \pi$. We denote by $\mathscr{O}^k(\mathfrak{sl}_n)$ the main block of the corresponding parabolic category $\mathscr{O}^{S_k}$. To make formulas easier, $\mathscr{O}^k(\mathfrak{sl}_n)$ denotes the zero category if $k < 0$ or $k > n+1$. We also use the notation $\mathscr{O}_i^k(\mathfrak{sl}_n)$ for the full subcategory of $\mathscr{O}_i(\mathfrak{sl}_n)$ defined by all locally $\mathfrak{p}_{S_k}$-finite modules. Let $\theta_0^i : \mathscr{O}_0(\mathfrak{sl}_n) \longrightarrow \mathscr{O}_i(\mathfrak{sl}_n)$ (resp., $\theta_i^0 : \mathscr{O}^i(\mathfrak{sl}_n) \longrightarrow \mathscr{O}_0(\mathfrak{sl}_n)$) denote the translation onto/out of the ith wall, and let $\theta_i = \theta_{s_i}$ denote the translation through the ith wall.

The following observation simplifies the proof of Theorem 5.1.

LEMMA 5.2
Let $n > 1$, and let $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $F = F_w \in \mathscr{P}$ be indecomposable with w not braid-avoiding. Restriction to $\mathscr{O}_0^{\mathfrak{p}}$ gives $F_w = 0$.

Proof
Let $w = b(sts)a$ be minimal with simple reflections s, t. Then $F_w \in \mathscr{P}$ is a direct summand of $\theta_{[a]}\theta_{[sts]}\theta_{[b]} \cong \theta_{[a]} F_{sts}\theta_{[b]} \oplus \theta_{[a]}\theta_s\theta_{[b]}$. If F_w occurs in the first summand, then $F_w = 0$ when restricted to $\mathscr{O}_0^{\mathfrak{p}}$ since F_{sts} becomes zero after restriction (see the proof of Theorem 4.1). On the other hand, by construction, it cannot occur in the second summand (see (3.6)) because $l(asb) \leq l(w) - 2$. The lemma follows. □

The following lemma describes $\mathrm{Supp}(F)$ for certain $F \in \mathscr{P}^{\mathfrak{p}}$.

LEMMA 5.3
Let $n > 1$, and let $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $x \in W^{\mathfrak{p}}$ and $w \in W$ with reduced expression $[w] = s_{i_1} s_{i_2} \cdot \ldots \cdot s_{i_r}$.

(1) *If $i_{j+1} = i_j + 1$ for all $1 \leq j < r$ or if $i_{j+1} = i_j - 1$ for all $1 \leq j < r$, then*

$$\theta_{[w]} L(x) \neq 0 \iff \theta_{s_{i_1}} L(x) \neq 0 \iff x > x s_{i_1}.$$

(2) *If $s_{i_j} s_{i_k} = s_{i_k} s_{i_j}$ for all $1 \leq j, k \leq r$, then*

$$\theta_{[w]} L(x) \neq 0 \iff (x s_{i_j} < x \text{ for } 1 \leq j \leq r).$$

Proof
(1) Let $F = \theta_{i_{r-1}} \theta_{s_{i_{r-2}}} \cdot \ldots \cdot \theta_{s_{i_1}}$. The definition of $\theta_{[w]}$ and Theorem 4.1 give

$$0 \neq \theta_{[w]} L(x) \Rightarrow 0 \neq F L(x) = \theta_{i_{r-1}} \theta_{i_r} F L(x) \Rightarrow \theta_{[w]} L(x) \neq 0.$$

Inductively, the first equivalence follows. The second is well known (see [J2, Formulas 4.12(3), 4.13(3)]).

(2) We have already verified the implication from left to right. If x satisfies the condition on the right-hand side, it is $[\theta_{[w]} L(x)] = [\theta_{s_{i_r}} \cdot \ldots \cdot \theta_{s_{i_2}} L(x)] + [\theta_{s_{i_r}} \cdot \ldots \cdot \theta_{s_{i_2}} M]$ for some module M (see [J2, Formulas 4.12(3), 4.13(3′)]. By the induction hypothesis, the first summand is nontrivial, and the statement follows. □

PROPOSITION 5.4
Let $n > 1$. Let $\mathfrak{p}_m \subset \mathfrak{sl}_n$ be a maximal parabolic subalgebra ($1 \leq m \leq n$). Let $w \in W$ be of the form described in Lemma 5.3(2). The following hold.

(a) *For any $x \in W^{\mathfrak{p}}$, $F_w(M^{\mathfrak{p}}(x))$ is indecomposable or zero.*
(b) *The restriction $F_w = \theta_{[w]}$ to $\mathcal{O}_0^{\mathfrak{p}}$ is indecomposable.*

Proof
(a) If $xs_{i_j} \notin W^{\mathfrak{p}}$ for some $j \in \{1, 2, \ldots, r\}$, then $F_w M^{\mathfrak{p}}(x) = 0$. By Proposition 1.5, we may assume $xs_{i_j} > x$ and $xs_{i_j} \in W^{\mathfrak{p}}$ for $1 \leq j \leq r$. In particular, xs_{i_j} is braid-avoiding (see Proposition A.2). Since all the s_{i_j} are pairwise commuting, the expression xw is minimal and braid-avoiding as well (see [BW, Lateral Convexity]). Hence, $F_w M^{\mathfrak{p}}(x)$ is a homomorphic image of $F_w P(x) \cong \theta_{[w]}\theta_{[x]}M(e) \cong \theta_{[xw]}M(e) \cong P(xw)$ (we use [BW, Theorem 1] and Theorem 3.1). In particular, $F_w M^{\mathfrak{p}}(x)$ is indecomposable.

(b) It remains to check that $X = \{x \in W^{\mathfrak{p}} \mid L(x) \in \operatorname{Supp}(F_w)\}$ satisfies property (2) of Theorem 3.9. Assume that there is a decomposition $X = X_1 \cup X_2$ such that $\operatorname{Comp}(X_1) \cap \operatorname{Comp}(X_2) \cap X = \emptyset$. We first consider the special case $r = 1$. Let $i_1 = i$; hence, $X = \{x \in W^{\mathfrak{p}}, xs_i < x\}$. With the notation of Proposition A.2, the elements of X are exactly those containing i but not $i - 1$. Let $x = k_1 \triangleright k_2 \triangleright \cdots \triangleright k_m \in X$ with $k_j = i$. We consider $x_o^j = (i + j - 1) \triangleright (i + j - 2) \triangleright \cdots \triangleright (i + 1) \triangleright i$. It is not difficult to see that there exists a chain $x_o^j, x_1^j, \ldots, x_p^j = x$, where $x_l^j \in X$ and $l(x_{l+1}^j) = l(x_l^j) + 1$ for $0 \leq l < p$. Therefore, $x_o^j \in X_a \Rightarrow x \in X_a$ for $a = 1, 2$.

We choose j minimal such that $x_o^j \neq e$. Let $x_o^j \in X_1$, say. We show that $X \subset X_1$ and hence that $X_2 = \emptyset$. If $i + j - 1 = n$, then j is also maximal such that $x_o^j \neq e$ and we are done. Otherwise, let $y = (i + j) \triangleright (i + j - 1) \triangleright \cdots \triangleright (i + 2) \triangleright i$; that is, let $y \in X_1$ and $x_o^{j+1} = ys_{i+1}s_i$. On the other hand, we have

$$[\theta_i\theta_{i+1}M^{\mathfrak{p}}(y)] = [M^{\mathfrak{p}}(x_o^{j+1})] + [M^{\mathfrak{p}}(ys_{i+1})] + [M^{\mathfrak{p}}(y)] + [M(ys_i)].$$

In particular,

$$0 \neq [\theta_{i+i}\theta_i P^{\mathfrak{p}}(y) \ : \ M^{\mathfrak{p}}(y)] = [P^{\mathfrak{p}}(x_o^{j+1}) \ : \ M^{\mathfrak{p}}(y)] = [M^{\mathfrak{p}}(y) \ : \ L(x_o^{j+1})].$$

(Note that the first equality uses the fact that x_o^{j+1} is braid-avoiding and [BW, Theorem 1].) Therefore, $x_o^{j+1} \in X_1$ because $y \in X_1$. Inductively, all x_o^j are contained in X_1. Hence $X \subseteq X_1$. This means that the assumptions of Theorem 3.9 are satisfied, and $F_w = \theta_i$ is indecomposable.

Let us now consider the general case. By Lemma 5.3, $L(x) \in X$ if and only if $xs_k < x$ for all simple reflections s_k occurring in a reduced expression of w. In the notation of Proposition A.2, the expression for x contains all such k but none of the $k - 1$. Arguments similar to those above show that a nontrivial decomposition $X = X_1 \cup X_2$ such that $\operatorname{Comp}(X_1) \cap \operatorname{Comb}(X_2) \cap X = \emptyset$ does not exist.

We omit most details, but we do not omit them completely. We assume $i_j > i_{j'}$

if $j < j'$. For J a sequence of numbers $n \geq j_1 > j_2 > \cdots > j_k \geq m$, let

$$X^J = \{x = x_1 \rhd x_2 \rhd \cdots \rhd x_m \in X \,|\, x_{j_k} = i_k, 1 \leq k \leq r\}$$

be the elements of X, where the important numbers occur exactly at the places given by J. Again, it is easy to see from Proposition A.2 that X^J is a finite set of the form $x_0^J < x_1^J < \cdots < x_{|X^J|}^J$, so that $l(x_{b+1}^J) = l(x_b^J) + 1$ for $0 \leq b < |X^J|$. This implies, in particular, $x_0^J \in X_1 \iff x_b^J \in X_1$ for any b. We fix now some J such that $X^J \neq \emptyset$. Assume $j_{l+1} \neq j_l + 1$ for some l. Without loss of generality let $x_{j_l+1} = i_l - 2$. We consider the following two cases.

(I) Assume that there exists $x \in X^J$ of the form $x = x_1 \rhd \cdots \rhd x_m$ such that $x_{j_l-1} > i_l + 1$. Then $y = xs_{i_l+1}s_{i_l-1}s_{i_l} \in X^{J'}$, where $j_l' = j_l + 1$, and $j_i' = j_i$ otherwise. On the other hand, $[\theta_k M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(x)] \neq 0$ for $k = i_{l+1}, i_{l-1}, i_l$. Hence, $[\theta_{i_l}\theta_{i_{l-1}}\theta_{i_l+1}M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(x)] \neq 0$. Since x, y are braid-avoiding, we get (see [BW]) $P^{\mathfrak{p}}(y) \cong \theta_{i_r}\theta_{i_{r-1}}\theta_{i_r+1}P^{\mathfrak{p}}(x)$. In particular, $0 \neq [P^{\mathfrak{p}}(y) : M^{\mathfrak{p}}(x)] = [M^{\mathfrak{p}}(x) : L(y)]$. We have that if $x \in X^J$, $y \in X^{J'}$, then $x \in X_1 \iff y \in X_1$.

(II) Let z be minimal such that $x_i = x_{i+1} + 1$ for $z \leq i \leq j_l$. Let $x' = s_{j_z+1}s_{j_z} \cdot \ldots \cdot s_{j_l+2}$. Note that $xx' \in X^J$ and $y = xx's_{j_l+1}s_{j_l-1}s_{j_l} \in X^{J'}$, as above. Direct calculations give $[\theta_{[x']}M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(xx')] \neq 0$. Applying the earlier arguments to xx' gives $[P^{\mathfrak{p}}(y) : M^{\mathfrak{p}}(xx')] = [M^{\mathfrak{p}}(xx') : L(y)] \neq 0$.

Inductively, it follows that $X_i = \emptyset$ for some $i \in \{1, 2\}$. Therefore, the assumptions of Theorem 3.9 are satisfied. The proposition follows. □

LEMMA 5.5

Let $\mathfrak{g} = \mathfrak{sl}_n$, and let $e \neq w \in W$ be braid-avoiding. Then there exists a reduced expression $w = s_{i_1}s_{i_2} \cdot \ldots \cdot s_{i_r}$ such that (at least) one of the following properties is satisfied:

(i) $s_{i_j}s_{i_k} = s_{i_k}s_{i_j}$ *for* $1 \leq j, k \leq r$*;*

(ii) $s_{i_1}s_{i_2} \neq s_{i_2}s_{i_1}$*;*

(iii) $s_{i_r}s_{i_{r-1}} \neq s_{i_{r-1}}s_{i_r}$.

