Categorification of tensor powers of the vector representation of $U_q(\mathfrak{gl}(1|1))$

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Abstract

We consider the monoidal subcategory of finite-dimensional representations of $U_q(\mathfrak{gl}(1|1))$ generated by the vector representation, and we provide a graphical calculus for the intertwining operators, which enables to compute explicitly the canonical basis, as well as the action of $U_q(\mathfrak{gl}(1|1))$. We construct a categorification using graded subquotient categories of the BGG category $\mathcal{O}(\mathfrak{gl}_n)$ and graded functors between them (translation, Zuckermann's and coapproximation functors). We describe then the regular blocks of these categories as modules over explicit diagram algebras, which are defined using Soergel modules and combinatorics of symmetric polynomials. We construct diagrammatically standard and proper standard modules for the properly stratified structure of these algebras.

Contents

Introduction

The *Jones polynomial* is a classical invariant of links in $\mathbb{R}^3$ defined using the vector representation of the Lie algebra $\mathfrak{sl}_2$ (or, more precisely, of the quantum algebra $U_q(\mathfrak{sl}_2)$). In his fundamental paper [Kho00], Khovanov constructed a graded homology theory for links whose graded Euler characteristic is the Jones polynomial. Khovanov homology has two main advantages over the Jones polynomial: first, it has been proved to be a finer invariant and second, it has values in a category of complexes and it also assigns to cobordisms between links chain maps between chain complexes. This categorical approach to classical invariants is often called *categorification*. Khovanov's work raised great interest in categorification, and since then a categorification program for representations of more general semisimple Lie algebras and even Kac-Moody algebras has been developed by several authors and motivated various generalizations (see for example [FKS06], [MS09], [Web13], [KL09], [KL11], [Rou08]). The main tools in all these works come from *representation theory* and geometry related to it.

Another very important invariant of knots is the *Alexander polynomial* [Ale28], which is much older than the Jones polynomial. Originally defined using the topology of the knot complement, the Alexander polynomial is not the quantum invariant corresponding to some complex semisimple Lie algebra, like the Jones polynomial. Instead, it can be defined using the representation theory of the general Lie *super*algebra $\mathfrak{gl}(1|1)$ (or, more precisely, its quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$; alternatively, one can use the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ where q is a root of unity, see [Vir06], but we will not consider this approach). A categorification of the Alexander polynomial exists, but comes from a very different area of mathematics: a homology theory, known as Heegard-Floer homology, whose Euler characteristic gives the Alexander polynomial, has been developed using symplectic geometry ([OS05], [MOST07]). This homology theory, however, does not have an interpretation or a counterpart in representation theory yet.

The present work is motivated by the attempt to construct/understand categorifications of Lie superalgebras (and hopefully a categorification of the Alexander polynomial) using tools from representation theory. In fact, there are only a few other recent works studying representation theoretical categorifications of Lie superalgebras and related structures ([Kho10], [FL13]). We hope that this thesis can be the starting point of a categorification program for $\mathfrak{gl}(1|1)$, beginning with a categorification of tensor powers of the vector representation and of their subrepresentations. We point out that a counterpart of our construction in the setting of symplectic and contact geometry has been developed by Tian [Tia12], [Tia13].

The main result of this thesis can be summarized as follows:

Main Theorem 1 (See Theorems 6.2.2 and 6.5.4). *Let V be the (complex) vector representation of $U_q(\mathfrak{gl}(1|1))$, fix $n > 0$ and consider the commuting actions of $U_q(\mathfrak{gl}(1|1))$ and of the Hecke algebra $\mathcal{H}_n = \mathcal{H}(\mathbb{S}_n)$ on $V^{\otimes n}$:*

$$U_q(\mathfrak{gl}(1|1)) \circlearrowright V^{\otimes n} \circlearrowleft \mathcal{H}_n. \qquad (\maltese)$$

For each $n > 0$ there exists a triangulated category $\mathcal{D}^\nabla\mathcal{Q}(\mathfrak{n})$ whose Grothendieck group is isomorphic to $V^{\otimes n}$ and two families of endofunctors $\{\mathcal{E}, \mathcal{F}\}$ and $\{\mathcal{C}_i \mid i = 1, \dots, n-1\}$ which commute with each other and which on the Grothendieck group level give the actions (✠) *of $U_q(\mathfrak{gl}(1|1))$ and of the Hecke algebra $\mathcal{H}_n$ on $V^{\otimes n}$ respectively:*

$$[\mathcal{E}], [\mathcal{F}] \circlearrowright \mathbf{K}^{\mathbb{C}(q)}(\mathcal{D}^\nabla\mathcal{Q}(\mathfrak{n})) \circlearrowleft [\mathcal{C}_i].$$

A remarkable property (and also a complication) of the finite-dimensional representations of $\mathfrak{gl}(1|1)$ (and more generally of $\mathfrak{gl}(m|n)$) is that they need not be semisimple. For example, if V is the vector representation of $\mathfrak{gl}(1|1)$, then $V \otimes V^*$ is a four-dimensional indecomposable non-irreducible representation. It is not clear how the lack of semisimplicity should affect the categorification, but it is plausible that this provides additional difficulties. What we can categorify in the present work is indeed only a semisimple monoidal subcategory of the representations of $\mathfrak{gl}(1|1)$, that contains the vector representation V, but not its dual V^*. We remark that in this thesis we will develop all the details for the quantum version, but in order to keep this introduction technically clean we avoid to introduce the quantum enveloping algebra now.

Our categorification relies on a very careful analysis of the representation theory of $\mathfrak{gl}(1|1)$ and its *canonical basis* (based on [Zha09]). In the categorification, indecomposable projective modules correspond to canonical basis elements, that we can compute explicitly via a diagram calculus, analogous to the diagram calculus developed in [FK97] for $\mathfrak{sl}_2$. The key-tool for our construction is the so-called *super Schur-Weyl duality* (originally studied in [BR87] and [Ser84]): the symmetric group algebra $\mathbb{C}[\mathbb{S}_n]$ acts on the tensor power $V^{\otimes n}$, and this action commutes with the action of $\mathfrak{gl}(1|1)$. The weight spaces of $V^{\otimes n}$ are modules for $\mathbb{C}[\mathbb{S}_n]$, and explicitly they are isomorphic to mixed induced modules of the form

$$(\dagger) \qquad (\mathrm{trv}_{\mathbb{S}_k} \boxtimes \mathrm{sgn}_{\mathbb{S}_{n-k}}) \otimes_{\mathbb{C}[\mathbb{S}_k \times \mathbb{S}_{n-k}]} \mathbb{C}[\mathbb{S}_n].$$

In particular, they can be equipped with a canonical basis coming from the action of symmetric group algebra. A crucial point is the following observation:

Theorem (See Proposition 3.2.5)**.** *Lusztig's canonical basis of $V^{\otimes n}$, defined using the action of $\mathfrak{gl}(1|1)$, agrees with the canonical basis defined in term of the symmetric group action.*

This Schur-Weyl duality is strictly related to a version of super skew Howe duality that connects representations of $\mathfrak{gl}(1|1)$, or more generally $\mathfrak{gl}(m|n)$, with representations of $\mathfrak{gl}_N$ [CW01]. In fact, the whole categorification process we develop works more generally for tensor powers of the vector representation of $\mathfrak{gl}(m|n)$. We will sketch the main ideas for the general case using super skew Howe duality in Appendix C. To develop the $\mathfrak{gl}(1|1)$–categorification theory we will use super Schur-Weyl duality instead of Howe duality, and hence reduce the problem to symmetric group categorification. The two approaches are equivalent, but we personally prefer to work out the detail based on the first one.

The fundamental tool used in our construction is the BGG category $\mathcal{O}$ (named after Bernšteĭn, Gel′fand and Gel′fand, who introduced it in [BGG76]), which plays already an important role in many other representation theoretical categorifications. In particular, we will construct a categorification of tensor powers of V and of their subrepresentations using some subquotient categories of $\mathcal{O}(\mathfrak{gl}_n)$. These categories are built in two steps: first one takes a parabolic subcategory and then a "$\mathfrak{q}$–presentable" quotient; the two steps can be reversed, and one gets the same result. The process is sketched in Figure 1, which is also helpful to remember how we index our categories. We will give the precise setup and definitions and discuss all the technical Lie-theoretical details in Chapter 5.

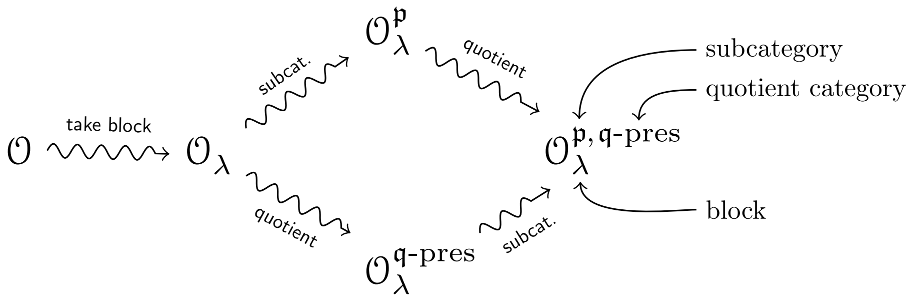

Figure 1: Subquotient categories of $\mathcal{O}$.

The construction of these subquotient categories is motivated by the following. Usually a semisimple module M is categorified via some abelian category $\mathcal{A}$. Now, M decomposes as direct sum of simple modules, but the category $\mathcal{A}$ is not supposed to decompose into blocks according to the decomposition of M. This is indeed one of the main points of the categorification: we want $\mathcal{A}$ to have more structure than M. When M is equipped with some *canonical basis*, the submodules generated by canonical basis elements in M give a filtration of M (but not a decomposition!); this corresponds to a filtration of $\mathcal{A}$ with subcategories. This principle has been applied in [MS08a] to categorify induced modules for the symmetric group: the category $\mathcal{O}_0(\mathfrak{gl}_n)$ is well-known to be a categorification of the regular representation of the symmetric group $\mathbb{S}_n$; these induced modules for the symmetric group are direct summands of the regular representation of $\mathbb{C}[\mathbb{S}_n]$; hence they can be categorified via subquotient categories of $\mathcal{O}_0(\mathfrak{gl}_n)$.

In particular, [MS08a] provide some categories, which we denote by $\mathcal{Q}_k(\mathbb{n})$, categorifying the induced modules (†) as vector spaces, and define on them a categorical action of $\mathbb{C}[\mathbb{S}_n]$ using translation functors. To categorify $V^{\otimes n}$ we take the direct sum of all these categories $\mathcal{Q}_k(\mathbb{n})$ for $k = 0, \ldots, n$. In addition, we consider also the corresponding singular blocks $\mathcal{Q}_k(\boldsymbol{a})$ of the same subquotient categories. Note that singular blocks do not appear in [MS08a] since they do not provide categorifications of $\mathbb{C}[\mathbb{S}_n]$–modules; in our picture, they categorify subrepresentations of $V^{\otimes n}$. The translation functors of category $\mathcal{O}(\mathfrak{gl}_n)$ restrict to all these subcategories $\mathcal{Q}_k(\boldsymbol{a})$ and finally categorify the action of the intertwining operators of the $\mathfrak{gl}(1|1)$–action.

What is left to complete the picture is to define functors that categorify the action of $\mathfrak{gl}(1|1)$ itself. There is a natural way to define adjoint functors $\mathcal{E}$ and $\mathcal{F}$ between $\mathcal{Q}_k(\boldsymbol{a})$ and $\mathcal{Q}_{k+1}(\boldsymbol{a})$, which portend to categorify the action of the generators E and F of $U(\mathfrak{gl}(1|1))$. Although $\mathcal{E}$ is exact, $\mathcal{F}$ is only right exact in general, and we need to derive our categories and functors in order to have an action on the Grothendieck groups. However, the following problem arises. The categories we consider are equivalent to categories of modules over some finite-dimensional algebras. Unfortunately, these algebras are not always quasi-hereditary; in general they are only *properly stratified* (the definition of standardly and properly stratified algebras has been modeled to describe the properties of some generalized parabolic subcategories of $\mathcal{O}$, introduced by [FKM02], that include as particular cases the categories that we consider). A properly stratified algebra does not have in general finite global dimension (this happens if and only if the algebra is quasi-hereditary). As a consequence, finite projective resolutions do not always exist, and we are forced to consider unbounded derived categories. But the Grothendieck groups of these unbounded derived categories vanish by some Eilenberg-swindle argument [Miy06]. A workaround to this problem has been developed in [AS13], using the additional structure of a *mixed Hodge structure*, which in our case is given by the grading. Given a graded abelian category, [AS13] define a proper subcategory of the left unbounded derived category of graded modules; this subcategory is big enough to contain projective resolutions, but small enough to prevent the Grothendieck group to vanish. In particular, the Grothendieck group of this triangulated subcategory is a q–adic completion of the Grothendieck group of the original graded abelian category. We will describe in detail how the categories we consider and the functors $\mathcal{E}$ and $\mathcal{F}$ can be derived using these techniques.

We remark that the categories $\mathcal{Q}_k(\boldsymbol{a})$ have a natural grading (inherited from the Koszul grading on $\mathcal{O}(\mathfrak{gl}_n)$) and all the functors we consider are actually graded functors between these categories. In fact, this grading was used in [MS08a] to get an action of the Hecke algebra instead of the symmetric group algebra for the induced modules (†). As a result, the categorification lifts to a categorification of representations of the quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$. We will work out all the details in the graded setting.

Of course at this point one would like to understand and describe these involved categories $\mathcal{Q}_k(\boldsymbol{a})$ explicitly. Very surprisingly (at least for us), this is indeed possible. To give an idea, let us present the categorification of $V^{\otimes 2}$. First, we notice that $V^{\otimes 2}$ has a weight space decomposition as given by the following picture:

$$
\begin{tikzcd}
\left(V^{\otimes 2}\right)_0 \arrow[r, bend left, "E"] \arrow[d, phantom, "\cong" rotate=-90] & \left(V^{\otimes 2}\right)_1 \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[d, phantom, "\cong" rotate=-90] & \left(V^{\otimes 2}\right)_2 \arrow[l, bend left, "F"] \arrow[d, phantom, "\cong" rotate=-90] \\
\mathrm{sgn}_{\mathbb{S}_2} & \mathbb{C}[\mathbb{S}_2] & \mathrm{trv}_{\mathbb{S}_2}
\end{tikzcd}
$$

where E and F are generators of $\mathfrak{gl}(1|1)$ and the vertical isomorphisms are isomorphisms of $\mathbb{C}[\mathbb{S}_2]$–modules. We let $R = \mathbb{C}[x]/(x^2)$ and $A = \mathrm{End}_R(\mathbb{C} \oplus R)$. The algebra A can be identified with the path algebra of the quiver

$$
\begin{tikzcd}
1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"]
\end{tikzcd}
\qquad \text{with the relation } ba = 0.
$$

We denote by e_1 and e_2 the two idempotents corresponding to the vertices of the quiver. Let us identify $\mathbb{C}$ with A/Ae_1A and notice that $\mathbb{C}$ becomes then naturally an $(A, \mathbb{C})$–bimodule. Moreover, notice that R is naturally isomorphic to the endomorphism ring of the projective module Ae_2, so that we can consider Ae_2 as an (A, R)–bimodule. The categorification of $V^{\otimes 2}$ is then given by the following picture:

$$
\begin{tikzcd}
\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{O}_0(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{O}_0^{\mathfrak{p}\text{-pres}}(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] \\
\mathbb{C}\text{–mod} \arrow[r, bend left, "\mathbb{C} \otimes \bullet"] & A\text{–mod} \arrow[l, bend left, "\mathrm{Hom}_A(\mathbb{C}, \bullet)"] \arrow[r, bend left, "\mathrm{Hom}_A(Ae_2, \bullet)"] & R\text{–mod} \arrow[l, bend left, "Ae_2 \otimes_R \bullet"]
\end{tikzcd}
$$

where $\mathfrak{p} = \mathfrak{gl}_2$. This should be compared with the standard categorification of $W^{\otimes 2}$ (see [FKS06]), where W is the vector representation of $\mathfrak{sl}_2$:

$$
\begin{tikzcd}
\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{O}_0(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_2) \arrow[d, phantom, "\cong" rotate=-90] \\
\mathbb{C}\text{–mod} \arrow[r, bend left, "\mathbb{C} \otimes \bullet"] & A\text{–mod} \arrow[l, bend left, "\mathrm{Hom}_A(\mathbb{C}, \bullet)"] \arrow[r, bend left, "\mathrm{Hom}_A(\mathbb{C}, \bullet)"] & \mathbb{C}\text{–mod} \arrow[l, bend left, "\mathbb{C} \otimes \bullet"]
\end{tikzcd}
$$

In particular, note that the first and the second leftmost weight spaces are categorified in the same way for $\mathfrak{gl}(1|1)$ and for $\mathfrak{sl}_2$. This will hold for all tensor powers $V^{\otimes n}$ and $W^{\otimes n}$ and is due to the fact that these weight spaces for $\mathfrak{gl}(1|1)$ and for $\mathfrak{sl}_2$ agree as modules for the symmetric group. The second leftmost weight space, in particular, is categorified using the well-known category of modules over the path algebra of the Khovanov-Seidel quiver [KS02]

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \cdots \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_{n-1}"] & n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd} \qquad \begin{array}{l} \text{with relations } b_1a_1 = 0 \text{ and} \\ b_ia_i = a_{i-1}b_{i-1} \text{ for all } i = 2, \dots, n-1. \end{array}$$

One should however notice the remarkable difference in the rightmost weight space of our example. Here our categorification differs from the $\mathfrak{sl}_2$ picture and leaves the world of highest weight categories. This is evident, since R has infinite global dimension.

In general, the description of our categories is slightly more involved, but still explicit. We will develop the instruments for that in Part III, where we will compute the endomorphism algebras of the projective generators using Soergel's functor $\mathbb{V}$ and Soergel modules [Soe90]. For this, we restrict ourselves for simplicity to regular blocks $\mathcal{Q}_k(\mathbb{n})$, although we believe that the same process can be applied more generally to the singular blocks. We determine the Soergel modules $\mathbb{V}P(w \cdot 0)$ corresponding to indecomposable projective modules in category $\mathcal{O}(\mathfrak{gl}_n)$, where w is in some subset D of $\mathbb{S}_n$ consisting of shortest/longest coset representatives. We compute then the homomorphism spaces $\operatorname{Hom}(\mathbb{V}P(w \cdot 0), \mathbb{V}P(w' \cdot 0))$ and the subspaces of morphisms that factor through some $\mathbb{V}P(z \cdot 0)$ for $z \notin D$; the quotient of the former by the latter gives the homomorphism space between the corresponding parabolic projective modules in the parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_n)$. We describe these homomorphism spaces diagrammatically using some *fork diagrams*. In particular, we construct in this way the endomorphism algebra $A_{n,k}$ of a projective generator of $\mathcal{Q}_k(\mathbb{n})$, and we get:

Main Theorem 2 (see Theorem 9.6.7)**.** *We have an equivalence of categories*

$$\text{mod}-A_{n,k} \cong \mathcal{Q}_k(\mathbb{n}).$$

Soergel bimodules and their diagrammatics were studied already from different angles; for the most recent treatment see [EW12] and [EW13]. However, as far as we know, this is the first work in which the Soergel functor is used to compute explicitly endomorphism algebras of indecomposable projective modules in the parabolic category $\mathcal{O}^{\mathfrak{p}}$. This is due to the fact that the standard approaches cannot be applied in the parabolic case. The crucial point that makes our computation work is the fact that the Soergel modules we consider are cyclic. This is equivalent to the corresponding Schubert varieties being rationally smooth (cf. [Str03b]), a property which was studied in detail in the non-parabolic case (see for example [Bil98] and [BW01]). In some sense, what we consider is a maximal subset of the symmetric group such that the corresponding Schubert varieties are all rationally smooth (cf. [GR02]).

Having provided a diagrammatic description of the algebra $A_{n,k}$, we reprove in purely elementary terms the fact, known from Lie theory, that $A_{n,k}$ is cellular and properly stratified, by explicitly constructing standard and proper standard modules. As a byproduct, we can describe the functors $\mathcal{E}$ and $\mathcal{F}$ as bimodules and compute their endomorphism rings, proving that they are indecomposable. We remark that one could expect an action of a KLR algebra (see [KL09], [KL11] and [Rou08]) on powers of $\mathcal{E}$ and $\mathcal{F}$. However, notice that since $\mathcal{E}^2 = 0$ and $\mathcal{F}^2 = 0$ it does not make sense to investigate the endomorphism spaces $\operatorname{End}(\mathcal{E}^k)$ and $\operatorname{End}(\mathcal{F}^k)$ for $k > 1$. At the moment it is not clear to us how one could get a 2-categorification for $\mathfrak{gl}(1|1)$–representations.

The Soergel functor and Soergel modules interplay the category $\mathcal{O}(\mathfrak{gl}_n)$ with the cohomology of the flag variety. In our case, since the category $\mathcal{Q}_k(\mathbb{n})$ is a quotient of the parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_n)$, where $\mathfrak{p}$ corresponds to a composition of n of type $(1, \dots, 1, n-k)$, one expects a connection with the cohomology of the Springer fiber of hook type sitting inside the full flag variety. Mimicking [SW12], we compute in Appendix B the cohomology rings of the closed attracting cells of this Springer fiber for the corresponding torus action and we prove that they are isomorphic to the endomorphism rings of the indecomposable projective modules of our categories $\mathcal{Q}_k(\mathbb{n})$. It should be possible to construct a convolution product on these cohomology rings as in [SW12] so that we recover the full algebra $A_{n,k}$. We believe that this interpretation could be used to establish a connection with the approach of Tian ([Tia12], [Tia13]).

Outline of the thesis

The thesis is divided into three parts. Although they are closely related, they are concerned with three different aspects of the story and have quite different points of view. In particular, the three parts can be read separately and we think each of them can be of independent interest.

In Part I we study in detail the representation theory of $U_q(\mathfrak{gl}(1|1))$. In Chapter 1 we define the Hopf superalgebra $U_q(\mathfrak{gl}(1|1))$ and classify its irreducible representations. In Chapter 2 we recall the definition of the Hecke algebra and of the Kazhdan-Lusztig basis, and study some mixed induced sign-trivial modules which arise as weight spaces of $U_q(\mathfrak{gl}(1|1))$–representations. In Chapter 3 we restrict to a semisimple subcategory Rep of representations, which contains the tensor powers of the vector representation. The main achievement of Part I is the construction of a graphical calculus for the category Rep, which we develop in §3.3 using webs, similar to the $\mathfrak{sl}_2$–diagram calculus of [FK97]. In particular, we define a diagrammatic category Web and a full functor

$$\mathscr{T} : \mathsf{Web} \to \mathsf{Rep}.$$

This allows to compute explicitly the canonical bases and the action of $U_q(\mathfrak{gl}(1|1))$. We point out that we can even define a quotient $\overline{\mathsf{Web}}$ of Web so that the functor $\mathscr{T}$ descends to an equivalence of categories $\overline{\mathsf{Web}} \cong \mathsf{Rep}$ (see Theorem 3.3.12).

In Part II we construct the categorification of this graphical calculus using the BGG category $\mathcal{O}$. Chapter 4 contains some facts about the graded version of $\mathcal{O}$ and graded lifts of translation functors, which are known in principle but cannot be found in the literature in full generality. Chapter 5 is the technical heart of the paper and contains the definitions of the subquotient categories $\mathcal{Q}_k(\boldsymbol{a})$ of $\mathcal{O}(\mathfrak{gl}_n)$; here we study in detail their properties and the functors between them. In Chapter 6 we then show how they can be used to construct a categorification of the representations in Rep, defining a functor $\mathscr{F} : \mathsf{Web} \to \mathcal{O}\mathsf{Cat}$, where $\mathcal{O}\mathsf{Cat}$ is a category containing all our categories $\mathcal{Q}_k(\boldsymbol{a})$. We prove then our Main Theorem 1, which can be restated as follows:

Theorem (See Theorems 6.2.2 and 6.5.4)**.** *There is a commuting diagram:*

$$\begin{tikzcd}
 & \mathcal{O}\mathsf{Cat} \arrow[d, "K^{\mathbb{C}(q)}"] \\
\mathsf{Web} \arrow[ur, "\mathscr{F}"] \arrow[r, "\mathscr{T}"'] & \mathsf{Rep}
\end{tikzcd}$$

At least on the level of derived categories, the $U_q(\mathfrak{gl}(1|1))$–action on representations in Rep *can be lifted to an action of functors on the corresponding categories $\mathcal{Q}(\boldsymbol{a})$.*

In Part III we realize the categories $\mathcal{Q}_k(\mathbb{n})$ as module categories over some diagram algebras. Chapter 7 contains some preliminary notions, in particular on the theory of symmetric polynomials. The ideals generated by complete symmetric polynomials play an important role in our diagrammatic algebras as well as the geometric interpretation in term of Springer fibers. In Chapter 8 we use them to describe the Soergel modules $\mathbb{V}P(w_k x \cdot 0)$, where x is a shortest coset representative for $\mathbb{S}_k \times \mathbb{S}_{n-k} \backslash \mathbb{S}_n$ and $w_k \in \mathbb{S}_k$ is the longest element, and morphisms between them. We determine moreover which morphisms die in the parabolic subcategory $\mathcal{O}^{\mathfrak{p}}$, where $\mathfrak{p}$ is a parabolic subalgebra with only one non-trivial block (cf. Theorem 8.3.5). Using these homomorphism spaces and some *fork diagrams* which remind of the web diagrams, we construct in Chapter 9 diagram algebras $A_{n,k}$. Moreover, we construct diagrammatically indecomposable projective, standard and proper standard modules, and we describe explicitly the properly stratified structure of $A_{n,k}$. Finally, we connect the diagram algebras $A_{n,k}$ with the categories $\mathcal{Q}_k(\boldsymbol{n})$, proving Main Theorem 2.

The thesis is completed by three appendices. In Appendix A we describe the connection between the category of $U_q(\mathfrak{gl}(1|1))$–representations and the Alexander polynomial, which motivates our interest in the whole categorification project. In Appendix B we compute the cohomology rings of some attracting varieties for a torus action inside the Springer fiber of hook type, and we prove that they are isomorphic to the endomorphism ring of the indecomposable projective modules in the categories $\mathcal{Q}_k(\mathfrak{n})$. In Appendix C, finally, we sketch how the whole categorification generalizes to $\mathfrak{gl}(m|n)$ for general $m, n \geq 0$.

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[...] perché è un modo comodo di vivere quello di credersi grande di una grandezza latente.

(Italo Svevo, La Coscienza di Zeno)

Part I

Representation theory of $U_q(\mathfrak{gl}(1|1))$

CHAPTER 1

The superalgebra $U_q(\mathfrak{gl}(1|1))$ and its representations

In this chapter, we define the Lie superalgebra $\mathfrak{gl}(1|1)$ and its quantum enveloping superalgebra $U_q = U_q(\mathfrak{gl}(1|1))$. We study then its representation theory. The material presented here is well-known, although we do not know a suitable reference for it.

1.1 The quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$

We will always work over the field of complex numbers $\mathbb{C}$.

In the following, as usual, by a *super* object (for example vector space, algebra, Lie algebra, module) we mean a $\mathbb{Z}/2\mathbb{Z}$–graded object. If X is such a super object we will use the notation $|x|$ to indicate the degree of a homogeneous element $x \in X$. Elements of degree 0 are called *even*, while elements of degree 1 are called *odd*. We stress that whenever we write $|x|$ we will always be assuming x to be homogeneous.

The Lie superalgebra $\mathfrak{gl}(1|1)$

Let $\mathbb{C}^{1|1}$ be the two-dimensional complex vector space on basis u_1, $u_{\bar{1}}$ viewed as a super vector space by setting $|u_1| = 0$ and $|u_{\bar{1}}| = 1$. The space of linear endomorphisms of $\mathbb{C}^{1|1}$ inherits a $\mathbb{Z}/2\mathbb{Z}$–grading and turns into a Lie superalgebra $\mathfrak{gl}(1|1)$ once equipped with the super commutator

$$[a, b] = ab - (-1)^{|a||b|} ba. \tag{1.1.1}$$

Evidently, $\mathfrak{gl}(1|1)$ is four-dimensional and as a vector space it is generated by the elements

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.1.2}$$

with $|h_1| = |h_2| = 0$ and $|e| = |f| = 1$. As Lie superalgebra elements, they are subject to the defining relations

$$\begin{array}{cccc} [h_1, e] = e, & [h_2, e] = -e, & [h_2, f] = f, & [h_1, f] = -f, \\ [h_1, h_2] = 0, & [e, f] = h_1 + h_2, & [e, e] = 0, & [f, f] = 0. \end{array} \tag{1.1.3}$$

Let $\mathfrak{h} \subset \mathfrak{gl}(1|1)$ be the Cartan subalgebra consisting of all diagonal matrices. Let $\varepsilon_1, \varepsilon_2$ be the dual basis to h_1, h_2 in $\mathfrak{h}^*$. We define a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ by setting on the basis

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} 1 & \text{if } i = j = 1, \\ -1 & \text{if } i = j = 2, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.1.4}$$

The *roots* of $\mathfrak{gl}(1|1)$ are $\alpha = \varepsilon_1 - \varepsilon_2$ and $-\alpha$; we choose α to be the positive *simple root*. We denote by $\mathsf{P} = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \subset \mathfrak{h}^*$ the *weight lattice* and by $\mathsf{P}^* = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \subset \mathfrak{h}$ its dual.

Remark 1.1.1. Note that in analogy with the classical Lie situation, we can set $\alpha^\vee = h_1 + h_2$. Then $e, f, \alpha^\vee$ generate the Lie superalgebra $\mathfrak{sl}(1|1)$ inside $\mathfrak{gl}(1|1)$. We work with $\mathfrak{gl}(1|1)$ and not with $\mathfrak{sl}(1|1)$ since the latter is not reductive, but nilpotent.

Hopf superalgebras

We recall that if A is a superalgebra then $A \otimes A$ can be given a superalgebra structure by declaring $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$. Analogously, if M and N are A–supermodules, than $M \otimes N$ becomes an $A \otimes A$–supermodule with action $(a \otimes b) \cdot (m \otimes n) = (-1)^{|b||m|} am \otimes bn$ for $a, b \in A$, $m \in M$, $n \in N$.

A *super bialgebra* B over a field $\Bbbk$ is a unital superalgebra which is also a coalgebra, such that the counit $\mathbf{u} : B \to \Bbbk$ and the comultiplication $\Delta : B \to B \otimes B$ are homomorphism of superalgebras (and are homogeneous of degree 0). A *Hopf superalgebra* H is a super bialgebra equipped with a $\Bbbk$-linear *antipode* $S : H \to H$ (homogeneous of degree 0) such that the usual diagram

$$\begin{tikzcd} H \otimes H \arrow[d, "S \otimes \mathrm{id}"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathbf{u}"] & H \otimes H \arrow[d, "\mathrm{id} \otimes S"] \\ {} & \Bbbk \arrow[d, "\mathbf{1}"] & {} \\ H \otimes H \arrow[r, "\nabla"] & H & H \otimes H \arrow[l, "\nabla"'] \end{tikzcd} \tag{1.1.5}$$

commutes, where $\nabla \colon H \otimes H \to H$ and $\mathbf{1} \colon \Bbbk \to H$ are the multiplication and unit of the algebra structure. We recall that S is then an anti-homomorphism $H \to H$.

If H is a Hopf superalgebra and M, N are (finite-dimensional) H–supermodules then the comultiplication Δ defines a map $H \to H \otimes H$ and hence makes it possible to give $M \otimes N$ an H–module structure by letting

$$x \cdot (m \otimes n) = \Delta(x)(m \otimes n) = \sum_{(x)} (-1)^{|x_{(2)}||m|} x_{(1)} m \otimes x_{(2)} n \tag{1.1.6}$$

for $x \in H$, $m \otimes n \in M \otimes N$, where we used Sweedler's notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$. Notice in particular that signs appear. The antipode S, moreover, allows to turn $M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ into an H–module via

$$(x\varphi)(v) = (-1)^{|\varphi||x|}\varphi(S(x)v) \tag{1.1.7}$$

for $x \in H$, $\varphi \in M^*$. We recall that the natural isomorphism $M \cong M^{**}$ for a super vector space is given by $x \longmapsto (\varphi \mapsto (-1)^{|x||\varphi|}\varphi(x))$.

Notice that in all formulas signs appear. A good rule to keep in mind is that a sign appears whenever an odd element steps over some other odd element. A good reference for sign issues is [Man97, Chapter 3].

The quantum enveloping superalgebra

The *quantum enveloping superalgebra* $U_q = U_q(\mathfrak{gl}(1|1))$ is defined to be the unital superalgebra over $\mathbb{C}(q)$ with generators E, F, $\mathbf{q}^h$ ($h \in \mathsf{P}^*$) in degrees $\left|\mathbf{q}^h\right| = 0$, $|E| = |F| = 1$ subject to the relations

$$\begin{aligned} &\mathbf{q}^0 = 1, && \mathbf{q}^h\mathbf{q}^{h'} = \mathbf{q}^{h+h'}, && \text{for } h, h' \in \mathsf{P}^*,\\ &\mathbf{q}^h E = q^{\langle h,\alpha\rangle} E\mathbf{q}^h, && \mathbf{q}^h F = q^{-\langle h,\alpha\rangle} F\mathbf{q}^h, && \text{for } h \in \mathsf{P}^*,\\ &E^2 = F^2 = 0, && EF + FE = \frac{K - K^{-1}}{q - q^{-1}}, && \text{where } K = \mathbf{q}^{h_1+h_2}. \end{aligned} \tag{1.1.8}$$

The elements $\mathbf{q}^h$, which as generators of U_q are just formal symbols, can be interpreted in terms of exponentials in the $\hbar$–version (see Appendix A). Notice that all elements $\mathbf{q}^h$ for $h \in \mathsf{P}^*$ are products of $\mathbf{q}^{h_1}$ and $\mathbf{q}^{h_2}$, so that U_q is finitely generated. Note also that K is a central element of U_q, very much in contrast to $U_q(\mathfrak{sl}_2)$.

The Hopf superalgebra structure

We define a *comultiplication* $\Delta\colon U_q \to U_q \otimes U_q$, a *counit* $\mathbf{u}\colon U_q \to \mathbb{C}(q)$ and an *antipode* $S\colon U_q \to U_q$ by setting on the generators

$$\begin{aligned} &\Delta(E) = E \otimes K^{-1} + 1 \otimes E, && \Delta(F) = F \otimes 1 + K \otimes F,\\ &S(E) = -EK, && S(F) = -K^{-1}F,\\ &\Delta(\mathbf{q}^h) = \mathbf{q}^h \otimes \mathbf{q}^h, && S(\mathbf{q}^h) = \mathbf{q}^{-h},\\ &\mathbf{u}(E) = \mathbf{u}(F) = 0, && \mathbf{u}(\mathbf{q}^h) = 1, \end{aligned} \tag{1.1.9}$$

and extending Δ and $\mathbf{u}$ to algebra homomorphisms and S to an algebra anti-homomorphism. We have then:

Proposition 1.1.2. *The maps Δ, $\mathbf{u}$ and S turn U_q into a Hopf superalgebra.*

Proof. This is a straightforward calculation. □

Notice that from the centrality of K it follows that $S^2 = \mathrm{id}$; this is a special property of U_q, that for instance does not hold in $U_q(\mathfrak{gl}(m|n))$ for general m, n (see [BKK00] for a definition of the general linear quantum supergroup).

We define a *bar involution* on U_q by setting:

$$\overline{E} = E, \qquad \overline{F} = F, \qquad \overline{\mathbf{q}^h} = \mathbf{q}^{-h}, \qquad \overline{q} = q^{-1}. \tag{1.1.10}$$

Note that $\overline{\Delta} = (^- \otimes ^-) \circ \Delta \circ ^-$ defines another comultiplication on U_q, and by definition $\overline{\Delta}(\overline{x}) = \overline{\Delta(x)}$ for all $x \in U_q$.

We define the following element $\Theta' \in U_q \otimes U_q$ which we will use later:

$$\Theta' = 1 + (q^{-1} - q)E \otimes F. \tag{1.1.11}$$

It is easy to show (see also Lemma A.2.3) that Θ' intertwines the comultiplication Δ and its barred version:

$$\Theta' \overline{\Delta}(x) = \Delta(x)\Theta' \qquad \text{for all } x \in U_q. \tag{1.1.12}$$

The following property

$$(\Delta \otimes 1)(\Theta')(\Theta' \otimes 1) = (1 \otimes \Delta)(\Theta')(1 \otimes \Theta'_{23}) \tag{1.1.13}$$

allows us to define $\Theta'^{(2)}$ as the expression (1.1.13). More generally, one can define $\Theta'^{(n)}$ for every n.

1.2 Representations

We define a parity function $|\cdot| : \mathsf{P} \to \mathbb{Z}/2\mathbb{Z}$ on the weight lattice by setting $|\varepsilon_1| = 0$, $|\varepsilon_2| = 1$ and extending additively. By a *representation* of U_q we mean a finite-dimensional U_q–supermodule with a decomposition into weight spaces $M = \bigoplus_{\lambda \in \mathsf{P}} M_\lambda$ with integral weights $\lambda \in \mathsf{P}$, such that $\mathbf{q}^h$ acts as $q^{\langle h, \lambda \rangle}$ on M_λ. We suppose further that M is $\mathbb{Z}/2\mathbb{Z}$–graded, and the grading is uniquely determined by the requirement that M_λ is in degree $|\lambda|$.

Irreducible representations

It is not difficult to find all simple representations of U_q: up to isomorphism they are indexed by their highest weight $\lambda \in \mathsf{P}$. If $\lambda \in \operatorname{Ann}(h_1 + h_2)$, then the simple representation with highest weight λ is one-dimensional, generated by a vector v^λ in degree $\left|v^\lambda\right| = |\lambda|$ with

$$Ev^\lambda = 0, \qquad Fv^\lambda = 0, \qquad \mathbf{q}^h v^\lambda = q^{\langle h, \lambda \rangle} v^\lambda, \qquad Kv^\lambda = v^\lambda. \tag{1.2.1}$$

We will denote this representation by $\mathbb{C}(q)_\lambda$, to emphasize that it is just the trivial module, but with the $\mathfrak{h}$–action twisted by the weight λ. In particular for $\lambda = 0$ we have the trivial representation $\mathbb{C}(q)_0$, that we will simply denote by $\mathbb{C}(q)$ in the following.

If $\lambda \notin \operatorname{Ann}(h_1 + h_2)$ then the simple representation $\mathsf{L}(\lambda)$ with highest weight λ is two-dimensional; we denote by v_1^λ its highest weight vector. Let us also introduce the following notation that will be useful later:

$$q^\lambda = q^{\langle h_1 + h_2, \lambda \rangle} \qquad \text{and} \qquad [\lambda] = [\langle h_1 + h_2, \lambda \rangle], \tag{1.2.2}$$

where, as usual, $[k]$ is the quantum number defined by

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{1.2.3}$$

Notice that if $k > 0$ then we have $[k] = q^{-k+1} + q^{-k+3} + \cdots + q^{k-3} + q^{k-1}$, and in general $[-k] = -[k]$.

As a vector space $\mathsf{L}(\lambda) = \mathbb{C}(q)\langle v_1^\lambda \rangle \oplus \mathbb{C}(q)\langle v_0^\lambda \rangle$ with $\left|v_1^\lambda\right| = |\lambda|$, $\left|v_0^\lambda\right| = |\lambda| + 1$ and the action of U_q is given by

$$\begin{aligned} &Ev_1^\lambda = 0, && Fv_1^\lambda = [\lambda] v_0^\lambda, && \mathbf{q}^h v_1^\lambda = q^{\langle h, \lambda \rangle} v_1^\lambda, && Kv_1^\lambda = q^\lambda v_1^\lambda, \\ &Ev_0^\lambda = v_1^\lambda, && Fv_0^\lambda = 0, && \mathbf{q}^h v_0^\lambda = q^{\langle h, \lambda - \alpha \rangle} v_0^\lambda, && Kv_0^\lambda = q^\lambda v_0^\lambda. \end{aligned} \tag{1.2.4}$$

REMARK 1.2.1. As a remarkable property of U_q, we notice that since $E^2 = F^2 = 0$ *all* simple U_q–modules (even the ones with non-integral weights) are finite-dimensional. In fact, formulas (1.2.4) define two-dimensional simple U_q–modules for all complex weights $\lambda \in \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2$ such that $\langle \lambda, h_1 + h_2 \rangle \neq 0$.

In the following, we set

$$\mathsf{P}' = \{\lambda \in \mathsf{P} \mid \lambda \notin \mathrm{Ann}(h_1 + h_2)\} \tag{1.2.5}$$

and we will mostly consider two-dimensional simple representations $\mathsf{L}(\lambda)$ for $\lambda \in \mathsf{P}'$. Also, $\mathsf{P}^{\pm} = \{\lambda \in \mathsf{P} \mid \langle \lambda, h_1 + h_2 \rangle \gtrless 0\}$ will be the set of positive/negative weights and $\mathsf{P}' = \mathsf{P}^+ \sqcup \mathsf{P}^-$.

Decomposition of tensor products

The following lemma is the first step to decompose a tensor product of U_q–representations:

Lemma 1.2.2. *Let $\lambda, \mu \in \mathsf{P}'$ and suppose also $\lambda + \mu \in \mathsf{P}'$. Then we have*

$$\mathsf{L}(\lambda) \otimes \mathsf{L}(\mu) \cong \mathsf{L}(\lambda + \mu) \oplus \mathsf{L}(\lambda + \mu - \alpha). \tag{1.2.6}$$

Proof. Under our assumptions, the vectors

$$E(v_0^\lambda \otimes v_0^\mu) = v_1^\lambda \otimes q^{-\mu} v_0^\mu + (-1)^{|\lambda|+1} v_0^\lambda \otimes v_1^\mu, \tag{1.2.7}$$

$$F(v_1^\lambda \otimes v_1^\mu) = [\lambda] v_0^\lambda \otimes v_1^\mu + (-1)^{|\lambda|} q^\lambda v_1^\lambda \otimes [\mu] v_0^\mu \tag{1.2.8}$$

are linearly independent. One can verify easily that $v_0^\lambda \otimes v_0^\mu$ and $E(v_0^\lambda \otimes v_0^\mu)$ span a module isomorphic to $\mathsf{L}(\lambda + \mu - \alpha)$, while $v_1^\lambda \otimes v_1^\mu$ and $F(v_1^\lambda \otimes v_1^\mu)$ span a module isomorphic to $\mathsf{L}(\lambda + \mu)$. □

On the other hand, we have:

Lemma 1.2.3. *Let $\lambda, \mu \in \mathsf{P}'$ and suppose $\lambda + \mu \in \mathrm{Ann}(h_1 + h_2)$. Then the representation $M = \mathsf{L}(\lambda) \otimes \mathsf{L}(\mu)$ is indecomposable and has a filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset M \tag{1.2.9}$$

with successive quotients

$$M_1 \cong \mathbb{C}(q)_\nu, \quad M_2/M_1 \cong \mathbb{C}(q)_{\nu-\alpha} \oplus \mathbb{C}(q)_{\nu+\alpha}, \quad M/M_2 \cong \mathbb{C}(q)_\nu \tag{1.2.10}$$

where $\nu = \lambda + \mu - \alpha$.

Moreover, $\mathsf{L}(\lambda') \otimes \mathsf{L}(\mu') \cong \mathsf{L}(\lambda) \otimes \mathsf{L}(\mu)$ for any $\lambda', \mu' \in \mathsf{P}'$ such that $\lambda' + \mu' = \lambda + \mu$.

Proof. Since $\lambda + \mu \in \mathrm{Ann}(h_1 + h_2)$ we have $q^\lambda = q^{-\mu}$ and $[\lambda] = -[\mu]$. Using (1.2.7) and (1.2.8) we get that

$$F(v_1^\lambda \otimes v_1^\mu) = (-1)^{|\lambda|+1} [\lambda] E(v_0^\lambda \otimes v_0^\mu). \tag{1.2.11}$$

In particular, since $E^2 = F^2 = 0$, the vector $F(v_1^\lambda \otimes v_1^\mu)$ generates a one-dimensional submodule $M_1 \cong \mathbb{C}(q)_{\lambda+\mu-\alpha}$ of M. It follows then that the images of $v_1^\lambda \otimes v_1^\mu$ and $v_0^\lambda \otimes v_0^\mu$ in M/M_1 generate two one-dimensional submodules isomorphic to $\mathbb{C}(q)_{\lambda+\mu}$ and $\mathbb{C}(q)_{\lambda+\mu-2\alpha}$ respectively. Let therefore M_2 be the submodule of M generated by $v_1^\lambda \otimes v_1^\mu$ and $v_0^\lambda \otimes v_0^\mu$. Then M/M_2 is a one-dimensional representation isomorphic to $\mathbb{C}(q)_\nu$.

The last assertion follows easily since both $\mathsf{L}(\lambda) \otimes \mathsf{L}(\mu)$ and $\mathsf{L}(\lambda') \otimes \mathsf{L}(\mu')$ are isomorphic as left U_q–modules to U_q/I where I is the left ideal generated by the elements $\mathbf{q}^h - q^{\langle h, \nu \rangle}$ for $h \in \mathsf{P}$. □

The dual of a representation

Let us now consider the dual $\mathsf{L}(\lambda)^*$ of the representation $\mathsf{L}(\lambda)$ for $\lambda \in \mathsf{P}'$ and let $(v_1^\lambda)^*, (v_0^\lambda)^*$ be the basis dual to the standard basis v_1^λ, v_0^λ. By explicit computation, the action of U_q on $\mathsf{L}(\lambda)^*$ is given by:

$$\begin{aligned} E(v_1^\lambda)^* &= -(-1)^{|\lambda|} q^\lambda (v_0^\lambda)^*, & E(v_0^\lambda)^* &= 0, \\ F(v_1^\lambda)^* &= 0, & F(v_0^\lambda)^* &= (-1)^{|\lambda|}[\lambda] q^{-\lambda} (v_1^\lambda)^*, \\ \mathbf{q}^h (v_1^\lambda)^* &= q^{-\langle h,\lambda\rangle} (v_1^\lambda)^*, & \mathbf{q}^h (v_0^\lambda)^* &= q^{-\langle h,\lambda-\alpha\rangle} (v_0^\lambda)^*. \end{aligned} \tag{1.2.12}$$

The assignment

$$\begin{aligned} \mathsf{L}(\alpha-\lambda) &\longrightarrow \mathsf{L}(\lambda)^* \\ v_1^{\alpha-\lambda} &\longmapsto -(-1)^{|\lambda|} q^\lambda (v_0^\lambda)^* \\ v_0^{\alpha-\lambda} &\longmapsto (v_1^\lambda)^* \end{aligned} \tag{1.2.13}$$

defines a $\mathbb{Q}(q)$–linear map which is in fact an isomorphism of U_q–modules

$$\mathsf{L}(\lambda)^* \cong \mathsf{L}(\alpha-\lambda). \tag{1.2.14}$$

Remark 1.2.4. Together with Lemma 1.2.3 it follows that $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$ is an indecomposable representation. In the filtration (1.2.9), the submodule M_1 is the image of the coevaluation map $\mathbb{C}(q) \to \mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$ while the submodule M_2 is the kernel of the evaluation map $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^* \to \mathbb{C}(q)$, see also (A.3.1) and (A.3.2) in Appendix A.

The vector representation

The *vector representation* V of U_q is isomorphic to $\mathsf{L}(\varepsilon_1)$. Its standard basis is $v_1^{\varepsilon_1}, v_0^{\varepsilon_1}$, the grading is given by $|v_1^{\varepsilon_1}| = 0, |v_0^{\varepsilon_1}| = 1$, and the action of U_q is determined by

$$\begin{aligned} Ev_1^{\varepsilon_1} &= 0, & Fv_1^{\varepsilon_1} &= v_0^{\varepsilon_1}, & \mathbf{q}^h v_1^{\varepsilon_1} &= q^{\langle h,\varepsilon_1\rangle} v_1^{\varepsilon_1}, & Kv_1^{\varepsilon_1} &= qv_1^{\varepsilon_1}, \\ Ev_0^{\varepsilon_1} &= v_1^{\varepsilon_1}, & Fv_0^{\varepsilon_1} &= 0, & \mathbf{q}^h v_0^{\varepsilon_1} &= q^{\langle h,\varepsilon_2\rangle} v_0^{\varepsilon_1}, & Kv_0^{\varepsilon_1} &= qv_0^{\varepsilon_1}. \end{aligned} \tag{1.2.15}$$

For $V^{\otimes n}$ we obtain directly from Lemma 1.2.2 the following decomposition:

Proposition 1.2.5 ([BM13, Theorem 6.4])**.** *The tensor powers of V decompose as*

$$V^{\otimes n} \cong \bigoplus_{\ell=0}^{n-1} \binom{n-1}{\ell} \mathsf{L}(n\varepsilon_1 - \ell\alpha). \tag{1.2.16}$$

Let us now consider mixed tensor products, involving also the dual V^*. By (1.2.14) we have that V^* is isomorphic to $\mathsf{L}(-\varepsilon_2)$. The following generalizes Proposition 1.2.5:

Theorem 1.2.6. *Suppose $m \neq n$. Then we have the following decomposition:*

$$V^{\otimes m} \otimes V^{*\otimes n} \cong \bigoplus_{\ell=0}^{m+n-1} \binom{m+n-1}{\ell} \mathsf{L}(m\varepsilon_1 - n\varepsilon_2 - \ell\alpha). \tag{1.2.17}$$

On the other hand, we have

$$V^{\otimes n} \otimes V^{*\otimes n} \cong \bigoplus_{i=1}^{2^{2n-2}} \big(V \otimes V^*\big) \tag{1.2.18}$$

and $V \otimes V^*$ is indecomposable but not irreducible.

Proof. The decomposition (1.2.17) follows from Lemma 1.2.2 by induction. To obtain (1.2.18) write $V^{\otimes n} \otimes V^{*\otimes n} \cong (V^{\otimes n} \otimes V^{*\otimes n-1}) \otimes V^*$ and use (1.2.17) together with Lemma 1.2.3. Finally, by Lemma 1.2.3 it follows also that $V \otimes V^*$ is indecomposable but not irreducible. $\square$

In particular, notice that $V^{\otimes m} \otimes V^{*\otimes n}$ is semisimple as long as $m \neq n$.

1.3 Lusztig's bar involution and canonical basis

We briefly recall from [Lus10] some facts about the bar involution and based modules. For a short but more detailed introduction see also [FK97, §1.5].

Bar involution

Recall that in §1.1 we defined a bar involution on U_q. It makes then sense to define a bar involution on a U_q–module to be an involution which is compatible with that:

Definition 1.3.1. *A* bar involution *on a U_q–module W is a q–anti-linear involution $^-$ such that $\overline{xv} = \overline{x} \cdot \overline{v}$ for all $x \in U_q$, $v \in W$.*

Note that $\overline{v_1^\lambda} = v_1^\lambda$, $\overline{v_0^\lambda} = v_0^\lambda$ define a bar involution on every simple representation $\mathsf{L}(\lambda)$, $\lambda \in \mathsf{P}'$, while $\overline{v^\mu} = v^\mu$ for $\mu \in \operatorname{Ann}(h_1 + h_2)$ defines a bar involution on $\mathbb{C}(q)_\mu$.

Assume we have bar involutions on U_q–modules W, W'. Then define on $W \otimes W'$

$$\overline{w \otimes w'} = \Theta'(\overline{w} \otimes \overline{w'}) \tag{1.3.1}$$

using the element Θ' from (1.1.11). It follows from (1.1.12) that this defines a bar involution on $W \otimes W'$. Moreover, (1.1.13) allows us to repeat the construction for bigger tensor products, and the result is independent of the bracketing.

Standard basis

We call $\mathbb{B}_\lambda = \{v_1^\lambda, v_0^\lambda\}$ the *standard basis* of $\mathsf{L}(\lambda)$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ be a sequence of weights $\lambda_i \in \mathsf{P}'$. On the tensor product $\mathsf{L}(\lambda_1) \otimes \cdots \otimes \mathsf{L}(\lambda_\ell)$ we have the *standard basis*

$$\mathbb{B}_{\boldsymbol{\lambda}} = \mathbb{B}_{\lambda_1} \otimes \cdots \otimes \mathbb{B}_{\lambda_\ell} = \{v_{\eta_1}^{\lambda_1} \otimes \cdots \otimes v_{\eta_\ell}^{\lambda_\ell} \mid \eta_i \in \{0,1\} \text{ for all } i\} \tag{1.3.2}$$

obtained by tensoring the elements of the standard basis of the factors.

On the elements of (1.3.2) we fix a partial order induced from the Bruhat order on permutations, as follows. The weight space of $\mathsf{L}(\lambda_1) \otimes \cdots \otimes \mathsf{L}(\lambda_\ell)$ of weight $\lambda_1 + \cdots + \lambda_\ell - (\ell - k)\alpha$ is spanned by the subset $(\mathbb{B}_{\boldsymbol{\lambda}})_k$ of the standard basis (1.3.2) consisting of elements such that $\sum_i \eta_i = k$. The symmetric group $\mathbb{S}_\ell$ acts from the right on the set of sequences $\{0,1\}^\ell$ by permutations, hence on $\mathbb{B}_{\boldsymbol{\lambda}}$. The action of $\mathbb{S}_\ell$ on each subset $(\mathbb{B}_{\boldsymbol{\lambda}})_k$ is transitive; mapping the identity $e \in \mathbb{S}_\ell$ to the *minimal element*

$$\underbrace{v_1^{\lambda_1} \otimes \cdots \otimes v_1^{\lambda_k}}_{k} \otimes \underbrace{v_0^{\lambda_{k+1}} \otimes \cdots \otimes v_0^{\lambda_\ell}}_{\ell-k} \tag{1.3.3}$$

determines a bijection

$$\left(\mathbb{S}_k \times \mathbb{S}_{\ell-k} \backslash \mathbb{S}_\ell\right)^{\text{short}} \xleftrightarrow{1-1} (\mathbb{B}_{\boldsymbol{\lambda}})_k, \tag{1.3.4}$$

where $({}^{\mathbb{S}_k \times \mathbb{S}_{\ell-k}}\backslash{}^{\mathbb{S}_\ell})^{\text{short}}$ is the set of shortest coset representatives for ${}^{\mathbb{S}_k \times \mathbb{S}_{\ell-k}}\backslash{}^{\mathbb{S}_\ell}$. The Bruhat order (see §2.1) of the latter induces a partial order on $(\mathbb{B}_{\boldsymbol{\lambda}})_k$ and hence on $\mathbb{B}_{\boldsymbol{\lambda}}$. Notice that the minimal element (1.3.3) is bar invariant.

Canonical basis

We have the following analogue of [Lus10, Theorem 27.3.2]:

Theorem 1.3.2. *In $\mathsf{L}(\lambda_1)\otimes\cdots\otimes\mathsf{L}(\lambda_\ell)$, for each standard basis element $v_{\eta_1}^{\lambda_1}\otimes\cdots\otimes v_{\eta_\ell}^{\lambda_\ell}\in\mathbb{B}_{\boldsymbol{\lambda}}$ there is a unique bar-invariant element*

$$v_{\eta_1}^{\lambda_1}\diamondsuit\cdots\diamondsuit v_{\eta_\ell}^{\lambda_\ell} \tag{1.3.5}$$

such that $v_{\eta_1}^{\lambda_1}\diamondsuit\cdots\diamondsuit v_{\eta_\ell}^{\lambda_\ell} - v_{\eta_1}^{\lambda_1}\otimes\cdots\otimes v_{\eta_\ell}^{\lambda_\ell}$ is a $q\mathbb{Z}[q]$–linear combination of elements of the standard basis that are smaller than $v_{\eta_1}^{\lambda_1}\otimes\cdots\otimes v_{\eta_\ell}^{\lambda_\ell}$.

Proof. The proof is completely analogous to [Lus10, Theorem 27.3.2]. □

Since the standard basis elements form a basis of $\mathsf{L}(\lambda_1)\otimes\cdots\otimes\mathsf{L}(\lambda_\ell)$, the condition in Theorem 1.3.2 implies that the canonical basis elements (1.3.5) form a basis as well.

Definition 1.3.3. *The elements* (1.3.5) *constitute the* canonical basis *of* $\mathsf{L}(\lambda_1)\otimes\cdots\otimes\mathsf{L}(\lambda_\ell)$.

Example 1.3.4. On the two-dimensional weight space of $V\otimes V$ the bar involution is given by

$$\begin{aligned}\overline{v_1^{\varepsilon_1}\otimes v_0^{\varepsilon_1}} &= v_1^{\varepsilon_1}\otimes v_0^{\varepsilon_1},\\ \overline{v_0^{\varepsilon_1}\otimes v_1^{\varepsilon_1}} &= v_0^{\varepsilon_1}\otimes v_1^{\varepsilon_1} + (q-q^{-1})v_1^{\varepsilon_1}\otimes v_0^{\varepsilon_1}.\end{aligned}$$

The canonical basis is then

$$\begin{aligned}v_1^{\varepsilon_1}\diamondsuit v_0^{\varepsilon_1} &= v_1^{\varepsilon_1}\otimes v_0^{\varepsilon_1},\\ v_0^{\varepsilon_1}\diamondsuit v_1^{\varepsilon_1} &= v_0^{\varepsilon_1}\otimes v_1^{\varepsilon_1} + qv_1^{\varepsilon_1}\otimes v_0^{\varepsilon_1}.\end{aligned}$$ ⊛

Example 1.3.5. On the two-dimensional weight space of $V^*\otimes V^*$ the bar involution is given by

$$\begin{aligned}\overline{v_1^{-\varepsilon_2}\otimes v_0^{-\varepsilon_2}} &= v_1^{-\varepsilon_2}\otimes v_0^{-\varepsilon_2},\\ \overline{v_0^{-\varepsilon_2}\otimes v_1^{-\varepsilon_2}} &= v_0^{-\varepsilon_2}\otimes v_1^{-\varepsilon_2} + (q-q^{-1})v_1^{-\varepsilon_2}\otimes v_0^{-\varepsilon_2}\end{aligned}$$

and its canonical basis is

$$\begin{aligned}v_1^{-\varepsilon_2}\diamondsuit v_0^{-\varepsilon_2} &= v_1^{-\varepsilon_2}\otimes v_0^{-\varepsilon_2},\\ v_0^{-\varepsilon_2}\diamondsuit v_1^{-\varepsilon_2} &= v_0^{-\varepsilon_2}\otimes v_1^{-\varepsilon_2} + qv_1^{-\varepsilon_2}\otimes v_0^{-\varepsilon_2}.\end{aligned}$$ ⊛

Example 1.3.6. On the two-dimensional weight space of $V\otimes V^*$ the bar involution is given by

$$\begin{aligned}\overline{v_1^{\varepsilon_1}\otimes v_0^{-\varepsilon_2}} &= v_1^{\varepsilon_1}\otimes v_0^{-\varepsilon_2},\\ \overline{v_0^{\varepsilon_1}\otimes v_1^{-\varepsilon_2}} &= v_0^{\varepsilon_1}\otimes v_1^{-\varepsilon_2} - (q-q^{-1})v_1^{\varepsilon_1}\otimes v_0^{-\varepsilon_2}\end{aligned}$$

and its canonical basis is

$$\begin{aligned}v_1^{\varepsilon_1}\diamondsuit v_0^{-\varepsilon_2} &= v_1^{\varepsilon_1}\otimes v_0^{-\varepsilon_2},\\ v_0^{\varepsilon_1}\diamondsuit v_1^{-\varepsilon_2} &= v_0^{\varepsilon_1}\otimes v_1^{-\varepsilon_2} - qv_1^{\varepsilon_1}\otimes v_0^{-\varepsilon_2}.\end{aligned}$$ ⊛

CHAPTER 2

The Hecke algebra and Hecke modules

Before continuing the study of U_q–representations, we need to introduce the Hecke algebra of the symmetric group. This will enter in the game in the next section, where we will use a super version of Schur-Weyl duality to connect the representation theory of U_q with the one of the Hecke algebra.

We recall first the definition of the Hecke algebra of the symmetric group, together with its bar involution and canonical basis. We study then in detail in §2.2 mixed induced sign-trivial modules; this generalizes work of Soergel [Soe97].

2.1 The Hecke algebra

Let $\mathbb{S}_n$ denote the symmetric group of permutations of n elements; it is generated by the simple reflections s_i for $i = 1, \dots, n-1$ subjected to the defining relations

(2.1.1a) $$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 2,$$

(2.1.1b) $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

(2.1.1c) $$s_i^2 = 1.$$

For $w \in \mathbb{S}_n$ we denote by $\ell(w)$ the *length* of w, which is the length of any reduced expression $w = s_{i_1} \cdots s_{i_r}$. Let $T = \{w s_i w^{-1} \mid w \in \mathbb{S}_n, i = 1, \dots, n-1\}$ be the set of transpositions; we will indicate by $\prec$ the *Bruhat order* on $\mathbb{S}_n$, which is the transitive closure of the relation $u \xrightarrow{t} w$ whenever $\ell(u) < \ell(w)$ and $w = ut$ for some $t \in T$.

Definition 2.1.1 ([KL79])**.** *The* Hecke algebra *of the symmetric group $\mathbb{S}_n$ is the unital associative $\mathbb{C}(q)$–algebra $\mathcal{H}_n$ generated by $\{H_i \mid i = 1, \dots, n-1\}$ with relations*

(2.1.2a) $$H_i H_j = H_j H_i \qquad \textit{if } |i-j| > 2,$$

(2.1.2b) $$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

(2.1.2c) $$H_i^2 = (q^{-1} - q) H_i + 1.$$

Notice that we use Soergel's normalization [Soe97], instead of the original one. However, we use the letter q as parameter in analogy with the quantum parameter of U_q.

It follows from (2.1.2c) that the elements H_i are invertible with $H_i^{-1} = H_i + q - q^{-1}$. For $w \in \mathbb{S}_n$ such that $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression, we define $H_w = H_{i_1} \cdots H_{i_r}$. It is a standard result (see for example [KT08, Lemma 4.16]) that this does not depend on the chosen reduced expression. The elements H_w for $w \in \mathbb{S}_n$ form a basis of $\mathcal{H}_n$ (see [KT08, Theorem 4.17]), called *standard basis*, and we have

$$H_w H_i = \begin{cases} H_{ws_i} & \text{if } \ell(ws_i) > \ell(w), \\ H_{ws_i} + (q^{-1} - q) H_w & \text{otherwise.} \end{cases} \tag{2.1.3}$$

We can define on $\mathcal{H}_n$ a *bar involution* by $\overline{H_w} = H_{w^{-1}}^{-1}$ and $\overline{q} = q^{-1}$; in particular $\overline{H_i} = H_i + q - q^{-1}$. We also have a non-degenerate $\mathbb{C}(q)$–*bilinear form* $\langle -, - \rangle$ on $\mathcal{H}_n$ such that the standard basis elements are orthonormal:

$$\langle H_w, H_{w'} \rangle = \delta_{w,w'} \qquad \text{for all } w, w' \in \mathbb{S}_n. \tag{2.1.4}$$

By standard arguments one can prove the following:

Proposition 2.1.2 ([KL79], in the normalization of [Soe97])**.** *There exists a unique basis* $\{\underline{H}_w \mid w \in \mathbb{S}_n\}$ *of* $\mathcal{H}_n$ *consisting of bar-invariant elements such that*

$$\underline{H}_w = H_w + \sum_{w' \prec w} \mathcal{P}_{w',w}(q) H_{w'} \tag{2.1.5}$$

with $\mathcal{P}_{w',w} \in q\mathbb{Z}[q]$ *for all* $w' \prec w$.

The basis $\underline{H}_w$ is called *Kazhdan-Lusztig basis.* We will also call it *canonical basis* of $\mathcal{H}_n$.

Remark 2.1.3. There is an inductive way to construct the canonical basis elements. First, note that by definition $\underline{H}_e = H_e$. Then set $\underline{H}_i = H_i + q$: since $\underline{H}_i$ is bar invariant, we must have $\underline{H}_{s_i} = \underline{H}_i$. Now suppose $w = w's_i \succ w'$: then $\underline{H}_{w'}\underline{H}_i$ is bar invariant and is equal to H_w plus a $\mathbb{Z}[q, q^{-1}]$–linear combination of some $H_{w''}$ for $w'' \prec w$. It follows that

$$\underline{H}_{w'}\underline{H}_i = \underline{H}_w + p \qquad \text{for some } p \in \bigoplus_{w'' \prec w} \mathbb{Z}\underline{H}_{w''}. \tag{2.1.6}$$

Anyway, it is in general impossible to give a closed formula for canonical basis elements. One of the rare examples in which this can be done is the following (cf. [Soe97, Prop. 2.9] – but see also §7.2):

Lemma 2.1.4. *Let* $W' \subseteq \mathbb{S}_n$ *be a parabolic subgroup (that is, a subgroup generated by simple transpositions) and let* $w_0 \in W'$ *be its longest element. Then the canonical basis element* $\underline{H}_{w_0}$ *is given by*

$$\underline{H}_{w_0} = \sum_{x \in W'} q^{\ell(w_0) - \ell(x)} H_x. \tag{2.1.7}$$

2.2 Induced Hecke modules

We will now consider induced Hecke modules which are a mixed version of the induced sign and induced trivial modules studied in [Soe97]. In the following, all modules over the Hecke algebra will be right modules.

Let $W_{\mathfrak{p}}, W_{\mathfrak{q}}$ be two parabolic subgroups[1] of $W = \mathbb{S}_n$ (that is, they are generated by simple transpositions) such that the elements of $W_{\mathfrak{p}}$ commute with the elements of $W_{\mathfrak{q}}$. Note that then $W_{\mathfrak{p}+\mathfrak{q}} = W_{\mathfrak{p}} \times W_{\mathfrak{q}}$ is also a parabolic subgroup of W. Let $\mathcal{H}_{\mathfrak{p}}$, $\mathcal{H}_{\mathfrak{q}}$ and $\mathcal{H}_{\mathfrak{p}+\mathfrak{q}}$ be the corresponding Hecke algebras; they are all naturally subalgebras of $\mathcal{H}_n$. We denote by $\mathrm{sgn}_{\mathfrak{p}}$ the *sign representation* of $\mathcal{H}_{\mathfrak{p}}$; this is the one-dimensional $\mathbb{C}(q)$–vector space on which each generator $H_i \in \mathcal{H}_{\mathfrak{p}}$ acts as $-q$. Moreover, we denote by $\mathrm{trv}_{\mathfrak{q}}$ the *trivial representation* of $\mathcal{H}_{\mathfrak{q}}$, which is the one-dimensional $\mathbb{C}(q)$–vector space on which each generator $H_i \in \mathcal{H}_{\mathfrak{q}}$ acts as q^{-1} (this makes sense because of the relation $(H_i + q)(H_i - q^{-1}) = 0$, which follows directly from (2.1.2c)). We define the *mixed induced Hecke module*

$$\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}} = \mathrm{Ind}^{\mathcal{H}_n}_{\mathcal{H}_{\mathfrak{p}+\mathfrak{q}}}(\mathrm{sgn}_{\mathfrak{p}} \boxtimes \mathrm{trv}_{\mathfrak{q}}) = (\mathrm{sgn}_{\mathfrak{p}} \boxtimes \mathrm{trv}_{\mathfrak{q}}) \otimes_{\mathcal{H}_{\mathfrak{p}+\mathfrak{q}}} \mathcal{H}_n. \tag{2.2.1}$$

If $W_{\mathfrak{p}}$ is trivial, we omit $\mathfrak{p}$ from the notation and we write $\mathcal{M}_{\mathfrak{q}}$. Analogously, if $W_{\mathfrak{q}}$ is trivial we omit $\mathfrak{q}$ and we write $\mathcal{M}^{\mathfrak{p}}$. Note that in $\mathcal{M}_{\mathfrak{q}}$ and $\mathcal{M}^{\mathfrak{p}}$ are denoted respectively $\mathcal{M}^{\mathfrak{q}}$ and $\mathcal{N}^{\mathfrak{p}}$ in [Soe97].

Let $W^{\mathfrak{p}}$, $W^{\mathfrak{q}}$ and $W^{\mathfrak{p}+\mathfrak{q}}$ be the set of shortest coset representatives for ${}_{W_{\mathfrak{p}}}\backslash W$, ${}_{W_{\mathfrak{q}}}\backslash W$ and ${}_{W_{\mathfrak{p}+\mathfrak{q}}}\backslash W$ respectively. Then a basis of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ is given by

$$\{N_w = 1 \otimes H_w \mid w \in W^{\mathfrak{p}+\mathfrak{q}}\} \tag{2.2.2}$$

(where 1 is some chosen generator of the $\mathbb{C}(q)$–vector space $\mathrm{sgn}_{\mathfrak{p}} \boxtimes \mathrm{trv}_{\mathfrak{q}}$).

The action of $\mathcal{H}_n$ on $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ is given explicitly by the following lemma:

Lemma 2.2.1. *For all $w \in W^{\mathfrak{p}+\mathfrak{q}}$ we have*

$$N_w \cdot H_i = \begin{cases} N_{ws_i} & \text{if } ws_i \in W^{\mathfrak{p}+\mathfrak{q}} \text{ and } \ell(ws_i) > \ell(w),\\ N_{ws_i} + (q^{-1} - q)H_w & \text{if } ws_i \in W^{\mathfrak{p}+\mathfrak{q}} \text{ and } \ell(ws_i) < \ell(w),\\ -qN_w & \text{if } ws_i = s_j w \text{ for } s_j \in W_{\mathfrak{p}},\\ q^{-1}N_w & \text{if } ws_i = s_j w \text{ for } s_j \in W_{\mathfrak{q}}. \end{cases} \tag{2.2.3}$$

The proof is analogous to the one in [Soe97, §3].

The module $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ inherits a bar involution by setting $\overline{N_w} = 1 \otimes \overline{H_w}$. Moreover, the bilinear form (2.1.4) induces a bilinear form on $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ by setting $\langle N_w, N_{w'} \rangle = \langle H_w, H_{w'} \rangle$.

A canonical basis on $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ can be defined by the following generalization of Proposition 2.1.2:

Proposition 2.2.2. *There exists a unique basis $\{\underline{N}_w \mid w \in W^{\mathfrak{p}+\mathfrak{q}}\}$ of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ consisting of bar-invariant elements satisfying*

$$\underline{N}_w = N_w + \sum_{w' \prec w} \mathcal{R}_{w',w}(q) N_{w'} \tag{2.2.4}$$

with $\mathcal{R}_{w',w} \in q\mathbb{Z}[q]$ for all $w' \prec w$.

As described in Remark 2.1.3, one can construct inductively the canonical basis of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$. In particular, for $W^{\mathfrak{p}+\mathfrak{q}} \ni ws_i \succ w$ one always has

$$\underline{N}_w \underline{H}_i = \underline{N}_{ws_i} + p \tag{2.2.5}$$

where p is a $\mathbb{Z}$–linear combination of $\underline{N}_{w'}$ for $w' \prec ws_i$.

[1] We use this notation because $W_{\mathfrak{p}}$ and $W_{\mathfrak{q}}$ will correspond later to two parabolic subalgebras $\mathfrak{p}, \mathfrak{q} \subset \mathfrak{gl}_n$.

Maps between Hecke modules (I)

We will now construct maps between induced modules $\mathcal{M}^{\mathsf{p}}_{\mathsf{q}}$ corresponding to different pairs of parabolic subgroups W_{p}, W_{q}. First, we consider the case in which we change the subgroup W_{q}.

Let $W_{\mathsf{q}'} \subset W_{\mathsf{q}}$ be also a parabolic subgroup of W. Let us define a map $\mathsf{i} = \mathsf{i}^{\mathsf{q}'}_{\mathsf{q}} : \mathcal{M}^{\mathsf{p}}_{\mathsf{q}} \to \mathcal{M}^{\mathsf{p}}_{\mathsf{q}'}$ by

$$\mathsf{i}\colon N_w \longmapsto \sum_{x \in W^{\mathsf{q}'} \cap W_{\mathsf{q}}} q^{\ell(w^{\mathsf{q}'}_{\mathsf{q}}) - \ell(x)} N_{xw} \tag{2.2.6}$$

where $w^{\mathsf{q}'}_{\mathsf{q}}$ is the longest element of $W^{\mathsf{q}'} \cap W_{\mathsf{q}} = \left({}_{W_{\mathsf{q}'}}\backslash{}^{W_{\mathsf{q}}}\right)^{\mathrm{short}}$. Note that for $w \in W^{\mathsf{p}+\mathsf{q}}$ and $x \in W^{\mathsf{q}'} \cap W_{\mathsf{q}}$ the product xw is an element of $W^{\mathsf{p}+\mathsf{q}'}$.

The map (2.2.6) is natural, in the sense that if $W_{\mathsf{q}''} \subset W_{\mathsf{q}'}$ is another parabolic subgroup of W reflections then $\mathsf{i}^{\mathsf{q}''}_{\mathsf{q}} = \mathsf{i}^{\mathsf{q}''}_{\mathsf{q}'} \circ \mathsf{i}^{\mathsf{q}'}_{\mathsf{q}}$; this follows because each element $y \in \left({}_{W_{\mathsf{q}''}}\backslash{}^{W_{\mathsf{q}}}\right)^{\mathrm{short}}$ factors in a unique way as the product $y''y'$ of an element $y'' \in \left({}_{W_{\mathsf{q}''}}\backslash{}^{W_{\mathsf{q}'}}\right)^{\mathrm{short}}$ and an element $y' \in \left({}_{W_{\mathsf{q}'}}\backslash{}^{W_{\mathsf{q}}}\right)^{\mathrm{short}}$.

Lemma 2.2.3. *The map* i *just defined is an injective homomorphism of* $\mathcal{H}_n$*–modules that commutes with the bar involution. Moreover it sends the canonical basis element* $\underline{N}_w$ *to the canonical basis element* $\underline{N}_{w^{\mathsf{q}'}_{\mathsf{q}} w}$.

Proof. The injectivity is clear, because $\mathsf{i}(N_w)$ is a linear combination of $N_{w'}$ for $w' \prec w^{\mathsf{q}'}_{\mathsf{q}} w$ and the coefficient of $N_{w^{\mathsf{q}'}_{\mathsf{q}} w}$ is 1. To prove that i is a homomorphism of $\mathcal{H}_n$–modules, it is sufficient to consider the case $W_{\mathsf{q}'} = \{e\}$. Indeed, we have a commutative diagram of injective maps

$$\begin{tikzcd} & \mathcal{M}^{\mathsf{p}} & \\ \mathcal{M}^{\mathsf{p}}_{\mathsf{q}} \arrow[ur, hook, "\mathsf{i}_{\mathsf{q}}"] \arrow[rr, hook, "\mathsf{i}^{\mathsf{q}'}_{\mathsf{q}}"] & & \mathcal{M}^{\mathsf{p}}_{\mathsf{q}'} \arrow[ul, hook', "\mathsf{i}_{\mathsf{q}'}"'] \end{tikzcd} \tag{2.2.7}$$

and if i_{q} and $\mathsf{i}_{\mathsf{q}'}$ are both $\mathcal{H}_n$–equivariant then so is $\mathsf{i}^{\mathsf{q}'}_{\mathsf{q}}$.

