Quiver Schur algebras and q-Fock space

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Abstract. We develop a graded version of the theory of cyclotomic q-Schur algebras, in the spirit of the work of Brundan-Kleshchev on Hecke algebras and of Ariki on q-Schur algebras. As an application, we identify the coefficients of the canonical basis on a higher level Fock space with q-analogues of the decomposition numbers of cyclotomic q-Schur algebras.

We present cyclotomic q-Schur algebras as a quotient of a convolution algebra arising in the geometry of quivers; hence we call these **quiver Schur algebras**. These algebras are also presented diagrammatically, similar in flavor to a recent construction of Khovanov and Lauda. They are also manifestly graded and so equip the cyclotomic q-Schur algebra with a non-obvious grading.

On the way we construct a *graded cellular basis* of this algebra, resembling the constructions for cyclotomic Hecke algebras by Mathas, Hu, Brundan and the first author.

The quiver Schur algebra is also interesting from the perspective of higher representation theory. The sum of Grothendieck groups of certain cyclotomic quotients is known to agree with a higher level Fock space. We show that our graded version defines a higher q-Fock space (defined as a tensor product of level 1 q-deformed Fock spaces). Under this identification, the indecomposable projective modules are identified with the canonical basis and the Weyl modules with the standard basis. This allows us to prove the already described relation between decomposition numbers and canonical bases.

Contents

1. Introduction

Recent years have seen remarkable advances in higher representation theory; the most exciting from the perspective of classical representation theory were probably the proof of Broué's conjecture for the symmetric groups by Chuang and Rouquier [CR08] and the introduction and study of graded versions of Hecke algebras by Brundan and Kleshchev [BK09a] with its Lie theoretic origins ([BK08],

[BS11]). At the same time, the question of finding categorical analogues of the usual structures of Lie theory has proceeded in the work of Khovanov, Lauda, Rouquier, Vazirani, the authors and others. In this paper, we address a question of interest from both perspectives, representation theory and higher categorical structures.

As classical (or quantum) representation theorists, we ask

Is there a natural graded version of the q-Schur algebra and its higher level analogues, the cyclotomic q-Schur algebra?

This question has been addressed already in the special case of level 1 by Ariki [Ari09]. We give here a more general construction that both illuminates connections to geometry and is more explicit.

As higher representation theorists, we ask

Is there a natural categorification of q-Fock space and its higher level analogues with a categorical action of $\widehat{\mathfrak{sl}}_n$*?*

We will show that these questions not only have natural answers; they have the *same* answer.

Our main theorem is a *graded version* with a *graded cellular basis* of the cyclotomic q-Schur algebra of Dipper, James and Mathas, [DJM98] and a combinatorics of *graded decomposition numbers* using higher Fock space.

To describe the results more precisely, let $\Bbbk$ be an algebraically closed field of characteristic zero and n, ℓ, e natural numbers with $e > 1$. Let $(q, Q_1, \dots, Q_\ell)$ be an $\ell+1$-tuple of e-roots of unities such that q is primitive. In particular, $Q_i = q^{\mathfrak{z}_i}$ for some $\mathfrak{z}_i$. The associated *cyclotomic Hecke algebra* or *Ariki-Koike algebra*

$$\mathfrak{H}(n; q, Q_1, \dots, Q_\ell) = \mathfrak{H}(S_n \wr \mathbb{Z}/\ell\mathbb{Z}; q, Q_1, \dots, Q_\ell)$$

is the associative unitary $\Bbbk$-algebra with generators T_i, $1 \le i \le n-1$ modulo the following relations, for $1 \le i < j-1 < n-1$,

$$(T_0 - Q_1)\cdots(T_0 - Q_\ell) = 1, \quad (T_i - q)(T_i + 1) = 1$$
$$T_iT_j = T_jT_i, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_0T_1T_0T_1 = T_1T_0T_1T_0$$

The *cyclotomic q-Schur algebra* we consider is the endomorphism ring

$$\mathbf{S}(n; q, Q_1, \dots, Q_\ell) \;=\; \mathrm{End}_{\mathfrak{H}(n;q,Q_1,\dots,Q_\ell)}\left(\bigoplus_{\mu} M(\hat{\mu})\right) \tag{1}$$

where the sum runs over all ℓ-multipartitions $\hat{\mu}$ of n and $M(\hat{\mu})$ denotes the (signed) permutation module associated to $\hat{\mu}$.

To make contact with the theory of quiver Hecke algebras we encode the parameters $(q, Q_1, \dots, Q_\ell)$ as a sequence $\underline{\boldsymbol{\nu}} = (w_{\mathfrak{z}_1}, \dots, w_{\mathfrak{z}_\ell})$ of fundamental weights for the affine Lie algebra $\widehat{\mathfrak{sl}}_e$. The corresponding cyclotomic Hecke algebra only depends on the multiplicities of each root of unity, that is, only on the weight $\nu = \sum_i \nu_i$. Thus, we write

$$H^\nu := \bigoplus_{n\ge 0} \mathfrak{H}(n; q, Q_1, \dots, Q_\ell) \qquad \mathscr{H}^{\underline{\boldsymbol{\nu}}} = \bigoplus_{n\ge 0} \mathbf{S}(n; q, Q_1, \dots, Q_\ell)$$

for the sums of corresponding Ariki-Koike and cyclotomic q-Schur algebras of all different ranks. To this data we introduce then a certain $\mathbb{Z}$-graded algebra

$A^{\underline{\boldsymbol{\nu}}}$ which we call the *cyclotomic quiver Schur algebra*. The name stems from the fact that the algebra is related to the cyclotomic quiver Hecke algebras R^ν and their tensor product analogues $T^{\underline{\boldsymbol{\nu}}}$ (defined in [Web10a]) like the finite Schur algebra is to the classical Hecke algebra. Our main result (Theorem 6.3) says that $A^{\underline{\boldsymbol{\nu}}}$ is a graded version of $\mathscr{H}^{\underline{\boldsymbol{\nu}}}$ given by an extension of the Brundan-Kleshchev isomorphism $\Phi^\nu : R^\nu \cong H^\nu$ from [BK09a] between the diagrammatic cyclotomic quiver Hecke algebras R^ν introduced in [KL09] and the sum of cyclotomic Hecke algebra H^ν.

Theorem A. *There is an isomorphism $\Phi^{\underline{\boldsymbol{\nu}}}$ from $A^{\underline{\boldsymbol{\nu}}}$ to the cyclotomic q-Schur algebra $\mathscr{H}^{\underline{\boldsymbol{\nu}}}$, extending the isomorphism $\Phi^\nu : R^\nu \cong H^\nu$. In particular, the algebra $\mathscr{H}^{\underline{\boldsymbol{\nu}}} = \bigoplus_{n\geq 0} \mathbf{S}(n; q, Q_1, \dots, Q_\ell)$ inherits a $\mathbb{Z}$-grading.*

Like the cyclotomic quiver Hecke algebra, the algebra $A^{\underline{\boldsymbol{\nu}}}$ can be realized as a natural quotient of a geometrically defined convolution algebra. This construction is based on the geometry of quivers using a certain category of flagged nilpotent representations (quiver partial flag varieties) of the cyclic affine type A quiver. It naturally extends the work of Varagnolo and Vasserot [VV09] and clarifies the origin of the grading.

The construction of $A^{\underline{\boldsymbol{\nu}}}$ (and its summand $A^{\underline{\boldsymbol{\nu}}}_n$ for fixed n) proceeds in three steps. We first define an infinite dimensional convolution algebra, which we call quiver Schur algebra, using flagged representations of the cyclic quiver Γ. This algebra only depend on e (and has summands depending on n). The second step is to add some extra shadow vertices to the quiver and define an involution algebra working with flagged representations of the extended quiver Γ depending on the parameters Q_i and ℓ. Finally the last step is to pass to a finite dimensional quotient which is the desired algebra $A^{\underline{\boldsymbol{\nu}}}$ (with the direct summand $A^{\underline{\boldsymbol{\nu}}}_n$ for fixed n).

To make explicit calculations we describe the quiver Schur algebra algebraically by considering a faithful representation on a direct sum of polynomial rings, extending the corresponding result for quiver Hecke algebras. Moreover, we give a diagrammatical description of the algebra by extending the diagram calculus of Khovanov and Lauda [KL09]. In contrast to their work, however, we are not able to give a complete list of relations diagrammatically. Still, we have enough information to construct (signed) permutation modules for the cyclotomic Hecke algebra and show that our algebra $A^{\underline{\boldsymbol{\nu}}}$ is isomorphic to the cyclotomic q-Schur algebra using the known three different description (geometric, algebraic and diagrammatical) of the quiver Hecke algebras.

Note that this can also be viewed as an extension of work of the second author [Web10a, §5.3], which showed that similar diagrammatic algebras were the endomorphism algebras of some, but not all, (signed) permutation modules.

Independently, Ariki [Ari09] introduced graded q-Schur algebras and studied in detail permutation modules. This is a special case of our results when $\ell = 1$ (and e and n are not too small) and indeed, we show that our grading coincides with Ariki's.

In the course of the proof we also establish, similar in spirit to the arguments in [BS11], the existence of a graded cellular basis (Theorem 5.7) in the sense of Mathas and Hu [HM10]:

Theorem B. *The cyclotomic quiver Schur algebra $A^{\underline{\nu}}$ is a graded cellular algebra. Moreover*

- *The cellular ideals coincide under $\Phi^{\underline{\nu}}$ with those of the Dipper-James-Mathas cellular ideals.*
- *The cell modules define graded lifts of the Weyl modules.*

After circulating a draft of this paper, we received a preprint of Hu and Mathas [HM] which gives an independent proof of Theorem B; while their construction doesn't follow exactly the same path as ours, a uniqueness argument analogous to Theorem 6.5 should show it is graded Morita equivalent to the grading we have defined. Their preprint also shows versions of Theorems C and D in the (special) case of a linear, rather than cyclic, quiver.

From both the geometric and the diagrammatic sides, our construction fits quite snugly inside Rouquier's program of categorical representation theory, [Rou08]. More precisely, the following holds

Theorem C. *The category of graded $A^{\underline{\nu}}$-modules carries a categorical action of $U_q(\widehat{\mathfrak{sl}}_e)$ in the sense of Rouquier. As a $U_q(\widehat{\mathfrak{sl}}_e)$-module, its complexified graded Grothendieck group is canonically isomorphic to the ℓ-fold tensor product*

$$\mathbb{F}_\ell = \mathbb{C}[q, q^{-1}, p_1, p_2, \ldots]^{\otimes \ell}$$

of level 1 Fock spaces with central charge $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell)$ given by $\underline{\boldsymbol{\nu}}$.

Here the parameter q corresponds to the effect of grading shift on the Grothendieck group, and this thus a formal variable, not a complex number.

Our constructions can also be described using an affine version of the "thick calculus" introduced by Khovanov, Lauda, Mackaay and Stošić for the upper half of $\widehat{\mathfrak{sl}}_e$. Unlike in [KLMS10] however, our category has objects which do not appear in the "thin calculus" and categorifies the upper half of $U_q(\widehat{\mathfrak{gl}}_e)$ instead of $U_q(\widehat{\mathfrak{sl}}_e)$.

Of course, it is tempting to think this could easily be extended to a graphical categorification of the Lie algebra $\widehat{\mathfrak{gl}}_e$, which is the direct sum of $\widehat{\mathfrak{sl}}_e$ and a Heisenberg Lie algebra $\mathbb{H}$ modulo an identification of their centers. This is especially intriguing and promising given the various interesting categorification of Heisenberg algebras ([CL10], [Kho10], [LS11]) which appeared recently in the literature. However, the connection cannot be as straightforward as one might hope at first, since the functors associated to the standard generators in the categorification of the upper half of $U_q(\widehat{\mathfrak{gl}}_e)$ simply do not have biadjoints (unlike those in $U_q(\widehat{\mathfrak{sl}}_e)$) as was pointed out already in [Sha08, 5.1,5.2]. In their action on the categories of $A^{\underline{\nu}}$-modules, they send projectives to projectives, but not injectives to injectives, so their right adjoints are exact, but not their left adjoints.

The Grothendieck group of graded $A^{\underline{\nu}}$-modules is naturally a $\mathbb{Z}[q, q^{-1}]$-module and comes also along with several distinguished lattices and bases. To describe them combinatorially we introduce a bar-involution on the tensor product $\mathbb{F}_\ell$ of Fock spaces appearing in Theorem C which allows us to define, apart from the standard basis, two other distinguished bases: the *canonical* and *dual canonical* bases. This canonical basis is a "limit" of that for higher level q-Fock spaces defined by Uglov, in a sense we describe later. Our canonical isomorphism induces

correspondences

bar involution	$\longleftrightarrow$	Serre-twisted duality
canonical basis	$\longleftrightarrow$	indecomposable projectives
dual canonical basis	$\longleftrightarrow$	simple modules
standard basis	$\longleftrightarrow$	Weyl modules

As a consequence we get (Theorem 7.10) information about (graded) decomposition numbers of cyclotomic q-Schur algebras.

Theorem D. *The graded decomposition numbers of the cyclotomic q-Schur algebra are the coefficients of the canonical basis in terms of the standard basis on the higher level q-Fock space $\mathbb{F}_\ell$.*

Again this problem was studied independently by Ariki [Ari09] for the level $\ell = 1$ case. The above theorem combines and generalizes therefore results of Ariki on Schur algebras and Brundan-Kleshchev-Wang [BKW11] on cyclotomic Hecke algebras describing graded decomposition numbers of cyclotomic Schur algebras. It is very similar in spirit to a conjecture of Yvonne [Yvo06, 2.13]; however, there are several small differences between Yvonne's conjecture and our results. The most important is that Yvonne used Jantzen filtrations to define a q-analogue of decomposition numbers instead of a grading. This approach using Jantzen filtrations has been worked out in level 1 by Ram-Tingley [RT10] and Shan [Sha08]. Their results show that the same q-analogue of decomposition numbers arise from counting multiplicities with respect to depth in Jantzen filtrations. We expect that this will hold in higher level as well; it should follow from the following

Conjecture E. *The cyclotomic quiver Schur algebra $A^{\underline{\nu}}$ is Koszul.*

Our approach has the advantage of working with gradings instead of filtrations which are easier to handle in practice. (A similar phenomenon appears for the classical category $\mathcal{O}$ for semi-simple complex Lie algebras, where the Jantzen filtration can also be described in terms of a grading, [BGS96], [Str03]. This grading is actually directly connected with the grading on the algebras $R^\nu \cong H^\nu$, see [BS11] for details).

We should emphasize that at the moment, this approach only allows us to understand the higher-level Fock spaces which are constructed as tensor products of level 1 Fock spaces (or their irreducible $\widehat{\mathfrak{gl}}_e$ constituents). This does not include the more elaborate higher level Fock spaces studied by Uglov [Ugl00]; these are categorified by category $\mathcal{O}$ of certain Cherednik algebras, see e.g. [Sha08]. Connections between these constructions have still to be clarified.

Let us briefly summarize the paper:

- Section 2 contains preliminaries of the geometry of quiver representations needed to define the quiver Schur algebra both as a geometric convolution algebra and in terms of an action on a polynomial ring which then is related to Demazure operators in Section 3. We connect its graded Grothendieck group with the generic nilpotent Hall algebra of the cyclic quiver.

- In Section 4, we discuss a generalization of this algebra using extended (or shadowed) quiver representations that will allow us to deal with higher level Fock spaces.
- In Section 5, we define cyclotomic quotients, equip them with a (graded) cellular structure and establish the isomorphism to cyclotomic q-Schur algebras in Section 6.
- In Section 7, we consider the connection of these constructions to higher representation theory, describe the categorical action of $\widehat{\mathfrak{sl}}_e$ on these categories, and show that they categorify q-Fock spaces. In particular, we consider the relationship between projective modules, canonical bases, and decomposition numbers.

Acknowledgment: We thank Geordie Williamson, Michela Varagnolo, Peter Tingley, Iain Gordon and Michael Ehrig for useful input and fruitful discussions and Bernard Leclerc and Peter Littelmann for helpful comments. We also would like to thank the Hausdorff Center for supporting B.W.'s visit to Bonn at the genesis of this work, and the organizers of the Oporto Meeting on Geometry, Topology and Physics 2010 for facilitating our collaboration.

B.W. was supported by an NSF Postdoctoral Research Fellowship and by the NSA under Grant H98230-10-1-0199.

2. The quiver Schur algebra

Throughout this paper, we will fix an integer $e > 1$. Let Γ be the Dynkin diagram for $\hat{\mathfrak{sl}}_e$ with the fixed clockwise orientation. Let $\mathbb{V} = \{1, 2, \dots, e\}$ be the set of vertices of Γ, identified with the set of remainders of integers modulo e. Let $h_{\overline{i}} : i \to i+1$ be the arrow from the vertex i to the vertex $i+1$ for $1 \leq i \leq e$ where here and in the following all formulas should be read taking indices corresponding to vertices of Γ modulo e.

Definition 2.1. *A* **representation** (V, f) *of* Γ *over a field* $\Bbbk$ *is*

- *a collection of finite dimensional* $\Bbbk$*-vector spaces* V_i*,* $1 \leq i \leq e$*, one attached to each vertex together with*
- $\Bbbk$*-linear maps* $f_i : V_i \to V_{i+1}$*.*

A **subrepresentation** *is a collection of vector subspaces* $W_i \subset V_i$ *such that* $f_i(W_i) \subset W_{i+1}$*. A representation* (V, f) *of* Γ *is called* **nilpotent** *if the map* $f_e \cdots f_2 f_1 : V_1 \to V_1$ *is nilpotent.*

2.1. **Quiver representations and quiver flag varieties.** The **dimension vector** of a representation (V, f) is the tuple $\mathbf{d} = (d_1, \dots, d_e)$, where $d_i = \dim V_i$. We let $|\mathbf{d}| = \sum d_i$. We denote by α_i the special dimension vector where $d_j = \delta_{ij}$. Mapping it to the corresponding simple root α_i of $\widehat{\mathfrak{sl}}_e$ identifies the set of dimension vectors with the positive cone in the root lattice of $\widehat{\mathfrak{sl}}_e$, and with semi-simple nilpotent representations of Γ:

Lemma 2.2. *There is a unique irreducible nilpotent representation* S_j *of dimension vector* α_j*. Any semi-simple nilpotent representation is of the form* (V, f) *with* $f_i = 0$ *for all* i*.*

Proof. Obviously S_j equals (V, f), where f_i and V_i are zero except of $V_j = \Bbbk$. Assume (V, f) is a non-trivial irreducible nilpotent representation. If $f_j \neq 0$ then

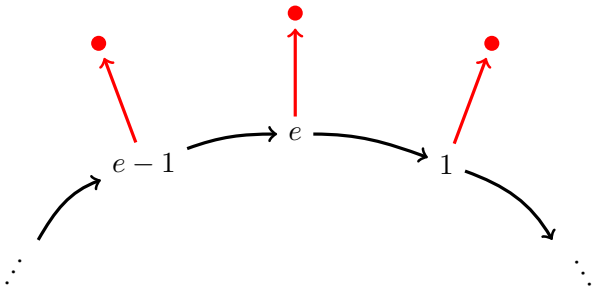


FIGURE 1. The oriented Dynkin quiver Γ and the extended Dynkin quiver $\tilde{\Gamma}$.

$\dim V_j = 1$, since otherwise $W_j = \ker f_j$, $W_i = \{0\}$ for $i \neq j$ defines a non-trivial proper subrepresentation. Not all f_i's are injective, since the representation is nilpotent and non-trivial. Say f_1 is not injective, then $\dim V_1 = 1$ and $f_1 = 0$. In particular S_1 is a subrepresentation and therefore (V, f) is isomorphic to S_1. Any representation (V, f) with $f_i = 0$ for all i is obviously semi-simple. Conversely, assume (V, f) is semi-simple, hence isomorphic to $\bigoplus_{i=1}^{e} S_i^{d_i}$. In particular, $S_i^{d_i}$ is a direct summand (for any i) which implies that $f_i = 0$. □

Let $\mathrm{Rep}_{\mathbf{d}}$ be the affine space of representations of Γ with dimension vector $\mathbf{d}$;

$$\mathrm{Rep}_{\mathbf{d}} = \bigoplus_{i \in \mathbb{V}} \mathrm{Hom}(V_i, V_{i+1}). \tag{2}$$

It has a natural algebraic action by conjugation of the algebraic group $G_{\mathbf{d}} = \mathrm{GL}(d_1) \times \cdots \times \mathrm{GL}(d_e)$, and we are interested in the **moduli space of representations**, i.e. the quotient $\mathrm{GRep}_{\mathbf{d}} = \mathrm{Rep}_{\mathbf{d}}/G_{\mathbf{d}}$ parametrizing isomorphism classes of representations. Since the $G_{\mathbf{d}}$-action is very far from being free, we must interpret this quotient in an intelligent way. One option is to consider it as an Artin stack. While this is perhaps the most elegant approach, it is more technical than necessary for our purposes. Instead the reader is encouraged to interpret this quotient as a formal symbol where, by convention for any complex algebraic G-variety X

- $H^*(X/G)$ is the G-equivariant cohomology $H^*_G(X)$, and $H^{BM}_*(X/G)$ is the G-equivariant Borel-Moore homology of X (for a discussion of equivariant Borel-Moore homology, see [Ful98, section 19], or [VV09, §1.2]).
- $D(X/G)$ (resp. $D^+(X/G)$) is the bounded (resp. bounded below) equivariant derived category of Bernstein-Lunts [BL94], with the usual six functor formalism described therein. See also [WW09] as an additional reference for our purposes.

Note that $H^*_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}) = H^*(BG_{\mathbf{d}})$, where $BG_{\mathbf{d}}$ denotes the classifying space of $G_{\mathbf{d}}$ (or the classifying space of its $\mathbb{C}$-points in the analytic topology for the topologically minded) since $\mathrm{Rep}_{\mathbf{d}}$ is contractible. Thus, we can use the usual Borel isomorphism to identify the $H^*(\mathrm{GRep}_{\mathbf{d}})$'s with polynomial rings.

We will consider the direct sum $H^*(\mathrm{GRep}) = \bigoplus_{\mathbf{d}} H^*(\mathrm{GRep}_{\mathbf{d}})$ which corresponds to taking the union of the quotients $\mathrm{GRep} = \bigsqcup_{\mathbf{d}} \mathrm{GRep}_{\mathbf{d}}$ as Artin stacks. One can think of this as a quotient by the groupoid $G = \bigsqcup_{\mathbf{d}} G_{\mathbf{d}}$, and we will

speak of the G-equivariant cohomology of $\mathrm{Rep} = \bigsqcup_{\mathbf{d}} \mathrm{Rep}_{\mathbf{d}}$, etc.

A **composition** of length r of $n \in \mathbb{Z}_{>0}$ is a tuple $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathbb{Z}_{>0}^r$ such that $\sum_{i=1}^r \mu_i = n$. In contrast, a **vector composition**[1] of type $m \in \mathbb{Z}_{>0}$ and length $r = r(\hat{\boldsymbol{\mu}})$ is a tuple $\hat{\boldsymbol{\mu}} = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \dots, \boldsymbol{\mu}^{(r)})$ of nonzero elements from $\mathbb{Z}_{\geq 0}^m$. If $m = e$ we call a vector composition also **residue data** and denote their set by Comp_e. Alternatively, $\hat{\boldsymbol{\mu}}$ can be viewed as an $r \times m$ matrix $\boldsymbol{\mu}[i,j]$ with the i-th row $\boldsymbol{\mu}^{(i)}$. We call the column sequence, i.e. the result of reading the columns, the **flag type sequence** $\mathrm{t}(\hat{\boldsymbol{\mu}})$, whereas the **residue sequence** $\mathrm{res}(\hat{\boldsymbol{\mu}})$ is the sequence

$$1^{\boldsymbol{\mu}_1^{(1)}}, 2^{\boldsymbol{\mu}_2^{(1)}}, \cdots, e^{\boldsymbol{\mu}_e^{(1)}} | 1^{\boldsymbol{\mu}_1^{(2)}}, 2^{\boldsymbol{\mu}_2^{(2)}}, \cdots, e^{\boldsymbol{\mu}_e^{(2)}} | \cdots | 1^{\boldsymbol{\mu}_1^{(r)}}, 2^{\boldsymbol{\mu}_2^{(r)}}, \cdots, e^{\boldsymbol{\mu}_e^{(r)}}. \tag{3}$$

The parts separated by the vertical lines | are called **blocks** of $\mathrm{res}(\hat{\boldsymbol{\mu}})$. We say that $\hat{\boldsymbol{\mu}}$ has **complete flag type** if every $\boldsymbol{\mu}^{(i)}$ is a unit vector which has exactly one non-zero entry, hence blocks of $\mathrm{res}(\hat{\boldsymbol{\mu}})$ contain at most one element.

