2-Segal spaces as invertible ∞-operads

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Abstract

We exhibit the simplex category Δ as an ∞-categorical localization of the category Ω_π of plane rooted trees introduced by Moerdijk and Weiss. As an application we obtain an equivalence of ∞-categories between 2-Segal simplicial spaces as introduced by Dyckerhoff and Kapranov and invertible non-symmetric ∞-operads. In addition, we prove analogous results where Δ is replaced by Connes' cyclic category Λ, the category of finite pointed sets or the category of non-empty finite sets; the corresponding categories of trees are given by plane trees, rooted trees and abstract trees, respectively.

Contents:

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1 Introduction

It is a well known fact that a simplicial set $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ is the nerve of a category if and only if the canonical map

$$\mathcal{X}_{[n]} \longrightarrow \mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \dots \times_{\mathcal{X}_{\{n-1\}}} \mathcal{X}_{\{n-1,n\}} \tag{1.1}$$

is a bijection; a simplicial set satisfying this condition is said to be **Segal**. The category corresponding to $\mathcal{X}$ has $\mathcal{X}_{[0]}$ as its set of objects and $\mathcal{X}_{[1]}$ as its set of morphisms; composition of morphisms is defined by the span

$$\mu\colon \mathcal{X}_{[1]} \times_{\mathcal{X}_{[0]}} \mathcal{X}_{[1]} \xleftarrow{\cong} \mathcal{X}_{[2]} \longrightarrow \mathcal{X}_{[1]}. \tag{1.2}$$

The Segal condition readily generalizes to simplicial objects $\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$ with values in any ∞-category $\mathcal{C}$ by replacing bijections of sets by equivalences and fiber products by their coherent counterpart.

In the literature we encounter two generalizations of this phenomenon to a multi-valued or operadic context:

- Dyckerhoff and Kapranov [DK] study the case where the first map in the span (1.2) is not an equivalence anymore; in this case one can still interpret μ as a "multi-valued composition law". This multi-valued composition is associative and unital precisely if the simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ satisfies the 2-*Segal condition*[1)]. There is a rich supply[2)] of 2-Segal simplicial objects and many of them carry additional structure in the form of a lift $\Delta^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{D}$ is one of Δ's "big brothers" like Connes' cyclic category Λ or the category $\mathbf{Fin}^{\mathrm{op}}_{(\star)}$ opposite to finite (pointed) sets.
- Moerdijk and Weiss [MW07] replace the simplex category Δ by an enlarged category Ω_{sym} of rooted trees[3)]; every element of Δ is seen as a linear tree in Ω_{sym}. Simplicial objects are then generalized to *symmetric dendroidal objects* $\Omega^{\mathrm{op}}_{\mathrm{sym}} \to \mathcal{C}$. In analogy to the case of categories, one can identify (colored) symmetric operads as those dendroidal sets $\mathcal{X}\colon \Omega^{\mathrm{op}}_{\mathrm{sym}} \to \mathbf{Set}$ that satisfy a dendroidal analogue of the Segal condition above; one can recover the set of n-ary operations as the value $\mathcal{X}(\mathrm{C}_n)$ at the tree C_n consisting of a single n-ary vertex. The category Ω_{sym} has various siblings; one example is the category Ω_π of *plane* rooted trees which describes *non-symmetric* operads.

The goal of this paper is to explain the relationship between these two theories. The key tool in this comparison are certain localization functors $\mathcal{L}$ which to each tree T associate an object that, roughly speaking, describes the boundary of T. In the case of a plane rooted tree, for instance, the boundary is described by the linearly ordered set of "areas" between the branches; this defines the functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ (see Section 2.2). The boundary of a non-plane rooted tree has less structure; the correct object in this case is the set of external edges pointed at the root, hence we obtain the functor $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}^{\mathrm{op}}_\star$ to the opposite (!) category of finite pointed sets (see Section 2.4). We also consider the category Ω_{cyc} of *cyclic trees* which are plane but unrooted; the corresponding boundary-functor $\mathcal{L}_{\mathrm{cyc}}\colon \Omega_{\mathrm{cyc}} \to \Lambda$ maps to the cyclic category (see Section 2.3).

By their very definition, the functors $\mathcal{L}$ send *boundary preserving* maps of trees to isomorphisms. The main result of this paper is that they are universal with this property in the strongest possible sense:

Theorem 1.0.1. The functor $\mathcal{L}_\pi$ exhibits Δ as an ∞-categorical localization of Ω_π at the set of boundary preserving maps. The same is true, mutatis mutandis, for each of the other functors

[1)] To be consistent with the original definition, we should say *unital* 2-Segal. However, we drop the word "unital" since non-unital 2-Segal objects don't play any role in this paper.

[2)] The main source of 2-Segal simplicial objects is Waldhausen's S-construction [Wal85]. We refer to Dyckerhoff's lecture notes [Dyc] for a survey of its many variants.

[3)] The category Ω_{sym} is usually just denoted by Ω; we add the subscript to clearly distinguish it from other categories of trees appearing in this paper.

$\mathcal{L}$ discussed in this paper. ♥

Theorem 1.0.1 implies that composition with $\mathcal{L}_\pi$ induces an equivalence of ∞-categories between

- simplicial objects $\Delta^{\mathrm{op}} \to \mathcal{C}$ and
- so-called *invertible* dendroidal objects $\Omega_\pi^{\mathrm{op}} \to \mathcal{C}$

with values in any given ∞-category $\mathcal{C}$. It follows directly from the explicit description of this equivalence that the dendroidal Segal condition corresponds precisely to the simplicial 2-Segal condition (see Lemma 4.1.10).

Using the equivalence between Segal dendroidal sets and operads[4)], we recover the following version of a result due to Dyckerhoff and Kapranov [DK, Thm. 3.6.8] proven with very different techniques:

Corollary 1.0.2. The functor $\mathcal{L}_\pi$ induces an equivalence of categories between 2-Segal simplicial sets and so-called *invertible* (non-symmetric) operads. □

Cisinski and Moerdijk [CM] proved that *complete* Segal symmetric dendroidal spaces are a model for symmetric ∞-operads. We expect the analogous result to hold in the non-symmetric case so that we can model non-symmetric ∞-operads as complete Segal planar dendroidal spaces. It turns out that the completeness condition is redundant in the case of *invertible* dendroidal spaces (see Lemma 4.3.10), hence our main result specializes to the following ∞-categorical version of Corollary 1.0.2 (see Section 4.3 for more details).

Corollary 1.0.3. The functor $\mathcal{L}_\pi$ induces an equivalence of ∞-categories between 2-Segal simplicial spaces and invertible (non-symmetric) ∞-operads. □

The remaining localization functors yield similar results between structured 2-Segal spaces and various flavors of invertible ∞-operads. For instance, the localization functor $\mathcal{L}_{\mathrm{cyc}}$ establishes an equivalence between the ∞-categories of cyclic 2-Segal spaces and invertible cyclic (non-symmetric) ∞-operads.

1.1 Acknowledgements

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[4)] This fact was established by Cisinski and Moerdijk [CM, Corollary 2.6] in the symmetric case; the non-symmetric case is analogous.

2 The localization functors

2.1 The category Ω_π of plane rooted trees

We recall some basic facts about (colored, non-symmetric) operads and the category Ω_π of plane rooted trees. See e.g. Weiss's thesis [Wei07] for more details.

Definition 2.1.1. A **(colored, non-symmetric) operad** $\mathcal{O} = (\mathcal{O}, O, \circ)$ consists of

- a collection O of **objects** (or **colors**),
- given colors $x_1, \dots, x_n, y \in O$, a set $\mathcal{O}(x_1, \dots, x_n; y)$ of n**-ary operations** from $(x_1, \dots, x_n)$ to y and
- for each $k, n_1, \dots, n_k \in \mathbb{N}$ and colors $x^i_{j_i}, z \in O$ (for $0 \leq j_i \leq n_i$, $0 \leq i \leq k$), a **composition map**

$$\coprod_{y_1,\dots,y_k \in O} \big(\mathcal{O}(x^1_1, \dots, x^1_{n_1}; y_1) \times \cdots \times \mathcal{O}(x^k_1, \dots, x^k_{n_k}; y_k)\big) \times \mathcal{O}(y_1, \dots, y_k; z) \xrightarrow{\circ} \mathcal{O}(x^1_1, \dots, x^1_{n_1}, \dots, x^k_1, \dots, x^k_{n_k}; z)$$

- for each color $x \in O$, a **unit map** $1 \colon \{x\} \longrightarrow \mathcal{O}(x, x)$.

such that the obvious associativity and unitality conditions are satisfied. There is an obvious notion of a morphism of operads, we denote the resulting category of operads by **Op**. ♣