Hence let $\mathsf{i} = \mathsf{i}_{\mathsf{q}}$ and let us show using (2.2.3) that $\mathsf{i}(N_w H_i) = \mathsf{i}(N_w) H_i$ for all $i = 1, \dots, n-1$ and for each basis element $N_w \in \mathcal{M}^{\mathsf{p}}_{\mathsf{q}}$. Note first that $W^{\mathsf{p}+\mathsf{q}} \subset W^{\mathsf{p}}$; moreover, if $ws_i \in W^{\mathsf{p}+\mathsf{q}}$ then $xws_i \in W^{\mathsf{p}}$ for all $x \in W_{\mathsf{q}}$, so that the first two cases of (2.2.3) are clear. Suppose then that we are in the fourth case, that is $ws_i = s_j w$ for some $s_j \in W_{\mathsf{p}}$; then $xws_i = xs_j w = s_j xw$ for every $x \in W_{\mathsf{q}}$, because elements of W_{p} commute with elements of W_{q}. We are left with the third case of (2.2.3), that we will now examine.

Pick an index i such that $ws_i = s_j w$ for some $s_j \in W_{\mathsf{q}}$, and let $A^{\gtrless} = \{x \in W_{\mathsf{q}} \mid \ell(xws_i) \gtrless \ell(xw)\}$; note that for $x \in A^{>}$ we have $\ell(xs_j) > \ell(x)$ and that the right multiplication by s_j is a bijection between $A^{>}$ and $A^{<}$ (unless $W_{\mathsf{q}} = \{e\}$, but this case is trivial since i is just the identity). Compute:

$$\begin{aligned} \mathsf{i}(N_w)H_i &= \sum_{x \in A^{>}} q^{\ell(w_{\mathsf{q}}) - \ell(x)} N_{xws_i} + \sum_{x \in A^{<}} q^{\ell(w_{\mathsf{q}}) - \ell(x)} (N_{xws_i} + (q^{-1} - q) N_{xw}) \\ &= \sum_{x \in A^{>}} q^{\ell(w_{\mathsf{q}}) - \ell(x)} N_{xs_j w} + \sum_{x \in A^{<}} q^{\ell(w_{\mathsf{q}}) - \ell(x)} (N_{xws_i} + (q^{-1} - q) N_{xw}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x' \in A^<} q^{\ell(w_{\mathfrak{q}})-\ell(x')+1} N_{x'w} + \sum_{x \in A^<} q^{\ell(w_{\mathfrak{q}})-\ell(x)} (N_{xws_i} + (q^{-1}-q) N_{xw}) \\
&= \sum_{x \in A^<} q^{\ell(w_{\mathfrak{q}})-\ell(x)} N_{xws_i} + q^{-1} \sum_{x \in A^<} q^{\ell(w_{\mathfrak{q}})-\ell(x)} N_{xw} \\
&= q^{-1} \sum_{x'' \in A^>} q^{\ell(w_{\mathfrak{q}})-\ell(x'')} N_{x''w} + q^{-1} \sum_{x \in A^<} q^{\ell(w_{\mathfrak{q}})-\ell(x)} N_{xw} \\
&= q^{-1} \mathsf{i}(N_w) = \mathsf{i}(q^{-1} N_w) = \mathsf{i}(N_w H_i).
\end{aligned}
$$

It remains to show the bar invariance. Again, by the commutativity of (2.2.7) it is sufficient to consider the case $W_{\mathfrak{q}'} = \{e\}$. It is enough to check it for a basis; in fact we will prove by induction that $\mathsf{i}(\underline{N}_w)$ is bar invariant for every $w \in W^{\mathfrak{p}+\mathfrak{q}}$. For $w = e$, we have $\mathsf{i}(\underline{N}_e) = \mathsf{i}(N_e) = \sum_{x \in W_{\mathfrak{q}}} q^{\ell(w_{\mathfrak{q}})-\ell(x)} N_x$, that by Lemma 2.1.4 is the canonical basis element of $\mathcal{H}_{\mathfrak{q}}$ corresponding to the longest element of $W_{\mathfrak{q}}$: hence it is bar invariant. For the inductive step, suppose $ws_i \succ w$ and use (2.2.5):

$$
\begin{aligned}
\overline{\mathsf{i}(\underline{N}_{ws_i})} &= \overline{\mathsf{i}(\underline{N}_w \underline{H}_i - p)} = \overline{\mathsf{i}(\underline{N}_w) \underline{H}_i - \mathsf{i}(p)} \\
&= \mathsf{i}(\underline{N}_w) \underline{H}_i - \mathsf{i}(p) = \mathsf{i}(\underline{N}_w \underline{H}_i - p) = \mathsf{i}(\underline{N}_{ws_i}).
\end{aligned} \tag{2.2.8}
$$

The last claim follows by the uniqueness of the canonical basis elements, because $\mathsf{i}(\underline{N}_w)$ is bar invariant and the coefficient of $N_{w'}$ in its standard basis expression is

- 1 if $w' = w_{\mathfrak{q}}^{\mathfrak{q}'} w$,
- a multiple of q if $w' = xw''$ for some $x \in W^{\mathfrak{q}'} \cap W_{\mathfrak{q}}$ and $w'' \in W^{\mathfrak{q}}$ with $w'' \preceq w$ (but $w' \neq w_{\mathfrak{q}}^{\mathfrak{q}'} w$),
- 0 otherwise. □

Now we define a left inverse $\mathsf{Q} \colon \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}'} \to \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ of i by setting

$$
\mathsf{Q}(N_e) = \frac{1}{c_{\mathfrak{q}}^{\mathfrak{q}'}} N_e, \qquad \text{where} \qquad c_{\mathfrak{q}}^{\mathfrak{q}'} = \sum_{x \in W^{\mathfrak{q}'} \cap W_{\mathfrak{q}}} q^{\ell(w_{\mathfrak{q}}^{\mathfrak{q}'})-2\ell(x)}. \tag{2.2.9}
$$

It is easy to show that Q is indeed well-defined (since $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ is a quotient of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}'}$, and Q is, up to a multiple, the quotient map). Moreover

$$
\begin{aligned}
\mathsf{Q} \circ \mathsf{i}(N_w) &= \mathsf{Q} \left(\sum_{x \in W^{\mathfrak{q}'} \cap W_{\mathfrak{q}}} q^{\ell(w_{\mathfrak{q}}^{\mathfrak{q}'})-\ell(x)} N_{xw} \right) \\
&= \frac{1}{c_{\mathfrak{q}}^{\mathfrak{q}'}} \sum_{x \in W^{\mathfrak{q}'} \cap W_{\mathfrak{q}}} q^{\ell(w_{\mathfrak{q}}^{\mathfrak{q}'})-2\ell(x)} N_w = N_w
\end{aligned} \tag{2.2.10}
$$

for all basis elements $N_w \in \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$.

Maps between Hecke modules (II)

Now let us examine the case in which we change the subgroup $W_{\mathfrak{p}}$. Namely let $W_{\mathfrak{p}'} \subset W_{\mathfrak{p}}$ be a parabolic subgroup of W, and define a linear map $\mathsf{j} = \mathsf{j}_{\mathfrak{p}}^{\mathfrak{p}'} : \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}} \to \mathcal{M}^{\mathfrak{p}'}_{\mathfrak{q}}$ by

$$
\mathsf{j} \colon N_w \longmapsto \sum_{x \in W^{\mathfrak{p}'} \cap W_{\mathfrak{p}}} (-q)^{\ell(x)} N_{xw} \tag{2.2.11}
$$

As for Lemma 2.2.3 it is easy to prove that j is an injective homomorphism of $\mathcal{H}_n$–modules. However, it does not commute with the bar involution and it does not send canonical basis elements to canonical basis elements. Instead, j sends the dual canonical basis (defined to be the basis that is dual to the canonical basis with respect to the bilinear form) to the dual canonical basis.

Define also a $\mathcal{H}_n$–module homomorphism $\mathsf{z}\colon \mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}'} \to \mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ by setting $\mathsf{z}(N_e) = N_e$. This gives a well-defined homomorphism because $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ is a quotient of $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}'}$ and it is cyclic.

Lemma 2.2.4. *The map z is bar-invariant and sends the canonical basis element $\underline{N}_w \in \mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}'}$ to $\underline{N}_w \in \mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ if $w \in W^{\mathfrak{p}+\mathfrak{q}}$ and to 0 otherwise. Moreover $\mathsf{z} \circ \mathsf{j} = \sum_{x \in W^{\mathfrak{p}'} \cap W_{\mathfrak{p}}} q^{2\ell(x)} \mathrm{id}$.*

Proof. The map z is bar invariant by definition: in fact obviously $\mathsf{z}(N_e) = \overline{\mathsf{z}(N_e)}$, and then by multiplying with the C_i's one can see that z is bar invariant on a set of generators.

If $w \in W^{\mathfrak{p}+\mathfrak{q}}$ then it is easily seen that $\mathsf{z}(\underline{N}_w) \in N_w + \sum_{w' \prec w} q\mathbb{Z}[q] N_{w'}$. By uniqueness of the canonical basis elements it has to be $\mathsf{z}(\underline{N}_w) = \underline{N}_w \in \mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$. If $w \notin W^{\mathfrak{p}+\mathfrak{q}}$ then by the same reasoning $\mathsf{z}(\underline{N}_w) = 0$.

Moreover

$$\mathsf{z} \circ \mathsf{j}(N_w) = \mathsf{z}\left(\sum_{x \in W^{\mathfrak{p}'} \cap W_{\mathfrak{p}}} (-q)^{\ell(x)} N_{xw} \right) = \sum_{x \in W^{\mathfrak{p}'} \cap W_{\mathfrak{p}}} q^{2\ell(x)} N_w, \tag{2.2.12}$$

hence the last assertion follows as well. □

CHAPTER 3

Graphical calculus for $U_q(\mathfrak{gl}(1|1))$–representations

In §1.2 we have seen explicitly the well-known fact that the category of representations of U_q is not semisimple. From now on, we will restrict ourselves to consider a semisimple subcategory of representations of U_q, that contains the tensor powers of the vector representation V. In particular, we will study in detail the intertwining operator. A key tool will be the so-called super Schur-Weyl duality for the tensor powers of the vector representation, which we will recall in §3.2. We will then develop in §3.3 a graphical calculus for $\mathfrak{gl}(1|1)$–representations, similar to the one in [FK97].

3.1 A semisimple subcategory of representations

Given a positive integer number a let $\mathsf{V}(a) = \mathsf{L}(a\varepsilon_1)$, and let $\mathsf{V}(0) = \mathbb{C}(q)$ be the trivial U_q–representation. For a sequence $\boldsymbol{a} = (a_1, \dots, a_\ell)$ of natural numbers let us denote $\mathsf{V}(\boldsymbol{a}) = \mathsf{V}(a_1) \otimes \cdots \otimes \mathsf{V}(a_\ell)$. Let Rep be the monoidal subcategory of finite-dimensional U_q–representations generated by $\mathsf{V}(a)$ for $a \in \mathbb{N}$: the objects of Rep are exactly $\{\mathsf{V}(\boldsymbol{a}) \mid \boldsymbol{a} \in \bigcup_{\ell \geq 0} \mathbb{N}^\ell\}$. Note that this category is not abelian (it is not even additive). However, by adding all direct sums and kernels we would get a monoidal abelian semisimple category, with simple modules $\mathsf{L}(m_1\varepsilon_1 + m_2\varepsilon_2)$ for $m_1, m_2 \in \mathbb{N}$.

Since $\mathsf{V}(0)$ is the trivial one-dimensional representation and hence the unit of the monoidal structure, it is enough to consider sequences $\boldsymbol{a}$ not containing 0; so, from now on, we will always suppose that our sequences consist of strictly positive integer numbers. If $a_1 + \cdots + a_\ell = n$, we will often call the sequence $\boldsymbol{a}$ a *composition* of n. The sequence

$$\mathbb{n} = (\underbrace{1, \dots, 1}_{n}) \tag{3.1.1}$$

will be called the *regular composition* of n. Any other composition of n will be called *singular*. Notice that $\mathsf{V}(\mathbb{n}) = \mathsf{V}(1)^{\otimes n} = V^{\otimes n}$.

We repeat formulas from §1.2 for the special case of the representations $\mathsf{V}(a)$. Let $v_1^a = v_1^{a\varepsilon_1}$ and $v_0^a = [a]v_0^{a\varepsilon_1}$. Then $\mathsf{V}(a)$ is a 2-dimensional vector space with basis vectors v_1^a in degree

0 and v_0^a in degree 1; the action of U_q is given by

$$\begin{array}{llll} Ev_1^a = 0, & Fv_1^a = v_0^a, & \mathbf{q}^h v_1^a = q^{\langle h, a\varepsilon_1\rangle} v_1^a, & Kv_1^a = q^a v_1^a, \\ Ev_0^a = [a]v_1^a, & Fv_0^a = 0, & \mathbf{q}^h v_0^a = q^{\langle h, a\varepsilon_1 - \alpha\rangle} v_0^a, & Kv_0^a = q^a v_0^a. \end{array} \tag{3.1.2}$$

Projections and embeddings

Let $a, b \geq 1$. By Lemma 1.2.2, $\mathsf{V}(a+b)$ is a subrepresentation of $\mathsf{V}(a) \otimes \mathsf{V}(b)$. Let us define explicit maps $\Phi_{a,b} \colon \mathsf{V}(a) \otimes \mathsf{V}(b) \to \mathsf{V}(a+b)$ and $\Phi^{a,b} \colon \mathsf{V}(a+b) \to \mathsf{V}(a) \otimes \mathsf{V}(b)$. We set

$$\begin{aligned} \Phi_{a,b} \colon \mathsf{V}(a) \otimes \mathsf{V}(b) &\longrightarrow \mathsf{V}(a+b) \\ v_0^a \otimes v_0^b &\longmapsto 0 \\ v_0^a \otimes v_1^b &\longmapsto q^{-b} \begin{bmatrix} a+b-1 \\ b \end{bmatrix} v^{a+b_0} \\ v_1^a \otimes v_0^b &\longmapsto \begin{bmatrix} a+b-1 \\ a \end{bmatrix} v_0^{a+b} \\ v_1^a \otimes v_1^b &\longmapsto \begin{bmatrix} a+b \\ a \end{bmatrix} v_1^{a+b} \end{aligned} \tag{3.1.3}$$

and

$$\begin{aligned} \Phi^{a,b} \colon \mathsf{V}(a+b) &\longrightarrow \mathsf{V}(a) \otimes \mathsf{V}(b) \\ v_0^{a+b} &\longmapsto v_0^a \otimes v_1^b + q^a v_1^a \otimes v_0^b \\ v_1^{a+b} &\longmapsto v_1^a \otimes v_1^b, \end{aligned} \tag{3.1.4}$$

where as usual we set

$$[k]! = [k][k-1]\cdots[1] \qquad \text{for all } k \geq 1, \tag{3.1.5}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \qquad \text{for all } n \geq 1,\ 1 \leq k \leq n. \tag{3.1.6}$$

Lemma 3.1.1. *The maps $\Phi_{a,b}$ and $\Phi^{a,b}$ are morphisms of U_q–representations which commute with the bar involution. Moreover, we have*

$$\Phi_{a,b}\Phi^{a,b} = \begin{bmatrix} a+b \\ a \end{bmatrix} \mathrm{id}. \tag{3.1.7}$$

Proof. It is a straightforward computation to check that $\Phi_{a,b}$ and $\Phi^{a,b}$ are U_q–equivariant. In order to show that they commute with the bar involution, it is then sufficient to check what happens on some generators (as U_q–representations). Notice first that from the definition (1.3.1) and the bar-invariance of v_0^a, v_1^a it follows that $v_0^a \otimes v_0^b$ and $v_1^a \otimes v_1^b$ are bar-invariant. Then $\Phi^{a,b}$ commutes with the bar involution since applied to the bar-invariant element v_1^{a+b} it gives the bar-invariant element $v_1^a \otimes v_1^b$. Analogously, $\Phi_{a,b}$ commutes with the bar involution since $\Phi_{a,b}(v_0^a \otimes v_0^b) = 0$ and $\Phi_{a,b}(v_1 \otimes v_1^b)$ are both bar-invariant. Finally, (3.1.7) is obviously true when applied to the vector v_1^{a+b} and so we are done by Schur's Lemma. □

Canonical basis and bilinear form

Let $a \in \mathbb{Z}_{>0}$. The elements v_0^a, v_1^a give the standard basis of $\mathsf{V}(a)$. Let now $\boldsymbol{a} = (a_1, \dots, a_\ell)$ be a sequence of (strictly) positive numbers. For any sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell) \in \{0,1\}^\ell$ we let $v^{\boldsymbol{a}}_{\boldsymbol{\eta}} = v^{a_1}_{\eta_1} \otimes \cdots \otimes v^{a_\ell}_{\eta_\ell}$. The elements $\{v^{\boldsymbol{a}}_{\boldsymbol{\eta}} \mid \boldsymbol{\eta} \in \{0,1\}^\ell\}$ are called the *standard basis vectors* of $\mathsf{V}(\boldsymbol{a})$.

According to Definition 1.3.3, for each standard basis vector $v^{\boldsymbol{a}}_{\boldsymbol{\eta}}$ there exists a corresponding *canonical basis vector*

$$v^{\diamondsuit\boldsymbol{a}}_{\boldsymbol{\eta}} = v^{a_1}_{\eta_1} \diamondsuit \cdots \diamondsuit v^{a_\ell}_{\eta_\ell}. \tag{3.1.8}$$

The bilinear form

Fix a sequence of positive numbers $\boldsymbol{a} = (a_1, \dots, a_\ell)$. We define a symmetric bilinear form on $\mathsf{V}(\boldsymbol{a})$ by setting

$$(v^{\boldsymbol{a}}_{\boldsymbol{\eta}}, v^{\boldsymbol{a}}_{\boldsymbol{\gamma}})_{\boldsymbol{a}} = q^{\sum_{i\neq j} \beta^{\boldsymbol{\eta}}_i \beta^{\boldsymbol{\eta}}_j} \begin{bmatrix} \beta^{\boldsymbol{\eta}}_1 + \cdots + \beta^{\boldsymbol{\eta}}_\ell \\ \beta^{\boldsymbol{\eta}}_1, \dots, \beta^{\boldsymbol{\eta}}_\ell \end{bmatrix} \delta^{\gamma_1}_{\eta_1} \cdots \delta^{\gamma_\ell}_{\eta_\ell} \tag{3.1.9}$$

where δ^j_i is the Kronecker delta,

$$\beta^{\boldsymbol{\eta}}_j = a_j - 1 + \eta_j = \begin{cases} a_j - 1 & \text{if } \eta_j = 0, \\ a_j & \text{otherwise} \end{cases} \tag{3.1.10}$$

and

$$\begin{bmatrix} h_1 + \cdots + h_\ell \\ h_1, \dots, h_\ell \end{bmatrix} = \frac{[h_1 + \cdots + h_\ell]!}{[h_1]! \cdots [h_\ell]!}. \tag{3.1.11}$$

Note that $q^{\sum_{i\neq j} \beta^{\boldsymbol{\eta}}_i \beta^{\boldsymbol{\eta}}_j}$ in (3.1.9) is exactly the factor needed so that the value of (3.1.9) is a polynomial in q with constant term 1. We introduce the following non-standard notation:

$$[h]_0 = q^{h-1}[h] \tag{3.1.12}$$

$$[h]_0! = q^{\frac{h(h-1)}{2}}[h]! \tag{3.1.13}$$

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_0 = q^{ab} \begin{bmatrix} a+b \\ a \end{bmatrix} \tag{3.1.14}$$

$$\begin{bmatrix} h_1 + \cdots + h_\ell \\ h_1, \dots, h_\ell \end{bmatrix}_0 = q^{\sum_{i\neq j} h_i h_j} \begin{bmatrix} h_1 + \cdots + h_\ell \\ h_1, \dots, h_\ell \end{bmatrix}. \tag{3.1.15}$$

These are rescaled versions of the quantum numbers and factorials and of the quantum binomial and multinomial coefficients so that they are actual polynomials in q with constant term 1. Hence we can rewrite (3.1.9) as

$$(v^{\boldsymbol{a}}_{\boldsymbol{\eta}}, v^{\boldsymbol{a}}_{\boldsymbol{\gamma}})_{\boldsymbol{a}} = \begin{bmatrix} \beta^{\boldsymbol{\eta}}_1 + \cdots + \beta^{\boldsymbol{\eta}}_\ell \\ \beta^{\boldsymbol{\eta}}_1, \dots, \beta^{\boldsymbol{\eta}}_\ell \end{bmatrix}_0 \delta^{\gamma_1}_{\eta_1} \cdots \delta^{\gamma_\ell}_{\eta_\ell}. \tag{3.1.16}$$

Notice that we have

$$[h]_0! = [h]_0[h-1]_0 \cdots [2]_0, \tag{3.1.17}$$

$$\begin{bmatrix} h_1 + \cdots + h_\ell \\ h_1, \dots, h_\ell \end{bmatrix}_0 = \frac{[h_1 + \cdots + h_\ell]_0!}{[h_1]_0! \cdots [h_\ell]_0!}. \tag{3.1.18}$$

Lemma 3.1.2. *For all $v \in \mathsf{V}(a) \otimes \mathsf{V}(b)$ and $v' \in \mathsf{V}(a+b)$ we have*

$$(\Phi_{a,b}(v), v')_{(a+b)} = (v, q^{-ab}\Phi^{a,b}(v'))_{(a,b)}. \tag{3.1.19}$$

Proof. This is a straightforward calculation on the basis vectors:

$$(\Phi_{a,b}(v_0^a \otimes v_1^b), v_0^{a+b})_{(a+b)} = q^{-b} \begin{bmatrix} a+b-1 \\ b \end{bmatrix} = (v_0^a \otimes v_1^b, q^{-ab}\Phi^{a,b} v_0^{a+b})_{(a,b)}$$
$$(\Phi_{a,b}(v_1^a \otimes v_0^b), v_0^{a+b})_{(a+b)} = \begin{bmatrix} a+b-1 \\ a \end{bmatrix} = (v_1^a \otimes v_0^b, q^{-ab}\Phi^{a,b} v_0^{a+b})_{(a,b)}$$
$$(\Phi_{a,b}(v_1^a \otimes v_1^b), v_1^{a+b})_{(a+b)} = \begin{bmatrix} a+b \\ a \end{bmatrix} = (v_1^a \otimes v_1^b, q^{-ab}\Phi^{a,b} v_1^{a+b})_{(a,b)}. \qquad \square$$

Lemma 3.1.3. *For all standard basis vectors $v_{\boldsymbol\eta}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\boldsymbol a} \in \mathsf{V}(\boldsymbol a)$ we have*

$$(Fv_{\boldsymbol\eta}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a} = \frac{q^{a_1+\cdots+a_\ell-1}}{[\beta_1^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta}]_0} (v_{\boldsymbol\eta}^{\boldsymbol a}, Ev_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a}. \tag{3.1.20}$$

Proof. Suppose that there exists an index r such that $\eta_i = \gamma_i$ for all $i \neq r$ and $\eta_r = 0$, $\gamma_r = 1$ (otherwise both sides of (3.1.20) are zero). Up to a sign (that we ignore, because it is the same in both formulas), we have

$$(Fv_{\boldsymbol\eta}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a} = (q^{a_1+\cdots+a_{r-1}} v_{\boldsymbol\gamma}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a} = q^{a_1+\cdots+a_{r-1}} \begin{bmatrix} \beta_1^{\boldsymbol\gamma}+\cdots+\beta_\ell^{\boldsymbol\gamma} \\ \beta_1^{\boldsymbol\gamma}, \ldots, \beta_\ell^{\boldsymbol\gamma} \end{bmatrix}_0$$

and

$$(v_{\boldsymbol\eta}^{\boldsymbol a}, Ev_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a} = (v_{\boldsymbol\eta}^{\boldsymbol a}, [a_r] q^{-a_{r+1}-\cdots-a_\ell} v_{\boldsymbol\eta}^{\boldsymbol a})_{\boldsymbol a} = [a_r] q^{-a_{r+1}-\cdots-a_\ell} \begin{bmatrix} \beta_1^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta} \\ \beta_1^{\boldsymbol\eta}, \ldots, \beta_\ell^{\boldsymbol\eta} \end{bmatrix}_0.$$

Since $\beta_i^{\boldsymbol\eta} = \beta_i^{\boldsymbol\gamma}$ for $i \neq r$ while $\beta_r^{\boldsymbol\eta} = \beta_r^{\boldsymbol\gamma} + 1 = a_r$, we have

$$\begin{aligned}
\frac{(v_{\boldsymbol\eta}^{\boldsymbol a}, Ev_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a}}{(Fv_{\boldsymbol\eta}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\boldsymbol a})_{\boldsymbol a}} &= [a_r] \frac{[\beta_1^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta}]}{[\beta_r^{\boldsymbol\eta}]} q^{\beta_1^{\boldsymbol\eta}+\cdots+\beta_{r-1}^{\boldsymbol\eta}+\beta_{r+1}^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta}} q^{-a_1-\cdots-a_{r-1}-a_{r+1}-\cdots-a_\ell} \\
&= [\beta_1^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta}]_0 \, q^{1-a_1-\cdots-a_\ell},
\end{aligned}$$

which proves the claim. $\square$

Remark 3.1.4. If we enlarge U_q with a new generator E' such that

$$E = q\frac{q^{2h_1}-1}{q^2-1} E' K^{-1} \tag{3.1.21}$$

then we get an adjuction between F and E'.

Dual standard and dual canonical basis

We define the *dual standard basis* $\{v_{\boldsymbol\eta}^{\clubsuit \boldsymbol a} \mid \eta \in \{0,1\}^\ell\}$ of $\mathsf{V}(\boldsymbol a)$ to be the basis dual to the standard basis with respect to the bilinear form $(\cdot,\cdot)_{\boldsymbol a}$:

$$(v_{\boldsymbol\eta}^{\boldsymbol a}, v_{\boldsymbol\gamma}^{\clubsuit \boldsymbol a})_{\boldsymbol a} = \begin{cases} 1 & \text{if } \boldsymbol\eta = \boldsymbol\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.22}$$

Of course, since the standard basis is already orthogonal, each $v_{\boldsymbol\eta}^{\clubsuit \boldsymbol a}$ is a multiple of $v_{\boldsymbol\eta}^{\boldsymbol a}$. In particular, one has

$$\begin{bmatrix} \beta_1^{\boldsymbol\eta}+\cdots+\beta_\ell^{\boldsymbol\eta} \\ \beta_1^{\boldsymbol\eta}, \ldots, \beta_\ell^{\boldsymbol\eta} \end{bmatrix}_0 v_{\boldsymbol\eta}^{\clubsuit \boldsymbol a} = v_{\boldsymbol\eta}^{\boldsymbol a}. \tag{3.1.23}$$

Moreover, we define the *dual canonical basis* $\{v_{\boldsymbol{\eta}}^{\heartsuit \boldsymbol{a}} \mid \eta \in \{0,1\}^\ell\}$ of $\mathsf{V}(\boldsymbol{a})$ to be the basis dual to the canonical basis with respect to the bilinear form $(\cdot,\cdot)_{\boldsymbol{a}}$:

$$(v_{\boldsymbol{\eta}}^{\diamondsuit \boldsymbol{a}}, v_{\boldsymbol{\gamma}}^{\heartsuit \boldsymbol{a}})_{\boldsymbol{a}} = \begin{cases} 1 & \text{if } \boldsymbol{\eta} = \boldsymbol{\gamma}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.24}$$

3.2 Super Schur-Weyl duality

Let us define a linear endomorphism $\check{H}$ of $V \otimes V$ by

$$\begin{aligned} &\check{H}(v_0^1 \otimes v_0^1) = -q v_0^1 \otimes v_0^1, \quad && \check{H}(v_0^1 \otimes v_1^1) = v_1^1 \otimes v_0^1 + (q^{-1} - q) v_0^1 \otimes v_1^1 \\ &\check{H}(v_1^1 \otimes v_0^1) = v_0^1 \otimes v_1^1 && \check{H}(v_1^1 \otimes v_1^1) = q^{-1} v_1^1 \otimes v_1^1. \end{aligned} \tag{3.2.1}$$

By an explicit computation it can be checked that $\check{H}$ can be expressed in terms of a projection (3.1.3) and an embedding (3.1.4):

$$\Phi^{1,1}\Phi_{1,1} = \check{H} + q. \tag{3.2.2}$$

It follows in particular that $\check{H}$ is U_q–equivariant.

We can consider on $V^{\otimes n}$ the operators

$$\check{H}_i = \mathrm{id}^{\otimes i-1} \otimes \check{H} \otimes \mathrm{id}^{\otimes n-i-1}. \tag{3.2.3}$$

By definition, they are intertwiners for the action of U_q. One can easily check that

$$\check{H}_i^2 = (q^{-1} - q)\check{H}_i + \mathrm{id}. \tag{3.2.4}$$

The category of U_q–representation is braided (see §A.3), and the endomorphism $\check{H}$ is just the inverse of the braiding $\check{R}_{V,V}$. From this it follows directly that $\check{H}$ is equivariant and that the braid relation $\check{H}_i\check{H}_{i+1}\check{H}_i = \check{H}_{i+1}\check{H}_i\check{H}_{i+1}$ holds for all $i = 1, \dots, n-1$. Since clearly $\check{H}_i\check{H}_j = \check{H}_j\check{H}_i$ for $|i-j| > 1$, it follows that the operators $\check{H}_i$ define on $V^{\otimes n}$ an action of the Hecke algebra $\mathcal{H}_n$, which we regard as a right action.

The following result is also known as super Schur-Weyl duality. The non-quantized version was originally proved by Berele and Regev [BR87] and independently by Sergeev [Ser84].

Proposition 3.2.1 ([Mit06])**.** *The map*

$$\begin{aligned} \mathcal{H}_n &\longrightarrow \mathrm{End}_{U_q}(V^{\otimes n}) \\ H_i &\longmapsto \check{H}_i \end{aligned} \tag{3.2.5}$$

is surjective. As a module for $\mathcal{H}_n$ we have

$$V^{\otimes n} = \bigoplus_{k=1}^{n} \big(S(\mu_{n,k}) \oplus S(\mu_{n,k})\big), \tag{3.2.6}$$

where $\mu_{n,k}$ is the hook partition $(k, 1^{n-k})$ and $S(\mu_{n,k})$ is the q–version of the corresponding Specht module.

By contrast, let us notice the following easy fact which we will use later:

Lemma 3.2.2. *If $n \neq m$ then* $\mathrm{Hom}_{U_q}(V^{\otimes m}, V^{\otimes n}) = \{0\}$.

Proof. This follows since K acts on $V^{\otimes m}$ by q^m and on $\mathbb{V}^{\otimes n}$ by q^n. Hence there is no U_q–equivariant map $V^{\otimes m} \to V^{\otimes n}$ if $m \neq n$. □

The Super Temperley-Lieb Algebra

It follows from Proposition 3.2.1 that the kernel of (3.2.5) is the two-sided ideal $\mathcal{I}_n$ generated by the idempotents projecting onto simple representations $S(\mu)$ of $\mathcal{H}_n$ corresponding to Young shapes μ with n boxes that are not hooks. For $n \leq 3$ there are no such Young shapes. For $n = 4$, the only Young shape that is not a hook is ⊞, and the corresponding idempotent is, up to a multiple,

$$(H_1 + q)(H_3 + q)H_2(H_1 - q^{-1})(H_3 - q^{-1}). \tag{3.2.7}$$

For $n \geq 4$ every Young shape that is not a hook contains some ⊞ and it is easy to prove that the ideal $\mathcal{I}_n$ is generated by

$$(H_{i-1} + q)(H_{i+1} + q)H_i(H_{i-1} - q^{-1})(H_{i+1} - q^{-1}) \tag{3.2.8}$$

for $i = 2, \dots, n-2$.

As often occurs with the Hecke algebra, it is more convenient to choose generators $C_i = H_i + q$. We introduce the Super Temperley-Lieb Algebra as follows:

Definition 3.2.3. *For $n \geq 1$, the* Super Temperley-Lieb Algebra STL_n *is the unital associative $\mathbb{C}(q)$–algebra generated by $\{C_i \mid i = 1, \dots, n-1\}$ subjected to the relations*

$$C_i^2 = (q + q^{-1})C_i, \tag{3.2.9a}$$

$$C_iC_j = C_jC_i \qquad \textit{for } |i - j| > 1, \tag{3.2.9b}$$

$$C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1}, \tag{3.2.9c}$$

and

$$C_{i-1}C_{i+1}C_i((q + q^{-1}) - C_{i-1})((q + q^{-1}) - C_{i+1}) = 0, \tag{3.2.9d}$$

$$((q + q^{-1}) - C_{i-1})((q + q^{-1}) - C_{i+1})C_iC_{i-1}C_{i+1} = 0. \tag{3.2.9e}$$

Since the first three relations are just the relations that the generators $C_i = H_i + q$ satisfy in the Hecke algebra, it follows that STL_n is a quotient of $\mathcal{H}_n$. Moreover, by the discussion above, we have

$$\mathrm{STL}_n \cong \mathrm{End}_{U_q}(V^{\otimes n}). \tag{3.2.10}$$

Canonical basis revisited

Consider the weight space decomposition

$$V^{\otimes n} = \bigoplus_{k=0}^{n} (V^{\otimes n})_k \tag{3.2.11}$$

where

$$(V^{\otimes n})_k = \{v \in V^{\otimes n} \mid \mathbf{q}^h v = q^{\langle h, k\varepsilon_1 + (n-k)\varepsilon_2\rangle} v\}. \tag{3.2.12}$$

Clearly, every weight space is a module for the Hecke algebra. We have:

Proposition 3.2.4. *Let $W_{\mathsf{q}} = \langle s_1, \dots, s_{k-1}\rangle$ and $W_{\mathsf{p}} = \langle s_{k+1}, \dots, s_{n-1}\rangle$ as subgroups of $\mathbb{S}_n$. With the notation of §2.2 we have*

$$(V^{\otimes n})_k \cong \mathcal{M}^{\mathsf{p}}_{\mathsf{q}} \tag{3.2.13}$$

as right $\mathcal{H}_n$–modules. The isomorphism is given explicitly by

$$\begin{aligned} \Psi\colon \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}} &\longrightarrow (V^{\otimes n})_k \\ N_w &\longmapsto v^{\boldsymbol{a}}_{\boldsymbol{\eta}_{min}\cdot w}, \end{aligned} \tag{3.2.14}$$

where

$$\boldsymbol{\eta}_{\min} = (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k}). \tag{3.2.15}$$

and $\mathbb{S}_n$ acts on sequences of $\{0,1\}^n$ from the right by permutations.

Proof. It is straightforward to check that, by the definition of the action of $\mathcal{H}_n$ on $V^{\otimes n}$ (3.2.1), we have $v_{\boldsymbol{\eta}_{\min}} \cdot H_w = v_{\boldsymbol{\eta}_{\min}\cdot w}$ whenever $w \in W^{\mathfrak{p}+\mathfrak{q}}$. In particular, (3.2.14) is a bijection. We need to show that the action of the Hecke algebra is the same on both sides. This follows by comparing (2.2.3) with (3.2.1). □

As a consequence, there is a second notion of canonical basis on $(V^{\otimes n})_k$, defined using the Hecke algebra action from Chapter 2. Not surprisingly, this coincides with Lusztig canonical basis (compare with [FKK98, Theorem 2.5], that deals with the case of $\mathfrak{sl}_2$):

Proposition 3.2.5. *Under the isomorphism Ψ (3.2.14), the canonical basis element $\underline{N}_w$ of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ is mapped to the canonical basis element $v^{\diamond \boldsymbol{a}}_{\boldsymbol{\eta}_{min}\cdot w}$.*

Proof. By the uniqueness results (Proposition 2.2.2 and Theorem 1.3.2), it is enough to show that the bar involution of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ is mapped to the bar involution of $(V^{\otimes n})_k$ under (3.2.14). On $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ the bar involution is uniquely determined by $\overline{N_e} = N_e$ and $\overline{XH_i} = \overline{X}H_i^{-1}$ for all $X \in \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$. It is enough to show that the same holds for Lusztig's bar involution on $(V^{\otimes n})_k$. Of course $\overline{v_{\boldsymbol{\eta}_{\min}}} = v_{\boldsymbol{\eta}_{\min}}$, and one can show by standard methods (cf. [Zha09, Lemma 2.3]) that

$$\overline{\check{H}_i(v_{\boldsymbol{\eta}})} = \check{H}_i^{-1}(\overline{v_{\boldsymbol{\eta}}}) \tag{3.2.16}$$

for all standard basis elements $v_{\boldsymbol{\eta}}$. □

The form $\langle\cdot,\cdot\rangle$ on $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ and the form $(\cdot,\cdot)_{\mathbb{m}}$ on $(V^{\otimes n})_k$ are proportional under Ψ:

Lemma 3.2.6. *Let Ψ be the isomorphism (3.2.14). Then*

$$(\Psi(X),\Psi(Y))_{\mathbb{m}} = [k]_0!\langle X,Y\rangle \qquad \textit{for all } X,Y \in \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}. \tag{3.2.17}$$

Proof. It is enough to check (3.2.17) on the standard basis $\{N_w \mid w \in W^{\mathfrak{p}+\mathfrak{q}}\}$ of $\mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$. We have

$$(\Psi(N_w),\Psi(N_z))_{\mathbb{m}} = (v_{\boldsymbol{\eta}_{\min}\cdot w}, v_{\boldsymbol{\eta}_{\min}\cdot z})_{\mathbb{m}} = \begin{cases} [k]_0! & \text{if } w = z, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.18}$$

By definition, this is the same as $[k]_0!\langle N_w, N_z\rangle$. □

3.3 Web diagrams and the intertwining operators

In this section we will provide a graphical calculus for the intertwining operators in the category Rep.

The category Web

We start by defining a diagrammatic category Web and a functor $\mathscr{T}:\mathsf{Web}\to\mathsf{Rep}$. We remark that the category Web is similar to the category of SL_n–spiders (see [Kup96], [Kim03], [Mor07] and [CKM12]) which describes intertwining operators of representations of $U_q(\mathfrak{gl}_n)$.

Web diagrams

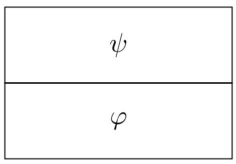
A *web diagram* is an oriented plane graph with edges labeled by positive integers. For simplicity we suppose that the edges do not have maxima or minima, and the orientation is then uniquely determined by orienting all edges upwards. Only single and triple vertices are allowed. Single vertices must lie on the bottom (respectively, top) line if they are sources (respectively, targets) for the corresponding edge. Around a triple vertex, the sum of the labels of the ingoing edges must agree with the sum of the labels of the outgoing vertices; this means that only the following labelings are allowed for arbitrary strictly positive numbers a, b:

(3.3.1)

We will not draw the orientation of the edges, because they are all oriented upwards. The *source* of a web is the sequence $\boldsymbol{a}=(a_1,\dots,a_\ell)$ of labels on the bottom line. The *target* is the sequence $\boldsymbol{a}'=(a'_1,\dots,a'_s)$ on the top line.

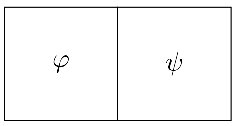
If we have two webs ψ,φ and the target of φ is the same as the source of ψ, then we can compose ψ and φ by concatenating vertically:

$\psi\circ\varphi=$



Additionally, we can always concatenate two webs φ,ψ horizontally, putting the second on the right of the first; in this case we use a tensor product symbol:

$\varphi\otimes\psi=$



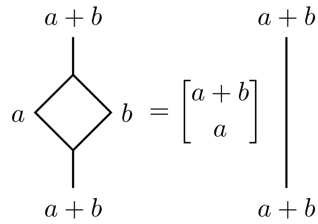
The categories Web′ and Web

Definition 3.3.1. *The category* Web' *is the category whose objects are sequences* $\boldsymbol{a}=(a_1,\dots,a_\ell)$ *of strictly positive integers; a morphism from* $\boldsymbol{a}$ *to* $\boldsymbol{a}'$ *is a* $\mathbb{C}(q)$*–linear combination of web diagrams with source* $\boldsymbol{a}$ *and target* $\boldsymbol{a}'$*. Composition of morphisms corresponds to vertical concatenation of web diagrams. Horizontal concatenation of web diagrams gives, on the other side, a monoidal structure on* Web'*, whose unit is the empty sequence* $()$*.*

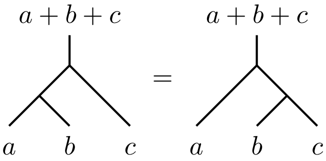
Definition 3.3.2. *We define the category* Web *to be the quotient of* Web' *by the following relations:*

(3.3.2a) *Orientation preserving isotopy (with source and target points fixed)*

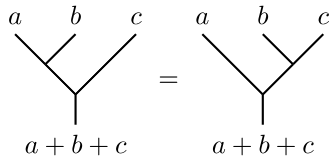
(3.3.2b)



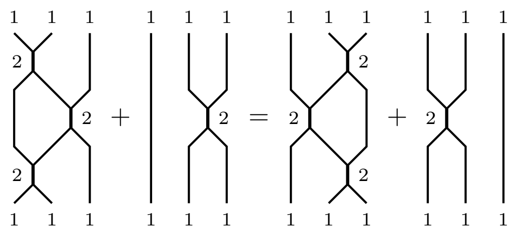
(3.3.2c)



(3.3.2d)

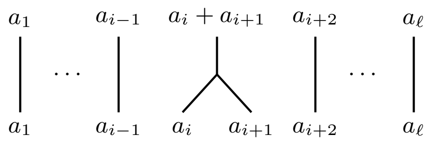


(3.3.2e)

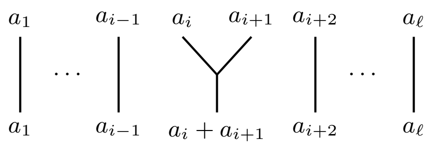


We define the two *elementary webs* $\curlywedge_{\boldsymbol{a},i}$ and $\curlyvee^{\boldsymbol{a},i}$ by the diagrams

(3.3.3) $\curlywedge_{\boldsymbol{a},i} =$



(3.3.4) $\curlyvee^{\boldsymbol{a},i} =$

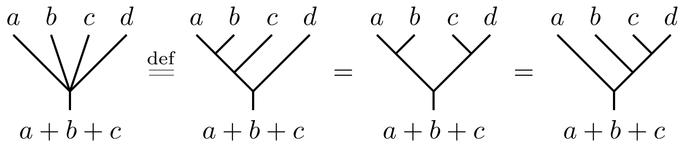


and notice that the category Web is generated by such elementary web diagrams. Given a sequence $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ we let also

(3.3.5)
$$\hat{\boldsymbol{a}}_i = (a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_\ell)$$

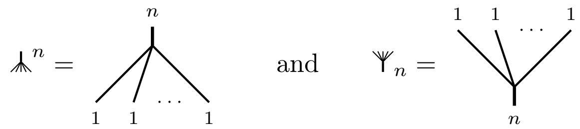
be the target of $\curlywedge_{\boldsymbol{a},i}$ (and the source of $\curlyvee^{\boldsymbol{a},i}$).

It will be useful to consider also multivalent vertices with only one outgoing (respectively, ingoing) edge: we define them to be equal to concatenations of trivalent vertices (3.3.1). For example:

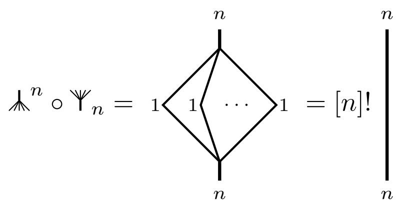
(3.3.6)


Notice that this is well-defined by (3.3.2c) and (3.3.2d).

Let us define the web diagrams

(3.3.7)


From (3.3.2b) it follows in particular that

(3.3.8)


Given a composition $\boldsymbol{a} = (a_1, \dots, a_\ell)$ of n, notice that we have a *standard inclusion*

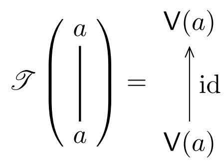
(3.3.9) $$\Upsilon_{a_1} \otimes \cdots \otimes \Upsilon_{a_\ell} \colon \boldsymbol{a} \to \mathbbm{n}$$

and a *standard projection*

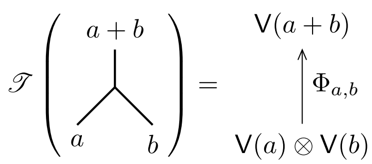
(3.3.10) $$\curlywedge^{a_1} \otimes \cdots \otimes \curlywedge^{a_\ell} \colon \mathbbm{n} \to \boldsymbol{a}.$$

Webs as intertwiners

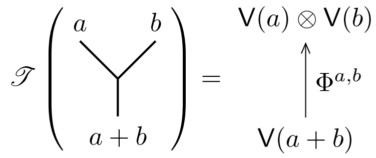
Now we are going to define a monoidal functor $\mathscr{T} \colon \mathsf{Web} \to \mathsf{Rep}$. On objects we set $\mathscr{T}(\boldsymbol{a}) = \mathsf{V}(\boldsymbol{a})$ and $\mathscr{T}(()) = \mathbb{C}(q)$, where $()$ is the empty sequence. To define $\mathscr{T}$ on morphisms, it suffices to consider elementary pieces of webs. An oriented edge is an identity morphism from the source a to the target a in Web, hence the functor $\mathscr{T}$ assigns to it the identity morphism of $\mathsf{V}(a)$:

(3.3.11)


To triple vertices we assign projections and inclusions of subrepresentations, as follows:

(3.3.12)


(3.3.13)



Proposition 3.3.3. *The assignments* (3.3.11)*,* (3.3.12)*,* (3.3.13) *define a dense full monoidal functor* $\mathscr{T}\colon \mathsf{Web} \to \mathsf{Rep}$.

Proof. First, we have to check that $\mathscr{T}$ satisfies the relations defining Web. It is straightforward to check that $\mathscr{T}$ assigns the same morphism to both sides of (3.3.2c) and (3.3.2d). Relation (3.3.2b) is satisfied thanks to (3.1.7). Relation (3.3.2e) is satisfied thanks to (3.2.9c).

The functors $\mathscr{T}$ is dense since, by definition, the objects of Rep are exactly the $\mathsf{V}(\boldsymbol{a})$ for all sequences $\boldsymbol{a}$ of positive integer numbers. We prove now that $\mathscr{T}$ is full. By Proposition 3.2.1 and (3.2.2) the map $\mathrm{Hom}_{\mathsf{Web}}(\mathbb{n}, \mathbb{n}) \to \mathrm{Hom}_{\mathsf{Rep}}(V^{\otimes n}, V^{\otimes n})$ induced by $\mathscr{T}$ is surjective. Together with Lemma 3.2.2, it follows more in general that the map $\mathrm{Hom}_{\mathsf{Web}}(\mathbb{m}, \mathbb{n}) \to \mathrm{Hom}_{\mathsf{Rep}}(V^{\otimes m}, V^{\otimes n})$ induced by $\mathscr{T}$ is surjective for all m, n. Now each representation $\mathsf{V}(\boldsymbol{a}) \in \mathsf{Rep}$ embeds in some $V^{\otimes n}$, and the corresponding inclusion and projection are images under $\mathscr{T}$ of the standard inclusion (3.3.9) and of the standard projection (3.3.10). Hence $\mathscr{T}$ induces a surjective map $\mathrm{Hom}_{\mathsf{Web}}(\boldsymbol{a}, \boldsymbol{a}') \to \mathrm{Hom}_{\mathsf{Rep}}(\mathsf{V}(\boldsymbol{a}), \mathsf{V}(\boldsymbol{a}'))$ for all sequences $\boldsymbol{a}, \boldsymbol{a}'$, hence $\mathscr{T}$ is full. $\square$

In what follows, we are often going to omit to write the functor $\mathscr{T}$ and consider a web just as a homomorphism of the corresponding representations.

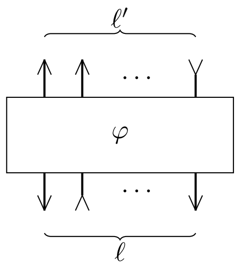
Matrix coefficients

Let φ be a web from $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ to $\boldsymbol{a}' = (a'_1, \ldots, a'_{\ell'})$. Given $\boldsymbol{\eta} \in \{0,1\}^\ell, \boldsymbol{\gamma} \in \{0,1\}^{\ell'}$, we can consider the matrix coefficient

(3.3.14) $$\langle \varphi(v^{\boldsymbol{a}}_{\boldsymbol{\eta}}), v^{\boldsymbol{a}'}_{\boldsymbol{\gamma}} \rangle,$$

which is the coefficient of $v^{\boldsymbol{a}'}_{\boldsymbol{\gamma}}$ in $\varphi(v^{\boldsymbol{a}}_{\boldsymbol{\eta}})$ when expressed in the standard basis. We represent it by a labeled web diagram

(3.3.15)



where the i–th line below is labeled by $\wedge$ if $\eta_i = 1$ and by $\vee$ if $\eta_i = 0$, and the i–th line above is labeled by $\wedge$ if $\gamma_i = 1$ and by $\vee$ if $\gamma_i = 0$.

Diagrams provide a convenient way to compute matrix coefficients, as we are going to explain. Let us fix a diagram φ and suppose that we want to compute the coefficient (3.3.14). We start with the picture (3.3.15). Then we label every internal edge of the graph with $\wedge$ and $\vee$, in all possible ways. Such a "completely labeled" graph is evaluated according to the local rules in Figure 3.1 (the missing label possibilities are evaluated to zero, and the total evaluation is obtained via multiplication). To evaluate the initial picture, sum the evaluations over all possible "complete labelings".

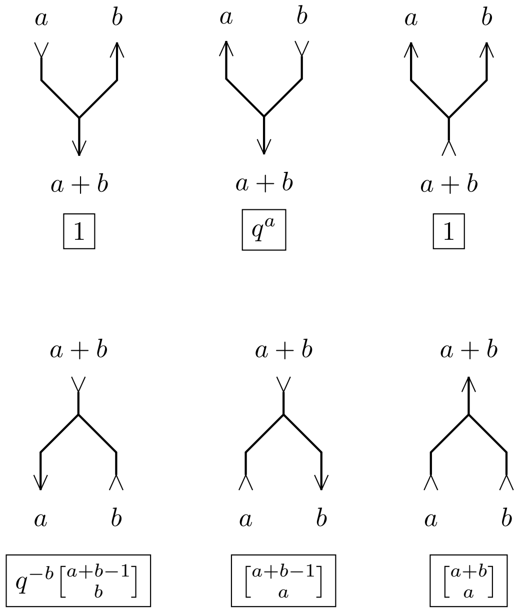
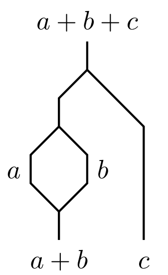


Figure 3.1: Evaluation of elementary diagrams.

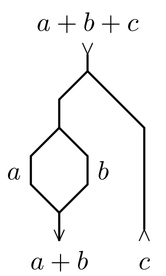
EXAMPLE 3.3.4. Consider the web diagram

(3.3.16) $\varphi =$



and suppose we want to compute the matrix coefficient $\langle \varphi(v^{(a+b,c)}_{(0,1)}), v^{(a+b+c)}_{(0)} \rangle$, which corresponds to the partially labeled diagram

(3.3.17)



This diagram has eight "complete labelings", but only the following two have non-zero

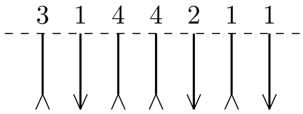
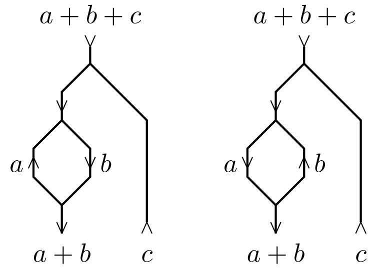


Figure 3.2: The standard basis diagram for $v^{(3,1,4,4,2,1,1)}_{(1,0,1,1,0,1,0)}$.

evaluation according to the rules of Figure 3.1:

(3.3.18)

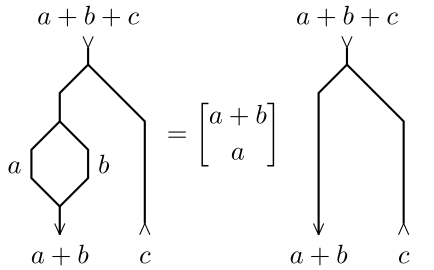


According to the evaluation rules, the first diagram evaluates to $q^a\begin{bmatrix}a+b-1\\a\end{bmatrix}q^{-c}\begin{bmatrix}a+b+c-1\\c\end{bmatrix}$, and the second one evaluates to $q^{-b}\begin{bmatrix}a+b-1\\b\end{bmatrix}q^{-c}\begin{bmatrix}a+b+c-1\\c\end{bmatrix}$. Hence we have

$$
\begin{aligned}
&\langle \varphi(v^{(a+b,c)}_{(0,1)}, v^{(a+b+c)}_{(0)}\rangle \\
&\quad = q^a\begin{bmatrix}a+b-1\\a\end{bmatrix}q^{-c}\begin{bmatrix}a+b+c-1\\c\end{bmatrix} + q^{-b}\begin{bmatrix}a+b-1\\b\end{bmatrix}q^{-c}\begin{bmatrix}a+b+c-1\\c\end{bmatrix} \\
&\quad = q^{-c}\begin{bmatrix}a+b\\a\end{bmatrix}\begin{bmatrix}a+b+c-1\\c\end{bmatrix}.
\end{aligned} \tag{3.3.19}
$$

Notice that we could have simplified the calculation using relation (3.3.2b):

(3.3.20)



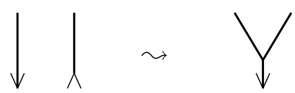
⊛

Webs and canonical basis

Fix a sequence $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ and consider a standard basis element $v^{\boldsymbol{a}}_{\boldsymbol{\eta}}$ of $\mathsf{V}(\boldsymbol{a})$. This standard basis element is represented by a (trivial) diagram, obtained as follows: take the identity web $\boldsymbol{a} \to \boldsymbol{a}$ and label the edges from the left to the right with an $\wedge$ if $\eta_i = 1$ and a $\vee$ if $\eta_i = 0$, (as in Figure 3.2). We call it the *standard basis diagram* corresponding to $v^{\boldsymbol{a}}_{\boldsymbol{\eta}}$.

Starting from this standard basis diagram, one can obtain the corresponding canonical basis element as follows. For each consecutive $\vee\wedge$ (in this order), join the corresponding two edges as follows:

(3.3.21)



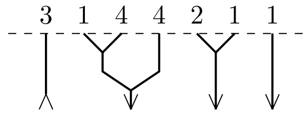


Figure 3.3: The canonical basis diagram for $v_{(1,0,1,1,0,1,0)}^{(3,1,4,4,2,1,1)}$.

Repeat this process using at each step also the $\vee$'s created in the previous steps, until no more $\vee\wedge$ is left. At the end, we will obtain some diagram $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ that we call the *canonical basis diagram* corresponding to $v_{\boldsymbol{\eta}}^{\boldsymbol{a}}$ (see Figure 3.3 and Example 3.3.7 below).

Remark 3.3.5. Note that this canonical basis diagram is obtained joining recursively each edge labeled by a $\vee$ with all immediately following edges labeled by $\wedge$'s. If we use multivalent vertices (as defined by (3.3.6)), we can construct the canonical basis diagram in just one step. In particular, the construction is independent of the order in which we consider the pairs $\vee\wedge$.

We claim that canonical basis diagrams correspond to canonical basis elements via $\mathscr{T}$. In fact, the diagram $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ has an underlying web that represents some embedding $\mathsf{V}(\boldsymbol{a}') \to \mathsf{V}(\boldsymbol{a})$, where $\boldsymbol{a}'$ is some composition that is refined by $\boldsymbol{a}$; this web carries on the bottom the labels of a basis element of $\mathsf{V}(\boldsymbol{a}')$, which is at the same time a standard basis element and a canonical basis element. Hence the diagram $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ is an "evaluated web", that gives a bar-invariant element of $\mathsf{V}(\boldsymbol{a})$ (since $\mathscr{T}(\varphi)$ sends bar-invariant elements to bar-invariant elements for all webs φ by Lemma 3.1.1). Examining the evaluation rules (Figure 3.1), one observes that the matrix coefficients of $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ are all in $q\mathbb{Z}[q]$ except for the coefficient of $v_{\boldsymbol{\eta}}^{\boldsymbol{a}}$, that is 1. Summarizing, we have:

Proposition 3.3.6. *The diagram $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ gives the canonical basis element $v_{\boldsymbol{\eta}}^{\diamondsuit\boldsymbol{a}}$ of $\mathsf{V}(\boldsymbol{a})$.*

Example 3.3.7. Let $\boldsymbol{a} = (3,1,4,4,2,1,1)$ and consider the element $v_{(1,0,1,1,0,1,0)}^{\boldsymbol{a}} \in \mathsf{V}(\boldsymbol{a})$. The corresponding standard and canonical basis diagrams are pictured in Figures 3.2 and 3.3. In particular, evaluating the canonical basis diagram according to the rules in Figure 3.1, we get the corresponding canonical basis element

$$\begin{aligned} v_{(1,0,1,1,0,1,0)}^{\diamondsuit\boldsymbol{a}} = v_{(1,0,1,1,0,1,0)}^{\boldsymbol{a}} + q v_{(1,1,0,1,0,1,0)}^{\boldsymbol{a}} + q^5 v_{(1,1,1,0,0,1,0)}^{\boldsymbol{a}} \\ + q^2 v_{(1,0,1,1,1,0,0)}^{\boldsymbol{a}} + q^3 v_{(1,1,0,1,1,0,0)}^{\boldsymbol{a}} + q^7 v_{(1,1,1,0,1,0,0)}^{\boldsymbol{a}}. \end{aligned} \tag{3.3.22}$$ $\circledast$

Remark 3.3.8. Notice that for the regular composition $\mathbb{n}$ of n and for some standard basis element $v_{\boldsymbol{\eta}}^{\mathbb{n}}$, the corresponding canonical basis diagram $C(v_{\boldsymbol{\eta}}^{\mathbb{n}})$ is a diagram of the type $\Upsilon_{a_1} \otimes \cdots \otimes \Upsilon_{a_\ell}$, where $\boldsymbol{a}$ is some composition of $\mathbb{n}$ depending on $\boldsymbol{\eta}$, and it is labeled at the bottom by a sequence of $\wedge$'s followed by a sequence of $\vee$'s.

Webs and the action of E and F

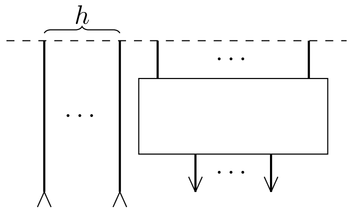
Using our diagram calculus we can easily compute the action of F on canonical basis elements (in an analogous way as [FK97] for $\mathfrak{sl}_2$).

Proposition 3.3.9. *Fix some representation $\mathsf{V}(\boldsymbol{a})$ and consider a canonical basis element $v_{\boldsymbol{\eta}}^{\diamondsuit\boldsymbol{a}}$. We have*

$$F(v_{\boldsymbol{\eta}}^{\diamondsuit\boldsymbol{a}}) = \begin{cases} v_0^{a_1} \diamondsuit v_{\eta_2}^{a_1} \diamondsuit \cdots \diamondsuit v_{\eta_\ell}^{a_\ell} & \textit{if } \eta_1 = 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.3.23}$$

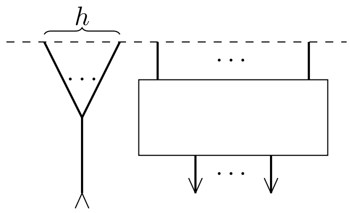
Proof. Suppose that $\eta_i = 1$ for $i = 1, \dots, h$, while $\eta_{h+1} = 0$ (possibly $h = 0$). The canonical basis diagram $C(v_{\boldsymbol{\eta}}^{\boldsymbol{a}})$ is

(3.3.24)



where there are some vertices in the box. We can also represent it as

(3.3.25)

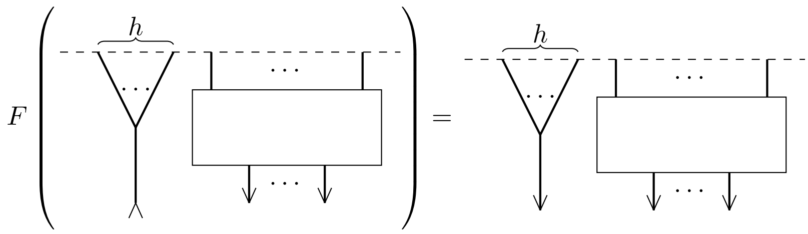


because this is the same element according to our diagrammatic calculus. On the bottom we have the labels of $v_{\boldsymbol{\gamma}}^{\boldsymbol{a}'}$ for some composition $\boldsymbol{a}'$ refining $\boldsymbol{a}$, where $\boldsymbol{\gamma} = (1, 0, \dots, 0)$. We have

$$F(v_1^{a'_1} \otimes v_0^{a'_2} \otimes \cdots \otimes v_0^{a'_{r'}}) = v_0^{a'_1} \otimes v_0^{a'_2} \otimes \cdots \otimes v_0^{a'_{r'}}. \tag{3.3.26}$$

Hence

(3.3.27)



which is exactly our assertion. ☐

By (3.1.20) it follows that E sends the dual canonical basis to the dual canonical basis (up to a multiple):

Proposition 3.3.10. *Fix some representation $\mathsf{V}(\boldsymbol{a})$ and consider a dual canonical basis element $v_{\boldsymbol{\eta}}^{\heartsuit \boldsymbol{a}}$. We have*

$$E(v_{\boldsymbol{\eta}}^{\heartsuit \boldsymbol{a}}) = \begin{cases} \dfrac{[\beta_1^{\boldsymbol{\eta}} + \cdots + \beta_\ell^{\boldsymbol{\eta}}]_0}{q^{a_1 + \cdots + a_\ell - 1}} v_1^{a_1} \heartsuit v_{\eta_2}^{a_1} \heartsuit \cdots \heartsuit v_{\eta_\ell}^{a_\ell} & \textit{if } \eta_1 = 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.3.28}$$

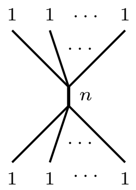
A faithful calculus

The functor $\mathscr{T} \colon \mathsf{Web} \to \mathsf{Rep}$ we constructed is full, but not faithful. In the following we will define a category $\overline{\mathsf{Web}}$ by adding more relations to Web, so that $\mathscr{T}$ descends to a faithful functor $\mathscr{T} \colon \overline{\mathsf{Web}} \to \mathsf{Rep}$. We point out that in Part II we will be able to categorify $\mathscr{T} \colon \mathsf{Web} \to \mathsf{Rep}$ (that is, we will define functors which satisfy categorical versions of the relations defining the category Web); although we believe that these functors satisfy also the additional relations defining $\overline{\mathsf{Web}}$, we will only be able to formulate a conjecture.

First, we need the following result:

Lemma 3.3.11. *Let $w_n \in \mathbb{S}_n$ be the longest element. Then the image of $\underline{H}_{w_n}$ under the map* (3.2.5) *is the endomorphism*

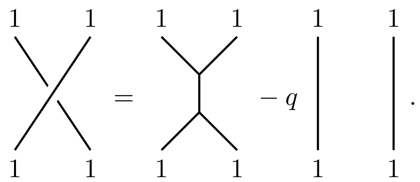
(3.3.29) $$\Xi_n = \quad \in \mathrm{End}_{\mathsf{Rep}}(V^{\otimes n}).$$



Proof. One can check by a standard calculation using (2.2.3) that the element $\underline{H}_{w_n}$ acts by 0 on $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ unless $W_{\mathfrak{p}}$ is trivial. In particular, by (3.2.13) it follows that $\underline{H}_{w_n}$ acts by 0 on the weight space $(V^{\otimes n})_k$ unless $k = n-1, n$. Now, $V^{\otimes n}$ decomposes as in (1.2.16) and the one copy of $\mathsf{L}(n\varepsilon_1)$ is the unique summand whose weight spaces are contained in $(V^{\otimes n})_{n-1}$ and $(V^{\otimes n})_n$. It follows that, up to a multiple, $\underline{H}_{w_n}$ acts on $V^{\otimes n}$ by projecting onto this summand $\mathsf{L}(n\varepsilon_1)$. Hence the homomorphisms defined by $\underline{H}_{w_n}$ and (3.3.29) coincide up to a multiple. A standard calculation shows that $\underline{H}_{w_n}^2 = [n]!\underline{H}_{w_n}$. By (3.3.8) the same holds for Ξ_n, hence the claim follows. ☐

From the web calculus we introduced we cannot see explicitly the action of the Hecke algebra. However, we can enhance our web calculus allowing edges labeled by 1 to cross, and define

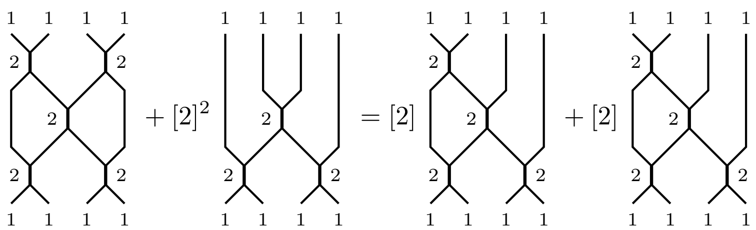
(3.3.30)



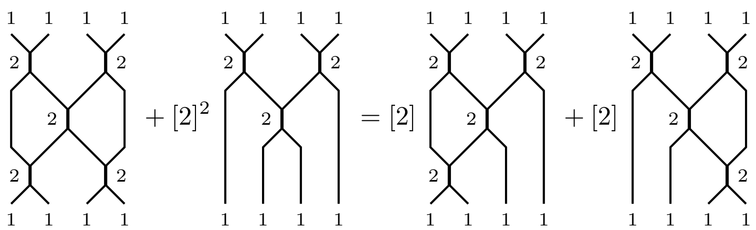
More generally let $\times_{\mathbb{n},i} = \curlyvee^{\mathbb{n},i} \circ \curlywedge_{\mathbb{n},i} - q\,\mathrm{id}_{\mathbb{n}}$ (so that (3.3.30) is just $\times_{\mathbb{n},i}$ for $n = 2$, $i = 1$). Since the $\times_{\mathbb{n},i}$ satisfy the relations of the Hecke algebra generators, we can set $\times_{\mathbb{n},w} = \times_{\mathbb{n},i_1} \circ \cdots \circ \times_{\mathbb{n},i_r}$ for all $w \in \mathbb{S}_n$, where $s_{i_1} \cdots s_{i_r}$ is a reduced expression for w. We have then:

Let $\overline{\mathsf{Web}}$ be the quotient of the category Web modulo the relations

(3.3.31a)



(3.3.31b)



and

(3.3.31c) $$\Xi_n = \sum_{w \in \mathbb{S}_n} q^{\ell(w_n) - \ell(w)} \times_{\mathbb{n},w}.$$

Theorem 3.3.12. *The functor $\mathscr{T}$ induces an equivalence of monoidal categories*

$$\mathscr{T} : \overline{\mathsf{Web}} \xrightarrow{\cong} \mathsf{Rep}. \tag{3.3.32}$$

Proof. First, the functor $\mathscr{T}$ respects relations (3.3.31a) and (3.3.31b) thanks to (3.2.9d) and (3.2.9e). Moreover, it respects relation (3.3.31c) by Lemma 3.3.11. Hence it descends to a functor on $\overline{\mathsf{Web}}$. Since this functor is obviously essentially surjective (by the definition of Rep), and is full by Proposition 3.3.3, it remains to show that it is faithful.

Consider a web diagram φ with source and target $\mathbb{n}$. Using multivalent vertices and using relation (3.3.8) to expand edges labeled by integers $i > 1$, we can suppose (up to a multiple) that φ is obtained by concatenating horizontally and vertically only identities and diagrams Ξ_r for $r = 2, \ldots, n$. Moreover, using the relation (3.3.31c) together with the definition of $\times_{\mathbb{n},i}$ to expand Ξ_r, we can write φ as linear combination of web diagrams whose edges are labeled only by 1 or 2. These correspond to generators of the Super Temperley-Lieb Algebra STL_n. Since all relations defining STL_n also hold for web diagrams, and since STL_n is isomorphic to $\mathrm{End}_{\mathsf{Rep}}(V^{\otimes n})$, it follows that the map $\mathrm{End}_{\mathsf{Web}}(\mathbb{n}) \to \mathrm{End}_{\mathsf{Rep}}(V^{\otimes n})$ induced by $\mathscr{T}$ is an isomorphism (and in particular injective).

Consider now a general web diagram φ with source $\boldsymbol{a}$ and target $\boldsymbol{a}'$, where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are compositions of n. As before we can suppose (up to a multiple) that φ is obtained by concatenating horizontally and vertically only identities and multivalent vertices of the type $\curlywedge^r$ and Ψ_r for $r > 1$. In particular, φ is the composition of the standard inclusion (3.3.9) $\boldsymbol{a} \to \mathbb{n}$, a web diagram with source and target $\mathbb{n}$, and the standard projection (3.3.10) $\mathbb{n} \to \boldsymbol{a}'$. It follows that the map $\mathrm{Hom}_{\mathsf{Web}}(\boldsymbol{a}, \boldsymbol{a}') \to \mathrm{Hom}_{\mathsf{Rep}}(\mathsf{V}(\boldsymbol{a}), \mathsf{V}(\boldsymbol{a}'))$ induced by $\mathscr{T}$ is an isomorphism for all compositions $\boldsymbol{a}, \boldsymbol{a}'$. $\square$

Part II

The categorification

CHAPTER 4

Graded category $\mathcal{O}$

The leading actor of our categorification construction is the BGG category $\mathcal{O}$, first introduced in [BGG76], and its graded version [BGS96]. After some generalities about grading and graded lifts in §4.1, we recall the definition and the main properties of category $\mathcal{O}$ in §4.2. We define then the graded version (§4.4) using Soergel's theorems, which we recall in §4.3. In §4.4 we prove then some relations between translation functors in the graded setting. We will follow quite closely [Str03a]; notice however that in [Str03a] only graded translation functors involving *regular* and *semi-regular* weights are studied, while we consider here the general case of arbitrary integral weights. Although the results of §4.4 are well-known to experts, we include them here since we do not know a good reference for them.

4.1 Gradings

If R is a ring we will denote from now on by mod$-R$ the category of finitely generated (right) R–modules. If moreover R is graded, then we will denote by gmod$-R$ the category of finitely generated graded R–modules. We stress that by *graded* we will always mean $\mathbb{Z}$–graded. We denote by $\mathfrak{f}\colon \mathrm{gmod}-R \to \mathrm{mod}-R$ the grading forgetting functor.

If $M \in \mathrm{gmod}-R$ then $M = \bigoplus_{i\in\mathbb{Z}} M_i$. For $m \in Z$ let $M\langle m\rangle$ be the graded module defined by $M\langle m\rangle_i = M_{i-m}$ with the same module structure as M, i.e. $\mathfrak{f}(M) = \mathfrak{f}(M\langle m\rangle)$. We will also use the notation $qM = M\langle 1\rangle$. Given two graded R–modules M and N we denote by $\mathrm{Hom}_R(M,N)$ the set of non-graded homomorphisms. This contains the set $\mathrm{Hom}_R(M,N)_i$ of all morphisms which are homogeneous of degree i. Notice that

$$\mathrm{Hom}_R(M\langle i\rangle, N)_0 = \mathrm{Hom}_R(M,N)_i = \mathrm{Hom}_R(M, N\langle -i\rangle)_0. \tag{4.1.1}$$

A module $M \in \mathrm{mod}-R$ will be called *gradable* if it has a *graded lift* $\tilde{M} \in \mathrm{gmod}-R$ such that $\mathfrak{f}(\tilde{M}) = M$. If S is another graded module, then a functor $F\colon \mathrm{mod}-R \to \mathrm{mod}-S$ will be called *gradable* if it has a *graded lift*, that is a functor $\tilde{F}\colon \mathrm{gmod}-R \to \mathrm{gmod}-S$ such that $\mathfrak{f}\tilde{F} = F\mathfrak{f}$. In general, not every module is gradable, see [Str03a] for an example.

Given an abelian category $\mathcal{A}$ which is equivalent to mod$-R$ for some graded ring R, we will say that ${}^{\mathbb{Z}}\mathcal{A} = \mathrm{gmod}-R$ is a *graded version* of $\mathcal{A}$.

We recall the following lemma:

Lemma 4.1.1 (See [Str03a, Lemma 3.4] and [Bas68, 2.2])**.** *Let R and S be any rings. There is an equivalence of categories*

$$\begin{aligned} \left\{ \begin{array}{c} \textit{right-exact, compatible with} \\ \textit{direct sums functors} \\ \text{(g)mod}-R \longrightarrow \text{(g)mod}-S \end{array} \right\} &\longleftrightarrow R\text{–(g)mod}-S \\ F &\longmapsto F(R) \\ \bullet \otimes_R X &\longleftarrow\!\shortmid X \end{aligned} \tag{4.1.2}$$

The Grothendieck group

We recall the definition of the Grothendieck group of an abelian category:

Definition 4.1.2. *Let $\mathcal{A}$ be an abelian category. Its* Grothendieck group $K(\mathcal{A})$ *is the quotient of the free $\mathbb{Z}$–module on generators $[M]$ for $M \in \mathcal{A}$ modulo the relation $[B] = [A] + [C]$ for each short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ in $\mathcal{A}$.*

If the category $\mathcal{A}$ is graded then $K(\mathcal{A})$ becomes a $\mathbb{Z}[q, q^{-1}]$–module under $q[M] = [qM] = [M\langle 1\rangle]$. For an abelian graded category $\mathcal{A}$ we let moreover

$$K^{\mathbb{C}(q)}(\mathcal{A}) = \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K(\mathcal{A}). \tag{4.1.3}$$

4.2 The BGG category $\mathcal{O}$

Let us fix a positive integer n. Let $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ be the general Lie algebra of $n \times n$ matrices with the standard Cartan decomposition $\mathfrak{gl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ into strictly lower diagonal, diagonal and strictly upper diagonal matrices respectively. Let also $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be its standard Borel subalgebra. We will indicate by $\Lambda \subset \mathfrak{h}^*$ the set of integral weights of $\mathfrak{gl}_n$ and by $\Lambda^+ \subset \Lambda$ the set of integral dominant weights (with our choice of Borel subalgebra $\mathfrak{b}$). We let $\rho \in \Lambda$ be half the sum of the positive roots.

Definition 4.2.1 ([BGG76])**.** *The BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n) = \mathcal{O}(\mathfrak{gl}_n, \mathfrak{b})$ is the full subcategory of $U(\mathfrak{gl}_n)$–modules which are*

- *($\mathcal{O}$1) finitely generated as $U(\mathfrak{gl}_n)$–modules,*
- *($\mathcal{O}$2) weight modules for the action of $\mathfrak{h}$ with* integral *weights, and*
- *($\mathcal{O}$3) locally $\mathfrak{n}^+$–finite.*

We stress that we consider here only modules with integral weights. The category $\mathcal{O}$ is abelian, Artinian and Noetherian and has enough projective objects. Moreover, it is obviously closed under tensoring with finite dimensional $\mathfrak{gl}_n$–modules. We recall some standard facts on the category $\mathcal{O}$; for more details we refer to [Hum08].