Proof

Write $w = d_1d_2 \cdots d_n$ minimal such that $d_1d_2 \cdots d_m \in \langle s_1, s_2, \ldots, s_m\rangle$ and $d_m \in \langle s_1, s_2, \ldots, s_{m-1}\rangle \backslash \langle s_1, s_2, \ldots, s_m\rangle$ is a distinguished coset representative of minimal length for any $m \in \{1, 2, \ldots, n\}$. By Proposition A.2, $d_m = s_m s_{m-1} \cdot \ldots \cdot s_{m-k}$ for some k or $d_m = e$. Pick (if it exists) $j \in \{1, 2, \ldots, n\}$ minimal such that d_j, $d_{j+1} \neq e$. By assumption, $d_{j-1} = e$, and hence, we get a minimal expression $w = ds_js_{j+1}w'd_{j+2} \cdots d_n$ for some $w' \in W$ and $d \in \langle s_1, s_2, \ldots, s_{j-2}\rangle$. Therefore, $w = s_js_{j+1}x$ for some $x \in W$, and w satisfies (2).

If j as above does not exist, we proceed by induction on the length of w. Without loss of generality, let $d_n \neq e$. If $l(d_n) > 1$, then obviously (iii) holds; otherwise, $d_n = s_n$ (and $d_{n-1} = e$), and therefore, $w = xs_n = s_n x$ for some $x \in \langle s_1, s_2, \ldots, s_{n-2} \rangle$. Certainly, x is braid-avoiding. The lemma follows from the induction hypothesis. □

Proof of Theorem 5.1

By Lemma 5.2, we may assume $w \in W$ to be braid-avoiding. If $w = e$ or if w satisfies the assumptions of Proposition 5.4, we are done. Otherwise, we prove the statement by induction on the length of w. Let us assume that w has a minimal expression of the form $w = w'ts$ with noncommuting $s, t \in \mathscr{S}$; in particular, $F_w \cong \theta_s \theta_t F_{w'}$. Let $F_w \cong G_1 \oplus G_2$ for some nontrivial G_i when restricted to $\mathscr{O}_0^{\mathfrak{p}}$. Considered as a functor on $\mathscr{O}_0^{\mathfrak{p}}$, we have $\theta_t F_w \cong \theta_t \theta_s \theta_t F_{w'} \cong \theta_t F_{w'} \cong F_{w't}$; hence, it is indecomposable by the induction hypothesis. This implies $\theta_t G_i = 0$ for $i = 1$, say. Note that $\theta_s F_w \cong F_w \oplus F_w$. We claim that

$$\theta_s G_1 \cong G_1 \oplus H \tag{5.1}$$

for some $H \in \mathscr{P}^{\mathfrak{p}}$ such that $\big[[H]\big] = \big[[G_1]\big]$. Let us believe this for a moment. Then $\theta_s \theta_t \theta_s G_1 \cong \theta_s G_1 \cong G_1 \oplus H \neq 0$. Hence, $\theta_t \theta_s G_1 \neq 0$. On the other hand, $\big[[\theta_t \theta_s G_1]\big] = \big[[\theta_t(G_1 \oplus G_1)]\big]$. Therefore, $\theta_t G_1 \neq 0$. This gives a contradiction. To prove (5.1), we fix an embedding $i : G_1 \to F_w$ together with a split j and consider the diagram

$$\begin{tikzcd}
G_1 \arrow[r, "i"'] \arrow[d, "\alpha"] & F_w \arrow[l, bend right, "j"'] \arrow[d, "\beta"'] \\
\theta_s G_1 \arrow[r, "\theta_s i"] & \theta_s F_w \arrow[l, bend left, "\theta_s j"] \arrow[u, bend right, "\beta'"']
\end{tikzcd}$$

The vertical maps α and β are the adjunction morphisms, so the inner diagram commutes. The isomorphism $\theta_s^2 \cong \theta_s \oplus \theta_s$ provides a split β' of β. The composition $\phi := j \circ \beta' \circ \theta_s i$ is then a split of α because $\phi \circ \alpha = j \circ \beta' \circ \theta_s i \circ \alpha = j \circ \beta' \circ \beta \circ i = j \circ i = \mathrm{id}$. This gives an isomorphism as in (5.1) for some $H \in \mathscr{P}^{\mathfrak{p}}$. Let $Q \in \mathscr{O}_0^{\mathfrak{p}}$ be projective; then $G_1(Q) \cong \bigoplus_{x \in W^{\mathfrak{p}}} P(x)^{\alpha_x}$ for some $\alpha_x \in \mathbb{N}$. Moreover, $\alpha_x = 0$ if $xs > x$. If $x, xs \in W^{\mathfrak{p}}$ such that $x > xs$, then $\theta_s P^{\mathfrak{p}}(x) \cong P^{\mathfrak{p}}(x) \oplus P^{\mathfrak{p}}(x)$. (Note that $\theta_s P(x)$ is projective and that its head is isomorphic to $L(x) \oplus L(x)$.) On the other hand, $\alpha_x \neq 0$ implies $x, xs \in W^{\mathfrak{p}}$ (as in the proof of Proposition 1.5). We get $[\theta_s G_1(Q)] = [(G_1 \oplus G_1)(Q)]$ for any projective object $Q \in \mathscr{O}_0^{\mathfrak{p}}$; hence, $\big[[\theta_s G_1]\big] = \big[[G_1 \oplus G_1]\big]$. The claim follows.

By Lemma 5.5, we are left with the case where $w = tsw'$ with noncommuting s, $t \in \mathscr{S}$. Arguments similar to those above prove the indecomposability. □

Remark 5.6

Applying the same induction arguments as in the proof of Theorem 5.1 shows the indecomposability of $F(M^{\mathfrak{p}}(x))$ for any indecomposable $F \in \mathscr{P}$ and $x \in W^{\mathfrak{p}}$. Moreover, using only the description from Proposition A.2, one can easily deduce that with the assumptions of Lemma 5.3(1), $\theta_{[w]} M^{\mathfrak{p}}(x) \cong \theta_{s_{i_1}} M^{\mathfrak{p}}(y)$ for some $y \in W^{\mathfrak{p}}$. So the indecomposability of F_w in this case follows directly from the proof of Proposition 5.4 using Lemma 5.3(1).

THEOREM 5.7
Let $\mathfrak{p} \subset \mathfrak{sl}_n$ be maximal parabolic. Then Conjecture 3.3 holds, and we have equality in (3.2).

Proof
Let $F = \bigoplus_{w \in W} (F_{w|\mathscr{O}_o^{\mathfrak{p}}})^{\alpha_w}$ and $G = \bigoplus_{w \in W} (F_{w|\mathscr{O}_o^{\mathfrak{p}}})^{\beta_w}$ such that $\big[[F]\big] = \big[[G]\big]$. By Lemma 5.2, we may assume $\alpha_w = 0 = \beta_w$ for non-braid-avoiding w. The specialisation of $\mathscr{H}/TL$ at $v = 1$ is semisimple (see, e.g., [W]); hence, $\mathscr{N}_{v=1}^{\mathfrak{p}} \cong \bigoplus_{i=1}^{r} L_i$ for some simple $\mathscr{H}/(TL)_{v=1}$-modules L_i. Since (see [W]) $\mathrm{Ann}_{\mathscr{H}/(TL)_{v=1}} L_i = \mathbb{C}\{\underline{H}_x \mid x \in W[i]\}$ for some $W[i] \subset W$, we get $\alpha_w = \beta_w$ for all $w \notin I := \bigcap_{i=1}^{r} W[i]$. Hence, $F \cong \bigoplus_{w \notin I} (F_{|\mathscr{O}_o^{\mathfrak{p}}})^{\alpha_w} \cong G$. On the other hand, it also shows that $\#\,\mathrm{Ind}(\mathfrak{g}, \mathfrak{p}) = |\{\underline{H}_w \mid w \notin I\} = \mathrm{R}(\mathfrak{g}, \mathfrak{p})$. The theorem follows. □

6. The Temperley-Lieb 2-category

In this section we describe a functor from the Temperley-Lieb category into a category given by projective functors with their natural transformations. Let $\mathscr{O}(\mathfrak{sl}_n)^{\max} = \bigoplus_{k=0}^{n} \mathscr{O}^k(\mathfrak{sl}_n)$. In [BFK], the authors consider functors

$$\bigcap_{i,n} : \mathscr{O}(\mathfrak{sl}_n)^{\max} \longrightarrow \mathscr{O}(\mathfrak{sl}_{n-2})^{\max},$$
$$\bigcup_{i,n} : \mathscr{O}(\mathfrak{sl}_n)^{\max} \longrightarrow \mathscr{O}(\mathfrak{sl}_{n+2})^{\max},$$

which are given on each summand as follows. Let

$$\bigcap_{i,n}^{k} : \mathscr{O}^k(\mathfrak{sl}_n) \longrightarrow \mathscr{O}^{k-1}(\mathfrak{sl}_{n-2}),$$
$$\bigcup_{i,n}^{k} : \mathscr{O}^k(\mathfrak{sl}_n) \longrightarrow \mathscr{O}^{k+1}(\mathfrak{sl}_{n+2})$$

be defined as

$$\bigcap_{i,n}^{k} = \zeta_{n,k}\theta_0^1\theta_2\theta_3\cdots\theta_i,$$

$$\bigcup_{i,n}^{k} = \theta_i\theta_{i-1}\cdots\theta_2\theta_1^0\zeta_{n+2,k}^{-1},$$

for any $0 < k \leq n$ (and $0 \leq k < n$, resp.) and zero otherwise. Here, $\zeta_{n,k} : \mathscr{O}_1^k(\mathfrak{sl}_n)\tilde{\to}\mathscr{O}^{k-1}(\mathfrak{sl}_{n-2})$ denotes the *Enright-Shelton equivalence* (see [ES, Chapter 11]). Let $\zeta_n = \bigoplus_k \zeta_{n,k}$. The next statement follows directly from the definitions.

LEMMA 6.1
There are adjoint pairs of functors $\left(\bigcap_{i,n}, \bigcup_{i,n-2}\right)$ *and* $\left(\bigcup_{i,n-2}, \bigcap_{i,n}\right)$.

We prove the following result conjectured in [BFK].

THEOREM 6.2
Let $j \geq i$. There are isomorphisms of functors

$$\bigcap_{i+1,n+2}\ \bigcup_{i,n} \cong \mathrm{ID}, \tag{6.1}$$

$$\bigcap_{i,n+2}\ \bigcup_{i+1,n} \cong \mathrm{ID}, \tag{6.2}$$

$$\bigcap_{j,n}\ \bigcap_{i,n+2} \cong \bigcap_{i,n}\ \bigcap_{j+2,n+2}, \tag{6.3}$$

$$\bigcup_{j,n-2}\ \bigcap_{i,n} \cong \bigcap_{i,n+2}\ \bigcup_{j+2,n}, \tag{6.4}$$

$$\bigcup_{i,n-2}\ \bigcap_{j,n} \cong \bigcap_{j+2,n+2}\ \bigcup_{i,n}, \tag{6.5}$$

$$\bigcup_{i,n+2}\ \bigcup_{j,n} \cong \bigcup_{j+2,n+2}\ \bigcup_{i,n}, \tag{6.6}$$

$$\bigcap_{i,n+2}\ \bigcup_{i,n} \cong \mathrm{ID}\oplus\mathrm{ID}\,. \tag{6.7}$$

Proof
By adjointness (see Lemma 6.1), it is enough to prove (6.1), (6.2), (6.3), (6.4), and (6.7). As already mentioned in [BFK], the isomorphisms (6.1) and (6.2) follow from the definitions of the functors and [BFK, Lemma 4]. The formula (6.7) can be verified

as follows:

$$\begin{aligned}\bigcap_{i,n+2}\bigcup_{i,n} &= \zeta_{n+2}\theta_0^1\theta_2\cdots\theta_{i-2}(\theta_{i-1}(\theta_i\theta_i)\theta_{i-1})\theta_{i-2}\cdots\theta_2\theta_1^0\zeta_{n+2}^{-1}\\ &\cong \zeta_{n+2}\theta_0^1\theta_2\theta_1^0 \oplus \zeta_{n+2}\theta_0^1\theta_2\theta_1^0\zeta_{n+2}^{-1}\\ &\cong \mathrm{ID}\oplus\mathrm{ID}\,.\end{aligned}$$

The first isomorphism follows from $\theta_i^2 \cong \theta_i \oplus \theta_i$ and relation (4.3). The second isomorphism follows from [BFK, Lemma 4]. The rest of the section is devoted to proving formulas (6.3) and (6.4) (see Propositions 6.6 and 6.4). □

