Hall Algebras via 2-periodic Complexes

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Introduction

It is well known that the complex simple Lie algebras can be classified via Dynkin diagrams. By the theorem of Serre ([Hum72, 17.3]) the Lie algebras can be defined via generators and relations. Moreover given a quiver Q the underlying diagram determines a Cartan matrix which yields a corresponding Kac-Moody Lie algebra $\mathfrak{g}$ (see [Kac94]). To study representation theory it is useful to introduce the universal enveloping algebra $U(\mathfrak{g})$ for a given Lie algebra $\mathfrak{g}$. Representations of $\mathfrak{g}$ are nothing else than $U(\mathfrak{g})$-modules.
The root space decomposition of $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ induces a triangular decomposition

$$U(\mathfrak{g}) = U(\mathfrak{n}^+) \otimes_{\mathbb{C}} U(\mathfrak{h}) \otimes_{\mathbb{C}} U(\mathfrak{n}^-)$$

for $U(\mathfrak{g})$. This algebra has a deformation $U_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ which can be defined again via generators and Serre type relations. Note that $U_v(\mathfrak{g})$ still has a triangular decomposition. The deformation parameter comes into the picture in a very subtle but at the same time natural way. This turns out by considering representations of the associated quiver defined over a finite field $\mathbb{F}_q$. The algebras $U_v(\mathfrak{g})$ are also called quantum groups and play an important role in representation theory as well as topology (see for example [Kas95]).
Now fix a finite field $k = \mathbb{F}_q$ with q elements and a quiver Q. Consider the category $\mathcal{R}ep_k(Q)$ of finite dimensional representations of Q. Then we build the (twisted) Hall algebra $\mathcal{H}_{tw}(\mathcal{R}ep_k(Q))$ of $\mathcal{R}ep_k(Q)$. This is the free $\mathbb{C}$-vector space generated by isomorphism classes of objects in $\mathcal{R}ep_k(Q)$. The multiplication encodes the structure of extensions between objects in $\mathcal{R}ep_k(Q)$ (for a detailed definition see Definition 3.1). Then by a famous theorem of Ringel [Rin90] one can specialize the corresponding quantum enveloping algebra $U_v(\mathfrak{g})$ at $t = +\sqrt{q}$ and obtain an embedding of $\mathbb{C}$-algebras

$$U_t(\mathfrak{n}^+) \hookrightarrow \mathcal{H}_{tw}(\mathcal{R}ep_k(Q)).$$

Moreover if Q is a simply-laced quiver with underlying Dynkin graph this is an isomorphism. Unfortunately this construction only gives the positive part. The construction of the whole universal enveloping algebra appears to be much harder and more involved (see the work of Peng and Xiao in [PX97] and [PX00]). Recently Bridgeland provides in [Bri13] a possibility to describe the whole quantum enveloping algebra $U_t(\mathfrak{g})$ in terms of Hall algebras. To be more specific he takes the Hall algebra of 2-periodic complexes of projective objects. Then he localizes this algebra at the acyclic complexes and introduces an additional relation (fot details see Definition 3.20). The result is the reduced localized Hall algebra $\mathcal{DH}_{red}(\mathcal{R}ep_k(Q))$ and he shows that there is an embedding

$$U_t(\mathfrak{g}) \hookrightarrow \mathcal{DH}_{red}(\mathcal{R}ep_k(Q))$$

which is an isomorphism in the simply-laced Dynkin case. Moreover the construction of the reduced localized Hall algebra is given for arbitrary k-linear abelian $\mathcal{A}$ categories with finite morphism spaces and finite global dimension. If $\mathcal{A}$ is hereditary we obtain a suitable description of a basis and of the multiplication in $\mathcal{DH}_{red}(\mathcal{A})$.
Now the main purpose of this thesis is to give a detailed overview of his work in [Bri13] where the embedding of $U_t(\mathfrak{g})$ is the main result. Moreover we discuss several examples of categories which satisfy the assumptions mentioned above. In particular we consider the category of representations of the quiver A_n

$$\underset{1}{\circ} \longrightarrow \circ \longrightarrow \dots \longrightarrow \underset{n}{\circ}$$

Another example is $k[X]$-$gmod^{fg}$, the category of finitely generated $\mathbb{Z}$-graded $k[X]$-modules. It turns out that the corresponding Hall algebras behave quite similar. This fact leads to the second important result of this thesis.

Theorem 0.1. *For every $n \in \mathbb{N}$ there is an embedding of $\mathbb{C}$-algebras:*

$$\mathcal{DH}_{red}(\mathcal{R}ep_k(A_n)) \hookrightarrow \mathcal{DH}_{red}(k[X]\text{-}gmod^{fg}).$$

Now we want to give a short overview on the contents of the particular sections.

Section 1 serves to introduce the reader to the world of quivers and their representations. We define the category $\mathcal{R}ep_k(Q)$ and describe the simple and projective objects. Furthermore we show that $\mathcal{R}ep_k(Q)$ is hereditary and of finite length.

Section 2 starts with the definition of Grothendieck groups and discuss several examples. Hereafter we formulate the following conditions on an abelian k-linear category $\mathcal{A}$:

(Ass1) $\mathcal{A}$ has finite morphism spaces.
(Ass2) $\mathcal{A}$ is of finite global dimension and has enough projectives.
(Ass3) $\mathcal{A}$ is hereditary.
(Ass4) Any non-zero object in $\mathcal{A}$ defines a non-zero class $K(\mathcal{A})$.

We check that the assumptions (Ass1)-(Ass4) hold for $\mathcal{R}ep_k(Q)$ and $k[X]$-$hmod^{fg}$. Then we introduce the non-hereditary category $\mathcal{G}$ (for a definition see Example 1.10). These tree categories are our main examples in this thesis. From this point on we restrict to categories $\mathcal{A}$ satisfying $(Ass1)-(Ass2)$. We consider the category of 2-periodic complexes of objects in $\mathcal{A}$ and show some useful properties of acyclic complexes and complexes of projectives. In particular for projective objects P we define complexes

$$\kappa_P = \; P \underset{0}{\overset{1}{\rightleftarrows}} P \quad \text{and} \quad \kappa_P^* = \; P \underset{1}{\overset{0}{\rightleftarrows}} P \, .$$

These complexes correspond to the generators K_i, K_i^{-1} of $U_t(\mathfrak{g})$ in the main Theorem (Theorem 4.37).

The focus of section 3 lies on the introduction of Hall algebras. We give the definition of the twisted and untwisted version and consider the examples $\mathcal{R}ep_k(A_n)$, $k[X]$-$gmod^{fg}$ and $\mathcal{G}$. We compute some Hall products in these categories and it turns out that one can embed the Hall algebra of $\mathcal{R}ep_k(A_n)$ into the Hall algebra of $k[X]$-$gmod^{fg}$. Hereafter we consider the (twisted) Hall algebra $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ of complexes of projecives and show some useful identities for Hall products in this algebra. In order to make the complexes κ_P invertible we localize $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ at the acyclic complexes and obtain the localized Hall algebra $\mathcal{DH}(\mathcal{A})$. Moreover to make the complexes κ_P and κ_P^* inverse to each other we define a reduced version $\mathcal{DH}_{red}(\mathcal{A})$ by imposing an additional relation.

The main part of section 4 is preparation for the proof of Theorem 4.37. Thus we restrict to categories $\mathcal{A}$ which satisfy all assumptions (Ass1)-(Ass4) like $\mathcal{R}ep_k(Q)$. We introduce the quantum enveloping algebra $U_t(\mathfrak{g})$ specialized at $t = +\sqrt{q}$ and state Ringel's theorem. This yields an injective linear map

$$\zeta\colon U_t(\mathfrak{g}) \cong U_t(\mathfrak{n}^+) \otimes U_t(\mathfrak{h}) \otimes U_t(\mathfrak{n}^-) \hookrightarrow \mathcal{H}_{tw}(\mathcal{A}) \otimes \mathbb{C}[K(\mathcal{A})] \otimes \mathcal{H}_{tw}(\mathcal{A})$$

for $\mathcal{A} = \mathcal{R}ep_k(Q)$. The main idea is now to define an isomorphism

$$\xi\colon \mathcal{H}_{tw}(\mathcal{A}) \otimes \mathbb{C}[K(\mathcal{A})] \otimes \mathcal{H}_{tw}(\mathcal{A}) \to \mathcal{DH}_{red}(\mathcal{A})$$

and to show that $\xi \circ \zeta$ is a morphism of $\mathbb{C}$-algebras. In order to do this we check that all defining relations of $U_t(\mathfrak{g})$ hold for the corresponding elements in $\mathcal{DH}_{red}(\mathcal{A})$. Hereafter we finally state the main theorem.

Section 5 serves to consider the reduced localized Hall algebras of $\mathcal{R}ep_k(A_n)$, $k[X]$-$gmod^{fg}$ and $\mathcal{G}$. In each case we give a description of the basis and compute several Hall products. Again it turns out that Hall products in $\mathcal{DH}_{red}(\mathcal{R}ep_k(A_n))$ behave quite similar to those in $\mathcal{DH}_{red}(k[X]$-$gmod^{fg})$. This leads to Theorem 0.1 and we use the main theorem to give a proof.

1 Representations of Quivers

In this section we introduce the basic concepts of quivers and their representation. The main reference is [Kra08]. Throughout this section we fix a field k.

1.1 Definitions and Notation

Definition 1.1. A *(finite) quiver* is a directed graph with finitely many vertices and finitely many arrows. More precisely, it is a quadrupel $Q = (V, E, s, t)$ consisting of a finite set V of vertices, a finite set E of edges or arrows and two functions $s, t\colon E \to V$ assigning each arrow its *source* and its *target*.

Definition 1.2. A *path* is a sequence of arrows $\alpha_l \dots \alpha_2\alpha_1$, where $s(\alpha_i) = t(\alpha_{i-1})$ holds for all $i \in \{2, \dots, l\}$. The number l is called the *length* of the path. In particular for each vertex i there exists a trivial path e_i of length zero.
A quiver has *no oriented cycles* if for each path $\alpha_l \dots \alpha_2\alpha_1$ and for all $i, j \in \{1, \dots, l\}$ holds: $i \neq j \Rightarrow t(\alpha_i) \neq t(\alpha_j)$.

Remark 1.3. By considering each vertex as an object and a path between two vertices as a morphism, each quiver forms a category.

Definition 1.4. Let Q be a quiver. A *representation* $(X, f) = (X_i, f_\alpha)_{i\in V,\alpha\in E}$ of Q consists of a vector space X_i for each vertex i and a k-linear map $f_\alpha\colon X_{s(\alpha)} \to X_{t(\alpha)}$ for each arrow α.
Let (X, f) and (Y, g) be two representations of Q. A morphism $\psi\colon (X, f) \to (Y, g)$ of representations is a tuple $\psi = (\psi_i)_{i\in V}$ of linear maps $\psi_i\colon X_i \to Y_i$ such that for each arrow α the following diagram commutes:

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_\alpha"] \arrow[d, "\psi_{s(\alpha)}"'] & X_{t(\alpha)} \arrow[d, "\psi_{t(\alpha)}"] \\ Y_{s(\alpha)} \arrow[r, "g_\alpha"] & Y_{t(\alpha)}. \end{tikzcd}$$

Definition 1.5. For a representation (X, f) of a quiver Q define

$$\dim(X, f) = \sum_{i\in V} \dim(X_i)$$

to be the *total dimension* of X.
We call (X, f) *finite dimensional* if $\dim(X, f) < \infty$.

From now on we only consider finite dimensional representations of finite quivers without oriented cycles. Throughout this thesis the following example will be our main example of a finite quiver without oriented cycles.

Example 1.6. For $n \in \mathbb{N}$ we define a quiver $A_n = (V, E, s, t)$ as follows:

$$V = \{1, \ldots, n\}, \quad E = \{\alpha_1, \ldots, \alpha_{n-1}\}, \quad s(\alpha_i) = i \quad \text{and} \quad t(\alpha_i) = i + 1.$$

We can represent A_n by a diagram:

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \longrightarrow \cdots \longrightarrow \underset{n}{\circ}$$

If (X, f) and (Y, g) are two representations of A_n, then $\psi_\bullet \colon X_\bullet \to Y_\bullet$ is a morphism of representations iff the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r, "f_{\alpha_1}"] \arrow[d, "\psi_1"] & X_2 \arrow[r, "f_{\alpha_2}"] \arrow[d, "\psi_2"] & X_3 \arrow[r, "f_{\alpha_3}"] \arrow[d, "\psi_3"] & \cdots \arrow[r, "f_{\alpha_{n-1}}"] & X_n \arrow[d, "\psi_n"] \\
Y_1 \arrow[r, "g_{\alpha_1}"'] & Y_2 \arrow[r, "g_{\alpha_2}"'] & Y_3 \arrow[r, "g_{\alpha_3}"'] & \cdots \arrow[r, "g_{\alpha_{n-1}}"'] & Y_n.
\end{tikzcd}$$

Remark 1.7. Considering a quiver Q as a category a representation can be viewed as a functor to the category of vector spaces with linear maps.

Notation 1.8. Let (X, f) and (Y, g) be two representations of a quiver Q.

- Instead of $(X, f) = (X_i, f_\alpha)_{i \in V}^{\alpha \in E}$ one often writes $X = (X_i, X_\alpha)_{i \in V}^{\alpha \in E}$.
- For some path $p = \alpha_l \ldots \alpha_2 \alpha_1$ define $f_p = f_{\alpha_l} \ldots f_{\alpha_2} f_{\alpha_1}$ and $f_{e_i} = \mathrm{id}_{X_i}$.
- We denote by $\mathcal{R}ep_k(Q)$ the category of finite dimensional representations of Q.
- We denote by $\mathrm{Hom}((X, f), (Y, g))$ the space of morphisms of representations from (X, f) to (Y, g).

Definition 1.9. Let Q be a quiver and $(X, f), (Y, g)$ two representations of Q. Then (X, f) is called a *subrepresentation* of (Y, g) if X_i is a subspace of Y_i for each vertex i and $f_\alpha = g_\alpha|_{X_{s(\alpha)}}$ holds for each arrow α. A non-zero representation is called *simple* if it has no proper subrepresentation.

Example 1.10. An example of a quiver with oriented cycles is the quiver G:

$$G = \begin{tikzcd} \underset{1}{\circ} \arrow[r, shift left, "\alpha"] & \underset{2}{\circ} \arrow[l, shift left, "\beta"] \end{tikzcd}$$

Now we want to introduce some kind of relation $\beta\alpha = 0$ to prevent the formation of an infinite number of paths. So the new set of all paths is given by

$$\{\ 0, e_1, e_2, \alpha, \beta, \alpha\beta\ \}.$$

A representation of G with respect to the relation $\beta\alpha = 0$ is then an object $(X, f) \in \mathcal{R}ep_k(G)$ with $f_\beta \circ f_\alpha = 0$. Define $\mathcal{G}$ to be the full subcategory of these objects in $\mathcal{R}ep_k(G)$.

1.2 Simple and Projective Representations

Lemma 1.11. *Let $Q = (V, E, s, t)$ be a quiver with no oriented cycles and (X, f) a simple representation of Q. Then there exists some $i \in V$ such that (X, f) is isomorphic to $S(i)$ with*

$$S(i)_j = \begin{cases} k, & \textit{if } j = i \\ 0, & \textit{if } j \neq i \end{cases}$$

and obviously $S(i)_\alpha = 0$ for all $j \in V$, $\alpha \in E$.

Proof. Let (X, f) be a simple representation of Q. Define $\tilde{Q} = (\tilde{V}, \tilde{E}, \tilde{s}, \tilde{t})$ to be the subquiver of Q with

$$\tilde{V} = \{i \in V \mid X_i \neq 0\}, \qquad \tilde{E} = \{\alpha \in E \mid s(\alpha), t(\alpha) \in \tilde{V}\}$$

and

$$\tilde{s} = s|_{\tilde{E}} : \tilde{E} \to \tilde{V}, \qquad \tilde{t} = t|_{\tilde{E}} : \tilde{E} \to \tilde{V}.$$

Suppose $\tilde{t}$ is surjective. Then one can easily construct a path p of length $|\tilde{V}| + 1$. Since at least one vertex has to occur twice in this path, p contains an oriented cycle. But this contradicts the assumption that Q has no oriented cycles. So we can assume, that there exists a vertex $i_0 \in \tilde{V}$ with $i_0 \notin \operatorname{im}(\tilde{t})$. But then $(\tilde{X}, \tilde{f})$ with

$$\tilde{X}_j = \begin{cases} 0, & \text{if } j = i_0 \\ X_j, & \text{if } j \neq i_0 \end{cases} \quad \text{and} \quad \tilde{f}_\alpha = f_\alpha|_{\tilde{X}_{s(\alpha)}}$$

for all $j \in V$, $\alpha \in E$ defines a subrepresentation of (X, f). Since (X, f) is simple, it follows that $(\tilde{X}, \tilde{f})$ is zero. Hence (X, f) is of the form

$$X_j = \begin{cases} X_{i_0}, & \text{if } j = i_0 \\ 0, & \text{if } j \neq i_0 \end{cases}$$

for all $j \in V$. And again since Q has no oriented cycles, we have that $f_\alpha = 0$ for all $\alpha \in E$. In this case any proper subspace of X_{i_0} defines a proper subrepresentation of (X, f). Thus X_{i_0} has to be one-dimensional. □

Definition 1.12. Let Q be a quiver. For every vertex $i \in V$ let $P(i)$ be the representation with:

$$\begin{aligned} P(i)_j &= \text{the free vector space over the set } \{p \mid p \text{ a path from i to j }\}, \\ P(i)_\alpha(p) &= \alpha p \end{aligned}$$

for all $j \in V$, $\alpha \in E$, and p a path from i to $s(\alpha)$.

Proposition 1.13. *Let (X, f) be a representation of Q and $i \in V$. Then there is a (natural) isomorphism of vector spaces* $\mathrm{Hom}(P(i), (X, f)) \cong X_i$. *In particular $P(i) \in \mathcal{R}ep_k(Q)$ is projective.*

Proof. Define

$$\begin{aligned} \Psi_i : X_i &\to \mathrm{Hom}(P(i), (X, f)) \qquad & \Psi_i^{-1} : \mathrm{Hom}(P(i), (X, f)) &\to X_i \\ x &\mapsto \varphi_x & \varphi &\mapsto \varphi_i(e_i) \end{aligned}$$

where $(\varphi_x)_j(p) = f_p(x)$ for p a path from i to j. Now by a direct calculation we see that for $x \in X_i$, $j \in V$ and p a path from i to j there holds:

$$\Psi_i^{-1}(\Psi_i(x)) = \Psi_i^{-1}(\varphi_x) = \varphi_x(e_i) = f_{e_i}(x) = \mathrm{id}_{X_i}(x) = x$$

and with $\psi \in \mathrm{Hom}(P(i), (X, f))$

$$\Psi_i(\Psi_i^{-1}(\psi))_j(p) = \Psi_i(\psi_i(e_i))_j(p) = (\varphi_{\psi_i(e_i)})_j(p) = f_p(\psi_i(e_i)) = \psi_j(p).$$

The last step follows because f_p is a composition of some f_α's, and ψ is a morphism of representations and $P(i)_\alpha$ is just composition of paths.
Now take $\omega\colon (X, f) \to (Y, g)$ a morphism of representations and ψ as above. Then naturality of Ψ_i^{-1} follows from

$$\begin{aligned} \Psi_{i,Y}^{-1}(\mathrm{Hom}(P(i), \omega)(\psi))) &= \Psi_{i,Y}^{-1}(\omega \circ \psi) = (\omega_i \psi_i)(e_i) \\ &= \omega_i \circ \Psi_{i,X}^{-1}(\psi) = \mathrm{Hom}(P(i), \omega)(\Psi_{i,X}^{-1}(\psi)). \end{aligned}$$

The Lemma follows. □

Example 1.14. Consider the quiver A_n defined in Example 1.6. We claim that every projective object in $\mathcal{R}ep_k(A_n)$ is a finite direct sum of some $P(i)$'s.

Proof. Let P be a projective object in $\mathcal{R}ep_k(A_n)$. Now choose a basis

$$B_1 = \{\ b_1^1, \ldots, b_{n_1}^1\ \} \subset P_1$$

of P_1. Then $P_{\alpha_1}(P_1)$ defines a subspace in P_2 and we can choose a set

$$B_2 = \{\ b_1^2, \ldots, b_{n_2}^2\ \} \subset P_2$$

which is a basis of a vector space complement of $P_{\alpha_1}(P_1)$ in P_2. Again $P_{\alpha_2}(P_2)$ defines a subspace in P_3 and we can continue inductively to define B_i for all $1 \le i \le n$. Define a map

$$\begin{aligned} f\colon \bigoplus_{i=1}^n \bigoplus_{b_j^i \in B_i} P(i) &\longrightarrow P \\ (e_i)_j &\longmapsto b_j^i \end{aligned}$$

and set $Q = \bigoplus_{i=1}^{n} \bigoplus_{b_j^i \in B_i} P(i)$. Since f is an epimorphism and P is projective there exists a split map $s\colon P \to Q$ such that $f \circ s = \mathrm{id}_P$. Since s is monomorphic it follows that every s_i is an injective linear map. But $\dim_k(Q_1) = \dim_k(P_1)$ and thus s_1 is an isomorphism. Now we do an induction on i. So assume that s_j is already an isomorphism for $j < i$. Then note that

$$Q_i = Q_{\alpha_{i-1}}(Q_{i-1}) \oplus \bigoplus_{b_j^i \in B_i} P(i)_i$$

and since s_{i-1} is an isomorphism we have that

$$s_i(P_{\alpha_{i-1}}(P_{i-1})) = Q_{\alpha_{i-1}}(s_{i-1}(P_{i-1})) = Q_{\alpha_{i-1}}(Q_{i-1}).$$

But by definition the set $B_i = \{ b_1^i, \ldots, b_{n_i}^i \}$ is a basis of a vector space complement of $P_{\alpha_{i-1}}(P_{i-1})$ and thus by taking the injectivity of s_i into account it follows that

$$\dim_k(s_i(P_i)) = \dim_k(s_i(P_{\alpha_{i-1}}(P_{i-1}))) + n_i$$

$$= \dim_k(Q_{\alpha_{i-1}}(Q_{i-1})) + \dim_k(\bigoplus_{j=1}^{n_i} P(i)_i) = \dim_k(Q_i).$$

Hence s_i is an isomorphism for $1 \leq i \leq n$ and $s\colon P \to Q$ defines an isomorphism of representations. □

1.3 Properties of $\mathcal{R}ep_k(Q)$

Theorem 1.15. *Let Q be a quiver with no oriented cycles. Then $\mathcal{R}ep_k(Q)$ has enough projectives. Moreover it is hereditary (i.e.* gldim ≤ 1*).*

Proof. Let (X, f) be any representation of Q. We have to show that (X, f) has a projective resolution of length 1. As a first step take for every vertex $i \in V$ the natural isomorphism Ψ_i from Proposition 1.13 and choose a basis $x_{i,1} \ldots x_{i,n_i}$ for each X_i. Then define

$$\coprod_{i\in V} \coprod_{j=1}^{n_i} \Psi_i(x_{i,j})\colon \bigoplus_{i\in V} \bigoplus_{j=1}^{n_i} P(i) \longrightarrow X.$$

The origin is projective as a finite sum of projectives. We first have to check that this map is epi. So let $y_i \in X_i$, then by definition $\Psi_i(y_i)$ is a morphism of representations with $e_i \mapsto y_i$. But since $\Psi_i(y_i) \in \mathrm{Hom}(P(i), (X, f))$ and Ψ_i takes a basis to a basis, we have that y_i is in the image of $(\coprod_{j=1}^{n_i} \Psi_i(x_{i,j}))_i$. Hence $(\coprod_{i\in V} \coprod_{j=1}^{n_i} \Psi_i(x_{i,j}))_i \to X_i$ is surjective and because this is true for

each $i \in V$ the whole map is epi.
Moreover every $(\coprod_{j=1}^{n_i} \Psi_i(x_{i,j}))_i$ is indeed an isomorphism from $(\bigoplus_{j=1}^{n_i} P(i))_i$ to X_i. Due to this fact we can construct the kernel of the map by summing up over all arrows and define the map

$$\Omega = \prod_{i \in V} \Omega^i \colon \bigoplus_{\alpha \colon l \to r} \bigoplus_{j=1}^{n_l} P(r) \to \bigoplus_{i \in V} \bigoplus_{j=1}^{n_i} P(i)$$

as follows: At first fix an $\alpha \colon l \to r$ and a vertex $v \in V$. We have that $x_{r,1} \dots x_{r,n_r}$ is a basis of X_r, so for each $1 \le j \le n_l$ there exist $\lambda^\alpha_{j,1} \dots \lambda^\alpha_{j,n_r}$ such that $f_\alpha(x_{l,j}) = \lambda^\alpha_{j,1} x_{r,1} + \dots + \lambda^\alpha_{j,n_r} x_{r,n_r}$. Now let p be a path from r to v. Then

$$z = (0, \dots, 0, \underset{j}{p}, 0, \dots, 0) \in \left(\bigoplus_{j=1}^{n_l} P(r) \right)_v$$

is mapped to

$$\Omega^q_w(z) = \begin{cases} -(\lambda^\alpha_{j,1} p, \dots, \lambda^\alpha_{j,n_r} p) & \text{if } q = r, w = v, \\ (0, \dots, 0, \underset{j}{p\alpha}, 0, \dots, 0) & \text{if } q = l, w = v, \\ 0 & \text{else.} \end{cases}$$

By construction the composition of Ω with $\displaystyle\coprod_{i \in V} \coprod_{j=1}^{n_i} \Psi_i(x_{i,j})$ is zero:

$$\begin{aligned} \psi_l(x_{l,j})(p\alpha) &= f_p(f_\alpha(x_{l,j})) = f_p(\lambda^\alpha_{j,1} x_{r,1} + \dots + \lambda^\alpha_{j,n_r} x_{r,n_r}) \\ &= \left(\coprod_{j=1}^{n_r} \Psi_r(x_{r,j}) \right) (\lambda^\alpha_{j,1} p, \dots, \lambda^\alpha_{j,n_r} p). \end{aligned}$$

To verify that Ω is mono, we will check that the induced maps on each vertex are injective. Fix a vertex v as above. Let $\sum_\alpha (y^\alpha_1, \dots, y^\alpha_{n_{s(\alpha)}})$ be any element in $(\bigoplus_{\alpha \colon l \to r} \bigoplus_{j=1}^{n_l} P(r))_v$ and assume it is mapped to zero. Since Q has no oriented cycles there is a vertex l which is not the target of any arrow of Q. Then

$$\Omega^l_v(\sum_\alpha (y^\alpha_1, \dots, y^\alpha_{n_{s(\alpha)}}) = \sum_{\alpha \colon l \to t(\alpha)} (y^\alpha_1 \alpha, \dots, y^\alpha_{n_l} \alpha).$$

Since two paths $\alpha_{m_1} \dots \alpha_1$ and $\beta_{m_2} \dots \beta_1$ can only coincide if $\alpha_1 = \beta_1$ and the paths from l to v build a basis of $P(l)_v$, we have that all y^α_j must be zero for those α whith $s(\alpha) = l$.

Again because Q has no oriented cycles, there exists a vertex l' which is not the target of any arrow except those with source l. Since we already know that the summands corresponding to those arrows are zero, we can conclude as above that

$$\Omega_v^{l'}(\sum_\alpha (y_1^\alpha, \dots, y_{n_{s(\alpha)}}^\alpha) = \sum_{\alpha\colon l' \to t(\alpha)} (y_1^\alpha \alpha, \dots, y_{n_{l'}}^\alpha \alpha).$$

Thus all y_j^α have to vanish for all α with $s(\alpha) = l'$. Moreover Q has a finite vertex set, so it follows by induction that $\sum_\alpha (y_1^\alpha, \dots, y_{n_{s(\alpha)}}^\alpha) = 0$ and thus Ω is a monomorphism.
To see that Ω is indeed the kernel we check the dimensions of the vector spaces:

$$\dim_k(X_i) = n_i, \qquad \dim_k \left(\left(\bigoplus_{l \in V} \bigoplus_{j=1}^{n_l} P(l) \right)_i \right) = \sum_{p\colon j \to i} n_j.$$

On the other hand:

$$\dim_k \left(\left(\bigoplus_{\alpha\colon l \to r} \bigoplus_{j=1}^{n_l} P(r) \right)_i \right) = \sum_{\alpha\colon j \to l} \sum_{p\colon l \to i} n_j = \sum_{\substack{p\colon j \to i, \\ p \neq e_i}} n_j = -n_i + \sum_{p\colon j \to i} n_j.$$

$\square$

Example 1.16. Consider the quiver

$$A_3 \quad = \quad \circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ.$$

We want to find a projective resolution of length 1 of

$$S(1) \quad = \quad k \longrightarrow 0 \longrightarrow 0.$$

Following the proof of Theorem 1.15 we have an epimorphism $P(1) \to S(1)$ and its kernel is given by $P(2)$ as follows:

$$\begin{tikzcd}
P(2) & = & 0 \arrow[r] \arrow[d] & k\langle e_2 \rangle \arrow[r, "\beta\circ"] \arrow[d, "\circ\alpha"] & k\langle \beta \rangle \arrow[d, "\circ\alpha"] \\
P(1) & = & k\langle e_1 \rangle \arrow[r, "\alpha\circ"] \arrow[d, "e_1 \mapsto 1"] & k\langle \alpha \rangle \arrow[r, "\beta\circ"] \arrow[d] & k\langle \beta\alpha \rangle \arrow[d] \\
S(1) & = & k \arrow[r] & 0 \arrow[r] & 0\,.
\end{tikzcd}$$

So we obtain a projective resolution

$$0 \to P(2) \to P(1) \to S(1) \to 0.$$

Lemma 1.17. *The category $\mathcal{R}ep_k(Q)$ has finite length i.e. every object has a finite composition series with simple subquotients.*

Proof. Let X be a finite dimensional representation of Q. We want to do an induction over $\dim(X)$:
If $\dim(X) = 1$ then X has to be isomorphic to a $S(i)$ for some $i \in V$ and hence X is already simple.
Let $\dim(X) = n$ and suppose that every representation Y with $\dim(Y) < n$ already has a finite composition series. Again we make use of the fact that Q has no oriented cycles. Like in Lemma 1.11 we can define $\tilde{Q}$ to be the full subquiver with exactly the vertices i of Q for which X_i is not zero. Again as in Lemma 1.11 we find a vertex $i_0 \in \tilde{V}$ with $i_0 \notin \mathrm{im}(\tilde{t})$. In particular we have $\dim(X_{i_0}) \neq 0$, so let U be a subspace of X_{i_0} of dimension $\dim(X_{i_0})$-1. But then $\tilde{X}$ defines a subrepresentation of X with

$$\tilde{X}_j = \begin{cases} U, & \text{if } j = i_0 \\ X_j, & \text{if } j \neq i_0 \end{cases} \quad \text{and} \quad \tilde{X}_\alpha = X_\alpha|_{\tilde{X}_{s(\alpha)}}$$

for all $j \in V$, $\alpha \in E$. Now by construction we have $\tilde{X} \subset X$ with $\dim\left({}^{X}\!/_{\tilde{X}}\right) = 1$. By induction $\tilde{X}$ already has a finite decomposition series so we are done. $\square$

Remark 1.18. The composition in Lemma 1.17 is unique up to isomorphism. This is true because for each $i \in V$ the number

$$[X : S(i)] = \dim_k(X_i)$$

is independent of the chosen composition series.