Highest weight modules. For each integral weight $\lambda \in \Lambda$ let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{h}$–module with generator $\mathbf{1}_\lambda$ and action given by $h \cdot \mathbf{1}_\lambda = \langle \lambda, h \rangle \mathbf{1}_\lambda$. By extending the action by zero to $\mathfrak{n}^+$, we can consider $\mathbb{C}_\lambda$ as a $\mathfrak{b}$–module. One defines then the *Verma module $M(\lambda)$* with highest weight λ to be

$$M(\lambda) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda. \tag{4.2.1}$$

The module $M(\lambda)$ has a unique simple quotient, which we denote by $L(\lambda)$. The objects $\{L(\lambda) \mid \lambda \in \Lambda\}$ form a complete set of pairwise non-isomorphic simple objects in $\mathcal{O}$. For each $\lambda \in \Lambda$ we denote by $P(\lambda)$ the projective cover of $L(\lambda)$.

Block decomposition. Consider the dot action of the Weyl group $\mathbb{S}_n$ on $\mathbb{A} \subset \mathfrak{h}^*$, defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in \mathbb{S}_n$, $\lambda \in \mathbb{A}$. Two simple objects $L(\lambda)$, $L(\mu)$ are in the same block of $\mathcal{O}$ if and only if λ and μ are in the same $\mathbb{S}_n$–orbit under the dot action. For an integral dominant weight $\lambda \in \mathbb{A}^+$ we let $\mathcal{O}_\lambda$ be the block of $\mathcal{O}$ containing $L(\lambda)$. We have then a block decomposition $\mathcal{O} = \bigoplus_{\lambda \in \mathbb{A}^+} \mathcal{O}_\lambda$. For a weight $\lambda \in \mathbb{A}$ let $\mathbb{S}_\lambda \subseteq \mathbb{S}_n$ be its stabilizer under the dot action. The weight λ is called *regular* if $\mathbb{S}_\lambda$ is the trivial group, otherwise it is called *singular*. The block $\mathcal{O}_\lambda$ is called regular (or singular) if λ is regular (respectively, singular).

Highest weight structure. Each block $\mathcal{O}_\lambda$ is a highest weight category ([CPS88]), where the poset of weights is the set of shortest coset representatives $\left({}^{\mathbb{S}_n}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ equipped with the Bruhat order; the *standard modules* are the Verma modules $M(\mu)$ for $\mu \in \mathbb{S}_n \cdot \lambda$.

Translation functors. Consider two weights $\lambda, \mu \in \mathbb{A}^+$. The *translation functor* $\mathbb{T}_\lambda^\mu \colon \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is defined by

$$\mathbb{T}_\lambda^\mu(M) = \mathrm{pr}_\mu(M \otimes E(\mu - \lambda)) \tag{4.2.2}$$

where pr_μ is the projection onto $\mathcal{O}_\mu$ and $E(\mu - \lambda)$ is the finite-dimensional $\mathfrak{gl}_n$–module with extremal weight $\mu - \lambda$. Translation functors are clearly exact. Moreover, the couple $(\mathbb{T}_\lambda^\mu, \mathbb{T}_\mu^\lambda)$ forms a pair of adjoint functors. If λ and μ are weights with stabilizers $\mathbb{S}_\lambda, \mathbb{S}_\mu$ with $\mathbb{S}_\lambda \subset \mathbb{S}_\mu$, we will use the expressions *translation onto the wall* and *translation out of the wall* to indicate the translation functors $\mathbb{T}_\lambda^\mu$ and $\mathbb{T}_\mu^\lambda$ respectively (note that in the literature these expressions are often used only when λ is regular and $\mathbb{S}_\mu$ has order 2).

4.3 Soergel's theorems

Soergel's theorems, which we now briefly recall, give a description of blocks of $\mathcal{O}$ via commutative algebra. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. The symmetric group $\mathbb{S}_n$ acts on R by permuting variables. We denote by $R_+^{\mathbb{S}_n} \subset R$ the ideal generated by symmetric polynomials without constant term. Then we have:

Theorem 4.3.1 ([Soe90, Endomorphismensatz 3]). *Let $\lambda \in \mathbb{A}^+$ and let w_0^λ be the longest element of $\left({}^{\mathbb{S}_n}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$. Then there is an isomorphism of algebras*

$$\mathrm{End}_{\mathcal{O}}(P(w_0^\lambda \cdot \lambda)) \cong \left(R/(R_+^{\mathbb{S}_n})\right)^{\mathbb{S}_\lambda}. \tag{4.3.1}$$

The algebra $R/(R_+^{\mathbb{S}_n})$ is called the *algebra of the coinvariants*; we will denote it by B. We denote its invariants under $\mathbb{S}_\lambda$ by $B^\lambda = B^{\mathbb{S}_\lambda}$.

Thanks to (4.3.1), one can define the functor $\mathbb{V} = \mathbb{V}_\lambda = \mathrm{Hom}_{\mathcal{O}}(P(w_0^\lambda \cdot \lambda), \bullet) \colon \mathcal{O}_\lambda \to B^\lambda$–mod. We have then:

Theorem 4.3.2 ([Soe90, Struktursatz 2]). *Let $\lambda \in \mathbb{A}^+$ be an integral dominant weight. The functor $\mathbb{V}_\lambda$ is fully faithful on projective objects. That is, $\mathbb{V}_\lambda$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_\lambda}(P, Q) \to \mathrm{Hom}_{B^\lambda}(\mathbb{V}_\lambda P, \mathbb{V}_\lambda Q) \tag{4.3.2}$$

for all projective modules $P, Q \in \mathcal{O}_\lambda$.

Translation functors and the functor $\mathbb{V}$ are related by the following:

Theorem 4.3.3 ([Soe90, Theorem 10]). *Let $\lambda, \mu \in \mathbb{A}^+$ with $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$. Then we have*

$$\mathbb{V}_\mu \mathbb{T}_\lambda^\mu \cong \mathrm{res}_\lambda^\mu \mathbb{V}_\lambda \qquad \textit{and} \qquad \mathbb{V}_\lambda \mathbb{T}_\mu^\lambda \cong B^\lambda \otimes_{B^\mu} \mathbb{V}_\mu, \tag{4.3.3}$$

where $\mathrm{res}_\lambda^\mu \colon B^\lambda$–mod $\to B^\mu$–mod and $B^\lambda \otimes_{B^\mu} \bullet \colon B^\mu$–mod $\to B^\lambda$–mod are the restriction and extension of scalars respectively.

4.4 Graded version

Let $\lambda \in \mathbb{A}^+$, and let $\mathscr{P}(\lambda) = \bigoplus_{x \in (\mathbb{S}_n/\mathbb{S}_\lambda)^{\text{short}}} P(x \cdot \lambda)$ be a (minimal) projective generator of $\mathcal{O}_\lambda$. Set $A_\lambda = \operatorname{End}_{\mathcal{O}}(\mathscr{P}(\lambda))$. Then we have an equivalence of categories

$$\begin{aligned} \mathcal{O}_\lambda &\longrightarrow \text{mod}-A_\lambda \\ M &\longmapsto \operatorname{Hom}_{\mathcal{O}}(\mathscr{P}(\lambda), M), \end{aligned} \tag{4.4.1}$$

see [Bas68, Theorem II.1.3]. Notice that by Theorem 4.3.2 we have

$$A_\lambda = \operatorname{End}_{\mathcal{O}}(\mathscr{P}(\lambda)) \cong \bigoplus_{x,y \in (\mathbb{S}_n/\mathbb{S}_\lambda)^{\text{short}}} \operatorname{Hom}_R(\mathbb{V}P(x\cdot\lambda), \mathbb{V}P(y\cdot\lambda)) \tag{4.4.2}$$

We consider now R as a graded ring with $\deg x_i = 2$ for all $i = 1, \dots, n$. Since the ideal $R_+^{\mathbb{S}_n}$ is homogeneous, B inherits a grading. Since the invariants are generated by homogeneous symmetric polynomials, all B^λ for $\lambda \in \mathbb{A}^+$ are graded rings. Moreover, it is not difficult to see that for $\lambda \in \mathbb{A}^+$ all $\mathbb{V}P(x\cdot\lambda)$, $x \in (\mathbb{S}_n/\mathbb{S}_\lambda)^{\text{short}}$ can be considered as graded B^λ–modules (see [Str03a, Theorem 2.1] and [BGS96]). We adopt the following usual convention: when we regard $\mathbb{V}P(x\cdot\lambda)$ as graded module, its highest degree is $\ell(x)$. We can then consider the algebra A_λ as a graded algebra, and define the *graded category* ${}^{\mathbb{Z}}\mathcal{O}_\lambda$ by

$${}^{\mathbb{Z}}\mathcal{O}_\lambda = \text{gmod}-A_\lambda \tag{4.4.3}$$

This grading is natural in the sense that it is the unique Koszul grading on $\mathcal{O}_\lambda$. Details can be found in [Soe00] and [BGS96].

In [Str03a] it is proved that projective, simple and Verma modules are gradable, and a graded shift is unique up to isomorphism and overall shift in the grading. We take their standard graded lifts to be determined by requiring that the simple head is concentrated in degree 0, and by a slight abuse of notation we will denote them again by $L(\lambda)$, $M(\lambda)$ and $P(\lambda)$. In particular notice that we have a decomposition of graded modules $A_\lambda \cong \bigoplus_{x \in (\mathbb{S}_n/\mathbb{S}_\lambda)^{\text{short}}} P(x\cdot\lambda)$, and the idempotent projecting onto $P(x\cdot\lambda)$ is homogeneous of degree 0.

Graded translation functors

Let $\lambda, \mu \in \mathbb{A}^+$. Then the translation functors $\mathbb{T}_\mu^\lambda$ corresponds under the equivalence of categories (4.4.1) to

$$\bullet \otimes_{A_\mu} \mathbb{T}_\mu^\lambda \mathscr{P}(\mu) \cong \bullet \otimes_{A_\mu} \operatorname{Hom}_{\mathcal{O}}(\mathscr{P}(\lambda), \mathbb{T}_\mu^\lambda \mathscr{P}(\mu)). \tag{4.4.4}$$

Let now $\lambda, \mu \in \mathbb{A}^+$ with $\mathbb{S}_\mu \subseteq \mathbb{S}_\lambda$. By Theorem 4.3.3 it follows that under the equivalence of categories (4.4.1)

$$\mathbb{T}_\lambda^\mu \quad \text{corresponds to} \quad \bullet \otimes_{A_\lambda} \operatorname{Hom}_{B^\mu}\big(\mathbb{V}\mathscr{P}(\mu), \operatorname{res}_\lambda^\mu \mathbb{V}\mathscr{P}(\lambda)\big), \tag{4.4.5}$$

$$\mathbb{T}_\mu^\lambda \quad \text{corresponds to} \quad \bullet \otimes_{A_\mu} \operatorname{Hom}_{B^\lambda}\big(\mathbb{V}\mathscr{P}(\lambda), B^\lambda \otimes_{B^\mu} \mathbb{V}\mathscr{P}(\mu)\big). \tag{4.4.6}$$

Hence these functors are gradable. We define their graded lifts to be

$$\mathbb{T}_\lambda^\mu = \bullet \otimes_{A_\lambda} \operatorname{Hom}_{B^\mu}\big(\mathbb{V}\mathscr{P}(\mu), \operatorname{res}_\lambda^\mu \mathbb{V}\mathscr{P}(\lambda)\big), \tag{4.4.7}$$

$$\mathbb{T}_\mu^\lambda = \bullet \otimes_{A_\mu} \operatorname{Hom}_{B^\lambda}\big(\mathbb{V}\mathscr{P}(\lambda), B^\lambda \otimes_{B^\mu} \mathbb{V}\mathscr{P}(\mu)\langle -\ell(x_0)\rangle\big), \tag{4.4.8}$$

where $x_0 = x_0^{\mu,\lambda}$ is the longest element in $(\mathbb{S}_\mu/\mathbb{S}_\lambda)^{\text{short}}$. The degree shift is such that $\mathbb{T}_\lambda^\mu M(\lambda) = P(\mu)$ and $\mathbb{T}_\mu^\lambda M(\mu) = P(x_0\cdot\lambda)$.

Lemma 4.4.1. *We have graded adjunctions*

$$\mathbb{T}_\mu^\lambda \dashv q^{\ell(x_0)}\mathbb{T}_\lambda^\mu \qquad \text{and} \qquad \mathbb{T}_\lambda^\mu \dashv q^{-\ell(x_0)}\mathbb{T}_\mu^\lambda. \tag{4.4.9}$$

Proof. By the classification theorem of projective functors in the category $\mathcal{O}$ (see [Hum08, Theorem 10.8]) and its graded version (see [Str05]), it suffices to check the degree shifts, hence to verify (4.4.9) on the dominant Verma modules. We have

$$\begin{aligned} q^{\ell(x_0)}\mathbb{C} &= \operatorname{Hom}(P(x_0\cdot\lambda), M(\lambda)) = \operatorname{Hom}(\mathbb{T}_\mu^\lambda P(\mu), M(\lambda)) \\ &= \operatorname{Hom}(P(\mu), q^{\ell(x_0)}\mathbb{T}_\lambda^\mu M(\lambda)) = \operatorname{Hom}(P(\mu), q^{\ell(x_0)}P(\mu)) = q^{\ell(x_0)}\mathbb{C} \end{aligned} \tag{4.4.10}$$

and

$$\begin{aligned} \mathbb{C} &= \operatorname{Hom}(P(\mu), M(\mu)) = \operatorname{Hom}(\mathbb{T}_\lambda^\mu M(\lambda), M(\mu)) \\ &= \operatorname{Hom}(M(\lambda), q^{-\ell(x_0)}\mathbb{T}_\mu^\lambda M(\mu)) = \operatorname{Hom}(M(\lambda), q^{-\ell(x_0)}P(x_0\cdot\lambda)) = \mathbb{C}. \end{aligned} \tag{4.4.11}$$

For the first calculation, we used the well-known fact that the composition factor $L(x_0\cdot\lambda)$ appears in $M(\lambda)$ only once in degree $\ell(x_0)$; for the second one, we used the also well-known fact that the shifted Verma module $q^{\ell(x_0)}M(\lambda)$ appears at the bottom of the projective module $P(x_0\cdot\lambda)$. □

4.5 Translation functors

We prove now some relations between translation functors in the graded setting, which correspond to the relations between web diagrams that we introduced in §3.3. We will use the following lemma:

Lemma 4.5.1. *Let $\lambda, \mu \in \mathbb{A}^+$ and let $F_1, F_2 : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ be two right-exact additive functors which send projective objects to projective objects. Suppose $\mathbb{V}_\mu F_1 = F_1'\mathbb{V}_\lambda$ and $\mathbb{V}_\mu F_2 = F_2'\mathbb{V}_\lambda$ for some functors $F_1', F_2' \colon B_\lambda{-}\mathrm{mod} \to B_\mu{-}\mathrm{mod}$. If the two functors F_1' and F_2' are isomorphic, then so are F_1 and F_2. The same holds in the graded setting.*

Proof. For $i = 1, 2$, the functor F_i corresponds to $\bullet \otimes_{A_\lambda} \operatorname{Hom}_{\mathcal{O}}(\mathscr{P}(\mu), F_i\mathscr{P}(\lambda))$ under the equivalence of categories (4.4.1). Now we have isomorphisms of (A_λ, A_μ)–bimodules:

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}}(\mathscr{P}(\mu), F_1\mathscr{P}(\lambda)) &\cong \operatorname{Hom}_{B^\mu}(\mathbb{V}\mathscr{P}(\mu), \mathbb{V}F_1\mathscr{P}(\lambda)) \\ &\cong \operatorname{Hom}_{B^\mu}(\mathbb{V}\mathscr{P}(\mu), F_1'\mathbb{V}\mathscr{P}(\lambda)) \\ &\cong \operatorname{Hom}_{B^\mu}(\mathbb{V}\mathscr{P}(\mu), F_2'\mathbb{V}\mathscr{P}(\lambda)) \\ &\cong \operatorname{Hom}_{B^\mu}(\mathbb{V}\mathscr{P}(\mu), \mathbb{V}F_2\mathscr{P}(\lambda)) \\ &\cong \operatorname{Hom}_{\mathcal{O}}(\mathscr{P}(\mu), F_2\mathscr{P}(\lambda)), \end{aligned} \tag{4.5.1}$$

hence the two functors F_1 and F_2 are isomorphic. □

Multivalent vertices. The following corresponds to (3.3.2c) and (3.3.2d):

Proposition 4.5.2. *Let $\lambda, \mu, \gamma \in \mathbb{A}^+$ and suppose $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu \subseteq \mathbb{S}_\gamma$. Then $\mathbb{T}_\mu^\gamma\mathbb{T}_\lambda^\mu \cong \mathbb{T}_\lambda^\gamma$ and $\mathbb{T}_\mu^\lambda\mathbb{T}_\gamma^\mu \cong \mathbb{T}_\gamma^\lambda$.*

Proof. We use Lemma 4.5.1 and the definitions (4.4.7) and (4.4.8) of the graded translation functors. First, we have $\mathrm{res}_\mu^\gamma \mathrm{res}_\lambda^\mu \cong \mathrm{res}_\lambda^\gamma$, hence $\mathbb{T}_\mu^\gamma\mathbb{T}_\lambda^\mu \cong \mathbb{T}_\lambda^\gamma$. Second, we have

$$B^\lambda \otimes_{B^\mu} B^\mu\langle -\ell(x_0)\rangle \otimes_{B^\gamma} \bullet \langle -\ell(y_0)\rangle \cong B^\lambda \otimes_{B^\gamma} \bullet \langle -(\ell(y_0)+\ell(x_0))\rangle, \tag{4.5.2}$$

where x_0, y_0 are the longest elements of $\left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ and $\left({}^{\mathbb{S}_\gamma}/_{\mathbb{S}_\mu}\right)^{\text{short}}$, respectively. Notice that y_0x_0 is the longest element of $\left({}^{\mathbb{S}_\gamma}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ and $\ell(y_0)\ell(x_0) = \ell(y_0) + \ell(x_0)$. Hence $\mathbb{T}_\mu^\lambda \mathbb{T}_\gamma^\mu \cong \mathbb{T}_\gamma^\lambda$. $\square$

Squares. The following is the counterpart of (3.3.2b)

Proposition 4.5.3. *Let $\lambda, \mu \in \mathbb{A}^+$ with $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$ and suppose $\mathbb{S}_\lambda = \mathbb{S}_{a_1} \times \cdots \times \mathbb{S}_{a_\ell}$, while*

$$\mathbb{S}_\mu = \mathbb{S}_{a_1} \times \cdots \times \mathbb{S}_{a_{i-1}} \times \mathbb{S}_{a_i+a_{i+1}} \times \mathbb{S}_{a_{i+2}} \times \cdots \times \mathbb{S}_{a_\ell}. \tag{4.5.3}$$

Then $\mathbb{T}_\lambda^\mu \mathbb{T}_\mu^\lambda \cong \left[\begin{smallmatrix} a_i+a_{i+1} \\ a_i \end{smallmatrix}\right]\mathrm{id}$.

Proof. Let x_0 be the longest element of $\left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$. Notice that $\ell(x_0) = a_i + a_{i+1}$. By Lemma 4.5.1 it suffices to show that

$$\mathrm{res}_\lambda^\mu B^\lambda \otimes_{B^\mu} \bullet \langle -(a_i + a_{i+1})\rangle \cong \begin{bmatrix} a_i + a_{i+1} \\ a_i \end{bmatrix} B^\mu \otimes_{B^\mu} \bullet. \tag{4.5.4}$$

Equivalently, it suffices to show that B^λ is free as left B^μ–module of (graded) rank $q^{a_i+a_{i+1}}\left[\begin{smallmatrix} a_i+a_{i+1} \\ a_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_i+a_{i+1} \\ a_i \end{smallmatrix}\right]_0$. This follows directly from [Wil11, Lemma 4.2, (1)] (see also [Dem73, Théorème 2, (c)]). $\square$

Isotopy invariance. Let now $n', n'' > 0$ with $n = n' + n''$, and consider the Lie algebras $\mathfrak{gl}_{n'}$ and $\mathfrak{gl}_{n''}$ with standard Cartan subalgebras $\mathfrak{h}'$ and $\mathfrak{h}''$. Consider the inclusion $\mathfrak{gl}_{n'} \oplus \mathfrak{gl}_{n''} \subset \mathfrak{gl}_n$, so that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Given a weight λ' for $\mathfrak{gl}_{n'}$ and a weight λ'' for $\mathfrak{gl}_{n''}$, let us denote by $\lambda' \oplus \lambda''$ the weight for $\mathfrak{gl}_n$ whose restriction to $\mathfrak{h}'$ is λ' and to $\mathfrak{h}''$ is λ''. Then we have:

Proposition 4.5.4. *Suppose λ', μ' are integral dominant weights for $\mathfrak{gl}_{n'}$ and λ'', μ'' are integral dominant weights for $\mathfrak{gl}_{n''}$. Suppose that either $\mathbb{S}_{\lambda'} \subset \mathbb{S}_{\mu'}$ or $\mathbb{S}_{\lambda'} \supset \mathbb{S}_{\mu'}$, and either $\mathbb{S}_{\lambda''} \subset \mathbb{S}_{\mu''}$ or $\mathbb{S}_{\lambda''} \supset \mathbb{S}_{\mu''}$. Let*

$$\lambda = \lambda' \oplus \lambda'', \quad \gamma_1 = \lambda' \oplus \mu'', \quad \gamma_2 = \mu' \oplus \lambda'', \quad \mu = \mu' \oplus \mu''. \tag{4.5.5}$$

Then $\mathbb{T}_{\gamma_1}^\mu \mathbb{T}_\lambda^{\gamma_1} \cong \mathbb{T}_{\gamma_2}^\mu \mathbb{T}_\lambda^{\gamma_2}$.

Proof. By Lemma 4.5.1 and Theorem 4.3.3 we need to check some commuting relations between restriction induction functors (4.3.3). If either $\mathbb{S}_{\lambda'} \subset \mathbb{S}_{\mu'}$ and $\mathbb{S}_{\lambda''} \subset \mathbb{S}_{\mu''}$, or $\mathbb{S}_{\lambda'} \supset \mathbb{S}_{\mu'}$ and $\mathbb{S}_{\lambda''} \supset \mathbb{S}_{\mu''}$ then this is obvious, since restriction functors commute with restriction functors, and induction functors commute with induction functors. So suppose $\mathbb{S}_{\lambda'} \subset \mathbb{S}_{\mu'}$ and $\mathbb{S}_{\lambda'''} \supset \mathbb{S}_{\mu''}$ (the remaining case is analogous). Then we need to check that for $M \in B^\lambda$–gmod we have a natural isomorphism of B^μ–modules $\mathrm{res}_{\gamma_1}^\mu B^{\gamma_1} \otimes_{B^\lambda} M \cong B^\mu \otimes_{B^{\gamma_2}} \mathrm{res}_\lambda^{\gamma_2} M$. Recall that B^{γ_1} is free as left B^μ–module (cf. the proof of Proposition 4.5.3); choose a basis $\xi_1 = 1, \xi_2, \ldots, \xi_N \in B^{\gamma_1}$. By our assumptions it follows immediately that this basis can be chosen in $B^{\gamma_1} \cap B^\lambda$ (since $B^{\lambda' \oplus \lambda''} \cong B^{\lambda'} \otimes B^{\lambda''}$). Then the isomorphism is given by

$$\begin{aligned} B^{\gamma_1} \otimes_{B^\lambda} M &\longrightarrow B^\mu \otimes_{B^{\gamma_2}} M \\ (b_1\xi_1 + \cdots + b_N\xi_N) \otimes m &\longmapsto b_1 \otimes \xi_1 m + \cdots + b_M \otimes \xi_M m \end{aligned} \tag{4.5.6}$$

Its inverse is just $b \otimes m \mapsto b \otimes m$. $\square$

Braid relation. Fix now a regular weight $\lambda \in \mathbb{A}^+$. Let $s_i \in \mathbb{S}_n$ be a simple reflection and choose a dominant weight $\mu \in \mathbb{A}^+$ such that the stabilizer $\mathbb{S}_\mu$ is the order two subgroup of $\mathbb{S}_n$ generated by s_i (μ is sometimes called *semi-regular*). Let us denote by $\theta_i = \theta_{s_i} : {}^{\mathbb{Z}}\mathcal{O}_\lambda \to {}^{\mathbb{Z}}\mathcal{O}_\lambda$ the composition $\mathbb{T}_\mu^\lambda \mathbb{T}_\lambda^\mu$. If we set $B_i = B_{s_i} = B \otimes_{B^{s_i}} B$ then

$$\theta_i \cong \bullet \otimes_{A_\lambda} \operatorname{Hom}_{B^\lambda}\big(\mathbb{V}\mathscr{P}(\lambda), B_i \otimes_B \mathbb{V}\mathscr{P}(\lambda)\big). \tag{4.5.7}$$

It follows that the functor θ_i does not depend on the choice of μ (up to natural isomorphism).

The following result is standard (cf. [Soe92]):

Proposition 4.5.5. *The functors θ_i satisfy the relations*

$$\theta_i\theta_j \cong \theta_j\theta_i \qquad \textit{if } |i-j| > 2 \tag{4.5.8}$$

$$\theta_i\theta_{i+1}\theta_i \oplus \theta_{i+1} \cong \theta_{i+1}\theta_i\theta_{i+1} \oplus \theta_i. \tag{4.5.9}$$

Proof. It follows from Lemma 4.5.1 that it suffices to check that

$$B_i \otimes_B B_j \cong B_j \otimes_B B_i \qquad \text{if } |i-j| > 2 \tag{4.5.10}$$

$$B_i \otimes_B B_{i+1} \otimes_B B_i \oplus B_{i+1}\langle 2\rangle \cong B_{i+1} \otimes_B B_i \otimes_B B_{i+1} \oplus B_i\langle 2\rangle. \tag{4.5.11}$$

This is well-known (see for example [EK09, §2.3] – notice that we need to quotient out the invariants in order to obtain (4.5.10) and (4.5.11) from [EK09]). $\square$

CHAPTER 5

Subquotient categories of $\mathcal{O}$

We define now the subquotient categories of $\mathcal{O}$ which we will use for the categorification. This chapter is purely Lie theoretical and is the technical heart of this part. We will start with a quick reminder about Serre quotient categories (§5.1). We will then give two equivalent definitions of the subquotient categories $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ (§5.2 and §5.3) and describe their properly stratified structure. Finally, in §5.4 and §5.5 we introduce and study the functors between these categories that we will use in the next chapter to categorify the action of U_q and of the intertwining operators.

5.1 Serre subcategories and Serre quotient categories

Let $\mathcal{A}$ be some abelian category which is equivalent to the category of finite-dimensional modules over some finite-dimensional $\mathbb{C}$–algebra. Let $\{L(\lambda) \mid \lambda \in \Lambda\}$ be the simple objects of $\mathcal{A}$ up to isomorphism. For all $\lambda \in \Lambda$ let $P(\lambda)$ be the projective cover of $L(\lambda)$. Let $P = \bigoplus_{\lambda\in\Lambda} P(\lambda)$ be a minimal projective generator and let $R = \operatorname{End}_{\mathcal{A}}(P)$. Then we have an equivalence of categories

$$\mathcal{A} \cong \text{mod}-R \tag{5.1.1}$$

via the functor $\operatorname{Hom}_{\mathcal{A}}(P, \bullet)$. We recall some standard facts about Serre subcategories and Serre quotient categories of $\mathcal{A}$.

Serre subcategories

A non-empty full subcategory $\mathcal{S} \subset \mathcal{A}$ is called a *Serre subcategory* if it is closed under subobjects, quotients and extensions. For a subset $\Gamma \subseteq \Lambda$ define $\mathcal{S}_\Gamma$ to be the full subcategory of $\mathcal{A}$ consisting of the modules with all composition factors of type $L(\gamma)$ for $\gamma \in \Gamma$. Then $\mathcal{S}_\Gamma$ is obviously a Serre subcategory of $\mathcal{A}$. Let I_Γ be the two-sided ideal of $R = \operatorname{End}_{\mathcal{A}}(P)$ generated by all endomorphisms which factor through some $P(\eta)$ for $\eta \notin \Gamma$. Notice that if we let e_λ for $\lambda \in \Lambda$ be the idempotent projecting onto $\operatorname{End}_{\mathcal{A}}(P(\lambda)) \subset R$ and $e_\Gamma^\perp = \sum_{\eta\notin\Gamma} e_\eta$ then $I_\Gamma = Re_\Gamma^\perp R$. Then

$$\mathcal{S}_\Gamma \cong \text{mod}-R/I_\Gamma. \tag{5.1.2}$$

A complete set of pairwise non-isomorphic simple objects in $\mathcal{S}_\Gamma$ is given by the $L(\gamma)$'s for $\gamma \in \Gamma$ and each of them has a projective cover $P^{\mathcal{S}_\Gamma}(\gamma)$ in $\mathcal{S}_\Gamma$, which is the biggest quotient of $P(\gamma)$ which lies in $\mathcal{S}_\Gamma$.

Serre quotient categories

Given a Serre subcategory $\mathcal{S} \subset \mathcal{A}$ as above one defines the *quotient category* $\mathcal{A}/\mathcal{S}$ to be the category with the same objects of $\mathcal{A}$ and with morphisms

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{S}}(M,N) = \varinjlim \operatorname{Hom}_{\mathcal{A}}(M', N/N') \tag{5.1.3}$$

where the direct limit is taken over all pairs $M' \subseteq M$, $N' \subseteq N$ such that $M/M' \in \mathcal{S}$ and $N' \in \mathcal{S}$. The quotient category turns out to be an abelian category, and comes with an exact quotient functor $Q \colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ (see [Gab62]).

Also in this case, we have an equivalence of categories

$$\mathcal{A}/\mathcal{S}_\Gamma \cong \operatorname{mod-}\operatorname{End}_{\mathcal{A}}(P_\Gamma^\perp), \tag{5.1.4}$$

where $P_\Gamma^\perp = \bigoplus_{\eta \in \Lambda - \Gamma} P(\eta)$ (see for example [AM11, Proposition 33]). The quotient functor is $Q = \operatorname{Hom}_{\mathcal{A}}(P_\Gamma^\perp, \bullet)$. In particular, we can deduce from (5.1.4) the abelian structure of $\mathcal{A}/\mathcal{S}_\Gamma$. Notice that $\operatorname{End}_{\mathcal{A}}(P_\Gamma^\perp) = e_\Gamma^\perp R e_\Gamma^\perp$ where $e_\Gamma^\perp = \sum_{\gamma \in \Lambda - \Gamma} e_\gamma$.

A complete set of pairwise non-isomorphic simple objects in $\mathcal{A}/\mathcal{S}_\Gamma$ is given by the $L(\eta)$'s for $\eta \in \Lambda - \Gamma$, with projective covers $P(\eta)$.

Presentable modules

Let $\mathcal{C}$ be an additive subcategory of the abelian category $\mathcal{A}$. We define the category of $\mathcal{C}$–*presentable objects* to be the full subcategory of $\mathcal{A}$ consisting of all objects $M \in \mathcal{A}$ having a presentation

$$Q_1 \longrightarrow Q_2 \twoheadrightarrow M \tag{5.1.5}$$

with $Q_1, Q_2 \in \mathcal{C}$. Given a projective object $P \in \mathcal{A}$ we let $\operatorname{Add}(P)$ be the additive full subcategory of $\mathcal{A}$ consisting of all objects which admit a direct sum decomposition with summands being direct summands of P, and we consider the category $\overline{\operatorname{Add}(P)}$ of P*–presentable* or $\operatorname{Add}(P)$*–presentable objects*. By [Aus74, Proposition 5.3], the category $\overline{\operatorname{Add}(P)}$ is equivalent to $\operatorname{mod-}\operatorname{End}_{\mathcal{A}}(P)$. In particular, if $P = P_\Gamma^\perp$ as in (5.1.4), then we have

$$\overline{\operatorname{Add}(P_\Gamma^\perp)} \cong \operatorname{mod-}\operatorname{End}_{\mathcal{A}}(P_\Gamma^\perp) \cong \mathcal{A}/\mathcal{S}_\Gamma. \tag{5.1.6}$$

Notice that this gives an equivalence between the quotient category $\mathcal{A}/\mathcal{S}_\Gamma$ and a full subcategory of $\mathcal{A}$.

Remark 5.1.1. If $M, N \in \mathcal{A}/\mathcal{S}_\Gamma$ then by definition M and N are also objects of $\mathcal{A}$ and we can consider both the homomorphism spaces $\operatorname{Hom}_{\mathcal{A}}(M,N)$ and $\operatorname{Hom}_{\mathcal{A}/\mathcal{S}_\Gamma}(M,N)$: they are in general different. But notice that if M and N, as objects of $\mathcal{A}$, are $P_\Gamma^\perp$–presentable, then the two homomorphism spaces coincide by (5.1.6). In the following, we will most of the time only deal with objects of Serre quotient categories which are also presentable.

5.2 Subquotient categories of $\mathcal{O}$

Let us fix a positive integer n. We consider the Lie algebra $\mathfrak{gl}_n$ and the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$. We will use the notation introduced in §4.2.

Parabolic category $\mathcal{O}$

Given a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}_n$ with Levi factor $\mathfrak{l}$, let $\mathcal{O}^{\mathfrak{p}}$ be the full subcategory of $\mathcal{O}$ consisting of modules that, viewed as $U(\mathfrak{l})$–modules, are direct sums of finite-dimensional simple $\mathfrak{l}$–modules. Let $W_{\mathfrak{p}} \subset \mathbb{S}_n$ be the standard parabolic subgroup corresponding to $\mathfrak{p}$, and let $W^{\mathfrak{p}}$ be the set of shortest coset representatives for ${}_{W_{\mathfrak{p}}}\backslash^{\mathbb{S}_n}$. Then $\mathcal{O}^{\mathfrak{p}}$ is also the full Serre subcategory of $\mathcal{O}$ generated by the simple objects $L(x \cdot \lambda)$ for λ dominant and $x \in W^{\mathfrak{p}}$ such that $x\mathbb{S}_\lambda \subseteq W^{\mathfrak{p}}$. We denote by $P^{\mathfrak{p}}(x \cdot \lambda)$ the projective cover of $L(x \cdot \lambda)$ in $\mathcal{O}^{\mathfrak{p}}$ and by $M^{\mathfrak{p}}(x \cdot \lambda)$ the corresponding parabolic Verma module. The block decomposition of $\mathcal{O}$ induces a block decomposition $\mathcal{O}^{\mathfrak{p}} = \bigoplus_\lambda \mathcal{O}^{\mathfrak{p}}_\lambda$.

Let $\lambda \in \mathbb{A}^+$ with stabilizer $\mathbb{S}_\lambda$, and recall that $\mathcal{O}_\lambda \cong \text{mod}-A_\lambda$. Let $e_{\mathfrak{p}}^{\perp} \in A_\lambda = \operatorname{End}(\mathscr{P}(\lambda))$ be the idempotent projecting onto the direct sum of the projective modules $P(x \cdot \lambda)$ for $x \in \mathbb{S}_n$ such that $x\mathbb{S}_\lambda \not\subseteq W^{\mathfrak{p}}$. Then $\operatorname{End}(\mathscr{P}^{\mathfrak{p}}(\lambda)) = A_\lambda / A_\lambda e_{\mathfrak{p}}^{\perp} A_\lambda$ and

$$\mathcal{O}^{\mathfrak{p}}_\lambda \cong \text{mod}-\big(A_\lambda / A_\lambda e_{\mathfrak{p}}^{\perp} A_\lambda\big). \tag{5.2.1}$$

Since the idempotent $e_{\mathfrak{p}}^{\perp}$ is homogeneous, the latter quotient algebra inherits a graded structure. In particular, there is a graded version ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p}}_\lambda = \text{gmod}-\big(A_\lambda / A_\lambda e_{\mathfrak{p}}^{\perp} A_\lambda\big)$.

Generalized parabolic subcategories of $\mathcal{O}$

Let now $\mathfrak{p}, \mathfrak{q}$ be two orthogonal standard parabolic subalgebras of $\mathfrak{gl}_n$ (by orthogonal we mean that the corresponding subsets $\Pi_{\mathfrak{p}}, \Pi_{\mathfrak{q}}$ of the simple roots Π of $\mathfrak{gl}_n$ are orthogonal; this is equivalent to imposing that $\mathfrak{p} + \mathfrak{q}$ is also a parabolic subalgebra of $\mathfrak{gl}_n$ and $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{b}$). Let $W_{\mathfrak{p}}, W_{\mathfrak{q}}$ be the corresponding parabolic subgroups of the Weyl group $\mathbb{S}_n$. Note that, since $\mathfrak{p}$ and $\mathfrak{q}$ are orthogonal, $W_{\mathfrak{p}} \times W_{\mathfrak{q}}$ is also a subgroup of $\mathbb{S}_n$. Consider the general Lie algebras $\mathfrak{gl}_{\mathfrak{p}}, \mathfrak{gl}_{\mathfrak{q}} \subset \mathfrak{gl}_n$ with Weyl groups $W_{\mathfrak{p}}$ and $W_{\mathfrak{q}}$ respectively, so that $\mathfrak{p} = \mathfrak{gl}_{\mathfrak{p}} + \mathfrak{b}$ and $\mathfrak{q} = \mathfrak{gl}_{\mathfrak{q}} + \mathfrak{b}$.

Following [MS08a], we let $\mathcal{P}_{\mathfrak{q}} = \operatorname{Add}(P(w_{\mathfrak{q}} \cdot 0))$ be the additive subcategory of $\mathcal{O}(\mathfrak{gl}_{\mathfrak{q}})$ generated by the anti-dominant indecomposable projective module $P(w_{\mathfrak{q}} \cdot 0)$, where $w_{\mathfrak{q}} \in W_{\mathfrak{q}}$ is the longest element. Let also $\overline{\mathcal{P}_{\mathfrak{q}}}$ be the category of $\mathcal{P}_{\mathfrak{q}}$–presentable modules (cf. §5.1).

Remark 5.2.1. The category $\overline{\mathcal{P}}_{\mathfrak{q}}$ is equivalent to the category of finitely generated modules over the endomorphism algebra of a projective generator of $\mathcal{P}_{\mathfrak{q}}$ (see §5.1), and therefore is an abelian category. In particular, if $W_{\mathfrak{q}} \cong \mathbb{S}_k$ then by Theorem 4.3.1 the category $\overline{\mathcal{P}}_{\mathfrak{q}}$ is equivalent to the category of finitely generated modules over the algebra of the coinvariants $R/(R_+^{\mathbb{S}_k})$, where $R = \mathbb{C}[x_1, \dots, x_k]$.

Let $\mathfrak{a} = \mathfrak{a}_{\mathfrak{p}+\mathfrak{q}} = (\mathfrak{gl}_{\mathfrak{p}} \oplus \mathfrak{gl}_{\mathfrak{q}}) + \mathfrak{h}$ and define $\mathfrak{n}_{\mathfrak{p}+\mathfrak{q}}$ by $\mathfrak{p} + \mathfrak{q} = \mathfrak{a} \oplus \mathfrak{n}_{\mathfrak{p}+\mathfrak{q}}$. Given a $\mathfrak{gl}_{\mathfrak{q}}$–module M, we denote by $\mathfrak{E}^{\mathfrak{a}} M$ the $\mathfrak{a}$–module obtained by extending the action by 0. Let $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ be the additive closure of the full subcategory of $\mathfrak{a}$–modules which have the form $E \otimes \mathfrak{E}^{\mathfrak{a}} P$, where E is a simple finite-dimensional $\mathfrak{a}$–module and $P \in \overline{\mathcal{P}_{\mathfrak{q}}}$ is a projective object. Finally, let $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}} = \overline{\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}}$ be the category of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$–presentable $\mathfrak{a}$–modules. In other words, $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ is the category

$$\langle E \otimes \mathfrak{E}^{\mathfrak{a}} P(w_{\mathfrak{q}} \cdot 0) \mid E \text{ is a simple finite-dimensional } \mathfrak{a}\text{–module} \rangle \tag{5.2.2}$$

and $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}} = \overline{\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}}$.

Definition 5.2.2. *We define $\mathcal{O}\{\mathfrak{p} + \mathfrak{q}, \mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ to be the full subcategory of $\mathfrak{gl}_n$–modules which are:*

(GP1) finitely generated;

(GP2) locally $\mathfrak{n}_{\mathfrak{p}+\mathfrak{q}}$–finite;

(GP3) direct sum of objects of $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$ as $\mathfrak{a}$–modules.

REMARK 5.2.3. The categories $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ fall into a more general family of categories that were first introduced in [FKM02] (called *generalized parabolic subcategories* of $\mathcal{O}$) and then generalized in [Maz04]. Our definition follows [MS08a], and in particular is a special case of [MS08a, Definition 32]. However, in [MS08a] only the trivial block is studied, while we are interested also in singular blocks. Notice that the category $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$ is admissible (in the sense of [MS08a, §6.3]) by [MS08a, Lemma 33].

Lemma 5.2.4. *The category $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ is a subcategory of $\mathcal{O}^{\mathfrak{p}}$.*

Proof. Conditions (GP2) and (GP3) together imply that modules of $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ are locally $\mathfrak{n}^+$–finite; condition (GP3) also implies that modules of $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ are weight modules for $\mathfrak{h}$; hence $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ is a subcategory of $\mathcal{O}$. By condition (GP3), moreover, objects of $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ are direct sums of finite-dimensional simple $\mathfrak{gl}_{\mathfrak{p}}$–modules. Hence $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$ is a subcategory of $\mathcal{O}^{\mathfrak{p}}$. $\square$

REMARK 5.2.5. If $\mathfrak{q}=\mathfrak{b}$ then by definition $\mathcal{O}\{\mathfrak{p},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{b}}\}$ is the parabolic category $\mathcal{O}^{\mathfrak{p}}$.

It follows in particular that the block decomposition $\mathcal{O}^{\mathfrak{p}}=\bigoplus_\lambda \mathcal{O}^{\mathfrak{p}}_\lambda$ induces a direct sum decomposition

$$\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}=\bigoplus_\lambda \mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda. \tag{5.2.3}$$

Lemma 5.2.6. *We have the following inclusions of full subcategories:*

(i) if $\mathfrak{p}'\subset\mathfrak{p}$ then $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}\subset\mathcal{O}\{\mathfrak{p}'+\mathfrak{q},\mathcal{A}^{\mathfrak{p}'}_{\mathfrak{q}}\}$;

(ii) if $\mathfrak{q}'\subset\mathfrak{q}$ then $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}\subset\mathcal{O}\{\mathfrak{p}+\mathfrak{q}',\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}'}\}$.

We warn the reader, however, that the second inclusion will not be an exact inclusion of abelian categories (once we will have defined the abelian structure on the categories $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$, see §5.3).

Proof. Let $M\in\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$. By definition, M is finitely generated and locally $\mathfrak{n}^+$–finite. Write $M=\bigoplus_\alpha M_\alpha$ as an $\mathfrak{a}_{\mathfrak{p}+\mathfrak{q}}$–module, with $M_\alpha\in\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$. Let $P_\alpha\to Q_\alpha\twoheadrightarrow M_\alpha$ be a $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$–presentation of M_α. Considering this as a sequence of $\mathfrak{a}_{\mathfrak{p}'+\mathfrak{q}}$–modules (respectively, $\mathfrak{a}_{\mathfrak{p}+\mathfrak{q}'}$–modules), we see that it is enough to show that

(i) every object of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ decomposes, as an $\mathfrak{a}_{\mathfrak{p}'+\mathfrak{q}}$–module, into a direct sum of objects of $\mathcal{P}^{\mathfrak{p}'}_{\mathfrak{q}}$;

(ii) every object of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ decomposes, as an $\mathfrak{a}_{\mathfrak{p}+\mathfrak{q}'}$–module, into a direct sum of objects of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}'}$.

Since (i) is straightforward (every object of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ is, as an $\mathfrak{a}_{\mathfrak{p}'+\mathfrak{q}}$–module, an object of $\mathcal{P}^{\mathfrak{p}'}_{\mathfrak{q}}$), let us verify (ii). For this it is enough to check that, for every dominant integral weight λ of $\mathfrak{gl}_{\mathfrak{q}}$, the anti-dominant projective module $P(w_{\mathfrak{q}}\cdot\lambda)\in\mathcal{O}(\mathfrak{gl}_{\mathfrak{q}})$ decomposes, as a $\mathfrak{gl}_{\mathfrak{q}'}$–module, as direct sum of objects of type $E\otimes P(w_{\mathfrak{q}'}\cdot\mu)$ for some weight μ of $\mathfrak{gl}_{\mathfrak{q}'}$ and some finite-dimensional $\mathfrak{gl}_{\mathfrak{q}'}$–module E. This follows because $\mathcal{O}(\mathfrak{gl}_{\mathfrak{q}})\ni P(w_{\mathfrak{q}'}\cdot\lambda)=U(\mathfrak{gl}_{\mathfrak{q}})\otimes_{U(\mathfrak{q}'\cap\mathfrak{gl}_{\mathfrak{q}})}P(w_{\mathfrak{q}'}\cdot\lambda|_{\mathfrak{gl}_{\mathfrak{q}'}})$, and $P(w_{\mathfrak{q}}\cdot\lambda)$ can be obtained from $P(w_{\mathfrak{q}'}\cdot\lambda)$ in $\mathcal{O}(\mathfrak{gl}_{\mathfrak{q}})$ by tensoring with finite-dimensional modules. $\square$

5.3 The parabolic category of $\mathfrak{q}$-presentable modules

We will give now another definition of the blocks of $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}$. Let λ be a dominant integral weight for $\mathfrak{gl}_n$ with stabilizer $\mathbb{S}_\lambda$ under the dot action. Define

$$\Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda) = \left\{ x \in \left(\mathbb{S}_n/\mathbb{S}_\lambda\right)^{\text{short}} \;\middle|\; \begin{array}{l} x\mathbb{S}_\lambda \subset W^{\mathfrak{p}} \\ x\mathbb{S}_\lambda \cap w_{\mathfrak{q}} W^{\mathfrak{q}} \neq \varnothing \end{array} \right\}. \tag{5.3.1}$$

Notice that $w_{\mathfrak{q}}W^{\mathfrak{q}}$ is simply the set of longest coset representatives for ${W_{\mathfrak{q}}}\backslash{\mathbb{S}_n}$. If $\mathfrak{p}=\mathfrak{b}$ or $\mathfrak{q}=\mathfrak{b}$ in the following we will omit them from the notation. If λ is regular then in particular $\Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)=\{w_{\mathfrak{q}}x \mid x\in W^{\mathfrak{p}+\mathfrak{q}}\}$ is the set of elements of $\mathbb{S}_n$ that are shortest coset representatives for ${W_{\mathfrak{p}}}\backslash{\mathbb{S}_n}$ and longest coset representatives for ${W_{\mathfrak{q}}}\backslash{\mathbb{S}_n}$. Let

$$\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda) = \bigoplus_{x\in\Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)} P^{\mathfrak{p}}(x\cdot\lambda) \tag{5.3.2}$$

and let $\operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}_\lambda$ consisting of all modules which admit a direct sum decomposition with summands being direct summands of $\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$.

Definition 5.3.1. *We define the category $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ to be the full subcategory of $\mathcal{O}^{\mathfrak{p}}_\lambda$ which consists of all $\operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$–presentable modules.*

As we already announced, these categories coincide with the generalized parabolic categories we defined in the previous section:

Proposition 5.3.2. *For all integral dominant weights λ, the categories $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda$ and $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ coincide as subcategories of $\mathcal{O}$.*

Proof. First we show the inclusion $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda \subseteq \mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda$. Consider the indecomposable projective module $P^{\mathfrak{p}}(w_{\mathfrak{q}}\cdot\lambda)$ in $\mathcal{O}^{\mathfrak{p}}_\lambda$. Let $L(\lambda|_{\mathfrak{gl}_{\mathfrak{p}}})\boxtimes P(w_{\mathfrak{q}}\cdot\lambda|_{\mathfrak{gl}_{\mathfrak{q}}})\in\mathcal{O}(\mathfrak{gl}_{\mathfrak{p}+\mathfrak{q}})$ denote the $(\mathfrak{gl}_{\mathfrak{p}}\oplus\mathfrak{gl}_{\mathfrak{q}})$–module obtained as external tensor product of the finite-dimensional simple $\mathfrak{gl}_{\mathfrak{p}}$–module $L(\lambda|_{\mathfrak{gl}_{\mathfrak{p}}})\in\mathcal{O}(\mathfrak{gl}_{\mathfrak{p}})$ and the anti-dominant indecomposable projective module $P(w_{\mathfrak{q}}\cdot\lambda|_{\mathfrak{gl}_{\mathfrak{q}}})\in\mathcal{O}(\mathfrak{gl}_{\mathfrak{q}})$. Consider it as an $\mathfrak{a}$–module by extending the action to $\mathfrak{h}$ with the weight λ, and then as a $(\mathfrak{p}+\mathfrak{q})$–module by letting $\mathfrak{n}_{\mathfrak{p}+\mathfrak{q}}$ act by zero. By the analogue of the BGG construction of projective modules in $\mathcal{O}$ [BGG76], we have

$$P^{\mathfrak{p}}(w_{\mathfrak{q}}\cdot\lambda) = U(\mathfrak{gl}_n)\otimes_{U(\mathfrak{p}+\mathfrak{q})}\left(L(\lambda|_{\mathfrak{gl}_{\mathfrak{p}}})\boxtimes P(w_{\mathfrak{q}}\cdot\lambda|_{\mathfrak{gl}_{\mathfrak{q}}})\right). \tag{5.3.3}$$

Since $U(\mathfrak{gl}_n)$ decomposes as direct sum of finite-dimensional modules for the adjoint action of $\mathfrak{gl}_{\mathfrak{p}}\oplus\mathfrak{gl}_{\mathfrak{q}}$, it follows that (5.3.3), as an $\mathfrak{a}$–module, decomposes as direct sum of objects of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$. By tensoring (5.3.3) with finite-dimensional $\mathfrak{gl}_n$–modules we can obtain all projective modules $P^{\mathfrak{p}}(x\cdot\lambda)$ for $x\in\Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$; since $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$ is closed under tensor product with finite-dimensional modules, it follows that each $P^{\mathfrak{p}}(x\cdot\lambda)$ for $x\in\Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$ decomposes as direct sum of objects of $\mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$. Now, if $M\in\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ then we have a presentation $Q_1\to Q_2\twoheadrightarrow M$ with $Q_1,Q_2\in\operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$. Considering this as a sequence of $\mathfrak{a}$–modules, it follows that M decomposes as a direct sum of objects of $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}=\overline{\mathcal{P}}^{\mathfrak{p}}_{\mathfrak{q}}$, and hence $M\in\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda$.

Now let us show the other inclusion $\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda\subseteq\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$. Let $M\in\mathcal{O}\{\mathfrak{p}+\mathfrak{q},\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}\}_\lambda$. By Lemma 5.2.4 we have $M\in\mathcal{O}^{\mathfrak{p}}_\lambda$. As an $\mathfrak{a}$–module, M is generated by elements of weight $x\cdot\lambda$ with $sx<x$ for any simple reflection $s\in W_{\mathfrak{q}}$ (i.e. $x\cdot\lambda$ is an anti-dominant weight for $\mathfrak{gl}_{\mathfrak{q}}$). Of course this is also true as a $\mathfrak{gl}_n$–module. Hence the projective cover Q of M in $\mathcal{O}^{\mathfrak{p}}_\lambda$ is an element of $\operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$. Let $K=\ker(Q\twoheadrightarrow M)$ in $\mathcal{O}^{\mathfrak{p}}_\lambda$, and consider the short exact sequence $K\hookrightarrow Q\twoheadrightarrow M$ as a sequence of $\mathfrak{a}$–modules. Since all objects of $\mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$ are finitely

generated, we may assume (taking direct summands) that $K \hookrightarrow Q \twoheadrightarrow M$ is a short exact sequence of finitely generated $\mathfrak{a}$–modules, that is, we can suppose $M \in \mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$ and, by the first paragraph, $Q \in \mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$. We can write $Q = Q_M \oplus Q'$ where Q_M is the projective cover of M, and $K = Q' \oplus \ker(Q_M \twoheadrightarrow M)$. Since $M \in \mathcal{A}^{\mathfrak{p}}_{\mathfrak{q}}$, we have a presentation $P_M \to Q_M \twoheadrightarrow M$ with $P_M, Q_M \in \mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$, hence we have a surjective map $P_M \twoheadrightarrow \ker(Q_M \twoheadrightarrow M)$ and therefore a surjective map $P' \twoheadrightarrow K$ for $P' = Q' \oplus P_M \in \mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}}$. Since as an $\mathfrak{a}$–module P' is generated by elements of weight $x \cdot \lambda$ with $sx < x$ for any simple reflection $s \in W_{\mathfrak{q}}$, the same holds for K. Hence we can apply the same construction we did for M to K and get a presentation $P \to Q \twoheadrightarrow M$ with $P, Q \in \operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$. □

For $\mathfrak{p} = \mathfrak{b}$ and $\lambda = 0$ we get the category $\mathcal{O}^{\mathfrak{q}\text{-pres}}_0$ of [MS05]. The results of [MS05, §2] carry over to the case of an arbitrary integral weight λ. For instance, we have:

Proposition 5.3.3 (see [MS05, §2])**.** *The category $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$ is an abelian category with a simple preserving duality and is equivalent to* $\operatorname{End}(\mathscr{P}_{\mathfrak{q}}(\lambda))$–mod*. For $x \in \Lambda_{\mathfrak{q}}(\lambda)$ the modules $P(x \cdot \lambda)$ are obviously objects of $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$. Each $P(x \cdot \lambda)$ has a unique simple quotient $S(x \cdot \lambda)$ in $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$, and the $S(x \cdot \lambda)$ for $x \in \Lambda_{\mathfrak{q}}(\lambda)$ give a full set of pairwise non isomorphic simple objects of $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$.*

We want to extend these results to the general case $\mathfrak{p} \neq \mathfrak{b}$. First, let us recall the definition of the *Zuckermann's functor* $\mathfrak{z} \colon \mathcal{O} \to \mathcal{O}^{\mathfrak{p}}$. Given $M \in \mathcal{O}$, the object $\mathfrak{z}M$ is the largest quotient of M that lies in $\mathcal{O}^{\mathfrak{p}}$. The functor $\mathfrak{z}$ is right exact and $\mathfrak{z}P(x \cdot \lambda) = P^{\mathfrak{p}}(x \cdot \lambda)$ for each $\lambda \in \Lambda^{\mathfrak{p}}(\lambda)$.

Lemma 5.3.4. *The category $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ coincides with the full subcategory of objects of $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$ that are in $\mathcal{O}^{\mathfrak{p}}_\lambda$.*

Proof. Since both are full subcategories of $\mathcal{O}(\mathfrak{gl}_n)$, we need only to prove that they have the same objects. Let $M \in \mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda \cap \mathcal{O}^{\mathfrak{p}}_\lambda$ and consider a presentation $P \to Q \to M \to 0$ with $P, Q \in \operatorname{Add}(\mathscr{P}_{\mathfrak{q}}(\lambda))$. Applying $\mathfrak{z}$ yields a presentation $\mathfrak{z}P \to \mathfrak{z}Q \to M \to 0$ with $\mathfrak{z}P, \mathfrak{z}Q \in \operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}}(\lambda))$.

The other inclusion follows from Proposition 5.3.2 and Lemma 5.2.6. □

Lemma 5.3.5. *The category $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ is the Serre subcategory of $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$ generated by the simple objects $S(x \cdot \lambda)$ for $x \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$.*

Proof. First let us prove that $S(x \cdot \lambda) \in \mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$ is in $\mathcal{O}^{\mathfrak{p}}_\lambda$ if $x \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$. Let $P \to Q \twoheadrightarrow S(x \cdot \lambda)$ be a presentation of $S(x \cdot \lambda)$ with $P, Q \in \operatorname{Add}(\mathscr{P}_{\mathfrak{q}})$. Applying the Zuckermann's functor $\mathfrak{z}$ yields a presentation of $\mathfrak{z}S(x \cdot \lambda)$ with $\mathfrak{z}P, \mathfrak{z}Q \in \operatorname{Add}(\mathscr{P}^{\mathfrak{p}}_{\mathfrak{q}})$. Since $\mathfrak{z}P(x \cdot \lambda) \neq 0$ (because $L(x \cdot \lambda)$ is a quotient of $P(x \cdot \lambda)$ in $\mathcal{O}$) and $S(x \cdot \lambda)$ is a quotient of $P(x \cdot \lambda)$, it follows that $\mathfrak{z}S(x \cdot \lambda) \neq 0$. On the other side, $\mathfrak{z}S(x \cdot \lambda) \in \mathcal{O}^{\mathfrak{q}\text{-pres}}$ by Lemma 5.3.4. But $\mathfrak{z}S(x \cdot \lambda)$ is a non-zero quotient in $\mathcal{O}^{\mathfrak{q}\text{-pres}}$ of the simple module $S(x \cdot \lambda)$, hence $\mathfrak{z}S(x \cdot \lambda) = S(x \cdot \lambda)$. It follows that $S(x \cdot \lambda) \in \mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$.

On the other side, if $x \in \Lambda_{\mathfrak{q}}(\lambda)$ but $x \notin \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$, then clearly $S(x \cdot \lambda) \notin \mathcal{O}^{\mathfrak{p}}_\lambda$. Since $\mathcal{O}^{\mathfrak{p}}_\lambda$ is closed under extensions, it follows that the objects of $\mathcal{O}^{\mathfrak{q}\text{-pres}}_\lambda$ that are also in $\mathcal{O}^{\mathfrak{p}}$ are exactly the objects whose composition factors are of type $S(x \cdot \lambda)$ for $x \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$. □

It follows that the modules $S(x \cdot \lambda)$ for $x \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$ give a full set of pairwise non-isomorphic simple objects of $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$. Moreover, the projective cover of $S(x \cdot \lambda)$ is $P^{\mathfrak{p}}(x \cdot \lambda)$.

Remark 5.3.6. We notice that in general the simple module $S(x \cdot \lambda) \in \mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$ is not irreducible as a $\mathfrak{gl}_n$–module (unless $\mathfrak{q} = \mathfrak{b}$).

The graded abelian structure

The category $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ is equivalent to the category of finitely generated (right) modules over $\mathrm{End}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}(\lambda))$:

$$\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \cong \text{mod}-\mathrm{End}\left(\mathscr{P}_\mathfrak{q}^\mathfrak{p}(\lambda)\right). \tag{5.3.4}$$

Via this equivalence we can define on $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ a natural abelian structure. However, as we already pointed out, this abelian structure is not induced by the abelian structure of $\mathcal{O}_\lambda$.

The algebra $\mathrm{End}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}(\lambda))$ can be obtained from $A_\lambda = \mathrm{End}(\mathscr{P}(\lambda))$ in two steps. First, let $e_\mathfrak{p}^\perp \in \mathrm{End}(\mathscr{P}(\lambda))$ be the idempotent projecting onto the direct sum of the projective modules $P(x \cdot \lambda)$ for $x \notin \Lambda^\mathfrak{p}(\lambda)$. Then $\mathrm{End}(\mathscr{P}^\mathfrak{p}(\lambda)) = A_\lambda / A_\lambda e^\mathfrak{p} A_\lambda$. Moreover, let $\overline{e}_\mathfrak{q} \in A_\lambda / A_\lambda e_\mathfrak{p}^\perp A_\lambda$ be the idempotent projecting onto the direct sum of the projective modules $P^\mathfrak{p}(x \cdot \lambda)$ for $x \in \Lambda_\mathfrak{q}^\mathfrak{p}(\lambda)$. Then $\mathrm{End}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}(\lambda)) = \overline{e}_\mathfrak{q}(A_\lambda / A_\lambda e_\mathfrak{p}^\perp A_\lambda)\overline{e}_\mathfrak{q}$.

By Lemma 5.3.5, the two steps can be done also in the inverse order: let $e_\mathfrak{q} \in A_\lambda$ be the idempotent projecting onto the direct sum of the projective modules $P(x \cdot \lambda)$ for $x \in \Lambda_\mathfrak{q}(\lambda)$ (notice that $\overline{e}_\mathfrak{q} = e_\mathfrak{q} + A_\lambda e_\mathfrak{p}^\perp A_\lambda$). Then $\mathrm{End}(\mathscr{P}_\mathfrak{q}(\lambda)) = e_\mathfrak{q} A_\lambda e_\mathfrak{q}$. Moreover, let $f_\mathfrak{p}^\perp = e_\mathfrak{q} e_\mathfrak{p}^\perp e_\mathfrak{q} \in e_\mathfrak{q} A_\lambda e_\mathfrak{q}$ be the idempotent projecting onto the direct sum of the projective modules $P(x \cdot \lambda)$ for $x \notin \Lambda_\mathfrak{q}^\mathfrak{p}(\lambda)$. Then $\mathrm{End}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}(\lambda)) = (e_\mathfrak{q} A_\lambda e_\mathfrak{q})/(e_\mathfrak{q} A_\lambda e_\mathfrak{q} f_\mathfrak{p}^\perp e_\mathfrak{q} A_\lambda e_\mathfrak{q})$. It follows that

$$(e_\mathfrak{q} A_\lambda e_\mathfrak{q})/(e_\mathfrak{q} A_\lambda e_\mathfrak{q} f_\mathfrak{p}^\perp e_\mathfrak{q} A_\lambda e_\mathfrak{q}) = \overline{e}_\mathfrak{q}(A_\lambda / A_\lambda e_\mathfrak{p}^\perp A_\lambda)\overline{e}_\mathfrak{q}. \tag{5.3.5}$$

As far as we understand, this is not a trivial result, but instead a consequence of Lemma 5.3.5.

Recall from §4.4 that the algebra A_λ has a natural grading. Since the idempotents $e_\mathfrak{p}^\perp$ and $\overline{e}_\mathfrak{q}$ are homogeneous, this induces a grading on the algebra (5.3.5). Summarizing, we have:

Proposition 5.3.7. *The category $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ is equivalent to the category of finite-dimensional (right) modules over a finite-dimensional positively graded algebra.*

We will denote by ${}^\mathbb{Z}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ the graded version of $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$, that is the category of finitely generated *graded* modules over the algebra (5.3.5). We remark that the techniques of [Str03a] ensure that simple and indecomposable projective modules are gradable, both as objects of $\mathcal{O}_\lambda$ and of $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ (although the grading is different). We take their standard graded lifts to be determined by requiring that the simple head is concentrated in degree 0.

The properly stratified structure

The results of [MS05, §2] extend to the categories $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Let us briefly sketch them.

Duality. First, we notice that the category $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ inherits from $\mathcal{O}_\lambda$ a simple-preserving duality:

Lemma 5.3.8. *The algebra* (5.3.5) *inherits from A_λ an anti-automorphism; this induces a simple-preserving duality on $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$.*

Proof. The category $\mathcal{O}$ has a simple-preserving duality (see for example [Hum08, §3.2]), which restricts to a simple-preserving duality on $\mathcal{O}_\lambda^\mathfrak{p}$. This defines an anti-automorphism on $\mathrm{End}_{\mathcal{O}^\mathfrak{p}}(\mathscr{P}^\mathfrak{p}(\lambda)) = A_\lambda / A_\lambda e_\mathfrak{p}^\perp A_\lambda$, which is the identity on the idempotents projecting onto the indecomposable projective modules $P^\mathfrak{p}(w \cdot \lambda)$. Hence this restricts to an anti-automorphism of $\overline{e}_\mathfrak{q}(A_\lambda / A_\lambda e_\mathfrak{p}^\perp A_\lambda)\overline{e}_\mathfrak{q}$, see (5.3.5), which induces, by the equivalence of categories (5.3.4), a simple-preserving duality on $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. □

Remark 5.3.9. One can also define the duality explicitly as in [MS05, Proposition 2.6].

Standard and proper standard modules. Let $x \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$. The module $P^{\mathfrak{p}}(x \cdot \lambda)$ is the projective cover of $S(x \cdot \lambda)$ in $\mathcal{O}_{\lambda}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Given two modules M, N the trace of M in N is defined to be $\operatorname{Tr}_M N = \bigcup_{f\colon M \to N} \operatorname{Im} f$. Then we have $S(x \cdot \lambda) = P^{\mathfrak{p}}(x \cdot \lambda) / \operatorname{Tr}_{\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)}(\operatorname{rad} P^{\mathfrak{p}}(x \cdot \lambda))$ as $\mathfrak{gl}_n$–modules. Let $P^{\mathfrak{p}}(\prec x) = \bigoplus_{w \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda), w \prec x} P^{\mathfrak{p}}(w \cdot \lambda)$ and set $\Delta(x \cdot \lambda) = P^{\mathfrak{p}}(x \cdot \lambda) / \operatorname{Tr}_{P^{\mathfrak{p}}(\prec x)} P^{\mathfrak{p}}(x \cdot \lambda)$. As in [MS05, Lemma 2.8], one can show that the modules $\Delta(x \cdot \lambda)$ satisfy a universal property, and as in [MS05, Proposition 2.9] this can be used to show that

$$\Delta(x \cdot \lambda) \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p}+\mathfrak{q})} P^{(\mathfrak{a})}(x \cdot \lambda), \tag{5.3.6}$$

where $P^{(\mathfrak{a})}(x \cdot \lambda)$ is the projective cover in $\mathcal{O}^{\mathfrak{p} \cap \mathfrak{a}}(\mathfrak{a})$ of the highest weight module with highest weight $x \cdot \lambda$. Moreover, one can define

$$\overline{\Delta}(x \cdot \lambda) \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p}+\mathfrak{q})} S^{(\mathfrak{a})}(x \cdot \lambda), \tag{5.3.7}$$

where $S^{(\mathfrak{a})}(x \cdot \lambda)$ is the simple module in $\mathcal{A}_{\mathfrak{q}}^{\mathfrak{p}}$ with highest weight $x \cdot \lambda$.

Properly stratified structure. We recall the definition of a *graded properly stratified algebra* in the sense of [Maz04] (see also [FKM02], [Fri07]).

Definition 5.3.10. *Let B be a finite-dimensional associative graded algebra over a field $\mathbb{K}$ with a simple-preserving duality and with equivalence classes of simple modules $\{\mathbb{L}(\lambda)\langle j \rangle \mid \lambda \in \Lambda, j \in \mathbb{Z}\}$ where $(\Lambda, \prec)$ is a partially ordered finite set. For each $\lambda \in \Lambda$ let:*

- *(i) $\mathbb{P}(\lambda)$ denote the projective cover of $\mathbb{L}(\lambda)$,*
- *(ii) $\mathbb{\Delta}(\lambda)$ be the maximal quotient of $\mathbb{P}(\lambda)$ such that $[\mathbb{\Delta}(\lambda) : \mathbb{L}(\mu)\langle i \rangle] = 0$ for all $\mu \succ \lambda$, $i \in \mathbb{Z}$,*
- *(iii) $\overline{\mathbb{\Delta}}(\lambda)$ be the maximal quotient of $\mathbb{\Delta}(\lambda)$ such that $[\operatorname{rad} \overline{\mathbb{\Delta}}(\lambda) : \mathbb{L}(\mu)\langle i \rangle] = 0$ for all $\mu \succeq \lambda$, $i \in \mathbb{Z}$.*

Then B is properly stratified *if the following conditions hold for every $\lambda \in \Lambda$:*

- *(PS1) the kernel of the canonical epimorphism $\mathbb{P}(\lambda) \twoheadrightarrow \mathbb{\Delta}(\lambda)$ has a filtration with subquotients isomorphic to graded shifts of $\mathbb{\Delta}(\mu)$, $\mu \succ \lambda$;*
- *(PS2) the kernel of the canonical epimorphism $\mathbb{\Delta}(\lambda) \twoheadrightarrow \overline{\mathbb{\Delta}}(\lambda)$ has a filtration with subquotients isomorphic to graded shifts of $\overline{\mathbb{\Delta}}(\lambda)$;*
- *(PS3) the kernel of the canonical epimorphism $\overline{\mathbb{\Delta}}(\lambda) \twoheadrightarrow \mathbb{L}(\lambda)$ has a filtration with subquotient isomorphic to graded shifts of $\mathbb{L}(\mu)$, $\mu \prec \lambda$.*

The modules $\mathbb{\Delta}(i)$ and $\overline{\mathbb{\Delta}}(i)$ are called *standard* and *proper standard* modules respectively. The same argument as for [MS05, Theorem 2.16] gives:

Theorem 5.3.11. *The algebra $\operatorname{End}(\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}}(\lambda))$ with the order induced by the Bruhat order on $\Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$ is a graded properly stratified algebra. The modules $\Delta(x \cdot \lambda)$ and $\overline{\Delta}(x \cdot \lambda)$ are the standard and proper standard modules respectively.*

It is easy to show that also the modules $\Delta(x \cdot \lambda)$ and $\overline{\Delta}(x \cdot \lambda)$ are gradable. They are indecomposable and hence a graded lift is unique up to isomorphism and overall shift. Again, we choose their standard lifts by requiring the simple heads to be concentrated in degree 0.

5.4 Functors between categories $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$

We examine now natural functors between the categories we have introduced. We still fix $\mathfrak{p},\mathfrak{q} \subseteq \mathfrak{gl}_n$ as above; for $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{q}' \subseteq \mathfrak{q}$ we will define functors

$$\begin{tikzcd} {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}} \arrow[r, shift left, "\mathfrak{z}"] & {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \arrow[l, shift left, hook', "\mathfrak{j}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}} \arrow[r, shift left, "\mathfrak{Q}"] & {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \arrow[l, shift left, hook', "\mathfrak{i}"] \end{tikzcd}.$$

Zuckermann's functors

Suppose $\mathfrak{p}'$ is also a standard parabolic subalgebra of $\mathfrak{gl}_n$ with $\mathfrak{p}' \subseteq \mathfrak{p}$. Let us fix an integral dominant weight λ. We have then an inclusion functor $\mathfrak{j}\colon \mathcal{O}_\lambda^{\mathfrak{p}} \to \mathcal{O}_\lambda^{\mathfrak{p}'}$. Since the abelian structure of $\mathcal{O}_\lambda^{\mathfrak{p}}$ is the restriction of the abelian structure of $\mathcal{O}_\lambda^{\mathfrak{p}'}$, this is an exact functor. Using Lemma 5.2.6, we see that this restricts to an exact functor $\mathfrak{j}\colon \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \to \mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$, which is just the inclusion functor of the Serre subcategory $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ into $\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$.

The left adjoint of $\mathfrak{j}\colon \mathcal{O}^{\mathfrak{p}} \to \mathcal{O}^{\mathfrak{p}'}$ is the *Zuckermann's functor* $\mathfrak{z}\colon \mathcal{O}_\lambda^{\mathfrak{p}'} \to \mathcal{O}_\lambda^{\mathfrak{p}}$, defined on $M \in \mathcal{O}_\lambda^{\mathfrak{p}'}$ by taking the maximal quotient that lies in $\mathcal{O}_\lambda^{\mathfrak{p}}$. Note that this is a generalization of the Zuckermann's functor that we defined and used in §5.3. The functor $\mathfrak{z}$ is right exact, but not exact in general. Being right exact, $\mathfrak{z}$ sends a presentation $P \to Q \twoheadrightarrow M$ with $P, Q \in \mathrm{Add}(\mathscr{P}_\mathfrak{q}^{\mathfrak{p}'}(\lambda))$ to a presentation $\mathfrak{z}P \to \mathfrak{z}Q \twoheadrightarrow \mathfrak{z}M$ of $\mathfrak{z}M$ with $\mathfrak{z}P, \mathfrak{z}Q \in \mathrm{Add}(\mathscr{P}_\mathfrak{q}^{\mathfrak{p}}(\lambda))$, hence it restricts to a functor $\mathfrak{z}\colon \mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}} \to \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$.

Notice that the definitions of $\mathfrak{j}$ and $\mathfrak{z}$ make sense in the graded setting too, and we have natural isomorphisms of graded vector spaces

$$\mathrm{Hom}_{{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}}(\mathfrak{z}M, N) \cong \mathrm{Hom}_{{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}}(M, \mathfrak{j}N) \tag{5.4.1}$$

for all $M \in {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$ and $N \in {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Hence we have also adjoint functors

$$\begin{tikzcd} {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}} \arrow[r, shift left, "\mathfrak{z}"] & {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \arrow[l, shift left, hook', "\mathfrak{j}"] \end{tikzcd}. \tag{5.4.2}$$

Coapproximation functors

Suppose that $\mathfrak{q}'$ is a standard parabolic subalgebra of $\mathfrak{gl}_n$ with $\mathfrak{q}' \subseteq \mathfrak{q}$ and let us fix an integral dominant weight λ. According to Lemma 5.2.6, we have an inclusion functor $\mathfrak{i}\colon \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \to \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}$. This is right exact but not left exact in general (cf. [MS05, Example 2.3] for an example).

Its right adjoint $\mathfrak{Q}\colon \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}} \to \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ is called *coapproximation*, and can be described Lie theoretically as follows. Take $M \in \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}$, and let $p\colon Q \twoheadrightarrow \mathrm{Tr}_{\mathscr{P}_\mathfrak{q}^{\mathfrak{p}}(\lambda)}(M)$ be a projective cover in $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}$ (notice that $\mathscr{P}_\mathfrak{q}^{\mathfrak{p}}(\lambda)$ is a direct summand of $\mathscr{P}_{\mathfrak{q}'}^{\mathfrak{p}}(\lambda)$ and in particular an object of $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}$). Then define $\mathfrak{Q}(M) = Q/\mathrm{Tr}_{\mathscr{P}_\mathfrak{q}^{\mathfrak{p}}(\lambda)}(\ker p)$. It is easy to verify that $\mathfrak{Q}$ is just a Serre quotient functor, and hence it is exact; indeed, it corresponds under the equivalence of categories (5.3.4) to $\mathrm{Hom}_{\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}}(\mathscr{P}_\mathfrak{q}^{\mathfrak{p}}(\lambda), \bullet)$. Its left adjoint $\mathfrak{i}$, on the other hand, corresponds to the induction functor $\bullet \otimes_{\operatorname{End}(\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}}(\lambda))} \operatorname{End}(\mathscr{P}_{\mathfrak{q}'}^{\mathfrak{p}}(\lambda))$. In particular, there are graded lifts

$$\mathfrak{i} = \bullet \otimes_{\operatorname{End}(\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}}(\lambda))} \operatorname{End}(\mathscr{P}_{\mathfrak{q}'}^{\mathfrak{p}}(\lambda)) \colon {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \longrightarrow {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}} \tag{5.4.3}$$

$$\mathfrak{Q} = \operatorname{Hom}_{{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}}(\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}}(\lambda), \bullet) \colon {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}} \longrightarrow {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}. \tag{5.4.4}$$

Our next goal is to compute the action of $\mathfrak{Q}$ on proper standard modules. We will need the following easy fact:

Lemma 5.4.1. *Let $\mathfrak{q}' \subset \mathfrak{q}$ and let $w \in \Lambda_{\mathfrak{q}'}^{\mathfrak{p}}(\lambda)$. Then there exists a unique $x \in W_{\mathfrak{q}}$ such that $xw \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$ and $\ell(xw) = \ell(x) + \ell(w)$.*

Proof. Let $\mathbb{S}_\lambda$ be the stabilizer of the weight λ. Since $\mathfrak{p}$ is orthogonal to $\mathfrak{q}$, we may assume $\mathfrak{p} = \mathfrak{b}$. Moreover, since $\Lambda_{\mathfrak{q}'}(\lambda) \subseteq \left(\mathbb{S}_n/\mathbb{S}_\lambda\right)^{\text{short}}$, it is clearly sufficient to prove the result for $w \in \left(\mathbb{S}_n/\mathbb{S}_\lambda\right)^{\text{short}}$. Then the lemma is simply a statement about double cosets. Let z be the shortest element in the double coset $W_{\mathfrak{q}} w \mathbb{S}_\lambda$. Then all shortest coset representatives for $\mathbb{S}_n/\mathbb{S}_\lambda$ contained in $W_{\mathfrak{q}} w \mathbb{S}_\lambda$ can be obtained as yz for $y \in W_{\mathfrak{q}}$ (and in particular $w = y_1 z$ for $y_1 \in W_{\mathfrak{q}}$). Let $y_0 \in W_{\mathfrak{q}}$ be the shortest element such that $y_0 z \mathbb{S}_\lambda \cap \left(W_{\mathfrak{q}} \backslash \mathbb{S}_n\right)^{\text{long}} \neq \varnothing$ (this exists, since this is the unique element such that $y_0 z w_\lambda$ is the longest element of the double coset $W_{\mathfrak{q}} w \mathbb{S}_\lambda$, where w_λ is the longest element of $\mathbb{S}_\lambda$). Setting $x = y_0 y_1^{-1}$ we get the claim. □

First we suppose that we are in the extreme case $\mathfrak{q}' = \mathfrak{b}$, and we compute the action of $\mathfrak{Q}$ on Verma modules.