LEMMA 6.3

Let $j \geq i$. Then

$$\bigcup_{j,n-2}\bigcap_{i,n} \cong \theta_j\theta_{j-1}\cdots\theta_i \qquad \textit{and} \qquad \bigcup_{i,n-2}\bigcap_{j,n} \cong \theta_i\theta_{i+1}\cdots\theta_j\,.$$

Proof

Using again Theorem 4.1, we get

$$\begin{aligned}\bigcup_{j,n-2}\bigcap_{i,n} &\cong \theta_j\theta_{j-1}\cdots\theta_2\theta_1^0\zeta_n^{-1}\zeta_n\theta_0^1\theta_2\theta_3\cdots\theta_i\\ &\cong \theta_j\theta_{j-1}\cdots\theta_2\theta_1\theta_2\cdots\theta_i\\ &\cong \theta_j\theta_{j-1}\cdots\theta_i;\\ \bigcup_{i,n-2}\bigcap_{j,n} &\cong \theta_i\theta_{i-1}\cdots\theta_2\theta_1^0\zeta_n^{-1}\zeta_n\theta_0^1\theta_2\theta_3\cdots\theta_j\\ &\cong \theta_i\theta_{i-1}\cdots\theta_2\theta_1\theta_2\cdots\theta_j\\ &\cong \theta_i\theta_{i+1}\cdots\theta_j\,.\end{aligned}$$

□

PROPOSITION 6.4

Let $j \geq i$. There exists an isomorphism

$$\bigcup_{j,n-2}\bigcap_{i,n} \cong \bigcap_{i,n+2}\bigcup_{j+2,n}\,.$$

Proof

First we claim

$$\bigcap_{j,n+2}\bigcup_{j+2,n}\bigcap_{i,n+2}\bigcup_{j+1,n} \cong \bigcap_{i,n+2}\bigcup_{j+2,n}\,. \tag{6.8}$$

By Lemma 6.3 and Theorem 4.1, we are able to handle the left-hand side of the formula:

$$\begin{aligned}
\mathrm{LHS} &\cong \zeta_{n+2}\theta_0^1\theta_2\theta_3\cdots\theta_j(\theta_{j+2}\theta_{j+1}\cdots\theta_i)\bigcup_{j+1,n}\\
&\cong \zeta_{n+2}\theta_0^1\theta_{j+2}\theta_2\theta_3\cdots\theta_{i-1}(\theta_i\theta_{i+1}\cdots\theta_j\theta_{j+1}\theta_j\cdots\theta_i)\bigcup_{j+1,n}\\
&\cong \zeta_{n+2}\theta_0^1\theta_{j+2}\theta_2\theta_3\cdots\theta_{i-1}(\theta_i)(\theta_{j+1}\theta_j\cdots\theta_2\theta_1^0\zeta_{n+2}^{-1})\\
&\cong \zeta_{n+2}\theta_0^1\theta_2\theta_3\cdots\theta_{i-1}(\theta_i)(\theta_{j+2}\theta_{j+1}\theta_j\cdots\theta_2\theta_1^0\zeta_{n+2}^{-1}).
\end{aligned}$$

The last line is by definition the right-hand side of formula (6.8). We now prove the statement by induction on $a = j - i$. By induction hypothesis and Lemma 6.3, we get

$$\begin{aligned}
\bigcap_{i,n+2}\bigcup_{j+2,n} &\cong \bigcap_{j,n+2}\bigcup_{j+2,n}\bigcap_{i,n+2}\bigcup_{j+1,n}\\
&\cong \bigcup_{j,n-2}\bigcap_{j,n}\bigcup_{j-1,n-2}\bigcap_{i,n}\\
&\cong \theta_j(\theta_{j-1}\cdots\theta_i)\\
&\cong \bigcup_{j,n-2}\bigcap_{i,n}.
\end{aligned}$$

It remains to check the starting point of the induction, that is, $\bigcap_{i,n+2}\bigcup_{i+2,n} \cong \theta_i$. We first note that the functor in question is a projective functor. To see this, consider the construction of $\zeta_{n,k}$ (see [ES]). It is a composite of four functors Λ_i. Two functors ($i = 1, 3$) are given by tensoring with a finite-dimensional representation, and two functors are given by compositions of parabolic induction and Zuckerman's functor. In particular, if E is a finite-dimensional $\mathfrak{g}$-module, then $\zeta_n(\bullet \otimes E) \cong (\bullet \otimes E')\zeta_n$ for some finite-dimensional module E'. That means that projective functors are sent to projective functors via the equivalence.

Direct calculations (using the explicit formula [ES, Section 11]) show that

$$\Big[\bigcap_{i,n+2}\bigcup_{j+2,n}\big(M^{\mathfrak{p}}(e)\big)\Big] = [M^{\mathfrak{p}}(e) \oplus M^{\mathfrak{p}}(s_i)]$$

if $\mathfrak{p} = \mathfrak{p}_i$ and zero otherwise. Hence, the projective functor $\bigcap_{i,n+2}\bigcup_{j+2,n}$ contains θ_i as a direct summand.

Hence, it is sufficient to show that $\bigcap_{i,n+2}\bigcup_{j+2,n}(M^{\mathfrak{p}}(x))$ has a generalised Verma flag whose length is equal to the length of a generalised Verma flag of $\theta_i(M^{\mathfrak{p}}(x))$ for any $x \in W^{\mathfrak{p}}$. We claim that $\bigcap_{i,n+2}\bigcup_{i+2,n} M^{\mathfrak{p}}(\lambda)$ has a generalised Verma flag of length 2 or 0 for any $M^{\mathfrak{p}} \in \mathscr{O}(\mathfrak{sl}_n)^{\max}$; equivalently, $\theta_0^1\theta_{i+2}\theta_2\theta_1^0 M^{\mathfrak{p}}(\lambda)$

has such a flag for any $M^{\mathfrak{p}}(\lambda) \in \mathscr{O}_1(\mathfrak{sl}_{n+2})^{\max}$. Since

$$\theta_1^0\theta_0^1\theta_{i+2}\theta_2\theta_1^0\theta_0^1 M^{\mathfrak{p}}(x) \cong \theta_{i+2}\theta_1\theta_2\theta_1 M^{\mathfrak{p}}(x) \cong \theta_{i+2}\theta_1 M^{\mathfrak{p}}(x)$$

always has a Verma flag of length 4 or 0, the claim follows (see [J2, Formulas 4.12(2), 4.13(1)]). To get an isomorphism $\bigcap_{i,n+2}\bigcup_{j+2,n} \cong \theta_i$, it is therefore enough to show that

$$\bigcap_{i,n+2}\bigcup_{j+2,n} M^{\mathfrak{p}}(\lambda) \neq 0$$

implies $\theta_i M^{\mathfrak{p}}(\lambda) \neq 0$ for any parabolic Verma module $M^{\mathfrak{p}}(\lambda)$.

Since $\zeta_{n,k}^{-1}$ is an equivalence, it induces a natural map ϕ such that $\zeta_{n,k}^{-1}L(x) \cong L(\phi(x))$. There is an explicit formula in [ES, Proposition 11.2], namely, $\phi(x) = wx^{\Rsh}r$ for a certain $w \in W$ (depending on k) and $r = s_n s_{n-1} \cdot \ldots \cdot s_2$. (The symbol $x^{\Rsh}$ means that we have to renumber the indices $i \rightsquigarrow i+1$ in a reduced expression of x.) In particular, xs_i is a distinguished coset representative if and only if

$$w\big((xs_i)^{\Rsh}\big)r = (wx^{\Rsh}r)r^{-1}(s_i)^{\Rsh}r = wx^{\Rsh}r(s_{i+1})^{\Rsh} = wx^{\Rsh}rs_{i+2}$$

is so. On the other hand,

$$\bigcap_{i,n+2}\bigcup_{j+2,n}\big(M^{\mathfrak{p}}(x)\big) \cong \zeta_n\theta_0^1\theta_{i+2}\theta_2\theta_1^0\zeta_{n+2}^{-1}\big(M^{\mathfrak{p}}(x)\big) \neq 0$$

implies $\theta_{i+2}M(\phi(x)) \neq 0$. Therefore, we get

$$\bigcap_{i,n+2}\bigcup_{j+2,n} M^{\mathfrak{p}}(\lambda) \neq 0 \Rightarrow \theta_i M^{\mathfrak{p}}(\lambda) \neq 0.$$

The theorem follows. □

LEMMA 6.5

Let $j \geq i$. There are isomorphisms of functors

$$\bigcup_{i,n-2}\bigcap_{j,n} \cong \bigcap_{j+2,n+2}\bigcup_{i,n}, \tag{6.9}$$

$$\bigcup_{j+2,n}\bigcup_{i,n-2}\bigcap_{i,n}\bigcap_{j+2,n+2} \cong \theta_i\theta_{j+2}, \tag{6.10}$$

$$\bigcup_{j+2,n}\bigcup_{i,n-2}\bigcap_{j,n}\bigcap_{i,n+2} \cong \theta_i\theta_{j+2}. \tag{6.11}$$

Proof

Formula (6.9) is clear since the adjoint functors are isomorphic (see Lemma 6.1 and

Proposition 6.4). Therefore, with Lemma 6.3 we get

$$\bigcup_{j+2,n}\Big(\bigcup_{i,n-2}\bigcap_{j,n}\Big)\bigcap_{i,n+2} \cong \bigcup_{j+2,n}\bigcap_{j+2,n+2}\bigcup_{i,n}\bigcap_{i,n+2} \cong \theta_{j+2}\theta_i .$$

This shows formula (6.11). Proposition 6.4, Lemma 6.3, and Theorem 4.1 imply

$$\begin{aligned}\bigcup_{j+2,n}\Big(\bigcup_{i,n-2}\bigcap_{i,n}\Big)\bigcap_{j+2,n+2} &\cong \bigcup_{j+2,n}\bigcap_{i,n+2}\bigcup_{i+2,n}\bigcap_{j+2,n+2}\\ &\cong (\theta_{j+2}\theta_{j+1}\cdots\theta_i)(\theta_{i+2}\theta_{i+3}\cdots\theta_{j+2})\\ &\cong \theta_{j+2}\theta_{j+1}\cdots\theta_{i+2}\theta_{i+1}\theta_{i+2}\cdots\theta_{j+2}\theta_i\\ &\cong \theta_{j+2}\theta_i .\end{aligned}$$

This proves formula (6.10). □

Finally, we can do the last step of proving Theorem 6.2.

PROPOSITION 6.6

Let $j \geq i$. There exists an isomorphism of functors

$$\bigcap_{j,n}\bigcap_{i,n+2} \cong \bigcap_{i,n}\bigcap_{j+2,n+2} .$$

Proof

Let $F = \bigcap_{i,n}\bigcap_{j+2,n+2}\bigcup_{j+2,n}\bigcup_{i,n-2}$. Applying relation (6.7) twice, we get $F \cong \bigoplus_{m=1}^{2}\bigcap_{i,n}\bigcup_{i,n-2} \cong \bigoplus_{m=1}^{4}$ ID. Lemma 6.5 implies

$$F\bigcap_{i,n}\bigcap_{j+2,n+2} \cong \bigcap_{i,n}\bigcap_{j+2,n+2}\theta_i\theta_{j+2} \cong F\bigcap_{j,n}\bigcap_{i,n+2} .$$

In other words,

$$\bigoplus_{l=1}^{4}\bigcap_{i,n}\bigcap_{j+2,n+2} \cong \bigoplus_{l=1}^{4}\bigcap_{j,n}\bigcap_{i,n+2} .$$

The proposition follows from Corollary 3.5. □

As a preparation for the next section, we prove the following result. (It contains, in fact, a refinement of formula (6.7).)

PROPOSITION 6.7

(1) *The functors $\bigcap_{i,n}$ and $\bigcup_{i,n}$ are gradable.*

(2) *There are graded lifts $\tilde{\bigcap}_{i,n}$, $\tilde{\bigcup}_{i,n}$ with isomorphisms of graded functors (where $j \geq i$)*

$$\tilde{\bigcup_{j,n-2}} \tilde{\bigcap_{i,n}} \cong \tilde{\theta}_j \tilde{\theta}_{j-1} \cdots \tilde{\theta}_i,$$

$$\tilde{\bigcup_{i,n-2}} \tilde{\bigcap_{j,n}} \cong \tilde{\theta}_i \tilde{\theta}_{i+1} \cdots \tilde{\theta}_j,$$

$$\tilde{\bigcap_{i,n+2}} \tilde{\bigcup_{i,n}} \cong \mathrm{ID}\langle 1\rangle \oplus \mathrm{ID}\langle -1\rangle.$$

With these choices, the remaining isomorphisms of Theorem 6.2 are compatible with the grading.