The transposed vector composition of $\hat{\boldsymbol{\mu}}$ of type $r(\boldsymbol{\mu})$ and length m is defined to be $\check{\boldsymbol{\mu}} = (\check{\boldsymbol{\mu}}^{(1)}, \check{\boldsymbol{\mu}}^{(2)}, \dots, \check{\boldsymbol{\mu}}^{(m)})$ where $\check{\boldsymbol{\mu}}_j^{(i)} = \boldsymbol{\mu}[j,i] = \hat{\boldsymbol{\mu}}_i^{(j)}$, which we call in case $m = e$ also the **flag data**. Given a composition μ of n of length r, we denote by $\mathcal{F}(\mu)$ the **variety of flags of type** μ, that is the variety of flags

$$F_1(\mu) \subset F_2(\mu) \subset \cdots \subset F_r(\mu) = \mathbb{C}^n$$

where $F_i(\mu)$ is a subspace of dimension $\sum_{j=1}^i \mu_i$. For a vector composition $\hat{\boldsymbol{\mu}}$, we let $\mathcal{F}(\hat{\boldsymbol{\mu}}) = \prod_i \mathcal{F}(\check{\boldsymbol{\mu}}^{(i)})$, a product of partial flag varieties inside $\mathbb{C}^d = \prod_{i=1}^e \mathbb{C}^{d_i}$ of type given by the flag data. Here $\mathbf{d} = \mathbf{d}(\hat{\boldsymbol{\mu}}) = (d_1, \dots, d_e)$ denotes the **dimension vector** of the vector composition $\hat{\boldsymbol{\mu}}$ which is simply the sum $\mathbf{d} = \sum_{i=1}^r \hat{\boldsymbol{\mu}}^{(i)}$, and d denotes the total dimension. For a given dimension vector $\mathbf{d}$ denote

$$\mathrm{VComp}_e(\mathbf{d}) = \{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e \mid \mathbf{d}(\hat{\boldsymbol{\mu}}) = \mathbf{d}\}.$$

Example 2.3. Let $e = 2$ and $\hat{\boldsymbol{\mu}} = ((2,1),(1,1),(2,3),(0,1))$, a residue data of length $r = 4$ with dimension vector $\mathbf{d} = (5,6)$. The flag data is

$$\check{\boldsymbol{\mu}} = ((2,1,2,0),(1,1,3,1)) \text{ and } \mathrm{res}(\hat{\boldsymbol{\mu}}) = 1,1,2|1,2|1,1,2,2,2|2.$$

There are $\binom{11}{5}$ elements of complete flag type in $\mathrm{VComp}_e(\mathbf{d})$.

Definition 2.4. *For a given vector composition $\hat{\boldsymbol{\mu}} \in \mathrm{Comp}_e$ (of length r) a* **representation with compatible flags of type** $\check{\boldsymbol{\mu}}$ *is a nilpotent representation (V, f) of Γ with dimension vector $\mathbf{d} = \mathbf{d}(\hat{\boldsymbol{\mu}})$ together with a flag $F(i)$ of type $\check{\boldsymbol{\mu}}^{(i)}$ inside V_i for each $1 \leq i \leq e$ such that $f_i(F(i)_j) \subset F(i+1)_{j-1}$ for $1 \leq j \leq r$. We denote by*

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}) = \left\{(V, f, F) \in \mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\boldsymbol{\mu}}) \mid f_i(F(i)_j) \subset F(i+1)_{j-1}\right\}$$

the subset of $\mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\boldsymbol{\mu}})$ of representations with compatible flag. It comes equipped with the obvious action of the group $G := G_{\mathbf{d}}$ by change of basis.

[1] Note that this differs from the notion of an r-multicomposition where the tuples could be of different lengths or an r-multipartition where the tuples are partitions but again not necessarily of the same length m.

Alternatively (see Lemma 2.2), a representation with compatible flags is a tuple $((V,f),F)$ consisting of a representation $(V,f) \in \mathrm{Rep}_{\mathbf{d}}$ equipped with a filtration by subrepresentations with semi-simple successive quotients (with dimension vectors given by $\hat{\boldsymbol{\mu}}$). In Example 2.3,

$$V(4) \cong S_2, V(3)/V(4) \cong S_1^2 \oplus S_2^3, V(2)/V(3) \cong S_1 \oplus S_2, V(1)/V(2) \cong S_1^2 \oplus S_2^1.$$

Forgetting either the flag or the representation defines two $G_{\mathbf{d}}$-equivariant proper morphisms of algebraic varieties

$$p: \quad \mathcal{Q}(\hat{\boldsymbol{\mu}}) \longrightarrow \mathrm{Rep}_{\mathbf{d}}, \qquad ((V,f),F) \mapsto (V,f) \in \mathrm{Rep}_{\mathbf{d}} \tag{4}$$

$$\pi: \quad \mathcal{Q}(\hat{\boldsymbol{\mu}}) \longrightarrow \mathcal{F}(\hat{\boldsymbol{\mu}}), \qquad ((V,f),F) \mapsto F \in \mathcal{F}(\hat{\boldsymbol{\mu}}) \tag{5}$$

Generalizing the notion of a quiver Grassmannian we call the fibres of p the **quiver partial flag varieties**.

Consider the $\Bbbk$-algebra of polynomials in an alphabet for each node in the affine Dynkin diagram of size given by the dimension at that node:

$$R(\mathbf{d}) = \bigotimes_{j=1}^{e} \Bbbk[x_{j,1}, \ldots, x_{j,d_j}] = \Bbbk[x_{1,1}, \ldots, x_{1,d_1}, \ldots, x_{e,1}, \ldots, x_{e,d_e}] \tag{6}$$

This algebra carries an action of the Coxeter group $S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_e}$ by permuting the variables in the same tensor factor. The flag data defines a Young subgroup $S_{\check{\boldsymbol{\mu}}} \subseteq S_{\mathbf{d}}$. Note that $\mathcal{F}(\hat{\boldsymbol{\mu}}) \cong G_{\mathbf{d}}/P_{\check{\boldsymbol{\mu}}}$ is the partial flag variety defined by the parabolic subgroup $P_{\check{\boldsymbol{\mu}}}$ of $G_{\mathbf{d}}$, given by upper triangular block matrices with block sizes determined by the flag sequence.

Thus, the Borel presentation identifies the rings

$$H^*(\mathcal{F}(\hat{\boldsymbol{\mu}})/G) \cong R(\mathbf{d})^{S_{\hat{\mu}}} =: \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}), \tag{7}$$

by sending Chern classes of tautological bundles to elementary symmetric functions, [Bri98, Proposition 1].

In the extreme case where $\mathbf{d} = (r, 0, \ldots, 0)$ we have $\hat{\boldsymbol{\mu}}^{(i)} = (1, 0, \ldots, 0)$ for all i and $S_{\hat{\boldsymbol{\mu}}}$ is trivial, hence we obtain the GL_d-equivariant cohomology of the variety of full flags in $\mathbb{C}^d$. If $\hat{\boldsymbol{\mu}} = (\mathbf{d})$ then $\check{\boldsymbol{\mu}} = ((d_1), (d_2), \ldots, (d_e))$, the variety $\mathcal{F}(\hat{\boldsymbol{\mu}})$ is just a point and $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}) = R(\mathbf{d})^{S_{\mathbf{d}}}$. We call this the **ring of total invariants**.

Lemma 2.5. *The map π is a vector bundle with affine fibre; in particular, we have a natural isomorphism $\pi^* : H^*(\mathcal{F}(\hat{\boldsymbol{\mu}})/G) \cong H^*(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G)$, and thus $H^*(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G) \cong \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}})$.*

Grading convention. Throughout this paper, we will use a somewhat unusual grading convention on cohomology rings: we shift the (equivariant) cohomology ring or equivariant Borel-Moore homology of a smooth variety X downward by the complex dimension of X, so that the identity class is in degree $-\dim_{\mathbb{C}} X$. This choice of grading has the felicitous effect that pull-back and push-forward maps in cohomology are more symmetric. Usually, for a map $f\colon X \to Y$ we have that pull-back has degree 0, and push-forward has degree $2\dim_{\mathbb{C}} Y - 2\dim_{\mathbb{C}} X$, whereas in our grading convention

$$\textit{both push-forward and pull-back have degree } \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X. \tag{8}$$

Those readers comfortable with the theory of the constructible derived category will recognize this as replacing the usual constant sheaf with the intersection cohomology sheaf $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G} \in D^+(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G)$ of $\mathcal{Q}(\hat{\boldsymbol{\mu}})/G$. Since $\mathcal{Q}(\hat{\boldsymbol{\mu}})$ is smooth,

$\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ is simply a homological shift of the usual constant sheaf on each individual component (but by different amounts on each component). A more conceptual explanation for the convention is the invariance of $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ under Verdier duality. We should also note that this grading shift provides a straightforward explanation of the grading on the representation $\mathcal{P}o\ell$ of the quiver Hecke algebra in [KL09], [KL11].

Since p is proper and the constant sheaf of geometric origin, the Beilinson-Bernstein-Deligne decomposition theorem from [BBD82] applies (in the formulation [dCM09, Theorem 4.22]), and p sends the $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ on $\mathcal{Q}(\hat{\boldsymbol{\mu}})/G$ to a direct sum L of shifts of simple perverse sheaves on $\mathrm{Rep}_{\mathbf{d}}$. Our first object of study in this paper will be the algebra of extensions of these sheaves.

2.2. **The convolution algebra.** Let $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}} \in \mathrm{Comp}_e$ be vector compositions with associated dimension vector $\mathbf{d}$, and consider the corresponding "Steinberg variety"

$$Z(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}) \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}(\hat{\boldsymbol{\lambda}}). \tag{9}$$

Its quotient modulo the diagonal $G_{\mathbf{d}}$-action is denoted by $\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$. Note that $\mathcal{Q}(\hat{\boldsymbol{\mu}})$ is smooth and p is proper, so by [CG97, Theorem 8.6.7], we can identify the algebra of self-extensions of L (although not as a graded algebra) with the Borel-Moore homology of our Steinberg variety: we have the natural identification

$$\mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}) = H^{BM,G}_*(Z(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})),$$

such that the Yoneda product

$$\begin{aligned} \mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}) \otimes \mathrm{Ext}^*_{D^b(\mathrm{Rep}_{\mathbf{d}})}(p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}, p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\nu}})}) \\ \to \quad \mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\nu}})}) \end{aligned}$$

agrees with the convolution product. This defines an associative non-unital graded algebra structure $A_{\mathbf{d}}$ on

$$\bigoplus_{(\hat{\mu},\hat{\lambda})} \mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}\left(p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}\right) = \bigoplus_{(\hat{\mu},\hat{\lambda})} H^{BM}_*(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})). \tag{10}$$

Here the sums are over all elements in $\mathrm{VComp}_e(\mathbf{d}) \times \mathrm{VComp}_e(\mathbf{d})$. For reasons of diagrammatic algebra, we call this product **vertical composition** and we denote it like the usual multiplication $(f, g) \mapsto fg$. We call the algebra $A_{\mathbf{d}}$ the **quiver Schur algebra**.

2.3. **A faithful polynomial representation.** the quiver Schur algebra acts, by [CG97, Proposition 8.6.15], naturally on the sum (over all vector partitions of dimension vector $\mathbf{d}$) of cohomologies

$$V_{\mathbf{d}} := \bigoplus_{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})} H^{BM}_*(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G) \cong \bigoplus_{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})} \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}). \tag{11}$$

compatible with the grading convention (8).

Proposition 2.6. *The $A_{\mathbf{d}}$-module $V_{\mathbf{d}}$ is faithful.*

Proof. By the decomposition theorem, the sheaves $p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ decompose into summands which are shifts of the intersection cohomology sheaves associated to nilpotent orbits (these orbits are $G_{\mathbf{d}}$-equivariantly simply connected, and thus these are the only simple perverse sheaves supported on the nilpotent locus). Now Borel-Moore homology $H_*^{BM}(X)$ of a smooth space X equals hypercohomology $\mathbb{H}^{-*}(X, \mathbb{D})$ of its dualizing sheaf $\mathbb{D}$ which is, up to a shift, the constant sheaf. Its push-forward $p_*\mathbb{D}_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ via the proper map p is a direct sum of simple IC-sheaves. The action of the algebra (10) is then induced by applying the functor j_* to $p_*\mathbb{D}_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ where j is the map from $\mathrm{Rep}_{\mathbf{d}}$ to a point.

We note that the spaces we deal with have good parity vanishing properties. Each orbit has even equivariant cohomology, since it has a transitive action of G with the stabilizer of a point given by a connected algebraic group ([Lib07, Lemma 78, Theorem 79]). Also, the stalks of intersection cohomology sheaves have even cohomology [Hen07, Theorem 5.2(3)].

Thus by [BGS96, Theorem 3.4.2], j_* is faithful on semi-simple G_d-equivariant perverse sheaves, and so we have a faithful action on

$$j_*p_*\mathbb{D}_{\mathcal{Q}(\hat{\boldsymbol{\mu}})} = H_*^{BM}(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G). \qquad \square$$

2.4. **Monoidal structure and a categorified nilpotent Hall algebra.** The algebra $A_{\mathbf{d}}$ comes along with distinguished idempotents $e_{\hat{\boldsymbol{\mu}}}$ indexed by vector compositions with dimension vector $\mathbf{d}$. The $A_{\mathbf{d}}$-module

$$P(\hat{\boldsymbol{\mu}}) = \bigoplus_{\hat{\boldsymbol{\lambda}} \in \mathrm{VComp}_e(\mathbf{d})} e_{\hat{\boldsymbol{\lambda}}} A_{\mathbf{d}} e_{\hat{\boldsymbol{\mu}}} \tag{12}$$

is a finitely generated indecomposable graded projective $A_{\mathbf{d}}$-module. Each finitely generated indecomposable graded projective $A_{\mathbf{d}}$-module is (up to a grading shift) isomorphic to one of this form. We denote by $A_{\mathbf{d}}-\mathrm{pmod}$ the category of graded finitely generated projective $A_{\mathbf{d}}$-modules.

Proposition 2.7. *The category* $A_{\mathbf{d}}-\mathrm{pmod}$ *is equivalent to the additive category of sums of shifts of semi-simple perverse sheaves in* $D^+(\mathrm{GRep})$ *which are pure of weight* 0 *with nilpotent support in* $\mathrm{GRep}_{\mathbf{d}}$.

Proof. There is a functor sending a finitely generated projective $A_{\mathbf{d}}$-module M to the perverse sheaf $\left(\bigoplus_{\mathbf{d}(\hat{\boldsymbol{\mu}})=\mathbf{d}} p_*\Bbbk_{\mathcal{Q}_{\hat{\boldsymbol{\mu}}}}\right) \otimes_{A_{\mathbf{d}}} M$. This map is fully faithful, since it induces an isomorphism on the endomorphisms of $A_{\mathbf{d}}$ itself. Thus, we need only show that every simple perverse sheaf on GRep with nilpotent support is a summand of $p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ for some $\hat{\boldsymbol{\mu}}$.

Every such simple perverse sheaf is $\mathbf{IC}(\bar{X})$ where X is the locus of modules isomorphic to a fixed module N. Consider the socle filtration on N. The dimension vectors of the successive quotients define a vector composition $\hat{\boldsymbol{\mu}}_N$. The map $\mathcal{Q}(\hat{\boldsymbol{\mu}}_N) \to \mathrm{GRep}$ is generically an isomorphism over X, and thus for dimension reasons, has image $\bar{X}$. In particular,

$$p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_N)} \cong \mathbf{IC}(\bar{X}) \oplus \mathbf{L}$$

where $\mathbf{L}$ is a finite direct sum of shifts of semi-simple perverse sheaves supported on $\bar{X} \setminus X$. Thus, every simple perverse sheaf with nilpotent support is a summand of such a pushforward, and we are done. $\square$

Using correspondences we describe a monoidal structure on $A_{\mathbf{d}}-$pmod. For $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}} \in \mathrm{VComp}_e$ the **join** $\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\nu}}$ is the vector composition

$$\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\nu}} = (\mu^{(1)}, \dots, \mu^{(r(\hat{\boldsymbol{\mu}}))}, \nu^{(1)}, \dots, \nu^{r(\hat{\boldsymbol{\nu}})})$$

obtained by joining the two tuples. Let $\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_2$ be vector compositions with associated dimension vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. Let

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2) \subseteq \mathcal{Q}(\hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2) \times_{\mathrm{Rep}_{\mathbf{c}}} \mathcal{Q}(\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2)$$

be the space of representations with dimension vector $\mathbf{c}+\mathbf{d}$ which carry a pair of compatible flags of type $\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2$ and $\hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2$ respectively, such that the subspaces of dimension vector $\mathbf{c}$ in the two flags coincide; let $\mathcal{H}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2)$ be the quotient by the diagonal $G_{\mathbf{d}}$-action.

Definition 2.8. *The **horizontal multiplication** is the map*

$$\begin{aligned} A_{\mathbf{c}} \times A_{\mathbf{d}} &\longrightarrow A_{\mathbf{c}+\mathbf{d}} \\ (a,b) &\longmapsto a|b \end{aligned} \tag{13}$$

induced on equivariant Borel-Moore homology by the correspondence (i.e. by pull-and-push on the following diagram)

$$\mathcal{H}(\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\lambda}}_1) \times \mathcal{H}(\hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_2) \longleftarrow \mathcal{H}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2) \longrightarrow \mathcal{H}(\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2). \tag{14}$$

Here the rightward map is the obvious inclusion, and the leftward is induced from the map $V \mapsto (W, V/W)$ of taking the common subrepresentation W of dimension vector $\mathbf{c}$ and the quotient by it.

We let $\mathbf{A} = \oplus_{\mathbf{d}}(A_{\mathbf{d}}-\mathrm{pmod})$ be the direct sum of the categories $A_{\mathbf{d}}-$pmod over all dimension vectors; that is its objects are formal direct sums of finitely many objects from these categories, with morphism spaces given by direct sums.

Proposition 2.9. *The assignment $\otimes : (P(\hat{\mu}), P(\hat{\nu})) \mapsto P(\hat{\mu} \cup \hat{\nu})$ extends to a monoidal structure $(\mathbf{A}, \otimes, \mathbf{1})$ with unit element $\mathbf{1} = P(\emptyset)$.*

Proof. In the ungraded case this follows directly from Lusztig's convolution product [Lus91, §3], by Proposition 2.7. More explicitly, we define

$$M \otimes N = A_{\mathbf{c}+\mathbf{d}} \otimes_{A_{\mathbf{c}} \times A_{\mathbf{d}}} M \boxtimes N,$$

meaning one first takes the outer tensor product of the graded $A_{\mathbf{c}}$-module M and the graded $A_{\mathbf{d}}$-module N. The resulting $A_{\mathbf{c}} \times A_{\mathbf{d}}$-module is then induced to a graded $A_{\mathbf{c}+\mathbf{d}}$-module via the horizontal multiplication (13). This is obviously functorial in both entries and defines the required tensor product with the asserted properties. □

2.5. **Categorified generic nilpotent Hall algebra.** Let $K_q^0(\mathbf{A})$ be the Grothendieck group of the additive Krull-Schmidt category $\mathbf{A}$ i. e. the free abelian group on isomorphism classes $[M]$ of objects in $A-$pmod modulo the relation $[M_1]+[M_2] = [M_1 \oplus M_2]$. This is a free $\mathbb{Z}[q, q^{-1}]$-module where the action of q is by grading shift (and has nothing to do with the parameter q from the introduction). For a graded vector space $W = \oplus_j W^j$, we define $W(d)$ by $W(d)^j = W^{d+j}$, and let $q^\ell[M] = [M(\ell)]$. The module $K_q^0(\mathbf{A})$ is of infinite rank, but is naturally a direct sum of the Grothendieck groups $K_q^0(A_{\mathbf{d}}-\mathrm{pmod})$, each of which is finite rank.

Let $\mathrm{VCompf}_e(\mathbf{d}) \subset \mathrm{VComp}_e$ be the set of vector compositions of $\mathbf{d}$ of complete flag type. For each $\mathbf{d}$, there is a subalgebra,

$$R_{\mathbf{d}} = \bigoplus_{\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}} \in \mathrm{VCompf}_e(\mathbf{d})} e_{\hat{\boldsymbol{\lambda}}} A_{\mathbf{d}} e_{\hat{\boldsymbol{\mu}}}. \tag{15}$$

There is also a corresponding monoidal subcategory $\mathbf{R}$ of $\mathbf{A}$ generated by the indecomposable projectives indexed by the $\hat{\boldsymbol{\mu}} \in \mathrm{VCompf}_e(\mathbf{d})$ for all $\mathbf{d}$. Both $A_{\mathbf{d}}$ and $R_{\mathbf{d}}$ can be defined for any quiver and the following proposition holds in general, though in this paper we only use these categories for the affine type A quiver. The algebra $R_{\mathbf{d}}$ appears as **quiver Hecke algebra** (associated with $\mathbf{d}$) in the literature:

Proposition 2.10 (Vasserot-Varagnolo/Rouquier [VV09, 3.6])**.**
As graded algebra, $\mathbf{R}_{\mathbf{d}}$ *is isomorphic to the quiver Hecke algebra* $R(\mathbf{d})$ *associated to the quiver* Γ *in* [Rou08]. *In particular,* $K_q^0(\mathbf{R})$ *is naturally isomorphic to the Lusztig integral form of* $U_q^-(\widehat{\mathfrak{sl}}_e)$ *by mapping the isomorphism classes of indecomposable projective objects to Lusztig's canonical basis.*

The idempotents in $R_{\mathbf{d}}$ get identified with those in $R(\mathbf{d})$ by viewing the residue sequence (3) as a sequence of simple roots α_i.

Remark 2.11. For a Dynkin quiver, the categories $\mathbf{R}$ and $\mathbf{A}$ are canonically equivalent. In fact, both are equivalent to the full category of semi-simple perverse sheaves on GRep. However, in affine type A (the case of interest in this paper), they differ. In terms of perverse sheaves, the IC-sheaves which appear in $p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ for $\hat{\boldsymbol{\mu}}$ having complete flag type are those whose Fourier transform has nilpotent support as well; for example, the constant sheaf on the trivial representation with dimension vector $(1,\dots,1)$ cannot appear. Thus, in this case there are objects in $\mathbf{A}$ which don't lie in $\mathbf{R}$.

Recall that the **nilpotent Hall algebra** of the quiver Γ is an algebra structure on the set of complex valued functions on the space of nilpotent representations, typically considered over a finite field. The structure constants are polynomial in the cardinality q of the field. Hence it makes sense to consider q as a formal parameter and define the **generic Hall algebra** over the ring of Laurent polynomials $\mathbb{C}[q,q^{-1}]$. Following Vasserot and Varagnolo, [VV99], we denote this algebra U_e^-. By work of Schiffmann [Sch00, §2.2] it is isomorphic as an algebra to $U_q^-(\widehat{\mathfrak{sl}}_e) \otimes \mathbf{\Lambda}(\infty)$, where $\mathbf{\Lambda}(\infty)$ denotes the ring of symmetric polynomials. Identifying $\mathbf{\Lambda}(\infty)$ with $U_q^-(\mathbb{H})$, the lower half of a Heisenberg algebra, this algebra can also be described as $U_q^-(\widehat{\mathfrak{gl}}_e)$. This generic Hall algebra has a basis given by characteristic functions on the isomorphism classes of nilpotent representations of Γ and is naturally generated as algebra by the characteristic functions $\mathbf{f}_{\mathbf{d}}$ on the classes of semi-simple representations (which we label by their dimension vectors $\mathbf{d}$ following Lemma 2.2). The integral form $U_{e,\mathbb{Z}}^-$ over $\mathbb{Z}[q,q^{-1}]$ is given here by the lattice generated by all $\mathbf{f}_{\mathbf{d}}$'s, analogous to Lusztig's integral form for quantum groups, see [Sch00].

Proposition 2.12. *There is an isomorphism* $K_q^0(\mathbf{A}) \cong U_{e,\mathbb{Z}}^-$ *of* $\mathbb{Z}[q,q^{-1}]$*-algebras sending* $[(\mathbf{d})] \mapsto \mathbf{f}_{\mathbf{d}}$ *from the graded Grothendieck ring of* $\mathbf{A}$ *to the integral form of the generic nilpotent Hall algebra of the cyclic quiver.*

Proof. Fixing a prime p, there is a natural map from $K_q^0(\mathbf{A})$ to $U_e^-|_{q=p}$. This is given by applying the equivalence of Proposition 2.7, and then sending the class of a semi-simple perverse sheaf to the function given by the super-trace of Frobenius on its stalks. This is a function on the points of Rep over the field $\mathbb{F}_p$ and hence defines an element of the Hall algebra. By the definition of the Hall multiplication and the Grothendieck trace formula, this is an algebra map.

Since these super-traces are polynomial in p (they are the Poincaré polynomials of the quiver partial flag varieties), the coefficients of the expansion of this function in terms of the characteristic functions of orbits are also polynomial, and this assignment can be lifted to an algebra map $K_q^0(\mathbf{A}) \to U_{e,\mathbb{Z}}^-$. This map is obviously surjective, since the function for each intersection cohomology sheaf on a nilpotent orbit, and thus the characteristic function on the orbit, is in its image. It is also injective, since when we expand any non-zero class in the Grothendieck group in terms of the classes of intersection cohomology sheaves, we must have a nonzero value of the corresponding function on the support of an intersection cohomology sheaf maximal (in the closure ordering) amongst those with non-zero coefficient.

Since $[(\mathbf{d})]$ corresponds to the skyscraper sheaf of the semi-simple representation of dimension $\mathbf{d}$, it is sent to the characteristic function of that point. □

The monoidal structure on $\mathbf{A}$ is compatible with the usual monoidal structure $(\mathsf{Vect}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ on the category of vector spaces in the following way:

Lemma 2.13. *Let $\Phi_{\mathbf{d}} : A_{\mathbf{d}} \to \mathrm{End}(V_{\mathbf{d}})$ be the representation from* (11)*. Then $\Phi_{\mathbf{c}+\mathbf{d}}((a|b))(v) = \Phi_{\mathbf{c}}(a)(v_1) \otimes \Phi_{\mathbf{c}}(b)(v_2)$ where v is the image of $v_1 \otimes v_2$ under the canonical map $V_{\hat{\boldsymbol{\mu}}} \otimes V_{\hat{\boldsymbol{\lambda}}} \to V_{\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\lambda}}}$.*

That is, the functor $\mathbf{V} \colon \mathbf{A} \to \mathsf{Vect}_{\Bbbk}$ given by $\hat{\boldsymbol{\mu}} \mapsto V_{\hat{\boldsymbol{\mu}}}$ is monoidal.