Remark 2.1.2. Each operad has an underlying category with objects $x \in O$ and morphism sets $\mathcal{O}(x, y)$. Conversely, each category can be viewed as an operad which has only 1-ary operations. More precisely, we have an adjunction **Cat** $\rightleftarrows$ **Op** with fully faithful left adjoint. ♢

Remark 2.1.3. If the reader were to encounter an "operad" in the literature, it might or might not be understood to be mono-colored, and it might or might not be understood to be symmetric. Throughout this paper, we use the word "operad" to mean "non-symmetric colored operad". ♢

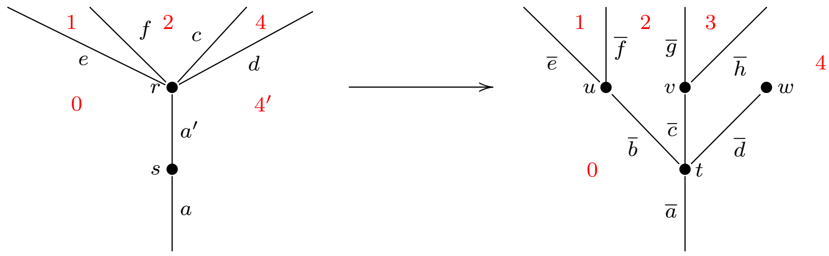
An object of Ω_π is called a **plane rooted tree** and consist of a finite plane rooted trees in the usual graph-theoretic sense together with a marking of some degree 1 vertices including the root-vertex. An edge between unmarked vertices is called internal, the other edges are called external. The unique external edge connected to the root-vertex is called the **root** (or output edge); an external edge attached to a marked non-root vertex is called a **leaf** (or input edge). The fact that a tree is plane means precisely that there is a designated linear order on the leaves.

Remark 2.1.4. From now on we completely ignore the marked vertices of a tree and never speak of them again. Thus "vertex" always means "unmarked vertex". When drawing trees, we omit the marked vertices and instead draw the external edges "towards infinity" (see Example 2.1.5 below). ♢

The number of leaves of a tree is its **arity**. Each vertex of a tree has some number (the **arity** of that vertex) of input edges and a unique output edge (which is the one that points in the direction of the root). We denote by η or $[0]$ the tree with only a single edge (which is both the root and a leaf); we denote by $\mathrm{C}_{[n]}$ or C_n the n-corolla, i.e. the unique n-ary tree with a single vertex.

Each plane rooted tree T gives rise to a free operad (also denoted by T); it has a color for each edge of T and its operations are freely generated by the vertices of T (an n-ary vertex is seen as an n-ary operation from its input edges to its output edge). A morphism in Ω_π between two trees is defined to be a morphism of the corresponding operads.

Example 2.1.5. Consider the following two plane rooted trees where the root is always drawn towards south. The operad associated to the left tree has colors $\{a', a, c, d, e, f\}$ and three non-unit operations $s \colon a' \to a$ and $r \colon (e, f, c, d) \to a'$ and $r \circ s \colon (e, f, c, d) \to a$. The other one has colors $\{\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}\}$ and eleven non-unit operations (t, u, v, w and all their composites).



The depicted morphism is described on colors by $a' \mapsto \overline{a}$, $a \mapsto \overline{a}$, $c \mapsto \overline{c}$ etc. and on generating operations by $s \mapsto 1_{\overline{a}}$ and $r \mapsto (u, 1_{\overline{c}}, 1_{\overline{d}}) \circ t$. (The red numbers are for later reference.) ♢

A **(planar) dendroidal object** in an ∞ category $\mathcal{C}$ is functor $\mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{C}$. We denote by $\mathbf{d}_\pi\mathbf{Set} \coloneqq [\Omega_\pi^{\mathrm{op}}, \mathbf{Set}]$ the category of (planar) dendroidal sets, i.e. dendroidal objects in **Set**. Given a plane rooted tree T, we denote by $\Omega_\pi[T]$ the dendroidal set represented by T. There is a canonical fully faithful embedding $\Delta \hookrightarrow \Omega_\pi$ of the simplex category Δ by interpreting every linearly ordered set as a linear tree. This embedding gives rise to an adjunction $\mathbf{sSet} \rightleftarrows \mathbf{d}_\pi\mathbf{Set}$ with fully faithful left adjoint. The inclusion $\Omega_\pi \hookrightarrow \mathbf{Op}$ (which is full by construction) gives rise to a realization/nerve-adjunction

$$\mathbf{d}_\pi\mathbf{Set} \rightleftarrows \mathbf{Op} \colon \mathrm{N}_d$$

which extends the usual adjunction $\mathbf{sSet} \rightleftarrows \mathbf{Cat} \colon \mathrm{N}$.

2.2 The functor $\mathcal{L}_\pi \colon \Omega_\pi \longrightarrow \Delta$

Let us introduce the main player in our game.

Construction 2.2.1. [Covariant description of $\mathcal{L}_\pi$] Each plane rooted tree $T \in \Omega_\pi$ (which we visualize with its external edges going towards infinity) partitions the plane into a set $\mathcal{L}_\pi T$ of "areas" which is linearly ordered clockwise starting from the root. It is straightforward to extend this assignment to a functor $\mathcal{L}_\pi \colon \Omega_\pi \to \Delta$. ♣

Remark 2.2.2. We give an alternative, more formal, construction of the functor $\mathcal{L}_\pi$ at the end of this section, see Construction 2.2.12 below. ♢

Example 2.2.3. The functor $\mathcal{L}_\pi$ sends the morphism depicted in Example 2.1.5 to the map $\{0, 1, 2, 4, 4'\} \to \{0, 1, 2, 3, 4\}$ in Δ. ♢

Remark 2.2.4. Specifying two adjacent "areas" of a plane rooted tree $T \in \Omega_\pi$ uniquely determines an external edge of T that separates them. If we write $[n] \coloneqq \mathcal{L}_\pi T$ (where n is the arity of T) then

- each minimal edge $\{i-1, i\} \hookrightarrow [n]$ (for $1 \leq i \leq n$) corresponds precisely to a leaf of T and
- the maximal edge $\{0, n\} \hookrightarrow [n]$ corresponds to the root of T. ♢

Remark 2.2.5. Usually the category of trees is related to the simplex category by the inclusion $\Delta \hookrightarrow \Omega_\pi$ of the linear trees. The composition $\Delta \hookrightarrow \Omega_\pi \xrightarrow{\mathcal{L}_\pi} \Delta$ is constant with value $[1] \in \Delta$. The two occurrences of the category Δ in relation to the category Ω_π are in some sense "orthogonal": the first is sensitive to the "depth" of a tree, the second measures the "width". ♢

Definition 2.2.6. A map of plane rooted trees is called **boundary preserving** if it maps the root to the root and each leaf to a leaf. ♣

Definition 2.2.7. A **collapse map** in Ω_π is a boundary preserving map $\mathrm{C}_{[n]} \to T$ out of a corolla (where n is the arity of T). A dendroidal object $\mathcal{X} \colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **invertible** if $\mathcal{X}$ maps all collapse maps to equivalences in $\mathcal{C}$. ♣

Remark 2.2.8. A boundary preserving map $\alpha\colon T \to S$ of plane rooted trees induces a bijection between the leaves of T and the leaves of S. Hence the functor $\mathcal{L}_\pi$ maps boundary preserving maps to isomorphisms. ♢

Remark 2.2.9. The motivation for the word "invertible" in Definition 2.2.7 will become apparent in Section 4.2 when we discuss invertible operads (in the sense of Dyckehoff and Kapranov [DK]) and show that an operad is invertible if and only if its nerve is an invertible dendroidal set (Lemma 4.2.4). ♢

Here is one version of our main result which we explain and prove in Section 3 below:

Theorem 2.2.10. The functor $\mathcal{L}_\pi$ exhibits Δ as an ∞-categorical localization of Ω_π at the set of collapse maps. ♥

Before going forward, we give a "contravariant" description of the functor $\mathcal{L}_\pi$. This description is useful because unlike the covariant one it can easily be adapted to the case of symmetric trees (see Section 2.4).

Lemma 2.2.11. The category Δ^{op} can be identified with the following category Δ_{bp}: objects are finite (possibly empty) linearly ordered sets; a morphism $f\colon N \to M$ consist of a monotone tri-partition $N = N^f_- \dot\cup N^f \dot\cup N^f_+$ (monotone means that $n_- < n < n_+$ for all $n_* \in N^f_*$) together with a weakly monotone map $f\colon N^f \to M$. ♥

Proof. Equivalently (by adding a minimal and a maximal element to each object) Δ_{bp} is the category whose objects are finite linearly ordered sets with at least two elements and whose morphisms preserve the boundary points. Using this description, the equivalence $\Delta^{\mathrm{op}} \simeq \Delta_{\mathrm{bp}}$ is given by the mutually inverse (contravariant) functors which send a linearly ordered set to its set of cuts (resp. non-degenerate cuts); in formulas:

$$\Delta \ni N \longmapsto \{N = L \dot\cup R \text{ monotone}\} \in \Delta_{\mathrm{bp}}$$
$$\Delta_{\mathrm{bp}} \ni M \longmapsto \{M = L \dot\cup R \text{ monotone} \mid L \neq \varnothing \neq R\} \in \Delta. \qquad \square$$

Using the identification $\Delta^{\mathrm{op}} \simeq \Delta_{\mathrm{bp}}$ we can give the following description of the functor $\mathcal{L}_\pi$, which is easily seen to be equivalent to Construction 2.2.1.