2 The Category of 2-Periodic Complexes

In this section we first give a short introduction to Grothendieck groups and discuss several examples. From this point on, we will restrict to certain abelian categories that we will show $\mathcal{R}ep_k(Q)$ to belong to. Hereafter we will define the category of 2-periodic complexes $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ and show some basic properties of acyclic complexes and complexes of projectives. The main reference is [Bri13].

2.1 Grothendieck Groups

Definition 2.1. Given an abelian category $\mathcal{A}$ we denote by $K(\mathcal{A})$ its *Grothendieck group*. That is the free abelian group generated by isomorphism classes $\hat{M}$ of objects M in $\mathcal{A}$ modulo short exact sequences. In formulas:

$$\hat{B} = \hat{A} + \hat{C}$$

if there is a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Example 2.2. Consider the abelian category Vect_k of finite dimensional vector spaces over k. Take a vector space $V \in \mathrm{Vect}_k$ with $2 \leq n = \dim_k(V)$. Then V is isomorphic to k^n and thus $\hat{V} = \hat{k}^n$ in $K(\mathrm{Vect}_k)$. There is a short exact sequence in Vect_k:

$$0 \longrightarrow k^{n-1} \longrightarrow k^n \longrightarrow k^n/k^{n-1} \longrightarrow 0.$$

Hence it follows that

$$\hat{k}^n = \hat{k}^{n-1} + (k^n \hat{/} k^{n-1}) \in K(\mathrm{Vect}_k)$$

with $\dim_k(k^{n-1}), \dim_k(k^n/k^{n-1}) < n$. So by induction over n we see that $\hat{V} = z \cdot \hat{k}$ for $z \in \mathbb{N}$ and thus we obtain

$$K(\mathrm{Vect}_k) = \mathbb{Z}\langle \hat{k} \rangle.$$

Example 2.3. We want to determine $K(\mathcal{R}ep_k(Q))$. Take a representation X of Q, then by Lemma 1.17 we know that X has a finite decomposition series

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_{r-1} \subset X_r = X$$

which yields short exact sequences

$$0 \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+1}/X_i \longrightarrow 0$$

for $0 \leq i < r - 1$. Hence we have that

$$\hat{X} = \sum_{i=0}^{r-1} (X_{i+1}/X_i)^{\hat{}} \in K(\mathcal{R}ep_k(Q))$$

where X_{i+1}/X_i are simple objects in $\mathcal{R}ep_k(Q)$. Now by Lemma 1.11 we obtain that

$$K(\mathcal{R}ep_k(Q)) = \mathbb{Z}\langle\{ \hat{S(i)} \mid i \in V \}\rangle.$$

In particular Example 2.2 follows if we set $Q = A_1$.

Lemma 2.4. *Let $\mathcal{A}$ be an abelian category with enough projectives and* $\mathrm{gldim}(\mathcal{A}) \leq 1$*. Then for every $\alpha \in K(\mathcal{A})$ exist projective objects $P, Q \in \mathcal{A}$ such that*

$$\alpha = \hat{P} - \hat{Q} \in K(\mathcal{A}).$$

Proof. Since $\mathcal{A}$ is hereditary every object $A \in \mathcal{A}$ has a projective resolution

$$0 \longrightarrow Q_A \longrightarrow P_A \longrightarrow A \longrightarrow 0$$

with projective objects Q_A and P_A. Moreover note that $(A \hat{\oplus} B) = \hat{A} + \hat{B}$ since

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

defines a short exact sequence for $B \in \mathcal{A}$.
Now let

$$\alpha = \sum_{i \in I} n_i \cdot \hat{A}_i - \sum_{j \in J} m_j \cdot \hat{B}_j, \qquad n_i, m_j \in \mathbb{N}, A_i, B_j \in \mathcal{A},$$

be any element in $K(\mathcal{A})$ where I and J are finite index sets. Then we can calculate:

$$\begin{aligned}
\alpha &= \sum_{i \in I} n_i \cdot \hat{A}_i - \sum_{j \in J} m_j \cdot \hat{B}_j \\
&= \sum_{i \in I} n_i \cdot (\hat{P}_{A_i} - \hat{Q}_{A_i}) - \sum_{j \in J} m_j \cdot (\hat{P}_{B_j} - \hat{Q}_{B_j}) \\
&= \left(\sum_{i \in I} n_i \cdot \hat{P}_{A_i} + \sum_{j \in J} m_j \cdot \hat{Q}_{B_j} \right) - \left(\sum_{i \in I} n_i \cdot \hat{Q}_{A_i} + \sum_{j \in J} m_j \cdot \hat{P}_{B_j} \right) \\
&= \left(\bigoplus_{i \in I} \bigoplus_{l=1}^{n_i} P_{A_i} \hat{\oplus} \bigoplus_{j \in J} \bigoplus_{l=1}^{m_j} Q_{B_j} \right) - \left(\bigoplus_{i \in I} \bigoplus_{l=1}^{n_i} Q_{A_i} \hat{\oplus} \bigoplus_{j \in J} \bigoplus_{l=1}^{m_j} P_{B_j} \right).
\end{aligned}$$

The lemma follows because direct sums of projective objects are again projective. $\square$

The following example shows that non-zero objects can define zero classes in the corresponding Grothendieck group. This motivates (Ass4) in the subsequent Setup 2.6.

Example 2.5. Denote by $\mathcal{N}$ the category of finitely generated modules over $k[X]$. Note that $\mathcal{N}$ is not of finite length (since for instance $k[X]$ has not finite length). Then

$$0 \longrightarrow k[X] \xrightarrow{X\cdot} k[X] \longrightarrow k = k[X]/(X) \longrightarrow 0$$

is a short exact sequence in $\mathcal{N}$ and hence

$$\hat{k[X]} = \hat{k[X]} + \hat{k} \in K(\mathcal{N}).$$

Thus $\hat{k} = 0 \in K(\mathcal{N})$ although $k \ncong 0 \in \mathcal{N}$.

2.2 General Setup: (Ass1)-(Ass4)

Setup 2.6. Fix now $k = \mathbb{F}_q$ a finite field with q elements, and set $t = +\sqrt{q}$. From now on let $\mathcal{A}$ be any abelian k-linear category which satisfies the following conditions:

(Ass1) $\mathcal{A}$ has finite morphism spaces.

(Ass2) $\mathcal{A}$ is of finite global dimension and has enough projectives.

From section 4 on we shall also assume:

(Ass3) $\mathcal{A}$ is hereditary.

(Ass4) Any non-zero object in $\mathcal{A}$ defines a non-zero class in the Grothendieck group $K(\mathcal{A})$.

Now we want to prove that $\mathcal{R}ep_k(Q)$ satisfies the assumptions of Setup 2.6. Since (Ass1)-(Ass3) follow from the results in section 1 and we already know that $\mathcal{R}ep_k(Q)$ is of finite length, the following lemma completes the proof.

Lemma 2.7. *Let $\mathcal{B}$ be a k-linear abelian category. If $\mathcal{B}$ is of finite length (i.e. every object has a finite composition series) then it satisfies assumption (Ass4).*

Proof. Let M be any object in $\mathcal{B}$. Since $\mathcal{B}$ is of finite length there exists $r \in \mathbb{N}$ and objects $M_1, \dots, M_r$ such that

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{r-1} \subset M_r = M$$

is a finite composition series of M with simple quotients. Like in Example 2.3 we obtain for $0 \leq i < r$ short exact sequences

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow {}^{M_{i+1}}\!/_{M_i} \longrightarrow 0$$

with $\hat{M}_{i+1} = \hat{M}_i + \widehat{\left({}^{M_{i+1}}\!/_{M_i}\right)}$ in the Grothendieck group. Hence in $K(\mathcal{B})$ the isomorphism class of M can be written as a sum of the isomorphism classes of its simple subquotients:

$$\hat{M} = \sum_{i=0}^{r-1} \widehat{\left({}^{M_{i+1}}\!/_{M_i}\right)}.$$

In particular the simple objects in $\mathcal{B}$ generate its Grothendieck group. Moreover we want to show that they form indeed a basis of $K(\mathcal{B})$ as a $\mathbb{Z}$-module. Define

$$\mathrm{Irr} = \{\text{isomorphism classes of simple objects}\},$$

and two maps

$$\begin{aligned} \Phi\colon \bigoplus_{[S]\in \mathrm{Irr}} \mathbb{Z} &\to K(\mathcal{B}) & \qquad \Phi^{-1}\colon K(\mathcal{B}) &\to \bigoplus_{[S]\in \mathrm{Irr}} \mathbb{Z} \\ 1_{[S]} &\mapsto \hat{S} & \hat{M} &\mapsto \sum_{[S]\in \mathrm{Irr}} m_{[S]} 1_{[S]}, \end{aligned}$$

where $m_{[S]} = [M : S]$ is the number of subquotients occuring in a composition series of M which are contained in $[S]$. This is unique by Remark 1.18. It is left to show that Φ^{-1} is well-defined. So consider a short exact sequence in $\mathcal{B}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Now choose a composition series of A and extend it by a composition series of ${}^{B}\!/_{A}$ to a composition series of B. Since $C \cong {}^{B}\!/_{A}$ it follows from the Jordan-Hölder Theorem that $b_S = a_S + c_S$ for all simple S in $\mathcal{B}$ and this shows that Φ^{-1} is well-defined and it is obviously the inverse of Φ. $\square$

Corollary 2.8. *$\mathcal{R}ep_k(Q)$ is a k-linear abelian category which satisfies the assumptions $(Ass1) - (Ass4)$.*

Proof. The condition $(Ass1)$ is satisfied because $k = \mathbb{F}_q$ is a finite field, Q a finite quiver and all representations are finite dimensional. Now $(Ass2)$ and $(Ass3)$ follow from Theorem 1.15 and $(Ass4)$ follows from Lemma 1.17 and Lemma 2.7. $\square$

2.3 The Category $k[X]$-gmod^{fg}

Although our main example satisfying (Ass1)-(Ass4) is $\mathcal{R}ep_k(Q)$ the following example provides another interesting category satisfying the assumptions of Setup 2.6.

Example 2.9. Consider $\mathcal{M} = k[X]\text{-gmod}^{fg}$, the category of finitely generated $\mathbb{Z}$-graded $k[X]$-modules with degree preserving morphisms. We will omit the proof of the fact that $\mathcal{M}$ is abelian (to see this note for example that $k[X]$ is noetherian and thus the kernel of every morphism is again finitely generated). We claim that the assumptions (Ass1)-(Ass4) of Setup 2.6 hold for $\mathcal{M}$. We want to introduce the *shift functor* $\langle z \rangle \colon \mathcal{M} \to \mathcal{M}$ for $z \in \mathbb{Z}$: Take two modules $M, N \in \mathcal{M}$ and $f \colon M \to N$. Then

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad N = \bigoplus_{i \in \mathbb{Z}} N_i \quad \text{and} \quad f = \bigoplus_{i \in \mathbb{Z}} f_i \colon \bigoplus_{i \in \mathbb{Z}} M_i \to \bigoplus_{i \in \mathbb{Z}} N_i$$

where all M_i and N_i are finite dimensional vector spaces over k, since M and N are finitely generated. Define

$$(M\langle z \rangle)_i = M_{i-z} \quad \text{and} \quad (f\langle z \rangle)_i = f_{i-z}.$$

Proposition 2.10. *The category $\mathcal{M}$ satisfies all assumptions of Setup 2.6.*

Proof. For (Ass1) take two modules $M, N \in \mathcal{M}$ as above. Now choose homogeneous generators $m_1, \dots, m_n$ of M with $m_j \in M_{i_j}$ for $1 \le j \le n$. Then $f = \bigoplus_{i \in \mathbb{Z}} f_i \colon M \to N$ is uniquely determined by the images of the m_j under f_{i_j}. But every $f_{i_j} \colon M_{i_j} \to N_{i_j}$ is a morphism of finite dimensional vector spaces. Thus there are only finitely many choices for f since k is a finite field.

To show that (Ass2) and (Ass3) hold we want to find a short exact projective resolution for every module in $\mathcal{M}$. Note that $k[X]\langle z \rangle$ is a projective object in $\mathcal{M}$ for every $z \in \mathbb{Z}$. Let M be as above. Then there is an epimorphism

$$\begin{aligned} p \colon \bigoplus_{j=1}^{n} k[X]\langle i_j \rangle &\longrightarrow M \\ 1_j &\longmapsto m_j \end{aligned}$$

where $Q = \bigoplus_{j=1}^{n} k[X]\langle i_j \rangle$ is projective as a finite direct sum of projectives. This shows that $\mathcal{M}$ has enough projectives. Now we have to show that $\ker(p) = \bigoplus_{i \in \mathbb{Z}} \ker(p_i)$ is projective. Let $m = \min\limits_{1 \le j \le n} (i_j)$. Then $Q_i = 0$ for $i < m$. Choose a basis

$$A_m = B_m = \{ \ b_1^m, \dots, b_{n_m}^m \ \} \subseteq Q_m$$

of $\ker(p_m)$. Since the action of X on Q is injective it follows that $X.A_m$ is a linearly independent subset of $\ker(p_{m+1})$. So we can choose a set

$$B_{m+1} = \{ \ b_1^{m+1}, \dots, b_{n_{m+1}}^{m+1} \ \} \quad \text{such that} \quad A_{m+1} = X.A_m \sqcup B_{m+1}$$

is a basis of $\ker(p_{m+1})$. Again $X.A_{m+1}$ is a linearly independent subset of $\ker(p_{m+2})$ and we can continue inductively to define B_i for all $m \leq i$. For $i < m$ set $B_i = \varnothing$. Since $\ker(p)$ is a finitely generated module we obtain that all B_i vanish for large enough i. Define a map

$$\begin{aligned} f \colon \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=1}^{n_i} k[X]\langle i \rangle &\longrightarrow Q \\ 1_{i,j} &\longmapsto b_j^i \end{aligned}$$

where $P = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=1}^{n_i} k[X]\langle i \rangle$ is projective as a finite sum of projective objects. Moreover by construction every $f_i \colon P_i \to \ker(p_i)$ is an isomorphism of vector spaces. Thus we have a projective resolution of M:

$$0 \longrightarrow P \xrightarrow{f} Q \xrightarrow{p} M \longrightarrow 0 \,.$$

This proves that $\mathcal{M}$ is hereditary.

Similar to Example 2.5 the category $\mathcal{M}$ is not of finite length. Thus we cannot use Lemma 2.7 to show that (Ass4) holds. Let M and N be modules as above and R another module in $\mathcal{M}$. First note that $M \cong N \in \mathcal{M}$ implies that $M_i \cong N_i \in \mathrm{Vect}_\mathrm{k}$ for all $i \in \mathbb{Z}$. Moreover every short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow R \longrightarrow 0$$

in $\mathcal{M}$ induces short exact sequences

$$0 \longrightarrow M_i \longrightarrow N_i \longrightarrow R_i \longrightarrow 0$$

in Vect_k for all $i \in \mathbb{Z}$. Hence $\hat{M} = 0 \in K(\mathcal{M})$ implies that $\hat{M_i} = 0 \in K(\mathrm{Vect}_k)$ for all $i \in \mathbb{Z}$. But since Vect_k is of finite length this implies that $M_i = 0$ for all $i \in \mathbb{Z}$ and thus $M = 0 \in \mathcal{M}$.
In view of Example 2.5 this shows that the condition that the modules are graded is quite strong. $\square$

Proposition 2.11. *Every projective object in $k[X]\text{-}\mathrm{gmod}^{fg}$ is a finite direct sum of some $k[X]\langle i \rangle$'s.*

Proof. This proof is very similar to the proof of Example 1.14. Let P be a projective object in $[X]\text{-}\operatorname{gmod}^{fg}$. Since P is finitely generated there is a minimal $m \in \mathbb{Z}$ such that $P_m \neq 0$. Choose a basis

$$B_m = \{\ b_1^m, \dots, b_{n_m}^m\ \} \subset P_m$$

of P_m. Then $X.(P_m)$ defines a subspace in P_{m+1} and we can choose a set

$$B_{m+1} = \{\ b_1^{m+1}, \dots, b_{n_{m+1}}^{m+1}\ \} \subset P_{m+1}$$

which is a basis of a vector space complement of $X.(P_m)$ in P_{m+1}. Again $X.(P_{m+1})$ defines a subspace in P_{m+2} and we can continue inductively to define B_i for all $m \leq i$. Since P is finitely generated all B_i vanish for large enough i. For $i < m$ set $B_i = \varnothing$ Define a map

$$f\colon \bigoplus_{i\in\mathbb{Z}} \bigoplus_{b_j^i \in B_i} k[X]\langle i\rangle \longrightarrow P$$

$$1_{i,j} \longmapsto b_j^i$$

and set $Q = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{b_j^i\in B_i} k[X]\langle i\rangle$. Since f is an epimorphism and P is projective there exists a split map $s\colon P \to Q$ such that $f \circ s = \mathrm{id}_P$. Since s is monomorphic it follows that every s_j is an injective linear map. Moreover $P_j = Q_j = 0$ for $j < m$ and $\dim_k(Q_m) = \dim_k(P_m)$. Thus s_j is an isomorphism for $j \leq m$. Now we do an induction on i. So assume that s_j is already an isomorphism for $j < i$. Then note that

$$Q_i = X.Q_{i-1} \oplus \bigoplus_{b_j^i \in B_i} k[X]\langle i\rangle_i$$

and since s_{i-1} is an isomorphism we have that

$$s_i(X.P_{i-1}) = X.(s_{i-1}(P_{i-1})) = X.(Q_{i-1}).$$

But by definition the set $B_i = \{\ b_1^i, \dots, b_{n_i}^i\ \}$ is a basis of a vector space complement of $X.P_{i-1}$ and thus it follows by taking the injectivity of s_i into account that

$$\dim_k(s_i(P_i)) = \dim_k(s_i(X.P_{i-1})) + n_i$$

$$= \dim_k(X.Q_{i-1}) + \dim_k(\bigoplus_{j=1}^{n_i} k[X]\langle i\rangle_i) = \dim_k(Q_i).$$

Hence s_i is an isomorphism for all $i \in \mathbb{Z}$ and $s\colon P \to Q$ defines an isomorphism of $k[X]\text{-}\operatorname{gmod}^{fg}$-modules . $\square$

2.4 The Category $\mathcal{G}$

Consider the category $\mathcal{G}$ defined as in Example 1.10. This category is an example for an abelian k-linear category satisfying (Ass1)-(Ass2) but which is not hereditary ($S(1)$ possesses no projective resolution of lenght 1).

Proposition 2.12. *The category $\mathcal{G}$ has enough projectives and* $\operatorname{gldim}(\mathcal{G}) = 2$.

Proof. Before we start with the proof, observe that $P(1)$ and $P(2)$ are projective objects in $\mathcal{G}$. $P(1)$ and $P(2)$ are given by

$$P(1) = \; k\langle e_1\rangle \underset{0=\beta\circ}{\overset{\alpha\circ}{\rightleftarrows}} k\langle\alpha\rangle \quad \text{and} \quad P(2) = \; k\langle\beta\rangle \underset{\beta\circ}{\overset{\alpha\circ}{\rightleftarrows}} k\langle e_2, \alpha\beta\rangle.$$

Although we use the same notation observe that objects in $\mathcal{G}$ are no 2-periodic complexes in general. Now consider an object

$$M = \; M_1 \underset{f_\beta}{\overset{f_\alpha}{\rightleftarrows}} M_2$$

and let $\{\; m^i_1, \ldots, m^i_{n_i} \;\}$ be a basis of M_i for $i = 1, 2$. Then the following map defines an epimorphism

$$\begin{aligned} p^1 \colon P^1 = \bigoplus_{i=1}^{2} \bigoplus_{j=1}^{n_i} P(i) &\longrightarrow M \\ (e_i)_j &\longmapsto m^i_j \end{aligned}$$

where P^1 is projective as a direct finite sum of projectives. This shows that $\mathcal{G}$ has enough projectives. Note that $P^1_\alpha \colon P^1_1 \to P^1_2$ is injective. Set $Q^1 = \ker p^1$. Then Q^1 is a subrepresentation of P^1 and thus $Q^1_\alpha \colon Q^1_1 \to Q^1_2$ is injective. Now choose a vector space complement K_2 of $\ker(Q^1_\beta)$ in Q^1_2 and let $\{\; k^2_1, \ldots, k^2_{l_2} \;\}$ be a basis of K_2. Then the maps

$$Q^1_\beta|_{K_2} \colon K_2 \to Q^1_\beta(K_2) \quad \text{and} \quad Q^1_\alpha|_{Q^1_\beta(K_2)} \colon Q^1_\beta(K_2) \to Q^1_{\alpha\beta}(K_2)$$

are both injective and since $Q^1_{\alpha\beta}(K_2) \subseteq \ker Q^1_\beta$ we have that $Q^1_{\alpha\beta}(K_2) \cap K_2 = \varnothing$ and in view of the definition of $P(2)$ we obtain an embedding:

$$\begin{aligned} \bigoplus_{j=1}^{l_2} P(2) &\longrightarrow Q^1 \\ (e_2)_j &\longmapsto k^2_j. \end{aligned}$$

Now choose a vector space complement K_1 of $Q^1_\beta(K_2)$ in Q^1_1 and let $\{ k^1_1, \dots, k^1_{l_1} \}$ be a basis of K_1. Since Q^1_α is injective and $Q_\alpha(K_1) \subseteq \ker(Q^1_\beta)$ it follows that $Q_\alpha(K_1) \cap Q^1_{\alpha\beta}(K_2) = \varnothing$ and $Q_\alpha(K_1) \cap K_2 = \varnothing$. Thus in view of the definition of $P(1)$ we have an embedding

$$\begin{aligned} \bigoplus_{i=1}^{2} \bigoplus_{j=1}^{l_i} P(i) &\longrightarrow Q^1 \\ (e_i)_j &\longmapsto k^i_j. \end{aligned}$$

Now choose a vector space complement K_3 of $Q^1_\alpha(Q^1)$ in $\ker(Q^1_\beta)$ and let $\{ k^2_{l_2+1}, \dots, k^2_{l_3} \}$ be a basis of K_3. Then the following map defines an epimorphism:

$$\begin{aligned} p_2 \colon P^2 = \bigoplus_{j=1}^{l_1} P(1) \oplus \bigoplus_{j=1}^{l_3} P(2) &\longrightarrow Q^1 \\ (e_i)_j &\longmapsto k^i_j. \end{aligned}$$

Set $Q^2 = \ker(p_2)$. Then by construction

$$\dim_k Q^2_1 = \dim_k K_3 = l_3 - l_2.$$

Since $P^2_\alpha \colon P^2_1 \to P^2_2$ is injectve and Q^2 is a subrepresentation of P^2 we know that $Q^2_\alpha \colon Q^2_1 \to Q^2_2$ is injective. Let $\{ q_1, \dots, 1_{l_3 - l_2} \}$ be a basis of Q_1 then in view of the definition of $P(1)$ the following map is an isomorphism:

$$\begin{aligned} p_3 \colon P^3 = \bigoplus_{j=1}^{l_3 - l_2} P(1) &\longrightarrow Q^2 \\ (e_1)_j &\longmapsto q_j. \end{aligned}$$

Altogether we obtain a projective resolution of M:

$$\begin{tikzcd}
0 \arrow[r] & P^3 \arrow[rr, tail] \arrow[dr, "p_3"', "\cong"] & & P^2 \arrow[rr] \arrow[dr, "p_2"'] & & P^1 \arrow[r, two heads, "p_1"] & M. \\
& & Q^2 \arrow[ur, tail] & & Q^1 \arrow[ur, tail] & &
\end{tikzcd}$$

$\square$

Lemma 2.13. *Every object $M \in \mathcal{G}$ has a (up to isomorphism unique) direct sum decomposition*

$$M = \bigoplus_{i=1}^{n} X_i \quad \textit{with } [X_i] \in \{ [S(1)], [S(2)], [P(1)], [P(2)], [I(1)] \},$$

where $I(1)$ is given by:

$$I(1) = \; k \underset{1}{\overset{0}{\rightleftarrows}} k.$$

Proof. Consider an object $M \in \mathcal{G}$

$$M = \; A \underset{f_\beta}{\overset{f_\alpha}{\rightleftarrows}} B.$$

Now we decompose A and B as follows: Let K_1 be a vector space complement of $\mathrm{im}(f_\beta)$ in A. Then let K_2 be a vector space complement of $\ker(f_\alpha) \cap \mathrm{im}(f_\beta)$ in $\mathrm{im}(f_\beta)$ and let K_3 be a vector space complement of $\ker(f_\alpha) \cap K_1$ in K_3. We have $A = A_1 \oplus A_2 \oplus A_3 \oplus A_4$ where

$$\begin{aligned} A_1 &= \ker(f_\alpha) \cap \mathrm{im}(f_\beta), & A_2 &= K_2, \\ A_3 &= K_3, & A_4 &= \ker(f_\alpha) \cap K_1. \end{aligned}$$

On the other hand choose a vector space complement L_1 of $\mathrm{im}(f_\alpha)$ in B. Since $f_\beta \circ f_\alpha = 0$ we already have that $\mathrm{im}(f_\alpha) = \ker(f_\beta \cap \mathrm{im}(f_\alpha))$. Then choose a vector space complement L_3 of $\ker(f_\beta) \cap L_1$ in L_1. We have $B = B_1 \oplus B_3 \oplus B_4$ where

$$\begin{aligned} B_1 &= \ker(f_\beta) \cap \mathrm{im}(f_\alpha), & B_3 &= L_2, \\ B_4 &= \ker(f_\beta) \cap L_1. \end{aligned}$$

Altogether we can write M as follows:

$$M = \; (A_1 \oplus A_2 \oplus A_3 \oplus A_4 \underset{f_\beta}{\overset{f_\alpha}{\rightleftarrows}} B_1 \oplus B_3 \oplus B_4)$$

$$= \; (A_4 \underset{0}{\overset{0}{\rightleftarrows}} B_4) \; \oplus \; (A_1 \oplus A_2 \oplus A_3 \underset{f_\beta|}{\overset{f_\alpha|}{\rightleftarrows}} B_1 \oplus B_3).$$

The first summand is a direct sum of $S(1)$'s and $S(2)$'s, so we can continue with the second one. Since B_1 lies in the kernel of f_β we know that $f_\beta|_{B_3} \colon B_3 \to \mathrm{im}(f_\beta)$ is surjective. Moreover since B_3 lies in the complement of the kernel it is an isomorphism. Hence B_3 decomposes as follows

$$B_3 \cong \mathrm{im}(f_\beta) = A_1 \oplus A_2.$$

On the other hand observe that $A = \ker(f_\alpha) \oplus (A_2 \oplus A_3)$. Hence $A_2 \oplus A_3$ is a vector space complement of the kernel of f_α. Thus $f_\alpha| \colon A_2 \oplus A_3 \to \mathrm{im}(f_\alpha)$ is an isomorphism. B_1 decomposes as follows

$$B_1 = \mathrm{im}(f_\alpha) \cong A_2 \oplus A_3.$$

Altogether we obtain

$$(A_1 \oplus A_2 \oplus A_3 \underset{f_\beta|}{\overset{f_\alpha|}{\rightleftarrows}} B_1 \oplus B_3)$$

$$\cong\ A_1 \oplus A_2 \oplus A_3 \underset{\tilde{f}_\beta}{\overset{\tilde{f}_\alpha}{\rightleftarrows}} (A_2 \oplus A_3) \oplus (A_1 \oplus A_2)$$

where

$$\tilde{f}_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{f}_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives the following decomposition:

$$A_1 \oplus A_2 \oplus A_3 \underset{\tilde{f}_\beta}{\overset{\tilde{f}_\alpha}{\rightleftarrows}} (A_2 \oplus A_3) \oplus (A_1 \oplus A_2)$$

$$= (\ A_1 \underset{1}{\overset{0}{\rightleftarrows}} A_1\ ) \oplus (\ A_3 \underset{0}{\overset{1}{\rightleftarrows}} A_3\ ) \oplus (\ A_2 \underset{\begin{pmatrix} 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\rightleftarrows}} A_2 \oplus A_2\ ).$$

The first summand is a direct sum of $I(1)$'s, the second summand is isomorphic to a direct sum of $P(1)$'s and the third summand is isomorphic to a direct sum of $P(2)$'s. This proves the lemma. □

Proposition 2.14. *Every projective object in $\mathcal{G}$ is a finite direct sum of some $P(i)$'s for $i = 1, 2$.*

Proof. Let P be projective. Like in the proof of Proposition 2.12 there is an epimorphism

$$P^1 = \bigoplus_{i=1}^{n} \bigoplus_{i=1}^{n_i} P(i) \longrightarrow P$$

for some $n_1, n_2 \in \mathbb{N}$. Since P is projective there exists a split map $s\colon P \to P^1$. In particular s_1 and s_2 are injective linear maps. Now we use Lemma 2.13 to obtain a direct sum decomposition

$$P = \bigoplus_{i=1}^{n} X_i \quad \textit{with } [X_i] \in \{\ [S(1)], [S(2)], [P(1)], [P(2)], [I(1)]\ \}.$$

Observe that $P^1_\alpha\colon P^1_1 \to P^1_2$ is injective. Suppose $[X_i] = [S(1)]$ for some i and take $x \neq 0 \in (X_i)_1$. Then

$$P^1_\alpha(s_1(x)) = s_1((X_i)_\alpha(x)) = s_1(0) = 0$$

and since P^1_α and s_1 are both injective this yields a contradiction to the assumption that $x \neq 0$. The same argument works for the case $[X_i] = [I(1)]$. Thus we have

$$P = \bigoplus_{i=1}^{n} X_i \quad \text{with } [X_i] \in \{\ [S(2)], [P(1)], [P(2)]\ \}.$$

Now suppose that $[X_i] = [S(2)]$ for some i and without loss of generality assume that $i = 1$. Then consider the following diagram:

$$\begin{tikzcd} & S(2) \oplus \bigoplus_{i=2}^{n} X_i \cong P \arrow[dl, dotted, "g"'] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix} = f"] \\ P(2) \arrow[r, "p", "e_2 \mapsto 1"'] & S(2). \end{tikzcd}$$

If P is projective there exists a map g which makes the diagram commute. But then for $1 \in S(2)$ we have that $g_2(1) = e_2$ and thus

$$\beta = P(2)_\beta(e_2) = P(2)_\beta(g_2(1)) = g_2(S(2)_\beta(1)) = g_2(0) = 0.$$

This is a contradiction and we finally obtain

$$P = \bigoplus_{i=1}^{n} X_i \quad \text{with } [X_i] \in \{\ [P(1)], [P(2)]\ \}.$$

$\square$

2.5 Definition of $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$

Definition 2.15. Let $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ be the abelian category of 2-periodic complexes of $\mathcal{A}$. Objects in this category are ordinary chain complexes of objects in $\mathcal{A}$ of the form:

$$\dots \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \dots, \tag{1}$$

where M_0 sits in the homological degree 0 and morphisms are ordinary chain maps. We abbreviate 2-periodic complexes (1) as follows:

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] & M_1 \arrow[l, shift left, "d_1"] \end{tikzcd}$$

with $M_0, M_1 \in Ob(\mathcal{A})$ and $d_1 \circ d_0 = 0$ and $d_0 \circ d_1 = 0$. A *morphism* $\psi_\bullet \colon M_\bullet \to \tilde{M}_\bullet$ in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ is then a diagram

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "\psi_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "\psi_1"] \\ \tilde{M}_0 \arrow[r, shift left, "\tilde{d}_0"] & \tilde{M}_1 \arrow[l, shift left, "\tilde{d}_1"] \end{tikzcd}$$

with $\psi_1 \circ d_0 = \tilde{d}_0 \circ \psi_0$ and $\psi_0 \circ d_1 = \tilde{d}_1 \circ \psi_1$.