Proof. This follows directly from the definitions, [CG97]. □

Thus, we can describe elements corresponding to vector compositions with a large number of parts by looking at (the action) of the ones with a very small number of parts.

3. Splitting and merging, Demazure operators and diagrams

In this section we describe a basis of the algebras $A_{\mathbf{d}}$ and elementary morphisms, called splits and merges. We give a geometric, algebraic and diagrammatical description of these maps.

Let $\hat{\boldsymbol{\lambda}}', \hat{\boldsymbol{\lambda}} \in \mathrm{Comp}_e$ be residue data.

Definition 3.1. *We say that $\hat{\boldsymbol{\lambda}}'$ is a* **merge** *of $\hat{\boldsymbol{\lambda}}$ (and $\hat{\boldsymbol{\lambda}}$ a* **split** *of $\hat{\boldsymbol{\lambda}}'$) at the index k if $\hat{\boldsymbol{\lambda}}' = (\boldsymbol{\lambda}^{(1)}, \cdots, \hat{\boldsymbol{\lambda}}^{(k)} + \hat{\boldsymbol{\lambda}}^{(k+1)}, \ldots, \hat{\boldsymbol{\lambda}}^{(r)})$.*

If $\hat{\boldsymbol{\lambda}}'$ is a merge of $\hat{\boldsymbol{\lambda}}$, then there is an associated correspondence

$$\begin{array}{ccccc} & & \mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) = \left\{ (V, f, F) \in \mathcal{Q}(\hat{\boldsymbol{\lambda}}) \mid f_i(F(i)_{k+1}) \subset F(i+1)_{k-1} \right\} & & \\ & \swarrow & & \searrow & \\ \mathcal{Q}(\hat{\boldsymbol{\lambda}}) & & & & \mathcal{Q}(\hat{\boldsymbol{\lambda}}') \end{array} \tag{16}$$

where the right map is just the obvious inclusion and the left map is forgetting $F_k(i)$ for all vertices $1 \leq i \leq e$ (and reindexing all subspaces in the flags with higher indices). Obviously the same variety defines also a correspondence in the opposite direction (reading from right to left) which we associate to the **split**. We are interested in the equivariant version:

Definition 3.2. *For $\hat{\boldsymbol{\lambda}}'$ a merge (resp. split) of $\hat{\boldsymbol{\lambda}}$ at k, we let $\hat{\boldsymbol{\lambda}} \stackrel{k}{\to} \hat{\boldsymbol{\lambda}}'$ or just $\hat{\boldsymbol{\lambda}} \to \hat{\boldsymbol{\lambda}}'$ denote the element of A given by multiplication with the equivariant fundamental class $[\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k)]$ (resp. $[\mathcal{Q}(\hat{\boldsymbol{\lambda}}', k)]$) pushed forward to $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\lambda}}', \hat{\boldsymbol{\lambda}}))$.*

Proposition 3.3. *Let $\hat{\boldsymbol{\lambda}}'$ be a merge or split of $\hat{\boldsymbol{\lambda}}$ at index k. Then $\hat{\boldsymbol{\lambda}} \to \hat{\boldsymbol{\lambda}}'$ is homogeneous of degree $\sum_{i=1}^{e} \boldsymbol{\lambda}_i^{(k)}(\boldsymbol{\lambda}_{i-1}^{(k+1)} - \boldsymbol{\lambda}_i^{(k+1)})$.*

Proof. In the most obvious choice of grading conventions, pull-back by a map is degree 0, and pushforward has degree given by minus the relative (real) dimension of the map (i.e. the dimension of the target minus the dimension of the domain). This normalization has the disadvantage of breaking the symmetry between splits and merges. It is, for example, carefully avoided in [KL09]. Instead, we use, (8), the perverse normalization of the constant sheaves which "averages" the degrees of pull-back and pushforward. Then the degree of convolving with the fundamental class of a correspondence is minus the sum of the relative (complex) dimensions of the two projection maps (note that for a correspondence over two copies of the same space, this agrees with the most obvious normalization).

Thus, it is clear that we will arrive at the same answer in the split and merge cases. If $\hat{\boldsymbol{\lambda}}'$ is a merge of $\hat{\boldsymbol{\lambda}}$, then

- the map $\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) \to \mathcal{Q}(\hat{\boldsymbol{\lambda}}')$ is a smooth surjection with fiber given by the product of Grassmannians of $\boldsymbol{\lambda}_i^{(k)}$-dimensional planes in $\boldsymbol{\lambda}_i^{(k)} + \boldsymbol{\lambda}_i^{(k+1)}$-dimensional space, which has dimension $\sum \boldsymbol{\lambda}_i^{(k)} \boldsymbol{\lambda}_i^{(k+1)}$ and
- the map $\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) \to \mathcal{Q}(\hat{\boldsymbol{\lambda}})$ is a closed inclusion of codimension

$$\sum_{i=1}^{e} \dim \mathrm{Hom}(F(i-1)_{k+1}/F(i-1)_k, F(i)_k/F(i)_{k-1}) = \sum \boldsymbol{\lambda}_i^{(k)} \boldsymbol{\lambda}_{i-1}^{(k+1)}.$$

The result follows. □

3.1. **Explicit formulas for merges and splits.** We give now explicit formulas for elementary merges and splits. Consider the particular choices for the vector compositions: $\mathcal{H}((\mathbf{c}, \mathbf{d}), (\mathbf{c}+\mathbf{d})) \cong \mathcal{F}(\mathbf{c}, \mathbf{d})/G_{\mathbf{c}+\mathbf{d}}$. The variety $\mathcal{Q}(\mathbf{c}+\mathbf{d})$ is just a point, but equipped with the action of $G = G_{\mathbf{c}+\mathbf{d}}$. We want to describe how the fundamental classes of $\mathcal{H}((\mathbf{c}, \mathbf{d}), (\mathbf{c}+\mathbf{d}))$ or $\mathcal{H}((\mathbf{c}+\mathbf{d}), (\mathbf{c}, \mathbf{d}))$ (which are isomorphic as varieties, but different as correspondences) act on V via Proposition 2.6 and determine in this way the merge and split map. They are given by pullback followed by pushforward in equivariant cohomology given by the diagram

$$H_*^{BM}(\mathcal{Q}(\mathbf{c}, \mathbf{d})/G) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} H_*^{BM}(\mathcal{F}(\mathbf{c}, \mathbf{d})/G) \underset{q^*}{\overset{q_*}{\rightleftarrows}} H_*^{BM}(\mathcal{Q}(\mathbf{c}+\mathbf{d})/G),$$

where $\iota : \mathcal{F}(\mathbf{c}, \mathbf{d}) \to \mathcal{Q}(\mathbf{c}, \mathbf{d})$ is the zero section of the G-equivariant fibre bundle $\pi : \mathcal{Q}(\mathbf{c}, \mathbf{d}) \longrightarrow \mathcal{F}(\mathbf{c}, \mathbf{d})$ and $q : \mathcal{F}(\mathbf{c}, \mathbf{d}) \to \mathcal{Q}(\mathbf{c}+\mathbf{d})$ is the proper G-equivariant map given by forgetting the subspaces of dimension c_i.

Proposition 3.4. *Let* $\mathbf{c}$, $\mathbf{d} \in \mathbb{Z}^e_{\geq 0}$ *non-zero. The following diagram*

$$\begin{array}{ccc} H^*(\mathcal{Q}(\mathbf{c},\mathbf{d})/G) & \underset{\iota_*\circ q^*}{\overset{q_*\circ\iota^*}{\rightleftarrows}} & H^*(\mathcal{Q}(\mathbf{c}+\mathbf{d})/G) \\ \Big\downarrow \text{Borel} & & \Big\downarrow \text{Borel} \\ \mathbf{\Lambda}(\mathbf{c},\mathbf{d}) & \underset{\text{E}}{\overset{\text{int}}{\rightleftarrows}} & \mathbf{\Lambda}(\mathbf{c}+\mathbf{d}) \end{array} \tag{17}$$

commutes where E *is the inclusion map from the total invariants* $\mathbf{\Lambda}(\mathbf{c}+\mathbf{d})$ *into the invariants* $\mathbf{\Lambda}(\mathbf{c},\mathbf{d})$ *followed by multiplication with the Euler class*

$$E := \prod_{i=1}^{e} \prod_{j=1}^{c_{i+1}} \prod_{k=c_i+1}^{d_i+c_i} (x_{i+1,j} - x_{i,k}) \tag{18}$$

and int *is the integration map which sends an element* f *to the total invariant*

$$\sum_{w\in S_{\mathbf{c}+\mathbf{d}}} (-1)^{l(w)} w(f) \prod_{i=1}^{e} \frac{1}{c_i! d_i!} \frac{w\Big(\prod_{1\leq j<k\leq c_i} (x_{i,j}-x_{i,k}) \prod_{c_i<\ell<m\leq c_i+d_i} (x_{i,\ell}-x_{i,m})\Big)}{\prod_{1\leq j<k\leq c_i+d_i} (x_{i,j}-x_{i,k})}$$

where ℓ *denotes the usual length function on the symmetric group.*

Remark 3.5. The degrees of the maps $q_* \circ \iota^*$ and $\iota_* \circ q^*$ are again not the degrees which one would naively guess, but rather given by the convention (8), in particular they are of the same degree. By convention, we set $E = 1$ if either one of the products in (18) is empty or one of the variables $x_{i-1,k}$ or $x_{i,j}$ does not exist.

Proof. Since π^* is an isomorphism and ι the inclusion of the zero section, ι^* is also an isomorphism. On the other hand q is the map to a point, hence q^* is just the inclusion of the total invariants. By the usual adjunction formula, $\iota^*\iota_*(a) = \mathbf{e} \cup a$ where $\mathbf{e}$ is the Euler class of the vector bundle π. To see that the map E is as asserted it is enough to verify the formula $E = \mathbf{e}$ for the Euler class. The map i is the inclusion of the zero section of the vector bundle

$$\bigoplus_{i\in\mathbb{V}} \mathrm{Hom}(\mathcal{V}_{i,d_i}, \mathcal{V}_{i+1,c_{i+1}}),$$

where $\mathcal{V} = \oplus_{i=1}^{e} \mathcal{V}_{i,d_i}$ is the tautological vector bundle on the moduli space of quiver representation with dimension vector $\mathbf{d}$. As equivariant vector bundles over the maximal torus of $G_{\mathbf{d}}$, we have a splitting into line bundles

$$\mathcal{V}_{i+1,\mathbf{c}} \cong \bigoplus_{j=1}^{c_{i+1}} \mathcal{L}_{i+1,j} \qquad \mathcal{V}_{i,\mathbf{d}} \cong \bigoplus_{k=c_i+1}^{c_i+d_i} \mathcal{L}_{i,k}$$

where $\mathcal{L}_{i,k}$ is the tautological line bundle for the corresponding weight space. Thus,

$$\mathrm{Hom}(\mathcal{V}_{i,\mathbf{d}}, \mathcal{V}_{i+1,\mathbf{c}}) \cong \bigoplus_{j=1}^{c_{i+1}} \bigoplus_{k=c_i+1}^{c_i+d_i} \mathcal{L}_{i+1,j} \otimes \mathcal{L}_{i,k}^*$$

and the formula (18) for the Euler class follows.

On the other hand, q is the projection from the partial flag variety $G_{\mathbf{a}}/P_{\mathbf{c},\mathbf{d}}$ to a point, where $\mathbf{a}=\mathbf{c}+\mathbf{d}$. The formula for equivariant integration on the full flag variety is given by

$$\int_{G_{\mathbf{a}}/B} f = \frac{\sum_{S_{\mathbf{a}}} (-1)^{l(w)} w\cdot f}{\prod_{i=1}^{e}\prod_{j<k\le a_i}(x_{i,j}-x_{i,k})} \tag{19}$$

Thus, we have that

$$\int_{\mathcal{Q}(\mathbf{c},\mathbf{d})} f = \int_{\mathcal{F}(\mathbf{c},\mathbf{d})}\int_{P_{\mathbf{c},\mathbf{d}}/B_{\mathbf{c}+\mathbf{d}}} \frac{1}{\prod_{i=1}^{e} c_i!d_i!} fD = \int_{G_{\mathbf{c}+\mathbf{d}}/B_{\mathbf{c}+\mathbf{d}}} \frac{1}{\prod_{i=1}^{e} c_i!d_i!} fD,$$

where $D=\prod_{j<k\le c_i}(x_{i,j}-x_{i,k})\prod_{c_i<\ell<m\le c_i+d_i}(x_{i,\ell}-x_{i,m})$. The integration formula follows then from (19). □

3.2. **Demazure operators.** The splitting and merging maps can be described algebraically via Demazure operators acting on polynomial rings. The i-th **Demazure operator** or **difference operator** $\Delta_i=\Delta_{s_i}$ acts on $\Bbbk[x_1,\dots x_n]$ by sending f to $\frac{f-s_i(f)}{x_i-x_{i+1}}$, where $s_i=(i,i+1)$ denotes the simple transposition acting by permuting the ith and $(i+1)$th variable. If $w=s_{i_1}s_{i_2}\dots s_{i_l}$ is a reduced expression of $w\in S_n$ we define $\Delta_w=\Delta_{i_1}\Delta_{i_2}\cdots s_{i_l}$. This is independent of the reduced expression, [Dem73]. If G is a product of symmetric groups we denote by $w_0\in G$ the longest element. Demazure operators satisfy the twisted derivation rule

$$\Delta_i(fg)=\Delta(f)g+s_i(f)\Delta_i(g)$$

and more generally for a reduced expression $w=s_{i_1}s_{i_2}\dots s_{i_l}$ the formula

$$\Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_l}(fg) \;=\; \sum A_{i_1}A_{i_2}\cdots A_{i_l}(f)B_{i_1}B_{i_2}\cdots B_{i_l}(g) \tag{20}$$

where the sum runs over all possible choices of either $A_j=\Delta_j$ and $B_j=\mathrm{id}$ or $A_j=s_j$ and $B_j=\Delta_j$, for each $1\le r$.

Proposition 3.6. *Let $(\mathbf{c},\mathbf{d})$ be a vector composition.*

(1) Assume $\mathbf{c}+\mathbf{d}=(c_i+d_i)\alpha_i$ for some i, then

$$\mathrm{int}(f)=\Delta_{w_0^{c_i,d_i}}$$

where $w_0^{c_i,d_i}\in S_{c_i+d_i}$ denotes the shortest coset representative of w_0 in $S_{c_i+d_i}/(S_{c_i}\times S_{d_i})$. In general, $\mathrm{int}(f)$ is a product of pairwise commuting Demazure operators $w_0^{c_i,d_i}$, one for each i.

(2) Merging successively from a vector composition $(\alpha_i,\alpha_i,\dots,\alpha_i)$ of length r to $r\alpha_i$ equals the Demazure operator for $w_0\in S_r$,

$$\Delta_{w_0}(f)=\sum_{w\in S_r}(-1)^{l(w)}w(f)\frac{1}{\prod_{1\le i<j\le r}(x_i-x_j)}. \tag{21}$$

(3) The split $\mathbf{c}+\mathbf{d}$ into $\mathbf{c}$ and $\mathbf{d}$ where either $c_{i+1}d_i=0$ for all i or $\mathbf{c}+\mathbf{d}=(c_i+d_i)\alpha_i$ for some i, is just the inclusion from $\mathbf{\Lambda}(\mathbf{c}+\mathbf{d})$ to $\mathbf{\Lambda}(\mathbf{c},\mathbf{d})$.

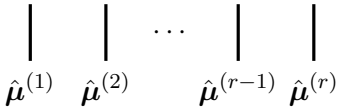
Proof. Part (3) is obvious, since $E = 1$ by Remark 3.5. The second statement is clear for $r = 1$ and $r = 2$. The successive merge of the first $r-1$ α_i's is given by induction hypothesis, hence we only have to merge with the last α_i and obtain

$$\begin{aligned} f &\mapsto \sum_{y\in S_{r-1}\times S_1} (-1)^{l(y)} \frac{y(f)}{(r-1)!} \frac{1}{\prod_{1\le i<j\le r-1}(x_i - x_j)} =: P \\ &\mapsto \sum_{z\in S_r/S_{r-1}\times S_1} (-1)^{l(z)} z(P) = \sum_{z,y} (-1)^{l(z)+l(y)} zy(f) \frac{1}{\prod_{1\le i<j\le r}(x_i - x_j)}. \end{aligned}$$

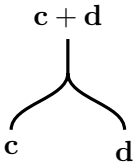
Hence (2) follows from the general formula for Δ_{w_0}, [Ful97, 10.12]. Associativity of the merges and formula (2) gives $\mathrm{int}(f)\Delta_{w_0(c_i,d_i)} = \Delta_{w_0(c_i+d_i)} = \Delta_{w_0(c_i,d_i)}\Delta_{w_0(c_i,d_i)}$, where $w_0(c_i, d_i)$ and $w_0(c_i+d_i)$ are the longest elements in $S_{c_i} \times S_{d_i}$ and $S_{c_i+d_i}$ respectively. Since $\Delta_{w_0(c_i,d_i)}$ surjects to the $S_{c_i} \times S_{d_i}$-invariants, $\mathrm{int}(f) = \Delta_{w_0(c_i,d_i)}$ and so (1) follows. □

3.3. **The pictorial interpretation.** As shown in [VV09], the quiver Hecke algebra $R(\mathbf{d})$ is isomorphic to the diagram algebra introduced by Khovanov and Lauda in [KL09]. This result allows to turn rather involved computations in the convolution algebra into a beautiful diagram calculus. Motivated by these ideas, we present now a graphical calculus for the algebra A where the split and merge maps from Proposition 3.4 are displayed as trivalent graphs. We will always read our diagrams from bottom to top. We represent

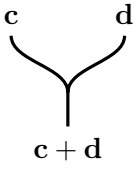
- the usual (vertical) algebra multiplication as vertical stacking of diagrams,
- horizontal multiplication as horizontal stacking of diagrams,
- the idempotent $e_{\hat{\boldsymbol{\mu}}}$ as a series of lines labeled with the parts of the vector composition or equivalently the blocks of the residue sequence,



- the morphism $(\mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{c} + \mathbf{d})$ as a joining of two strands $\mathbf{c}, \mathbf{d}$,



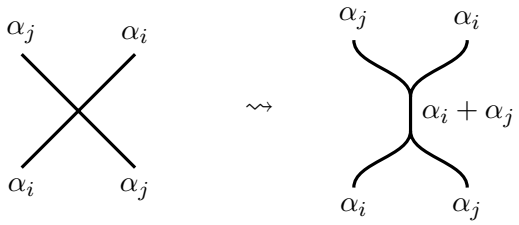
- the morphism $(\mathbf{c} + \mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d})$ as its mirror image,



- multiplication by a polynomial is displayed by putting a box containing the polynomial.

A typical element of A is obtained by horizontally and vertically composing these morphisms. The composition of a merge followed by a split of the form $(\mathbf{c},\mathbf{d}) \to (\mathbf{c}+\mathbf{d}) \to (\mathbf{d},\mathbf{c})$ is also abbreviated as a **crossing** and denoted $(\mathbf{c},\mathbf{d}) \to (\mathbf{d},\mathbf{c})$.

Remark 3.7. Our calculus is an extension of the graphical calculus of [KL09]: given a crossing as in the Khovanov-Lauda picture, we interpret it as merge-split of the kth and $(k+1)$th strands:



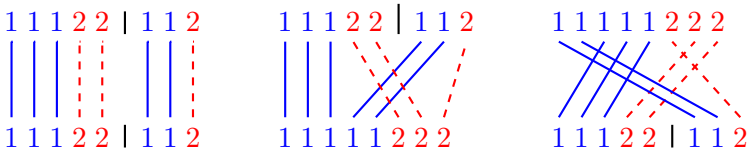
Assuming it is the ath α_i and bth α_j in the residue sequence.

- If $j \neq i, i+1$ then our map just flips the tensor factors $\Bbbk[x_{i,a}] \otimes \Bbbk[x_{j,b}] \mapsto \Bbbk[x_{j,b}] \otimes \Bbbk[x_{i,a}]$.
- If $i = j$ then we associate the Demazure operator Δ_k as in [KL09].
- For $j = i+1$, we multiply by $x_{i+1,a} - x_{i,b}$ followed by flipping the tensor factors.

In each case, this agrees with the action $\mathcal{P}o\ell$ defined in [KL11], though Khovanov and Lauda have a single alphabet of variables, which they index by their left-position, as opposed to having separate alphabets for each node of the Dynkin diagram. We believe that when one incorporates non-unit dimension vectors, the latter convention is more convenient. Lemma 3.3 implies that crossings have degree 1, -2, or 0 according to the cases $j = i \pm 1$, $i = j$ or $j \neq i, i \pm 1$ respectively. Given a single strand labeled with the a-th α_j we denote, following [KL09] multiplication with $x_{j,a}^R$ also by decorating the strand with R dots.

To keep track of the permutation w appearing in the Demazure operators Δ_w, it will be useful to work also with residue sequences, interpreting the numbers occurring in a residue sequence as colors from the chart $\{1,\dots,e\}$. If the sequence is of length d, then permutations in S_d permuting only inside the colors can be viewed as elements of $S_{\mathbf{d}}$, where d_i is the number of i's appearing. We want to associate to each idempotent, elementary split or merge such a permutation as follows:

(22)



To an idempotent corresponding to a residue sequence of length d we attach the identity element in S_d which corresponds to the identity element in $S_{\mathbf{d}}$. To a split $(\mathbf{c}+\mathbf{d}) \longrightarrow (\mathbf{c},\mathbf{d})$ we associate also the identity element in $S_{\mathbf{c}+\mathbf{d}}$, which when viewed as a permutation in S_{c+d} is sending the a to b if on place a of the residue sequence of $(\mathbf{c}+\mathbf{d})$ we find the k-th j, whereas the k-th j in the residue

sequence for $(\mathbf{c},\mathbf{d})$ is on place b. To an elementary merge $M:(\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{c}+\mathbf{d})$ we associate the product $w=w(M)=\prod_{i=1}^{e}w_0^{c_i,d_i}$. In the example (22) we have the permutations (written in cycle decomposition) $(1,3,5,2,4)(6,8,7)\in S_{\mathbf{d}}=S_5\times S_3\subset S_8$.

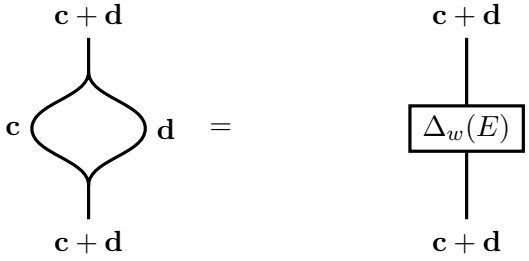
Note that $\Delta_{\omega(M)}$ for a merge M is precisely the Demazure operator from Lemma 3.6. By (18), a split corresponds to an inclusion followed by multiplication with some polynomial. For a composition $X=M_1M_2\cdots M_t$ of splits and merges let $\omega(X)=\omega(M_1)\omega(M_2)\cdots\omega(M_t)$, where ω of a split is, compatible with the definition above, just the identity permutation. For a finite linear combination X' of such X's we let $\omega(X')$ be the the sum of all $\omega(X)$'s of maximal length. The following relations are basic relations in the algebra A which we call **braid relations**:

Proposition 3.8. *Let $\mathbf{b},\mathbf{c},\mathbf{d}$ be vector compositions.*

(1) Let E be the Euler class attached to the splitting $(\mathbf{c}+\mathbf{d})\longrightarrow(\mathbf{c},\mathbf{d})$ and $\Delta_w=\Delta_{w_0^{c_1,d_1}}\cdots\Delta_{w_0^{c_e,d_e}}$. Then

$$(\mathbf{c}+\mathbf{d})\longrightarrow(\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{c}+\mathbf{d}) \;=\; \Delta_w(E)(\mathbf{c}+\mathbf{d})\xrightarrow{\mathrm{id}}(\mathbf{c}+\mathbf{d})$$

or equivalently in terms of diagrams:

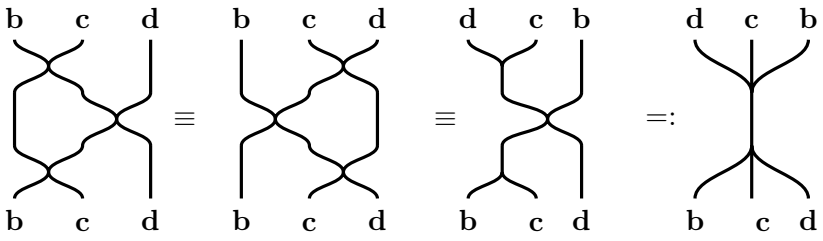


In particular, if $X:=(\mathbf{c},\mathbf{d})\to(\mathbf{d},\mathbf{c})$ is a crossing then $X^2=fX$ for some (possibly zero) polynomial f.