Construction 2.2.12 (Contravariant description of $\mathcal{L}_\pi$)**.** To each plane rooted tree $T \in \Omega_\pi$ we associate the (possibly empty) linearly ordered set $\mathcal{L}_\pi T \in \Delta_{\mathrm{bp}}$ of its leaves. This association extends to a functor $\Omega_\pi^{\mathrm{op}} \to \Delta_{\mathrm{bp}}$ in the following way: Given a map $\alpha\colon T \leftarrow S$ of trees, we can (monotonely) partition the leaves of T as $\mathcal{L}_\pi T = (\mathcal{L}_\pi T)^\alpha_- \dot\cup (\mathcal{L}_\pi T)^\alpha \dot\cup (\mathcal{L}_\pi T)^\alpha_+$, where $\mathcal{L}_\pi T^\alpha$ (resp. $\mathcal{L}_\pi T^\alpha_-$, resp. $\mathcal{L}_\pi T^\alpha_+$) consist of those leaves of T which lie over (resp. to the left of, resp. to the right of) the image under α of the root r_S of S. Given a leaf l of T that lies over $\alpha(r_S)$, there is a unique leaf l' of S such that $\alpha(l') \leq l$; we define $(\mathcal{L}_\pi\alpha)(l) \coloneqq l'$ to be this unique leaf. ♣

2.3 Variant: plane unrooted trees and the cyclic category Λ

We recall the definition of Connes' cyclic category Λ.

Definition 2.3.1. [Con83] To each natural number $n \in \mathbb{N}$ corresponds an object $[n] \in \Lambda$ which we interpret as the unit circle S^1 in the complex plane with $n+1$ many equidistant marked points. The morphisms are homotopy classes of weakly monotone maps $S^1 \to S^1$ of degree 1 that send marked points to marked points. ♣

Remark 2.3.2. We fix the inclusion $\Delta \hookrightarrow \Lambda$ which arranges the $n+1$ many elements of an object $[n] \in \Delta$ as marked points on a circle. This inclusion is dense and faithful but not full. ♢

We define the category Ω_{cyc} of **plane *unrooted* trees**. In analogy to how Ω_π is a full subcategory of the category **Op** of operads, we define Ω_{cyc} as a full subcategory of the category of cyclic operads which we now define.

Definition 2.3.3. A **cyclic structure** on an operad $(\mathcal{O}, O, \circ)$ consists of

- an involution $(-)^\vee \colon O \to O$ on colors (called **duality**) and
- a system of **rotation isomorphisms**

$$\mathcal{O}(x_1, \ldots, x_n; y) \xrightarrow{\cong} \mathcal{O}(y^\vee, x_1, \ldots, x_{n-1}; x_n^\vee)$$

  which is compatible with the composition of operations;

such that for each $n \in \mathbb{N}$ the $(n+1)$-fold composition

$$\mathcal{O}(x_1, \ldots, x_n; y) \xrightarrow{\cong} \mathcal{O}(y^\vee, x_1, \ldots, x_{n-1}; x_n^\vee) \xrightarrow{\cong} \mathcal{O}(x_n^\vee, y^\vee, x_1, \ldots, x_{n-2}; x_{n-1}^\vee)$$
$$\xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathcal{O}(x_2, \ldots, x_n, y^\vee; x_1^\vee) \xrightarrow{\cong} \mathcal{O}(x_1, \ldots, x_n; y)$$

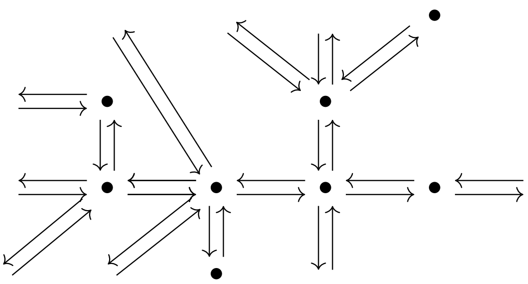
of rotation isomorphisms is equal to the identity.

A **cyclic operad** is an operad together with a cyclic structure. The cyclic operads are assembled into a category **cycOp** where the morphisms are required to be compatible with the additional structure in the obvious way. ♣

Remark 2.3.4. We have an adjunction $\mathbf{Op} \rightleftarrows \mathbf{cycOp}$ where the right adjoint forgets the cyclic structure and the left adjoint adds a cyclic structure freely. ♢

Definition 2.3.5. A **plane (unrooted) tree** consists of vertices and (unoriented) edges arranged in the plane, where an edge can connect two vertices or go to infinity in one or (in the case of the unique tree η with no vertices) both directions. We require our trees to have at least one external edge. We think of each unoriented edge as a pair of anti-parallel arrows. ♣

Example 2.3.6. A typical example of a plane tree looks as follows:



♢

We call an arrow a **leaf** if comes from infinity and a **root** if it goes to infinity. An arrow a is called a **direct predecessor** of an arrow b (and b is then a **direct successor** of a) if there is a vertex which is both the target $t(a)$ of a and the source $s(b)$ of b. We say a is a **predecessor** of b (or b is a successor of a), written $a \geq b$, if a is an iterated direct predecessor of b (this includes the case $a = b$). The **arity** of a tree (resp. a vertex) is n, where $n+1$ is the number of arrows leaving (or, equivalently, entering) the tree (resp. the vertex).

Remark 2.3.7. For every arrow b in a tree T, the set of predecessors of b in T forms a plane rooted tree (the root is b itself). In particular there is a preferred linear order (clockwise along the boundary) on the set of those leaves a of T which are predecessors of b. ♢

Construction 2.3.8. Each plane tree T gives rise to a cyclic operad (also denoted T) as follows:

- Each arrow is a color.
- Each pair (v, a) consisting of an n-ary vertex $v \in T$ and an arrow a starting in v gives rise to an n-ary operation

$$v_a \colon (a_1, \ldots, a_n) \longrightarrow a$$

  where the a_i's are the direct predecessors of a (hence $t(a_i) = v$) in clockwise order. All other operations are freely generated by these v_a's.

- The involution on the colors exchanges the two anti-parallel arrows associated to a single edge.
- The rotation isomorphisms are given on generators by $v_a \mapsto v_{a_n^\vee}$. ♣

Definition 2.3.9. We define the category $\Omega_{\mathrm{cyc}} \subset$ **cycOp** of plane trees to be the full subcategory spanned by the cyclic operads T constructed as above. ♣

Remark 2.3.10. Our category Ω_{cyc} is very close to the category of plane unrooted trees introduced by Joyal and Kock [JK]; the only difference is that we require our trees to have at least one external edge. For instance, we do not allow the tree • which consists only of a single vertex, since this tree can not be interpreted as a cyclic operad in a meaningful way. ♢

Remark 2.3.11. The free-cyclic-structure functor **Op** → **cycOp** induces an inclusion $\Omega_\pi \to \Omega_{\mathrm{cyc}}$ which replaces each edge with two anti-parallel arrows and forgets the root. ♢

Remark 2.3.12. The cyclic operad corresponding to the tree η (which has no vertices and exactly two mutually anti-parallel arrows) consists of two colors which are dual to each other and no non-identity operations. This cyclic operad η has an involution given by exchanging the two colors, i.e. the two arrows. A morphism $\eta \to \mathcal{O}$ to some cyclic operad $\mathcal{O}$ corresponds to a color of $\mathcal{O}$; the involution on the colors of $\mathcal{O}$ is induced by the involution on η. ♢

Remark 2.3.13. It is easy to check that an operation in the cyclic operad $T \in \Omega_{\mathrm{cyc}}$ is uniquely determined by its input and output colors. Hence a map $S \to T$ between such operads is uniquely determined by the value at each arrow. Such a map would not, however, be determined by its values on unoriented edges; for instance, every unoriented edge e of a tree T gives rise to two different maps $\eta \to T$ in Ω_{cyc} corresponding to the two mutually dual colors described by e.