Proof
(1) Let G be one of the functors in question. In [Ry], it is proved that the Enright-Shelton equivalence is compatible with the grading. All the other functors occurring in the definition of G are gradable by the results of [St]. This defines graded lifts $\tilde{\bigcap}_{i,n}$ and $\tilde{\bigcup}_{i,n}$. In the following, concerning the notation, we do not distinguish between the Enright-Shelton equivalence and its graded lift.

(2) The first two isomorphisms follow from the proofs of Lemma 6.3 and Theorem 4.1 since we have canonically $\tilde{\theta}_1^0 \tilde{\theta}_0^1 \cong \tilde{\theta}_1$ (see [St, Corollary 8.3]). To get the third isomorphism, we first note that $\tilde{\bigcap}_{i,n} \tilde{\bigcup}_{i,n-2} \cong \mathrm{ID}\langle j\rangle \oplus \mathrm{ID}\langle k\rangle$ for certain $j, k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
\tilde{\theta}_i \langle j\rangle \oplus \tilde{\theta}_i \langle k\rangle &\cong \tilde{\bigcup_{i,n-2}} \tilde{\bigcap_{i,n}} \langle j\rangle \oplus \tilde{\bigcup_{i,n-2}} \tilde{\bigcap_{i,n}} \langle k\rangle \\
&\cong \tilde{\bigcup_{i,n-2}} (\mathrm{ID}\langle j\rangle \oplus \mathrm{ID}\langle k\rangle) \tilde{\bigcap_{i,n}} \\
&\cong \tilde{\bigcup_{i,n-2}} \tilde{\bigcap_{i,n}} \tilde{\bigcup_{i,n-2}} \tilde{\bigcap_{i,n}} \\
&\cong (\tilde{\theta}_i)^2 \cong \tilde{\theta}_i \langle 1\rangle \oplus \tilde{\theta}_i \langle -1\rangle.
\end{aligned}$$

(The last isomorphism is given by Theorem 4.1.) This implies $\{j, k\} = \{-1, 1\}$ and provides the third isomorphism. We have to check the compatibility of (6.1). Since a graded lift of an indecomposable exact functor is unique up to isomorphism and grading shift (see [St, Lemma 1.5]), we may assume

$$F = (\zeta_{n,k} \tilde{\theta}_0^1 \tilde{\theta}_2 \cdots \tilde{\theta}_{i+1})(\tilde{\theta}_i \cdots \tilde{\theta}_2 \tilde{\theta}_1^0 \zeta_{n,k}^{-1}) \cong \mathrm{ID}\langle l\rangle$$

for some $l \in \mathbb{Z}$. Using again Theorem 4.1 and the isomorphism $\tilde{\theta}_1^0\tilde{\theta}_0^1 \cong \tilde{\theta}_1$, we get

$$\begin{aligned}\tilde{\theta}_1\langle l\rangle &\cong \tilde{\theta}_1^0\tilde{\theta}_0^1\langle l\rangle\\ &\cong \tilde{\theta}_0^1\zeta_{n,k}^{-1}F\zeta_{n,k}\tilde{\theta}_0^1\\ &\cong \tilde{\theta}_1\tilde{\theta}_2\cdots\tilde{\theta}_{i+1}\tilde{\theta}_i\cdots\tilde{\theta}_2\tilde{\theta}_1\\ &\cong \tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 \cong \tilde{\theta}_1.\end{aligned}$$

The indecomposability of θ_1 therefore implies $l = 0$. The compatibility with the grading for isomorphism (6.2) can be proved in an analogous way. To see that isomorphism (6.4) is compatible with the grading, it is by induction sufficient to consider the case $i = j$ (see formula (6.8) and its proof). Then $\tilde{\bigcup}_{i,n-2}\tilde{\bigcap}_{i,n} \cong \tilde{\theta}_i$ is self-adjoint (see [St, Corollary 8.5]). Assume $\tilde{\theta}_i \cong \tilde{\bigcap}_{i,n+2}\tilde{\bigcup}_{i+2,n}\langle l\rangle$ for some $l \in \mathbb{Z}$. The adjointness properties of $\tilde{\theta}_0^1$ and $\tilde{\theta}_1^0$ (see [St, Corollary 8.3]) directly imply that $\tilde{\bigcap}_{i,n+2}\tilde{\bigcup}_{i+2,n}$ is self-adjoint; hence, $l = 0$.

Let us consider (6.6). We fix $0 \le k \le n$. We first claim that the restriction, call it R, of $F = \bigcup_{i,n+2}\bigcup_{j,n}$ to $\mathscr{O}^k(\mathfrak{sl}_n)$ is indecomposable. Assume $R = F_1 \oplus F_2$. There are isomorphisms of functors

$$\begin{aligned}F\bigcap_{j,n+2}\bigcap_{i,n+4} &\cong \bigcup_{i,n+2}\bigcup_{j,n}\bigcap_{j,n+2}\bigcap_{i,n+4}\\ &\cong \bigcup_{i,n+2}\bigcap_{j,n+4}\bigcup_{j+2,n+2}\bigcap_{i,n+4}\\ &\cong (\theta_i\theta_{i+1}\cdots\theta_j)(\theta_{j+2}\theta_{j+1}\cdots\theta_i)\\ &\cong (\theta_i\theta_{i+1}\cdots\theta_j)(\theta_{j+1}\cdots\theta_i)\theta_{j+2}\\ &\cong \theta_i\theta_{j+2}.\end{aligned}$$

In particular, its restriction to $\mathscr{O}^{k+2}(\mathfrak{sl}_{n+4})$ is indecomposable; hence,

$$F_m \bigcap_{j,n+2}^{k+1}\bigcap_{i,n+4}^{k+2} = 0$$

for $m = 1$, say. This implies

$$F_1 \cong F_1 \bigcap_{j,n+2}^{k+1}\bigcap_{i,n+4}^{k+2}\bigcup_{i+1,n+2}^{k+1}\bigcup_{j+1,n}^{k} = 0.$$

Hence, R is indecomposable; therefore, there exists $l \in \mathbb{Z}$ such that

$$\tilde{\bigcup_{i,n+2}}\tilde{\bigcup_{j,n}} \cong \tilde{\bigcup_{j+2,n+2}}\tilde{\bigcup_{i,n}}\langle l\rangle.$$

Using the graded versions of (6.1) and (6.4), we get isomorphisms

$$\begin{aligned}\tilde{\bigcup_{j,n}} &\cong \tilde{\bigcap_{i+1,n+4}} \tilde{\bigcup_{j+2,n+2}} \tilde{\bigcup_{i,n}}\langle l\rangle \\ &\cong \tilde{\bigcup_{j,n}} \tilde{\bigcap_{i+1,n+2}} \tilde{\bigcup_{i,n}}\langle l\rangle \\ &\cong \tilde{\bigcup_{j,n}}\langle l\rangle.\end{aligned}$$

Hence, $l = 0$. The compatibility with the grading of the remaining two isomorphisms (6.3) and (6.5) then follows easily by adjointness properties. □

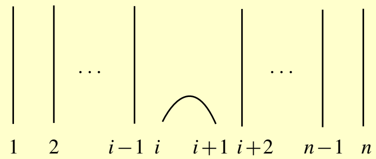
7. Tangles and knot invariants

Any tangle in $\mathbb{R}^3$ has a generic plane projection that is isomorphic to a concatenation of elementary tangles t_i^1, t_i^2, t_i^3, as depicted below, and the right basic braid t_i^4. We associate now to each tangle diagram a certain complex of projective functors and prove that this assignment is compatible with concatenation and well defined up to isomorphism.

We consider $\mathrm{D}^b(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max})$, the bounded derived category of the graded version of $\mathscr{O}(\mathfrak{sl}_n)^{\max}$. (More precisely, for $0 \leq k \leq n$, let P_n^k be a minimal projective generator of $\mathscr{O}^k(\mathfrak{sl}_n)$ with endomorphism ring A_n^k equipped with the grading from [BGS] or [B1]. Then $\bigoplus_k \mathscr{O}^k(\mathfrak{sl}_n) \cong \bigoplus_k \mathrm{mof}-A_n^k$, and $\mathrm{D}^b(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max})$ denotes the bounded derived category of $\bigoplus_k \mathrm{gmof}-A_n^k$.)

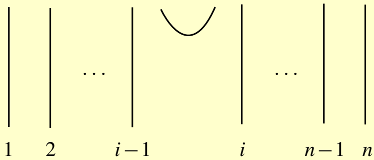
For an exact endofunctor F of $\mathscr{O}(\mathfrak{sl}_n)^{\max}$, we also denote by F its extension to $\mathrm{D}^b(\mathscr{O}(\mathfrak{sl}_n)^{\max})$. As suggested in [BFK], we associate functors to elementary tangles as follows:

t_i^1 :



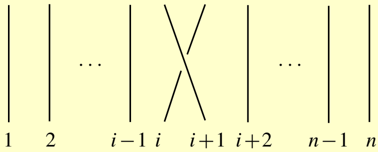
$$\rightsquigarrow \qquad \tilde{\bigcap}_{i,n} : \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_{n-2})^{\max}\big),$$

t_i^2 :



$$\rightsquigarrow \quad \tilde{\bigcup}_{i,n} : \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_{n+2})^{\max}\big),$$

t_i^3 :



$$\rightsquigarrow \quad \mathscr{C}_i := \mathrm{Cone}(\mathrm{ID}\langle 1\rangle \xrightarrow{\text{adj}} \tilde{\theta}_i)\langle 1\rangle : \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max}\big).$$

Note that $\mathscr{C}_i$ is the left derived functor of the graded version of the shuffling functor studied by Irving [I2]. These derived shuffling functors also occur in the context of tilting complexes (see [R]). Let $\mathscr{K}_i$ be the adjoint functor of $\mathscr{C}_i$. The main properties of these functors are the following (see [MS]).

(P1) They define auto-equivalences of derived categories; that is, $\mathscr{C}_i\mathscr{K}_i \cong \mathrm{ID} \cong \mathscr{K}_i\mathscr{C}_i$.

(P2) Let $w = s_1 s_2 \cdot \ldots \cdot s_r$ be a reduced expression. Up to isomorphism, the composition $\mathscr{C}_w = \mathscr{C}_{s_1}\mathscr{C}_{s_2} \cdot \ldots \cdot \mathscr{C}_{i_r}$ is independent of the choice of the reduced expression.

We associate to the right basic braid t_i^4 the functor $\mathscr{K}_i$. We call a tangle with m bottom and n top points an (m,n)-tangle. To a presentation t_α of a tangle t as a composition of elementary tangles, we associate $\mathscr{T}(t_\alpha)$, the corresponding composition of functors. (If t' is an (m,n)-tangle and t is an (n,n')-tangle, the composition tt' is given by putting t above t'.) We state the main result (see [BFK, Conjecture 4]).

THEOREM 7.1

Let t be an (m,n)-tangle with representations t_α, t_β and corresponding functors $\mathscr{T}(t_\alpha)$, $\mathscr{T}(t_\beta)$. Then

$$\mathscr{T}(t_\alpha) \cong \mathscr{T}(t_\beta)\langle 3r\rangle[r] : \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_m)^{\max}\big) \longrightarrow \mathscr{D}^b\big(\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max}\big)$$

for some $r \in \mathbb{Z}$. In particular, up to grading and degree shifts, $\mathscr{T}(t_\alpha t'_{\alpha'}) \cong \mathscr{T}(t_\alpha)\mathscr{T}(t'_{\alpha'})$ for any two tangles t, t' with representations t_α and $t_{\alpha'}$, respectively, so that the concatenation corresponds to the composition of functors.

Proof

In Theorem 6.2 we proved that for tangles without crossings, $\mathscr{T}(t_\alpha) \cong \mathscr{T}(t_\beta)$ if $\alpha \cong \beta$ via isotopies of plane diagrams. It remains to check compatibility with the isotopies depicted in Figure 1, its vertical flip, and whether the assignment is stable under Reidemeister moves (see, e.g., [K], [Tu]).