(2) Let

$$\begin{aligned}X_1 &:= (\mathbf{b},\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{c},\mathbf{b},\mathbf{d})\longrightarrow(\mathbf{c},\mathbf{d},\mathbf{b})\longrightarrow(\mathbf{d},\mathbf{c},\mathbf{b})\\ X_2 &:= (\mathbf{b},\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{b},\mathbf{d},\mathbf{c})\longrightarrow(\mathbf{d},\mathbf{b},\mathbf{c})\longrightarrow(\mathbf{d},\mathbf{c},\mathbf{b})\\ X_3 &:= (\mathbf{b},\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{b}+\mathbf{c}+\mathbf{d})\longrightarrow(\mathbf{d},\mathbf{c},\mathbf{b})\end{aligned}$$

Then $X_1=X_2+R$, and $X_1=X_3+R'$ where $\omega(R),\omega(R')<\omega(X_1)=\omega(X_2)$. Hence up to lower order terms $X_1\equiv X_2\equiv X_3$, in diagrams



Proof. Let $x=w_0^{c_1,d_1}\cdot w_0^{c_e,d_e}$. Recall that $\Delta_i(gf)=\Delta_i(g)f+s_i(g)\Delta_i(f)$ for any f,g and $\Delta_i(f)=0$ if f is s_i-invariant. Hence $\Delta_x(Ef)=\Delta_x(E)f$ for any total invariant polynomial f and the first claim follows. Let now $x_1=w(X_1)$ and

$x_2 = w(X_2)$. Then obviously $x_1 = x_2$ and $X_1 = \alpha\Delta_{x_1} + R_1$ and $X_2 = \beta\Delta_{x_2} + R_2$ for some polynomials α, β and R_1, R_2 with $\omega(R_1), \omega(R_2) < \omega(X_1) = \omega(X_2)$. We still have to verify that $\alpha = \beta$ up to terms with ω of smaller length and the definition of (18). Let X_1 be the composition $E_3 \circ M_3 \circ E_2 \circ M_2 \circ E_1 \circ M_1$ where the M_i denotes the i-th merge and E_i denotes the multiplication with an Euler class $E(i)$. We verify the claim by invoking the definitions and the equality $\Delta_i(gf) = s_i(g)\Delta_i(f) + \Delta(g)f$. First note that up to lower order terms

$$M_2 \circ E_1 \quad = \quad w \left(\prod_{i=1}^{e} \prod_{\substack{1 \le r \le c_{i+1} \\ c_i+1 \le s \le b_i+c_i}} (x_{i+1,r} - x_{i,s}) \right) M_2$$

where w is the permutation associated with M_2 via (22), hence

$$M_2 \circ E_1 \quad = \quad \prod_{\substack{1 \le r \le c_{i+1} \\ c_i+d_i+1 \le s \le b_i+c_i+d_i}} (x_{i+1,r} - x_{i,s}) M_2.$$

Repeating these type of calculations we obtain

$$\begin{aligned} X_1 \quad = \quad & \prod_{i=1}^{e} \left(\prod_{\substack{1 \le r \le d_{i+1} \\ d_i+1 \le s \le c_i+d_i}} (x_{i+1,r} - x_{i,s}) \prod_{\substack{1 \le r \le d_{i+1} \\ c_i+d_i+1 \le s \le b_i+c_i+d_i}} (x_{i+1,r} - x_{i,s}) \right. \\ & \left. \times \prod_{\substack{d_{i+1}+1 \le r \le c_{i+1}+d_{i+1} \\ c_i+d_i+1 \le s \le b_i+c_i+d_i}} (x_{i+1,r} - x_{i,s}) \right) M_3 \circ M_2 \circ M_1 \\ X_2 \quad = \quad & \prod_{i=1}^{e} \left(\prod_{\substack{d_{i+1}+1 \le r \le c_{i+1}+d_{i+1} \\ c_i+d_i+1 \le s \le b_i+c_i+d_i}} (x_{i+1,r} - x_{i,s}) \prod_{\substack{1 \le r \le d_{i+1} \\ c_i+d_i+1 \le s \le b_i+c_i+d_i}} (x_{i+1,r} - x_{i,s}) \right. \\ & \left. \times \prod_{\substack{1 \le r \le d_{i+1} \\ d_i+1 \le s \le c_i+d_i}} (x_{i+1,r} - x_{i,s}) \right) M_3 \circ M_2 \circ M_1 \end{aligned}$$

and therefore $\alpha = \beta$. Note also that if $M_i = \Delta_{w_i}$ then $M_3M_2M_1$ is the Demazure operator corresponding to the product $w = w_3w_2w_1$ and the last claim follows as well. □

Example 3.9. *Let* $e = 3$ *and* $\mathbf{b} = (2,1,0), \mathbf{c} = (1,1,0), \mathbf{d} = (1,1,0)$. *Then* $X_1 \equiv X_2 \equiv X_3 : (\mathbf{b},\mathbf{c},\mathbf{d}) \longrightarrow (\mathbf{d},\mathbf{c},\mathbf{b})$ *equal*

$$(x_{2,1} - x_{1,2})(x_{2,1} - x_{1,3})(x_{2,1} - x_{1,4})(x_{2,2} - x_{1,3})(x_{2,2} - x_{1,4})\Delta_{s_1s_2s_3s_1s_2}\Delta_{s_{\overline{1}}s_{\overline{2}}s_{\overline{1}}}$$

up to lower order terms where s_i *(respectively* $\overline{s}_i$*) denotes the* i*th transposition for the strands colored by* 1 *(respectively by* 2*) in* $S_{\mathbf{d}} = S_5 \times S_3$.

3.4. **An explicit basis of A.** We construct and describe explicitly a basis of A by geometric arguments and then interpret it diagrammatically. We again fix a dimension vector $\mathbf{d}$.

Fix $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}} \in \mathrm{VComp}_e(\mathbf{d})$ with residue sequences $\operatorname{res}\hat{\boldsymbol{\mu}}$ and $\operatorname{res}\hat{\boldsymbol{\lambda}}$; and let $S^{\hat{\boldsymbol{\mu}}}$ and $S^{\hat{\boldsymbol{\lambda}}}$ be the subgroups of $S_{\mathbf{d}}$ preserving the blocks. Note that $\hat{\boldsymbol{\mu}}$ is determined uniquely by $\operatorname{res}\hat{\boldsymbol{\mu}}$ and $S^{\hat{\boldsymbol{\mu}}}$. Now, fix a permutation $p \in S_d$ such that $p(\operatorname{res}\hat{\boldsymbol{\mu}}) = \operatorname{res}\hat{\boldsymbol{\lambda}}$ (hence $p \in S_{\mathbf{d}}$) and let p_- be its shortest double coset representative in $S^{\hat{\boldsymbol{\lambda}}}\backslash S_{\mathbf{d}}/S^{\hat{\boldsymbol{\mu}}}$. Note that in the graphical pictures (22) an element $p \in S_{\mathbf{d}}$ is a shortest coset representative in $S^{\hat{\boldsymbol{\mu}}}\backslash S_{\mathbf{d}}/S^{\hat{\boldsymbol{\lambda}}}$ if and only if strands with the same color which end or start in the same block do not cross. In particular, idempotents, splits and merges correspond to shortest double coset representatives.

To the triple $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}, p_-$, we associate a composition of merges and splits from $\hat{\boldsymbol{\mu}}$ to $\hat{\boldsymbol{\lambda}}$ as follows: First, let $\hat{\boldsymbol{\mu}}'$ be the unique vector composition with residue sequence

$$\operatorname{res}(\hat{\boldsymbol{\mu}}') = p_-^{-1}(\operatorname{res}\hat{\boldsymbol{\lambda}}) = p_-^{-1}p(\operatorname{res}\hat{\boldsymbol{\mu}})$$

such that $S^{\hat{\boldsymbol{\mu}}'} = S^{\hat{\boldsymbol{\mu}}} \cap p_- S^{\hat{\boldsymbol{\lambda}}} p_-^{-1}$, and similarly, $\hat{\boldsymbol{\lambda}}'$ the unique vector composition so that

$$\operatorname{res}(\hat{\boldsymbol{\lambda}}') = p_-(\operatorname{res}\hat{\boldsymbol{\mu}}) = p_- p^{-1}(\operatorname{res}\hat{\boldsymbol{\lambda}})$$

and $S^{\hat{\boldsymbol{\lambda}}'} = S^{\hat{\boldsymbol{\lambda}}} \cap p_-^{-1} S^{\hat{\boldsymbol{\mu}}} p_-$.

In other words we first refine the blocks of $\operatorname{res}\hat{\boldsymbol{\mu}}$ into $\operatorname{res}\hat{\boldsymbol{\mu}}'$ as coarse as possible such that p_- sends all elements in a block of $\operatorname{res}\hat{\boldsymbol{\mu}}$ to the same block of $\operatorname{res}\hat{\boldsymbol{\lambda}}$. Similarly refine $\hat{\boldsymbol{\lambda}}$ into $\hat{\boldsymbol{\lambda}}'$ again as coarse as possible such that p_-^{-1} sends all elements in a block to the same block.

Proposition 3.10. *The vector compositions $\hat{\boldsymbol{\mu}}', \hat{\boldsymbol{\lambda}}'$ have the same length (i.e. number of blocks), say ℓ, and $\hat{\boldsymbol{\lambda}}'$ can be obtained from $\hat{\boldsymbol{\mu}}'$ by some permutation $q \in S_\ell$ of its blocks inducing p_- on the residue sequences.*

Proof. Each part of $\hat{\boldsymbol{\mu}}'$ is given by the numbers of 1's, 2's, etc. in a given block of the residue sequence for $\hat{\boldsymbol{\mu}}$ which are sent to a fixed block of $\hat{\boldsymbol{\lambda}}$. It is obtained by refining $\hat{\boldsymbol{\mu}}$ according to the blocks of $\operatorname{res}\hat{\boldsymbol{\lambda}}$ (read from left to right) they are sent to. On the other hand, the parts $\hat{\boldsymbol{\lambda}}'$ have the same description, just reversing the roles of $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$; thus, we just change the order of the blocks, which gives the permutation $q \in S_\ell$ as asserted. □

For each $p \in S_{\hat{\boldsymbol{\lambda}}}\backslash S_{\mathbf{d}}/S_{\hat{\boldsymbol{\mu}}}$, we find the associated permutation q as in Proposition 3.10. Fix a reduced decomposition $q = s_{i_1} \cdots s_{i_\ell}$ with the corresponding morphism $\hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}'$ given by composition of crossings and define

$$\hat{\boldsymbol{\mu}} \overset{p;1}{\Longrightarrow} \hat{\boldsymbol{\lambda}} := \hat{\boldsymbol{\mu}} \longrightarrow \hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}' \longrightarrow \hat{\boldsymbol{\lambda}}.$$

Of course, this definition depends on our choice of reduced decomposition for q. For $h \in \Lambda(\hat{\boldsymbol{\mu}}')$ set let $\hat{\boldsymbol{\mu}}' \xrightarrow{h} \hat{\boldsymbol{\mu}}'$ be multiplication with h and set

$$\hat{\boldsymbol{\mu}} \overset{p;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}} := \hat{\boldsymbol{\mu}} \longrightarrow \hat{\boldsymbol{\mu}}' \xrightarrow{h} \hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}' \longrightarrow \hat{\boldsymbol{\lambda}}.$$

Theorem 3.11. *The morphisms $\hat{\boldsymbol{\mu}} \overset{p;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ generate A as a $\Bbbk$-vector space. In fact, if we let range*

- $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ *over all ordered pairs of vector compositions of type* e,
- p *over the minimal coset representatives* w *in* $S_{\hat{\boldsymbol{\lambda}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\mu}}}$ *and*
- h *over a basis for* $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}')$,

then the morphisms $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ *form a basis of* A.

Remark 3.12. If we choose different reduced decompositions for one q, the resulting elements will differ by a linear combination of maps corresponding to shorter double cosets (Proposition 3.8) which also shows that non-reduced decompositions can be replaced by reduced ones by changing h.

Let q as in Proposition 3.10 and $q = s_{i_k} \cdots s_{i_1}$ a reduced decomposition. where $s_i = (i, i+1) \in S_\ell$ is a simple transposition; let $q^{(j)} = s_{i_j} \cdots s_{i_1}$. From this reduced decomposition, we can read off a string diagram of q, as in Khovanov-Lauda [KL09]. We let $\hat{\boldsymbol{\mu}}_{2j} = q^{(j)}(\hat{\boldsymbol{\mu}}')$ and $\hat{\boldsymbol{\mu}}_{2j+1}$ be the merge of $\hat{\boldsymbol{\mu}}_{2j}$ at index i_{j+1}. Then, of course, $\hat{\boldsymbol{\mu}}_{2j}$ is a split of $\hat{\boldsymbol{\mu}}_{2j}$ at index i_j. In particular, $\hat{\boldsymbol{\mu}}_{2k} = \hat{\boldsymbol{\lambda}}'$.

For each $w \in S_{\hat{\boldsymbol{\mu}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\lambda}}}$, let $p(w)$ be the unique permutation in S_d which acts as w does on each individual color, where $[1, d]$ is colored according to $\hat{\boldsymbol{\mu}}$ for the domain and $\hat{\boldsymbol{\lambda}}$ for the image. Put another way, if we think of $S_{\mathbf{d}}$ as acting on colored alphabets $1^1, \ldots, d_1^1, 1^2, \cdots d_2^2, \ldots$, and $p(\hat{\boldsymbol{\mu}})$ is the unique minimal length permutation sending this sequence to one colored according to the residue sequence of $\hat{\boldsymbol{\mu}}$, then $p(w) = p(\hat{\boldsymbol{\lambda}}) \cdot w \cdot p(\hat{\boldsymbol{\mu}})^{-1}$ (where $p(\hat{\boldsymbol{\lambda}})$ is the unique permutation sending a totally ordered sequence of 1's, 2's etc. to $\operatorname{res} \hat{\boldsymbol{\lambda}}$).

Proof of Theorem 3.11. Our method of proof is to show that this remains a basis when we take associated graded of our algebra. The filtration we require is geometric in nature.

Forgetting the representation (and only keeping the flags) defines a canonical map $\Phi : \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) \to \mathcal{F}(\hat{\boldsymbol{\mu}}) \times \mathcal{F}(\hat{\boldsymbol{\lambda}})$ which is $G_{\mathbf{d}}$-equivariant. Every fiber is the space of quiver representations on a vector space preserving a particular pair of flags, which is naturally an affine space (one can assume both flags are spanned by vectors in a fixed basis, and thus the condition of preserving the pair of flags is simply requiring certain matrix coefficients to vanish). Thus, over each $G_{\mathbf{d}}$-orbit, the map is an affine bundle, but with a different fiber over each orbit.

The $G_{\mathbf{d}}$-orbits on $\mathcal{F}(\hat{\boldsymbol{\mu}}) \times \mathcal{F}(\hat{\boldsymbol{\lambda}})$ are in bijection with the double coset space $S_{\hat{\boldsymbol{\mu}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\lambda}}}$; if we choose completions of the partial flags to complete flags, their relative position is an element of $S_{\mathbf{d}}$, and its double coset is independent of the completion chosen.

That is, using the Bruhat decomposition with fixed lifts of elements from $S_{\mathbf{d}}$ to $G_{\mathbf{d}}$ it sends an element $(xP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, yP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})})$, $x, y \in S_{\mathbf{d}}$ to $x^{-1}y$, and a shortest double coset representative w defines the orbit $\mathcal{F}_w$ of all elements of the form $(gP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, gwP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})})$, $g \in G_{\mathbf{d}}$. A pair of flags contained in $\mathcal{F}_w$ is said to have *relative position* w.

The orbit $\mathcal{F}_w$ is an affine bundle over $G_{\mathbf{d}}/(P_{\hat{\boldsymbol{\mu}}} \cap wP_{\hat{\boldsymbol{\lambda}}}w^{-1})$ via the bundle map

$$(gP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, gwP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})}) \mapsto g(P_{\hat{\boldsymbol{\mu}}} \cap wP_{\hat{\boldsymbol{\lambda}}}w^{-1}),$$

and so both the orbit and its preimage in $G_{\mathbf{d}}$ have (equivariant) homology concentrated in even degrees.

Using the surjection Φ we can partition $\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ into the preimages of the orbits and filter the space by letting $\mathcal{H}_k$ be the union of orbits with length of the shortest representative $\leq k$. The space $\mathcal{H}_k \setminus \mathcal{H}_{k-1}$ is a disjoint union of affine bundles over partial flag varieties, and in particular has even homology. The usual spectral sequence for a filtered topological space shows that $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}))$ has a filtration whose associated graded is the sum of the homologies of these orbits (the spectral sequence degenerates immediately for degree reasons because of the concentrations in even degrees).

Thus, it suffices to show that the morphisms $\hat{\boldsymbol{\mu}} \stackrel{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ pass to a basis of the associated graded. That is, if we fix w, that the classes $\hat{\boldsymbol{\mu}} \stackrel{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ pullback to a basis of $H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$.

It is enough to show that $\hat{\boldsymbol{\mu}} \stackrel{w;1}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ goes to the fundamental class of $\Phi^{-1}(\mathcal{F}_w)$, since adding the h simply has the effect of multiplying by classes that range over a basis of $H^*(\Phi^{-1}(\mathcal{F}_w)/G)$.

Consider the iterated fiber product

$$\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) =$$
$$\mathcal{Q}(\hat{\boldsymbol{\mu}}_0, \hat{\boldsymbol{\mu}}) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_0)} \mathcal{Q}(\hat{\boldsymbol{\mu}}_0, i_1) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_1)} \mathcal{Q}(\hat{\boldsymbol{\mu}}_2, i_1) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_2)} \mathcal{Q}(\hat{\boldsymbol{\mu}}_2, i_2) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_3)} \cdots$$
$$\times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-2})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-2}, i_k) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-1})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k}, i_k) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k}, \hat{\boldsymbol{\lambda}})$$

where $\mathcal{Q}(\hat{\boldsymbol{\mu}}_0, \hat{\boldsymbol{\mu}})$ denotes the subset of $\mathcal{Q}(\hat{\boldsymbol{\mu}}_0)$ such that the associated graded remains semi-simple after coarsening the flag to one of type $\hat{\boldsymbol{\mu}}$. Recall that by the definition of $\hat{\boldsymbol{\mu}}_{2k+1}$ we have that

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\mu}}_{k+1}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, i_{k+1}) \qquad \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\mu}}_{k-1}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, i_k).$$

We can think of this fiber product as the space of representations with compatible flags $F, F_0, \ldots, F_{2k}, F'$ of dimension vectors $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\mu}}_0, \ldots, \hat{\boldsymbol{\mu}}_{2k}, \hat{\boldsymbol{\lambda}}$ respectively such that if any two consecutive flags have subspaces of the same size, those spaces must coincide. We have a map $b^w : \mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) \to \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ which must remembers the flags F and F' and $\hat{\boldsymbol{\mu}} \stackrel{w;1}{\Longrightarrow} \hat{\boldsymbol{\lambda}} = b_*[\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})]$ is the push-forward of the fundamental class of $\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})$.

More generally, we can identify $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}_{2k})$ with the classes in $H^*(\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}))$ generated by the Chern classes of the tautological bundles corresponding to the successive quotients of the flag F_{2k}. Then $\hat{\boldsymbol{\mu}} \stackrel{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}} = b_*(h \cap [\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})])$ is the pushforward of the cap product of h with the fundamental class (the Poincaré dual of the class h).

Lemma 3.13. *The image of the map b^w lies in $\mathcal{H}_{\ell(w)}$ and induces an isomorphism between the locus in $\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})$ where F and F' have relative position w and $\Phi^{-1}(\mathcal{F}_w)$.*

Proof. Let $p^{(j)}$ be the element of S_d induced by acting on the blocks of res $\hat{\boldsymbol{\mu}}'$ with $q^{(j)}$. By the usual Bruhat decomposition, we must have that F_{2j}, considered as a flag on $\oplus_{i=1}^e V_i$, has relative position $\leq p^{(j)}$ with respect to F and $\leq p^{(j)}p(w)^{-1}$ with respect to F'. Since $p^{(j)} \leq p(w)$ in right Bruhat order (its inversions are a subset of those of $p(w)$), this condition uniquely specifies F_{2j} (which also uniquely specifies F_{2j-1}, since this is a coarsening of F_{2j}).

Thus, we need only show that this flag is compatible with the decomposition $\oplus_{i=1}^{e} V_i$ and strictly preserved by the accompanying quiver representation f. The former is clear, since we can choose a basis compatible with this decomposition such that both F and F' contain only coordinate subspaces; thus, the spaces of F_j are coordinate for this basis as well. It follows that they are also compatible with the decomposition. The latter is a well-known property of Bruhat decomposition: if a nilpotent preserves two different flags F and F', then it preserves any other flag F'' that fits into a chain where the relative position of F and F'' is v, that of F'' to F' is w, and $\ell(vw) = \ell(v) + \ell(w)$. We simply apply this to f thought of as a nilpotent endomorphism of $\oplus_{i=1}^{e} V_i$. □

Thus, we have a Cartesian diagram

$$\begin{array}{ccc} \Phi^{-1}(\mathcal{F}_w) & \hookrightarrow & \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) \\ \wr \uparrow & & \uparrow b \\ b^{-1}(\Phi^{-1}(\mathcal{F}_w)) & \hookrightarrow & \mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) \end{array}$$

If we push-forward and pull-back $h \cap [\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})]$ from the bottom right to upper left, we obtain the class of $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ in the associated graded with respect to the geometric filtration, thought of as a Borel-Moore class on $\Phi^{-1}(\mathcal{F}_w)$. We can also calculate this class by pull-back and pushforward; this goes to $h \cap [\Phi^{-1}(\mathcal{F}_w)]$ where we consider h now as a class on $\Phi^{-1}(\mathcal{F}_w)$ by pull-back. The tautological bundles for F_{2k} pulled back to $\Phi^{-1}(\mathcal{F}_w)$ induce an isomorphism $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}_{2k}) \cong H^*(\Phi^{-1}(\mathcal{F}_w)/G)$, and cap product with the fundamental class induces an isomorphism $H^*(\Phi^{-1}(\mathcal{F}_w)/G) \cong H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$. Thus, the classes $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ pull-back to a basis of $H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$; thus ranging over all w, we obtain a basis of the associated graded. These vectors must thus have been a basis of $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}))$. □

Remark 3.14. Theorem 3.11 could also be proved diagrammatically. The linearly independence follows analogously to the arguments in [KL09] as follows: Let $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}$ be fixed. By construction, $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ is zero if $w \neq w_-$. Let S be the split from $\hat{\boldsymbol{\lambda}}$ into unit vectors and M the merge from unit vectors to $\hat{\boldsymbol{\mu}}$. Then a linear combination $\sum \alpha_{w,h}(\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}})$ is zero if and only if $\sum \alpha_{w,h} S \circ (\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}) \circ M = 0$ (because M is surjective and composing with S is injective. The linear independence is therefore given by [KL09, Theorem 2.5].

Theorem 3.11 provides a basis which is natural diagrammatically as well as geometrically, and Proposition 3.8 provides some fairly obvious relations, we do not have a complete set of relations defining our algebras. Although we have reduced the question to one of linear algebra, it appears to be quite difficult linear algebra and we are left with the following problem:

Problem 3.15. *Find an explicit presentation with generators and defining relations of the algebras $A_{\mathbf{d}}$ (or at least Morita equivalent one) and A.*

The following result describes its center (which is a Morita invariant):

Lemma 3.16. *The ring of total invariants $R(\mathbf{d})^{S_d}$ is naturally isomorphic to the center of $\mathbf{A_d}$.*

Proof. The proof is essentially identical to [KL09, 2.9]. Restricting the action maps simultaneously to the total invariants, defines a map α from $R(\mathbf{d})^{S_d}$ into the center of $\mathbf{A_d}$. For each $\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})$, we can obtain a new vector composition $\hat{\boldsymbol{\mu}}' \in \mathrm{VComp}_e(\mathbf{d})$ by splitting each dimension vector between each colors. By Proposition 3.4, composing with this defines an inclusion from $Ae_{\hat{\boldsymbol{\mu}}}$ to $Ae_{\hat{\boldsymbol{\mu}}'}$. Now, we obtain a third vector composition $\hat{\boldsymbol{\mu}}'' \in \mathrm{VComp}_e(\mathbf{d})$ by reordering all the colors in the residue sequence, the smallest to the left, by permuting the blocks by applying crossings. In this case the Demazure operator from Proposition 3.4 is the identity and we get an inclusion from $Ae'_{\hat{\boldsymbol{\mu}}}$ to $Ae_{\hat{\boldsymbol{\mu}}''}$. It follows that $Ae_{\hat{\boldsymbol{\mu}}}$ is a submodule (not necessarily direct summand) of $Ae_{\hat{\boldsymbol{\mu}}''}$. In particular, every central element acts non-trivially on some $P(\hat{\boldsymbol{\mu}}'') = Ae_{\hat{\boldsymbol{\mu}}''}$. Furthermore, every $P(\hat{\boldsymbol{\mu}}'')$ is just obtained by inducing the direct sum of modules of the form $P((d_1, 0, \dots)), P((0, d_2, 0, \dots)), \dots) \dots$. We thus have an injective homomorphism $\beta\colon Z(A_\mathbf{d}) \to e_{\mu_\mathbf{d}} A e_{\mu_\mathbf{d}} \cong R(\mathbf{d})^{S_d}$ given by $\psi(z) = e_{\mu_\mathbf{d}}$. Furthermore, the composition $\beta \circ \alpha$ is the identity on $R(\mathbf{d})^{S_d}$. Thus, β is surjective as well, and gives the desired isomorphism. □

4. Extended quiver representations and convolution algebra

In this section we extend our convolution algebras to a "higher level version." Just as in [Web10a, §4], we consider not only quiver representations of Γ, but equip them with extra data, motivated by the construction of quiver varieties of Nakajima [Nak94], [Nak98], [Nak01]. We equip the quiver Γ with shadow vertices, one for each $i \in \mathbb{V}$ together with an arrow pointing from the ith vertex to the i-th shadow vertex, see Figure 1 were the shadow vertices are drawn in red/grey. For given $\nu : \mathbb{V} \to \mathbb{Z}_{\geq 0}$, we extend the affine space (2) of representations of a fixed dimension vector $\mathbf{d}$ to the affine space

$$\mathrm{Rep}_{\mathbf{d};\nu} := \mathrm{Rep}_\mathbf{d} \times \bigoplus_i \mathrm{Hom}_K\left(V_i, K^{\nu(i)}\right), \qquad \mathrm{Rep}_\nu = \bigsqcup_\mathbf{d} \mathrm{Rep}_{\mathbf{d};\nu}$$

of representations (V, f, γ) shadowed by vector spaces $K^{\nu(i)}$. It comes endowed with the product $G_\mathbf{d}$-action.