If one were only interested in mono-colored cyclic operads or, more generally, cyclic operads with trivial duality (i.e. every color is self-dual), then it would be enough to consider unoriented edges. This point of view is taken by Hackney, Robertson and Yau [HRY]. ♢

Definition 2.3.14. A map of plane trees is called **boundary preserving** if it maps leaves to leaves and roots to roots. A **collapse map** in Ω_π is a boundary preserving map $\mathrm{C} \to T$ out of a corolla. A dendroidal object $\mathcal{X} \colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **invertible** if $\mathcal{X}$ maps all collapse maps to equivalences in $\mathcal{C}$. ♣

As the notation suggests, the category Ω_{cyc} of plane trees has a close relationship to the cyclic category: the latter is a localization of the former.

Construction 2.3.15 (Covariant description of $\mathcal{L}_{\mathrm{cyc}}$)**.** Analogously to the case of rooted plane trees, an unrooted plane tree partitions the plane into "areas" which are arranged clockwise around a circle. This assignment is a functor $\mathcal{L}_{\mathrm{cyc}} \colon \Omega_{\mathrm{cyc}} \to \Lambda$ which extends the functor $\mathcal{L}_\pi \colon \Omega_\pi \to \Delta$. ♣

Construction 2.3.16 (Contravariant description of $\mathcal{L}_{\mathrm{cyc}}$)**.** Using the self-duality $\Lambda \cong \Lambda^{\mathrm{op}}$ (which interchanges marked points and intervals on a circle) we can define the functor $L \colon \Omega_{\mathrm{cyc}} \to \Lambda^{\mathrm{op}}$ instead:

A tree T gets mapped to its set of leaves which are naturally arranged around a circle. The image of a morphism $\alpha \colon S \to T$ sends each leaf a of T to the unique leaf b of S such that $\alpha(b)$ is a successor of a. This assignment does not yet uniquely determine $L\alpha$ as a morphism in Λ; we still need to specify a linear order on the pre-images $(L\alpha)^{-1}(b)$ (for every leaf b of S) but this is taken care of by Remark 2.3.7. ♣

We will prove the following result in Section 3 below:

Theorem 2.3.17. The functor $\mathcal{L}_{\mathrm{cyc}} \colon \Omega_{\mathrm{cyc}} \to \Lambda$ exhibits Λ as an ∞-categorical localization of Ω_{cyc} at the set of collapse maps. ♥

2.4 Variant: symmetric (rooted) trees and finite (pointed) sets

Denote by Ω_{sym} the category of symmetric rooted trees (i.e. trees without a plane embedding). It is defined as a full subcategory of **symOp**, the category of symmetric operads. A symmetric operad is an operad equipped with an action of the symmetric groups which interchanges the input colors. All the notions from Section 2.1 have an obvious analogue which we shall not describe here again.

We can also define a functor $\mathcal{L}_{\mathrm{sym}}$, which is analogous to $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ by adapting the contravariant construction of the latter.

Construction 2.4.1 (The functor $\mathcal{L}_{\mathrm{sym}}$)**.** We define the functor $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ to the (opposite) category of finite pointed sets as follows: To each tree T we assign the set of external edges which is pointed at the root. Given a morphism $\alpha\colon S \to T$ of rooted trees and a leaf e of T there is at most one external edge d of S such that $\alpha(d) \leq e$. We define $(\mathcal{L}_{\mathrm{sym}}\alpha)(e) \coloneqq d$ if such a d exists and $(\mathcal{L}_{\mathrm{sym}}\alpha)(e) \coloneqq \star$ otherwise. ♣

It is straightforward to show that $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ is well defined and extends the functor $\mathcal{L}_\pi$ in the sense that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbf{Op} & \hookleftarrow & \Omega_\pi & \xrightarrow{\mathcal{L}_\pi} & \Delta & \underset{\mathrm{cut}}{\overset{\simeq}{\longleftrightarrow}} & \Delta_{\mathrm{bp}}^{\mathrm{op}} \\
\downarrow{\scriptstyle \mathrm{sym}} & & \downarrow & & & \swarrow & \\
\mathbf{symOp} & \hookleftarrow & \Omega_{\mathrm{sym}} & \xrightarrow{\mathcal{L}_{\mathrm{sym}}} & \mathbf{Fin}_\star^{\mathrm{op}} & &
\end{array}$$

where the leftmost arrow is the symmetrization functor and the rightmost diagonal arrow forgets the linear ordering and adds a basepoint.

Remark 2.4.2. By combining the ideas from Section 2.3 and Section 2.4 we can construct a category of abstract (i.e. non-plane and unrooted) trees as a full subcategory of cyclic symmetric operads[5)]. The corresponding functor $\mathcal{L}_{\mathrm{abs}}\colon \Omega_{\mathrm{abs}} \to \mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$ maps a tree to its nonempty set of leaves (i.e. incoming arrows). Given a morphism $\alpha\colon S \to T$ of rooted trees and a leaf a of T we define $(\mathcal{L}_{\mathrm{sym}}\alpha)(a)$ to be the unique leaf b of S such that $\alpha(b) \leq a$ (i.e. $\alpha(b)$ is a successor of a). ♢

Remark 2.4.3. Unlike $\mathcal{L}_\pi$ and $\mathcal{L}_{\mathrm{cyc}}$, which can be described concretely in terms of "areas" between branches, it appears that the functors $\mathcal{L}_{\mathrm{sym}}$ and $\mathcal{L}_{\mathrm{abs}}$ do not admit a nice covariant description. ♢

We have the following localization result (see Section 3):

Theorem 2.4.4. The functor $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ (resp. $\mathcal{L}_{\mathrm{abs}}\colon \Omega_{\mathrm{abs}} \to \mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$) exhibits $\mathbf{Fin}_\star^{\mathrm{op}}$ (resp. $\mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$) as an ∞-categorical localization of Ω_{sym} (resp. Ω_{abs}) at the set of collapse maps. ♥

[5)] Such operads have both a cyclic and a symmetric structure which are compatible when regarding the symmetric group S_n and the cyclic group $\mathbb{Z}\,/\,(n+1)$ as a subgroup of S_{n+1}.

3 Proof of the localization theorems

We collect here the main results we want to prove.

Theorem 3.0.1. For every ∞-category $\mathcal{C}$, the functor $\mathcal{L}_\pi \colon \Omega_\pi \to \Delta$ induces a fully faithful functor

$$\mathcal{L}_\pi^\star \colon \operatorname{Fun}(\mathrm{N}(\Delta), \mathcal{C}) \longrightarrow \operatorname{Fun}(\mathrm{N}(\Omega_\pi), \mathcal{C})$$

of ∞-categories with essential image spanned by those functors $\mathrm{N}(\Omega_\pi) \to \mathcal{C}$ which map all collapse maps $\mathrm{C} \to T$ to equivalences. The analogous statement is true for the functors $\mathcal{L}_{\mathrm{cyc}} \colon \Omega_{\mathrm{cyc}} \to \Lambda$, $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ and $\mathcal{L}_{\mathrm{abs}} \colon \Omega_{\mathrm{abs}} \to \mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$. ♥

Remark 3.0.2. The notation $\operatorname{Fun}(\mathcal{C}', \mathcal{C})$ denotes the ∞-category of functors $\mathcal{C} \to \mathcal{C}'$ [Lur09, Notation 1.2.7.2]. ♢

3.1 The general situation

Our strategy to prove Theorem 3.0.1 is to apply the following general lemma which we will prove separately in Section 3.2 below.

Lemma 3.1.1. Let $L \colon W \to D$ be a functor of (ordinary) categories and for each $n \in D$ let $B_n \subset W_n$ be a subcategory of the weak fiber W_n of L such that (with the notation of Remark 3.1.2 below)

- B_n has an initial object c_n and
- the inclusion $\mathrm{N}(B_n) \hookrightarrow \mathrm{N}(W)_{/n}$ is cofinal.

Then for every ∞-category $\mathcal{C}$, composition with L induces a fully faithful functor

$$L^\star \colon \operatorname{Fun}(\mathrm{N}(D), \mathcal{C}) \longrightarrow \operatorname{Fun}(\mathrm{N}(W), \mathcal{C})$$

of ∞-categories with the essential image spanned by those functors $\mathrm{N}(W) \to \mathcal{C}$ which send all the edges of the form $c_n \to t$ in $\mathrm{N}(B_n)$ (for $n \in D$) to equivalences. ♥

Remark 3.1.2. Recall that the weak fiber W_n (also called 2-fiber) of $L \colon W \to D$ is the category whose objects consist of an object $t \in W$ and an isomorphism $t \xrightarrow{\cong} n$ in D. The left fiber $W_{/n} \supset W_n$ has objects $(t, f \colon t \to n)$ where f is not required to be an isomorphism. ♢

Let Ω be any one of the categories Ω_π, Ω_{cyc}, Ω_{sym}, Ω_{abs}; let $\mathcal{L}$ be the corresponding functor (among $\mathcal{L}_\pi$, $\mathcal{L}_{\mathrm{cyc}}$, $\mathcal{L}_{\mathrm{sym}}$, $\mathcal{L}_{\mathrm{abs}}$) and denote its target (which is either Δ, Λ, $\mathbf{Fin}_\star^{\mathrm{op}}$ or $\mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$) by $\mathcal{D}$. For every object $[n] \in \mathcal{D}$ we denote by $\Omega_{/[n]}$ the left fiber, by $\Omega_{[n]}$ the weak fiber and by $\mathrm{bp}_{[n]} \subset \Omega_n$ the subcategory of $\Omega_{[n]}$ with the same objects but only boundary preserving morphisms. We shall now show that the functors $\mathcal{L}$ satisfy the requirements for Lemma 3.1.1, thus concluding the proof of Theorem 3.0.1.

