Quiver Schur algebras and q-Fock space

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Abstract. We develop a graded version of the theory of cyclotomic q-Schur algebras, in the spirit of the work of Brundan-Kleshchev on Hecke algebras and of Ariki on q-Schur algebras. As an application, we identify the coefficients of the canonical basis on a higher level Fock space with q-analogues of the decomposition numbers of cyclotomic q-Schur algebras.

We present cyclotomic q-Schur algebras as a quotient of a convolution algebra arising in the geometry of quivers - we call it **quiver Schur algebra** - and also diagrammatically, similar in flavor to a recent construction of Khovanov and Lauda. They are manifestly graded and so equip the cyclotomic q-Schur algebra with a non-obvious grading. On the way we construct a **graded cellular basis** of this algebra, resembling similar constructions for cyclotomic Hecke algebras.

The quiver Schur algebra is also interesting from the perspective of higher representation theory. The sum of Grothendieck groups of certain cyclotomic quotients is known to agree with a higher level Fock space. We show that our graded version defines a higher q-Fock space (defined as a tensor product of level 1 q-deformed Fock spaces). Under this identification, the indecomposable projective modules are identified with the canonical basis and the Weyl modules with the standard basis. This allows us to prove the already described relation between decomposition numbers and canonical bases.

Contents

1. Introduction

Recent years have seen remarkable advances in higher representation theory; the most exciting from the perspective of classical representation theory were probably the proof of Broué's conjecture for the symmetric groups by Chuang and Rouquier [CR08] and the introduction and study of graded versions of Hecke algebras by Brundan and Kleshchev [BK09a] with its Lie theoretic origins ([BK08], [BS11]). At the same time, the question of finding categorical analogues of the usual structures of Lie theory has proceeded in the work of Khovanov, Lauda, Rouquier, Vazirani, the authors and others.

In this paper, we address a question of interest from both perspectives, representation theory and higher categorical structures.

As classical (or quantum) representation theorists, we ask

Is there a natural graded version of the q-Schur algebra and its higher level analogues, the cyclotomic q-Schur algebra?

This question has been addressed already in the special case of level 1 by Ariki [Ari09]. We give here a more general construction that both illuminates connections to geometry and is more explicit. Our construction includes the case of ordinary Schur algebras, see Remark 6.4 and the explicit example at the end of the paper.

As higher representation theorists, we ask

Is there a natural categorification of q-Fock space and its higher level analogues with a categorical action of $\widehat{\mathfrak{sl}}_n$?

We will show that the above two questions not only have natural answers; they have the *same* answer.

Our main theorem is a *graded version* with a **graded cellular basis** of the cyclotomic q-Schur algebra of Dipper, James and Mathas, [DJM98] and a combinatorics of *graded decomposition numbers* using higher Fock space.

To describe the results more precisely, let $\Bbbk$ be an algebraically closed field and n, ℓ, e natural numbers with $e > 1$ (we will allow the possibility that $e = \infty$). Let $q \in \Bbbk$ be a scalar with e the smallest integer such that $1 + q + \cdots + q^{e-1} = 0$ and $(Q_1, \ldots, Q_\ell)$ an ℓ-tuple of elements of $\Bbbk$ satisfying the same equation. In particular, $Q_i = q^{\mathfrak{z}_i}$ for some $\mathfrak{z}_i \in \mathbb{Z}/e\mathbb{Z}$ (since $q^e = 1$), unless $q = 1$, in which case $\Bbbk$ has characteristic e, and we set $Q_i = \mathfrak{z}_i$. The associated **cyclotomic Hecke algebra** or **Ariki-Koike algebra**

$$\mathfrak{H}(n; q, Q_1, \ldots, Q_\ell) = \mathfrak{H}(S_n \wr \mathbb{Z}/\ell\mathbb{Z}; q, Q_1, \ldots, Q_\ell)$$

is the associative unitary $\Bbbk$-algebra with generators T_i, $1 \leq i \leq n-1$ modulo the following relations, for $1 \leq i < j-1 < n-1$,

$$(T_0 - Q_1)\cdots(T_0 - Q_\ell) = 1, \qquad (T_i - q)(T_i + 1) = 1,$$

$$T_iT_j = T_jT_i, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_0T_1T_0T_1 = T_1T_0T_1T_0.$$

The **cyclotomic q-Schur algebra** we consider is the endomorphism ring

$$\mathbf{S}(n; q, Q_1, \ldots, Q_\ell) \quad = \quad \operatorname{End}_{\mathfrak{H}(n;q,Q_1,\ldots,Q_\ell)}\left(\bigoplus_{\hat{\mu}\in\Lambda} M(\hat{\mu})\right) \tag{1.1}$$

where the sum runs over all *admissible* ℓ-multi-compositions $\hat{\mu}$ of n and $M(\hat{\mu})$ denotes the (signed) permutation module associated to $\hat{\mu}$. The admissibility condition we choose (see Lemma 5.4) defines a certain subset Λ of all ℓ-multi-compositions; in this way we pick via (1.1) a specific representative out of the family of cyclotomic q-Schur algebras from [DJM98, §6].

To make contact with the theory of quiver Hecke algebras, we encode the parameters $(q, Q_1, \ldots, Q_\ell)$ as a sequence $\underline{\boldsymbol{\nu}} = (w_{\mathfrak{z}_1}, \ldots, w_{\mathfrak{z}_\ell})$ of fundamental weights for the affine Lie algebra $\widehat{\mathfrak{sl}}_e$. The corresponding cyclotomic Hecke algebra only depends on the multiplicities of each charge, that is, only on the weight $\nu = \sum_i \nu_i$. Thus, we write

$$\mathbf{S}^{\underline{\nu}} = \bigoplus_{n \geq 0} \mathbf{S}(n; q, Q_1, \ldots, Q_\ell) \quad \text{resp.} \quad \mathfrak{H}^{\nu} := \bigoplus_{n \geq 0} \mathfrak{H}(n; q, Q_1, \ldots, Q_\ell) \tag{1.2}$$

for the sums of corresponding cyclotomic q-Schur and Ariki-Koike algebras of all different ranks. To this data we introduce then a certain $\mathbb{Z}$-graded algebra $A^{\underline{\nu}}$ which we call the **cyclotomic quiver Schur algebra**. The name stems from the fact that the algebra is related to the cyclotomic quiver Hecke algebras R^{ν} and their tensor product analogues $T^{\underline{\nu}}$ (defined in [Web10]) like the finite Schur algebra is to the classical Hecke algebra. Our main result (Theorem 6.2) says that $A^{\underline{\nu}}$ is a graded version of $\mathbf{S}^{\underline{\nu}}$ given by an extension of the Brundan-Kleshchev isomorphism $\Phi^{\nu} : R^{\nu} \cong \mathfrak{H}^{\nu}$ from [BK09a] between the diagrammatic cyclotomic quiver Hecke algebras R^{ν} introduced in [KL09] and the sum of cyclotomic Hecke algebra $\mathfrak{H}^{\nu}$.

Theorem A. *There is an isomorphism $\Phi^{\underline{\nu}}$ from $A^{\underline{\nu}}$ to the cyclotomic q-Schur algebra $\mathbf{S}^{\underline{\nu}}$, extending the isomorphism $\Phi^{\nu} : R^{\nu} \cong \mathfrak{H}^{\nu}$. In particular, the cyclotomic q-Schur algebra $\mathbf{S}^{\underline{\nu}}$ from* (1.2) *inherits a $\mathbb{Z}$-grading.*

Like the cyclotomic quiver Hecke algebra, the algebra $A^{\underline{\nu}}$ can be realized as a natural quotient of a geometrically defined convolution algebra. This construction is based on the geometry of quivers using a certain category of flagged nilpotent representations (quiver partial flag varieties) of the cyclic affine type A quiver. It naturally extends the work of Varagnolo and Vasserot [VV11] and clarifies the origin of the grading.

The construction of $A^{\underline{\nu}}$ (and its summand $A^{\underline{\nu}}_n$ for fixed n) proceeds in three steps. We first define an infinite dimensional convolution algebra, which we call **quiver Schur algebra**, using flagged representations of the cyclic quiver Γ. This algebra only depends on e (and has summands depending on n). The second step is to add some extra shadow vertices to the quiver and define an convolution algebra working with flagged representations of the extended quiver Γ depending on the parameters Q_i and ℓ. Finally the last step is to pass to a certain finite dimensional quotient which is the desired algebra $A^{\underline{\nu}}$ (with the direct summand $A^{\underline{\nu}}_n$ for fixed n).

Although we focus here on the cyclotomic quotients, we want to stress that the quiver Schur algebra appears naturally in representation theory. It is, after completion, isomorphic to Vigneras' Schur algebra [Vig03] for the general linear p-adic group, see [MS14].

To make explicit calculations we describe the quiver Schur algebra algebraically by considering a faithful representation on a direct sum of polynomial rings, extending the corresponding result for quiver Hecke algebras. Moreover, we give a diagrammatical description of the algebra by extending the diagram calculus of Khovanov and Lauda [KL09]. In contrast to their work, however, we are not able to give a complete list of relations diagrammatically. Still, we have enough information to construct (signed) permutation modules for the cyclotomic Hecke algebra and show that our algebra $A^{\underline{\nu}}$ is isomorphic to the cyclotomic q-Schur algebra using the known three different description (geometric, algebraic and diagrammatical) of the quiver Hecke algebras.

Note that this can also be viewed as an extension of work of the second author [Web10, 5.31], which showed that similar diagrammatic algebras were the endomorphism algebras of some, but not all, (signed) permutation modules.

Independently, Ariki [Ari09] introduced graded q-Schur algebras and studied their permutation modules. This is a special case of our results when $\ell = 1$ (and e and n are not too small) and indeed, we show that our grading coincides with Ariki's.

In the course of the proof we also establish, similar in spirit to the arguments in [BS11], the existence of a *graded cellular basis* (Theorem 5.8) in the sense of Hu and Mathas [HM10]:

To avoid case by case arguments for the combinatorics in the special case $e = 2$ we restrict ourselves in the combinatorial part (starting in Section 5.1) of the paper and for the remaining results of the introduction to the case $e > 2$, although the main theorems extend to the case $e = 2$ as well.

Theorem B. *The cyclotomic quiver Schur algebra* $A^{\underline{\boldsymbol{\nu}}}$ *is a graded cellular algebra. Moreover,*

- *The cellular ideals coincide under* $\Phi^{\underline{\boldsymbol{\nu}}}$ *with those of Dipper-James-Mathas.*
- *The cell modules define graded lifts of the Weyl modules.*

The grading on the basis vectors comes from a degree function defined on semistandard multitableaux extending known degree functions on standard tableaux. We show that this grading comes naturally from geometry and extends the grading on the tensor algebras from [Web10].

After circulating a draft of this paper, we received a preprint of Hu and Mathas [HM11] which gives a definition of a different, but closely related algebra which they also call a quiver Schur algebra. They prove Theorems B, C and D in this context for the (special) case of a linear, rather than cyclic, quiver (that is, when $e = \infty$). In [Web10, 5.31], the second author shows that Hu and Mathas's algebra is Morita equivalent to certain tensor product algebras $T^{\boldsymbol{\lambda}}$; these algebras are, in turn, Morita equivalent to those defined here when $e = \infty$ by Proposition 4.12.

From both the geometric and the diagrammatic sides, our construction fits quite snugly inside Rouquier's program of categorical representation theory, [Rou08]:

Theorem C. *The category of graded* $A^{\underline{\boldsymbol{\nu}}}$*-modules carries a categorical action of* $U_q(\widehat{\mathfrak{sl}}_e)$ *in the sense of Rouquier. As a* $U_q(\widehat{\mathfrak{sl}}_e)$*-module, its complexified graded Grothendieck group is canonically isomorphic to the* ℓ*-fold tensor product*

$$\mathbb{F}_\ell = \mathbb{F}_1(\mathfrak{z}_\ell) \otimes \cdots \otimes \mathbb{F}_1(\mathfrak{z}_1),$$

of level 1 *(fermionic) Fock space with central charge* $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell)$ *given by* $\underline{\boldsymbol{\nu}}$*.*

Here the parameter q corresponds to the effect of grading shift on the Grothendieck group, and is thus a formal variable not a complex number. The action of the standard Chevalley generators E_i, F_i is given by i-induction and i-restriction functors. In the ungraded case these functors and their connection to undeformed Fock space were independently studied by [Wad11].

Our constructions can also be described using an affine version of the "thick calculus" introduced by Khovanov, Lauda, Mackaay and Stošić for the upper half of $\widehat{\mathfrak{sl}}_e$. Unlike in

[KLMS10] however, our category has objects which do not appear in the "thin calculus" and categorifies the upper half of $U_q(\widehat{\mathfrak{gl}}_e)$ instead of $U_q(\widehat{\mathfrak{sl}}_e)$.

It is tempting to think this could easily be extended to a graphical categorification of the Lie algebra $\widehat{\mathfrak{gl}}_e$, which is the direct sum of $\widehat{\mathfrak{sl}}_e$ and a Heisenberg Lie algebra $\mathbb{H}$ modulo an identification of their centers. This is especially intriguing and promising given the various interesting categorification of Heisenberg algebras ([CL12], [Kho10], [LS11], [LS12], [CLS12]) which appeared recently in the literature. However, the connection cannot be as straightforward as one might hope at first, since the functors associated to the standard generators in the categorification of the upper half of $U_q(\widehat{\mathfrak{gl}}_e)$ simply do not have biadjoints (unlike those in $U_q(\widehat{\mathfrak{sl}}_e)$). This was already pointed out in [Sha11, 5.1,5.2]. In their action on the categories of $A^{\underline{\nu}}$-modules, they send projectives to projectives, but not injectives to injectives, so their right adjoints are exact, but not their left adjoints.

The Grothendieck group of graded $A^{\underline{\nu}}$-modules is naturally a $\mathbb{Z}[q, q^{-1}]$-module and comes also along with several distinguished lattices and bases. To describe them combinatorially we introduce a bar-involution on the tensor product $\mathbb{F}_\ell$ of Fock spaces appearing in Theorem C which allows us to define, apart from the standard basis, two other distinguished bases: the **canonical** and **dual canonical** bases. This canonical basis is a "limit" of that for higher level q-Fock spaces defined by Uglov, [Ugl00], in a sense we describe later. Our canonical isomorphism induces correspondences

bar involution	$\longleftrightarrow$	Serre-twisted duality	
canonical basis	$\longleftrightarrow$	indecomposable projectives	$(\mathrm{char}(\Bbbk) = 0)$
dual canonical basis	$\longleftrightarrow$	simple modules	$(\mathrm{char}(\Bbbk) = 0)$
standard basis	$\longleftrightarrow$	Weyl modules	

As a consequence we get information about (graded) decomposition numbers of cyclotomic q-Schur algebras:

Theorem D (Theorem 7.20)**.** *If $\Bbbk$ has characteristic 0, the graded decomposition numbers of the cyclotomic q-Schur algebra are the coefficients of the canonical basis in terms of the standard basis on the higher level q-Fock space $\mathbb{F}_\ell$.*

Again this problem was studied independently by Ariki [Ari09] for the level $\ell = 1$ case. The above theorem combines and generalizes therefore results of Ariki on Schur algebras and Brundan-Kleshchev-Wang [BKW11] on cyclotomic Hecke algebras. It is very similar in spirit to a conjecture of Yvonne [Yvo06, 2.13]; however, there are several small differences between Yvonne's conjecture and our results. The most important is that Yvonne used Jantzen filtrations to define a q-analogue of decomposition numbers instead of a grading. This approach has been worked out in level 1 by Ram-Tingley [RT10] and Shan [Sha11]. Their results show that the same q-analogue of decomposition numbers arise from counting multiplicities with respect to depth in Jantzen filtrations. We expect that this will hold in higher level as well; it should follow from the following fact (which was conjectured in a first draft of this paper and) proved in [Mak14]:

Theorem E. *The (underlying basic algebra of) the cyclotomic quiver Schur algebra $A^{\underline{\nu}}$ is Koszul.*

We prefer working with gradings instead of filtrations, since they are easier to handle in practice. (A similar phenomenon appears for the classical category $\mathcal{O}$ for semi-simple complex Lie algebras, where the Jantzen filtration can also be described in terms of a grading, [BGS96], [Str03]. This grading is actually directly connected with the grading on the algebras $R^\nu \cong H^\nu$ in case $e = \infty$, see [BS11], [HM10]). Again, for level $\ell = 1$, the Theorem E was already known to be true, [CM12]; an elementary argument for $e = \infty$ is given in [BS10].

We should emphasize that at the moment, this approach only allows us to understand the higher-level Fock spaces which are constructed as tensor products of level 1 Fock spaces (or their irreducible $\widehat{\mathfrak{gl}}_e$ constituents). This does not include the twisted higher level Fock spaces studied by Uglov, [Ugl00], which will require a generalization of the algebras we consider here. The same Fock spaces are categorified by category $\mathcal{O}$ of certain Cherednik algebras, see e.g. [Sha11], [GL14]. The action is again given by induction and restriction functors as in [Wad11], but it remains to be clarified how our work fits into this framework.

Let us briefly summarize the paper.

- Section 2 contains preliminaries of the geometry of quiver representations needed to define the quiver Schur algebra both as a geometric convolution algebra and in terms of an action on a polynomial ring which then is related to Demazure operators in Section 3. We connect its graded Grothendieck group with the generic nilpotent Hall algebra of the cyclic quiver.
- In Section 4, we discuss a generalization of this algebra using extended (or shadowed) quiver representations that will allow us to deal with higher level Fock spaces.
- In Section 5, we define cyclotomic quotients, equip them with a (graded) cellular structure and establish the isomorphism to cyclotomic q-Schur algebras in Section 6.
- In Section 7, we consider the connection of these constructions to higher representation theory, describe the categorical action of $\widehat{\mathfrak{sl}}_e$ on these categories, and show that they categorify q-Fock spaces. In particular, we consider the relationship between projective modules, canonical bases, and decomposition numbers.

Acknowledgment: We thank Geordie Williamson, Michela Varagnolo, Peter Tingley, Iain Gordon, Michael Ehrig, Bernard Leclerc and Peter Littelmann for useful input and fruitful discussions. We are in particularly grateful to Peng Shan, Vanessa Miemitz and Daniel Tubbenhauer for their careful reading and probing questions. We also would like to thank the Hausdorff Center for supporting B.W.'s visit to Bonn at the genesis of this work, and the organizers of the Oporto Meeting on Geometry, Topology and Physics 2010 for facilitating our collaboration. B.W. was supported by an NSF Postdoctoral Research Fellowship and by the NSA under Grant H98230-10-1-0199.

2. The quiver Schur algebra

Throughout this paper, we will fix an integer $e > 1$; as mentioned before, we also allow the possibility that $e = \infty$. (We will assume for simplicity $e > 2$ at some point later in the text.) Let Γ be the Dynkin diagram for $\hat{\mathfrak{sl}}_e$; with the fixed clockwise orientation if

e is finite and with the fixed linear orientation if $e = \infty$, Figure 1. Let $\mathbb{V} = \{1, 2, \dots, e\}$ (or $\mathbb{V} = \mathbb{Z}$ for $e = \infty$) be the set of vertices of Γ, identified with the set of remainders of integers modulo e. Let $h_{\bar{i}} : i \to i + 1$ be the arrow from the vertex i to the vertex $i + 1$ where here and in the following all formulas should be read taking indices corresponding to vertices of Γ modulo e when $e < \infty$.

Definition 2.1. *A (finite-dimensional)* **representation** (V, f) *of* Γ *over a field* $\Bbbk$ *is*

- *a collection of* $\Bbbk$*-vector spaces* $V_i, i \in \mathbb{V}$ *such that* $\sum_i \dim V_i < \infty$*, together with*
- $\Bbbk$*-linear maps* $f_i : V_i \to V_{i+1}$.

A **subrepresentation** *is a collection of vector subspaces* $W_i \subset V_i$ *such that* $f_i(W_i) \subset W_{i+1}$. *A representation* (V, f) *of* Γ *is called* **nilpotent** *if the map* $f_e \cdots f_2 f_1 : V_1 \to V_1$ *is nilpotent (when* $e = \infty$*, all representations are called nilpotent).*

2.1. **Quiver representations and quiver flag varieties.** The **dimension vector** of a representation (V, f) is the tuple $\mathbf{d} = (d_1, \dots, d_e)$, where $d_i = \dim V_i$. We let $|\mathbf{d}| = \sum d_i$ and denote by α_i the special dimension vector where $d_j = \delta_{ij}$. Mapping it to the simple root α_i of $\widehat{\mathfrak{sl}}_e$ identifies the set of dimension vectors with the positive cone in the root lattice of $\widehat{\mathfrak{sl}}_e$, and with semi-simple nilpotent representations of Γ:

Lemma 2.2. *There is a unique irreducible nilpotent representation* S_j *of dimension vector* α_j*. Any semi-simple nilpotent representation is of the form* (V, f) *with* $f_i = 0$ *for all* i*.*

Proof. Obviously S_j equals (V, f), where f_i and V_i are zero except of $V_j = \Bbbk$. Assume (V, f) is a non-trivial irreducible nilpotent representation. If, for given j, $f_j \neq 0$ then f_j is injective, since otherwise $W_j = \ker f_j$ and $W_r = \{0\}$ for $r \neq j$ defines a non-trivial proper subrepresentation. Not all f_i's are injective, since the representation is nilpotent and non-trivial. Pick i such that f_i is not injective with $V_i \neq 0$. Then $f_i = 0$ and hence (V, f) is isomorphic to S_i, since S_i is a subrepresentation. Any representation (V, f) with $f_i = 0$ for all i is obviously semi-simple. Conversely, assume (V, f) is semi-simple, hence isomorphic to $\bigoplus_{i=1}^{e} S_i^{d_i}$. In particular, $S_i^{d_i}$ is a direct summand (for any i) which implies that $f_i = 0$. □

Let $\mathrm{Rep}_{\mathbf{d}}$ be the affine space of representations of Γ with dimension vector $\mathbf{d}$, i.e.

$$\mathrm{Rep}_{\mathbf{d}} = \bigoplus_{i \in \mathbb{V}} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_{i+1}}). \tag{2.1}$$

This space has a natural algebraic action by conjugation of the algebraic group $G_{\mathbf{d}} = \mathrm{GL}(d_1) \times \cdots \times \mathrm{GL}(d_e)$, and we are interested in the **moduli space of representations**, i.e. the quotient $\mathrm{GRep}_{\mathbf{d}} = \mathrm{Rep}_{\mathbf{d}} / G_{\mathbf{d}}$ parametrizing isomorphism classes of representations. Since the $G_{\mathbf{d}}$-action is very far from being free, we must interpret this quotient in an intelligent way. One option is to consider it as an Artin stack. While this is perhaps the most elegant approach, it is more technical than necessary for our purposes. Instead the reader is encouraged to interpret this quotient as a formal symbol where, by convention for any complex algebraic G-variety X.

- $H^*(X/G)$ is the G-equivariant cohomology $H^*_G(X)$, and $H^{BM}_*(X/G)$ is the G-equivariant Borel-Moore homology of X (for a discussion of equivariant Borel-Moore homology, see [Ful98, section 19], or [VV11, §1.2]).

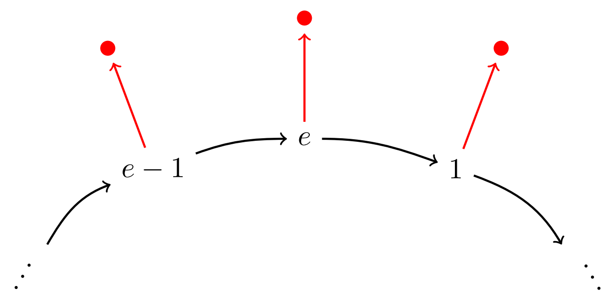


FIGURE 1. The oriented Dynkin quiver Γ and the extended Dynkin quiver $\tilde{\Gamma}$.

- $D(X/G)$ (resp. $D^+(X/G)$) is the bounded (resp. bounded below) equivariant derived category of Bernstein-Lunts [BL94], with the usual six functor formalism described therein. See also [WW09] as an additional reference for our purposes.

Note that $H^*_{G_{\mathbf{d}}}(\mathrm{Rep}_{\mathbf{d}}) = H^*(BG_{\mathbf{d}})$, where $BG_{\mathbf{d}}$ denotes the classifying space of $G_{\mathbf{d}}$ (or the classifying space of its $\mathbb{C}$-points in the analytic topology for the topologically minded) since $\mathrm{Rep}_{\mathbf{d}}$ is contractible. Thus, we can use the usual Borel isomorphism to identify the $H^*(\mathrm{GRep}_{\mathbf{d}})$'s with polynomial rings.

We will consider the direct sum $H^*(\mathrm{GRep}) = \bigoplus_{\mathbf{d}} H^*(\mathrm{GRep}_{\mathbf{d}})$ which corresponds to taking the union of the quotients $\mathrm{GRep} = \bigsqcup_{\mathbf{d}} \mathrm{GRep}_{\mathbf{d}}$ as Artin stacks. One can think of this as a quotient by the groupoid $G = \bigsqcup_{\mathbf{d}} G_{\mathbf{d}}$, and we will speak of the G-equivariant cohomology of $\mathrm{Rep} = \bigsqcup_{\mathbf{d}} \mathrm{Rep}_{\mathbf{d}}$, etc.

We'll be interested in spaces of quiver representations equipped with compatible flags. These have appeared several times in the literature, most importantly in work of Lusztig, e.g. [Lus91], on the geometric construction of the canonical basis; our work builds on his ideas. Now, we introduce the combinatorics underlying these spaces.

A **composition** of length r of $n \in \mathbb{Z}_{>0}$ is a tuple $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathbb{Z}^r_{>0}$ such that $\sum_{i=1}^r \mu_i = n$. In contrast, a **vector composition**[1] of type $m \in \mathbb{Z}_{>0}$ and length $r = r(\hat{\boldsymbol{\mu}})$ is a tuple $\hat{\boldsymbol{\mu}} = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \dots, \boldsymbol{\mu}^{(r)})$ of nonzero elements from $\mathbb{Z}^m_{\geq 0}$. If $m = e$ we call a vector composition also **residue data** and denote their set by Comp_e. Alternatively, $\hat{\boldsymbol{\mu}}$ can be viewed as an $r \times m$ matrix $\boldsymbol{\mu}[i,j]$ with the ith row $\boldsymbol{\mu}^{(i)}$. We call the column sequence, i.e. the result of reading the columns, the **flag type sequence** $\mathrm{t}(\hat{\boldsymbol{\mu}})$, whereas the **residue sequence** $\mathrm{res}(\hat{\boldsymbol{\mu}})$ is the sequence

$$1^{\boldsymbol{\mu}_1^{(1)}}, 2^{\boldsymbol{\mu}_2^{(1)}}, \cdots, e^{\boldsymbol{\mu}_e^{(1)}} | 1^{\boldsymbol{\mu}_1^{(2)}}, 2^{\boldsymbol{\mu}_2^{(2)}}, \cdots, e^{\boldsymbol{\mu}_e^{(2)}} | \cdots | 1^{\boldsymbol{\mu}_1^{(r)}}, 2^{\boldsymbol{\mu}_2^{(r)}}, \cdots, e^{\boldsymbol{\mu}_e^{(r)}}. \tag{2.2}$$

The parts separated by the vertical lines $|$ are called **blocks** of $\mathrm{res}(\hat{\boldsymbol{\mu}})$. We say that $\hat{\boldsymbol{\mu}}$ has **complete flag type** if every $\boldsymbol{\mu}^{(i)}$ is a unit vector which has exactly one non-zero entry. Hence blocks of $\mathrm{res}(\hat{\boldsymbol{\mu}})$ contain in this case at most one element.

The transposed vector composition of $\hat{\boldsymbol{\mu}}$ of type $r(\boldsymbol{\mu})$ and length m is defined to be $\check{\boldsymbol{\mu}} = (\check{\boldsymbol{\mu}}^{(1)}, \check{\boldsymbol{\mu}}^{(2)}, \dots, \check{\boldsymbol{\mu}}^{(m)})$ where $\check{\boldsymbol{\mu}}^{(i)}_j = \boldsymbol{\mu}[j,i] = \hat{\boldsymbol{\mu}}^{(j)}_i$, which we call in case $m = e$ also

[1] This differs from the notion of an r-multi-composition where the tuples could be of different lengths or an r-multi-partition where the tuples are partitions, but again not necessarily of the same length m.

the **flag data**. Given a composition μ of n of length r, we denote by $\mathcal{F}(\mu)$ the **variety of flags of type** μ, that is the variety of flags

$$F^1(\mu) \subset F^2(\mu) \subset \cdots \subset F^r(\mu) = \mathbb{C}^n,$$

where $F^i(\mu)$ is a subspace of dimension $\sum_{j=1}^i \mu_j$. For a vector composition $\hat{\boldsymbol{\mu}}$ we let $\mathcal{F}(\hat{\boldsymbol{\mu}}) = \prod_i \mathcal{F}(\check{\boldsymbol{\mu}}^{(i)})$, a product of partial flag varieties inside $\mathbb{C}^d = \prod_{i=1}^e \mathbb{C}^{d_i}$ of type given by the flag data. Here $\mathbf{d} = \mathbf{d}(\hat{\boldsymbol{\mu}}) = (d_1, \dots, d_e)$ denotes the **dimension vector** of the vector composition $\hat{\boldsymbol{\mu}}$ which is simply the sum $\mathbf{d} = \sum_{i=1}^r \hat{\boldsymbol{\mu}}^{(i)}$, and d denotes the total dimension. For a given dimension vector $\mathbf{d}$ denote

$$\mathrm{VComp}_e(\mathbf{d}) = \{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e \mid \mathbf{d}(\hat{\boldsymbol{\mu}}) = \mathbf{d}\}.$$

Example 2.3. Let $e = 2$ and $\hat{\boldsymbol{\mu}} = ((2,1),(1,1),(2,3),(0,1))$, a residue data of length $r = 4$ with dimension vector $\mathbf{d} = (5,6)$. The flag data is

$$\check{\boldsymbol{\mu}} = ((2,1,2,0),(1,1,3,1)) \text{ and } \mathrm{res}(\hat{\boldsymbol{\mu}}) = 1,1,2|1,2|1,1,2,2,2|2.$$

There are $\binom{11}{5}$ elements of complete flag type in $\mathrm{VComp}_e(\mathbf{d})$.

Definition 2.4. *For a given vector composition $\hat{\boldsymbol{\mu}} \in \mathrm{Comp}_e$ (of length r) a* **representation with compatible flags of type** $\check{\boldsymbol{\mu}}$ *is a nilpotent representation (V,f) of Γ with dimension vector $\mathbf{d} = \mathbf{d}(\hat{\boldsymbol{\mu}})$ together with a flag $F(i)$ of type $\check{\boldsymbol{\mu}}^{(i)}$ inside V_i for each $1 \le i \le e$ such that $f_i(F(i)^j) \subset F(i+1)^{j-1}$ for $1 \le j \le r$. We denote by*

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}) = \left\{(V,f,F) \in \mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\boldsymbol{\mu}}) \mid f_i(F(i)^j) \subset F(i+1)^{j-1}\ \forall\, i,j\right\}$$

the subset of $\mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\boldsymbol{\mu}})$ of representations with compatible flag. It comes equipped with the obvious action of the group $G := G_{\mathbf{d}}$ by change of basis.

Alternatively (see Lemma 2.2), a representation with compatible flags is a tuple $((V,f),F)$ consisting of a representation $(V,f) \in \mathrm{Rep}_{\mathbf{d}}$ equipped with a filtration by subrepresentations with semi-simple successive quotients (with dimension vectors given by $\hat{\boldsymbol{\mu}}$). In Example 5.7, we have a quiver Γ with two vertices and simple representations S_1, S_2 and the residue data or flag data gives flags of the form

$$F_2^1(1) \subseteq F_3^2(1) \subseteq F_5^3(1) \subseteq F_5^4(1) = \mathbb{C}^5, \qquad F_1^1(2) \subseteq F_2^2(2) \subseteq F_5^3(2) \subseteq F_6^4(2) = \mathbb{C}^6,$$

where the subindex denotes the dimension of the subspaces. In particular, if we set $V_i = F^i(1) \oplus F^i(2)$, then the semisimple subquotients are the representations given by the dimension vector $\hat{\boldsymbol{\mu}}$, namely

$$V_4 \cong S_2, \quad V_3/V_4 \cong S_1^2 \oplus S_2^3, \quad V_2/V_3 \cong S_1 \oplus S_2, \quad V_1/V_2 \cong S_1^2 \oplus S_2^1.$$

The flag data defines a Young subgroup $S_{\check{\boldsymbol{\mu}}} \subseteq S_{\mathbf{d}}$. Note that $\mathcal{F}(\hat{\boldsymbol{\mu}}) \cong G_{\mathbf{d}}/P_{\check{\boldsymbol{\mu}}}$ is the partial flag variety defined by the parabolic subgroup $P_{\check{\boldsymbol{\mu}}}$ of $G_{\mathbf{d}}$, given by upper triangular block matrices with block sizes determined by the flag sequence. The flags which appear will be complete if and only if $\hat{\boldsymbol{\mu}}$ has complete flag type, as suggested by the name. This special case is particularly important; it played a key role in earlier papers of Lusztig [Lus91] and Varagnolo and Vasserot [VV11].

Forgetting either the flag or the representation defines two $G_{\mathbf{d}}$-equivariant morphisms

$$p: \quad \mathcal{Q}(\hat{\boldsymbol{\mu}}) \longrightarrow \mathrm{Rep}_{\mathbf{d}}, \qquad ((V,f),F) \mapsto (V,f) \in \mathrm{Rep}_{\mathbf{d}}, \tag{2.3}$$

$$\pi: \quad \mathcal{Q}(\hat{\boldsymbol{\mu}}) \longrightarrow \mathcal{F}(\hat{\boldsymbol{\mu}}), \qquad ((V,f),F) \mapsto F \in \mathcal{F}(\hat{\boldsymbol{\mu}}). \tag{2.4}$$

of algebraic varieties. Generalizing the notion of a quiver Grassmannian we call the fibres of p the **quiver partial flag varieties**.

To each $i \in \mathbb{V}$ assign a polynomial ring over $\Bbbk$ in d_i variables $x_{i,1}, \dots, x_{i,d_i}$ and set

$$R(\mathbf{d}) = \bigotimes_{j=1}^{e} \Bbbk[x_{j,1}, \dots, x_{j,d_j}] = \Bbbk[x_{1,1}, \dots, x_{1,d_1}, \dots, x_{e,1}, \dots, x_{e,d_e}]. \tag{2.5}$$

This algebra carries an action of the Coxeter group $S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_e}$ by permuting the variables in the same tensor factor. Thus, the Borel presentation identifies the rings

$$H^*(\mathcal{F}(\hat{\boldsymbol{\mu}})/G) \cong R(\mathbf{d})^{S_{\hat{\boldsymbol{\mu}}}} =: \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}), \tag{2.6}$$

by sending Chern classes of tautological bundles to elementary symmetric functions, [Bri98, Proposition 1].