Definition 2.16. Two morphisms $\phi_\bullet, \psi_\bullet \colon M_\bullet \to \tilde{M}_\bullet$ are called *homotopic* if there are morphisms $h_0 \colon M_0 \to \tilde{M}_1$ and $h_1 \colon M_1 \to \tilde{M}_0$ such that:

$$\psi_0 - \phi_0 = \tilde{d}_1 \circ h_0 + h_1 \circ d_0 \quad \text{and} \quad \psi_1 - \phi_1 = \tilde{d}_0 \circ h_1 + h_0 \circ d_1.$$

Define $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})$ to be the category with the same objects as $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ and morphisms the equivalence classes modulo homotopy.

Definition 2.17. Define $* \colon \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A}) \to \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ to be the shift functor which shifts the homological grading of the complex by one and changes the sign of the morphisms:

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "\psi_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "\psi_1"] \\ \tilde{M}_0 \arrow[r, shift left, "\tilde{d}_0"] & \tilde{M}_1 \arrow[l, shift left, "\tilde{d}_1"] \end{tikzcd} \quad \overset{*}{\longmapsto} \quad \begin{tikzcd} M_1 \arrow[r, shift left, "d_1"] \arrow[d, "-\psi_1"'] & M_0 \arrow[l, shift left, "d_0"] \arrow[d, "-\psi_0"] \\ \tilde{M}_1 \arrow[r, shift left, "\tilde{d}_1"] & \tilde{M}_0. \arrow[l, shift left, "\tilde{d}_0"] \end{tikzcd}$$

Note that the shift functor defines an involution on $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$.

Notation 2.18. Let $\mathcal{P}$ denote the full additive subcategory of projective objects of $\mathcal{A}$ and define $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ and $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{P})$ to be the corresponding full subcategories in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ respectively $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})$ whose objects are complexes of objects in $\mathcal{P}$.

2.6 Acyclic Complexes and Complexes of Projectives

Definition 2.19. We call a complex $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ *acyclic* if $H^*(M_\bullet) = 0$. For $P \in \mathcal{P}$ there are two distinguished acyclic complexes:

$$\kappa_P = (P \underset{0}{\overset{1}{\rightleftarrows}} P), \qquad \kappa_P^* = (P \underset{1}{\overset{0}{\rightleftarrows}} P).$$

The complexes κ_P and κ_P^* are quite important because they are in correspondence to the generators K_i, K_i^-1 of $U_v(\mathfrak{g})$ in the main theorem (Theorem 4.37). Moreover they are the building blocks of all acyclic complexes in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ in the following sense.

Lemma 2.20. *Let $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ be an acyclic complex of projectives. Then there are projective objects $P, Q \in \mathcal{P}$ such that $M_\bullet \cong \kappa_P \oplus \kappa_Q^*$.*

Proof. Suppose

$$M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1$$

is an acyclic complex of projectives. Define $P = \ker(d_1) = \operatorname{im}(d_0)$ and $Q = \ker(d_0) = \operatorname{im}(d_1)$. Then there are two short exact sequences

$$0 \to Q \to M_0 \to P \to 0 \qquad 0 \to P \to M_1 \to Q \to 0.$$

Now let $i > 0$ and consider the long exact $\operatorname{Ext}_{\mathcal{A}}$-sequences induced by the short exact sequences above:

$$\to \operatorname{Ext}^i_{\mathcal{A}}(M_0, -) \to \operatorname{Ext}^i_{\mathcal{A}}(P, -) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(Q, -) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(M_0, -) \to$$

$$\to \operatorname{Ext}^{i+1}_{\mathcal{A}}(M_1, -) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(Q, -) \to \operatorname{Ext}^{i+2}_{\mathcal{A}}(P, -) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(M_1, -) \to$$

Since M_0 and M_1 are already projective we have that $\operatorname{Ext}^l_{\mathcal{A}}(M_j, -) = 0$ for $l > 0$ and $j = 1, 2$. So it follows that

$$\operatorname{Ext}^i_{\mathcal{A}}(P, -) \cong \operatorname{Ext}^{i+1}_{\mathcal{A}}(Q, -) \cong \operatorname{Ext}^{i+2}_{\mathcal{A}}(P, -)$$

and analogously

$$\operatorname{Ext}^i_{\mathcal{A}}(Q, -) \cong \operatorname{Ext}^{i+1}_{\mathcal{A}}(P, -) \cong \operatorname{Ext}^{i+2}_{\mathcal{A}}(Q, -).$$

But since $\mathcal{A}$ has finite global dimension P and Q have finite projective resolutions. So for large enough i the Ext^i-groups must all vanish and hence all these Ext-groups vanish. It follows that P and Q are projective and the short exact sequences split with:

$$s \colon Q \to M_1 \quad \text{and} \quad t \colon P \to M_0.$$

Thus we get the desired result:

$$\begin{array}{ccc}
(M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1) & \cong & (t(P) \oplus Q \underset{d_1 \oplus d_1}{\overset{d_0 \oplus d_0}{\rightleftarrows}} P \oplus s(Q)) \\
 & & = \\
(P \oplus Q \underset{0 \oplus 1}{\overset{1 \oplus 0}{\rightleftarrows}} P \oplus Q) & \cong & (t(P) \oplus Q \underset{0 \oplus d_1}{\overset{d_0 \oplus 0}{\rightleftarrows}} P \oplus s(Q)).
\end{array}$$

□

Corollary 2.21. *A complex $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ is acyclic if and only if $M_\bullet \cong 0$ in $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{P})$.*

Proof. Let $M_\bullet$ be a complex in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$
Assume $M_\bullet$ is acyclic. By Lemma 2.20 there are projective objects P and Q such that $M_\bullet \cong \kappa_P \oplus \kappa_Q^*$. Now define

$$h_0 = (0 \oplus 1) \colon P \oplus Q \to P \oplus Q \quad \text{and} \quad h_1 = (1 \oplus 0) \colon P \oplus Q \to P \oplus Q.$$

Hence we get that

$$d_1^{\kappa_P \oplus \kappa_Q^*} \circ h_0 + h_1 \circ d_0^{\kappa_P \oplus \kappa_Q^*} = (0 \oplus 1) \circ (0 \oplus 1) + (1 \oplus 0) \circ (1 \oplus 0) = \mathrm{id}_{\kappa_P \oplus \kappa_Q^*}$$

and

$$d_0^{\kappa_P \oplus \kappa_Q^*} \circ h_1 + h_0 \circ d_1^{\kappa_P \oplus \kappa_Q^*} = (1 \oplus 0) \circ (1 \oplus 0) + (0 \oplus 1) \circ (0 \oplus 1) = \mathrm{id}_{\kappa_P \oplus \kappa_Q^*}.$$

But this shows that $\mathrm{id}_{\kappa_P \oplus \kappa_Q^*} \simeq 0$ and thus $\kappa_P \oplus \kappa_Q^* \cong 0$ in $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{P})$.
Now assume $M_\bullet \cong 0$ in $\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{P})$. Hence $\mathrm{id}_{M_\bullet} \simeq 0$ and there are $h_0 \colon M_0 \to M_1$ and $h_1 \colon M_1 \to M_0$ such that:

$$\mathrm{id}_{M_0} = h_1 \circ d_0 + d_1 \circ h_0 \qquad \text{and} \qquad \mathrm{id}_{M_1} = h_0 \circ d_1 + d_0 \circ h_1.$$

It follows that

$$\ker(d_0) = \mathrm{id}_{M_0}(\ker(d_0)) \subseteq \mathrm{im}(d_1 \circ h_0) \subseteq \mathrm{im}(d_1)$$

and

$$\ker(d_1) = \mathrm{id}_{M_1}(\ker(d_1)) \subseteq \mathrm{im}(d_0 \circ h_1) \subseteq \mathrm{im}(d_0)$$

and thus $M_\bullet$ is acyclic. □

Example 2.22. Assumption (Ass2) of Setup 2.6 is necessary for Corollary 2.21:
Let $k = \mathbb{F}_q$ and set $R = {}^{k[X]}\!/_{(X^2)}$. Consider the category *R-mod* of R-modules. *R-mod* has infinite global dimension. Then the complex

$$R \underset{\cdot X}{\overset{\cdot X}{\rightleftarrows}} R$$

is acyclic but not zero in $\mathcal{H}o_{\mathbb{Z}_2}(R\text{-}mod)$. Suppose it were then there exist $h_0, h_1 \colon R \to R$ such that

$$\mathrm{id}_R = h_1 \circ (\cdot X) + (\cdot X) \circ h_0 \qquad \text{and} \qquad \mathrm{id}_R = h_0 \circ (\cdot X) + (\cdot X) \circ h_1.$$

In particular it follows that

$$X = (h_1 \circ (\cdot X))(X) + ((\cdot X) \circ h_0)(X) = h_1(X^2) + X \cdot h_0(X) = h_0(X^2) = 0.$$

Hence there cannot be such h_0, h_1 and thus this complex is acyclic but not zero in $\mathcal{H}o_{\mathbb{Z}_2}(R\text{-}mod)$.

Note that statements like in Corollary 2.21 are usually stated for bounded complexes and the standard proofs make use of this condition. Later on the following result will be very useful to compute extension groups of complexes.

Lemma 2.23. *If $M_\bullet, N_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ then*

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(N_\bullet, M_\bullet) \cong \mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})}(N_\bullet, M_\bullet^*).$$

Proof. Let

$$0 \to M_\bullet \to P_\bullet \to N_\bullet \to 0 \tag{2}$$

be a short exact sequence in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ with $M_\bullet, N_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$. This leads to two short exact sequences in $\mathcal{A}$ which both split because M_i and N_i are projective and thus $P_i = M_i \oplus N_i$ for $i = 1, 2$. So (2) is as an extension isomorphic to a diagram of the form

$$\begin{tikzcd}
M_0 \arrow[r, shift left, "d_0^M"] \arrow[d, "i_0"'] & M_1 \arrow[l, shift left, "d_1^M"] \arrow[d, "i_1"] \\
M_0 \oplus N_0 \arrow[r, shift left, "d_0^P"] \arrow[d, "p_0"'] & M_1 \oplus N_1 \arrow[l, shift left, "d_1^P"] \arrow[d, "p_1"] \\
N_0 \arrow[r, shift left, "d_0^N"] & N_1 \arrow[l, shift left, "d_1^N"]
\end{tikzcd}$$

with i_0, i_1 and p_0, p_1 the canonical inclusions and projections. From the commutativity of the above diagram we can write d_0^P, d_1^P as follows:

$$d_0^P = \begin{pmatrix} d_0^M & s_0 \\ 0 & d_0^N \end{pmatrix}, \qquad d_1^P = \begin{pmatrix} d_1^M & \tilde{s}_1 \\ 0 & d_1^N \end{pmatrix}$$

with morphisms $s_0 \colon N_0 \to M_1$ and $\tilde{s}_1 \colon N_1 \to M_0$ such that $d_0^P \circ d_1^P = 0$ and $d_1^P \circ d_0^P = 0$. This condition is equivalent to

$$d_1^M \circ s_0 = -\tilde{s}_1 \circ d_0^N \quad \text{and} \quad d_0^M \circ \tilde{s}_1 = -s_0 \circ d_1^N.$$

But this condition is precisely the condition that

$$s_\bullet \colon N_\bullet \to M_\bullet^* \tag{3}$$

is a morphism of complexes for $s_1 = -\tilde{s}_1$. So any morphism (3) determines an extensions (2) and any extension gives a morphism (3) (note that we did not consider extension classes yet!).
Now suppose there are two morphisms $s_\bullet, t_\bullet \colon N_\bullet \to M^*_\bullet$. Then they determine two extensions with middle term $P_\bullet$ and $\bar{P}_\bullet$ and differentials given by

$$d_0^P = \begin{pmatrix} d_0^M & s_0 \\ 0 & d_0^N \end{pmatrix}, \qquad d_1^P = \begin{pmatrix} d_1^M & -s_1 \\ 0 & d_1^N \end{pmatrix}$$

and

$$d_0^{\bar{P}} = \begin{pmatrix} d_0^M & t_0 \\ 0 & d_0^N \end{pmatrix}, \qquad d_1^{\bar{P}} = \begin{pmatrix} d_1^M & -t_1 \\ 0 & d_1^N \end{pmatrix}.$$

The corresponding extensions agree in $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(N_\bullet, M_\bullet)$ precisely if there is an isomorphism $\psi_\bullet \colon P_\bullet \to \bar{P}_\bullet$ such that the following diagram commutes:

$$\begin{tikzcd}
M_\bullet \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & (M_0 \oplus N_0 \arrow[r, shift left, "d_0^P"] \arrow[d, "\psi_0"'] & M_1 \oplus N_1)) \arrow[l, shift left, "d_1^P"] \arrow[r, "p"] \arrow[d, "\psi_1"] & N_\bullet \arrow[d, "\mathrm{id}"] \\
M_\bullet \arrow[r, "i"] & (M_0 \oplus N_0 \arrow[r, shift left, "d_0^{\bar{P}}"] & M_1 \oplus N_1)) \arrow[l, shift left, "d_1^{\bar{P}}"] \arrow[r, "p"] & N_\bullet
\end{tikzcd}$$

This commutativity means that $\psi_\bullet$ is of the form

$$\psi_0 = \begin{pmatrix} 1 & h_0 \\ 0 & 1 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} 1 & \tilde{h}_1 \\ 0 & 1 \end{pmatrix}$$

with morphisms $h_0 \colon N_0 \to M_0$ and $\tilde{h}_1 \colon N_1 \to M_1$ such that $\psi_\bullet$ is a morphism of complexes. But this last condition is equivalent to

$$s_0 - t_0 = d_0^M \circ h_0 - \tilde{h}_1 \circ d_0^N \quad \text{and} \quad s_1 - t_1 = h_0 \circ d_1^N - d_1^M \circ \tilde{h}_1.$$

Now by setting $h_1 = -\tilde{h}_1$ this leads to $s_\bullet \simeq t_\bullet$ via h_0 and h_1 and we finally get that these two extensions with middle term P and $\bar{P}$ agree in $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(N_\bullet, M_\bullet)$ precisely if $s_\bullet$ and $t_\bullet$ agree in $\mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})}(N_\bullet, M^*_\bullet)$. □

3 The Hall Algebra of Complexes

At the beginning of this section we define the twisted Hall algebra $\mathcal{H}_{tw}(\mathcal{A})$ and the twisted Hall algebra of complexes $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$. Then we first compute Hall products for some concrete categories and later on in the general case for complexes. Finally we are ready to define the reduced localized Hall algebra $\mathcal{DH}_{red}(\mathcal{A})$ and give a first example. The main reference is [Bri13].

3.1 Hall Algebras

Definition 3.1. Given objects $A, B, C \in \mathcal{A}$ define $\mathrm{Ext}^1_{\mathcal{A}}(A, C)_B$ to be the subset of $\mathrm{Ext}^1_{\mathcal{A}}(A, C)$ parameterising extensions with middle term isomorphic to B. Then the *Hall algebra* $\mathcal{H}(\mathcal{A})$ is the free $\mathbb{C}$-vector space over the isomorphism classes $[A] \in \mathrm{Iso}(\mathcal{A})$ of objects $A \in \mathcal{A}$ and with associative multiplication given by

$$[A] \diamond [C] = \sum_{[B] \in \mathrm{Iso}(\mathcal{A})} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(A, C)_B|}{|\mathrm{Hom}_{\mathcal{A}}(A, C)|} \cdot [B].$$

Remark 3.2. The sum in Definition 3.1 is well-defined: Let A, C be two objects in $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives we can choose a projective resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to A \to 0$$

of A. Now we obtain $\mathrm{Ext}^1_{\mathcal{A}}(A, C)$ by taking the first homology of the following sequence:

$$0 \to \mathrm{Hom}(P_0, C) \to \mathrm{Hom}(P_1, C) \to \mathrm{Hom}(P_2, C) \to \ldots \; .$$

But since all Hom-spaces are finite it follows that $\mathrm{Ext}_{\mathcal{A}}(A, C)$ is finite.

Definition 3.3. For objects $A, B \in \mathcal{A}$ define

$$\langle A, B \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k(\mathrm{Ext}^i_{\mathcal{A}}(A, B)).$$

Since $\mathcal{A}$ is of finite global dimension and has enough projectives it turns out, that every object has a finite projective resolution. Hence $\mathrm{Ext}^i_{\mathcal{A}}$ vanishes for large enough i and the sum is finite.
Note that

$$\langle -, - \rangle \colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$$

is a well-defined bilinear form called the *Euler form.* Moreover we want to define the *symmetrised Euler form*

$$(-, -) \colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

Remark 3.4. Since for $P \in \mathcal{P}$ all *Ext*-groups vanish we have $\langle \hat{P}, - \rangle = \dim_k(\mathrm{Hom}_{\mathcal{A}}(P, -))$.

In order to have a natural bialgebra structure on $\mathcal{H}(\mathcal{A})$ we have to twist the multiplication. Moreover if we extend the Hall algebra by symbols K_α for $\alpha \in K(\mathcal{A})$ together with imposing some relations we even obtain a Hopf algebra structure. For further details see [Sch06, Lecture 1].

Definition 3.5. Now define the *twisted Hall algebra* $\mathcal{H}_{tw}(\mathcal{A})$ as the same vector space as $\mathcal{H}(\mathcal{A})$, but with twisted multiplication given by

$$[A] * [C] = t^{\langle \hat{A}, \hat{C} \rangle} \cdot [A] \diamond [C].$$

Moreover we can define the *extended twisted Hall algebra* $\mathcal{H}^e_{tw}(\mathcal{A})$ by adjoining symbols K_α for $\alpha \in K(\mathcal{A})$, with respect to the following relations for $\alpha, \beta \in K(\mathcal{A})$ and $A \in \mathcal{A}$:

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \qquad K_\alpha * [A] = t^{(\alpha, \hat{A})} \cdot [A] * K_\alpha. \tag{4}$$

Remark 3.6. Note that $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ is usually not of finite global dimension. But since $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ is closed under extensions it follows from Lemma 2.23 that it makes perfect sense to define $\mathcal{H}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ as in Definition 3.1.

Definition 3.7. Define $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ to be the same vector space as $\mathcal{H}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$, but with twisted multiplication given by

$$[M_\bullet] * [N_\bullet] = t^{\langle \hat{M}_0, \hat{N}_0 \rangle + \langle \hat{M}_1, \hat{N}_1 \rangle} \cdot [A] \diamond [C].$$

3.2 The Hall Algebra of $\mathcal{R}ep_k(A_n)$

Example 3.8. Consider the category $\mathcal{R} = \mathcal{R}ep_k(A_n)$ where A_n is defined as in Example 1.6. We want to compute some Hall products in $\mathcal{H}_{tw}(\mathcal{R})$.

- $[S(i)] * [S(i)]$ for $i < n$:
Choose a projective resolution of $S(i)$ as in Example 1.16

$$0 \longrightarrow P(i+1) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

and consider the corresponding $\mathrm{Hom}_{\mathcal{R}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i), S(i)) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i+1), S(i)) \longrightarrow 0.$$

Since $P(i+1)_i = 0$ it follows that $\mathrm{Hom}_{\mathcal{R}}(P(i+1), S(i)) = 0$ and thus $\mathrm{Ext}^1_{\mathcal{R}}(S(i), S(i)) = 0$. Hence

$$[S(i)] * [S(i)] = t^{\langle \hat{S(i)}, \hat{S(i)} \rangle} \cdot |\,\mathrm{Hom}_{\mathcal{R}}(S(i), S(i))|^{-1} \cdot [S(i) \oplus S(i)]$$

$$= t \cdot q^{-1} \cdot [S(i) \oplus S(i)] = t^{-1} \cdot [S(i) \oplus S(i)].$$

- $[S(i)] * [S(j)]$ for $j < i < n$:
As above we obtain a $\mathrm{Hom}_{\mathcal{R}}$-sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i), S(j)) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i+1), S(j)) \longrightarrow 0$$

and we can conclude that $\mathrm{Ext}^1_{\mathcal{R}}(S(i), S(j)) = 0$ since $P(i+1)_j = 0$. Moreover $\mathrm{Hom}_{\mathcal{R}}(S(i), S(j)) = 0$. Thus

$$[S(i)] * [S(j)] = t^{\langle \hat{S(i)}, \hat{S(j)} \rangle} \cdot |\mathrm{Hom}_{\mathcal{R}}(S(i), S(j))|^{-1} \cdot [S(i) \oplus S(j)]$$

$$= t^0 \cdot 1^{-1} \cdot [S(i) \oplus S(j)] = [S(i) \oplus S(j)].$$

Note that we omitted the case $i = n$. Then $S(i) = P(i)$ is already projective and all occuring $\mathrm{Ext}^1_{\mathcal{R}}$-groups vanish. Thus we obtain the same results as above.

- $[S(i)] * [S(j)]$ for $i < j$:
As above we have a $\mathrm{Hom}_{\mathcal{R}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i), S(j)) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i+1), S(j)) \longrightarrow 0.$$

If $i+1 < j$ we have that $\mathrm{Hom}_{\mathcal{R}}(P(i+1), S(j)) = 0$ because every morphism $f\colon P(i+1) \to S(j)$ maps e_{i+1} to zero and thus

$$f_j(p) = f_j(P(i+1)_p(e_{i+1})) = Q(i+1)_p(f_{i+1}(e_{i+1})) = 0$$

where p is the path from $i+1$ to j. We can conclude as above that

$$[S(i)] * [S(j)] = [S(i) \oplus S(j)].$$

Now suppose $i+1 = j$. Then $\mathrm{Hom}_{\mathcal{R}}(P(i+1), S(j)) = k$. Because every morphism $f\colon P(i) \to S(j)$ maps e_i to zero it follows as above that $\mathrm{Hom}_{\mathcal{R}}(P(i), S(j)) = 0$. Thus $\mathrm{Ext}^1_{\mathcal{R}}(S(i), S(j)) = \mathrm{Hom}_{\mathcal{R}}(P(i+1), S(j)) = k$. Now define a representation M as follows:

$$M_l = \begin{cases} k, & \text{if } l \in \{i, j\}, \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad M_{\alpha_i} = \mathrm{id}_k .$$

Moreover define for every $z \in k$ a map $f^z\colon M \to S(i)$ with

$$\begin{aligned} f^z_i\colon k &\to k \\ 1 &\mapsto z \end{aligned}$$

and consider for non-zero elements $z_1, z_2 \in k$ the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & S(j) \arrow[r, hook] \arrow[d, "\mathrm{id}"] & M \arrow[r, "f^{z_1}"] \arrow[d, dotted, "g"] & S(i) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & S(j) \arrow[r, hook] & M \arrow[r, "f^{z_2}"'] & S(i) \arrow[r] & 0. \end{tikzcd}$$

Suppose there exists a map g for which this diagram commutes. Since the left hand square commutes it follows that $g_j = \mathrm{id}_k$ and since $M_{\alpha_i} = \mathrm{id}_k$ we have that $g = \mathrm{id}_M$. But then the right hand square only commutes if $z_1 = z_2$. Thus every $z \neq 0 \in k$ defines an extension of $S(i)$ by $S(j)$ and two such extensions are only isomorphic if the corresponding z coincide. Note moreover that $\mathrm{Hom}_{\mathcal{R}}(S(i), S(j)) = 0$. Since there are only q many non-isomorphic extensions in total we have that

$$[S(i)]*[S(j)] = t^{\langle \hat{S(i)}, \hat{S(j)} \rangle} \cdot |\,\mathrm{Hom}_{\mathcal{M}}(S(i), S(j))|^{-1} \cdot ((q-1)\cdot[M] + [S(i) \oplus S(j)])$$

$$= t^{0-1} \cdot 1^{-1} \cdot ((q-1)\cdot[M] + [S(i) \oplus S(j)]) = t^{-1} \cdot ((q-1)\cdot[M] + [S(i) \oplus S(j)]).$$

3.3 The Hall Algebra of $k[X]$-gmod^{fg}

Example 3.9. Consider $\mathcal{M} = k[X]\text{-gmod}^{fg}$ as in Example 2.9. Recall the definition of the shift functor and note that for $i \in \mathbb{N}$ we have the identity $k\langle i \rangle = (X^i)$. We want to compute some Hall products in $\mathcal{H}_{tw}(\mathcal{M})$.