Proposition 3.1.3. Fix an object $[n] \in \mathcal{D}$.

(1) The n-corolla $\mathrm{C}_{[n]}$ (together with any identification $\mathcal{L}\mathrm{C}_{[n]} \xrightarrow{\cong} [n]$) is an initial object in the category $\mathrm{bp}_{[n]}$.

(2) The inclusion $\mathrm{bp}_{[n]} \subset \Omega_{[n]} \hookrightarrow \Omega_{/[n]}$ has a left adjoint.

♥

Corollary 3.1.4. The inclusion $\mathrm{bp}_{[n]} \hookrightarrow \Omega_{/[n]}$ is cofinal in the sense of Joyal [Joy08, 8.11] [Lur09, Theorem 4.1.3.1]. □

Proof (of Proposition 3.1.3)**.** The first statement is obvious.
The functor $\Omega_{/[n]} \to \mathrm{bp}_{[n]}$ is constructed as follows: Given an object $(T, f \colon \mathcal{L}T = [m] \to [n])$ we define the tree T_f by glueing some corollas to T along its outer edges (see also Figure 1). We only describe this process explicitly for $\mathcal{L} = \mathcal{L}_\pi$ but the construction is essentially the same in the other cases.

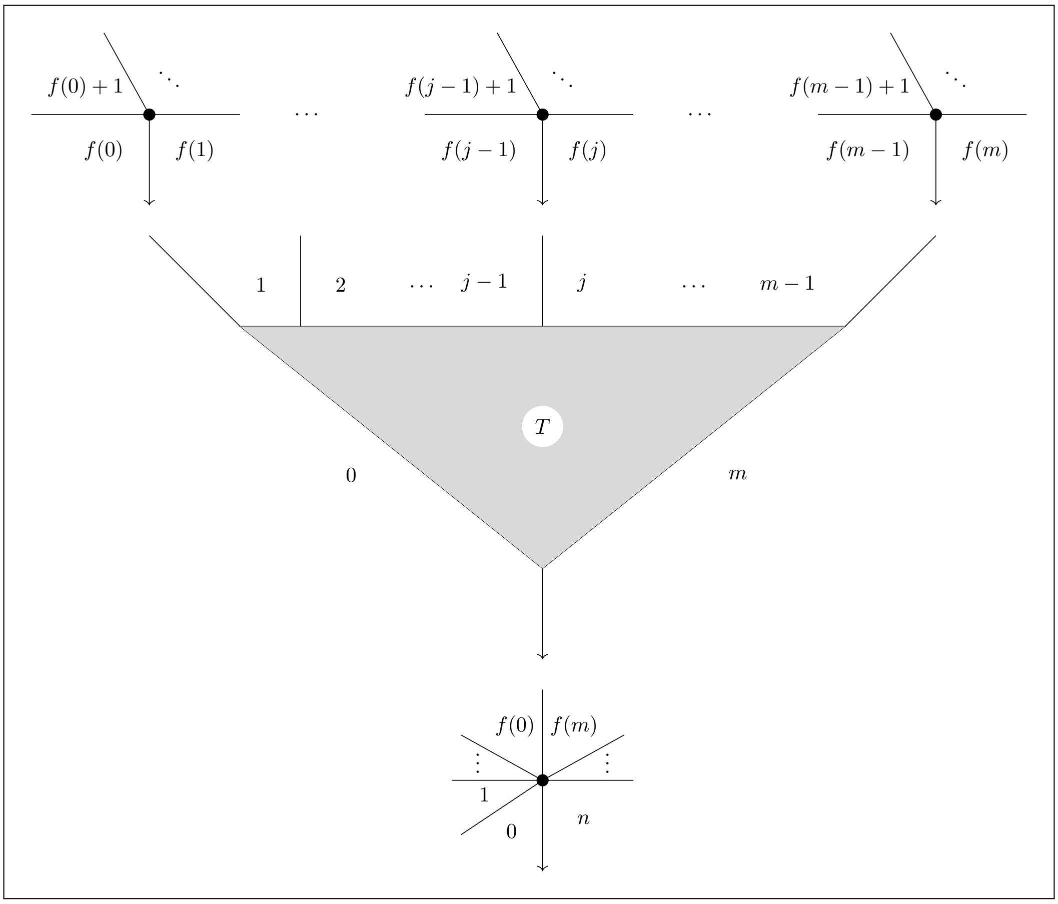


Figure 1: The construction of the tree T_f in the case $\mathcal{L} = \mathcal{L}_\pi$. The little arrows decorate the roots of the various trees. Forgetting the root and/or the plane embedding describes the analogous construction in the cases $\mathcal{L} = \mathcal{L}_{\mathrm{cyc}}, \mathcal{L}_{\mathrm{sym}}, \mathcal{L}_{\mathrm{abs}}$

- To a leaf of T corresponding to the minimal edge $\{j-1, j\} \hookrightarrow [m]$ we glue a corolla $\mathrm{C}^f_{j-1,j}$ (of arity $f(j) - f(j-1)$) with leaves $\{i-1, i\}$ for $f(j-1) < i \leq f(j)$ (this might be a 0-corolla if $f(j-1) = f(j)$).
- To the root (corresponding to the maximal edge $\{0, m\} \hookrightarrow [m]$) we glue a corolla $\mathrm{C}^f_{\max}$ with leaves

$$\{0,1\}, \{1,2\}, \ldots, \{f(0)-1, f(0)\}, \{f(0), f(m)\}, \{f(m), f(m)+1\}, \ldots, \{n-1, n\}$$

along the special leaf $\{f(0), f(m)\}$ of $\mathrm{C}^f_{\max}$.

The adjunction unit at (T, f) is the inclusion $T \hookrightarrow T_f$ which we denote by f_T. We need to prove that given a morphism of trees $\alpha\colon T \to S$ over $f\colon [m] \to [n]$ there is a unique factorization $T \xrightarrow{f_T} T_f \xrightarrow{\alpha^{\mathrm{bp}}} S$ with α^{bp} in $\mathrm{bp}_{[n]}$. We have no other choice than to define α^{bp} as α on the subtree $T \hookrightarrow T_f$ and to make it the identity on the boundary; hence uniqueness is clear. It is straightforward to verify that this map of trees is indeed well defined. □

3.2 Proof of Lemma 3.1.1

Let M be defined as the Grothendieck construction of the functor $\Delta^1 \to \mathbf{Cat}$ which parametrizes the functor $L\colon W \to D$. Explicitly, an object in M is either an object $t \in W$ or an object $n \in D$; for $s, t \in W$ and $m, n \in D$ we put $M(t, s) = W(t, s)$ and $M(n, m) = D(n, m)$ and $M(t, n) = D(Lt, n)$ and $M(n, t) = \varnothing$. We have a factorization $L\colon W \hookrightarrow M \xrightarrow{\overline{L}} D$ where the first arrow is the obvious fully faithful inclusion and the second arrow has a fully faithful right adjoint $D \hookrightarrow M$. We identify D with its image in M and we denote by $\eta\colon \mathrm{id}_M \to \overline{L}$ the unit of the adjunction $\overline{L}\colon M \rightleftarrows D$; it is an isomorphism (in fact the identity) at exactly those objects in M that belong to D.[6)]

We deal with the two components of $L\colon W \hookrightarrow M \rightleftarrows D$ individually by using standard techniques from Higher Topos Theory [Lur09]. Lemma 3.1.1 is a direct consequence of Corollary 3.2.4 and Corollary 3.2.9 below.

Remark 3.2.1. For each $n \in D$ the forgetful functor $B_n \subset W_n \to W$ extends to a functor ${B_n}^{\triangleright} \hookrightarrow M$ by sending the new vertex v to n and the new arrow $(t, f) \to v$ (for $(t, f) \in B_n$) to the arrow $f\colon t \to n$ of W. ◇

Fix an ∞-category $\mathcal{C}$. We recall the following result:

Lemma 3.2.2. [Lur09, Proposition 5.2.7.12] Let $\overline{L}\colon \mathcal{M} \to \mathcal{D}$ be a localization functor of ∞-categories (i.e. $\overline{L}$ has a fully faithful right adjoint) and let $\mathcal{C}$ be another ∞-category. Then composition with $\overline{L}$ induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{C})$$

with essential image consisting of those functors that map an edge f in $\mathcal{M}$ to an equivalence in $\mathcal{C}$ provided that $\overline{L}f$ is an equivalence in $\mathcal{D}$. ♥

Lemma 3.2.3. Let $F\colon \mathrm{N}(M) \to \mathcal{C}$ be a functor of ∞-categories. The following are equivalent:

(1) For every edge f in $\mathrm{N}(M)$, if $\overline{L}f$ is an equivalence in D then Ff is an equivalence in $\mathcal{C}$.