In the extreme case where $\mathbf{d} = (r, 0, \dots, 0)$ we have $\hat{\boldsymbol{\mu}}^{(i)} = (1, 0, \dots, 0)$ for all i and $S_{\hat{\boldsymbol{\mu}}}$ is trivial. Hence we obtain the GL_d-equivariant cohomology of the variety of full flags in $\mathbb{C}^d$. If $\hat{\boldsymbol{\mu}} = (\mathbf{d})$ then $\hat{\boldsymbol{\mu}} = ((d_1), (d_2), \dots, (d_e))$, the variety $\mathcal{F}(\hat{\boldsymbol{\mu}})$ is just a point and $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}) = R(\mathbf{d})^{S_{\mathbf{d}}}$. We call this the **ring of total invariants**.

Lemma 2.5. *The map π is a vector bundle with affine fibre; in particular, we have a natural isomorphism $\pi^* : H^*(\mathcal{F}(\hat{\boldsymbol{\mu}})/G) \cong H^*(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G)$, and thus $H^*(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G) \cong \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}})$.*

Grading convention. Throughout this paper, we will use a somewhat unusual grading convention on cohomology rings: we shift the (equivariant) cohomology ring or equivariant Borel-Moore homology of a smooth variety X downward by the complex dimension of X, so that the identity class is in degree $-\dim_{\mathbb{C}} X$. This choice of grading has the felicitous effect that pull-back and push-forward maps in cohomology are more symmetric. Usually, for a map $f\colon X \to Y$ we have that pull-back has degree 0, and push-forward has degree $2\dim_{\mathbb{C}} Y - 2\dim_{\mathbb{C}} X$, whereas in our grading convention

(2.7) *both push-forward and pull-back have degree* $\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$.

Those readers comfortable with the theory of the constructible derived category will recognize this as replacing the usual constant sheaf with the intersection cohomology sheaf, denoted $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G} \in D^+(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G)$, of $\mathcal{Q}(\hat{\boldsymbol{\mu}})/G$.

Since $\mathcal{Q}(\hat{\boldsymbol{\mu}})$ is smooth, $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ is simply a homological shift of the usual constant sheaf on each individual component (but by different amounts on each component). A more conceptual explanation for the convention is the invariance of $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ under Verdier duality. This grading shift also provides a straight-forward explanation of the grading on the representation $\mathcal{P}o\ell$ of the quiver Hecke algebra in [KL09], [KL11].

Since p is proper and the constant sheaf of geometric origin, the Beilinson-Bernstein-Deligne decomposition theorem from [BBD82] applies (in the formulation [dCM09, Theorem 4.22]), and p sends the $\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})/G}$ on $\mathcal{Q}(\hat{\boldsymbol{\mu}})/G$ to a direct sum L of shifts of simple perverse sheaves on $\mathrm{Rep}_{\mathbf{d}}$. Our first object of study in this paper will be the algebra of extensions of these sheaves.

2.2. The convolution algebra.

Let $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}} \in \mathrm{Comp}_e$ be vector compositions with associated dimension vector $\mathbf{d}$, and consider the corresponding "Steinberg variety"

$$Z(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}) \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}(\hat{\boldsymbol{\lambda}}). \tag{2.8}$$

Let $G_{\mathbf{d}}$ act diagonally and abbreviate $\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) = Z(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})/G_{\mathbf{d}}$. Note that $\mathcal{Q}(\hat{\boldsymbol{\mu}})$ is smooth and p is proper. So, by [CG97, Theorem 8.6.7], we can identify the algebra of self-extensions of L (although not as a graded algebra) with the equivariant Borel-Moore homology $H_*^{BM,G}$ of our Steinberg variety: we have the natural identification

$$\mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}) = H_*^{BM,G}(Z(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})),$$

such that the Yoneda product

$$\begin{aligned}\mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}) \otimes \mathrm{Ext}^*_{D^b(\mathrm{Rep}_{\mathbf{d}})}(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\lambda}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\nu}})}) \\ \to \quad \mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\nu}})})\end{aligned}$$

agrees with the convolution product. This defines an associative non-unital graded algebra structure $A_{\mathbf{d}}$ on

$$\bigoplus_{(\hat{\mu},\hat{\lambda})} \mathrm{Ext}^*_{D^b(\mathrm{GRep}_{\mathbf{d}})}\left(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}\right) = \bigoplus_{(\hat{\mu},\hat{\lambda})} H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})). \tag{2.9}$$

Here the sums are over all elements in $\mathrm{VComp}_e(\mathbf{d}) \times \mathrm{VComp}_e(\mathbf{d})$. For reasons of diagrammatic algebra, we call this product **vertical composition** and we denote it like the usual multiplication $(f, g) \mapsto fg$. We call the algebra $A_{\mathbf{d}}$ the **quiver Schur algebra**.

2.3. **A faithful polynomial representation.** The quiver Schur algebra $A_{\mathbf{d}}$ acts, by [CG97, Proposition 8.6.15], naturally on the sum (over all vector partitions of dimension vector $\mathbf{d}$) of cohomologies

$$V_{\mathbf{d}} \quad := \bigoplus_{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})} H_*^{BM}(\mathcal{Q}(\hat{\boldsymbol{\mu}})/G) \cong \bigoplus_{\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})} \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}). \tag{2.10}$$

Note that this is compatible with the grading convention (2.7).

Proposition 2.6. *If* $\mathrm{char}(\Bbbk) = 0$*, the* $A_{\mathbf{d}}$*-module* $V_{\mathbf{d}}$ *is faithful.*

In fact, the hypothesis on characteristic is unnecessary, since this is a special case of Proposition 4.7, which is characteristic independent; however, there is a more geometric proof in this special case, which we give here.

Proof. Recall that the ring $A_{\mathbf{d}}$ is just the Ext-algebra of $\oplus_{|\hat{\boldsymbol{\mu}}|=\mathbf{d}} p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$. The space $V_{\mathbf{d}}$ is the hypercohomology of $\oplus_{|\hat{\boldsymbol{\mu}}|=\mathbf{d}} p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}}))}$ up to shifts of grading, since Borel-Moore homology $H_*^{BM}(X)$ of a smooth space X equals hypercohomology $\mathbb{H}^{-*}(X, \mathbb{D})$ of its dualizing sheaf $\mathbb{D}$ which is, up to a shift, the constant sheaf. Hence, if we let j be the map from $\mathrm{Rep}_{\mathbf{d}}$ to a point, faithfulness is equivalent to j_* being injective,

$$j_* : \quad \mathrm{Ext}^\bullet(p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}}')}) \to \mathrm{Ext}^\bullet(j_* p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}, j_* p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}}')}). \tag{2.11}$$

We only need to check that the same property holds when $p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ is replaced by a summand. The nilpotent orbits of $G_{\mathbf{d}}$ in $\mathrm{Rep}_{\mathbf{d}}$ are equivariantly simply connected; thus every simple perverse sheaf supported on the nilpotent locus is the intermediate extension of a trivial local system. Thus, by the decomposition theorem, the sheaves $p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ decompose into summands which are shifts of the intersection cohomology sheaves of these orbits. We note that the spaces we deal with have good parity vanishing properties. Each orbit has even equivariant cohomology, since it has a transitive action of G with the stabilizer of a point given by a connected algebraic group ([Lib07, Lemma 78, Theorem 79]). Also, the stalks of intersection cohomology sheaves have even cohomology [Hen07, Theorem 5.2(3)]. Thus, by [BGS96, Theorem 3.4.2], j_* is faithful on semi-simple G_d-equivariant perverse sheaves, and so the map (2.11) is injective. □

2.4. **Monoidal structure.** The algebra $A_{\mathbf{d}}$ comes along with distinguished idempotents $e_{\hat{\boldsymbol{\mu}}}$ indexed by vector compositions with dimension vector $\mathbf{d}$. The $A_{\mathbf{d}}$-module

$$P(\hat{\boldsymbol{\mu}}) = \bigoplus_{\hat{\boldsymbol{\lambda}} \in \mathrm{VComp}_e(\mathbf{d})} e_{\hat{\boldsymbol{\lambda}}} A_{\mathbf{d}} e_{\hat{\boldsymbol{\mu}}} \tag{2.12}$$

is a finitely generated indecomposable graded projective $A_{\mathbf{d}}$-module. Each finitely generated indecomposable graded projective $A_{\mathbf{d}}$-module is (up to a grading shift) isomorphic to one of this form. We denote by $A_{\mathbf{d}}-\mathrm{pmod}$ the category of graded finitely generated projective $A_{\mathbf{d}}$-modules.

Proposition 2.7. *Assume* $\mathrm{char}(\Bbbk) = 0$. *The category* $A_{\mathbf{d}}-\mathrm{pmod}$ *is equivalent to the additive category of sums of shifts of semi-simple perverse sheaves in* $D^+(\mathrm{GRep})$ *which are pure of weight* 0 *with nilpotent support in* $\mathrm{GRep}_{\mathbf{d}}$.

Again, the characteristic 0 hypothesis could be avoided, but at the cost of some difficulties we prefer to avoid. The corresponding perverse sheaves in the characteristic p case will not be semi-simple, but rather parity sheaves, in the sense of [JMW09].

Proof. There is a functor sending a finitely generated projective $A_{\mathbf{d}}$-module M to the perverse sheaf $\left(\bigoplus_{\mathbf{d}(\hat{\boldsymbol{\mu}})=\mathbf{d}} p_* \Bbbk_{\mathcal{Q}_{\hat{\mu}}}\right) \otimes_{A_{\mathbf{d}}} M$. This map is fully faithful, since it induces an isomorphism on the endomorphisms of $A_{\mathbf{d}}$ itself. Thus, we only need to show that every simple perverse sheaf on GRep with nilpotent support is a summand of $p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ for some $\hat{\boldsymbol{\mu}}$.

Every such simple perverse sheaf is $\mathbf{IC}(\bar{X})$ where X is the locus of modules isomorphic to a fixed module N. Consider the socle filtration on N. The dimension vectors of the successive quotients define a vector composition $\hat{\boldsymbol{\mu}}_N$. The map $\mathcal{Q}(\hat{\boldsymbol{\mu}}_N) \to \mathrm{GRep}$ is generically an isomorphism over X, and thus for dimension reasons, has image $\bar{X}$. In particular,

$$p_* \Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_N)} \cong \mathbf{IC}(\bar{X}) \oplus \mathbf{L}$$

where $\mathbf{L}$ is a finite direct sum of shifts of semi-simple perverse sheaves supported on $\bar{X} \setminus X$. Thus, every simple perverse sheaf with nilpotent support is a summand of such a pushforward, and we are done. □

We are going to define a monoidal structure on $A_{\mathbf{d}}-\mathrm{pmod}$ using correspondences. For $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}} \in \mathrm{VComp}_e$ the **join** $\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\nu}}$ is the vector composition

$$\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\nu}} = (\mu^{(1)}, \dots, \mu^{(r(\hat{\boldsymbol{\mu}}))}, \nu^{(1)}, \dots, \nu^{r(\hat{\boldsymbol{\nu}})})$$

obtained by joining the two tuples. Let $\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_2$ be vector compositions with associated dimension vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. Let

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2) \subseteq \mathcal{Q}(\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2) \times_{\mathrm{Rep}_{\mathbf{c}}} \mathcal{Q}(\hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2)$$

be the space of representations with dimension vector $\mathbf{c} + \mathbf{d}$ which carry a pair of compatible flags of type $\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2$ and $\hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2$ respectively, such that the subspaces of dimension vector $\mathbf{c}$ in the two flags coincide. Let $\mathcal{H}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2)$ be the quotient by the diagonal $G_{\mathbf{d}}$-action.

Definition 2.8. *The **horizontal multiplication** is the map*

$$\begin{array}{rcl} A_{\mathbf{c}} \times A_{\mathbf{d}} & \longrightarrow & A_{\mathbf{c}+\mathbf{d}} \\ (a,b) & \longmapsto & a|b \end{array} \tag{2.13}$$

induced on equivariant Borel-Moore homology by the correspondence (i.e. by pull-and-push on the following diagram)

$$\mathcal{H}(\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\lambda}}_1) \times \mathcal{H}(\hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_2) \longleftarrow \mathcal{H}(\hat{\boldsymbol{\mu}}_1; \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1; \hat{\boldsymbol{\lambda}}_2) \longrightarrow \mathcal{H}(\hat{\boldsymbol{\mu}}_1 \cup \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\lambda}}_1 \cup \hat{\boldsymbol{\lambda}}_2). \tag{2.14}$$

Here the rightward map is the obvious inclusion, and the leftward is induced from the map $V \mapsto (W, V/W)$ of taking the common subrepresentation W of dimension vector $\mathbf{c}$ and the quotient by it.

We let $\mathbf{A} = \oplus_{\mathbf{d}}(A_{\mathbf{d}}-\text{pmod})$ be the direct sum of the categories $A_{\mathbf{d}}-\text{pmod}$ over all dimension vectors; that is its objects are formal direct sums of finitely many objects from these categories, with morphism spaces given by direct sums.

Proposition 2.9. *The assignment $\otimes : (P(\hat{\mu}), P(\hat{\nu})) \mapsto P(\hat{\mu} \cup \hat{\nu})$ extends to a monoidal structure $(\mathbf{A}, \otimes, \mathbf{1})$ with unit element $\mathbf{1} = P(\emptyset)$.*

Proof. In the ungraded case this follows directly from Lusztig's convolution product [Lus91, §3], by Proposition 2.7. More explicitly, we define

$$M \otimes N = A_{\mathbf{c}+\mathbf{d}} \otimes_{A_{\mathbf{c}} \times A_{\mathbf{d}}} M \boxtimes N,$$

meaning one first takes the outer tensor product of the graded $A_{\mathbf{c}}$-module M and the graded $A_{\mathbf{d}}$-module N. The resulting $A_{\mathbf{c}} \times A_{\mathbf{d}}$-module is then induced to a graded $A_{\mathbf{c}+\mathbf{d}}$-module via the horizontal multiplication (2.13). This is functorial in both entries and defines the required tensor product with the asserted properties. □

2.5. **Categorified generic nilpotent Hall algebra.** Let $K_q^0(\mathbf{A})$ be the split Grothendieck group of the additive Krull-Schmidt category $\mathbf{A}$, i.e. the free abelian group on isomorphism classes $[M]$ of objects in $A-\text{pmod}$ modulo the relation $[M_1] + [M_2] = [M_1 \oplus M_2]$. This is a free $\mathbb{Z}[q, q^{-1}]$-module where the action of q is by grading shift (and has nothing to do with the parameter q from the introduction).

For a graded vector space $W = \oplus_j W^j$, we define its grading shifts $W\langle d \rangle$, $d \in \mathbb{Z}$, by $(W\langle d \rangle)^j = W^{d+j}$, and let $q^d[M] = [M\langle d \rangle]$. The module $K_q^0(\mathbf{A})$ is of infinite rank, but is naturally a direct sum of the Grothendieck groups $K_q^0(A_{\mathbf{d}}-\text{pmod})$, each of which is finite rank.

Let $\text{VCompf}_e(\mathbf{d}) \subset \text{VComp}_e$ be the set of vector compositions of $\mathbf{d}$ of complete flag type. For each $\mathbf{d}$, there is a subalgebra,

$$R_{\mathbf{d}} = \bigoplus_{\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}} \in \text{VCompf}_e(\mathbf{d})} e_{\hat{\boldsymbol{\lambda}}} A_{\mathbf{d}} e_{\hat{\boldsymbol{\mu}}}. \tag{2.15}$$

There is also a corresponding monoidal subcategory $\mathbf{R}$ of $\mathbf{A}$ generated by the indecomposable projectives indexed by the $\hat{\boldsymbol{\mu}} \in \text{VCompf}_e(\mathbf{d})$ for all $\mathbf{d}$. Both $A_{\mathbf{d}}$ and $R_{\mathbf{d}}$ can be defined for any quiver and the following proposition holds in general, though in this paper we only use these categories for the affine type A quiver. The algebra $R_{\mathbf{d}}$ appears as **quiver Hecke algebra** (associated with $\mathbf{d}$) in the literature:

Proposition 2.10 (Vasserot-Varagnolo/Rouquier [VV11, 3.6])**.**
As a graded algebra, $\mathbf{R}_{\mathbf{d}}$ is isomorphic to the quiver Hecke algebra $R(\mathbf{d})$ associated to the quiver Γ in [Rou08]. *In particular, $K_q^0(\mathbf{R})$ is naturally isomorphic to the Lusztig integral form of $U_q^-(\widehat{\mathfrak{sl}}_e)$ by mapping the isomorphism classes of indecomposable projective objects to Lusztig's canonical basis.*

The idempotents in $R_{\mathbf{d}}$ get identified with those in $R(\mathbf{d})$ by viewing the residue sequence (2.2) as a sequence of simple roots α_i. We should note that Proposition 2.10 uses the signed version of the quiver Hecke algebra appearing in [VV11], [BK09a] which differs from the first paper of Khovanov and Lauda [KL09].

Remark 2.11. For a Dynkin quiver, the categories $\mathbf{R}$ and $\mathbf{A}$ are canonically equivalent. In fact, both are equivalent to the full category of semi-simple perverse sheaves on GRep. However, in affine type A (the case of interest in this paper), they differ. In terms of perverse sheaves, the IC-sheaves which appear in $p_*\Bbbk_{\mathcal{Q}(\hat{\boldsymbol{\mu}})}$ for $\hat{\boldsymbol{\mu}}$ having complete flag type are those whose Fourier transform has nilpotent support as well; for example, the constant sheaf on the trivial representation with dimension vector $(1,\dots,1)$ cannot appear. Thus, in this case there are objects in $\mathbf{A}$ which don't lie in $\mathbf{R}$.

Recall that the **nilpotent Hall algebra** of the quiver Γ is an algebra structure on the set of complex valued functions on the space of (isomorphism classes of) nilpotent representations, typically considered over a finite field. The structure constants are polynomial in the cardinality q' of the field. If $[M]$ denotes the constant function on the class of the representation M with dimension vector $\mathbf{d}(M)$ then

$$[M]\cdot[N] = q^{\{\mathbf{d}(M),\mathbf{d}(N)\}}F^Q_{M,N}[Q],$$

where $\{\mathbf{d}',\mathbf{d}''\} = \sum_{i=1}^e \mathbf{d}'_i(\mathbf{d}''_i - \mathbf{d}''_{i+1})$ denotes the Euler form, $q' = q^2$ and the $F^Q_{M,N}$ are the Hall numbers. Hence it makes sense to consider q as a formal parameter and define the **generic Hall algebra** over the ring of Laurent polynomials $\mathbb{C}[q,q^{-1}]$. Following Vasserot and Varagnolo, [VV99], we denote this algebra U_e^-. By work of Schiffmann [Sch00, §2.2] it is isomorphic as an algebra to $U_q^-(\widehat{\mathfrak{sl}}_e)\otimes\mathbf{\Lambda}(\infty)$, where $\mathbf{\Lambda}(\infty)$ denotes the ring of symmetric polynomials. Identifying $\mathbf{\Lambda}(\infty)$ with $U_q^-(\mathbb{H})$, the lower half of a Heisenberg algebra, this algebra can also be described as $U_q^-(\widehat{\mathfrak{gl}}_e)$ as in work of Hubery [Hub05], [Hub10]. This generic Hall algebra has a basis given by characteristic functions on the isomorphism classes of nilpotent representations of Γ and is naturally generated as algebra by the characteristic functions $\mathbf{f}_{\mathbf{d}}$ on the classes of semi-simple representations (which we label by their dimension vectors $\mathbf{d}$ following Lemma 2.2). Note that for instance $\mathbf{f}_i := \mathbf{f}_{\alpha_i} = [S_i]$ and $\mathbf{f}_{\alpha_i+\alpha_{i+1}} = \mathbf{f}_{i+1}\mathbf{f}_i = [S_{i+1}\oplus S_i]$, whereas $\mathbf{f}_{(1,\dots,1)}$ is not in the subalgebra generated by the $[S_i]$'s, cf. Remark 2.11. The integral form $U^-_{e,\mathbb{Z}}$ over $\mathbb{Z}[q,q^{-1}]$ is given here by the lattice generated by all $\mathbf{f}_{\mathbf{d}}$'s, analogous to Lusztig's integral form for quantum groups, see [Sch00].

Proposition 2.12. *If* $\mathrm{char}(\Bbbk) = 0$, *then there is an isomorphism $K_q^0(\mathbf{A}) \cong U^-_{e,\mathbb{Z}}$, $[(\mathbf{d})]\mapsto\mathbf{f}_{\mathbf{d}}$, of $\mathbb{Z}[q,q^{-1}]$-algebras from the graded Grothendieck ring of $\mathbf{A}$ to the integral form of the generic nilpotent Hall algebra of the cyclic quiver.*

The result is in fact also true for $\Bbbk$ of positive characteristic and can be proved using the usual technique of deforming to a characteristic 0 discrete valuation ring. Since the general result is not needed here, we omit it, but refer to [Mak13].

Proof. Fixing a prime p, there is a natural map from $K_q^0(\mathbf{A})$ to $U_e^-|_{q=p}$. This is given by applying the equivalence of Proposition 2.7, and then sending the class of a semi-simple perverse sheaf to the function given by the super-trace of Frobenius on its stalks. This is a function on the points of Rep over the field $\mathbb{F}_p$ and hence defines an element of the Hall algebra. By the definition of the Hall multiplication and the Grothendieck trace formula, this is an algebra map. Since these super-traces are polynomial in p (they are the Poincaré polynomials of the quiver partial flag varieties), the coefficients of the expansion of this function in terms of the characteristic functions of orbits are also polynomial, and this assignment can be lifted to an algebra map $K_q^0(\mathbf{A}) \to U_{e,\mathbb{Z}}^-$. This map is obviously surjective, since the function for each intersection cohomology sheaf on a nilpotent orbit, and thus the characteristic function on the orbit, is in its image. It is also injective, since when we expand any non-zero class in the Grothendieck group in terms of the classes of intersection cohomology sheaves, we must have a nonzero value of the corresponding function on the support of an intersection cohomology sheaf maximal (in the closure ordering) amongst those with non-zero coefficient. Since $[(\mathbf{d})]$ corresponds to the skyscraper sheaf of the semi-simple representation of dimension $\mathbf{d}$, it is sent to the characteristic function of that point. □

The monoidal structure on $\mathbf{A}$ and the usual monoidal structure $(\mathsf{Vect}_\Bbbk, \otimes_\Bbbk, \Bbbk)$ on the category of vector spaces are compatible in the following way:

Lemma 2.13. *Let $\Phi_\mathbf{d} : A_\mathbf{d} \to \mathrm{End}(V_\mathbf{d})$ be the representation from (2.10). Then*

$$\Phi_{\mathbf{c}+\mathbf{d}}((a|b))(v) = \Phi_\mathbf{c}(a)(v_1) \otimes \Phi_\mathbf{c}(b)(v_2),$$

where v is the image of $v_1 \otimes v_2$ under the canonical map $V_{\hat{\boldsymbol{\mu}}} \otimes V_{\hat{\boldsymbol{\lambda}}} \to V_{\hat{\boldsymbol{\mu}} \cup \hat{\boldsymbol{\lambda}}}$.

That is, the functor $\mathbf{V} \colon \mathbf{A} \to \mathsf{Vect}_\Bbbk$ given by $\hat{\boldsymbol{\mu}} \mapsto V_{\hat{\boldsymbol{\mu}}}$ is monoidal.

Proof. This follows directly from the definitions, see [CG97]. □

Thus, we can describe elements corresponding to vector compositions with a large number of parts by looking at (the action) of the ones with a small number of parts.

3. Diagrams and Demazure operators

In this section we describe a basis of the algebras $A_\mathbf{d}$ and elementary morphisms, called splits and merges. We give a geometric, algebraic and diagrammatical description of these maps.

Let $\hat{\boldsymbol{\lambda}}', \hat{\boldsymbol{\lambda}} \in \mathrm{Comp}_e$ be residue data.

Definition 3.1. *We say that $\hat{\boldsymbol{\lambda}}'$ is a* **merge** *of $\hat{\boldsymbol{\lambda}}$ (and $\hat{\boldsymbol{\lambda}}$ a* **split** *of $\hat{\boldsymbol{\lambda}}'$) at the index k if $\hat{\boldsymbol{\lambda}}' = (\boldsymbol{\lambda}^{(1)}, \cdots, \hat{\boldsymbol{\lambda}}^{(k)} + \hat{\boldsymbol{\lambda}}^{(k+1)}, \ldots, \hat{\boldsymbol{\lambda}}^{(r)})$.*

If $\hat{\boldsymbol{\lambda}}'$ is a merge of $\hat{\boldsymbol{\lambda}}$, then there is an associated correspondence

$$\begin{array}{ccc} & \mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) = \left\{(V, f, F) \in \mathcal{Q}(\hat{\boldsymbol{\lambda}}) \mid f_i(F(i)^{k+1}) \subset F(i+1)^{k-1}\right\} & \\ \swarrow & & \searrow \\ \mathcal{Q}(\hat{\boldsymbol{\lambda}}) & & \mathcal{Q}(\hat{\boldsymbol{\lambda}}') \end{array} \tag{3.1}$$

where the left map is just the obvious inclusion and the right map is forgetting $F_k(i)$ for all vertices $1 \leq i \leq e$ (and reindexing all subspaces in the flags with higher indices). Obviously the same variety defines also a correspondence in the opposite direction (reading from right to left) which we associate to the **split**. We are interested in the equivariant version:

Definition 3.2. *For $\hat{\boldsymbol{\lambda}}'$ a merge (resp. split) of $\hat{\boldsymbol{\lambda}}$ at k, we let $\hat{\boldsymbol{\lambda}} \xrightarrow{k} \hat{\boldsymbol{\lambda}}'$ or just $\hat{\boldsymbol{\lambda}} \to \hat{\boldsymbol{\lambda}}'$ denote the element of A given by multiplication with the equivariant fundamental class $[\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k)]$ (resp. $[\mathcal{Q}(\hat{\boldsymbol{\lambda}}', k)]$) pushed forward to $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\lambda}}', \hat{\boldsymbol{\lambda}}))$.*

In the most obvious choice of grading conventions, pull-back by a map is of degree 0, and pushforward has degree given by minus the relative (real) dimension of the map (i.e. the dimension of the target minus the dimension of the domain). This normalization has the disadvantage of breaking the symmetry between splits and merges. It is, for example, carefully avoided in [KL09]. Instead, we use, (2.7), the perverse normalization of the constant sheaves which "averages" the degrees of pull-back and pushforward. Then the degree of convolving with the fundamental class of a correspondence is minus the sum of the relative (complex) dimensions of the two projection maps (note that for a correspondence over two copies of the same space, this agrees with the most obvious normalization). In particular, we get the same answer in the split and merge cases.

Proposition 3.3. *Let $\hat{\boldsymbol{\lambda}}'$ be a merge or split of $\hat{\boldsymbol{\lambda}}$ at index k. Then $\hat{\boldsymbol{\lambda}} \to \hat{\boldsymbol{\lambda}}'$ is homogeneous of degree*

$$\sum_{i=1}^{e} \boldsymbol{\lambda}_i^{(k)} (\boldsymbol{\lambda}_{i-1}^{(k+1)} - \boldsymbol{\lambda}_i^{(k+1)}) =: -\{\boldsymbol{\lambda}^{(k+1)}, \boldsymbol{\lambda}^{(k)}\}. \tag{3.2}$$

Proof. If $\hat{\boldsymbol{\lambda}}'$ is a merge of $\hat{\boldsymbol{\lambda}}$, then

- the map $\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) \to \mathcal{Q}(\hat{\boldsymbol{\lambda}}')$ is a smooth surjection with fiber given by the product of Grassmannians of $\boldsymbol{\lambda}_i^{(k)}$-dimensional planes in $\boldsymbol{\lambda}_i^{(k)} + \boldsymbol{\lambda}_i^{(k+1)}$-dimensional space, which has dimension $\sum \boldsymbol{\lambda}_i^{(k)} \boldsymbol{\lambda}_i^{(k+1)}$, and
- the map $\mathcal{Q}(\hat{\boldsymbol{\lambda}}, k) \to \mathcal{Q}(\hat{\boldsymbol{\lambda}})$ is a closed inclusion of codimension
$$\sum_{i=1}^{e} \dim \operatorname{Hom}(F(i-1)^{k+1}/F(i-1)^k, F(i)^k/F(i)^{k-1}) = \sum_{i=1}^{e} \boldsymbol{\lambda}_i^{(k)} \boldsymbol{\lambda}_{i-1}^{(k+1)}.$$

The result follows for merges and hence also for splits. □

3.1. **Explicit formulas for merges and splits.** We give now explicit formulas for elementary merges and splits. Consider the particular choices for the vector compositions: $\mathcal{H}((\mathbf{c}, \mathbf{d}), (\mathbf{c} + \mathbf{d})) \cong \mathcal{F}(\mathbf{c}, \mathbf{d})/G_{\mathbf{c}+\mathbf{d}}$. The variety $\mathcal{Q}(\mathbf{c} + \mathbf{d})$ is just a point, but equipped with the action of $G = G_{\mathbf{c}+\mathbf{d}}$. We want to describe how the fundamental classes of $\mathcal{H}((\mathbf{c},\mathbf{d}),(\mathbf{c}+\mathbf{d}))$ or $\mathcal{H}((\mathbf{c}+\mathbf{d}),(\mathbf{c},\mathbf{d}))$ (which are isomorphic as varieties, but different as correspondences) act on V via Proposition 2.6 and determine in this way the merge and split map. They are given by pullback followed by pushforward in equivariant cohomology via the diagram

$$H_*^{BM}(\mathcal{Q}(\mathbf{c},\mathbf{d})/G) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} H_*^{BM}(\mathcal{F}(\mathbf{c},\mathbf{d})/G) \underset{q^*}{\overset{q_*}{\rightleftarrows}} H_*^{BM}(\mathcal{Q}(\mathbf{c}+\mathbf{d})/G),$$

where $\iota : \mathcal{F}(\mathbf{c},\mathbf{d}) \to \mathcal{Q}(\mathbf{c},\mathbf{d})$ is the zero section of the G-equivariant fibre bundle $\pi : \mathcal{Q}(\mathbf{c},\mathbf{d}) \longrightarrow \mathcal{F}(\mathbf{c},\mathbf{d})$ and $q : \mathcal{F}(\mathbf{c},\mathbf{d}) \to \mathcal{Q}(\mathbf{c}+\mathbf{d})$ is the proper G-equivariant map given by forgetting the subspaces of dimension c_i for any i.

Proposition 3.4. *Let* $\mathbf{c}$, $\mathbf{d} \in \mathbb{Z}^e_{\geq 0}$ *non-zero. The following diagram commutes*

$$\begin{array}{ccc} H^*(\mathcal{Q}(\mathbf{c},\mathbf{d})/G) & \underset{\iota_*\circ q^*}{\overset{q_*\circ\iota^*}{\rightleftarrows}} & H^*(\mathcal{Q}(\mathbf{c}+\mathbf{d})/G) \\ \text{Borel}\downarrow & & \downarrow\text{Borel} \\ \mathbf{\Lambda}(\mathbf{c},\mathbf{d}) & \underset{\mathrm{E}}{\overset{\mathrm{int}}{\rightleftarrows}} & \mathbf{\Lambda}(\mathbf{c}+\mathbf{d}) \end{array} \tag{3.3}$$

where E is the inclusion map from the total invariants $\mathbf{\Lambda}(\mathbf{c}+\mathbf{d})$ into the invariants $\mathbf{\Lambda}(\mathbf{c},\mathbf{d})$ followed by multiplication with the Euler class

$$E := \prod_{i=1}^{e} \prod_{j=1}^{c_{i+1}} \prod_{k=c_i+1}^{d_i+c_i} (x_{i+1,j} - x_{i,k}), \tag{3.4}$$

and int *is the integration map which sends an element f to the total invariant*

$$\sum_{w\in S_{\mathbf{c}+\mathbf{d}}} (-1)^{l(w)} w(f) \prod_{i=1}^{e} \frac{1}{c_i! d_i!} \frac{w\Big(\prod_{1\leq j<k\leq c_i} (x_{i,j}-x_{i,k}) \prod_{c_i<\ell<m\leq c_i+d_i} (x_{i,\ell}-x_{i,m})\Big)}{\prod_{1\leq j<k\leq c_i+d_i} (x_{i,j}-x_{i,k})} \tag{3.5}$$

where ℓ denotes the usual length function on the symmetric group.

Those readers who are interested in the case where $\Bbbk$ is of small positive characteristic might be worried about (3.5), since it involves division by a scalar which may not be invertible in $\Bbbk$; However, like divided difference operators, these operations preserve integer valued polynomials, and so are well-defined maps modulo p for any prime p.

Remark 3.5. By convention, we set $E = 1$ if either one of the products in (3.4) is empty or one of the variables $x_{i-1,k}$ or $x_{i,j}$ does not exist. The degrees of the maps $q_* \circ \iota^*$ and $\iota_* \circ q^*$ are again not the degrees which one would naively guess, but rather given by the convention (2.7), in particular they are of the same degree.

Proof. Since π^* is an isomorphism and ι the inclusion of the zero section, ι^* is also an isomorphism. On the other hand q is the map to a point, hence q^* is just the inclusion of the total invariants. By the usual adjunction formula, $\iota^*\iota_*(a) = \mathbf{e} \cup a$, where $\mathbf{e}$ is the Euler class of the vector bundle π. To see that the map E is as asserted it is enough to verify the formula $E = \mathbf{e}$ for the Euler class. The map i is the inclusion of the zero section of the vector bundle

$$\bigoplus_{i\in\mathbb{V}} \mathrm{Hom}(\mathcal{V}_{i,d_i}, \mathcal{V}_{i+1,c_{i+1}}),$$

where $\mathcal{V} = \oplus_{i=1}^{e} \mathcal{V}_{i,d_i}$ is the tautological vector bundle on the moduli space of quiver representation with dimension vector $\mathbf{d}$. As equivariant vector bundles over the maximal torus of $G_{\mathbf{d}}$, we have a splitting into line bundles

$$\mathcal{V}_{i+1,\mathbf{c}} \cong \bigoplus_{j=1}^{c_{i+1}} \mathcal{L}_{i+1,j}, \qquad \mathcal{V}_{i,\mathbf{d}} \cong \bigoplus_{k=c_i+1}^{c_i+d_i} \mathcal{L}_{i,k},$$

where $\mathcal{L}_{i,k}$ is the tautological line bundle for the corresponding weight space. Thus,

$$\mathrm{Hom}(\mathcal{V}_{i,\mathbf{d}}, \mathcal{V}_{i+1,\mathbf{c}}) \;\cong\; \bigoplus_{j=1}^{c_{i+1}} \bigoplus_{k=c_i+1}^{c_i+d_i} \mathcal{L}_{i+1,j} \otimes \mathcal{L}_{i,k}^*$$

and the formula (3.4) for the Euler class follows.