Proposition 5.4.2. *Consider the coapproximation functor $\mathfrak{Q} \colon {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}} \to {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Let λ be a dominant integral weight, let $w \in \Lambda^{\mathfrak{p}}(\lambda)$, and let $x \in W_{\mathfrak{q}}$ be the element given by Lemma 5.4.1 such that $xw \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$. Then we have $\mathfrak{Q} M^{\mathfrak{p}}(w \cdot \lambda) = q^{\ell(x)} \overline{\Delta}(xw \cdot \lambda)$.*

For the proof we will need some preliminary results in the ungraded setting.

Lemma 5.4.3. *Suppose $w \in \Lambda^{\mathfrak{p}}(\lambda)$, let $M(w \cdot \lambda)$ be a Verma module in $\mathcal{O}_\lambda$ and $M^{\mathfrak{p}}(w \cdot \lambda)$ be its parabolic quotient in $\mathcal{O}_\lambda^{\mathfrak{p}}$. Then for every simple reflection $s \in W_{\mathfrak{q}}$ such that $\ell(sw) > \ell(w)$ the map $M^{\mathfrak{p}}(sw \cdot \lambda) \to M^{\mathfrak{p}}(w \cdot \lambda)$ induced from the inclusion $M(sw \cdot \lambda) \hookrightarrow M(w \cdot \lambda)$ after taking the parabolic quotients is injective.*

Example 5.4.4. Notice that in the statement of the lemma it is essential to assume that the simple reflection s is orthogonal to the parabolic subalgebra $\mathfrak{p}$. As a counterexample when this is not true, consider the regular block $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_3)$, where $\mathfrak{p} \subset \mathfrak{gl}_3$ is the standard parabolic subalgebra corresponding to the composition $(2, 1)$. Then the inclusion $M(s_2 \cdot 0) \hookrightarrow M(0)$ of Verma modules in $\mathcal{O}(\mathfrak{gl}_3)$ induces a map $M^{\mathfrak{p}}(s_2 \cdot 0) \to M^{\mathfrak{p}}(0)$ which is not injective (the kernel is isomorphic to the simple module $L(s_2 s_1 \cdot 0)$). ⊛

Proof of Lemma 5.4.3. Let v_{sw}, v_w be the highest weight vectors of $M(sw \cdot \lambda)$ and $M(w \cdot \lambda)$ respectively. Then (cf. [Hum08, §1.4]) the inclusion $M(sw \cdot \lambda) \hookrightarrow M(w \cdot \lambda)$ is determined by $v_{sw} \mapsto f_{\alpha_s}^k v_w$ for some $k \in \mathbb{N}$, where $f_{\alpha_s} \in \mathfrak{n}^-$ is the standard generator of $U(\mathfrak{gl}_n)$ corresponding to the simple root α_s. This indeed defines an injective map because the Verma modules are free as $U(\mathfrak{n}^-)$–modules and $U(\mathfrak{n}^-)$ has no zero-divisors, both by the PBW Theorem.

Let $\mathfrak{gl}_n = \mathfrak{p} \oplus \mathfrak{u}_{\mathfrak{p}}^-$. The parabolic Verma modules $M^{\mathfrak{p}}(x \cdot \lambda) = \mathfrak{z} M(x \cdot \lambda)$ can alternatively be defined through parabolic induction, hence they are free as $U(\mathfrak{u}_{\mathfrak{p}}^-)$–modules (although in general not of rank one). Since the simple reflection s is orthogonal to the set of reflections

$W_{\mathfrak{p}}$, the element f_{α_s} lies in $U(\mathfrak{u}_{\mathfrak{p}}^-)$ and the map on the parabolic quotients is again given by multiplication by it. By the same argument as before, this map has to be injective. □

Lemma 5.4.5. *With the same notation as before,* $\operatorname{coker}\big(M^{\mathfrak{p}}(sw\cdot\lambda)\hookrightarrow M^{\mathfrak{p}}(w\cdot\lambda)\big)$ *has only composition factors of type* $L(y\cdot\lambda)$ *with* $sy>y$.

Proof. The inclusion is given by multiplication by $f_{\alpha_s}^k$. By the PBW Theorem, it follows immediately that the cokernel is locally $\langle f_{\alpha_s}^k\rangle_{k\in\mathbb{N}}$–finite, hence all its composition factors are indexed by elements of $\mathbb{S}_n$ that are shortest coset representatives for ${}_{\langle s\rangle}\backslash^{\mathbb{S}_n}$. □

Lemma 5.4.6. *Let* $\lambda\in\mathbb{A}^+$*. For every* $w\in\Lambda^{\mathfrak{p}}(\lambda)$ *and* $x\in W_{\mathfrak{q}}$ *such that* $xw\in\Lambda^{\mathfrak{p}}(\lambda)$ *we have*

$$q^{\ell(x)}\mathfrak{Q}M^{\mathfrak{p}}(xw\cdot\lambda)=\mathfrak{Q}M^{\mathfrak{p}}(w\cdot\lambda). \tag{5.4.5}$$

Proof. Of course, it is sufficient to prove the claim for a simple reflection $s\in W_{\mathfrak{q}}$. Then the result follows from Lemma 5.4.5 if we apply the exact functor $\mathfrak{Q}$ to the short exact sequence

$$0\longrightarrow qM^{\mathfrak{p}}(sw\cdot\lambda)\longrightarrow M^{\mathfrak{p}}(w\cdot\lambda)\longrightarrow Q\longrightarrow 0. \tag{5.4.6}$$ □

Lemma 5.4.7. *Let* $\lambda\in\mathbb{A}^+$ *and* $w\in\Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$*. Then* $\mathfrak{Q}M^{\mathfrak{p}}(w\cdot\lambda)=\overline{\Delta}(w\cdot\lambda)$.

Proof. The projective module $P^{\mathfrak{p}}(w\cdot\lambda)$ has a filtration by parabolic Verma modules in $\mathcal{O}_\lambda^{\mathfrak{p}}$. Hence the projective module $P^{\mathfrak{p}}(w\cdot\lambda)=\mathfrak{Q}P^{\mathfrak{p}}(w\cdot\lambda)$ in $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ has a filtration by modules $\mathfrak{Q}M^{\mathfrak{p}}(y\cdot\lambda)$ for $y\in\Lambda^{\mathfrak{p}}(\lambda)$, $y\preceq w$.

Now, the proper standard module $\overline{\Delta}(w\cdot\lambda)$ is defined to be the maximal quotient Q of $P^{\mathfrak{p}}(w\cdot\lambda)$ in $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ satisfying

$$[\operatorname{rad}Q:S(z\cdot\lambda)]=0\qquad\text{for all } z\preceq w. \tag{5.4.7}$$

Obviously the quotient $\mathfrak{Q}M^{\mathfrak{p}}(w\cdot\lambda)$ at the top of $P^{\mathfrak{p}}(w\cdot\lambda)$ satisfies (5.4.7). Any bigger quotient contains the simple head of some $\mathfrak{Q}M^{\mathfrak{p}}(y\cdot\lambda)$ for $y\prec w$. Consider such a y and let $x'\in W_{\mathfrak{q}}$ be the element given by Lemma 5.4.1 for y. By Lemma 5.4.6 the simple head in $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ of $\mathfrak{Q}M^{\mathfrak{p}}(y\cdot\lambda)$ is the simple head of $\mathfrak{Q}M^{\mathfrak{p}}(x'y\cdot\lambda)$; but this is the simple head of $\mathfrak{Q}P^{\mathfrak{p}}(x'y\cdot\lambda)$, that is $S(x'y\cdot\lambda)$. Notice that $x'y\preceq w$ (this follows because $y\prec w$ and both $x'y,w\in\Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$). Hence $\mathfrak{Q}M^{\mathfrak{p}}(w\cdot\lambda)$ is indeed the maximal quotient satisfying (5.4.7). □

The proof of the proposition follows now easily:

Proof of Proposition 5.4.2. By Lemma 5.4.6 we have $\mathfrak{Q}M^{\mathfrak{p}}(w\cdot\lambda)=q^{\ell(x)}\mathfrak{Q}M^{\mathfrak{p}}(xw\cdot\lambda)$ and by Lemma 5.4.7 this is $q^{\ell(x)}\overline{\Delta}(xw\cdot\lambda)$. □

From Proposition 5.4.2 one can directly deduce:

Corollary 5.4.8. *Let* $\mathfrak{q}'$ *be a standard parabolic subalgebra of* $\mathfrak{gl}_n$ *with* $\mathfrak{q}'\subset\mathfrak{q}$ *and consider the coapproximation functor* $\mathfrak{Q}\colon{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}\to{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$*. Let* $w\in\Lambda_{\mathfrak{q}'}^{\mathfrak{p}}(\lambda)$ *and let* $x\in W_{\mathfrak{q}}$ *be the element given by Lemma 5.4.1. Then we have* $\mathfrak{Q}\overline{\Delta}(w\cdot\lambda)\cong q^{\ell(x)}\overline{\Delta}(xw\cdot\lambda)$.

Proof. Let $\mathfrak{Q}_{\mathfrak{q}'}\colon{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}\to{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}'\text{-pres}}$ and $\mathfrak{Q}_{\mathfrak{q}}:{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}\to{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ be the coapproximation functors. It follows from the definition that $\mathfrak{Q}\circ\mathfrak{Q}_{\mathfrak{q}'}=\mathfrak{Q}_{\mathfrak{q}}$. By Proposition 5.4.2 we have $\mathfrak{Q}_{\mathfrak{q}'}M^{\mathfrak{p}}(w\cdot\lambda)=\overline{\Delta}(w\cdot\lambda)$ and $\mathfrak{Q}_{\mathfrak{q}}M^{\mathfrak{p}}(w\cdot\lambda)=q^{\ell(x)}\overline{\Delta}(xw\cdot\lambda)$, and the claim follows. □

The coapproximation functor $\mathfrak{Q}$ enables us to compute proper standard filtrations of standard modules:

Proposition 5.4.9. *Suppose that $\mathfrak{q}$ has only one block (that is, $W_\mathfrak{q} \cong \mathbb{S}_k$ for some integer k) and let λ be a dominant regular weight. Then*

(i) for all $w \in \Lambda_\mathfrak{q}^\mathfrak{p}(\lambda)$ the proper standard filtration of the standard module $\Delta(w \cdot \lambda) \in \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ has length $k!$

(ii) in the Grothendieck group of ${}^\mathbb{Z}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ we have $\Delta(w \cdot \lambda)] = [k]_0! \,[\overline{\Delta}(w \cdot \lambda)$.

Proof. Since λ is regular, w is a longest coset representative for $W_\mathfrak{q} \backslash \mathbb{S}_n$, hence $w = w_\mathfrak{q} w'$. It is well-known that in a Verma flag of the projective module $P^\mathfrak{p}(w \cdot \lambda)$ all Verma modules $M^\mathfrak{p}(xw' \cdot \lambda)$ for $x \in W_\mathfrak{q}$ appear exactly once. Applying $\mathfrak{Q}$, by Proposition 5.4.2 we get a filtration of $P^\mathfrak{p}(w \cdot \lambda)$ in $\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ with $\overline{\Delta}(w \cdot \lambda)$ appearing exactly $k!$ times. Of course, this is the part of the filtration that builds the standard module $\Delta(w \cdot \lambda)$. By the Kazhdan-Lusztig conjecture, in the Grothendieck group of ${}^\mathbb{Z}\mathcal{O}^\mathfrak{p}$ we have

$$[P^\mathfrak{p}(w \cdot \lambda)] \in \sum_{x \in W_\mathfrak{q}} q^{\ell(w_\mathfrak{q}) - \ell(x)} [M^\mathfrak{p}(xw' \cdot \lambda)] + \sum_{x \in W_\mathfrak{q}, z \prec w'} q\mathbb{Z}[q][M^\mathfrak{p}(xz \cdot \lambda)]. \tag{5.4.8}$$

Applying $\mathfrak{Q}$ and considering only the part of the filtration that builds $\Delta(w \cdot \lambda)$ we get

$$[\Delta(w \cdot \lambda)] = \sum_{x \in W_\mathfrak{q}} q^{2(\ell(w_\mathfrak{q}) - \ell(x))} [\overline{\Delta}(w \cdot \lambda)], \tag{5.4.9}$$

which gives (ii). $\square$

5.5 Translation functors on $\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$

We study now graded translation functors (cf. Chapter 4) restricted to the categories ${}^\mathbb{Z}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$.

Restriction of translation functors

Translation functors preserve the subcategories we have introduced:

Lemma 5.5.1. *Given two dominant weights λ, μ, the translation functor $\mathbb{T}_\lambda^\mu$ restricts to a functor $\mathbb{T}_\lambda^\mu : \mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \to \mathcal{O}_\mu^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Moreover, translation functors commute with the functors $\mathfrak{j}, \mathfrak{z}, \mathfrak{i}, \mathfrak{Q}$.*

Proof. It follows directly from the definition that tensoring with a finite-dimensional $\mathfrak{gl}_n$–module defines an exact endofunctor of the category $\mathcal{O}\{\mathfrak{p} + \mathfrak{q}, \mathcal{A}_\mathfrak{q}^\mathfrak{p}\}$. In particular, the translation functor $\mathbb{T}_\lambda^\mu$ preserves the category $\mathcal{O}\{\mathfrak{p} + \mathfrak{q}, \mathcal{A}_\mathfrak{q}^\mathfrak{p}\}$.

Since $\mathfrak{j}, \mathfrak{i}$ are inclusions, it follows that $\mathbb{T}_\lambda^\mu$ commutes with them. By adjunction, it commutes with $\mathfrak{z}, \mathfrak{Q}$ as well. $\square$

Of course we also have the graded versions

$$\mathbb{T}_\lambda^\mu \colon {}^\mathbb{Z}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}} \to {}^\mathbb{Z}\mathcal{O}_\mu^{\mathfrak{p},\mathfrak{q}\text{-pres}}. \tag{5.5.1}$$

We will need the following easy result to compute the action of translation functors (5.5.1) in the category ${}^\mathbb{Z}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$:

Lemma 5.5.2. *Let $\mathbb{S}_\lambda, \mathbb{S}_\mu$ be standard parabolic subgroups of $\mathbb{S}_n$ with $\mathbb{S}_\lambda \subset \mathbb{S}_\mu$. Then for every $w \in \left({}^{\mathbb{S}_n}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ there exist unique elements $w' \in \left({}^{\mathbb{S}_n}/_{\mathbb{S}_\mu}\right)^{\text{short}}$ and $x \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ such that $w = w'x$. Moreover $\ell(w) = \ell(w') + \ell(x)$.*

Proof. The element w determines some coset $w\mathbb{S}_\mu$, in which there is a unique shortest coset representative w'. Hence $w = w'x$ for some $x \in \mathbb{S}_\mu$ with $\ell(w) = \ell(w') + \ell(x)$. Since $w \in \left({}^{\mathbb{S}_n}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ we have $\ell(wt) > \ell(w)$ for all $t \in \mathbb{S}_\lambda$; but then also $\ell(xt) > \ell(x)$ for all $t \in \mathbb{S}_\lambda$, hence $x \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$. ☐

Translation of proper standard modules

Now we compute how translation functors act on proper standard modules.

Translation onto the wall. First, we consider translation onto the wall:

Proposition 5.5.3. *Let λ, μ be dominant weights with stabilizers $\mathbb{S}_\lambda, \mathbb{S}_\mu$ respectively, and suppose $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$. Let $w \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda)$, and write $w = w'x$ as given by Lemma 5.5.2. Then we have*

$$\mathbb{T}^\mu_\lambda \overline{\Delta}(w \cdot \lambda) \cong \begin{cases} q^{-\ell(x)} \overline{\Delta}(w' \cdot \mu) & \text{if } w' \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\mu), \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.2}$$

Proof. First, we compute in the usual category $\mathcal{O}(\mathfrak{gl}_n)$. It is well-known that translating a Verma module to the wall gives a Verma module. In fact if we forget the grading then $\mathbb{T}^\mu_\lambda M(w \cdot \lambda) \cong M(w' \cdot \mu)$ (cf. [Hum08, Theorem 7.6]). The graded version can be computed generalizing [Str03a, Theorem 8.1], and is $\mathbb{T}^\mu_\lambda M(w \cdot \lambda) \cong q^{-\ell(x)} M(w' \cdot \mu)$.

Now since the functors $\mathfrak{z}$ and $\mathfrak{Q}$ commute with $\mathbb{T}^\mu_\lambda$, using Proposition 5.4.2 we have

$$\mathbb{T}^\mu_\lambda \overline{\Delta}(w \cdot \lambda) \cong \mathbb{T}^\mu_\lambda \mathfrak{Q}\mathfrak{z} M(w \cdot \lambda) \cong \mathfrak{Q}\mathfrak{z}\mathbb{T}^\mu_\lambda M(w \cdot \lambda) \cong q^{-\ell(x)} \mathfrak{Q}\mathfrak{z} M(w' \cdot \mu). \tag{5.5.3}$$

If $w' \notin \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\mu)$ then $\mathfrak{z}M(w' \cdot \mu) \cong 0$. Otherwise we get $\mathbb{T}^\mu_\lambda \overline{\Delta}(w \cdot \lambda) \cong q^{-\ell(x)} \overline{\Delta}(w' \cdot \mu)$. ☐

Translation out of the wall. Now let us compute translation of proper standard modules out of the wall:

Proposition 5.5.4. *Let λ, μ be dominant weights with stabilizers $\mathbb{S}_\lambda, \mathbb{S}_\mu$ respectively, and suppose $\mathbb{S}_\lambda \subset \mathbb{S}_\mu$. Then for every $w \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\mu)$ we have*

$$[\mathbb{T}^\lambda_\mu \overline{\Delta}(w \cdot \mu)] = \sum_{y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}} q^{\ell(y_0) - \ell(y) + \ell(x_y)} [\overline{\Delta}(x_y w y \cdot \lambda)], \tag{5.5.4}$$

where y_0 is the longest element of $\left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$, and for every $y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ the element x_y is the element given by Lemma 5.4.1 for $wy \in \Lambda^{\mathfrak{p}}(\lambda)$.

Note that $w \in \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\mu)$ implies that $w\mathbb{S}_\mu \subseteq W^{\mathfrak{p}}$; but as $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$ we have then $wy\mathbb{S}_\lambda \subseteq W^{\mathfrak{p}}$, and in particular $wy \in \Lambda^{\mathfrak{p}}(\lambda)$ for all $y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$.

Proof. Consider $M(w \cdot \mu)$ in ${}^{\mathbb{Z}}\mathcal{O}$. Then $\mathbb{T}^\lambda_\mu M(w \cdot \mu)$ has a Verma flag and we have

$$[\mathbb{T}^\lambda_\mu M(w \cdot \mu)] = \sum_{y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}} q^{\ell(y_0) - \ell(y)} [M(wy \cdot \lambda)]. \tag{5.5.5}$$

This is well-known in the ungraded setting (see for example [Hum08, Theorem 7.12]); the graded version follows as in [Str05]. Notice that $wy\mathbb{S}_\lambda \subseteq w\mathbb{S}_\mu \subseteq W^{\mathfrak{p}}$ for all $y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$. In particular, $\mathfrak{z}M(wy\cdot\lambda) \ncong 0$ for all $y \in \left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$. Hence, by Lemma 5.5.5 below, we can apply $\mathfrak{z}$ to either side of (5.5.5) and we get in ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p}}$:

$$[\mathbb{T}_\mu^\lambda M^{\mathfrak{p}}(w\cdot\mu)] = \sum_{y\in\left({}^{\mathbb{S}_\mu}/_{\mathbb{S}_\lambda}\right)^{\text{short}}} q^{\ell(y_0)-\ell(y)}[M^{\mathfrak{p}}(wy\cdot\lambda)]. \tag{5.5.6}$$

Now we can apply the exact functor $\mathfrak{Q}$ to both sides. Using Proposition 5.4.2 and the commutativity of $\mathfrak{Q}$ with $\mathbb{T}_\mu^\lambda$ we obtain the claim. $\square$

We include the technical lemma which we used in the previous proof:

Lemma 5.5.5. *The Zuckermann's functor* $\mathfrak{z}\colon \mathcal{O}_\lambda \to \mathcal{O}_\lambda^{\mathfrak{p}}$ *is exact on modules which admit a Verma flag with Verma modules* $M(z\cdot\lambda)$ *such that* $\mathfrak{z}M(z\cdot\lambda)\ncong 0$.

Proof. Let $\mathbb{L}^i\mathfrak{z}$ denote the left derived functor of $\mathfrak{z}$. We need to show that for each module $X \in \mathcal{O}_\lambda$ which satisfies the hypothesis of the lemma we have $\mathbb{L}^i\mathfrak{z}X = 0$ for all $i > 0$. By induction on the length of a Verma flag, it is enough to check this for the Verma modules $M(z\cdot\lambda)$ such that $\mathfrak{z}M(z\cdot\lambda) = M^{\mathfrak{p}}(z\cdot\lambda) \ncong 0$.

First, let us suppose that λ is a regular weight. We proceed then by induction on the Bruhat order. Since the dominant Verma module $M(\lambda)$ is projective, we have of course $\mathbb{L}^i\mathfrak{z}M(\lambda) \cong 0$ for all $i > 0$ (notice, on the other side, that $\mathfrak{z}M(\lambda)$ is never zero, since it has as quotient the finite-dimensional simple module $L(\lambda)$, which is in $\mathcal{O}_\lambda^{\mathfrak{p}}$). Now let us consider some $M(z\cdot\lambda)$ for $z \neq e$ with $\mathfrak{z}M(z\cdot\lambda) = M^{\mathfrak{p}}(z\cdot\lambda) \ncong 0$. Notice that the last condition is equivalent to $z \in \left({}^{W^{\mathfrak{p}}}\backslash{}^{\mathbb{S}_n}\right)^{\text{short}}$. Let $s_j \in \mathbb{S}_n$ be some simple reflection such that $zs_j \prec z$ and $zs_j \in \left({}^{W^{\mathfrak{p}}}\backslash{}^{\mathbb{S}_n}\right)^{\text{short}}$. Then we have a short exact sequence (see for example [Irv85, Proposition 2.2 (i)])

$$0 \longrightarrow M(zs_j\cdot\lambda) \longrightarrow \theta_j M(zs_j\cdot\lambda) \longrightarrow M(z\cdot\lambda) \longrightarrow 0. \tag{5.5.7}$$

By induction we can suppose that $\mathbb{L}^i\mathfrak{z}M(zs_j\cdot\lambda)\cong 0$ for all $i > 0$. Moreover, since $\mathfrak{z}$ and θ_j commute, we can also suppose that $\mathbb{L}^i\mathfrak{z}\theta_jM(zs_j\cdot\lambda)\cong 0$ for all $i>0$. From the long exact sequence for $\mathbb{L}^\bullet\mathfrak{z}$ corresponding to (5.5.7) we deduce then immediately that $\mathbb{L}^i\mathfrak{z}M(zs_j\cdot\lambda)\cong 0$ for all $i > 1$. Moreover, since the short exact sequence (5.5.7) induces, after applying $\mathfrak{z}$, the short exact sequence

$$0 \longrightarrow M^{\mathfrak{p}}(zs_j\cdot\lambda) \longrightarrow \theta_j M^{\mathfrak{p}}(zs_j\cdot\lambda) \longrightarrow M^{\mathfrak{p}}(z\cdot\lambda) \longrightarrow 0 \tag{5.5.8}$$

(see [Irv85, Proposition 2.2 (v)]), it follows that also $\mathbb{L}^1\mathfrak{z}M(zs_j\cdot\lambda)\cong 0$, and we are done.

It remains to consider the case in which λ is a singular weight. Let $z \in \left({}^{\mathbb{S}_n}/_{\mathbb{S}_\lambda}\right)^{\text{short}}$ be such that $\mathfrak{z}M(z\cdot\lambda)\ncong 0$. Now, we have $\mathbb{T}_0^\lambda M(z\cdot 0)\cong M(z\cdot\lambda)$ (see for example [Hum08, Theorem 7.6]) and $\mathfrak{z}M(z\cdot 0)\ncong 0$ because $\mathbb{T}_0^\lambda\mathfrak{z}M(z\cdot 0)\cong\mathfrak{z}\mathbb{T}_0^\lambda M(z\cdot 0)\cong\mathfrak{z}M(z\cdot\lambda)$. Since 0 is a regular weight, we have then $\mathbb{L}^i\mathfrak{z}M(z\cdot\lambda)\cong\mathbb{L}^i\mathfrak{z}\mathbb{T}_0^\lambda M(z\cdot 0)\cong\mathbb{T}_0^\lambda\mathbb{L}^i\mathfrak{z}M(z\cdot 0)\cong 0$ for all $i > 0$, and we are done. $\square$

Translation of projective and simple modules

Now we compute translations of projective modules out of the wall:

Proposition 5.5.6. *Let* λ, μ *be dominant weights with stabilizers* $\mathbb{S}_\lambda, \mathbb{S}_\mu$ *respectively, and suppose that* $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$. *Then for every* $w \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\mu)$ *we have in* ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$*:*

$$\mathbb{T}_\mu^\lambda P^{\mathfrak{p}}(w \cdot \mu) = P^{\mathfrak{p}}(wy_0 \cdot \lambda) \tag{5.5.9}$$

where y_0 *is the longest element of* $\left({}^{\mathbb{S}_\mu}\!/\!{}_{\mathbb{S}_\lambda}\right)^{\text{short}}$.

Proof. Let $P(w \cdot \lambda) \in {}^{\mathbb{Z}}\mathcal{O}$. By [Hum08, Theorem 7.11] we have $\mathbb{T}_\mu^\lambda P(w \cdot \mu) = P(wy_0 \cdot \lambda)$ as ungraded modules. By (5.5.6), the top Verma module is not shifted under translation, hence this also holds as graded modules. Applying the Zuckermann's functor $\mathfrak{z}$ we get (5.5.9) in ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p}}$, hence also in ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Notice that we get for free that $wy_0 \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$ (although it would be easy to check it directly). $\square$

Using the adjunctions (4.4.9) we can then compute translations of simple modules onto the wall:

Proposition 5.5.7. *Let* λ, μ *be dominant weights with stabilizers* $\mathbb{S}_\lambda, \mathbb{S}_\mu$ *respectively, and suppose that* $\mathbb{S}_\lambda \subseteq \mathbb{S}_\mu$. *Let* y_0 *be the longest element of* $\left({}^{\mathbb{S}_\mu}\!/\!{}_{\mathbb{S}_\lambda}\right)^{\text{short}}$. *Then for every* $w \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$ *we have in* ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}$*:*

$$\mathbb{T}_\lambda^\mu S(w \cdot \lambda) = \begin{cases} q^{-\ell(y_0)} S(z \cdot \mu) & \text{if } w = zy_0 \text{ for some } z \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\mu) \in \mathbb{S}_\mu \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.10}$$

Proof. We use the previous result together with the adjunction $\mathbb{T}_\mu^\lambda \dashv q^{\ell(y_0)}\mathbb{T}_\lambda^\mu$. For every projective module $P^{\mathfrak{p}}(z \cdot \mu) \in {}^{\mathbb{Z}}\mathcal{O}_\mu^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ we have

$$\operatorname{Hom}(\mathbb{T}_\mu^\lambda P^{\mathfrak{p}}(z \cdot \mu), S(w \cdot \lambda)) \cong \operatorname{Hom}(P^{\mathfrak{p}}(z \cdot \mu), q^{\ell(y_0)}\mathbb{T}_\lambda^\mu S(w \cdot \lambda)). \tag{5.5.11}$$

The left hand side is 0 unless $w = zy_0$, in which case it is $\mathbb{C}$, and the claim follows. $\square$

CHAPTER 6

The categorification

This chapter is devoted to the construction of the categorification of the representations studied in Chapter 3. We will define the categorification itself in §6.1 and construct the action of the intertwining operators in §6.2. We will prove in §6.3 that the indecomposable projective modules categorify the canonical basis. In §6.4, moreover, we will categorify the bilinear form (3.1.9). Finally, in §6.5 we will construct the action of the generators of U_q on the categorification.

Notation. For every composition $\boldsymbol{a}$ of some n we fix, once and forever, a dominant integral weight $\lambda_{\boldsymbol{a}}$ for $\mathfrak{gl}_n$ with stabilizer $\mathbb{S}_{\boldsymbol{a}}$ under the dot action. We suppose for future notational convenience that if $\mathbb{n}$ is the regular composition of n (3.1.1) then $\lambda_{\mathbb{n}} = 0$. Fix now a positive integer n and $k \in \{0, \ldots, n\}$. If $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is the set of the simple roots of $\mathfrak{gl}_n$, we let $\mathfrak{p}$ and $\mathfrak{q}$ be the standard parabolic subalgebras of $\mathfrak{gl}_n$ with corresponding sets of simple roots $\Pi_{\mathfrak{q}} = \{\alpha_1, \ldots, \alpha_{k-1}\}$ and $\Pi_{\mathfrak{p}} = \{\alpha_k, \ldots, \alpha_{n-1}\}$, so that $\mathbb{S}_k \times \mathbb{S}_{n-k} \cong W_{\mathfrak{p}+\mathfrak{q}} \subseteq \mathbb{S}_n$. We set

$$\Lambda_k(\boldsymbol{a}) = \Lambda^{\mathfrak{p}}_{\mathfrak{q}}(\lambda_{\boldsymbol{a}}) \qquad \text{and} \qquad \mathcal{Q}_k(\boldsymbol{a}) = {}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_{\lambda_{\boldsymbol{a}}}. \tag{6.0.1}$$

From now on, for $w \in \Lambda_k(\boldsymbol{a})$ we denote by $S_{\boldsymbol{a},k}(w) \in \mathcal{Q}_k(\boldsymbol{a})$ the simple module $S(w \cdot \lambda_{\boldsymbol{a}})$ and by $Q_{\boldsymbol{a},k}(w)$ its projective cover $P^{\mathfrak{p}}(w \cdot \lambda_{\boldsymbol{a}})$. We let also $\Delta_{\boldsymbol{a},k}(w)$ and $\overline{\Delta}_{\boldsymbol{a},k}(w)$ be the corresponding standard and proper standard module. We will sometimes omit the subscripts k and $\boldsymbol{a}$ when there will be no risk of confusion.

6.1 Categorification of the representation $\mathsf{V}(\boldsymbol{a})$

Fix a positive integer n and a composition $\boldsymbol{a}$ of n.

Combinatorics of tableaux

Given an integer k with $0 \leq k \leq n$, recall that a *hook partition* of shape $(n-k, k)$ is made of a row of length $n-k$ and a column of length k, arranged as shown in Figure 6.1. We call the first row just the *row* and the first column just the *column* of the hook partition. Keep in mind that for us the box in the corner belongs to the row, but not to the column. Therefore

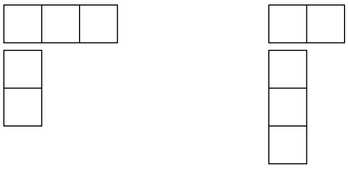


Figure 6.1: Hook partitions of shape (3,2) and (2,3).

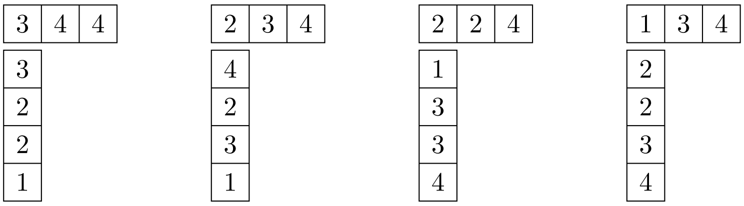


Figure 6.2: These are $(3,4)$–tableaux of type $(1,2,2,2)$. The leftmost tableau is the minimal one. Notice that only the last one is admissible.

we display the column slightly detached from the row. If $\boldsymbol{a} = (a_1, \dots, a_\ell)$ is a composition of n, a $(n-k,k)$*–tableau of type* $\boldsymbol{a}$ is a tableau filled with the integers

$$\underbrace{1,\dots,1}_{a_1 \text{ times}}, \underbrace{2,\dots,2}_{a_2 \text{ times}}, \dots, \underbrace{\ell,\dots,\ell}_{a_\ell \text{ times}}. \tag{6.1.1}$$

If we number the boxes of the hook partition of shape $(n-k,k)$ from 1 to n starting with the column from the bottom to the top and ending with the row from the left to the right, then the permutation group $\mathbb{S}_n$ acts from the left on the set of $(n-k,k)$–tableaux of type $\boldsymbol{a}$ permuting the boxes. The stabilizer of this action is $\mathbb{S}_{\boldsymbol{a}}$.

Define the *minimal* $(n-k,k)$–tableau $T^{\min}_{\boldsymbol{a}}$ of type $\boldsymbol{a}$ to be the tableau obtained putting the numbers (6.1.1) in order first in the column, from the bottom to the top, then in the row, from the left to the right (see Figure 6.2). Set also

$$T_{\boldsymbol{a}}(w) = w \cdot T^{\min}_{\boldsymbol{a}} \tag{6.1.2}$$

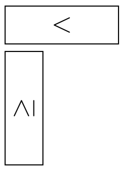
for each $w \in \mathbb{S}_n$. Then we can define a bijection $w \mapsto T_{\boldsymbol{a}}(w)$ between $\left({}^{\mathbb{S}_n}/_{\mathbb{S}_{\boldsymbol{a}}}\right)^{\text{short}}$ and $(n-k,k)$–tableaux of type $\boldsymbol{a}$.

We say that a tableau is *admissible* if:

(a) the entries in the row are strictly increasing (from left to right),

(b) the entries in the column are non-increasing (from the bottom to the top),

as shown in the picture on the right. For an example see the last tableau in Figure 6.2.



Proposition 6.1.1. *The bijection*

$$\begin{aligned} \left({}^{\mathbb{S}_n}/_{\mathbb{S}_{\boldsymbol{a}}}\right)^{\text{short}} &\xleftrightarrow{1-1} \{(n-k,k)\text{–}\textit{tableaux of type } \boldsymbol{a}\} \\ w &\longmapsto T_{\boldsymbol{a}}(w) \end{aligned} \tag{6.1.3}$$

restricts to a bijection

$$\Lambda_k(\boldsymbol{a}) \xleftrightarrow{1-1} \{\textit{admissible } (n-k,k)\text{–}\textit{tableaux of type } \boldsymbol{a}\}. \tag{6.1.4}$$

Proof. Given $w \in \left({}^{\mathbb{S}_n}/_{\mathbb{S}_{\boldsymbol{a}}}\right)^{\text{short}}$, it is enough to observe that the condition (a) is equivalent to $w \in W^{\mathfrak{p}}$ and the condition (b) is equivalent to $w\mathbb{S}_{\boldsymbol{a}} \cap w_{\mathfrak{q}} W^{\mathfrak{q}} \neq \varnothing$. $\square$

The Grothendieck group of $\mathcal{Q}_k(\boldsymbol{a})$

Fix an integer $0 \leq k \leq n$. It follows directly from the Jordan-Hölder Theorem that a basis of the Grothendieck group $K(\mathcal{Q}_k(\boldsymbol{a}))$ as a $\mathbb{Z}[q, q^{-1}]$–module is given by the simple modules $S_{\boldsymbol{a},k}(w)$ for $w \in \Lambda_k(\boldsymbol{a})$. Since $\mathcal{Q}_k(\boldsymbol{a})$ is properly stratified, the matrix which expresses the proper standard modules in the basis given by the simple modules is lower triangular (with respect to the ordering $\prec$), with ones on the diagonal. Hence equivalence classes of the proper standard modules also give a basis. On the other side, the standard modules do not give a basis over $\mathbb{Z}[q, q^{-1}]$ in general (although they always give a basis of $K^{\mathbb{C}(q)}(\mathcal{Q}_k(\boldsymbol{a}))$ over $\mathbb{C}(q)$).

According to Proposition 6.1.1, the set $\Lambda_k(\boldsymbol{a})$ is in bijection with the set of admissible $(n-k, k)$–tableaux of type $\boldsymbol{a}$. For $w \in \Lambda_k(\boldsymbol{a})$ let $v_{(w)} = v^{\boldsymbol{a}}_{\boldsymbol{\eta}} \in \mathsf{V}(\boldsymbol{a})$, where

$$\eta_i = \begin{cases} 0 & \text{if the number } i \text{ appears in the row of } T_{\boldsymbol{a}}(w), \\ 1 & \text{otherwise.} \end{cases} \tag{6.1.5}$$

We write also $v_{(T_{\boldsymbol{a}}(w))} = v_{(w)}$. We can then define an isomorphism

$$\begin{aligned} K^{\mathbb{C}(q)}(\mathcal{Q}_k(\boldsymbol{a})) &\longrightarrow \mathsf{V}(\boldsymbol{a})_k \\ [\overline{\Delta}_{\boldsymbol{a},k}(w)] &\longmapsto \frac{1}{(v_{(w)}, v_{(w)})_{\boldsymbol{a}}} v_{(w)}. \end{aligned} \tag{6.1.6}$$

Notice that if $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ then for $k < n - \ell$ the category $\mathcal{Q}_k(\boldsymbol{a})$ is empty. We set

$$\mathcal{Q}(\boldsymbol{a}) = \bigoplus_{k=n-\ell}^{n} \mathcal{Q}_k(\boldsymbol{a}) \tag{6.1.7}$$

and we get an isomorphism

$$K^{\mathbb{C}(q)}(\mathcal{Q}(\boldsymbol{a})) \cong \mathsf{V}(\boldsymbol{a}). \tag{6.1.8}$$

6.2 Categorification of the intertwining operators

Let $\mathcal{O}\mathsf{Cat}$ be the category whose objects are finite direct sums of the categories $\mathcal{Q}_k(\boldsymbol{a})$ for all $n \geq 0$, $0 \leq k \leq n$ and for all compositions $\boldsymbol{a}$ of n, and whose morphisms are all functors between these categories up to natural isomorphism. We define a functor $\mathscr{F} \colon \mathsf{Web} \to \mathcal{O}\mathsf{Cat}$ as follows. If $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ is an object of Web, then we set

$$\mathscr{F}(\boldsymbol{a}) = \mathcal{Q}(\boldsymbol{a}). \tag{6.2.1}$$

If $\lambda_{\boldsymbol{a}}, \lambda_{\boldsymbol{a}'}$ with $n = \sum a_i = \sum a'_j$ are the fixed dominant weights of $\mathfrak{gl}_n$ with stabilizers $\mathbb{S}_{\boldsymbol{a}}, \mathbb{S}_{\boldsymbol{a}'}$ let us denote $\mathbb{T}^{\boldsymbol{a}'}_{\boldsymbol{a}} = \mathbb{T}^{\lambda_{\boldsymbol{a}'}}_{\lambda_{\boldsymbol{a}}}$. Then we define $\mathscr{F}$ on the elementary webs (3.3.3) and (3.3.4) by

$$\mathscr{F}(\curlywedge_{\boldsymbol{a},i}) = \mathbb{T}^{\hat{\boldsymbol{a}}_i}_{\boldsymbol{a}} \qquad \text{and} \qquad \mathscr{F}(\curlyvee^{\boldsymbol{a},i}) = \mathbb{T}^{\boldsymbol{a}}_{\hat{\boldsymbol{a}}_i} \tag{6.2.2}$$

where $\hat{\boldsymbol{a}}_i$ was defined in (3.3.5).

Lemma 6.2.1. *The assignment* (6.2.2) *defines a functor* $\mathscr{F} \colon \mathsf{Web} \to \mathcal{O}\mathsf{Cat}$*.*

Proof. We need to check that translation functors satisfy isotopy invariance and the relations (3.3.2b-3.3.2e). By Propositions 4.5.4, 4.5.3, 4.5.2 and 4.5.5 respectively, these relations are satisfied by translation functors on ${}^{\mathbb{Z}}\mathcal{O}$. Recall from §5.5 that the translation functors restrict to the subquotient categories $\mathcal{Q}_k(\boldsymbol{a})$. Of course these restricted translation functors also satisfy the relations (3.3.2b-3.3.2e). $\square$

The functor $\mathscr{F}$ categorifies the functor $\mathscr{T}$ (cf. §3.3):

Theorem 6.2.2. *The following diagram commutes:*

(6.2.3)
$$
\begin{tikzcd}
 & \mathcal{O}\mathsf{Cat} \arrow[d, "K^{\mathbb{C}(q)}"] \\
\mathsf{Web} \arrow[ur, "\mathscr{F}"] \arrow[r, "\mathscr{T}"'] & \mathsf{Rep}
\end{tikzcd}
$$

Proof. Let $\boldsymbol{a} = (a_1, \dots, a_\ell)$ and $\boldsymbol{a}' = \hat{\boldsymbol{a}}_i$. We need to show that $K^{\mathbb{C}(q)}(\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'}) = \mathscr{T}(\curlywedge_{\boldsymbol{a},i})$ and $K^{\mathbb{C}(q)}(\mathbb{T}_{\boldsymbol{a}'}^{\boldsymbol{a}}) = \mathscr{T}(\curlyvee_{\boldsymbol{a},i})$. Of course it is sufficient to check this on the basis of proper standard modules. Hence it suffices to check that

$$[\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w)] = \mathscr{T}(\curlywedge_{\boldsymbol{a},i})[\overline{\Delta}_{\boldsymbol{a},k}(w)], \tag{6.2.4}$$
$$[\mathbb{T}_{\boldsymbol{a}'}^{\boldsymbol{a}} \overline{\Delta}_{\boldsymbol{a}',k}(w')] = \mathscr{T}(\curlyvee^{\boldsymbol{a},i})[\overline{\Delta}_{\boldsymbol{a}',k}(w')] \tag{6.2.5}$$

for all $w \in \Lambda_k(\boldsymbol{a})$ and $w' \in \Lambda_k(\boldsymbol{a}')$ (for all possible values of k).

Let us fix k and start with (6.2.4). Fix $w \in \Lambda_k(\boldsymbol{a})$ and write $w = w'x$ with $w' \in \left(\mathbb{S}_n/\mathbb{S}_{\boldsymbol{a}'}\right)^{\mathrm{short}}$, $x \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\mathrm{short}}$ as given by Lemma 5.5.2. By Proposition 5.5.3 we have

$$\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w) = \begin{cases} q^{-\ell(x)} \overline{\Delta}_{\boldsymbol{a}',k}(w') & \text{if } w' \in \Lambda_k(\boldsymbol{a}'), \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.6}$$

In what follows, we only write the i–th and $(i+1)$–th tensor factors of $v_{(w)}$ and the i–th tensor factor of $v_{(w')}$, since the other ones are clearly the same. Let $T_{\boldsymbol{a}}(w)$ be the $(n-k,k)$–tableau of type $\boldsymbol{a}$ corresponding to w, and notice that the tableau $T_{\boldsymbol{a}'}(w)$ can be obtained from $T_{\boldsymbol{a}}(w)$ by decreasing by one all entries greater or equal to $i+1$.

We have four cases (see Figure 6.3):

(a) If $v_{(w)} = v_0^{a_i} \otimes v_0^{a_{i+1}}$ then $T_{\boldsymbol{a}}(w)$ has both an entry i and an entry $i+1$ in the row. Then $T_{\boldsymbol{a}'}(w)$ has two entries i in the row, and is not admissible; of course this also holds for $T_{\boldsymbol{a}'}(w')$ since $w' = wx^{-1}$. Hence $w' \notin \Lambda_k(\boldsymbol{a}')$ and $\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w) = 0$.

(b) If $v_{(w)} = v_0^{a_i} \otimes v_1^{a_{i+1}}$ then $T_{\boldsymbol{a}}(w)$ has an entry i but no entry $i+1$ in the row. It is easy to see that in this case x is a permutation of length a_{i+1} composed with the longest element of $\left(\mathbb{S}_{a_i+a_{i+1}-1}/\mathbb{S}_{a_i-1} \times \mathbb{S}_{a_{i+1}}\right)^{\mathrm{short}}$ and therefore

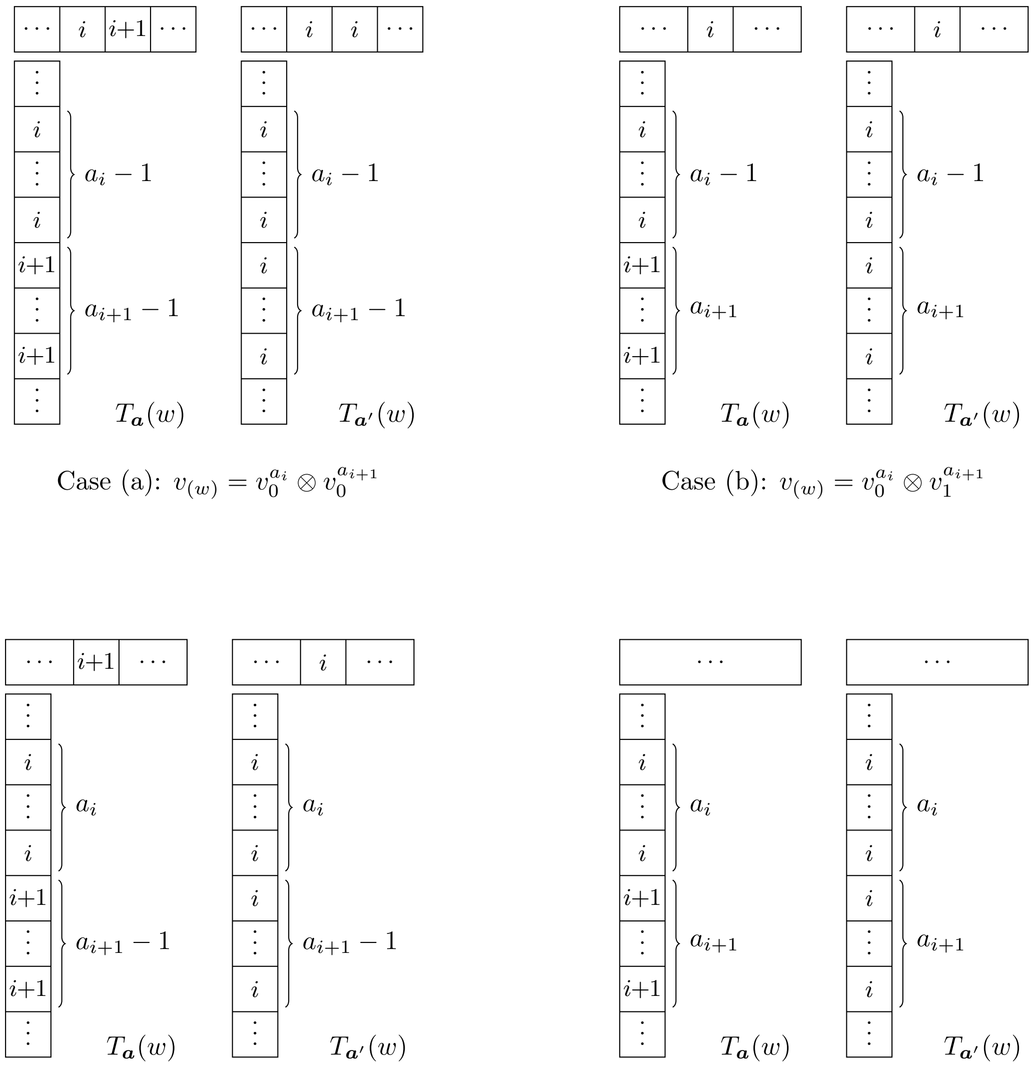
$$\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w) = q^{-a_{i+1}} q^{-(a_i-1)a_{i+1}} \overline{\Delta}_{\boldsymbol{a}',k}(w'). \tag{6.2.7}$$

(c) If $v_{(w)} = v_1^{a_i} \otimes v_0^{a_{i+1}}$ then $T_{\boldsymbol{a}}(w)$ has an entry $i+1$ but no entry i in the row. Then x is the longest element of $\left(\mathbb{S}_{a_i+a_{i+1}-1}/\mathbb{S}_{a_i} \times \mathbb{S}_{a_{i+1}-1}\right)^{\mathrm{short}}$ and therefore

$$\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w) = q^{-a_i(a_{i+1}-1)} \overline{\Delta}_{\boldsymbol{a}',k}(w'). \tag{6.2.8}$$

(d) If $v_{(w)} = v_1^{a_i} \otimes v_1^{a_{i+1}}$ then all entries i and $i+1$ of $T_{\boldsymbol{a}}(w)$ are in the column. Then x is the longest element of $\left(\mathbb{S}_{a_i+a_{i+1}}/\mathbb{S}_{a_i} \times \mathbb{S}_{a_{i+1}}\right)^{\mathrm{short}}$ and hence

$$\mathbb{T}_{\boldsymbol{a}}^{\boldsymbol{a}'} \overline{\Delta}_{\boldsymbol{a},k}(w) = q^{-a_i a_{i+1}} \overline{\Delta}_{\boldsymbol{a}',k}(w'). \tag{6.2.9}$$



Case (a): $v_{(w)} = v_0^{a_i} \otimes v_0^{a_{i+1}}$

Case (b): $v_{(w)} = v_0^{a_i} \otimes v_1^{a_{i+1}}$

Case (c): $v_{(w)} = v_1^{a_i} \otimes v_0^{a_{i+1}}$

Case (d): $v_{(w)} = v_1^{a_i} \otimes v_1^{a_{i+1}}$

Figure 6.3: Here are depicted the tableaux $T_{\boldsymbol{a}}(w)$ and $T_{\boldsymbol{a}'}(w)$ appearing in each of the four cases of the proof of Theorem 6.2.2.

In cases (b) and (c) the tableau $T_{\boldsymbol{a}'}(w')$ has one entry i in the row, hence $v_{(w')} = v_0^{a_i+a_{i+1}}$, while in case (d) the tableau $T_{\boldsymbol{a}'}(w')$ has all entries i in the column and hence $v_{(w')} = v_1^{a_i+a_{i+1}}$. Hence in all four cases we have that (6.2.4) holds up to a multiple, and we only need to verify that the coefficients fit. For example in case (b) comparing with (3.1.3) we must check that

$$q^{-a_{i+1}} q^{-(a_i-1)a_{i+1}} \frac{(v_{(w)}, v_{(w)})_{\boldsymbol{a}}}{(v_{(w')}, v_{(w')})_{\boldsymbol{a}'}} = q^{-a_{i+1}} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}. \tag{6.2.10}$$

Using the formula (3.1.16) for the bilinear form and the notation as in (3.1.10), we compute the l.h.s. of (6.2.10):

$$q^{-a_{i+1}} q^{-(a_i-1)a_{i+1}} \frac{[\beta_1 + \cdots + \beta_\ell]_0!}{[\beta_1]_0! \cdots [\beta_\ell]_0!} \frac{[\beta_1']_0! \cdots [\beta_{\ell-1}']_0!}{[\beta_1' + \cdots + \beta_{\ell-1}']_0!}, \tag{6.2.11}$$

where if $v_{(w)} = v_{\boldsymbol{\eta}}^{\boldsymbol{a}}$ and $v_{(w')} = v_{\boldsymbol{\gamma}}^{\boldsymbol{a}}$ we set $\beta_j = \beta_j^{\boldsymbol{\eta}}$ and $\beta_j' = \beta_j^{\boldsymbol{\gamma}}$. Substituting $\beta_j' = \beta_j$ for $j < i$, $\beta_j' = \beta_{j+1}$ for $j > i$, $\beta_i' = a_i + a_{i+1} - 1$, $\beta_i = a_i - 1$, $\beta_i = a_i$ we get exactly the r.h.s. of (6.2.10). Similarly we can handle cases (c) and (d).

Now let us consider (6.2.5). Let $w' \in \Lambda_k(\boldsymbol{a}')$, and consider the corresponding tableau $T = T_{\boldsymbol{a}'}(w')$. Suppose first that $v_{(w')} = v_{(T)} = v_0^{a_i+a_{i+1}}$: then T has exactly one entry i in the row, and we can apply Lemma 6.2.3 below. Note that the tableaux T'' and T' of Lemma 6.2.3 correspond to $v_0^{a_i} \otimes v_1^{a_{i+1}}$ and $v_1^{a_i} \otimes v_0^{a_{i+1}}$ respectively. Hence we just need to check that the coefficients are the right ones. Let us start with the first term of the r.h.s. of (6.2.18): comparing (6.2.18) with (3.1.4), using the isomorphism defined by (6.1.6), we must show that

$$\begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}_0 \frac{(v_{(T)}, v_{(T)})_{\boldsymbol{a}'}}{(v_{(T'')}, v_{(T'')})_{\boldsymbol{a}}} = 1 \tag{6.2.12}$$

or equivalently

$$\begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}_0 (v_{(T)}, v_{(T)})_{\boldsymbol{a}'} = (v_{(T'')}, v_{(T'')})_{\boldsymbol{a}}. \tag{6.2.13}$$

Using the formula (3.1.16) for the bilinear form and the notation as in (3.1.10), we compute the r.h.s. of (6.2.13):

$$\begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}_0 \begin{bmatrix} \beta_1' + \cdots + \beta_{\ell-1}' \\ \beta_1', \ldots, \beta_{\ell-1}' \end{bmatrix}_0 = \frac{[a_i + a_{i+1} - 1]_0!}{[a_{i+1}]_0! [a_i - 1]_0!} \frac{[\beta_1' + \cdots + \beta_{\ell-1}']_0!}{[\beta_1']_0! \cdots [\beta_{\ell-1}']_0!}, \tag{6.2.14}$$

where as before if $v_{(T)} = v_{\boldsymbol{\eta}}^{\boldsymbol{a}}$ and $v_{(T')} = v_{\boldsymbol{\gamma}}^{\boldsymbol{a}}$ we set $\beta_j' = \beta_j^{\boldsymbol{\eta}}$ and $\beta_j = \beta_j^{\boldsymbol{\gamma}}$. Since $\beta_i' = a_i + a_{i+1} - 1$, $a_{i+1} = \beta_{i+1}$, $a_i - 1 = \beta_i$, $\beta_j' = \beta_j$ for $j < i$ and $\beta_j' = \beta_{j+1}$ for $j > i$ we see that (6.2.14) is equal to

$$\frac{[\beta_1 + \cdots + \beta_\ell]_0!}{[\beta_1]_0! \cdots [\beta_\ell]_0!} \tag{6.2.15}$$

and we are done. Analogously for the second term of the r.h.s. of (6.2.18) we have that

$$\begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix}_0 (v_{(T)}, v_{(T)})_{\boldsymbol{a}'} = (v_{(T')}, v_{(T')})_{\boldsymbol{a}}. \tag{6.2.16}$$

Now suppose instead that $v_{(w')} = v_{(T)} = v_1^{a_i+a_{i+1}}$: then T has all entries i in the column, and we can apply Lemma 6.2.4 below. The tableau T' of Lemma 6.2.4 corresponds to $v_1^{a_i} \otimes v_1^{a_{i+1}}$, and we just need to check that

$$\begin{bmatrix} a_i + a_{i+1} \\ a_i \end{bmatrix}_0 \frac{(v_{(T)}, v_{(T)})_{\boldsymbol{a}'}}{(v_{(T')}, v_{(T')})_{\boldsymbol{a}}} = 1, \tag{6.2.17}$$

that follows as before. □

Lemma 6.2.3. *Let $\boldsymbol{a}, \boldsymbol{a}'$ as in the proof of Theorem 6.2.2. Let T be an admissible tableau of type $\boldsymbol{a}'$ with exactly one entry i in the row. Construct admissible tableaux T', T'' of type $\boldsymbol{a}$ as follows: first increase by 1 all entries of T greater than i; then substitute the first a_{i+1} entries i with $i+1$ (here first means, as always for our hook diagrams, that we first go through the column from the bottom to the top and then through the row from the left to the right). Call the result T'. Moreover, let $T'' = x_0 \cdot T'$ where x_0 is the longest element of $\left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}$. Then we have*

$$[\mathbb{T}_{\boldsymbol{a}'}^{\boldsymbol{a}} \overline{\Delta}(T)] = \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}_0 [\overline{\Delta}(T'')] + q^{a_i} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix}_0 [\overline{\Delta}(T')], \tag{6.2.18}$$

where for an admissible tableau $T_{\boldsymbol{a}}(w)$ we wrote $\overline{\Delta}(T_{\boldsymbol{a}}(w))$ for $\overline{\Delta}(w)$.

Proof. We just need to translate Proposition 5.5.4 into the combinatorics of tableaux. Let $w \in \Lambda_k(\boldsymbol{a}')$ be such that $T = T_{\boldsymbol{a}'}(w)$. Consider the sum on the r.h.s. of (5.5.4). First consider the set $\left\{T_{\boldsymbol{a}}(wy) \mid y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}\right\}$: this consists of all tableaux obtained by permuting the entries i and $i+1$ of T'. Notice now that for all $y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}$ the tableau $T_{\boldsymbol{a}}(x_y wy)$ is obtained from $T_{\boldsymbol{a}}(wx)$ permuting the entries i and $i+1$ in the column so that it becomes admissible; in particular $\ell(x_y) + \ell(w) + \ell(y) = \ell(x_y wy)$ and the set $\left\{T_{\boldsymbol{a}}(x_y wy) \mid y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}\right\}$ consists of the two tableaux T' and T''. Notice also that for each $y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}$ we have $x_y wy = wx'_y y$ for a unique $x'_y \in \mathbb{S}_{\boldsymbol{a}'}$ with $\ell(x'_y) = \ell(x_y)$; in particular $\ell(x'_y) + \ell(y) = \ell(x'_y y)$. Let

$$\begin{aligned} \boldsymbol{b}' &= (a_1, \ldots, a_i + a_{i+1} - 1, 1, a_{i+2}, \ldots, a_\ell), \\ \boldsymbol{b} &= (a_1, \ldots, a_i, a_{i+1} - 1, 1, a_{i+2}, \ldots, a_\ell). \end{aligned} \tag{6.2.19}$$

Then we have $T' = T_{\boldsymbol{a}}(wy'_0)$ and $T'' = T_{\boldsymbol{a}}(wy_0)$ where y'_0 is the longest element of $\left(\mathbb{S}_{\boldsymbol{b}'}/\mathbb{S}_{\boldsymbol{b}}\right)^{\text{short}}$ and y_0 is the longest element of $\left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}}$. Now we can compute the two coefficients of (6.2.18); the second coefficient is

$$\begin{aligned} \sum_{\substack{y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}} \\ x'_y y = y'_0}} q^{\ell(y_0) - \ell(y) + \ell(x'_y)} &= \sum_{\substack{y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}} \\ x'_y y = y'_0}} q^{\ell(y_0) - 2\ell(y) + \ell(y'_0)} \\ &= q^{\ell(y_0) - \ell(y'_0)} \sum_{y \in \left(\mathbb{S}_{\boldsymbol{b}'}/\mathbb{S}_{\boldsymbol{b}}\right)^{\text{short}}} q^{2\ell(y'_0) - 2\ell(y)} = q^{a_i} \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_i \end{bmatrix}_0, \end{aligned} \tag{6.2.20}$$

while the first coefficient is

$$\begin{aligned} \sum_{\substack{y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}} \\ x'_y y = y_0}} q^{\ell(y_0) - \ell(y) + \ell(x'_y)} &= \sum_{\substack{y \in \left(\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}}\right)^{\text{short}} \\ x'_y y = y_0}} q^{2\ell(y_0) - \ell(y)} \\ &= \sum_{z \in \left(\mathbb{S}_{a_i + a_{i+1}}/\mathbb{S}_{a_i - 1} \times \mathbb{S}_{a_{i+1}}\right)^{\text{short}}} q^{2\ell(z_0) - 2\ell(z)} = \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix}_0 \end{aligned} \tag{6.2.21}$$

where we restricted to $\mathbb{S}_{a_i + a_{i+1}}$ (since the permutations act trivially elsewhere) and we substituted $y = zz'$ for $z' = s_{a_i + a_{i+1} - 1} \cdots s_{a_i + 1} s_{a_i}$; the element z_0 is the longest element of $\left(\mathbb{S}_{a_i + a_{i+1}}/\mathbb{S}_{a_i - 1} \times \mathbb{S}_{a_{i+1}}\right)^{\text{short}}$. □

Lemma 6.2.4. *Let $\boldsymbol{a}, \boldsymbol{a}'$ for fixed i as in the proof of Theorem 6.2.2. Let T be an admissible tableau of type $\boldsymbol{a}'$ with all entries equal to i in the column. Construct an admissible tableaux T'*

of type $\boldsymbol{a}$ as follows: first increase of 1 all entries of T greater than i; then substitute the first a_{i+1} entries i with $i+1$ (here first means, as always for our hook diagrams, that we first go through the column from the bottom to the top and then through the row from the left to the right). Then we have

$$[\mathbb{T}_{\boldsymbol{a}'}^{\boldsymbol{a}} \overline{\Delta}(T)] = \begin{bmatrix} a_i + a_{i+1} \\ a_i \end{bmatrix}_0 [\overline{\Delta}(T')], \tag{6.2.22}$$

where for an admissible tableau $T_{\boldsymbol{a}}(w)$ we wrote $\overline{\Delta}(T_{\boldsymbol{a}}(w))$ for $\overline{\Delta}(w)$.

Proof. The proof is similar to the previous one, but easier. We just need to compute

$$\sum_{y \in (\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}})^{\text{short}}} q^{\ell(y_0)-\ell(y)+\ell(x_y)} = \sum_{x \in (\mathbb{S}_{\boldsymbol{a}'}/\mathbb{S}_{\boldsymbol{a}})^{\text{short}}} q^{2\ell(y_0)-2\ell(y)} = \begin{bmatrix} a_i + a_{i+1} \\ a_i \end{bmatrix}_0. \tag{6.2.23}$$ □

Let us consider in particular the regular composition $\mathbb{n}$ of n. Let $\boldsymbol{a} = \mathbb{n}$ and for every $i = 1, \dots, n-1$ consider $\hat{\boldsymbol{a}}_i$ as defined in (3.3.5). Define $\theta_i = \mathbb{T}_{\hat{\boldsymbol{a}}_i}^{\boldsymbol{a}} \circ \mathbb{T}_{\boldsymbol{a}}^{\hat{\boldsymbol{a}}_i}$ as a functor $\theta_i \colon \mathcal{Q}(\mathbb{n}) \to \mathcal{Q}(\mathbb{n})$ (this is just the functor θ_i defined in §4.5 restricted to $\mathcal{Q}(\mathbb{n})$). As a consequence of Theorem 6.2.2 we have:

Corollary 6.2.5. *The endofunctors θ_i on $\mathcal{Q}(\mathbb{n})$ categorify (i.e. give, at the level of the Grothendieck group) the action of the Super Temperley-Lieb Algebra* STL_n *(see Definition 3.2.3).*

It follows by Lemma 6.2.1 that the functors θ_i satisfy the relations

$$\theta_i^2 \cong \theta_i\langle 1\rangle \oplus \theta_i\langle -1\rangle, \tag{6.2.24a}$$
$$\theta_i\theta_j \cong \theta_j\theta_i, \qquad \text{for } |i-j|>1 \tag{6.2.24b}$$
$$\theta_i\theta_{i+1}\theta_i \oplus \theta_{i+1} \cong \theta_{i+1}\theta_i\theta_{i+1} \oplus \theta_i. \tag{6.2.24c}$$

In fact, these relations are the categorical versions of the relations of the Hecke algebra (3.2.9a-3.2.9c) and are satisfied by the endofunctors θ_i of ${}^{\mathbb{Z}}\mathcal{O}$ (cf. §4.5). By Corollary 6.2.5, the relations (3.2.9d-3.2.9e) are satisfied in the Grothendieck group. We conjecture that their categorical versions are satisfied by the functors θ_i:

Conjecture 6.2.6. *The functors θ_i on $\mathcal{Q}(\mathbb{n})$ satisfy the relations*

$$\begin{aligned} &\theta_{i-1}\theta_{i+1}\theta_i\theta_{i-1}\theta_{i+1} \oplus [2]^2\theta_{i-1}\theta_{i+1}\theta_i \\ &\qquad \cong [2](\theta_{i-1}\theta_{i+1}\theta_i\theta_{i-1} \oplus \theta_{i-1}\theta_{i+1}\theta_i\theta_{i+1}), \end{aligned} \tag{6.2.24d}$$

$$\begin{aligned} &\theta_{i-1}\theta_{i+1}\theta_i\theta_{i-1}\theta_{i+1} \oplus [2]^2\theta_i\theta_{i-1}\theta_{i+1} \\ &\qquad \cong [2](\theta_{i-1}\theta_i\theta_{i-1}\theta_{i+1} \oplus \theta_{i+1}\theta_i\theta_{i-1}\theta_{i+1}) \end{aligned} \tag{6.2.24e}$$

for all $i = 2, \dots, n-2$, where we used the abbreviations $[2]\theta_i = \theta_i\langle 1\rangle \oplus \theta_i\langle -1\rangle$ and $[2]^2\theta_i = \theta_i\langle 2\rangle \oplus \theta_i \oplus \theta_i \oplus \theta_i\langle -2\rangle$.

Although apparently harmful, we believe Conjecture 6.2.6 to be quite hard. The difficulty is due to the lack of a classification of projective functors on the parabolic category $\mathcal{O}^{\mathfrak{p}}$ if $\mathfrak{p}$ is not the Borel subalgebra $\mathfrak{b}$.

6.3 Categorification of the canonical basis

Now we give a categorical interpretation of the canonical basis of $\mathsf{V}(\boldsymbol{a})$. First we restrict ourselves to consider the regular composition $\mathbb{n}$. Recall that by Proposition 3.2.5 the canonical basis of $(V^{\otimes n})_k$ can be interpreted as a canonical basis for the Hecke algebra action. In this section we will use the Hecke module structure of the Grothendieck groups of our categories.

Let $\mathfrak{p}, \mathfrak{q} \subset \mathfrak{gl}_n$ be the parabolic subalgebras defined at the beginning of the chapter, such that $\mathcal{Q}_k(\mathbb{n}) = {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Using the notation introduced in Chapter 2, we fix isomorphisms

$$
\begin{aligned}
K^{\mathbb{C}(q)}\big({}^{\mathbb{Z}}\mathcal{O}_\lambda\big) &\to \mathcal{H}_n & K^{\mathbb{C}(q)}\big({}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}\big) &\to \mathcal{M}^{\mathfrak{p}}\\
[M(w\cdot\lambda)] &\mapsto H_w & [M^{\mathfrak{p}}(w\cdot\lambda)] &\mapsto N_w.
\end{aligned}
\tag{6.3.1}
$$

As well-known, by the Kazhdan-Lusztig conjecture projective modules are sent to the canonical basis elements of $\mathcal{H}_n$ and $\mathcal{M}^{\mathfrak{p}}$ by the two isomorphisms.

Composing the isomorphism (6.1.6) with the isomorphism (3.2.13) we get an isomorphism

$$
\begin{aligned}
K_0\big({}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}\big) &\to \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}\\
[\Delta(w_{\mathfrak{q}}w\cdot\lambda)] &\mapsto N_w
\end{aligned}
\tag{6.3.2}
$$

for $w \in W^{\mathfrak{p}+\mathfrak{q}}$, where $w_{\mathfrak{q}} \in W_{\mathfrak{q}}$ is the longest element.

Lemma 6.3.1. *The coapproximation functor $\mathfrak{Q}\colon {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}} \to {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ categorifies the map $\mathsf{Q}\colon \mathcal{M}^{\mathfrak{p}} \to \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$ (defined in §2.2).*

Proof. Let $w \in \Lambda^{\mathfrak{p}}(\mathbb{n}) = W^{\mathfrak{p}}$. By Proposition 5.4.2 we have $\mathfrak{Q}M^{\mathfrak{p}}(w\cdot 0) = q^{\ell(x)}\overline{\Delta}(xw\cdot 0)$ where $x \in W_{\mathfrak{q}}$ is given by Lemma 5.4.1. Now $[M^{\mathfrak{p}}(w\cdot 0)] = N_w \in \mathcal{M}^{\mathfrak{p}}$ and $[\overline{\Delta}(xw\cdot 0)] = \frac{1}{[k]_0!}N_{w_{\mathfrak{q}}xw} \in \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$. On the other side, by definition $\mathsf{Q}N_w = c_{\mathfrak{q}}^{-1}q^{-\ell(w_{\mathfrak{q}})+\ell(x)}N_{w_{\mathfrak{q}}xw}$. The claim follows since

$$
c_{\mathfrak{q}}^{-1}q^{-\ell(w_{\mathfrak{q}})+\ell(x)} = \frac{1}{[k]!q^{\ell(w_{\mathfrak{q}})}}q^{\ell(x)} = \frac{1}{[k]_0!}q^{\ell(x)}. \tag{6.3.3}
$$

□

Lemma 6.3.2. *Under the isomorphism* (6.1.6) *we have $[Q(w_{\mathfrak{q}}w)] \mapsto \underline{N}_w$ for all $w \in W^{\mathfrak{p}+\mathfrak{q}}$.*

Proof. By Lemma 2.2.3 and the discussion after it, it follows that Q sends the canonical basis element $\underline{N}_{w_{\mathfrak{q}}w} \in \mathcal{M}^{\mathfrak{p}}$ to $\underline{N}_w \in \mathcal{M}^{\mathfrak{p}}_{\mathfrak{q}}$. By Lemma 6.3.1 we have

$$
[Q(w_{\mathfrak{q}}w)] = [\mathfrak{Q}P^{\mathfrak{p}}(w_{\mathfrak{q}}w\cdot 0)] = \mathsf{Q}[P^{\mathfrak{p}}(w_{\mathfrak{q}}w\cdot 0)] = \mathsf{Q}\underline{N}_{w_{\mathfrak{q}}w} = \underline{N}_w. \tag{6.3.4}
$$

□

Now let us consider a general composition $\boldsymbol{a}$.

Proposition 6.3.3. *Under the isomorphism* (6.1.6) *the class of the indecomposable projective module $Q(w)$ maps to the canonical basis element $v^{\diamond}_{(w)} \in \mathsf{V}(\boldsymbol{a})$ corresponding to the standard basis element $v_{(w)}$.*

Proof. By Lemma 6.3.2 we know the result for the regular composition $\mathbb{n}$. Consider the standard inclusion $\mathsf{V}(\boldsymbol{a}) \to V^{\otimes n}$ given by the web diagram $\varphi = \Upsilon_{a_1} \otimes \cdots \otimes \Upsilon_{a_\ell}$, see (3.3.9). We know that $\mathscr{F}(\varphi)\colon \mathcal{Q}_k(\boldsymbol{a}) \to \mathcal{Q}_k(\mathbb{n})$, that categorifies φ, sends indecomposable projective modules to indecomposable projective modules (Proposition 5.5.6). On the other side, it follows immediately from our diagrammatic calculus that what φ sends to a canonical basis element is a canonical basis element (cf. Remark 3.3.8). □

6.4 Categorification of the bilinear form

We give now a categorical interpretation of the bilinear form (3.1.9). Given a $\mathbb{Z}$–graded complex vector space $M = \bigoplus_{i\in\mathbb{Z}} M^i$, let $h(M) = \sum_{i\in\mathbb{Z}}(\dim_{\mathbb{C}} M_i)q^i \in \mathbb{Z}[q,q^{-1}]$ be its graded dimension. Now let M, N be objects of $\mathcal{Q}_k(\boldsymbol{a})$. Set

$$h(\operatorname{Ext}(M,N)) = \sum_{j\in\mathbb{Z}}(-1)^j h(\operatorname{Ext}^j(M,N)). \tag{6.4.1}$$

Let also $^{-}$ be the involution of $\mathbb{Z}[q,q^{-1}]$ given by $\overline{q} = q^{-1}$.

Fix now a composition $\boldsymbol{a} = (a_1,\dots,a_\ell)$ of n and an integer $n-\ell \le k \le n$, and consider the category $\mathcal{Q}_k(\boldsymbol{a})$.

Proposition 6.4.1. *For $M, N \in \mathcal{Q}_k(\boldsymbol{a})$ we have*

$$\overline{h(\operatorname{Ext}(M,N^*))} = ([M],[N])_{\boldsymbol{a}}. \tag{6.4.2}$$

Proof. First, note that the l.h.s. of (6.4.2) defines a bilinear form on the Grothendieck group. Hence we only need to prove that the two sides coincide on a basis.

By the properties of properly stratified algebras (cf. [Fri07, Lemma 4]) we have

$$\operatorname{Ext}^i\left(\Delta(z),(\overline{\Delta}(w))^*\right) = \begin{cases}\mathbb{C} & \text{if } z = w \text{ and } i = 0,\\ 0 & \text{otherwise.}\end{cases} \tag{6.4.3}$$

Hence we are left to prove that

$$\frac{([\Delta(z)],v_{(w)})_{\boldsymbol{a}}}{(v_{(w)},v_{(w)})_{\boldsymbol{a}}} = \delta_{z,w} \qquad \text{for all } w,z \in \Lambda_k(\boldsymbol{a}) \tag{6.4.4}$$

or equivalently that

$$[\Delta(z)] = v_{(z)} = (v_{(z)},v_{(z)})_{\boldsymbol{a}}[\overline{\Delta}(z)] \qquad \text{for all } z \in \Lambda_k(\boldsymbol{a}). \tag{6.4.5}$$

By the properties of a properly stratified algebra, it suffices for that to prove that the proper standard module $\overline{\Delta}(z)$ appears $(v_{(z)},v_{(z)})$–times in some proper standard filtration of the indecomposable projective $P(z)$. Since by (6.1.6) and by Proposition 6.3.3 we know which basis the proper standard and the indecomposable projective modules categorify, this follows. $\square$

By Proposition 6.4.1, and since

$$\operatorname{Ext}^i(\Delta(z),(\overline{\Delta}(w))^*) = \begin{cases}\mathbb{C} & \text{if } z = w \text{ and } i = 0,\\ 0 & \text{otherwise,}\end{cases} \tag{6.4.6}$$

$$\operatorname{Ext}^i(Q(z),(S(w))^*) = \begin{cases}\mathbb{C} & \text{if } z = w \text{ and } i = 0,\\ 0 & \text{otherwise,}\end{cases} \tag{6.4.7}$$

we have:

Theorem 6.4.2. *Under the isomorphism* (6.1.6) *we have the following correspondences:*

$$\begin{aligned} \{\textit{standard modules}\} &\longleftrightarrow \textit{standard basis,}\\ \{\textit{proper standard modules}\} &\longleftrightarrow \textit{dual standard basis,}\\ \{\textit{indecomposable projective modules}\} &\longleftrightarrow \textit{canonical basis,}\\ \{\textit{simple modules}\} &\longleftrightarrow \textit{dual canonical basis.}\end{aligned}$$

Note than an analog of this theorem for tensor products of representations of $U_q(\mathfrak{g})$ was established in [FKS06] for $\mathfrak{g} = \mathfrak{sl}_2$ and in [Web13] for the a general semisimple Lie algebra $\mathfrak{g}$.