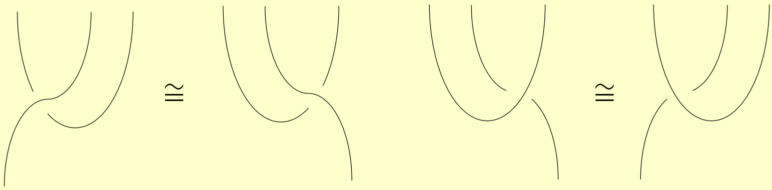


Figure 1. Tangle isotopies

(I) *Addition/removal of a left-twisted curl.* Using Proposition 6.7, we get isomorphisms

$$\begin{aligned}\mathscr{T}(t_{i,n}^1 t_{i,n}^3) &= \widetilde{\bigcap_{i,n}} \circ \mathscr{C}_{i,n} \cong \operatorname{Cone}\Big(\widetilde{\bigcap_{i,n}}\langle 1\rangle \xrightarrow{\text{adj}} \widetilde{\bigcap_{i,n}}\, \tilde{\theta}_i\Big)\langle 1\rangle \\ &\cong \operatorname{Cone}\Big(\widetilde{\bigcap_{i,n}}\langle 1\rangle \xrightarrow{\text{adj}} \widetilde{\bigcap_{i,n}}\langle 1\rangle \oplus \widetilde{\bigcap_{i,n}}\langle -1\rangle\Big)\langle 1\rangle \\ &\cong \widetilde{\bigcap_{i,n}}\langle -1\rangle\langle 1\rangle \\ &\cong \mathscr{T}(t_{i,n}^1).\end{aligned}$$

(II) *Addition/removal of a right-twisted curl.* We have

$$\begin{aligned}\mathscr{T}(t_{i,n}^1 t_{i,n}^4) &= \widetilde{\bigcap_{i,n}} \circ \mathscr{K}_{i,n}\langle -1\rangle \cong \operatorname{Cone}\Big(\widetilde{\bigcap_{i,n}}\, \tilde{\theta}_i \xrightarrow{\text{adj}} \widetilde{\bigcap_{i,n}}\langle -1\rangle\Big)\langle -1\rangle \\ &\cong \widetilde{\bigcap_{i,n}}\langle 1\rangle\langle -1\rangle \\ &\cong \mathscr{T}(t_{i,n}^1).\end{aligned}$$

(III) *Tangency moves.*

$$\mathscr{T}(t_{i,n}^3 t_{i,n}^4) = \mathscr{C}_i \mathscr{K}_i \cong \mathrm{ID} \qquad \text{and} \qquad \mathscr{T}(t_{i,n}^4 t_{i,n}^3) = \mathscr{K}_i \mathscr{C}_i \cong \mathrm{ID}$$

by property (P1).

(IV) *Triple point move.* We have $\mathscr{T}(t^3_{i,n}t^3_{i+1,n}t^3_{i,n}) \cong \mathscr{T}(t^3_{i+1,n}t^3_{i,n}t^3_{i+1,n})$ by property (P2) and therefore also $\mathscr{T}(t^4_{i,n}t^4_{i+1,n}t^4_{i,n}) \cong \mathscr{T}(t^4_{i+1,n}t^4_{i,n}t^4_{i+1,n})$ by property (P1).

(V) *Height shifting.* Property (P2) implies $\mathscr{C}_j\mathscr{C}_i \cong \mathscr{C}_i\mathscr{C}_j$ and implies also isomorphisms such as $\mathscr{K}_j\mathscr{C}_i \cong \mathscr{C}_i\mathscr{K}_j$ if $|i-j| \geq 2$ by property (P1).

To see compatibility with Figure 1, we recall the equivalence

$$\tilde{\mathscr{O}}(\mathfrak{sl}_n)^{\max} \cong \bigoplus_k \mathrm{gmof}-A^k_n.$$

The functors $\mathscr{C}_i$ and $\mathscr{K}_i$ are given by the tilting complexes $(\mathrm{ID}\langle 1\rangle \xrightarrow{\mathrm{adj}} \theta_i)\langle 1\rangle$ and $(\theta_i \xrightarrow{\mathrm{adj}} \mathrm{ID}\langle -1\rangle)\langle -1\rangle$, respectively (see [R], [MS]). Let us consider the first image from Figure 1. By (6.1) and (6.2), it is sufficient to verify $\mathscr{K}_i\tilde{\theta}_{i+1} \cong \mathscr{C}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1}$ up to shifts. The left-hand side is described by tensoring with the tilting complex $\mathbf{T}$ given by

$$(\mathbf{T}_0 = \tilde{\theta}_i\tilde{\theta}_{i+1} \xrightarrow{f:=\mathrm{adj}_{\tilde{\theta}_{i+1}}} \tilde{\theta}_{i+1}\langle -1\rangle = \mathbf{T}_{-1})\langle -1\rangle,$$

whereas the right-hand side is given by tensoring with $\mathbf{G}$ defined as

$$(\mathbf{G}_1 = \theta_i\theta_{i+1}\langle 1\rangle \xrightarrow{g:=\mathrm{adj}_{\tilde{\theta}_i\tilde{\theta}_{i+1}}} \mathbf{G}_0 = \tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1})\langle 1\rangle.$$

We claim that (with Theorem 4.1) $\mathbf{T}\langle 3\rangle[1] \cong \mathbf{G}$. To avoid explicit calculations, let us consider for a moment the translation functors as endofunctors of gmof$-A$, the graded version of $\mathscr{O}_0$. The isomorphism $\tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1} \cong \tilde{F}_{s_{i+1}s_is_{i+1}} \oplus \tilde{\theta}_{i+1}$ gives a natural transformation $p : \tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1} \to \tilde{\theta}_{i+1}$, homogeneous of degree zero. Using Corollary 8.8, which is proved later, we get that $R = \mathrm{Hom}(\tilde{\theta}_i\tilde{\theta}_{i+1}, \tilde{\theta}_{i+1}) \cong \mathrm{Hom}(\tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1}, \mathrm{ID})$ is strictly positively graded and $R_1 \cong \mathbb{C}$. Hence, $g \circ p = \lambda \cdot f$ for some $\lambda \in \mathbb{C}$. Restricting to the parabolic categories, p becomes an isomorphism (see Theorem 4.1), and the maps $\lambda^{-1} \cdot \mathrm{id}$ and p define the required isomorphism. Compatibility with the second image in Figure 1 and the vertically flipped images is proved in an analogous way. In any case, we get an isomorphism up to a shift $\langle 3\rangle[1]$. (Compatibility with the vertically flipped images follows also by adjointness properties.) □

Remark 7.2 (Oriented tangles)

The $\mathbb{Z}$-indeterminacy in Theorem 7.1 can be removed by working with oriented tangles (an analogy to the approach in [Kh2]). For this we assign to the right basic braid t^4_i the functor $\mathscr{K}_i\langle 3\rangle[1]$, and then for any representation t_α of a tangle t in terms of elementary tangles, we have a corresponding composition of functors, say, $\hat{\mathscr{T}}(t_\alpha)$. Now, any *oriented* tangle $\tilde{t}$ can be presented as a concatenation $\tilde{t}_\alpha$ of elementary oriented tangles. Assume $\tilde{t}_\alpha = t_\alpha$ after forgetting the orientation. Then we define $\mathscr{T}_{or}(\tilde{t}_\alpha) = \hat{\mathscr{T}}(\tilde{\alpha})\langle -3p\rangle[-p]$, where p denotes the number of positively oriented crossings (see [Kh1, Figure 49]). Using Theorem 7.1 and the relations in [Tu], one easily checks that the functor $\mathscr{T}_{or}(\tilde{t}_\alpha)$ (now up to isomorphism!) does not depend on the chosen representation $\tilde{t}_\alpha$. Hence, it defines an invariant of oriented tangles.

Remark 7.3 (Kauffman bracket and Jones polynomial)
If we renormalise by taking $\mathscr{C}_i := \mathrm{Cone}(\mathrm{ID}\langle 1\rangle \overset{\mathrm{adj}}{\to} \tilde{\theta}_i)$ and $\mathscr{K}_i := \mathscr{C}_i^{-1}[1]\langle 1\rangle$, then we have the following equalities in the Grothendieck group: $\big[\tilde{\bigcap}_{i,n+2}\tilde{\bigcup}_{i,n}\big] = (v + v^{-1})[\mathrm{ID}]$, $[\mathscr{C}_i] = [\tilde{\theta}_i] - v[\mathrm{ID}]$, and $[\mathscr{K}_i] = [\mathrm{ID}] - v[\tilde{\theta}_i]$. These can be considered as Kauffman brackets in the normalisation of [Kh1]. Given a tangle t with c_j crossings of type t_i^j for $j = 3, 4$, we can define $K(t) := \big[\mathscr{T}(t)\langle c_4 - 2c_3\rangle[c_3]\big]$. Then $K(t)$ satisfies the skein relations for the scaled Kauffman bracket (as in [Kh1]), which is, up to a normalisation, the Jones polynomial.

Remark 7.4
Using the main result of [Ry], that translation and Zuckerman's functors are Koszul dual to each other, [BFK, Conjecture 1] follows directly. On the other hand, it is not clear if one really needs Ryom-Hansen's result to prove the conjecture. All our arguments can easily be transferred to the singular case with one exception. It does not seem to be obvious how to translate the starting point for the induction in the proof of Proposition 6.4.

8. Cobordisms and natural transformations

To each tangle, hence in particular to a closed loop/circle, we assign a functor. The goal of this section now is to describe a functor from the category $\mathscr{COB}$ of 2-cobordisms into a category given by projective functors. The objects of $\mathscr{COB}$ are disjoint unions of labelled oriented closed one-dimensional manifolds, that is, a disjoint union of labelled oriented circles. (We also allow the empty set, i.e., no circle.) A *surface between* n oriented circles $\mathbf{n}$ and m oriented circles $\mathbf{m}$ is a surface S with an orientation-preserving isomorphism ϕ_S between the boundary δS of S and the union $\mathbf{n}^r \sqcup \mathbf{m}$, where $\mathbf{n}^r$ denotes the manifold $\mathbf{n}$ but with reversed orientation. Two surfaces S and T between $\mathbf{n}$ and $\mathbf{m}$ are *equivalent* if there is an isomorphism of surfaces (resp., a diffeomorphism) $\psi : S \tilde{\to} T$ such that the following diagram commutes:

$$\begin{tikzcd} \delta S \arrow[r, "\phi_S"] \arrow[d, "\psi_{|\delta S}"] & \mathbf{n}^r \sqcup \mathbf{m} \\ \delta T \arrow[ur, "\phi_T"'] & \end{tikzcd}$$

A *morphism* $\Sigma : \mathbf{n} \to \mathbf{m}$ in $\mathscr{COB}$ is an equivalence class of surfaces $S : \mathbf{n} \to \mathbf{m}$ between $\mathbf{n}$ and $\mathbf{m}$. The morphisms in $\mathscr{COB}$ are generated by gluing copies of the six basic surfaces depicted in Figure 2 subject to certain relations. For details we refer to [A, Section 4].