On the other hand, q is the projection from the partial flag variety $G_{\mathbf{a}}/P_{\mathbf{c},\mathbf{d}}$ to a point, where $\mathbf{a} = \mathbf{c} + \mathbf{d}$. The formula for equivariant integration on the full flag variety is given by

$$\int_{G_{\mathbf{a}}/B} f \;=\; \frac{\displaystyle\sum_{S_{\mathbf{a}}} (-1)^{l(w)} w \cdot f}{\displaystyle\prod_{i=1}^{e} \prod_{j<k\leq a_i} (x_{i,j} - x_{i,k})}. \tag{3.6}$$

Thus, we have that

$$\int_{\mathcal{Q}(\mathbf{c},\mathbf{d})} f = \int_{\mathcal{F}(\mathbf{c},\mathbf{d})} \int_{P_{\mathbf{c},\mathbf{d}}/B_{\mathbf{c}+\mathbf{d}}} \frac{1}{\prod_{i=1}^{e} c_i! d_i!} fD = \int_{G_{\mathbf{c}+\mathbf{d}}/B_{\mathbf{c}+\mathbf{d}}} \frac{1}{\prod_{i=1}^{e} c_i! d_i!} fD,$$

where $D = \prod_{j<k\leq c_i}(x_{i,j} - x_{i,k}) \prod_{c_i<\ell<m\leq c_i+d_i}(x_{i,\ell} - x_{i,m})$. The integration formula follows then from (3.6). □

3.2. **Demazure operators.** The splitting and merging maps can be described algebraically via Demazure operators acting on polynomial rings. The ith **Demazure operator** or **difference operator** $\Delta_i = \Delta_{s_i}$ acts on $\Bbbk[x_1, \dots x_n]$ by sending f to $\frac{f - s_i(f)}{x_i - x_{i+1}}$, where $s_i = (i, i+1)$ denotes the simple transposition acting by permuting the ith and $(i+1)$th variable. If $w = s_{i_1} s_{i_2} \dots s_{i_l}$ is a reduced expression of $w \in S_n$ we define $\Delta_w = \Delta_{i_1} \Delta_{i_2} \cdots s_{i_l}$. This is independent of the reduced expression, see [Dem73]. If G is a product of symmetric groups we denote by $w_0 \in G$ the longest element. Demazure operators satisfy the twisted derivation rule

$$\Delta_i(fg) = \Delta(f)g + s_i(f)\Delta_i(g)$$

and more generally for a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_l}$ the formula

$$\Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_l}(fg) \;=\; \sum A_{i_1} A_{i_2} \cdots A_{i_l}(f) B_{i_1} B_{i_2} \cdots B_{i_l}(g), \tag{3.7}$$

where the sum runs over all possible choices of either $A_j = \Delta_j$ and $B_j = \mathrm{id}$ or $A_j = s_j$ and $B_j = \Delta_j$ for each $1 \leq r$.

Proposition 3.6. *Let* $(\mathbf{c}, \mathbf{d})$ *be a vector composition.*

1. *Assume* $\mathbf{c} + \mathbf{d} = (c_i + d_i)\alpha_i$ *for some* i*, then*
$$\mathrm{int}(f) = \Delta_{w_0^{c_i,d_i}},$$
where $w_0^{c_i,d_i} \in S_{c_i+d_i}$ *denotes coset representative of* w_0 *in* $S_{c_i+d_i}/(S_{c_i} \times S_{d_i})$ *of minimal length. In general,* $\mathrm{int}(f)$ *is a product of pairwise commuting Demazure operators* $w_0^{c_i,d_i}$*, one for each* i*.*
2. *Merging successively from a vector composition* $(\alpha_i, \alpha_i, \ldots, \alpha_i)$ *of length* r *to* $r\alpha_i$ *equals the Demazure operator for* $w_0 \in S_r$*, in formulas*

$$\Delta_{w_0}(f) = \sum_{w \in S_r} (-1)^{l(w)} w(f) \frac{1}{\prod_{1 \leq i < j \leq r}(x_i - x_j)}. \tag{3.8}$$

3. *The split* $\mathbf{c}+\mathbf{d}$ *into* $\mathbf{c}$ *and* $\mathbf{d}$*, where either* $c_{i+1}d_i = 0$ *for all* i *or* $\mathbf{c}+\mathbf{d} = (c_i+d_i)\alpha_i$ *for some* i*, is just the inclusion from* $\mathbf{\Lambda}(\mathbf{c}+\mathbf{d})$ *to* $\mathbf{\Lambda}(\mathbf{c}, \mathbf{d})$*.*

Proof. Part (3) is obvious, since $E = 1$ by Remark 3.5. The second statement is clear for $r = 1$ and $r = 2$. The successive merge of the first $r-1$ α_i's is given by induction hypothesis, hence we only have to merge with the last α_i and obtain

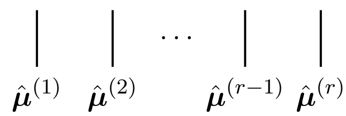
$$\begin{aligned} f \quad &\mapsto \quad \sum_{y \in S_{r-1} \times S_1} (-1)^{l(y)} \frac{y(f)}{(r-1)!} \frac{1}{\prod_{1 \leq i < j \leq r-1}(x_i - x_j)} =: P \\ &\mapsto \quad \sum_{z \in S_r / S_{r-1} \times S_1} (-1)^{l(z)} z(P) = \sum_{z,y} (-1)^{l(z)+l(y)} zy(f) \frac{1}{\prod_{1 \leq i < j \leq r}(x_i - x_j)}. \end{aligned}$$

Then (2) follows from the general formula for Δ_{w_0}, see [Ful97, 10.12]. Associativity of the merges and formula (2) gives $\mathrm{int}(f)\Delta_{w_0(c_i,d_i)} = \Delta_{w_0(c_i+d_i)} = \Delta_{w_0(c_i,d_i)}\Delta_{w_0(c_i,d_i)}$, where $w_0(c_i, d_i)$ and $w_0(c_i + d_i)$ are the longest elements in $S_{c_i} \times S_{d_i}$ and $S_{c_i+d_i}$ respectively. We have $\mathrm{int}(f) = \Delta_{w_0(c_i,d_i)}$, since $\Delta_{w_0(c_i,d_i)}$ surjects to the $S_{c_i} \times S_{d_i}$-invariants, and so (1) follows. □

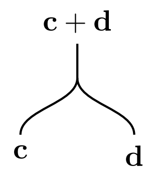
3.3. **The pictorial interpretation.** As shown in [VV11], the quiver Hecke algebra $R(\mathbf{d})$ is isomorphic to the diagram algebra introduced by Khovanov and Lauda in [KL09] (modulo the mentioned small differences in signs). This result allows to turn rather involved computations in the convolution algebra into a beautiful diagram calculus. Motivated by these ideas, we present now a graphical calculus for the algebra A where the split and merge maps from Proposition 3.4 are displayed as trivalent graphs. We will always read our diagrams from bottom to top. We represent

- the usual (vertical) algebra multiplication as vertical stacking of diagrams,
- horizontal multiplication as horizontal stacking of diagrams,

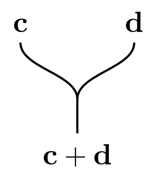
- the idempotent $e_{\hat{\boldsymbol{\mu}}}$ as a series of lines labeled with the parts of the vector composition or equivalently the blocks of the residue sequence (2.2),



- the morphism $(\mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{c} + \mathbf{d})$ as a joining of two strands $\mathbf{c}, \mathbf{d}$,



- the morphism $(\mathbf{c} + \mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d})$ as its mirror image,

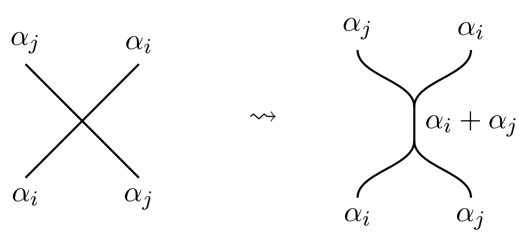


- multiplication by a polynomial is displayed by putting a box containing the polynomial.

A typical element of A is obtained by horizontally and vertically composing these morphisms. The composition of a merge followed by a split of the form $(\mathbf{c}, \mathbf{d}) \to (\mathbf{c} + \mathbf{d}) \to (\mathbf{d}, \mathbf{c})$ is also abbreviated as a **crossing** and denoted $(\mathbf{c}, \mathbf{d}) \to (\mathbf{d}, \mathbf{c})$.

Remark 3.7. Our calculus is an extension of the graphical calculus of [KL09]: given a crossing as in the Khovanov-Lauda picture, we interpret it as merge-split of the kth and $(k+1)$th strands:

(3.9)



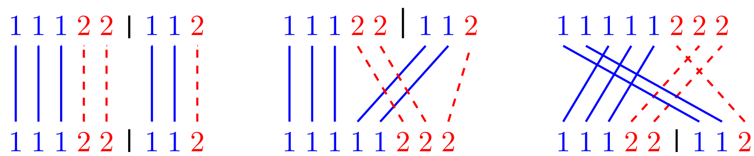
Assume it involves the ath α_i and bth α_j in the residue sequence.

- If $j \neq i, i+1$, then our map just flips the tensor factors $\mathbb{k}[x_{i,a}] \otimes \mathbb{k}[x_{j,b}] \mapsto \mathbb{k}[x_{j,b}] \otimes \mathbb{k}[x_{i,a}]$.
- If $i = j$, then we associate the Demazure operator Δ_k as in [KL09].
- For $j = i+1$, we multiply by $x_{i+1,a} - x_{i,b}$ followed by flipping the tensor factors.

In each case, this agrees with the action on $\mathcal{P}o\ell$ defined in [KL11], though Khovanov and Lauda have a single alphabet of variables, which they index by their left-position, as opposed to having separate alphabets for each node of the Dynkin diagram. We believe that when one incorporates non-unit dimension vectors, the latter convention is more convenient. Lemma 3.3 implies that crossings have degree 1, -2, or 0 according to the cases $j = i \pm 1$, $i = j$ or $j \neq i, i \pm 1$ respectively. Given a single strand labeled with the ath α_j we denote, following [KL09] multiplication with $x_{j,a}^R$ also by decorating the strand with R dots.

To keep track of the permutation w appearing in the Demazure operators Δ_w, it will be useful to interpret the numbers occurring in a residue sequence as colors from the chart $\{1, \ldots, e\}$. If the sequence is of length d, then permutations in S_d permuting only inside the colors can be viewed as elements of $S_{\mathbf{d}}$, where d_i is the number of i's appearing. We want to associate to each idempotent, elementary split or merge such a permutation as follows:

(3.10)



To an idempotent corresponding to a residue sequence of length d we attach the identity element in S_d which corresponds to the identity element in $S_{\mathbf{d}}$. To a split $(\mathbf{c}+\mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d})$ we associate the identity element in $S_{\mathbf{c}+\mathbf{d}}$. When viewed as a permutation in S_{c+d} it sends a to b if we find the kth j on place a in the residue sequence of $(\mathbf{c}+\mathbf{d})$, and on place b in the residue sequence for $(\mathbf{c}, \mathbf{d})$. To an elementary merge $M : (\mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{c}+\mathbf{d})$ we associate the product $w = w(M) = \prod_{i=1}^{e} w_0^{c_i, d_i}$; e.g. $(1,3,5,2,4)(6,8,7) \in S_{\mathbf{d}} = S_5 \times S_3 \subset S_8$ (written in cycle decomposition) to the merge in (3.10). Note that $\Delta_{\omega(M)}$ for a merge M is precisely the Demazure operator from Lemma 3.6.

Whereas we see this diagrams just as a tool of bookkeeping, they get interpreted in [Web12b, §3.3] as maps from the quiver Schur algebra into a **weighted KLR algebra**.

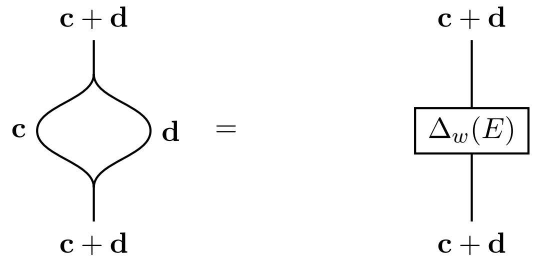
By (3.4), a split corresponds to an inclusion followed by multiplication with some polynomial. For a composition $X = M_1 M_2 \cdots M_t$ of splits and merges let $\omega(X) = \omega(M_1)\omega(M_2)\cdots\omega(M_t)$, where ω of a split is, compatible with the definition above, just the identity permutation. For a finite linear combination X' of such X's we let $\omega(X')$ be the sum of all $\omega(X)$'s of maximal length. The following relations are basic relations in the algebra A which we call **braid relations**.

Proposition 3.8. *Let* $\mathbf{b}, \mathbf{c}, \mathbf{d}$ *be vector compositions.*

(1) Let E be the Euler class attached to the splitting $(\mathbf{c}+\mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d})$ and $\Delta_w = \Delta_{w_0^{c_1,d_1}} \cdots \Delta_{w_0^{c_e,d_e}}$. Then

$$(\mathbf{c}+\mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{c}+\mathbf{d}) \quad = \quad \Delta_w(E)\big((\mathbf{c}+\mathbf{d}) \xrightarrow{\text{id}} (\mathbf{c}+\mathbf{d})\big)$$

or equivalently in terms of diagrams:

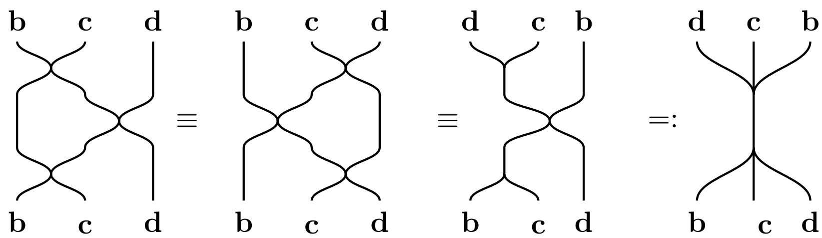


In particular, if $X := (\mathbf{c}, \mathbf{d}) \to (\mathbf{d}, \mathbf{c})$ *is a crossing then* $X^2 = fX$ *for some (possibly zero) polynomial* f.

(2) *Let*

$$\begin{aligned} X_1 &:= (\mathbf{b}, \mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{b}, \mathbf{d}) \longrightarrow (\mathbf{c}, \mathbf{d}, \mathbf{b}) \longrightarrow (\mathbf{d}, \mathbf{c}, \mathbf{b}) \\ X_2 &:= (\mathbf{b}, \mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{b}, \mathbf{d}, \mathbf{c}) \longrightarrow (\mathbf{d}, \mathbf{b}, \mathbf{c}) \longrightarrow (\mathbf{d}, \mathbf{c}, \mathbf{b}) \\ X_3 &:= (\mathbf{b}, \mathbf{c}, \mathbf{d}) \longrightarrow (\mathbf{b} + \mathbf{c} + \mathbf{d}) \longrightarrow (\mathbf{d}, \mathbf{c}, \mathbf{b}) \end{aligned}$$

Then $X_1 = X_2 + R$ *and* $X_1 = X_3 + R'$, *where* $\omega(R), \omega(R') < \omega(X_1) = \omega(X_2)$. *Hence, up to lower order terms* $X_1 \equiv X_2 \equiv X_3$, *in diagrams*



At the moment, the lower order terms are beyond any satisfactory control.

Proof. Let $x = w_0^{c_1,d_1} \cdot w_0^{c_e,d_e}$. Recall that $\Delta_i(gf) = \Delta_i(g)f + s_i(g)\Delta_i(f)$ for any f, g and $\Delta_i(f) = 0$ if f is s_i-invariant. Hence $\Delta_x(Ef) = \Delta_x(E)f$ for any total invariant polynomial f and the first claim follows.

Let now $x_1 = w(X_1)$ and $x_2 = w(X_2)$. Then obviously $x_1 = x_2$ and $X_1 = \alpha\Delta_{x_1} + R_1$ and $X_2 = \beta\Delta_{x_2} + R_2$ for some polynomials α, β and R_1, R_2 with $\omega(R_1), \omega(R_2) < \omega(X_1) = \omega(X_2)$. We still have to verify that $\alpha = \beta$ up to terms with ω of smaller length and the definition of (3.4). Let X_1 be the composition $E_3 \circ M_3 \circ E_2 \circ M_2 \circ E_1 \circ M_1$ where the M_i denotes the ith merge and E_i denotes the multiplication with an Euler class $E(i)$. We verify the claim by invoking the definitions and the equality $\Delta_i(gf) = s_i(g)\Delta_i(f) + \Delta(g)f$. First note that up to lower order terms

$$M_2 \circ E_1 = w\left(\prod_{i=1}^{e} \prod_{\substack{1 \le r \le c_{i+1} \\ c_i+1 \le s \le b_i + c_i}} (x_{i+1,r} - x_{i,s})\right) M_2,$$

where w is the permutation associated with M_2 via (3.10), hence

$$M_2 \circ E_1 = \prod_{\substack{1 \le r \le c_{i+1} \\ c_i + d_i + 1 \le s \le b_i + c_i + d_i}} (x_{i+1,r} - x_{i,s}) M_2.$$

Repeating these type of calculations we obtain

$$\begin{aligned} X_1 &= \prod_{i=1}^{e} \Bigg(\prod_{\substack{1\le r\le d_{i+1}\\ d_i+1\le s\le c_i+d_i}} (x_{i+1,r}-x_{i,s}) \prod_{\substack{1\le r\le d_{i+1}\\ c_i+d_i+1\le s\le b_i+c_i+d_i}} (x_{i+1,r}-x_{i,s}) \\ &\times \prod_{\substack{d_{i+1}+1\le r\le c_{i+1}+d_{i+1}\\ c_i+d_i+1\le s\le b_i+c_i+d_i}} (x_{i+1,r}-x_{i,s}) \Bigg) M_3\circ M_2\circ M_1 \\ X_2 &= \prod_{i=1}^{e} \Bigg(\prod_{\substack{d_{i+1}+1\le r\le c_{i+1}+d_{i+1}\\ c_i+d_i+1\le s\le b_i+c_i+d_i}} (x_{i+1,r}-x_{i,s}) \prod_{\substack{1\le r\le d_{i+1}\\ c_i+d_i+1\le s\le b_i+c_i+d_i}} (x_{i+1,r}-x_{i,s}) \\ &\times \prod_{\substack{1\le r\le d_{i+1}\\ d_i+1\le s\le c_i+d_i}} (x_{i+1,r}-x_{i,s}) \Bigg) M_3\circ M_2\circ M_1 \end{aligned}$$

and therefore $\alpha=\beta$. Note also that if $M_i=\Delta_{w_i}$ then $M_3M_2M_1$ is the Demazure operator corresponding to the product $w=w_3w_2w_1$ and the last claim follows as well. □

Example 3.9. *Let $e=3$ and $\mathbf{b}=(2,1,0),\mathbf{c}=(1,1,0),\mathbf{d}=(1,1,0)$. Then $X_1\equiv X_2\equiv X_3:(\mathbf{b},\mathbf{c},\mathbf{d})\longrightarrow(\mathbf{d},\mathbf{c},\mathbf{b})$ equal*

$$(x_{2,1}-x_{1,2})(x_{2,1}-x_{1,3})(x_{2,1}-x_{1,4})(x_{2,2}-x_{1,3})(x_{2,2}-x_{1,4})\Delta_{s_1s_2s_3s_1s_2}\Delta_{s_{\overline{1}}s_{\overline{2}}s_{\overline{1}}}$$

up to lower order terms where s_i (respectively $\overline{s}_i$) denotes the ith transposition for the strands colored by 1 (respectively by 2) in $S_{\mathbf{d}}=S_5\times S_3$.

3.4. **An explicit basis of A.** We construct and describe explicitly a basis of A by geometric arguments and then interpret it diagrammatically.

We again fix a dimension vector $\mathbf{d}$. Let $\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\lambda}}\in\mathrm{VComp}_e(\mathbf{d})$ with residue sequences $\operatorname{res}\hat{\boldsymbol{\mu}}$ and $\operatorname{res}\hat{\boldsymbol{\lambda}}$; and let $S^{\hat{\boldsymbol{\mu}}}$ and $S^{\hat{\boldsymbol{\lambda}}}$ be the subgroups of $S_{\mathbf{d}}$ preserving the blocks. Note that $\hat{\boldsymbol{\mu}}$ is determined uniquely by $\operatorname{res}\hat{\boldsymbol{\mu}}$ and $S^{\hat{\boldsymbol{\mu}}}$. Now, fix a permutation $p\in S_d$ such that $p(\operatorname{res}\hat{\boldsymbol{\mu}})=\operatorname{res}\hat{\boldsymbol{\lambda}}$ (hence $p\in S_{\mathbf{d}}$) and let p_- be its shortest double coset representative in $S^{\hat{\boldsymbol{\lambda}}}\backslash S_{\mathbf{d}}/S^{\hat{\boldsymbol{\mu}}}$.

Note that in the graphical pictures (3.10) an element $p_-\in S_{\mathbf{d}}$ is a shortest coset representative in $S^{\hat{\boldsymbol{\mu}}}\backslash S_{\mathbf{d}}/S^{\hat{\boldsymbol{\lambda}}}$ if and only if strands with the same color which end or start in the same block do not cross. In particular, idempotents, splits and merges correspond to shortest double coset representatives.

To the triple $\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\lambda}},p_-$ we associate a composition of merges and splits from $\hat{\boldsymbol{\mu}}$ to $\hat{\boldsymbol{\lambda}}$ as follows: First, let $\hat{\boldsymbol{\mu}}'$ be the unique vector composition with residue sequence

$$\operatorname{res}(\hat{\boldsymbol{\mu}}') \;=\; p_-^{-1}(\operatorname{res}\hat{\boldsymbol{\lambda}})=p_-^{-1}p(\operatorname{res}\hat{\boldsymbol{\mu}}),$$

such that $S^{\hat{\boldsymbol{\mu}}'} = S^{\hat{\boldsymbol{\mu}}} \cap p_- S^{\hat{\boldsymbol{\lambda}}} p_-^{-1}$, and similarly $\hat{\boldsymbol{\lambda}}'$ the unique vector composition so that

$$\operatorname{res}(\hat{\boldsymbol{\lambda}}') = p_-(\operatorname{res}\hat{\boldsymbol{\mu}}) = p_- p^{-1}(\operatorname{res}\hat{\boldsymbol{\lambda}})$$

and $S^{\hat{\boldsymbol{\lambda}}'} = S^{\hat{\boldsymbol{\lambda}}} \cap p_-^{-1} S^{\hat{\boldsymbol{\mu}}} p_-$. In other words we first refine the blocks of $\operatorname{res}\hat{\boldsymbol{\mu}}$ into $\operatorname{res}\hat{\boldsymbol{\mu}}'$ as coarse as possible such that p_- sends all elements in a block of $\operatorname{res}\hat{\boldsymbol{\mu}}$ to the same block of $\operatorname{res}\hat{\boldsymbol{\lambda}}$. Similarly refine $\hat{\boldsymbol{\lambda}}$ into $\hat{\boldsymbol{\lambda}}'$ again as coarse as possible such that p_-^{-1} sends all elements in a block to the same block. For instance, if $\operatorname{res}\hat{\boldsymbol{\mu}} = 1,2|3,3|2|1,2|1$ and $\operatorname{res}\hat{\boldsymbol{\lambda}} = 1,1,2,3|1,2,3|2$ then $\operatorname{res}\hat{\boldsymbol{\mu}}' = 1,2|3|3|2|1,2|1$ and $\hat{\boldsymbol{\lambda}}' = 1,2|3|1|3|1.2|2$ with the permutation p_- displayed in Figure 6.

Lemma 3.10. *The vector compositions $\hat{\boldsymbol{\mu}}', \hat{\boldsymbol{\lambda}}'$ have the same length (i.e. number of blocks), say ℓ, and $\hat{\boldsymbol{\lambda}}'$ can be obtained from $\hat{\boldsymbol{\mu}}'$ by some permutation $q \in S_\ell$ of its blocks inducing p_- on the residue sequences.*

Proof. Each part of $\hat{\boldsymbol{\mu}}'$ is given by the numbers of 1's, 2's, etc. in a given block of the residue sequence for $\hat{\boldsymbol{\mu}}$ which are sent to a fixed block of $\hat{\boldsymbol{\lambda}}$. It is obtained by refining $\hat{\boldsymbol{\mu}}$ according to the blocks of $\operatorname{res}\hat{\boldsymbol{\lambda}}$ (read from left to right) they are sent to. On the other hand, the parts $\hat{\boldsymbol{\lambda}}'$ have the same description, just reversing the roles of $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$; thus, we just change the order of the blocks, which gives the permutation $q \in S_\ell$ as asserted. □

For each $p \in S_{\hat{\boldsymbol{\lambda}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\mu}}}$, we have the associated permutation q as in Lemma 3.10 and from now on fix a reduced decomposition $q = s_{i_1} \cdots s_{i_\ell}$ in terms of simple transpositions $s_i = (i, i+1) \in S_\ell$. We also fix the morphism $\hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}'$ given by the corresponding composition of crossings and define

$$\hat{\boldsymbol{\mu}} \overset{p;1}{\Longrightarrow} \hat{\boldsymbol{\lambda}} := \hat{\boldsymbol{\mu}} \longrightarrow \hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}' \longrightarrow \hat{\boldsymbol{\lambda}}. \tag{3.11}$$

Of course, this definition depends on our choice of reduced expression for q. For $h \in \Lambda(\hat{\boldsymbol{\mu}}')$ let $\hat{\boldsymbol{\mu}}' \xrightarrow{h} \hat{\boldsymbol{\mu}}'$ be multiplication with h and set

$$\hat{\boldsymbol{\mu}} \overset{p;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}} := \hat{\boldsymbol{\mu}} \longrightarrow \hat{\boldsymbol{\mu}}' \xrightarrow{h} \hat{\boldsymbol{\mu}}' \xrightarrow{q} \hat{\boldsymbol{\lambda}}' \longrightarrow \hat{\boldsymbol{\lambda}}.$$

Theorem 3.11. *The morphisms $\hat{\boldsymbol{\mu}} \overset{p;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ generate A as a $\Bbbk$-vector space. In fact, if we let range*

- *$(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ over all ordered pairs of vector compositions of type e,*
- *p over the minimal coset representatives w in $S_{\hat{\boldsymbol{\lambda}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\mu}}}$ and*
- *h over a basis for $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}')$,*

then the morphisms $\hat{\boldsymbol{\mu}} \overset{p;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ form a basis of A.

Remark 3.12. If we choose different reduced decompositions for q as in Lemma 3.10, the resulting elements (3.11) will differ by a linear combination of maps corresponding to shorter double cosets (Proposition 3.8) which also shows that non-reduced decompositions can be replaced by reduced ones by changing h.

For a permutation q as in Lemma 3.10 with its fixed reduced expression $q = s_{i_1} \cdots s_{i_\ell}$ set $q^{(j)} = s_{i_j} \cdots s_{i_1}$. From this we can read off a string diagram of q, as in Khovanov-Lauda [KL09]. We let $\hat{\boldsymbol{\mu}}_{2j} = q^{(j)}(\hat{\boldsymbol{\mu}}')$ and $\hat{\boldsymbol{\mu}}_{2j+1}$ be the merge of $\hat{\boldsymbol{\mu}}_{2j}$ at index i_{j+1}. Then, of course, $\hat{\boldsymbol{\mu}}_{2j}$ is a split of $\hat{\boldsymbol{\mu}}_{2j}$ at index i_j. In particular, $\hat{\boldsymbol{\mu}}_{2k} = \hat{\boldsymbol{\lambda}}'$.

For each $w \in S_{\hat{\boldsymbol{\mu}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\lambda}}}$, let $p(w)$ be the unique permutation in S_d of minimal length which acts as w does on each individual color, where $[1, d]$ is colored according to $\hat{\boldsymbol{\mu}}$ for the domain and $\hat{\boldsymbol{\lambda}}$ for the image. Put another way, if we think of $S_{\mathbf{d}}$ as acting on colored alphabets $1^1, \dots, d_1^1, 1^2, \cdots d_2^2, \dots, 1^e, \dots, d_e^e$, and let $p(\hat{\boldsymbol{\mu}})$ be the unique permutation of minimal length sending this sequence to one colored according to the residue sequence of $\hat{\boldsymbol{\mu}}$, then $p(w) = p(\hat{\boldsymbol{\lambda}}) \cdot w \cdot p(\hat{\boldsymbol{\mu}})^{-1}$ (where $p(\hat{\boldsymbol{\lambda}})$ is the unique permutation sending a totally ordered sequence of 1's, 2's etc. to $\operatorname{res} \hat{\boldsymbol{\lambda}}$).

Proof of Theorem 3.11. Our method of proof is to show that this remains a basis when we take the associated graded of our algebra with respect to a geometrically defined filtration.

Forgetting the representation (and only keeping the flags) defines a canonical map

$$\Phi : \quad \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) \to \mathcal{F}(\hat{\boldsymbol{\mu}}) \times \mathcal{F}(\hat{\boldsymbol{\lambda}}) \tag{3.12}$$

which is $G_{\mathbf{d}}$-equivariant. Every fiber is the space of quiver representations on a vector space preserving a particular pair of flags, which is naturally an affine space (one can assume both flags are spanned by vectors in a fixed basis, and thus the condition of preserving the pair of flags is simply requiring certain matrix coefficients to vanish). Thus, over each $G_{\mathbf{d}}$-orbit, the map is an affine bundle, but with a different fiber over each orbit. The $G_{\mathbf{d}}$-orbits on $\mathcal{F}(\hat{\boldsymbol{\mu}}) \times \mathcal{F}(\hat{\boldsymbol{\lambda}})$ are in bijection with the double coset space $S_{\hat{\boldsymbol{\mu}}} \backslash S_{\mathbf{d}} / S_{\hat{\boldsymbol{\lambda}}}$, namely if we choose completions of the partial flags to complete flags, their relative position is an element of $S_{\mathbf{d}}$, and its double coset is independent of the completion chosen. That is, using the Bruhat decomposition with fixed lifts of elements from $S_{\mathbf{d}}$ to $G_{\mathbf{d}}$ it sends an element $(xP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, yP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})})$, $x, y \in S_{\mathbf{d}}$ to $x^{-1}y$, and a shortest double coset representative w defines the orbit $\mathcal{F}_w$ of all elements of the form $(gP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, gwP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})})$, $g \in G_{\mathbf{d}}$. A pair of flags contained in $\mathcal{F}_w$ is said to have relative position w.

The orbit $\mathcal{F}_w$ is an affine bundle over $G_{\mathbf{d}}/(P_{\hat{\boldsymbol{\mu}}} \cap wP_{\hat{\boldsymbol{\lambda}}}w^{-1})$ via the bundle map

$$(gP_{\mathrm{t}(\hat{\boldsymbol{\mu}})}, gwP_{\mathrm{t}(\hat{\boldsymbol{\lambda}})}) \mapsto g(P_{\hat{\boldsymbol{\mu}}} \cap wP_{\hat{\boldsymbol{\lambda}}}w^{-1}),$$

and so both the orbit and its preimage in $G_{\mathbf{d}}$ have (equivariant) homology concentrated in even degrees.

Using the surjection Φ we can partition $\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ into the preimages of the orbits and filter the space by letting $\mathcal{H}_k = \bigcup_{l(w) \le k} \Phi^{-1}(\mathcal{F}_w)$ be the union of orbits with length of the shortest representative at most k. The space $\mathcal{H}_k \setminus \mathcal{H}_{k-1} = \bigcup_{l(w)=k} \Phi^{-1}(\mathcal{F}_w)$ is a disjoint union of affine bundles over partial flag varieties, and in particular has even homology. The usual spectral sequence for a filtered topological space shows that $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}))$ has a filtration whose associated graded is the sum of the homologies of these orbits (the spectral sequence degenerates immediately for degree reasons because of the concentrations in even degrees).

Thus, it suffices to show that the morphisms $\hat{\boldsymbol{\mu}} \xRightarrow{w;h} \hat{\boldsymbol{\lambda}}$ pass to a basis of the associated graded. That is, if we fix w, that the classes $\hat{\boldsymbol{\mu}} \xRightarrow{w;h} \hat{\boldsymbol{\lambda}}$ pullback to a basis of $H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$.

It is enough to show that $\hat{\boldsymbol{\mu}} \xRightarrow{w;1} \hat{\boldsymbol{\lambda}}$ goes to the fundamental class of $\Phi^{-1}(\mathcal{F}_w)$, since adding the h simply has the effect of multiplying by classes that range over a basis of $H^*(\Phi^{-1}(\mathcal{F}_w)/G)$. Consider the iterated fiber product

$$\begin{aligned}\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) &= \mathcal{Q}(\hat{\boldsymbol{\mu}}_0, \hat{\boldsymbol{\mu}}) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_0)} \mathcal{Q}(\hat{\boldsymbol{\mu}}_0, i_1) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_1)} \mathcal{Q}(\hat{\boldsymbol{\mu}}_2, i_1) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_2)} \cdots \\ &\quad \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-2})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-2}, i_k) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k-1})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k}, i_k) \times_{\mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k})} \mathcal{Q}(\hat{\boldsymbol{\mu}}_{2k}, \hat{\boldsymbol{\lambda}}),\end{aligned}$$

where $\mathcal{Q}(\hat{\boldsymbol{\mu}}_0, \hat{\boldsymbol{\mu}})$ denotes the subset of $\mathcal{Q}(\hat{\boldsymbol{\mu}}_0)$ such that the associated graded remains semi-simple after coarsening the flag to one of type $\hat{\boldsymbol{\mu}}$. Recall that by the definition of $\hat{\boldsymbol{\mu}}_{2k+1}$ we have that

$$\mathcal{Q}(\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\mu}}_{k+1}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, i_{k+1}) \qquad \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\mu}}_{k-1}) = \mathcal{Q}(\hat{\boldsymbol{\mu}}_k, i_k).$$

We can think of the above fiber product as the space of representations with compatible flags $F, F_0, \dots, F_{2k}, F'$ of dimension vectors $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\mu}}_0, \dots, \hat{\boldsymbol{\mu}}_{2k}, \hat{\boldsymbol{\lambda}}$ such that if any two consecutive flags have subspaces of the same size, those spaces must coincide. We have a map

$$b^w : \qquad \mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) \to \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$$

which remembers the flags F and F' and $\hat{\boldsymbol{\mu}} \xRightarrow{w;1} \hat{\boldsymbol{\lambda}} = b_*[\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})]$ is the push-forward of the fundamental class of $\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})$.