We conclude with an example of how the bilinear form can be used to compute combinatorially dimensions of homomorphism spaces.

Lemma 6.4.3. *Let $w, z \in \Lambda_k(\mathbb{n})$. Then the dimension of $\mathrm{Hom}(Q_{\mathbb{n},k}(w), Q_{\mathbb{n},k}(z))$ is $k!$ times the number of elements $x \in \Lambda_k(\mathbb{n})$ such that both the canonical basis diagrams $C(v^\diamondsuit_{(w)})$ and $C(v^\diamondsuit_{(z)})$ have nonzero value when labeled with the standard basis diagram $v_{(x)}$ (the evaluation is computed according to the rules in Figure 3.1).*

Proof. Since the modules $Q(w)$ and $Q(z)$ are projective, we can compute the dimension of $\mathrm{Hom}(Q(w), Q(z))$ using Proposition 6.4.1:

$$\dim_{\mathbb{C}} \mathrm{Hom}(Q(w), Q(z)) = ([Q(w)], [Q(z)^*])^{q=1}_{\mathbb{n}}, \tag{6.4.8}$$

where $(\cdot,\cdot)^{q=1}_{\mathbb{n}}$ is the form $(\cdot,\cdot)_{\mathbb{n}}$ evaluated at $q = 1$. By the orthogonality of the standard basis elements $v_{(w)}$ for $w \in \Lambda_k(\mathbb{n})$ we can write

$$([Q(w)], [Q(z)^*])_{\mathbb{n}} = \frac{1}{[k]!} \sum_{x \in \Lambda_k(\mathbb{n})} \big([Q(w)], v_{(x)}\big)_{\mathbb{n}} \big(v_{(x)}, [Q(z)^*]\big)_{\mathbb{n}}. \tag{6.4.9}$$

Since $[Q(z)^*]$ coincides with $[Q(z)]$ after substituting q with q^{-1} in the Grothendieck group, we can also write

$$\begin{aligned} ([Q(w)], [Q(z)^*])^{q=1}_{\mathbb{n}} &= \frac{1}{[k]!} \sum_{x \in \Lambda_k(\mathbb{n})} \big([Q(w)], v_{(x)}\big)^{q=1}_{\mathbb{n}} \big(v_{(x)}, [Q(z)]\big)^{q=1}_{\mathbb{n}} \\ &= \frac{1}{[k]!} \sum_{x \in \Lambda_k(\mathbb{n})} \Big(v^\diamondsuit_{(w)}, v_{(x)}\Big)^{q=1}_{\mathbb{n}} \Big(v^\diamondsuit_{(z)}, v_{(x)}\Big)^{q=1}_{\mathbb{n}}. \end{aligned} \tag{6.4.10}$$

Let $C(v_{(w)})$, $C(v_z)$ be the canonical basis diagrams corresponding to $v_{(w)}$ and $v_{(z)}$ respectively. By the definition of the bilinear form, $\big(v^\diamondsuit_{(w)}, v_{(x)}\big)_{\mathbb{n}}$ is equal to $[k]!$ times the evaluation of the diagram $\mathcal{D}_x$ obtained by labeling the canonical basis diagram $C(v_{(w)})$ with $\wedge$'s and $\vee$'s according to the standard basis diagram of $v_{(x)}$. If one analyzes the evaluation rules (Figure 3.1), one sees immediately that the evaluation of $\mathcal{D}_x$ is a monomial in q if the corresponding diagram $C(v_{(w)})$ labeled by x is oriented, and zero otherwise. Hence the claim follows. □

6.5 Categorification of the action of $U_q(\mathfrak{gl}(1|1))$

We want now to define functors that categorify the action of U_q. As happens in the case of $\mathfrak{sl}_2$, we are not able to categorify both the action of the intertwiners and the action of U_q via exact functors; hence we will need to consider the derived categories.

Functors $\mathcal{E}$ and $\mathcal{F}$

Fix an integer n, a composition $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ of n and an integer $n - \ell \leq k < n$. Let $\lambda = \lambda_{\boldsymbol{a}}$, and let $\mathfrak{p}, \mathfrak{q}, \mathfrak{p}', \mathfrak{q}'$ be the parabolic subalgebras of $\mathfrak{gl}_n$ such that $\mathcal{Q}_k(\boldsymbol{a}) = {}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p},\mathfrak{q}\text{-pres}}_\lambda$

and $\mathcal{Q}_{k+1}(\boldsymbol{a}) = {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}'\text{-pres}}$. Notice that $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{q} \subseteq \mathfrak{q}'$. We have a diagram

(6.5.1)
$$\begin{tikzcd} & {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}} \arrow[dl, bend left, "\mathfrak{z}"] \arrow[dr, bend left, "\mathfrak{Q}"] & \\ \mathcal{Q}_k(\boldsymbol{a}) \arrow[ur, bend left, "\mathfrak{j}"] & & \mathcal{Q}_{k+1}(\boldsymbol{a}) \arrow[ul, bend left, "\mathfrak{i}"] \end{tikzcd}$$

Let us define $\mathcal{E}_k = \mathfrak{Q} \circ \mathfrak{j}$ and $\mathcal{F}_k = \mathfrak{z} \circ \mathfrak{i}$. We get then a pair of adjoint functors $\mathcal{F}_k \dashv \mathcal{E}_k$:

(6.5.2)
$$\begin{tikzcd} \mathcal{Q}_k(\boldsymbol{a}) \arrow[r, bend left, "\mathcal{E}_k"] & \mathcal{Q}_{k+1}(\boldsymbol{a}) \arrow[l, bend left, "\mathcal{F}_k"] \end{tikzcd}$$

We remark that the functors $\mathcal{E}_k, \mathcal{F}_k$ commute with translation functors by Lemma 5.5.1.

We can compute explicitly the action of $\mathcal{F}_k$ on projective modules and of $\mathcal{E}_k$ on simple modules:

Proposition 6.5.1. *For $w \in \Lambda_{k+1}(\boldsymbol{a})$ we have*

(6.5.3)
$$\mathcal{F}_k Q_{\boldsymbol{a},k+1}(w) = \begin{cases} Q_{\boldsymbol{a},k}(w) & \text{if } w \in \Lambda_k(\boldsymbol{a}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the diagram (6.5.1). Of course $\Lambda_k(\boldsymbol{a}) = \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda) \subseteq \Lambda_{\mathfrak{q}}^{\mathfrak{p}'}(\lambda)$, and we have $\mathfrak{i}Q_{\boldsymbol{a},k+1}(w) = P^{\mathfrak{p}'}(w \cdot \lambda) \in {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$. By the definition of the Zuckermann's functor we have then $\mathfrak{z}P^{\mathfrak{p}'}(w \cdot \lambda) = P^{\mathfrak{p}}(w \cdot \lambda) = Q_{\boldsymbol{a},k}(w) \in \mathcal{Q}_k(\boldsymbol{a})$ if $w \in \Lambda_{\mathfrak{q}}^{\mathfrak{p}}(\lambda)$, or 0 otherwise. □

Proposition 6.5.2. *For $w \in \Lambda_k(\boldsymbol{a})$ we have*

(6.5.4)
$$\mathcal{E}_k S_{\boldsymbol{a},k}(w) = \begin{cases} S_{\boldsymbol{a},k+1}(w) & \text{if } w \in \Lambda_{k+1}(\boldsymbol{a}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the diagram (6.5.1). By Lemma 5.3.5, the simple objects of $\mathcal{Q}_k(\boldsymbol{a})$ are the simple objects $S(w \cdot \lambda)$ of ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$ such that $w \in \Lambda_k(\boldsymbol{a})$. In particular, $\mathfrak{j}S_{\boldsymbol{a},k}(w) = S(w \cdot \lambda)$ for each $w \in \Lambda_k(\boldsymbol{a})$. Let $\mathfrak{Q}_{\mathfrak{q}'} : {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}'} \to {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}'\text{-pres}}$ and $\mathfrak{Q}_{\mathfrak{q}} : {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}'} \to {}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$ be the corresponding coapproximation functors. As we already noticed, it follows from the definition that $\mathfrak{Q}_{\mathfrak{q}'} = \mathfrak{Q} \circ \mathfrak{Q}_{\mathfrak{q}}$. Since $S(w \cdot \lambda) = \mathfrak{Q}_{\mathfrak{q}} L(w \cdot \lambda)$, we have $\mathfrak{Q}S(w \cdot \lambda) = \mathfrak{Q}_{\mathfrak{q}'} L(w \cdot \lambda)$. This is $S_{\boldsymbol{a},k+1}(w) \in \mathcal{Q}_{k+1}(\boldsymbol{a})$ if $w \in \Lambda_{k+1}(\boldsymbol{a})$, or 0 otherwise. □

Unbounded derived categories

Being the composition of exact functors, the functor $\mathcal{E}_k$ is exact. On the other side, being the composition of right-exact functors, $\mathcal{F}_k$ is right exact, but not exact in general. Therefore, $\mathcal{F}_k$ does not induce a map between the Grothendieck groups, unless we pass to the derived categories. Unfortunately, properly stratified algebras do not have, in general, finite global dimension (this happens if and only if they are quasi-hereditary). Hence, we shall consider unbounded derived categories. The main problem with unbounded derived categories is that their Grothendieck group is trivial (see [Miy06]). A workaround to this problem has been developed by Achar and Stroppel in [AS13]. We recall briefly their main definitions and results, adapted to our setting.

Consider a finite-dimensional positively graded $\mathbb{C}$–algebra $A = \bigoplus_{i \leq 0} A_i$ with semisimple A_0, and let $\mathcal{A} = A-\mathrm{gmod}$. Each simple object of $\mathcal{A}$ is concentrated in one degree. Achar and Stroppel define a full subcategory $\mathcal{D}^\nabla \mathcal{A}$ of the unbounded derived category $\mathcal{D}^- \mathcal{A}$ by

$$\mathcal{D}^\nabla \mathcal{A} = \left\{ X \in \mathcal{D}^- \mathcal{A} \;\middle|\; \begin{array}{c} \text{for each } m \in \mathbb{Z} \text{ only finitely many of the } H^i(X) \\ \text{contain a composition factor of degree} < m \end{array} \right\}. \tag{6.5.5}$$

Recall that the Grothendieck group $K(\mathcal{T})$ of a small triangulated category $\mathcal{T}$ is defined to be the free abelian group on isomorphism classes $[X]$ for $X \in \mathcal{T}$ modulo the relation $[B] = [A] + [C]$ whenever there is a distinguished triangle of the form $A \to B \to C \to A[1]$. As for abelian categories, if $\mathcal{T}$ is graded then $K(\mathcal{T})$ is naturally a $\mathbb{Z}[q, q^{-1}]$–module. Let

$$I = \{x \in \mathcal{D}^\nabla(\mathcal{A}) \mid [\beta_{\leq m}]x = 0 \text{ in } K(\mathcal{D}^\nabla(\mathcal{A})) \text{ for all } m \in \mathbb{Z}\}, \tag{6.5.6}$$

where $\beta_{\leq m} \colon D^\nabla \mathcal{A} \to D^\nabla \mathcal{A}$ is induced by the exact functor $\beta_{\leq m} \colon \mathcal{A} \to \mathcal{A}$ defined on the graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ by $\beta_{\leq m} M = \bigoplus_{i \leq m} M_i$. Then $\mathbf{K}(\mathcal{D}^\nabla \mathcal{A}) = K(\mathcal{D}^\nabla \mathcal{A})/I$ is the *topological Grothendieck group* of $\mathcal{D}^\nabla \mathcal{A}$. The names is motivated by the fact that one can define on $\mathbf{K}(\mathcal{D}^\nabla \mathcal{A})$ a (q)–adic topology with respect to which $\mathbf{K}(\mathcal{D}^\nabla \mathcal{A})$ is complete. It follows that $\mathbf{K}(\mathcal{D}^\nabla \mathcal{A})$ is a $Z[[q]][q^{-1}]$–module.

On the other side, let $\hat{K}(\mathcal{A})$ be the completion of the $\mathbb{Z}[q, q^{-1}]$–module $K(\mathcal{A})$ with respect to the (q)–adic topology ([AS13, §2.3]). Then the natural map $K(\mathcal{A}) \to \mathbf{K}(\mathcal{D}^\nabla \mathcal{A})$ is injective and induces an isomorphism of $\mathbb{Z}[[q]][q^{-1}]$–modules

$$\hat{K}(\mathcal{A}) \cong \mathbf{K}(\mathcal{D}^\nabla \mathcal{A}). \tag{6.5.7}$$

Moreover, if $\{L_i \mid i \in I\}$, with I finite, is a full set of pairwise non-isomorphic simple objects of $\mathcal{A}$ concentrated in degree 0 and P_i is the projective cover of L_i, then both $\{L_i \mid i \in I\}$ and $\{P_i \mid i \in I\}$ give a $\mathbb{Z}[[q]][q^{-1}]$–basis for $\hat{K}(\mathcal{A})$. In particular $\hat{K}(\mathcal{A}) \cong \mathbb{Z}[[q]][q^{-1}] \otimes_{\mathbb{Z}[q,q^{-1}]} K(\mathcal{A})$.

In our setting, we have for each category ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ naturally

$$\begin{aligned} K^{\mathbb{C}(q)}({}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}) &\cong \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K({}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}) \\ &\cong \mathbb{C}(q) \otimes_{\mathbb{Z}[[q]][q^{-1}]} \hat{K}({}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p},\mathfrak{q}\text{-pres}}). \end{aligned} \tag{6.5.8}$$

In particular, the same holds for $\mathcal{Q}_k(\boldsymbol{a})$. We define also

$$\mathbf{K}^{\mathbb{C}(q)}(\mathcal{D}^\nabla \mathcal{A}) = \mathbb{C}(q) \otimes_{\mathbb{Z}[[q]][q^{-1}]} \mathbf{K}(\mathcal{D}^\nabla \mathcal{A}). \tag{6.5.9}$$

Let $\mathcal{A}_{\geq m}$ be the full subcategory of $\mathcal{A}$ consisting of objects $M = \bigoplus_{i \geq m} M_i$. An additive functor $G \colon \mathcal{A} \to \mathcal{A}'$ is said to be of *finite degree amplitude* if there exists some $\alpha \geq 0$ such that $G(\mathcal{A}_{\geq m}) \subset \mathcal{A}'_{\geq m-\alpha}$ for all $m \in \mathbb{Z}$. Let $G \colon \mathcal{A} \to \mathcal{A}'$ be a right-exact functor that commutes with the degree shift. If G has finite degree amplitude, then the left-derived functor $\mathbb{L}G$ induces a continuous homomorphism of $\mathbb{Z}[[q]][q^{-1}]$–modules $[\mathbb{L}G] \colon \hat{K}(\mathcal{A}) \to \hat{K}(\mathcal{A}')$.

Derived functors $\mathcal{E}$ and $\mathcal{F}$

Let us now go back to our functors $\mathcal{E}_k$ and $\mathcal{F}_k$. Being exact, $\mathcal{E}_k$ induces a functor $\mathcal{E}_k \colon \mathcal{D}^\nabla(\mathcal{Q}_k(\boldsymbol{a})) \to \mathcal{D}^\nabla(\mathcal{Q}_{k+1}(\boldsymbol{a}))$. On the other side, it is immediate to check that the functors $\mathfrak{i}$ and $\mathfrak{z}$, and therefore also $\mathcal{F}_k$, have finite degree amplitude. Hence $\mathbb{L}\mathcal{F}_k$ restricts to a functor $\mathbb{L}\mathcal{F}_k \colon \mathcal{D}^\nabla(\mathcal{Q}_{k+1}(\boldsymbol{a})) \to \mathcal{D}^\nabla(\mathcal{Q}_k(\boldsymbol{a}))$. Since $\mathcal{E}_k$ is exact, it follows by standard arguments that we have a pair of adjoint functors $\mathbb{L}\mathcal{F}_k \dashv E_k$:

$$\begin{tikzcd} \mathcal{D}^\nabla \mathcal{Q}_k(\boldsymbol{a}) \arrow[r, bend left, "\mathcal{E}_k"] & \mathcal{D}^\nabla \mathcal{Q}_{k+1}(\boldsymbol{a}) \arrow[l, bend left, "\mathbb{L}\mathcal{F}_k"] \end{tikzcd} \tag{6.5.10}$$

Remark 6.5.3. Since $\mathfrak{i}$ sends projective modules to projective modules, it follows from [Wei94, Corollary 10.8.3] that $\mathbb{L}\mathcal{F}_k = \mathbb{L}\mathfrak{z} \circ \mathbb{L}\mathfrak{i}$.

Theorem 6.5.4. *The functors $\mathbb{L}\mathcal{F}_k$ and $\mathcal{E}_k$ categorify F and E' respectively, that is, the following diagrams commute:*

$$\begin{tikzcd} \mathcal{D}^\nabla \mathcal{Q}_k(\boldsymbol{a}) \arrow[d, "\mathbf{K}^{\mathbb{C}(q)}"] & \mathcal{D}^\nabla \mathcal{Q}_{k+1}(\boldsymbol{a}) \arrow[l, "\mathbb{L}\mathcal{F}_k"'] \arrow[d, "\mathbf{K}^{\mathbb{C}(q)}"] \\ \mathsf{V}(\boldsymbol{a})_k & \mathsf{V}(\boldsymbol{a})_{k+1} \arrow[l, "F"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}^\nabla \mathcal{Q}_k(\boldsymbol{a}) \arrow[r, "\mathcal{E}_k"] \arrow[d, "\mathbf{K}^{\mathbb{C}(q)}"] & \mathcal{D}^\nabla \mathcal{Q}_{k+1}(\boldsymbol{a}) \arrow[d, "\mathbf{K}^{\mathbb{C}(q)}"] \\ \mathsf{V}(\boldsymbol{a})_k \arrow[r, "E'"] & \mathsf{V}(\boldsymbol{a})_{k+1} \end{tikzcd}$$

Proof. We use Proposition 6.5.1 to check that the first diagram commutes on the basis given by indecomposable projective modules. Let $w \in \Lambda_{k+1}(\boldsymbol{a})$ and write $v_{(w)} = v_{\boldsymbol{\eta}}^{\boldsymbol{a}}$. Then in $\mathbf{K}^{\mathbb{C}(q)}(\mathcal{D}^\nabla \mathcal{Q}_{k+1}(\boldsymbol{a}))$ we have $[Q_{\boldsymbol{a},k+1}(w \cdot \lambda)] = v_{\boldsymbol{\eta}}^{\diamondsuit \boldsymbol{a}}$. Now w is in $\Lambda_k(\boldsymbol{a})$ if and only if it is a shortest coset representative for ${}_{W_\mathfrak{p}}\backslash \mathbb{S}_n$. Let $T_{\boldsymbol{a}}^{k+1}(w)$ (respectively, $T_{\boldsymbol{a}}^k(w)$) be the $(n-k-1, k+1)$–tableau (respectively, $(n-k,k)$–tableau) of type $\boldsymbol{a}$ corresponding to w. Obviously $T_{\boldsymbol{a}}^k(w)$ can be obtained from $T_{\boldsymbol{a}}^{k+1}(w)$ by removing the upper box b of the column and adding it to the row in the leftmost position. Clearly $T_{\boldsymbol{a}}^k(w)$ is admissible if and only if the entry of this box b is 1. Hence $w \in \Lambda_k(\boldsymbol{a})$ if and only if $\eta_1 = 1$, and in this case we have $[Q_{\boldsymbol{a},k}(w)] = v_0^{a_1} \diamondsuit v_{\eta_2}^{a_2} \diamondsuit \cdots \diamondsuit v_{\eta\ell}^{a_\ell}$ in $\mathbf{K}^{\mathbb{C}(q)}(\mathcal{D}^\nabla \mathcal{Q}_k(\boldsymbol{a}))$. By Proposition 3.3.9, this is the action of F.

Since $\mathcal{E}_k$ is the adjoint functor of $\mathbb{L}\mathcal{F}_k$, the commutativity of the second diagram follows from the adjunction (3.1.20) and Proposition 6.4.1 (of course we could also argue as for $\mathbb{L}\mathcal{F}_k$ and check directly the commutativity of the second diagram above using Proposition 6.5.2). $\square$

We define $\mathcal{E} = \bigoplus_{k=n-\ell}^{n-1} \mathcal{E}_k$ and $\mathcal{F} = \bigoplus_{k=n-\ell}^{n-1} \mathcal{F}_k$ as endofunctors of $\mathcal{Q}(\boldsymbol{a})$. We have the following categorical version of the relation $E^2 = F^2 = 0$:

Proposition 6.5.5. *The functors $\mathcal{E}$ and $\mathcal{F}$ satisfy $\mathcal{E} \circ \mathcal{E} = \mathcal{F} \circ \mathcal{F} = 0$.*

Proof. Let $S \in \mathcal{Q}(\boldsymbol{a})$ be a simple module. It follows from Proposition 6.5.2 that $\mathcal{E}^2 S = 0$. Since $\mathcal{E}$ is exact, this implies that $\mathcal{E} = 0$.

On the other side, it follows from 6.5.1 that $\mathcal{F}^2$ is zero on projective modules. Since $\mathcal{F}$ is right exact, hence in particular preserves surjective maps, and any object of $\mathcal{Q}(\boldsymbol{a})$ has a projective presentation, it follows that $\mathcal{F}^2$ is the zero functor. $\square$

Since $\mathcal{L}$ sends projective modules to projective modules, it follows (cf. [Wei94, Corollary 10.8.3]) that $\mathbb{L}\mathcal{F} \circ \mathbb{L}\mathcal{F} = \mathbb{L}(\mathcal{F} \circ \mathcal{F}) = 0$.

We summarize the results of this section in the following:

Theorem 6.5.6. *Let φ be a web defining a morphism $\mathsf{V}(\boldsymbol{a}) \to \mathsf{V}(\boldsymbol{a}')$. Then the diagram*

$$\begin{tikzcd} \mathcal{D}^\nabla \mathcal{Q}(\boldsymbol{a}') \arrow[r, "{\mathcal{E}, \mathbb{L}\mathcal{F}}"] & \mathcal{D}^\nabla \mathcal{Q}(\boldsymbol{a}') \\ \mathcal{D}^\nabla \mathcal{Q}(\boldsymbol{a}) \arrow[r, "{\mathcal{E}, \mathbb{L}\mathcal{F}}"] \arrow[u, "\mathscr{F}(\varphi)"] & \mathcal{D}^\nabla \mathcal{Q}(\boldsymbol{a}) \arrow[u, "\mathscr{F}(\varphi)"'] \end{tikzcd} \tag{6.5.11}$$

commutes and categorifies (i.e. gives, after applying the completed Grothendieck group $\mathbf{K}^{\mathbb{C}(q)}$*) the diagram*

$$\begin{tikzcd} \mathsf{V}(\boldsymbol{a}') \arrow[r, "{E', F}"] & \mathsf{V}(\boldsymbol{a}') \\ \mathsf{V}(\boldsymbol{a}) \arrow[u, "\mathscr{T}(\varphi)"] \arrow[r, "{E', F}"] & \mathsf{V}(\boldsymbol{a}) \arrow[u, "\mathscr{T}(\varphi)"'] \end{tikzcd} \tag{6.5.12}$$

In particular, for $\boldsymbol{a} = \mathbb{n}$ *we have two families of endofunctors* $\{\mathcal{E}, \mathbb{L}\mathcal{F}\}$ *and* $\{\mathcal{C}_i \mid i = 1, \ldots, n-1\}$ *of* $\mathcal{D}^{\nabla}\mathcal{Q}(\mathbb{n})$ *which commute with each other and which on the Grothendieck group level give the actions of* U_q *and of the Hecke algebra* $\mathcal{H}_n$ *on* $V^{\otimes n}$ *respectively.*

Part III

Diagram algebra

CHAPTER 7

Preliminaries

In this chapter we collect some general notions which we will need in the following.

First, in §7.1 we introduce some combinatorics for the shortest coset representatives for the quotient ${}_{\mathbb{S}_k \times \mathbb{S}_{n-k}}\backslash^{\mathbb{S}_n}$ of the symmetric group. In particular, we will describe some different ways of parametrizing such cosets; the notation we introduce here will be omnipresent later. In §7.2 we compute explicitly some canonical basis elements of the Hecke algebra; these will be used in Chapter 8 to determine the dimension of the corresponding Soergel modules.

In §7.3 we introduce complete symmetric polynomials and Demazure operators and recall some formulas for them. In §7.4 we define a class of ideals of a polynomial ring which are generated by some complete symmetric polynomials in a subset of the variables, and we determine homomorphism between the corresponding quotient modules using the machinery of Groebner basis. The Soergel modules which we will study in the next chapter will turn out to be of this type. Finally, in §7.5 we recall the definition of Schubert polynomials, which will be used later in Chapter 9.

7.1 Combinatorics of coset representatives

Let us fix an integer $0 \leq k \leq n$. If $s_1, \ldots, s_{n-1}$ are the simple reflections in $\mathbb{S}_n$, let W_k be the subgroup generated by $s_1, \ldots, s_{k-1}$ and $W_k^\perp$ be the subgroup generated by $s_{k+1}, \ldots, s_{n-1}$. Notice that $\mathbb{S}_k \times \mathbb{S}_{n-k} \cong W_k \times W_k^\perp \subseteq \mathbb{S}_n$. Let w_k be the longest element of W_k, and let $D = D_{n,k}$ be the set of shortest coset representatives $\left({}_{W_k \times W_k^\perp}\backslash^{\mathbb{S}_n}\right)^{\mathrm{short}}$.

Remark 7.1.1. The notation agrees with the more general one introduced in §2.2. Indeed, here we are considering only the particular case $W_{\mathfrak{q}} = W_k = \langle s_1, \ldots, s_{k-1} \rangle$, $W_{\mathfrak{p}} = W_k^\perp = \langle s_{k+1}, \ldots, s_{n-1} \rangle$. Accordingly, we have $w_k = w_{\mathfrak{q}}$ and $D = W^{\mathfrak{p}+\mathfrak{q}}$.

The set D is in natural bijection with $\wedge\vee$–sequences consisting of k $\wedge$'s and $n-k$ $\vee$'s, by mapping the identity $e \in \mathbb{S}_n$ to the sequence

$$e = \underbrace{\wedge \cdots \wedge}_{k} \underbrace{\vee \cdots \vee}_{n-k} \tag{7.1.1}$$

and letting $\mathbb{S}_n$ act by permutation of positions; in order to obtain a bijection with right coset representatives we regard this as a right action. From now on, we identify an element $z \in D$ with the corresponding $\wedge\vee$–sequence.

There are a few different ways to encode an element $z \in D$, which we explain now.

1) The position sequences. In an $\wedge\vee$–sequence $z \in D$, we number the $\wedge$'s (resp. the $\vee$'s) from 1 to k (resp. from 1 to $n-k$) from the left to the right. Moreover, we number the positions of an $\wedge\vee$–sequence from 1 to n from the left to the right. We let $\wedge_i^z$ be the position of the i–th $\wedge$ and $\vee_j^z$ be the position of the j–th $\vee$ in z. For example, in the sequence

$$z = \wedge\vee\vee\wedge\vee\wedge\wedge$$

we have $\wedge_2^z = 4$ and $\vee_1^z = 2$. Notice that either of the sequences $(\wedge_1^z, \dots, \wedge_k^z)$ and $(\vee_1^z, \dots, \vee_{n-k}^z)$ uniquely determines z.

2) The $\wedge$–distance sequence. We set

$$z_i^\wedge = \wedge_i^z - i \qquad \text{for } i = 1, \dots, k, \tag{7.1.2}$$

so that

$$(\wedge_1^z, \dots, \wedge_k^z) = (1 + z_1^\wedge, \dots, k + z_k^\wedge). \tag{7.1.3}$$

In other words, $z_i^\wedge$ measures how many steps the i–th $\wedge$ of the initial sequence e has been moved to the right by the permutation z. This defines a bijection $z \mapsto \boldsymbol{z}^\wedge$ between D and the set

$$\{\boldsymbol{z}^\wedge = (z_1^\wedge, \dots, z_k^\wedge) \mid 0 \leq z_1^\wedge \leq \cdots \leq z_k^\wedge \leq n-k\}. \tag{7.1.4}$$

Define the permutation

$$t_{i,\ell}^\wedge = s_i s_{i+1} \cdots s_{i+\ell-1} \tag{7.1.5}$$

for all $i = 1, \dots, n-1$ and $\ell = 1, \dots, n-i$ (and set $t_{i,0}^\wedge = e$). Then we have a reduced expression for z:

$$z = t_{k,z_k^\wedge}^\wedge t_{k-1,z_{k-1}^\wedge}^\wedge \cdots t_{1,z_1^\wedge}^\wedge. \tag{7.1.6}$$

3) The $\vee$–distance sequence. Analogously, set

$$z_i^\vee = i - \vee_{k-i}^z \qquad \text{for } i = k+1, \dots, n \tag{7.1.7}$$

so that

$$(\vee_1^z, \dots, \vee_{n-k}^z) = (k + 1 - z_{k+1}^\vee, \dots, n - z_{n-k}^\vee). \tag{7.1.8}$$

In other words, $z_i^\vee$ measures how many steps the $(i-k)$–th $\vee$ of e has been moved to the left by the permutation z. This defines a bijection $z \mapsto \boldsymbol{z}^\vee$ between D and the set

$$\{\boldsymbol{z}^\vee = (z_{k+1}^\vee, \dots, z_n^\vee) \mid k \geq z_{k+1}^\vee \geq \cdots \geq z_n^\vee \geq 0\}. \tag{7.1.9}$$

Define

$$t_{k+i,\ell}^\vee = s_{k+i-1} s_{k+i-2} \cdots s_{k+i-\ell} \tag{7.1.10}$$

for $i = 1, \dots, n-k$ and $\ell = 1, \dots, k$ (and set $t_{k+i,0}^\vee = e$). Then we have another reduced expression for z:

$$z = t_{k+1,z_{k+1}^\vee}^\vee t_{k+2,z_{k+2}^\vee}^\vee \cdots t_{n,z_n^\vee}^\vee. \tag{7.1.11}$$

4) The $\boldsymbol{b}$–sequence. Finally we want to assign to the element $z \in D$ its $\boldsymbol{b}$–sequence $\boldsymbol{b}^z$. Let

$$\mathscr{B}_{n,k} = \left\{ \boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n \;\middle|\; \begin{array}{l} k+1 \geq b_1 \geq \cdots \geq b_n = 1, \\ b_i \leq b_{i+1} + 1 \text{ for all } i = 1, \ldots, n-1 \end{array} \right\} \tag{7.1.12}$$

and define $\boldsymbol{b}^z \in \mathscr{B}_{n,k}$ by

$$b_i^z = \#\{j \mid \wedge_j^z > i\} + 1. \tag{7.1.13}$$

In other words, $b_i^z - 1$ is the number of $\wedge$'s on the right of position i. It is clear that $\boldsymbol{b}^z$ uniquely determines the element $z \in D$. In fact, this defines a bijection between D and $\mathscr{B}_{n,k}$.

Example 7.1.2. Let $n = 8$, $k = 4$ and consider the element $z = s_4 s_5 s_6 s_3 \in D$. The corresponding $\wedge\vee$–sequence and the $\boldsymbol{b}$–sequences are:

$$\begin{array}{r} \wedge \; \wedge \; \vee \; \wedge \; \vee \; \vee \; \wedge \; \vee \\ \mathbf{b}^z = 4 \;\; 3 \;\; 3 \;\; 2 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \end{array}$$

We also have $\boldsymbol{z}^\wedge = (0,0,1,3)$ and $\boldsymbol{z}^\vee = (2,1,1,0)$. Note that we can write either $z = s_4 s_5 s_6 \cdot s_3$ as in (7.1.6) or $z = s_4 s_3 \cdot s_5 \cdot s_6$ as in (7.1.11). $\circledast$

7.2 Some canonical basis elements

As we anticipated, in this section we will compute some canonical basis elements of the Hecke algebra $\mathcal{H}_n$ (for the definition of which we refer to §2.1).

Applying Lemma 2.1.4 to the parabolic subgroup $W_k \subseteq \mathbb{S}_n$ we get

$$\underline{H}_{w_k} = \sum_{x \in W_k} v^{\ell(w_k) - \ell(x)} H_x. \tag{7.2.1}$$

In the next proposition we will generalize (7.2.1) and give explicit formulas for the canonical basis elements $\underline{H}_{w_k z}$ for $z \in D$. But first we introduce the following notation: we set

$$\sum_{w' \in \mathbb{S}_k}^{(q)} f(w') = \sum_{w' \in S_k} q^{-\ell(w')} f(w') \quad \text{and} \quad \sum_{i=0}^{h}{}^{(q)} g(i) = \sum_{i=0}^{h} q^{-i} g(i) \tag{7.2.2}$$

for whatever functions f defined on $\mathbb{S}_k$ and g defined on $\{0, \ldots, h\}$.

Proposition 7.2.1. *Let $z \in D$, and write $z = t^\vee_{k+1, z^\vee_{k+1}} \cdots t^\vee_{n, z^\vee_n}$. Then*

$$\underline{H}_{w_k z} = \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}}{}^{(q)} \cdots \sum_{i_n=0}^{z^\vee_n}{}^{(q)} q^{\ell(w_k z)} H_{w' t^\vee_{k+1, i_{k+1}} \cdots t^\vee_{n, i_n}}. \tag{7.2.3}$$

Proof. First, we note that all words $w' t^\vee_{k+1, i_{k+1}} \cdots t^\vee_{n, i_n}$ that appear in the expression on the right are actually reduced words. This is clear if we look at the action of this permutation on the string

$$\wedge_1 \cdots \wedge_k \vee_{k+1} \cdots \vee_n \tag{7.2.4}$$

from the right: the length of the permutation is the cardinality of the set X of the couples of symbols of this string that have been inverted. To X belong $\ell(w')$ couples consisting of two $\wedge$'s; moreover, every $\vee_{k+\alpha}$ appears in X exactly i_α times coupled with some $\wedge$ or some $\vee_{k+\beta}$ for $\beta < \alpha$. Hence the length of the permutation $w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{n,i_n}$ is exactly $\ell(w') + i_{k+1} + \cdots + i_n$, and therefore this is a reduced expression.

Now, in the r.h.s. of (7.2.3) the coefficient of $H_{w_k z}$ is one, while the coefficient of every other basis element $H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{n,i_n}}$ is divisible by q. Hence the only thing we have to prove is that the r.h.s of (7.2.3) is bar invariant.

We proceed by induction on the length of z, the case $z = 0$ being given by (7.2.1). Let h be the greatest index such that $z^\vee_h \neq 0$. Hence we have $z = t^\vee_{k+1,z^\vee_{k+1}} \cdots t^\vee_{h,z^\vee_h}$. First suppose that $z^\vee_h \geq 2$. Set $z' = t^\vee_{k+1,z^\vee_{k+1}} \cdots t^\vee_{h,z^\vee_h - 1}$ and $j = h - z^\vee_h$ so that $z = z' s_j$. We compute:

$$\underline{H}_{w_k z'} \underline{H}_j = \left(\sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_h=0}^{z^\vee_h - 1 (q)} q^{\ell(w_k z')} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h,i_h}} \right) (H_j + q) \tag{7.2.5}$$

$$= \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_{h-1}=0}^{z^\vee_{h-1}(q)} q^{\ell(w_k z') - z^\vee_h + 1} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h-1,i_{h-1}} t^\vee_{h,z^\vee_h}} \tag{7.2.6}$$

$$+ \left(\sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_h=0}^{z^\vee_h - 2 (q)} q^{\ell(w_k z')} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h,i_h}} \right) H_j \tag{7.2.7}$$

$$+ \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_h=0}^{z^\vee_h - 1 (q)} q^{\ell(w_k z') + 1} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h,i_h}}. \tag{7.2.8}$$

The element $\underline{H}_{w_k z'} \underline{H}_j$ is obviously bar-invariant. Moreover, the sum of (7.2.6) and (7.2.8) gives exactly the r.h.s. of (7.2.3) for $\underline{H}_{w_k z}$; hence we only need to prove that (7.2.7) is bar invariant. It is easy to check that in (7.2.7) the term H_j on the right acts as q^{-1}; hence (7.2.7) is equal to the r.h.s. of (7.2.3) for $\underline{H}_{w_k z''}$, where $z'' = t^\vee_{k+1,z^\vee_{k+1}} \cdots t^\vee_{h,z^\vee_h - 2}$, and this is bar-invariant by induction.

Now suppose instead that $z^\vee_h = 1$. Set $z' = t^\vee_{k+1,z^\vee_{k+1}} \cdots t^\vee_{h-1,z^\vee_{h-1}}$ so that $z = z' s_{h-1}$, and compute:

$$\underline{H}_{w_k z'} \underline{H}_{h-1} = \left(\sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_{h-1}=0}^{z^\vee_{h-1}(q)} q^{\ell(w_k z')} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h-1,i_{h-1}}} \right) \underline{H}_{h-1} \tag{7.2.9}$$

$$= \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_{h-1}=0}^{z^\vee_{h-1}(q)} q^{\ell(w_k z')} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h-1,i_{h-1}} s_{h-1}} \tag{7.2.10}$$

$$+ \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z^\vee_{k+1}(q)} \cdots \sum_{i_{h-1}=0}^{z^\vee_{h-1}(q)} q^{\ell(w_k z') + 1} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{h-1,i_{h-1}}}. \tag{7.2.11}$$

This is exactly the r.h.s. of (7.2.3) for $\underline{H}_{w_k z}$; hence this is also bar invariant. □

We will need some other canonical basis elements that we now compute.

Proposition 7.2.2. *Let $z \in D$, with $z = t^\vee_{k+1,z^\vee_{k+1}} \cdots t^\vee_{n,z^\vee_n}$. Suppose that for some index j*

we have $z_j^\vee = z_{j+1}^\vee$. *Then* $\underline{H}_{s_j w_k z}$ *is equal to*

$$\sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z_{k+1}^\vee (q)} \cdots \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{i_j (q)} \cdots \sum_{i_n=0}^{z_n^\vee (q)} q^{\ell(w_k z)} H_{s_j w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots t^\vee_{n,i_n}}$$
$$+ \sum_{w' \in \mathbb{S}_k}^{(q)} \sum_{i_{k+1}=0}^{z_{k+1}^\vee (q)} \cdots \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{i_j (q)} \cdots \sum_{i_n=0}^{z_n^\vee (q)} q^{\ell(w_k z)+1} H_{w' t^\vee_{k+1,i_{k+1}} \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots t^\vee_{n,i_n}}. \tag{7.2.12}$$

Proof. Analogously as for the previous proof, we only need to check that (7.2.12) is bar-invariant. We prove this claim by induction on the length of z, using Proposition 7.2.1 for the expression of $\underline{H}_{w_k z}$. In the following computation, we do not write the sums over $w' \in \mathbb{S}_k$ and over the indices i_h for $h \neq j, j+1$.

$$\underline{H}_j \underline{H}_{w_k z} = \underline{H}_j \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{z_{j+1}^\vee (q)} q^{\ell(w_k z)} H_{w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} \tag{7.2.13}$$

$$= \underline{H}_j \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{i_j (q)} q^{\ell(w_k z)} H_{w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} \tag{7.2.14}$$

$$+ \underline{H}_j \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=i_j+1}^{z_{j+1}^\vee (q)} q^{\ell(w_k z)} H_{w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} \tag{7.2.15}$$

Permutations $w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots$ occurring in (7.2.14) become longer when multiplied on the left with s_j. Hence (7.2.14) becomes

$$\sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{i_j (q)} q^{\ell(w_k z)} H_{s_j w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} + \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=0}^{i_j (q)} q^{\ell(w_k z)+1} H_{w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} \tag{7.2.16}$$

This is exactly (7.2.12). Hence we are left to show that (7.2.15) is bar invariant.

For the permutations occurring in (7.2.15) we have

$$w' \cdots t^\vee_{j,i_j} t^\vee_{j,i_{j+1}} \cdots = s_j w' \cdots t^\vee_{j,i_{j+1}-1} t^\vee_{j+1,i_j} \cdots . \tag{7.2.17}$$

Hence (7.2.15) is equal to

$$\sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=i_j+1}^{z_{j+1}^\vee (q)} q^{\ell(w_k z)-1} H_{s_j w' \cdots t^\vee_{j,i_{j+1}-1} t^\vee_{j+1,i_j} \cdots} + \sum_{i_j=0}^{z_j^\vee (q)} \sum_{i_{j+1}=i_j+1}^{z_{j+1}^\vee (q)} q^{\ell(w_k z)} H_{w' \cdots t^\vee_{j,i_{j+1}-1} t^\vee_{j+1,i_j} \cdots} \tag{7.2.18}$$

Note that for $i_j = z_j^\vee$ the second sum runs over an empty set of indices. Therefore we can rewrite (7.2.18) as

$$\sum_{i_j=0}^{z_j^\vee-1}{}^{(q)} \sum_{i_{j+1}=i_j}^{z_{j+1}^\vee-1}{}^{(q)} q^{\ell(w_k z)-2} H_{s_j w' \cdots t^\vee_{j,i_{j+1}} t^\vee_{j+1,i_j} \cdots} + \sum_{i_j=0}^{z_j^\vee-1}{}^{(q)} \sum_{i_{j+1}=i_j}^{z_{j+1}^\vee-1}{}^{(q)} q^{\ell(w_k z)-1} H_{w' \cdots t^\vee_{j,i_{j+1}} t^\vee_{j+1,i_j} \cdots} \tag{7.2.19}$$

or, renaming the indices and swapping the sums,

$$\sum_{i_j=0}^{z_j^\vee-1}{}^{(q)} \sum_{i_{j+1}=0}^{i_j}{}^{(q)} q^{\ell(w_k z)-2} H_{s_j w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots} + \sum_{i_j=0}^{z_j^\vee-1}{}^{(q)} \sum_{i_{j+1}=0}^{i_j}{}^{(q)} q^{\ell(w_k z)-1} H_{w' \cdots t^\vee_{j,i_j} t^\vee_{j+1,i_{j+1}} \cdots}. \tag{7.2.20}$$

Let $z' \in D$ be determined by $z'^\vee_h = z^\vee_h$ for $h \neq j, j+1$ while $z'^\vee_j = z'^\vee_{j+1} = z^\vee_j - 1$. By induction (7.2.20) is $\underline{H}_{s_j w_k z'}$, hence it is bar-invariant. □

We will not need the explicit expression (7.2.12), but only the following

Corollary 7.2.3. *Let $z \in D$, with $z = t^\vee_{k+1, z^\vee_{k+1}} \cdots t^\vee_{n, z^\vee_n}$. Suppose that for some index j we have $z^\vee_j = z^\vee_{j+1}$. Then the canonical basis element $\underline{H}_{s_j w_k z}$ is a sum of*

$$k!(z^\vee_{k+1}+1)\cdots(z^\vee_j+1)(z^\vee_{j+1}+2)(z^\vee_{j+2}+1)\cdots(z^\vee_n+1) \tag{7.2.21}$$

standard basis elements with monomial coefficient in q.

7.3 Complete symmetric polynomials

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring. The *complete symmetric polynomials* are defined as

$$h_j(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} x_{i_1} \cdots x_{i_j} \tag{7.3.1}$$

for every $j \geq 1$ so that for example $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$. We set also $h_0(x_1, \ldots, x_n) = 1$, while if $n = 0$ (i.e., we have zero variables), we let $h_i() = 0$ for every $i \geq 1$. The symmetric group $\mathbb{S}_n$ acts on R permuting the variables, and the polynomials $h_i(x_1, \ldots, x_n)$ are invariant under this action; in fact, they generate the whole algebra $R^{\mathbb{S}_n}$ of invariant polynomials (see [Ful97, Section 6]).

We will consider complete symmetric polynomials in some subset of the variables of R. The following formula helps us to decompose a complete symmetric polynomial in k variables as complete symmetric polynomials in ℓ and $k - \ell$ variables, for every $\ell = 1, \ldots, k-1$:

$$h_j(x_1, \ldots, x_k) = \sum_{n=0}^{j} h_n(x_1, \ldots, x_\ell) h_{j-n}(x_{\ell+1}, \ldots, x_k). \tag{7.3.2}$$

It is also possible to express a complete symmetric polynomials in $k-1$ variables in terms of complete symmetric polynomials in k variables:

$$h_j(x_1, \ldots, x_{k-1}) = h_j(x_1, \ldots, x_k) - x_k h_{j-1}(x_1, \ldots, x_k). \tag{7.3.3}$$

Both (7.3.2) and (7.3.3) can be verified easily by comparing which monomials appear on both sides.

Demazure operators. For $1 \leq i \leq n-1$ let R^{s_i} be the subring of R consisting of polynomials invariant under the simple transposition s_i. We recall from [Dem73] the definition of the classical *Demazure operator* $\partial_i \colon R \to R^{s_i}$, given by

$$\partial_i \colon f \longmapsto \frac{f - s_i(f)}{x_i - x_{i+1}}. \tag{7.3.4}$$

The operator ∂_i is linear, vanishes on R^{s_i} and satisfies

$$\partial_i(fg) = f\partial_i g \qquad \text{whenever } f \in R^{s_i}. \tag{7.3.5}$$

Let also $P_i \colon R \to R$ be defined by $P_i(f) = f - x_i\partial_i(f)$. It is easy to show that P_i has also values in R^{s_i}. The following commutation rules hold:

$$[P_i, x_{i+1}] = -x_i s_i, \qquad [\partial_i, x_i] = s_i. \tag{7.3.6}$$

The operators ∂_i and P_i can be used to define the decomposition $R \cong R^{s_i} \oplus x_i R^{s_i}$ as a R^{s_i}–module, by

$$f \mapsto P_i f \oplus x_i \partial_i f. \tag{7.3.7}$$

Demazure operators have the nice property of sending complete symmetric polynomials to other complete symmetric polynomials:

Lemma 7.3.1. *For all $j \geq 1$ we have*

$$\partial_k h_j(x_1, \dots, x_k) = h_{j-1}(x_1, \dots, x_{k+1}). \tag{7.3.8}$$

Proof. Using (7.3.2) for $\ell = k-1$ we compute

$$\begin{aligned}
\partial_k h_j(x_1, \dots, x_k) &= \partial_k \left(\sum_{\ell=0}^{j} h_{j-\ell}(x_1, \dots, x_{k-1}) x_k^\ell \right) \\
&= \sum_{\ell=0}^{j} h_{j-\ell}(x_1, \dots, x_{k-1}) \partial_k(x_k^\ell) \\
&= \sum_{\ell=1}^{j} h_{j-\ell}(x_1, \dots, x_{k-1}) \frac{x_k^\ell - x_{k+1}^\ell}{x_k - x_{k+1}} \\
&= \sum_{\ell=1}^{j} h_{j-\ell}(x_1, \dots, x_{k-1}) h_{\ell-1}(x_k, x_{k+1}) \\
&= h_{j-1}(x_1, x_2, \dots, x_{k+1}).
\end{aligned} \tag{7.3.9}$$

Notice that in the last equality we used (7.3.2) again. □

7.4 Ideals generated by complete symmetric polynomials

We are going to study quotients rings of R generated by some of the h_i's. Let

$$\mathscr{B}' = \{ \boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{N}^n \mid b_i \geq b_{i+1} \geq b_i - 1 \}. \tag{7.4.1}$$

In other words, $\mathscr{B}'$ is the set of weakly decreasing sequences of positive numbers such that the difference between two consecutive items is at most one. Notice also that for all $k \geq 0$ we have $\mathscr{B}_{n,k} \subseteq \mathscr{B}'$, see (7.1.12). For every sequence $\boldsymbol{b} \in \mathscr{B}'$ let $I_{\boldsymbol{b}} \subset R$ be the ideal generated by

$$h_{b_1}(x_1), h_{b_2}(x_1, x_2), \dots, h_{b_n}(x_1, \dots, x_n). \tag{7.4.2}$$

Set also $R_{\boldsymbol{b}} = R/I_{\boldsymbol{b}}$.

We recall shortly the definition of Groebner basis, which are a useful tool for studying ideals in polynomial rings; for a complete reference see [CLO07, Chapter 2]. Let us fix a lexicographic monomial order on R with

$$x_n > x_{n-1} > \cdots > x_1. \tag{7.4.3}$$

With respect to this ordering, each polynomial $p \in R$ has a leading term $\mathrm{LT}(p)$. Given an ideal $I \subseteq R$, let $\mathrm{LT}(I) = \{\mathrm{LT}(p) \mid p \in I\}$ be the set of leading terms of elements of I and let $\langle \mathrm{LT}(I) \rangle$ be the ideal they generate. We recall that a finite subset $\{p_1, \dots, p_r\}$ of an ideal I of R is called a *Groebner basis* if the leading terms of the $p_1, \dots, p_r$ generate $\langle \mathrm{LT}(I) \rangle$. Then we have:

Lemma 7.4.1. *The polynomials* (7.4.2) *are a Groebner basis for $I_{\boldsymbol{b}}$ with respect to the order* (7.4.3).

Proof. By [CLO07, Theorem 2.9.3 and Proposition 2.9.4] it is enough to check that the leading monomials of the polynomials (7.4.2) are pairwise relatively prime. This is obvious. □

Proposition 7.4.2. *Let $\boldsymbol{b} \in \mathscr{B}'$. The quotient ring $R_{\boldsymbol{b}} = R/I_{\boldsymbol{b}}$ has dimension $b_1 \cdots b_n$, and a $\mathbb{C}$–basis is given by*

$$\{\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} \cdots x_n^{j_n} \mid 0 \leq j_i < b_i\}. \tag{7.4.4}$$

Proof. By the theory of Groebner bases (cf. [CLO07, Proposition 2.6.1]), any $f \in R$ can be written uniquely as $f = g + r$, with $g \in I_{\boldsymbol{b}}$ and r such that no term of r is divisible by any of the leading terms of the Groebner basis (7.4.2); that is, r is a linear combination of the monomials (7.4.4). This means exactly that the monomials (7.4.4) are a basis of $R_{\boldsymbol{b}}$. □

Example 7.4.3. Let $\boldsymbol{b} = (1, \dots, 1)$. Then $x_i = h_1(x_1, \dots, x_i) - h_1(x_1, \dots, x_{i-1})$ lies in $I_{\boldsymbol{b}}$ for each i, hence $I_{\boldsymbol{b}} = (x_1, \dots, x_n)$ and $R_{\boldsymbol{b}} \cong \mathbb{C}$ is one-dimensional. ⊛

Example 7.4.4. Let $\boldsymbol{b} = (n, n-1, \dots, 1)$. Then it is easy to show that the ideal $I_{\boldsymbol{b}}$ is the ideal generated by the symmetric polynomials in n variables with zero constant term, and $R_{\boldsymbol{b}}$ is the ring of the coinvariants $B = R/(R_+^{\mathbb{S}_n})$, cf. also §4.3, isomorphic to the cohomology of the full flag variety of $\mathbb{C}^n$ (see [Ful97, §10.2, Proposition 3]). ⊛

Morphisms between quotient rings

Next, we are going to determine all R–module homomorphisms between rings $R_{\boldsymbol{b}}$. This section will be devoted to the proof of the following proposition:

Proposition 7.4.5. *Let $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{B}'$, and let $c_i = \max\{b'_i - b_i, 0\}$. Then a $\mathbb{C}$–basis of $\mathrm{Hom}_R(R_{\boldsymbol{b}}, R_{\boldsymbol{b}'})$ is given by*

$$\{1 \mapsto x_1^{j_1} \cdots x_n^{j_n} \mid c_i \leq j_i < b'_i\}. \tag{7.4.5}$$

The proof consists of several lemmas.

Lemma 7.4.6. *Let $\boldsymbol{b} \in \mathscr{B}'$. Then $h_a(x_1, \ldots, x_i) \in I_{\boldsymbol{b}}$ for every $a \geq b_i$.*

Proof. We prove by induction on $\ell \geq 0$ that $h_{b_i+\ell}(x_1, \ldots, x_i) \in I_{\boldsymbol{b}}$ for every $i = 1, \ldots, n$. For $\ell = 0$ the statement follows from the definition. For the induction step, choose an index i and pick $j < i$ maximal such that $b_j = b_i + 1$ (or let $j = 0$ if such an index does not exist) and write using iteratively (7.3.3):

$$\begin{aligned} h_{b_i+\ell}(x_1, \ldots, x_i) = h_{b_i+\ell}(x_1, \ldots, x_j) + x_{j+1} h_{b_i+\ell-1}(x_1, \ldots, x_{j+1}) \\ + \cdots + x_{i-1} h_{b_i+\ell-1}(x_1, \ldots, x_{i-1}) + x_i h_{b_i+\ell-1}(x_1, \ldots, x_i). \end{aligned}$$

Since $b_i + \ell = b_j + \ell - 1$, the terms on the right all lie in $I_{\boldsymbol{b}}$ by the inductive hypothesis. $\square$

Lemma 7.4.7. *Let $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathscr{B}'$ and*

$$\boldsymbol{b}' = (b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_n) \tag{7.4.6}$$

for some i. Suppose that also $\boldsymbol{b}' \in \mathscr{B}'$. Then $I_{\boldsymbol{b}'} \subseteq I_{\boldsymbol{b}}$ while $x_i I_{\boldsymbol{b}} \subseteq I_{\boldsymbol{b}'}$.

Proof. It follows directly from Lemma 7.4.6 that $I_{\boldsymbol{b}'} \subseteq I_{\boldsymbol{b}}$. For the other assertion, since $h_{b_j}(x_1, \ldots, x_j) \in I_{\boldsymbol{b}'}$ for all $j \neq i$, we only need to prove that $x_i h_{b_i}(x_1, \ldots, x_i) \in I_{\boldsymbol{b}'}$. By (7.3.3) we have

$$x_i h_{b_i}(x_1, \ldots, x_i) = h_{b_i+1}(x_1, \ldots, x_i) - h_{b_i+1}(x_1, \ldots, x_{i-1}). \tag{7.4.7}$$

Since we suppose $\boldsymbol{b}' \in \mathscr{B}'$, it follows that $b_{i-1} = b_i + 1$, hence the r.h.s. of (7.4.7) lies in $I_{\boldsymbol{b}'}$. $\square$

We will say that two sequences $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{B}'$ that satisfy the hypothesis of Lemma 7.4.7 (without regarding the order) are *near each other.*

Lemma 7.4.8. *Let $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{B}'$ and set $c_i = \max\{b'_i - b_i, 0\}$. Then $x_1^{c_1} \cdots x_n^{c_n} I_{\boldsymbol{b}} \subseteq I_{\boldsymbol{b}'}$.*

Proof. We can find a sequence $\boldsymbol{b} = \boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(N)} = \boldsymbol{b}'$ with $\boldsymbol{b}^{(k)} \in \mathscr{B}'$ for each k and $N = \sum_i |b_i - b'_i|$ such that $\boldsymbol{b}^{(i)}$ and $\boldsymbol{b}^{(i+1)}$ are near each other. Then the claim follows applying iteratively Lemma 7.4.7. $\square$

Lemma 7.4.9. *Let $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{B}'$. Let $c_i = \max\{b'_i - b_i, 0\}$. Suppose $p \in R$ is such that $pI_{\boldsymbol{b}} \subseteq I_{\boldsymbol{b}'}$. Then $x_1^{c_1} \cdots x_n^{c_n} \mid p$.*

Proof. We prove the claim by induction on the leading term of p, using the lexicographic order (7.4.3). Let p' be the leading term of p and pick an index $1 \leq i \leq n$. By assumption, $ph_{b_i}(x_1, \ldots, x_i) \in I_{\boldsymbol{b}'}$. By the theory of Groebner basis, the leading term of $ph_{b_i}(x_1, \ldots, x_i)$ is divisible by $x_1^{b'_1} \cdots x_n^{b'_n}$, and this leading term is just $p' x_i^{b_i}$. It follows immediately that $x_1^{c_1} \cdots x_n^{c_n} \mid p'$. By Lemma 7.4.8 we then know that $p' I_{\boldsymbol{b}} \subseteq I_{\boldsymbol{b}'}$, hence also $(p - p')I_{\boldsymbol{b}} \subseteq I_{\boldsymbol{b}'}$. By induction, we may assume that $x_1^{c_1} \cdots x_n^{c_n} \mid (p - p')$, and we are done. $\square$

Proof of Proposition 7.4.5. It follows from Lemma 7.4.8 that the elements of (7.4.5) indeed define morphisms $R_{\boldsymbol{b}} \to R_{\boldsymbol{b}'}$. By Proposition 7.4.2 they are linearly independent, and by Lemma 7.4.9 they are a set of generators. $\square$

Duality

The category of finite-dimensional R–modules has a duality, given by

$$M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}). \tag{7.4.8}$$

In fact, the vector space M^* is endowed with an R–action by setting $(r \cdot f)(m) = f(r \cdot m)$ for all $f \in M^*$, $m \in M$, $r \in R$ (since R is commutative).

We will consider R as a graded ring, with the variables x_i in degree 2. If the module M is graded, the dual inherits a grading declaring $(M^*)_j = (M_{-j})^*$.

Now consider some $\boldsymbol{b} \in \mathscr{B}'$. Notice that $I_{\boldsymbol{b}}$ is a homogeneous ideal and hence $R_{\boldsymbol{b}}$ is a graded R–module. The monomial basis (7.4.4) of $R_{\boldsymbol{b}}$ has a unique element of maximal degree $b = 2(b_1 + \cdots + b_n - n)$, namely $\boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{1}}$ where $\boldsymbol{1} = (1, \ldots, 1)$ and $\boldsymbol{b} - \boldsymbol{1}$ is the sequence $(b_1 - 1, \ldots, b_n - 1)$. We define a symmetric bilinear form $(\cdot, \cdot)$ on $R_{\boldsymbol{b}}$ by letting

$$(\boldsymbol{x}^{\boldsymbol{j}}, \boldsymbol{x}^{\boldsymbol{j}'}) = \begin{cases} 1 & \text{if } \boldsymbol{j} + \boldsymbol{j}' = \boldsymbol{b} - \boldsymbol{1}, \\ 0 & \text{otherwise} \end{cases} \tag{7.4.9}$$

on the monomial basis (7.4.4), where sequences are added termwise. In other words, (p, q) is the coefficient of $\boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{1}}$ in the expression of $pq \in R_{\boldsymbol{b}}$ as a linear combination of elements of the basis (7.4.4). Since this form is clearly non-degenerate, we get an isomorphism of graded R–modules

$$R_{\boldsymbol{b}} \cong R_{\boldsymbol{b}}^*\langle -b \rangle \quad \text{for every } \boldsymbol{b} \in \mathscr{B}'. \tag{7.4.10}$$

The degree shift comes out because the bilinear form pairs the degree i component of $R_{\boldsymbol{b}}$ with its component of degree $b - i$.

By the properties of a duality, we have

$$\mathrm{Hom}_R(R_{\boldsymbol{b}}, R_{\boldsymbol{b}'}) \cong \mathrm{Hom}_R(R_{\boldsymbol{b}'}^*, R_{\boldsymbol{b}}^*) \cong \mathrm{Hom}_R(R_{\boldsymbol{b}'}, R_{\boldsymbol{b}})\langle b' - b \rangle \tag{7.4.11}$$

for any $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{B}'$. It is not difficult to see that the composite isomorphism is given explicitly by

$$\Xi \colon (1 \mapsto p) \longmapsto \left(1 \mapsto \frac{\boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{1}}}{\boldsymbol{x}^{\boldsymbol{b}'-\boldsymbol{1}}} p\right). \tag{7.4.12}$$

7.5 Schubert polynomials

We recall some basic facts about Schubert polynomials, referring to [Mac91] for more details. Let $w \in \mathbb{S}_n$ be a permutation; then the operator $\partial_w = \partial_{i_1} \cdots \partial_{i_r}$, where $w = s_{i_1} \cdots s_{i_r}$ is some reduced expression, does not depend on the particular chosen reduced expression and is hence well-defined. Let $w_n \in \mathbb{S}_n$ be the longest element. Then one defines the *Schubert polynomial*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \partial_{w^{-1}w_n}\big(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\big) \tag{7.5.1}$$

for each $w \in \mathbb{S}_n$. The Schubert polynomials give a basis of $R/(R_+^{\mathbb{S}_n})$ [Dem73]. It follows from the definition that $\deg \mathfrak{S}_w(x_1, \ldots, x_n) = 2\ell(w)$.

For our purposes, it will be more convenient to have a monomial basis of $R/(R_+^{\mathbb{S}_n})$, indexed by permutations $w \in \mathbb{S}_n$.

Definition 7.5.1. *For each $w \in \mathbb{S}_n$ we define $\mathfrak{S}'_w(x_1, \ldots, x_n)$ to be the leading term of $\mathfrak{S}_w(x_1, \ldots, x_n)$ in the lexicographic order* (7.4.3).

Being the leading terms of a basis of $R/(R_+^{\mathbb{S}_n})$, it follows by the theory of Groebner bases (see §7.4) that also the monomials $\mathfrak{S}'_w(x_1, \ldots, x_n)$ give a basis. Indeed, this is the basis

$$\{x_1^{j_1} \cdots x_n^{j_n} \mid 0 \leq j_i \leq n-i\} \tag{7.5.2}$$

given by Proposition 7.4.2 (see also Example 7.4.4). The advantage of using Schubert polynomials is that they give us a way to index the basis elements through permutations.

There is an easy way to construct the monomials $\mathfrak{S}'_w(x_1, \ldots, x_n)$ (cf. [BJS93]): let $c_{w(i)} = \#\{j < i \mid w(j) > w(i)\}$; then $\mathfrak{S}'_w(x_1, \ldots, x_n) = x_1^{c_1} \cdots x_{n-1}^{c_{n-1}}$.

Example 7.5.2. The following table contains the Schubert polynomials and the polynomials $\mathfrak{S}'_w$ in the case $n = 3$.

$w \in \mathbb{S}_3$	$\mathfrak{S}_w$	$\mathfrak{S}'_w$
e	1	1
s	x_1	x_1
t	$x_1 + x_2$	x_2
st	x_1x_2	x_1x_2
ts	x_1^2	x_1^2
w_3	$x_1^2x_2$	$x_1^2x_2$

CHAPTER 8

Soergel modules

In this chapter we will describe some Soergel modules as quotient rings $R_{\boldsymbol{b}}$ (defined in §7.4). The strategy is the following: given a Soergel module M, we prove that the ideal $I_{\boldsymbol{b}}$ is contained in the annihilator of M; we use then a dimension argument comparing the dimension of $R_{\boldsymbol{b}}$ (Proposition 7.4.2) with the dimension of M (given by the corresponding canonical basis element computed in §7.2).

In the homomorphism spaces between these Soergel modules we will define *illicit* morphisms, which are the morphisms which factor through some "wrong" Soergel modules. We will determine explicitly the homomorphism spaces between Soergel modules modulo illicit morphisms in §8.3.

Of course there is a connection between this chapter and §4.3, where we recalled Soergel's theorems. In §8.1 we explain the commutative algebra side of the picture from [Soe90], but we postpone the connection with Lie theory to §9.6. Here we anticipate only that taking the quotient by illicit morphisms corresponds, in the Lie-theoretical setting, to considering a parabolic subcategory of the category $\mathcal{O}$.

8.1 Soergel modules

Fix a positive integer n and let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. Let moreover $B = R/(R_+^{\mathbb{S}_n})$ be the ring of the coinvariants (cf. also Example 7.4.4). For a simple reflection $s_i \in \mathbb{S}_n$, let B^{s_i} denote the invariants under s_i, that is

$$B^{s_i} = \mathbb{C}[x_1, \ldots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n]/(R_+^{\mathbb{S}_n}). \tag{8.1.1}$$

In the following, we will abbreviate $\otimes_{B^{s_i}}$ by $\otimes_i$ while $\otimes_B$ will be simply $\otimes$. We let also $B_i = B \otimes_i B$.

We define now Soergel modules for the symmetric group $\mathbb{S}_n$ by recursion on the Bruhat ordering. First we set $\mathsf{C}_e = \mathbb{C}$. Let then $w \in \mathbb{S}_n$ be a permutation and choose some reduced expression $w = s_{i_1} \cdots s_{i_r}$ where $s_{i_1}, \ldots, s_{i_r} \in \mathbb{S}_n$ are simple reflections. We have:

Theorem 8.1.1 ([Soe90])**.** *The B–module $B_{i_r} \otimes \cdots \otimes B_{i_1} \otimes \mathbb{C}$ has a unique indecomposable direct summand C_w which is not isomorphic to some $\mathsf{C}_{w'}$ for $w' \prec w$. This is the unique* *indecomposable summand containing $1 \otimes \cdots \otimes 1$. Up to isomorphism, C_w does not depend on the particular reduced expression chosen for w.*

We call the C_w's for $w \in \mathbb{S}_n$ *Soergel modules.*

EXAMPLE 8.1.2. Consider a simple reflection $s_i \in \mathbb{S}_n$. According to the theorem, the indecomposable object $\mathsf{C}_i = \mathsf{C}_{s_i}$ is a summand of $B_i \otimes \mathbb{C}$. But it is immediate to check that the two dimensional B–module $B_i \otimes \mathbb{C}$ is indecomposable, hence $\mathsf{C}_i = B_i \otimes \mathbb{C}$. This module is in fact isomorphic to $R/(x_1, \ldots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n)$. ⊛

Notice that since B is a quotient of R we have

$$\operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_R(M, N) \tag{8.1.2}$$

for all $M, N \in B-\mathrm{mod}$. In other words, the category of B–modules embeds as a full subcategory into the category of R–modules. Hence, it is harmful to consider B–modules as R–modules.

To compute Soergel modules we will need to know their dimension. This is given by Proposition 9.6.2, which we postpone because we will need to use Lie theory and the Kazhdan-Lusztig conjecture for the proof.

8.2 Some Soergel modules

We determine now explicitly some modules C_w. In the following, we use the notation introduced in §7.1. We recall the following well-known fact:

Lemma 8.2.1. *As a $\mathbb{C}$–vector space, a basis of $B \otimes_{i_1} B \otimes_{i_2} \cdots \otimes_{i_{r-1}} B \otimes_{i_r} \mathbb{C}$ is given by*

$$\{x_{i_1}^{\varepsilon_1} \otimes \cdots \otimes x_{i_r}^{\varepsilon_r} \otimes 1 \mid \varepsilon_j \in \{0, 1\}\}. \tag{8.2.1}$$

Proof. The claim follows since each polynomial $f \in R$ can be written uniquely as $f = P_i(f) + x_i \partial_i(f)$, with $P_i(f), \partial_i(f) \in R^{s_i}$, cf. (7.3.7). □

A key-tool to determine the Soergel modules is given by the next proposition; its proof is based on a lemma which uses facts about the BGG category $\mathcal{O}$, and that we hence postpone to §9.6.

Proposition 8.2.2. *For all $z \in D$ the module $\mathsf{C}_{w_k z}$ is cyclic. In particular, we have*

$$\mathsf{C}_{w_k z} \cong R/\operatorname{Ann}_R \mathsf{C}_{w_k z} \cong B/\operatorname{Ann}_B \mathsf{C}_{w_k z}. \tag{8.2.2}$$

Proof. By Proposition 7.2.1, H_e appears exactly once with coefficient $q^{\ell(w_k z)}$ in the canonical basis element $\underline{H}_{w_k z}$. By Lemma 9.6.4, this implies that $\mathsf{C}_{w_k z}$ is cyclic. □

Lemma 8.2.3. *For every $z \in D$ the dimension of $\mathsf{C}_{w_k z}$ over $\mathbb{C}$ is given by*

$$\dim_{\mathbb{C}} \mathsf{C}_{w_k z} = k!(z_{k+1}^{\vee} + 1) \cdots (z_n^{\vee} + 1) = b_1^z \cdots b_n^z. \tag{8.2.3}$$

Proof. The first equality follows directly from Proposition 9.6.2 and Proposition 7.2.1. We want to show the second equality. As in Example 7.1.2, we imagine the $\boldsymbol{b}$–sequence written on top of the $\wedge\vee$–sequence for z. In order to compute $b_1^z \cdots b_n^z$ we have to take the product of all the numbers appearing. Over the $\wedge$'s we have the numbers between 1 and k, each appearing once: hence their contribute is $k!$. Over the j–th $\vee$, we have a number measuring how many $\wedge$'s are on its right, plus one: this coincides with how many times this $\vee$ has been moved to the left plus one, that is, $z_{k+j}^{\vee} + 1$. The claim follows immediately. $\square$

Lemma 8.2.4. *The module C_{w_k} is isomorphic to $R_{\boldsymbol{b}^e}$, where $e \in \mathbb{S}_n$ is the identity element.*

Proof. By Proposition 8.2.2, the module C_{w_k} is cyclic over B. Choose any reduced expression $s_{i_1} \cdots s_{i_N}$ for w_k and build the corresponding module $B_{w_k} = B_{i_N} \otimes \cdots \otimes B_{i_1} \otimes \mathbb{C}$. It is easy to show that, as in Example 7.4.4, the ideal $I_{\boldsymbol{b}^e}$ is generated by the polynomials with zero constant term which are symmetric in the variables $x_1, \dots, x_k$. It follows immediately that $I_{\boldsymbol{b}^e} \subseteq \operatorname{Ann}_R B_{w_k} \subseteq \operatorname{Ann}_R \mathsf{C}_{w_k}$, hence C_{w_k} is a quotient of $R/I_{\boldsymbol{b}^e}$. By Lemma 8.2.3 and Proposition 7.4.2, $\dim_{\mathbb{C}} \mathsf{C}_{w_k} = \dim_{\mathbb{C}} R/I_{\boldsymbol{b}^e}$, hence $\mathsf{C}_{w_k} = R_{\boldsymbol{b}^e}$. $\square$

As we said, we will use the same notation introduced in §7.1. For $t_{i,\ell}^{\wedge}$, see (7.1.5), let

$$B_{t_{i,\ell}^{\wedge}} = B_{i+\ell-1} \otimes B_{i+\ell-2} \otimes \cdots \otimes B_i \tag{8.2.4}$$

and for $z = t_{k,z_k^{\wedge}}^{\wedge} \cdots t_{1,z_1^{\wedge}}^{\wedge}$ let

$$B_z^{\wedge} = B_{t_{1,z_1^{\wedge}}^{\wedge}} \otimes \cdots \otimes B_{t_{k,z_k^{\wedge}}^{\wedge}}. \tag{8.2.5}$$

Moreover, for $t_{i,\ell}^{\vee}$, see (7.1.10), let

$$B_{t_{i,\ell}^{\vee}} = B_{i-\ell} \otimes B_{i-\ell+1} \otimes \cdots \otimes B_{i-1} \tag{8.2.6}$$

and for $z = t_{k+1,z_{k+1}^{\vee}}^{\vee} \cdots t_{n,z_n^{\vee}}^{\vee}$ let

$$B_z^{\vee} = B_{t_{n,z_n^{\vee}}^{\vee}} \otimes \cdots \otimes B_{t_{k+1,z_{k+1}^{\vee}}^{\vee}}. \tag{8.2.7}$$

From Soergel's Theorem 8.1.1 and Proposition 8.2.2, it follows that $\mathsf{C}_{w_k z}$ is isomorphic both to the B–submodule of $B_z^{\wedge} \otimes \mathsf{C}_{w_k}$ generated by $\underline{1} = 1 \otimes \cdots \otimes 1$ and to the B–submodule of $B_z^{\vee} \otimes \mathsf{C}_{w_k}$ generated by $\underline{1} = 1 \otimes \cdots \otimes 1$.

The following lemma is the crucial step to determine the annihilator of $\mathsf{C}_{w_k z}$.

Lemma 8.2.5. *Let $z \in D$, and let m be the number of nonzero $z_i^{\wedge}$'s. Then*

$$h_\ell(x_1, \dots, x_{k-m+z_{k-m+1}^{\wedge}}) \in \operatorname{Ann} \mathsf{C}_{w_k z} \tag{8.2.8}$$

for all $\ell > m$.

Proof. Let us prove the assertion by induction on the sum N of the $z_i^{\wedge}$'s (that is also the length of z). The case $N = 0$ is given by Lemma 8.2.4. (Notice that $h_\ell(x_1, \dots, x_{k-m}) \in I_{\boldsymbol{b}^e}$ for $\ell > m$ by Lemma 7.4.6.)

For the induction step, let $i = k - m + z_{k-m+1}^{\wedge} = \wedge_{k-m+1}^{z} - 1$, write $z = z' s_i$ and compute in $B \otimes_i (B_{z'}^{\wedge} \otimes \mathsf{C}_{w_k})$:

$$\begin{aligned} &h_{\ell+1}(x_1, \dots, x_i) \cdot (1 \otimes \underline{1}) \\ &\qquad = \big(P_i(h_{\ell+1}(x_1, \dots, x_i)) + x_i \partial_i(h_{\ell+1}(x_1, \dots, x_i))\big) \cdot 1 \otimes \underline{1} \\ &\qquad = 1 \otimes (P_i(h_{\ell+1}(x_1, \dots, x_i)) \cdot \underline{1}) + x_i \otimes (\partial_i(h_{\ell+1}(x_1, \dots, x_i)) \cdot \underline{1}). \end{aligned} \tag{8.2.9}$$

Since $\mathsf{C}_{w_k z}$ is a summand of $B \otimes_i (B^{\vee}_{z'} \otimes \mathsf{C}_{w_k z})$, it is sufficient to show that (8.2.9) is zero. In fact, we prove that both terms $P_i(h_{\ell+1}(x_1, \dots, x_i))$ and $\partial_i(h_{\ell+1}(x_1, \dots, x_i))$ act as 0 on $B^{\wedge}_{z'}$.

Let us start considering the second term. Let $y \in D$ be determined by $y^{\wedge}_i = z^{\wedge}_i$ for $i \neq k-m+1, k-m+2$, while $y^{\wedge}_{k-m+1} = 0$ and $y^{\wedge}_{k-m+2} = z^{\wedge}_{k-m+1}$. Notice that our chosen reduced expression (7.1.6) for z splits as $z' = yz''$, so that

$$B^{\wedge}_{z'} = B_{i-1} B_{i-2} \cdots B_{k-m+1} \, B_j B_{j-1} \cdots B_{i+2} \, B^{\wedge}_y = B_{z''} B^{\wedge}_y \tag{8.2.10}$$

for $j = k-m+1+z^{\wedge}_{k-m+2} = \wedge^z_{k-m+2} - 1$, where we omitted the tensor product signs. By (7.3.8), $\partial_i(h_{\ell+1}(x_1, \dots, x_i)) = h_\ell(x_1, \dots, x_{i+1})$; being symmetric in the variables x_a for $a \neq i, i+1$, this steps over $B_{z''}$ and acts on $B^{\wedge}_y \otimes \mathsf{C}_{w_k}$. By induction, this action is zero.