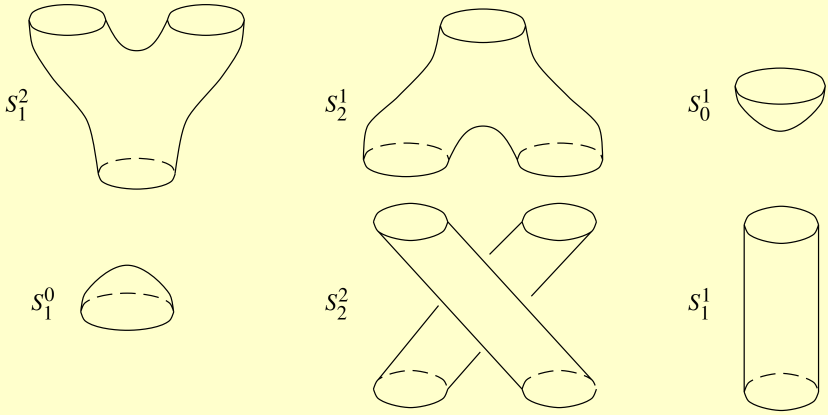


Figure 2. Basic cobordisms

8.1. Basic cobordisms

We fix $n \in \mathbb{N}$ and write $\zeta_{n+2} = \zeta$. Recall that we assigned to an occurring circle the composition

$$\bigcap_{i,n+2} \bigcup_{i,n} = (\zeta\theta_0^1\theta_2 \cdots \theta_i)(\theta_i\theta_{i-1} \cdots \theta_2\theta_1^0\zeta^{-1}). \tag{8.1}$$

Since this is (up to isomorphism) independent of i, we choose $i = 2$ and set $G = \zeta\theta_0^1$, $F = \theta_1^0\zeta^{-1}$. Note that $G\theta_2 F \cong \mathrm{ID}$ and $\theta_2 F G \theta_2 \cong \theta_2\theta_1^0\zeta\zeta^{-1}\theta_0^1\theta_2 \cong \theta_2\theta_1\theta_2 \cong \theta_2$ as endofunctors on $\mathscr{O}^{\max}(\mathfrak{sl}_n)$. Let $\mathrm{adj} : \mathrm{ID} \to \theta_2$ and $\overline{\mathrm{adj}} : \theta_2 \to \mathrm{ID}$ denote the adjunction morphisms. Since $\tilde{\theta}_2\tilde{\theta}_2 \cong \tilde{\theta}_2\langle 1\rangle \oplus \tilde{\theta}_2\langle -1\rangle$, there are a monomorphism $\tilde{\alpha}' : \tilde{\theta}_2 \to \tilde{\theta}_2\tilde{\theta}_2$ and also an epimorphism $\tilde{\beta}' : \tilde{\theta}_2\tilde{\theta}_2 \to \tilde{\theta}_2$ of degree -1. Let α' and β' denote the corresponding morphisms of functors after forgetting the grading. There is an isomorphism of graded functors $\tilde{\sigma}' : \tilde{\theta}_2\tilde{\theta}_2\tilde{\theta}_2 \cong \tilde{\theta}_2\langle 2\rangle \oplus \tilde{\theta}_2 \oplus \tilde{\theta}_2 \oplus \tilde{\theta}_2\langle -2\rangle$ obtained by switching the two middle summands. Let σ' denote the corresponding isomorphism after forgetting the grading. To each basic cobordism we assign a natural transformation:

$$\Phi(S_1^2) = \Delta : \theta_2\theta_2 \xrightarrow{\theta_2\,\mathrm{adj}_{\theta_2(\bullet)}} \theta_2\theta_2\theta_2,$$

$$\Phi(S_2^1) = \mu : \theta_2\theta_2\theta_2 \xrightarrow{\theta_2\overline{\mathrm{adj}}_{\theta_2(\bullet)}} \theta_2\theta_2,$$

$$\Phi(S_0^1) = i : \theta_2 \xrightarrow{\alpha'} \theta_2\theta_2,$$

$$\Phi(S_1^0) = \epsilon : \theta_2\theta_2 \xrightarrow{\beta'} \theta_2,$$

$$\Phi(S_2^2) = \sigma' : \theta_2\theta_2\theta_2 \xrightarrow{\sigma'} \theta_2\theta_2\theta_2,$$

$$\Phi(S_1^1) = \mathrm{id} : \theta_2\theta_2 \xrightarrow{\mathrm{id}} \theta_2\theta_2. \tag{8.2}$$

If $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r$ is a disjoint union of basic cobordisms $S_i : \mathbf{m}_i \to \mathbf{n}_i$, then $\Phi(S) : (\theta_2)^{m_1+m_2+\cdots+m_r+1} \longrightarrow (\theta_2)^{n_1+n_2+\cdots+n_r+1}$ is inductively defined as the composite

$$(\theta_2)^{n_1}\big(\Phi(S_2 \sqcup \cdots \sqcup S_r)\big) \circ \Phi(S_1)_{(\theta_2)^{m_2+m_3+\cdots+m_r+1}}.$$

Let $\mathscr{P}_n^{\max}$ denote the category of projective functors on $\mathscr{O}^{\max}(\mathfrak{sl}_n)$.

8.2. A functor from $\mathscr{C}\mathscr{O}\mathscr{B}$ to the functor category

With the notation above, we get the following result.

THEOREM 8.1 (Cobordisms as natural transformations)
There is a functor $\mathrm{CAT} = \mathrm{CAT}_n : \mathscr{C}\mathscr{O}\mathscr{B} \to \mathscr{P}_n^{\max}$ *given by*

$$\mathbf{m} \longmapsto G(\theta_2)^{m+1}F$$

on objects and on disjoint unions of basic morphisms as

$$\mathrm{CAT}(S) = G\Phi(S)_{F(\bullet)}.$$

To make computations easier, we use Soergel's functor $\mathbb{V} : \mathscr{O}_0 \longrightarrow C - \mathrm{mof}$, where $C - \mathrm{mof}$ denotes the category of finitely generated modules over the coinvariant algebra $C = S(\mathfrak{h})/(S(\mathfrak{h})_+)^W$. We recall its main properties, give explicit formulas, and then prove Theorem 8.1 (for details, see [S1], [S3]). The functor $\mathbb{V}$ is exact and fully faithful on projectives. For a simple reflection s, there is a natural isomorphism $\mathbb{V}\theta_s \cong C \otimes_{C^s} \mathbb{V}$, where C^s denotes the invariants of C under s. Note that C is a free C^s-module. A basis is given by 1 and by X, the coroot corresponding to s.

LEMMA 8.2
The adjunction morphisms correspond under $\mathbb{V}$ to the natural transformations given

by the following morphisms of C-modules:

$$\begin{aligned} m_N : C \otimes_{C^s} N &\longrightarrow N, \quad c \otimes n \longmapsto cn, \\ \delta_N : N &\longrightarrow C \otimes_{C^s} N, \quad 1 \otimes n \longmapsto 1 \otimes Xn + X \otimes n, \end{aligned}$$

for $c \in C$, $n \in N \in C -$ mof.

Proof

The first adjunction morphism is given as the preimage of the identity under the canonical isomorphism

$$\begin{aligned} \operatorname{Hom}_C(C \otimes_{C^s} N, N) &\longrightarrow \operatorname{Hom}_{C^s}(N, N), \\ f &\longmapsto \big(n \mapsto f(1 \otimes n)\big), \\ \big(c \otimes n \mapsto cf(n)\big) &\longleftarrow f. \end{aligned} \tag{8.3}$$

Hence $m_N(c \otimes n) = cn$. To prove the second statement, we use the isomorphisms

$$\begin{aligned} \operatorname{Hom}_{C^s}(N, N) &\cong \operatorname{Hom}_{C^s}(N^*, N^*) \cong \operatorname{Hom}_C(C \otimes_{C^s} N^*, N^*) \\ &\overset{\Psi}{\cong} \operatorname{Hom}_C\big((C \otimes_{C^s} N)^*, N^*\big) \cong \operatorname{Hom}_C(N, C \otimes_{C^s} N). \end{aligned}$$

The second isomorphism is given by (8.3). According to [S3, Lemma 2.9.2], there is an isomorphism $\psi : C \otimes_{C^s} N^* \tilde{\to} (C \otimes_{C^s} N)^*$ of C-modules given by $\psi(1 \otimes f)(1 \otimes n) = 0$ and $\psi(1 \otimes f)(X \otimes n) = f(n)$ for $f \in N^*, n \in N$. This defines Ψ. All the other maps are given by duality. Since $m_{N^*}(1 \otimes f) = f$, we get $\Psi(m_{N^*})(\psi(1 \otimes f)) = f$. On the other hand, $\delta^*(\psi(1 \otimes f)(n)) = \psi(1 \otimes f)(\delta(n)) = \psi(1 \otimes f)(1 \otimes Xn + X \otimes n) = \psi(1 \otimes f)(X \otimes n) = f(n)$. This proves, in fact, that δ is the adjunction morphism. □

Via $\mathbb{V}$, the isomorphism $\theta_s\theta_s \cong \theta_s \oplus \theta_s$ becomes the following.

LEMMA 8.3

There are natural isomorphisms of C-modules

$$\begin{aligned} Q_N : \quad C \otimes_{C^s} C \otimes_{C^s} N &\longrightarrow C \otimes_{C^s} N \oplus C \otimes_{C^s} N, \\ 1 \otimes 1 \otimes n &\longmapsto (1 \otimes n, 0), \\ X \otimes 1 \otimes n &\longmapsto (X \otimes n, 0), \\ 1 \otimes X \otimes n &\longmapsto (-X \otimes n, 1 \otimes n), \\ X \otimes X \otimes n &\longmapsto (-X^2 \otimes n, X \otimes n). \end{aligned}$$

Hence, β' corresponds to $\beta = p_2 \circ Q$, where p_2 denotes the projection onto the second summand, and α' corresponds to $\alpha = Q^{-1} \circ i_1$, where i_1 denotes the inclusion of the first summand.

Proof
The inverse map is defined by $(1 \otimes n, 0) \mapsto 1 \otimes 1 \otimes n$ and $(0, 1 \otimes n) \mapsto 1 \otimes X \otimes n + X \otimes 1 \otimes n$. $\square$

The permutation morphism σ becomes under $\mathbb{V}$ the following isomorphism.

LEMMA 8.4
There is an isomorphism of functors

$$\sigma : C \otimes_{C^s} C \otimes_{C^s} (C \otimes_{C^s} \bullet) \longrightarrow C \otimes_{C^s} C \otimes_{C^s} (C \otimes_{C^s} \bullet)$$

given by $\sigma = Q^{-1}_{C \otimes_{C^s}(\bullet)} \circ (Q^{-1} \oplus Q^{-1}) \circ (\mathrm{id} \oplus \overline{\sigma} \oplus \mathrm{id}) \circ (Q \oplus Q) \circ Q_{C \otimes_{C^s}(\bullet)}$, *where* $\overline{\sigma} : (C \otimes_{C^s} N) \oplus (C \otimes_{C^s} N) \tilde{\rightarrow} (C \otimes_{C^s} N) \oplus (C \otimes_{C^s} N)$, $\overline{\sigma}(x, y) = (y, x)$.

Proof
This follows directly from Lemma 8.3. $\square$

Proof of Theorem 8.1
By [A, Proposition 12], we first have to check that Δ, μ, ϵ, i, and σ satisfy formally the properties of a (co)associative and (co)commutative (co)multiplication, a (co)unit, and a permutation map. Second, we have to show that $\theta_2 \mu \circ \Delta_{\theta_2} = \Delta \circ \mu : \theta_2 \theta_2 \theta_2 \longrightarrow \theta_2 \theta_2 \theta_2$.

• *Associativity*, that is, $\mathrm{CAT}(S^1_2 \circ (S^1_2 \sqcup S^1_1)) = \mathrm{CAT}(S^1_2 \circ (S^1_1 \sqcup S^1_2))$. It is enough to verify

$$\overline{\mathrm{adj}}\ \overline{\mathrm{adj}}_{\theta_2(\bullet)} = \overline{\mathrm{adj}}\ \theta_2\ \overline{\mathrm{adj}}(\bullet) : (\theta_2)^2 \to \mathrm{ID}\,.$$

We claim that this holds even on $\mathscr{O}_0$. Let $N \in C\mathrm{-mof}$. Let $c \otimes d \otimes n \in C \otimes_{C^s} C \otimes_{C^s} N$. We calculate $m \circ (m \otimes \mathrm{id})(c \otimes d \otimes n) = m(cd \otimes n) = cdn$. On the other hand, $m \circ (\mathrm{id} \otimes m)(c \otimes d \otimes N) = m(c \otimes dn) = cdn$. The associativity follows.

• *Coassociativity*. Since

$$\begin{aligned}
(\delta \otimes \mathrm{id}) \circ \delta(1 \otimes n) &= \delta \otimes \mathrm{id}\big(1 \otimes (X \otimes n) + X \otimes (1 \otimes n)\big) \\
&= 1 \otimes X \otimes X \otimes n + X \otimes 1 \otimes X \otimes n \\
&\quad + 1 \otimes X^2 \otimes 1 \otimes n + X \otimes X \otimes 1 \otimes n \\
&= 1 \otimes 1 \otimes X^2 \otimes n + 1 \otimes X \otimes X \otimes n \\
&\quad + X \otimes 1 \otimes X \otimes n + X \otimes X \otimes 1 \otimes n \\
&= \mathrm{id} \otimes \delta(1 \otimes X \otimes n + X \otimes 1 \otimes n) \\
&= (\mathrm{id} \otimes \delta) \circ \delta(1 \otimes n),
\end{aligned}$$

it follows that $\mathrm{CAT}((S^2_1 \sqcup S^1_1) \circ S^2_1) = \mathrm{CAT}((S^1_1 \sqcup S^2_1) \circ S^2_1)$.