More generally, we can identify $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}_{2k})$ with the classes in $H^*(\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}))$ generated by the Chern classes of the tautological bundles corresponding to the successive quotients of the flag F_{2k}. Then $\hat{\boldsymbol{\mu}} \xRightarrow{w;h} \hat{\boldsymbol{\lambda}} = b_*(h \cap [\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})])$ is the pushforward of the cap product of h with the fundamental class (the Poincaré dual of the class h).

To establish the linearly independence the following Lemma will be used.

Lemma 3.13. *The image of the map b^w lies in $\mathcal{H}_{\ell(w)}$ and induces an isomorphism between the locus in $\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})$ where F and F' have relative position w and $\Phi^{-1}(\mathcal{F}_w)$.*

Proof. Let $p^{(j)}$ be the element of S_d induced by acting on the blocks of $\operatorname{res} \hat{\boldsymbol{\mu}}'$ with $q^{(j)}$. By the usual Bruhat decomposition, we must have that F_{2j}, considered as a flag on $\oplus_{i=1}^e V_i$, has relative position $\leq p^{(j)}$, i.e. smaller or equal $p^{(j)}$ in Bruhat order, with respect to F and $\leq p^{(j)} p(w)^{-1}$ with respect to F'. Since $p^{(j)} \leq p(w)$ in right Bruhat order (its inversions are a subset of those of $p(w)$), this condition uniquely specifies F_{2j} (which also uniquely specifies F_{2j-1}, since this is a coarsening of F_{2j}).

Thus, we only need to show that this flag is compatible with the decomposition $\oplus_{i=1}^e V_i$ and strictly preserved by the accompanying quiver representation f. The former is clear, since we can choose a basis compatible with this decomposition such that both F and F' contain only coordinate subspaces; thus, the spaces of F_j are coordinate for this basis as well. It follows that they are also compatible with the decomposition. The latter is a well-known property of the Bruhat decomposition: if a nilpotent preserves two different flags F and F', then it preserves any other flag F'' that fits into a chain where the relative position of F and F'' is v, that of F'' to F' is w, and $\ell(vw) = \ell(v) + \ell(w)$. We simply apply this to f thought of as a nilpotent endomorphism of $\oplus_{i=1}^{e} V_i$. ☐

Thus, we have a Cartesian diagram

$$\begin{array}{ccc} \Phi^{-1}(\mathcal{F}_w) & \hookrightarrow & \mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) \\ \wr\uparrow & & \uparrow b \\ b^{-1}(\Phi^{-1}(\mathcal{F}_w)) & \hookrightarrow & \mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}}) \end{array}$$

If we push-forward and pull-back $h \cap [\mathcal{H}(\hat{\boldsymbol{\mu}}, w, \hat{\boldsymbol{\lambda}})]$ from the bottom right to the upper left, we obtain the class of $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ in the associated graded with respect to the geometric filtration, thought of as a Borel-Moore class on $\Phi^{-1}(\mathcal{F}_w)$. We can also calculate this class by pull-back and pushforward; this goes to $h \cap [\Phi^{-1}(\mathcal{F}_w)]$, where we consider h now as a class on $\Phi^{-1}(\mathcal{F}_w)$ by pull-back. The tautological bundles for F_{2k} pulled back to $\Phi^{-1}(\mathcal{F}_w)$ induce an isomorphism $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}_{2k}) \cong H^*(\Phi^{-1}(\mathcal{F}_w)/G)$, and cap product with the fundamental class induces an isomorphism $H^*(\Phi^{-1}(\mathcal{F}_w)/G) \cong H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$. Thus, the classes $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ pull-back to a basis of $H_*^{BM}(\Phi^{-1}(\mathcal{F}_w)/G)$. Ranging over all w, we obtain a basis of $H_*^{BM}(\mathcal{H}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}))$, since it is a basis of the associated graded. ☐

Remark 3.14. It is worth noting that Theorem 3.11 could also be proved diagrammatically. The easy part is the linearly independence which follows analogously to the arguments in [KL09] as follows: Let $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}$ be fixed. By construction, $\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}$ is zero if $w \neq w_-$. Let S be the split from $\hat{\boldsymbol{\lambda}}$ into unit vectors and M the merge from unit vectors to $\hat{\boldsymbol{\mu}}$. Then a linear combination $\sum \alpha_{w,h}(\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}})$ is zero if and only if $\sum \alpha_{w,h} S \circ (\hat{\boldsymbol{\mu}} \overset{w;h}{\Longrightarrow} \hat{\boldsymbol{\lambda}}) \circ M = 0$ (because M is surjective and composing with S is injective). The linear independence is therefore given by [KL09, Theorem 2.5].

Theorem 3.11 provides a basis which appears natural from the diagrammatical as well as geometrical point of view. Proposition 3.8 provides some fairly obvious relations, but unfortunately we do not have a complete set of relations defining our algebras. Although we have reduced the question to one of linear algebra, it appears to be quite difficult linear algebra and we are left with the following problem:

Problem 3.15. *Find an explicit presentation with generators and defining relations of the algebras $A_{\mathbf{d}}$ (or a Morita equivalent algebra).*

The following result at least describes its center (which is a Morita invariant):

Lemma 3.16. *The ring of total invariants $R(\mathbf{d})^{S_{\mathbf{d}}}$ is isomorphic to the center of $\mathbf{A}_{\mathbf{d}}$.*

Proof. The proof is essentially identical to [KL09, 2.9]. Restricting the action maps simultaneously to the total invariants, defines a map α from $R(\mathbf{d})^{S_d}$ into the center of $\mathbf{A}_{\mathbf{d}}$. For each $\hat{\boldsymbol{\mu}} \in \mathrm{VComp}_e(\mathbf{d})$, we can obtain a new vector composition $\hat{\boldsymbol{\mu}}' \in \mathrm{VComp}_e(\mathbf{d})$ by splitting each dimension vector between each colors. By Proposition 3.4, composing with this defines an inclusion from $Ae_{\hat{\boldsymbol{\mu}}}$ to $Ae_{\hat{\boldsymbol{\mu}}'}$. Now, we obtain a third vector composition $\hat{\boldsymbol{\mu}}'' \in \mathrm{VComp}_e(\mathbf{d})$ by reordering all the colors in the residue sequence, the smallest to the left, by permuting the blocks by applying crossings. In this case the Demazure operator from Proposition 3.4 is the identity and we get an inclusion from $Ae'_{\hat{\boldsymbol{\mu}}}$ to $Ae_{\hat{\boldsymbol{\mu}}''}$. It follows that $Ae_{\hat{\boldsymbol{\mu}}}$ is a submodule (not necessarily direct summand) of $Ae_{\hat{\boldsymbol{\mu}}''}$. In particular, every central element acts non-trivially on some $P(\hat{\boldsymbol{\mu}}'') = Ae_{\hat{\boldsymbol{\mu}}''}$. Furthermore, every $P(\hat{\boldsymbol{\mu}}'')$ is just obtained by inducing the direct sum of modules of the form $P((d_1, 0, \dots)), P((0, d_2, 0, \dots)), \dots), \dots$; we thus have an injective homomorphism $\beta\colon Z(A_{\mathbf{d}}) \to e_{\mu_{\mathbf{d}}} A e_{\mu_{\mathbf{d}}} \cong R(\mathbf{d})^{S_d}$ given by $\psi(z) = e_{\mu_{\mathbf{d}}}$. Furthermore, the composition $\beta \circ \alpha$ is the identity on $R(\mathbf{d})^{S_d}$. Thus, β is surjective as well, and gives the desired isomorphism. □

4. Higher level generalizations

In this section we extend our convolution algebras to a "higher level version." Motivated by the construction of quiver varieties of Nakajima [Nak94], [Nak98], [Nak01] we consider not only quiver representations of Γ, but enrich them with extra data. For a further development of our approach, see also [Web12b].

We equip the quiver Γ with shadow vertices, one for each $i \in \mathbb{V}$ together with an arrow pointing from the ith vertex to the ith shadow vertex, see Figure 1 were the shadow vertices are drawn in red/grey. For given $\nu : \mathbb{V} \to \mathbb{Z}_{\geq 0}$, we extend the affine space (2.1) of representations of a fixed dimension vector $\mathbf{d}$ to the affine space

$$\operatorname{Rep}_{\mathbf{d};\nu} := \operatorname{Rep}_{\mathbf{d}} \times \bigoplus_i \operatorname{Hom}_K\left(V_i, K^{\nu(i)}\right), \qquad \operatorname{Rep}_\nu = \bigsqcup_{\mathbf{d}} \operatorname{Rep}_{\mathbf{d};\nu}$$

of representations (V, f, γ) shadowed by vector spaces $K^{\nu(i)}$. It comes endowed with the product $G_{\mathbf{d}}$-action.

Definition 4.1. *Assume we are given a* **weight data** *consisting out of*

- *an ℓ-tuple $\underline{\boldsymbol{\nu}} = (\nu_1, \dots, \nu_\ell)$ of maps $\mathbb{V} \to \mathbb{Z}_{\geq 0}$, called an ℓ-***weight***;*
- *an $\ell{+}1$-tuple $\grave{\mu} = (\hat{\boldsymbol{\mu}}(0), \hat{\boldsymbol{\mu}}(1), \dots, \hat{\boldsymbol{\mu}}(\ell))$ of vector compositions of no fixed length, but all of type e.*

Then we call $\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell) \in \operatorname{VComp}_e(\mathbf{d})$ the **associated vector composition** *and denote by $\mathfrak{Q}(\grave{\mu})$ the subspace of $\mathcal{Q}(\hat{\boldsymbol{\mu}}) \times \bigoplus_i \operatorname{Hom}_K(V_i, K^{\nu(i)})$ defined as*

$$\mathfrak{Q}(\grave{\mu}) = \Big\{ ((V, f, F), \{\gamma_i\}) \mid \gamma_i(\grave{W}_i(k)) \subset K^{\nu_1(i) + \cdots + \nu_k(i)} \Big\},$$

where as usual $K^a \subset K^b$ for $a \leq b$ is the subspace spanned by the first a unit vectors, and $\grave{W}_i(1) \subset \grave{W}_i(2) \subset \cdots \subset \grave{W}_i(\ell+1) = V_i$ is the partial flag at the vertex i coarsening F_i and obtained by picking out the largest subspace corresponding to each part of $\grave{\mu}$.

Just as an unadorned representation carries a canonical socle filtration (i.e. a filtration starting with the maximal semi-simple submodule and proceeding such that the successive quotients are maximal semi-simple), an extended representation $(V, f, \gamma) \in \operatorname{Rep}_{\mathbf{d};\nu}$ carries a slightly modified filtration which starts with $\{0\} \subset R_1 = (W, f, \gamma)$ such that (W, f) is the largest subrepresentation (V, f) for which $\gamma(W_i) \subset K^{\nu_1(i) + \cdots + \nu_k(i)}$ and proceeds inductively by considering the corresponding first step in $(V/W, \overline{f}, \overline{\gamma})$ pulled back to (V, f). The dimension vectors of these subquotients define a weight data $(\underline{\boldsymbol{\nu}}, \grave{\mu})$ where $\grave{\mu}$ denotes the type of the multi-flag R_i/R_{i-1} induced by the filtration. We call this the **extended socle filtration**.

4.1. **Extended convolution algebras.** Generalizing (2.3), we have a projection map

$$p\colon \mathfrak{Q}(\grave{\mu}) \to \operatorname{Rep}_{\mathbf{d};\nu}/G_{\mathbf{d}}$$

where $\nu := \nu_1 + \cdots + \nu_\ell$, and we can study the convolution algebra

$$\tilde{A}^{\underline{\boldsymbol{\nu}}} := \operatorname{Ext}^*_{D(\operatorname{Rep}_\nu/G)}\Big(\bigoplus_{\grave{\mu}} p_*\Bbbk_{\mathfrak{Q}(\grave{\mu})}, \bigoplus_{\grave{\mu}} p_*\Bbbk_{\mathfrak{Q}(\grave{\mu})}\Big) \cong \bigoplus_{\grave{\mu},\grave{\nu}} H^{BM}_*(\mathfrak{H}(\grave{\mu},\grave{\nu})),$$

where $\mathfrak{H}(\grave{\mu},\grave{\nu}) \cong \mathfrak{Q}(\grave{\mu}) \times_{\operatorname{Rep}_{\mathbf{d};\nu}} \mathfrak{Q}(\grave{\nu})/G$ and the sum runs over all weight data $(\underline{\boldsymbol{\nu}}, \grave{\mu})$ with associated vector composition $\hat{\boldsymbol{\mu}} \in \operatorname{VComp}_e(\mathbf{d})$.

No individual algebra (with vertical multiplication given by convolution) in this family has a notion of horizontal multiplication. Instead, there is a horizontal multiplication $\tilde{A}^{\underline{\boldsymbol{\nu}}} \times \tilde{A}^{\underline{\boldsymbol{\nu}}'} \to \tilde{A}^{\underline{\boldsymbol{\nu}} \cup \underline{\boldsymbol{\nu}}'}$ which concatenates the tuples of weights. This can, of course, be organized in a single algebra $\tilde{A}$, but for our purposes it is more profitable to think of the separate algebras $\tilde{A}^{\underline{\boldsymbol{\nu}}}$. The following is a direct consequence of our definitions.

Proposition 4.2. *For each fixed dimension vector* $\mathbf{d}$*, horizontal composition induces a right* $\mathbf{A}$*-module structure on* $\mathbf{A}^{\underline{\boldsymbol{\nu}}}$*.*

Using horizontal and vertical composition, the algebra $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ is generated by a small number of elements (like $\mathbf{A}$ is) which are defined in some sense locally and also have an easy diagrammatic description. We mimic this construction now in the extended case incorporating the shadow vertices. Of course, we still have the old merges and splits not involving the shadow vertices, but also have a new "move" on ℓ-tuples of compositions.

Definition 4.3. *We call* $\grave{\lambda}$ *a* **left shift** *of* $\grave{\mu}$ *(and* $\grave{\mu}$ *a* **right shift** *of* $\grave{\lambda}$*) by* $\mathbf{c}$ *if for some index* m*, we have that*

$$\grave{\lambda}(m) = \grave{\mu}(m) \cup \mathbf{c} \text{ and } \grave{\mu}(m+1) = \mathbf{c} \cup \grave{\lambda}(m+1).$$

In words, if a vector $\mathbf{c}$ *has been shifted from the start of the* $m+1$*-st composition of* $\grave{\mu}$ *to the end of the* m*th composition for* $\grave{\lambda}$*.*

If $\grave{\lambda}$ is a left shift of $\grave{\mu}$, then $\mathfrak{Q}(\grave{\lambda})$ is naturally a subspace of $\mathfrak{Q}(\grave{\mu})$. Thus, generalizing the construction after Definition 3.1, we can think of it as a correspondence (read from right to left)

$$\begin{array}{ccc} & \mathfrak{Q}(\grave{\lambda}) & \\ \swarrow & & \searrow \\ \mathfrak{Q}(\grave{\lambda}) & & \mathfrak{Q}(\grave{\mu}) \end{array}$$

Similarly, for right shifts we can reverse this correspondence and read from left to right. Thus, the fundamental class of this correspondence gives elements of $A^{\underline{\boldsymbol{\nu}}}$ corresponding to left and right shifts, which we denote by $\grave{\mu} \to \grave{\lambda}$ and $\grave{\lambda} \to \grave{\mu}$.

Proposition 4.4. *The degree of the map* $\grave{\lambda} \to \grave{\mu}$ *or* $\grave{\mu} \leftarrow \grave{\lambda}$ *associated to right or left shift by* $\mathbf{c}$ *at the index* m *equals* $\sum_i c_i \nu_m(i)$*.*

Proof. This follows from a direct calculation. □

Theorem 3.11 generalizes immediately to the algebra $\tilde{A}^{\underline{\boldsymbol{\nu}}}$. We wish to define elements $\grave{\mu} \xRightarrow{w;h} \grave{\nu}$ in this case, generalizing those for A. Consider a minimal length coset representative w in $S_{\grave{\mu}} \backslash S_{\mathbf{d}} / S_{\grave{\nu}}$. We let $\grave{\mu} \xRightarrow{w;h} \grave{\lambda}$ be an arbitrarily chosen composition of merges, splits and left and right shifts such that if we ignore the left and right shifts the resulting element represents w, cf. Remark 3.12. If we let $S_{\grave{\mu}}$ be the Young subgroup associated to the concatenation of the parts of $\grave{\mu}$, then the following holds.

Proposition 4.5. *The morphisms $\grave{\mu} \xRightarrow{w;h} \grave{\lambda}$ form a basis of $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ where we let range*

- *$(\grave{\mu}, \grave{\nu})$ over ordered pairs of $(\ell + 1)$-tuples of vector compositions $\grave{\mu}$, $\grave{\lambda}$ of type e,*
- *w over minimal coset representatives in $S_{\grave{\mu}} \backslash S_{\mathbf{d}} / S_{\grave{\lambda}}$, and*
- *h over a basis for $\boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}_j)$.*

Proof. Analogous to the proof of Theorem 3.11; the important points are to note that

- the diagrams $\grave{\mu} \xRightarrow{w;h} \grave{\lambda}$ give Borel-Moore classes supported on pairs of flags with relative position $\leq w$, and
- modulo classes supported on pairs with relative position $< w$, the elements $\grave{\mu} \xRightarrow{w;h} \grave{\lambda}$ give a basis of the Borel-Moore classes of the subset with relative position w as we let h range over a basis of the appropriate polynomial ring.

This immediately shows that these elements are a basis. □

As in Section 3.1, we can also calculate how this morphism acts on the polynomial ring $\tilde{V}^{\underline{\boldsymbol{\nu}}}$ from (4.1). Using horizontal composition, it suffices to consider the case where $\underline{\boldsymbol{\nu}} = (\lambda)$ (so $\ell = 1$) and look at pairs $(\emptyset, \mathbf{d})$ and $(\mathbf{d}, \emptyset)$ of vector compositions of type e. Both $\mathfrak{H}((\emptyset, \mathbf{d}), (\mathbf{d}, \emptyset))$ and $\mathfrak{H}((\mathbf{d}, \emptyset), (\emptyset, \mathbf{d}))$ are simply $\mathcal{Q}(\mathbf{d})$, and thus their Borel-Moore homology is a rank 1 free module over $H^*(BG_{\mathbf{d}}) \cong \boldsymbol{\Lambda}(\mathbf{d})$ generated by the corresponding fundamental class. Their actions on $\tilde{V}^{\underline{\boldsymbol{\nu}}}$ are simply the actions of ι^* and ι_* where $\iota \colon \mathfrak{Q}(\mathbf{d}, \emptyset) \to \mathfrak{Q}(\emptyset, \mathbf{d})$ is the obvious inclusion map.

Proposition 4.6. *The following diagram commutes,*

$$\begin{array}{ccc} H_*^{BM}(\mathfrak{Q}(\mathbf{d}, \emptyset)) & \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} & H_*^{BM}(\mathfrak{Q}(\emptyset, \mathbf{d})) \\ \Big\downarrow \text{Borel} & & \Big\downarrow \text{Borel} \\ \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell)) & \underset{1}{\overset{\prod_{k=1}^{e} t_k^{\lambda(k)}}{\rightleftarrows}} & \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell)) \end{array}$$

where $t_k = x_{k,1} \cdots x_{k,d_k}$ is the highest degree elementary symmetric function in the alphabet (2.5) *for the kth vertex in Γ.*

Proof. The map ι is the inclusion of the 0-section of a $G_{\mathbf{d}}$-equivariant vector bundle, whose underlying vector bundle is trivial, but equals $\oplus_{k=1}^{e} \operatorname{Hom}(\mathcal{V}_k, K^{\lambda(k)})$ equivariantly. As the second vector space is trivial, the Euler class of this bundle is $\prod_{k=1}^{e} c_{top}(\mathcal{V}_k^*)^{\lambda(k)}$.

Since the Borel isomorphism sends the Chern class of maximal possible degree $c_{top}(\mathcal{V}_k^*)$ to t_k, the result follows. □

To make the algebra more concrete we first generalize Proposition 2.6:

Proposition 4.7. *The algebra $\tilde{A}^{\boldsymbol{\nu}}$ has a natural faithful action on*

$$\tilde{V}^{\boldsymbol{\nu}} = \bigoplus_{\grave{\mu}} H^*(\mathfrak{Q}(\grave{\mu})) \cong \boldsymbol{\Lambda}(\hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell)). \tag{4.1}$$

Proof. Let $\tilde{\mathfrak{Q}}$ and $\tilde{\mathfrak{H}}$ denote the varieties corresponding to $\mathfrak{Q}$ and $\mathfrak{H}$ respectively where we do not take the quotient by $G_{\mathbf{d}}$. Consider the inclusion maps of $T_{\mathbf{d}}$-fixed points

$$\iota_{\grave{\mu}} \colon \tilde{\mathfrak{Q}}(\grave{\mu})^{T_{\mathbf{d}}}/T_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\mu})/T_{\mathbf{d}}, \qquad \iota_{\grave{\mu};\grave{\nu}} \colon \mathfrak{H}(\grave{\mu},\grave{\nu})^{T_{\mathbf{d}}\times T_{\mathbf{d}}}/T_{\mathbf{d}} \times T_{\mathbf{d}} \to \mathfrak{H}(\grave{\mu},\grave{\nu})/T_{\mathbf{d}} \times T_{\mathbf{d}},$$

and the natural map of quotients

$$\beta_{\grave{\mu}} \colon \tilde{\mathfrak{Q}}(\grave{\mu})/T_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\mu})/G_{\mathbf{d}} = \mathfrak{Q}(\grave{\mu}), \quad \beta_{\grave{\mu};\grave{\nu}} \colon \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/T_{\mathbf{d}}\times T_{\mathbf{d}} \to \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/G_{\mathbf{d}}\times G_{\mathbf{d}} = \mathfrak{H}(\grave{\mu},\grave{\nu}).$$

We also have the proper projection maps

$$p_1 : \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/T_{\mathbf{d}} \times T_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\mu})/T_{\mathbf{d}}, \quad p_2 : \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/T_{\mathbf{d}} \times T_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\nu})/T_{\mathbf{d}}$$

and similarly

$$\overline{p}_1 : \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/G_{\mathbf{d}} \times G_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\mu})/G_{\mathbf{d}}, \quad \overline{p}_2 : \tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/G_{\mathbf{d}} \times G_{\mathbf{d}} \to \tilde{\mathfrak{Q}}(\grave{\nu})/G_{\mathbf{d}},$$

and also the maps p_1^T, p_2^T for the T-fixed points. We abbreviate $U = H^*(BG_{\mathbf{d}})$ and $V = H^*(BT_{\mathbf{d}})$. Since $G_{\mathbf{d}}$ is a product of general linear groups, the pull-back by $\beta : BT_{\mathbf{d}} \to BG_{\mathbf{d}}$ is an isomorphism between U and invariants of the Weyl group $S_{\mathbf{d}}$ in V. In particular, V is a free U-algebra of finite rank. (Note that the maps β are smooth, hence pullbacks exist.)

Now, consider the Leray-Serre spectral sequence applied to the map $\beta_{\grave{\mu}}$. It degenerates at the E^2-page for parity reasons, so as a V-module, we have that $H^*(\tilde{\mathfrak{Q}}(\grave{\mu})/T_{\mathbf{d}})$ is isomorphic to $V \otimes_U H^*(\tilde{\mathfrak{Q}}(\grave{\mu})/G_{\mathbf{d}}) = V \otimes_U H^*(\mathfrak{Q}(\grave{\mu}))$; similarly for $\mathfrak{H}$.

We claim there is a (up to signs) commutative diagram

$$\begin{array}{ccc}
 & & \mathrm{Hom}_U(H^*(\mathfrak{Q}(\grave{\mu})), H^*(\mathfrak{Q}(\grave{\nu}))) \\
H_*^{BM}(\mathfrak{H}(\grave{\mu},\grave{\nu})) & \xrightarrow{\star-} \nearrow & \Big\downarrow \mathrm{id}_V \otimes - \\
\Big\downarrow \beta^*_{\grave{\mu};\grave{\nu}} & & \mathrm{Hom}_V(V \otimes_U H^*(\mathfrak{Q}(\grave{\mu})), V \otimes_U H^*(\mathfrak{Q}(\grave{\nu}))) \\
H_*^{BM}(\tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})/T_{\mathbf{d}} \times T_{\mathbf{d}}) & \xrightarrow{\star-} \searrow & \Big\downarrow \wr \\
\Big\downarrow \iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_*\iota^*_{\grave{\mu};\grave{\nu}} & & \mathrm{Hom}_V(H^*(\tilde{\mathfrak{Q}}(\grave{\mu})/T_{\mathbf{d}}), H^*(\tilde{\mathfrak{Q}}(\grave{\nu})/T_{\mathbf{d}})) \\
H_*^{BM}(\tilde{\mathfrak{H}}(\grave{\mu},\grave{\nu})^{T_{\mathbf{d}}\times T_{\mathbf{d}}}/T_{\mathbf{d}} \times T_{\mathbf{d}}) & \xrightarrow{\star-} \searrow & \Big\downarrow \iota^*_{\grave{\nu}} \circ - \circ (\iota_{\grave{\mu}})_* \\
 & & \mathrm{Hom}_V(H^*(\tilde{\mathfrak{Q}}(\grave{\mu})^{T_{\mathbf{d}}}/T_{\mathbf{d}}), H^*(\tilde{\mathfrak{Q}}(\grave{\nu})^{T_{\mathbf{d}}}/T_{\mathbf{d}}))
\end{array}$$

where $\star$ denotes the convolution in equivariant Borel-Moore homology. (Note that the ι^*'s exist by [CG97, 2.6.21], since the variety $\iota_{\grave{\mu}}$ can be embedded into $\mathcal{Q}(\hat{\boldsymbol{\mu}}) \times \bigoplus_i \operatorname{Hom}_K(V_i, K^{\nu(i)})$ with smooth fixed point set, similarly for the other ι's.) We first show that the top square is commutative. For $a \in H^*(\mathfrak{Q}(\grave{\mu})/T_{\mathbf{d}})$ and $b \in H_*^{BM}(\mathfrak{H}(\grave{\mu}, \grave{\nu}))$ we can find $v \in V$ and $x \in H^*(\mathfrak{Q}(\grave{\mu}))$ such that

$$\begin{aligned} a \star \beta^*_{\grave{\mu};\grave{\nu}}(b) &= p_{2*}(p_1^*(a) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)) \\ &\qquad = p_{2*}(p_1^*(v \cdot \beta^*_{\grave{\mu}}(x)) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)) = p_{2*}(v \cdot p_1^*(\beta^*_{\grave{\mu}}(x)) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)), \end{aligned}$$

where we used that p_{1*} is V-equivariant. Hence

$$\begin{aligned} a \star \beta^*_{\grave{\mu};\grave{\nu}}(b) &= p_{2*}(v \cdot (\beta_{\grave{\mu}} \circ p_1)^*(x) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)) = p_{2*}(v \cdot (p_1 \circ \beta_{\grave{\mu},\grave{\nu}})^*(x) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)) \\ &= p_{2*}(v \cdot \beta^*_{\grave{\mu},\grave{\nu}}(\overline{p}_1^*(x)) \cap \beta^*_{\grave{\mu};\grave{\nu}}(b)) = v \cdot p_{2*}(\beta^*_{\grave{\mu},\grave{\nu}}(\overline{p}_1^*(x) \cap b) \\ &= v \beta^*_{\grave{\nu}} \overline{p}_{2*}(\overline{p}_1^*(x) \cap b) = v \beta^*_{\grave{\nu}}(x \star b), \end{aligned}$$

where we used $\beta_\mu \circ p_1 = \overline{p}_1 \circ \beta^*_{\grave{\mu};\grave{\nu}}$, base change and the fact that p_{2*} is V-equivariant. Altogether

$$a \star \beta^*_{\grave{\mu};\grave{\nu}}(b) = \beta^*_{\grave{\nu}}(v \cdot x \star b)$$

and hence the top square is commutative. To see that the bottom square is commutative up to sign, consider $a \in H^*(\mathfrak{Q}(\grave{\mu})/T_{\mathbf{d}})$ and $b \in H_*^{BM}(\tilde{\mathfrak{H}}(\grave{\mu}, \grave{\nu})/T_{\mathbf{d}} \times T_{\mathbf{d}})$. Then

$$\begin{aligned} \iota^*_{\grave{\nu}}((\iota_{\grave{\mu}})_* a \star b) &= \iota^*_{\grave{\nu}}(p_{2*}(p_1^*(\iota_{\grave{\mu}*}(a)) \cap b)) && \text{(def. of conv.)} \\ &= p^T_{2\,*}(\iota^*_{\grave{\mu};\grave{\nu}}(p_1^*(\iota_{\grave{\mu}*}(a)) \cap b)) && \text{(base change)} \\ &= p^T_{2\,*}(\iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_*(p_1^{T*}(a)) \cap b) && \text{(base change)} \\ &= \pm p^T_{2*}(\iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_* p_1^{T*}(a)) \cap \iota^*_{\grave{\mu};\grave{\nu}}(b)) && \text{(projection)} \\ &= \pm p^T_{2*}(q \cdot (p_1^{T*}(a)) \cap \iota^*_{\grave{\mu};\grave{\nu}}(b)) && (\iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_* = q\cdot) \\ &= \pm p^T_{2*}(p_1^{T*}(a) \cap \iota^*_{\grave{\mu};\grave{\nu}} \iota_{\grave{\mu};\grave{\nu}\,*} \iota^*_{\grave{\mu};\grave{\nu}}(b)) && (\iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_* = q\cdot) \\ &= \pm a \star \iota^*_{\grave{\mu};\grave{\nu}} \iota_{\grave{\mu};\grave{\nu}\,*} \iota^*_{\grave{\mu};\grave{\nu}}(b). && \text{(def. of conv.)} \end{aligned}$$

where in addition to the definition of convolution and the base change formula, we used the projection formula [FA07, Remark 5.4] and the property from [CG97, Corollary 2.6.44] that $\iota^*_{\grave{\mu};\grave{\nu}}(\iota_{\grave{\mu};\grave{\nu}})_*$ is multiplication with some Euler class q.

The map $\beta^*_{\grave{\mu},\grave{\nu}}$ is injective, since $\mathrm{id}_V \otimes \_$ and the map $M \to V \otimes_U M, m \mapsto 1 \otimes m$ is injective for any free U module. Furthermore, the ι^*'s and ι_*'s are injective, because all varieties that appear are equivariantly formal and the local Euler classes are not zerodivisors, see [Bri98, Theorem 3 and Lemma 4]. The bottom horizontal arrow is injective, since the torus fixed points are isolated.

Thus any element in the kernel of the top horizontal map is in the kernel of the map from the top left to the bottom right. We have seen already that going first vertically and then horizontally is injective, so the kernel is trivial. This completes the proof. □

Just as A contains an idempotent which projects down to the quiver Hecke algebra, the algebra $\tilde{\mathbf{A}}^{\boldsymbol{\nu}}$ contains an analogous idempotent e_T. Let

(4.2) $$e_T \;=\; \sum e_{\grave{\mu}}$$

denote the idempotent in $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$ defined as the sum over all primitive idempotents $e_{\grave{\mu}}$ indexed by all vector multi-compositions $\grave{\mu}$ where each composition appearing is of the form α_i, that is, the corresponding flag is complete. We call $e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$ the **geometric tensor algebra**. The name and construction is based on a diagrammatically defined tensor algebra $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ defined in [Web10, Def. 4.5] to categorify tensor products of representations of quantum groups. To connect the two algebras we first extend our graphical calculus from Section 3.3 to the algebras $\tilde{A}^{\underline{\boldsymbol{\nu}}}$.

4.2. **Extended graphical calculus.** To extend our graphical calculus to the algebras $\tilde{A}^{\underline{\boldsymbol{\nu}}}$ we have to encode the data attached to the shadow vertices. To do so we add a **red band** labeled with ν_i between the black lines representing $\hat{\boldsymbol{\mu}}(i-1)$ and $\hat{\boldsymbol{\mu}}(i)$. For instance, for fixed $\underline{\boldsymbol{\nu}}$, the idempotent $e_{(\underline{\boldsymbol{\nu}},\grave{\mu})}$ corresponding to the weight data $(\underline{\boldsymbol{\nu}},\grave{\mu})$ and another element of the algebra is graphically described in Figure 2 below.

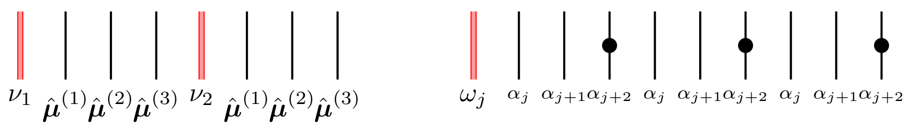


FIGURE 2. An idempotent and $t^{(j;9)}$ from Lemma 5.23 for $e=3$.

The splitting and merging morphisms defined earlier are displayed as in Section 3.3, except of additional red lines separating the multi-compositions. We have also diagrams associated to the new morphisms, that is to left and right shifts. They are denoted by a (left resp. right) crossing of red and black strands, as shown below:

(4.3)

a left shift a right shift

For instance Figure 5 (read as usual from bottom to top) shows a merge and split followed by a left shift. Proposition 4.4 assigns a degree to each crossing of a red band with a black strand, see Remark 5.15 for explicit formulas.

Following [KL09] and [Web10] we define now a diagram algebra, much like that defined above, but only allowing lines labeled by simple roots, and only allowing crossings, rather than splits and merges. Let m be a fixed integer with its integral points $\mathcal{P} = \{(x,y) \mid x \in \{1,2,\dots,m\}, y \in \{1,2\}\}$. A **strand diagram** is a collections of labeled decorated arcs (or strands) on the plane connecting points from $\mathcal{P}$ with $y=0$ with those with $y=1$ satisfying the following properties. Arcs are assumed to have no critical points and each of them is colored black or red. Black arcs are labeled by elements from $\mathbb{V}$ (or alternatively simple roots of $\mathfrak{g} = \hat{\mathfrak{sl}}_e$) and red arcs are labeled by weights of $\mathfrak{g}$. Moreover, black arcs can carry a finite number of dots. Arcs can intersect but triple intersections are not allowed. We consider strand diagrams up to isotopies that do not change the combinatorial type (labels and decorations) of the diagram and do not create critical points.

Locally around each point of an arc a strand diagram is either a single line or a crossing of two black strands as in (3.9) or a crossing of a black with a red line as in (4.3). In particular, two red arcs never cross and no pair of two lines is allowed to end at the same point. We consider the vector space $\mathbb{D}$ spanned by isotopy classes of such diagrams for varying m. It is an algebra, called the **free strand algebra**, by vertical concatenation of diagrams. for instance, the first element in Figure 2 is an idempotent.

To define an interesting quotient of this free strand algebra we need to introduce some extra data, similar to the quiver Hecke algebras from [KL09], [Rou08], [BK09a].