- $[k] * [k]$, where $k = k[X]/(X)$:

The short exact sequence

$$0 \longrightarrow k[X]\langle 1 \rangle = (X) \longrightarrow k[X] \longrightarrow k \longrightarrow 0$$

yields a projective resolution of k. To compute $\mathrm{Ext}^1_{\mathcal{M}}(k,k)$ consider the corresponding $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X], k) \longrightarrow \mathrm{Hom}_{\mathcal{M}}((X), k) \longrightarrow 0.$$

Since $(X)_0 = 0$ it follows that $\mathrm{Hom}_{\mathcal{M}}((X), k) = 0$ and thus $\mathrm{Ext}^1_{\mathcal{M}}(k,k) = 0$. Hence

$$[k] * [k] = t^{\langle \hat{k}, \hat{k} \rangle} \cdot |\,\mathrm{Hom}_{\mathcal{M}}(k,k)|^{-1} \cdot [k \oplus k]$$

$$= t \cdot q^{-1} \cdot [k \oplus k] = t^{-1} \cdot [k \oplus k].$$

- $[k]^n$, for $n \in \mathbb{N}$:
We first compute $[k] * [\bigoplus_{i=1}^m k]$ for $m \in \mathbb{N}$. As above we obtain a $\mathrm{Hom}_{\mathcal{M}}$-sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X], \bigoplus_{i=1}^m k) \longrightarrow \mathrm{Hom}_{\mathcal{M}}((X), \bigoplus_{i=1}^m k) \longrightarrow 0$$

and we can conclude that $\mathrm{Ext}^1_{\mathcal{M}}(k, \bigoplus_{i=1}^m k) = 0$. Thus

$$[k] * [\bigoplus_{i=1}^m k] = t^{\langle \hat{k}, m \cdot \hat{k} \rangle} \cdot |\mathrm{Hom}_{\mathcal{M}}(k, \bigoplus_{i=1}^m k)|^{-1} \cdot [\bigoplus_{i=1}^{m+1} k]$$

$$= t^m \cdot q^{-m} \cdot [\bigoplus_{i=1}^{m+1} k] = t^{-m} \cdot [\bigoplus_{i=1}^{m+1} k].$$

Hence we can conclude by induction that

$$[k]^n = t^{(-\sum_{i=1}^{n-1} i)} \cdot [\bigoplus_{i=1}^n k] = t^{-\frac{n(n-1)}{2}} \cdot [\bigoplus_{i=1}^n k].$$

- $[k] * [k[X]/(X^n)]$, for $n > 1$:
As above we have a $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X], k[X]/(X^n)) \xrightarrow{\mathrm{incl}^*} \mathrm{Hom}_{\mathcal{M}}((X), k[X]/(X^n)) \longrightarrow 0.$$

Now $\mathrm{Hom}_{\mathcal{M}}((X), k[X]/(X^n))$ is not zero in general, but incl^* is surjective. Thus we can conclude again that $\mathrm{Ext}^1_{\mathcal{M}}(k, k[X]/(X^n)) = 0$. Moreover we have for $f \in \mathrm{Hom}_{\mathcal{M}}(k, k[X]/(X^n))$ and $\lambda \in k$ that

$$X.f(\lambda) = f(X.\lambda) = f(0) = 0.$$

Since $n > 1$ we have that $X.f(\lambda) = 0$ implies that $f(\lambda) = 0$ and it follows that $f = 0$. Hence

$$[k]*[k[X]/(X^n)] = t^{\langle \hat{k}, (k[X]\hat{/}(X^n)) \rangle} \cdot |\mathrm{Hom}_{\mathcal{M}}(k, k[X]/(X^n))|^{-1} \cdot [k \oplus k[X]/(X^n)]$$

$$= t^0 \cdot 1^{-1} \cdot [k \oplus k[X]/(X^n)] = [k \oplus k[X]/(X^n)].$$

- $[k[X]/(X^n)] * [k]$, for $n > 1$:
Now consider the short exact sequence

$$0 \longrightarrow k[X]\langle n \rangle = (X^n) \longrightarrow k[X] \longrightarrow k[X]/(X^n) \longrightarrow 0$$

which yields a projective resolution of $k[X]/(X^n)$. Thus we obtain a $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X],k) \longrightarrow \mathrm{Hom}_{\mathcal{M}}((X^n),k) \longrightarrow 0.$$

Since $(X^n)_0 = 0$ it follows that $\mathrm{Hom}_{\mathcal{M}}((X^n),k) = 0$ and thus $\mathrm{Ext}^1_{\mathcal{M}}(k[X]/(X^n),k) = 0$. Hence

$$[k[X]/(X^n)]*[k] = t^{\langle (k[X]\hat{/}(X^n)),\hat{k}\rangle} \cdot |\,\mathrm{Hom}_{\mathcal{M}}(k[X]/(X^n),k)|^{-1} \cdot [k[X]/(X^n) \oplus k]$$

$$= t \cdot q^{-1} \cdot [k \oplus k[X]/(X^n)] = t^{-1} \cdot [k \oplus k[X]/(X^n)].$$

This shows that $\mathcal{H}_{tw}(\mathcal{M})$ is a non-commutative algebra.

- $[k] * [(X)]$:

As above we obtain a $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X],(X)) \longrightarrow \mathrm{Hom}_{\mathcal{M}}((X),(X)) \longrightarrow 0.$$

Now $\mathrm{Hom}_{\mathcal{M}}(k[X],(X)) = 0$ because every morphism $f\colon k[X] \to (X)$ maps 1 into $(X)_0 = 0$. Thus $\mathrm{Ext}^1_{\mathcal{M}}(k,(X)) = \mathrm{Hom}_{\mathcal{M}}((X),(X)) = k$. For $z \in k$ define

$$\begin{aligned} f^z\colon (X) &\to k[X] \\ X &\mapsto zX \end{aligned}$$

and consider for non-zero elements $z_1, z_2 \in k$ the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & (X) \arrow[r, "f^{z_1}"] \arrow[d, "\mathrm{id}"] & k[X] \arrow[r] \arrow[d, dotted, "g"] & k \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & (X) \arrow[r, "f^{z_2}"'] & k[X] \arrow[r] & k \arrow[r] & 0. \end{tikzcd}$$

Suppose there exists a map g for which this diagram commutes. Since the right hand square commutes it follows that $g(1) = 1$ and hence $g = \mathrm{id}$. But then the left hand square only commutes if $z_1 = z_2$. Thus every $z \neq 0 \in k$ defines an extension of k by (X) and two such extensions are only isomorphic if the corresponding z coincide. Note moreover that $\mathrm{Hom}_{\mathcal{M}}(k,(X)) = 0$ since $(X)_0 = 0$. Since there are only q many non-isomorphic extensions in total we have that

$$[k] * [(X)] = t^{\langle \hat{k},(\hat{X})\rangle} \cdot |\,\mathrm{Hom}_{\mathcal{M}}(k,(X))|^{-1} \cdot ((q-1)\cdot[k[X]] + [k \oplus (X)])$$

$$= t^{0-1} \cdot 1^{-1} \cdot ((q-1)\cdot[k[X]] + [k \oplus (X)]) = t^{-1} \cdot ((q-1)\cdot[k[X]] + [k \oplus (X)]).$$

- $[k\langle i\rangle] * [k\langle j\rangle]$ for $2 \leq |i-j|$:
Consider the short exact sequence

$$0 \longrightarrow k[X]\langle i+1\rangle \longrightarrow k[X]\langle i\rangle \longrightarrow k\langle i\rangle \longrightarrow 0$$

which yields a projective resolution of $k\langle i\rangle$. We obtain a $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X]\langle i\rangle, k\langle j\rangle) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X]\langle i+1\rangle, k\langle j\rangle) \longrightarrow 0.$$

Since $2 \leq |i-j|$ we know that $1 \in k[X]\langle i+1\rangle$ must be mapped to zero by any map $k[X]\langle i+1\rangle \to k\langle j\rangle$. We can conclude that $\mathrm{Hom}_{\mathcal{M}}(k[X]\langle i+1\rangle, k\langle j\rangle) = 0$ and hence $\mathrm{Ext}^1_{\mathcal{M}}(k\langle i\rangle, k\langle j\rangle) = 0$. Moreover $\mathrm{Hom}_{\mathcal{M}}(k\langle i\rangle, k\langle j\rangle) = 0$ since $i \neq j$ and thus

$$[k\langle i\rangle] * [k\langle j\rangle] = t^{\langle \widehat{k\langle i\rangle}, \widehat{k\langle j\rangle}\rangle} \cdot |\mathrm{Hom}_{\mathcal{M}}(k\langle i\rangle, k\langle j\rangle)|^{-1} \cdot [k\langle i\rangle \oplus k\langle j\rangle]$$

$$= t^{0-0} \cdot 1^{-1} \cdot [k\langle i\rangle \oplus k\langle j\rangle] = [k\langle i\rangle \oplus k\langle j\rangle].$$

3.4 The Hall Algebra of $\mathcal{G}$

We know by Proposition 2.12 that $\mathrm{gldim}(\mathcal{G}) = 2$. Thus in contrast to the previous examples we have to take the $\mathrm{Ext}^2_{\mathcal{G}}$-groups into account.

Example 3.10. Consider the category $\mathcal{G}$ defined as in Example 1.10. Proposition 2.12 shows that it makes perfect sense to define the Hall Algebra of $\mathcal{G}$. We want to compute some Hall products in $\mathcal{H}_{tw}(\mathcal{M})$.

- $[S(1)] * [S(1)]$:
The following exact sequence

$$\begin{tikzcd} P(1) \arrow[r, tail] & P(2) \arrow[rr] \arrow[dr, two heads] & & P(2) \arrow[r, two heads] & S(1) \\ & & S(2) \arrow[ur, tail] & & \end{tikzcd}$$

yields a projective resolution of $S(1)$. Consider the corresponding $\mathrm{Hom}_{\mathcal{G}}$-sequence:

$$\mathrm{Hom}_{\mathcal{G}}(P(1), S(1)) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(2), S(1)) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(1), S(1)) \longrightarrow 0.$$

Observe that $\mathrm{Hom}_{\mathcal{G}}(P(1), S(1)) = k$ and $\mathrm{Hom}_{\mathcal{G}}(P(2), S(1)) = 0$. Hence we can conclude that $\mathrm{Ext}^1_{\mathcal{G}}(S(1), S(1)) = 0$ and $\mathrm{Ext}^2_{\mathcal{G}}(S(1), S(1)) = k$ and we obtain

$$[S(1)] * [S(1)] = t^{\langle \hat{S(1)}, \hat{S(1)} \rangle} \cdot |\mathrm{Hom}_{\mathcal{G}}(S(1), S(1))|^{-1} \cdot [S(1) \oplus S(1)]$$

$$= t^{1-0+1} \cdot q^{-1} \cdot [S(1) \oplus S(1)] = [S(1) \oplus S(1)].$$

- $[S(1)] * [S(2)]$:
As above we obtain a $\mathrm{Hom}_{\mathcal{G}}$-sequence

$$\mathrm{Hom}_{\mathcal{G}}(P(1), S(2)) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(2), S(2)) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(1), S(2)) \longrightarrow 0$$

where $\mathrm{Hom}_{\mathcal{G}}(P(1), S(2)) = 0$ and $\mathrm{Hom}_{\mathcal{G}}(P(2), S(2)) = k$. Then it follows that $\mathrm{Ext}^1_{\mathcal{G}}(S(1), S(2)) = k$ and $\mathrm{Ext}^2_{\mathcal{G}}(S(1), S(2)) = 0$. Define for every $z \in k$ a map $f^z \colon S(2) \to P(1)$ with

$$\begin{aligned} f^z_2 \colon k &\to k\langle \alpha \rangle \\ 1 &\mapsto z \cdot \alpha \end{aligned}$$

and consider for non-zero elements $z_1, z_2 \in k$ the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & S(2) \arrow[r, "f^{z_1}"] \arrow[d, "\mathrm{id}"] & P(1) \arrow[r, two heads] \arrow[d, dotted, "g"] & S(1) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & S(2) \arrow[r, "f^{z_2}"'] & P(1) \arrow[r, two heads] & S(1) \arrow[r] & 0. \end{tikzcd}$$

Suppose there exists a map g for which this diagram commutes. Since the right hand square commutes it follows that $g_1 = \mathrm{id}$ and since $g_2(\alpha) = g_2(P(1)_\alpha(e_1)) = P(1)_\alpha(e_1) = \alpha$ we have that $g = \mathrm{id}_{P(1)}$. But then the left hand square only commutes if $z_1 = z_2$. Thus every $z \neq 0 \in k$ defines an extension of $S(1)$ by $S(2)$ and two such extensions are only isomorphic if the corresponding z coincide. Note moreover that $\mathrm{Hom}_{\mathcal{G}}(S(1), S(2)) = 0$. Since there are only q many non-isomorphic extensions in total we have that

$$[S(1)]*[S(2)] = t^{\langle \hat{S(1)}, \hat{S(2)} \rangle} \cdot |\mathrm{Hom}_{\mathcal{G}}(S(1), S(2))|^{-1} \cdot ((q-1) \cdot [P(1)] + [S(1) \oplus S(2)])$$

$$t^{0-1+0} \cdot 1^{-1} \cdot ((q-1) \cdot [P(1)] + [S(1) \oplus S(2)]) = t^{-1} \cdot ((q-1) \cdot [P(1)] + [S(1) \oplus S(2)]).$$

- $[S(2)] * [I(1)]$:
The short exact sequence

$$0 \longrightarrow P(1) \xrightarrow{i} P(2) \longrightarrow S(2) \longrightarrow 0$$

yields a projective resolution of $S(2)$. Consider the corresponding $\mathrm{Hom}_{\mathcal{G}}$-sequence:

$$\mathrm{Hom}_{\mathcal{G}}(P(2), I(1)) \xrightarrow{i^*} \mathrm{Hom}_{\mathcal{G}}(P(1), I(1)) \longrightarrow 0.$$

Observe that i^* is surjective. It follows that $\mathrm{Ext}_{\mathcal{G}}(S(2), I(1)) = 0$ (this follows immediately if you already know that $I(1)$ and $I(2) = P(2)$ are the indecomposable injective objects in $\mathcal{G}$). Moreover we have that $\mathrm{Hom}_{\mathcal{G}}(S(2), I(1)) = 0$ and thus

$$[S(2)] * [I(1)] = t^{\langle \hat{S(2)}, \hat{I(1)} \rangle} \cdot |\,\mathrm{Hom}_{\mathcal{G}}(S(2), I(1))|^{-1} \cdot [S(2) \oplus I(1)]$$

$$= t^{0-0} \cdot 1^{-1} \cdot [S(2) \oplus I(1)] = [S(2) \oplus I(1)].$$

3.5 Embedding of $\mathcal{H}_{tw}(\mathcal{R}ep_k(A_n))$ into $\mathcal{H}_{tw}(k[X]\text{-gmod}^{fg})$

In view of the previous examples note that Hall products in $\mathcal{H}_{tw}(\mathcal{R}ep_k(A_n))$ behave quite similar to Hall products in $\mathcal{H}_{tw}(k[X]\text{-gmod}^{fg})$ where the representations $S(i)$ and $P(i)$ correspond to the modules $k\langle i \rangle$ and $k[X]\langle i \rangle$ respectively. Indeed the following lemma shows that we can embed $\mathcal{H}_{tw}(\mathcal{R}ep_k(A_n))$ in $\mathcal{H}_{tw}(k[X]\text{-gmod}^{fg})$.

Proposition 3.11. *There is an embedding of $\mathbb{C}$-algebras:*

$$J\colon \mathcal{H}_{tw}(\mathcal{R}ep_k(A_n)) \hookrightarrow \mathcal{H}_{tw}(k[X]\text{-}\,\mathrm{gmod}^{fg}).$$

Proof. First set $\mathcal{R} = \mathcal{R}ep_k(A_n)$ and $\mathcal{M} = k[X]\text{-}\,\mathrm{gmod}^{fg}$ as above and define a functor

$$\begin{aligned} \iota\colon \mathcal{R} &\to \mathcal{M} \\ Y &\mapsto \iota(Y) \\ f &\mapsto \iota(f) \end{aligned}$$

where

$$\iota(Y)_i = \begin{cases} Y_i & \text{if } 1 \le i \le n, \\ 0 & \text{else} \end{cases} \qquad \text{with} \quad X.y = Y_{\alpha_i}(y) \quad \text{for } y \in Y_i$$

and

$$\iota(f)_i = \begin{cases} f_i & \text{if } 1 \le i \le n, \\ 0 & \text{else} \end{cases} \qquad \text{for a morphism } f \text{ in } \mathcal{R}.$$

Then define $J([Y]) = [\iota(Y)]$. Consider two objects Y, Z in $\mathcal{R}$. If $f\colon Y \to Z$ is an isomorphism then $\iota(f)\colon \iota(Y) \to \iota(Z)$ is obviously again an isomorphism and every isomorphism $g\colon \iota(Y) \to \iota(Z)$ in $\mathcal{M}$ yields an isomorphism $\tilde{g}\colon Y \to Z$ in $\mathcal{R}$ where $\tilde{g}_i = g_i$. Thus J is well-defined and maps a basis of $\mathcal{R}$ to linearly independent elements in $\mathcal{M}$. Hence J defines an embedding of vector spaces. It is left to show that J is a morphism of algebras. Recall the definition of the multiplication in $\mathcal{H}_{tw}(\mathcal{A})$ for an abelian category $\mathcal{A}$ satisfying (Ass1)-(Ass4):

$$[A] * [C] = t^{\dim_k \operatorname{Hom}_{\mathcal{A}}(A,C) - \dim_k \operatorname{Ext}^1_{\mathcal{A}}(A,C)} \cdot \sum_{[B] \in \operatorname{Iso}(\mathcal{A})} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A,C)_B|}{|\operatorname{Hom}_{\mathcal{A}}(A,C)|} \cdot [B]$$

where $A, C \in \mathcal{A}$. Hence it is enough to show that ι is fully faithful and induces an isomorphism

$$\operatorname{Ext}^1_{\mathcal{R}}(Y, Z) \cong \operatorname{Ext}^1_{\mathcal{M}}(\iota(Y), \iota(Z)) \quad \text{for } Y, Z \in \mathcal{R}. \tag{5}$$

These are all easy observations so we will not give a detailed proof. Just note that every extension

$$0 \longrightarrow Z \longrightarrow W \longrightarrow Y \longrightarrow 0$$

in $\mathcal{R}$ yields an extension

$$0 \longrightarrow \iota(Z) \longrightarrow \iota(W) \longrightarrow \iota(Y) \longrightarrow 0$$

in $\mathcal{M}$ and every extension of $\iota(Y)$ by $\iota(Z)$ in $\mathcal{M}$ is already of this form. □

In particular J induces an isomorphism between $\mathcal{H}_{tw}(\mathcal{R}ep_k(A_n))$ and the subalgebra of $\mathcal{H}_{tw}(k[X]\text{-}\operatorname{gmod}^{fg})$ generated by the elements $[k\langle 1\rangle], \ldots, [k\langle n\rangle]$. This is true because by Theorem 4.28 the elements $[S(1)], \ldots, [S(n)]$ generate $\mathcal{H}_{tw}(\mathcal{R}ep_k(A_n))$ as an algebra.

Proposition 3.12. *There is an embedding of* $\mathbb{C}$*-algebras:*

$$J^e\colon \mathcal{H}^e_{tw}(\mathcal{R}ep_k(A_n)) \hookrightarrow \mathcal{H}^e_{tw}(k[X]\text{-}\operatorname{gmod}^{fg}).$$

Proof. Set $\mathcal{R} = \mathcal{R}ep_k(A_n)$ and $\mathcal{M} = k[X]\text{-}\operatorname{gmod}^{fg}$ as above. In view of (4) it is enough to show that

$$\begin{aligned} \hat{\iota}\colon K(\mathcal{R}) &\to K(\mathcal{M}) \\ \hat{A} &\mapsto \iota(\hat{A}) \end{aligned}$$

defines an injective morphism of groups. Take $\alpha \in K(\mathcal{R})$ and suppose that $\hat{\iota}(\alpha) = 0 \in K(\mathcal{M})$. By Lemma 2.4 there are projective objects $P, Q \in \mathcal{R}$ such that $\alpha = \hat{P} - \hat{Q}$. Thus we have that

$$\iota(\hat{P}) - \iota(\hat{Q}) = 0 \in K(\mathcal{M}).$$

With the same argument as in the proof of (Ass4) in Proposition 2.10 we can conclude that $\iota(\hat{P})_i = \iota(\hat{Q})_i \in K(\mathrm{Vect}_k)$ for all $1 \leq i \leq n$. But since

$$\dim_k(P_i) \cdot \hat{k} = \iota(\hat{P})_i \quad \text{and} \quad \dim_k(Q_i) \cdot \hat{k} = \iota(\hat{Q})_i$$

in $K(\mathrm{Vect}_k)$ we get that $P_i \cong Q_i$ for all $1 \leq i \leq n$. Then it follows with Example 1.14 that $[P] = [Q]$ and hence $\alpha = 0 \in K(\mathcal{R})$. This proves the injectivity of $\hat{\iota}$. $\square$

3.6 Computation of some Hall-Products in $\mathcal{H}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$

Definition 3.13. Every object $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ defines a class

$$\hat{M}_\bullet = \hat{M}_0 - \hat{M}_1 \in K(\mathcal{A}).$$

In particular we have that $\hat{M}_\bullet^* = -\hat{M}$.

Note that by definition $\hat{M}_\bullet$ lies in $K(\mathcal{A})$ and not in $K(\mathcal{C}_{\mathbb{Z}_2})(A))$. For the following lemma recall that $t = +\sqrt{q}$ and $k = \mathbb{F}_q$.

Lemma 3.14. *For $P \in \mathcal{P}$ and $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ there are the following identities in $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$:*

$$[\kappa_P] * [M_\bullet] = t^{-\langle \hat{P}, \hat{M}_\bullet \rangle} \cdot [\kappa_P \oplus M_\bullet], \qquad [\kappa_P^*] * [M_\bullet] = t^{\langle \hat{P}, \hat{M}_\bullet \rangle} \cdot [\kappa_P^* \oplus M_\bullet]$$

$$[M_\bullet] * [\kappa_P] = t^{\langle \hat{M}_\bullet, \hat{P} \rangle} \cdot [\kappa_P \oplus M_\bullet], \qquad [M_\bullet] * [\kappa_P^*] = t^{-\langle \hat{M}_\bullet, \hat{P} \rangle} \cdot [\kappa_P^* \oplus M_\bullet].$$

Proof. Since κ_p and κ_P^* are acyclic complexes we know by Corollary 2.21 that they are homotopy equivalent to the zero complex and it follows from Lemma 2.23 that $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(\kappa_P, M_\bullet)$, $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(\kappa_P^*, M_\bullet)$, $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(M_\bullet, \kappa_P)$, and $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(M_\bullet, \kappa_P^*)$ are all trivial. Now all morphisms $\psi_\bullet \colon \kappa_P \to M_\bullet$ and $\varphi_\bullet \colon \kappa_P^* \to M_\bullet$ have the form:

$$\begin{tikzcd} P \arrow[r, shift left, "1"] \arrow[d, "\psi_0"'] & P \arrow[l, shift left, "0"] \arrow[d, "\psi_1 = d_0 \circ \psi_0"] \\ M_0 \arrow[r, shift left, "d_0"] & M_1 \arrow[l, shift left, "d_1"] \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, shift left, "0"] \arrow[d, "\varphi_0 = d_1 \circ \varphi_1"'] & P \arrow[l, shift left, "1"] \arrow[d, "\varphi_1"] \\ M_0 \arrow[r, shift left, "d_0"] & M_1 \arrow[l, shift left, "d_1"] \end{tikzcd}.$$

Hence we have $\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(\kappa_P, M_\bullet) \cong \operatorname{Hom}_{\mathcal{A}}(P, M_0)$ and $\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(\kappa_P^*, M_\bullet) \cong \operatorname{Hom}_{\mathcal{A}}(P, M_1)$. Moreover all morphisms $\xi_\bullet \colon M_\bullet \to \kappa_P$ and $\zeta_\bullet \colon M_\bullet \to \kappa_P^*$ have the form:

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "\xi_0=\xi_1\circ d_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "\xi_1"] \\ P \arrow[r, shift left, "1"] & P \arrow[l, shift left, "0"] \end{tikzcd} \qquad \begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "\zeta_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "\zeta_1=\zeta_0\circ d_1"] \\ P \arrow[r, shift left, "0"] & P. \arrow[l, shift left, "1"] \end{tikzcd}$$

Hence we have $\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(M_\bullet, \kappa_P) \cong \operatorname{Hom}_{\mathcal{A}}(M_1, P)$ and $\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(M_\bullet, \kappa_P^*) \cong \operatorname{Hom}_{\mathcal{A}}(M_0, P)$. Now taking Remark 3.4 into account we can calculate:

$$[\kappa_P] * [M_\bullet] = t^{\langle \hat{P}, \hat{M}_0\rangle + \langle \hat{P}, \hat{M}_1\rangle} \cdot |\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(\kappa_P, M_\bullet)|^{-1} \cdot [\kappa_P \oplus M_\bullet]$$

$$= t^{\langle \hat{P}, \hat{M}_0\rangle + \langle \hat{P}, \hat{M}_1\rangle} \cdot q^{-\dim_k \operatorname{Hom}_{\mathcal{A}}(P, M_0)} \cdot [\kappa_P \oplus M_\bullet]$$

$$= t^{\langle \hat{P}, \hat{M}_0\rangle + \langle \hat{P}, \hat{M}_1\rangle} \cdot t^{-2\langle \hat{P}, \hat{M}_0\rangle} \cdot [\kappa_P \oplus M_\bullet]$$

$$= t^{-\langle \hat{P}, \hat{M}_\bullet\rangle} \cdot [\kappa_P \oplus M_\bullet].$$

All other equalities follow analogously. $\square$

Corollary 3.15. *For $P, Q \in \mathcal{P}$ and $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ there are the following identities in $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$:*

$$[\kappa_P] * [M_\bullet] = t^{-(\hat{P}, \hat{M}_\bullet)} \cdot [M_\bullet] * [\kappa_P] \tag{6}$$

$$[\kappa_P^*] * [M_\bullet] = t^{(\hat{P}, \hat{M}_\bullet)} \cdot [M_\bullet] * [\kappa_P^*] \tag{7}$$

$$[\kappa_P] * [\kappa_Q] = [\kappa_P \oplus \kappa_Q] \tag{8}$$

$$[\kappa_P^*] * [\kappa_Q^*] = [\kappa_P^* \oplus \kappa_Q^*] \tag{9}$$

$$[\kappa_P^*] * [\kappa_Q] = [\kappa_Q] * [\kappa_P^*] = [\kappa_P^* \oplus \kappa_Q]. \tag{10}$$

Proof. Equation (6) and (7) follow immediately from Lemma 3.14. For (8)-(10) note that for $A \in \mathcal{P}$ we have that $\hat{\kappa}_A = \hat{A} - \hat{A} = 0 \in K(\mathcal{A})$ and hence $\langle \hat{A}, - \rangle = \langle -, \hat{A} \rangle = 0$. $\square$

3.7 The Localized Hall Algebra

As mentioned above we want that the objects $[\kappa_P]$ and $[\kappa_P^*]$ in $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ correspond to the elements K_i and K_i^{-1} in the quantum enveloping algebra. In order to make them invertible we want to localize $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ at the following set:

Definition 3.16. Define $\mathcal{S}$ to be the vector subspace of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ generated by elements in $\{[M_\bullet] \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}) \mid H_*(M_\bullet) = 0\}$.

Remark 3.17. Note that $\mathcal{S}$ is a multiplicative closed subset of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$. This follows immediately from Lemma 2.20 and Lemma 3.14.

Observe that $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ is a non commutative algebra in general. To obtain a well-defined localization we have to check the Ore Condition.

Lemma 3.18. *$\mathcal{S}$ satisfies the right Ore Condition, i.e. for generators $[N_\bullet] \in \mathcal{S}$ and $[M_\bullet] \in \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ the intersection $([M_\bullet] * \mathcal{S}) \cap ([N_\bullet] * \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})))$ is non empty.*

Proof. Let $[N_\bullet] \in \mathcal{S}$ and $[M_\bullet] \in \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$. By Lemma 2.20 we have $[N_\bullet] = [\kappa_P \oplus \kappa_Q^*]$ for some $P, Q \in \mathcal{P}$. Now we use Corollary 3.15 to calculate:

$$[M_\bullet] * \underbrace{[\kappa_P \oplus \kappa_Q^*]}_{\in \mathcal{S}} = [M_\bullet] * [\kappa_P] * [\kappa_Q^*]$$

$$= [\kappa_P] * (t^{(\hat{P}, \hat{M}_\bullet)} \cdot [M_\bullet]) * [\kappa_Q^*]$$

$$= [\kappa_P] * [\kappa_Q^*] * (t^{(\hat{P}, \hat{M}_\bullet) - (\hat{Q}, \hat{M}_\bullet)} \cdot [M_\bullet])$$

$$= [\kappa_P \oplus \kappa_Q^*] * (t^{(\hat{P}, \hat{M}_\bullet) - (\hat{Q}, \hat{M}_\bullet)} \cdot [M_\bullet])$$

$$= [N_\bullet] * \underbrace{(t^{(\hat{P}, \hat{M}_\bullet) - (\hat{Q}, \hat{M}_\bullet)} \cdot [M_\bullet])}_{\in \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))}.$$

□

Definition 3.19. Define the *localized Hall algebra* $\mathcal{DH}(\mathcal{A})$ as the twisted Hall algebra $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ localized at $\mathcal{S}$:

$$\mathcal{DH}(\mathcal{A}) = \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))[[M_\bullet]^{-1} \colon H_*(M_\bullet) = 0].$$

Since $\mathcal{S}$ satisfies the right Ore condition this is a well-defined localization.