Definition 4.1. *Assume we are given a* **weight data** *consisting out of*

- *an ℓ-tuple $\underline{\boldsymbol{\nu}} = (\nu_1, \dots, \nu_\ell)$ of maps $\mathbb{V} \to \mathbb{Z}_{\geq 0}$, called an ℓ-***weight***,*
- *an $\ell + 1$-tuple $\grave{\mu} = (\hat{\boldsymbol{\mu}}(0), \hat{\boldsymbol{\mu}}(1), \dots, \hat{\boldsymbol{\mu}}(\ell))$ of vector compositions of no fixed length, but all of type e.*

The we call $\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}(0) \cup \dots \cup \hat{\boldsymbol{\mu}}(\ell) \in \mathrm{VComp}_e(\mathbf{d})$ the **associated vector composition** *and denote by $\mathfrak{Q}(\grave{\mu})$ the subspace of $\mathcal{Q}(\hat{\boldsymbol{\mu}}) \times \bigoplus_i \mathrm{Hom}_K(V_i, K^{\nu(i)})$ defined as*

$$\mathfrak{Q}(\grave{\mu}) = \left\{ ((V, f, F), \{\gamma_i\}) \mid \gamma_i(\grave{W}_i(k)) \subset K^{\nu_1(i) + \dots + \nu_k(i)} \right\}$$

where as usual $K^a \subset K^b$ for $a \leq b$ is the subspace spanned by the first a unit vectors, and $\grave{W}_i(1) \subset \grave{W}_i(2) \subset \dots \subset \grave{W}_i(\ell + 1) = V_i$ is the partial flag at the vertex i coarsening F_i and obtained by picking out the largest subspace corresponding to each part of $\grave{\mu}$.

Just as an unadorned representation carries a canonical socle filtration (i.e. a filtration starting with the maximal semi-simple submodule and proceeding such that the successive quotients are maximal semi-simple), an extended representation $(V, f, \gamma) \in \mathrm{Rep}_{\mathbf{d};\nu}$ carries a slightly modified filtration which starts with $\{0\} \subset R_1 = (W, f, \gamma)$ such that (W, f) is the largest subrepresentation (V, f) for which $\gamma(W_i) \subset K^{\nu_1(i)+\cdots+\nu_k(i)}$ and proceeds inductively by considering the corresponding first step in $(V/W, \overline{f}, \overline{\gamma})$ pulled back to (V, f). The dimension vectors of these subquotients define a weight data $(\underline{\boldsymbol{\nu}}, \grave{\mu})$ where $\grave{\mu}$ denotes the type of the multi flag R_i/R_{i-1} induced by the filtration.

4.1. **Extended convolution algebras.** Generalizing (4), we have a projection map

$$p\colon \mathfrak{Q}(\grave{\mu}) \to \mathrm{Rep}_{\mathbf{d};\nu}$$

where $\nu := \nu_1 + \cdots + \nu_\ell$, and we can study the convolution algebra

$$\tilde{A}^{\underline{\boldsymbol{\nu}}} := \mathrm{Ext}^*_{D(\mathrm{Rep}_\nu/G)}\Big(\bigoplus_{\grave{\mu}} p_*\Bbbk_{\mathfrak{Q}(\grave{\mu})}, \bigoplus_{\grave{\mu}} p_*\Bbbk_{\mathfrak{Q}(\grave{\mu})}\Big) \cong \bigoplus_{\grave{\mu},\grave{\nu}} H^{BM}_*(\mathfrak{H}(\grave{\mu}, \grave{\nu}))$$

where $\mathfrak{H}(\grave{\mu}, \grave{\nu}) \cong \mathfrak{Q}(\grave{\mu}) \times_{\mathrm{Rep}_{\mathbf{d};\nu}} \mathfrak{Q}(\grave{\nu})/G$ and the sum runs over all weight data $(\underline{\boldsymbol{\nu}}, \grave{\mu})$ with associated vector composition $\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})$.

Proposition 2.6 generalizes directly to the statement that

Proposition 4.2. *The algebra $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ has a natural faithful action on*

$$\tilde{V}^{\underline{\boldsymbol{\nu}}} \;=\; \bigoplus_{\grave{\mu}} H^*(\mathfrak{Q}(\grave{\mu})) \cong \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell)). \tag{23}$$

No individual algebra (with vertical multiplication given by convolution) in this family has a notion of horizontal multiplication. Instead, there is a horizontal multiplication $\tilde{A}^{\underline{\boldsymbol{\nu}}} \times \tilde{A}^{\underline{\boldsymbol{\nu}}'} \to \tilde{A}^{\underline{\boldsymbol{\nu}} \cup \underline{\boldsymbol{\nu}}'}$ which concatenates the tuples of weights. This can, of course, be organized in a single algebra $\tilde{A}$, but for our purposes, it is more profitable to think of the separate algebras $\tilde{A}^{\underline{\boldsymbol{\nu}}}$. The following is a direct consequence of our definitions:

Proposition 4.3. *For each fixed dimension vector $\mathbf{d}$, horizontal composition induces a right $\mathbf{A}$-module structure on $\mathbf{A}^{\underline{\boldsymbol{\nu}}}$.*

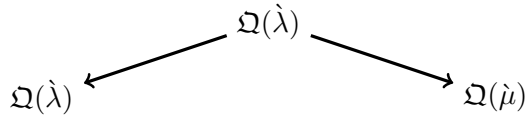
Using horizontal and vertical composition, the algebra $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ is generated by a small number of elements, like $\mathbf{A}$ is, which are defined in some sense locally and also have an easy diagrammatic description. We mimic this construction now in the extended case incorporating the shadow vertices. Of course, we still have the old merges and splits not involving the shadow vertices, but also have a new "move" on ℓ-tuples of compositions.

Definition 4.4. *We call $\grave{\lambda}$ a* **left shift** *of $\grave{\mu}$ (and $\grave{\mu}$ a* **right shift** *of $\grave{\lambda}$) by $\mathbf{c}$ if for some index m, we have that*

$$\grave{\lambda}(m) = \grave{\mu}(m) \cup \mathbf{c} \text{ and } \grave{\mu}(m+1) = \mathbf{c} \cup \grave{\lambda}(m+1).$$

In words, if a vector $\mathbf{c}$ has been shifted from the start of the $m+1$-st composition of $\grave{\mu}$ to the end of the mth composition for $\grave{\lambda}$.

If $\grave{\lambda}$ is a left shift of $\grave{\mu}$, then $\mathfrak{Q}(\grave{\lambda})$ is naturally a subspace of $\mathfrak{Q}(\grave{\mu})$. Thus, generalizing the construction after Definition 3.1, we can think of it as a correspondence (read from right to left)



Similarly, for right shifts we can reverse this correspondence and read from left to right. Thus, the fundamental class of this correspondence gives elements of $A^{\underline{\boldsymbol{\nu}}}$ corresponding to left and right shifts, which we denote by $\grave{\mu} \to \grave{\lambda}$ and $\grave{\lambda} \to \grave{\mu}$.

Proposition 4.5. *The degree of the map $\grave{\lambda} \to \grave{\mu}$ or $\grave{\mu} \leftarrow \grave{\lambda}$ associated to right or left shift by $\mathbf{c}$ at the index m equals $\sum_i c_i \nu_m(i)$.*

Proof. This follows from a direct calculation. □

Theorem 3.11 generalizes immediately to the algebra $\tilde{A}^{\underline{\boldsymbol{\nu}}}$: If we let $S_{\grave{\mu}}$ be the Young subgroup associated to the concatenation of the parts of $\grave{\mu}$, then the following holds:

Proposition 4.6. *If we let*

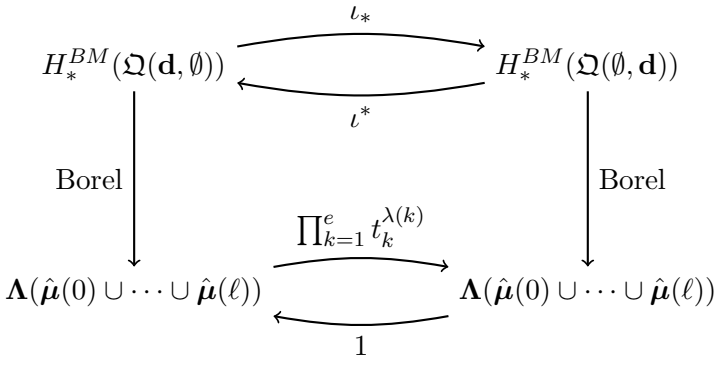
- *$(\grave{\mu}, \grave{\nu})$ range over all ordered pairs of $\ell+1$-tuples of vector compositions of type e,*
- *w range over minimal coset representatives in $S_{\grave{\mu}} \backslash S_{\mathbf{d}} / S_{\grave{\nu}}$ and*
- *h range over a basis for $\mathbf{\Lambda}(\hat{\boldsymbol{\mu}}_j)$,*

then the morphisms $\grave{\mu} \overset{w;h}{\Longrightarrow} \grave{\nu}$ form a basis of $\tilde{A}^{\underline{\boldsymbol{\nu}}}$.

Proof. Completely analogous to the proof of Theorem 3.11. □

As in Section 3.1, we can also calculate how this morphism acts on the polynomial ring $\tilde{V}^{\underline{\boldsymbol{\nu}}}$ from (23). Using horizontal composition, we need only consider the case where $\underline{\boldsymbol{\nu}} = (\lambda)$ (so $l = 1$) and look at pairs $(\emptyset, \mathbf{d})$ and $(\mathbf{d}, \emptyset)$ of vector compositions of type e. Both $\mathfrak{H}((\emptyset, \mathbf{d}), (\mathbf{d}, \emptyset))$ and $\mathfrak{H}((\mathbf{d}, \emptyset), (\emptyset, \mathbf{d}))$ are simply $\mathcal{Q}(\mathbf{d})$, and thus their Borel-Moore homology is a rank 1 free module over $H^*(BG_{\mathbf{d}}) \cong \mathbf{\Lambda}(\mathbf{d})$ generated by the corresponding fundamental class. Their actions on $\tilde{V}^{\underline{\boldsymbol{\nu}}}$ are simply the actions of ι^* and ι_* where $\iota \colon \mathfrak{Q}(\mathbf{d}, \emptyset) \to \mathfrak{Q}(\emptyset, \mathbf{d})$ is the obvious inclusion map.

Proposition 4.7. *The following diagram commutes,*



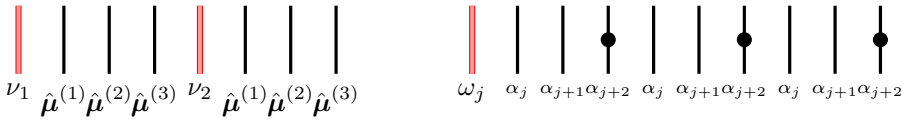


FIGURE 2. An idempotent and $t^{(j;9)}$ from Lemma 5.15 for $e = 3$.

where $t_k = x_{k,1} \cdots x_{k,d_k}$ is the highest degree elementary symmetric function in the alphabet (6) *for the kth vertex in Γ.*

Proof. The map ι is the inclusion of the 0-section of a $G_{\mathbf{d}}$-equivariant vector bundle, whose underlying vector bundle is trivial, but equals $\oplus_{k=1}^{e} \mathrm{Hom}(\mathcal{V}_k, K^{\lambda(k)})$ equivariantly. As the second vector space is trivial, the Euler class of this bundle is $\prod_{k=1}^{e} c_{top}(\mathcal{V}_k^*)^{\lambda(k)}$. Since the Borel isomorphism sends the Chern class of maximal possible degree $c_{top}(\mathcal{V}_k^*)$ to t_k, the result follows. □

4.2. **Extended graphical calculus.** We can extend our graphical calculus to the algebras $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ quite easily. We simply add a **red band** labeled with ν_i between the black lines representing $\hat{\boldsymbol{\mu}}(i-1)$ and $\hat{\boldsymbol{\mu}}(i)$. For instance, for fixed $\underline{\boldsymbol{\nu}}$, the idempotent $e_{(\underline{\boldsymbol{\nu}},\grave{\mu})}$ corresponding to the weight data $(\underline{\boldsymbol{\nu}}, \grave{\mu})$ and another element of the algebra is graphically described in Figure 2.

The splitting and merging morphisms defined earlier are displayed as in Section 3.3, except of additional red lines separating the multicompositions. We have also diagrams associated to the new morphisms, that is to left and right shifts. They are denoted by a (left resp. right) crossing of red and black strands, as shown below:

(24)

a left shift

a right shift

Hence Figure 3 (read as usual from bottom to top) is a merge and split followed by a left shift.

Remark 4.8. Diagrammatically, Proposition 4.5 is assigning a degree to each crossing of a red band with a black strand which agrees with the definitions in [Web10a, 2.1].

4.3. **The tensor product algebra and cyclotomic quotients.** Just as A contains an idempotent which projects down to the quiver Hecke algebra, the algebra $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$ contains an analogous idempotent e_T. Let

$$e_T \quad = \quad \sum e_{\grave{\mu}} \tag{25}$$

denote the idempotent in $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$ defined as the sum over all primitive idempotents $e_{\grave{\mu}}$ indexed by all vector multicompositions $\grave{\mu}$ where each composition appearing is of the form α_i, that is, the corresponding flag is complete.

The **tensor product algebra** $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ from [Web10a, Def. 2.2] is then the centralizer algebra:

Proposition 4.9. $e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T = \tilde{T}^{\underline{\boldsymbol{\nu}}}$

Proof. By Proposition 4.2, we have a faithful action of $e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$ on

$$e_T\tilde{V}^{\underline{\boldsymbol{\nu}}} \cong \bigoplus_{\hat{\boldsymbol{\mu}}(i)\in \mathrm{VCompf}_e(\mathbf{d})} H^*(\mathfrak{Q}(\grave{\mu})) \cong \mathbf{\Lambda}(\hat{\boldsymbol{\mu}}(0)\cup\cdots\cup\hat{\boldsymbol{\mu}}(\ell)).$$

This is a sum of polynomial rings, corresponding to $\ell+1$-tuples of sequences of simple roots. By [Web10a, Proof of Prop. 2.4], $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ acts on a sum of polynomial rings with the same indexing set. (We can obtain the indexing used in [Web10a] by taking $\mathbf{i}$ to be the concatenation of the sequences of roots in the description in this paper, and the function κ to send j to $\sum_{k<j}|\hat{\boldsymbol{\mu}}(k)|$.) Comparing the explicit formulas of this action [Web10a, Figure 9] with the formulas from Propositions 3.4 and 4.7 shows that the action of a crossing or dot in the two calculi agree, except in the case where $e=2$. Thus, the images of $e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$ and $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ in the endomorphisms of this module are identified, giving the desired isomorphism.

The case of $e=2$ is easily dealt with by a slight generalization of the representation given in [Web10a], following Rouquier [Rou08, 3.12]; if $R_{ij}(u,v)$ denotes a polynomial such that $Q_{ij}(u,v)=R_{ij}(u,v)R_{ji}(v,u)$ (with Q_{ij} as in [Web10a]), then we can adjust the action to send a crossing of the k and $k+1$st black strands to the operator $R_{i_{k+1}i_k}(y_k,y_{k+1})(k,k+1)\cdot -$ when $i_k\neq i_{k+1}$. One can easily check that the relations between these operators only depend on Q_{ij} and that the representations of $e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$ and $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ in the case where $e=2$ just correspond to different choices of R_{ij} with the same Q_{ij}. In this way we get the asserted identification. □

The quiver Hecke algebra $R(\mathbf{d})$ has a very interesting family of finite dimensional quotients, the cyclotomic quiver Hecke algebras T^λ (implicitly depending on our choice of $\mathbf{d}$) where λ is an integer valued function on Γ. We will usually think of this function as a weight of the affine Lie algebra $\widehat{\mathfrak{sl}}_e$. As shown in [BK09a], blocks of the diagrammatically defined cyclotomic quotients of the quiver Hecke algebra are isomorphic to blocks of cyclotomic Hecke algebras for symmetric groups. The latter are Hecke algebra which behave as though they are of "characteristic e" (i.e. the cyclotomic quotient is defined using powers of an element q from the ground field and e is the smallest number such that $1+q+q^2+\cdots+q^{e-1}=0$).

We want now to define similar quotients $A^{\underline{\boldsymbol{\nu}}}$ of $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$:

Definition 4.10. *Let $I=I^{\underline{\boldsymbol{\nu}}}$ be the ideal of $A^{\underline{\boldsymbol{\nu}}}$ generated by all elements of the form $e_{\alpha_i}|a$. We call this ideal the* **violating ideal** *and the quotient $A^{\underline{\boldsymbol{\nu}}}=\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}/I$* **cyclotomic quotient** *of $A^{\underline{\boldsymbol{\nu}}}$.*

Pictorially, I is generated by all diagrams where at some point the left-most strand is black and labeled with some simple root, see Figure 3 for an example.

Obviously, the idempotent (2) associated to an $\ell+1$-tuple will be 0 in this quotient unless it begins with the empty set (hence a red strand to the left). To avoid much tedious writing of $\emptyset$, we shall henceforth just deal with ℓ-tuples, and leave the initial $\emptyset$ as given.

Note that if $\underline{\boldsymbol{\nu}}=(\lambda)$, and r is the element which is just a red strand labeled with λ, then $a\mapsto r|a$ followed by projection defines an algebra map $\tilde{A}^\lambda\to A^\lambda$ which is surjective. The kernel of this map is called the **cyclotomic ideal** of A. An argument analogous to [Web10a, Theorem 2.8] shows that this ideal has a definition looking more like a traditional cyclotomic ideal.

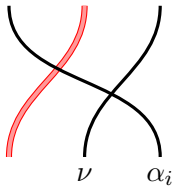


FIGURE 3. An example of an element of the violating ideal.

Let $T^{\underline{\boldsymbol{\nu}}} = \tilde{T}^{\underline{\boldsymbol{\nu}}}/J$ be the cyclotomic quotient of the tensor algebra as defined in [Web10a, Def. 2.2].

Proposition 4.11. *$e_T A^{\underline{\boldsymbol{\nu}}} e_T \cong T^{\underline{\boldsymbol{\nu}}}$*

Proof. We first claim that any element x of $e_T \tilde{A}^{\underline{\boldsymbol{\nu}}} e_T$ which is in the violating ideal in $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$ is mapped to an element in the violating ideal of $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ via the isomorphism of Proposition 4.9. Now x is a sum of elements of the form abc where $a \in e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}, c \in \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$ and $b = e_{\alpha_i}|b'$ is a generator of I as above. First, we apply Theorem 3.11 to a and c, so we may assume that $a = a_1 a_2$ and $c = c_2 c_1$ where $a_1, c_1 \in e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$, and a_2 just joins and c_2 just splits strands. In both cases the leftmost strand is kept unchanged. Thus both commute with b, so $abc = a_1 b a_2 c \in e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T \cdot b \cdot e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$. Similarly, $abc = a c_2 b c_1 \in \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T \cdot b \cdot e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$. The claim follows and implies the proposition by definition of the violating ideal J. □

5. A graded cellular basis

We specialize in the remaining sections to the case where $\underline{\boldsymbol{\nu}}$ is a sequence of fundamental weights, so $\underline{\boldsymbol{\nu}} = (\omega_{\jmath_1}, \ldots, \omega_{\jmath_\ell})$, where the j-th fundamental weight is of the form $\omega_j(k) = \delta_{j,k}$. We wish to show that $A^{\underline{\boldsymbol{\nu}}}$ has a cellular basis, turning it into a graded cellular algebra. Let us first recall the relevant definitions from [GL96].

Definition 5.1. *A* **cellular algebra** *is an associative unital algebra H together with a* **cell datum** *$(\Lambda, M, C, *)$ such that*

1. *Λ is a partially ordered set and $M(\lambda)$ is a finite set for each $\lambda \in \Lambda$;*
2. *$C : \bigcup_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to H, (T, S) \mapsto C^\lambda_{T,S}$ is an injective map whose image is a basis for H;*
3. *the map $* : H \to H$ is an algebra anti-automorphism such that $(C^\lambda_{T,S})^* = C^\lambda_{S,T}$ for all $\lambda \in \Lambda$ and $\alpha, \beta \in M(\lambda)$;*
4. *if $\xi \in \Lambda$ and $\mathsf{S}, \mathsf{T} \in M(\lambda)$ then for any $x \in H$ we have that*

$$x C^\xi_{\mathsf{S},\mathsf{T}} \equiv \sum_{\mathsf{S}' \in M(\xi)} r_x(\mathsf{S}', \mathsf{S}) C^\xi_{\mathsf{S}',\mathsf{T}} \pmod{H(> \xi)}$$

where the scalar $r_x(\mathsf{S}', \mathsf{S})$ is independent of T and $H(> \mu)$ denotes the subspace of H generated by $\{C^\nu_{\mathsf{S}'',\mathsf{T}''} \mid \nu > \xi, \mathcal{S}'', \mathsf{T}'' \in M(\nu)\}$.

The basis consisting of the $C^\lambda_{T,S}$ is then a **cellular basis** *of H.*

If moreover there is a function $\deg : \bigcup_{\lambda \in \lambda} M(\lambda) \to \mathbb{Z}$, $S \mapsto \deg^\lambda_S$ such that for $(S, T) \in M(\lambda) \times M(\lambda)$ putting $C_{S,T}$ in degree $\deg^\lambda_S + \deg^\lambda_T$ turns H into a

graded algebra, H is called a **graded cellular algebra** with graded cell datum $(\Lambda, M, C, *, \deg)$, see [HM10].

We want to construct now such a basis, indexed by pairs (T, S) of semi-standard Young tableaux of the same shape λ (but, not necessarily of the same type) with the shape λ ranging over the set Λ of all ℓ-multipartitions (i.e. an ℓ-multicomposition where the parts are partitions). The involution $*$ will be given by a vertical reflection of the diagram in the diagrammatical realization.

Definition 5.2. *Given an ℓ-multipartition $\hat{\lambda}$, a* **semi-standard $\hat{\lambda}$-tableau** *is a filling of the boxes of its ℓ-tuple of Young diagrams with numbers from the ℓ-fold disjoint union of $\mathbb{Z}_{\geq 0}$, usually denoted with a subscript to show which of the ℓ copies it comes from, subject to the following rules (when the partitions are drawn in the English style):*

- *the entries in each component are in each row weakly increasing from left to right and in each columns strictly increasing from top to bottom,*
- *the ith copy of $\mathbb{Z}$ cannot only be used in the first i partitions of the multipartition.*
- *We will also always assume that our tableaux have no gaps, i.e. if j_k appears, then i_k for all $i \leq j$ also appear in the tableau.*

Here $\leq$ denotes the lexicographic order, first in the subscripts and then the numbers themselves.

The **type** of S is $\boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \dots, \boldsymbol{\mu}^{(\ell)})$ where $\boldsymbol{\mu}_i^{(k)}$ denotes the multiplicity of the number i coming from the kth copy of the alphabet.

Example 5.3. *The example from* [DJM98, 4.9] *of shape* $((4,3),(2,1),(2,1))$ *,*

$$\mathsf{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3} \end{array} , \begin{array}{|c|c|} \hline 1_2 & 3_3 \\ \hline 2_2 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1} \end{array} \right)$$

is a semistandard multitableau of type $((3,3,1),(1,1,0),(2,1,1))$.

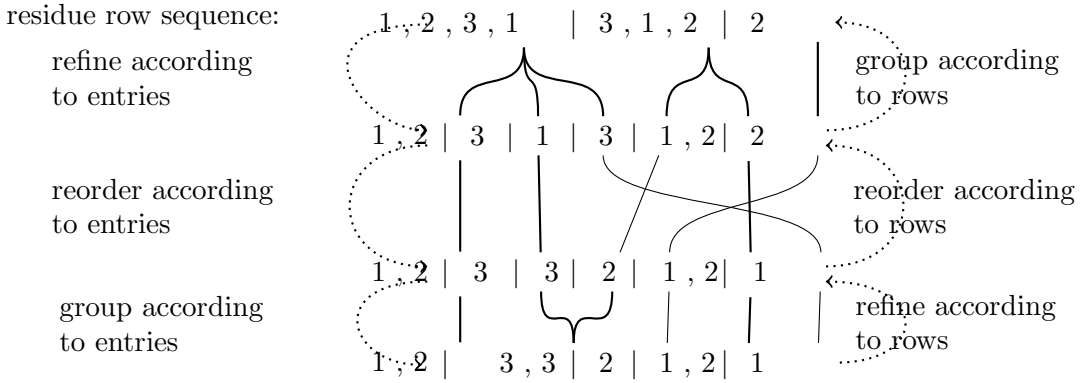
A semi-standard $\hat{\lambda}$-tableau S is called **standard** or a **standard multitableau**, if each of the entries only appears once and **super-standard** if it contains only the numbers $1, \dots n$ from the last alphabet.