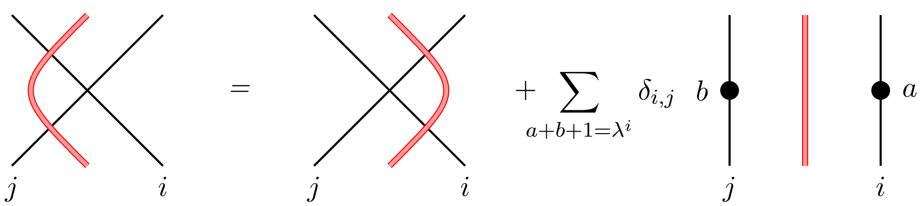
For a weight λ let $\lambda^i = \alpha_i^\vee(\lambda)$ be its Dynkin labels. For $i \neq j \in \mathbb{V}$ we set

$$Q_{ij}(u,v) \quad = \quad \begin{cases} 1 & \text{if } i \neq j \pm 1 \\ u - v & \text{if } i = j-1 \quad (e \neq 2) \\ v - u & \text{if } i = j+1 \quad (e \neq 2) \\ (u-v)(v-u) & \text{if } i = j-1 = j+1 \quad (e = 2). \end{cases} \tag{4.4}$$

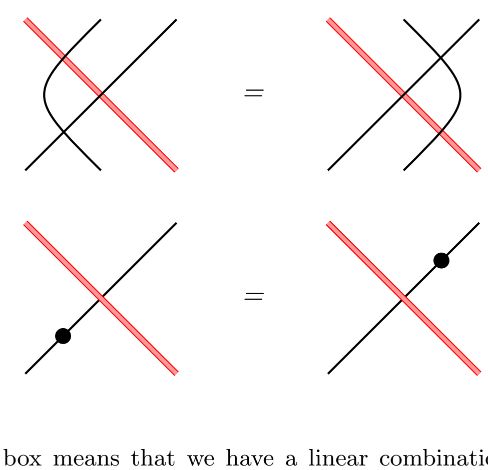
Definition 4.8. *For an ℓ-tuple of weights, $\underline{\boldsymbol{\nu}}$, the* **tensor algebra** $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ *is the subquotient algebra of the free strand algebra spanned by all diagrams which have exactly ℓ red arcs labeled by $\nu_1, \dots, \nu_\ell$ in this order modulo the following relations (R1)-(R3) with all its possible mirror images:*

(R1) The usual KLR relations from Figure 4.[2]

(R2) All black crossings and dots pass through red lines, with possibly a correction term:



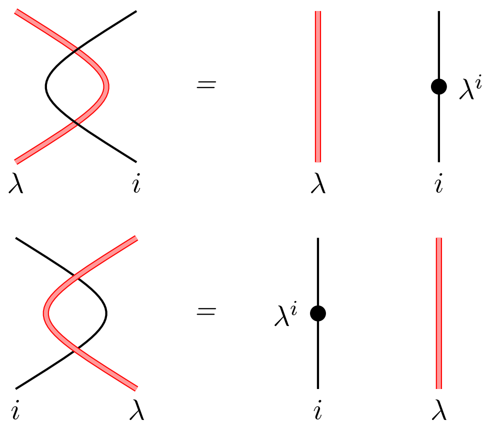
(4.5)



[2]The polynomial in the box means that we have a linear combination of diagrams, one for each monomial appearing. The diagram attached to the monomial is the identity diagram as indicated equipped with a_i dots on the ith strand, where a_i is the exponent of y_i. To match it with [BK09a] note that our y's are the negatives of the y's there.

(R3) A red line labeled λ_i and a black line labeled α_i can be separated by adding $\lambda^i = \alpha_i^\vee(\lambda)$ dots to the black strand:

(4.6)



This algebra is graded by setting: each black crossing with label i and j is of degree -2 if $i = j$, of degree 1 if $j = \mp 1$ and of degree zero otherwise. A crossing of a red line labeled λ and a black line labeled α_i is of degree $\lambda^i = \alpha_i^\vee(\lambda)$, and a dot is of degree 2.

The geometrical tensor algebra agrees with the diagrammatical tensor algebra:

Proposition 4.9. *We have an isomorphism of graded rings*

$$\tilde{\varepsilon}\colon \tilde{T}^{\underline{\boldsymbol{\nu}}} \cong e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$$

sending each crossings (3.9) *of black strands to the corresponding composition of merge and split and the crossings of a red with a black strand to the corresponding shift map in $e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$, and finally matching a dot on the kth strand from the left that is labeled i with the polynomial generators $x_{i,k}$.*

Proof. In order to check that this map is well-defined, we must check the relations of $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ in $e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$. By Proposition 4.7, we have a faithful action of $e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$ on

$$e_T \tilde{V}^{\underline{\boldsymbol{\nu}}} \;\cong \bigoplus_{\hat{\boldsymbol{\mu}}(i) \in \mathrm{VCompf}_e(\mathbf{d})} H^*(\mathfrak{Q}(\grave{\mu})) \cong \mathbf{\Lambda}(\hat{\boldsymbol{\mu}}(0) \cup \cdots \cup \hat{\boldsymbol{\mu}}(\ell)). \tag{4.7}$$

This is a sum of polynomial rings, corresponding to $\ell + 1$-tuples of sequences of simple roots. The explicit formulas for this action from Propositions 3.4 and 4.6 are shown in Figure 3. (There we always display the relevant part of the diagrams indicating the geometric convolution operation. We depict the kth strand and, apart from the last picture, the $k+1$th strand and write y_j for the variable attached to the jth strand.) An easy direct calculation shows that the assignments satisfy the relations (R1)-(R3), hence $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ acts on (4.7), too, by the assignments from Figure 3. By [Web10, Lem. 4.17] this action of $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ is faithful. (We can obtain the indexing used in [Web10] by taking $\mathbf{i}$ to be the concatenation of the sequences of roots in the description in this paper, and the function κ to send j to $\sum_{k<j} |\hat{\boldsymbol{\mu}}(k)|$.) This shows we have an injection $\tilde{\varepsilon} : \tilde{T}^{\underline{\boldsymbol{\nu}}} \hookrightarrow e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$ since we matched faithful representations of these algebras. This map satisfies by construction all the assertions of the proposition. Moreover, it sends

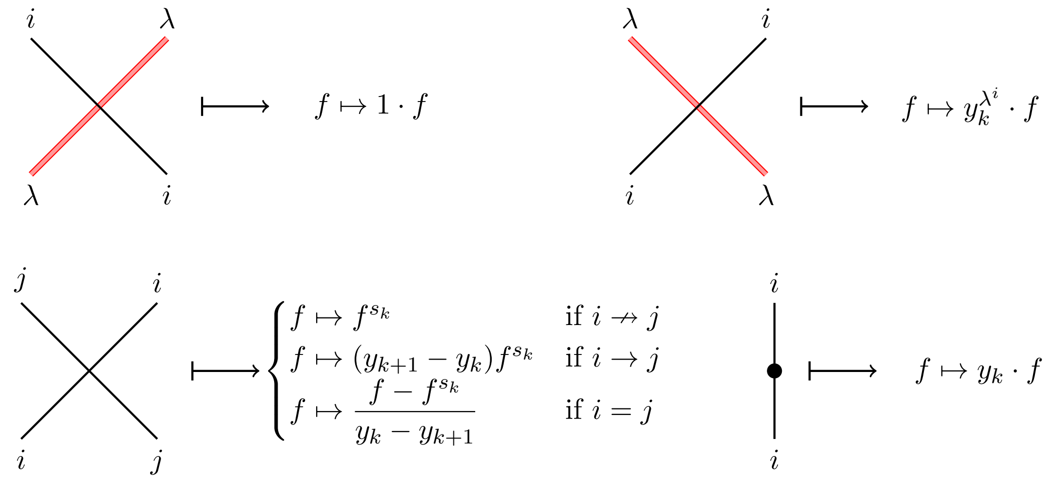


FIGURE 3. The polynomial representation

the basis of [Web10, 4.17] for $\tilde{T}^{\underline{\boldsymbol{\nu}}}$ to a basis of the type given by $e_T \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}} e_T$ in Proposition 4.5. Thus, this map is an isomorphism. □

4.3. **Cyclotomic quotients.** The quiver Hecke algebra $R(\mathbf{d})$ has a very interesting family of finite dimensional quotients, the cyclotomic quiver Hecke algebras T^λ (implicitly depending on our choice of $\mathbf{d}$) where λ is an integer valued function on Γ. We will usually think of this function as a weight of the affine Lie algebra $\widehat{\mathfrak{sl}}_e$. As shown in [BK09a], blocks of the diagrammatically defined cyclotomic quotients of the quiver Hecke algebra are isomorphic to blocks of cyclotomic Hecke algebras for symmetric groups. The latter are Hecke algebra which behave as though they are of "characteristic e" (i.e. the cyclotomic quotient is defined using powers of an element q from the ground field and e is the smallest number such that $1 + q + q^2 + \cdots + q^{e-1} = 0$).

We want now to define similar quotients $A^{\underline{\boldsymbol{\nu}}}$ of $\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}$:

Definition 4.10. *Let $I = I^{\underline{\boldsymbol{\nu}}}$ be the ideal of $A^{\underline{\boldsymbol{\nu}}}$ generated by all elements of the form $e_\alpha | a$, where α is any root. We call this ideal the* **violating ideal** *and the quotient $A^{\underline{\boldsymbol{\nu}}} = \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}/I$ the* **cyclotomic quotient** *of $A^{\underline{\boldsymbol{\nu}}}$ and call $A^{\underline{\boldsymbol{\nu}}}$ the* **cyclotomic quiver Schur algebra**.

Pictorially, I is generated by all diagrams where at some point the left-most strand is black and labeled with some root, see Figure 5 for an example. Obviously, the idempotent (2) associated to an $\ell + 1$-tuple will be 0 in this quotient unless it begins with the empty set (hence a red strand to the left). To avoid much tedious writing of $\emptyset$, we shall henceforth just deal with ℓ-tuples, and leave the initial $\emptyset$ as given.

Note that if $\underline{\boldsymbol{\nu}} = (\lambda)$, and r is the element which is just a red strand labeled with λ, then $a \mapsto r | a$ followed by projection defines an algebra map $\tilde{A}^\lambda \to A^\lambda$ which is

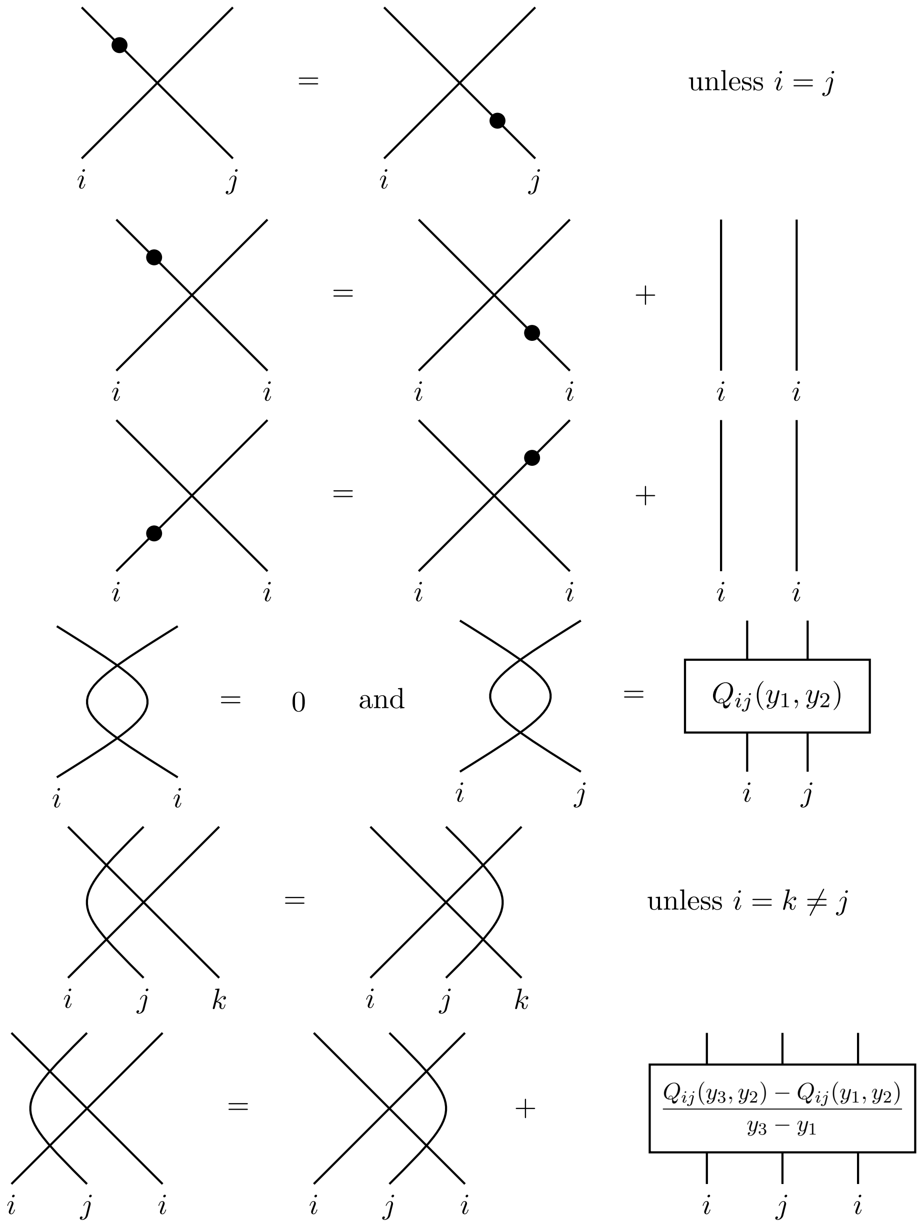


FIGURE 4. The relations of the quiver Hecke algebra.

surjective. The kernel of this map is called the **cyclotomic ideal** of A. An argument analogous to [Web10, Th. 4.20] shows that this ideal has a definition looking more like a traditional cyclotomic ideal.

Let $T^{\underline{\nu}} = \tilde{T}^{\underline{\nu}}/J$ be the cyclotomic quotient of the tensor algebra as defined in [Web10, Def. 4.6].

Proposition 4.11. $e_T A^{\underline{\nu}} e_T \cong T^{\underline{\nu}}$

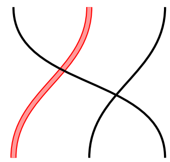


FIGURE 5. An example of an element of the violating ideal.

Proof. The surjection $\Psi\colon e_T\tilde{A}^{\underline{\boldsymbol{\nu}}}e_T \to e_T A^{\underline{\boldsymbol{\nu}}}e_T$ induces a surjective map $\Psi\circ\tilde{\varepsilon}\colon \tilde{T}^{\underline{\boldsymbol{\nu}}} \to e_T A^{\underline{\boldsymbol{\nu}}}e_T$. This map obviously kills the violating ideal J, and thus induces a surjective map $\varepsilon\colon T^{\underline{\boldsymbol{\nu}}} \to e_T A^{\underline{\boldsymbol{\nu}}}e_T$. Thus in order to check that this is an isomorphism, we need to check that $\ker(\Psi\circ\tilde{\varepsilon}) \subset J$.

By definition, any element $x \in \ker(\Psi\circ\tilde{\varepsilon})$ is a sum of elements of the form $x' = abc$ where $a \in e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}, c \in \tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$ and $b = e_\alpha|b'$ is a generator of I as above. First, we apply Proposition 4.5 to a and c, so we may assume that $a = a_1a_2$ and $c = c_2c_1$ where $a_1, c_1 \in e_T\tilde{\mathbf{A}}^{\underline{\boldsymbol{\nu}}}e_T$, and a_2 just joins and c_2 just splits strands. The element a_2 does not move any black strands to the right of reds, so at its top, there is still at least one black strand still left of all reds. This shows immediately that the element a_1 is in the violating ideal. Thus, the factorization $x' = a_1(a_2bc)$ shows that $x' \in J$. This completes the proof. □

Proposition 4.12. *When $e = \infty$, this idempotent induces a Morita equivalence between $A^{\underline{\boldsymbol{\nu}}}$ and $T^{\underline{\boldsymbol{\nu}}}$*

Proof. We need only note that the idempotent $e_{\mathbf{d}}$ factors through the horizontal product $\cdots|e_{d_1\alpha_1}|e_{d_0\alpha_0}|e_{d_{-1}\alpha_{-1}}|\cdots$ of the idempotents $e_{d_i\alpha_i}$ in order (this is a finite product, since $|\mathbf{d}| < \infty$); the split and merge just gives the identity. □

5. A GRADED CELLULAR BASIS

5.1. **Cellular basis of cyclotomic quotients.** We specialize in the remaining sections to the case where $\underline{\boldsymbol{\nu}}$ is a sequence of fundamental weights, so $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}_1}, \dots, \omega_{\mathfrak{z}_\ell})$, where the jth fundamental weight is of the form $\omega_j(\alpha_k) = \delta_{j,k}$. *We also assume for simplicity from now on $e > 2$.* We wish to show that $A^{\underline{\boldsymbol{\nu}}}$ has a cellular basis, turning it into a graded cellular algebra. Let us first recall the relevant definitions from [GL96].

Definition 5.1. *A* **cellular algebra** *is an associative unital algebra H together with a* **cell datum** $(\Lambda, M, C, *, >)$ *such that*

(C1) $(\Lambda, >)$ *is a partially ordered set and $M(\lambda)$ is a finite set for each $\lambda \in \Lambda$;*
(C2) $C : \dot{\bigcup}_{\lambda\in\Lambda} M(\lambda)\times M(\lambda) \to H, (\mathsf{S},\mathsf{T}) \mapsto C^\lambda_{\mathsf{S},\mathsf{T}}$ *is an injective map whose image is a basis for H;*
(C3) *the map $* : H \to H$ is an algebra anti-automorphism such that $(C^\lambda_{\mathsf{S},\mathsf{T}})^* = C^\lambda_{\mathsf{T},\mathsf{S}}$ for all $\lambda \in \Lambda$ and $\mathsf{S},\mathsf{T} \in M(\lambda)$;*
(C4) *if $\lambda \in \Lambda$ and $\mathsf{S},\mathsf{T} \in M(\lambda)$ then for any $x \in H$ we have that*

$$xC^\lambda_{\mathsf{S},\mathsf{T}} \equiv \sum_{\mathsf{S}'\in M(\xi)} r_x(\mathsf{S}',\mathsf{S})C^\lambda_{\mathsf{S}',\mathsf{T}} \pmod{H(>\lambda)}$$

where the scalar $r_x(\mathsf{S}', \mathsf{S})$ is independent of T and $H(> \lambda)$ denotes the subspace of H generated by $\{C^{\nu}_{\mathsf{S}'',\mathsf{T}''} \mid \nu > \lambda, \mathsf{S}'', \mathsf{T}'' \in M(\nu)\}$.

The basis consisting of the $C^{\lambda}_{T,S}$ is then called a **cellular basis** *of H.*

If there is additionally a degree function $\deg : \bigcup_{\lambda \in \lambda} M(\lambda) \to \mathbb{Z}$, $S \mapsto \deg^{\lambda}_{S}$ with the property $\deg(C_{S,T}) = \deg^{\lambda}_{S} + \deg^{\lambda}_{T}$ (for $(S,T) \in M(\lambda) \times M(\lambda)$) such that H turns into a graded algebra, then H is called a **graded cellular algebra** with graded cell datum $(\Lambda, M, C, *, <, \deg)$, see [HM10].

We want to construct now such a basis diagrammatically, indexed by pairs (T,S) of semi-standard Young tableaux of the same shape λ (but, not necessarily of the same type) with the shape λ ranging over the set Λ of all ℓ-multi-partitions (i.e. an ℓ-multi-composition where the parts are partitions) ordered lexicographically. The involution $*$ will be given by a vertical reflection of the diagram in the diagrammatical realization.

Definition 5.2. *Given an ℓ-multi-partition $\hat{\lambda}$, a* **semi-standard $\hat{\lambda}$-tableau** *is a filling of the boxes of its ℓ-tuple of Young diagrams with numbers from the ℓ-fold disjoint union of $\mathbb{Z}_{\geq 0}$, usually denoted with a subscript to show which of the ℓ copies it comes from, subject to the following rules (when the partitions are drawn in the English style):*

- *the entries in each component are in each row weakly increasing from left to right and in each column strictly increasing from top to bottom, (with respect to the lexicographic order first in the subscripts and then the numbers themselves);*
- *the ith copy of $\mathbb{Z}$ can only be used in the first i partitions of the multi-partition;*
- *We will also always assume that our tableaux have no gaps, i.e. if j_k appears, then the i_k for all $i \leq j$ also appear in the tableau.*

A semi-standard $\hat{\lambda}$-tableau S is called **standard** or a **standard multitableau**, if each of the entries only appears once and **super-standard** if it contains only the numbers $1, \ldots, n$ from the last alphabet.

The **type** of S is $\boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \ldots, \boldsymbol{\mu}^{(\ell)})$ where $\boldsymbol{\mu}^{(k)}_i$ denotes the multiplicity of the number i coming from the kth copy of the alphabet.

Example 5.3. *The example from* [DJM98, 4.9] *of shape* $((4,3),(2,1),(2,1))$,

$$\mathsf{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3} \end{array}\ ,\ \begin{array}{|c|c|} \hline 1_2 & 3_3 \\ \hline 2_2 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|} \hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1} \end{array} \right)$$

is a semistandard multitableau of type $((3,3,1),(1,1,0),(2,1,1))$.

Note that the no-gap condition in Definition 5.2 restricts the possible types of semi-standard tableaux to ℓ-multi-compositions, where $\boldsymbol{\mu}^{(k)}_i = 0$ implies $\boldsymbol{\mu}^{(k)}_j = 0$ for all $j > i$. We call such multi-compositions **admissible**. Note that multi-partitions are automatically admissible and we have the following obvious fact:

Lemma 5.4. *Let $\boldsymbol{\mu}$ be an admissible ℓ-multi-composition of n and $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ be an ℓ-multi-partition of n larger than $\boldsymbol{\mu}$ in the dominance ordering. Then $\boldsymbol{\lambda}$ is (viewed as a multi-composition) also admissible.*

In particular, following [DJM98, §6], we have the cyclotomic q-Schur algebra associated to the set Λ of admissible ℓ-multi-compositions of n defined in (1.1) as the endomorphism ring of the direct sum of signed permutation modules associated with $\mu \in \Lambda$. As shown in [DJM98], the Ariki-Koike algebra $\mathfrak{H}(q, Q_1, \ldots, Q_\ell)$ from the introduction is a cellular algebra with basis labeled by pairs of standard $\hat{\lambda}$-tableaux of the same shape, but varying over all $\hat{\lambda}$.

We fix an ℓ-tuple of vertices $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell)$ in $\mathbb{Z}/e\mathbb{Z}$, which is called the **charge**. Given a Young diagram, the **content** of a box in the ith row and jth column of is $j - i$. If it appears in the kth Young diagram of an ℓ-multi-partition, then its **residue** is $\mathfrak{z}_k + j - i \pmod e$.

Let S be a semi-standard $\hat{\lambda}$-tableau. The **residue row-sequence** of S is the sequence obtained by reading the residues along the rows from top to bottom starting in the first partition followed by the second partition etc. Hereby the rows are separated by vertical black lines $|$ and the partitions themselves with vertical red lines (see e.g. Figure 2). The **residue entry-sequence** is the sequence obtained by reading the the residues along the entries starting from 1_1, 1_2, etc. ending with the residue of the box with the largest entry. In case an entry appears more than once we order them according to their appearance in the residue sequence.

Then w_{S} denotes the permutation of minimal length such that w_{S} applied to the row reading word of S is increasing. In particular, w_{S} is a shortest coset representative for the Young subgroup S_λ attached to the rows (since the entries of each row are already weakly ordered) acting from the right, and the Young subgroup S_μ associated to the multiplicities of the entries in S (that is, the Young subgroup that fixes w_{S} times the row reading word) acting from the left.

The compositions λ and μ get refined by the following two vector ℓ-multicompositions $\grave{\lambda}_{\mathsf{S}} = (\hat{\boldsymbol{\lambda}}_{\mathsf{S}}(1), \ldots, \hat{\boldsymbol{\lambda}}_{\mathsf{S}}(\ell))$ and $\grave{\mu}_{\mathsf{S}} = (\hat{\boldsymbol{\mu}}_{\mathsf{S}}(1), \ldots, \hat{\boldsymbol{\mu}}_{\mathsf{S}}(\ell))$:

$$\begin{aligned} \grave{\lambda}_{\mathsf{S}}(k)[g,h] &= \#\{\text{boxes of residue } h \text{ in the } g\text{th row of the } k\text{th Young diagram}\} \\ \grave{\mu}_{\mathsf{S}}(k)[g,h] &= \#\{\text{boxes of residue } h \text{ and entry } g_k\} \end{aligned}$$

Note that the first does not depend on the entries of S, but only on the shape. In particular, if ξ is a shape, then we will use $\grave{\lambda}_\xi$ to denote $\grave{\lambda}_{\mathsf{S}}$ for a tableau of shape ξ.

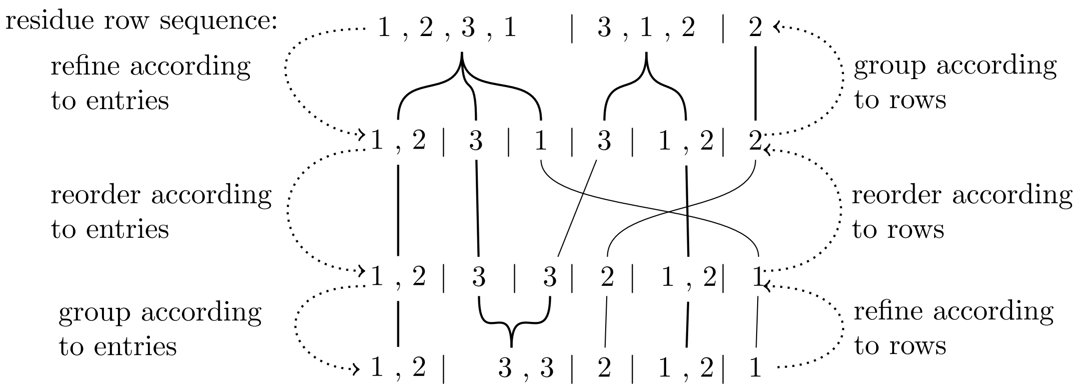
Example 5.5. *Let $e = 3$, $l = 1$ and* $\hat{\lambda} = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \cline{1-4} \ & \ \\ \cline{1-2} \end{array}$. *Then* $\mathsf{S} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$ *is a semi-standard $\hat{\lambda}$-tableau of type $\mu = (2, 2, 1, 2, 1)$. The row reading word is $1, 1, 2, 5, 2, 4, 4, 3$ and therefore* $w_{\mathsf{S}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 8 & 4 & 6 & 7 & 5 \end{pmatrix} \in S_8$, *a shortest coset representative for $S_\mu \backslash S_8 / S_\lambda$, where $S_\lambda = S_4 \times S_3 \times S_2$ and $S_\mu = S_2 \times S_2 \times S_1 \times S_2 \times S_1$. If $\mathfrak{z} = (1)$, then the residues are* $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array}$, *the residue sequence is $1, 2, 3, 1|3, 1, 2|2$ and we have*

$$\begin{aligned} \grave{\mu}_{\mathsf{S}} &= (((1,1,0), (0,0,2), (0,1,0), (1,2,0), (1,0,0))), \\ \grave{\lambda}_{\mathsf{S}} &= (((2,1,1), (1,1,1), (0,1,0)). \end{aligned}$$

By construction w_{S} is a shortest length representative in $S_{\grave{\lambda}_{\mathsf{S}}} \backslash S_d / S_{\grave{\mu}_{\mathsf{S}}}$, hence we can invoke Proposition 4.5 and associate a basis vector in $A^{\underline{\nu}}$:

Definition 5.6. *Given* S*, a semi-standard* $\hat{\mu}$*-tableau, denote by* B_{S} *the element* $\grave{\mu}_{\mathsf{S}} \overset{w_{\mathsf{S}};1}{\Longrightarrow} \grave{\lambda}_{\mathsf{S}}$ *and by* B^*_{S} *the element obtained by flipping the associated diagram, cf. Figure 6, vertically. Set* $C_{\mathsf{S},\mathsf{T}} = B^*_{\mathsf{S}} B_{\mathsf{T}}$ *for* S *and* T *semi-standard tableaux.*

Since the shape of S is determined by the idempotent attached to $\grave{\lambda}_{\mathsf{S}}$, applying the definition of $C_{\mathsf{S},\mathsf{T}} = B^*_{\mathsf{S}} B_{\mathsf{T}}$ to tableaux which are not the same shape gives 0. Also note that $C^*_{\mathsf{S},\mathsf{T}} = (B^*_{\mathsf{S}} B_{\mathsf{T}})^* = (B_{\mathsf{T}} B^*_{\mathsf{S}}) = C_{\mathsf{T},\mathsf{S}}$.



The bottom line represents $\grave{\mu}_{\mathsf{S}}$, the top line, $\grave{\lambda}_{\mathsf{S}}$.

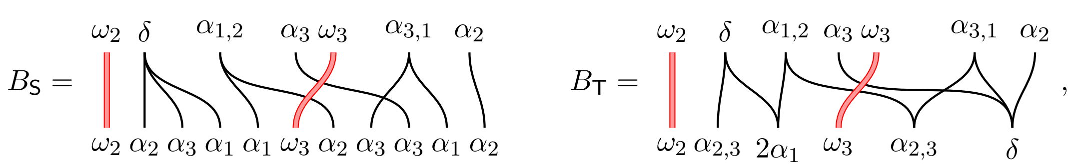
FIGURE 6. Construction of the element B_{S}

Now, let Λ be the set of ℓ-multi-partitions ordered lexicographically, and for $\xi \in \Lambda$ let $M(\xi)$ be the set of ξ-semi-standard tableaux (which is finite thanks to the last condition in Definition 5.2). Let $C : M(\xi) \times M(\xi) \to A^{\underline{\nu}}$ be the map $(\mathsf{S},\mathsf{T}) \mapsto C_{\mathsf{S},\mathsf{T}}$ and the involution $*$ from Definition 5.6. Let $\deg(B_{\mathsf{S}})$ be the degree of the homogeneous element B_{S}.

Example 5.7. *For* $e = 3$ *consider the semistandard* 2*-tableaux*

$$\mathsf{S} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 2_1 & 3_1 \\ \hline 4_1 & 1_2 \\ \cline{1-2} 3_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 2_2 & 4_2 \\ \hline 5_2 \\ \cline{1-1} \end{array} \right) \qquad \mathsf{T} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 1_2 \\ \cline{1-2} 2_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1_2 & 2_2 \\ \hline 2_2 \\ \cline{1-1} \end{array} \right)$$

Denoting the residue and row sequences using simple roots we have



abbreviating $\delta = \alpha_1 + \alpha_2 + \alpha_3$*, in case* $\mathfrak{z} = (2,3)$*. Then*

$$\deg(B_{\mathsf{S}}) = 0+0+(-2+1)+0+0+0+1+0+0 = 0, \qquad \deg(B_{\mathsf{T}}) = -1+1+0+0 = 0$$

where we listed the contributions for each strand starting with the one corresponding to the largest entry. Changing the charge for instance to $\mathfrak{z} = (0,0)$ *gives the degrees*

$$\deg(B_{\mathsf{S}}) = 0+0+1+0+1+0+1+0+0 = 3, \qquad \deg(B_{\mathsf{T}}) = 0+0+0+0 = 0.$$

Theorem 5.8. *The collection* $(\Lambda, M, C, *, \deg)$ *is a graded cell datum for* $A^{\underline{\boldsymbol{\nu}}}$.

5.2. **Cellular basis of the quiver Schur algebra.** Before we give the proof of this theorem we deduce a couple of easy extensions to related algebras and then introduce the combinatorics of the degrees. First, we can remove the restriction that the weights ν_i must be fundamental. Consider an arbitrary list of weights $\underline{\boldsymbol{\nu}}'$ and a list of fundamental weights $\underline{\boldsymbol{\nu}}$ such that $\nu'_k = \nu_{j_{k-1}+1} + \cdots + \nu_{j_k}$ for some $j_1 < \cdots < j_g$. In this case, we can find an idempotent $e_{\underline{\boldsymbol{\nu}}'}$ such that $A^{\underline{\boldsymbol{\nu}}'} = e_{\underline{\boldsymbol{\nu}}'} A^{\underline{\boldsymbol{\nu}}} e_{\underline{\boldsymbol{\nu}}'}$, namely the idempotent defined by

$$e_{\underline{\boldsymbol{\nu}}'} e_{\hat{\mu}} x = \begin{cases} 0 & \text{if } \hat{\boldsymbol{\mu}}(j_m) \neq 0 \text{ for some } m, \\ e_{\hat{\mu}} x & \text{otherwise.} \end{cases}$$

In terms of diagrams, it kills all diagrams in which a black strand ends between the k and $(k+1)$st strands for some $k \neq j_m$.

This idempotent is compatible with the basis of Theorem 5.8. We let $M(\lambda, \underline{\boldsymbol{\nu}}')$ be the set of standard tableaux of shape λ which only use entries from the union of the alphabets indexed by j_m for all m. We let $\mathring{M}(\lambda, \underline{\boldsymbol{\nu}}')$ be the complement of this set in standard tableaux of shape λ, that is, those which contain at least one entry from the alphabet $1_k, 2_k, \dots$ for $k \neq j_m$.

Immediately from the definition, we have that

$$e_{\underline{\boldsymbol{\nu}}'} C_{\mathsf{S},\mathsf{T}} e_{\underline{\boldsymbol{\nu}}'} = \begin{cases} C_{\mathsf{S},\mathsf{T}} & \text{if } \mathsf{S}, \mathsf{T} \in M(\lambda, \underline{\boldsymbol{\nu}}'), \\ 0 & \text{if } \mathsf{S} \text{ or } \mathsf{T} \in \mathring{M}(\lambda, \underline{\boldsymbol{\nu}}'). \end{cases}$$

We can thus define a graded cell datum $(\Lambda, M(-, \underline{\boldsymbol{\nu}}'), C, *, \deg)$ by restricting the maps C and deg to $M(-, \underline{\boldsymbol{\nu}}')$ and $*$ to $A^{\underline{\boldsymbol{\nu}}'}$.

Corollary 5.9. *The collection* $(\Lambda, M(-, \underline{\boldsymbol{\nu}}'), C, *, \deg)$ *is a graded cell datum for* $A^{\underline{\boldsymbol{\nu}}'}$.

The cellular bases from Corollary 5.9 can be used to define a cellular basis of the quiver Schur algebra $A_{\mathbf{d}}$ introduced in Section 2 by realizing the latter as an inverse limit of cyclotomic quotients. This limit is compatible with the cellular bases of these quotients, and thus induces a cellular basis on $A_{\mathbf{d}}$.