Definition 3.20. Define the *reduced localized Hall algebra* $\mathcal{DH}_{red}(\mathcal{A})$ by taking $\mathcal{DH}(\mathcal{A})$ and setting $[M_\bullet] = 1$ whenever $M_\bullet$ is an acyclic complex, invariant under the shift functor, in formulas:

$$\mathcal{DH}_{red}(\mathcal{A}) = \mathcal{DH}(\mathcal{A})/([M_\bullet] - 1\colon H_*(M_\bullet) = 0,\ M_\bullet \cong M_\bullet^*).$$

Remark 3.21. Setting $[M_\bullet] = 1$ for acyclic complexes $M_\bullet$, invariant under the shift functor, is the same as setting $[\kappa_P] * [\kappa_P^*] = 1$ for all $P \in \mathcal{P}$:
Let $M_\bullet$ be such a complex. Then by Lemma 2.20 there are objects $P, Q \in P$ such that $[M_\bullet] = [\kappa_P \oplus \kappa_Q^*]$. But since $M_\bullet$ is invariant under $*$ it follows that $P \cong Q$ and by taking Corollary 3.15 into account we have $[M_\bullet] = [\kappa_P \oplus \kappa_P^*]$ and $[\kappa_P \oplus \kappa_P^*] = [\kappa_P] * [\kappa_P^*]$. On the other hand every complex $\kappa_P \oplus \kappa_P^*$ is already acyclic and invariant under $*$.
In particular if we set $[M_\bullet] = 1$ for all acyclic complexes invariant under $*$, this already forces κ_P to be invertible so we have

$$\mathcal{DH}_{red}(\mathcal{A}) = \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))/([M_\bullet] - 1\colon H_*(M_\bullet) = 0,\ M_\bullet \cong M_\bullet^*).$$

Example 3.22. Consider the category of finite dimensional vector spaces $\mathrm{Vect}_k = \mathcal{R}ep_k(A_1)$. We want to find a basis of $\mathcal{DH}_{red}(\mathrm{Vect}_k)$. Note that all elements in Vect_k are projective. Let

$$M_\bullet = \ M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1\ , \qquad M_0, M_1 \in \mathrm{Vect}_k,$$

be any element in $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathrm{Vect}_k)$. Now choose vector space complements K_0 and K_1 of $\ker(d_0)$ in M_0 and of $\ker(d_1)$ in M_1 respectively. Since $\mathrm{im}(d_0) \subseteq \ker(d_1)$ and $\mathrm{im}(d_1) \subseteq \ker(d_0)$ we have that $M_\bullet$ decomposes as follows:

$$\begin{aligned} M_\bullet = \ (M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1) \ &= \ (\ker(d_0) \oplus K_0 \underset{(d_1|_{K_1})\oplus 0}{\overset{0\oplus(d_0|_{K_0})}{\rightleftarrows}} K_1 \oplus \ker(d_1)) \\ &= \ (\ker(d_0) \underset{d_1|_{K_1}}{\overset{0}{\rightleftarrows}} K_1) \ \oplus \ (K_0 \underset{0}{\overset{d_0|_{K_0}}{\rightleftarrows}} \ker(d_1))\ . \end{aligned}$$

Now choose vector space complements C_0 and C_1 of $d_1(K_1)$ in $\ker(d_0)$ and of $d_0(K_0)$ in $\ker(d_1)$ respectively. Then

$$(\ker(d_0) \underset{d_1|_{K_1}}{\overset{0}{\rightleftarrows}} K_1) \ \oplus \ (K_0 \underset{0}{\overset{d_0|_{K_0}}{\rightleftarrows}} \ker(d_1))$$

$$= \ (C_0 \underset{0}{\overset{0}{\rightleftarrows}} 0) \ \oplus \ (d_1(K_1) \underset{d_1|_{K_1}}{\overset{0}{\rightleftarrows}} K_1) \ \oplus \ (0 \underset{0}{\overset{0}{\rightleftarrows}} C_1) \ \oplus \ (K_0 \underset{0}{\overset{d_0|_{K_0}}{\rightleftarrows}} d_0(K_0))$$

$$= \ (C_0 \underset{0}{\overset{0}{\rightleftarrows}} C_1) \ \oplus \ (d_1(K_1) \underset{d_1|_{K_1}}{\overset{0}{\rightleftarrows}} K_1) \ \oplus \ (K_0 \underset{0}{\overset{d_0|_{K_0}}{\rightleftarrows}} d_0(K_0)) \ .$$

Since the maps $d_i|_{K_i} \colon K_i \to d_i(K_i)$ are isomorphisms for $i = 1, 2$, it follows with Lemma 3.14 that

$$[M_\bullet] = [\ (C_0 \underset{0}{\overset{0}{\rightleftarrows}} C_1) \ \oplus \kappa_{K_0} \oplus \kappa^*_{K_1}]$$

$$= t^n \cdot [\ (C_0 \underset{0}{\overset{0}{\rightleftarrows}} C_1)\ ] * [\kappa_{K_0} \oplus \kappa^*_{K_1}]$$

for some $n \in \mathbb{Z}$. For every vector space A we have that $A \cong k^{\dim_k(A)}$. Thus the set

$$\{\ [\ (k^{n_0} \underset{0}{\overset{0}{\rightleftarrows}} k^{n_1})\ ] * [\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_Q}}] \mid n_0, n_1, n_P, n_Q \in \mathbb{N}\ \}$$

forms a basis of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathrm{Vect}_k)$. Now consider a basis element

$$[\ (k^{n_0} \underset{0}{\overset{0}{\rightleftarrows}} k^{n_1})\ ] * [\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_Q}}]$$

and assume that $n_Q \leq n_P$. Then by using Corollary 3.15 we have the following equation in $\mathcal{DH}_{red}(\mathrm{Vect}_k)$:

$$[\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_Q}}] = [\kappa_{k^{(n_P - n_Q)}} \oplus \kappa_{k^{n_Q}} \oplus \kappa^*_{k^{n_Q}}]$$

$$= [\kappa_{k^{(n_P - n_Q)}}] \quad * \quad \underbrace{[\kappa_{k^{n_Q}} \oplus \kappa^*_{k^{n_Q}}]}_{\text{acyclic and invariant under } *} \quad = \quad [\kappa_{k^{(n_P - n_Q)}}].$$

On the other hand if we assume that $n_P \leq n_Q$ we obtain

$$[\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_Q}}] = [\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_P}} \oplus \kappa^*_{k^{(n_Q - n_P)}}]$$

$$= [\kappa_{k^{n_P}} \oplus \kappa^*_{k^{n_P}}] * [\kappa^*_{k^{(n_Q - n_P)}}] = [\kappa^*_{k^{(n_Q - n_P)}}] \quad \text{in } \mathcal{DH}_{red}(\mathrm{Vect}_k).$$

Hence again by using Lemma 3.14 we can state the following lemma:

Lemma 3.23. *A basis of $\mathcal{DH}_{red}(\mathrm{Vect}_k)$ is given by elements*

$$[\ (k^{n_0} \underset{0}{\overset{0}{\rightleftarrows}} k^{n_1})\ \oplus \kappa_{k^{n_P}}] \quad \textit{and} \quad [\ (k^{n_0} \underset{0}{\overset{0}{\rightleftarrows}} k^{n_1})\ \oplus \kappa^*_{k^{n_Q}}]$$

for $n_0, n_1, n_P, n_Q \in \mathbb{N}$.

4 The Hereditary Case

In order to show our main theorem (Theorem 4.37) we now restrict to categories with properties similar to those of $\mathcal{R}ep_k(Q)$. Thus from now on we assume that $\mathcal{A}$ also satisfies the assumptions (Ass3)-(Ass4) of Setup 2.6. The main reference is again [Bri13]. We start by showing that there is a direct sum decomposition of elements in $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ which yields a basis of $\mathcal{DH}(\mathcal{A})$ later on. Using (Ass3) we see that every object $A \in \mathcal{A}$ has a projective resolution of lenght 1 which defines a corresponding element $E_A \in \mathcal{DH}(\mathcal{A})$. By using results from the theory of derived categories it turns out that the assignments

$$\begin{aligned} I_+ \colon \mathcal{H}_{tw}(\mathcal{A}) &\to \mathcal{DH}(\mathcal{A}), & I_- \colon \mathcal{H}_{tw}(\mathcal{A}) &\to \mathcal{DH}(\mathcal{A}) \\ [A] &\mapsto E_A & [A] &\mapsto F_A = E_A^* \end{aligned}$$

define injective morphisms of rings. Moreover we use (Ass4) and the description of a basis of $\mathcal{DH}(\mathcal{A})$ to show that the assignment $a \otimes b \mapsto I_+(a) * I_-(b)$ defines an isomorphism of vector spaces. We even obtain an isomorphism

$$\mathcal{H}_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A}) \longrightarrow \mathcal{DH}_{red}(\mathcal{A}).$$

At this point we set $\mathcal{A} = \mathcal{R}ep_k(Q)$ and consider the corresponding quantum enveloping algebra $U_t(\mathfrak{g})$ specialized at $t = +\sqrt{q}$. We use its triangular decomposition and Ringel's Theorem to define an embedding of vector spaces

$$U_t(\mathfrak{g}) \longrightarrow \mathcal{H}_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A}) \longrightarrow \mathcal{DH}_{red}(\mathcal{A}).$$

Knowing that I_+ and I_- are already morphisms of rings we check all defining relations of $U_t(\mathfrak{g})$ and obtain that this composition is already a morphism of $\mathbb{C}$-algebras. This is the main theorem of this thesis.

4.1 Minimal Projective Resolutions

The condition that $\mathcal{A}$ is hereditary implies that all subobjects of projective objects are again projective. Since $\mathcal{A}$ has enough projectives every object A has a projective resolution of the form

$$0 \longrightarrow P \xrightarrow{f} Q \xrightarrow{g} A \longrightarrow 0 \tag{11}$$

with $P \cong \ker(g)$. Condition (Ass1) ensures that $\mathcal{A}$ is a Krull-Schmidt category so we can decompose P and Q into finite direct sums of indecomposable objects $P = \bigoplus_{i \in I} P_i$, $Q = \bigoplus_{j \in J} Q_j$. Hence we may write $f = (f_{i,j})_{i \in I, j \in J}$ in matrix form for certain morphisms $f_{i,j} \colon P_i \to Q_j$.

Definition 4.1. Let $A \in \mathcal{A}$ a resolution (11) is then called *minimal* if none of the morphisms $f_{i,j}$ is an isomorphism.

Lemma 4.2. *Any resolution* (11) *is isomorphic to a resolution of the form*

$$0 \longrightarrow R \oplus P' \xrightarrow{1 \oplus f'} R \oplus Q' \xrightarrow{0 \oplus g'} A \longrightarrow 0$$

with $R \in \mathcal{P}$, *and some minimal resolution*

$$0 \longrightarrow P' \xrightarrow{f'} Q' \xrightarrow{g'} A \longrightarrow 0.$$

Proof. Suppose (11) is not minimal then some $f_{i,j}$ is an isomorphism. Without loss of generality we can assume that $P_i = Q_j$ and $f_{i,j} = \mathrm{id}$. Set $R = P_i = Q_j$, $P' = P/P_i$, and $Q' = Q/Q_j$. Now the short exact sequence of complexes

$$\begin{tikzcd} R \arrow[r, "1"] \arrow[d] & R \arrow[d] \\ P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\ P' \arrow[r, "f'"] & Q' \end{tikzcd}$$

splits and by choosing the correct split we have $f = 1 \oplus f'$. Hence (11) is isomorphic to

$$0 \longrightarrow R \oplus P' \xrightarrow{1 \oplus f'} R \oplus Q' \xrightarrow{0 \oplus g'} A \longrightarrow 0.$$

Now we can repeat this process. It terminates because the sets I and J are finite. □

4.2 Direct Sum Decomposition

We want to generalize the decomposition given in Example 3.22. To do this we have to define the occuring components for an arbitrary category $\mathcal{A}$ satisfying (Ass1)-(Ass4).

Definition 4.3. Given $A \in \mathcal{A}$ choose a minimal projective resolution (11) and define $C_A \in \mathrm{Iso}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ as follows:

$$C_A = \; [Q \underset{f}{\overset{0}{\rightleftarrows}} P]. \tag{12}$$

Remark 4.4. Note that C_A is well-defined by Lemma 4.2 but P, Q, and f are only unique up to isomorphism. By an abuse of notation we will also denote a representative in $[Q \underset{f}{\overset{0}{\rightleftarrows}} P]$ by C_A.

Definition 4.5. Let $\alpha \in K(\mathcal{A})$. Then by Lemma 2.4 there are projective objects $P, Q \in \mathcal{P}$ with $\alpha = \hat{P} - \hat{Q}$. Define:

$$K_\alpha = [\kappa_P^*] * [\kappa_Q^*]^{-1}.$$

To see that K_α is well-defined we prove the following lemma.

Lemma 4.6. *The map*

$$\begin{aligned} \Lambda\colon K(\mathcal{A}) &\to \mathcal{DH}(\mathcal{A})^\times \\ \alpha &\mapsto [\kappa_P^*] * [\kappa_Q^*]^{-1} \quad \textit{for } \alpha = \hat{P} - \hat{Q} \end{aligned}$$

is a well-defined morphism of groups.

Proof. Define a map

$$\begin{aligned} \xi\colon \operatorname{Iso}(\mathcal{A}) &\to \mathcal{DH}(\mathcal{A}) \\ [A] &\mapsto [\kappa_P^*] * [\kappa_Q^*]^{-1} \end{aligned}$$

where

$$0 \longrightarrow Q \longrightarrow P \longrightarrow A \longrightarrow 0$$

is a projective resolution of A. This assignment is independent of the chosen resolution: By Lemma 4.2 there exists a minimal projective resolution

$$0 \longrightarrow Q' \longrightarrow P' \longrightarrow A \longrightarrow 0$$

and an object $R \in \mathcal{A}$ such that $Q \cong Q' \oplus R$ and $P \cong P' \oplus R$. Thus we obtain by Corollary 3.15 that

$$[\kappa_P^*] * [\kappa_Q^*]^{-1} = [\kappa_{P' \oplus R}^*] * [\kappa_{Q' \oplus R}^*]^{-1}$$

$$= [\kappa_{P'}^*] * [\kappa_R^*] * [\kappa_R^*]^{-1} * [\kappa_{Q'}^*]^{-1} = [\kappa_{P'}^*] * [\kappa_{Q'}^*]^{-1}.$$

Since two minimal resolutions are isomorphic we can conclude that ξ is well-defined. Now consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and projective resolutions

$$0 \longrightarrow Q_A \longrightarrow P_A \xrightarrow{p_A} A \longrightarrow 0\,,$$

$$0 \longrightarrow Q_C \longrightarrow P_C \xrightarrow{p_C} C \longrightarrow 0\,.$$

Since g is an epimorphism and P_C is a projective object there exists a morphism $p_C^*\colon P_C \to B$ such that $g \circ p_C^* = p_C$. We obtain the following commutative diagram:

$$\begin{tikzcd}
Q_A \arrow[d] & & & & Q_C \arrow[d] \\
P_A \arrow[rr] \arrow[drr, "f \circ p_A"'] \arrow[d, "p_A"'] & & P_A \oplus P_C \arrow[d, "\zeta"] & & P_C \arrow[ll] \arrow[dll, "p_C^*"] \arrow[d, "p_C"] \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}$$

where $\zeta = (f \circ p_A) \oplus p_C^*$ is an epimorphism. Define $\bar{p}_C^*\colon P_C \to B/f(A)$ and observe that

$$\ker(\zeta) \cong \ker(f \circ p_A) \oplus \ker(\bar{p}_C^*) = \ker(p_A) \oplus \ker(\bar{p}_C^*).$$

On the other hand we have that

$$Q_C \cong \ker(p_C) = \ker(g \circ p_C^*) = \ker(\bar{p}_C^*)$$

and it follows that $\ker(\zeta) \cong Q_A \oplus Q_C$. Hence by taking Corollary 3.15 into account we have that

$$\xi([B]) = [\kappa^*_{P_A \oplus P_C}] * [\kappa^*_{Q_A \oplus Q_C}]^{-1} = [\kappa^*_{P_A}] * [\kappa^*_{P_C}] * [\kappa^*_{Q_A}]^{-1} * [\kappa^*_{Q_C}]^{-1}$$

$$= [\kappa^*_{P_A}] * [\kappa^*_{Q_A}]^{-1} * [\kappa^*_{P_C}] * [\kappa^*_{Q_C}]^{-1} = \xi([A]) * \xi([C]).$$

This proves that Λ is well-defined. $\square$

Remark 4.7. There are some simple but useful identities:

$$K_{\hat{P}} = [\kappa_P^*] = [\kappa_P]^{-1} = K^{-1}_{-\hat{P}} \in \mathcal{DH}_{red}(\mathcal{A})$$

$$K_{P \hat{\oplus} Q} = [\kappa_P^* \oplus \kappa_Q^*] \in \mathcal{DH}(\mathcal{A})$$

$$\hat{C}_A = \hat{Q} - \hat{P} = \hat{A} \in K(A)$$

with P, Q and A as above.

Lemma 4.8. *Every object $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ has a direct sum decomposition*

$$M_\bullet = C_A \oplus C_B^* \oplus \kappa_P \oplus \kappa_Q^*$$

with objects $A, B \in \mathcal{A}$ and $P, Q \in \mathcal{P}$ uniquely determined up to isomorphism.

Proof. Because $\mathcal{A}$ is a Krull-Schmidt category we can assume that

$$M_\bullet = \ (M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1)$$

is already indecomposable. Consider the short exact sequences

$$0 \longrightarrow \ker d_0 \xrightarrow{j} M_0 \xrightarrow{p_0} \operatorname{im}(d_0) \longrightarrow 0$$
$$0 \longrightarrow \ker d_1 \xrightarrow{i_1} M_1 \xrightarrow{q} \operatorname{im}(d_1) \longrightarrow 0.$$

Since $\mathcal{A}$ is hereditary, all objects in these sequences are projective and thus we get two splits

$$i_0\colon \operatorname{im}(d_0) \to M_0, \qquad p_1\colon M_1 \to \ker(d_1)$$

with $p_0 \circ i_0 = \mathrm{id}$ and $p_1 \circ i_1 = \mathrm{id}$. Set

$$N_\bullet = \ (\operatorname{im}(d_0) \underset{0}{\overset{m}{\rightleftarrows}} \ker(d_1)),$$

where m is the obvious inclusion. Now $i_\bullet$ and $p_\bullet$ define morphisms of complexes

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "p_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "p_1"] \\ N_0 \arrow[r, shift left, "m"] & N_1 \arrow[l, shift left, "0"] \end{tikzcd} \qquad \begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] & M_1 \arrow[l, shift left, "d_1"] \\ N_0 \arrow[r, shift left, "m"] \arrow[u, "i_0"] & N_1 \arrow[l, shift left, "0"] \arrow[u, "i_1"'] \end{tikzcd}$$

and moreover $i_\bullet\colon N_\bullet \to M_\bullet$ defines a split of the short exact sequence of complexes

$$0 \longrightarrow \ker p_\bullet \longrightarrow M_\bullet \xrightarrow{p_\bullet} N_\bullet \longrightarrow 0.$$

Hence we have that $M_\bullet \cong N_\bullet \oplus \ker(p_\bullet)$ and since $M_\bullet$ is indecomposable we can conclude that either

$$\ker(d_1) = \operatorname{im}(d_0) = 0 \quad \text{or} \quad \ker(d_0) = \operatorname{im}(d_1) = 0. \tag{13}$$

We assume that the first one holds, so $d_0 = 0$ and d_1 is mono. Then

$$0 \longrightarrow M_1 \xrightarrow{d_1} M_0 \longrightarrow H_0(M_\bullet) \longrightarrow 0 \tag{14}$$

is a projective resolution of $H_0(M_\bullet)$. If it is minimal we have $M_\bullet = C_{H_0(M_\bullet)}$. Now suppose it is not minimal then by Lemma 4.2 there exists a minimal resolution (11) and $R \not\cong 0$ such that (14) is isomorphic to

$$0 \longrightarrow R \oplus P \xrightarrow{1 \oplus f} R \oplus Q \xrightarrow{0 \oplus g} H_0(M_\bullet) \longrightarrow 0.$$

But then it follows that

$$M_\bullet \cong \left(R \underset{1}{\overset{0}{\rightleftarrows}} R\right) \oplus \left(Q \underset{f}{\overset{0}{\rightleftarrows}} P\right),$$

and since $M_\bullet$ is indecomposable and $R \not\cong 0$ we get $P \cong Q \cong 0$ and $M_\bullet \cong \kappa_R^*$. If the second case in (13) holds we have that d_0 is mono and $d_1 = 0$ and we obtain that $M_\bullet$ is either $C^*_{H_0(M_\bullet)}$ or κ_R. $\square$

4.3 The Root Category

Definition 4.9. Define $\mathcal{D}^b(\mathcal{A})$ the ($\mathbb{Z}$-graded) *bounded derived category* of $\mathcal{A}$ to be its bounded homotopy category $\mathcal{H}o^b_{\mathbb{Z}}(\mathcal{A})$ localized at the morphism set of quasi-isomorphisms.

Definition 4.10. For $i \in \mathbb{Z}$ define the *shift functor* $[i]$ on $\mathcal{D}^b(\mathcal{A})$ as follows:

$$\begin{tikzcd}
{} \arrow[r] & A_j \arrow[r, "d_j^A"] \arrow[d, "f_j"] & A_{j+1} \arrow[r] \arrow[d, "f_{j+1}"] & {} \arrow[r, mapsto, "{[i]}"] & {} \arrow[r] & A_{j+i} \arrow[r, "(-1)^i d^A_{j+i}"] \arrow[d, "f_{j+i}"] & A_{j+i+1} \arrow[r] \arrow[d, "f_{j+i+1}"] & {} \\
{} \arrow[r] & B_j \arrow[r, "d_j^B"'] & B_{j+1} \arrow[r] & {} & {} \arrow[r] & B_{j+i} \arrow[r, "(-1)^i d^B_{j+i}"'] & B_{j+i+1} \arrow[r] & {}
\end{tikzcd}$$

for a morphism $f_\bullet \colon A_\bullet \to B_\bullet$ in $\mathcal{D}^b(\mathcal{A})$.

Definition 4.11. Let $\mathcal{R}(\mathcal{A}) = \mathcal{D}^b(\mathcal{A})/[2]$ be the *root category* of $\mathcal{A}$. This has the same objects as $\mathcal{D}^b(\mathcal{A})$, but the morphisms are given by

$$\mathrm{Hom}_{\mathcal{R}(\mathcal{A})}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[2i]).$$

Theorem 4.12. *The category $\mathcal{D}^b(\mathcal{A})$ is equivalent to the bounded homotopy category $\mathcal{H}o^b_{\mathbb{Z}}(\mathcal{P})$ of projectives.*

Proof. See appendix 6.1. $\square$

In particular $\mathcal{R}(\mathcal{A})$ is well defined as the orbit category of $\mathcal{H}o^b_{\mathbb{Z}}(\mathcal{P})$, i.e. $\mathcal{R}(\mathcal{A}) = \mathcal{H}o^b_{\mathbb{Z}}(\mathcal{P})/[2]$.

Lemma 4.13. *There is a fully faithful functor*

$$F \colon \mathcal{R}(\mathcal{A}) \to \mathcal{H}o^b_{\mathbb{Z}_2}(\mathcal{P})$$

sending a $\mathbb{Z}$-graded complex $(P_i)_{i \in \mathbb{Z}}$ of projectives to the $\mathbb{Z}_2$-graded complex

$$\bigoplus_{i \in \mathbb{Z}} P_{2i} \underset{\bigoplus_i d_{2i+1}}{\overset{\bigoplus_i d_{2i}}{\rightleftarrows}} \bigoplus_{i \in \mathbb{Z}} P_{2i+1}.$$

Proof. Consider a morphism $f_\bullet$:

$$\begin{tikzcd}
\bigoplus\limits_{i\in\mathbb{Z}} P_{2i} \arrow[r, shift left, "\bigoplus_i d^P_{2i}"] \arrow[d, "f_0"'] & \bigoplus\limits_{i\in\mathbb{Z}} P_{2i+1} \arrow[l, shift left, "\bigoplus_i d^P_{2i+1}"] \arrow[d, "f_1"] \\
\bigoplus\limits_{i\in\mathbb{Z}} Q_{2i} \arrow[r, shift left, "\bigoplus_i d^Q_{2i}"] & \bigoplus\limits_{i\in\mathbb{Z}} Q_{2i+1}. \arrow[l, shift left, "\bigoplus_i d^Q_{2i+1}"]
\end{tikzcd}$$

Then we can write $f_1 = (f_{2k,2l})_{k,l}$ and $f_2 = (f_{2k+1,2l+1})_{k,l}$ with $f_{i,j}\colon P_i \to Q_j$. Since $f_\bullet$ is a morphism of complexes the following diagram commutes for all $k,l \in \mathbb{Z}$:

$$\begin{tikzcd}
\dots \arrow[r] & P_{2k} \arrow[r, "d^P_{2k}"] \arrow[d, "f_{2k,2l}"] & P_{2k+1} \arrow[r, "d^P_{2k+1}"] \arrow[d, "f_{2k+1,2l+1}"] & P_{2k+2} \arrow[r] \arrow[d, "f_{2k+2,2l+2}"] & \dots \\
\dots \arrow[r] & Q_{2k} \arrow[r, "d^Q_{2k}"'] & Q_{2k+1} \arrow[r, "d^Q_{2k+1}"'] & Q_{2k+2} \arrow[r] & \dots
\end{tikzcd}$$

This shows exactly that $f_{\bullet,2i+\bullet}$ is a morphism of complexes from $P_\bullet$ to $Q_\bullet[2i]$ for all $i \in \mathbb{Z}$. Thus $(f_{\bullet,2i+\bullet})_{i\in\mathbb{Z}}$ is a preimage of $f_\bullet$ under F.
On the other hand suppose that $f_\bullet = F((f_{\bullet,2i+\bullet})_{i\in\mathbb{Z}}) = F((g_{\bullet,2i+\bullet})_{i\in\mathbb{Z}}) = g_\bullet$ in $\mathcal{H}o^b_{\mathbb{Z}_2}(\mathcal{P})$, i.e. there are maps $h_0 = (h_{2k,2l+1})_{k,l}$ and $h_1 = (h_{2k-1,2l})_{k,l}$ such that

$$f_0 - g_0 = (\bigoplus_i d^Q_{2i+1}) \circ h_0 + h_1 \circ (\bigoplus_i d^P_{2i})$$

and

$$f_1 - g_1 = (\bigoplus_i d^Q_{2i}) \circ h_1 + h_0 \circ (\bigoplus_i d^P_{2i+1}).$$

But this is exactly the condition that $f_{\bullet,2i+\bullet}$ is homotopic to $g_{\bullet,2i+\bullet}$ via $h_{\bullet,2i+1+\bullet}$ and thus $(f_{\bullet,2i+\bullet})_{i\in\mathbb{Z}} = (g_{\bullet,2i+\bullet})_{i\in\mathbb{Z}}$ in $\mathcal{R}(\mathcal{A})$. □

Notation 4.14. For $X \in A$ let $X_\bullet \in \mathcal{D}(\mathcal{A})$ denote the complex with

$$X_i = \begin{cases} X, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

Lemma 4.15. *Given $X, Y \in A$ and $i \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X_\bullet, Y_\bullet[i]) = \mathrm{Ext}^i_{\mathcal{A}}(X, Y).$$

Proof. See appendix 6.2. □

4.4 Embedding of $\mathcal{H}(\mathcal{A})$ into $\mathcal{DH}(\mathcal{A})$

Definition 4.16. For $A \in \mathcal{A}$ define

$$E_A = t^{\langle \hat{P}, \hat{A} \rangle} \cdot K_{-\hat{P}} * [C_A] \in \mathcal{DH}(\mathcal{A}). \tag{15}$$

Remark 4.17. In $\mathcal{DH}_{red}(\mathcal{A})$ the following equation holds:

$$E_A = t^{\langle \hat{P}, \hat{A} \rangle} \cdot K_{-\hat{P}} * [C_A] = [\kappa_P \oplus C_A].$$

This follows from Lemma 3.14 and the fact that $K_{-\hat{P}} = [\kappa_P] \in \mathcal{DH}_{red}(\mathcal{A})$ and $\hat{C}_A = \hat{Q} - \hat{P} = \hat{A} \in K(\mathcal{A})$.