As shown in [DJM98], the Ariki-Koike algebra $\mathfrak{H}(q, Q_1, \dots, Q_\ell)$ from the introduction is a cellular algebra with basis labeled by pairs of standard $\hat{\lambda}$-tableaux of the same shape, but varying over all $\hat{\lambda}$.

We fix an ℓ-tuple of vertices $\mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_\ell)$ in $\mathbb{Z}/e\mathbb{Z}$, which is called the **charge**. The **content** of a box in the ith row and jth column is $i - j$. If it appears in the kth Young diagram then its **residue** is $\mathfrak{z}_k + j - i \pmod{e}$.

Definition 5.4. *Let S be a semi-standard $\hat{\lambda}$-tableau. Then w_{S} denotes the permutation of minimal length such that w_{S} applied to the row reading word of S is increasing.*

In particular, w_{S} is a shortest coset representative for the Young subgroup S_λ attached to the rows (since the entries of each row are already weakly ordered) acting from the right, and the Young subgroup S_μ associated to the multiplicities of the entries in S (that is, the Young subgroup that fixes w_{S} times the row reading word) acting from the left.



The bottom line represents $\grave{\mu}_{\mathsf{S}}$, to the top line, $\grave{\lambda}_{\mathsf{S}}$.

FIGURE 4. Construction of the element B_{S}

The compositions λ and μ get refined by the following two vector ℓ-multicompositions $\grave{\lambda}_{\mathsf{S}} = (\hat{\boldsymbol{\lambda}}_{\mathsf{S}}(1), \ldots, \hat{\boldsymbol{\lambda}}_{\mathsf{S}}(\ell))$ and $\grave{\mu}_{\mathsf{S}} = (\hat{\boldsymbol{\mu}}_{\mathsf{S}}(1), \ldots, \hat{\boldsymbol{\mu}}_{\mathsf{S}}(\ell))$:

$$\begin{aligned} \grave{\lambda}_{\mathsf{S}}(k)[g,h] &= \#\{\text{boxes of residue } h \text{ in the } g\text{th row of the } k\text{th Young diagram}\} \\ \grave{\mu}_{\mathsf{S}}(k)[g,h] &= \#\{\text{boxes of residue } h \text{ and entry } g_k\} \end{aligned}$$

Note that the first does not depend on the entries of S, but only on the shape. In particular, if ξ is a shape, then we will use $\grave{\lambda}_{\xi}$ to denote $\grave{\lambda}_{\mathsf{S}}$ for a tableau of shape ξ.

Example 5.5. *Let $e = 3$, $l = 1$ and $\hat{\lambda} =$* $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2}\end{array}$. *Then* $\mathsf{S} =$ $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$ *is a semistandard $\hat{\lambda}$-tableau of type $\mu = (2,2,1,2,1)$. The row reading word is $1,1,2,5,2,4,4,3$ and therefore $w_{\mathsf{S}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 8 & 4 & 6 & 7 & 5 \end{pmatrix} \in S_8$, a shortest coset representative for $S_\mu \backslash S_8 / S_\lambda$, where $S_\lambda = S_4 \times S_3 \times S_2$ and $S_\mu = S_2 \times S_2 \times S_1 \times S_2 \times S_1$. If $\mathfrak{z} = (1)$ then the residues are* $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}$ *and we have*

$$\begin{aligned} \grave{\lambda}_{\mathsf{S}} &= (((2,1,1),(1,1,1),(0,1,0))) \\ \grave{\mu}_{\mathsf{S}} &= (((1,1,0),(0,0,2),(0,1,0),(1,2,0),(1,0,0))). \end{aligned}$$

By construction w_{S} is a shortest length representative in $S_{\grave{\lambda}_{\mathsf{S}}} \backslash S_d / S_{\grave{\mu}_{\mathsf{S}}}$, hence we can invoke Proposition 4.6 and associate a basis vector in $A^{\underline{\boldsymbol{\nu}}}$:

Definition 5.6. *Given S, a semi-standard $\hat{\mu}$-tableau, denote by B_{S} the element $\grave{\mu}_{\mathsf{S}} \overset{w_{\mathsf{S}};1}{\Longrightarrow} \grave{\lambda}_{\mathsf{S}}$ and by B_{S}^* the element obtained by flipping the diagram vertically. Set $C_{\mathsf{S},\mathsf{T}} = B_{\mathsf{S}}^* B_{\mathsf{T}}$ for S and T semi-standard tableaux.*

Since the shape of S is determined by the idempotent attached to $\grave{\lambda}_{\mathsf{S}}$, applying the definition of $C_{\mathsf{S},\mathsf{T}} = B_{\mathsf{S}}^* B_{\mathsf{T}}$ to tableaux which are not the same shape gives 0. Also note that $C_{\mathsf{S},\mathsf{T}}^* = (B_{\mathsf{S}}^* B_{\mathsf{T}})^* = (B_{\mathsf{T}}^* B_{\mathsf{S}}) = C_{\mathsf{T},\mathsf{S}}$.

Now, let Λ be the set of ℓ-multipartitions ordered lexicographically, and for $\xi \in \Lambda$ let $M(\xi)$ be the set of ξ-semi-standard tableaux (which is finite thanks

to the last condition in Definition 5.2. Let $C : M(\xi) \times M(\xi) \to A^{\underline{\nu}}$ be the map $(\mathsf{S}, \mathsf{T}) \mapsto C_{\mathsf{S},\mathsf{T}}$ and the involution $*$ from Definition 5.6. Let $\deg(B_{\mathsf{S}})$ be the degree of the homogeneous element B_{S}.

In the setup of Example 5.5, the element B_{S} is displayed in Figure 4 and of degree $0 + 0 + 2 = 2$.

Theorem 5.7. *The data* $(\Lambda, M, C, *, \deg)$ *is a graded cell datum for* $A^{\underline{\nu}}$.

Before giving the proof we want to emphasize that our construction can be used to define a grading on the set of semistandard multitableaux S by setting the $\deg(\mathsf{S}) := \deg(B_{\mathsf{S}})$.

This degree function also has a combinatorial definition. For a semi-standard tableau S let $\mathsf{S}(< i_j)$ (or $\mathsf{S}(\leq i_j)$) be the subdiagram of all boxes with entries smaller (or equal) i_j respectively.

Definition 5.8. *The* **degree** $d_{\mathsf{S}}(b)$ *of a box* b *with entry* i_j *in a semi-standard tableau* S *is the number of boxes strictly below* b *of the same residue as* b *that are are addable in the diagram* $\mathsf{S}(\leq i_j)$*, minus the number of such boxes which removable in* $\mathsf{S}(< i_j)$*. The* **degree** $\mathrm{Deg}(\mathsf{S})$ *of a tableau is the sum of the degrees of the boxes,* $\sum_b d_{\mathsf{S}}(b)$.

Note that this definition generalizes the usual definitions for standard tableaux from [LLT96, §4.2], [AM00, Definition 2.4], [BKW11, §3.5].

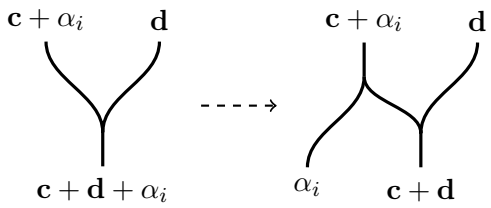
Proposition 5.9. $\deg(\mathsf{S}) = \mathrm{Deg}(\mathsf{S})$

Proof. First, we note that this formula is correct for super-standard tableaux. Indeed, adding a box of residue i changes the geometrical degree by the number of boxes in the same row with residue $i+1$ minus those with residue i (Proposition 3.3), hence equals zero if i is not the residue of the unique addable box strictly below and 1 otherwise. (Alternatively, it follows from [HM10, 5.3], since the element $C_{\mathsf{S},\mathsf{T}}$ is equal to Hu and Mathas's element $\psi_{\mathsf{S},\mathsf{T}}$ plus other homogeneous elements of the same degree.) We can easily extend this to standard tableaux as follows. Let $\theta \in A^{\underline{\nu}}$ be the element which sweeps all black strands to the far right and red strands to the left, while introducing no crossings between elements of the same color. This element is discussed more extensively in [Web10a, §3.4]. Then the claim is just [BKW11, Corollary 3.14].

Finally, we consider the passage from standard tableaux to semi-standard. We work by induction on the number of boxes in the diagram which share an entry with another box. Consider the "highest" box that shares an entry with another (the first encountered in the reading word), and consider the effect of raising by 1 the entries of all other boxes with greater or equal entries in the same alphabet. This tableau has fewer boxes that share entries, and so, by induction, its degrees $\mathrm{Deg}(\mathsf{S})$ agrees with $\deg(\mathsf{S})$. Let i be the residue of this box, and k the entry in it.

The effect that this transformation has on the degree of the associated morphism is as follows. The degree on the second part (the permutation) in Figure 4 stays the same, only the split and merge parts could change the degree; Thus,

we are left considering the transformation:



where $\mathbf{c}$ is the dimension vector of the k's in the same row as our chosen box, and $\mathbf{d}$ is the dimension vector of those in higher rows. The degrees of these morphisms are

$$d_{i-1} - d_i + \sum c_j(d_{j-1} - d_j) \quad \text{and} \quad c_{i-1} - c_i + \sum c_j(d_{j-1} - d_j)$$

Hence the difference in degree equals $c_{i-1} - c_i - d_{i-1} + d_i$.

On the other hand, consider the change in $\mathrm{Deg}(\mathsf{S})$. Nothing changes for the degrees of the boxes below our chosen one b. The degree b could be affected by every row containing k's. If that row has equal numbers of boxes labeled k with residue i and $i-1$ there are no changes; however, if there is one more with residue i than $i-1$, then an addable box of residue i was created by removing the strip consisting of all boxes whose entry was changed from k to $k+1$. If there is one more $i-1$ than i, then an addable box of residue i was destroyed. Thus, the net change is exactly $d_i - d_{i-1}$. On the other hand, for each box in the same row as our chosen one, a removable box below with residue i has been created and one with residue $i-1$ has been destroyed. Thus, the net change from these is $c_{i-1} - c_i$. Thus, we arrive at the same difference, and by induction, the proposition is proved. □

5.1. **The proof of cellularity.** We will organize our proof into several chunks. The first step only concerns the tensor product algebras $T^{\underline{\boldsymbol{\nu}}}$ and extends Hu and Mathas' basis from [HM10] to these algebras, the second proves that the proposed cellular basis for the Schur algebra spans, the four shows that it is indeed a cellular basis and the final fourth step compares the ideals with the Dipper-James-Mathas cellular ideals.

Step I: the case of T. We start with the following easy observation

Lemma 5.10. $e_T C_{\mathsf{S},\mathsf{T}} e_T = \begin{cases} C_{\mathsf{S},\mathsf{T}} & \text{if } \mathsf{S} \text{ and } \mathsf{T} \text{ are standard} \\ 0 & \text{otherwise} \end{cases}$

Proof. Recall from (25) that e_T corresponds to the vector compositions of complete flag type, which in turn correspond precisely to the standard tableaux. □

Thus, Theorem 5.7 would imply that the elements $C_{\mathsf{S},\mathsf{T}}$ for standard tableaux give a basis of $T^{\underline{\boldsymbol{\nu}}}$ from Proposition 4.11. However we will prove this result first as a stepping stone to the full proof. Let N denote the set of standard tableaux.

Theorem 5.11. *The data $(\Lambda, N, C|_N, *)$ defines a cell datum for $T^{\underline{\boldsymbol{\nu}}}$.*

Proof. By restricting our construction further to tableaux ν (instead of multi-tableaux) and only using the last copy of $\mathbb{Z}$ we obtain an algebra

$$T^{\nu} \subset T^{\underline{\boldsymbol{\nu}}} \tag{26}$$

with a distinguished basis which is precisely the cellular basis from [HM10].

If one fixes the idempotents $\grave{\mu}_{\mathsf{S}}$ and $\grave{\mu}_{\mathsf{T}}$, that is, fixes the multiset of entries in the tableaux, then the map $(\mathsf{S},\mathsf{T}) \mapsto (\mathsf{S}^\diamond,\mathsf{T}^\diamond)$ is injective (because of our no gap condition). Since we already know from [HM10] that the $C_{\mathsf{S},\mathsf{T}}$ for pairs of tableaux which only use the last copy of $\mathbb{Z}$ are linearly independent, the map given by $a \mapsto \theta a \theta^*$ is injective, and the elements $C_{\mathsf{S},\mathsf{T}}$ for standard tableaux are linearly independent. They are thus a basis by the dimension formula from [Web10a, 3.6]. This establishes conditions (1) and (2) of cellularity, and (3) is obvious from the definition as mentioned above.

Finally to see (4), let $x \in T^{\underline{\nu}}$ be an arbitrary element. Consider xB_{S}; since the $C_{\mathsf{S},\mathsf{T}}$'s are a basis we must have $xB_{\mathsf{S}}^* = \sum_{\mathsf{S}',\mathsf{T}'} r_x(\mathsf{S}',\mathsf{S},\mathsf{T}) C_{\mathsf{S}',\mathsf{T}'}$ for some uniquely defined coefficients $r_x(\mathsf{S}',\mathsf{S},\mathsf{T}')$. For this coefficient to be not 0 we must have $\grave{\mu}'_{\mathsf{T}} = \grave{\lambda}_{\mathsf{S}}$; in other words, T' is a semi-standard tableaux whose type is the shape of S.

Thus, either T' is the ground state tableau T^{S} of the shape of S (by which we mean the tableau where the entries are just the row numbers) or the shape of T' is above that of S in dominance order of ℓ-multipartitions, hence contained in $H(\geq \mu)$, where μ is the shape of S. If we let $r_x(\mathsf{S}',\mathsf{S}) = r_x(\mathsf{S}',\mathsf{S},T^{\mathsf{S}})$ then $xC_{\mathsf{S},\mathsf{T}} = xB_{\mathsf{S}}^* B_{\mathsf{T}} = \sum_{\mathsf{S}'} r_x(\mathsf{S}',\mathsf{S}) C_{\mathsf{S}',T^{\mathsf{S}}} B_{\mathsf{T}}$. Note that $C_{\mathsf{S}',T^{\mathsf{S}}} B_{\mathsf{T}} = B_{\mathsf{S}}'^* B_{T^{\mathsf{S}}} B_{\mathsf{T}} = B_{\mathsf{S}}'^* B_{\mathsf{T}} = C_{\mathsf{S}',\mathsf{T}}$ since the shape of T and S' equal the type of T^{S} and $B_{T^{\mathsf{S}}}$ is then just the identity. Hence we have precisely condition (4). □

Corollary 5.12. $(\Lambda, N, C|_N, *, \deg)$ *defines a graded cellular algebra.*

Proof. By definition $\deg(C_{\mathsf{S},\mathsf{T}}) = \deg(B_{\mathsf{S}}) + \deg(B_{\mathsf{T}})$. □

Step 2: signed permutation modules and spanning set. Let now n be a fixed natural number and let

$$R_n = \bigoplus_{|\mathbf{d}|=n} R_{\mathbf{d}}$$

be the subalgebra of the quiver Hecke algebra of diagrams with n strands. The summand T_n^ν of the algebra T^ν from (26) corresponding to partitions of n is then a cyclotomic quotient R_n^ν of R_n.

In [BK09a, (4.36)], an isomorphism between R_n^ν and a cyclotomic Hecke algebra H_n^ν of S_n with parameters $(\zeta, \zeta^{\mathfrak{z}_1}, \dots, \zeta^{\mathfrak{z}_\ell})$ was established, where ζ is an element of the separable algebraic closure of $\Bbbk$ which satisfies $\zeta + \zeta^2 + \dots + \zeta^{e-1}$ and e is the smallest integer where this holds.

This isomorphism $R_n^\nu \cong H_n^\nu$ from [BK09a, (4.36)] depends on the choice of certain polynomials $Q_r(\mathbf{i})$ which we want to fix now. It will turn out to be convenient not to follow the suggested choice of [BK09a, (4.36)], but instead fix

$$Q_r(\mathbf{i})(y_r, y_{r+1}) = \begin{cases} 1 - \zeta + \zeta y_{r+1} - y_r & i_r = i_{r-1} \\ \frac{P_r(\mathbf{i})-1}{y_r - y_{r+1}} & i_r = i_{r+1}+1 \\ P_r(\mathbf{i}) - 1 & i_r \neq i_{r+1}, i_{r+1}+1 \end{cases} \tag{27}$$

with the notations from [BK09a, (4.27)]. The equation [BK09a, (4.28)] implies that $(1 - P_r(\mathbf{i}))^{s_r} = \zeta + P_r(\mathbf{i})$, so the equations [BK09a, (4.33-35)] follow immediately and this choice indeed defines an isomorphism of algebras

$$R_n^\nu \cong H_n^\nu \tag{28}$$

In the following we will identify

$$T_n^\nu = R_n^\nu = H_n^\nu \qquad T^\nu = R^\nu := \bigoplus_{n\geq 0} R_n^\nu = \bigoplus_{n\geq 0} H_n^\nu =: H^\nu \tag{29}$$

Analogously, we will write $A_n^{\underline{\boldsymbol{\nu}}} = \bigoplus_{|\mathbf{d}|=n} A_{\mathbf{d}}^{\underline{\boldsymbol{\nu}}}$.

In [Web10a, §5.3] the isomorphism 28 was extended to an isomorphism

$$T_n^{\underline{\boldsymbol{\nu}}} \;\cong\; \mathrm{End}_{H_n^{\underline{\boldsymbol{\nu}}}}\left(\bigoplus_{\sum_{i=1}^{\ell} a_i = n} H_n^\nu u_{\mathbf{a}}^+\right) \tag{30}$$

where $u_{\mathbf{a}}^+$, for $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$, is the element defined by Dipper, James and Mathas, [DJM98, Definition 3.1], as

$$u_{\mathbf{a}}^+ = \prod_{s=1}^{\ell} \prod_{k=1}^{a_k} (L_k - \zeta^{\mathfrak{d}_s}).$$

For an ℓ-multicomposition ξ, we let $\mathbf{a}_\xi = \big(0, |\xi^{(1)}|, |\xi^{(1)}| + |\xi^{(2)}|, \dots, |\xi^{(1)}| + \cdots + |\xi^{(\ell-1)}|\big)$ and, following [DJM98, (3.2)(ii)], will assume from now on that

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell \leq n.$$

Thus, we can describe the cyclotomic Schur algebra $\mathscr{H}_n^{\underline{\boldsymbol{\nu}}}$ from the introduction (for the parameters $(q, Q_1, Q_2, \dots Q_\ell) := (\zeta, \zeta^{\mathfrak{d}_1}, \dots, \zeta^{\mathfrak{d}_\ell})$) as the endomorphism algebra of the $T^{\underline{\boldsymbol{\nu}}}$ module

$$\mathrm{Hom}_{H_n^{\underline{\boldsymbol{\nu}}}}\Big(\bigoplus_{|\mathbf{a}|=n} H_n^{\underline{\boldsymbol{\nu}}} u_{\mathbf{a}}^+, \bigoplus_{\xi} H_n^{\underline{\boldsymbol{\nu}}} u_\xi^+ x_\xi\Big) = \bigoplus_{|\xi|=n} T^{\underline{\boldsymbol{\nu}}} x_\xi \tag{31}$$

where x_ξ is the projection to vectors which transform under the sign representation for the Young subgroup S_ξ. We refer to the modules $T^{\underline{\boldsymbol{\nu}}} x_\xi$ as **signed permutation modules** for $T^{\underline{\boldsymbol{\nu}}}$. Using the cellular basis of $\mathscr{H}_n^{\underline{\boldsymbol{\nu}}}$, one can easily deduce, see [DJM98, (4.14)], that

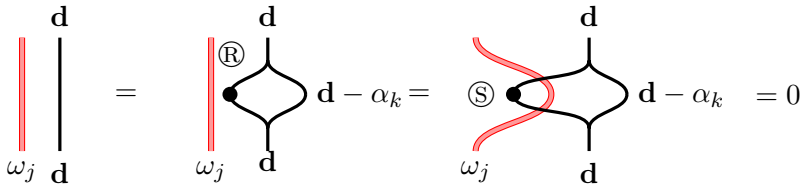
Lemma 5.13. *The dimension of $T^{\underline{\boldsymbol{\nu}}} x_\xi$ is precisely the number of pairs (S, T) of tableaux on ℓ-multipartitions with n boxes with the same shape satisfying*

- T *is of type ξ (i.e. the number of occurrences of i_j is the length of the ith row in the jth diagram of ξ), and*
- S *is standard.*

Step 3: graded cellular basis. We start with some preparatory lemmata. For $j \in \mathbb{Z}/e\mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we let $e_{j;n}$ be the idempotent for the vector composition $\hat{\boldsymbol{\mu}}_{j;n} = (\alpha_j, \alpha_{j+1}, \dots, \alpha_{j+n})$, and let $\mathbf{d}_{j;n}$ be its dimension vector. Note that for fixed j and varying n such dimension vectors $\mathbf{d}$ are characterized by $d_k + 1 \geq d_{k-1} \geq d_k$ for $k \neq j$ and $d_{j-1} \leq d_j \leq d_{j-1} + 1$.

Lemma 5.14. *Let $j \in \mathbb{Z}/e\mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Let $e_{(\mathbf{d})}$ be an idempotent corresponding to a vector composition corresponding to step 1 flags. In A^{ω_j}, we have $e_{(\mathbf{d})} = 0$ unless $\mathbf{d} = \mathbf{d}_{j;n}$ for some n. Furthermore, $e_{(\mathbf{d}_{j;n})} A^{\omega_j} e_{(\mathbf{d}_{j;n})} \cong \Bbbk$*

Proof. Assume $\mathbf{d} \neq \mathbf{d}_{j;n}$ for any n. If there exists $k \neq j$ such that $d_k > d_{k-1}$ or $d_k > d_{k-1} + 1$ for $k = j$ then set $R = d_k - d_{k-1} - 1$. In either case we claim that



where the labels Ⓡ and Ⓢ with $S = R - \delta_{j,k}$ denote the number of dots on the strand (following Remark 3.7). Since the two cases correspond to $S = R \geq 0$ respectively $S + 1 = R \geq 1$, the last equality holds. The second relation is [Web10a, (2.2)] with Proposition 4.9. To see the first equality in case $k \neq j$ note that the second diagram corresponds to

$$f \mapsto X(d_{k-1}, d_k)(f) := \Delta_w(\prod_{r=1}^{d_{k-1}} (x_{k,1} - x_{k-1,r}) x_{k,1}^R f), \tag{32}$$

with $w = s_{d_{k-1}} \cdots s_2 s_1$, where s_t denotes the transposition swapping the variables $x_{k,t}$ and $x_{k,t+1}$; hence it is enough to see that $X(d_{k-1}, d_k) = \mathrm{id}$ or even $X(d_{k-1}, d_k)(1) = 1$. If $d_{k-1} = 1, 0$ this is easily verified, and so we proceed by induction. Using formula (20) we obtain

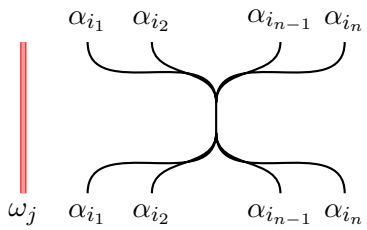
$$\begin{aligned} X(d_{k-1}, d_k) &= \Delta_w(\prod_{r=1}^{d_{k-1}-1} (x_{k,1} - x_{k-1,r}) x_{k,1}^R) \\ &\quad + \Delta_{ws_1}(\prod_{r=1}^{d_{k-1}-1} (x_{k,2} - x_{k-1,r}) x_{k,1}^R) \Delta_{s_1}(x_{3,1} - x_{2,d_{k-1}}) \\ &= 0 + 1 = 1 \end{aligned}$$

using twice the induction hypothesis and the fact $\Delta_{s_1}(x_{3,1} - x_{2,d_{k-1}}) = 1$. In case $k = j$, the split in the diagram (just an inclusion) is followed by multiplication with $x_{j,1}^R$ and the operator Δ_w for $w = s_{d_{j-1}} \cdots s_2 s_1$. Now if $d_k \geq d_{k-1}$ for all $k \neq j$ and $d_j \leq d_{j-1} + 1$ we automatically have $d_k > d_{k-1} - 1$ and $d_j < d_{j-1}$. Hence $e_{(\mathbf{d})} A^{\omega_j} e_{(\mathbf{d})} \neq 0$ then $\mathbf{d} = (\mathbf{d}_{j;n})$ for some j, n.

In this case all elements $e_{(\mathbf{d}_{j;n})} x e_{(\mathbf{d}_{j;n})}$ where x is of the form $\grave{\mu} \overset{w;h}{\Longrightarrow} \grave{\nu}$ (as in Proposition 4.6) and $w \neq \mathrm{id}$ or $w = \mathrm{id}$ and $h \neq 1$ can be written in terms of similar diagrams where a single strand is pulled off, so they are again zero and only the scalars remain. □

Now, we consider the element $t^{(j;n)} := (\prod_{k=1}^{\lfloor n/e \rfloor} x_{j-1,k}) e_{\hat{\boldsymbol{\mu}}_{j;n}}$, that is, $e_{\hat{\boldsymbol{\mu}}_{j;n}}$ with a dot on every strand labeled by $j - 1$ and no dots on any others, see Figure 2. This element plays an important role in the basis of Hu and Mathas, [HM10].