We can describe this system diagrammatically. For any finite sequence $\underline{\boldsymbol{\nu}}$, we have inclusions $\tilde{A}^{(\nu_\ell)} \hookrightarrow \tilde{A}^{(\nu_{\ell-1},\nu_\ell)} \hookrightarrow \cdots \hookrightarrow \tilde{A}^{\underline{\boldsymbol{\nu}}}$. Thus, for an infinite sequence $\underline{\boldsymbol{\nu}}_\infty = (\dots, \nu_{-1}, \nu_0)$ we can construct a tower of algebras containing each of its finite truncations; we let $\tilde{A}^{\underline{\boldsymbol{\nu}}_\infty}$ denote the direct limit of this tower. We can imagine this represented by having infinitely many red strands which continue off to the left of the page from that labeled ν_0. If we let $e_{\mathbf{d}}$ be the idempotent that acts by 1 on sequences where only the last vector composition is non-zero and 0 on all others (pictorially, all black strands are to the right of all red strands), then we have that $e_{\mathbf{d}} \tilde{A}^{\underline{\boldsymbol{\nu}}} e_{\mathbf{d}} = A_{\mathbf{d}}$ for any finite list $\underline{\boldsymbol{\nu}}$. Thus, it follows that $e_{\mathbf{d}} \tilde{A}^{\underline{\boldsymbol{\nu}}_\infty} e_{\mathbf{d}} = A_{\mathbf{d}}$ as well.

Now, consider the ideal of $\tilde{A}^{\underline{\boldsymbol{\nu}}_\infty}$ generated by the idempotents for every sequence where a vector composition outside the last $m+1$ is non-zero (i.e. the ideal consisting of diagrams with a black strand that is left of the $(m+1)$th red strand). The quotient by this ideal is just $A^{(\nu_{-m},\dots,\nu_0)}$, and the image of $A_{\mathbf{d}}$ in this quotient is $A^{\nu(m)}$ where $\nu(m) = \sum_{i=-m+1}^{0} \nu_i$. This nested system of ideals induces an inverse system

$$\cdots \to A_{\mathbf{d}}^{\nu(m+1)} \to A_{\mathbf{d}}^{\nu(m)} \to \cdots . \tag{5.1}$$

From now on, we consider the infinite sequence $\underline{\boldsymbol{\nu}}_c$ which cycles through the fundamental weights in cyclic order $(\dots, \omega_0, \omega_1, \dots, \omega_{e-1}, \omega_0)$; any sequence in which all fundamental weights occur infinitely many times will suffice.

Proposition 5.10. *For $\underline{\boldsymbol{\nu}}_c$, the inverse limit of* (5.1) *in the category of graded algebras is $A_{\mathbf{d}}$. The cellular bases of Corollary 5.9 are compatible under the maps of* (5.1)*; that is, each basis vector in the target is the image of a unique one in the source, and all other basis vectors in the source are sent to 0. These bases thus induce a basis on $A_{\mathbf{d}}$.*

Proof. Since the degrees of elements of $A_{\mathbf{d}}$ are bounded below by some integer k, any two-sided ideal generated by elements of degree $\geq p$ only contains elements of degree $\geq p+2k$. In particular, For each degree h, if we choose $g > h-2k$, the cyclotomic ideal I^{μ_g} for the weight $\mu_g = g\omega_0 + \cdots + g\omega_{e-1}$ is generated by elements of degree g, and thus has trivial intersection with the elements of degree h. Thus, the map $A_{\mathbf{d}} \to A_{\mathbf{d}}^{\mu_g}$ is an isomorphism in degree h for all $g > h-2k$. This shows that each graded degree of $A_{\mathbf{d}}$ is the inverse limit of the graded degrees of the inverse system. This is precisely equivalent to $A_{\mathbf{d}}$ being the desired inverse limit in the category of graded algebras.

Now consider the effect of the projection map $p\colon A_{\mathbf{d}}^{\nu(m)} \to A_{\mathbf{d}}^{\nu(m-1)}$ on the basis of elements $C^{\lambda}_{\mathsf{S},\mathsf{T}}$ where S and T are standard tableaux of the same shape λ on m-multipartitions. If the first partitions of λ is non-empty, then $p(C^{\lambda}_{\mathsf{S},\mathsf{T}}) = 0$. If the first partition of λ is empty, then $p(C_{\mathsf{S},\mathsf{T}}) = C^{\lambda'}_{\mathsf{S}',\mathsf{T}'}$ where λ', S' and T' are obtained by just stripping out the empty partition. Thus, we have the desired result. □

This basis of $A_{\mathbf{d}}$ has a cellular structure, with the cell data essentially given by taking the union of the cell data for the inverse system. For $A_{\mathbf{d}}^{\nu(m)}$, the cell datum is $(\Lambda_m, M(-,\nu(m)), C, *, \deg)$ where Λ_m is the m-multi-partitions of residue $\mathbf{d}$, and $M(\lambda,\nu(m))$ is the set of standard tableaux in the alphabet $\{1_0, 2_0, \dots\}$. We have an inclusion $\Lambda_m \hookrightarrow \Lambda_{m+1}$ by adding an empty partition in the first slot. Let Λ_∞ be the direct limit of these maps, which we can think of as the sequences of partitions indexed by non-positive integers with total residue $\mathbf{d}$.

Note that the inclusion above induces a bijection $M(\lambda,\nu(m)) \to M(\lambda,\nu(m+1))$. For each $\lambda \in \Lambda_\infty$, we let $\lambda(m) \in \Lambda_m$ be its truncation to the first m partitions. Thus, for all integers k large enough that all non-empty partitions in λ occur in the last k slots, the sets $M(\lambda(k),\nu(k))$ are in canonical bijection. In view of this, we define $M(\lambda,\infty) := M(\lambda(k),\nu(k))$ for $k \gg 0$. We therefore obtain

Corollary 5.11. $(\Lambda_\infty, M(-,\infty), C, *, \deg)$ *is a graded cell datum for the algebra $A_{\mathbf{d}}$.*

5.3. **The combinatorics of degrees.** We want to emphasize that our construction can be used to define a grading on the set of semistandard multitableaux S by setting $\deg(\mathsf{S}) := \deg(B_{\mathsf{S}})$. This degree function extends nicely a known combinatorial definition on standard tableaux as follows:

For a semi-standard tableau S let $\mathsf{S}(< i_j)$ (or $\mathsf{S}(\leq i_j)$) be the subdiagram of all boxes with entries smaller (or equal) i_j respectively. Given a box b in S with entry say i_j any column in S with the same entry i_j is *blocked* with respect to b.

Definition 5.12. *The* **degree** $d_{\mathsf{S}}(b)$ *of b in S is the number of addable minus the number of removable boxes in the diagram $\mathsf{S}(< i_j)$ which are strictly below or to the* *right of b in the first j components, not in blocked columns and of the same residue as b. The* **degree** $\mathrm{Deg}(\mathsf{S})$ *of a tableau is the sum of the degrees of its boxes, $\sum_b d_{\mathsf{S}}(b)$.*

Example 5.13. We compute $\mathrm{Deg}(\mathsf{S})$ for S as in Example 5.7 with $\mathfrak{z} = (0, 0)$ starting with the largest entry. There is no addable or removable box below the entry 5_2 and no addable or removable box of the correct residue below 4_2. But there is then an addable box below 3_2, but no contribution for 2_2. For 1_2 there is one addable box to count (in the second component). The box b with 1_4 has degree zero and for 1_3 we have now an addable box (at the position of b). Finally 2_1 and 1_1 do not contribute anything. Hence, $\mathrm{Deg}(\mathsf{S}) = 3 = \deg(\mathsf{S})$. In case $\mathfrak{z} = (2, 3)$ we have no contribution for 5_2 and 4_2. There is a removable box for 3_2 and no contributions for 2_2, 1_2 and 4_1. There is an addable box for 3_1, and no contribution for 2_1 and 1_1. Hence $\mathrm{Deg}(\mathsf{S}) = 0 = \deg(\mathsf{S})$.

Our degree function generalizes the degree for standard tableaux from [LLT96, §4.2], [AM00, Definition 2.4], [BKW11, §3.5] with the following (easy to verify) property:

Lemma 5.14. *Let S be an ordinary standard tableaux, i.e. 1-standard tableaux, with $\mathfrak{z}_1 = c$. Assume there are n_i boxes with residue i. Then the number of addable minus the number of removable boxes of residue i equals $\langle \omega_c - \sum_{j=1}^{e} n_j \alpha_j, \alpha_i \rangle$.*

Proposition 5.15. *We have* $\deg(\mathsf{S}) = \mathrm{Deg}(\mathsf{S})$.

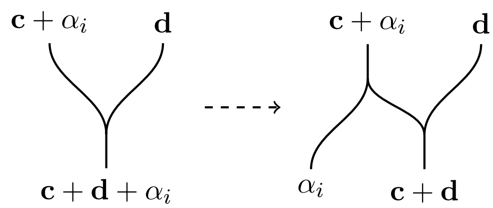
Proof. First we show the claim for standard tableaux by induction on the number of boxes. (The base of induction being obvious). Assume we are given a standard tableau and let b be the box with the largest entry, say of residue i, and with entry m_p. We assume that b is in the kth component, in particular the $k+1$st through ℓth components are empty. By assumption, the claim holds for the tableau T with b removed.

In terms of diagrams, removing b means we remove the black strand L which is rightmost at the bottom of the diagram. The geometric degree deg will increase by

$$\left(\sum_{j=k+1}^{p} \langle \omega_{\mathfrak{z}_j}, \alpha_i \rangle \right) + (-2n_i + n_{i+1} + n_{i-1}) + y. \tag{5.2}$$

Here the $\omega_{\mathfrak{z}_j}$ are precisely the labels of the red lines crossed by L, and n_s is the number of black lines crossed by L which are labelled (by the simple root attached to) s, and y is a correction term if the topmost split changes degree when removing L. The black strands crossed are those corresponding to rows below the box b (necessarily in the kth component of S). Since the rth component for $k+1 \leq r \leq p$ is empty, it has no removable box, and only a single addable box of residue $\mathfrak{z}_r$. That is, the number of addable boxes of residue i in the k'th component for $k' > k$ is equal to the first bracket expression in (5.2). Now, consider in the kth component the (ordinary) standard tableau, say ξ, given by the rows below b and denote $c = \mathfrak{z}_k$. Note that the boxes in ξ correspond precisely to the black lines which L crosses. These lines contribute precisely the second bracket in (5.2) to the degree deg. Finally note that the term $\langle \omega_c, \alpha_i \rangle = \delta_{c,i}$ supplies the correction y for the top split, since $c = i$ if and only if the residue of the row containing b is a multiple of δ. Thus by Lemma 5.14, removing b has the same effect on deg and Deg and the proof is completed for standard tableaux.

Finally, we consider the passage from standard tableaux to semi-standard. We work by induction on the number of boxes in the diagram which share an entry with another box. Consider the "highest" box that shares an entry with another (the first encountered in the reading word), and consider the effect of raising by 1 the entries of all other boxes with greater or equal entries in the same alphabet. This tableau has fewer boxes that share entries, and so, by induction, its degree $\deg(\mathsf{S})$ agrees with $\mathrm{Deg}(\mathsf{S})$. Let i be the residue of this box, and k its entry. The effect of this transformation for the corresponding diagram and its degree deg is as follows. The degree on the second part (the permutation) in Figure 6 stays the same, only the split and merge parts could change the degree. Thus it suffices to consider the transformation



where $\mathbf{c}$ is the dimension vector of the k's in the same row as our chosen box, and $\mathbf{d}$ is the dimension vector of those in higher rows. The degrees of these morphisms are

$$d_{i-1} - d_i + \sum c_j(d_{j-1} - d_j) \quad \text{and} \quad c_{i-1} - c_i + \sum c_j(d_{j-1} - d_j).$$

Hence the difference in degree equals $c_{i-1} - c_i - d_{i-1} + d_i$.

On the other hand, consider the change in $\mathrm{Deg}(\mathsf{S})$. Nothing changes for the degrees of the boxes below our chosen one b. The degree of b can be affected by every row below b containing k's. If that row has equal numbers of boxes labeled k with residue i and $i-1$, there are no changes; however, if there is one more with residue i than $i-1$, then an addable box of residue i was created by removing the strip consisting of all boxes whose entry was changed from k to $k+1$, hence the combinatorial degree changes by 1. If there is one more $i-1$ than i, then an addable box of residue i was destroyed changing the degree by -1. Thus, the net change is exactly $d_i - d_{i-1}$. Moreover, the degrees of the boxes b' whose entry was moved from k to $k+1$ in the same row as b could be changed. If b' has residue i a removable box has been created if it has $i-1$ an addable box has been removed. Hence the total difference is $c_{i-1} - c_i$. Thus, we arrive at the same difference, and by induction, the proposition is proved. □

5.4. **The proof of cellularity.** We will organize our proof into four parts. The first step only concerns the tensor product algebras $T^{\underline{\nu}}$ and extends Hu and Mathas' basis from [HM10] to these algebras, the second proves that the proposed cellular basis for $A^{\underline{\nu}}$ spans, the third shows that it is indeed a cellular basis and the final fourth step compares the ideals with the Dipper-James-Mathas cellular ideals.

Step I: the case of $T^{\underline{\nu}}$. We start with the following easy observation:

Lemma 5.16. *We have* $e_T C_{\mathsf{S},\mathsf{T}} e_T = \begin{cases} C_{\mathsf{S},\mathsf{T}} & \textit{if } \mathsf{S} \textit{ and } \mathsf{T} \textit{ are standard,} \\ 0 & \textit{otherwise.} \end{cases}$

Proof. Recall from (4.2) that e_T corresponds to the vector compositions of complete flag type, which in turn correspond precisely to the standard tableaux. □

Thus, Theorem 5.8 would imply that the elements $C_{\mathsf{S},\mathsf{T}}$ for standard tableaux give a basis of $T^{\underline{\boldsymbol{\nu}}}$ from Proposition 4.11. However we will prove this result first as a stepping stone to the full proof. Let N denote the set of standard tableaux.

Theorem 5.17. *The data* $(\Lambda, N, C|_N, <, *)$ *defines a cell datum for* $T^{\underline{\boldsymbol{\nu}}}$*.*

Proof. By restricting our construction further to tableaux ν (instead of multitableaux) and only using the last copy of $\mathbb{Z}$ we obtain an algebra

$$T^{\nu} \subset T^{\underline{\boldsymbol{\nu}}} \tag{5.3}$$

with a distinguished basis which is precisely the cellular basis from [HM10].

If one fixes the idempotents $\grave{\mu}_{\mathsf{S}}$ and $\grave{\mu}_{\mathsf{T}}$, that is, fixes the multiset of entries in the tableaux, then the map $(\mathsf{S},\mathsf{T}) \mapsto (\mathsf{S}^{\diamond},\mathsf{T}^{\diamond})$ is injective (because of our no gap condition). Since we already know from [HM10] that the $C_{\mathsf{S},\mathsf{T}}$ for pairs of tableaux which only use the last copy of $\mathbb{Z}$ are linearly independent, the map given by $a \mapsto \theta a \theta^*$ is injective, and the elements $C_{\mathsf{S},\mathsf{T}}$ for standard tableaux are linearly independent. They are thus a basis by the dimension formula from [Web10, Lem. 4.36]. This establishes conditions (C1) and (C2) of cellularity, and (C3) is obvious from the definition as mentioned above.

Finally to see (C4), let $x \in T^{\underline{\boldsymbol{\nu}}}$ be an arbitrary element. Consider xB_{S}; since the $C_{\mathsf{S},\mathsf{T}}$'s are a basis we must have $xB^*_{\mathsf{S}} = \sum_{\mathsf{S}',\mathsf{T}'} r_x(\mathsf{S}',\mathsf{S},\mathsf{T})C_{\mathsf{S}',\mathsf{T}'}$ for some uniquely defined coefficients $r_x(\mathsf{S}',\mathsf{S},\mathsf{T}')$. For this coefficient to be not 0 we must have $\grave{\mu}'_{\mathsf{T}} = \grave{\lambda}_{\mathsf{S}}$; in other words, T' is a semi-standard tableaux whose type is the shape of S. Thus, either T' is the super standard tableau T^{S} of the shape of S (by which we mean the tableau where the entries are just the row numbers) or the shape of T' is above that of S in dominance order of ℓ-multi-partitions, hence contained in $H(\geq \mu)$, where μ is the shape of S. If we let $r_x(\mathsf{S}',\mathsf{S}) = r_x(\mathsf{S}',\mathsf{S},T^{\mathsf{S}})$ then

$$xC_{\mathsf{S},\mathsf{T}} = xB_{\mathsf{S}}B^*_{\mathsf{T}} = \sum_{\mathsf{S}'} r_x(\mathsf{S}',\mathsf{S})C_{\mathsf{S}',T^{\mathsf{S}}}B^*_{\mathsf{T}}.$$

Note that $C_{\mathsf{S}',T^{\mathsf{S}}}B^*_{\mathsf{T}} = B'_{\mathsf{S}}B_{T^{\mathsf{S}}}B^*_{\mathsf{T}} = B'_{\mathsf{S}}B^*_{\mathsf{T}} = C_{\mathsf{S}',\mathsf{T}}$ since the shape of T and S' equal the type of T^{S} and $B_{T^{\mathsf{S}}}$ is then just the identity. Hence we have precisely condition (C4). □

Corollary 5.18. $(\Lambda, N, C|_N, *, \deg)$ *defines a graded cellular algebra.*

Proof. By definition $\deg(C_{\mathsf{S},\mathsf{T}}) = \deg(B_{\mathsf{S}}) + \deg(B_{\mathsf{T}})$. □

Step 2: signed permutation modules and spanning set. Let now n be a fixed natural number and let

$$R_n = \bigoplus_{|\mathbf{d}|=n} R_{\mathbf{d}}$$

be the subalgebra of the quiver Hecke algebra of diagrams with n strands. The summand T^{ν}_n of the algebra T^{ν} from (5.3) corresponding to partitions of n is then a cyclotomic quotient R^{ν}_n of R_n.

In [BK09a, (4.36)], an isomorphism between R^{ν}_n and a cyclotomic Hecke algebra $\mathfrak{H}^{\nu}_n$ of S_n with parameters $(\zeta, \zeta^{\mathfrak{z}_1}, \dots, \zeta^{\mathfrak{z}_\ell})$ was established, where ζ is an element of the separable algebraic closure of $\Bbbk$ which satisfies $\zeta + \zeta^2 + \dots + \zeta^{e-1}$ and e is the smallest integer where this holds.

This isomorphism $R_n^\nu \cong \mathfrak{H}_n^\nu$ from [BK09a, (4.36)] depends on the choice of certain polynomials $Q_r(\mathbf{i})$ which we want to fix now. It will turn out to be convenient not to follow the suggested choice of [BK09a, (4.36)], but instead fix

$$Q_r(\mathbf{i}) \quad = \quad \begin{cases} 1-\zeta-\zeta y_{r+1}+y_r & \text{if } i_r = i_{r-1}, \\ \frac{P_r(\mathbf{i})-1}{y_r-y_{r+1}} & \text{if } i_{r+1} = i_r+1, \\ P_r(\mathbf{i})-1 & \text{if } i_{r+1} \neq i_r, i_r+1, \end{cases} \tag{5.4}$$

with the notations from [BK09a, (4.27)] except that, as in Section 4.2 (R1), our ζ is q there and our y_r, y_{r+1} are the $-y_r, -y_{r+1}$ there. The equation [BK09a, (4.28)] implies that ${}^{s_r}(1-P_r(s_r(\mathbf{i}))) = \zeta + P_r(\mathbf{i})$, so the equations [BK09a, (4.33-35)] follow immediately and this choice indeed defines an isomorphism of algebras

$$R_n^\nu \cong \mathfrak{H}_n^\nu. \tag{5.5}$$

In the following we will identify

$$T_n^\nu = R_n^\nu = \mathfrak{H}_n^\nu, \quad \text{and} \quad T^\nu = R^\nu := \bigoplus_{n\geq 0} R_n^\nu = \bigoplus_{n\geq 0} \mathfrak{H}_n^\nu =: \mathfrak{H}^\nu. \tag{5.6}$$

Analogously, we will write $A_n^{\boldsymbol{\nu}} = \bigoplus_{|\mathbf{d}|=n} A_{\mathbf{d}}^{\boldsymbol{\nu}}$.

Let M be a representation of $\mathfrak{H}_n^\nu = T_n^\nu$. Recall, [BK09a, 4.1], that $\mathfrak{H}_n^\nu$ contains the finite dimensional Iwahori-Hecke algebra as a natural subalgebra. A vector $v \in M$ **generates a sign representation** if it generates a 1-dimensional sign representation for this finite Hecke algebra (i.e. $T_r v = -v$ for all generators T_r in the notation of [BK09a]). We first express this condition in the standard generators ψ_r of $R_n^{\boldsymbol{\nu}}$:

Lemma 5.19. *A vector $v \in M$ generates a sign representation if and only if it transforms under the action of ψ_r by*

$$\psi_r e_{\mathbf{i}} v = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ (y_r - y_{r+1}) e_{\mathbf{i}} v & \text{if } i_{r+1} = i_r + 1, \\ e_{\mathbf{i}} v & \text{if } i_{r+1} \neq i_r, i_r+1. \end{cases}$$

Proof. This is immediate by plugging the choices (5.4) into [BK09a, (4.38)] using the fact that $T_i v = -v$: the first case follows since in this case $P_r(\mathbf{i}) = 1$, so $(T_r + P_r(\mathbf{i}))e(\mathbf{i})v = 0$; the last two follow immediately from the substitution of -1 for T_r. □

It might seem strange that we use here anti-invariant vectors instead of invariant vectors. This could be fixed by picking a different isomorphism to the Hecke algebra, but it is "hard-coded" into the isomorphism chosen in [BK09a], since the defining quadratic relation for the Hecke algebra is $(T_r + 1)(T_r - q) = 0$, hence it comes along with a sign representation (but not a trivial representation where T_r acts by 1).

We are interested in a version of Lemma 5.19 for right T_n^ν-modules M (recall the obvious identification of $\mathfrak{H}_n^\nu = T_n^\nu$ with its opposite algebra):

Proposition 5.20. *A vector $v \in M$ generates a sign representation (for the right action) if and only if it transforms under the action of ψ_r by*

$$ve_{\mathbf{i}}\psi_r = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ ve_{s_r(\mathbf{i})}(y_r - y_{r+1}) & \text{if } i_r = i_{r+1}+1, \\ ve_{s_r(\mathbf{i})} & \text{if } i_r \neq i_{r+1}, i_{r+1}+1, \end{cases}$$

where s_r acts on $\mathbf{i}$ by swapping the rth and $r+1$th entry.

Proof. This is an easy consequence from Lemma 5.19 using the defining relations. □

In [Web10, §9.2] the isomorphism (5.5) was extended to an isomorphism

$$T_n^{\boldsymbol{\nu}} \;\cong\; \mathrm{End}_{\mathfrak{H}_n^{\boldsymbol{\nu}}}\left(\bigoplus_{\sum_{i=1}^{\ell} a_i = n} \mathfrak{H}_n^{\nu} u_{\mathbf{a}}^+ \right) \tag{5.7}$$

where $u_{\mathbf{a}}^+$, for $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$, is the element defined by Dipper, James and Mathas, [DJM98, Definition 3.1], as

$$u_{\mathbf{a}}^+ = \prod_{s=1}^{\ell} \prod_{k=1}^{a_k} (L_k - \zeta^{\mathfrak{z}_s}).$$

For an ℓ-multi-composition ξ, we let $\mathbf{a}_\xi = \big(0, |\xi^{(1)}|, |\xi^{(1)}| + |\xi^{(2)}|, \dots, |\xi^{(1)}| + \cdots + |\xi^{(\ell-1)}|\big)$ and, following [DJM98, (3.2)(ii)], will assume from now on that

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell \leq n.$$

Thus, we can describe the cyclotomic Schur algebra $H_n^{\boldsymbol{\nu}}$ from the introduction (for the parameters $(q, Q_1, Q_2, \dots Q_\ell) := (\zeta, \zeta^{\mathfrak{z}_1}, \dots, \zeta^{\mathfrak{z}_\ell})$) as the endomorphism algebra of the $T^{\boldsymbol{\nu}}$ module

$$\mathrm{Hom}_{\mathfrak{H}_n^{\boldsymbol{\nu}}}\Big(\bigoplus_{|\mathbf{a}|=n} \mathfrak{H}_n^{\boldsymbol{\nu}} u_{\mathbf{a}}^+, \bigoplus_{\xi} \mathfrak{H}_n^{\boldsymbol{\nu}} u_\xi^+ x_\xi \Big) = \bigoplus_{|\xi|=n} T^{\boldsymbol{\nu}} x_\xi \tag{5.8}$$

where x_ξ is the projection to vectors which transform under the sign representation for the Young subgroup S_ξ. We refer to the modules $T^{\boldsymbol{\nu}} x_\xi$ as **signed permutation modules** for $T^{\boldsymbol{\nu}}$.

Using the cellular basis of $H_n^{\boldsymbol{\nu}}$, one can directly deduce, see [DJM98, (4.14)] the fact:

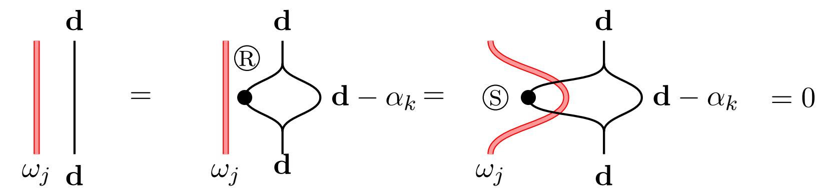
Lemma 5.21. *The dimension of $T^{\boldsymbol{\nu}} x_\xi$ is precisely the number of pairs (S, T) of tableaux on ℓ-multi-partitions with n boxes with the same shape satisfying*

- T *is of type ξ (i.e. the number of occurrences of i_j is the length of the ith row in the jth diagram of ξ), and*
- S *is standard.*

Step 3: graded cellular basis. We start with some preparatory lemmata. For $j \in \mathbb{Z}/e\mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we let $e_{j;n}$ be the idempotent for the vector composition $\hat{\boldsymbol{\mu}}_{j;n} = (\alpha_j, \alpha_{j+1}, \dots, \alpha_{j+n})$, and let $\mathbf{d}_{j;n}$ be its dimension vector. Note that for fixed j and varying n such dimension vectors $\mathbf{d}$ are characterized by $d_k + 1 \geq d_{k-1} \geq d_k$ for $k \neq j$ and $d_{j-1} \leq d_j \leq d_{j-1} + 1$.

Lemma 5.22. *Let $j \in \mathbb{Z}/e\mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Let $e_{(\mathbf{d})}$ be an idempotent corresponding to a vector composition corresponding to step 1 flags. In A^{ω_j} we have $e_{(\mathbf{d})} = 0$ unless $\mathbf{d} = \mathbf{d}_{j;n}$ for some n. Furthermore, $e_{(\mathbf{d}_{j;n})} A^{\omega_j} e_{(\mathbf{d}_{j;n})} \cong \Bbbk$.*

Proof. Assume $\mathbf{d} \neq \mathbf{d}_{j;n}$ for any n. If there exists $k \neq j$ such that $d_k > d_{k-1}$ or $d_k > d_{k-1} + 1$ for $k = j$, then set $R = d_k - d_{k-1} - 1$. In either case we claim that



where the labels Ⓡ and Ⓢ with $S = R - \delta_{j,k}$ denote the number of dots on the strand (following Remark 3.7). Since the two cases correspond to $S = R \geq 0$ respectively $S + 1 = R \geq 1$, the last equality holds. The second relation is [Web10, (4.2)] with Proposition 4.9. To see the first equality in case $k \neq j$ note that the second diagram corresponds to

$$f \mapsto X(d_{k-1}, d_k)(f) := \Delta_w\Big(\prod_{r=1}^{d_{k-1}} (x_{k,1} - x_{k-1,r}) x_{k,1}^R f\Big), \tag{5.9}$$

with $w = s_{d_{k-1}} \cdots s_2 s_1$, where s_t denotes the transposition swapping the variables $x_{k,t}$ and $x_{k,t+1}$; hence it is enough to see that $X(d_{k-1}, d_k) = \mathrm{id}$ or even $X(d_{k-1}, d_k)(1) = 1$. If $d_{k-1} = 0, 1$ this is easily verified, and so we proceed by induction. Using formula (3.7) we obtain

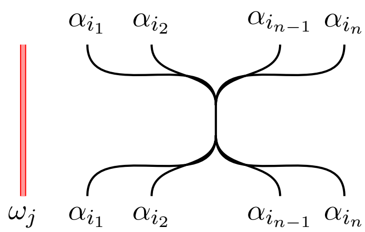
$$\begin{aligned} X(d_{k-1}, d_k) &= \Delta_w\Big(\prod_{r=1}^{d_{k-1}-1} (x_{k,1} - x_{k-1,r}) x_{k,1}^R\Big) \\ &\quad + \Delta_{ws_1}\Big(\prod_{r=1}^{d_{k-1}-1} (x_{k,2} - x_{k-1,r}) x_{k,1}^R\Big) \Delta_{s_1}(x_{3,1} - x_{2,d_{k-1}}) \\ &= 0 + 1 \quad = \quad 1 \end{aligned}$$

using twice the induction hypothesis and the fact $\Delta_{s_1}(x_{3,1} - x_{2,d_{k-1}}) = 1$. In case $k = j$, the split in the diagram (just an inclusion) is followed by multiplication with $x_{j,1}^R$ and the operator Δ_w for $w = s_{d_{j-1}} \cdots s_2 s_1$. Now if $d_k \geq d_{k-1}$ for all $k \neq j$ and $d_j \leq d_{j-1} + 1$, we automatically have $d_k > d_{k-1} - 1$ and $d_j < d_{j-1}$. Hence if $e_{(\mathbf{d})} A^{\omega_j} e_{(\mathbf{d})} \neq 0$ then $\mathbf{d} = (\mathbf{d}_{j;n})$ for some j, n.

In this case all elements $e_{(\mathbf{d}_{j;n})} x e_{(\mathbf{d}_{j;n})}$ where x is of the form $\grave{\mu} \overset{w;h}{\Longrightarrow} \grave{\nu}$ (as in Proposition 4.5) and $w \neq \mathrm{id}$ or $w = \mathrm{id}$ and $h \neq 1$ can be written in terms of similar diagrams where a single strand is pulled off, so they are again zero and only the scalars remain. □

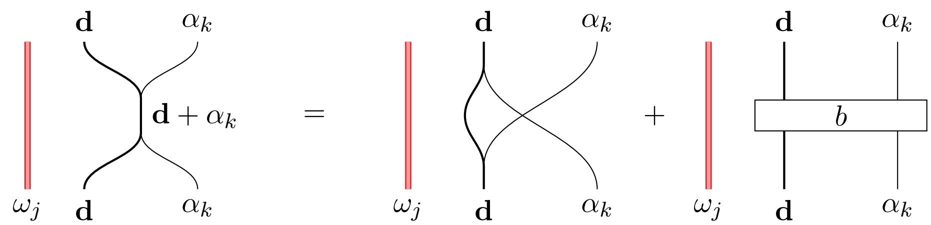
Now, we consider the element $t^{(j;n)} := (\prod_{k=1}^{\lfloor n/e \rfloor} x_{j-1,k}) e_{\hat{\boldsymbol{\mu}}_{j;n}}$, that is, $e_{\hat{\boldsymbol{\mu}}_{j;n}}$ with a dot on every strand labeled by $j - 1$ and no dots on any others, see Figure 2. This element plays an important role in the basis of Hu and Mathas, [HM10].

Lemma 5.23. *Let $\alpha_{i_1}, \alpha_{i_2}, \ldots \alpha_{i_n}$ be an arbitrary sequence of simple roots. Then*



$$= \begin{cases} \mp t^{(j;n)} & \textit{if } \alpha_{i_g} = \alpha_{j-g+1} \textit{ for all } g, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The fact that if $\alpha_{i_g} \neq \alpha_{j-g+1}$ for some g then the element is 0 follows easily from Lemma 5.22. Now let $\mathbf{d} = \mathbf{d}_{j;n}$, and $k = j + n$. The proof is then by induction on n. When $n < e$ the claim follows directly from Lemma 3.4. Otherwise, we claim that



(5.10)

for some polynomial $b \in \mathbf{\Lambda}(\mathbf{d}, \alpha_k)$. To see this let G and F denote the first and second morphism. Abbreviating $a = d_k$ and setting $E = \prod_{m=1}^{a}(x_{k+1,m} - x_{k,a+1})$ we have

$$G(f) = \begin{cases} E\Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}\Delta_a^{k}(f) & \text{if } d_k - d_{k+1} = 1, \\ E\Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}(f) & \text{if } d_k = d_{k-1}, \end{cases} \tag{5.11}$$

where Δ_i^k denotes the Demazure operator involving the variables $x_{k,i}, x_{k,i+1}$. Note that F is a composition of first multiplying with an Euler class E_1, followed by a merge M_1, an inclusion and finally a merge M_2 as follows

$$F = (\mathbf{d}, \alpha_k) \to (\mathbf{d} - \alpha_k, \alpha_k, \alpha_k) \to (\mathbf{d} - \alpha_k, 2\alpha_k) \to (\mathbf{d} - \alpha_k, \alpha_k, \alpha_k) \to (\mathbf{d}, \alpha_k)(1)$$

so we get

$$F(f) = \begin{cases} \Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}\Delta_a^{k}(E_1 f) & \text{if } d_k - d_{k+1} = 1, \\ \Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}(E_1 f) & \text{if } d_k = d_{k-1}. \end{cases}$$

where $E_1 = \prod_{m=1}^{a}(x_{k+1,m} - x_{k,a})$. The latter is a polynomial in $x_{k,a}$ of degree a. Using (3.7) we obtain

$$F(f) = \begin{cases} \Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}\Delta_a^{k}(E_1)f + G(f) & \text{if } d_k - d_{k+1} = 1, \\ \Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}(E_1)f + G(f) & \text{if } d_k = d_{k-1}. \end{cases} \tag{5.12}$$

since all the other terms vanish. Hence the claim follows with $b = F(1)$. By an easy induction one can show

$$\Delta_1^{(k)}\Delta_2^{(k)} \cdots \Delta_{a-1}^{(k)}\Delta_a^{k}(x_{k,a}^{n}) = \begin{cases} 1 & a = n, \\ 0 & \text{otherwise,} \end{cases} \tag{5.13}$$

and so we are only interested in the leading term $(-1)^a x_{k,a}^a$ of E_1. On the other hand, again an easy induction shows that

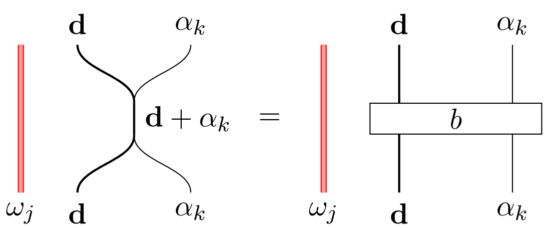
$$\Delta_1^{(k)}\Delta_2^{(k)}\cdots\Delta_{a-1}^{(k)}(x_{k,a}^n) \quad = \quad (-1)^{a-1}(e_1(k+1,a)-e_1(k,a)), \tag{5.14}$$

where $e_1(p,a)$ denotes the first elementary symmetric polynomial in the variable $x_{p,q}$, $1 \le q \le a$. Altogether we obtain

$$b = F(1) \quad = \quad \begin{cases} (-1)^a & \text{if } d_k - d_{k+1} = 1, \\ e_1(k+1,a)-e_1(k,a) & \text{if } d_k = d_{k-1}. \end{cases} \tag{5.15}$$

Note that $d_k = d_{k-1}$ if and only if $k+1 = j$.