Now let us consider the action of the term $P_i(h_{\ell+1}(x_1, \dots, x_i))$. Write

$$\begin{aligned} P_i(h_{\ell+1}(x_1, \dots, x_i)) &= h_{\ell+1}(x_1, \dots, x_i) - x_i \partial_i h_{\ell+1}(x_1, \dots, x_i) \\ &= h_{\ell+1}(x_1, \dots, x_i) - x_i h_\ell(x_1, \dots, x_{i+1}). \end{aligned} \tag{8.2.11}$$

Of these two summands, the second acts as zero exactly as before. For the first one, write $y' s_{i+1} = y$ so that $B^{\wedge}_y = B \otimes_{i+1} B^{\wedge}_{y'}$. Then $h_{\ell+1}(x_1, \dots, x_i)$ steps over the first tensor product, and by induction acts as zero on $B^{\wedge}_{y'}$. □

Proposition 8.2.6. *Let $z \in D$ with corresponding $\boldsymbol{b}$–sequence $\boldsymbol{b}^z$. Then the complete symmetric polynomial $h_{b^z_i}(x_1, \dots, x_i)$ lies in $\operatorname{Ann} \mathsf{C}_{w_k z}$ for all $i = 1, \dots, n$.*

Proof. We subdivide the indices $i \in \{1, \dots, n\}$ corresponding to the positions in the $\wedge\vee$–sequence of z in three subsets: we call an index i such that $z^{\wedge}_i = 0$ *initial*, we call an index i for which $b^z_i = 1$ *final*, and we call all other indices in between *in the middle*:

$$\underbrace{\wedge \cdots \wedge}_{\textit{initial}} \underbrace{\vee \times \cdots \times}_{\textit{middle}} \underbrace{\wedge \vee \cdots \vee}_{\textit{final}}$$

where a $\times$ stands for a $\wedge$ or a $\vee$. Notice that it can happen that an index i is both *initial* and *final* if and only if there are no $\vee$'s, that is $k = n$. Since in this case we already know the claim, we can exclude it.

If i is *final*, then $w_k z \in \mathbb{S}_i \subseteq \mathbb{S}_n$ (where $\mathbb{S}_i$ is the subgroup generated by the first $i-1$ simple transpositions) and obviously $h_1(x_1, \dots, x_i)$ annihilates $\mathsf{C}_{w_k z}$.

If z is not the identity (in which case there are no indexes *in the middle*), then $i = k - h + z^{\wedge}_{k-h+1}$ is *in the middle*, and Lemma 8.2.5 states that $h_{b^z_i}(x_1, \dots, x_i) \in \operatorname{Ann} \mathsf{C}_{w_k z}$. For the other indexes *in the middle*, we can use Lemma 8.2.5 after letting $h_{b^z_i}(x_1, \dots, x_i)$ step over some initial tensor symbols of $B^{\wedge}_z$.

If i is *initial*, then z is a permutation in the subgroup of $\mathbb{S}_n$ generated by $s_{i+1}, \dots, s_{n-1}$, hence $h_{b^z_i}(x_1, \dots, x_i)$, when acting on $B^{\wedge}_z \otimes \mathsf{C}_{w_k}$, can step over $B^{\wedge}_z$, and we only need to prove that $h_{b^z_i}(x_1, \dots, x_i) \in \operatorname{Ann} \mathsf{C}_{w_k}$. This follows by Lemma 8.2.5. □

We now identify the Soergel modules with the rings $R_{\boldsymbol{b}} = R/I_{\boldsymbol{b}}$ defined in §7.4.

Theorem 8.2.7. *Let $z \in D$ with corresponding $\boldsymbol{b}$–sequence $\boldsymbol{b}^z$. Then $\operatorname{Ann} \mathsf{C}_{w_k z} = I_{\boldsymbol{b}^z}$ and $\mathsf{C}_{w_k x} \cong R_{\boldsymbol{b}^z}$. A basis of $R_{\boldsymbol{b}^z}$ is given by*

$$\left\{ x_1^{c_1} \cdots x_{n-1}^{c_{n-1}} \cdot \mid 0 \leq c_i < b^z_i \right\}. \tag{8.2.12}$$

Proof. Let $\boldsymbol{b} = \boldsymbol{b}^z$. By Proposition 8.2.6, $I_{\boldsymbol{b}} \subseteq \operatorname{Ann} \mathsf{C}_{w_k z}$, so we have a surjective map $R/I_{\boldsymbol{b}} \twoheadrightarrow R/(\operatorname{Ann} \mathsf{C}_{w_k z})$. By Proposition 7.4.2 and Lemma 8.2.3 the dimensions agree, hence $I_{\boldsymbol{b}} = \operatorname{Ann} \mathsf{C}_{w_k z}$. The basis of $R_{\boldsymbol{b}}$ is given by Proposition 7.4.2. □

As an application, by translating Proposition 7.4.5 into the setup of Theorem 8.2.7, we can determine the homomorphism spaces between the Soergel modules $\mathsf{C}_{w_k z}$:

Corollary 8.2.8. *Let $z, z' \in D$ with $\boldsymbol{b}$–sequences $\boldsymbol{b}^z, \boldsymbol{b}^{z'}$. Let $c_i = \max\{b_i^{z'} - b_i^z, 0\}$ for $i = 1, \dots, n-1$. Then a $\mathbb{C}$–basis of $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ is given by*

$$\{1 \mapsto x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} \mid c_i \leq j_i < b_i^{z'}\}. \tag{8.2.13}$$

We will need in the following some other Soergel modules, corresponding to elements $w' \in \mathbb{S}_n$ which differ from some $w_k z$ only by a simple reflection, as in Proposition 7.2.2.

Proposition 8.2.9. *Let $z \in D$ with corresponding $\boldsymbol{b}$–sequence $\boldsymbol{b}^z$. Suppose $z_j^\vee = z_{j+1}^\vee$ for some index j. Let $\ell = j - z_j^\vee$, so that $s_j z = z s_\ell$. Then $\mathsf{C}_{s_j w_k z}$ is the quotient of R modulo the ideal generated by the complete symmetric polynomials*

$$h_{a_i}(x_1, \dots, x_i) \qquad \text{for } i = 1, \dots, n, \tag{8.2.14}$$

where $a_i = \boldsymbol{b}_i^z$ for $i \neq \ell$ while $a_\ell = \boldsymbol{b}_{\ell+1}^z$.

Notice that the sequence $\boldsymbol{a} = (a_1, \dots, a_n)$ is not an element of $\mathscr{B}'$, since $a_\ell = a_{\ell-1} + 1$.

Proof. The proof is analogous to the proof of Theorem 8.2.7, hence we will only give a sketch. By Corollary 7.2.3, the module $\mathsf{C}_{s_j w_k z}$ is cyclic. In particular, it is the submodule generated by $\underline{1}$ inside $B \otimes_\ell \mathsf{C}_{w_k z}$. First, let us prove that the polynomials (8.2.14) lie in $\mathrm{Ann}\, \mathsf{C}_{s_j w_k z}$, or equivalently that they vanish on $B \otimes_\ell \mathsf{C}_{w_k z}$. This is clear for $i \neq \ell$: in this case, these polynomials can step over the first tensor product, and then they vanish because they lie in $\mathrm{Ann}\, \mathsf{C}_{w_k z}$ by Theorem 8.2.7. For the remaining case, we have

$$\begin{split} h_{a_\ell}(x_1, \dots, x_\ell) \cdot (1 \otimes \underline{1}) \\ = 1 \otimes (P_\ell(h_{a_\ell}(x_1, \dots, x_\ell)) \cdot \underline{1}) + x_\ell \otimes (\partial_\ell(h_{a_\ell}(x_1, \dots, x_\ell)) \cdot \underline{1}). \end{split} \tag{8.2.15}$$

By (7.3.8) both $P_\ell(h_{a_\ell}(x_1, \dots, x_\ell))$ and $\partial_\ell(h_{a_\ell}(x_1, \dots, x_\ell))$ are contained in the ideal generated by $h_{a_\ell}(x_1, \dots, x_\ell)$ and $h_{a_\ell - 1}(x_1, \dots, x_{\ell+1})$, which both lie in $\mathrm{Ann}\, \mathsf{C}_{w_k z}$, and we are done.

It remains to prove that the polynomials (8.2.14) are a set of generators. Let I be the ideal generated by them. We know that $\mathsf{C}_{s_j w_k z}$ is a quotient of R/I. As for Lemma 7.4.1, the polynomials (8.2.14) are a basis of I. As for Proposition 7.4.2, the quotient R/I has dimension $a_1 \cdots a_n$. By Corollary 7.2.3 and an argument similar to the proof of Lemma 8.2.3, this coincides with the dimension of $\mathsf{C}_{s_j w_k z}$, and we are done. □

8.3 Morphisms between Soergel modules

In each basis set (8.2.13) there is exactly one morphism of minimal degree, which we call the *minimal degree morphism* $\mathsf{C}_{w_k z} \to \mathsf{C}_{w_k z'}$. For each $z \in D$, the homomorphism space $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z})$ is a ring that is naturally isomorphic to $\mathsf{C}_{w_k z}$. Moreover, for $z, z' \in D$ the homomorphism space $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ is naturally a $(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z})$–bimodule. It follows directly from Corollary 8.2.8 that this bimodule is cyclic (even more, it is cyclic both as a left and as a right module), generated by the minimal degree morphism. In what follows, we will often refer to this fact saying that the minimal degree morphism *divides* all other morphisms.

We let D' be the set of shortest coset representatives for $W_k^\perp \backslash \mathbb{S}_n$. In particular, for every $z \in D$ we have $z, w_k z \in D'$.

Definition 8.3.1. *For $z, z' \in D$ we say that a morphism $\mathsf{C}_{w_k z} \to \mathsf{C}_{w_k z'}$ is* illicit *if it factors through some C_y, where y is a longest coset representative for ${}_{W_k}\backslash^{\mathbb{S}_n}$ with $y \notin D'$.*

We let $\mathsf{W}_{z,z'}$ be the vector subspace of $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ consisting of all illicit morphisms. Since it is a $(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z})$–submodule, we can define the quotient bimodule

$$\mathsf{Z}_{z,z'} = \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\mathsf{W}_{z,z'}. \tag{8.3.1}$$

We are going to determine all the subspaces $\mathsf{W}_{z,z'}$, and consequently the quotients $\mathsf{Z}_{z,z'}$.

Lemma 8.3.2. *Let $z, z' \in D$, and suppose that for some index j we have*

$$z_i'^{\vee} = \begin{cases} z_i^{\vee} + 1 & \text{for } i = j, j+1, \\ z_i^{\vee} & \text{otherwise.} \end{cases} \tag{8.3.2}$$

In particular $z' = z s_\ell s_{\ell+1}$ for $\ell = j - z_j^{\vee} - 1$, and the corresponding $\wedge\vee$–sequence in positions $\ell, \ell+1, \ell+2$ are

$$z = \cdots \wedge\vee\vee \cdots \qquad \text{and} \qquad z' = \cdots \vee\vee\wedge \cdots . \tag{8.3.3}$$

Then

$$\mathsf{W}_{z,z'} = \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{s_k z'}) \quad \text{and} \quad \mathsf{W}_{z',z} = \mathrm{Hom}_R(\mathsf{C}_{w_k z'}, \mathsf{C}_{s_k z}). \tag{8.3.4}$$

Proof. It is enough to show that $\varphi \in \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$, $\varphi\colon 1 \mapsto x_j x_{j+1}$ and $\psi \in \mathrm{Hom}_R(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z})$, $\psi\colon 1 \mapsto 1$ are illicit, since they divide all other morphisms. First of all, note that by construction

$$b_i^{z'} = \begin{cases} b_i^z + 1 & \text{for } i = \ell, \ell+1, \\ b_i^z & \text{otherwise.} \end{cases} \tag{8.3.5}$$

Let $y = s_j z = z s_{\ell+1}$, and note that $y \notin D'$. We computed $\mathsf{C}_{w_k y}$ in Proposition 8.2.9. Since $\mathrm{Ann}(\mathsf{C}_{w_k z}) \subseteq \mathrm{Ann}(\mathsf{C}_{w_k y}) \subseteq \mathrm{Ann}(\mathsf{C}_{w_k z'})$, the morphism ψ can be written as the composition of the natural quotient maps

$$\mathsf{C}_{w_k z'} \xrightarrow{1} \mathsf{C}_{w_k y} \xrightarrow{1} \mathsf{C}_{w_k z}, \tag{8.3.6}$$

hence it is illicit.

On the other side, $x_{\ell+1} \mathrm{Ann}(\mathsf{C}_{w_k z}) \subseteq \mathrm{Ann}(\mathsf{C}_{w_k y})$ because by (7.3.3)

$$x_{\ell+1} h_{b^z_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = h_{b^z_{\ell+1}+1}(x_1, \ldots, x_{\ell+1}) - h_{b^z_{\ell+1}+1}(x_1, \ldots, x_\ell) \tag{8.3.7}$$

and $h_{b^z_{\ell+1}+1}(x_1, \ldots, x_\ell) \in \mathrm{Ann}(\mathsf{C}_{w_k y})$ by the argument of the proof of Lemma 7.4.6. Moreover, $x_\ell \mathrm{Ann}(\mathsf{C}_{w_k y}) \subseteq \mathrm{Ann}(\mathsf{C}_{w_k z'})$ because by (7.3.3) we have

$$x_\ell h_{b^z_\ell}(x_1, \ldots, x_\ell) = h_{b^z_\ell+1}(x_1, \ldots, x_\ell) - h_{b^z_\ell+1}(x_1, \ldots, x_{\ell-1}) \tag{8.3.8}$$

and this is in $\mathrm{Ann}(\mathsf{C}_{w_k z'})$ by Lemma 7.4.6. Hence φ can be written as the composition

$$\mathsf{C}_{w_k z} \xrightarrow{x_{\ell+1}} \mathsf{C}_{w_k y} \xrightarrow{x_\ell} \mathsf{C}_{w_k z'}, \tag{8.3.9}$$

and therefore is illicit. $\square$

Lemma 8.3.3. *Let $z \in D$ and suppose $z_j^{\vee} = z_{j+1}^{\vee}$ for some index j. Let $\ell = j - z_j^{\vee}$, so that $s_j z = z s_\ell$. Then the endomorphism $1 \mapsto x_\ell$ of $\mathsf{C}_{w_k z}$ is illicit.*

Proof. Let $y = s_j w_k z$ and notice that $y \notin D'$. We claim that $x_\ell \operatorname{Ann}(\mathsf{C}_{w_k z}) \subseteq \operatorname{Ann}(\mathsf{C}_y)$ and hence that $1 \mapsto x_\ell$ defines a morphism $\mathsf{C}_{w_k z} \to \mathsf{C}_y$. By Theorem 8.2.7 and Proposition 8.2.9 the only thing to check is that $x_\ell h_{b_\ell^z}(x_1, \ldots, x_\ell) \in \operatorname{Ann}(\mathsf{C}_y)$. By (7.3.3) we have

$$x_\ell h_{b_\ell^z}(x_1, \ldots, x_\ell) = h_{b_\ell^z+1}(x_1, \ldots, x_\ell) - h_{b_\ell^z+1}(x_1, \ldots, x_{\ell-1}) \in \operatorname{Ann}(\mathsf{C}_y). \tag{8.3.10}$$

On the other side, again by Theorem 8.2.7 and Proposition 8.2.9, it is clear that $1 \mapsto 1$ defines a morphism $\mathsf{C}_y \to \mathsf{C}_{w_k z}$. Hence the endomorphism $1 \mapsto x_\ell$ of $\mathsf{C}_{w_k z}$ factors through C_y and is therefore illicit. $\square$

More generally we have:

Lemma 8.3.4. *Let $z \in D$. For every j between $k+1$ and $n-1$ the endomorphism of $\mathsf{C}_{w_k z}$*

$$1 \longmapsto x_\ell x_{\ell+1} \cdots x_{\ell'}, \tag{8.3.11}$$

where $\ell = j - z_j^\vee$ and $\ell' = (j+1) - z_{j+1}^\vee - 1$, is illicit.

Proof. Let $y \in D$ be defined by $y_i^\vee = z_i^\vee$ for $i \neq j$, while $y_j^\vee = z_{j+1}^\vee$. From Corollary 8.2.8 we have that $1 \mapsto 1$ and $1 \mapsto x_\ell x_{\ell+1} \cdots x_{\ell'-1}$ define morphisms $\mathsf{C}_{w_k z} \to \mathsf{C}_{w_k y}$ and $\mathsf{C}_{w_k y} \to \mathsf{C}_{w_k z}$ respectively. By Lemma 8.3.3 the endomorphism $1 \to x_{\ell'}$ of $\mathsf{C}_{w_k y}$ is illicit, and so is (8.3.11), since it can be expressed as composition of these three morphism. $\square$

Theorem 8.3.5. *For all $z, z' \in D$ define a subbimodule $\widetilde{\mathsf{W}}_{z,z'}$ of the homomorphism space* $\operatorname{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ *as follows:*

(i) if for some index $1 \leq j \leq n-k-1$ we have $\vee_j^z \geq \vee_{j+1}^{z'}$ or $\vee_j^{z'} \geq \vee_{j+1}^z$, then we set $\widetilde{\mathsf{W}}_{z,z'} = \operatorname{Hom}(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$;

(ii) otherwise we define $\widetilde{\mathsf{W}}_{z,z'}$ to be the subbimodule generated by the morphisms

$$1 \mapsto (x_{\vee_j^z} x_{\vee_j^z+1} \cdots x_{\beta(j)})(x_1^{c_1} \cdots x_{n-1}^{c_{n-1}}) \quad \text{for } 1 \leq j \leq n-k, \tag{8.3.12}$$

where $c_i = \max\{b_i^{z'} - b_i^z, 0\}$ and

$$\beta(j) = \begin{cases} \min\{\vee_{j+1}^z, \vee_{j+1}^{z'}\} - 1 & \text{if } j < n-k, \\ n & \text{if } j = n-k. \end{cases} \tag{8.3.13}$$

Then we have $\widetilde{\mathsf{W}}_{z,z'} = \mathsf{W}_{z,z'}$.

Example 8.3.6. Let us consider the following example:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\boldsymbol{b}^z$	10	10	9	8	7	6	5	5	5	4	3	2	2	1
z	$\wedge$	$\boxed{\vee_1}$	$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\boxed{\vee_2}$	$\boxed{\vee_3}$	$\wedge$	$\wedge$	$\wedge$	$\boxed{\vee_4}$	$\wedge$
z'	$\vee_1$	$\wedge$	$\wedge$	$\wedge$	$\boxed{\vee_2}$	$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\vee_3$	$\wedge$	$\wedge$	$\boxed{\vee_4}$
$\boldsymbol{b}^{z'}$	11	10	9	8	8	7	6	5	4	3	3	2	1	1
		$\underbrace{\qquad}_{x_2x_3x_4}$ (2–5)							$\underbrace{\qquad}_{x_9x_{10}x_{11}x_{12}}$ (9–13)					

For convenience we have written the subscripts of the $\vee$'s, indicating their progressive number. We are in case (ii), and the generating morphisms (8.3.12) of $\widetilde{\mathsf{W}}_{z,z'}$ are

j	morphism
1	$1 \longmapsto (x_2x_3x_4)(x_1x_5x_6x_7)$
2	$1 \longmapsto (x_8)(x_1x_5x_6x_7)$
3	$1 \longmapsto (x_9x_{10}x_{11}x_{12})(x_1x_5x_6x_7)$
4	$1 \longmapsto (x_{13})(x_1x_5x_6x_7)$

In the picture, the case $j = 1$ is highlighted in solid and the case $j = 3$ is highlighted in dashed. ⊛

Proof of Theorem 8.3.5. First, we assume that $\vee^z_j \geq \vee^{z'}_{j+1}$ for some index $1 \leq j < n-k$. Pick j minimal with this property. Notice that by the minimality of j we have $\vee^z_{j-1} < \vee^{z'}_j$ (if $j > 1$), and hence on the left of the j–th $\vee$ of z there is an $\wedge$ (this remains true also if $j = 1$, since in this case $\vee^z_1 \geq \vee^{z'}_2 > \vee^{z'}_1 \geq 1$). Let $\alpha = \vee^z_j$ and $\ell = \vee^z_{j+1} - \vee^z_j$, and define $z^{(1)}$ and $z^{(2)}$ as

$$
\begin{array}{llr}
 & z & \wedge\vee\overbrace{\wedge\cdots\wedge\vee}^{\ell} \\
(8.3.14) & z^{(1)} = zs_{\alpha+\ell-1}s_{\alpha+\ell-2}\cdots s_{\alpha+1} & \wedge\vee\vee\cdots\wedge\wedge \\
(8.3.15) & z^{(2)} = z^{(1)}s_{\alpha-1}s_h & \vee\vee\wedge\cdots\wedge\wedge
\end{array}
$$

where on the right we pictured the corresponding $\wedge\vee$–sequences between positions $\alpha - 1$ and $\alpha + \ell$ (and we included z for clarity). The composition

$$\mathsf{C}_{w_kz} \xrightarrow{1} \mathsf{C}_{w_kz^{(1)}} \xrightarrow{x_{\ell-1}x_\ell} \mathsf{C}_{w_kz^{(2)}} \tag{8.3.16}$$

is illicit by Lemma 8.3.2. Composing with the minimal degree morphism $\mathsf{C}_{w_kz^{(2)}} \to \mathsf{C}_{w_kz'}$ we obtain the minimal degree morphism $\mathsf{C}_{w_kz} \to \mathsf{C}_{w_kz'}$, which is therefore illicit. It follows that $\mathrm{Hom}_R(\mathsf{C}_{w_kz}, \mathsf{C}_{w_kz'}) = \mathsf{W}_{z,z'}$.

A straightforward dual argument (cf. §7.4) proves that $\mathrm{Hom}_R(\mathsf{C}_{w_kz'}, \mathsf{C}_{w_kz}) = \mathsf{W}_{z',z}$. Swapping z and z' implies that $\mathrm{Hom}_R(\mathsf{C}_{w_kz}, \mathsf{C}_{s_kz'}) = \mathsf{W}_{z,z'}$ if $\vee^{z'}_j \geq \vee^z_{j+1}$.

Now assume we are in case (ii) and fix an index j. First, let us consider the case $\vee^{z'}_{j+1} < \vee^z_{j+1}$, so that $\beta(j) = \vee^{z'}_{j+1}$. Let $\gamma = \vee^z_j$, $\delta = \vee^{z'}_{j+1}$, $\varepsilon = \vee^z_{j+1}$. Define $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$ by

$$
\begin{array}{llr}
 & z & \vee\wedge\cdots\wedge\wedge\cdots\wedge\vee \\
(8.3.17) & z^{(1)} = zs_\gamma s_{\gamma+1}\cdots s_{\delta-1} & \wedge\cdots\wedge\vee\wedge\cdots\wedge\vee \\
(8.3.18) & z^{(2)} = z^{(1)}s_{\varepsilon-1}s_{\varepsilon-2}\cdots s_{\delta+1} & \wedge\cdots\wedge\vee\vee\wedge\cdots\wedge \\
(8.3.19) & z^{(3)} = z^{(2)}s_{\delta-1}s_\delta & \wedge\cdots\vee\vee\wedge\wedge\cdots\wedge
\end{array}
$$

where on the right we pictured the corresponding $\wedge\vee$–sequences between positions γ and ε. The composition

$$\mathsf{C}_{w_kz} \xrightarrow{1} \mathsf{C}_{w_kz^{(1)}} \xrightarrow{x_{\delta+1}x_{\delta+2}\cdots x_{\varepsilon-1}} \mathsf{C}_{w_kz^{(2)}} \xrightarrow{x_{\delta-1}x_\delta} \mathsf{C}_{w_kz^{(3)}} \tag{8.3.20}$$

is illicit by Lemma 8.3.2. By construction, the composition of (8.3.20) with the minimal degree morphism $\mathsf{C}_{w_kz^{(3)}} \to \mathsf{C}_{w_kz'}$ equals the morphism (8.3.12) from C_{w_kz} to $\mathsf{C}_{w_kz'}$, that is therefore illicit.

Let us now consider the other case $\vee^z_{j+1} \leq \vee^{z'}_{j+1}$. By Lemma 8.3.4 the endomorphism of C_{w_kz} defined by

$$1 \mapsto x_{\vee^z_j}x_{\vee^z_j+1}\cdots x_{\vee^z_{j+1}} \tag{8.3.21}$$

is illicit. This morphism divides the morphism (8.3.12), which is therefore illicit.

To conclude the proof we are left to check that in case (ii) $\mathsf{W}_{z,z'} \subseteq \widetilde{\mathsf{W}}_{z,z'}$. Unfortunately we are not able to check this directly. Instead, by Lemma 9.2.2 in the next chapter we have that the dimensions of the quotients of $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{z_k z'})$ by $\mathsf{W}_{z,z'}$ and $\widetilde{\mathsf{W}}_{z,z'}$ agree. This implies that $\mathsf{W}_{z,z'} = \widetilde{\mathsf{W}}_{z,z'}$. □

Grading

In order to keep the computations more transparent, we decided to postpone the introduction of the grading until now. The ring R is graded with $\deg x_i = 2$. Since the ideal $(R_+^{\mathbb{S}_n})$ is homogeneous, B is also graded, and the graded definition of the module B_i is $B_i = B \otimes_i B\langle -1 \rangle$. By Soergel's theorems all C_w for $w \in \mathbb{S}_n$ are gradable (cf. [Str03a, Lemma 1.4]). The grading is unique up to isomorphism and overall degree shift, hence we fix the standard graded lift of the module $\mathsf{C}_{w_k z}$, which by a slight abuse of notation we will again denote by $\mathsf{C}_{w_k z}$, so that the cyclic generator is in degree $-\ell(w_k z)$.

By our discussion in §7.4, and with the opportune degree shifting we put on the modules $\mathsf{C}_{w_k z}$, it follows that all modules $\mathsf{C}_{w_k z}$ are graded self-dual. In particular

$$\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'}) \cong \mathrm{Hom}_R(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z}) \tag{8.3.22}$$

as graded vector spaces for all $z, z' \in D$. An explicit isomorphism was described in (7.4.12).

Finally, by Theorem 8.3.5 the spaces $\mathsf{W}_{z,z'}$ are homogeneous subbimodules, and the quotients $\mathsf{Z}_{z,z'}$ are then graded bimodules.

CHAPTER 9

The diagram algebra

We want now to define diagram algebras $A_{n,k}$ over $\mathbb{C}$, which are isomorphic to the endomorphism rings of the minimal projective generators of the categories $\mathcal{Q}_k(\mathfrak{n})$. They are analogous to the generalized Khovanov algebras defined in [BS11], which instead are isomorphic to the endomorphism rings of the minimal projective generators of the maximal parabolic categories $\mathcal{O}^{\mathfrak{p}}$ used for categorifying representations of $\mathfrak{sl}_2$. We will use some diagrams which represent morphisms between the Soergel modules we studied in the previous section. We remark that the diagrams will remind of the graphical calculus of Chapter 3, exactly as the diagrams of [BS11] reminds of the graphical calculus [FK97].

We point out that the major difficulty is the definition of the multiplication of two basis diagrams, which is not simply stacking one on the top of the other (as in many other diagram algebras), but instead a quite involved process. In [BS11], Brundan and Stroppel use Khovanov's TQFT to define this multiplication. Since there is not an analogous of such a TQFT in our case, we construct the multiplication in an indirect way using composition of morphisms between Soergel modules. A drawback of our definition of the multiplication is that it is not clear how to define diagrammatically bimodules for the diagram algebra, as in [BS10].

We will introduce the diagrams in §9.1 and we will define the algebras $A_{n,k}$ in §9.2. Mimicking the techniques of [BS11] we will describe explicitly the graded cellular and properly stratified structure (§9.3). In §9.4 we determine and study indecomposable projective-injective modules using a bilinear form on our diagram algebras. In §9.5 we define diagrammatic versions of the functors $\mathcal{E}$ and $\mathcal{F}$ from §6.5.

Finally, in §9.6 we explain the connection between Part II and Part III by establishing an equivalence of categories between $\mathcal{Q}_k(\mathfrak{n})$ and $A_{n,k}$−gmod. As a consequence, we will be able to determine the endomorphism rings of the functors $\mathcal{E}_k$ and $\mathcal{F}_k$, proving that they are indecomposable.

9.1 Diagrams

We start introducing the diagrams on which our algebras will be built. We will redefine some keywords that are commonly used in Lie theory (such as *weight* and *block*) in a diagrammatic sense.

Weights and blocks

Weights. A *number line* $\mathbf{L}$ is a horizontal line containing a finite number of *vertices* indexed by a set of consecutive integers in increasing order from left to right. Given a number line, a *weight* is obtained by labeling each of the vertices by $\wedge$ or $\vee$. On the set of weights there is the partial order called *Bruhat order*, generated by $\wedge\vee \succ \vee\wedge$.

Blocks. For weights λ, μ declare $\lambda \sim \mu$ if μ can be obtained from λ by permuting $\wedge$'s and $\vee$'s. A *block* Γ is a $\sim$–equivalence class of weights. From now on, let us fix a block Γ. Let also k be the number of $\wedge$'s and $n-k$ be the number of $\vee$'s of any weight of Γ. The weights of Γ can be identified with $\wedge\vee$–sequences in the sense of §7.1, and hence with elements of $D_{n,k}$. For a weight λ, we can then define as in §7.1 the position sequences $(\wedge_1^\lambda, \ldots, \wedge_k^\lambda)$ and $(\vee_1^\lambda, \ldots, \vee_{n-k}^\lambda)$ and the $\boldsymbol{b}$–sequence $\boldsymbol{b}^\lambda$.

Enhanced weights. An *enhanced weight* λ^σ is a weight λ together with a bijection σ between the vertices labeled $\wedge$ in λ and the set $\{1, \ldots, k\}$. By numbering the $\wedge$'s from the left to the right we may view σ as an element in $\mathbb{S}_k$ and call it the underlying permutation. We call λ the *underlying weight.* We will also say that we obtain the enhanced weight λ^σ by enhancing the weight λ with the permutation σ. Notice that there are exactly $k!$ enhanced weights with the same underlying weight.

We define a partial order on the set of enhanced weights by the following rule:

$$\lambda^\sigma \preceq \mu^\tau \iff \lambda \prec \mu \quad \text{or} \quad \big(\lambda = \mu \text{ and } \ell(\sigma) \leq \ell(\tau)\big). \tag{9.1.1}$$

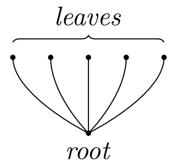
Example 9.1.1. Consider a block Γ consisting of weights with two $\wedge$'s and one $\vee$. The order on the set of enhanced weights of Γ is then

$$(\wedge\wedge\vee)^s \succ (\wedge\wedge\vee)^e \succ (\wedge\vee\wedge)^s \succ (\wedge\vee\wedge)^e \succ (\vee\wedge\wedge)^s \succ (\vee\wedge\wedge)^e \tag{9.1.2}$$

where $s \in \mathbb{S}_2$ is the unique transposition. ⊛

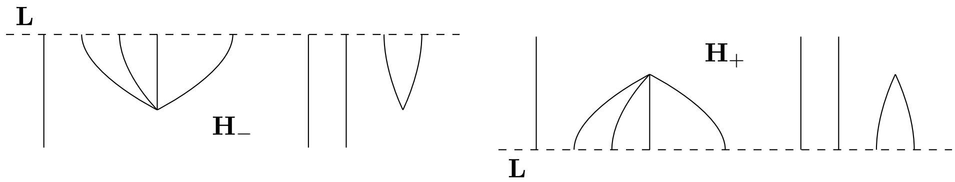
Fork diagrams

Forks. An *m–fork* is a tree with a unique branching point (the *root*) of valency m; the other m vertices of the tree are called the *leaves.* A 1–fork will be also called a *ray.* Here is an example of a 5–fork:



Lower fork diagrams. Let $\mathbf{V}$ be the set of vertices of the number line $\mathbf{L}$, and let $\mathbf{H}_-$ (resp. $\mathbf{H}_+$) be the half-plane below (resp. above) $\mathbf{L}$. A *lower fork diagram* is a diagram made out of the number line $\mathbf{L}$ together with some forks contained in $\mathbf{H}_-$, such that the leaves of each m–fork are m distinct consecutive vertices in $\mathbf{V}$; we require each vertex of $\mathbf{V}$ to be a leaf of some fork. The forks and rays of a lover fork diagram will be also called *lower forks* and *lower rays.*

Upper rays, *upper forks* and *upper fork diagrams* are defined in an analogous way. If c is a lower fork diagram, the mirror image c^* through the horizontal number line is an upper fork diagram, and vice versa. The following are examples of a lower fork diagram c and its mirror image c^*:



Oriented diagrams

If c is a lower fork diagram and λ is a weight with the same underlying number line, we can glue them to obtain a diagram $c\lambda$. We call $c\lambda$ an *unenhanced oriented lower fork diagram* if:

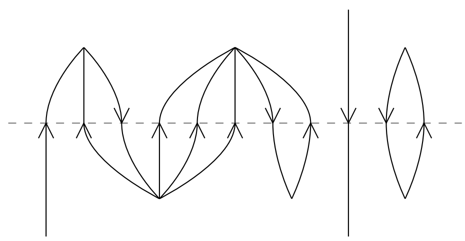
- each m–fork for $m \geq 1$ is labeled with exactly one $\vee$ and $m-1$ $\wedge$'s;
- the diagram begins at the left with a (possibly empty) sequence of rays labeled $\wedge$, and there are no other rays labeled $\wedge$ in c.

Notice that by definition each $\wedge$ and $\vee$ of λ labels some fork of c. Analogously, we call μd an *unenhanced oriented upper fork diagram* if $d^*\mu$ is an unenhanced oriented lower fork diagram. The *orientation* of an unenhanced oriented lower (or upper) fork diagram is the corresponding weight.

An (enhanced) *oriented lower fork diagram* $c\lambda^\sigma$ is an unenhanced oriented lower fork diagram $c\lambda$ together with a permutation $\sigma \in \mathbb{S}_k$ such that λ^σ is an enhanced weight. Similarly we define an (enhanced) *oriented upper fork diagram.* If not explicitly specified, our oriented lower/upper fork diagrams will always be enhanced.

For $m \geq 1$ and $1 \leq i \leq m$ we define $\lambda(m, i)$ to be the weight formed by one $\vee$ and $m-1$ $\wedge$'s, where the $\vee$ is at the i–th place. Note that a lower fork diagram c consisting of only a lower m–fork admits exactly $m!$ orientations, and they are exactly $\lambda(m, i)^\sigma$ for $i \in \{1, \ldots, m\}$, $\sigma \in \mathbb{S}_{m-1}$.

By a *fork diagram* we mean a diagram of the form ab obtained by gluing a lower fork diagram a underneath an upper fork diagram b, assuming that they have the same underlying number lines. An *unenhanced oriented fork diagram* is a fork diagram $a\lambda b$ obtained by gluing an oriented lower fork diagram $a\lambda$ and an oriented upper fork diagram λb, as in the picture:



An *(enhanced) oriented fork diagram* is obtained by additionally enhancing the corresponding weight.

Degrees

Define the *degree* of an unenhanced oriented lower (or upper) m–fork by setting

$$\deg(c\lambda(m, i)) = \deg(\lambda(m, i)c^*) = (i-1). \tag{9.1.3}$$

Define then the degree of an unenhanced oriented lower (resp. upper) fork diagram to be the sum of the degrees of all the lower (resp. upper) forks. Finally, we define the degree of an unenhanced oriented fork diagram $a\lambda b$ to be

$$\deg(a\lambda b) = \deg(a\lambda) + \deg(\lambda b). \tag{9.1.4}$$

Moreover, define the degree of a permutation σ as $\deg(\sigma) = 2\ell(\sigma)$. Then we define the degree of enhanced oriented diagrams by

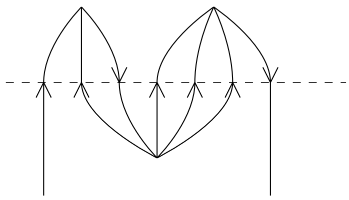
$$\deg(a\lambda^\sigma) = \deg(a\lambda) + \deg(\sigma), \tag{9.1.5}$$

$$\deg(\lambda^\sigma b) = \deg(\lambda b) + \deg(\sigma), \tag{9.1.6}$$

$$\deg(a\lambda^\sigma b) = \deg(a\lambda b) + \deg(\sigma) = \deg(a\lambda) + \deg(\lambda b) + \deg(\sigma) \tag{9.1.7}$$

In particular, enhancing with the neutral element $e \in \mathbb{S}_k$ preserves the degree.

Example 9.1.2. Consider the fork diagram $a\lambda b$ given by:



We have $\deg(a\lambda) = 1$ and $\deg(\lambda b) = 2 + 3 = 5$, so that $\deg(a\lambda b) = 6$. We can enhance the diagram with any permutation $\sigma \in \mathbb{S}_5$, and then $\deg(a\lambda^\sigma b) = 6 + 2\ell(\sigma)$. ⊛

The lower fork diagram associated to a weight

There is a natural way to associate a lower fork diagram to a weight λ:

Lemma 9.1.3. *For each weight λ there is a unique lower fork diagram, denoted $\underline{\lambda}$, such that $\underline{\lambda}\lambda^e$ is an oriented lower fork diagram of degree* 0.

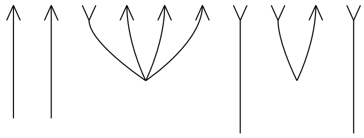
Proof. Suppose that some oriented lower fork diagram $c\lambda^e$ of degree 0 exists. Recall that, by the definition of orientation, each fork of c is labeled by at most one $\vee$ of λ; by the assumption on the degree, this $\vee$ has to be the leftmost label of the corresponding fork. As a consequence, each m–fork of c, with the only exception of some initial rays labeled by $\wedge$, has to be labeled by the weight $\lambda(m,1)$. In other words, the lower fork diagram c is obtained in the following way: examine the weight λ from the left to the right and find all maximal subsequences consisting of a $\vee$ followed by some (eventually empty) set of $\wedge$'s; draw a lower fork under each of these subsequences, and then draw lower rays under the remaining $\wedge$'s which are at the beginning of λ. It follows at once that c exists and is uniquely determined. □

Analogously we let $\overline{\lambda} = (\underline{\lambda})^*$ be the unique upper fork diagram such that $\lambda^e\overline{\lambda}$ is an oriented upper fork diagram of degree 0.

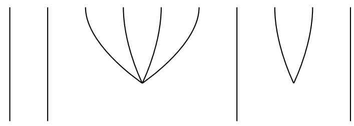
Example 9.1.4. As an example, let us illustrate the procedure of constructing $\underline{\lambda}$ for $\lambda = \wedge\wedge\vee\wedge\wedge\wedge\vee\vee\wedge\vee$. First, we circle all maximal subsequences consisting of a $\vee$ followed by $\wedge$'s:

$$\wedge \;\; \wedge \;\; \vee\;\wedge\;\wedge\;\wedge \;\; \vee \;\; \vee\;\wedge \;\; \vee$$

Then we draw a lower fork under each of such subsequences, and lower rays under the remaining $\wedge$'s at the beginning of λ.



The resulting lower fork diagram is then

$\underline{\lambda} =$  ⊛

Remark 9.1.5. Notice the analogy with the construction of the canonical basis diagram (§3.3.

For weights μ and λ, we use the notation $\mu \subset \lambda$ to indicate that $\mu \sim \lambda$ and $\underline{\mu}\lambda^e$ is an oriented lower fork diagram.

Lemma 9.1.6. *Let λ, μ be two weights in the same block Γ. If $\underline{\lambda} = \underline{\mu}$ then $\lambda = \mu$. If $\underline{\mu}\lambda$ is oriented then $\mu \preceq \lambda$ in the Bruhat order.*

Proof. Being in the same block, the weights λ and μ have the same number of $\wedge$'s and $\vee$'s; let h be the number of $\vee$'s. Consider the h rightmost forks of $\underline{\lambda}$ and let $a_1, \ldots, a_h$ be their initial positions; then λ is uniquely determined by the condition of having $\vee$'s in the positions $a_1, \ldots, a_h$ and $\wedge$'s elsewhere. Hence the first claim follows.

Now, given the lower fork diagram $\underline{\mu}$, let $F_1, \ldots, F_h$ denote its h rightmost forks. Let also $\Gamma_\mu = \{\lambda \in \Gamma \mid \underline{\mu}\lambda \text{ is oriented}\}$. Then $\lambda \in \Gamma_\mu$ if and only if each $\vee$ of λ labels exactly one of the F_i's. Since μ is the weight of Γ_μ with the $\vee$'s in the leftmost positions, it follows that μ is the minimal element in Γ_μ with respect to the Bruhat order. □

In particular, given our fixed block Γ, it follows that every lower fork diagram a (such that $a\mu$ is oriented for some $\mu \in \Gamma$) determines a unique weight λ with $\underline{\lambda} = a$. In what follows, we will sometime interchange a and λ in the notation: for example, we will write $\vee_j^a$ for $\vee_j^\lambda$ or $\boldsymbol{b}^a$ for $\boldsymbol{b}^\lambda$ and so on.

We collect now some lemmas that we will need later.

Lemma 9.1.7. *Let λ, μ be two weights in the same block Γ.*

(i) The lower fork diagram $\underline{\lambda}\mu$ is oriented if and only if

$$\vee_i^\lambda \leq \vee_i^\mu < \vee_{i+1}^\lambda \quad \text{for all } i \in 1, \ldots, n-k-1. \tag{9.1.8}$$

(ii) There exists an oriented fork diagram $\underline{\lambda}\eta\overline{\mu}$ for some $\eta \in \Gamma$ if and only if

$$\vee_i^\lambda < \vee_{i+1}^\mu \quad \text{and} \quad \vee_i^\mu < \vee_{i+1}^\lambda \quad \text{for all } i \in 1, \ldots, n-k-1. \tag{9.1.9}$$

Proof. It is clear that (ii) follows from (i), so let us prove (i). It is easy to see that the lower fork diagram $\underline{\lambda}\mu$ is oriented if and only if each lower fork of $\underline{\lambda}$ is labeled by exactly one $\vee$; this is exactly the same condition as (9.1.8). □

Lemma 9.1.8. *Consider weights $\lambda, \mu \in \Gamma$ with the corresponding $\boldsymbol{b}$–sequences $\boldsymbol{b}^\lambda, \boldsymbol{b}^\mu$.*

(a) If $\mu \succeq \lambda$, then $b_i^\mu \leq b_i^\lambda$ for all $i = 1, \ldots, n$.

(b) If $\underline{\lambda}\mu$ is oriented, then $b_i^\lambda - b_i^\mu \leq 1$ for all $i = 1, \ldots, n$.

(c) If $\underline{\lambda}\eta^e\overline{\mu}$ is oriented (for some weight $\eta \in \Gamma$), then $\left|b_i^\lambda - b_i^\mu\right| \leq 1$ for all $i = 1, \ldots, n$.

Proof. If $\mu \succeq \lambda$ then the i–th $\vee$ of μ is not on the right of the i–th $\vee$ of λ, and the first claim follows.

Let $\underline{\lambda}\mu$ be oriented. By Lemma 9.1.7 we have $\vee_i^\lambda \leq \vee_i^\mu < \vee_{i+1}^\lambda$. This means that for every vertex $v \in \mathbf{V}$ there is at most one $\wedge$ more to the right of v in λ than in μ. This is exactly (b).

The last claim follows from the second: if $\underline{\lambda}\eta^\sigma\overline{\mu}$ is oriented (for some weight η with $\boldsymbol{b}$–sequence $\boldsymbol{b}^\eta$), then $b_i^\lambda - b_i^\mu = b_i^\lambda - b_i^\eta + b_i^\eta - b_i^\mu \in \{1-1, 1+0, 0-1, 0+0\}$. □

Since we have identified Γ with $D_{n,k}$, we can define the length $\ell(\lambda)$ of any weight $\lambda \in \Gamma$ to be the length of the corresponding permutation in $D_{n,k}$.

Lemma 9.1.9. *Consider weights λ, η in the same block Γ. Then*

$$\deg(\underline{\lambda}\eta^\sigma) = \ell(\lambda) - \ell(\eta) + 2\ell(\sigma). \tag{9.1.10}$$

Proof. Since $\underline{\lambda}\eta$ is oriented, the weight η is obtained from λ permuting the $\wedge$'s and $\vee$'s on each lower fork of $\underline{\lambda}$. The degree of $\underline{\lambda}\eta$ is the sum of how much each $\vee$ of λ has been moved to the right to reach the corresponding $\vee$ of η; hence it is just the length of this permutation. In other words, if we let $z, z' \in D_{n,k}$ be the permutations corresponding to λ, η respectively, then we have $z = z'y$ for some $y \in \mathbb{S}_n$ with $\ell(z') = \ell(z) + \ell(y)$, and $\deg(\underline{\lambda}\eta) = \ell(y)$. □

9.2 The algebra structure

We connect now our diagrams with the commutative algebra from Chapter 8. Let us fix a block Γ with k $\wedge$'s and $n-k$ $\vee$'s.

Relations with polynomial rings

We associate to the weight λ the ring $R_\lambda = R_{\boldsymbol{b}^\lambda} = R/I_{\boldsymbol{b}^\lambda}$ (defined in §7.4), and we want to describe $\mathsf{Z}_{z,z'}$ from (8.3.1) diagrammatically.

Given an oriented lower fork diagram $\underline{\lambda}\eta^\sigma$, we define the polynomial

$$p_{\underline{\lambda}\eta^\sigma} = \mathfrak{S}'_\sigma(x_{\wedge_1^\eta}, \ldots, x_{\wedge_k^\eta}) \cdot \prod_{j=1}^{n-k} x_{\vee_j^\lambda} x_{\vee_j^\lambda + 1} \cdots x_{\vee_j^\eta - 1} \in R. \tag{9.2.1}$$

with $\mathfrak{S}'_\sigma(x_{\wedge_1^\eta}, \ldots, x_{\wedge_k^\eta})$ as defined in §7.5. Notice that the terms on the right always make sense because, since $\underline{\lambda}\eta^\sigma$ is oriented, $\vee_j^\eta \geq \vee_j^\lambda$ for all indices j (cf. Lemma 9.1.7). Often we will consider $p_{\underline{\lambda}\varepsilon^\sigma}$ as an element in the quotient R_λ, but it will be convenient to have a chosen lift in R. Notice that we have

$$\deg(p_{\underline{\lambda}\eta^\sigma}) = 2(\ell(\sigma) + \ell(\lambda) - \ell(\eta)). \tag{9.2.2}$$

Proposition 9.2.1. *Let $\lambda, \mu \in \Gamma$ be weights, and let z, z' be the corresponding elements of D. Let $\mathcal{Z}_{\mu,\lambda}$ be the graded vector space with homogeneous basis*

$$\{\underline{\mu}\eta^\sigma\overline{\lambda} \mid \underline{\mu}\eta^\sigma\overline{\lambda} \text{ is an oriented fork diagram}\}. \tag{9.2.3}$$

With $\widetilde{\mathsf{W}}_{z,z'}$ as defined in Theorem 8.3.5 we have an isomorphism of graded vector spaces

$$\begin{aligned} \Psi\colon \mathcal{Z}_{\mu,\lambda} &\longrightarrow \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\widetilde{\mathsf{W}}_{z,z'} \\ \underline{\mu}\eta^\sigma\overline{\lambda} &\longmapsto (1 \mapsto p_{\underline{\mu}\eta^\sigma}) + \widetilde{\mathsf{W}}_{z,z'}. \end{aligned} \tag{9.2.4}$$

Proof. First, note that $p_{\underline{\mu}\eta^\sigma} = p_{\underline{\mu}\eta^e}\mathfrak{S}'_\sigma(x_{\wedge_1^\eta}, \ldots, x_{\wedge_k^\eta})$. We have $p_{\underline{\mu}\eta^e} = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ by definition, where $\varepsilon_j = b_j^\mu - b_j^\eta$. By Lemma 9.1.8, $b_j^\lambda \geq b_j^\eta$ for every j, hence $\varepsilon_j \geq b_j^\mu - b_j^\lambda$. By Corollary 8.2.8, the map $1 \mapsto p_{\underline{\mu}\eta^\sigma}$ induces a well-defined morphism in $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$, hence also in the quotient.

Let us show that (9.2.4) is homogeneous of degree 0. The degree of the morphism $1 \mapsto p_{\underline{\mu}\eta^\sigma}$ in $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ is $\deg(p_{\underline{\mu}\eta^\sigma}) - \ell(w_k z') + \ell(w_k z)$, that is the same as $\deg(p_{\underline{\mu}\eta^\sigma}) - \ell(z') + \ell(z) = \deg(p_{\underline{\mu}\eta^\sigma}) - \ell(\mu) + \ell(\lambda)$. By (9.2.2) this is $\ell(\lambda) + \ell(\mu) - 2\ell(\eta) + 2\ell(\sigma)$. By Lemma 9.1.9, this is the same as $\deg(\underline{\mu}\eta^\sigma\overline{\lambda})$.

Next, we want to see that $p_{\underline{\mu}\eta^\sigma}$ is always a monomial of the basis (8.2.13). For this, note that we have $\varepsilon_j = 1$ exactly when $b_j^\mu = b_j^\eta + 1$. Moreover, the monomial $\mathfrak{S}'_\sigma(x_{\wedge_1^\eta}, \ldots, x_{\wedge_k^\eta}) = x_1^{i_1} \cdots x_n^{i_n}$ is by construction in the basis of R_η, that means that $i_j < b_j^\eta$ for every j. It follows that $i_j + \varepsilon_j < b_j^\mu$, hence $p_{\underline{\mu}\eta^\sigma}$ is a monomial of the basis (8.2.13).

We claim now that none of the $p_{\underline{\mu}\eta^\sigma}$ is in $\widetilde{\mathsf{W}}_{z,z'}$. Note that by construction the indeterminate $x_{\vee_j^\eta}$ does not appear in $p_{\underline{\mu}\eta^\sigma}$. By Lemma 9.1.7 we have $\vee_j^\lambda \leq \vee_j^\eta$ and both $\vee_j^\eta < \vee_{j+1}^\lambda$ and $\vee_j^\eta < \vee_{j+1}^\mu$. This means that both $x_{\vee_j^\lambda} \cdots x_{\vee_{j+1}^\lambda - 1}$ and $x_{\vee_j^\lambda} \cdots x_{\vee_{j+1}^\mu - 1}$ do not divide $p_{\underline{\mu}\eta^\sigma}$.

To conclude the proof, we need to construct an inverse of Ψ. Take a basis monomial $\mathfrak{m} = x_1^{i_1} \cdots x_1^{i_n} \in \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})$ that does not lie in $\widetilde{\mathsf{W}}_{z,z'}$. For every j, let ℓ_j be the maximum such that $x_{\vee_j^\mu} x_{\vee_j^\mu + 1} \cdots x_{\ell_j - 1}$ divide $\mathfrak{m}$. Since $\mathfrak{m}$ does not lie in $\widetilde{\mathsf{W}}_{z,z'}$, we obtain $\ell_j < \vee_{j+1}^\lambda$ and $\ell_j < \vee_{i+1}^\mu$. Form a weight η in the same block of λ and μ with the $\vee$'s in positions $\ell_1, \ldots, \ell_{n-k}$. By Lemma 9.1.7 the diagram $\underline{\mu}\eta\overline{\lambda}$ is oriented. Let $\mathfrak{m}'$ be the quotient of $\mathfrak{m}$ by $p_{\underline{\mu}\eta^e}$. By construction, $b_j^\mu = b_j^\eta$ if x_j does not appear in $p_{\underline{\mu}\eta^e}$, and $b_j^\mu = b_j^\eta + 1$ if x_j appears (with coefficient 1) in $p_{\underline{\mu}\eta^e}$. Hence, it is clear that $\mathfrak{m}'$ is a monomial $\mathfrak{S}'_\sigma(x_{\wedge_1^\eta}, \ldots, x_{\wedge_k^\eta})$. By construction, we get an inverse of the map (9.2.4), that is hence an isomorphism. □

As a consequence we obtain the following result, which completes the proof of Theorem 8.3.5:

Lemma 9.2.2. *For all $z, z' \in D$ we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\widetilde{\mathsf{W}}_{z,z'} = \dim_{\mathbb{C}} \mathsf{Z}_{z,z'}. \tag{9.2.5}$$

Proof. By Proposition 9.2.1, the dimension of $\mathrm{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\widetilde{\mathsf{W}}_{z,z'}$ is the same as $\dim_{\mathbb{C}} \mathcal{Z}_{\mu,\lambda}$, where $\lambda, \mu \in \Gamma$ are the weights corresponding to z, z'. This dimension is simply $k!$ times the number of unenhanced weights η such that $\underline{\mu}\eta\overline{\lambda}$ is oriented. By Lemma 9.6.6 this is the same as $\dim \mathsf{Z}_{z,z'}$. □

Being Γ and $D_{n,k}$ identified, we will often write C_λ for $\mathsf{C}_{w_k z}$, where $z \in D_{n,k}$ is the element corresponding to λ. If $a = \underline{\lambda}$ and $b = \overline{\lambda}$ we will even write C_a or C_b instead of C_λ. We will do similarly for $\mathsf{W}_{z,z'}$ and $\mathsf{Z}_{z,z'}$.

The algebra structure

Thanks to Proposition 9.2.1, we can define a graded algebra $A = A_\Gamma$ over $\mathbb{C}$. As a graded vector space, it has as homogeneous basis the elements

$$\{(\underline{\alpha}\lambda^\sigma\overline{\beta}) \mid \text{for all } \alpha, \lambda, \beta \in \Gamma, \sigma \in \mathbb{S}_k \text{ such that } \alpha \supset \lambda \subset \beta\} \tag{9.2.6}$$

that is the same as

$$\{(a\lambda^\sigma b) \mid \text{for all oriented fork diagrams } a\lambda b \text{ with } \lambda \in \Gamma\}. \tag{9.2.7}$$

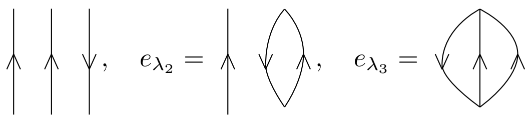
The degree on this basis is given by the degree function on fork diagrams. For $\lambda \in \Gamma$ we write e_λ for $(\underline{\lambda}\lambda\overline{\lambda})$. Note that the vectors e_λ give a basis of the degree 0 component of A.

Example 9.2.3. Let us consider a block Γ of weights with 2 $\wedge$'s and 1 $\vee$, that is

$$\Gamma = \{\lambda_1 = \wedge\wedge\vee,\ \lambda_2 = \wedge\vee\wedge,\ \lambda_3 = \vee\wedge\wedge\}. \tag{9.2.8}$$

Then the basis $\{e_{\lambda_i}\}$ of the degree 0 component is given by

(9.2.9) $e_{\lambda_1} =$, $e_{\lambda_2} =$, $e_{\lambda_3} =$



⊛

From Proposition 9.2.1 we get the following:

Corollary 9.2.4. *There is an isomorphism of graded vector spaces*

$$A \cong \bigoplus_{z,z' \in D} \mathrm{Hom}(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\mathsf{W}_{z,z'}. \tag{9.2.10}$$

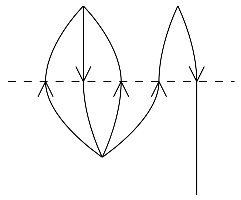
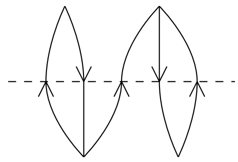
In particular, A inherits from the r.h.s. a graded algebra structure.

Proof. We just notice that the r.h.s. of (9.2.10) is indeed an algebra under composition of homomorphisms. In fact, if $f \in \mathrm{Hom}(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\mathsf{W}_{z,z'}$ and $g \in \mathrm{Hom}(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z''})/\mathsf{W}_{z',z''}$ then $g \circ f \in \mathrm{Hom}(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z''})/\mathsf{W}_{z,z''}$ is well-defined because a morphism which is divided by an illicit morphism is itself illicit. □

The product of two basis vectors of A can be computed explicitly using the isomorphism (9.2.10) as explained in details in the following Remark 9.2.5. Unfortunately, we are not able to describe the multiplication in the algebra A purely in terms of diagrams. Nevertheless, the diagrammatic description proves useful to find other properties of the algebra A, as we will explain in the following.

Remark 9.2.5. Explicitly, the multiplication of the basis vectors $(a\lambda^\sigma b)$ and $(c\mu^\tau d)$ can be computed in the following way. First, if $b^* \neq c$ then set it to be zero. Now suppose $b = c^*$. Then take $p_{c\mu^\tau}$ and $p_{a\lambda^\sigma}$ in R and multiply them. By construction, the result gives a well defined morphism of the corresponding Soergel modules: write it as a linear combination of the basis (8.2.13) and translate it in the diagrammatic algebra A using the isomorphism from Proposition 9.2.1.

EXAMPLE 9.2.6. Let

$a\lambda b =$  and $c\mu d =$ 

Let also $\sigma = s_1 \in \mathbb{S}_3$, $\tau = e \in \mathbb{S}_3$. We want to compute the product $(a\lambda^\sigma b)(c\mu^\tau d)$. First notice that $b^* = c$ (otherwise the product would be trivially zero). By (9.2.1) we have

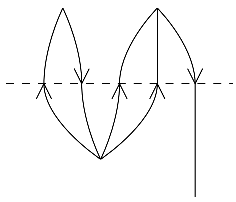
$$p_{a\lambda^\sigma} = x_1 \cdot x_1 x_4 \tag{9.2.11}$$

$$p_{c\mu^\tau d} = 1 \cdot x_1 \tag{9.2.12}$$

(for the computation of the polynomials $\mathfrak{S}'_\sigma$ and $\mathfrak{S}'_\tau$ we refer to Example 7.5.2). The product is $p_{a\lambda^\sigma} p_{c\mu^\tau} = x_1^3 x_4$. The $\boldsymbol{b}$–sequence of a is $(4,3,2,1,1)$, hence $x_1^3 x_4$ is not an element of the monomial basis (8.2.13) of R_a. We need to do some computations in the ring R_a: using the relations $x_1 + x_2 + x_3 + x_4 \equiv 0$ and $x_1^4 \equiv 0$ we have

$$x_1^3 x_4 \equiv -x_1^4 - x_1^3 x_2 - x_1^3 x_3 \equiv -x_1^3 x_2 - x_1^3 x_3. \tag{9.2.13}$$

This is now a linear combination of monomials of the basis (8.2.13). The monomial $-x_1^3 x_2$, although not zero in R_a, is of type (8.3.12), hence defines an illicit morphism and is zero in the quotient. We are left only with the monomial $\mathfrak{m} = x_1^3 x_3$. This is an element of (8.2.13) and, according to Theorem 8.3.5, does not define an illicit morphism. We need to translate it into a diagram via Proposition 9.2.1. The $\wedge\vee$–sequence corresponding to a is $\vee\wedge\wedge\wedge\vee$; in particular, the indices of the $\vee$'s are $1, 5$. Now, x_5 does not divide $\mathfrak{m}$, and the biggest index i such that $x_1 x_2 \cdots x_i \mid \mathfrak{m}$ is 1. Hence the monomial $\mathfrak{m}$ corresponds to a diagram $a\eta^\pi d$ where η has $\vee$'s in positions $2, 5$. Moreover, the permutation π is determined by $\mathfrak{S}'_\pi(x_1, x_3, x_4) = x_1^2 x_3$. By Example 7.5.2, π is the longest element of $\mathbb{S}_3$. Hence $(a\lambda^\sigma b)(c\mu^\tau d) = -(a\eta^\pi d)$, where

$a\eta d =$  ⊛

By construction, $p_{\underline{\lambda}\lambda^e} = 1$ for any $\lambda \in \Gamma$. Under the isomorphism of Proposition 9.2.1, the element e_λ is sent to $\mathrm{id}_{\mathsf{C}_{w_k z}} \in \mathrm{End}_R(\mathsf{C}_{s_k z})$, where $z \in D$ corresponds to λ; hence the elements e_λ satisfy

$$e_\lambda(a\mu^\sigma b) = \begin{cases} a\mu^\sigma b & \text{if } a = \overline{\lambda}, \\ 0 & \text{otherwise,} \end{cases} \qquad (a\mu^\sigma b)e_\lambda = \begin{cases} a\mu^\sigma b & \text{if } b = \underline{\lambda}, \\ 0 & \text{otherwise} \end{cases} \tag{9.2.14}$$

for any basis element $a\mu^\sigma b \in A$. That is, the vectors $\{e_\lambda \mid \lambda \in \Gamma\}$ are pairwise orthogonal idempotents whose sum is the identity $1 \in A$. The decomposition (9.2.10) can be written as

$$A = \bigoplus_{\lambda,\mu \in \Gamma} e_\lambda A e_\mu. \tag{9.2.15}$$

A basis of the summand $e_\lambda A e_\mu$ is

$$\{\underline{\lambda}\eta^\sigma\overline{\mu} \mid \text{ for all } \eta \in \Gamma, \sigma \in \mathbb{S}_k \text{ such that } \lambda \supset \eta \subset \mu\}. \tag{9.2.16}$$

Duality

Recall from §7.4 that for every $z, z' \in D$ we have an isomorphism

$$\begin{aligned} \Xi\colon \operatorname{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'}) &\longrightarrow \operatorname{Hom}_R(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z}), \\ (1 \mapsto p) &\longmapsto (1 \mapsto \boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{b}'} p), \end{aligned} \tag{9.2.17}$$

where $\boldsymbol{b}$ and $\boldsymbol{b}'$ are the $\boldsymbol{b}$–sequences of z and z', respectively, $\boldsymbol{b} - \boldsymbol{b}' = (b_1 - b'_1, \dots, b_n - b'_n)$ and the notation is as in (7.4.12).

Lemma 9.2.7. *Let $\lambda, \mu \in \Gamma$ and let z, z' be the corresponding elements of $D_{n,k}$. We have $\Xi(\mathsf{W}_{z,z'}) = \mathsf{W}_{z',z}$. Therefore the isomorphism Ξ descends to an isomorphism $\Xi\colon \mathsf{Z}_{z,z'} \to \mathsf{Z}_{z',z}$ such that*

$$\Xi(\Psi(\underline{\mu}\eta^\sigma\overline{\lambda})) = \Psi(\underline{\lambda}\eta^\sigma\overline{\mu}) \tag{9.2.18}$$

for all enhanced weights η^σ such that $\underline{\mu}\eta^\sigma\overline{\lambda}$ is oriented.

Proof. Let $\boldsymbol{b}$, $\boldsymbol{b}'$ be the $\boldsymbol{b}$–sequences of λ and μ, respectively. Note that

$$\frac{\boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{1}}}{\boldsymbol{x}^{\boldsymbol{b}'-\boldsymbol{1}}} = \boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{b}'} = \prod_{\vee^\lambda_j < \vee^\mu_j} (x_{\vee^\lambda_j} \cdots x_{\vee^\mu_j - 1}) \prod_{\vee^\mu_j < \vee^\lambda_j} (x^{-1}_{\vee^\mu_j} \cdots x^{-1}_{\vee^\lambda_j - 1}) \tag{9.2.19}$$

as an element in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. If $(1 \mapsto \mathfrak{m})$ is a monomial morphism of the basis (8.2.13) of $\operatorname{Hom}_R(\mathsf{C}_{w_k z'}, \mathsf{C}_{w_k z})$, it follows immediately that $(1 \mapsto \mathfrak{m}) \in \mathsf{W}_{z',z}$ if and only if $(1 \mapsto \boldsymbol{x}^{\boldsymbol{b}'-\boldsymbol{b}}\mathfrak{m}) \in \mathsf{W}_{z,z'}$, hence $\Xi(\mathsf{W}_{z',z}) = \mathsf{W}_{z,z'}$.

Moreover, it follows from the definition (9.2.1) of the polynomials $p_{\underline{\lambda}\eta^\sigma}$ and $p_{\underline{\mu}\eta^\sigma}$ that $p_{\underline{\mu}\eta^\sigma} = x^{\boldsymbol{b}-\boldsymbol{b}'} p_{\underline{\lambda}\eta^\sigma}$, hence $\Xi(\Psi(\underline{\mu}\eta^\sigma\overline{\lambda})) = \Psi(\underline{\lambda}\eta^\sigma\overline{\mu})$. □

Recall that for a fork diagram a we denote by a^* its mirror image. We define then a linear map $\star\colon A \to A$ by

$$(a\lambda b)^\star = (b^*\lambda a^*). \tag{9.2.20}$$

As a direct corollary of Lemma 9.2.7 we have:

Corollary 9.2.8. *The map $\star\colon A \to A$ is an anti-isomorphism, hence gives an isomorphism $A \cong A^{\mathrm{opp}}$.*

As follows from the definition, the algebra A only depends on the number of $\wedge$'s and $\vee$'s in the block Γ.

Definition 9.2.9. *We define $A_{n,k} = A_\Gamma$ for some block Γ with k $\wedge$'s and $n-k$ $\vee$'s.*

9.3 Cellular and properly stratified structure

This section is devoted to prove that the algebras $A_{n,k}$ are graded cellular and properly stratified, by constructing explicitly standard and proper standard modules. As before, we fix n and k and we let $A = A_{n,k}$. The following is inspired by [BS11].

Graded cellular structure

The key-step for proving that A is graded cellular is the following result:

Proposition 9.3.1. *Let $(a\lambda b)$ and $(c\mu d)$ be basis vectors of A. The product $(a\lambda^\sigma b)(c\mu^\tau d)$ is equal to:*

$$
(9.3.1) \qquad \begin{cases} 0 & \text{if } b \neq c^*, \\ (a\mu^\tau d) & \text{if } b = c^* = \overline{\lambda},\ \sigma = e \text{ and } (a\mu d) \text{ is oriented,} \\ \displaystyle\sum_{\ell(\tau')>\ell(\tau)} t^{\tau'}_{(a\lambda^\sigma c)}(\mu^\tau)\cdot(a\mu^{\tau'} d) + (\dagger) & \text{if } b = c^*,\ (a\mu d) \text{ is oriented, and either } b \neq \overline{\lambda} \text{ or } \sigma \neq e, \\ (\dagger) & \text{otherwise,} \end{cases}
$$

where:

(i) the scalars $t^{\tau'}_{(a\lambda^\sigma c)}(\mu^\tau)$ are independent of d;

(ii) $(\dagger)$ denotes a linear combination of basis vectors of A of the form $(a\nu^\chi d)$ with $\nu \succ \mu$.

Proof. If $b \neq c^*$ the claim is obvious, so let us suppose $b = c^*$. Suppose moreover that there is some weight ν such that $a\nu d$ is oriented (or equivalently that $\mathsf{Z}_{d,a}$ is not trivial) otherwise the claim is also obvious.

Of course we have

$$
(9.3.2) \qquad (a\lambda^\sigma b)(c\mu^\tau d) = \sum_{\nu\in\Gamma,\chi\in\mathbb{S}_k} C(\nu^\chi)\,(a\nu^\chi d).
$$

for some coefficients $C(\nu^\chi) \in \mathbb{C}$. Let us first prove that only terms with $\nu^\chi \succeq \mu^\tau$ occur in the sum, i.e. if $C(\nu^\chi) \neq 0$ then $\nu^\chi \succeq \mu^\tau$.

Before continuing, let us stress the subtlety in the argument. We want to understand which element of A corresponds to the morphism $1 \mapsto p_{a\lambda^\sigma} p_{c\mu^\tau}$: in general this morphism is not a monomial morphism of the basis (8.2.13), and we have to use the relations defining R_a to rewrite it as a linear combination of the monomial morphisms (8.2.13).

Let us fix some ν^χ such that $C(\nu^\chi) \neq 0$. First, let us prove that $\nu \succeq \mu$. By definition, $\nu \succeq \mu$ is equivalent to $\vee^\nu_j \geq \vee^\mu_j$ for all $j = 1,\dots,n-k$. Fix an index j. If $\vee^a_j \geq \vee^\mu_j$, then also $\vee^\nu_j \geq \vee^\mu_j$ by Lemma 9.1.7 (i). Hence suppose $\vee^a_j < \vee^\mu_j$. By construction, the monomial

$$
(9.3.3) \qquad (x_{\vee^a_j} x_{\vee^a_j+1}\cdots x_{\vee^\lambda_j-1})(x_{\vee^c_j} x_{\vee^c_j+1}\cdots x_{\vee^\mu_j-1})
$$

divides $p_{a\lambda^\sigma} p_{c\mu^\tau}$. In particular, since $\vee^\lambda_j \geq \vee^b_j = \vee^c_j$, also $x_{\vee^a_j} x_{\vee^a_j+1}\cdots x_{\vee^\mu_j-1}$ divides $p_{a\lambda^\sigma} p_{c\mu^\tau}$. Hence, if $p_{a\lambda^\sigma} p_{c\mu^\tau}$ is a monomial of the basis (8.2.13), we can conclude that $\vee^\nu_j \geq \vee^\mu_j$. Otherwise, we get the same conclusion using the technical Lemma 9.3.2 below.

Now, to check that $\nu^\chi \succeq \mu^\tau$ we have to show that in the case $\nu = \mu$ we have $\ell(\chi) \geq \ell(\tau)$. So let us suppose $\nu = \mu$. Since the multiplication is graded, we must have

$$
(9.3.4) \qquad \deg(a\lambda^\sigma b) + \deg(c\mu^\tau d) = \deg(a\mu^\chi d).
$$

If $a = \underline{\eta}$ we write $\ell(a)$ for $\ell(\eta)$, and similarly for b, c, d. Then, using Lemma 9.1.9, we get from (9.3.4)

$$
(9.3.5) \qquad 2\ell(\chi) = 2\ell(\tau) + 2\ell(\sigma) + 2\ell(b) - 2\ell(\lambda).
$$

Since $\lambda^\sigma b$ is oriented, by Lemma 9.1.6 the diagram b corresponds to some weight that is smaller or equal than λ in the Bruhat order. This implies that $\ell(\lambda) \leq \ell(b)$ (notice that under the identification of Γ with $D_{n,k}$, the Bruhat order on weights corresponds to the opposite of the usual Bruhat order on permutations). It follows that $\ell(\chi) \geq \ell(\tau)$. Hence we have shown that

$$(a\lambda^\sigma b)(c\mu^\tau d) = \sum_{\ell(\tau') \geq \ell(\tau)} C(\mu^{\tau'})\,(a\mu^{\tau'} d) + \sum_{\nu \succ \mu, \chi \in \mathbb{S}_k} C(\nu^\chi)\,(a\nu^\chi d). \tag{9.3.6}$$

Now suppose that $C(\mu^\xi) \neq 0$ for some $\xi \in \mathbb{S}_k$ with $\ell(\xi) = \ell(\tau)$. If we substitute in (9.3.5) $\chi = \xi$, we get $2\ell(\sigma) + 2\ell(b) - 2\ell(\lambda) = 0$. Since $\ell(b) \geq \ell(\lambda)$, we must have $\ell(\sigma) = 0$ and $\ell(b) = \ell(\lambda)$. This implies $\sigma = e$ and $b = \overline{\lambda}$. It is easy to see that in this case the morphism $1 \mapsto p_{a\lambda^\sigma} p_{c\mu^\tau}$ is an element of the monomial basis (8.2.13), and hence we have exactly $(a\lambda^\sigma b)(c\mu^\tau d) = (a\mu^\tau d)$. This shows the second case of (9.3.1) and also that if either $b \neq \overline{\lambda}$ or $\sigma \neq e$ then we can rewrite (9.3.6) as

$$(a\lambda^\sigma b)(c\mu^\tau d) = \sum_{\ell(\tau') > \ell(\tau)} C(\mu^{\tau'})\,(a\mu^{\tau'} d) + \sum_{\nu \succ \mu, \chi \in \mathbb{S}_k} C(\nu^\chi)\,(a\nu^\chi d). \tag{9.3.7}$$

Since $C(\mu^{\tau'})$ is automatically zero unless $a\mu d$ is oriented, this concludes the proof of (9.3.1) and (ii)

We are left to show (i). In order to determine the coefficients of (9.3.2), consider the expression of the polynomial $p_{a\lambda^\sigma} p_{c\mu^\tau}$ in the basis (8.2.12) of R_a:

$$p_{a\lambda^\sigma} p_{c\mu^\tau} = \sum_{\boldsymbol{j} \in J} \alpha_{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{j}}. \tag{9.3.8}$$

Define $J'' \subseteq J$ to be the subset of tuples $\boldsymbol{j}$ such that the morphism $(1 \mapsto \boldsymbol{x}^{\boldsymbol{j}}) \in \mathrm{Hom}_R(\mathsf{C}_d, \mathsf{C}_a)$ dies in the quotient $\mathsf{Z}_{d,a}$, since it is divided by some morphism of the type (ii) of Theorem 8.3.5. Let also $J' = J \setminus J''$. Fix some $j \in J'$; by Proposition 9.2.1, the basis morphism $(1 \mapsto \boldsymbol{x}^{\boldsymbol{j}}) \in \mathsf{Z}_{d,a}$ corresponds to a diagram $a\nu^\chi d$: then we have $C(\nu^\chi) = \alpha_j$. Notice that the unique dependence on d is in determining the subset $J'' \subseteq J$.

Now suppose $a\mu d$ is oriented, fix some $\tau' \in \mathbb{S}_k$ and let $(1 \mapsto \boldsymbol{x}^{\boldsymbol{j}}) \in \mathsf{Z}_{d,a}$ be the morphism of the basis 8.2.13 corresponding to the diagram $a\mu^{\tau'} d$. By the definition of orientation, for all i we have $\vee^d_i \leq \vee^\mu_i < \vee^d_{i+1}$ and $\vee^a_i \leq \vee^\mu_i < \vee^a_{i+1}$. Since $\boldsymbol{x}^{\boldsymbol{j}} \in \mathbb{C}[x_{\wedge^\mu_1}, \ldots, x_{\wedge^\mu_k}]$, neither $x_{\vee^d_i} x_{\vee^d_i + 1} \cdots x_{\vee^d_{i+1}}$ nor $x_{\vee^d_i} x_{\vee^d_i + 1} \cdots x_{\vee^a_{i+1}}$ can divide $\boldsymbol{x}^{\boldsymbol{j}}$. Hence for all d such that $a\mu d$ is oriented we have $(1 \mapsto \boldsymbol{x}^{\boldsymbol{j}}) \notin \mathsf{W}_{d,a}$ and with the notation of the preceding paragraph $\boldsymbol{j} \in J'$. Hence $C(\mu^{\tau'})$ is independent of d, proving (i). □

Lemma 9.3.2. *Fix some $\boldsymbol{b} \in \mathscr{B}$ and let m be an index such that $b_{m-1} = b_m$. Suppose that $x_m x_{m+1} \cdots x_{m+\ell}$ divides some polynomial $p \in R$. Write $p = \sum_{\boldsymbol{i}} \gamma_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}}$ in $R_{\boldsymbol{b}}$, where $\boldsymbol{x}^{\boldsymbol{i}}$ are monomials of the basis* (8.2.13). *Then $x_m x_{m+1} \cdots x_{m+\ell}$ divides all monomials $\boldsymbol{x}^{\boldsymbol{i}}$ for which $\gamma_{\boldsymbol{i}} \neq 0$.*

Proof. We will use the relations defining the ideal $I_{\boldsymbol{b}}$ to write the expression of p as a linear combination of basis monomials. Of course, it is sufficient to examine the case in which $p = \boldsymbol{x}^{\boldsymbol{j}}$ is a monomial.