• *Unit*, that is, $\mathrm{CAT}(S^1_2 \circ (S^1_1 \sqcup S^1_0)) = \mathrm{CAT}(S^1_1) = \mathrm{CAT}(S^1_2 \circ (S^1_1 \circ S^1_0))$. For the first equality, it is enough to check $\overline{\mathrm{adj}}_{\theta_2} \circ \alpha' = \mathrm{id} : \theta_2 \to \theta_2$. This, however, is true since $(m \otimes \mathrm{id}) \circ \alpha(1 \otimes n) = (m \otimes \mathrm{id})(1 \otimes 1 \otimes n) = 1 \otimes n$ by Lemma 8.3. Similarly, $(\mathrm{id} \otimes m) \circ \alpha(1 \otimes n) = \mathrm{id} \otimes m(1 \otimes 1 \otimes n) = 1 \otimes n$, proving the second equality.

• *Counit*. We omit showing explicitly the dual statement for the counit.

• *Commutativity*, that is, $\mu \circ \sigma' = \mu$. This follows from the commutativity of the diagram

$$\begin{tikzcd}
\theta_2\theta_2\theta_2 \arrow[rr, "\sigma'"] \arrow[drr, "\theta_2\overline{\mathrm{adj}}_{\theta_2}"'] & & \theta_2\theta_2\theta_2 \arrow[d, "\theta_2\overline{\mathrm{adj}}_{\theta_2}"] \\
& & \theta_2\theta_2 = \theta_2 \, \mathrm{ID} \, \theta_2
\end{tikzcd}$$

or, more generally, from the commutativity of

$$\begin{tikzcd}
C \otimes_{C^s} C \otimes_{C^s} (C \otimes_{C^s} N) \arrow[rr, "\sigma"] \arrow[drr, "\mathrm{id} \otimes m \otimes \mathrm{id}"'] & & C \otimes_{C^s} C \otimes_{C^s} (C \otimes_{C^s} N) \arrow[d, "\mathrm{id} \otimes m \otimes \mathrm{id}"] \\
& & C \otimes_{C^s} C \otimes_{C^s} \otimes N
\end{tikzcd}$$

Direct calculations using Lemmas 8.3 and 8.4 show that $\sigma(c_1 \otimes c_2 \otimes c_3 \otimes n) = c_1 \otimes c_3 \otimes c_2 \otimes n$ for $c_i \in \{1, X\}$, $n \in N$. Therefore,

$$\begin{aligned}
(\mathrm{id} \otimes m \otimes \mathrm{id}) \circ \sigma(c_1 \otimes c_2 \otimes c_3 \otimes n) &= c_1 \otimes c_3 c_2 \otimes n = c_1 \otimes c_2 c_3 \otimes n \\
&= \mathrm{id} \otimes m \otimes \mathrm{id}(c_1 \otimes c_2 \otimes c_3 \otimes n).
\end{aligned}$$

The commutativity follows.

• *Cocommutativity*. This follows from the calculations

$$\begin{aligned}
\sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id})(c \otimes 1 \otimes n) &= \sigma(c \otimes 1 \otimes X \otimes n + c \otimes X \otimes 1 \otimes n) \\
&= c \otimes X \otimes 1 \otimes n + c \otimes 1 \otimes X \otimes n \\
&= (\mathrm{id} \otimes \delta \otimes \mathrm{id})(c \otimes 1 \otimes n)
\end{aligned}$$

and

$$\begin{aligned}
\sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id})(c \otimes X \otimes n) &= \sigma(c \otimes X \otimes X \otimes n) + c \otimes 1 \otimes X^2 \otimes n \\
&= c \otimes X \otimes X \otimes n + c \otimes 1 \otimes X^2 \otimes n \\
&= (\mathrm{id} \otimes \delta \otimes \mathrm{id})(c \otimes 1 \otimes n).
\end{aligned}$$

• To prove the remaining relation, it is enough to check the commutativity of

$$\begin{tikzcd}
\theta_2\theta_2\theta_2 \arrow[r, "\mu = \theta_2\overline{\mathrm{adj}}_{\theta_2}"] \arrow[d, "\Delta_{\theta_2} = \theta_2\, \mathrm{adj}_{\theta_2\theta_2}"'] & \theta_2\theta_2 \arrow[d, "\Delta = \theta_2\, \mathrm{adj}_{\theta_2}"] \\
\theta_2\theta_2\theta_2\theta_2 \arrow[r, "\theta_2\mu = \theta_2\theta_2\overline{\mathrm{adj}}_{\theta_2}"] & \theta_2\theta_2\theta_2
\end{tikzcd}$$

or just the commutativity of one of the following diagrams (with arbitrary $N \in C-$ mof):

$$\begin{tikzcd}
\theta_2 \arrow[r, "\overline{\mathrm{adj}}"] \arrow[d, "\mathrm{adj}_{\theta_2}"] & \mathrm{ID} \arrow[d, "\mathrm{adj}"] \\
\theta_2\theta_2\theta_2\overline{\mathrm{adj}} \arrow[r, "\theta_2\overline{\mathrm{adj}}"] & \theta_2
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes_{C^s} N \arrow[r, "m"] \arrow[d, "\delta_{C\otimes_{C^s}}"] & N \arrow[d, "\delta"] \\
C \otimes_{C^s} C \otimes_{C^s} N \arrow[r, "\mathrm{id} \otimes m"] & C \otimes_{C^s} N
\end{tikzcd}$$

Let $1 \otimes n \in C \otimes_{C^s} N$. Since $\delta \circ m(1 \otimes n) = \delta(n) = 1 \otimes Xn + X \otimes n$ and $(\mathrm{id} \otimes m) \circ \delta)(1 \otimes n) = (\mathrm{id} \otimes m)(1 \otimes X \otimes n + X \otimes 1 \otimes n) = 1 \otimes Xn + X \otimes n$, the last diagram above commutes. This proves $\theta_2\mu \circ \Delta_{\theta_2(\bullet)} = \Delta \circ \mu$.

Therefore, the assignment of the theorem is well defined and defines a functor as described. □

Remark 8.5 (Gradings and Euler characteristic)
All the occurring functors assigned to closed oriented labelled 1-manifolds are gradable. Choosing the standard lifts, by construction the natural transformations assigned to the basic cobordisms become homogeneous with the degrees $\deg\tilde{\Delta} = \deg\tilde{\mu} = 1$, $\deg(\tilde{i}) = \deg(\tilde{\epsilon}) = -1$, $\deg\tilde{\sigma}' = \deg\mathrm{id} = 0$. Let $S = S_1S_2\cdots S_r : \mathbf{n} \to \mathbf{m}$ be a surface between two disjoint unions of labelled oriented closed 1-manifolds given as a product of disjoint unions of surfaces S_i $(1 \le i \le r)$ from Figure 2 with corresponding natural transformations $\Phi(S_i)$. Set $\deg(S) = \sum_{i=1}^{r} \deg\Phi(S_i)$. The relations in [A] directly imply that $\deg(S)$ is well defined, that is, constant on equivalence classes. If $\chi(S)$ denotes the Euler characteristic of S, we get

$$\chi(S) = -\deg(S).$$

Remark 8.6
If a surface between two closed oriented 1-manifolds contains a punctured genus $> l(w_o)$ surface, where w_o is the longest element in the Weyl group corresponding to $\mathfrak{sl}_n$, then $\mathrm{CAT}_n(S) = 0$. To verify this, one has to consider the composition $g = (m \circ \delta)^{l(w_o)}$. Since $\mathbb{V}P$ for any projective module $P \in \mathcal{O}_0(\mathfrak{sl}_n)$ has a natural grading (see [S1]), g induces a homogeneous endomorphism of degree $(l(w_o) \cdot \deg(X))$ on $C \otimes_{C^s} \mathbb{V}P$ for any $P \in \mathcal{O}_0(\mathfrak{sl}_n)$. On the other hand, however, $\mathbb{V}P_i \neq 0 \Rightarrow l \le i \le l + l(w_0) \cdot (\deg(X))$ for some $l \in \mathbb{Z}$ (e.g., by [S1] again).

We finish with a small result describing homomorphisms between translation functors on the graded version of the main block of $\mathcal{O}$ via bimodules over the coinvariant algebra (see also [B2]). Let C be given the *even* grading induced from $S(\mathfrak{h})$, where $S(\mathfrak{h})^2 = \mathfrak{h}$. Let $x \in W$ with a reduced expression $[x] = s_1s_2 \cdot \ldots \cdot s_r$. Let $\tilde{\theta}_{[x]} =$

$\tilde{\theta}_{s_r} \cdots \tilde{\theta}_{s_2} \tilde{\theta}_{s_1}$ be considered as an endofunctor of gmof $-A$. We denote $\mathbf{C}_{[x]} = C \otimes_{C^{s_r}} C \otimes_{C^{s_{r-1}}} \cdots \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \bullet$ as a functor on graded C-modules.

PROPOSITION 8.7

Let $x, w \in W$ with fixed reduced expressions $[x]$ and $[w]$, respectively. There is a natural isomorphism of graded vector spaces

$$\operatorname{Hom}(\tilde{\theta}_{[x]}, \tilde{\theta}_{[w]}) \cong \operatorname{Hom}_{C-\mathrm{gmof}-C}\left(\mathbf{C}_{[x]}(C)\langle -l(x)\rangle, \mathbf{C}_{[w]}(C)\langle -l(w)\rangle\right).$$

Proof

The results of [S1] give a natural map

$$\begin{aligned} \Phi : \operatorname{Hom}(\tilde{\theta}_{[x]}, \tilde{\theta}_{[w]}) &\to \operatorname{Hom}_{C-\mathrm{gmof}}\left(\mathbf{C}_{[x]}\langle -l(x)\rangle, \mathbf{C}_{[w]}\langle -l(w)\rangle\right), \\ f &\mapsto \hat{\mathbb{V}} f_{\mathbb{V}\tilde{P}(w_0)}, \end{aligned}$$

where $\hat{\mathbb{V}}$ denotes the functor $\operatorname{Hom}_{\mathrm{gmof}-A}(\tilde{P}(w_0), \cdot) : \mathrm{gmof}-A \to \mathrm{gmof}-C$. Since f is a natural transformation, we have $f \circ \mathbf{C}_{[x]}(g) = \mathbf{C}_{[w]}(g) \circ f$ for any endomorphism $g \in \operatorname{End}_{\mathrm{gmof}-A}(\tilde{P}(w_o)) = C$. Hence, $\Phi(f)$ is a morphism of graded C-*bi*modules. The morphism Φ is injective since any projective object $Q \in \mathrm{gmof}-A$ has a copresentation of the form

$$Q \hookrightarrow \bigoplus_{i \in I_1} \tilde{P}(w_0)\langle i\rangle \to \bigoplus_{i \in I_2} \tilde{P}(w_0)\langle i\rangle$$

for some finite multisets I_1, I_2. Any homomorphism in the target space of Φ defines a natural transformation between functors $\mathbf{C}_{[x]}\langle -l(x)\rangle$ and $\mathbf{C}_{[w]}\langle -l(w)\rangle$ on the category of graded C-modules. By Soergel's structure theorem [S1, Struktursatz 9], we therefore get a natural transformation g between the functors $\tilde{\theta}_{[x]}$ and $\tilde{\theta}_{[w]}$ restricted to projective objects. For arbitrary $N \in \mathrm{gmof}-A$, we choose a projective resolution $P^\bullet$. Since g is a natural transformation, it provides a morphism of resolutions $\tilde{\theta}_{[x]} P^\bullet \to \tilde{\theta}_{[w]} P^\bullet$ inducing a unique morphism $g_N : \tilde{\theta}_{[x]} N \to \tilde{\theta}_{[w]} N$. By standard arguments, g_N does not depend on the actual choice of the projective resolution, and these maps define a natural transformation of functors. Hence, Φ is surjective, and the statement of the proposition follows. □

We give the following example needed in the proof of Theorem 7.1.

COROLLARY 8.8

Let $x = sts = tst$ for noncommuting simple reflections s and t. Fix $[x] = tst$. Then $R = \operatorname{Hom}(\tilde{\theta}_{[x]}, \mathrm{ID})$ is strictly positively graded (i.e., $R_i = 0$ for $i \leq 0$) and $R_1 = \mathbb{C}$.

Proof

Direct calculations show that $\mathbf{C}_{[x]}(C)$ is generated as a C-bimodule by $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes X \otimes 1 \otimes 1$, where X denotes the coroot corresponding to s. Hence, $\mathbf{C}_{[x]}(C)\langle -3 \rangle$ is generated in degrees -3 and -1. Since C is positively graded with $C_0 = \mathbb{C}$, the statement follows because there is a nontrivial transformation of degree 1 (namely, $p \circ \mathrm{adj}_{\tilde{\theta}_s \tilde{\theta}_t}$ occurring in the proof of Theorem 7.1). □

A. Appendix. Explicit calculations in Type A

We consider the special example where $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{p} = \mathfrak{p}_m \cong \mathfrak{sl}_m \times \mathfrak{sl}_{n-m}$ is a maximal parabolic subalgebra.