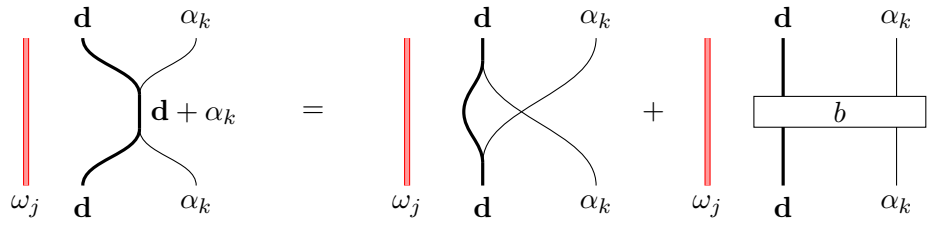
Lemma 5.15. *Let $\alpha_{i_1}, \alpha_{i_2}, \dots \alpha_{i_n}$ be an arbitrary sequence of simple roots. Then*



$$= \begin{cases} \mp t^{(j;n)} & \text{if } \alpha_{i_g} = \alpha_{j-g+1} \text{ for all } g, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The fact that if $\alpha_{i_g} \neq \alpha_{j-g+1}$ for some g then the element is 0 follows easily from Lemma 5.14. Now let $\mathbf{d} = \mathbf{d}_{j;n}$, and $k = j + n$. The proof is then by induction on n. When $n < e$ the claim follows directly from Lemma 3.4. Otherwise, we claim that

(33)



for some polynomial $b \in \mathbf{\Lambda}(\mathbf{d}, \alpha_k)$. To see this let G and F denote the first and second morphism. Abbreviating $a = d_k$ and setting $E = \prod_{m=1}^{a}(x_{k+1,m} - x_{k,a+1})$ we have

$$G(f) = \begin{cases} E\Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}\Delta_a^k(f) & \text{if } d_k - d_{k+1} = 1 \\ E\Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}(f) & \text{if } d_k = d_{k-1}, \end{cases} \tag{34}$$

where Δ_i^k denotes the Demazure operator involving the variables $x_{k,i}, x_{k,i+1}$. Note that F is a composition of first multiplying with an Euler class E_1, followed by a merge M_1, an inclusion and finally a merge M_2 as follows

$$F = (\mathbf{d}, \alpha_k) \to (\mathbf{d} - \alpha_k, \alpha_k, \alpha_k) \to (\mathbf{d} - \alpha_k, 2\alpha_k) \to (\mathbf{d} - \alpha_k, \alpha_k, \alpha_k) \to (\mathbf{d}, \alpha_k)(1)$$

so we get

$$F(f) = \begin{cases} \Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}\Delta_a^k(E_1 f) & \text{if } d_k - d_{k+1} = 1 \\ \Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}(E_1 f) & \text{if } d_k = d_{k-1}. \end{cases}$$

where $E_1 = \prod_{m=1}^{a}(x_{k+1,m} - x_{k,a})$. The latter is a polynomial in $x_{k,a}$ of degree a. Using (20) we obtain

$$F(f) = \begin{cases} \Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}\Delta_a^k(E_1)f + G(f) & \text{if } d_k - d_{k+1} = 1 \\ \Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}(E_1)f + G(f) & \text{if } d_k = d_{k-1}. \end{cases} \tag{35}$$

since all the other terms vanish. Hence the claim follows with $b = F(1)$. By an easy induction one can show

$$\Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}\Delta_a^k(x_{k,a}^n) = \begin{cases} 1 & a = n \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

and so we are only interested in the leading term $(-1)^a x_{k,a}^a$ of E_1. On the other hand, again an easy induction shows that

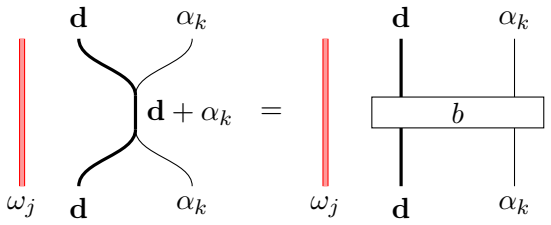
$$\Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}(x_{k,a}^n) \quad = \quad (-1)^{a-1}(e_1(k+1,a)-e_1(k,a)), \tag{37}$$

where $e_1(p,a)$ denotes the first elementary symmetric polynomial in the variable $x_{p,q}$, $1\le q\le a$. Altogether we obtain

$$b = F(1) \quad = \quad \begin{cases} (-1)^a & \text{if } d_k - d_{k+1} = 1 \\ e_1(k+1,a)-e_1(k,a) & \text{if } d_k = d_{k-1}, \end{cases} \tag{38}$$

Note that $d_k = d_{k-1}$ if and only if $k+1=j$.

So far we did not use the cyclotomic condition, which gives the even stronger relation

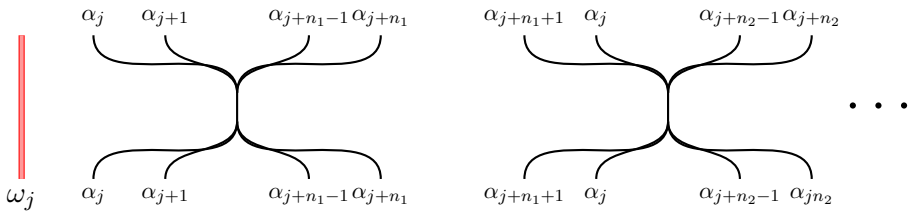


(39)

by noting that the middle term in (33) is in fact zero by Lemma 5.15 (since the idempotent after splitting of the two α_k causes it to vanish). Since all positive degree endomorphisms of $e_{(\mathbf{d})}$ are 0, we have $e_1(j,a)-e_1(j-1,a)e_{\mathbf{d},\alpha_k} = x_{j-1;a}e_{\mathbf{d},\alpha_j}$ and the Lemma follows therefore by induction □

Given an vector composition $\hat{\boldsymbol{\mu}}$, there is an induced usual composition of length r given by $\mathfrak{c}(\hat{\boldsymbol{\mu}}) = \{\sum_{j=1}^{e}\mu[-,j]\}$. These compositions are endowed with the usual lexicographic order on parts (note, this is a refinement of dominance order). We can define a filtration of A^{ω_j} by letting $A^{\omega_j}_{>\mathfrak{c}}$ be the two-sided ideal generated by $e_{\hat{\boldsymbol{\mu}}}$ for $\mathfrak{c}(\hat{\boldsymbol{\mu}}) > \mathfrak{c}$ in lexicographic order.

Lemma 5.16. *For a composition $\mathfrak{n} = (n_1, n_2, \dots)$, we have*



$$\equiv \quad \pm t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\cdots \pmod{A^{\omega_j}_{>\mathfrak{c}(\mathbf{d}_{j;n_1},\mathbf{d}_{j+1;n_2},\dots)}}$$

Proof. Obviously, if the composition $\mathfrak{n}$ has 1 part, then we are done by Lemma 5.15. We assume we have $k+1$ parts and the statement is true for k parts. Thus the displayed diagram is equivalent to

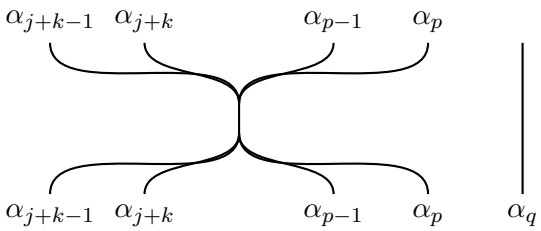
$$t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\cdots|t^{(j+k-1;n_k)}|m$$

where m is the last of the double headed pitchforks, modulo $I := A^{\omega_j}_{>\mathfrak{c}}$.

By Lemma 5.15, this is the same as the desired element, plus

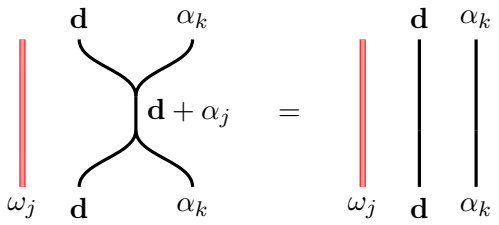
$$t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\dots|t^{(j+k-1;n_k)}|m'$$

where m' lies in the cyclotomic ideal for $A^{\omega_{j+k}}$. That is, it can be written so that every diagram in it has a strand at the left labeled with a single root α_q, which in addition carries a dot if that root is α_{j+k-1}, hence we have for instance a situation

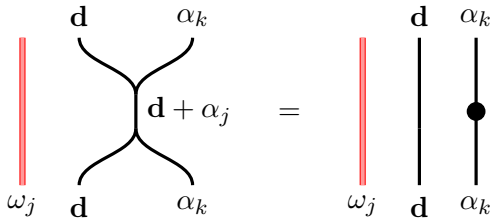


In order to complete the induction, it is sufficient to show that this can be written as a sum of elements each of which factors through the join of all strands (and hence is contained in I) or is contained in the cyclotomic ideal $A^{\omega_{j+k-1}}$ (and hence we can repeat our argument until we finally obtain only elements in I or in the cyclotomic ideal J for A^{ω_j}.)

If $q \not\equiv p+1 \mod e$, then the diagram above is already in J by Lemma 5.14. If $q \equiv p+1 \not\equiv j+k-1 \mod e$ then by (39) we have



If $k = p+n+1 = p+1 \mod e$ then



since the middle term of (33) vanishes (consider the idempotent after splitting of the two α_k and apply Lemma 5.15). Hence our element factors through the join of all strands and hence lies in $A^{\omega_j}_{>\mathfrak{c}}$. The lemma follows. □

Proposition 5.17. *The vectors $C_{\mathsf{S},\mathsf{T}}$ span $A^{\underline{\nu}}$.*

Proof. By Proposition 4.6, we need only show that vectors of the form

$$\grave{\mu}_1 \overset{w_1^{-1};h_1}{\Longrightarrow} \grave{\lambda} \overset{w_2;h_2}{\Longrightarrow} \grave{\mu}_2$$

can be written in terms of the $C_{\mathsf{S},\mathsf{T}}$'s. As usual, we induct on $\mathfrak{c}(\grave{\lambda})$.

By Lemma 5.16, if $\grave{\lambda}$ is not of the form $\grave{\lambda}_{\mathsf{S}}$ for some semi-standard tableaux, then we can rewrite our vector to factor through $\grave{\eta}$ which is higher in lexicographic order. Thus, we need only consider show that elements of the form

$$\grave{\mu}_1 \overset{w_1^{-1};h_1}{\Longrightarrow} \grave{\lambda}_{\mathsf{S}} \overset{w_2;h_2}{\Longrightarrow} \grave{\mu}_2$$

can be written in terms of the $C_{\mathsf{S},\mathsf{T}}$'s (which are the special case where $h_1 = h_2 = 1$). At the center of this diagram, the picture looks precisely like that shown in the statement of Lemma 5.16, except that some of the legs may not split all the way down to unit vectors. We assume that we add the action of h_1 at the bottom of that portion of the diagram, which is after applying all crossings coming from w_1. By Lemma 5.16, the central portion of the diagram lies in $A_{>\mathfrak{c}(\grave{\lambda})}$, and so by induction, this element can be written as a linear combination of $C_{\mathsf{S},\mathsf{T}}$'s. By induction, the result follows. □

Given a semi-standard tableau S, we can associate a standard tableaux S° of the same shape with the following properties:

- Any box contains labels from the same alphabets in S and S°.
- Any pair of boxes with different entries in S has entries in S° in the same order.
- The row reading word of S° is maximal in Bruhat order amongst the standard tableau satisfying the first 2 conditions.

While the map $\mathsf{S} \mapsto \mathsf{S}^\circ$ is obviously not injective, it is injective on the set of tableaux with a fixed type.

To $\grave{\mu}$ we associate a vector $\varphi_{\grave{\mu}}$ of $A^{\underline{\boldsymbol{\nu}}}e_{\grave{\mu}}$ defined as follows: let K be the set of vector compositions whose corresponding flags are complete refinements of that for $\grave{\mu}$. For each $\grave{\lambda} \in K$, there is a diagram $\grave{\mu} \longrightarrow \grave{\lambda}$ which looks like a bunch of chicken feet where the top is labeled with the sequence $\hat{\boldsymbol{\lambda}}$.

Definition 5.18. *We let $\varphi_{\grave{\mu}} = \sum_K \grave{\mu} \longrightarrow \grave{\lambda}$ and $\varphi = \sum_{\grave{\mu}} \varphi_{\grave{\mu}}$. We call them* **chicken feet** *vectors and their duals $\varphi_{\grave{\mu}}*$, φ^** **pitchfork vectors**.

Now, consider the map $A^{\underline{\boldsymbol{\nu}}} \to T^{\underline{\boldsymbol{\nu}}}$, $a \mapsto \varphi a \varphi^*$. This map is obviously not injective, but it is on $e_{\mathfrak{n}} A^{\underline{\boldsymbol{\nu}}} e_{\mathfrak{m}}$ for a fixed pair of compositions $\mathfrak{n}, \mathfrak{m}$.

Lemma 5.19. *For all S, T, we have*

$$\varphi C_{\mathsf{S},\mathsf{T}} \varphi^* = C_{\mathsf{S}^\circ,\mathsf{T}^\circ} + \sum_{\substack{\mathsf{S}'<\mathsf{S}^\circ \\ \mathsf{T}'<\mathsf{T}^\circ}} a_{\mathsf{S}',\mathsf{T}'} C_{\mathsf{S}',\mathsf{T}'} \pmod{A^{\underline{\boldsymbol{\nu}}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}}. \tag{40}$$

In particular, the elements $C_{\mathsf{S},\mathsf{T}}$ are all linearly independent.

Proof. First, we note that

$$\varphi B^*_{\mathsf{S}} = B^*_{\mathsf{S}^\circ} + \sum_{\mathsf{S}'<\mathsf{S}} a_{\mathsf{S}'} B^*_{\mathsf{S}'} \pmod{A^{\underline{\boldsymbol{\nu}}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}}.$$

The first term of the RHS comes from the term $\grave{\mu}_{\mathsf{S}^\circ} \longrightarrow \grave{\mu}_{\mathsf{S}} \cdot B^*_{\mathsf{S}} = B_{\mathsf{S}^\circ}$ of the product on the LHS; for any other pitchfork terms, we will either have a standard tableaux where $\ell(w_{\mathsf{S}'}) < \ell(w_{\mathsf{S}^\circ})$ (by assumption), or a non-standard tableau, in which case the term lies in $A^{\underline{\boldsymbol{\nu}}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}$. Thus, the equality follows. If we have a non-trivial relation between $C_{\mathsf{S},\mathsf{T}}$'s, then by multiplying by idempotents $e_{\mathfrak{n}}$ on the left and right, we may assume that all tableaux which appear are of the same

type. Since the $a \mapsto \varphi a \varphi^*$ is injective on such elements, we have a non-trivial relation between the right hand sides of (40); which is impossible because of the upper-triangularity in (40) and since the vectors $C_{\mathsf{S}^\circ,\mathsf{T}^\circ}$ are linearly independent modulo the image of $A^{\underline{\boldsymbol{\nu}}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}$ by Theorem 5.11. □

Proof of Theorem 5.7. We must check the conditions of a graded cell datum for $(\Lambda, M, C, *)$.

(1) Clear.
(2) This is the claim that the vectors $C_{\mathsf{S},\mathsf{T}}$ are a basis. They span by Lemma 5.17 and are linearly independent by Lemma 5.19.
(3) By definition
$$C^*_{\mathsf{S},\mathsf{T}} = (B^*_{\mathsf{S}} B_{\mathsf{T}})^* = B^*_{\mathsf{T}} B_{\mathsf{S}} = C_{\mathsf{T},\mathsf{S}}.$$
(4) This is essentially identical to the proof of Theorem 5.11. Consider S of shape ξ. Since the $C_{\mathsf{S},\mathsf{T}}$ are a basis,
$$x B^*_{\mathsf{S}} = \sum_{\mathsf{S}',\mathsf{T}} r_x(\mathsf{S},\mathsf{S}',\mathsf{T}) C_{\mathsf{S}',\mathsf{T}}$$
for some coefficients $r_x(\mathsf{S},\mathsf{S}',\mathsf{T})$. Since T must be a semi-standard tableau of type ξ, we must have that the shape of T is above ξ in dominance order, unless T is the super-standard tableau. So,
$$x B^*_{\mathsf{S}} = \sum_{\mathsf{S}'} r_x(\mathsf{S},\mathsf{S}') B^*_{\mathsf{S}'} \pmod{A^{\underline{\boldsymbol{\nu}}}(> \xi)}.$$
(5) The degree function deg obviously satisfies the required conditions.

□

6. Dipper-James-Mathas cellular ideals and graded Weyl modules

Let again $\underline{\boldsymbol{\nu}}$ be of the form $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}1}, \ldots, \omega_{\mathfrak{z}\ell})$; recall that from the introduction that having fixed ζ, a primitive nth root of unity in $\Bbbk$, we have an induced choice of parameters for the cyclotomic Hecke algebra, given by $q = \zeta, Q_i = \zeta^{\mathfrak{z}_I}$.

Let V be a representation of $H^\nu_n = T^\nu_n$ from (28). Recall, [BK09a, 4.1], that H^ν_n contains the finite dimensional Iwahori-Hecke algebra as a natural subalgebra. A vector $v \in V$ *generates a sign representation* if it generates a 1-dimensional sign representation for this finite Hecke algebra (i.e. $T_i v = -v$ for all generators T_i in the notation of [BK09a]). We first express this condition in the standard generators ψ_r of $R^{\underline{\boldsymbol{\nu}}}_n$

Proposition 6.1. *A vector $v \in V$ generates a sign representation if and only if it transforms under the action of ψ_r by*
$$\psi_r e_{\mathbf{i}} v = \begin{cases} 0 & i_r = i_{r+1} \\ (y_r - y_{r+1}) e_{\mathbf{i}^{s_r}} v & i_r = i_{r+1} + 1 \\ e_{\mathbf{i}^{s_r}} v & i_r \neq i_{r+1}, i_{r+1}+1 \end{cases}$$

Proof. This is immediate by plugging the choices (27) into [BK09a, (4.38)] using the fact that $T_i v = -v$: the first case follows since in this case $P_r(\mathbf{i}) = 1$, so $(T_r + P_r(\mathbf{i}))e(\mathbf{i})v = 0$; the last two follow immediately from the substitution of -1 for T_i. □

It might seem strange that we use anti-invariant vectors for the this proposition, instead of invariant. This could be fixed by picking a different isomorphism to the Hecke algebra, but it is "hard-coded" into the isomorphism chosen in [BK09a].

Proposition 6.2. *The chicken feet vector $\varphi_{\grave{\mu}}$ transforms according to the sign representation for the Young subgroup $S_{\mathfrak{c}(\grave{\mu})}$.*

Proof. This follows directly from the definition [BK09a, (4.38)], using Proposition 3.4 and Proposition 6.1. □

We can consider $A^{\underline{\boldsymbol{\nu}}}e_T$ as a $T^{\underline{\boldsymbol{\nu}}}$-representation (acting from the right). For each ℓ-multicomposition ξ, we let $\varphi_\xi = \sum_{\mathfrak{c}(\grave{\mu})=\xi} \varphi_{\grave{\mu}}$, and let $e_\xi = \sum_{\mathfrak{c}(\grave{\mu})=\xi} e_{\grave{\mu}}$. Then φ_ξ is a vector in $e_\xi A^{\underline{\boldsymbol{\nu}}}e_T$ which generates a sign representation for S_ξ. Thus, it induces a map $h_\xi \colon x_\xi T^\nu \to e_\xi A^{\underline{\boldsymbol{\nu}}}e_T$, by $x_\xi a \mapsto \varphi_\xi a$.

Theorem 6.3. *The map h_ξ is an isomorphism. In particular, $A^{\underline{\boldsymbol{\nu}}}e_T \cong \bigoplus_\xi x_\xi T^{\underline{\boldsymbol{\nu}}}$ as right $T^{\underline{\boldsymbol{\nu}}}$−modules, and there is an isomorphism $\Phi^{\underline{\boldsymbol{\nu}}}\colon A^{\underline{\boldsymbol{\nu}}} \to \mathscr{H}^{\underline{\boldsymbol{\nu}}}$. On the cyclotomic q-Schur algebra of rank n, we obtain an induced isomorphism*

$$\Phi^{\boldsymbol{\nu}}_n\colon \quad A^{\boldsymbol{\nu}}_n \to \mathbf{S}(n; q, Q_1, \dots, Q_\ell).$$

Proof. The surjectivity of h_ξ is clear; every element of $e_T A^{\underline{\boldsymbol{\nu}}}e_\xi$ is of the form

$$\sum a_{\grave{\lambda}} \cdot (\grave{\mu} \longrightarrow \grave{\lambda}) = \sum a_{\grave{\lambda}} e_{\grave{\lambda}} \varphi_{\grave{\mu}},$$

where $\grave{\lambda}$ only contains compositions of type e which correspond to simple roots α_i, and $a_{\grave{\lambda}} \in T^{\underline{\boldsymbol{\nu}}}A^{\underline{\boldsymbol{\nu}}}e_T$.

Thus, the map is an isomorphism if and only if the spaces have the same dimension. We know that the dimension of $x_\xi T^\nu$ by Lemma 5.13 is the number of pairs of tableaux (S, T) where S is standard and T is type ξ. By Lemma 5.19, the elements $C_{\mathsf{S},\mathsf{T}}$ for the same set of pairs are linearly independent vectors in $e_T A^{\underline{\boldsymbol{\nu}}}e_\xi$, so it must have at least this dimension. Thus, h_ξ is an isomorphism. The surjective map $\Phi^{\underline{\boldsymbol{\nu}}}_n\colon A^{\underline{\boldsymbol{\nu}}}_n \to \mathbf{S}(n; q, Q_1, \dots, Q_\ell)$ arises because we always have a surjective map $A \to \mathrm{End}_{eAe}(eA)$ for any algebra A and idempotent $e \in A$, applied to $e_n e_T$. Since the vectors $C_{\mathsf{S},\mathsf{T}}$ span $A^{\underline{\boldsymbol{\nu}}}_n$, and there is the same number of them as the dimension of $\mathbf{S}(n; q, Q_1, \dots, Q_\ell)$, this map must be an isomorphism. Summing over all n, we obtain that $\Phi^{\underline{\boldsymbol{\nu}}}$ is an isomorphism. □

The following holds under the fixed isomorphism (28).

Theorem 6.4. *The cellular data from Theorem 5.7 satisfies the following:*

1. *The cellular ideals coincide with those of* [DJM98, Def. 6.7] *with respect to the lexicographic partial ordering on multipartitions.*
2. *The cell modules $W^\xi = A^{\underline{\boldsymbol{\nu}}}(\geq \xi)/A^{\underline{\boldsymbol{\nu}}}(> \xi)$ are graded lifts of the Weyl modules of $\mathscr{H}^{\underline{\boldsymbol{\nu}}}$, and the $F^\xi = W^\xi/\operatorname{rad} W^\xi$ form a complete, irredundant set of graded lifts of simple modules.*
3. *Any other graded lift of a Weyl module differs (up to isomorphism) only by an overall shift in the grading.*

Proof. The cellular ideals of either our basis or the Dipper-James-Mathas basis can be defined in terms of maps between permutation modules factoring through those greater in lexicographic order. That is, identifying $e_\xi \mathscr{H}^{\underline{\boldsymbol{\nu}}}e_{\xi'}$ with

$\mathrm{Hom}(\mathscr{H}^{\underline{\boldsymbol{\nu}}}e_\xi, \mathscr{H}^{\underline{\boldsymbol{\nu}}}e_{\xi'})$, the intersection with $e_\xi \mathscr{H}^{\underline{\boldsymbol{\nu}}}(> \vartheta)e_{\xi'}$ is just the maps factoring through $\mathscr{H}^{\underline{\boldsymbol{\nu}}}e_{\vartheta'}$ with $\vartheta' < \vartheta$ is the lexicographic order. Of course, it is clear that the same definition works for our cellular structure. The second statement is then clear and the third follows by standard arguments (e.g. [Str03, Lemma 1.5]), since the Weyl modules are the standard modules of a highest weight structure, and thus indecomposable. □

In the case where ν is itself a fundamental weight, $\mathscr{H}^\nu$ is the sum of the usual q-Schur algebras for all different ranks and except for some degenerate cases, a grading on this algebra has already been defined by Ariki.

Theorem 6.5. *The grading defined on $\mathscr{H}^\nu$ via the isomorphism Φ^ν agrees with Ariki's up to graded Morita equivalence.*

Proof. Ariki's grading is defined uniquely (up to Morita equivalence) by the fact that there is a graded version of the Schur functor compatible with the Brundan-Kleshchev grading on H^ν. The image of an indecomposable projective of the q-Schur algebra under the Schur functor is a graded lift of an indecomposable summand of a permutation module; since this object is indecomposable, its graded lift is unique up to shift by the standard argument, [Str03, Lemma 1.5]). This shows that such a grading is unique, and both Ariki's and our gradings satisfy this condition. □

7. Higher representation theory and q-Fock space

7.1. **The categorical action.** As promised in the introduction, we now draw the connection between the algebras $A^{\underline{\boldsymbol{\nu}}}$ and the theory of higher representation theory as in the work of Rouquier [Rou08] and Khovanov-Lauda [KL08]. We still assume that $\underline{\boldsymbol{\nu}}$ is of the form $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}_1}, \dots, \omega_{\mathfrak{z}_\ell})$,

We have proven that $A^{\underline{\boldsymbol{\nu}}}$ and $T^{\underline{\boldsymbol{\nu}}}$ satisfy the double centralizer property for the bimodule $A^{\underline{\boldsymbol{\nu}}}e_T$. This implies, see e.g. [MS08, Remark 2.8], that the functor $M \mapsto Me_T$ is a full and faithful functor from the category of projective $A^{\underline{\boldsymbol{\nu}}}$-modules to the category of all $T^{\underline{\boldsymbol{\nu}}}$ modules.