(2) For every $n \in D$, the functor F maps all edges in $\mathrm{N}(B_n)^{\triangleright}$ to equivalences in $\mathcal{C}$.

(3) F sends every component $\eta_t\colon t \to \overline{L}t$ of the unit to an equivalence in $\mathcal{C}$.

We denote by K^+ the full subcategory of $\mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ spanned by such functors. ♥

Proof. Clearly (1) implies (2) (because $\overline{L}(f)$ is the identity for each edge f of $\mathrm{N}(B_n)^{\triangleright}$) and (2) trivially implies (3).
Observe that if $f\colon t \to s$ is a morphism in M then we have a commutative naturality square

$$\begin{array}{ccc} t & \xrightarrow{\eta_t} & \overline{L}t \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{L}f} \\ s & \xrightarrow{\eta_s} & \overline{L}s \end{array}$$

Hence (3) implies (1) by the two-out-of-three property for equivalences in $\mathcal{C}$. □

Corollary 3.2.4. Composition with the functor $\overline{L}\colon M \to D$ induces a fully faithful functor $\mathrm{Fun}(\mathrm{N}(D), \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ with essential image K^+. □

Let us recall the following result:

Lemma 3.2.5. [Lur09, Proposition 4.3.1.12] Let $\mathcal{C}$ be an ∞-category and let $\overline{F}\colon B^{\triangleright} \to \mathcal{C}$ be a diagram where B is a weakly contractible simplicial set and $\overline{F}$ carries each edge of B to an equivalence in $\mathcal{C}$. Then $\overline{F}$ is a colimit diagram in $\mathcal{C}$ if and only if it carries every edge in $B^{\triangleright}$ to an equivalence in $\mathcal{C}$. ♥

Lemma 3.2.6. Let $F\colon \mathrm{N}(W) \to \mathcal{C}$ be a functor. The following are equivalent:

(1) The functor F admits a left Kan extension along $W \hookrightarrow M$ and the resulting functor $\mathrm{N}(M) \to \mathcal{C}$ lies in K^+.

(2) For every $n \in D$ the functor F maps every edge of $\mathrm{N}(B_n)$ to an equivalence in $\mathcal{C}$.

(3) For every $n \in D$ and every $t \in B_n$ the functor F maps the unique edge $c_n \to t$ in $\mathrm{N}(B_n)$ to an equivalence in $\mathcal{C}$.

6) The components $\eta_t\colon t \to \overline{L}t$ of the adjunction are precisely the coCartesian morphisms of the coCartesian fibration $M \to \Delta^1$.

We denote by K the full subcategory of $\mathrm{Fun}(\mathrm{N}(W), \mathcal{C})$ spanned by such functors. ♥

Proof. The equivalence between (2) and (3) is obvious because c_n is an initial element in B_n. Using description (2) of Lemma 3.2.3 it is clear that (1) implies (2).

Let us prove the converse: By the pointwise construction of Kan extensions [Lur09, Lemma 4.3.2.13], a left Kan extension of F along $W \hookrightarrow M$ can be assembled from colimit cones for the diagrams $\mathrm{N}(W)_{/n} \to \mathrm{N}(W) \xrightarrow{F} \mathcal{C}$ (for $n \in D$). Recall that $B_n \hookrightarrow W_{/n}$ is cofinal, hence we can reduce to finding colimits for the diagrams $\mathrm{N}(B_n) \hookrightarrow \mathrm{N}(W_{/n}) \to \mathrm{N}(W) \xrightarrow{F} \mathcal{C}$. All edges of these diagrams are equivalences by condition (2) and $\mathrm{N}(B_n)$ is contractible (because B_n has an initial element). Therefore by Lemma 3.2.5 these colimits exists and the corresponding colimit cones $\mathrm{N}(B_n)^{\triangleright} \to \mathcal{C}$ map all edges to equivalences in $\mathcal{C}$, thus verifying condition (2) of Lemma 3.2.3. □

Fix the following notation:

- Denote by H^+ the full subcategory of $\mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ spanned by those functors which are the left Kan extension of their restriction to $W \subset M$.
- Denote by H the full subcategory of $\mathrm{Fun}(\mathrm{N}(W), \mathcal{C})$ spanned by those functors which admit a left Kan extension along $W \hookrightarrow M$.

Recall the following result:

Lemma 3.2.7. [Lur09, Proposition 4.3.2.15] The restriction functor along $\mathrm{N}(W) \hookrightarrow \mathrm{N}(M)$ is a trivial fibration $H^+ \to H$ of simplicial sets. ♥

Lemma 3.2.8. We have inclusions $K^+ \subset H^+$ and $K \subset H$ and a pullback square

$$\begin{array}{ccc} K^+ & \hookrightarrow & H^+ \\ \downarrow & & \downarrow \\ K & \hookrightarrow & H \end{array}$$

of simplicial sets with vertical arrows given by restriction along $W \hookrightarrow M$. ♥

Proof. This follows directly from Lemma 3.2.3 and Lemma 3.2.6 □

Since trivial fibrations of simplicial sets are stable under pullbacks we obtain:

Corollary 3.2.9. The restriction functor along the inclusion $W \hookrightarrow M$ is a trivial fibration $K^+ \to K$ of simplicial sets. □

This concludes the proof of Lemma 3.1.1 and therefore of Theorem 3.0.1.

4 Applications

Consider the category **sSet** $:= [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets equipped with the classical (Kan-Quillen) left proper combinatorial simplicial model structure. Denote by $\mathcal{S} := \mathrm{N}_\Delta(\mathbf{sSet}^\circ)$ the corresponding **∞-category of spaces** obtained as the simplicial nerve of the subcategory of fibrant-cofibrant objects. A dendroidal (resp. simplicial) object in $\mathcal{S}$ is called a dendroidal (resp. simplicial) space.

4.1 2-Segal simplicial objects and Segal dendroidal objects

In this section we compare the dendroidal Segal condition due to Cisinski and Moerdijk [CM] and the simplicial 2-Segal condition due to Dyckerhoff and Kapranov [DK].

Definition 4.1.1. [CM, Definition 2.2] The **Segal core** of a tree $\eta \neq T \in \Omega_{\mathrm{sym}}$ is the union

$$\mathrm{Sc}[T] := \bigcup_v \Omega_{\mathrm{sym}}[\mathrm{C}_{n(v)}]$$

where v runs over all vertices of T and $\mathrm{C}_{n(v)} \hookrightarrow T$ denotes the subtree with vertex v. We use the convention $\mathrm{Sc}[\eta] := \Omega_{\mathrm{sym}}[\eta]$ for the trivial tree.

A symmetric dendroidal space $\mathcal{X} \colon \mathrm{N}(\Omega_{\mathrm{sym}}^{\mathrm{op}}) \to \mathcal{S}$ is **Segal** if for any tree $T \in \Omega_{\mathrm{sym}}$ the map

$$\mathcal{X}_T = \mathrm{Hom}(\Omega_{\mathrm{sym}}(T), \mathcal{X}) \longrightarrow \mathrm{Hom}(\mathrm{Sc}[T], \mathcal{X})$$

is a trivial fibration. ♣

We adapt this definition as follows.