Distinguished coset representatives

We first explicitly describe distinguished coset representatives. Let $W(n) = \langle s_1, \dots, s_n \rangle$ be the Weyl group of type A_n.

LEMMA A.1

Let $n \leq 1$. Then

$$W(n)^{\mathfrak{p}_1} = \{e, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_n\}, \tag{A.1}$$

and all the expressions are reduced.

Proof

The expressions in (A.1) are obviously reduced since no braid relation or commutator relation can be applied. For $n = 1$ or $n = 2$, the assertion is true. Let us assume the lemma to be true for type A_{n-1}. For $2 < j \leq n$, we get

$$\begin{aligned} l\big(s_j(s_1 s_2 \cdots s_k)\big) &= l(s_1 s_j s_2 \cdots s_k) = 1 + l(s_j s_2 \cdots s_k) \\ &> 1 + l(s_2 \cdots s_k) = l(s_1 s_2 \cdots s_k) \end{aligned}$$

by the induction hypothesis. On the other hand, $l(s_2(s_1 s_2 \cdots s_k)) = l(s_2 s_1 s_2) + l(s_3 s_4 \cdots s_k) = 3 + l(s_3 \cdots s_k) = 1 + l(s_1 s_2 s_3 \cdots s_k)$. Hence, the elements of the set (A.1) are distinguished coset representatives. Since $|W(n)^{\mathfrak{p}_1}| = \frac{|W(n)|}{|W(n)_{\mathfrak{p}_1}|}$, the lemma follows. □

Let $S(n+1, m)$ be the set of all subsets of order m of $\{0, 1, \dots, n\}$. We write $i_1 \triangleright i_2 \triangleright \cdots \triangleright i_k$ to denote the element $\{i_1, \dots, i_k\} \in S(n, k)$ with $i_1 > i_2 > \cdots > i_k$.

PROPOSITION A.2
Let $m \in \{1, \dots, n\}$. There is a bijection of sets

$$\begin{aligned}\Psi(n,m):\ S(n+1,m) &\longrightarrow W(n)^{\mathfrak{p}_m},\\ i_1 \triangleright i_2 \triangleright \cdots \triangleright i_k &\longmapsto (s_m s_{m+1}\cdots s_{i_1})(s_{m-1}s_m\cdots s_{i_2})\cdots(s_1 s_2\cdots s_{i_m}),\end{aligned} \tag{A.2}$$

where, by definition, $s_j s_{j+1}\cdot\ldots\cdot s_r = e$ if $r < j$. All the expressions occurring in the image of this map are reduced.

We write just $w = i_1 \triangleright i_2 \triangleright \cdots \triangleright i_k$ if they correspond via the bijection above. Moreover, we abuse notation and write just $i_1 \triangleright i_2 \triangleright \cdots \triangleright i_l$ with $l < m$ if $s_j s_{j-1}\cdot\ldots\cdot s_r = e$ for $j > l$.

Proof
For $n = 2$, or for n arbitrary but $m = 1$, the proposition holds by Lemma A.1. Now let $1 \leq m < n$. We assume that the claim holds for $\Psi(n', m')$ if either $n' < n$ or $n' = n$ and $m' < m$. Lemma A.1 successively shows that the occurring expressions are reduced.

Let

$$\begin{aligned} w &= (s_m s_{m+1}\cdots s_{i_1})(s_{m-1}s_m\cdots s_{i_2})\cdots(s_1 s_2\cdots s_{i_m})\\ &= (s_m s_{m+1}\cdots s_{i_1})y = w'(s_1 s_2\cdots s_{i_m}).\end{aligned}$$

To show that $w \in W^{\mathfrak{p}_m}$, we consider two cases.

- For $j \in \{2, 3, \dots, n\}\backslash\{m\}$, we have

  $$l(s_j w) = l(s_j w') + i_k = l(w') + 1 + i_k = l(w) + 1$$

  by the induction hypothesis.
- For $j = 1$, by the induction hypothesis,

  $$l(s_1 w) = l(s_1 s_m s_{m+1}\cdots s_{i_1}) + l(y) = 1 + l(s_m s_{m+1}\cdots s_{i_1}) + l(y) = 1 + l(w).$$

Hence, all the elements occurring in the image of $\Psi(n,m)$ are distinguished coset representatives. The remaining thing we have to prove is the injectivity of the map. Let us assume $\Psi(n,m)(i_1 \triangleright \cdots \triangleright i_m\}) = \Psi(n,m)(j_1 \triangleright \cdots \triangleright j_m)$. Since $s_{\max\{i_1,j_1\}}$ has to occur on both sides, we conclude that $i_1 = j_1$; hence,

$$(s_{m-1}s_m\cdots s_{i_2})\cdots(s_1 s_{n-1}\cdots s_{i_k}) = (s_{m-1}s_m\cdots s_{i_2})\cdots(s_1 s_{n-1}\cdots s_{i_k}).$$

The same argumentation gives successively $i_2 = j_2, \dots, i_k = j_k$. The theorem follows. □

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References

[A] L. ABRAMS, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5**, (1996), 569 – 587. MR 1414088 585, 588, 590

[AJS] H. H. ANDERSEN, J. C. JANTZEN, and W. SOERGEL, *Representations of Quantum Groups at a pth Root of Unity and of Semisimple Groups in Characteristic p: Independence of p*, Astérisque **220**, Soc. Math. France, Montrouge, 1994. MR 1272539 564

[AS] H. H. ANDERSEN and C. STROPPEL, *Twisting functors on $\mathcal{O}_0$*, Represent. Theory **7** (2003), 681 – 699. MR 2032059 563

[B1] E. BACKELIN, *Koszul duality for parabolic and singular category $\mathcal{O}$*, Represent. Theory **3** (1999), 139 – 152. MR 1703324 556, 580

[B2] ———, *The Hom-spaces between projective functors*, Represent. Theory **5** (2001), 267 – 283. MR 1857082 548, 562, 590

[Ba] H. BASS, *Algebraic K-Theory*, Benjamin, New York, 1968. MR 0249491 564

[BGS] A. BEILINSON, V. GINZBURG, and W. SOERGEL, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473 – 527. MR 1322847 548, 555, 556, 580

[BFK] J. BERNSTEIN, I. FRENKEL, and M. KHOVANOV, *A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors*, Selecta Math. (N.S.) **5** (1999), 199 – 241. MR 1714141 547, 548, 549, 550, 566, 567, 572, 573, 574, 580, 581, 584

[BG] J. N. BERNSTEIN and S. I. GEL′FAND, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), 245 – 285. MR 0581584 548, 558

[BGG] I. N. BERNSTEIN, I. M. GEL′FAND, and S. I. GEL′FAND, *A certain category of $\mathfrak{g}$-modules* (in Russian), Funkcional. Anal. i Priložen. **10**, no. 2 (1976), 1 – 8. MR 0407097 550, 551

[BW] S. C. BILLEY and G. S. WARRINGTON, *Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111 – 136. MR 1826948 567, 569, 570

[BK] T. BRZEZIŃSKI and J. KATRIEL, *Representation-theoretic derivation of the Temperley-Lieb-Martin algebras*, J. Phys. A **28** (1995), 5305 – 5312. MR 1364137 566

[D] V. V. DEODHAR, *On some geometric aspects of Bruhat orderings, II: The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987), 483 – 506. MR 0916182 553

[Di] T. TOM DIECK, *Temperley-Lieb algebras associated to the root system D*, Arch. Math. (Basel) **71** (1998), 407 – 416. MR 1649348 565

[ES] T. J. ENRIGHT and B. SHELTON, *Categories of highest weight modules: Applications to classical Hermitian symmetric pairs*, Mem. Amer. Math. Soc. **67** (1987), no. 367. MR 0888703 573, 575, 576

[FK] I. B. FRENKEL and M. G. KHOVANOV, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$*, Duke Math. J. **87** (1997), 409 – 480. MR 1446615 548

[G] R. M. GREEN, *Generalized Temperley-Lieb algebras and decorated tangles*, J. Knot Theory Ramifications **7** (1998), 155 – 171. MR 1618912 549, 567

[H] M. HÄRTERICH, *Murphy bases of generalized Temperley-Lieb algebras*, Arch. Math. (Basel) **72** (1999), 337 – 345. MR 1680384 566, 567

[I1] R. IRVING, *Projective modules in the category $\mathcal{O}_S$: Self-duality*, Trans. Amer. Math. Soc. **291** (1985), 701 – 732. MR 0800259 555

[I2] ———, *Shuffled Verma modules and principal series modules over complex semisimple Lie algebras*, J. London Math. Soc. (2) **48** (1993), 263 – 277. MR 1231714 549, 581

[J1] J. C. JANTZEN, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Math. **750**, Springer, Berlin, 1979. MR 0552943 551, 554

[J2] ———, *Einhüllende Algebren halbeinfacher Lie-Algebren*, Ergeb. Math. Grenzgeb. (3) **3**, Springer, Berlin, 1983. MR 0721170 551, 554, 566, 568, 576

[K] L. H. KAUFFMAN, *Knots and Physics*, 3rd ed., Ser. Knots Everything **1**, World Sci., River Edge, N.J., 2001. MR 1858113 548, 582

[KL] D. KAZHDAN and G. LUSZTIG, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165 – 184. MR 0560412 553

[Kh1] M. KHOVANOV, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), 359 – 426. MR 1740682 548, 584

[Kh2] ———, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002), 665 – 741. MR 1928174 549, 583

[L] T. Y. LAM, *A First Course in Noncommutative Rings*, Grad. Texts in Math. **131**, Springer, New York, 1991. MR 1125071 559, 560

[Li] J. LIPP, *Vergleich von Basen von Hecke-Algebren vom Typ A*, Diplomarbeit, Universität Stuttgart, Stuttgart, Germany, 1994. 566, 567

[M] P. MARTIN, "On Schur-Weyl duality, A_n Hecke algebras and quantum sl(N) on $\bigotimes^{n+1} \mathbf{C}^N$" in *Infinite Analysis, Part A, B (Kyoto, 1991)*, Adv. Ser. Math. Phys., World Sci., River Edge, N.J., 1992, 645 – 673. MR 1187568 566, 567

[MS] V. MAZORCHUK and C. STROPPEL, *Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module*, to appear in Trans. Amer. Math. Soc., http://ams.org/tran/0000-000-00/S0002-9947-04-03650-5/home.html; PII: S 0002-9947(04)03650-5 581, 583

[R] J. RICKARD, "Translation functors and equivalences of derived categories for blocks of algebraic groups" in *Finite-Dimensional Algebras and Related Topics (Ottawa, 1992)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **424**, Kluwer, Dordrecht, Netherlands, 1994, 255 – 264. MR 1308990 581, 583

[Ro] A. ROCHA-CARIDI, *Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Berňsteĭn-Gel′fand-Gel′fand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module*, Trans. Amer. Math. Soc. **262** (1980), 335 – 366. MR 0586721 552

[Ry] S. RYOM-HANSEN, *Koszul duality of translation and Zuckerman functors*, J. Lie Theory **14** (2004), 151 – 163. MR 2040174 549, 578, 584

[S1] W. SOERGEL, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), 421 – 445. MR 1029692 586, 590, 591

[S2] ———, *Kazhdan-Lusztig polynomials and a combinatorics for tilting modules*, Represent. Theory **1** (1997), 83 – 114. MR 1444322 553, 554, 566

[S3] ———, *On the relation between intersection cohomology and representation theory in positive characteristic*, J. Pure Appl. Algebra **152** (2000), 311 – 335. MR 1784005 586, 587

[St] C. STROPPEL, *Category $\mathcal{O}$: Gradings and translation functors*, J. Algebra **268** (2003), 301 – 326. MR 2005290 548, 550, 555, 556, 557, 564, 565, 578, 579

[TL] H. N. V. TEMPERLEY and E. H. LIEB, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A **322** (1971), 251 – 280. MR 0498284 565

[Tu] V. G. TURAEV, *Quantum Invariants of Knots and* 3*-Manifolds*, de Gruyter Stud. Math. **18**, de Gruyter, Berlin, 1994. MR 1292673 582, 584

[W] B. W. WESTBURY, *The representation theory of the Temperley-Lieb algebras*, Math. Z. **219** (1995), 539 – 565. MR 1343661 572

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