Consider the map $\gamma_i \colon A^{\underline{\boldsymbol{\nu}}} \to A^{\underline{\boldsymbol{\nu}}}$ defined by $a \mapsto a|e_i$ (where e_i is the idempotent corresponding to the sequence (α_i)).

Definition 7.1. *Extension and restriction of scalars along the map γ defines the i-***induction functor** $\mathfrak{F}_i$ *and the i-***restriction functor** $\mathfrak{E}_i$*,*

$$\mathfrak{F}_i, \mathfrak{E}_i : \quad A^{\underline{\boldsymbol{\nu}}}\text{−mod} \to A^{\underline{\boldsymbol{\nu}}}\text{−mod}.$$

Viewing $A^{\underline{\boldsymbol{\nu}}}$−mod as the representation category of $A^{\underline{\boldsymbol{\nu}}}$, these functors are induced by the monoidal action of A described earlier and its adjoints.

Theorem 7.2. *The isomorphism $\Phi^{\underline{\boldsymbol{\nu}}}$ intertwines $\mathfrak{F}_i$ and $\mathfrak{E}_i$ with the usual functors of i-induction and i-restriction of modules over the cyclotomic q-Schur algebra.*

In particular, these functors are biadjoint (when the grading is not considered).

Proof. This follows from the fact that $\Phi^{\underline{\boldsymbol{\nu}}}$ intertwines the map $\sum_i \gamma_i$ with the inclusion $\mathscr{H}_d^{\underline{\boldsymbol{\nu}}} \to \mathscr{H}_{d+1}^{\underline{\boldsymbol{\nu}}}$ induced by the map $H_d^\nu \to H_{d+1}^\nu$ as the inclusion of a Young subalgebra. Furthermore, the decomposition into the summands γ_i's is simply the decomposition of this functor according to eigenvalues of the deformed Jucys-Murphys element under the isomorphism (28); see [BK09a, (4.21)]. □

At the heart of higher representation theory, there is a 2-category $\mathcal{U}$ categorifying $\widehat{\mathfrak{sl}}_e$; here we use the definition of [Web10a, §1.1], which is only a minor variation on that of Rouquier [Rou08] and Khovanov-Lauda [KL08]. As before, recall that we are using the sign convention of [BK09a, (1.14)]. By [Web10a, 2.11], the category of $T^{\underline{\boldsymbol{\nu}}}$–modules carries an action of this 2-category $\mathcal{U}$; that is, there is a 2-functor from $\mathcal{U}$ to the 2-category of $\Bbbk$-linear categories sending an object λ (which means integral weight λ for $\widehat{\mathfrak{sl}}_e$) to the category $T^{\underline{\boldsymbol{\nu}}}_{\lambda}$–mod and the generating morphisms $\mathcal{E}_i : \lambda \to \lambda + \alpha_i$ and $\mathcal{F}_i : \lambda \to \lambda - \alpha_i$ to i-restriction and i-induction for cyclotomic Hecke algebras.

Theorem 7.3. *The functors $\mathfrak{E}_i$ and $\mathfrak{F}_i$ define an action of $\mathcal{U}$ on the category of representations of $A^{\underline{\boldsymbol{\nu}}}$.*

Proof. First, note that the functor $A^{\underline{\boldsymbol{\nu}}}$–mod $\to T^{\underline{\boldsymbol{\nu}}}$–mod defined by $M \mapsto e_T M$ commutes with $\mathfrak{F}_i$ and $\mathfrak{E}_i$. Furthermore, this functor is faithful on projectives since the socle of no projective is killed by e_T (this follows immediately from Theorem 5.19) and hence fully faithful by Proposition 4.11.

Thus, projective $A^{\underline{\boldsymbol{\nu}}}$-modules are equivalent to a full subcategory of all $T^{\underline{\boldsymbol{\nu}}}$-modules, which is closed under the action of $\mathcal{U}$, and thus itself carries a $\mathcal{U}$-module category structure. The action on all $A^{\underline{\boldsymbol{\nu}}}$-modules follows since this is the category of representations of $\mathbf{A}^{\underline{\boldsymbol{\nu}}}$, that is, the category of functors from $\mathbf{A}^{\underline{\boldsymbol{\nu}}}$ to finite dimensional vector spaces. More concretely, the action on projective modules and on all modules are defined by tensor product with the same bimodules, and checking that this is a $\mathcal{U}$-action involves checking equalities of bimodule morphisms, which can be done in either category. □

7.2. **Decategorification.** Now, we consider how our construction decategorifies, that is, we study its effect on the Grothendieck group. Assume again that $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}_1}, \dots, \omega_{\mathfrak{z}_\ell})$ (although all of our results can be extended to an arbitrary sequence of weights by replacing the Fock space by any highest weight representation of $U_q(\widehat{\mathfrak{gl}}_e)$). Recall the construction of the quantized level 1 Fock space from [Hay90] and [MM90] with its Bosonic realization $\mathbb{C}[q, p_1, p_2, \dots]$ via Boson-Fermion correspondence, [KR87].

Definition 7.4. *Let $\mathbf{F}_\ell$ be the ℓ-fold level 1 bosonic Fock space*

$$\mathbf{F}_\ell = (\mathbb{C}[q, p_1, p_2, \dots])^{\otimes \ell},$$

with charge $\{\mathfrak{z}_1, \dots, \mathfrak{z}_\ell\}$ equipped with its usual basis u_ξ where ξ ranges over ℓ-multipartitions and the usual inner product where the basis u_ξ is orthonormal. The elements u_ξ are called **standard basis vectors** *(and are just products of Schur functions $u_\xi = s_{\xi_1} s_{\xi_2} \cdots s_{\xi_\ell}$ in ℓ different alphabets).*

As we noted in the introduction, we are *not* considering the higher level Fock space studied by Uglov [Ugl00], but the more naive tensor product of level 1 Fock spaces. This distinction is discussed extensively in [BK09b, §3], see also Theorem 7.9. The Hayashi tensor product action of $U_q(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_\ell$ extends to an action of the generic Hall algebra U_e^- of nilpotent representations of Γ, [VV99, 6.2]. If, as in Section 2.5, $\mathbf{f}_{\mathbf{d}}$ denotes the characteristic function of the trivial $\mathbf{d}$-dimensional representation then this action is defined by

$$\mathbf{f}_{\mathbf{d}} u_\xi = \sum_{\mathrm{res}(\eta/\xi)=\mathbf{d}} q^{-m(\eta/\xi)} u_\eta$$

where η ranges over ℓ-multipartitions such that η/ξ has no two boxes in any column, and $m(\eta/\xi)$ is the sum over boxes x in η/ξ of the number of boxes with the same residue below that box which are addable in η minus the number below that box removable in ξ. (By "below" we mean as usual below in the same component or in a later component.)

The action of $U_q(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_\ell$ is then given by the operators $\mathbf{f}_{\alpha_i}$ and their adjoints $\mathbf{e}_{\alpha_i}$ under $(-,-)$. Of course, the operators $\mathbf{e}_{\alpha_i}$ have a description just as above based on removing boxes, see [BK09b, (3.22)] for explicit formulas.

Definition 7.5. *Given a residue data $\hat{\boldsymbol{\mu}}$, define $\mathbf{f}_{\hat{\boldsymbol{\mu}}} = \mathbf{f}_{\mu^{(r)}} \cdots \mathbf{f}_{\mu^{(1)}} \in U_e^-$. The associated* **canonical basis vector** *$h_{\grave{\mu}}$ is defined as*

$$h_{\grave{\mu}} = \mathbf{f}_{\hat{\boldsymbol{\mu}}(\ell)} \cdot (h_{(\hat{\boldsymbol{\mu}}\ell-1,\dots,\hat{\boldsymbol{\mu}}(1))}) \otimes u_\emptyset)$$

We let $K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$ be the split Grothendieck group of the graded category of graded projective modules over $A^{\underline{\boldsymbol{\nu}}}$. This is a natural module over $\mathbb{Z}[q,q^{-1}]$ by the action of grading shift, where q shifts the grading down by 1. Also, it is endowed with a q-bilinear pairing by

$$([P],[P']) \quad = \quad \dim_q \left(\dot{P} \otimes_{A^{\underline{\boldsymbol{\nu}}}} P'\right). \tag{41}$$

Hereby $\dot{P}$ is P considered as a right module using the $*$-antiautomorphism and $\dim_q M = \sum \dim M_i q^i$ denotes the graded Poincare polynomial for any finite dimensional $\mathbb{Z}$-graded $A^{\underline{\boldsymbol{\nu}}}$-module $M = \oplus_{i\in Z} M_i$. We will often identify $K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$ with the Grothendieck group of all finitely generated $A^{\underline{\boldsymbol{\nu}}}$-modules or its bounded derived category. Then the pairing (41) extends to objects M and N by taking the derived tensor product $([M],[N]) = \dim_q \left(\dot{M} \otimes^{\mathbb{L}}_{A^{\underline{\boldsymbol{\nu}}}} N\right)$.

Theorem 7.6. *There is an isomorphism of $U_q(\widehat{\mathfrak{sl}}_e)$-representations $\mathbb{C}\otimes_{\mathbb{Z}} K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}}) \cong \mathbf{F}_\ell$ satisfying*

$$[A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}}] \mapsto h_{\grave{\mu}}, \qquad [W^\xi] \mapsto u_\xi,$$

and intertwining the inner product (41) with the inner product on $\mathbf{F}_\ell$.

Proof. First, note that applying the formula for the action inductively together with Proposition 5.9 we obtain

$$(h_{\grave{\mu}}, u_\xi) = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi \\ \mathrm{type}(\mathsf{S})=\grave{\mu}}} q^{-\deg(\mathsf{S})}$$

where the sum is over semistandard ℓ-multitableaux S of shape $\mathrm{sh}(\mathsf{S}) = \xi$ and type $\grave{\mu}$. Thus, we have that

$$(h_{\grave{\mu}}, h_{\grave{\lambda}}) = \sum_\xi (h_{\grave{\mu}}, u_\xi)(u_\xi, h_{\grave{\lambda}}) = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\mathrm{sh}(\mathsf{T}) \\ \mathrm{type}(\mathsf{S})=\grave{\mu} \\ \mathrm{type}(\mathsf{S})=\grave{\lambda}}} q^{-\deg(\mathsf{S})-\deg(\mathsf{T})}$$

Since $\deg(\mathsf{S}) + \deg(\mathsf{T})$ is the degree of cellular basis vector $C_{\mathsf{S},\mathsf{T}}$, this shows that

$$([A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}}], [A^{\underline{\boldsymbol{\nu}}} e_{\grave{\lambda}}]) = \dim_q e_{\grave{\mu}} A^{\underline{\boldsymbol{\nu}}} \otimes_{A^{\underline{\boldsymbol{\nu}}}} A^{\underline{\boldsymbol{\nu}}} e_{\grave{\lambda}} = \dim_q e_{\grave{\mu}} A^{\underline{\boldsymbol{\nu}}} e_{\grave{\lambda}} = (h_{\grave{\mu}}, h_{\grave{\lambda}}).$$

The form $(-,-)$ is non-degenerate on $\mathbf{F}_\ell$, hence any relation that holds in $K^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$ between the vectors $[A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}}]$ also holds between the vectors $h_{\grave{\mu}}$ in $\mathbf{F}_\ell$. Since these vectors span $\mathbf{F}_\ell$, this defines a surjective map $K^0(A^{\underline{\boldsymbol{\nu}}}) \to \mathbf{F}_\ell$. This

preserves types, in the sense that if the type of $\grave{\mu}$ is $\mathbf{d}$, then $[A^{\underline{\boldsymbol{\nu}}}e_{\grave{\mu}}]$ is sent to the span of u_ξ where ξ ranges over Young diagrams of type ξ. Since we already show that the dimension of $K^0(A^{\underline{\boldsymbol{\nu}}}_{\mathbf{d}})$ is the number of shapes with this residue, this shows that the map is an isomorphism.

Note that

$$(h_{\grave{\mu}}, [W^\xi]) = W^\xi \otimes^{\mathbb{L}}_{A^{\underline{\boldsymbol{\nu}}}} A^{\underline{\boldsymbol{\nu}}}e_{\grave{\mu}} = \dim_q W^\xi e_{\grave{\mu}} = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi \\ \mathrm{type}(\mathsf{S})=\grave{\mu}}} q^{-d^{\mathsf{S}}},$$

by Theorem 6.4, so the identification of $[W^\xi]$ with u_ξ follows by the non-degeneracy of the inner product. □

For any left $A^{\underline{\boldsymbol{\nu}}}$-module M, the space $H = \mathrm{Hom}_{A^{\underline{\boldsymbol{\nu}}}}(M, A^{\underline{\boldsymbol{\nu}}})$ is naturally a right $A^{\underline{\boldsymbol{\nu}}}$-module via the action $(f \cdot a)(m) = f(m).a$ which can be turned into a left $A^{\underline{\boldsymbol{\nu}}}$-module via $(f \cdot a)(m) = f(m).a^*$ for $f \in H, m \in M, a \in A^{\underline{\boldsymbol{\nu}}}$. Of course, this extends to the derived functor

$$\mathfrak{D} = \mathrm{RHom}_{A^{\underline{\boldsymbol{\nu}}}}(-, A^{\underline{\boldsymbol{\nu}}}) \colon \quad D^b(A^{\underline{\boldsymbol{\nu}}}\text{−mod}) \to D^b(A^{\underline{\boldsymbol{\nu}}}\text{−mod}).$$

We refer to this functor as **Serre-twisted duality**. This name can be explained by the following alternate description of the same functor. Let $\star : A^{\underline{\boldsymbol{\nu}}}\text{−mod} \to A^{\underline{\boldsymbol{\nu}}}\text{−mod}$ be the duality functor given by taking vector space dual, and then twisting the action of $A^{\underline{\boldsymbol{\nu}}}$ by the anti-automorphism $*$. Let $\mathfrak{S} : D^b(A^{\underline{\boldsymbol{\nu}}}\text{−mod}) \to D^b(A^{\underline{\boldsymbol{\nu}}}\text{−mod})$ be the graded version of the Serre functor (in the sense of [MS08]). Since $A^{\underline{\boldsymbol{\nu}}}_{\mathbf{d}}$ is finite-dimensional and has finite global dimension, this is simply derived tensor product with the bimodule $(A^{\underline{\boldsymbol{\nu}}}_{\mathbf{d}})^*$; thus, $\star \circ \mathfrak{S} = \mathfrak{D}$.

The following gives a natural construction of a bar-involution on $\mathbf{F}_\ell$ which we will show coincides with the construction in [BK09b]. Let $g \mapsto \overline{g}$ be the q-antilinear automorphism of U_e^- which fixes the standard generators $\mathbf{f}_{\mathbf{d}}$.

Theorem 7.7. *The system of maps $\Psi : \mathbf{F}_\ell \to \mathbf{F}_\ell$ for all multi-charges induced on Grothendieck groups by $\mathfrak{D}$ satisfy the relations*

(B1) $\Psi(g \cdot v) = \bar{g} \cdot \Psi(v)$ *for* $g \in U_e^-$.
(B2) $\Psi(v \otimes u_\emptyset) = \Psi(v) \otimes u_\emptyset$.
(B3) $\Psi(u_\xi) = u_\xi + \sum_{\eta<\xi} a_{\eta,\xi} u_\eta$ *for* $a_{\eta,\xi} \in \mathbb{Z}[q, q^{-1}]$

and is uniquely map characterized by the first two properties. The vectors $h_{\grave{\mu}}$ are invariant under this involution.

Proof. First, we note that $h_{\grave{\mu}}$ are invariant since for any algebra A, anti-automorphism $*$ and idempotent e, we have that $\mathfrak{D}(Ae) = Ae^*$. Of course $e^*_{\grave{\mu}} = e_{\grave{\mu}}$, so

$$\mathfrak{D}(A^{\underline{\boldsymbol{\nu}}}e_{\grave{\mu}}) \cong A^{\underline{\boldsymbol{\nu}}}e_{\grave{\mu}}.$$

For (1), it's enough to prove that it is true for a bar-invariant spanning set of U_e^-. Such a spanning set is given by the monomials $\mathbf{f}_{\hat{\boldsymbol{\mu}}}$. Since $\mathbf{f}_{\hat{\boldsymbol{\mu}}} h_{\grave{\lambda}} = h_{\hat{\boldsymbol{\mu}} \cup \grave{\lambda}}$, the result follows. For (2), it is enough to note that $h_{\grave{\mu}} \otimes u_\emptyset = h_{\grave{\mu}'}$ where $\grave{\mu}' = (\hat{\boldsymbol{\mu}}(1), \ldots, \hat{\boldsymbol{\mu}}(\ell), \emptyset)$. Since these properties determine the behavior on a spanning set, the uniquely characterize the map.

Finally, we prove (3) by induction. Since

$$h_{\grave{\lambda}_\xi} = \sum_{\mathrm{type}(\mathsf{S})=\xi} q^{-d^{\mathsf{S}}} u_{\mathrm{sh}(\mathsf{S})}$$

we have that $h_{\grave{\lambda}_\xi} = u_\xi + \sum_{\eta<\xi} b_{\eta,\xi}(q)u_\eta$ for $b_{\eta,\xi} \in \mathbb{Z}[q,q^{-1}]$. Thus, for ξ minimal, $\overline{u_\xi} = \overline{h_{\grave{\lambda}_\xi}} = u_\xi$. Now, assume this for $\eta < \xi$; thus we have that

$$\overline{u_\xi} = \overline{h_{\grave{\lambda}_\xi} - \sum_{\eta<\xi} b_{\eta,\xi}(q)u_\eta} = h_{\grave{\lambda}_\xi} - \overline{\sum_{\eta<\xi} b_{\eta,\xi}(q)u_\eta} = u_\xi + \sum_{\eta<\xi} a'_{\eta,\xi}(q)u_\eta$$

where again $a_{\eta,\xi} \in \mathbb{Z}[q,q^{-1}]$. □

Choosing integers $(\tilde{\mathfrak{z}}_1, \ldots, \tilde{\mathfrak{z}}_\ell) \in [0, e-1]^\ell$ such that $\tilde{\mathfrak{z}}_i \equiv \mathfrak{z}_i \pmod e$, one has a natural vector space isomorphism β taking standard vectors to standard vectors between our Fock space $\mathbf{F}_\ell$ and Uglov's Fock space $\tilde{\mathbf{F}}_\ell$. It is an isomorphism of U_e^--modules on the weight spaces of height $\leq m$, [BK09b, Lemma 3.20]. Using this isomorphism one can also define a bar-type involution Ψ' on the tensor product by pulling back the bar involution Ψ_U of Uglov.

Definition 7.8. *A multi-charge is* m**-dominant** *if for each* i*, we have* $\tilde{\mathfrak{z}}_i - \tilde{\mathfrak{z}}_{i+1} \geq m$.

Theorem 7.9. *If the multicharge* $(\tilde{\mathfrak{z}}_1, \ldots, \tilde{\mathfrak{z}}_\ell)$ *is* m*-dominant, then on the weight spaces of height* $\leq m$*, we have that* $\Psi = \Psi'$.

Proof. Since we have given uniquely characterizing properties (B1)-(B3) of Ψ, we need only to show that Ψ' satisfies these. Since β is an isomorphism of U_e^--modules on the weight spaces of height $\leq m$,

$$\Psi'(g\cdot v) = \beta^{-1}(\Psi_U(\beta(g.v))) = \beta^{-1}(\bar{g}\cdot(\Psi_U(\beta(v))) = \bar{g}\cdot\beta^{-1}(\Psi_U(\beta(v))) = \bar{g}\cdot\Psi'(v)$$

for weight vectors v of height $\leq m$, $g \in U_e^-$ (using [Ugl00, 3.31]); hence (B1) holds.

To see (B2) we must consider the effect of tensoring with $u_\emptyset$ in Uglov's language: we must add n new variables in our bosonic between every group of $n\ell$. The requirement that the multicharge is m-dominant assures that these are charged so that they are all to the right of any variables which are "vacant." Thus Uglov's formula [Ugl00, 3.23] for the bar involution in terms of wedges simply leaves this part of the semi-infinite wedge unchanged and thus the bar involution commutes with adding the new variables. This shows that

$$\Psi'(v \otimes u_\emptyset) = \Psi'(v) \otimes u_\emptyset$$

and completes the proof, because (B3) is clear. □

In the setup of Theorem 6.4 let P^ξ be the graded projective cover of F^ξ (which is the same as the projective cover of W^ξ). Obviously, since we have a surjective map $A^{\underline{\nu}}e_{\grave{\lambda}_\xi} \to F^\xi$, the projective P^ξ is a summand of this module with multiplicity 1. We let $p_\xi = [P^\xi]$, and $\phi_\xi = [F^\xi]$.

Theorem 7.10. *We have*

$$p_\xi = u_\xi + \sum_{\eta<\xi} e_{\xi\eta}(q)u_\eta \quad \textit{for polynomials } e_{\xi\eta}(q) \in q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}].$$

By BGG-reciprocity, we also have $u_\xi = \phi_\xi + \sum_{\xi>\eta} e_{\xi\eta}(q)\phi_\eta$*, and so the coefficients* $e_{\xi\eta}$ *are the graded decomposition numbers of* $A^{\underline{\boldsymbol{\nu}}}$*.*

Proof. Since the complementary summand of $e_{\grave{\lambda}_\xi} A^{\underline{\boldsymbol{\nu}}}$ to P_ξ is filtered by Weyl modules W^η for $\eta < \xi$, we must have an expression as above for some Laurent polynomials $e_{\xi\eta}(q)$; the coefficients of these polynomials are manifestly non-negative integral, since they are graded multiplicities of Weyl modules in the standard filtration of P^ξ.

The negativity of powers of q follows from the fact that there is a surjective map $\mathrm{Ext}^*(\mathbf{IC}_\xi, \mathbf{IC}_{\xi'}) \to \mathrm{Hom}(P^\xi, P^{\xi'})$ where $\mathbf{IC}_\xi$ is the intersection cohomology sheaf of the orbit of extended Γ-representations whose socle filtration is of type $\grave{\lambda}_\xi$. Since perverse sheaves are a t-structure, this Ext group is positively graded, so only positive shifts of Weyl modules can appear in the standard filtration of P^ξ.

Finally, the BGG-reciprocity applies, since $A^{\underline{\boldsymbol{\nu}}}$ is quasi-hereditary by [DJM98]. The graded version follows then by general arguments as for instance in [MS05, §8]. □

This property of upper-triangularity with respect to a standard basis is one of the hallmarks of a *canonical basis* (in the sense of Lusztig), the other being fixed under a bar-involution, such as the Ψ defined above. In fact, the basis $\{p_\xi\}$ does satisfy these properties and thus can be thought of a canonical basis, justifying our naming.

Theorem 7.11. *The basis* p_ξ *is the unique basis of* $\mathbf{F}_\ell$ *such that:*

- $\Psi(p_\xi) = p_\xi$ *and*
- $p_\xi = u_\xi + \sum_{\eta<\xi} e_{\xi\eta}(q)u_\eta$ *for* $e_{\xi\eta}(q) \in q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}]$.

That is, p_ξ *is the "canonical basis" of the involution* Ψ *with "dual canonical basis"* ϕ_ξ*.*

As with the bar involution Ψ, we can view this basis as a "limit" of Uglov's canonical basis G^-_* as defined in [Ugl00, 3.25], in the sense that the transition matrix from the standard basis to Uglov's basis stabilizes to the transition matrix for our basis on the weight spaces of height $\leq m$ once the multicharge is m-dominant.

Proof. We have already shown that the second property holds. For the first, we note that $\mathfrak{D}$ fixes the projective $A^{\underline{\boldsymbol{\nu}}}_{\grave{\lambda}_\xi}$, and thus fixes each of its "isotypic components" (the maximal summands of $e_{\grave{\lambda}_\xi} A^{\underline{\boldsymbol{\nu}}}$ which are direct sums of the same indecomposable projective). Since P_ξ appears with multiplicity 1, it is also invariant under $\mathfrak{D}$, so $\Psi(p_\xi) = p_\xi$.

Thus one only needs to prove uniqueness, and this holds by the usual standard arguments: if $\{p'_\xi\}$ is another such basis, then we have that

$$p_\xi - p'_\xi = \sum (e_{\xi\eta}(q) - e'_{\xi\eta}(q))u_\eta$$

is Ψ-invariant, which is impossible by Theorem 7.7(3). Finally Theorem 7.6 together with the definition (41) show that the classes of the simple objects are dual to the classes of indecomposable projectives. □

It is a general phenomenon that canonical bases appear naturally from categorifications with positivity and integrality properties. For example, in all simply-laced finite dimensional Lie algebras, Lusztig's canonical bases for tensor products of irreducible representations were shown by the second author to arise from the projectives. In the special cases [FSS11, Theorem 45, Proposition 76] explicit formulas are available which play an important role in the context of link homology theories developed therein and more generally in [Web10b, §1.3]. The general interplay between canonical bases and higher representation theory will be discussed in greater detail in forthcoming work [Web11] by the second author. Also note that our result is moreover yet another striking example of how $\widehat{\mathfrak{gl}}_e$ behaves like a Kac-Moody algebra, even though it is not covered by the theory of categorification for quantum groups as described in [KL08], [Rou08].

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