Definition 4.1.2. A dendroidal object $\mathcal{X} \colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **Segal** if $\mathcal{X}$ sends the diagram

$$\begin{array}{ccc} T & \hookleftarrow & T_2 \\ \uparrow & & \uparrow \\ T_1 & \hookleftarrow & e \end{array} \tag{4.1}$$

to a coherent pullback square in $\mathcal{C}$ whenever the tree $T \in \Omega_\pi$ arises by grafting two trees T_1 and T_2 along a common edge e. ♣

Remark 4.1.3. Clearly Definition 4.1.1 and Definition 4.1.2 make sense, mutatis mutandis, for planar, symmetric, cyclic, and cyclic symmetric dendroidal objects. ♢

Remark 4.1.4. If a tree T arises by grafting two trees T_1 and T_2 along a common edge e then clearly $\mathrm{Sc}[T] = \mathrm{Sc}[T_1] \sqcup_e \mathrm{Sc}[T_2]$. By successively decomposing a tree along its inner edges we therefore see that Definition 4.1.1 and Definition 4.1.2 agree for dendroidal objects in the ∞-category $\mathcal{S}$ of spaces. ♢

The importance of the dendroidal Segal condition is highlighted by the following result:

Proposition 4.1.5. [CM, Corollary 2.6] The symmetric dendroidal nerve functor

$$\mathrm{N}_d \colon \mathbf{symOp} \longrightarrow \mathbf{dSet}$$

is fully faithful and the essential image consists precisely of the Segal symmetric dendroidal sets. ♥

Remark 4.1.6. Proposition 4.1.5 directly generalizes to all types of operads discussed in this paper: non-symmetric operads, symmetric operads, cyclic operads, cyclic symmetric operads. ♢

Definition 4.1.7. [DK, Proposition 2.3.2] A simplicial object $\mathcal{X}\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **(unital) 2-Segal** if for each $0 \le i \le j \le m$ it maps the square

$$\begin{array}{ccc} \{0,\dots,m\} & \hookleftarrow & \{i,\dots,j\} \\ \uparrow & & \uparrow \\ \{0,\dots,i,j,\dots m\} & \hookleftarrow & \{i,j\} \end{array} \tag{4.2}$$

in Δ to a coherent pullback square square in $\mathcal{C}$. ♣

Remark 4.1.8. We always interpret the elements i and j in the lower row of Diagram 4.2 as distinct; thus in the case $i = j$ the vertical arrows are degeneracy maps. ♢

Remark 4.1.9. Since non-unital 2-Segal objects never make an appearance in this paper, we just write "2-Segal" and leave the adjective "unital" implicit. ♢

Lemma 4.1.10. A simplicial object $\mathcal{X}\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is 2-Segal if and only if the composition $\mathcal{L}_\pi^\star\mathcal{X}\colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \xrightarrow{\mathcal{L}_\pi} \mathrm{N}(\Delta^{\mathrm{op}}) \xrightarrow{\mathcal{X}} \mathcal{C}$ is a Segal dendroidal object. ♥

Proof. Let $T = T_1 \cup_e T_2$ be a grafting of trees where e is the root of T_2 and a leaf of T_1. Put $[m] \coloneqq \mathcal{L}_\pi T$. Applying $\mathcal{L}_\pi$ to the inclusion $e \hookrightarrow T$ defines a map $[1] = \mathcal{L}_\pi e \xrightarrow{f} [m]$, so we can define $i \coloneqq f(0)$ and $j \coloneqq f(1)$. It is easy to see that with this notation $\mathcal{L}_\pi$ sends Diagram (4.1) to Diagram (4.2) and that every instance of Diagram (4.2) arises this way. □

Remark 4.1.11. The dendroidal Segal condition (which reduces to the simplicial 2-Segal condition under the functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$) is defined for all dendroidal spaces, not just invertible ones. Hence one might wonder if we can put a reasonable condition on dendroidal objects (without requiring invertibility) which reduces to the (ordinary) simplicial Segal condition. The naive way certainly fails, since the commutative square

$$\begin{array}{ccc} [2] & \hookleftarrow & \{1,2\} \\ \uparrow & & \uparrow \\ \{0,1\} & \hookleftarrow & \{1\} \end{array}$$

in Δ (which describes an instance of the simplicial Segal condition) does not admit a lift to Ω_π. ♢

4.2 Equivalence: 2-Segal simplicial sets and invertible operads

Definition 4.2.1. [DK, Def. 3.6.7] An operad $\mathcal{O}$ is called **invertible** if all the composition and unit maps (as in Definition 2.1.1) are invertible. ♣

Remark 4.2.2. If an operad $\mathcal{O}$ is invertible then its underlying category is discrete. ♢

Proposition 4.2.3. [DK, Thm. 3.6.8] Fix a set B of colors. Then there is an equivalence of categories between invertible B-colored operads and 2-Segal simplicial sets $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ with $\mathcal{X}_{[1]} = B$. ♥

We can characterize invertibility of an operad in terms of its dendroidal nerve.

Lemma 4.2.4. Let $\mathcal{O}$ be an operad and let $\mathrm{N}_d(\mathcal{O})\colon \Omega_\pi^{\mathrm{op}} \to \mathbf{Set}$ be its dendroidal nerve. The following are equivalent:

(1) The dendroidal set $\mathrm{N}_d(\mathcal{O})$ maps all boundary preserving maps to isomorphisms.
(2) The dendroidal set $\mathrm{N}_d(\mathcal{O})$ is invertible, i.e. it inverts all collapse maps. ♥
(3) The operad $\mathcal{O}$ is invertible.

Proof. If $\alpha\colon T \to S$ is boundary preserving, then clearly the collapse map for S factors through the collapse map for T as $\mathrm{C} \to T \xrightarrow{\alpha} S$. Hence (1) and (2) are equivalent by the 2-out-of-3-property for isomorphisms.

Taking the coproduct over all the unit maps in Definition 2.1.1 yields precisely the image under $\mathrm{N}(\mathcal{O})$ of the collapse map $\mathrm{C}_1 \to \eta$. Taking the coproduct over all the composition maps for fixed $k, n_1, \ldots, n_k \in \mathbb{N}$ yields (putting $n := \sum_{i=1}^{k} n_i$) precisely the image of the collapse map $\mathrm{C}_n \to T_k^{n_1,\ldots,n_k}$, where $T_k^{n_1,\ldots,n_k}$ is tree obtained by glueing (for all $0 \leq i \leq k$) the corolla C_{n_i} to the i-th leaf of the corolla C_k. Hence (2) implies (3). The converse holds because every "generalized composition map" represented by a collapse map $\mathrm{C} \to T$ can be written as the composition of unit and composition maps as in Definition 2.1.1. □

Using

- the characterization of operads as Segal dendroidal sets (the non-symmetric analogue of Proposition 4.1.5),
- the characterization of invertible operads (Lemma 4.2.4),
- our main result (Theorem 2.2.10) in the case $\mathcal{C} = \mathbf{Set}$ and
- the corresponcence between Segal dendroidal objects and 2-Segal simplicial objects (Lemma 4.1.10)

we recover the following more elegant version of Proposition 4.2.3.

Corollary 4.2.5. The composition $\mathbf{sSet} \xrightarrow{\mathcal{L}_\pi^\star} \mathbf{d}_\pi\mathbf{Set} \longrightarrow \mathbf{Op}$ restricts to an equivalence of categories between the full subcategories of 2-Segal simplicial sets on one side and invertible operads on the other. □

4.3 Equivalence: 2-Segal simplicial spaces and invertible ∞-operads

As a direct consequence of Theorem 2.2.10 and Lemma 4.1.10 we obtain the following comparison result:

Corollary 4.3.1. Composition with $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ induces an equivalence between the ∞-category of 2-Segal simplicial spaces and the ∞-category of invertible Segal dendroidal spaces. □

The goal of this Section 4.3 is to give an interpretation of this result by identifying the ∞-category of invertible Segal dendroidal spaces as a full subcategory of the ∞-category of complete Segal dendroidal spaces. We treat the latter as a model for (non-symmetric) ∞-operads (in analogy to results due to Cisinski and Moerdijk [CM] in the symmetric case) so that we can rephrase Corollary 4.3.2 as follows:

Corollary 4.3.2. Composition with $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ induces an equivalence between the ∞-category of 2-Segal simplicial spaces and the ∞-category of invertible (non-symmetric) ∞-operads. □

The theory of complete Segal dendroidal spaces was developed by Cisinski and Moerdijk [CM] for *symmetric* dendroidal spaces. They prove that complete Segal symmetric dendroidal spaces are a model for symmetric ∞-operads (see Theorem 4.3.4 below). We briefly retrace their main definitions in the world of non-symmetric operads but we do not re-prove their theorem in this setting. We will use the resulting model category of complete Segal *planar* dendroidal spaces (or rather, its underlying ∞-category) as a model for (non-symmetric) ∞-operads even though, strictly speaking, this is not motivated by the current state of the literature.

Construction 4.3.3. [CM, Sections 5. and 6.] We build the simplicial model category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}$ of **complete Segal dendroidal spaces** (also called **dendroidal Rezk model category**) as constructed by Cisinski and Moerdijk in the symmetric case:
Take the Reedy model structure[7)] on the functor category $\mathbf{dsSet} := [\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]$ and then Bousfield-localize [Lur09, Proposition A.3.7.3] two times:

[7)] Cisinski and Moerdijk actually use a generalized version of the Reedy model structure since the category Ω_{sym} of symmetric rooted trees is not a Reedy category (unlike Ω_π, which is).