Remark 4.18. Note that (15) is independent of the chosen resolution of A. Suppose we take a different, not necessarily minimal, projective resolution. By Lemma 4.2 there exists an $R \in \mathcal{P}$ such that this resolution is isomorphic to

$$0 \longrightarrow R \oplus P \xrightarrow{1 \oplus f} R \oplus Q \xrightarrow{0 \oplus g} A \longrightarrow 0.$$

But then, by Lemma 3.14, we get

$$t^{\langle P \hat{\oplus} R, \hat{A} \rangle} \cdot K_{-P \hat{\oplus} R} * [\kappa_R^* \oplus C_A] = t^{\langle \hat{P}, \hat{A} \rangle} \cdot K_{-\hat{P}} * [C_A].$$

Corollary 4.19. *For $A_1, A_2 \in \mathcal{A}$ there are identities:*

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}) \cong \mathrm{Ext}^1_{\mathcal{A}}(A_1, A_2)$$
$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C^*_{A_2}) \cong \mathrm{Hom}_{\mathcal{A}}(A_1, A_2).$$

Proof. Note that $C_{A_2}[-1] = C_{A_2}[1] = C^*_{A_2}$, then by Lemma 2.23 we have for $i \in \{-1, 0\}$

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}[i]) \cong \mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}[i+1])$$

and by Lemma 4.13

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}[i+1]) \\ &= \mathrm{Hom}_{\mathcal{R}(\mathcal{A})}(C_{A_1}, C_{A_2}[i+1]) \\ &= \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(C_{A_1}, C_{A_2}[i+1]). \end{aligned}$$

Now by Lemma 4.15 we obtain

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(C_{A_1}, C_{A_2}[i+1]) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}((A_1)_\bullet, (A_2)_\bullet[i+1]) \cong \mathrm{Ext}^{i+1}_{\mathcal{A}}(A_1, A_2)$$

since C_{A_j} is quasi-isomorphic to $(A_j)_\bullet$ and $C^*_{A_j}$ is quasi-isomorphic to $(A_j)_\bullet[1]$. $\square$

Lemma 4.20. *For $A_1, A_2 \in \mathcal{A}$ there is a short exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Q_1, P_2) \xrightarrow{\zeta} \mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}) \xrightarrow{\xi} \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to 0.$$

Proof. Assume without loss of generality that f_1 and f_2 are inclusions and define ζ and ξ as follows:

$$\begin{tikzcd}
C_{A_1} \arrow[d, phantom, ""] & = & (Q_1 \arrow[r, shift left, "0"] \arrow[d, "\zeta(h)_0 = f_2 \circ h"'] \arrow[dr, "h"] & P_1) \arrow[l, shift left, "f_1"] \arrow[d, "\zeta(h)_1 = h \circ f_1"] \\
C_{A_2} & = & (Q_2 \arrow[r, shift left, "0"] & P_2), \arrow[l, shift left, "f_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \cong Q_1/P_1 \arrow[d, "\xi(\psi_\bullet) = \bar{\psi}_0"] & Q_1 \arrow[l, "g_1"'] \arrow[r, shift left, "0"] \arrow[d, "\psi_0"] & P_1 \arrow[l, shift left, "f_1"] \arrow[d, "\psi_1"] \\
A_2 \cong Q_2/P_2 & Q_2 \arrow[l, "g_2"] \arrow[r, shift left, "0"] & P_2. \arrow[l, shift left, "f_2"]
\end{tikzcd}$$

Now suppose $\zeta(h) = 0$ then we have $0 = \zeta(h)_0 = f_2 \circ h$ and since f_2 is mono we get $h = 0$. Hence ζ is injective.
On the other hand consider morphisms from $A_1 \cong Q_1/P_1$ to $A_2 \cong Q_2/P_2$. These are exactly those morphisms $\psi\colon Q_1 \to Q_2$ which send P_1 to P_2. But then $\psi_\bullet$ with $\psi_0 = \psi$ and $\psi_1 = \psi|_{P_1}$ defines a morphism from C_{A_1} to C_{A_2}. Hence ξ is surjective.
So finally suppose that $\xi(\psi_\bullet) = 0$. This is equivalent to the fact that ψ_0 factors through P_2 like $\psi_0 = f_2 \circ h$ for some $h\colon Q_1 \to P_2$. But this shows exactly that $\psi_\bullet = \zeta(h)$. Hence $\ker(\xi) = \mathrm{im}(\zeta)$ and thus the sequence is exact. □

Theorem 4.21. *There is an injective morphism of $\mathbb{C}$-algebras*

$$I_+\colon \mathcal{H}_{tw}(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{A}), \quad [A] \mapsto E_A.$$

Proof. We first check that I_+ is a morphism of rings: For $A_1, A_2 \in \mathcal{A}$ define objects $C_{A_1}, C_{A_2} \in \mathcal{C}(\mathcal{P})$ as above. Then by Corollary 3.15 and Remark 4.7

$$\begin{aligned}
E_{A_1} * E_{A_2} &= t^{\langle \hat{P}_1, \hat{A}_1 \rangle + \langle \hat{P}_2, \hat{A}_2 \rangle} \cdot K_{-\hat{P}_1} * [C_{A_1}] * K_{-\hat{P}_2} * [C_{A_2}] \\
&= t^{(\hat{P}_2, \hat{A}_1) + \langle \hat{P}_1, \hat{A}_1 \rangle + \langle \hat{P}_2, \hat{A}_2 \rangle} \cdot K_{-(\hat{P}_1 + \hat{P}_2)} * [C_{A_1}] * [C_{A_2}].
\end{aligned}$$

Set $n = (\hat{P}_2, \hat{A}_1) + \langle \hat{P}_1, \hat{A}_1 \rangle + \langle \hat{P}_2, \hat{A}_2 \rangle$. We claim that any extension of C_{A_1} by C_{A_2} is a complex C_{A_3} where A_3 is the corresponding extension of A_1 by

A_2. So consider the following diagram:

$$\begin{tikzcd}
C_{A_2} \arrow[r, phantom, "="] & (Q_2 \arrow[r, shift left, "0"] \arrow[d, "i"'] & P_2) \arrow[l, shift left, "f_2"] \arrow[d, "i"] \\
& (Q_1 \oplus Q_2 \arrow[r, shift left, "v"] \arrow[d, "p"'] & P_1 \oplus P_2) \arrow[l, shift left, "u"] \arrow[d, "p"] \\
C_{A_1} \arrow[r, phantom, "="] & (Q_1 \arrow[r, shift left, "0"] & P_1). \arrow[l, shift left, "f_1"]
\end{tikzcd}$$

If it is commutative then u is of the form

$$u = \begin{pmatrix} f_1 & 0 \\ s & f_2 \end{pmatrix}$$

and since f_1 and f_2 are mono so is u. But then $u \circ v = 0$ implies that $v = 0$ and this proves the claim.
Now by Lemma 4.20 we have that

$$|\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2})| = |\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)| \cdot |\operatorname{Hom}_{\mathcal{A}}(Q_1, P_2)|$$

$$= |\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)| \cdot t^{2\langle \hat{Q}_1, \hat{P}_2 \rangle}.$$

By putting all this together and taking Corollary 4.19 into account we get:

$$E_{A_1} * E_{A_2} = t^n \cdot K_{-(\hat{P}_1 + \hat{P}_2)} * [C_{A_1}] * [C_{A_2}]$$

$$= t^{n+\langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle} \cdot K_{-(\hat{P}_1+\hat{P}_2)} * \sum_{[C_{A_3}] \in \operatorname{Iso}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))} \frac{|\operatorname{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2})|}{|\operatorname{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2})|} \cdot [C_{A_3}]$$

$$= t^{n+\langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle} \cdot K_{-(\hat{P}_1+\hat{P}_2)} * \sum_{[A_3] \in \operatorname{Iso}(\mathcal{A})} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A_1, A_2)|}{|\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)|} \cdot [C_{A_3}]$$

$$= t^{n+\langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1 + \hat{P}_2, \hat{A}_3 \rangle} \cdot \sum_{[A_3] \in \operatorname{Iso}(\mathcal{A})} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A_1, A_2)|}{|\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)|} \cdot E_{A_3}$$

$$\overset{!}{=} t^{\langle \hat{A}_1, \hat{A}_2 \rangle} \cdot \sum_{[A_3] \in \operatorname{Iso}(\mathcal{A})} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A_1, A_2)|}{|\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)|} \cdot E_{A_3} \tag{16}$$

$$= I_+ \left(t^{\langle \hat{A}_1, \hat{A}_2 \rangle} \cdot \sum_{[A_3] \in \mathrm{Iso}(\mathcal{A})} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A_1, A_2)|}{|\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)|} \cdot [A_3] \right)$$

$$= I_+([A_1] * [A_2]).$$

So it is left to show that equation (16) holds and thus we have to compute the power of t. Note that $\hat{A}_i = \hat{Q}_i - \hat{P}_i \in K(\mathcal{A})$:

$$n + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1 + \hat{P}_2, \hat{A}_3 \rangle$$

$$= (\hat{P}_2, \hat{A}_1) + \langle \hat{P}_1, \hat{A}_1 \rangle + \langle \hat{P}_2, \hat{A}_2 \rangle + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle$$
$$- \langle \hat{P}_1 + \hat{P}_2, \hat{A}_1 + \hat{A}_2 \rangle$$

$$= \langle \hat{P}_2, \hat{A}_1 \rangle + \langle \hat{A}_1, \hat{P}_2 \rangle + \langle \hat{P}_1, \hat{A}_1 \rangle + \langle \hat{P}_2, \hat{A}_2 \rangle + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle$$
$$- \langle \hat{P}_1, \hat{A}_1 \rangle - \langle \hat{P}_1, \hat{A}_2 \rangle - \langle \hat{P}_2, \hat{A}_1 \rangle - \langle \hat{P}_2, \hat{A}_2 \rangle$$

$$= \langle \hat{A}_1, \hat{P}_2 \rangle + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{A}_2 \rangle$$

$$= \langle \hat{Q}_1 - \hat{P}_1, \hat{P}_2 \rangle + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{Q}_2 - \hat{P}_2 \rangle$$

$$= \langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{P}_2 \rangle + \langle \hat{Q}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle - 2\langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle$$

$$= \langle \hat{Q}_1, \hat{Q}_2 \rangle - \langle \hat{Q}_1, \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{Q}_2 \rangle + \langle \hat{P}_1, \hat{P}_2 \rangle$$

$$= \langle \hat{Q}_1, \hat{Q}_2 - \hat{P}_2 \rangle - \langle \hat{P}_1, \hat{Q}_2 - \hat{P}_2 \rangle$$

$$= \langle \hat{Q}_1 - \hat{P}_1, \hat{Q}_2 - \hat{P}_2 \rangle = \langle \hat{A}_1, \hat{A}_2 \rangle.$$

To show injectivity we define a linear map

$$\begin{aligned} Q \colon \mathcal{DH}_{red}(\mathcal{A}) &\to \mathcal{H}_{tw}(\mathcal{A}) \\ [M_\bullet] &\mapsto H_0(M_\bullet) \end{aligned}$$

and compute:

$$Q(I_+([A])) = Q(E_A) = H_0([\kappa_P \oplus C_A])$$

$$= H_0([\ P \oplus Q \underset{0\oplus f}{\overset{1\oplus 0}{\rightleftarrows}} P \oplus P\ ]) = [Q/\operatorname{Im}(f)] = [A].$$

□

Remark 4.22. By composing I_+ with the involution $*$ we obtain another injective ring homomorphism

$$I_- \colon \mathcal{H}_{tw}(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{A}), \quad [A] \mapsto F_A$$

where $F_A = E_A^*$.

Corollary 4.23. *There is an embedding of algebras*

$$I_+^e \colon \mathcal{H}_{tw}^e(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{A})$$

mapping $[A]$ *to* E_A *and* K_α *to* K_α.

Proof. This follows immediately from Theorem 4.21 and the fact that Corollary 3.15 shows that the relations (4) hold in $\mathcal{DH}(\mathcal{A})$. □

Remark 4.24. Again by composing I_+^e with $*$ we obtain another embedding of algebras

$$I_-^e \colon \mathcal{H}_{tw}^e(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{A})$$

sending $[A]$ to F_A and K_α to K_α^*.

4.5 Quantum Enveloping Algebra

Let Γ be a finite graph with vertices $\{1, \dots, n\}$. Let n_{ij} be the number of edges connecting i and j. Assume that $n_{ii} = 0$ for all i. Let $a_{ij} = 2\delta_{ij} - n_{ij}$. Now $(a_{ij})_{ij}$ is a symmetric generalised Cartan matrix with corresponding Kac-Moody Lie algebra $\mathfrak{g}$.

Definition 4.25. Define the quantum enveloping algebra $U_v(\mathfrak{g})$ as an associative algebra over $\mathbb{Q}(v)$, the rational functions in v, generated by symbols E_i, F_i, K_i, K_i^{-1} with respect to the following relations:

$$K_i * K_i^{-1} = K_i^{-1} * K_i = 1, \qquad [K_i, K_j] = 0, \tag{17}$$

$$K_i * E_j = v^{a_{ij}} \cdot E_j * K_i, \qquad K_i * F_j = v^{-a_{ij}} \cdot F_j * K_i, \tag{18}$$

$$[E_i, F_i] = \frac{K_i - K_i^{-1}}{v - v^{-1}}, \qquad [Ei, F_j] = 0 \ \text{ for } i \neq j, \tag{19}$$

and the quantum Serre relations:

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} \cdot E_i^n * E_j * E_i^{1-a_{ij}-n} = 0 \ \text{ for } i \neq j, \tag{20}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix} \cdot F_i^n * F_j * F_i^{1-a_{ij}-n} = 0 \ \text{ for } i \neq j, \tag{21}$$

where the coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! \cdot [n-k]!}, \qquad [n]! = \prod_{i=1}^{n} [i], \qquad [n] = \frac{t^n - t^{-n}}{t - t^{-1}},$$

are quantum binomials.

Definition 4.26. Define subalgebras

$$U_v(\mathfrak{n}^+), U_v(\mathfrak{h}), U_v(\mathfrak{n}^-) \subset U_v(\mathfrak{g})$$

generated by the E_i, the $K_i^{\pm}$, and the F_i respectively. Moreover define

$$U_v(\mathfrak{b}^+), U_v(\mathfrak{b}^-) \subset U_v(\mathfrak{g})$$

generated by $E_i, K_i^{\pm}$ and $F_i, K_i^{\pm}$ respectively.

Lemma 4.27. *The multiplication map*

$$U_v(\mathfrak{n}^+) \otimes_{\mathbb{C}} U_v(\mathfrak{h}) \otimes_{\mathbb{C}} U_v(\mathfrak{n}^-) \to U_v(\mathfrak{g})$$

is an isomorphism of vector spaces.

Proof. See [Lus10, Corollary 3.2.5]. □

4.6 Ringel's Theorem

Let Q be a finite quiver without oriented cycles and Γ its underlying graph. Let $U_t(\mathfrak{g})$ be the quantum enveloping algebra corresponding to the Cartan matrix of Γ but specialized at $t = +\sqrt{q}$ (i.e. the same elements and relations as $U_v(\mathfrak{g})$ but evaluated at $v = t$).

Theorem 4.28. *There are injective morphisms of $\mathbb{C}$-algebras*

$$R\colon U_t(\mathfrak{n}^+) \hookrightarrow \mathcal{H}_{tw}(\mathcal{R}ep_k(Q))$$
$$R^e\colon U_t(\mathfrak{b}^+) \hookrightarrow \mathcal{H}_{tw}(\mathcal{R}ep_k(Q))$$

defined by

$$R(E_i) = R^e(E_i) = \frac{[S_i]}{q-1}, \quad R^e(K_i) = K_{\hat{S}_i}.$$

These maps are isomorphisms precisely if the underlying graph of Q is a simply-laced Dynkin diagram.

Proof. See [Sch06, Theorem 3.16]. □

4.7 Relations

Recall that $F_A = E_A^*$ and $E_A = t^{\langle \hat{P},\hat{A}\rangle} \cdot K_{-\hat{P}} * [C_A]$. We now want to show that the relations between the E_i's and the F_j's in the definition of the quantum enveloping algebra also hold in $\mathcal{DH}(\mathcal{A})$ for the E_A's and the F_B's respectively.

Lemma 4.29. *Suppose $A_1, A_2 \in \mathcal{A}$ satisfy*

$$\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) = 0 = \mathrm{Hom}_{\mathcal{A}}(A_2, A_1)$$

then $[E_{A_1}, F_{A_2}] = 0$.

Proof. Note that $\psi_\bullet \in \mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}^*)$ is of the form

$$\begin{tikzcd} Q_1 \arrow[r, shift left, "0"] \arrow[d, "0"'] & P_1 \arrow[l, shift left, "f_1"] \arrow[d, "\psi_1"] \\ P_2 \arrow[r, shift left, "f_2"] & Q_2 \arrow[l, shift left, "0"] \end{tikzcd}$$

and thus we have

$$\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}^*) = \mathrm{Hom}_{\mathcal{A}}(P_1, Q_2). \tag{22}$$

Moreover it follows from Corollary 4.19 that

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_{A_1}, C_{A_2}^*) \cong \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) = 0.$$

Then by Corollary 3.15 and Remark 4.7 we have

$$E_{A_1} * F_{A_2} = t^{\langle \hat{P}_1,\hat{A}_1\rangle + \langle \hat{P}_2,\hat{A}_2\rangle} \cdot K_{-\hat{P}_1} * [C_{A_1}] * K^*_{-\hat{P}_2} * [C^*_{A_2}]$$

$$= t^{\langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - (\hat{P}_2, \hat{A}_1)} \cdot K_{-\hat{P}_1} * K^*_{-\hat{P}_2} * [C_{A_1}] * [C^*_{A_2}] \tag{23}$$

$$= t^{\langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - (\hat{P}_2, \hat{A}_1) + \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle} \cdot K_{-\hat{P}_1} * K^*_{-\hat{P}_2} * \frac{[C_{A_1} \oplus C^*_{A_2}]}{|\operatorname{Hom}_{\mathcal{A}}(P_1, Q_2)|}$$

$$= t^{\langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - (\hat{P}_2, \hat{A}_1) + \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle} \cdot K_{-\hat{P}_1} * K^*_{-\hat{P}_2} * \frac{[C_{A_1} \oplus C^*_{A_2}]}{t^{2\langle \hat{P}_1, \hat{Q}_2\rangle}}$$

$$= t^{\langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - (\hat{P}_2, \hat{A}_1) - \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle} \cdot K_{-\hat{P}_1} * K^*_{-\hat{P}_2} * [C_{A_1} \oplus C^*_{A_2}].$$

Note that $\hat{A}_i = \hat{Q}_i - \hat{P}_i$ then the total power of t can be rewritten as

$$\langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - (\hat{P}_2, \hat{A}_1) - \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle$$

$$= \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{A}_1, \hat{P}_2\rangle - \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle$$

$$= \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{Q}_1 - \hat{P}_1, \hat{P}_2\rangle - \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle$$

$$= \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{Q}_1, \hat{P}_2\rangle + \langle \hat{P}_1, \hat{P}_2\rangle - \langle \hat{P}_1, \hat{Q}_2\rangle + \langle \hat{Q}_1, \hat{P}_2\rangle$$

$$= \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{P}_1, \hat{Q}_2 - \hat{P}_2\rangle$$

$$= \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{P}_1, \hat{A}_2\rangle$$

and this is invariant under exchanging the indices 1 and 2. Set

$$n = \langle \hat{P}_1, \hat{A}_1\rangle + \langle \hat{P}_2, \hat{A}_2\rangle - \langle \hat{P}_2, \hat{A}_1\rangle - \langle \hat{P}_1, \hat{A}_2\rangle$$

then

$$F_{A_2} * E_{A_1} = (E_{A_2} * F_{A_1})^* = t^n \cdot K^*_{-\hat{P}_2} * K_{-\hat{P}_1} * [C_{A_2} \oplus C^*_{A_1}]^*$$

$$= t^n \cdot K_{-\hat{P}_1} * K^*_{-\hat{P}_2} * [C_{A_1} \oplus C^*_{A_2}] = E_{A_1} * F_{A_2}.$$

$\square$

Observe that the result of the following lemma looks quite different from relation (19). But in view of the definition of

$$R\colon U_t(\mathfrak{g}) \hookrightarrow \mathcal{DH}_{tw}(Rep_k(Q))$$

in Theorem 4.37, this turns out to be exactly the relation we need.

Lemma 4.30. *Suppose $A \in \mathcal{A}$ satisfies* $\mathrm{End}_{\mathcal{A}}(A) = k$*, then*

$$[E_A, F_A] = (q-1) \cdot (K^*_{\hat{A}} - K_{\hat{A}}).$$

Proof. It follows from Corollary 4.19 that

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_A, C_A^*) = \mathrm{Hom}_{\mathcal{A}}(A, A) = k.$$

Note that any endomorphism of A is either an isomorphism or zero. Now any extension of C_A by C_A^* is isomorphic to an extension $M_\bullet$ of the form:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
P \arrow[r, shift left, "f"] \arrow[d, "i_P"'] & Q \arrow[l, shift left, "0"] \arrow[d, "i_Q"] \\
P \oplus Q \arrow[r, shift left, "d_0^M = \begin{bmatrix} f & s_0 \\ 0 & 0 \end{bmatrix}"] \arrow[d, "\pi_Q"'] & Q \oplus P \arrow[l, shift left, "d_1^M = \begin{bmatrix} 0 & s_1 \\ 0 & f \end{bmatrix}"] \arrow[d, "\pi_P"] \\
Q \arrow[r, shift left, "0"] & P. \arrow[l, shift left, "f"]
\end{tikzcd}$$

Since f is mono it follows from $d_0^M \circ d_1^M = 0$ that $s_1 = 0$. Now consider the long exact sequence in cohomology of this extension:

$$\begin{tikzcd}
H^0(C_A^*) = 0 \arrow[r] & H^0(M_\bullet) = \ker(s_0)/\operatorname{im}(f) \arrow[r, "\pi_Q^*"] & H^0(C_A) = A \arrow[dll, "\zeta"] \\
H^1(C_A^*) = A \arrow[r, "i_Q^*"'] & H^1(M_\bullet) = Q/\operatorname{im}([f \;\; s_0]) \arrow[r] & H^1(C_A) = 0.
\end{tikzcd}$$

Note that π_Q^* is mono and i_Q^* is epi. Since $\zeta \in \mathrm{Hom}_{\mathcal{A}}(A, A) = k$ is either an isomorphism or zero it follows that π_Q^* and i_Q^* are either zero or isomorphisms respectively. If they are isomorphisms it follows that $s_0 = 0$ and hence

$M_\bullet = C_A^* \oplus C_A$ is the trivial extension. If they are zero it follows that $M_\bullet$ is acyclic because π_Q^* is mono and i_Q^* is epi. Moreover it turns out that

$$\ker(d_0^M) = \ker(s_0) = P, \quad \text{and} \quad \ker(d_1^M) = Q$$

and by Lemma 2.20 we have $M_\bullet = \kappa_Q \oplus \kappa_P^*$. Thus by (22) and (23)

$$E_A * F_A = t^{\langle \hat{P}, \hat{A} \rangle - \langle \hat{A}, \hat{P} \rangle} \cdot K_{-\hat{P}} * K^*_{-\hat{P}} * [C_A] * [C_A^*]$$

$$= t^{\langle \hat{P}, \hat{A} \rangle - \langle \hat{A}, \hat{P} \rangle + \langle \hat{Q}, \hat{P} \rangle + \langle \hat{P}, \hat{Q} \rangle} \cdot K_{-\hat{P}} * K^*_{-\hat{P}} * \frac{[C_A \oplus C_A^*] + (q-1) \cdot [\kappa_Q \oplus \kappa_P^*]}{t^{2\langle \hat{P}, \hat{Q} \rangle}} \quad (24)$$

$$= t^{\langle \hat{P}, \hat{A} \rangle - \langle \hat{A}, \hat{P} \rangle + \langle \hat{Q}, \hat{P} \rangle - \langle \hat{P}, \hat{Q} \rangle} \cdot K_{-\hat{P}} * K^*_{-\hat{P}} * ([C_A \oplus C_A^*] + (q-1) \cdot K^*_{\hat{Q}} * K_{\hat{P}}).$$

Taking $\hat{A} = \hat{Q} - \hat{P}$ into account the total power of t is:

$$\langle \hat{P}, \hat{A} \rangle - \langle \hat{A}, \hat{P} \rangle + \langle \hat{Q}, \hat{P} \rangle - \langle \hat{P}, \hat{Q} \rangle = \langle \hat{P}, \hat{A} - \hat{Q} \rangle + \langle \hat{Q} - \hat{A}, \hat{P} \rangle$$

$$= -\langle \hat{P}, \hat{P} \rangle + \langle \hat{P}, \hat{P} \rangle = 0.$$

Applying $*$ gives:

$$F_A * E_A = (E_A * F_A)^* = K^*_{-\hat{P}} * K_{-\hat{P}} * ([C_A^* \oplus C_A] + (q-1) \cdot K_{\hat{Q}} * K^*_{\hat{P}})$$

$$= K_{-\hat{P}} * K^*_{-\hat{P}} * ([C_A \oplus C_A^*] + (q-1) \cdot K_{\hat{Q}} * K^*_{\hat{P}}).$$

So finally we get

$$E_A * F_A - F_A * E_A = (q-1) \cdot K_{-\hat{P}} * K^*_{-\hat{P}} * (K^*_{\hat{Q}} * K_{\hat{P}} - K_{\hat{Q}} * K^*_{\hat{P}})$$

$$= (q-1) \cdot (K^*_{\hat{Q}-\hat{P}} - K_{\hat{Q}-\hat{P}}) = (q-1) \cdot (K^*_{\hat{A}} - K_{\hat{A}}).$$

□

4.8 Decomposition Statements

Lemma 4.31. *$\mathcal{DH}(\mathcal{A})$ has a basis*

$$\{\ [C_A \oplus C_B^*] * K_\alpha * K_\beta^* \mid [A],[B] \in \mathrm{Iso}(\mathcal{A}) \textit{ and } \alpha,\beta \in K(\mathcal{A})\ \}.$$

Proof. The basis of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ consists of isomorphism classes $[M_\bullet]$ where $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$. To obtain a basis of $\mathcal{DH}(\mathcal{A})$ we have to take the inverses of the isomorphism classes of acyclic complexes into account. Lemma 2.20 shows that every acyclic complex decomposes into a direct sum $\kappa_P \oplus \kappa_Q^*$ for two objects $P, Q \in \mathcal{P}$. Now by Corollary 3.15 we know that

$$[\kappa_P \oplus \kappa_Q^*]^{-1} = [\kappa_P]^{-1} * [\kappa_Q^*]^{-1}.$$

Moreover we know from Lemma 3.14 that $[\kappa_P]$ and $[\kappa_Q^*]$ commute with other elements $[M_\bullet] \in \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}_2}(\mathcal{P}))$ up to some factors t^n with $n \in \mathbb{Z}$. So the elements $[\kappa_P]^{-1}$ and $[\kappa_Q^*]^{-1}$ do in $\mathcal{DH}(\mathcal{A})$.
Altogether we obtain that a basis of $\mathcal{DH}(\mathcal{A})$ consists of elements

$$[M_\bullet] * [\kappa_{Q_\alpha}^*]^{-1} * [\kappa_{Q_\beta}]^{-1}$$

where $M_\bullet \in \mathcal{C}_{\mathbb{Z}_2}(\mathcal{P})$ and $Q_\alpha, Q_\beta \in \mathcal{P}$. Now by Lemma 4.8 there are elements $A, B \in \mathcal{A}$, and P_α, P_β such that

$$M_\bullet = C_A \oplus C_B^* \oplus \kappa_{P_\alpha}^* \oplus \kappa_{P_\beta}.$$

So finally this gives

$$[M_\bullet] * [\kappa_{Q_\alpha}^*]^{-1} * [\kappa_{Q_\beta}]^{-1} = [C_A \oplus C_B^* \oplus \kappa_{P_\alpha}^* \oplus \kappa_{P_\beta}] * [\kappa_{Q_\alpha}^*]^{-1} * [\kappa_{Q_\beta}]^{-1}$$

$$= t^n \cdot [C_A \oplus C_B^*] * [\kappa_{P_\alpha}^*] * [\kappa_{P_\beta}] * [\kappa_{Q_\alpha}^*]^{-1} * [\kappa_{Q_\beta}]^{-1} = t^n \cdot [C_A \oplus C_B^*] * K_\alpha * K_\beta^*$$

for some $n \in \mathbb{Z}$, $\alpha = P_\alpha - Q_\alpha$ and $\beta = P_\beta - Q_\beta$ in $K(\mathcal{A})$. So a basis of $\mathcal{DH}(\mathcal{A})$ consists of elements

$$[C_A \oplus C_B^*] * K_\alpha * K_\beta^*.$$

□

Definition 4.32. Assumption $(Ass4)$ of Setup 2.6 implies that we have a partial order on $K(\mathcal{A})$ given by:

$$\alpha \leq \beta \Leftrightarrow \exists A \in \mathcal{A}\colon \beta - \alpha = \hat{A}$$

for $\alpha, \beta \in K(\mathcal{A})$.

Definition 4.33. For $\gamma \in K(\mathcal{A})$ define $\mathcal{DH}(\mathcal{A})_{\leq \gamma}$ to be the subspace spanned by those basis elements

$$[C_A \oplus C_B^*] * K_\alpha * K_\beta^*, \qquad A, B \in \mathcal{A}, \quad \alpha, \beta \in K(\mathcal{A})$$

for which $\hat{A} + \hat{B} \leq \gamma$ holds.

Lemma 4.34. *For $\alpha, \beta \in K(\mathcal{A})$ we have*

$$\mathcal{DH}(\mathcal{A})_{\leq \alpha} * \mathcal{DH}(\mathcal{A})_{\leq \beta} \subseteq \mathcal{DH}(\mathcal{A})_{\leq \alpha+\beta}$$

so that this defines a filtration on $\mathcal{DH}(\mathcal{A})$.