Consider the maximum r for which $j_r \geq b_r$: if there is no such r, then p is a monomial of the basis (8.2.13) and we are done. If $r < m$ or $r > m + \ell$ then using the relation $h_{b_r}(x_1, \ldots, x_r) \equiv 0$ we can rewrite p as a linear combination of monomials $\boldsymbol{x}^{\boldsymbol{j}'}$ with $j'_r < j_r$

and $x_m x_{m+1} \cdots x_{m+\ell} \mid \boldsymbol{x}^{\boldsymbol{j}'}$: so by an induction argument we may suppose $m < r < m+\ell$. If $\ell \geq 1$ we can write

$$x_{r-1}x_r^{j_r} = x_{r-1}h_{j_r}(x_1,\dots,x_r) - \sum_{s=0}^{j_\ell-1} x_{\ell-1}x_r^s h_{j_r-s}(x_1,\dots,x_{r-1}). \tag{9.3.9}$$

Since $h_{j_r}(x_1,\dots,x_r) \in I_{\boldsymbol{b}}$ because $j_r \geq b_r$, and also $x_{r-1}h_{j_r}(x_1,\dots,x_{r-1}) \in I_{\boldsymbol{b}}$ by (7.3.3), the expression (9.3.9) gives in $R_{\boldsymbol{b}}$

$$x_{r-1}x_r^{j_r} \equiv \sum_{s=1}^{j_r-1} x_{r-1}x_r^s h_{j_r-s}(x_1,\dots,x_{r-1}) \mod I_{\boldsymbol{b}}. \tag{9.3.10}$$

In the special case $\ell = 0$, $r = m$, we write instead

$$x_m^{j_m} = h_{j_m}(x_1,\dots,x_m) - \sum_{s=0}^{j_m-1} x_m^s h_{j_m-s}(x_1,\dots,x_{m-1}), \tag{9.3.11}$$

that in $R_{\boldsymbol{b}}$ is

$$x_m^{j_m} \equiv -\sum_{s=1}^{j_m-1} x_m^s h_{j_m-s}(x_1,\dots,x_{m-1}) \mod I_{\boldsymbol{b}}, \tag{9.3.12}$$

since $j_m \geq b_{m-1}, b_m$. Both in (9.3.10) and (9.3.12), on the r.h.s. we have a sum of monomials $\boldsymbol{x}^{\boldsymbol{j}'}$ with $1 \leq j'_r < j_r$: by an induction argument on j_r, the claim follows. □

The main result of this subsection is the graded cellular algebra structure of A in the sense of [GL96], [HM10]. A *graded cellular algebra* is an associative unital algebra H together with a *graded cell datum* $(X, I, C, \deg)$ such that:

(GC1) X is a finite partially ordered set;

(GC2) $I(\lambda)$ is a finite set for each $\lambda \in X$;

(GC3) $C\colon \bigsqcup_{\lambda\in X} I(\lambda)\times I(\lambda) \to H$, $(i,j) \mapsto C^\lambda_{i,j}$ is an injective map whose image is a basis of H;

(GC4) the map $H \to H$, $C^\lambda_{i,j} \mapsto C^\lambda_{j,i}$ is an algebra anti-automorphism;

(GC5) if $\lambda \in X$ and $i,j \in I(\lambda)$ then for any $x \in H$ we have that

$$xC^\lambda_{i,j} \equiv \sum_{i'\in I(\lambda)} r_x(i',i)C^\lambda_{i',j} \pmod{H_{>\lambda}}, \tag{9.3.13}$$

where the scalar $r_x(i',i)$ is independent of j and $H_{>\lambda}$ is the subspace of H spanned by $\{C^\mu_{h,l} \mid \mu > \lambda \text{ and } k,l \in I(\mu)\}$;

(GC6) $\deg\colon \bigsqcup_{\lambda\in X} I(\lambda) \to \mathbb{Z}$, $i \mapsto \deg^\lambda_i$ is a function such that the $\mathbb{Z}$–grading on H defined by declaring $\deg C^\lambda_{i,j} = \deg^\lambda_i + \deg^\lambda_j$ makes H into a graded algebra.

We have:

Proposition 9.3.3. *The algebra $A = A_{k,n}$ is a graded cellular algebra with graded cell datum $((\Gamma\times\mathbb{S}_k, \preceq), I, C, \deg)$ where:*

(a) $I(\lambda^\sigma) = \{\alpha \in \Gamma \mid \alpha \subset \lambda\}$*;*

(b) C is defined by setting $C^{\lambda^\sigma}_{\alpha,\beta} = (\underline{\alpha}\lambda^\sigma\overline{\beta})$*;*

(c) $\deg^{\lambda^\sigma}_\alpha = \deg(\underline{\alpha}\lambda^\sigma) - \ell(\sigma)$.

Proof. Conditions (GC1-3) and (GC6) are direct consequences of the definitions. Condition (GC4) follows from Lemma 9.2.7. Condition (GC5) follows from Proposition 9.3.1. □

Properly stratified structure

As before, let us fix a block Γ and let $A = A_\Gamma$. We construct now explicitly a properly stratified structure on A (Theorem 9.3.7). The construction is similar to the one of [BS11].

An A–module will always be a finite-dimensional graded left A–module. Let $A-$gmod be the category of such modules. If $M = \bigoplus M_i$ is a graded A–module then we will write $M\langle j\rangle$ for the same module structure but with new grading defined by $(M\langle j\rangle)_i = M_{i-j}$. If M, N are graded A–modules then $\mathrm{Hom}_A(M, N)$ is a graded vector space.

Irreducible and projective A-modules

As we already noticed, the algebra A is unital with $1 = \sum_{\lambda\in\Gamma} e_\lambda$. Let $A_{>0}$ be the sum of all components of A of strictly positive degree. Then

$$A/A_{>0} = \bigoplus_{\lambda} e_\lambda \mathbb{C} e_\lambda \cong \bigoplus_{\lambda\in\Gamma} \mathbb{C} \tag{9.3.14}$$

is a split semisimple algebra, with a basis given by the images of the idempotents e_λ. The image of e_λ spans a one-dimensional $A/A_{>0}$–modules, and hence also a one dimensional A–module which we denote $L(\lambda)$. Thus $L(\lambda)$ is a copy of the field concentrated in degree 0, and $(a\mu^\sigma b) \in A$ acts on it as 1 if $(a\mu^\sigma b) = (\underline{\lambda}\lambda^e\overline{\lambda})$ and as 0 otherwise. The modules

$$\{L(\lambda)\langle j\rangle \mid \lambda \in \Gamma, j \in \mathbb{Z}\} \tag{9.3.15}$$

give a complete set of isomorphism classes of irreducible graded A–modules.

For any finite-dimensional graded A–module M, let M^* denote its graded dual. That is, $(M^*)_j = \mathrm{Hom}_\mathbb{C}(M_{-j}, \mathbb{C})$ and $x \in A$ acts on $f \in M^*$ by $xf(m) = f(x^\star m)$. As $e_\lambda^\star = e_\lambda$ we have that

$$L(\lambda)^* \cong L(\lambda) \tag{9.3.16}$$

for each $\lambda \in \Gamma$.

For each $\lambda \in \Gamma$ let also $P(\lambda) = Ae_\lambda$. This is a graded A–module with basis

$$\{(\underline{\nu}\mu^\sigma\overline{\lambda}) \mid \text{for all } \nu, \mu \in \Gamma \text{ and } \sigma \in \mathbb{S}_k \text{ with } \nu \subset \mu \supset \lambda\}. \tag{9.3.17}$$

The module $P(\lambda)$ is a projective module; in fact, it is the projective cover of $L(\lambda)$ in $A-$gmod. The modules

$$\{P(\lambda)\langle j\rangle \mid \lambda \in \Gamma, j \in \mathbb{Z}\} \tag{9.3.18}$$

give a complete set of isomorphism classes of indecomposable projective A–modules.

Cell modules and standard modules

We introduce now *standard modules.* The terminology will be motivated at the end of the section. For $\mu \in \Gamma$, define $\Delta(\mu)$ to be the vector space with basis

$$\{(\underline{\lambda}\mu^\tau| \mid \text{for all } \lambda \in \Gamma, \tau \in \mathbb{S}_k \text{ such that } \lambda \subset \mu\} \tag{9.3.19}$$

or, equivalently,

$$\{(c\mu^\tau| \mid \text{for all oriented lower fork diagrams } c\mu^\tau\}. \tag{9.3.20}$$

We put a grading on $\Delta(\mu)$ by defining the degree of $(c\mu^\tau|$ to be $\deg(c\mu^\tau)$, and we make it into an A–module through

$$(a\lambda^\sigma b)(c\mu^\tau| = \begin{cases} \sum_{\tau'\in\mathbb{S}_k} t^{\tau'}_{(a\lambda^\sigma b)}(\mu^\tau)(a\mu^{\tau'}| & \text{if } b=c^* \text{ and } (a\mu) \text{ is oriented,} \\ 0 & \text{otherwise,} \end{cases} \tag{9.3.21}$$

where $t^{\tau'}_{(a\lambda^\sigma b)}(\mu^\tau)$ is the scalar defined by Proposition 9.3.1. This is well-defined by the axiom (GC5). Note that $t^{\tau'}_{(a\lambda^\sigma b)}(\mu^\tau)$ was defined only for $\tau'=\tau$ or for $\ell(\tau')>\ell(\tau)$; otherwise we set $t^{\tau'}_{(a\lambda^\sigma b)}(\mu^\tau)=0$.

Proposition 9.3.4. *For $\lambda\in\Gamma$ enumerate the distinct elements of the set $\{\mu\in\Gamma \mid \mu\supset\lambda\}$ as $\mu_1,\mu_2,\dots,\mu_m=\lambda$ so that if $\mu_i\prec\mu_j$ then $i>j$. Set $M(0)=\{0\}$ and for $i=1,\dots,m$ define $M(i)$ to be the subspace of $P(\lambda)$ generated by $M(i-1)$ and the vectors*

$$\{(c\mu_i^\tau\overline{\lambda}) \mid \textit{for all oriented lower fork diagrams } c\mu_i^\tau\}. \tag{9.3.22}$$

Then

$$\{0\}=M(0)\subset M(1)\subset\cdots\subset M(m)=P(\lambda) \tag{9.3.23}$$

is a filtration of $P(\lambda)$ as an A–module such that

$$M(i)/M(i-1)\cong\Delta(\mu_i)\langle\deg\mu_i\overline{\lambda}\rangle \tag{9.3.24}$$

for each $i=1,\dots,m$.

Proof. It follows from Proposition 9.3.1 that $M(i)$ is indeed a submodule of $P(\lambda)$. The map

$$\begin{aligned} f_i\colon \Delta(\mu_i)\langle\deg\mu_i\overline{\lambda}\rangle &\longrightarrow M(i)/M(i-1) \\ (c\mu_i^\tau| &\longmapsto (c\mu_i^\tau\overline{\lambda})+M(i-1) \end{aligned} \tag{9.3.25}$$

gives an isomorphism of graded vector spaces. This map is of degree zero because

$$\deg(c\mu_i^\tau\overline{\lambda})=\deg(c\mu_i^\tau)+\deg(\mu_i\overline{\lambda}). \tag{9.3.26}$$

Through the vector space isomorphism (9.3.25) we can transport the A–module structure of $M(i)/M(i-1)$ to $\Delta(\mu_i)$. Using Proposition 9.3.1 we see that the module structure we get on $\Delta(\mu_i)$ is given by (9.3.21). Hence (9.3.21) defines indeed an A–module structure on $\Delta(\mu_i)$ and (9.3.25) is an isomorphism of A–modules. Since any weight μ arises as μ_i for some λ as in the statement of the theorem (take for example $\lambda=\mu$, $i=m$), we conclude also that (9.3.21) defines an A–module structure for every μ. ☐

Let us now define *cell modules* and *proper standard modules*. Let $\mu^\tau\in\Gamma\times\mathbb{S}_k$ be an enhanced weight and define $V(\mu^\tau)$ to be the vector space on basis

$$\{(\underline{\lambda}\mu^\tau\rfloor \mid \text{for all } \lambda\in\Gamma \text{ such that } \lambda\subset\mu\} \tag{9.3.27}$$

or, equivalently,

$$\{(c\mu^\tau\rfloor \mid \text{for all oriented lower fork diagrams } c\mu^\tau\}. \tag{9.3.28}$$

We remark that the difference with (9.3.19) and (9.3.20) is that now the permutation τ is fixed. As before, we put a grading on $V(\mu^\tau)$ by defining the degree of $(c\mu^\tau\rfloor$ to be $\deg(c\mu^\tau)$, and we make it into an A–module through

$$(a\lambda^\sigma b)(c\mu^\tau\rfloor = \begin{cases} t^{\tau}_{(a\lambda^\sigma b)}(\mu^\tau)\cdot(a\mu^\tau\rfloor & \text{if } b=c^* \text{ and } (a\mu) \text{ is oriented,} \\ 0 & \text{otherwise.} \end{cases} \tag{9.3.29}$$

From Proposition 9.3.1 we have that $t^{\tau}_{(a\lambda^{\sigma}b)}(\mu^{\tau})$ does not depend on τ. Hence (9.3.29) is the same as

$$(a\lambda^{\sigma}b)(c\mu^{\tau}\rfloor = \begin{cases}(a\mu^{\tau}\rfloor & \text{if } b = c^* = \overline{\lambda},\ \sigma = e \text{ and } (a\mu) \text{ is oriented,}\\ 0 & \text{otherwise.}\end{cases} \tag{9.3.30}$$

It will follow from Proposition 9.3.5 that this indeed defines an A–module structure. It is clear from (9.3.30) that all cell modules $V(\mu^{\tau})$ for a fixed μ are isomorphic (up to a degree shift). Explicitly we have $V(\mu^{\tau}) \cong V(\mu^{e})\langle \deg(\tau)\rangle$. We recall that $\deg(\tau) = 2\ell(\tau)$. Therefore for a weight $\mu \in \Gamma$ we define the *proper standard module* $\overline{\Delta}(\mu)$ to be the vector space with basis

$$\{(\underline{\lambda}\mu\rfloor \mid \text{for all } \lambda \in \Gamma \text{ such that } \lambda \subset \mu\} \tag{9.3.31}$$

or, equivalently,

$$\{(c\mu\rfloor \mid \text{for all unenhanced oriented lower fork diagrams } c\mu\}. \tag{9.3.32}$$

We put a grading on $\overline{\Delta}(\mu)$ by defining the degree of $(c\mu\rfloor$ to be $\deg(c\mu)$, and we make it into an A–module through

$$(a\lambda^{\sigma}b)(c\mu\rfloor = \begin{cases}(a\mu\rfloor & \text{if } b = c^* = \overline{\lambda},\ \sigma = e \text{ and } (a\mu) \text{ is oriented,}\\ 0 & \text{otherwise.}\end{cases} \tag{9.3.33}$$

Of course we have an isomorphism $\overline{\Delta}(\mu) \cong V(\mu^{e})$.

Proposition 9.3.5. *Let $\mu \in \Gamma$. Enumerate the elements of $\mathbb{S}_k$ as $\sigma_1, \sigma_2, \dots, \sigma_{k!} = e$ in such a way that if $\ell(\sigma_i) > \ell(\sigma_j)$ then $i < j$. Let $N(0) = \{0\}$ and for $i = 1, \dots, k!$ define $N(i)$ to be the subspace of $\Delta(\mu)$ generated by $N(i-1)$ and the vectors*

$$\{(c\mu^{\sigma_i}| \mid \text{ for all oriented lower fork diagrams } c\mu^{\sigma_i}\}. \tag{9.3.34}$$

Then

$$\{0\} = N(0) \subset N(1) \subset \dots \subset N(k!) = \Delta(\mu) \tag{9.3.35}$$

is a filtration of $\Delta(\mu)$ as an A–module such that

$$N(i)/N(i-1) \cong \overline{\Delta}(\mu)\langle 2\ell(\sigma_i)\rangle. \tag{9.3.36}$$

Proof. It follows from Proposition 9.3.1 that $N(i)$ is indeed a submodule of $\Delta(\mu)$. The map

$$\begin{aligned} f_i \colon \overline{\Delta}(\mu)\langle 2\ell(\sigma_i)\rangle &\longrightarrow N(i)/N(i-1)\\ (c\mu\rfloor &\longmapsto (c\mu^{\sigma_i}| + N(i-1)\end{aligned} \tag{9.3.37}$$

gives an isomorphism of graded vector spaces. The degree shift comes from

$$\deg(c\mu^{\sigma_i}) = \deg(c\mu) + 2\ell(\sigma_i). \tag{9.3.38}$$

Through f_i we can transport the A–module structure of $N(i)/N(i-1)$ to $\overline{\Delta}(\mu)$. The module structure on $N(i)/N(i-1)$ is described by (9.3.21). It follows that $\overline{\Delta}(\mu)\langle 2\ell(\sigma_i)\rangle$ is endowed with the module structure of $V(\mu^{\sigma_i})$ described by (9.3.29); this shows in particular that (9.3.29) defines indeed an A–module. We have already argued that this is the same as the module structure described by (9.3.33) on $\overline{\Delta}(\mu)$. $\square$

Proposition 9.3.6. *For $\mu \in \Gamma$, let $Q(j)$ be the submodule of $\overline{\Delta}(\mu)$ spanned by all homogeneous vectors of degree $\geq j$. Then*

$$\overline{\Delta}(\mu) = Q(0) \supseteq Q(1) \supseteq Q(2) \supseteq \cdots \tag{9.3.39}$$

is a (finite) filtration of $\overline{\Delta}(\mu)$ as an A–module such that

$$Q(j)/Q(j+1) \cong \bigoplus_{\substack{\lambda \subset \mu \text{ with} \\ \deg(\underline{\lambda}\mu)=j}} L(\lambda)\langle j \rangle \tag{9.3.40}$$

for all $j \geq 0$.

Proof. Since A is positively graded, it is clear that each $Q(j)$ is a submodule. The quotient $Q(j)/Q(j+1)$ has basis

$$\{(\underline{\lambda}\mu| + Q(j+1) \mid \text{for all } \lambda \in \Gamma \text{ such that } \lambda \subset \mu \text{ and } \deg(\underline{\lambda}\mu) = j\}. \tag{9.3.41}$$

We need to show that for each λ which occurs the one-dimensional subspace $Q'(\lambda)$ of $Q(j)/Q(j+1)$ spanned by $(\underline{\lambda}\mu| + Q(j+1)$ is an A–module isomorphic to $L(\lambda)\langle j\rangle$. It is clear where the degree shift comes from. If $x \in A$ has $\deg(x) > 0$ then obviously x vanishes on $Q(j)/Q(j+1)$. So let us consider $e_\nu \in A$. It follows from (9.3.33) that

$$e_\nu \cdot (\underline{\lambda}\mu| = \begin{cases} (\underline{\lambda}\mu| & \text{if } \nu = \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{9.3.42}$$

Hence $Q'(\lambda)$ is isomorphic to $L(\lambda)\langle j\rangle$. □

The Grothendieck group

The Grothendieck group $K(A-\mathrm{gmod})$ of $A-\mathrm{gmod}$ is a free $\mathbb{Z}$–module with basis given by equivalence classes of simple modules. The group $K(A-\mathrm{gmod})$ becomes a $\mathbb{Z}[q, q^{-1}]$–module if we set $q[M] = [M\langle 1\rangle]$ for all graded A–modules M. It is also free as a $\mathbb{Z}[q, q^{-1}]$–module, with basis $\{[L(\lambda)] \mid \lambda \in \Gamma\}$.

For $\lambda, \mu \in \Gamma$, define

$$d_{\lambda,\mu} = \begin{cases} q^{\deg(\underline{\lambda}\mu)} & \text{if } \lambda \subset \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{9.3.43}$$

By Propositions 9.3.4, 9.3.6 and 9.3.5 respectively we have that

$$[P(\lambda)] = \sum_{\mu \in \Gamma} d_{\lambda,\mu}[\Delta(\mu)], \tag{9.3.44}$$

$$[\overline{\Delta}(\mu)] = \sum_{\lambda \in \Gamma} d_{\lambda,\mu}[L(\lambda)], \tag{9.3.45}$$

$$[\Delta(\mu)] = [k]_0! \cdot [\overline{\Delta}(\mu)], \tag{9.3.46}$$

Since $d_{\lambda,\lambda} = 1$, the matrix $(d_{\lambda,\mu})$ is upper triangular with determinant 1, hence it is invertible over $\mathbb{Z}[q, q^{-1}]$. In particular, the proper standard modules give also a $\mathbb{Z}[q, q^{-1}]$–basis of $K_0(A-\mathrm{gmod})$. On the other side, notice that the matrix $[k]_0!\,\mathrm{Id}$ is not invertible over $\mathbb{Z}[q, q^{-1}]$ unless $k = 0, 1$. In particular, standard and projective modules do not give a basis of the Grothendieck group in general (cf. also §6.1).

Our algebra is graded properly stratified in the sense of Definition 5.3.10:

Theorem 9.3.7. *For every block Γ the algebra A_Γ is a graded properly stratified algebra. The partially ordered set indexing the simple modules is $(\Gamma, \prec)$. The modules $\Delta(\mu)$ and $\overline{\Delta}(\mu)$ are the standard and proper standard modules respectively. Moreover, the diagonal matrix of the multiplicity numbers of the proper standard modules in the filtrations of the standard modules is a multiple of the identity, given by* (9.3.46).

Proof. We already noticed that $A = A_\Gamma$ is a finite-dimensional associative unital graded algebra over $\mathbb{C}$ with a duality with respect to which the simple modules are self-dual. For $\lambda \in \Gamma$ let $\mathbb{L}(\lambda) = L(\lambda)$ and define $\mathbb{P}(\lambda)$, $\mathbb{\Delta}(\lambda)$ and $\overline{\mathbb{\Delta}}(\lambda)$ as in Definition 5.3.10 (i), (ii), (iii). By the uniqueness of the projective cover we have $P(\lambda) \cong \mathbb{P}(\lambda)$. From (9.3.46) and (9.3.45) we have that $\Delta(\lambda)$ is a quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = 0$ for every $\mu \succ \lambda$; from Proposition 9.3.4 it follows that it is maximal with this property, hence $\Delta(\lambda) \cong \mathbb{\Delta}(\lambda)$. By the same argument using (9.3.45) and Proposition 9.3.5 we get that $\overline{\Delta}(\lambda) \cong \overline{\mathbb{\Delta}}(\lambda)$. Hence we need to show that properties (PS1-3) are satisfied. But this follows immediately from Propositions 9.3.4, 9.3.5 and 9.3.6. $\square$

9.4 A bilinear form and self-dual projective modules

We define now a bilinear form on A and we determine which projective modules are self-dual. As above, we fix a block Γ with k $\wedge$'s and $n-k$ $\vee$'s.

Defect

Let λ be a weight in some block Γ. We say that an $\wedge$ of λ is *initial* if it has no $\vee$'s on its left. Let us define the *defect* of λ to be

$$\operatorname{def}(\lambda) = \#\{\text{non initial } \wedge\text{'s of } \lambda\}. \tag{9.4.1}$$

We have the following elementary result:

Lemma 9.4.1. *Let $\lambda \in \Gamma$. The maximal degree of $e_\lambda A e_\lambda$ is $k(k-1) + 2\operatorname{def}(\lambda)$ and the homogeneous subspace of maximal degree of $e_\lambda A e_\lambda$ is one dimensional.*

Proof. It is straightforward to notice that the homogeneous subspace of maximal degree of $e_\lambda A e_\lambda$ is one dimensional: the diagram of maximal degree is $\underline{\lambda}\eta^\sigma\overline{\lambda}$, where η orients every fork of $\underline{\lambda}$ with maximal degree (that is, each $\vee$ is at the rightmost position) and σ is the longest element of $\mathbb{S}_k$. By definition, the degree of this diagram is obtained by adding $2\ell(\sigma)$ to the sum of $2(m-1)$ for every m–fork of $\underline{\lambda}$. Hence, this degree is $2\ell(\sigma)$ plus twice the number of non-initial $\wedge$'s of λ. $\square$

Lemma 9.4.2. *Consider $\lambda, \mu \in \Gamma$ and suppose that $e_\lambda A e_\mu$ is not trivial. Then the homogeneous subspaces of minimal and maximal degree of $e_\lambda A e_\mu$ are one dimensional. The minimal degree is*

$$\sum_{i=1}^{n-k} \left|\vee_i^\lambda - \vee_i^\mu\right| \tag{9.4.2}$$

and the maximal degree is

$$k(k-1) + \sum_{i=1}^{n-k} \left|\vee_{i+1}^{\min} - 1 - \vee_i^\lambda\right| + \left|\vee_{i+1}^{\min} - 1 - \vee_i^\mu\right| \tag{9.4.3}$$

where $\vee_i^{\min} = \min\{\vee_i^\lambda, \vee_i^\mu\}$ *and* $\vee_{n-k+1}^\lambda = n+1$.

If $\operatorname{def}(\lambda) \geq \operatorname{def}(\mu)$ *then the sum of* (9.4.2) *and* (9.4.3) *is equal to the maximal degree of* $e_\lambda A e_\lambda$.

Proof. We use the condition (9.1.8) to determine if a diagram is oriented. The minimal degree diagram is $\underline{\lambda}\eta^e\overline{\mu}$ where $\vee_i^\eta = \max\{\vee_i^\lambda, \vee_i^\mu\}$. The maximal degree diagram is $\underline{\lambda}\eta^{w_k}\overline{\mu}$ where $w_k \in \mathbb{S}_k$ is the longest element and $\vee_i^\eta = \min\{\vee_{i+1}^\lambda, \vee_{i+1}^\mu\} - 1$. Computing their degrees we obtain exactly (9.4.2) and (9.4.3).

Let us now check the last assertion. The sum of (9.4.2) and (9.4.3) is

$$k(k-1) + \sum_{i=1}^{n-k} 2\left(\vee_{i+1}^{\min} - 1 - \vee_i^{\min}\right). \tag{9.4.4}$$

This is the maximal degree of $e_\eta A e_\eta$ where $\eta \in \Gamma$ is the weight with $\vee_i^\eta = \vee_i^{\min}$. Of course $\operatorname{def}(\eta) = \max\{\operatorname{def}(\lambda), \operatorname{def}(\mu)\}$, and by Lemma 9.4.1 the maximal degrees of $e_\lambda A e_\lambda$ and $e_\eta A e_\eta$ are the same. □

Notice that a weight λ is of maximal defect if and only if it starts with a $\vee$. If λ is not of maximal defect, let $\tilde\lambda$ be obtained from λ by swapping the first $\vee$ and the first $\wedge$. Otherwise, let $\tilde\lambda = \lambda$. In particular, $\tilde\lambda$ is always of maximal defect.

Lemma 9.4.3. *For every* $\lambda \in \Gamma$ *the socle of* $P(\lambda)$ *contains a copy of* $L(\tilde\lambda)$*, possibly shifted in the degree.*

In fact, the socle of $P(\lambda)$ is simple, hence it is isomorphic to a degree shift of $L(\tilde\lambda)$, but we will not need this in what follows.

Proof. It is straightforward to check that the diagram of maximal degree in Ae_λ is of type $\underline{\tilde\lambda}\eta^\sigma\overline{\lambda}$. The claim follows. □

A bilinear form

Let $\Gamma^\vee \subseteq \Gamma$ be the subset consisting of weights of maximal defect. For every $\lambda \in \Gamma^\vee$, let us choose a non-zero element $\xi_\lambda^{\max} \in e_\lambda A e_\lambda$ of maximal degree (for example, we can choose it to be the diagram $\underline{\lambda}\eta^\sigma\overline{\lambda}$ of the previous proof). For every element $z \in A$ write $e_\lambda z e_\lambda = t\xi_\lambda^{\max} +$ terms of lower degree, and set $\Theta_\lambda(z) = t$. Moreover, define

$$\Theta(z) = \sum_{\lambda \in \Gamma^\vee} \Theta_\lambda(z). \tag{9.4.5}$$

Finally, define a bilinear form $\theta \colon A \times A \to \mathbb{C}$ by setting $\theta(y,z) = \Theta(yz)$. Obviously, this form is associative in the sense that $\theta(y, zw) = \theta(yz, w)$ for all $y, z, w \in A$.

Lemma 9.4.4. *For every* λ*, the form* θ *restricted to* $e_\lambda A e_\lambda$ *is associative, symmetric and non-degenerate.*

Proof. Let λ correspond to $z \in D$. Up to a degree shift, $e_\lambda A e_\lambda \cong \mathsf{Z}_{z,z}$. Since $\mathsf{Z}_{z,z}$ is commutative, the form θ is symmetric on $e_\lambda A e_\lambda$. Consider the monomial basis $\{1 \mapsto \boldsymbol{x}^{\boldsymbol{i}}\}$ that consists of the elements of (8.2.13) that are not divisible by (8.3.12). It is clear that for every element φ in that basis there exists exactly one element φ^T in the same basis with $\theta(\varphi, \varphi^T) \neq 0$. This proves that the form is non-degenerate. □

Let $e^\vee = \sum_{\lambda\in\Gamma^\vee} e_\lambda$.

Lemma 9.4.5. *The form θ restricted to $e^\vee A \times Ae^\vee$ is non-degenerate.*

Proof. We may take $t \in e_\mu A e_\lambda$ for some λ of maximal defect and suppose $\theta(y,t) = 0$ for all $y \in e_\lambda A e_\mu$. Let y_0 be a generator of the minimal-degree subspace of $e_\lambda A e_\mu$ (which by Lemma 9.4.2 is one-dimensional). In particular, $\theta(y', y_0 t) = \theta(y' y_0, t) = 0$ for all $y' \in e_\lambda A e_\lambda$. By Lemma 9.4.4, this implies that $y_0 t = 0$. From the following Lemma 9.4.6 it follows then that $t = 0$.

The vice versa follows because $\theta(y,t) = \theta(t^\star, y^\star)$. □

Lemma 9.4.6. *Suppose λ is of maximal defect and let $0 \neq t \in e_\mu A e_\lambda$. Let also $0 \neq y_0 \in e_\lambda A e_\mu$ be of minimal degree. Then $y_0 t \neq 0$.*

Proof. First, let $0 \neq t_0 \in e_\mu A e_\lambda$ be of minimal degree, and let us prove that $y_0 t_0 \neq 0$. By definition, $y_0 t_0 \colon 1 \mapsto \boldsymbol{x}^{\boldsymbol{h}}$, where $h_i = \left| b_i^\lambda - b_i^\mu \right| \in \{0,1\}$. First let us suppose that $1 \mapsto \boldsymbol{x}^{\boldsymbol{h}}$ is an element of the basis (8.2.13), that is $h_i < b_i^\lambda$ for every i. It is quite easy to argue that for every i there exist an index j with $\vee_i^\lambda \leq j < \vee_{i+1}^\lambda$ and $b_i^\lambda = b_i^\mu$; in fact it is sufficient to choose $j = \vee_i^\mu$ if $\vee_i^\mu \geq \vee_i^\lambda$ or $j = \vee_i^\lambda$ otherwise. This means that $1 \mapsto \boldsymbol{x}^{\boldsymbol{h}}$ is not illicit (cf. Theorem 8.3.5), hence it is not zero.

We should now consider the case in which $1 \mapsto \boldsymbol{x}^{\boldsymbol{h}}$ is not an element of the basis (8.2.13). This happens if $h_i = 1$ for some i with $b_i^\lambda = 1$ and $b_i^\mu = 2$. Let j be such that $\vee_j^\lambda$ is the rightmost $\vee$ in a position $\vee_j^\lambda \leq i$. It is easy to argue that for $e_\mu A e_\lambda$ to be non-trivial we must actually have $\vee_j^\lambda < i$. Let also $i' = \vee_j^{\max} = \max\{\vee_j^\lambda, \vee_j^\mu\} < i$. Then we have $b_{i'}^\lambda = b_{i'}^\mu \geq 2$. Using the relation $h_1(x_1, \ldots, x_i) \equiv 0$ to write $\boldsymbol{x}^{\boldsymbol{h}}$ in our fixed monomial basis we get in particular a term divided by $x_{i'}$. Applying the techniques of the previous paragraph to this term we get that $y_0 t_0 \neq 0$: the only thing to notice is that $x_{\vee_j^\lambda} x_{\vee_j^\lambda + 1} \cdots x_{\vee_{j+1}^\lambda - 1}$ never divides a monomial basis element, since $b^\lambda_{\vee_{j+1}^\lambda - 1} = 1$.

Now, it follows from the proof of Lemma 9.4.4 that there is some element $u \in R$ such that $y_0 t_0 u$ generates the maximal degree subspace of $e_\lambda A e_\lambda$. In particular $y_0 t_0 u \neq 0$. By Lemma 9.4.2, $t_0 u$ is of maximal degree in $e_\mu A e_\lambda$. It is then clear by our characterization of $e_\mu A e_\lambda$ that there exists an element $u' \in R$ such that $u' t = t_0 u$. Now $y_0 t u' = y_0 u' t = y_0 t_0 u \neq 0$ implies that $y_0 t \neq 0$. □

Self-dual projective modules

Finally, we can determine which indecomposable projective modules are self-dual.

Lemma 9.4.7. *Let λ be of maximal defect. Then $P(\lambda)$ is self-dual up to a degree shift. In particular, it is an injective module.*

Proof. By Lemma 9.4.5, the map

$$y \longmapsto \theta(y^\star, \cdot) \tag{9.4.6}$$

defines an isomorphism between $P(\lambda)$ and its dual up to a degree shift. □

Theorem 9.4.8. *Let $\lambda \in \Gamma$. Then $P(\lambda)$ is an injective module if and only if λ is of maximal defect.*

Proof. By Lemma 9.4.7 if λ is of maximal defect then $P(\lambda)$ is injective. On the other side, suppose $P(\lambda)$ is injective. Then $P(\lambda)$ is a tilting module, and by standard theory it is self dual (as an ungraded module). In particular, the socle of $P(\lambda)$ is $L(\lambda)$. By Lemma 9.4.3, λ has to be of maximal defect. □

EXAMPLE 9.4.9. Consider the block $\Gamma = \Gamma_{3,2}$. The weights of Γ are $\wedge\wedge\vee$, $\wedge\vee\wedge$ and $\vee\wedge\wedge$, and have defect 2, 1 and 0 respectively. In particular, the only indecomposable projective-injective module in $A_{3,2}$–mod is $P(\vee\wedge\wedge)$. Indeed, one can explicitly compute that the socle of both $P(\wedge\wedge\vee)$, $P(\wedge\vee\wedge)$ and $P(\vee\wedge\wedge)$ is isomorphic to $L(\vee\vee\wedge)$ (up to a degree shift), and that only $P(\vee\wedge\wedge)$ is self-dual. ⊛

REMARK 9.4.10. Let w^0 be the longest element of D. The weights λ of maximal defect are exactly the ones that correspond to permutations $w_k z$, $z \in D$ which are in the same right Kazhdan-Lusztig cell of $w_k w^0$. This can be easily checked using the equivalence between Kazhdan-Lusztig cells and Knuth equivalence (see [KL79, §5]), and either applying directly the definition of Knuth equivalence or using its description through the Robinson-Schensted correspondence (cf. [Knu73, §5.1.4] and also [Du05]). This gives another proof of a particular case of [MS08b, Theorem 5.1] (for the relation with the category $\mathcal{O}$ see §9.6 below).

9.5 Diagrammatic functors $\mathbf{E}_k$ and $\mathbf{F}_k$

The goal of this section is to construct functors

(9.5.1)
$$
\begin{tikzcd} A_{n,k}\text{–gmod} \arrow[r, bend left, "\mathbf{F}_k \otimes \bullet"] & A_{n,k+1}\text{–gmod}, \arrow[l, bend left, "\mathbf{E}_k \otimes \bullet"] \end{tikzcd}
$$

which will turn out to be the diagrammatic version of the functors $\mathcal{F}_k$ and $\mathcal{E}_k$ defined in §6.5 (see §9.6 below).

Let us fix an integer n. For all $k = 0, \dots, n$ let us set in this section $A_k = A_{n,k}$. Let $\Gamma_k^\vee$ be the subset of weights of Γ_k of maximal defect, and let $\Gamma_k^\wedge = \Gamma_k - \Gamma_k^\vee$. Notice that given $\lambda \in \Gamma_k$ we have $\lambda \in \Gamma_k^\vee$ if and only if the leftmost symbol of λ is a $\vee$, and conversely $\lambda \in \Gamma_k^\wedge$ if and only if the leftmost symbol of λ is an $\wedge$. Let also

(9.5.2)
$$
e_k^\vee = \sum_{\lambda \in \Gamma_k^\vee} e_\lambda \qquad \text{and} \qquad e_k^\wedge = \sum_{\lambda \in \Gamma_k^\wedge} e_\lambda.
$$

Consider now $P_k^\vee = A_k e_k^\vee$, that is the sum of all indecomposable projective-injective A_k–modules (cf. Theorem 9.4.8). Our next goal is to describe a right A_{k+1}–action on it.

For any $\lambda \in \Gamma_k^\vee$ let $\lambda^{(\wedge)} \in \Gamma_{k+1}^\wedge$ be the weight obtained from λ by substituting the leftmost symbol, which by assumption is a $\vee$, with an $\wedge$. Conversely, given $\mu \in \Gamma_{k+1}^\wedge$ let $\mu^{(\vee)} \in \Gamma_k^\vee$ be the weight obtained from μ after substituting the leftmost symbol, which by assumption is an $\wedge$, with a $\vee$. Clearly the map $\lambda \mapsto \lambda^{(\wedge)}$ defines a bijection $\Gamma_k^\vee \to \Gamma_{k+1}^\wedge$ with inverse $\mu \mapsto \mu^{(\vee)}$.

Lemma 9.5.1. *Let $\lambda, \mu \in A_{k+1}^\wedge$. Then we have an isomorphism of R–modules*

(9.5.3)
$$
\operatorname{Hom}_R(\mathsf{C}_\lambda, \mathsf{C}_\mu) \cong \operatorname{Hom}_R(\mathsf{C}_{\lambda^{(\vee)}}, \mathsf{C}_{\mu^{(\vee)}})
$$

that induces a surjective algebra homomorphism

(9.5.4)
$$
e_\mu A_{k+1} e_\lambda \longrightarrow e_{\mu^{(\vee)}} A_k e_{\lambda^{(\vee)}}.
$$

Proof. Since the $\boldsymbol{b}$–sequences of λ and $\lambda^{(\vee)}$ are the same, the first claim follows. By Theorem 8.3.5 the bimodule $\mathsf{W}_{\lambda^{(\vee)},\mu^{(\vee)}}$ is generated by $\mathsf{W}_{\lambda,\mu}$ together with the morphism $1 \mapsto x_1 \cdots x_j$ where $j = \min\{\vee_1^\lambda, \vee_1^\mu\}$. Hence $e_{\mu^{(\vee)}} A_k e_{\lambda^{(\vee)}}$ is a quotient of $e_\mu A_{k+1} e_\lambda$. □

Corollary 9.5.2. *There exists a surjective algebra homomorphism*

$$\Psi\colon e_{k+1}^\wedge A_{k+1} e_{k+1}^\wedge \longrightarrow e_k^\vee A_k e_k^\vee. \tag{9.5.5}$$

Proposition 9.5.3. *The homomorphism Ψ induces a well-defined surjective algebra homomorphism*

$$\begin{aligned} A_{k+1}/A_{k+1}e_{k+1}^\vee A_{k+1} &\longrightarrow e_k^\vee A_k e_k^\vee \\ [x] &\longmapsto \Psi(e_{k+1}^\wedge x e_{k+1}^\wedge) \end{aligned} \tag{9.5.6}$$

for $x \in A_{k+1}$.

Proof. We need to show that (9.5.6) does not depend on the particular representative x chosen, or equivalently that $\Psi(e_{k+1}^\wedge x e_{k+1}^\wedge) = 0$ for all $x \in A_{k+1}e_{k+1}^\vee A_{k+1}$. By linearity, it suffices to consider the case $x \in A_{k+1}e_\nu A_{k+1}$ for some $\nu \in \Gamma_{k+1}^\vee$. Pick such an x and fix $\lambda, \mu \in \Gamma_{k+1}^\wedge$. Choose some morphism $f \in \mathrm{Hom}_R(\mathsf{C}_\lambda, \mathsf{C}_\mu)$ which corresponds to $e_\mu x e_\lambda$ in the quotient $\mathrm{Hom}_R(\mathsf{C}_\lambda, \mathsf{C}_\mu)/\mathsf{W}_{\lambda,\mu}$. Since $x \in A_{k+1}e_\nu A_{k+1}$, we can write f as the composition $f_2 \circ f_1$ with $f_1 \in \mathrm{Hom}_R(\mathsf{C}_\lambda, \mathsf{C}_\nu)$ and $f_2 \in \mathrm{Hom}_r(\mathsf{C}_\nu, \mathsf{C}_\mu)$. By Corollary 8.2.8, f_1 is divisible by $x_1 \cdots x_{\vee_1^\lambda}$, hence f is as well. By Theorem 8.3.5 (cf. also the proof of Lemma 9.5.1 above) we have $f \in \mathsf{W}_{\lambda^{(\vee)},\mu^{(\vee)}}$, and hence $\Psi(e_\mu x e_\lambda) = 0$. Since λ and μ were chosen arbitrarily in $\Gamma_{k+1}^\wedge$, it follows that $\Psi(e_{k+1}^\wedge x e_{k+1}^\wedge) = 0$.

The surjectivity of (9.5.6) is a direct consequence of the surjectivity of (9.5.5). □

The functor $\mathbf{F}_k$

Let us now define $\mathbf{F}_k$ to be the (A_k, A_{k+1})–bimodule $P_k^\vee$, where the right A_{k+1}–structure is induced by the quotient map $A_{k+1} \to A_{k+1}/A_{k+1}e_{k+1}^\vee A_{k+1}$ composed with (9.5.6). The bimodule $\mathbf{F}_k$ defines a right-exact functor

$$A_{k+1}\text{–gmod} \xrightarrow{\mathbf{F}_k \otimes_{A_{k+1}} \bullet} A_k\text{–gmod}. \tag{9.5.7}$$

Notice that for each indecomposable projective module $P(\mu) = A_{k+1}e_\mu$ we have

$$\mathbf{F}_k \otimes_{A_{k+1}} (A_{k+1}e_\mu) \cong \begin{cases} A_k e_\lambda & \text{if } \lambda^{(\wedge)} = \mu \text{ for some } \lambda \in \Gamma_k, \\ 0 & \text{otherwise.} \end{cases} \tag{9.5.8}$$

The functor $\mathbf{E}_k$

The usual hom-tensor adjunction gives a natural isomorphism

$$\mathrm{Hom}_{A_k}(\mathbf{F}_k \otimes_{A_{k+1}} M, N) \cong \mathrm{Hom}_{A_{k+1}}(M, \mathrm{Hom}_{A_k}(\mathbf{F}_k, N)) \tag{9.5.9}$$

for all $M \in A_{k+1}$–gmod, $N \in A_k$–gmod. Notice that we have a natural isomorphism $\mathrm{Hom}_{A_k}(\mathbf{F}_k, N) \cong \mathrm{Hom}_{A_k}(\mathbf{F}_k, A_k) \otimes_{A_k} N$, where $\mathrm{Hom}_{A_k}(\mathbf{F}_k, A_k)$ is regarded as a (A_{k+1}, A_k)–bimodule. Let us therefore define $\mathbf{E}_k$ to be the (A_{k+1}, A_k)–bimodule $\mathrm{Hom}_{A_k}(\mathbf{F}_k, A_k)$. The functor

$$A_k\text{–gmod} \xrightarrow{\mathbf{E}_k \otimes_{A_k}} A_{k+1}\text{–gmod}. \tag{9.5.10}$$

is right adjoint to (9.5.7). Since $\mathbf{F}_k$ is a projective A_k–module, the functor (9.5.10) is exact.

REMARK 9.5.4. Since $\mathbf{F}_k = A_k e_k^\vee$ as a left A_k–module, we have $\mathrm{Hom}_{A_k}(\mathbf{F}_k, A_k) \cong e_k^\vee A_k$ as a right A_k–module. Notice that the anti-isomorphism $\star$ of A, see (9.2.20), restricts to an isomorphism of vector spaces $\psi\colon \mathbf{F}_k \to \mathbf{E}_k$ such that $\psi(\alpha x \beta) = \beta^\star \psi(x) \alpha^\star$ for all $x \in \mathbf{F}_k$, $\alpha \in A_k$, $\beta \in A_{k+1}$.

9.6 Diagram algebra and category $\mathcal{O}$

The goal of this final section is to prove that the diagram algebra $A_{n,k}$ is isomorphic to the endomorphism ring of a minimal projective generator of the category $\mathcal{Q}_k(\mathfrak{n})$ defined in Chapter 6. We will need some of the notation introduced in Part II.

Soergel modules and category $\mathcal{O}$

Fix a positive integer n. The following result of Soergel connects category $\mathcal{O}$ with Soergel modules, and in particular §4.3 with §8.1.

Theorem 9.6.1 ([Soe90, Zerlegungssatz 1 and Theorem 4])**.** *For each $z \in \mathbb{S}_n$ the B–module $\mathbb{V}P(z \cdot 0)$ is isomorphic to the Soergel module C_z defined in §8.1.*

We prove now two results which we used in Chapter 8. We postponed the proofs until now because we need the connection with category $\mathcal{O}$.

Proposition 9.6.2. *For all $w \in \mathbb{S}_n$ we have*

$$\dim_{\mathbb{C}} \mathsf{C}_w = \sum_{w' \preceq w} \mathcal{P}_{w',w}(1), \tag{9.6.1}$$

where the $\mathcal{P}_{w',w}$'s are the Kazhdan-Lusztig polynomials (2.1.5).

Proof. By Theorems 4.3.1 and 9.6.1 we have $\mathsf{C}_w \cong \mathrm{Hom}_B(B, \mathsf{C}_w) \cong \mathrm{Hom}_{\mathcal{O}}(P(w_0 \cdot 0), P(w \cdot 0))$, where w_0 is the longest element of $\mathbb{S}_n$. Hence

$$\dim_{\mathbb{C}} \mathsf{C}_w = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}}(P(w_0 \cdot 0), P(w \cdot 0)) = [P(w \cdot 0) : L(w_0 \cdot 0)], \tag{9.6.2}$$

where the latter denotes the multiplicity of the simple module $L(w_0 \cdot 0)$ in some composition series of $P(w \cdot 0)$. Since $P(w \cdot 0)$ has a Verma filtration, and since $[M(z \cdot 0) : L(w_0 \cdot 0)] = 1$ for all Verma modules $M(z \cdot 0)$, we have further that $[P(w \cdot 0) : L(w_0 \cdot 0)] = \sum_{z \in \mathbb{S}_n} (P(w \cdot 0) : M(z \cdot 0))$, where $(P(w \cdot 0) : M(z \cdot 0))$ denotes the multiplicity of $M(z \cdot 0)$ in some Verma filtration of $P(w \cdot 0)$. The Kazhdan-Lusztig conjecture [KL79] (see [EW12] for a proof) states precisely that $(P(w \cdot 0) : M(z \cdot 0)) = \mathcal{P}_{z,w}(1)$, and this concludes the proof (notice that $\mathcal{P}_{z,w}(1) = 0$ unless $z \preceq w$). □

REMARK 9.6.3. There is also the following graded version of (9.3.1):

$$\operatorname{gr}\dim_{\mathbb{C}} \mathsf{C}_z = q^{-\ell(w_k z)} \sum_{w' \preceq z} \mathcal{P}_{w',z}(q^2). \tag{9.6.3}$$

Lemma 9.6.4. *The module C_z is cyclic if and only if $\mathcal{P}_{e,z} = q^{\ell(z)}$, i.e. if and only if H_e appears exactly once with coefficient $q^{\ell(z)}$ in the expression of the canonical basis element $\underline{H}_z$.*

Proof. Let $\mathcal{P}_{e,z}$ be the Kazhdan-Lusztig polynomial which gives the coefficient of H_e in the expression of $\underline{H}_z$ in the standard basis. Let $(P(z \cdot 0) : M(0))$ denote the multiplicity of the dominant Verma module $M(0)$ in some Verma flag of the indecomposable projective module $P(z \cdot 0)$ in the category $\mathcal{O}(\mathfrak{gl}_n)$. By the Kazhdan-Lusztig conjecture we have $\mathcal{P}_{e,z}(1) = (P(z \cdot 0) : M(0))$. By [Str03b, Lemma 7.3], $(P(z \cdot 0) : M(0))$ is the cardinality of a minimal system of generators for C_z. □

The algebra $A_{n,k}$ and the category $\mathcal{Q}_k(\mathfrak{m})$

Fix two integers $n \geq 0$ and $0 \leq k \leq n$. Let W_k, $W_k^\perp$ be the parabolic subgroups of $\mathbb{S}_n$ defined in §7.1. Let $\mathfrak{q}, \mathfrak{p} \subseteq \mathfrak{gl}_n$ be the standard parabolic subalgebras with $W_\mathfrak{q} = W_k$ and $W_\mathfrak{p} = W_k^\perp$ so that $\mathbb{S}_k \times \mathbb{S}_{n-k} \cong W_\mathfrak{q} \times W_\mathfrak{p} \subseteq \mathbb{S}_n$. As before, let $w_\mathfrak{q}$ be the longest element of $W_\mathfrak{q}$.

As in §7.1, let D be the set of shortest coset representatives for ${}_{\mathbb{S}_k \times \mathbb{S}_{n-k}}\backslash^{\mathbb{S}_n}$, that is $D = W^\mathfrak{q} \cap W^\mathfrak{p} = W^{\mathfrak{p}+\mathfrak{q}}$. We let $\mathscr{P}_\mathfrak{q}^\mathfrak{p} = \mathscr{P}_\mathfrak{q}^\mathfrak{p}(0)$ be a minimal projective generator of $\mathcal{O}_0^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Recall the Definition 8.3.1 of illicit morphisms.

Proposition 9.6.5. *We have an isomorphism of graded algebras*

$$\operatorname{End}_{\mathbb{Z}\mathcal{O}}\left(\mathscr{P}_\mathfrak{q}^\mathfrak{p}\right) \cong \operatorname{End}_R\Big(\bigoplus_{z\in D} \mathsf{C}_{w_k z}\Big)\Big/\{\textit{illicit morphisms}\}. \tag{9.6.4}$$

Proof. By (5.3.5) we have $\operatorname{End}_{\mathbb{Z}\mathcal{O}}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}) \cong \operatorname{End}_{\mathbb{Z}\mathcal{O}}(\mathscr{P}_\mathfrak{q})/\overline{I}_\mathfrak{p}$, where $\mathscr{P}_\mathfrak{q} = \bigoplus_{w\in w_\mathfrak{q} W^\mathfrak{q}} P(w \cdot 0)$ and $\overline{I}_\mathfrak{p}$ is the ideal of all morphisms which factor through some $P(y \cdot 0)$ for $y \notin W^\mathfrak{p}$. Since $\operatorname{End}_{\mathbb{Z}\mathcal{O}}(\mathscr{P}_\mathfrak{q}) = \bigoplus_{w,z\in w_\mathfrak{q} W^\mathfrak{q}} \operatorname{Hom}_{\mathbb{Z}\mathcal{O}}(P(w \cdot 0), P(z \cdot 0))$ and any morphisms with source or target some $P(y \cdot 0)$ for $y \notin W^\mathfrak{p}$ lies in the ideal $\overline{I}_\mathfrak{p}$, we have

$$\operatorname{End}_{\mathbb{Z}\mathcal{O}}(\mathscr{P}_\mathfrak{q}^\mathfrak{p}) \cong \operatorname{End}_{\mathbb{Z}\mathcal{O}}\Big(\bigoplus_{w\in D} P(w_\mathfrak{q} w \cdot O)\Big)\Big/\overline{I}_\mathfrak{p}. \tag{9.6.5}$$

After applying the isomorphism (4.3.2), the ideal $\overline{I}_\mathfrak{p}$ becomes exactly the ideal generated by illicit morphisms. Hence the claim follows from Soergel's Theorem 4.3.2. □

It follows also that

$$\operatorname{Hom}_{\mathbb{Z}\mathcal{O}}(Q(w_k z), Q(w_k z')) \cong \operatorname{Hom}_R(\mathsf{C}_{w_k z}, \mathsf{C}_{w_k z'})/\mathsf{W}_{z,z'} = \mathsf{Z}_{z,z'} \tag{9.6.6}$$

for all $z, z' \in D$. We deduce then the following lemma, which completes the proof of Lemma 9.2.2.

Lemma 9.6.6. *Let Γ be a block of weights with k $\wedge$'s and $n-k$ $\vee$'s. Let $z, z' \in D_{n,k}$, and let $\lambda, \mu \in \Gamma$ be the corresponding weights. The dimension of $\mathsf{Z}_{z,z'}$ is $k!$ times the number of unenhanced weights η such that $\underline{\mu}\eta\overline{\lambda}$ is oriented.*

Proof. We computed the dimension of the homomorphism space (9.6.6) in Lemma 6.4.3 in terms of evaluation of canonical basis diagrams labeled by standard basis diagrams. This translates immediately in terms of oriented fork diagrams (notice that a canonical basis diagram is the same as a lower fork diagram and a standard basis diagram is the same as an unenhanced weight, cf. also Remark 9.1.5). □

We can now state the main theorem of this whole part:

Theorem 9.6.7. *We have an isomorphism of graded algebras*

$$A_{n,k} \cong \operatorname{End}_{\mathcal{Q}_k(\mathfrak{n})}(\mathscr{P}_{\mathfrak{p}}^{\mathfrak{q}}), \tag{9.6.7}$$

and in particular we have an equivalence of categories

$$\text{gmod}-A_{n,k} \cong \mathcal{Q}_k(\mathfrak{n}). \tag{9.6.8}$$

Proof. We just need to identify the quotient of the endomorphism algebra appearing in the r.h.s. of (9.6.4) with $A_{n,k}$. This follows from Corollary 9.2.4. □

In the previous sections we focused on left $A_{n,k}$–modules, but the whole chapter could be rewritten for right modules. Alternatively, one could use the anti-automorphism $\star$, see (9.2.20), to identify A with its opposite algebra A^{opp} and hence identify the categories of right and left graded $A_{n,k}$–modules. Therefore we actually have an equivalence

$$A_{n,k}-\text{gmod} \cong \mathcal{Q}_k(\mathfrak{n}). \tag{9.6.9}$$

Although the equivalence (9.6.8) implies that working with right $A_{n,k}$–modules is conceptually more correct, we personally prefer to work with left $A_{n,k}$–modules.

The functors $\mathcal{F}_k$ and $\mathcal{E}_k$

We want now to relate the diagrammatic functors $\mathbf{F}_k$ and $\mathbf{E}_k$ defined in §9.5 with their Lie theoretical versions from §6.5.

Proposition 9.6.8. *Under the equivalence of categories* (9.6.9) *the functor* $\mathbf{F}_k \otimes_{A_{k+1}} \bullet$ *corresponds to the functor* $\mathcal{F}_k$*.*

Proof. Let $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{gl}_n$, be the parabolic subalgebras corresponding to k and $\mathfrak{p}', \mathfrak{q}' \subseteq \mathfrak{gl}_n$ be the parabolic subalgebras corresponding to $k+1$. Recall that the functor $\mathcal{F}_k$ is defined as the composition of the inclusion $\mathfrak{i}\colon {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}',\mathfrak{q}'\text{-pres}} \to {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$ and the Zuckermann's functor $\mathfrak{z}\colon {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}',\mathfrak{q}\text{-pres}} \to {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p},\mathfrak{q}\text{-pres}}$. Let $H = \operatorname{End}_{{}^{\mathbb{Z}}\mathcal{O}}(\mathscr{P}_{\mathfrak{q}}^{\mathfrak{p}'})$. Let also $f_{\mathfrak{q}'} \in H$ be the idempotent projecting onto the direct sum of the projective modules $P^{\mathfrak{p}'}(x \cdot 0)$ for $x \in w_{\mathfrak{q}'} W^{\mathfrak{q}'} \cap W^{\mathfrak{p}'}$ and $f_{\mathfrak{p}}^{\perp} \in H$ be the idempotent projecting onto the direct sum of the indecomposable projective modules $P^{\mathfrak{p}'}(x \cdot 0) \in {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}',\mathfrak{q}\text{-pres}}$ for $x \in w_{\mathfrak{q}'} W^{\mathfrak{q}'} \cap W^{\mathfrak{p}}$ but $x \notin w_{\mathfrak{q}} W^{\mathfrak{q}}$. Then we have (using the transitive property of taking parabolic subcategories and presentable quotient categories discussed in Chapter 5)

$$A_k \cong H/Hf_{\mathfrak{p}}^{\perp}H \qquad \text{and} \qquad A_{k+1} \cong f_{\mathfrak{q}'}Hf_{\mathfrak{q}'}. \tag{9.6.10}$$

Moreover, the inclusion functor $\mathfrak{i}$ corresponds to $H \otimes_{f_{\mathfrak{q}'}Hf_{\mathfrak{q}'}} \bullet$ while the Zuckermann's functor corresponds to $(H/Hf_{\mathfrak{p}}^{\perp}H) \otimes_H \bullet$. Hence the functor $\mathcal{F}_k$ corresponds to

$$M \longmapsto (H/Hf_{\mathfrak{p}}^{\perp}H) \otimes_{f_{\mathfrak{q}'}Hf_{\mathfrak{q}'}} M, \tag{9.6.11}$$

that is the same as

$$M \longmapsto (H/Hf_{\mathfrak{p}}^{\perp}H)\overline{f}_{\mathfrak{q}'} \otimes_{f_{\mathfrak{q}'}Hf_{\mathfrak{q}'}} M, \tag{9.6.12}$$

where $\overline{f}_{\mathfrak{q}'}$ is the image of $f_{\mathfrak{q}'}$ in $H/Hf_{\mathfrak{p}}^{\perp}H$. Obviously $(H/Hf_{\mathfrak{p}}^{\perp}H)\overline{f}_{\mathfrak{q}'} = P_k^{\vee}$ as a left A_k–module. It is easy to notice that also the right A_{k+1}–module structure is the same, since in both cases it is the natural structure induced by the bigger algebra $\operatorname{End}_{{}^{\mathbb{Z}}\mathcal{O}}(\mathscr{P})$, where $\mathscr{P}$ is a minimal projective generator of ${}^{\mathbb{Z}}\mathcal{O}_0$. □

By the uniqueness of the adjoint functor we get:

Proposition 9.6.9. *Under the equivalence of categories* (9.6.9) *the functor* $\mathbf{E}_k \otimes \bullet$ *corresponds to the functor* $\mathcal{E}_k$.

Using our diagrammatic description of the functors $\mathcal{E}_k$ and $\mathcal{F}_k$ together with their Lie theoretical interpretation we can compute their endomorphism rings:

Theorem 9.6.10. *We have* $\mathrm{End}(\mathcal{E}_k) \cong \mathrm{End}(\mathcal{F}_k) \cong \mathbb{C}[x_1, \ldots, x_n]/I_k$ *where* I_k *is the ideal generated by the complete symmetric functions*

$$\begin{array}{rll} h_{k+1}(x_{i_1}, \ldots, x_{i_m}) & \textit{for all} & 1 \leq m \leq n-k, \\ h_{n-m+1}(x_{i_1}, \ldots, x_{i_m}) & \textit{for all} & n-k+1 \leq m \leq n. \end{array} \tag{9.6.13}$$

In particular, $\mathcal{E}_k$ *and* $\mathcal{F}_k$ *are indecomposable functors.*

Proof. Let us first compute $\mathrm{End}(\mathcal{F}_k)$. By Proposition 9.6.8, we have an isomorphism $\mathrm{End}(\mathcal{F}_k) \cong \mathrm{End}_{A_k \otimes A_{k+1}^{\mathrm{op}}}(\mathbf{F}_k)$. Since the structure of right A_{k+1}–module is induced by the surjective map (8.1.1), this is the same as $\mathrm{End}_{A_k \otimes (e_k^\vee A_k e_k^\vee)^{\mathrm{op}}}(\mathbf{F}_k)$, that is the center of $e_k^\vee A_k e_k^\vee$. This algebra is the endomorphism algebra of the indecomposable projective-injective modules of $\mathcal{O}_0^{\mathfrak{p},\mathfrak{q}\text{-pres}}$, where as before $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{gl}_n$ are the parabolic subalgebras corresponding to k. Since the projective-injective modules of $\mathcal{O}_0^{\mathfrak{p},\mathfrak{q}\text{-pres}}$ are the same as the projective-injective modules of $\mathcal{O}_0^{\mathfrak{p}}$, it is also the endomorphism algebra of the indecomposable projective-injective modules of $\mathcal{O}_0^{\mathfrak{p}}$. By a standard argument using the parabolic version of Soergel's functor $\mathbb{V}$ (see [Str03b, Section 10]) it follows that this endomorphism algebra is isomorphic to the center of $\mathcal{O}_0^{\mathfrak{p}}$. Brundan [Bru08, Main Theorem] showed that this center is canonically isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/I_k$, where I_k is the ideal generated by

$$\begin{array}{rlll} h_r(x_{i_1}, \ldots, x_{i_m}) & \text{for all} & 1 \leq m \leq n-k, & r > k \\ h_r(x_{i_1}, \ldots, x_{i_m}) & \text{for all} & n-k+1 \leq m \leq n, & r > n-m. \end{array} \tag{9.6.14}$$

Notice that this result builds on a conjecture of Khovanov [Kho04, Conjecture 3] (proved in [Bru08, Main Theorem], [Str09, Theorem 1]), that the center of $\mathcal{O}_0^{\mathfrak{p}}$ agrees with the cohomology ring of a Springer fiber. Under this identification, the presentation (9.6.14) can be deduced from Tanisaki presentation [Tan82] of the cohomology of the Springer fiber. Using (7.3.3) one can easily prove that the polynomials (9.6.14) generate the same ideal as (9.6.13).

For $\mathcal{E}_k$, by Proposition 9.6.9 we have $\mathrm{End}(\mathcal{E}_k) \cong \mathrm{End}_{A_{k+1} \otimes A_k^{\mathrm{op}}}(\mathbf{E}_k)$. By Remark 9.5.4, it follows that

$$\mathrm{End}(\mathcal{E}_k) \cong \mathrm{End}_{A_{k+1} \otimes A_k^{\mathrm{op}}}(\mathbf{E}_k) \cong \mathrm{End}_{A_k \otimes A_{k+1}^{\mathrm{op}}}(\mathbf{F}_k) \cong \mathrm{End}(\mathcal{F}_k). \tag{9.6.15}$$

The middle isomorphism can be explained as follows: $\mathbf{E}_k$ and $\mathbf{F}_k$ have the same underlying vector space V; since the action of $A_{k+1} \otimes A_k^{\mathrm{op}}$ on $\mathbf{E}_k$ is just the action of $A_k \otimes A_{k+1}^{\mathrm{op}}$ on $\mathbf{F}_k$ twisted (see Remark 9.5.4), a $\mathbb{C}$–linear endomorphism of V is $(A_{k+1} \otimes A_k^{\mathrm{op}})$–equivariant (i.e. it is an endomorphism of $\mathbf{E}_k$ as a (A_{k+1}, A_k)–bimodule) exactly when it is $(A_k \otimes A_{k+1}^{\mathrm{op}})$–equivariant (i.e. it is an endomorphism of $\mathbf{F}_k$ as a (A_k, A_{k+1})–bimodule).

The fact that the functors $\mathcal{E}_k$ and $\mathcal{F}_k$ are indecomposable follows since $\mathrm{End}(\mathcal{E}_k) \cong \mathrm{End}(\mathcal{F}_k)$ is a graded local ring (cf. [Str05] and references therein). $\square$

Appendices

APPENDIX A

The Alexander polynomial

One of the main motivation for us for studying the problem of categorification of representations of $U_q(\mathfrak{gl}(1|1))$ was the aim of constructing a representation-theoretical categorification of the Alexander polynomial. Relations between the representation theory of $U_q(\mathfrak{gl}(1|1))$, or more generally $U_q(\mathfrak{gl}(n|n))$, and the Alexander polynomial have been noticed, studied and generalized by lots of authors (see for example [Deg89], [Sal90], [KS91], [GLZ96], [DWIL05], [GPM07], [GPM10], [Vir06]). The purpose of this appendix is to provide, from a purely representation theoretical point of view, a short but complete and self-contained explanation of how the Alexander polynomial arises as quantum invariant corresponding to the vector representation of $U_q(\mathfrak{gl}(1|1)$.

A.1 Introduction

The Alexander polynomial is a classical invariant of links in the three-dimensional space, defined first in the 1920s by Alexander [Ale28]. Constructed originally in combinatorial terms, it can be defined in modern language using the homology of a cyclic covering of the link complement (see for example [Lic97]).

The Alexander polynomial can also be defined using the Burau representation of the braid group (see for example [KT08, Chapter 3]). As well-known to experts, this representation can be constructed using a solution of the Yang-Baxter equation, which comes from the action of the R–matrix of $U_q(\mathfrak{gl}(1|1))$ [KS91] (or alternatively of $U_q(\mathfrak{sl}_2)$ for q a root of unity; see [Vir06] for the parallel between $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}_2$).

In other words, the key-point of the construction is the *braided* structure of the monoidal category of finite dimensional representations of $U_q(\mathfrak{gl}(1|1))$, that is, there is an action of an R–matrix satisfying the braid relation. This can obviously be used to construct representations of the braid group. Considering tensor powers of the vector representation of $U_q(\mathfrak{gl}(1|1))$, one obtains in this way the Burau representation of the braid group. Given a representation of the braid group, one can extend it to an invariant of links considered as closures of braids by defining a Markov trace.

Here we exploit this construction a bit further, proving that the category of finite-dimensional $U_q(\mathfrak{gl}(1|1))$–representations is not only braided, but actually *ribbon*. A ribbon category is exactly what one needs to use the Reshetikhin-Turaev construction [RT90] to get invariants of oriented framed tangles. The advantage of the ribbon structure is that one can consider arbitrary diagrams of links, and not just braid diagrams.

To construct a ribbon structure on the category of modules over some algebra, a possible strategy is to prove that the algebra is actually a ribbon Hopf algebra. Unfortunately, similarly to the case of a classical semisimple Lie algebra, the Hopf algebra $U_q(\mathfrak{gl}(1|1))$ is not ribbon. We consider hence another version of the quantum enveloping algebra, which we call $U_\hbar(\mathfrak{gl}(1|1))$, and which is a topological algebra over $\mathbb{C}[[\hbar]]$. The price of working with power series pays off, since $U_\hbar(\mathfrak{gl}(1|1))$ is in fact a ribbon Hopf algebra. By a standard argument, we see that the R–matrix and the ribbon element of $U_\hbar(\mathfrak{gl}(1|1))$ act on finite-dimensional representations of $U_q(\mathfrak{gl}(1|1))$ and deduce hence the ribbon structure of this category.

Given an oriented framed tangle T and a labeling $\boldsymbol{\ell}$ of the strands of T by finite-dimensional irreducible $U_q(\mathfrak{gl}(1|1))$–representations, we get then an invariant $Q^{\boldsymbol{\ell}}(T)$, which is some $U_q(\mathfrak{gl}(1|1))$–equivariant map. In particular, restricting to oriented framed links (viewed as special cases of tangles), we obtain a $\mathbb{C}(q)$–valued invariant.

If we label all the strands by the vector representation of $U_q(\mathfrak{gl}(1|1))$, an easy calculation shows that the corresponding invariant of oriented framed tangles is actually independent of the framing and hence is an invariant of oriented tangles (as well-known, the same happens for the ordinary $\mathfrak{sl}_n$–invariant).

Unfortunately, when considering invariants of closed links, there is a little problem we have to take care of. Namely, it follows from the fact that the category of finite-dimensional $U_q(\mathfrak{gl}(1|1))$–modules is not semisimple (cf. §1.2) that the invariant $Q^{\boldsymbol{\ell}}(L)$ is zero for all closed links L (see Proposition A.3.4). The work-around to this problem is to choose a strand of the link L, cut it and consider the invariant of the framed 1–tangle that is obtained in this way (Theorem A.3.6). The resulting invariant will be an element of the endomorphism ring of an irreducible representation (the one that labels the strand being cut); since this ring can be naturally identified with $\mathbb{C}(q)$, the invariant that we obtain in this way is actually a rational function. The construction does not depend on the strand we cut, but rather on the representation labeling the strand. In particular for a constant labeling $\boldsymbol{\ell}$ of all the components of L we get a true invariant of framed links.

Applying this construction to the constant labeling by the vector representation, one obtains as before an invariant of links. In fact, it is easy to prove that this coincides with the Alexander polynomial (see Theorem A.3.10).

A.2 The $\hbar$–version of the quantum enveloping superalgebra

Our goal is to construct a ribbon category of representations of U_q, so that we can define link invariants. The main ingredient is the R-matrix. Unfortunately, as usual, it is not possible to construct a universal R-matrix for U_q; instead, we need to consider the $\hbar$–version of the quantum enveloping superalgebra, which we will denote by $U_\hbar$ and which is a $\mathbb{C}[[\hbar]]$–superalgebra completed with respect to the $\hbar$–adic topology. We will prove that $U_\hbar$ is a ribbon algebra. Then, using a standard argument of Tanisaki [Tan92], we obtain a ribbon structure on the category of finite-dimensional U_q–representations. For details about topological $\mathbb{C}[[\hbar]]$–algebras we refer to [Kas95, Chapter XVI]. We will denote by the symbol $\hat{\otimes}$ the *completed* tensor product of topological $\mathbb{C}[[\hbar]]$–algebras.

Throughout the section we will use some standard facts about Hopf superalgebras. The analogous statements in the non-super setting can be found for example in [CP94], [Kas95], [Oht02]. The proofs carry directly over to the super case.

The Hopf superalgebra $U_\hbar$

We define $U_\hbar = U_\hbar(\mathfrak{gl}(1|1))$ to be the unital $\mathbb{C}[[\hbar]]$–algebra topologically generated by the elements E, F, H_1, H_2 in degrees $|H_1| = |H_2| = 0$, $|E| = |F| = 1$ subject to the relations

$$\begin{gathered} H_1H_2 = H_2H_1, \\ H_iE - EH_i = \langle H_i, \alpha\rangle E, \qquad H_iF - FH_i = -\langle H_i, \alpha\rangle F, \\ EF + FE = \frac{e^{\hbar(H_1+H_2)} - e^{-\hbar(H_1+H_2)}}{e^\hbar - e^{-\hbar}}, \qquad E^2 = F^2 = 0. \end{gathered} \tag{A.2.1}$$

Note that although $e^\hbar - e^{-\hbar}$ is not invertible, it is the product of $\hbar$ and an invertible element of $\mathbb{C}[[\hbar]]$, hence the fourth relation makes sense.

Although the relation between U_q and $U_\hbar$ is technically not easy to formalize (see [CP94] for details), one should keep in mind the following picture:

$$\begin{aligned} q &\longleftrightarrow e^\hbar, \\ \mathbf{q}^{h_i} &\longleftrightarrow e^{\hbar H_i}. \end{aligned} \tag{A.2.2}$$

This also explains why we used the symbols $\mathbf{q}^h$ as generators for U_q in §1.1. In the following, we set $q = e^\hbar$ as an element of $\mathbb{C}[[\hbar]]$ and $K = e^{\hbar(H_1+H_2)}$ as an element of $U_\hbar$.

As for U_q, we define a *comultiplication* $\Delta\colon U_\hbar \to U_\hbar \hat{\otimes} U_\hbar$, a *counit* $\mathbf{u}\colon U_\hbar \to \mathbb{C}[[\hbar]]$ and an *antipode* $S\colon U_\hbar \to U_\hbar$ by setting on the generators

$$\begin{aligned} \Delta(E) &= E \otimes K^{-1} + 1 \otimes E, & \Delta(F) &= F \otimes 1 + K \otimes F, \\ S(E) &= -EK, & S(F) &= -K^{-1}F, \\ \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, & S(H_i) &= -H_i, \\ \mathbf{u}(E) &= \mathbf{u}(F) = 0, & \mathbf{u}(H_i) &= 0, \end{aligned} \tag{A.2.3}$$

and extending Δ and $\mathbf{u}$ to algebra homomorphisms and S to an algebra anti-homomorphism. We have then:

Proposition A.2.1. *The maps Δ, $\mathbf{u}$ and S turn $U_\hbar$ into a Hopf superalgebra.*

The proof requires precisely the same calculations as the proof of Proposition 1.1.2.

As for U_q, we define a *bar involution* on $U_\hbar$ by setting:

$$\overline{E} = E, \qquad \overline{F} = F, \qquad \overline{H_i} = H_i, \qquad \overline{\hbar} = -\hbar. \tag{A.2.4}$$

Again, $\overline{\Delta} = (^{-} \otimes {}^{-}) \circ \Delta \circ {}^{-}$ defines another comultiplication on $U_\hbar$, and by definition $\overline{\Delta}(\overline{x}) = \overline{\Delta(x)}$ for all $x \in U_\hbar$.

The braided structure

We are going to recall the *braided Hopf superalgebra structure* (cf. [Zha02], [Oht02]) of $U_\hbar$. The main ingredient is the universal R–matrix, which has been explicitly computed by Khoroshkin and Tolstoy (cf. [KT91]). We adapt their definition to our notation.[1]

[1] Our comultiplication is the opposite of [KT91], hence we have to take the opposite R–matrix, cf. also [Kas95, Chapter 8].

We define $R = \Theta\Upsilon \in U_\hbar \hat{\otimes} U_\hbar$ where

$$\Upsilon = e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}, \tag{A.2.5}$$

$$\Theta = 1 + (q - q^{-1})F \otimes E. \tag{A.2.6}$$

Notice that the expression for Υ makes sense as an element of the completed tensor product $U_\hbar \hat{\otimes} U_\hbar$. Recall that a vector w in some representation W of $U_\hbar$ is said to be a *weight vector* of *weight* μ if $H_i w = \langle H_i, \mu\rangle w$ for $i = 1, 2$. The element Υ is then characterized by the property that it acts on a weight vector $w_1 \otimes w_2$ by $q^{(\mu_1,\mu_2)} = e^{\hbar(\mu_1,\mu_2)}$, if w_1 and w_2 have weights μ_1 and μ_2 respectively.

The element Θ is called the *quasi R-matrix*; it is easy to check that it satisfies

$$\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1 \otimes 1. \tag{A.2.7}$$

It follows in particular that R is invertible with inverse $R^{-1} = \Upsilon^{-1}\Theta^{-1} = \Upsilon^{-1}\overline{\Theta}$.

Recall that a bialgebra B is called *quasi-cocommutative* ([Kas95, Definition VIII.2.1]) if there exists an invertible element $R \in B \otimes B$ such that for all $x \in B$ we have $\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}$, where Δ^{op} is the opposite comultiplication $\Delta^{\mathrm{op}} = \sigma \circ \Delta$ with $\sigma(a \otimes b) = (-1)^{|a||b|}(b \otimes a)$.

Lemma A.2.2. *For all $x \in U_\hbar$ we have*

$$R\Delta(x) = \Delta^{\mathrm{op}}(x)R. \tag{A.2.8}$$

Hence the Hopf algebra $U_\hbar$ is quasi-cocommutative.