(1) by the Segal core inclusions $\mathrm{Sc}[T] \longrightarrow \Omega_\pi[T]$ and
(2) by the maps $\Omega_\pi[T] \otimes J_d \longrightarrow \Omega_\pi[T]$, where J_d is the dendroidal nerve of the category $\bullet \xrightarrow{\cong} \bullet$ with two objects and a single isomorphism between them. ♣

The Reedy model category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{Reedy}}$ has a canonical simplicial enrichment [RV14, Theorem 10.3] which is maintained by the Bousfield localization processes [Lur09, Proposition A.3.7.3]. Therefore we can construct what we call the **∞-category of ∞-operads** as the simplicial nerve of the fibrant-cofibrant objects:

$$\mathcal{O}\mathrm{p} \coloneqq \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}^\circ)$$

The name is justified by the following result:

Theorem 4.3.4. [CM, Corollary 6.8] The inclusion $\mathbf{dSet} \hookrightarrow [\Omega_{\mathrm{sym}}, \mathbf{sSet}]_{\mathrm{cS}}$ is a left Quillen equivalence between the model category of symmetric ∞-operads as defined by Cisinski, Moerdijk and Weiss [MW07,CM11] and the model category of complete Segal symmetric dendroidal spaces. ♥

Definition 4.3.5. We denote by $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{iS}}$ the Bousfield localization of $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}$ by the collapse maps

$$\Omega_\pi[\mathrm{C}_n] \longrightarrow \Omega_\pi[T]$$

for each n-ary tree T; we call it the **model category of invertible Segal dendroidal spaces**. We denote the corresponding **∞-category of invertible ∞-operads** by

$$\mathrm{i}\mathcal{O}\mathrm{p} \coloneqq \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{iS}}^\circ). \qquad ♣$$

Remark 4.3.6. It is immediate from the characterization of Bousfield localization that $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{iS}}^\circ$ is a full simplicial subcategory of $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}^\circ$. Hence the ∞-category $\mathrm{i}\mathcal{O}\mathrm{p}$ of invertible ∞-operads is a full subcategory of the ∞-category $\mathcal{O}\mathrm{p}$ of (all) ∞-operads. ♢

Lemma 4.3.7. The ∞-category $\mathrm{i}\mathcal{O}\mathrm{p}$ of invertible ∞-operads is equivalent to the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Omega_\pi^{\mathrm{op}}), \mathcal{S})$ consisting of those dendroidal spaces $\mathcal{X} \colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{S}$ which are invertible Segal and satisfy the following **completeness** condition:

- For each tree T, the maps $\Omega_\pi[T] \otimes J_d \to \Omega_\pi[T]$ from Construction 4.3.3 induce equivalences

$$\mathrm{Hom}(\Omega_\pi[T] \otimes J_d, \mathcal{X}) \xrightarrow{\simeq} \mathcal{X}_T. \tag{4.3}$$

♥

To prove Lemma 4.3.7 we use the following result:

Proposition 4.3.8. [Lur09, Proposition 4.2.4.4.] Let $\mathbf{A}$ be a combinatorial simplicial model category, $\mathcal{C}$ a small simplicial category and S a simplicial set equipped with an equivalence $\mathfrak{C}[S] \xrightarrow{\simeq} \mathcal{C}$. Then the induced map

$$\mathrm{N}_\Delta([\mathcal{C}, \mathbf{A}]^\circ) \longrightarrow \mathrm{Fun}(S, \mathrm{N}_\Delta(A^\circ))$$

is a categorical equivalence of simplicial sets. ♥

Remark 4.3.9. In Proposition 4.3.8 it does not matter whether we equip $[\mathcal{C}, \mathbf{A}]$ with the injective, projective or (if $\mathcal{C}$ is a Reedy category) with the Reedy model structure, since they are all Quillen equivalent [Lur09, Remark A.2.9.23]. ♢

Proof (of Lemma 4.3.7)**.** We specialize Proposition 4.3.8 to $\mathbf{A} \coloneqq \mathbf{sSet}$ and $\mathcal{C} \coloneqq \Omega_\pi^{\mathrm{op}}$ (seen as a discrete simplicial category); we put $S \coloneqq \mathrm{N}(\Omega_\pi^{\mathrm{op}}) = \mathrm{N}_\Delta(\Omega_\pi^{\mathrm{op}})$ equipped with the adjunction counit $\mathfrak{C}[\mathrm{N}_\Delta(\Omega_\pi^{\mathrm{op}})] \xrightarrow{\simeq} \Omega_\pi$. We obtain an equivalence

$$\mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{Reedy}}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{N}(\Omega_\pi^{\mathrm{op}}), \mathcal{S}) \tag{4.4}$$

of ∞-categories. Passing to Bousfield localizations replaces the simplicial category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{Reedy}}$ by the full subcategory of the new fibrant-cofibrant objects. Therefore the equivalence (4.4) restricts to an equivalence between $\mathrm{i}\mathcal{O}\mathrm{p} := \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{iS}})$ and some full subcategory of $\mathrm{Fun}(\mathrm{N}(\Omega_\pi^{\mathrm{op}}), \mathcal{S})$ whose objects are determined by the fibrancy conditions in the three localization steps. Each of these steps corresponds precisely to one of the three conditions (invertibility, Segal, completeness) in Lemma 4.3.7. □

We will now see that the completeness condition in Lemma 4.3.7 is redundant.

Lemma 4.3.10. An invertible Segal dendroidal space is automatically complete. ♥

Proof. Let $\mathcal{X}\colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{S}$ be an invertible Segal space. We need to show that for each plane rooted tree $T \in \Omega_\pi$ the map $\mathcal{X}_T \to \mathrm{Hom}(\Omega_\pi[T] \otimes J_d, \mathcal{X})$ is a week equivalence. By invoking the Segal condition and decomposing T along its interior edges we can reduce to the case where $T = \mathrm{C}_{[n]}$ is a corolla. The dendroidal set $\Omega_\pi[\mathrm{C}_{[n]}] \otimes J_d$ is the dendroidal nerve of the operad $\mathrm{C}_{[n]} \otimes_{\mathrm{BV}} \left(\bullet \xrightarrow{\cong} \bullet\right)$ (where $\otimes_{\mathrm{BV}}$ denotes the tensor product of operads introduced by Boardman-Vogt [BV73]), which admits the following explicit construction:

First, take two copies of the operad $T = \mathrm{C}_{[n]}$, join them by ($n+1$ many) new arrows (a.k.a. 1-ary operations) between the respective colors and require the resulting "square" to commute; this constructs $T \otimes_{\mathrm{BV}} [1]$. Second, adjoin inverses to the new connecting arrows.

We can express this construction as the colimit of the following diagram:

$$\begin{array}{ccccccccc}
\eta \longleftarrow & \{0,2\} & & T_+ & & T^+ & & \coprod\{0,2\} & \longrightarrow \coprod\eta \\
& \downarrow & & \uparrow & \nwarrow \quad \nearrow & \uparrow & & \downarrow & \\
& [3] & \longleftarrow & \{1,2\} & \mathrm{C}_{[n]} & \coprod\{1,2\} & \longrightarrow & \coprod[3] & \\
& \uparrow & & & & & & \uparrow & \\
\eta \longleftarrow & \{1,3\} & & & & & & \coprod\{1,3\} & \longrightarrow \coprod\eta
\end{array} \tag{4.5}$$

where T_+ (resp. T^+) are trees that arise from T by glueing an 1-corolla to the root (resp. an 1-corolla to each leaf); the maps $\{1,2\} \hookrightarrow T_+$ and $\coprod\{1,2\} \hookrightarrow T^+$ are the inclusions of the new vertices and the diagonal maps are the unique boundary preserving maps. Indeed, the triangle in the middle describes[8)] the operad $T \otimes \Delta^1$; the pieces on the outside add a left and a right inverse to each of the new arrows.

Since $\mathcal{X}\colon \mathrm{N}(\Omega_\pi^{\mathrm{op}}) \to \mathcal{S}$ is invertible, it maps almost all of the arrows in Diagram (4.5) to equivalences; the only exceptions are the inclusions $\{1,2\} \hookrightarrow T_+$ and $\coprod\{1,2\} \hookrightarrow T^+$. The map $T \otimes J_d \to T$ is described by the degeneracy maps $T^+ \to T$ and $T_+ \to T$ which contract the new 1-corollas; these are also sent to equivalences by $\mathcal{X}$. Hence the map $\mathcal{X}_T \to \mathrm{Hom}(\Omega_\pi[T] \otimes J_d, \mathcal{X})$ is equivalent to the map $\mathcal{X}_T \to \mathrm{Hom}(\mathrm{colim}(\eta \hookrightarrow T \hookleftarrow \coprod\eta), X)$ which is an equivalence. □

Lemma 4.3.10 motivates the name "invertible Segal" (rather than "invertible complete Segal") in Definition 4.3.5 and completes the transition from Corollary 4.3.1 to Corollary 4.3.2.

8) This is analogous to the description of $\Delta^1 \times \Delta^1$ as the pushout of $\Delta^2 \hookleftarrow \Delta^{\{0,2\}} \hookrightarrow \Delta^2$.

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