Proof. First note that for $A, B \in \mathcal{A}$ and $P, Q \in \mathcal{P}$ we have the following identities:

$$\begin{aligned} &H_0(C_A) = A, \quad H_1(C_A) = 0, \quad H_0(C_B) = 0, \quad H_1(C_B) = B, \\ &H_0(\kappa_P) = 0, \quad H_0(\kappa_Q^*) = 0, \quad H_1(\kappa_P) = 0, \quad H_1(\kappa_Q^*) = 0. \end{aligned}$$

and hence

$$H_0(C_A \oplus C_B^* \oplus \kappa_P \oplus \kappa_Q^*) = A, \quad \text{and} \quad H_1(C_A \oplus C_B^* \oplus \kappa_P \oplus \kappa_Q^*) = B.$$

Now take two elements

$$[C_{A_M} \oplus C_{B_M}^*] * K_{\alpha_M} * K_{\beta_M}^* \in \mathcal{DH}(\mathcal{A})_{\leq \beta}$$

$$[C_{A_N} \oplus C_{B_N}^*] * K_{\alpha_N} * K_{\beta_N}^* \in \mathcal{DH}(\mathcal{A})_{\leq \alpha}$$

where $A_M, A_N, B_M, B_N \in \mathcal{A}$ and $\alpha_M, \alpha_N, \beta_M, \beta_N \in K(\mathcal{A})$. Then multiplication gives

$$([C_{A_M} \oplus C_{B_M}^*] * K_{\alpha_M} * K_{\beta_M}^*) * ([C_{A_N} \oplus C_{B_N}^*] * K_{\alpha_N} * K_{\beta_N}^*)$$

$$= t^n \cdot [C_{A_M} \oplus C_{B_M}^*] * [C_{A_N} \oplus C_{B_N}^*] * K_{\alpha_M+\alpha_N} * K_{\beta_M+\beta_N}^*$$

for some $n \in \mathbb{Z}$. Set

$$M_\bullet = C_{A_M} \oplus C_{B_M}^* \quad \text{and} \quad N_\bullet = C_{A_N} \oplus C_{B_N}^*.$$

Then we have to show that for every extension $P_\bullet \in \mathcal{C}(\mathcal{P})$

$$0 \to M_\bullet \xrightarrow{i_\bullet} P_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0 \tag{25}$$

the following holds:

$$H_0(\hat{P_\bullet}) + H_1(\hat{P_\bullet}) \leq \alpha + \beta.$$

Now (25) gives a long exact sequence in homology which can be split to give two long exact sequences:

$$0 \to K \hookrightarrow H_0(M_\bullet) \xrightarrow{i_0^*} H_0(P_\bullet) \xrightarrow{\pi_0^*} H_0(N_\bullet) \twoheadrightarrow Q \to 0 \tag{26}$$

$$0 \to Q \hookrightarrow H_1(M_\bullet) \xrightarrow{i_1^*} H_1(P_\bullet) \xrightarrow{\pi_1^*} H_1(N_\bullet) \twoheadrightarrow K \to 0$$

where $K = \ker(i_0^*)$ and $Q = H_0(N_\bullet)/\operatorname{im}(\pi_0^*)$. But this gives the following equalities in $K(\mathcal{A})$:

$$\hat{Q} - H_0(\hat{N}_\bullet) + H_0(\hat{P}_\bullet) - H_0(\hat{M}_\bullet) + \hat{K} = 0$$

$$\hat{K} - H_1(\hat{N}_\bullet) + H_1(\hat{P}_\bullet) - H_1(\hat{M}_\bullet) + \hat{Q} = 0.$$

Putting this together gives

$$2(\hat{K} + \hat{Q}) = (H_0(\hat{N}_\bullet) + H_1(\hat{N}_\bullet)) + (H_0(\hat{M}_\bullet) + H_1(\hat{M}_\bullet)) - (H_0(\hat{P}_\bullet) + H_1(\hat{P}_\bullet))$$

$$= (\hat{A}_N + \hat{B}_N) + (\hat{A}_M + \hat{B}_M) - (H_0(\hat{P}_\bullet) + H_1(\hat{P}_\bullet)).$$

Thus we finally get

$$H_0(\hat{P}_\bullet) + H_1(\hat{P}_\bullet) \leq (\hat{A}_N + \hat{B}_N) + (\hat{A}_M + \hat{B}_M) \leq \alpha + \beta.$$

$\square$

Lemma 4.35. *The multiplication map $\mu\colon a \otimes b \mapsto I_+^e(a) * I_-^e(b)$ defines an isomorphism of vector spaces*

$$\mu\colon \mathcal{H}_{tw}^e(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{H}_{tw}^e(\mathcal{A}) \to \mathcal{DH}(\mathcal{A}).$$

Proof. By Corollary 4.19 we have

$$\operatorname{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_A, C_B^*) = \operatorname{Hom}_{\mathcal{A}}(A, B)$$

for $A, B \in \mathcal{A}$. Like in the proof of Lemma 4.30 an extension $L_\bullet$ looks like:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
P_B \arrow[r, shift left, "f_B"] \arrow[d, "i_{P_B}"'] & Q_B \arrow[l, shift left, "0"] \arrow[d, "i_{Q_B}"] \\
P_B \oplus Q_A \arrow[r, shift left, "d_0^L = \begin{bmatrix} f_B & s_0 \\ 0 & 0 \end{bmatrix}"] \arrow[d, "\pi_{Q_A}"'] & Q_B \oplus P_A \arrow[l, shift left, "d_1^L = \begin{bmatrix} 0 & 0 \\ 0 & f_A \end{bmatrix}"] \arrow[d, "\pi_{P_A}"] \\
Q_A \arrow[r, shift left, "0"] & P_A \arrow[l, shift left, "f_A"]
\end{tikzcd}$$

and we obtain a long exact sequence

$$\begin{tikzcd}
H^0(C_A^*) = 0 \arrow[r] & H^0(L_\bullet) = \ker(s_0)/\operatorname{im}(f_A) \arrow[r, "\pi_{Q_A}^*"] & H^0(C_A) = A \arrow[dll, "\zeta"] \\
H^1(C_B^*) = B \arrow[r, "i_{Q_B}^*"'] & H^1(L_\bullet) = Q_B/\operatorname{im}([f_B \ \ s_0]) \arrow[r] & H^1(C_A) = 0
\end{tikzcd}$$

where $\zeta = \bar{s}_0 \colon Q_A/f_A(P_A) \to Q_B/f_B(P_B)$. This yields a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_A \arrow[r, "f_A"] \arrow[d, "0=s_1"'] & Q_A \arrow[r, two heads] \arrow[d, "s_0"] & A = Q_A/f_A(P_A) \arrow[d, "\zeta=\bar{s}_0"] \\
0 \arrow[r] & P_B \arrow[r, "f_B"'] & Q_B \arrow[r, two heads] & B = Q_B/f_B(P_B).
\end{tikzcd}$$

Then for fixed ζ we have by the Comparison Theorem (see [Wei95, Theorem 2.2.6]) that $s_\bullet$ is unique up to chain homotopy. Thus by Lemma 2.23 we can conclude that an extension class in $\operatorname{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})}(C_A, C_B^*)$ is completely determined by the corresponding connecting homomorphism.

Now set $N_\bullet = C_A$ and $M_\bullet = C_B^*$ and consider (26). Then $K = 0$ and it follows from the previous part and condition $(Ass4)$ in Setup 2.6 that $\hat{Q} = 0$ exactly when the extension is trivial. Since $\mathcal{DH}(\mathcal{A})$ is a filtered algebra by Lemma 4.34 we can build the associated graded algebra. On the other hand

$$\mathcal{H}_{tw}(\mathcal{A}) = \bigoplus_{\alpha = \hat{A} \in K(\mathcal{A})} \mathcal{H}_\alpha(\mathcal{A})$$

is already graded where $\mathcal{H}_\alpha(\mathcal{A})$ denotes the subspace spanned by elements $[A]$ with $\hat{A} = \alpha$. Then by definition μ respects these gradings and thus induces a well-defined map of graded objects. We obtain the follwing equation:

$$\mu([A] * K_\alpha \otimes [B] * K_\beta) = t^n [C_A \oplus C_B^*] * K_{\alpha - \hat{P}_A} * K^*_{\beta - \hat{P}_B}$$

for some integer n. Hence in every degree μ takes a basis to a basis and thus $\mu\colon \mathcal{H}^e_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{H}^e_{tw}(\mathcal{A}) \to \mathcal{DH}(\mathcal{A})$ is an isomorphism. □

Lemma 4.36. *The multiplication map* $[A] \otimes \alpha \otimes [B] \mapsto E_A * K_\alpha * F_B$ *defines an isomorphism of vector spaces*

$$\mathcal{H}_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A}) \to \mathcal{DH}_{red}(\mathcal{A}).$$

Proof. This is exactly the same argument as in Lemma 4.35. □

4.9 Main Theorem

Let Q be a finite quiver without oriented cycles, and vertex set $\{1, \ldots, n\}$, and underlying graph Γ. Let $U_t(\mathfrak{g})$ be the quantum enveloping algebra specialized at $t = +\sqrt{q}$ corresponding to the Cartan matrix of Γ. Set $\mathcal{R} = \mathcal{R}ep_k(Q)$.

Theorem 4.37. *There is an injective morphism of* $\mathbb{C}$*-algebras*

$$R\colon U_t(\mathfrak{g}) \hookrightarrow \mathcal{DH}_{red}(\mathcal{R})$$

defined on generators by

$$R(E_i) = (q-1)^{-1} \cdot E_{S_i}, \quad R(F_i) = (-t) \cdot (q-1)^{-1} \cdot F_{S_i},$$

$$R(K_i) = K_{\hat{S}_i}, \qquad\qquad R(K^{-1}) = K^*_{\hat{S}_i}.$$

The map R *is an isomorphism precisely if the underlying graph of* Q *is a simply-laced Dynkin diagram.*

Proof. We first have to check that all defining relations of the quantum enveloping algebra are mapped to zero to have a well-defined map. Now Corollary 4.23 and Remark 4.24 show together with Ringel's Theorem 4.28 that this already holds for the relations (17), (18), (20) and (21). Now we compute with Lemma 4.30:

$$[R(E_i), R(F_i)] = (-t) \cdot (q-1)^{-2} \cdot [E_{S_i}, F_{S_i}]$$

$$= (-t) \cdot (t^2-1)^{-1} \cdot (K^*_{\hat{S}_i} - K_{\hat{S}_i}) = \frac{K_{\hat{S}_i} - K^*_{\hat{S}_i}}{t - t^{-1}} = R\left(\frac{K_i - K_i^{-1}}{t - t^{-1}}\right).$$

This shows together with Lemma 4.29 that the relations (19) are also mapped to zero. Thus R is a well-defined morphism of rings. There is a commutative diagram of vector spaces

$$\begin{tikzcd}
U_t(\mathfrak{n}^+) \otimes_{\mathbb{C}} U_t(\mathfrak{h}) \otimes_{\mathbb{C}} U_t(\mathfrak{n}^-) \arrow[r, "A"] \arrow[d, "\cong"'] & \mathcal{H}_{tw}(\mathcal{R}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{R})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{R}) \arrow[d, "\cong"] \\
U_t(\mathfrak{g}) \arrow[r, "R"'] & \mathcal{DH}_{red}(\mathcal{R})
\end{tikzcd}$$

where the vertical arrows are the isomorphisms given in Lemma 4.27 and Lemma 4.36 and A is built out of Theorem 4.28. Hence it follows from Theorem 4.28 that R is injective in general and an isomorphism precisely if Γ is a simply-laced Dynkin diagram. $\square$

5 Examples of reduced localized Hall Algebras

5.1 The Quiver A_n^τ

Example 5.1. For $n > 3$ define a quiver A_n^τ as follows:

$$A_n^\tau \quad = \quad \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \longrightarrow \dots \longrightarrow \underset{n}{\circ}.$$

Note that A_n^τ and A_n have the same underlying graph Γ. Thus A_n and A_n^τ yield the same quantum enveloping algebra $U_t(\mathfrak{g})$. Since Γ is a simply-laced Dynkin diagram it follows from Theorem 4.37 that

$$\mathcal{DH}_{red}(\mathcal{R}ep_k(A_n)) \cong U_t(\mathfrak{g}) \cong \mathcal{DH}_{red}(\mathcal{R}ep_k(A_n^\tau))$$

as $\mathbb{C}$-algebras. So we can observe in general that two quivers with the same underlying simply-laced Dynkin diagram have isomorphic reduced localized Hall algebras no matter how the arrows are oriented.

5.2 The reduced localized Hall Algebra of $\mathcal{R}ep_k(A_n)$

Example 5.2. Consider the category $\mathcal{R} = \mathcal{R}ep_k(A_n)$. We want to describe a basis of $\mathcal{DH}_{red}(\mathcal{R})$. We can use Lemma 4.31 to see that a basis of $\mathcal{DH}(\mathcal{R})$ is given by

$$\{\ [C_A \oplus C_B^*] * K_\alpha * K_\beta^* \mid [A],[B] \in \mathrm{Iso}(\mathcal{R}) \text{ and } \alpha,\beta \in K(\mathcal{R})\ \}.$$

Now fix $[A],[B] \in \mathrm{Iso}(\mathcal{R})$ and $\alpha,\beta \in K(\mathcal{R})$. By Lemma 2.4 there exist projective objects P_α, Q_α, P_β, $Q_\beta \in \mathcal{R}$ such that $\alpha = \hat{P}_\alpha - \hat{Q}_\alpha$ and $\beta = \hat{P}_\beta - \hat{Q}_\beta$. By taking Remark 3.21, Corollary 3.15 and Lemma 3.14 into account we can calculate in $\mathcal{DH}_{red}(\mathcal{R})$ that

$$[C_A \oplus C_B^*] * K_\alpha * K_\beta^* = [C_A \oplus C_B^*] * [\kappa_{P_\alpha}^*] * [\kappa_{Q_\alpha}^*]^{-1} * [\kappa_{P_\beta}^*] * [\kappa_{Q_\beta}^*]^{-1}$$

$$= [C_A \oplus C_B^*] * [\kappa_{P_\alpha}^*] * [\kappa_{Q_\alpha}] * [\kappa_{P_\beta}^*] * [\kappa_{Q_\beta}] = [C_A \oplus C_B^*] * [\kappa_{P_\alpha}^* \oplus \kappa_{P_\beta}^* \oplus \kappa_{Q_\beta} \oplus \kappa_{Q_\alpha}]$$

$$= [C_A \oplus C_B^*] * [\kappa_{P_\alpha \oplus P_\beta}^* \oplus \kappa_{Q_\alpha \oplus Q_\beta}] = t^n \cdot [C_A \oplus C_B^* \oplus \kappa_{P_\alpha \oplus P_\beta}^* \oplus \kappa_{Q_\alpha \oplus Q_\beta}]$$

for some $n \in \mathbb{Z}$. Moreover the following equation holds in $\mathcal{DH}_{red}(\mathcal{R})$:

$$[\kappa_P \oplus \kappa_P^*] = [\kappa_P] * [\kappa_P^*] = 1$$

for projective objects $P \in \mathcal{R}$. Thus with Example 1.14 we can state the following lemma:

Lemma 5.3. *A basis of $\mathcal{DH}_{red}(\mathcal{R})$ is given by elements*

$$[C_A \oplus C_B^* \oplus \kappa_P \oplus \kappa_Q^*]$$

for $[A],[B] \in \mathrm{Iso}(\mathcal{R})$ and

$$P = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} P(i) \quad \text{and} \quad Q = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} P(i)$$

where $(n_1, \ldots, n_n), (m_1, \ldots, m_n) \in \mathbb{N}^n$ with $0 \in \{n_i, m_i\}$ for all $1 \leq i \leq n$.

Now we want to compute some Hall products in $\mathcal{DH}_{red}(\mathcal{R})$.

- $[C_{S(i)}] * [C_{S(i)}]$:

We use Theorem 4.21 and the results of Example 3.8 to see that

$$t^{(\widehat{P(i+1)},\widehat{S(i)})+2\langle \widehat{P(i+1)},\widehat{S(i)}\rangle} \cdot K_{-2\widehat{P(i+1)}} * [C_{S(i)}] * [C_{S(i)}] = E_{S(i)} * E_{S(i)}$$

$$= I_+([S(i)]) * I_+([S(i)]) = I_+([S(i)] * [S(i)]) = I_+(t^{-1} \cdot [S(i) \oplus S(i)])$$

$$= t^{-1} \cdot E_{S(i) \oplus S(i)} = t^{-1+\langle 2\widehat{P(i+1)}, 2\widehat{S(i)}\rangle} \cdot K_{-2\widehat{P(i+1)}} * [C_{S(i)\oplus S(i)}].$$

Since $\langle \widehat{P(i+1)}, \widehat{S(i)} \rangle = \dim_k \mathrm{Hom}_{\mathcal{R}}(P(i+1), S(i)) = 0$ we just have to compute $\langle \widehat{S(i)}, \widehat{P(i+1)} \rangle$. Choose a short exact projective resolution of $S(i)$

$$0 \longrightarrow P(i+1) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

and consider the corresponding $\mathrm{Hom}_{\mathcal{R}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i), P(i+1)) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(P(i+1), P(i+1)) \longrightarrow 0.$$

Observing that $\mathrm{Hom}_{\mathcal{G}}(P(i), P(i+1)) = 0$ we can conclude that $\mathrm{Ext}^i_{\mathcal{R}}(S(i), P(i+1)) = k$. Moreover $\mathrm{Hom}_{\mathcal{R}}(S(i), P(i+1)) = 0$ and thus $\langle \widehat{S(i)}, \widehat{P(i+1)} \rangle = -1$. Since $K_{-2\widehat{P(i)}}$ is invertible in $\mathcal{DH}_{red}(\mathcal{R})$ we obtain that

$$[C_{S(i)}] * [C_{S(i)}] = [C_{S(i)\oplus S(i)}].$$

- $[C_{S(i)}] * [C^*_{S(j)}]$ for $i > j$:
Since $i \neq j$ we have the same setup as in Lemma 4.29. Thus we can use Equation (23) to see that

$$[C_{S(i)}] * [C_{S(j)}] = t^{\langle P(\hat{i+1}), P(\hat{j})\rangle + \langle P(\hat{i}), P(\hat{j+1})\rangle} \cdot \frac{[C_{S(i)} \oplus C^*_{S(j)}]}{|\operatorname{Hom}_{\mathcal{R}}(P(i+1), P(j))|}.$$

Because $j < i$ we observe that

$$\dim_k \operatorname{Hom}_{\mathcal{R}}(P(i+1), P(j)) = \dim_k \operatorname{Hom}_{\mathcal{R}}(P(i), P(j+1)) = 1$$

and thus

$$[C_{S(i)}] * [C_{S(j)}] = [C_{S(i)} \oplus C^*_{S(j)}].$$

5.3 The reduced localized Hall Algebra of $k[X]$-gmod^{fg}

Example 5.4. Consider the category $\mathcal{M} = k[X]\text{-gmod}^{fg}$. It follows from Proposition 2.11 that one can show analogously to Example 5.2 that the following lemma holds:

Lemma 5.5. *A basis of $\mathcal{DH}_{red}(\mathcal{M})$ is given by elements*

$$[C_A \oplus C^*_B \oplus \kappa_P \oplus \kappa^*_Q]$$

for $[A], [B] \in \operatorname{Iso}(\mathcal{M})$ and

$$P = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=1}^{n_i} k[X]\langle i\rangle \quad \textit{and} \quad Q = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=1}^{m_i} k[X]\langle i\rangle$$

where $(n_i)_{i\in\mathbb{Z}}, (m_i)_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ with $0 \in \{n_i, m_i\} \subset \mathbb{N}$ for all $i \in \mathbb{Z}$.

Now we want to compute some Hall products in $\mathcal{DH}_{red}(\mathcal{M})$.

- $[C_{k\langle i\rangle}] * [C_{k\langle i\rangle}]$:
Recall the embedding

$$J\colon \mathcal{H}_{tw}(\mathcal{R}ep_k(A_n)) \hookrightarrow \mathcal{H}_{tw}(k[X]\text{- gmod}^{fg})$$

of Proposition 3.11. Then use Theorem 4.21 and the results of Example 3.8 to see that

$$t^{((\hat{X})\langle i\rangle, k\hat{\langle i\rangle}) + 2\langle (\hat{X})\langle i\rangle, k\hat{\langle i\rangle}\rangle} \cdot K_{-2((\hat{X})\langle i\rangle)} * [C_{k\langle i\rangle}] * [C_{k\langle i\rangle}] = E_{k\langle i\rangle} * E_{k\langle i\rangle}$$

$$= I_+([k\langle i\rangle]) * I_+([k\langle i\rangle]) = I_+([k\langle i\rangle] * [k\langle i\rangle]) = I_+(J([S(i)] * [S(i)]))$$

$$= I_+(J(t^{-1}\cdot[S(i)\oplus S(i)])) = I_+(t^{-1}\cdot[k\langle i\rangle\oplus k\langle i\rangle])$$

$$= t^{-1}\cdot E_{k\langle i\rangle\oplus k\langle i\rangle} = t^{-1+\langle 2(\widehat{(X)\langle i\rangle}), 2(\widehat{k\langle i\rangle})\rangle}\cdot K_{-2(\widehat{(X)\langle i\rangle})} * [C_{k\langle i\rangle\oplus k\langle i\rangle}].$$

Since $\langle \widehat{(X)\langle i\rangle}, \widehat{k\langle i\rangle}\rangle = \dim_k \mathrm{Hom}_{\mathcal{M}}((X)\langle i\rangle, k\langle i\rangle) = 0$ we just have to compute $\langle \widehat{k\langle i\rangle}, \widehat{(X)\langle i\rangle}\rangle$. Choose a projective resolution of $k\langle i\rangle$

$$0 \longrightarrow (X)\langle i\rangle \longrightarrow k[X]\langle i\rangle \longrightarrow k\langle i\rangle \longrightarrow 0$$

and consider the corresponding $\mathrm{Hom}_{\mathcal{M}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(k[X]\langle i\rangle, (X)\langle i\rangle) \longrightarrow \mathrm{Hom}_{\mathcal{M}}((X)\langle i\rangle, (X)\langle i\rangle) \longrightarrow 0.$$

Observing that $\mathrm{Hom}_{\mathcal{M}}(k[X]\langle i\rangle, (X)\langle i\rangle) = 0$ we can conclude that $\mathrm{Ext}^i_{\mathcal{M}}(k\langle i\rangle, (X)\langle i\rangle) = k$. Moreover $\mathrm{Hom}_{\mathcal{M}}(k\langle i\rangle, (X)\langle i\rangle) = 0$ and thus $\langle \widehat{k\langle i\rangle}, \widehat{(X)\langle i\rangle}\rangle = -1$. Since $K_{-2\widehat{(X)\langle i\rangle}}$ is invertible in $\mathcal{DH}_{red}(\mathcal{M})$ we obtain that

$$[C_{k\langle i\rangle}] * [C_{k\langle i\rangle}] = [C_{k\langle i\rangle\oplus k\langle i\rangle}].$$

- $[C_k] * [C_k^*]$:
Since $\mathrm{Hom}_{\mathcal{M}}(k,k) = k$ we have the same setup as in Lemma 4.30. Thus we can use Equation (24) to see that

$$[C_k] * [C_k^*] = t^{\langle \widehat{k[X]}, (\hat{X})\rangle + \langle (\hat{X}), \widehat{k[X]}\rangle} \cdot \frac{[C_k \oplus C_k^*] + (q-1)\cdot[\kappa_{k[X]} \oplus \kappa^*_{(X)}]}{t^{2\langle (\hat{X}), \widehat{k[X]}\rangle}}.$$

Observe that

$$\dim_k \mathrm{Hom}_{\mathcal{M}}(k[X], (X)) = 0 \quad \text{and} \quad \dim_k \mathrm{Hom}_{\mathcal{M}}((X), k[X]) = 1$$

and thus

$$[C_k] * [C_k^*] = t^{-1}\cdot([C_k \oplus C_k^*] + (q-1)\cdot[\kappa_{k[X]} \oplus \kappa^*_{(X)}]).$$

5.4 The reduced localized Hall Algebra of $\mathcal{G}$

In contrast to the previous examples we cannot apply the results of section 4 to the category $\mathcal{G}$. We will see that it is quite difficult to find a basis of $\mathcal{DH}_{red}(\mathcal{G})$ and to describe the multiplication.

Notation 5.6. Let $f\colon X \to Y$ be a morphism of representations and $x \in X_i$. In order to make the following part more readable we sometimes write $f(x)$ instead of $f_i(x)$.

Example 5.7. Consider the category $\mathcal{G}$. We want to describe a basis of $\mathcal{DH}_{red}(\mathcal{G})$. So take any complex of projectives

$$M_\bullet = \; P \underset{d_1}{\overset{d_0}{\rightleftarrows}} Q.$$

We want to find all $\kappa_{P'}$ and $\kappa^*_{P'}$ which are direct summands in $M_\bullet$. By Proposition 2.14 we know that P and Q are finite direct sums of $P(1)$'s and $P(2)$'s. In particular we have that P_α and Q_α are injective maps. Now by Lemma 2.13 we have that

$$d_0(P) = \bigoplus_{i=1}^{l} X_i \quad \text{with } [X_i] \in \{[I(1)], [S(1)], [S(2)], [P(1)], [P(2)]\}$$

for some $l \in \mathbb{N}$. Moreover since P_α and Q_α are injective we can conclude that

$$d_0(P) = \bigoplus_{i=1}^{n_s} S(2) \oplus \bigoplus_{i=1}^{n_1} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2)$$

for some $n_s, n_1, n_2 \in \mathbb{N}$. Define a projection map

$$p\colon d_0(P) \twoheadrightarrow \bigoplus_{i=1}^{n_1} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2)$$

and observe that $p \circ d_0$ is an epimorphism. Since $\bigoplus_{i=1}^{n_1} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2)$ is projective this map splits and we obtain

$$P = \ker(p \circ d_0) \oplus \bigoplus_{i=1}^{n_1} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2).$$

These $P(i)$'s are the candidates for the desired direct summands in $M_\bullet$. Let $(e_2)_1$ be the generator of the first $P(2)$-summand. As mentioned above there are $n, m \in \mathbb{N}$ such that

$$Q = \bigoplus_{i=1}^{n} P(1) \oplus \bigoplus_{i=1}^{m} P(2).$$

Thus we can write

$$d_0((e_2)_1) = (\lambda_1^1\alpha, \ldots, \lambda_n^1\alpha, \mu_1^1 e_2 + \nu_1^1\alpha\beta, \ldots, \mu_m^1 e_2 + \nu_m^1\alpha\beta)$$

for some $\lambda_i^1, \mu_i^1, \nu_i^1 \in k$. Since $d_0((e_2)_1)$ is supposed to be a generator of a submodule isomorphic to $P(2)$ in Q there has to be a $\mu_j \neq 0$. Without loss of generality we can assume that $j = m$ and we obtain a new direct sum decomposition

$$Q = \bigoplus_{i=1}^{n} P(1) \oplus \bigoplus_{i=1}^{m-1} P(2) \oplus \langle \zeta_1^2 \rangle_{\mathcal{G}}$$

where $\zeta_i^2 = d_0((e_2)_i)$ and $\langle z \rangle_{\mathcal{G}}$ is the submodule generated by z. Now consider

$$\zeta_2^2 = (\lambda_1^2 \alpha, \dots, \lambda_n^2 \alpha, \mu_1^2 e_2 + \nu_1^2 \alpha\beta, \dots, \mu_{m-1}^2 e_2 + \nu_{m-1}^2 \alpha\beta, \mu_m^2 \zeta_1^2 + \nu_m^2 Q_{\alpha\beta}(\zeta_1^2))$$

for some $\lambda_i^2, \mu_i^2, \nu_i^2 \in k$. Then as above there exists a $\mu_j^2 \neq 0$ and moreover we claim that there is such a j with $j < m$. This is true because otherwise we obtain that

$$Q_{\alpha\beta}(\zeta_2^2) = \mu_m^2 Q_{\alpha\beta}(\zeta_1^2)$$

and this is a contradiction to the direct sum decomposition of $d_0(P)$. Without loss of generality assume that $j = m - 1$ and we have a new direct sum decomposition

$$Q = \bigoplus_{i=1}^{n} P(1) \oplus \bigoplus_{i=1}^{m-2} P(2) \oplus \bigoplus_{i=1}^{2} \langle \zeta_i^2 \rangle_{\mathcal{G}}$$

and by induction we can decompose Q as follows

$$Q = \bigoplus_{i=1}^{n} P(1) \oplus \bigoplus_{i=1}^{m-n_2} P(2) \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}.$$

Note that one can embed $P(1)$ in $P(2)$ via $e_1 \mapsto \beta$ for example. Thus we have to pay a bit more attention to the $P(1)$-summands. Define $\zeta_i^1 = d_0((e_1)_i)$ as above and write

$$\zeta_i^1 = (x_1^i e_1, \dots, x_n^i e_1, y_1^i \beta, \dots, y_{m-n_2}^i \beta, z_1^i Q_\beta(\zeta_i^2), \dots, z_{n_2}^i Q_\beta(\zeta_i^2))$$

where $x_j^i, y_j^i, z_j^i \in k$. Consider the vectors $x^i = (x_1^i, \dots, x_n^i) \in k^n$ and choose a maximal linearly independent set of these vectors. Without loss of generality assume that this is given by

$$\{x^1, \dots, x^{\tilde{n}}\}$$

for some $\tilde{n} \in \mathbb{N}$. In particular we have that $x_j^1 \neq 0$ for some $1 \leq j \leq n$. Without loss of generality assume that $j = m$. Then we have a new direct sum decomposition

$$Q = \bigoplus_{i=1}^{n-1} P(1) \oplus \langle \zeta_1^1 \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{m-n_2} P(2) \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}$$

and with the same arguments as above we obtain

$$Q = \bigoplus_{i=1}^{n-\tilde{n}} P(1) \oplus \bigoplus_{i=1}^{\tilde{n}} \langle \zeta_i^1 \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{m-n_2} P(2) \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}.$$

Our goal is to show that

$$\bigoplus_{i=1}^{\tilde{n}} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2) \underset{0}{\overset{d_0}{\rightleftarrows}} \bigoplus_{i=1}^{\tilde{n}} \langle \zeta_i^1 \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}$$

is a direct summand in $M_\bullet$. Now consider

$$\zeta_{\tilde{n}+1}^1 = (\tilde{x}^{\tilde{n}+1} e_1, \tilde{w}^{\tilde{n}+1} \cdot \zeta^1, \tilde{y}^{\tilde{n}+1} \beta, \tilde{z}^{\tilde{n}+1} \cdot Q_\beta(\zeta^2)) \in Q$$

where

$$\tilde{x}^{\tilde{n}+1} \in k^{n-\tilde{n}}, \tilde{w}^{\tilde{n}+1} \in k^{\tilde{n}}, \tilde{y}^{\tilde{n}+1} \in k^{m-n_2}, \tilde{z}^{\tilde{n}+1} \in k^{n_2}$$

and

$$\zeta^1 = (\zeta_1^i, \ldots, \zeta_{\tilde{n}}^1) \quad \text{and} \quad Q_\beta(\zeta^2) = (Q_\beta(\zeta_1^2), \ldots, Q_\beta(\zeta_{n_2}^2)).$$