So far we did not use the cyclotomic condition, which gives the even stronger relation

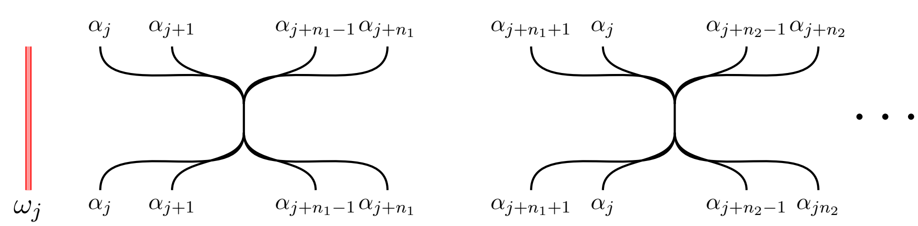


(5.16)

by noting that the middle term in (5.10) is in fact zero by Lemma 5.23 (since the idempotent after splitting of the two α_k causes it to vanish). Since all positive degree endomorphisms of $e_{(\mathbf{d})}$ are 0, we have $e_1(j,a) - e_1(j-1,a)e_{\mathbf{d},\alpha_k} = x_{j-1;a}e_{\mathbf{d},\alpha_j}$ and the lemma follows therefore by induction. ☐

Given a vector composition $\hat{\boldsymbol{\mu}}$, there is an induced usual composition of length r given by $\mathfrak{c}(\hat{\boldsymbol{\mu}}) = \{\sum_{j=1}^{e} \mu[-,j]\}$. These compositions are endowed with the usual lexicographic order on parts (this is a refinement of dominance order). We can define a filtration of A^{ω_j} by letting $A^{\omega_j}_{>\mathfrak{c}}$ be the two-sided ideal generated by $e_{\hat{\boldsymbol{\mu}}}$ for $\mathfrak{c}(\hat{\boldsymbol{\mu}}) > \mathfrak{c}$ in lexicographic order.

Lemma 5.24. *For a composition $\mathfrak{n} = (n_1, n_2, \dots)$, we have*



$$\equiv \quad \pm t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\cdots \quad (\text{mod } A^{\omega_j}_{>\mathfrak{c}(\mathbf{d}_{j;n_1},\mathbf{d}_{j+1;n_2},\dots)}.)$$

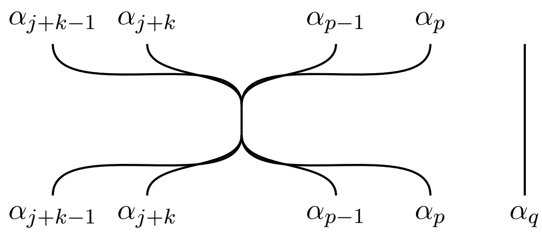
Proof. If the composition $\mathfrak{n}$ has 1 part, then we are done by Lemma 5.23. We assume we have $k+1$ parts and the statement is true for k parts. Thus the displayed diagram is equivalent to

$$t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\cdots|t^{(j+k-1;n_k)}|m$$

where m is the last of the double headed pitchforks, modulo $I := A^{\omega_j}_{>\mathfrak{c}}$. By Lemma 5.23, this is the same as the desired element, plus

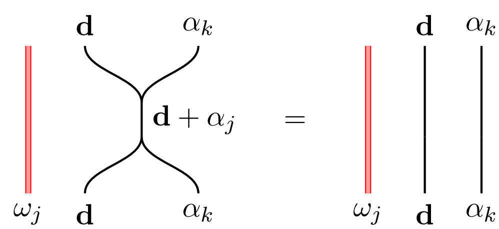
$$t^{(j;n_1)}|t^{(j+1;n_2)}|t^{(j+2;n_3)}|\cdots|t^{(j+k-1;n_k)}|m'$$

where m' lies in the cyclotomic ideal for $A^{\omega_{j+k}}$. That is, it can be written so that every diagram in it has a strand at the left labeled with a single root α_q, which in addition carries a dot if that root is α_{j+k-1}. Hence we have for instance a situation

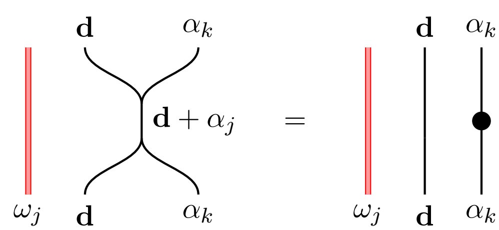


In order to complete the induction, it suffices to show that this can be written as a sum of elements each of which factors through the join of all strands (and hence is contained in I) or is contained in the cyclotomic ideal $A^{\omega_{j+k-1}}$ (and we can repeat our argument until we finally obtain only elements in I or in the cyclotomic ideal J for A^{ω_j}.)

If $q \not\equiv p+1 \mod e$, then the diagram above is already in J by Lemma 5.22. If $q \equiv p+1 \not\equiv j+k-1 \mod e$, then by (5.16) we have



If $k = p+n+1 = p+1 \mod e$, then



since the middle term of (5.10) vanishes (consider the idempotent after splitting of the two α_k and apply Lemma 5.23). Hence our element factors through the join of all strands and hence lies in $A^{\omega_j}_{>\mathfrak{c}}$. The lemma follows. □

Proposition 5.25. *The vectors $C_{\mathsf{S},\mathsf{T}}$ span $A^{\underline{\nu}}$.*

Proof. By Proposition 4.5, we need only show that vectors of the form

$$\grave{\mu}_1 \overset{w_1^{-1};h_1}{\Longrightarrow} \grave{\lambda} \overset{w_2;h_2}{\Longrightarrow} \grave{\mu}_2$$

can be written in terms of the $C_{\mathsf{S},\mathsf{T}}$'s. As usual, we induct on $\mathfrak{c}(\grave{\lambda})$.

By Lemma 5.24, if $\grave{\lambda}$ is not of the form $\grave{\lambda}_{\mathsf{S}}$ for some semi-standard tableaux, then we can rewrite our vector to factor through $\grave{\eta}$ which is higher in lexicographic order. Thus, we need only to show that elements of the form

$$\grave{\mu}_1 \overset{w_1^{-1};h_1}{\Longrightarrow} \grave{\lambda}_{\mathsf{S}} \overset{w_2;h_2}{\Longrightarrow} \grave{\mu}_2$$

can be written in terms of the $C_{\mathsf{S},\mathsf{T}}$'s (which are the special case where $h_1 = h_2 = 1$). At the center of this diagram, the picture looks precisely like that shown in the statement of Lemma 5.24, except that some of the legs may not split all the way down to unit vectors. We assume that we add the action of h_1 at the bottom of that portion of the diagram, which is after applying all crossings coming from w_1. By Lemma 5.24, the central portion of the diagram lies in $A_{>\mathfrak{c}(\grave{\lambda})}$, and so by induction, this element can be written as a linear combination of $C_{\mathsf{S},\mathsf{T}}$'s. By induction, the result follows. □

Given a semi-standard tableau S, we can associate a standard tableaux S° of the same shape with the following properties:

- Any box contains labels from the same alphabets in S and S°.
- Any pair of boxes with different entries in S has entries in S° in the same order.
- The row reading word of S° is maximal in Bruhat order amongst the standard tableau satisfying the first 2 conditions.

While the map $\mathsf{S} \mapsto \mathsf{S}^\circ$ is obviously not injective, it is injective on the set of tableaux with a fixed type.

To $\grave{\mu}$ we associate a vector $\varphi_{\grave{\mu}}$ of $A^{\boldsymbol{\nu}} e_{\grave{\mu}}$ defined as follows: let K be the set of vector compositions whose corresponding flags are complete refinements of that for $\grave{\mu}$. For each $\grave{\lambda} \in K$, there is a diagram $\grave{\mu} \longrightarrow \grave{\lambda}$ which looks like a bunch of chicken feet where the top is labeled with the sequence $\hat{\boldsymbol{\lambda}}$.

Definition 5.26. *We let* $\varphi_{\grave{\mu}} = \sum_{\grave{\lambda} \in K} \grave{\mu} \longrightarrow \grave{\lambda}$ *and* $\varphi = \sum_{\grave{\mu}} \varphi_{\grave{\mu}}$. *We call them* **chicken feet** *vectors and their duals* $\varphi^*_{\grave{\mu}}$, φ^* **pitchfork vectors**.

Now, consider the map $A^{\boldsymbol{\nu}} \to T^{\boldsymbol{\nu}}$, $a \mapsto \varphi a \varphi^*$. This map is obviously not injective, but it is on $e_{\mathfrak{n}} A^{\boldsymbol{\nu}} e_{\mathfrak{m}}$ for a fixed pair of compositions $\mathfrak{n}, \mathfrak{m}$.

Lemma 5.27. *For all* S, T*, we have*

$$\varphi C_{\mathsf{S},\mathsf{T}} \varphi^* = C_{\mathsf{S}^\circ,\mathsf{T}^\circ} + \sum_{\substack{\mathsf{S}' < \mathsf{S}^\circ \\ \mathsf{T}' < \mathsf{T}^\circ}} a_{\mathsf{S}',\mathsf{T}'} C_{\mathsf{S}',\mathsf{T}'} \pmod{A^{\boldsymbol{\nu}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}}. \tag{5.17}$$

In particular, the elements $C_{\mathsf{S},\mathsf{T}}$ *are all linearly independent.*

Proof. First, we note that

$$\varphi B_{\mathsf{S}} = B_{\mathsf{S}^\circ} + \sum_{\mathsf{S}' < \mathsf{S}} a_{\mathsf{S}'} B_{\mathsf{S}'} \pmod{A^{\boldsymbol{\nu}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}}.$$

The first term of the RHS comes from the term $\grave{\mu}_{\mathsf{S}^\circ} \longrightarrow \grave{\mu}_{\mathsf{S}} \cdot B_{\mathsf{S}} = B_{\mathsf{S}^\circ}$ of the product on the LHS; for any other pitchfork terms, we will either have a standard tableaux where $\ell(w_{\mathsf{S}'}) < \ell(w_{\mathsf{S}^\circ})$ (by assumption), or a non-standard tableau, in which case the term lies in $A^{\boldsymbol{\nu}}_{>\mathfrak{c}(\grave{\lambda}_{\mathsf{S}})}$. Thus, the equality follows. If we have a non-trivial relation between $C_{\mathsf{S},\mathsf{T}}$'s, then (by multiplying by idempotents $e_{\mathfrak{n}}$ on the left and right) we may assume that all tableaux which appear are of the same type. Since the $a \mapsto \varphi a \varphi^*$ is injective on such elements, we have a non-trivial relation between the right hand sides of (5.17). This is impossible because of the upper-triangularity in (5.17) and since the vectors $C_{\mathsf{S}^\circ,\mathsf{T}^\circ}$ are linearly independent modulo the image of $A^{\underline{\boldsymbol{\nu}}}_{>\mathfrak{c}}(\grave{\lambda}_{\mathsf{S}})$ by Theorem 5.17. □

Proof of Theorem 5.8. We verify that $(\Lambda, M, C, *, <, \deg)$ is a graded cell datum.

(C1) Clear.

(C2) This is the claim that the vectors $C_{\mathsf{S},\mathsf{T}}$ are a basis. They span by Lemma 5.25 and are linearly independent by Lemma 5.27.

(C3) By definition $C^*_{\mathsf{S},\mathsf{T}} = (B_{\mathsf{S}} B^*_{\mathsf{T}})^* = B_{\mathsf{T}} B^*_{\mathsf{S}} = C_{\mathsf{T},\mathsf{S}}$.

(C4) This is essentially identical to the proof of Theorem 5.17. Consider S of shape ξ. Since the $C_{\mathsf{S},\mathsf{T}}$ are a basis,

$$xB_{\mathsf{S}} = \sum_{\mathsf{S}',\mathsf{T}} r_x(\mathsf{S}, \mathsf{S}', \mathsf{T}) C_{\mathsf{S}',\mathsf{T}}$$

for some coefficients $r_x(\mathsf{S}, \mathsf{S}', \mathsf{T})$. Since T must be a semi-standard tableau of type ξ, we must have that the shape of T is above ξ in dominance order, unless T is the super-standard tableau. So,

$$xB_{\mathsf{S}} = \sum_{\mathsf{S}'} r_x(\mathsf{S}, \mathsf{S}') B_{\mathsf{S}'} \pmod{A^{\underline{\boldsymbol{\nu}}}(> \xi)}.$$

(C5) The degree function deg clearly satisfies the required conditions. □

6. Dipper-James-Mathas cellular ideals and graded Weyl modules

Let again $\underline{\boldsymbol{\nu}}$ be of the form $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}_1}, \ldots, \omega_{\mathfrak{z}_\ell})$; recall from the introduction that having for fixed ζ, a primitive nth root of unity in $\Bbbk$, we have an induced choice of parameters for the cyclotomic Hecke algebra, given by $q = \zeta, Q_i = \zeta^{\mathfrak{z}_I}$.

Proposition 6.1. *The chicken feet vector $\varphi_{\grave{\mu}}$ from Definition 5.26 transforms according to the sign representation for the Young subgroup $S_{\mathfrak{c}(\grave{\mu})}$.*

Proof. We verify the formulas from Proposition 5.20 using Proposition 3.4 or more precisely Remark 3.7. The case $i_r = r_{r+1}$ is obvious, since it corresponds to the fact that ${\Delta_r}^2 = 0$, whereas the case $i_r = i_{r+1} + 1$ corresponds to the last bullet point in Remark 3.7. The other cases are treated by the first bullet point. The claim follows. □

We can consider $A^{\underline{\boldsymbol{\nu}}} e_T$ as a $T^{\underline{\boldsymbol{\nu}}}$-representation (acting from the right). For each ℓ-multi-composition ξ, we let $\varphi_\xi = \sum_{\mathfrak{c}(\grave{\mu})=\xi} \varphi_{\grave{\mu}}$, and let $e_\xi = \sum_{\mathfrak{c}(\grave{\mu})=\xi} e_{\grave{\mu}}$. Then φ_ξ is a vector in $e_\xi A^{\underline{\boldsymbol{\nu}}} e_T$ which generates a sign representation for S_ξ. Thus, it induces a map $h_\xi \colon x_\xi T^{\underline{\boldsymbol{\nu}}} \to e_\xi A^{\underline{\boldsymbol{\nu}}} e_T$ such that $x_\xi a \mapsto \varphi_\xi a$.

Theorem 6.2. *The map h_ξ is an isomorphism. In particular, $A^{\underline{\boldsymbol{\nu}}} e_T \cong \bigoplus_\xi x_\xi T^{\underline{\boldsymbol{\nu}}}$ as right $T^{\underline{\boldsymbol{\nu}}}$-modules, and there is an isomorphism $\Phi^{\underline{\boldsymbol{\nu}}} \colon A^{\underline{\boldsymbol{\nu}}} \to \mathbf{S}^{\underline{\boldsymbol{\nu}}}$. On the cyclotomic q-Schur algebra of rank n, we obtain an induced isomorphism*

$$\Phi^{\underline{\boldsymbol{\nu}}}_n \colon \quad A^{\underline{\boldsymbol{\nu}}}_n \to \mathbf{S}(n; q, Q_1, \ldots, Q_\ell).$$

Proof. The surjectivity of h_ξ is clear; every element of $e_T A^{\underline{\boldsymbol{\nu}}} e_\xi$ is of the form

$$\sum a_{\grave{\lambda}} \cdot (\grave{\mu} \longrightarrow \grave{\lambda}) = \sum a_{\grave{\lambda}} e_{\grave{\lambda}} \varphi_{\grave{\mu}},$$

where $\grave{\lambda}$ only contains compositions of type e which correspond to simple roots α_i, and $a_{\grave{\lambda}} \in T^{\underline{\boldsymbol{\nu}}} A^{\underline{\boldsymbol{\nu}}} e_T$. Thus, the map is an isomorphism if and only if the spaces have the same dimension. Thanks to Lemma 5.21, the dimension of $x_\xi T^\nu$ is the number of pairs of tableaux (S, T) where S is standard and T is type ξ. By Lemma 5.27, the elements $C_{\mathsf{S},\mathsf{T}}$ for the same set of pairs are linearly independent vectors in $e_T A^{\underline{\boldsymbol{\nu}}} e_\xi$, so it must have at least this dimension. Thus, h_ξ is an isomorphism and we obtain $\Phi^{\underline{\boldsymbol{\nu}}}\colon A^{\underline{\boldsymbol{\nu}}} \to \mathbf{S}^{\underline{\boldsymbol{\nu}}}$ by Proposition 4.11 and the definition of $\mathbf{S}^{\underline{\boldsymbol{\nu}}}$. The theorem follows. □

Theorem 5.8 shows that $A_n^{\underline{\boldsymbol{\nu}}}$ comes with a natural grading and hence Theorem 6.2 shows that $\Phi_n^{\underline{\boldsymbol{\nu}}}\colon \quad A_n^{\underline{\boldsymbol{\nu}}}$ is a graded version of $\mathbf{S}(n; q, Q_1, \dots, Q_\ell)$. Following [Str03], we say that a $\mathbf{S}(n; q, Q_1, \dots, Q_\ell)$-module $\overline{M}$ *has a graded lift* M if there is a graded $A_n^{\underline{\boldsymbol{\nu}}}$-module which is isomorphic to $\overline{M}$ after forgetting the grading. Any such M is then a *graded lift* of $\overline{M}$. By [Str03, Lemma 1.5], graded lifts of indecomposable modules are unique up to isomorphism and overall grading shifts.

Theorem 6.3. *Under the isomorphism* (5.5)*, the cellular structure from Theorem 5.8 is intertwined with the Dipper-James-Mathas cellular structure* [DJM98, Def. 6.7] *on the cyclotomic q-Schur algebra in that sense that:*

1. *The cellular ideals coincide when the order in the Dipper-James-Mathas structure is weakened to the lexicographic ordering on multi-partitions.*
2. *The cell modules W^ξ are graded lifts of the Weyl modules of $\mathbf{S}^{\underline{\boldsymbol{\nu}}}$, and the $F^\xi = W^\xi / \operatorname{rad} W^\xi$ form (up to overall grading shift) a complete, irredundant set of graded lifts of simple modules.*
3. *Any other graded lift of a Weyl module differs (up to isomorphism) only by an overall shift in the grading.*

Proof. The cellular ideals of either our basis or the Dipper-James-Mathas basis can be defined in terms of maps between permutation modules factoring through those greater in lexicographic order. That is, identifying $e_\xi \mathbf{S}^{\underline{\boldsymbol{\nu}}} e_{\xi'}$ with $\operatorname{Hom}_{\mathbf{S}^{\underline{\boldsymbol{\nu}}}}(\mathbf{S}^{\underline{\boldsymbol{\nu}}} e_\xi, \mathbf{S}^{\underline{\boldsymbol{\nu}}} e_{\xi'})$, the intersection with $e_\xi \mathbf{S}^{\underline{\boldsymbol{\nu}}}(> \vartheta) e_{\xi'}$ is just the maps factoring through $\mathbf{S}^{\underline{\boldsymbol{\nu}}} e_{\vartheta'}$ with $\vartheta' < \vartheta$ in the lexicographic order. It is clear that the same definition works for our cellular structure. The second statement is then clear and the third follows by standard arguments (e.g. [Str03, Lemma 1.5]), since the Weyl modules are the standard modules of a highest weight structure, and thus indecomposable. □

Remark 6.4. In the case where ν is itself a fundamental weight, $\mathbf{S}^\nu$ is the sum of the usual q-Schur algebras for all different ranks. Except for some degenerate cases, a grading on this algebra has already been defined by Ariki, [Ari09].

Theorem 6.5. *The grading defined on $\mathbf{S}^\nu$ via the isomorphism Φ^ν agrees with Ariki's,* [Ari09]*, up to graded Morita equivalence.*

Proof. Ariki's grading is defined uniquely (up to Morita equivalence) by the fact that there is a graded version of the Schur functor compatible with the Brundan-Kleshchev grading on $\mathfrak{H}^\nu$. The image of an indecomposable projective of the q-Schur algebra under the Schur functor is a graded lift of an *indecomposable* summand of a permutation module; hence a graded lift is unique up to shift by [Str03, Lemma 1.5]. This shows that such a grading is unique, and both Ariki's and our gradings satisfy this condition. □

Remark 6.6. From our identification of the cyclotomic quiver Schur algebras with the cyclotomic Schur algebras, Theorem 6.2, it follows that $A^{\underline{\boldsymbol{\nu}}}$ and $A^{\underline{\boldsymbol{\nu}}}_{\mathbf{d}}$ have finite global dimension. The indecomposable projective modules form (in the nongraded and also in the graded version) a $\mathbb{Z}$-basis of the Grothendieck group. We will describe the underlying combinatorics in the next section.

7. Graded multiplicities and q-Fock space

As promised in the introduction, we now draw the connection between the algebras $A^{\underline{\boldsymbol{\nu}}}$ and the theory of higher representation theory as in the work of Rouquier [Rou08] and Khovanov-Lauda [KL10]. We start with the combinatorics of Fock space, obtain the main result on decomposition numbers and finish with a categorification result.

7.1. **Induction and restriction functors.** We still assume that $\underline{\boldsymbol{\nu}} = (\omega_{\mathfrak{z}_1}, \dots, \omega_{\mathfrak{z}_\ell})$ and also $e > 2$ (although all of our results can be extended to an arbitrary sequence of weights by replacing the Fock space by any highest weight representation of $U_q(\widehat{\mathfrak{gl}}_e)$).

As before, we identify the dimension vector $\mathbf{d}$ with the element $\sum d_i\alpha_i$ of the root lattice of $\widehat{\mathfrak{sl}}_e$; we let $\langle -, - \rangle$ denote the usual pairing between the root and weight lattice of $\widehat{\mathfrak{sl}}_e$, in particular $\langle \omega_i, \mathbf{d} \rangle = \mathbf{d}_i$. In terms of the bilinear Euler form $\{\mathbf{d}', \mathbf{d}''\} = \sum_{i=1}^{e} \mathbf{d}'_i(\mathbf{d}''_i - \mathbf{d}''_{i+1})$ we have that

$$\langle \mathbf{d}', \mathbf{d}'' \rangle = \{\mathbf{d}', \mathbf{d}''\} + \{\mathbf{d}'', \mathbf{d}'\}.$$

Consider the inclusion of algebras $\gamma_{\mathbf{d}} \colon A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}} \to A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}}$ defined by $a \mapsto a|e_{\mathbf{d}}$ and $A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}|\mathbf{d}} = \gamma_{\mathbf{d}} A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}} \gamma_{\mathbf{d}}$ with the appropriate idempotent $\gamma_{\mathbf{d}}$ (the sum of all $e_{\hat{\mu}}$ where the last multicomposition ends with $\mathbf{d}$). The most interesting case is when $\mathbf{d} = \alpha_i$ for which we use the abbreviation $\gamma_i = \gamma_{\mathbf{d}}$ and denote the image of the inclusion also by $A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}|i}$.

Definition 7.1. *Define, for any* $\mathbf{c}$*, the graded* $\mathbf{d}$*-induction and* $\mathbf{d}$*-restriction functors*

$$\mathfrak{F}_{\mathbf{d}} \colon \quad A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}}\text{-mod} \to A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}}\text{-mod}, \qquad M \mapsto \mathfrak{F}_{\mathbf{d}} M = A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}} \gamma_{\mathbf{d}} \otimes_{A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}}} M,$$
$$\mathfrak{E}_{\mathbf{d}} \colon \quad A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}}\text{-mod} \to A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}}\text{-mod}, \qquad N \mapsto \mathfrak{E}_{\mathbf{d}} N = \operatorname{Hom}_{A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}}}(\gamma_{\mathbf{d}} A^{\underline{\boldsymbol{\nu}}}_{\mathbf{c}+\mathbf{d}}, N)\langle s(\mathbf{d}, \mathbf{c}) \rangle.$$

where $\langle \_ \rangle$ *denotes the grading shift as in Section 2.5 and* $s(\mathbf{d}, \mathbf{c}) = \{\mathbf{d}, \mathbf{d}\} + \{\mathbf{c}, \mathbf{d}\} + \{\mathbf{d}, \mathbf{c}\} - \sum_{i=1}^{\ell} \mathbf{d}_{\mathfrak{z}_i}$.

Lemma 7.2. *The functors* $\mathfrak{F}_{\mathbf{d}}$ *send projective objects to projective objects, and the functors* $\mathfrak{E}_{\mathbf{d}}$ *are exact.*

Proof. The vector space underlying $\mathfrak{E}_{\mathbf{d}} M$ is $\gamma_{\mathbf{d}} N$, the image of an idempotent acting on M, so $\mathfrak{E}_{\mathbf{d}}$ is exact. The left adjoint of an exact functor always sends projectives to projectives, see also the proof of Lemma 7.16. □

Note that viewing $A^{\boldsymbol{\nu}}$−mod as the representation category of $A^{\boldsymbol{\nu}}$, these functors are induced by the monoidal action of A described earlier and its adjoints. In the special case $\mathbf{d} = \alpha_i$ we obtain after forgetting the grading via the isomorphism, Theorem 6.2,

$$\Phi_n^{\boldsymbol{\nu}}: \quad A_n^{\boldsymbol{\nu}} \to \mathbf{S}(n; q, Q_1, \ldots, Q_\ell)$$

the ordinary i-induction and i-restriction functors for the cyclotomic q-Schur algebras,

$$\begin{aligned} \overline{\mathfrak{F}}_i &: \mathbf{S}(n; q, Q_1, \ldots, Q_\ell)\text{−mod} \to \mathbf{S}(n+1; q, Q_1, \ldots, Q_\ell)\text{−mod}, \\ \overline{\mathfrak{E}}_i &: \mathbf{S}(n+1; q, Q_1, \ldots, Q_\ell)\text{−mod} \to \mathbf{S}(n; q, Q_1, \ldots, Q_\ell)\text{−mod}. \end{aligned}$$

A detailed study of the ordinary i-induction and i-restriction functors can be found in [Wad11, §5]. The functors $\overline{\mathfrak{F}}_i$ and $\overline{\mathfrak{E}}_i$ are biadjoint, hence in particular exact, [Wad11, Theorem 4.14]; however, when the grading is taken into account the left and right adjoints of $\overline{\mathfrak{F}}_i$ will differ by a shift in the grading. For general $\mathbf{d}$, the functors $(\overline{\mathfrak{F}}_{\mathbf{d}}, \overline{\mathfrak{E}}_{\mathbf{d}})$ form an adjoint pair, but are not biadjoint.

7.2. **Combinatorics of higher Fock space.** We now introduce the combinatorics which will control graded versions of induction and restriction functors.

Fix $\mathfrak{z} \in \mathbb{Z}$ and let $V = \mathbb{Z}[q, q^{-1}]^{\mathbb{Z}}$ with basis $u_i, i \in \mathbb{Z}$. For a partition λ we denote by λ' its transposed partition.

Definition 7.3. *The* **level 1 quantized Fock space** $\mathbb{F}_1(\mathfrak{z})$ *of charge* $\mathfrak{z}$ *is the* $\mathbb{Z}[q, q^{-1}]$*-module freely generated by a symbol* λ *for each partition* λ*.*

Recall the realization of Fock space in terms of the free $\mathbb{C}[q, q^{-1}]$-module $\wedge^{\frac{\infty}{2}} V$ of semi-infinite wedges in V on basis $u_{i_1} \wedge u_{i_2} \wedge \cdots$ where the indices $i_k \in \mathbb{Z}$ form an increasing sequence $i_1 < i_2 < \ldots$ such that $i_k = \mathfrak{z} + k - 1$ for $k \gg 1$:

Lemma 7.4. *Fix a charge* $\mathfrak{z}$*. There is an isomorphism of* $\mathbb{Z}[q, q^{-1}]$*-modules*

$$\begin{aligned} \mathbb{F}_1(\mathfrak{z}) &\cong \wedge^{\frac{\infty}{2}} V \\ \lambda &\mapsto u_\lambda := u_{\mathfrak{z}-\lambda'_1} \wedge u_{\mathfrak{z}-\lambda'_2-1} \wedge u_{\mathfrak{z}-\lambda'_3-2} \wedge \cdots. \end{aligned} \tag{7.1}$$

Under this identification, there is an addable box in column k *of* λ *iff* $i_k > i_{k-1} + 1$*. In this case the index* i_k *taken modulo* e *is the residue of this unique addable box.*

Proof. This follows directly from the definitions. □

Th space (7.1) carries an action of $U_q(\widehat{\mathfrak{sl}}_e)$ defined by Hayashi [Hay90]. The explicit formulas crucially depend on the identification (7.1) and the choice of coproduct on the Hall algebra discussed in Section 2.5. We chose a slightly unusual identification (7.1), but better suitable for our purpose than for instance [VV11] (where the partition gets not transposed and the indices form a decreasing sequence). We choose the coproduct

$$\Delta(f_i) = 1 \otimes f_i + f_i \otimes k_i, \quad \Delta(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1, \quad \Delta(k_i) = k_i \otimes k_i.$$

for $U_q(\widehat{\mathfrak{sl}}_e)$. It extends by [VV99, (10)] and the Drinfeld double construction, see e.g. [Xia97], to a coproduct of $U_q(\widehat{\mathfrak{gl}}_e)$ by setting on the standard generators, Section 2.5,

$$\Delta(\mathbf{f_d}) = \sum_{\mathbf{d}=\mathbf{d}'+\mathbf{d}''} q^{-\{\mathbf{d}',\mathbf{d}''\}} \mathbf{f}_{\mathbf{d}''} \otimes \mathbf{f}_{\mathbf{d}'} \mathbf{k}_{\mathbf{d}''}, \quad \Delta(\mathbf{e_d}) = \sum_{\mathbf{d}=\mathbf{d}'+\mathbf{d}''} q^{\{\mathbf{d}'',\mathbf{d}'\}} \mathbf{k}_{\mathbf{d}'}^{-1} \mathbf{e}_{\mathbf{d}''} \otimes \mathbf{e}_{\mathbf{d}'},$$

$$\Delta(\mathbf{k_d}) = \Delta(\mathbf{k_d}) \otimes \Delta(\mathbf{k_d}), \tag{7.2}$$

where $U_q(\widehat{\mathfrak{gl}}_e)$ is the Drinfeld double of U_e^-. By [VV99, 6.2], the Hayashi action extends to an action of $U_q(\widehat{\mathfrak{gl}}_e)$, such that acting with $\mathbf{f_d}$ on a partition λ produces a $\mathbb{Z}[q,q^{-1}]$-linear combination $\mathbf{f_d}\lambda = \sum_\mu q^{m(\mu/\lambda)_-}\mu$ of all partitions μ obtained from λ by adding $|\mathbf{d}|$ boxes *but at most one per column* and exactly d_i of residue i for $i \in \mathbb{V}$. We write $\mathrm{res}(\mu/\lambda) = \mathbf{d}$ for all such partitions μ. Similarly $\mathbf{e_d}$ acts by removing $|\mathbf{d}|$ boxes, again at most one per column, and exactly d_i of residue i for $i \in \mathbb{V}$. To describe the coefficients $q^{m(\mu/\lambda)_-}$ assume $\mathrm{res}(\mu/\lambda) = \mathbf{d}$ and let $m_j = 1$ if there was a box added in column j to obtain μ from λ and set $m_j = 0$ otherwise for all $j \geq 1$. Using the identification (7.1) for λ define

$$m(\mu/\lambda)_- \quad = \quad \sum_{i_s<i_t} m_t(1-m_s)(\delta_{\overline{i_s},\overline{i_t}} - \delta_{\overline{i_s+1},\overline{i_t}}).$$

Here t denotes a column where a box b was added and then $\delta_{\overline{i_s},\overline{i_t}} - -\delta_{\overline{i_s+1},\overline{i_t}}$ counts the number of addable minus the number of removable boxes below b with the same residue as b, excluding the blocked columns (see Definition 5.12) thanks to the factor $(1-m_s)$. Similarly, we have the 'reversed' statistics involving boxes above

$$m(\mu/\lambda)_+ \quad = \quad \sum_{i_s>i_t} m_t(1-m_s)(\delta_{\overline{i_s},\overline{i_t}} - \delta_{\overline{i_s+1},\overline{i_t}}).$$

Lemma 7.5. *There is an $U_q(\widehat{\mathfrak{gl}}_e)$-action on Fock space $\mathbb{F}_1(\mathfrak{z})$ given by*

$$\mathbf{f_d}\lambda = \textstyle\sum_{\mathrm{res}(\mu/\lambda)=\mathbf{d}} q^{m(\mu/\lambda)_-}\mu, \qquad \mathbf{e_d}\lambda = \sum_{\mathrm{res}(\lambda/\mu)=\mathbf{d}} q^{m(\mu/\lambda)_+}u_\lambda,$$
$$\mathbf{k_d}\lambda = q^{\langle \mathbf{d}, \mathrm{wt}(\lambda)\rangle}\lambda.$$

Here $\langle \mathbf{d}, \mathrm{wt}(\lambda)\rangle = \sum_{i=1}^e d_i(\mathrm{add}_i - \mathrm{rem}_i)$, where add_i respectively rem_i denotes the number of addable respectively removable boxes in λ of residue i.

Proof. This follows by straightforward calculations from our definitions, see [VV99] for details (remembering that we work with the transposed partitions). □

Remark 7.6. We should note that conventions vary from author to author. Our conventions are transposed to [VV11] and also [Ugl00, 2.1] and designed so as to match the projectives of our categorification with the canonical basis of Fock space; different conventions can match it instead with the tilting modules or other collections of modules.