Proof. Using Lemma A.2.3 below we compute

$$R\Delta(x) = \Theta\Upsilon\Delta(x) = \Theta\overline{\Delta}^{\mathrm{op}}(x)\Upsilon = \Delta^{\mathrm{op}}(x)\Theta\Upsilon = \Delta^{\mathrm{op}}(x)R. \qquad \square$$

Lemma A.2.3. *The following properties hold for all $x \in U_\hbar$:*

$$\Theta\overline{\Delta}^{\mathrm{op}}(x) = \Delta^{\mathrm{op}}(x)\Theta \tag{A.2.9}$$

$$\Upsilon\Delta(x) = \overline{\Delta}^{\mathrm{op}}(x)\Upsilon. \tag{A.2.10}$$

Proof. It is enough to check (A.2.9) and (A.2.10) on the generators. We have

$$\begin{aligned}
\Theta\overline{\Delta}^{\mathrm{op}}(E) &= \Theta(K \otimes E + E \otimes 1)\\
&= K \otimes E + E \otimes 1 + (q - q^{-1})FK \otimes E^2 - (q - q^{-1})FE \otimes E\\
&= K \otimes E + E \otimes 1 + (q - q^{-1})EF \otimes E - (K - K^{-1}) \otimes E\\
&= K^{-1} \otimes E + E \otimes 1 + (q - q^{-1})EF \otimes E\\
&= (K^{-1} \otimes E + E \otimes 1)\Theta = \Delta^{\mathrm{op}}(E)\Theta
\end{aligned}$$

and

$$\begin{aligned}
\Theta\overline{\Delta}^{\mathrm{op}}(F) &= \Theta(1 \otimes F + F \otimes K^{-1})\\
&= 1 \otimes F + F \otimes K^{-1} + (q - q^{-1})F \otimes EF - (q - q^{-1})F^2 \otimes EK^{-1}\\
&= 1 \otimes F + F \otimes K^{-1} - (q - q^{-1})F \otimes FE + F \otimes (K - K^{-1})\\
&= 1 \otimes F + F \otimes K - (q - q^{-1})F \otimes FE\\
&= (1 \otimes F + F \otimes K)\Theta = \Delta^{\mathrm{op}}(F)\Theta
\end{aligned}$$

and for $i = 1, 2$

$$\begin{aligned}
\Theta\overline{\Delta}^{\mathrm{op}}(H_i) &= \Theta(1 \otimes H_i + H_i \otimes 1) \\
&= 1 \otimes H_i + H_i \otimes 1 + (q - q^{-1})F \otimes EH_i + (q - q^{-1})FH_i \otimes E \\
&= 1 \otimes H_i + H_i \otimes 1 - (q - q^{-1})\langle H_i, \alpha\rangle F \otimes E + (q - q^{-1})F \otimes H_i E + \\
&\qquad + (q - q^{-1})\langle H_i, \alpha\rangle F \otimes E + (q - q^{-1})H_i F \otimes E \\
&= 1 \otimes H_i + H_i \otimes 1 + (q - q^{-1})F \otimes H_i E + (q - q^{-1})H_i F \otimes E \\
&= (1 \otimes H_i + H_i \otimes 1)\Theta = \Delta^{\mathrm{op}}(H_i)\Theta.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\Upsilon\Delta(E) &= e^{\hbar(H_1 \otimes H_1 - H_2 \otimes H_2)}(E \otimes K^{-1} + 1 \otimes E) \\
&= (E \otimes K^{-1})e^{\hbar((H_1+1)\otimes H_1 - (H_2-1)\otimes H_2)} + (1 \otimes E)e^{\hbar(H_1 \otimes (H_1+1) - H_2 \otimes (H_2-1))} \\
&= (E \otimes 1 + K \otimes E)e^{\hbar(H_1 \otimes H_1 - H_2 \otimes H_2)} = \overline{\Delta}^{\mathrm{op}}(E)\Upsilon
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon\Delta(F) &= e^{\hbar(H_1 \otimes H_1 - H_2 \otimes H_2)}(F \otimes 1 + K \otimes F) \\
&= (F \otimes 1)e^{\hbar((H_1-1)\otimes H_1 - (H_2+1)\otimes H_2)} + (K \otimes F)e^{\hbar(H_1 \otimes (H_1-1) - H_2 \otimes (H_2+1))} \\
&= (F \otimes K^{-1} + 1 \otimes F)e^{\hbar(H_1 \otimes H_1 - H_2 \otimes H_2)} = \overline{\Delta}^{\mathrm{op}}(F)\Upsilon.
\end{aligned}$$

Finally, for $i = 1, 2$ we have $\Upsilon\Delta(H_i) = \Delta(H_i)\Upsilon$ since the elements H_1, H_2 commute with each other. Since $\overline{\Delta}^{\mathrm{op}}(H_i) = \Delta(H_i)$ we get $\Upsilon\Delta(H_i) = \overline{\Delta}^{\mathrm{op}}(H_i)\Upsilon$ and we are done. □

Given an element $X = \sum x_{(1)} \otimes x_{(2)} \in U_\hbar \hat{\otimes} U_\hbar$ and two indices $0 \le i < j \le n$, let us denote by $X_{ij} \in U_\hbar^{\hat{\otimes} n}$ the element

(A.2.11) $$X_{ij} = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes x_{(1)} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j-i-1} \otimes x_{(2)} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}.$$

A quasi-cocommutative Hopf algebra is called *braided* or *quasi-triangular* if the following quasi-triangular identities hold:

(A.2.12) $$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

In this case, the element R is called *universal R-matrix.*

Proposition A.2.4. *The Hopf superalgebra $U_\hbar$ is braided.*

Proof. Since

$$(\Delta \otimes \mathrm{id})(\Upsilon) = e^{\hbar(H_1 \otimes 1 \otimes H_1 + 1 \otimes H_1 \otimes H_1 - H_2 \otimes 1 \otimes H_2 - 1 \otimes H_2 \otimes H_2)} = \Upsilon_{13}\Upsilon_{23}$$

we can compute using Lemma A.2.5 below

$$(\Delta \otimes \mathrm{id})(R) = (\Delta \otimes \mathrm{id})(\Theta) \cdot (\Delta \otimes \mathrm{id})(\Upsilon) = \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1}\Upsilon_{13}\Upsilon_{23} = R_{13}R_{23}.$$

Similarly we get $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$. □

Lemma A.2.5. *In $U_\hbar$ the following identities hold:*

(A.2.13) $$(\Delta \otimes \mathrm{id})(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1},$$

(A.2.14) $$(\mathrm{id} \otimes \Delta)(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{12}\Upsilon_{13}^{-1}.$$

Proof. The two computations are similar, so let us check (A.2.13) and leave (A.2.14) to the reader. The l.h.s. is simply

$$(\Delta \otimes \mathrm{id})(\Theta) = 1 + (q-q)^{-1} F \otimes 1 \otimes E + (q-q^{-1}) K \otimes F \otimes E. \tag{A.2.15}$$

We will now compute the r.h.s. First we have

$$\begin{aligned} \Upsilon_{13}(1 \otimes F \otimes E)\Upsilon_{13}^{-1} &= \Upsilon_{13}(1 \otimes F \otimes E) e^{-\hbar(H_1 \otimes 1 \otimes H_1)} e^{\hbar(H_2 \otimes 1 \otimes H_2)} \\ &= \Upsilon_{13} e^{-\hbar(H_1 \otimes 1 \otimes (H_1 - 1))} (1 \otimes F \otimes E) e^{\hbar(H_2 \otimes 1 \otimes H_2)} \\ &= \Upsilon_{13} e^{-\hbar(H_1 \otimes 1 \otimes (H_1 - 1))} e^{\hbar(H_2 \otimes 1 \otimes (H_2 + 1))} (1 \otimes F \otimes E) \\ &= K \otimes F \otimes E. \end{aligned} \tag{A.2.16}$$

Therefore

$$\Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1} = (1 + (q-q)^{-1} F \otimes 1 \otimes E)(1 + (q-q)^{-1} K \otimes F \otimes E) \tag{A.2.17}$$

coincides with (A.2.15) since $E^2 = 0$. □

As an easy consequence of the braided structure, the following Yang-Baxter equation holds (see [Kas95, Theorem VIII.2.4] or [CP94, Proposition 4.2.7]):

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{A.2.18}$$

The ribbon structure

Write $R = \sum_r a_r \otimes b_r$ and define

$$u = \sum_r (-1)^{|a_r||b_r|} S(b_r) a_r \in U_\hbar. \tag{A.2.19}$$

Then (cf. [CP94, Proposition 4.2.3]) u is invertible and we have

$$S^2(x) = u x u^{-1} \qquad \text{for all } x \in U_\hbar. \tag{A.2.20}$$

In our case, in particular, since $S^2 = \mathrm{id}$, the element u is central. By an easy explicit computation, we have

$$u = (1 + (q - q^{-1}) EKF) e^{\hbar(H_2^2 - H_1^2)} \tag{A.2.21}$$

and

$$S(u) = e^{\hbar(H_2^2 - H_1^2)} (1 - (q - q^{-1}) F K^{-1} E). \tag{A.2.22}$$

We recall that a braided Hopf superalgebra A is called *ribbon* (cf. [Oht02, Chapter 4] or [CP94, §4.2.C]) if there is an even central element $v \in A$ such that

$$\begin{gathered} v^2 = u S(u), \qquad \mathbf{u}(v) = 1, \qquad S(v) = v, \\ \Delta(v) = (R_{21} R_{12})^{-1} (v \otimes v). \end{gathered} \tag{A.2.23}$$

In $U_\hbar$ let

$$v = K^{-1} u = u K^{-1} = (K^{-1} + (q - q^{-1}) EF) e^{\hbar(H_2^2 - H_1^2)}. \tag{A.2.24}$$

Then we have:

Proposition A.2.6. *With v as above, $U_\hbar$ is a ribbon Hopf superalgebra.*

Proof. Since both u and K^{-1} are central, so is v. Let us check that $S(u) = uK^{-2}$. Indeed we have

$$\begin{aligned} u &= (1 + (q - q^{-1})EFK)e^{\hbar(H_2^2 - H_1^2)} \\ &= e^{\hbar(H_2^2 - H_1^2)}(1 + (q - q^{-1})EFK) \\ &= e^{\hbar(H_2^2 - H_1^2)}(1 + (K - K^{-1})K - (q - q^{-1})FEK) \\ &= e^{\hbar(H_2^2 - H_1^2)}(K^2 - (q - q^{-1})FEK) = S(u)K^2. \end{aligned} \tag{A.2.25}$$

It follows then immediately that $v^2 = u^2K^{-2} = uS(u)$ and $S(v) = S(u)K = uK^{-1} = v$.

The relations $\Delta(v) = (R_{21}R_{12})^{-1}(v \otimes v)$ and $\mathbf{u}(v) = 1$ follow from analogous relations for u, that hold for every quasi-triangular Hopf superalgebra (see [Oht02, Proposition 4.3]). □

A.3 Invariants of links

In this section we define the ribbon structure on the category of representations of U_q and derive the corresponding invariants of oriented framed tangles and links. We refer to §1.2 for the representation theory of U_q.

Recall that if W is an n–dimensional complex super vector space the *evaluation maps* are defined by

$$\begin{aligned} \mathrm{ev}_W \colon W^* \otimes W &\longrightarrow \mathbb{C}(q), & \widehat{\mathrm{ev}}_W \colon W \otimes W^* &\longrightarrow \mathbb{C}(q), \\ \varphi \otimes w &\longmapsto \varphi(w), & w \otimes \varphi &\longmapsto (-1)^{|\varphi||w|}\varphi(w), \end{aligned} \tag{A.3.1}$$

and the *coevalutaion maps* are defined by

$$\begin{aligned} \mathrm{coev}_W \colon \mathbb{C}(q) &\longrightarrow W \otimes W^*, & \widehat{\mathrm{coev}}_W \colon \mathbb{C}(q) &\longrightarrow W^* \otimes W, \\ 1 &\longmapsto \sum_{i=1}^n w_i \otimes w_i^*, & 1 &\longmapsto \sum_{i=1}^n (-1)^{|w_i|} w_i^* \otimes w_i, \end{aligned} \tag{A.3.2}$$

where w_i is a basis of W and w_i^* is the corresponding dual basis of W^*. Note that if $\sigma_{Z,W}$ denotes the map

$$\begin{aligned} \sigma_{Z,W} \colon Z \otimes W &\longrightarrow W \otimes Z \\ z \otimes w &\longmapsto (-1)^{|z||w|} w \otimes z. \end{aligned} \tag{A.3.3}$$

then $\widehat{\mathrm{ev}}_W = \mathrm{ev}_W \circ \sigma_{W^*,W}$ and $\widehat{\mathrm{coev}}_W = \sigma_{W,W^*} \circ \mathrm{coev}_W$.

Ribbon structure on U_q–representations

Following the arguments of Tanisaki [Tan92] (see also [CP94, §10.1.D]), we can construct a ribbon structure on the category of U_q–representations using the ribbon superalgebra structure on $U_\hbar$. We indicate now the main steps of those arguments.

The key observation is that, although Υ does not make sense as an element of $U_q \otimes U_q$, it acts on every tensor product $Z \otimes W$ of two finite-dimensional U_q–modules. In other words, there is a well-defined operator $\Upsilon_{Z,W} \in \mathrm{End}_{\mathbb{C}(q)}(Z \otimes W)$ determined by setting

$\Upsilon_{Z,W}(z_\lambda \otimes w_\mu) = q^{(\lambda,\mu)}(z_\lambda \otimes w_\mu)$ if z_λ and w_μ have weights λ and μ respectively. Note however that $\Upsilon_{Z,W}$ is not U_q–equivariant, since Υ satisfies $\Upsilon\Delta(x) = \overline{\Delta}^{\mathrm{op}}(x)\Upsilon$ (see Lemma A.2.3).

On the other hand, notice that the definition (A.2.6) of Θ makes sense also in U_q, and (A.2.9) holds in U_q. Moreover, one has the following counterpart of equations (A.2.13) and (A.2.14):

$$(\Delta \otimes \mathrm{id})(\Theta) = \Theta_{13}(\Upsilon_{Z,Y})_{13}\Theta_{23}(\Upsilon^{-1}_{Z,Y})_{13} \tag{A.3.4}$$

$$(\mathrm{id} \otimes \Delta)(\Theta) = \Theta_{13}(\Upsilon_{Z,Y})_{13}\Theta_{12}(\Upsilon^{-1}_{Z,Y})_{13}. \tag{A.3.5}$$

This is now an equality of linear endomorphisms of $Z \otimes W \otimes Y$ for all finite-dimensional U_q–representations Z, W, Y. Setting

$$R_{Z,W} = \Theta\Upsilon_{Z,W} \in \mathrm{End}_{\mathbb{C}(q)}(Z \otimes W) \tag{A.3.6}$$

one gets an operator which satisfies the Yang-Baxter equation. Note that $R_{Z,W}$ is invertible, since Θ and $\Upsilon_{Z,W}$ both are. Because of (A.2.8), if we define $\check{R}_{Z,W} = \sigma \circ R_{Z,W}$, then we get an U_q–equivariant isomorphism $\check{R}_{Z,W} \in \mathrm{Hom}_{U_q}(Z \otimes W, W \otimes Z)$.

Analogously, although the elements u and v do not make sense in U_q, they act on each finite-dimensional U_q–representation Z as operators $u_V, v_V \in \mathrm{End}_{U_q}(Z)$ (they are U_q–equivariant because u, v are central in $U_\hbar$). In the following, we will forget the subscripts of the operators $\check{R}$, u and v.

For convenience, we give explicit formulas for the (inverse of the) operator $R_{\mathsf{L}(\lambda),\mathsf{L}(\mu)}$ for $\lambda, \mu \in \mathsf{P}'$:

$$\begin{aligned}
\check{R}^{-1}(v_0^\lambda \otimes v_0^\mu) &= (-1)^{(|\lambda|+1)(|\mu|+1)} q^{-(\mu-\alpha,\lambda-\alpha)} v_0^\mu \otimes v_0^\lambda, \\
\check{R}^{-1}(v_0^\lambda \otimes v_1^\mu) &= (-1)^{(|\lambda|+1)|\mu|} \big(q^{-(\mu,\lambda-\alpha)} v_1^\mu \otimes v_0^\lambda, \\
&\qquad + (-1)^{|\mu|} q^{-(\mu-\alpha,\lambda)} (q^{-1} - q)[\mu] v_0^\mu \otimes v_1^\lambda \big), \\
\check{R}^{-1}(v_1^\lambda \otimes v_0^\mu) &= (-1)^{|\lambda|(|\mu|+1)} q^{-(\mu-\alpha,\lambda)} v_0^\mu \otimes v_1^\lambda \\
\check{R}^{-1}(v_1^\lambda \otimes v_1^\mu) &= (-1)^{|\lambda||\mu|} q^{-(\mu,\lambda)} v_1^\mu \otimes v_1^\lambda.
\end{aligned} \tag{A.3.7}$$

Invariants of tangles

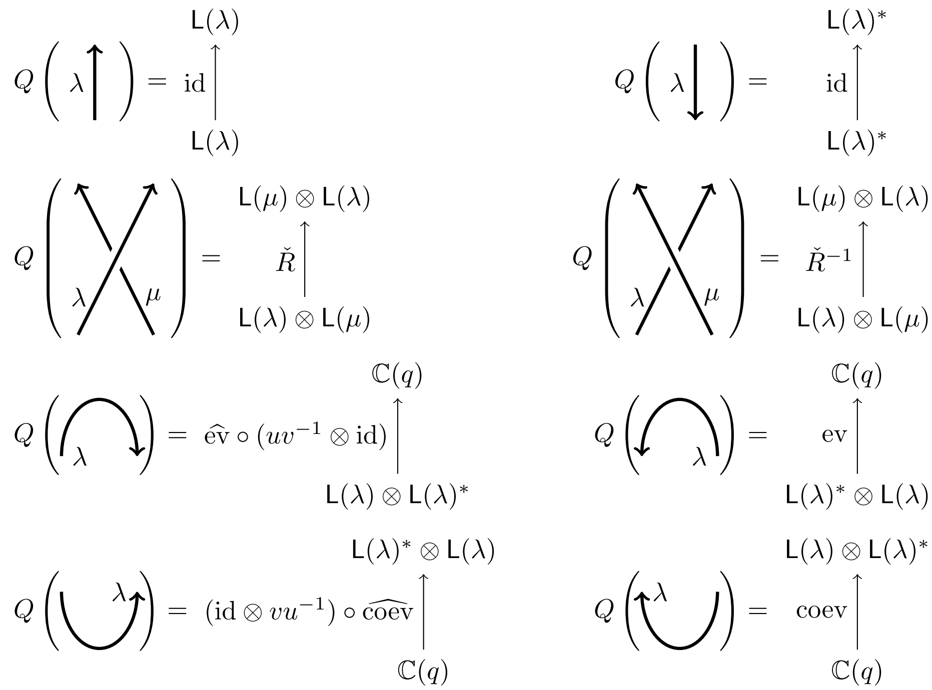
Let D be an oriented framed tangle diagram. We will not draw the framing because we will always suppose that it is the *blackboard framing*. (Recall that a framing is a trivialization of the normal bundle: since the tangle is oriented, such a trivialization is uniquely determined by a section of the normal bundle; the blackboard framing is the trivialization determined by the unit vector orthogonal to the plane – or to the blackboard – pointing outwards.)

We assume $D \subset \mathbb{R} \times [0,1]$ and we let $\boldsymbol{s}(D) = D \cap (\mathbb{R} \times 0) = \{s_1^D, \dots, s_a^D\}$ with $s_1^D < \cdots < s_a^D$ be the source points of D and $\boldsymbol{t}(D) = D \cap (\mathbb{R} \times 1) = \{t_1^D, \dots, t_b^D\}$ with $t_1^D < \cdots < t_b^D$ be the target points of D. Let also $\boldsymbol{\ell}$ be a labeling of the strands of D by simple two-dimensional representations of U_q (that is, a map from the set of strands of D to P'). We indicate by $\ell_1^s, \dots, \ell_a^s$ the labeling of the strands at the source points of D and by $\ell_1^t, \dots, \ell_b^t$ the labeling at the target points. Moreover, we let $\gamma_1^s, \dots, \gamma_a^s$ and $\gamma_1^t, \dots, \gamma_b^t$ be the signs corresponding to the orientations of the strands at the source and target points (where $+1$ corresponds to a strand oriented upwards and -1 to a strand oriented downwards).

Given this data, one can define a U_q–equivariant map

$$Q^{\boldsymbol{\ell}}(D) \colon \mathsf{L}(\ell_1^s)^{\gamma_1^s} \otimes \cdots \otimes \mathsf{L}(\ell_a^s)^{\gamma_a^s} \longrightarrow \mathsf{L}(\ell_1^t)^{\gamma_1^t} \otimes \cdots \otimes \mathsf{L}(\ell_b^t)^{\gamma_b^t}, \tag{A.3.8}$$

where $\mathsf{L}(\lambda)^{-1} = \mathsf{L}(\lambda)^*$, by decomposing D into elementary pieces as shown below and assigning the corresponding morphisms as displayed.



As we already mentioned, although U_q itself is not a ribbon superalgebra, its representation category is a ribbon category. Hence we have:

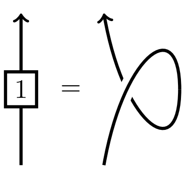
Theorem A.3.1. *The map $Q^{\boldsymbol{\ell}}(D)$ just defined is an isotopy invariant of oriented framed tangles.*

The proof, for which we refer to [Oht02, Theorem 4.7], is a direct check of the Reidemeister moves (or, more precisely, of the analogues of the Reidemeister moves for framed tangles). In fact, the axioms of a ribbon category are equivalent to the validity of these moves.

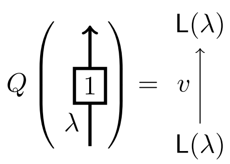
If all strands are labeled by the same simple representation $\mathsf{L}(\lambda)$ (i.e. $\boldsymbol{\ell}$ is the constant map with value λ), then we write $Q^{\lambda}(D)$ instead of $Q^{\boldsymbol{\ell}}(D)$.

Let us indicate a full $+1$ twist by the symbol

(A.3.9)



Then we have (cf. [Oht02, §4.2])



Lemma A.3.2. *The element v acts by the identity on the vector representation $\mathsf{L}(\varepsilon_1)$ and on its dual $\mathsf{L}(\varepsilon_1)^*$.*

Proof. Recall that we denote by $v_1^{\varepsilon_1}$, $v_0^{\varepsilon_1}$ the standard basis of $\mathsf{L}(\varepsilon_1)$. We have

$$\begin{aligned} vv_1^{\varepsilon_1} &= (K^{-1} + (q-q^{-1})EF)\mathbf{q}^{-(h_1+h_2)(h_1-h_2)}v_1^{\varepsilon_1} \\ &= (K^{-1} + (q-q^{-1})EF)q^{-\langle h_1+h_2,\varepsilon_1\rangle\langle h_1-h_2,\varepsilon_1\rangle}v_1^{\varepsilon_1} \\ &= (q^{-1}+q-q^{-1})q^{-1}v_1^{\varepsilon_1} = v_1^{\varepsilon_1}. \end{aligned} \tag{A.3.10}$$

Since $\mathsf{L}(\varepsilon_1)$ is irreducible and v acts in an U_q–equivariant way, it follows that v acts by the identity on $\mathsf{L}(\varepsilon_1)$. Since $S(v) = v$, the element v acts by the identity also on $\mathsf{L}(\varepsilon_1)^*$. □

As a consequence, if we label all strands of our tangles by the vector representation then we do not have to worry about the framing any more:

Corollary A.3.3. *The assignment $D \mapsto Q^{\varepsilon_1}(D)$ is an invariant of oriented tangles.*

Invariants of links

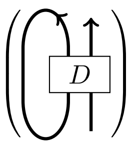
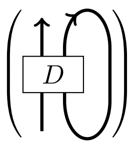
Since links are in particular tangles, we obtain from $Q^{\boldsymbol{\ell}}$ an invariant of oriented framed links; unfortunately, this invariant is always zero:

Proposition A.3.4. *Let L be a closed link diagram and $\boldsymbol{\ell}$ a labeling of its strands. Then $Q^{\boldsymbol{\ell}}(L) = 0$.*

Proof. The invariant associated to L is some endomorphism φ of the trivial representation $\mathbb{C}(q)$. Up to isotopy, we can assume that there is some level at which the link diagram L has only two strands, one oriented upwards and the other one downwards, labeled by the same weight λ. Without loss of generality suppose that the leftmost is oriented upwards. Slice the diagram at this level, so that we can write φ as the composition $\varphi_2 \circ \varphi_1$ of two U_q–equivariant maps $\varphi_1 \colon \mathbb{C}(q) \to \mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$ and $\varphi_2 \colon \mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^* \to \mathbb{C}(q)$. If $\varphi = \varphi_2 \circ \varphi_1$ is not zero, then we have an inclusion φ_1 of $\mathbb{C}(q)$ inside $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$ and a projection φ_2 of the latter onto $\mathbb{C}(q)$, so that $\mathbb{C}(q)$ would be a direct summand of $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$. But this is not possible, since $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)^*$ is indecomposable (by Lemma 1.2.3); hence $\varphi = 0$. □

To get non-trivial invariants of closed links we need to cut the links, as we are going to explain now. First, we need the following result:

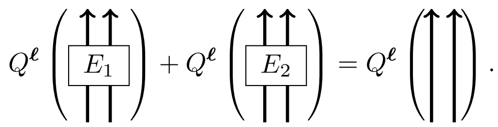
Proposition A.3.5. *Let D be an oriented tangle diagram with two source points and two target points. Let $\boldsymbol{\ell}$ be a labeling of the strands of D such that $\ell_1^s = \ell_2^s = \ell_1^t = \ell_2^t = \lambda$ for some $\lambda \in \mathsf{P}'$. Then*

(A.3.11) $Q^{\boldsymbol{\ell}}$  $= Q^{\boldsymbol{\ell}}$ 

Proof. Then $Q^{\boldsymbol{\ell}}(D) = \varphi$ where $\varphi \colon \mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda) \to \mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)$. By Lemma 1.2.2 the representation $\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda)$ is isomorphic to the direct sum $\mathsf{L}(2\lambda) \oplus \mathsf{L}(2\lambda - \alpha)$. Let e_1, e_2 be the two orthogonal idempotents corresponding to this decomposition.

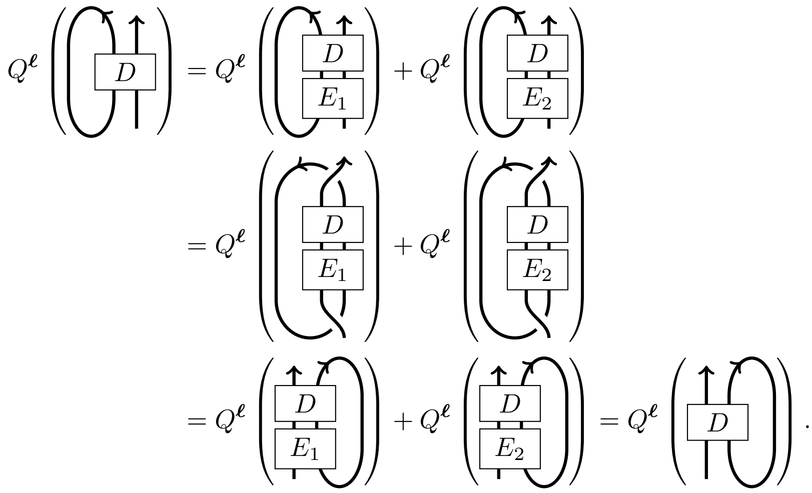
We consider formal $\mathbb{C}(q)$–linear combinations of tangle diagrams, and we extend $Q^{\boldsymbol{\ell}}$ to them. Since $\mathrm{End}_{U_q}(\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda))$ is a two-dimensional $\mathbb{C}(q)$–vector space and $\check{R}_{\lambda,\lambda}$ is not a multiple of the identity by (A.3.7), there are some $\mathbb{C}(q)$–linear combinations of tangle diagrams E_1 and E_2 such that $Q^{\boldsymbol{\ell}}(E_1) = e_1$ and $Q^{\boldsymbol{\ell}}(E_2) = e_2$. Hence we can write

(A.3.12)



Now we have

(A.3.13)



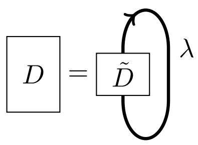
The second equality here follows because we must have

(A.3.14) $$\check{R}e_1 = e_1\check{R} = a_1e_1 \qquad \text{and} \qquad \check{R}e_2 = e_2\check{R} = a_2e_2$$

for some $a_1, a_2 \in \mathbb{C}(q)$, since the elements e_1 and e_2 project onto one-dimensional subspaces of $\mathrm{End}_{U_q}(\mathsf{L}(\lambda) \otimes \mathsf{L}(\lambda))$. The penultimate equality follows by isotopy invariance. $\square$

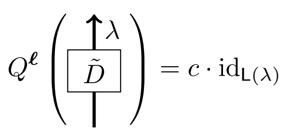
Let now D be an oriented framed link diagram, $\boldsymbol{\ell}$ a labeling of its strands and $\lambda \in \mathsf{P}'$ some weight which labels some strand of D. By cutting one of the strands labeled by λ, we can suppose that D is the closure of a tangle $\tilde{D}$ with one source and one target point, as in the picture

(A.3.15)



Then we define $\hat{Q}^{\boldsymbol{\ell},\lambda}(D) = c \in \mathbb{C}(q)$ where

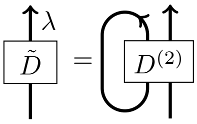
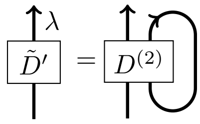
(A.3.16)



We have:

Theorem A.3.6. *The assignment $D \mapsto \hat{Q}^{\boldsymbol{\ell},\lambda}(D) \in \mathbb{C}(q)$ is an invariant of oriented framed links.*

Proof. Since $Q^{\boldsymbol{\ell}}(\tilde{D})$ is an invariant of oriented framed tangles, we need only to show that $\hat{Q}^{\boldsymbol{\ell},\lambda}$ is independent of how we cut D to get $\tilde{D}$. If $\tilde{D}'$ is obtained by some different cutting, but always along some strand labeled by λ, then after some isotopy we must have

(A.3.17)  and 

for some tangle $D^{(2)}$. By Proposition A.3.5 we have then $Q^{\boldsymbol{\ell}}(\tilde{D}) = Q^{\boldsymbol{\ell}}(\tilde{D}')$. □

If $\boldsymbol{\ell}$ is the constant labeling by the weight λ, we write $\hat{Q}^{\lambda}$ instead of $\hat{Q}^{\boldsymbol{\ell},\lambda}$. For $\lambda = \varepsilon_1$ we write simply $\hat{Q}$. As a consequence of Corollary A.3.3 and Theorem A.3.6 we obtain:

Corollary A.3.7. *The assignment* $D \mapsto \hat{Q}(D) \in \mathbb{C}(q)$ *is an invariant of oriented links.*

Recovering the Alexander polynomial

If we compute the action of the R-matrix on $\mathsf{L}(\varepsilon_1) \otimes \mathsf{L}(\varepsilon_1)$ we get by (A.3.7), setting $v_0 = v_0^{\varepsilon_1}$ and $v_1 = v_1^{\varepsilon_1}$:

$$\begin{aligned} \check{R}^{-1}(v_0 \otimes v_0) &= -qv_0 \otimes v_0, & \check{R}^{-1}(v_0 \otimes v_1) &= v_1 \otimes v_0 + (q^{-1} - q)v_0 \otimes v_1 \\ \check{R}^{-1}(v_1 \otimes v_0) &= v_0 \otimes v_1 & \check{R}^{-1}(v_1 \otimes v_1) &= q^{-1}v_1 \otimes v_1. \end{aligned} \tag{A.3.18}$$

On can easily check that

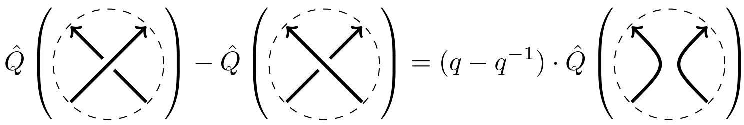
$$(\check{R}^{-1})^2 = (q^{-1} - q)\check{R}^{-1} + \mathrm{Id}. \tag{A.3.19}$$

and hence

$$\check{R} = \check{R}^{-1} + q - q^{-1}. \tag{A.3.20}$$

It follows:

Proposition A.3.8. *The invariant of links* $\hat{Q}$ *satisfies the following skein relation*

(A.3.21) 

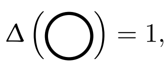
where the pictures represent three links that differ only inside a small neighborhood of a crossing.

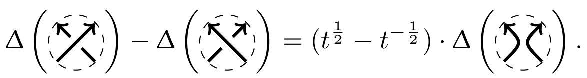
We recall one of the equivalent definitions of the Alexander-Conway polynomial ([Ale28], [Con70]):

Definition A.3.9. *The* Alexander-Conway polynomial *is the value of the assignment*

$$\Delta\colon \text{Links} \to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \tag{A.3.22}$$

defined by the following skein relations:

(A.3.23) 

(A.3.24) 

Notice that obviously $\hat{Q}(\bigcirc) = 1$ since $Q^{\varepsilon_1}(\uparrow) = \mathrm{Id}$. As a consequence, we have that Q is essentially the Alexander-Conway polynomial:

Theorem A.3.10. *For all oriented links L in $\mathbb{R}^3$ we have*

$$\Delta(L) = \hat{Q}(L)\big|_{q=t^{\frac{1}{2}}}. \tag{A.3.25}$$

In particular, $\hat{Q}(L) \in \mathbb{Z}[q, q^{-1}]$ is a Laurent polynomial in q.

APPENDIX B

Cohomology of the Springer fiber

In this appendix we prove that the endomorphism rings of the indecomposable projective modules Ae_λ over the diagrammatic algebra $A_{n,k}$ defined in §9.2 are isomorphic to the cohomology rings of some subvarieties of the Springer fiber. Conjecturally, it should be possible to describe the whole algebra $A_{n,k}$ using a convolution product on the direct sum of cohomologies. This would be the counterpart for $\mathfrak{gl}(1|1)$ of the result of Stroppel and Webster [SW12] for $\mathfrak{sl}_2$.

We warn the reader that in this appendix we will use an ad hoc notation, which differs sometimes from the one used in the rest of the thesis.

B.1 The Springer fiber of hook type

Let us fix a positive integer n and an integer $0 \leq \ell \leq n$. Let $G = GL(n)$ be the general linear group of invertible $n \times n$ matrices, B the Borel subgroup of upper triangular matrices, T the torus of invertible diagonal matrices. Let N be the standard nilpotent matrix of Jordan type $(\ell, 1^{n-\ell})$. If $\{e_1, \dots, e_\ell, f_1, \dots, f_{n-\ell}\}$ is the standard basis of $\mathbb{C}^n$, then $Ne_i = e_{i-1}$ for $i = 2, \dots, \ell$, and $Ne_1 = Nf_i = 0$. Let $\mathscr{B}_N = (G/B)^N$ be the Springer fiber consisting of all flags fixed by $\mathrm{Id} + N$.

To keep the connection with the notation of Part III, we think $\ell = n - k$. In Chapter 8 we described the Soergel modules for the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$, where $\mathfrak{p}$ was of type $(1, \dots, 1, n-k)$. But dealing with the Springer fiber, we prefer to follow the standard convention and to "reorder variables, indices and positions" so that the composition $(1, \dots, 1, n-k)$ becomes a partition $(n-k, 1, \dots, 1)$. This is the reason why in this appendix we are using a somehow 'dual' notation.

Tableaux of hook shape

We consider a Young diagram of hook shape $(\ell, 1^{n-\ell})$. This shape is formed by a row with ℓ boxes and a column with $n - \ell$ boxes; according to our convention, the box in the corner belongs to the row and not to the column: note that this makes a difference between the

$$
\begin{array}{|c|c|c|}\hline 7 & 3 & 6 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{1} \\ \cline{1-1} \multicolumn{1}{|c|}{5} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 6 & 5 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \multicolumn{1}{|c|}{1} \\ \cline{1-1} \multicolumn{1}{|c|}{7} \\ \cline{1-1} \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 6 & 5 & 3 \\ \hline \multicolumn{1}{|c|}{7} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 7 & 5 & 3 \\ \hline \multicolumn{1}{|c|}{6} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array}
$$

Figure B.1: These are tableaux of shape $(3, 4)$. The second one is row-strict, the third one is row-strict-column-strict and the fourth one is standard.

hook shape $(1, 1^{n-1})$ and the hook shape $(0, 1^n)$. A tableaux of shape $(\ell, 1^{n-\ell})$ is obtained by filling the row with numbers $r_\ell, \ldots, r_1$ from the left to the right and the column with numbers $c_1, \ldots, c_{n-\ell}$ from the top to the bottom, such that $\{r_i, c_j\} = \{1, \ldots, n\}$:

$$
\begin{array}{|c|c|c|}\hline r_\ell & \cdots & r_1 \\ \hline \multicolumn{1}{|c|}{c_1} \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{c_{n-\ell}} \\ \cline{1-1} \end{array}
\tag{B.1.1}
$$

Definition B.1.1. *We say that a tableau of shape* $(\ell, 1^{n-\ell})$ *is*

- row-strict *if* $r_\ell > r_{\ell-1} > \cdots > r_1$,
- row-strict-column-strict *if moreover* $c_1 > c_2 > \cdots > c_{n-\ell}$,
- standard *if moreover* $r_\ell = n$.

We denote by $\mathrm{Rs}(n, \ell)$, $\mathrm{RsCs}(n, \ell)$, $\mathrm{St}(n, \ell)$ *respectively the sets of row-strict, row-strict-column-strict and standard tableaux of shape* $(\ell, 1^{n-\ell})$.

Note that this is not the usual definition (although there is a straightforward correspondence with the usual definition).

Irreducible components of the Springer fiber

Let $\tau \in \mathrm{St}(n, \ell)$. Define Y_τ to be the subset of $\mathscr{B}_N$ consisting of all flags $F_\bullet$ such that

$$
\begin{aligned}
\operatorname{Im} N^{\ell-1} &\subseteq F_{r_1} \subseteq \ker N, \\
\operatorname{Im} N^{\ell-2} &\subseteq F_{r_2} \subseteq \ker N^2, \\
&\vdots \\
\operatorname{Im} N^{1} &\subseteq F_{r_{\ell-1}} \subseteq \ker N^{\ell-1}.
\end{aligned}
\tag{B.1.2}
$$

Then (cf. [Fun03, Theorem 2.1]) Y_τ is a locally closed subset of $\mathscr{B}_N$ whose closure is an irreducible component.

For future convenience, we rewrite the conditions (B.1.2) in the following equivalent way:

$$\begin{gathered}\langle e_1\rangle \subseteq F_{r_1} \subseteq \langle e_1\rangle + Q,\\ \langle e_1, e_2\rangle \subseteq F_{r_2} \subseteq \langle e_1, e_2\rangle + Q,\\ \vdots \\ \langle e_1, \dots, e_{\ell-1}\rangle \subseteq F_{r_{\ell-1}} \subseteq \langle e_1, \dots, e_{\ell-1}\rangle + Q,\\ \langle e_1, \dots, e_\ell\rangle \subseteq F_{r_\ell} \subseteq \langle e_1, \dots, e_\ell\rangle + Q\end{gathered} \tag{B.1.3}$$

where $Q = \langle f_1, \dots, f_{n-\ell}\rangle$. Of course, the last condition is unnecessary since for a standard tableau we have $F_{r_\ell} = F_n = \mathbb{C}^n$.

B.2 Fixed points and attracting varieties

Let $S \subset T \subset GL(n)$ be the centralizer of N in T. One can easily see that S is a $(n-\ell+1)$–dimensional torus and consists of all invertible diagonal matrices whose first ℓ elements are all equal. The action of T on G/B induces an action of S on $\mathscr{B}_N$.

Lemma B.2.1. *We have a bijection $\tau \mapsto F_\bullet(\tau)$ between $\mathrm{Rs}(n,\ell)$ and the set of fixed points for the action of S on $\mathscr{B}_N$, given by*

$$F_i(\tau) = \langle e_p \mid p \leq R_i\rangle + \langle f_q \mid c_q \leq i\rangle \tag{B.2.1}$$

where R_i is the number of elements r_j in the row of τ that are smaller than or equal to i.

Proof. It is clear that if $F_\bullet \in \mathscr{B}_N$ is fixed by S then each F_i is generated by some of the standard basis vectors. Conversely, every flag generated by basis vectors is obviously fixed by S. Such a flag is in $\mathscr{B}_N$ if and only if whenever $e_j \in F_i$ then also $e_{j-1} \in F_i$. □

Fix the cocharacter

$$\begin{aligned}\mathbb{C}^\times &\longrightarrow S\\ t &\longmapsto \mathrm{diag}(\underbrace{t^{-1}, \dots, t^{-1}}_{\ell}, t, t^2, \dots, t^{n-\ell}).\end{aligned} \tag{B.2.2}$$

This determines an action of the one-dimensional torus $\mathbb{C}^\times$ on $\mathscr{B}_N$. For $\tau \in \mathrm{Rs}(n,\ell)$ let us define the *attracting variety*

$$\mathcal{Y}_\tau^\circ = \{F_\bullet \in \mathscr{B}_N \mid \lim_{t\to\infty} t \cdot F_\bullet = F_\bullet(\tau)\} \tag{B.2.3}$$

and let $\mathcal{Y}_\tau = \overline{\mathcal{Y}_\tau^\circ}$ be its closure.

We connect now the combinatorics of tableaux with the diagrammatic weights from §9.1. We number the first n vertices on the number line from n to 1 from the left to the right. Then we have obviously:

Lemma B.2.2. *There is a bijection between $\mathrm{RsCs}(n,\ell)$ and a block $\Gamma_{n-\ell}$ consisting of weights with $n-\ell$ $\wedge$'s and ℓ $\vee$'s, given by putting the $\vee$'s in positions $r_\ell, r_{\ell-1}, \dots, r_1$ and the $\wedge$'s in positions $c_1, c_2, \dots, c_{n-\ell}$.*

Recall that in §9.4 we defined for every weight λ a weight $\tilde{\lambda}$ of maximal defect. This assignment together with the lemma above gives for every row-strict-column-strict tableau τ a standard tableau $\tilde{\tau}$.

Proposition B.2.3. *Let $\tau \in \mathrm{RsCs}(n,\ell)$. The set $\mathcal{Y}_\tau$ is the set of flags $F_\bullet \in \mathscr{B}_N$ such that*

$$\begin{aligned} \langle e_1 \rangle \subseteq F_{r_1} &\subseteq \langle e_1 \rangle + Q, \\ \langle e_1, e_2 \rangle \subseteq F_{r_2} &\subseteq \langle e_1, e_2 \rangle + Q, \\ &\vdots \\ \langle e_1, \dots, e_{\ell-1} \rangle \subseteq F_{r_{\ell-1}} &\subseteq \langle e_1, \dots, e_{\ell-1} \rangle + Q, \\ \langle e_1, \dots, e_\ell \rangle \subseteq F_{r_\ell} &\subseteq \langle e_1, \dots, e_\ell \rangle + Q \end{aligned} \tag{B.2.4}$$

where $Q = \langle f_1, \dots, f_{n-\ell} \rangle$. In particular $\mathcal{Y}_\tau \subset \mathcal{Y}_{\tilde{\tau}}$ and if τ is a standard tableau then $\mathcal{Y}_\tau = Y_\tau$.

Proof. First observe that since $P = \langle e_1, \dots, e_\ell \rangle$ has minimal weight for the action of $\mathbb{C}^\times$, if $v \notin P$ then $\lim_{t\to\infty} v \notin P$. Hence we have

$$\dim \lim_{t\to\infty} (t \cdot F_i) \cap P \leq \dim F_i \cap P \tag{B.2.5}$$

for all i.

Now let $F_\bullet \in \mathscr{B}_N$ and suppose $\langle e_1, \dots, e_i \rangle \not\subseteq F_{r_i}$ for some i. Then $\dim F_{r_i} \cap P < i$. By (B.2.5) this also holds for the limit. Hence $t \cdot F_\bullet \not\to F_\bullet(\tau)$. On the other side, it is clear that if (B.2.4) holds then generically $t \cdot F_\bullet \to F_\bullet(\tau)$. □

B.3 The cohomology rings

In the following we will denote by H^* the cohomology with complex coefficients. Our next goal is to compute the cohomology rings $H^*(\mathcal{Y}_\tau)$ for all $\tau \in \mathrm{RsCs}(n,\ell)$.

Dimension

In [Fun03, Theorem 3.2] the dimension of $H^*(\mathcal{Y}_\tau)$ for τ a standard tableau is computed. We generalize it now to $\tau \in \mathrm{RsCs}(n,\ell)$. We will need the following lemma:

Lemma B.3.1. *For every $\tau \in \mathrm{RsCs}(n,\ell)$ we have a fibration*

$$\mathcal{Y}_\sigma \longrightarrow \mathcal{Y}_\tau \longrightarrow \mathcal{G}(n-\ell, (r_\ell, 1^{n-\ell-r_\ell})) \tag{B.3.1}$$

where σ is the standard tableau obtained from τ after removing all boxes containing entries $i > r_\ell$, while $\mathcal{G}(n-\ell, (r_\ell, 1^{n-\ell-r_\ell}))$ is the partial flag variety of $\mathbb{C}^{n-\ell}$ consisting of flags

$$F_\bullet : \quad \{0\} = F_0 \subset F_{r_\ell - \ell} \subset F_{r_\ell - \ell + 1} \subset \cdots \subset F_{n-\ell} = \mathbb{C}^{n-\ell}. \tag{B.3.2}$$

Proof. The fibration $\mathcal{Y}_\tau \longrightarrow \mathcal{G}(n-\ell, (r_\ell, 1^{n-\ell-r_\ell}))$ is defined by

$$F_\bullet \longmapsto \{0\} = F_0 \subset F_{r_\ell}/P \subset F_{r_\ell+1}/P \subset \cdots \subset F_{n-\ell} = \mathbb{C}^n/P = \mathbb{C}^{n-\ell} \tag{B.3.3}$$

where as before $P = \langle e_1, \dots, e_\ell \rangle$. □

Proposition B.3.2. *For all $\tau \in \mathrm{RsCs}(n,\ell)$ we have*

$$\dim H^*(\mathcal{Y}_\tau) = (n-\ell)! \cdot r_1(r_2 - r_1) \cdots (r_\ell - r_{\ell-1}). \tag{B.3.4}$$

Proof. For $\tau \in \mathrm{St}(n,\ell)$, since $\mathcal{Y}_\tau = Y_\tau$, this is just [Fun03, Theorem 3.2]. If τ is not standard, we use the fibration (B.3.1). Since we are dealing with complex varieties, the dimension of the cohomology of the total space is just the product of the dimensions of the fiber and of the base space. Notice that the tableau σ is standard ($\sigma \in \mathrm{St}(r_\ell,\ell)$), hence we know already that

$$\dim H^*(\mathcal{Y}_\sigma) = (r_\ell - \ell)! \cdot r_1(r_2 - r_1)\cdots(r_\ell - r_{\ell-1}). \tag{B.3.5}$$

Since the dimension of the cohomology of $\mathcal{G}(n-\ell,(r_\ell,1^{n-\ell-r_\ell}))$ is

$$(n-\ell)(n-\ell-1)\cdots(r_\ell-\ell+1), \tag{B.3.6}$$

the claim follows. □

Surjectivity

We want now to find a set of generators. The following argument is inspired by [DCP81].

Let $\tau \in \mathrm{RsCs}(n,\ell)$. Let $p\colon \mathcal{Y}_\tau \to \mathbb{P}(\ker N)$ be the projection $F_\bullet \mapsto F_1$. We fix the following complete flag of $\ker N$:

$$\begin{aligned} W_0 &= \{0\} \\ W_1 &= \langle e_1 \rangle \\ W_2 &= \langle e_1, f_1 \rangle \\ &\vdots \\ W_{n-\ell} &= \langle e_1, f_1, \ldots, f_{n-\ell-1} \rangle \\ W_{n-\ell+1} &= \ker N. \end{aligned} \tag{B.3.7}$$

We let $\Delta^j = \mathbb{P}(W_j) - \mathbb{P}(W_{j-1})$; this is of course an open affine cell of $\mathbb{P}(\ker N)$, isomorphic to $\mathbb{C}^{j-1}$. Let moreover $V_\tau^j = p^{-1}(\mathbb{P}(W_j))$.

Given a tableau τ and an entry a of τ, we define $\check{\tau}^a$ to be the tableau obtained from τ by removing the box containing a and then subtracting 1 to all entries bigger than a. Note that if $\tau \in \mathrm{RsCs}(n,\ell)$ then $\check{\tau}^a$ is also a row-strict-column-strict tableau.

Lemma B.3.3. *The set $V_\tau^j - V_\tau^{j-1}$ is either empty or isomorphic to $\Delta^j \times \mathcal{Y}_{\check{\tau}^1}$ for $j > 1$ and to $\Delta^j \times \mathcal{Y}_{\check{\tau}^{r_1}}$ for $j = 1$.*

Proof. Let $U = \mathbb{P}(\ker N) - \mathbb{P}(W_1)$ and $U' = \mathbb{P}(W_1)$, so that $U \cup U' = \mathbb{P}(\ker N)$. Notice that p is surjective onto $\mathbb{P}(\ker N)$ if and only if 1 is not in the row of τ, that is $r_1 \neq 1$; otherwise p is onto $\mathbb{P}(W_1)$. Now $p|_{p^{-1}(U')}$ is a locally trivial fibration with fiber isomorphic to $\mathcal{Y}_{\check{\tau}^{r_1}}$ (in this specific case, the base space is even a point), while $p|_{p^{-1}(U)}$ is a locally trivial fibration with fiber isomorphic to $\mathcal{Y}_{\check{\tau}^1}$ (if non-empty). In particular, for every j the projection p restricted to $V_\tau^j - V_\tau^{j-1}$ is a locally trivial fibration; since the base space is isomorphic to $\mathbb{C}^{j-1}$, the fibration has to be trivial, hence isomorphic (if non-empty) to the product of Δ^j and the fiber. □

Thanks to Lemma B.3.3, we have a recursive construction of a cell decomposition of $\mathcal{Y}_\tau$ with even dimensional cells.

Proposition B.3.4. *For every $\tau \in \mathrm{RsCs}(n,\ell)$ the inclusion $\mathcal{Y}_\tau \hookrightarrow G/B$ of $\mathcal{Y}_\tau$ into the full flag variety induces a surjective homomorphism $H^*(G/B) \twoheadrightarrow H^*(\mathcal{Y}_\tau)$ in cohomology.*

Proof. We prove by induction on n that it is possible to construct a cell decomposition of G/B with even dimensional cells such that $\mathcal{Y}_\tau$, with the CW-structure that we have defined, is a subcomplex of it. For $n = 0$ there is nothing to prove, so let us consider $n > 0$. Notice that $G/B = \mathcal{Y}_\sigma$ where σ is the unique element of $\mathrm{RsCs}(n, 0)$. Complete the flag $W_\bullet$ of (B.3.7) for τ to a full flag of $\mathbb{C}^n$. Then by Lemma B.3.3 $V_\sigma^j - V_\sigma^{j-1} \cong \Delta^j \times \mathcal{Y}_{\breve{\sigma}^1}$, where $\breve{\sigma}^1 \in \mathrm{RsCs}(n-1, 0)$, while $V_\tau^j - V_\tau^{j-1}$ is either isomorphic to $\Delta^j \times \mathcal{Y}_{\breve{\tau}^{a_j}}$ for some a_j or empty. By induction, we can suppose that $\mathcal{Y}_{\breve{\tau}^{a_j}}$ is a subcomplex of $\mathcal{Y}_{\breve{\sigma}^1}$; then the claim for $\mathcal{Y}_\tau$ follows.

Since the cells are even dimensional, they give a basis of the cohomology as a vector space. It follows that the homomorphism $H^*(G/B) \to H^*(\mathcal{Y}_\tau)$ in cohomology is surjective. □

The isomorphism with $\mathsf{Z}_{z,z}$

For $\tau \in \mathrm{RsCs}(n, \ell)$ let us define an ideal I_τ of $R = \mathbb{C}[x_1, \dots, x_n]$ as follows. Let $\boldsymbol{b}$ be the $\boldsymbol{b}$–sequence of the weight corresponding to τ. Let

$$I'_\tau = (h_{b_i}(x_n, x_{n-1}, \dots, x_i))_{i=n,\dots,1} \tag{B.3.8}$$

and

$$I''_\tau = (x_{r_i} x_{r_i - 1} \cdots x_{r_{i-1}+1})_{i=h,\dots,1} \tag{B.3.9}$$

where $r_0 = 0$. Set

$$I_\tau = I'_\tau + I''_\tau. \tag{B.3.10}$$

Finally, set

$$R_\tau = \mathbb{C}[x_1, \dots, x_n]/I_\tau. \tag{B.3.11}$$

Note that according to Theorems 8.2.7 and 8.3.5 and Corollary 9.2.4 we have

$$R_\tau \cong \mathsf{Z}_{z,z} = e_\lambda A_{n,n-\ell} e_\lambda \tag{B.3.12}$$

where $z \in D_{n,n-\ell}$ and $\lambda \in \Gamma_{n-\ell}$ are the permutation and the weight corresponding to the tableau τ. Since we work in the dual pictures (with reordered indices), the isomorphism is given by $x_i \mapsto x_{n-i}$.

We recall that the *elementary symmetric polynomials* are defined as

$$\mathsf{e}_j(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} \tag{B.3.13}$$

for $0 \le j \le n$.

We are now ready to state the main theorem of this appendix.

Theorem B.3.5. *For every $\tau \in \mathrm{RsCs}(n, \ell)$ the cohomology ring of $\mathcal{Y}_\tau$ is isomorphic to R_τ. The Chern class of the canonical bundle F_i/F_{i-1} over $\mathcal{Y}_\tau$ is sent to the class of x_i under this isomorphism.*

The proof will consist of several reduction steps. Let us remark that by Proposition B.3.4 we know that the cohomology ring of $\mathcal{Y}_\tau$ is generated by the Chern classes of its canonical line bundles F_i/F_{i-1} (since this holds for the full flag variety). Moreover, by Proposition B.3.2 we already know that the dimensions agree. Hence it suffices to prove that for every $\tau \in \mathrm{RsCs}(n, \ell)$ the Chern classes of the canonical bundles F_i/F_{i-1} on $\mathcal{Y}_\tau$ satisfy the relations of the ideal I_τ.

Lemma B.3.6. *Let τ be the row-strict-column-strict tableau of shape (1^n). Then Theorem B.3.5 holds for $\mathcal{Y}_\tau$.*

Proof. In this case, R_τ is the cohomology ring of the full flag variety. But $\mathcal{Y}_\tau$ is the full flag variety, since conditions (B.2.4) are void for it (the row of τ is empty). □

We recall that the isomorphism is given by sending the Chern class of the canonical bundle F_i/F_{i-1} to x_i.

Lemma B.3.7. *Suppose $\lambda \in \Gamma^\vee_{n-\ell}$ is a weight starting with a $\vee$, and let $\lambda^{(\wedge)}$ as defined in §9.5. Let τ, σ be the tableaux corresponding to λ and $\lambda^{(\wedge)}$ respectively. If Theorem B.3.5 holds for σ, then it holds for τ.*

Proof. For notation convenience, let $a = r_{\ell-1}$. We have $I_\tau = I_\sigma + (x_n \cdots x_{a+1})$. A flag $F_\bullet \in \mathcal{Y}_\tau$ obviously satisfies the relations (B.2.4) also for σ. Moreover, if it is invariant for the nilpotent N_τ of shape $(\ell, 1^{n-\ell})$, it is a fortiori invariant for the nilpotent N_σ of shape $(\ell-1, 1^{n-\ell+1})$. Hence we have an inclusion map $\mathcal{Y}_\tau \hookrightarrow \mathcal{Y}_\sigma$, and the relations that $x_1, \ldots, x_n$ satisfy in $H^*(\mathcal{Y}_\sigma)$ are also satisfied in $H^*(\mathcal{Y}_\tau)$.

We are left to prove that the relation $x_n \cdots x_{a+1}$ holds on $H^*(\mathcal{Y}_\tau)$. By (B.2.4) for τ, we know that $F_a \subset K = \langle e_1, \ldots, e_{\ell-1}\rangle + Q$. Let us work in K-theory for bundles over $\mathcal{Y}_\tau$ and write $[\mathbb{C}^n/F_a] = [\mathbb{C}^n/K] + [K/F_a]$. Since the bundle $\mathbb{C}^n/K$ is a one-dimensional trivial bundle, the $(n-a)$–th Chern class of $\mathbb{C}^n/F_a$ is trivial. But this class is equal to the elementary symmetric function $\mathsf{e}_{n-a}(x_n, \ldots, x_{a+1}) = x_n \cdots x_{a+1}$ by the Whitney sum formula, and we are done. □

Lemma B.3.8. *Suppose τ is a row-strict-column-strict tableau that is not standard. If Theorem B.3.5 holds for the standard tableau $\tilde\tau$ (defined in §9.4), then it also holds for τ.*

Proof. Remember that $\tilde\tau$ is obtained permuting the leftmost $\wedge$ with the leftmost $\vee$ of the $\wedge\vee$–sequence corresponding to τ. As before, since $\mathcal{Y}_\tau \subset \mathcal{Y}_{\tilde\tau}$, all relations of $H^*(\mathcal{Y}_{\tilde\tau})$ also hold in $H^*(\mathcal{Y}_\tau)$. Hence we need to prove that in $H^*(\mathcal{Y}_\tau)$ the relations $h_{b_i}(x_n, \ldots, x_i)$ for $i > r_\ell$ hold.

The variety $\mathcal{Y}_\tau$ consists of all flags $F_\bullet$ in $\mathcal{Y}_{\tilde\tau}$ that satisfy also $P \subseteq F_{r_\ell}$. Let $a \geq r_\ell$. We argue as in the previous proof: we have in K-theory $[F_a] = [F_a/P] + [P]$; since P is a trivial bundle, by the Whitney sum formula we have

$$\mathsf{e}_i(x_a, \ldots, x_1) = 0 \qquad \text{for all } i > a - \ell. \tag{B.3.14}$$

Note that $a - \ell$ is equal to the number of $\wedge$'s that are on the right of position a, that is $b_a - 1$. Let us consider the following identity of symmetric functions:

$$\begin{aligned} &h_{a-\ell+1}(x_n, \ldots, x_{a+1}) \\ &\quad = (-1)^{a-\ell+1}\Bigg(\sum_{i=0}^{a-\ell} (-1)^i h_i(x_n, \ldots, x_{a+1}) \mathsf{e}_{a-\ell-i+1}(x_n, \ldots, x_1) \\ &\qquad\qquad - \mathsf{e}_{a-\ell+1}(x_a, \ldots, x_1)\Bigg). \end{aligned} \tag{B.3.15}$$

It follows that $h_{a-\ell+1}(x_n, \ldots, x_{a+1}) = 0$. □

Proof of Theorem B.3.5. Let $\tau \in$ RsCs. Applying repeatedly Lemmas B.3.7 and B.3.8 we can restrict to the case in which τ is a sequence with $\wedge$'s only. Then the theorem holds by Lemma B.3.6. □

REMARK B.3.9. It is also possible to derive Theorem B.3.5 from [GR02, Theorem 3.1], where cohomology rings of varieties defined by inclusions are computed. The varieties $\mathcal{Y}_\tau$ are particular cases of such varieties, and to get Theorem B.3.5 one could check algebraically that the quotient ring given by [GR02, Theorem 3.1] is isomorphic to R_τ. We included nevertheless a direct proof of Theorem B.3.5 for two reasons: first, because our particular case is quite easier than the more general one treated in [GR02], and second, because the proof we presented suggests an inductive way to construct such cohomology rings, and conjecturally the whole algebra $A_{n,n-\ell}$, starting with the cohomology of the full flag variety.

Let $\tau, \tau' \in \mathrm{RsCs}(n,\ell)$, and let $z, z' \in D_{n,n-\ell}$ and $\lambda, \lambda' \in \Gamma_{n-\ell}$ be the permutations and the weights corresponding to them. Set $\ell(\tau) = \ell(z)$ and $\ell(\tau') = \ell(z')$. Generalizing Proposition B.3.2 (using the techniques of [Fun03]), it is actually possible to check that the graded dimension of $H^*(\mathcal{Y}_\tau \cap \mathcal{Y}_{\tau'})$ is the same, up to a degree shift of $|\ell(\tau) - \ell(\tau')|$, as the graded dimension of $\mathsf{Z}_{z',z}$, which is the graded dimension of $e_\lambda A_{n,n-\ell} e_{\lambda'}$. As a consequence, we have an isomorphism of vector spaces

$$A_{n,n-\ell} \cong \bigoplus_{\tau,\tau' \in \mathrm{RsCs}(n,\ell)} H^*(\mathcal{Y}_\tau \cap \mathcal{Y}_{\tau'}) \left\langle\, |\ell(\tau) - \ell(\tau')| \,\right\rangle. \tag{B.3.16}$$

We conjecture that, as in [SW12], it is possible to define a convolution product on the direct sum

$$\bigoplus_{\tau,\tau' \in \mathrm{RsCs}(n,\ell)} H^*(\mathcal{Y}_\tau \cap \mathcal{Y}_{\tau'}) \left\langle\, |\ell(\tau) - \ell(\tau')| \,\right\rangle \tag{B.3.17}$$

such that the resulting algebra is isomorphic to the algebra $A_{n,n-\ell}$. This would give a geometric realization of the endomorphism algebras coming from Lie theory and of their diagrammatic versions (9.2.10).

APPENDIX C

Categorification of representations of $\mathfrak{gl}(m|n)$

The construction we presented in Part II for $U_q(\mathfrak{gl}(1|1))$ can be extended to the case of $U_q(\mathfrak{gl}(m|n))$ for $m, n \geq 1$. However, as we have seen, the combinatorics for $U_q(\mathfrak{gl}(1|1))$ is already quite involved; developing the analogous combinatorics for general $U_q(\mathfrak{gl}(m|n))$ would make this work unreadable.

Nevertheless, in order to be complete, we want to present in this appendix a categorification result for $\mathfrak{gl}(m|n)$, avoiding some of the technicalities. In order to do that, we make the following simplifications:

- we consider the classical (non-quantum) version;
- we consider only tensor powers of the vector representation (and not their subrepresentations);
- we categorify only the action of the intertwining operators (and not of $U_q(\mathfrak{gl}(m|n))$.

We derive the categorification from super skew Howe duality instead of from Schur-Weyl duality, although the two approaches are equivalent.

C.1 Super skew Howe duality

Let $I_{m|n} = \{1, \ldots, m+n\}$ with a parity function $|\cdot| : I_{m|n} \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$|i| = \begin{cases} 0 & \text{if } i \leq m \\ 1 & \text{if } i > m \end{cases} \tag{C.1.1}$$

for each $i \in I_{m|n}$. Let also $\mathbb{C}^{m|n}$ be a $(m+n)$–dimensional super vector space on basis $\{e_i \mid i \in I_{m|n}\}$ such that $|e_i| = |i|$, where as usual $|v|$ denotes the degree of an homogeneous element $v \in \mathbb{C}^{m|n}$. Then the Lie superalgebra $\mathfrak{gl}(m|n)$ is the super vector space of matrices $\mathrm{End}(\mathbb{C}^{m|n})$ equipped with the Lie super bracket

$$[x, y] = xy - (-1)^{|x||y|} yx. \tag{C.1.2}$$

In particular note that $\mathfrak{gl}(m|0) \cong \mathfrak{gl}(0|m) \cong \mathfrak{gl}_m$. The Lie superalgebra $\mathfrak{gl}(m|n)$ acts by matrix multiplication on $\mathbb{C}^{m|n}$: this is the vector representation of $\mathfrak{gl}(m|n)$.

If V is a super vector space, we define an action of the symmetric group $\mathbb{S}_N$ on the tensor power $\bigotimes^N V$ by setting

$$s_\ell \cdot (x_1 \otimes \cdots \otimes x_N) = (-1)^{|x_\ell||x_{\ell+1}|} x_1 \otimes \cdots \otimes x_{\ell+1} \otimes x_\ell \otimes \cdots \otimes x_N \tag{C.1.3}$$

for every simple reflection $s_\ell \in \mathbb{S}_N$. Let $\pi^{\mathsf{S}}, \pi^{\wedge} \in \mathbb{C}[\mathbb{S}_n]$ be the idempotents projecting onto the trivial and sign representations respectively. We set then

$$\mathsf{S}^N V = \pi^{\mathsf{S}} \cdot (\textstyle\bigotimes^N V) \qquad \text{and} \qquad \textstyle\bigwedge^N V = \pi^{\wedge} \cdot (\textstyle\bigotimes^N V). \tag{C.1.4}$$

In particular, notice that if V is a vector space concentrated in degree zero then this definitions coincide with the usual symmetric and exterior powers of V.

Remark C.1.1. Notice that $\mathsf{S}^N(\mathbb{C}^{m|n}) \cong \bigwedge^N(\mathbb{C}^{n|m})$. It follows in particular that, in contrast to the classical case, $\bigwedge^N V$ can be non-zero also for $N \gg 0$.

If $v_1, \ldots, v_r$ is a basis of V, then a basis of $\bigwedge^N V$ is given by

$$v_{i_1} \wedge \cdots \wedge v_{i_N} = \pi^{\wedge} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_N}) \tag{C.1.5}$$

for all sequences $(i_1, \ldots, i_N)$ of indices $i_\ell \in \{1, \ldots, r\}$ such that $i_1 \leq i_2 \leq \cdots \leq i_N$ and if $i_\ell = i_{\ell+1}$ then $|v_{i_\ell}| = 1$. Moreover a basis of $\mathrm{S}^N V$ is given by

$$v_{i_1} \odot \cdots \odot v_{i_N} = \pi^{\mathsf{S}} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_N}) \tag{C.1.6}$$

for all sequences $(i_1, \ldots, i_N)$ of indices $i_\ell \in \{1, \ldots, r\}$ such that $i_1 \leq i_2 \leq \cdots \leq i_N$ and if $i_\ell = i_{\ell+1}$ then $|v_{i_\ell}| = 0$.

We have the following result (cf. [CW01], [CW10]):

Proposition C.1.2 (Super skew Howe duality). *Let $p, m, N \in \mathbb{Z}_{>0}$ be positive integers and $q, n \in \mathbb{Z}_{\geq 0}$. The natural actions of $\mathfrak{gl}(p|q)$ and $\mathfrak{gl}(m|n)$ on $\bigwedge^N(\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n})$ commute with each other and generate each other's centralizer. As a $\mathfrak{gl}(m|n)$–module, $\bigwedge^N(\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n})$ decomposes as the direct sum*

$$\bigoplus_{i_1+\cdots+i_{p+q}=N} \textstyle\bigwedge^{i_1} \mathbb{C}^{m|n} \otimes \cdots \otimes \bigwedge^{i_p} \mathbb{C}^{m|n} \otimes \mathsf{S}^{i_{p+1}} \mathbb{C}^{m|n} \otimes \cdots \otimes \mathsf{S}^{i_{p+q}} \mathbb{C}^{m|n}. \tag{C.1.7}$$

Note that inverting the roles of $p|q$ and $m|n$ we have a similar decomposition (C.1.7) as a $\mathfrak{gl}(p|q)$–module.

Proof. The first part is [CW01, Theorem 3.3 and Corollary 3.2]. We check the decomposition (C.1.7).

Let $\{e_1, \ldots, e_{p+q}\}$ and $\{f_1, \ldots, f_{m+n}\}$ be the standard bases of $\mathbb{C}^{p|q}$ and $\mathbb{C}^{m|n}$ respectively. We fix the following ordered basis of $\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n}$:

$$e_1 \otimes f_1, \ldots, e_1 \otimes f_{m+n}, \ldots, e_{p+q} \otimes f_1, \ldots, e_{p+q} \otimes f_{m+n}. \tag{C.1.8}$$

We get then a basis of $\bigwedge^N(\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n})$ as in (C.1.5). Let M be equal to (C.1.7). We define an isomorphism Ψ from $\bigwedge^N(\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n})$ to M in the following way. Given a basis vector $w = (e_{i_1} \otimes f_{j_1}) \wedge \cdots \wedge (e_{i_N} \otimes f_{j_N})$ of $\bigwedge^N(\mathbb{C}^{p|q} \otimes \mathbb{C}^{m|n})$, define functions $a, b\colon \{1, \ldots, p+q\} \to \{\bullet, 1, \ldots, N\}$ by $a(h) = \min\{\ell \mid i_\ell = h\}$ and $b(h) = \max\{\ell \mid i_\ell = h\}$ or

$a(h) = b(h) = \bullet$ if this set is empty. Set also $c(h) = b(h) - a(h) + 1$, with the convention $\bullet - \bullet = -1$. Then we define

$$\Psi(w) \in \textstyle\bigwedge^{c(1)} \mathbb{C}^{m|n} \otimes \cdots \otimes \bigwedge^{c(p)} \mathbb{C}^{m|n} \otimes \mathrm{S}^{c(p+1)} \otimes \cdots \otimes \mathrm{S}^{c(q)}\, \mathbb{C}^{m|n} \tag{C.1.9}$$

to be the element

$$\begin{aligned}(f_{j_{a(1)}} \wedge \cdots \wedge f_{j_{b(1)}}) \otimes \cdots \otimes (f_{j_{a(m)}} \wedge \cdots \wedge f_{j_{b(m)}})\\ \otimes (f_{j_{a(m+1)}} \odot \cdots \odot f_{j_{b(m+1)}}) \otimes \cdots \otimes (f_{j_{a(m+n)}} \odot \cdots \odot f_{j_{b(m+n)}}).\end{aligned} \tag{C.1.10}$$

It is straightforward to check that this is indeed an element of the basis, and that Ψ is bijective and $\mathfrak{gl}(m|n)$–equivariant. □

Remark C.1.3. Another kind of duality, called super Schur-Weyl duality, relates $\mathfrak{gl}(m|n)$ and the symmetric group $\mathbb{S}_N$: the natural action of $\mathbb{C}[\mathbb{S}_N]$ on $V^{\otimes N}$ is $\mathfrak{gl}(m|n)$–equivariant; moreover, the map $\mathbb{C}[\mathbb{S}_N] \to \mathrm{End}_{\mathfrak{gl}(m|n)}(V^{\otimes N})$ is always surjective, and it is injective if and only if $N \leq (m+1)(n+1)$ (see [BR87], [Ser84]).

C.2 Categorification of $\mathfrak{gl}(m|n)$

Set now $V = \mathbb{C}^{m|n}$. Our goal is to construct a categorification of $V^{\otimes N}$ for $N > 0$.

Set $p = N$ and $q = 0$ in Proposition C.1.2. We have then that $\bigwedge^N(\mathbb{C}^N \otimes V)$ decomposes as a $\mathfrak{gl}(m|n)$–module as

$$\bigoplus_{i_1+\cdots+i_N=N} \textstyle\bigwedge^{i_1} V \otimes \cdots \otimes \bigwedge^{i_N} V \tag{C.2.1}$$

and as a $\mathfrak{gl}_N$–module as

$$\bigoplus_{j_1+\cdots+j_{m+n}=N} \textstyle\bigwedge^{j_1} \mathbb{C}^N \otimes \cdots \otimes \bigwedge^{j_m} \mathbb{C}^N \otimes \mathrm{S}^{j_{m+1}}\, \mathbb{C}^N \otimes \cdots \otimes \mathrm{S}^{j_{m+n}}\, \mathbb{C}^N. \tag{C.2.2}$$

Notice that one summand of (C.2.1) is in particular $V^{\otimes N}$. A categorification of the $\mathfrak{gl}_N$–module (C.2.2), although not written in the literature, is in principle known to experts, and is what we are going to use to categorify the $\mathfrak{gl}(m|n)$–module (C.2.1).

In order to state the categorification theorem, we need some notation. Let us fix the standard basis $\{v_1, \ldots, v_{m+n}\}$ of $V = \mathbb{C}^{m|n}$, with

$$|v_i| = \begin{cases} 0 & \text{for } i = 1, \ldots, m, \\ 1 & \text{for } i = m+1, \ldots, n. \end{cases} \tag{C.2.3}$$

Let $\mathfrak{h} \subset \mathfrak{gl}(m|n)$ be the subalgebra of diagonal matrices. Then $V^{\otimes N}$ decomposes as direct sum of weight spaces for the action of $\mathfrak{h}$. Let Λ be the set of compositions $\lambda = (\lambda_1, \ldots, \lambda_{m+n})$ of N with at most $m+n$ parts (that is, we allow $\lambda_i = 0$ for some indices i). Then the weight spaces of $V^{\otimes N}$ are indexed by Λ, and the correspondence is given by

$$(V^{\otimes N})_\lambda = \mathrm{span}\{v_{\sigma(a_1^\lambda)} \otimes \cdots \otimes v_{\sigma(a_N^\lambda)} \mid \sigma \in \mathbb{S}_N\}, \tag{C.2.4}$$

where

$$a^\lambda = (\underbrace{1, \ldots, 1}_{\lambda_1}, \underbrace{2, \ldots, 2}_{\lambda_2}, \ldots, \underbrace{m+n, \ldots, m+n}_{\lambda_{m+n}}). \tag{C.2.5}$$

We can now state our main result:

Theorem C.2.1. *Given $\lambda \in \Lambda$, let $\mathfrak{q}_\lambda \subset \mathfrak{gl}_N$ be the standard parabolic subalgebra corresponding to the composition $(\lambda_1, \dots, \lambda_m, 1, \dots, 1)$ and $\mathfrak{p}_\lambda \subset \mathfrak{gl}_N$ be the standard parabolic subalgebra corresponding to the composition $(1, \dots, 1, \lambda_{m+1}, \dots, \lambda_{m+n})$. Then there is an isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{O}_0^{\mathfrak{p}_\lambda, \mathfrak{q}_\lambda\text{-pres}}(\mathfrak{gl}_N)) \longmapsto (V^{\otimes N})_\lambda \tag{C.2.6}$$

sending equivalence classes of standard modules to standard basis vectors.

The translation functors θ_i give a categorical action of the generators $s_i + 1$ of $\mathbb{C}[\mathbb{S}_N]$, which descends to the action (C.1.3) *at the level of the Grothendieck group.*

We refer to Chapter 5 for the definitions of the categories appearing in (C.2.6) and of the translation functors θ_i.

Proof. The first claim follows from the definition of the categories $\mathcal{O}_0^{\mathfrak{p}_\lambda, \mathfrak{q}_\lambda\text{-pres}}(\mathfrak{gl}_N)$ (cf. §5.3). The second claim can be proved generalizing the proof of Theorem 6.2.2. $\square$

Remark C.2.2. Combining Zuckermann's/coapproximation functors and their adjoints (see §5.4 for the definitions) one can define functors $\mathcal{E}_j$, $\mathcal{F}_j$ for $j = 1, \dots, m+n-1$ between some opportune unbounded derived categories, as in §6.5. These functors commute with the functors θ_i and give an action of $\mathfrak{gl}(m|n)$ at the level of the Grothendieck groups.

We remark that for $n = 0$ Theorem C.2.1 gives exactly the categorification of $(\mathbb{C}^m)^{\otimes N}$ developed in [MS09].

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