Definition 7.7 (Higher Level Fock space)**.** *Let $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell\}$ be a fixed ℓ-tuple of charges. Then $\mathbb{F}_\ell$ denotes the ℓ-fold level 1 fermionic Fock space*

$$\mathbb{F}_\ell = \mathbb{F}_1(\mathfrak{z}_1) \otimes \cdots \otimes \mathbb{F}_1(\mathfrak{z}_\ell),$$

with charges $\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell$ equipped with the basis $u_\xi = u_{\xi^{(1)}} \otimes \cdots \otimes u_{\xi^{(\ell)}}$ where ξ ranges over all ℓ-multi-partitions. It comes with the usual $\mathbb{C}[q,q^{-1}]$-bilinear inner product $(-,-)$ where the basis u_ξ is orthonormal. The elements u_ξ are called **standard basis vectors***.*

Remark 7.8. In the bosonic realization of Fock space, [KR87], our standard basis is just the products of Schur functions $u_\xi = s_{\xi^{(1)}}s_{\xi^{(2)}} \cdots s_{\xi^{(\ell)}}$ in ℓ different alphabets. As we pointed out already in the introduction, we are *not* considering the higher level Fock space studied by Uglov [Ugl00], but the more naive tensor product of level 1 Fock spaces. This distinction is discussed extensively in [BK09b, §3], see also Theorem 7.19.

We can easily generalize our description of the Hayashi action to this tensor product. For boxes (i,j,k) and (i',j',k') in the diagram of a multi-partition, we say that (i',j',k') is **below** (i,j,k) if $k'>k$ or $k'=k$ and $i'<i$. The careful reader should note that this doesn't match some writers' conventions (for example, from [HM10]; our convention will match theirs if one indexes partitions in opposite order). Then, we can define $m(\mu/\lambda)_-$ for multi- partitions exactly as before, namely by going through all added boxes b and count the number of addable minus the number of removable boxes below b with the same residue as b, excluding the blocked columns. The definitions imply

Lemma 7.9. *The action is given by*

$$\mathbf{f_d}u_\xi = \sum_{\mathrm{res}(\eta/\xi)=\mathbf{d}} q^{m(\eta/\xi)_+}u_\eta, \qquad \mathbf{e_d}u_\eta = \sum_{\mathrm{res}(\eta/\xi)=\mathbf{d}} q^{-m(\eta/\xi)_-}u_\xi \tag{7.3}$$

where η and ξ ranges over pairs of ℓ-multi-partitions such that η/ξ has no two boxes in the same column.

Example 7.10. *For instance, if $e=3$, $\ell=2$ and $\mathfrak{z}=(0,0)$, then we have the following (where we abuse notation and write the residues into the boxes)*

$$\begin{aligned} e_0f_0\cdot\left(\begin{ytableau}0&1&2\\2\end{ytableau},\emptyset\right) &= e_0\cdot\left(q\cdot q\left(\begin{ytableau}0&1&2&0\\2\end{ytableau},\emptyset\right)+1\cdot q\left(\begin{ytableau}0&1&2\\2&0\end{ytableau},\emptyset\right)+1\left(\begin{ytableau}0&1&2\\2\end{ytableau},\begin{ytableau}0\end{ytableau}\right)\right). \\ &= (q^2\cdot 1+q\cdot q^{-1}+1\cdot q^{-2})\left(\begin{ytableau}0&1&2\\2\end{ytableau},\emptyset\right). \end{aligned}$$

where the numbers indicate the residues. Hence $e_of_0-f_0e_0$ acts by multiplication with $(q^2\cdot 1+q\cdot q^{-1}+1\cdot q^{-2})=\frac{(q^3-q^{-3})}{(q-q^{-1})}$ which agrees with the action of $\frac{k_i-k_i^{-1}}{q-q^{-1}}$.

$$f_{(1,0,1)}\cdot\left(\begin{ytableau}0&1&2\\2\end{ytableau},\emptyset\right) = q^2\left(\begin{ytableau}0&1&2&0\\2\\1\end{ytableau},\emptyset\right)+q\left(\begin{ytableau}0&1&2\\2&0\\1\end{ytableau},\emptyset\right)+\left(\begin{ytableau}0&1&2\\2\\1\end{ytableau},\begin{ytableau}0\end{ytableau}\right)$$

(Note that the two boxes can't be put both in the first column.)

Lemma 7.11. *We have the following adjunction formula for the action on $\mathbb{F}_\ell$:*

$$(\mathbf{f_d}u_\xi, u_\eta) \;=\; (u_\xi, q^{\{\mathbf{d},\mathbf{d}\}}\mathbf{e_d}\mathbf{k_d}u_\eta) \tag{7.4}$$

Proof. This follows directly from (7.3) and the formula

$$m(\eta/\xi)_+ + m(\eta/\xi)_- = \{\mathbf{d},\mathbf{d}\} + \langle\mathbf{d},\mathrm{wt}(u_\eta)\rangle,$$

where $\langle -,-\rangle$ denotes the usual pairing between roots and weights of $\widehat{\mathfrak{sl}}_e$ as in Lemma 7.5 summed over all occurring partitions, since $\mathbf{k_d}u_\eta = q^{\langle\mathbf{d},\mathrm{wt}(u_\eta)\rangle}$. □

7.3. **A weak categorification.** Let $K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$ be the split Grothendieck group of the graded category of graded projective modules over $A^{\underline{\boldsymbol{\nu}}}$; since the algebra $A^{\underline{\boldsymbol{\nu}}}$ has finite global dimension by Remark 6.6, this is canonically isomorphic to the Grothendieck group of all $A^{\underline{\boldsymbol{\nu}}}$-modules, not just the projectives. By Theorem 6.3(2), the Grothendieck group of $A^{\underline{\boldsymbol{\nu}}}-$mod has a basis over $\mathbb{Z}[q,q^{-1}]$ given by the classes of the cell modules W^ξ with the chosen standard lift in the grading such that the head is in degree zero. The action of grading shift induces a $\mathbb{Z}[q,q^{-1}]$-module structure on $K_q^0(\mathbf{A}^{\underline{\nu}})$ where q shifts the grading up by 1. We'll be interested in the isomorphism

$$\psi\colon \quad \mathbb{Z}K_q^0(\mathbf{A}^{\underline{\nu}}) \xrightarrow{\sim} \mathbb{F}_\ell, \quad \text{defined by} \quad [W^\xi] \mapsto u_\xi. \tag{7.5}$$

Remark 7.12. *Note that under the isomorphism ψ, isomorphism classes from $A^{\underline{\nu}}_{\mathbf{c}}$–mod get sent to elements of weight* $\mathrm{wt}(\mathbf{c}) = \sum_{i=1}^{\ell} \lambda_i - \sum_{i=1}^{e} c_i\alpha_i$.

We define the following grading shifting functors for any $\mathbf{c}, \mathbf{d}$:

$$\mathfrak{K}_{\mathbf{d}} = \langle \mathbf{d}, \mathrm{wt}(c)\rangle : \qquad A^{\underline{\nu}}_{\mathbf{c}}\text{–mod} \to A^{\underline{\nu}}_{\mathbf{c}}\text{–mod}. \tag{7.6}$$

By the above remark they are well-defined and their induced action on the Grothendieck group agrees with $\mathbf{k_d}$ via ψ. Moreover, we have the following:

Theorem 7.13. *The functors $\mathfrak{F}_{\mathbf{d}}$ and $\mathfrak{E}_{\mathbf{d}}$ induce an action of $U_q(\widehat{\mathfrak{gl}}_e)$ on $\mathbb{C}(q)\otimes_{\mathbb{Z}[q,q^{-1}]} K_q^0(\mathbf{A}^{\underline{\nu}})$; the map ψ defines an isomorphism of representations $K_q^0(\mathbf{A}^{\underline{\nu}}) \cong \mathbb{F}_\ell$.*

Before giving the proof of this theorem, we need some preparation.

Definition 7.14. *Given a residue datum $\hat{\boldsymbol{\mu}}$, define $\mathbf{f}_{\hat{\boldsymbol{\mu}}} = \mathbf{f}_{\mu^{(r)}}\cdots\mathbf{f}_{\mu^{(1)}} \in U_e^-$ and inductively the vectors $h_{\grave{\mu}} \in \mathbb{F}_\ell$ as*

$$h_{\grave{\mu}} = \mathbf{f}_{\hat{\boldsymbol{\mu}}(\ell)}\left(u_\emptyset \otimes h_{(\hat{\boldsymbol{\mu}}(\ell-1),\dots,\hat{\boldsymbol{\mu}}(1))}\right).$$

(*The jth factor acts here on elements of the jth-fold level* 1 *Fock space* $\mathbb{F}_j$.)

Proposition 7.15. *We have that*

$$h_{\grave{\mu}} = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi\\ \mathrm{type}(\mathsf{S})=\grave{\mu}}} q^{\mathrm{Deg}(\mathsf{S})}u_\xi, \qquad \text{and} \qquad [A^{\underline{\nu}}_{\grave{\mu}}] = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi\\ \mathrm{type}(\mathsf{S})=\grave{\mu}}} q^{\deg(\mathsf{S})}[W^\xi].$$

In particular, it follows from (7.5) *and Proposition 5.15 that* $\psi([A^{\underline{\nu}}e_{\grave{\mu}}]) = h_{\grave{\mu}}$.

Proof. We start with the first displayed equality and prove this by induction on ℓ. For $\ell = 1$ this is clear from the definitions and Lemma 7.5. We fix now $\ell > 1$ and assume the result holds for shorter $\hat{\boldsymbol{\mu}}$. In particular we only need to consider the case where $\hat{\boldsymbol{\mu}}^{(\ell)} \neq \emptyset$ and can assume the claim to hold for $\grave{\nu}$ obtained from $\grave{\mu}$ with the last part $\hat{\boldsymbol{\mu}}^{(\ell)}$ of $\hat{\boldsymbol{\mu}}$ removed. Let this part be $\mathbf{d}$. Then $h_{\grave{\mu}} = \mathbf{f_d}h_{\grave{\nu}}$ and

$$h_{\grave{\nu}} = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi\\ \mathrm{type}(\mathsf{S})=\grave{\nu}}} q^{-\deg(\mathsf{S})}u_\xi$$

Thus, we have that

$$\mathbf{f_d}h_{\grave{\nu}} = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi\\ \mathrm{type}(\mathsf{S})=\grave{\nu}}} q^{\deg(\mathsf{S})}\mathbf{f_d}u_\xi = \sum_{\substack{\mathrm{sh}(\mathsf{S})=\xi\\ \mathrm{type}(\mathsf{S})=\grave{\mu}}} q^{\deg(\mathsf{S})}u_\xi.$$

For the last equality we used Lemma 7.3 and the definition of the comultiplication (7.2) in comparison with the Definition 5.12 of the combinatorial degree Deg. This completes the proof of the first formula.

For the second equality, we apply the result [HM10, 2.14]; the projective module $A^{\underline{\nu}}e_{\grave{\mu}}$ has a filtration by W_ξ with multiplicity spaces $\dot{W}_\xi \otimes_{A^{\underline{\nu}}} A^{\underline{\nu}}e_{\grave{\mu}} \cong e_{\grave{\mu}}W_\xi$ and hence we are done by Theorem 5.8: the space $e_{\grave{\mu}} W_\xi$ has a homogeneous basis indexed by semi-standard tableaux of type $\grave{\mu}$ and shape ξ. □

The Grothendieck group $K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$ is endowed with a q-bilinear pairing defined by

$$([P],[P']) \;=\; \dim_q\left(\dot{P} \otimes_{A^{\underline{\boldsymbol{\nu}}}} P'\right). \tag{7.7}$$

Here $\dot{P}$ is P considered as a right module using the $*$-antiautomorphism and $\dim_q M = \sum \dim M_i q^i$ denotes the graded Poincare polynomial for any finite dimensional $\mathbb{Z}$-graded vector space $M = \oplus_{i\in Z} M_i$. Then the pairing (7.7) extends to objects M and N by taking the derived tensor product $([M],[N]) = \dim_q\left(\dot{M} \otimes^{\mathbb{L}}_{A^{\underline{\boldsymbol{\nu}}}} N\right)$. Since $\{u_\xi\}$ and $\{[W^\xi]\}$ are orthonormal bases for the natural q-bilinear forms on the two spaces, the map ψ intertwines the inner product of $\mathbb{F}_\ell$ and the form $(-,-)$ on $K_q^0(\mathbf{A}^{\underline{\boldsymbol{\nu}}})$.

Let $\mathbf{F_d} = [\mathcal{L}\mathbf{F_d}]$ and $\mathbf{E_d} = [\mathcal{L}\mathfrak{E}_\mathbf{d}] = [\mathfrak{E}_\mathbf{d}]$ be the endomorphisms on the Grothendieck groups induced by the derived functors $\mathcal{L}\mathfrak{F}_\mathbf{d}$ and $\mathcal{R}\mathfrak{E}_\mathbf{d}$, and $\mathbf{K_d}$ the endomorphism induced by $\mathfrak{K}_\mathbf{d}$ (which is just multiplication with a certain power of q). We have then the following categorification of the adjunction formula (7.4)

Lemma 7.16. *With the pairing from* (7.7) *we have*

$$(\mathbf{F_d}[M],[N]) \;=\; q^{\{\mathbf{d},\mathbf{d}\}}\left([M],\mathbf{E_d}\mathbf{K_d}[N]\right). \tag{7.8}$$

Proof. By Remark 6.6 it is enough to check the formula on projectives, where all the functors are exact. Abbreviate $B = A^{\boldsymbol{\nu}}_{\mathbf{c}+\mathbf{d}}$ and $A = A^{\boldsymbol{\nu}}_{\mathbf{c}} = \gamma_\mathbf{d} B \gamma_\mathbf{d}$ and and let $M = Ae_{\grave{\mu}}$ and $N = Be_{\grave{\lambda}}$. Then the definitions imply

$$\begin{aligned}
(\mathbf{F_d}[M],[N]) &= \dim_q((\dot{\mathfrak{F}_\mathbf{d} M}) \otimes_B N) = \dim_q(B\gamma_\mathbf{d} \dot{\otimes}_A M) \otimes_B Be_{\grave{\lambda}} \\
&= \dim_q(e_{\grave{\mu}} A \otimes_A \gamma_\mathbf{d} B \otimes_B) Be_{\grave{\lambda}} = \dim_q(e_{\grave{\mu}|\mathbf{d}} Be_{\grave{\lambda}}). \\
([M],\mathbf{E_d}[N]) &= \dim_q(\dot{M} \otimes_A \mathfrak{E}_\mathbf{d} N) = q^{s(\mathbf{c},\mathbf{d})} \dim_q(e_{\grave{\mu}} A \otimes_A \gamma_\mathbf{d} Be_{\grave{\lambda}} \\
&= q^{s(\mathbf{c},\mathbf{d})} \dim_q(e_{\grave{\mu}|\mathbf{d}} Be_{\grave{\lambda}}).
\end{aligned}$$

Then the right hand side of (7.8) equals q^c times the left hand side, where $c = \{\mathbf{d},\mathbf{d}\} + s(\mathbf{c},\mathbf{d}) + \sum_{i=1}^{\ell} \mathbf{d}_{\mathfrak{z}i} - \langle \mathbf{d}, \mathbf{c}+\mathbf{d}\rangle = 0$. □

Proof of Theorem 7.13. From Proposition 7.15 we have $\mathbf{f_d}\psi([A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}}]) = \psi([A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}|\mathbf{d}}])$. On the other hand $\mathbf{F_d}([A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}}]) = [A^{\underline{\boldsymbol{\nu}}} e_{\grave{\mu}|\mathbf{d}}]$ by definition of $\mathfrak{F}_\mathbf{d}$. Hence ψ intertwines $\mathbf{F_d}$ with $\mathbf{f_d}$ by Remark 6.6 and as we know already $\mathfrak{K}_\mathbf{d}$ with $\mathbf{k_d}$. Then it also intertwines $\mathbf{E_d}$ with $\mathbf{e_d}$ by Lemma 7.16, since it intertwines the two nondegenerate bilinear forms. □

7.4. **The involution induced by Serre twisted duality.** For any left $A^{\underline{\boldsymbol{\nu}}}$-module M, the space $H = \mathrm{Hom}_{A^{\underline{\boldsymbol{\nu}}}}(M, A^{\underline{\boldsymbol{\nu}}})$ is naturally a right $A^{\underline{\boldsymbol{\nu}}}$-module via the action $(f\cdot a)(m) = f(m)\cdot a$. Using the anti-involution $*$, this can be turned into a left $A^{\underline{\boldsymbol{\nu}}}$-module via $(f\cdot a)(m) = f(m)\cdot a^*$ for $f \in H, m \in M, a \in A^{\underline{\boldsymbol{\nu}}}$. This extends to the derived functor

$$\mathfrak{D} = \mathrm{RHom}_{A^{\underline{\boldsymbol{\nu}}}}(-, A^{\underline{\boldsymbol{\nu}}}): \quad D^b(A^{\underline{\boldsymbol{\nu}}}-\mathrm{mod}) \to D^b(A^{\underline{\boldsymbol{\nu}}}-\mathrm{mod}).$$

We refer to this functor as **Serre-twisted duality**.

This name can be explained by the following alternate description of the same functor. Let $\star : A^{\underline{\boldsymbol{\nu}}}-\mathrm{mod} \to A^{\underline{\boldsymbol{\nu}}}-\mathrm{mod}$ be the duality functor given by taking vector space dual, and then twisting the action of $A^{\underline{\boldsymbol{\nu}}}$ by the anti-automorphism $*$. Let

$\mathfrak{S} : D^b(A^{\boldsymbol{\nu}}\text{-mod}) \to D^b(A^{\boldsymbol{\nu}}\text{-mod})$ be the graded version of the Serre functor. Since $A^{\boldsymbol{\nu}}_{\mathbf{d}}$ is finite-dimensional and has finite global dimension (Remark 6.6), this is simply the derived functor of tensor product with the bimodule $(A^{\boldsymbol{\nu}}_{\mathbf{d}})^*$; thus, $\star \circ \mathfrak{S} = \mathfrak{D}$, see [Hap88, pg. 37] and [MS08].

The following gives a natural construction of a bar-involution on $\mathbb{F}_\ell$ which we will show coincides with the construction in [BK09b]. Let $g \mapsto \overline{g}$ be the unique q-antilinear automorphism of U_e^- which fixes the standard generators $\mathbf{f_d}$.

Theorem 7.17. *For any ℓ and any fixed charge let $\Psi : \mathbb{F}_\ell \to \mathbb{F}_\ell$ be the map induced by $\mathfrak{D}$ on the Grothendieck group. These maps satisfy the compatibility properties*
(B1) $\Psi(g \cdot v) = \bar{g} \cdot \Psi(v)$ *for* $g \in U_e^-$.
(B2) $\Psi(v \otimes u_\emptyset) = \Psi(v) \otimes u_\emptyset$.
(B3) $\Psi(u_\xi) = u_\xi + \sum_{\eta<\xi} a_{\eta,\xi} u_\eta$ *for some* $a_{\eta,\xi} \in \mathbb{Z}[q, q^{-1}]$
and are uniquely characterized by the first two properties. Moreover, the vectors $h_{\grave{\mu}}$ are invariant under this involution.

Proof. By Proposition 7.15, the vectors $h_{\grave{\mu}}$ correspond to (the standard lifts of) projective modules which are obviously fixed by $\mathfrak{D}$, hence the last statement follows. To prove (B1), it suffices to check that it is true for a bar-invariant spanning set of U_e^-. Such a spanning set is given by the monomials $\mathbf{f}_{\hat{\boldsymbol{\mu}}}$. Since $\mathbf{f}_{\hat{\boldsymbol{\mu}}} h_{\grave{\lambda}} = h_{\hat{\boldsymbol{\mu}} \cup \grave{\lambda}}$, the result follows. For (B2), it is enough to note that $h_{\grave{\mu}} \otimes u_\emptyset = h_{\grave{\mu}'}$ where $\grave{\mu}' = (\hat{\boldsymbol{\mu}}(1), \dots, \hat{\boldsymbol{\mu}}(\ell), \emptyset)$. Since properties (B1) and (B2) determine the behavior on a spanning set, they uniquely characterize the map.

Finally, we prove (B3) by induction. Since $h_{\grave{\lambda}_\xi} = \sum_{\text{type}(\mathsf{S})=\xi} q^{\text{Deg}(\mathsf{S})} u_{\text{sh}(\mathsf{S})}$ we have that $h_{\grave{\lambda}_\xi} = u_\xi + \sum_{\eta<\xi} b_{\eta,\xi}(q) u_\eta$ for $b_{\eta,\xi} \in \mathbb{Z}[q, q^{-1}]$. Thus, for ξ minimal, $\Psi(u_\xi) = \Psi(h_{\grave{\lambda}_\xi}) = h_{\grave{\lambda}_\xi} = u_\xi$. Now, we assume the claim holds for $\eta < \xi$ and get

$$\Psi(u_\xi) = \Psi h_{\grave{\lambda}_\xi} - \sum_{\eta<\xi} b_{\eta,\xi}(q) u_\eta = h_{\grave{\lambda}_\xi} - \Psi \sum_{\eta<\xi} b_{\eta,\xi}(q) u_\eta = u_\xi + \sum_{\eta<\xi} a'_{\eta,\xi}(q) u_\eta$$

where again $a_{\eta,\xi} \in \mathbb{Z}[q, q^{-1}]$. □

Choosing integers $(\tilde{\mathfrak{z}}_1, \dots, \tilde{\mathfrak{z}}_\ell)$ such that $\tilde{\mathfrak{z}}_i \equiv \mathfrak{z}_i \pmod{e}$, one has a natural vector space isomorphism β taking standard vectors to standard vectors between our Fock space $\mathbb{F}_\ell$ and Uglov's Fock space $\tilde{\mathbb{F}}_\ell$ (in [Ugl00], this is denoted $\mathbf{F}_q[(\tilde{\mathfrak{z}}_1, \dots, \tilde{\mathfrak{z}}_\ell)]$. On the level of ℓ-multipartition this isomorphism is just given by reading the components in opposite order. Using the isomorphism β, one can also define a q-antilinear involution Ψ' on $\mathbb{F}_\ell$ by pulling back Uglov's bar involution Ψ_U. We wish to compare Ψ' with Ψ.

Definition 7.18. *A multi-charge is m-**dominant** if for each i, we have $\tilde{\mathfrak{z}}_{i+1} - \tilde{\mathfrak{z}}_i \geq m$.*

If our charge is m-dominant, then the map β is an isomorphism of U_e^--modules on the weight spaces of height $\leq m$, [BK09b, Lemma 3.20]; we should note that we have several differences of convention [BK09b], but in this case, they felicitously cancel.

Theorem 7.19. *If the multicharge $(\tilde{\mathfrak{z}}_1, \dots, \tilde{\mathfrak{z}}_\ell)$ is m-dominant, then on the weight spaces of height $\leq m$, we have that $\Psi = \Psi'$.*

Proof. Since we have given uniquely characterizing properties (B1)-(B2) of Ψ, we need only to show that Ψ' satisfies these. Since β is an isomorphism of U_e^--modules on the weight spaces of height $\leq m$, we have that

$$\Psi'(g \cdot v) = \beta^{-1}(\Psi_U(\beta(g.v))) = \beta^{-1}(\bar{g} \cdot (\Psi_U(\beta(v))) = \bar{g} \cdot \beta^{-1}(\Psi_U(\beta(v))) = \bar{g} \cdot \Psi'(v)$$

for weight vectors v of height $\leq m$ and $g \in U_e^-$ (using [Ugl00, 3.31]); hence (B1) holds.

To see (B2) we must consider the effect of tensoring with $u_\emptyset$ in Uglov's language: we add a new variable at every ℓth place in the bosonic realization of the Fock space, and include all of these variable up to the charge in our wedge, which is larger than any of the variables corresponding to any other partitions that appear in the wedge. Applying Uglov's formula [Ugl00, 3.23] for the bar involution in terms of wedges, we must reverse the order of all terms in the wedge, and use the rules [Ugl00, 3.16] to reorder them.

First, we pull all of the new variables to the right end of the wedge. We claim that no non-zero correction terms appear when we do this, so the expression remains a pure wedge. Let u_i be a new variable, and u_j any other. By [Ugl00, 3.16], $u_i \wedge u_j = \pm q^? u_j \wedge u_i$ modulo wedges that replace this with $u_{i+h\ell} \wedge u_{j-h\ell}$ or $u_{j-h\ell} \wedge u_{i+h\ell}$ for some h such that $i + h\ell$ is strictly between i and j. Thus, all terms which arise have the form

$$\cdots \wedge u_{i+h\ell} \wedge \cdots \wedge u_{i+(h-1)\ell} \wedge \cdots \wedge u_{i+\ell} \wedge \cdots \wedge u_{i+h\ell} \wedge \cdots$$

if $j > i$ or

$$\cdots \wedge u_{i+h\ell} \wedge \cdots \wedge u_{i-\ell} \wedge \cdots \wedge u_{i+(h+1)\ell} \wedge \cdots \wedge u_{i+h\ell} \wedge \cdots$$

if $i < j$. Both of these wedges are clearly 0 by the straightening rules if $h = 1$, and by induction, we can reduce to this case. That is,

$$\cdots \wedge u_i \wedge u_j \wedge \cdots = \cdots \wedge \pm q^? u_j \wedge u_i \wedge \cdots \tag{7.9}$$

as long as all new variables between i and j appear somewhere in the wedge. Now, reorder the old variables; again using [Ugl00, 3.16], we see that exactly the same correction terms appear here. Now, applying (7.9), we see that reordering the new variables and shuffling them back into place introduces no correction terms. Note that the we need not worry about the sign or power of q appearing in (7.9); by [Ugl00, 3.24], they must be correct. This shows that

$$\Psi'(v \otimes u_\emptyset) = \Psi'(v) \otimes u_\emptyset$$

and completes the proof. □

7.5. **Decomposition numbers and canonical bases.** For the remainder of this article, we assume that $\mathrm{char}(\Bbbk) = 0$. In the setup of Theorem 6.3, let P^ξ be the graded projective cover of F^ξ (which is the same as the projective cover of W^ξ). Obviously, since we have a surjective map $A^{\underline{\nu}} e_{\grave{\lambda}_\xi} \to F^\xi$, the projective P^ξ is a summand of this module with multiplicity 1. We let $p_\xi = [P^\xi]$, and $\phi_\xi = [F^\xi]$.

Theorem 7.20. *Assume that* $\mathrm{char}(\Bbbk) = 0$. *We have*

$$p_\xi \quad = \quad u_\xi + \sum_{\eta < \xi} a_{\xi\eta}(q) u_\eta \quad \textit{for polynomials } a_{\xi\eta}(q) \in q\mathbb{Z}_{\geq 0}[q]. \tag{7.10}$$

By BGG-reciprocity, we also have $u_\xi = \phi_\xi + \sum_{\xi > \eta} a_{\xi\eta}(q)\phi_\eta$, *and so the coefficients* $a_{\xi\eta}$ *are the graded decomposition numbers of* $A^{\underline{\nu}}$.

Proof. First note that BGG reciprocity applies in the nongraded version, since $A^{\underline{\nu}}$ is quasi-hereditary by [DJM98] and Theorem 6.3. The graded version follows then by general arguments as for instance in [MS05, §8]. Therefore it remains to show (7.10). Since the complementary summand of $e_{\grave{\lambda}_\xi} A^{\underline{\nu}}$ to P_ξ is filtered by Weyl modules W^η for $\eta < \xi$, we must have an expression as above for some Laurent polynomials $b_{\xi\eta}(q) \in \mathbb{Z}[q]$, see Proposition 7.15 (the coefficients of these polynomials are manifestly non-negative integral, since they are graded multiplicities of Weyl modules in the standard filtration of P^ξ).

The negativity of powers of q is equivalent to showing that $\mathrm{End}_{\tilde{A}^{\underline{\nu}}}(\bigoplus P^\xi)$ is a positively graded algebra. For this, it suffices to show the same for $\mathrm{End}_{\tilde{A}^{\underline{\nu}}}(\bigoplus \tilde{P}^\xi)$, the corresponding modules over $\tilde{A}^{\underline{\nu}}$, since this ring surjects onto $\mathrm{End}_{A^{\underline{\nu}}}(\bigoplus P^\xi) = \mathrm{End}_{\tilde{A}^{\underline{\nu}}}(\bigoplus P^\xi)$ by the universal property of projectives. This is the ring of self-extensions of the sum of shifts of simple perverse sheaves $\bigoplus_{\grave{\mu}} p_* \Bbbk_{\mathfrak{Q}(\grave{\mu})}$, so the indecomposable projectives are all of the form $\mathrm{Ext}_{\mathrm{GRep}_{\mathbf{d}}}\big(L_\xi, \bigoplus_{\grave{\mu}} p_* \Bbbk_{\mathfrak{Q}(\grave{\mu})}\big)$ where L_ξ is a shift of a simple perverse sheaf appearing in $\bigoplus_{\grave{\mu}} p_* \Bbbk_{\mathfrak{Q}(\grave{\mu})}$. The projective $\tilde{P}^\xi$ appears with multiplicity 1 in $A^{\underline{\nu}} e_\xi$, so L_ξ appears with multiplicity 1 in $p_* \Bbbk_{\mathfrak{Q}(\grave{\lambda}_\xi)}$. Since $p_* \Bbbk_{\mathfrak{Q}(\grave{\lambda}_\xi)}$ is Verdier self-dual, so is L_ξ and thus L_ξ is perverse. Thus, we have an isomorphism

$$\mathrm{Ext}^*_{\mathrm{GRep}_{\mathbf{d}}}(L_\xi, L_{\xi'}) \cong \mathrm{Hom}_{\tilde{A}^{\underline{\nu}}}(\tilde{P}^\xi, \tilde{P}^{\xi'})$$

and a surjective map

$$\mathrm{Ext}^*_{\mathrm{GRep}_{\mathbf{d}}}(L_\xi, L_{\xi'}) \to \mathrm{Hom}_{A^{\underline{\nu}}}(P^\xi, P^{\xi'}).$$

Since perverse sheaves are the heart of a t-structure, this Ext group is positively graded, so only positive shifts of Weyl modules can appear in the standard filtration of P^ξ. □

This property of upper-triangularity with respect to a standard basis is one of the hallmarks of a **canonical basis** (in the sense of Lusztig), the other being fixed under a bar-involution, such as the Ψ defined above; in fact, the basis $\{p_\xi\}$ does satisfy these properties and thus can be thought of a canonical basis, justifying our naming (A general definition of canonical bases including this one as a special case is given in [Web12a]):

Theorem 7.21. *Assume* $\mathrm{char}(\Bbbk) = 0$. *The basis* p_ξ *is the unique basis of* $\mathbb{F}_\ell$ *such that:*

- $\Psi(p_\xi) = p_\xi$ *and*
- $p_\xi = u_\xi + \sum_{\eta<\xi} a_{\xi\eta}(q) u_\eta$ *for* $e_{\xi\eta}(q) \in q\mathbb{Z}_{\geq 0}[q]$.

That is, p_ξ *is the "canonical basis" of the involution* Ψ *with "dual canonical basis"* ϕ_ξ.

Remark 7.22. As with the bar involution Ψ, we can view this basis as a "limit" of Uglov's canonical basis G^+_* defined in [Ugl00, 3.25], in the sense that the transition matrix from the standard basis to Uglov's basis stabilizes to the transition matrix for our basis on the weight spaces of height $\leq m$ once the multicharge is m-dominant.

Proof of Theorem 7.21. We have already shown that the second property holds. For the first, we note that $\mathfrak{D}$ fixes the projective $A^{\underline{\nu}}_{\grave{\lambda}_\xi}$, and thus fixes each of its "isotypic components" (the maximal summands of $e_{\grave{\lambda}_\xi} A^{\underline{\nu}}$ which are direct sums of the same indecomposable projective). Since P_ξ appears with multiplicity 1, it is also invariant under $\mathfrak{D}$, so $\Psi(p_\xi) = p_\xi$.

Uniqueness holds by the usual standard arguments: if $\{p'_\xi\}$ is another such basis, then we have that

$$p_\xi - p'_\xi = \sum (a_{\xi\eta}(q) - a'_{\xi\eta}(q))u_\eta$$

is Ψ-invariant and has only coefficients in $q\mathbb{Z}[q]$ with respect to the standard basis. This is impossible by Theorem 7.17(B3) (consider a maximal standard basis vector).

Finally Theorem 7.13 together with the definition (7.7) show that the classes of the simple objects are dual to the classes of indecomposable projectives. □

Corollary 7.23. *Assume* $\mathrm{char}(\Bbbk) = 0$. *The graded decomposition numbers* $a_{\xi\eta}$ *are the entries of the matrix expressing the canonical basis in terms of the standard basis, hence given by Proposition 7.15.*

When $e = \infty$, a number of interpretations of this canonical basis as classes of indecomposable projectives have already appeared in the literature, for instance:

- in category $\mathcal{O}$ of type A, e.g. [Sus08], [BS11], [FKS06], [MS09], [SS14].
- for generalized Khovanov algebras, [BS11],
- with projective modules over the algebra $T^{\underline{\nu}}$ constructed earlier in [Web12a, 6.8], which we have already shown is Morita equivalent to $A^{\underline{\nu}}$, and
- with projective modules over what Hu and Mathas call a "quiver Schur algebra" [HM11, §7.4] (which is different from our definition).

Unsurprisingly, all of these categories are graded Morita equivalent; in fact, a direct proof of the equivalence of each pair has appeared for certain parabolic category $\mathcal{O}$ and generalized Khovanov algebras in [BS11], for $T^{\underline{\nu}}$ and Hu and Mathas's algebra in [Web10, Th. 5.31], for $T^{\underline{\nu}}$ and category $\mathcal{O}$ in [Web10, §9.2], and for Hu and Mathas's algebra and category $\mathcal{O}$ by [HM11, Th. C], and in abstract terms in [BLW13].

In the special cases [FSS12, Th. 45, Prop. 76] explicit formulas are available which play an important role in the context of link homology theories developed therein. In general it is a nontrivial task to compute these bases explicitly.

It is a general phenomenon that canonical bases appear naturally from categorifications with positivity and integrality properties. For example, in all simply-laced finite dimensional Lie algebras, Lusztig's canonical bases for tensor products of irreducible representations were shown by the second author to arise from the projectives in the so-called tensor algebra. This and the general interplay between canonical bases and higher representation theory is discussed in greater detail in [Web12a].

Also note that our result is another striking example of how $\widehat{\mathfrak{gl}}_e$ behaves like a Kac-Moody algebra, even though it is not covered by the theory of categorification for quantum groups from [KL10], [Rou08]. One theme in recent years has been a recognition that theorems like Theorem 7.13, which prove the existence of a *weak* categorification, can be considerably strengthened to a *strong* categorification or strong action by considering 2-morphisms, [CR08], [Rou08], [KL10], [CL12]. At the moment, no consensus definition of such a strong action for $\widehat{\mathfrak{gl}}_e$ exists. Some examples and partial constructions have appeared in the literature: for example, some pieces occur in [HY13, §9], in the context of polynomial functors and [SV12], in the context of representations of rational Cherednik algebras; but even in these situations, the notion of a strong action remains unclear and a formal definition is so far not available. We expect that our construction should provide such a $\widehat{\mathfrak{gl}}_e$-action.

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