Moreover $\cdot$ is defined as follows:

$$\tilde{w}^{\tilde{n}+1} \cdot \zeta^1 = (\tilde{w}_1^{\tilde{n}+1} \zeta_1^1, \ldots, \tilde{w}_{\tilde{n}}^{\tilde{n}+1} \zeta_{\tilde{n}}^1).$$

We will stick to this notation. Since $\{x^1, \ldots, x^{\tilde{n}}\}$ is a maximal linear independent set, we know that $\tilde{x}^{\tilde{n}+1} = 0$ and $\tilde{w}^{\tilde{n}+1}$ lies in the span of $x^1, \ldots, x^{\tilde{n}}$. Thus there are $\gamma_1^{\tilde{n}+1}, \ldots, \gamma_{\tilde{n}}^{\tilde{n}+1} \in k$ such that

$$\tilde{w}^{\tilde{n}+1} = \gamma_1^{\tilde{n}+1} x^1 + \ldots + \gamma_{\tilde{n}}^{\tilde{n}+1} x^{\tilde{n}}$$

and define

$$g_{\tilde{n}+1} = (e_1)_{\tilde{n}+1} - (\gamma_1^{\tilde{n}+1} (e_1)_1 + \ldots + \gamma_{\tilde{n}}^{\tilde{n}+1} (e_1)_{\tilde{n}}) \in P.$$

We obtain a new decomposition of P

$$P = \ker(p \circ d_0) \oplus \bigoplus_{i=1}^{\tilde{n}} P(1) \oplus \langle g_{\tilde{n}+1} \rangle_{\mathcal{G}} \oplus \bigoplus_{i=\tilde{n}+2}^{n_1} P(1) \oplus \bigoplus_{i=1}^{n_2} P(2).$$

By induction we can decompose P as follows:

$$P = \ker(p \circ d_0) \oplus \bigoplus_{i=1}^{\tilde{n}} P(1) \oplus \bigoplus_{i=\tilde{n}+1}^{n_1} \langle g_i \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_2} P(2)$$

where the g_i are defined analogously to $g_{\tilde{n}+1}$. By construction we know that

$$\xi_i = d_0(g_i) \in \bigoplus_{i=1}^{m-n_2} P(2) \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}.$$

Thus we can write

$$\xi_{\tilde{n}+1} = (0, 0, \bar{y}^{\tilde{n}+1}\beta, \bar{z}^{\tilde{n}+1} \cdot Q_\beta(\zeta^2))$$

where $\bar{y}^{\tilde{n}+1} \in k^{m-n_2}$ and $\bar{z}^{\tilde{n}+1} \in k^{n_2}$. We claim that there is a $\bar{y}_j^{\tilde{n}+1} \neq 0$ for $1 \leq j \leq m - n_2$. This is true because otherwise we have that

$$\xi_{\tilde{n}+1} \in \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}$$

and this is a contradiction to the direct sum decomposition of $d_0(P)$. Assume without loss of generality that $j = m - n_2$. Now we have the problem that $\langle \xi_{\tilde{n}+1} \rangle_{\mathcal{G}}$ is no direct summand in Q. Thus we define an element $\omega_{\tilde{n}+1}$ corresponding to $\xi_{\tilde{n}+1}$ as follows:

$$\omega_{\tilde{n}+1} = (0, 0, \bar{y}^{\tilde{n}+1} e_2, \bar{z}^{\tilde{n}+1} \cdot \zeta^2).$$

Then $\langle \omega_{\tilde{n}+1} \rangle_{\mathcal{G}}$ is isomorphic to $P(2)$ and due to the direct sum decomposition of $d_0(P)$ it is a direct summand in Q:

$$Q = \bigoplus_{i=1}^{n-\tilde{n}} P(1) \oplus \bigoplus_{i=1}^{\tilde{n}} \langle \zeta_i^1 \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{m-n_2-1} P(2) \oplus \langle \omega_{\tilde{n}+1} \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}.$$

By induction we obtain with the same arguments as above that

$$Q = \bigoplus_{i=1}^{n-\tilde{n}} P(1) \oplus \bigoplus_{i=1}^{\tilde{n}} \langle \zeta_i^1 \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{\tilde{m}} P(2) \oplus \bigoplus_{i=\tilde{n}+1}^{n_1} \langle \omega_i \rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_2} \langle \zeta_i^2 \rangle_{\mathcal{G}}$$

where $\tilde{m} = (m - n_2) - (n_1 - \tilde{n})$. Now we have to take care of the direct summand $\bigoplus_{i=0}^{n_s} S(2)$ in $d_0(P)$. Due to the direct sum decomposition of $d_0(P)$ we can choose a basis $\sigma_1, \ldots, \sigma_{\bar{n}}, \tau_1, \ldots, \tau_{n_s - \bar{n}}$ of $(\bigoplus_{i=0}^{n_s} S(2))_2$ for some $\bar{n} \in \mathbb{N}$ which is of the following form:

$$\sigma_i = (a^i \alpha, 0 \cdot Q_\alpha(\zeta^1), 0\alpha\beta, 0 \cdot Q_\alpha(\xi), 0 \cdot Q_{\alpha\beta}(\zeta^2))$$

and

$$\tau_i = (0\alpha, 0 \cdot Q_\alpha(\zeta^1), b^i\alpha\beta, 0 \cdot Q_\alpha(\xi), 0 \cdot Q_{\alpha\beta}(\zeta^2))$$

where

$$a^i \in k^{n-\tilde{n}} \quad \text{and } b^i \in k^{\tilde{m}}.$$

Again $\langle\sigma_i\rangle_{\mathcal{G}}$ and $\langle\tau_i\rangle_{\mathcal{G}}$ define no direct summands in Q. Hence define corresponding elements

$$\bar{\sigma}_i = (a^i e_1, 0, 0, 0, 0)$$

and

$$\bar{\tau}_i = (0, 0, b^i e_2, 0, 0).$$

Then $\langle\bar{\sigma}_i\rangle_{\mathcal{G}} \cong P(1)$ and $\langle\bar{\tau}_i\rangle_{\mathcal{G}} \cong P(2)$ and without loss of generality and with the same arguments as always we obtain our final decomposition of Q:

$$\bigoplus_{i=1}^{\bar{n}}\langle\bar{\sigma}_i\rangle \oplus \bigoplus_{i=\bar{n}+1}^{n-\tilde{n}} P(1) \oplus \bigoplus_{i=1}^{\tilde{n}}\langle\zeta_i^1\rangle \oplus \bigoplus_{i=1}^{n_s-\bar{n}}\langle\bar{\tau}_i\rangle \oplus \bigoplus_{i=n_s-\bar{n}+1}^{\tilde{m}} P(2) \oplus \bigoplus_{i=\tilde{n}+1}^{n_1}\langle\omega_i\rangle \oplus \bigoplus_{i=1}^{n_2}\langle\zeta_i^2\rangle.$$

Now consider

$$P = \ker(p \circ d_0) \oplus \bigoplus_{i=1}^{\tilde{n}} P(1) \oplus \bigoplus_{i=\tilde{n}+1}^{n_1}\langle g_i\rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_2} P(2) \xrightarrow{d_0} Q\ .$$

Altogether we know by construction that

$$d_0(\ker(p \circ d_0)) \subseteq \bigoplus_{i=1}^{\bar{n}}\langle\bar{\sigma}_i\rangle_{\mathcal{G}} \oplus \bigoplus_{i=1}^{n_s-\bar{n}}\langle\bar{\tau}_i\rangle_{\mathcal{G}},$$

$$d_0\colon \bigoplus_{i=1}^{\tilde{n}} P(1) \xrightarrow{\cong} \bigoplus_{i=1}^{\tilde{n}}\langle\zeta_i^1\rangle_{\mathcal{G}},$$

$$d_0(\bigoplus_{i=\tilde{n}+1}^{n_1}\langle g_i\rangle_{\mathcal{G}}) \subseteq \bigoplus_{i=\tilde{n}+1}^{n_1}\langle\omega_i\rangle_{\mathcal{G}},$$

$$d_0\colon \bigoplus_{i=1}^{n_2} P(2) \xrightarrow{\cong} \bigoplus_{i=1}^{n_2}\langle\zeta_i^2\rangle_{\mathcal{G}}.$$

Since $M_\bullet$ is a complex we have that $d_1 \circ d_0 = 0$ and thus

$$d_1(\bigoplus_{i=1}^{\tilde{n}}\langle\zeta_i^1\rangle_{\mathcal{G}}) = 0 \quad \text{and} \quad d_1(\bigoplus_{i=1}^{n_2}\langle\zeta_i^2\rangle_{\mathcal{G}}) = 0.$$

Hence we can finally conclude that

$$\bigoplus_{i=1}^{\tilde{n}} P(1) \underset{0}{\overset{d_0}{\rightleftarrows}} \bigoplus_{i=1}^{\tilde{n}}\langle\zeta_i^1\rangle_{\mathcal{G}}$$

and

$$\bigoplus_{i=1}^{n_2} P(2) \underset{0}{\overset{d_0}{\rightleftarrows}} \bigoplus_{i=1}^{n_2}\langle\zeta_i^2\rangle_{\mathcal{G}}$$

are direct summands in $M_\bullet$ which are isomorphic to some $\kappa_{P'}$. Analogously we can find direct summands which are isomorphic to some $\kappa^*_{P'}$.
Now consider the complement in P:

$$K = \ker(p \circ d_0) \oplus \bigoplus_{i=\tilde{n}+1}^{n_1} \langle g_i\rangle_{\mathcal{G}}.$$

Then by construction

$$(d_0)_1(K_1) \subseteq Q_\beta(Q_2) \quad \text{and} \quad (d_0)_2(K_2) \subseteq Q_\alpha(Q_1).$$

This shows that we found all direct summands which are isomorphic to some $\kappa_{P'}$. Hence we can formulate the following lemma:

Lemma 5.8. *A basis of $\mathcal{DH}_{red}(\mathcal{G})$ is given by elements*

$$[(M \underset{d_1}{\overset{d_0}{\rightleftarrows}} N) \oplus \kappa_P \oplus \kappa_Q^*]$$

for isomorphism classes $[\ M \underset{d_1}{\overset{d_0}{\rightleftarrows}} N\ ]$ of those complexes $M \underset{d_1}{\overset{d_0}{\rightleftarrows}} N$ of projectives with

$$(d_0)_1(M_1) \subseteq N_\beta(N_2), \qquad (d_0)_2(M_2) \subseteq N_\alpha(N_1),$$

$$(d_1)_1(N_1) \subseteq M_\beta(M_2), \qquad (d_1)_2(N_2) \subseteq M_\alpha(M_1)$$

and for

$$P = \bigoplus_{i=1}^{2}\bigoplus_{j=1}^{n_i} P(i) \quad \textit{and} \quad Q = \bigoplus_{i=1}^{2}\bigoplus_{j=1}^{m_i} P(i)$$

where $(n_1, n_2), (m_1, m_2) \in \mathbb{N}^2$ *with* $0 \in \{n_i, m_i\}$ *for* $i = 1, 2$.

Now we want to compute a Hall product in $\mathcal{DH}_{red}(\mathcal{G})$. Since $S(2)$ is the only indecomposable object with projective dimension 1 we compute

- $[C_{S(2)}] * [C_{S(2)}]$:
We first compute $[S(2)] * [S(2)] \in \mathcal{H}_{tw}(\mathcal{G})$. Consider the projective resolution

$$0 \longrightarrow P(1) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0\,.$$

This yields a corresponding $\mathrm{Hom}_{\mathcal{G}}$-sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(2), S(2)) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P(1), S(2)) \longrightarrow 0\,.$$

Since every morphism $f\colon P(1) \to S(2)$ sends the generator e_1 to zero it follows that $\mathrm{Hom}_{\mathcal{G}}(P(1), S(2)) = 0$ and hence $\mathrm{Ext}^1_{\mathcal{G}}(S(2), S(2)) = 0$. Moreover we have that $\mathrm{Hom}_{\mathcal{G}}(S(2), S(2)) = k$ and thus

$$[S(2)] * [S(2)] = t^{\langle \hat{S(2)}, \hat{S(2)} \rangle} \cdot |\,\mathrm{Hom}_{\mathcal{G}}(S(2), S(2))|^{-1} \cdot [S(2) \oplus S(2)]$$

$$= t^{1-0} \cdot q^{-1} \cdot [S(2) \oplus S(2)] = t^{-1} \cdot [S(2) \oplus S(2)].$$

Since $P(1)$ and $P(2)$ are projective every extension of $C_{S(2)}$ by $C_{S(2)}$ is of the form

$$\begin{array}{ccc}
C_{S(2)} & = & \left(P(2) \underset{f}{\overset{0}{\rightleftarrows}} P(1)\right) \\
 & & \quad {\scriptstyle i}\downarrow \qquad\qquad \downarrow{\scriptstyle i} \\
 & & \left(P(2) \oplus P(2) \underset{u}{\overset{v}{\rightleftarrows}} P(1) \oplus P(1)\right) \\
 & & \quad {\scriptstyle p}\downarrow \qquad\qquad \downarrow{\scriptstyle p} \\
C_{S(2)} & = & \left(P(2) \underset{f}{\overset{0}{\rightleftarrows}} P(1)\right)
\end{array}$$

where u is given by

$$u = \begin{pmatrix} f & 0 \\ s & f \end{pmatrix}$$

for some $s\colon P(1) \to P(2)$. Since f is monomorph we can conclude that u is monomorphic and hence $v = 0$ since $u \circ v = 0$. Thus every extension of $C_{S(2)}$ by $C_{S(2)}$ is given by C_A where A is an extension of $S(2)$ by $S(2)$. As shown above this is only the trivial extension. Moreover it follows from the proof that Lemma 4.20 applies for all objects A_1, A_2 with projective dimension 1. Thus we have that

$$|\operatorname{Hom}_{\mathcal{G}}(C_{S(2)}, C_{S(2)})| = |\operatorname{Hom}_{\mathcal{G}}(S(2), S(2))| \cdot |\operatorname{Hom}_{\mathcal{G}}(P(2), P(1))| = q^2.$$

Then we can compute

$$[C_{S(2)}] * [C_{S(2)}] = t^{\langle P(2),P(2)\rangle + \langle P(1),P(1)\rangle} \cdot |\operatorname{Hom}_{\mathcal{G}}(C_{S(2)}, C_{S(2)})|^{-1} \cdot [C_{S(2)\oplus S(2)}]$$

$$= t^{2+1} \cdot q^{-2} \cdot [C_{S(2)\oplus S(2)}] = t^{-1} \cdot [C_{S(2)\oplus S(2)}].$$

5.5 Embedding of $\mathcal{DH}_{red}(\mathcal{R}ep_k(A_n))$ into $\mathcal{DH}_{red}(k[X]\text{-}\operatorname{gmod}^{fg})$

Observering that the calculations in Example 5.2 are quite similar to those in Example 5.4 one could ask if there is a statement like Proposition 3.11 for reduced localized Hall algebras. And indeed we can finally state the following theorem.

Theorem 5.9. *Let Γ be the underlying graph of A_n. Let $U_t(\mathfrak{g})$ be the quantum enveloping algebra specialized at $t = +\sqrt{q}$ corresponding to the Cartan matrix of Γ. There is an injective morphism of $\mathbb{C}$-algebras*

$$L\colon U_t(\mathfrak{g}) \hookrightarrow \mathcal{DH}_{red}(k[X]\text{-}\operatorname{gmod}^{fg})$$

defined on generators by

$$L(E_i) = (q-1)^{-1} \cdot E_{k\langle i\rangle}, \quad L(F_i) = (-t)\cdot (q-1)^{-1} \cdot F_{k\langle i\rangle},$$

$$L(K_i) = K_{k\hat{\langle i\rangle}}, \qquad\qquad L(K^{-1}) = K^*_{k\hat{\langle i\rangle}}.$$

In particular L yields an embedding

$$\mathcal{DH}_{red}(\mathcal{R}ep_k(A_n)) \hookrightarrow \mathcal{DH}_{red}(k[X]\text{-}\operatorname{gmod}^{fg}).$$

Proof. It follows from Proposition 2.10 that we can apply the results of sections 4 for $k[X]\text{-}\operatorname{gmod}^{fg}$ and thus like in Theorem 4.37 we can conclude that L is a well-define morphism of rings. Set $\mathcal{M} = k[X]\text{-}\operatorname{gmod}^{fg}$ and

$\mathcal{R} = \mathcal{R}ep_k(A_n)$. Now there is a commutative diagram of vector spaces

$$\begin{tikzcd}
U_t(\mathfrak{n}^+) \otimes_{\mathbb{C}} U_t(\mathfrak{h}) \otimes_{\mathbb{C}} U_t(\mathfrak{n}^-) \arrow[r, "A"] \arrow[d, "\cong"'] & \mathcal{H}_{tw}(\mathcal{R}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{R})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{R}) \arrow[d, "(J \otimes \hat{i} \otimes J)"] \\
U_t(\mathfrak{g}) \arrow[d, "R"'] \arrow[dr, "L"] & \mathcal{H}_{tw}(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[K(\mathcal{M})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{M}) \arrow[d, "\cong"] \\
\mathcal{DH}_{red}(\mathcal{R}) \arrow[r, "L \circ R^{-1}"'] & \mathcal{DH}_{red}(\mathcal{M})
\end{tikzcd}$$

where A, R and the vertical isomorphisms are defined as in Theorem 4.37, and J and $\hat{\imath}$ are defined as in Proposition 3.11 and Proposition 3.12 respectively. Since Γ is a simply-laced Dynkin diagramm it follows from Theorem 4.28 and Theorem 4.37 that the maps R and A are isomorphisms. Now taking Proposition 3.11 and Proposition 3.12 into account we can conclude that $(J \otimes \hat{i} \otimes J)$ is injective and thus L is injective. $\square$

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6 Appendix

In this section we will give the prove of two commonly known results from the theory of derived categories. Since the proofs are quite lengthy and they are of no further interest to the rest of this thesis it was decided to outsource them.
The main reference for part A6.1 is [Bue07, Lemma 9.7] whereas the main reference for part A6.2 is [Mur06, Lemma 28].

A 6.1. *The category $\mathcal{D}^b(\mathcal{A})$ is equivalent to the bounded homotopy category $\mathcal{H}o^b_{\mathbb{Z}}(\mathcal{P})$ of projectives.*

Proof. We prove that the functor $\iota\colon \mathcal{H}o^b_{\mathbb{Z}}(\mathcal{P}) \to \mathcal{D}^b(\mathcal{A})$ is full, faithful and essential surjective.

(a) ι is essential surjective:

Proof of (a). Let

$$\ldots \longrightarrow 0 \longrightarrow A_m \xrightarrow{d^A_m} A_{m+1} \longrightarrow \ldots \longrightarrow A_{n-1} \xrightarrow{d^A_{n-1}} A_n \longrightarrow 0 \longrightarrow \ldots$$

be a bounded complex with $A_i = 0$ if $i < m$ or $i > n$. Since $\mathcal{A}$ has enough projectives we can choose $P_n \in \mathcal{P}$ with $\bar{\alpha}_n\colon P_n \twoheadrightarrow A_n$ and build the following pullback square:

$$\begin{tikzcd}
& A_{n-1} \arrow[dr, "d^A_{n-1}"] & \\
\tilde{A}_{n-1} \arrow[ur, "\tilde{\alpha}_{n-1}"] \arrow[dr, "\tilde{d}_{n-1}"'] & & A_n \\
& P_n \arrow[ur, "\bar{\alpha}_n"'] &
\end{tikzcd}$$

Since $d^A_{n-1} \circ d^A_{n-2} = 0$ the universal property of the pullback square yields a unique morphism $\bar{d}_{n-2}\colon A_{n-2} \to \tilde{A}_{n-1}$ such that $\tilde{\alpha}_{n-1} \circ \bar{d}_{n-2} = d^A_{n-2}$ and $\tilde{d}_{n-1} \circ \bar{d}_{n-2} = 0$. Again since $\mathcal{A}$ has enough projectives we can choose $P_{n-1} \in \mathcal{P}$ with $\bar{\alpha}_{n-1}\colon P_{n-1} \twoheadrightarrow \tilde{A}_{n-1}$ and build the pullback square as above. By repeating this process we obtain:

$$\begin{tikzcd}
& A_{n-3} \arrow[rr, "d^A_{n-3}"] \arrow[dr, "\bar{d}_{n-3}"] & & A_{n-2} \arrow[rr, "d^A_{n-2}"] \arrow[dr, "\bar{d}_{n-2}"] & & A_{n-1} \arrow[dr, "d^A_{n-1}"] & \\
\cdots & & \tilde{A}_{n-2} \arrow[ur, "\tilde{\alpha}_{n-2}"] \arrow[dr, "\tilde{d}_{n-2}"] & & \tilde{A}_{n-1} \arrow[ur, "\tilde{\alpha}_{n-1}"] \arrow[dr, "\tilde{d}_{n-1}"] & & A_n = \tilde{A}_n \\
& P_{n-2} \arrow[rr, "d^P_{n-2}"'] \arrow[ur, "\bar{\alpha}_{n-2}"] & & P_{n-1} \arrow[rr, "d^P_{n-1}"'] \arrow[ur, "\bar{\alpha}_{n-1}"] & & P_n \arrow[ur, "\bar{\alpha}_n"] &
\end{tikzcd}$$

with $d_i^P = \tilde{d}_i \circ \bar{\alpha}_i$. The maps d_i^P define indeed a differential:

$$d_{i+1}^P \circ d_i^P = \tilde{d}_{i+1} \circ (\bar{\alpha}_{i+1} \circ \tilde{d}_i) \circ \bar{\alpha}_i = \tilde{d}_{i+1} \circ (\bar{d}_i \circ \tilde{\alpha}_i) \circ \bar{\alpha}_i = 0$$

since $\tilde{d}_{i+1} \circ \bar{d}_i = 0$. Because $A_\bullet$ is a bounded complex and every object in $\mathcal{A}$ has a finite projective resolution this process terminates and we obtain a bounded complex $P_\bullet$ of projectives by setting all undefined P_i to zero. In particular if $\mathcal{A}$ is hereditary the lower boundary looks like this:

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & & \\
0 \arrow[ur] \arrow[dr] & & \ker \bar{\alpha}_m \arrow[ur] \arrow[dr, "\mathrm{incl}"] & & \tilde{A}_m & \dots \\
 & \ker \bar{\alpha}_m \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{incl}"'] & & P_m \arrow[ur, "\bar{\alpha}_m"'] & &
\end{tikzcd}$$

and we set $P_i = 0$ for $i > n$ and $i < m-1$.
Now we define $\alpha_i = \tilde{\alpha}_i \circ \bar{\alpha}_i$ and claim that $\alpha_\bullet \colon P_\bullet \to A_\bullet$ is an quasi-isomorphism. Thus we consider its mapping cone

$$\begin{tikzcd}[column sep=small]
P_{n-2} \oplus A_{n-3} \arrow[rr, "\begin{bmatrix} -d_{n-2}^P & 0 \\ \alpha_{n-2} & d_{n-3}^A \end{bmatrix}"] \arrow[dr, two heads, "{\begin{bmatrix} \bar{\alpha}_{n-2} & \bar{d}_{n-3} \end{bmatrix}}"'] & & P_{n-1} \oplus A_{n-2} \arrow[rr, "\begin{bmatrix} -d_{n-1}^P & 0 \\ \alpha_{n-1} & d_{n-2}^A \end{bmatrix}"] \arrow[dr, two heads] & & P_n \oplus A_{n-1} \arrow[d, "{\begin{bmatrix} \alpha_n & d_{n-1}^A \end{bmatrix}}"] \\
 & \tilde{A}_{n-2} \arrow[ur, tail, "\begin{bmatrix} -\tilde{d}_{n-2} \\ \tilde{\alpha}_{n-2} \end{bmatrix}"'] & & \tilde{A}_{n-1} \arrow[ur, tail] & A_n = \tilde{A}_n
\end{tikzcd}$$

which is exact because of the exactness of the sequences $\tilde{A}_i \rightarrowtail P_{i+1} \oplus A_i \twoheadrightarrow \tilde{A}_{i+1}$. This follows immediately from the universal property of the pushout squares. Thus the mapping cone is acyclic and $\alpha_\bullet$ defines a quasi-isomorphism.

(b) Claim: Let $P_\bullet$ be a bounded complex of projectives and $f_\bullet \colon A_\bullet \to P_\bullet$ a quasi-isomorphism. Then $f_\bullet$ has right inverse in $Ho_{\mathbb{Z}}^b(\mathcal{A})$:

(c) Let $P_\bullet, Q_\bullet$ be two bounded complexes of projectives. Then

$$\mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}}^b(\mathcal{P})}(P_\bullet, Q_\bullet) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(P_\bullet, Q_\bullet),$$

i.e. ι is full and faithful:

Proof of (c). Consider an element $P_\bullet \xleftarrow{f_\bullet} A_\bullet \xrightarrow{g_\bullet} Q_\bullet$ in $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(P_\bullet, Q_\bullet)$ where $f_\bullet$ is a quasi-isomorphism. Then by part (b) there exists a right inverse $r_\bullet$ of $f_\bullet$. We have

$$\begin{tikzcd}
P_\bullet \arrow[d, equal] & A_\bullet \arrow[l, "f_\bullet"'] \arrow[r, "g_\bullet"] & Q_\bullet \arrow[d, equal] \\
P_\bullet & P_\bullet \arrow[l, "\mathrm{id}"] \arrow[u, "r_\bullet"] \arrow[r, "g_\bullet r_\bullet"] & Q_\bullet
\end{tikzcd}$$

and thus $P_\bullet \xleftarrow{f_\bullet} A_\bullet \xrightarrow{g_\bullet} Q_\bullet = \iota(g_\bullet r_\bullet)$.
On the other hand suppose that $\iota(g_\bullet) = \iota(\tilde{g}_\bullet)$ for two chain maps $g_\bullet, \tilde{g}_\bullet \colon P_\bullet \to Q_\bullet$. Then there exists a quasi-isomorphism $f_\bullet \colon A_\bullet \to P_\bullet$ such that $g_\bullet \circ f_\bullet = \tilde{g}_\bullet \circ f_\bullet$. Again by part (b) we can choose a right inverse $r_\bullet \colon P_\bullet \to A_\bullet$ of $f_\bullet$ and we can calculate:

$$g_\bullet = g_\bullet \circ f_\bullet \circ r_\bullet = \tilde{g}_\bullet \circ f_\bullet \circ r_\bullet = \tilde{g}_\bullet.$$

This proves (c) and finally completes the proof that ι is an equivalence of categories.
Note that for (c) we never used that $Q_\bullet$ is a complex of projectives so (c) holds even if we assume that $Q_\bullet$ is in $\mathcal{C}_{\mathbb{Z}}(\mathcal{A})$.

Proof of (b). Consider the chain maps

$$0, \begin{bmatrix} 0 \\ 1 \end{bmatrix} : P_\bullet \to \mathrm{cone}(f_\bullet).$$

Because $f_\bullet$ is a quasi-isomorphism, $\mathrm{cone}(f_\bullet)$ is acyclic and we can apply the comparison Theorem (see [Wei95, Theorem 2.2.6] and [Wei95, Proism 2.2.7]) to conclude that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is null-homotopic by a map $\begin{bmatrix} r \\ h \end{bmatrix}$:

$$\begin{tikzcd}[column sep=large]
\dots \arrow[r] & A_{n-1} \oplus P_{n-2} \arrow[r, "\begin{bmatrix} -d^A_{n-1} & 0 \\ f_{n-1} & d^P_{n-2} \end{bmatrix}"] & A_n \oplus P_{n-1} \arrow[r, "\begin{bmatrix} f_n & d^P_{n-1} \end{bmatrix}"] & P_n \arrow[r] & 0 \\
\dots \arrow[r] & P_{n-2} \arrow[u, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] \arrow[r, "d^P_{n-2}"'] & P_{n-1} \arrow[u, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] \arrow[ul, "\begin{bmatrix} r_{n-1} \\ h_{n-1} \end{bmatrix}"'] \arrow[r, "d^P_{n-1}"'] & P_n \arrow[u, "1"] \arrow[ul, "\begin{bmatrix} r_n \\ h_n \end{bmatrix}"'] \arrow[r] & 0. \arrow[u]
\end{tikzcd}$$

In formulas:

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}r_{i+1}\\h_{i+1}\end{bmatrix} \circ d_i^P + \begin{bmatrix}-d_i^A & 0\\ f_i & d_{i-1}^P\end{bmatrix} \circ \begin{bmatrix}r_i\\h_i\end{bmatrix}$$

$$= \begin{bmatrix} r_{i+1} \circ d_i^P - d_i^A \circ r_i \\ h_{i+1} \circ d_i^P + f_i r_i + d_{i-1}^P \circ h_i \end{bmatrix}. \tag{27}$$

The first coordinate of (27) shows that $r_\bullet \colon P_\bullet \to A_\bullet$ defines a chain map and the second coordinate shows that $f_\bullet r_\bullet$ is homotopic to the identity via $h_\bullet$. □

A 6.2. *Given $X, Y \in \mathcal{A}$ and $i \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X_\bullet, Y_\bullet[i]) = \mathrm{Ext}^i_{\mathcal{A}}(X, Y). \tag{28}$$

Proof. At first we define a complex $\mathrm{Hom}^\bullet_{\mathcal{C}(\mathcal{A})}(A_\bullet, B_\bullet)$ for two complexes $A_\bullet, B_\bullet \in \mathcal{C}(\mathcal{A})$ as follows:

$$\mathrm{Hom}^i_{\mathcal{C}(\mathcal{A})}(A_\bullet, B_\bullet) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A_j, B_{j+i})$$

and the differential:

$$\partial^i(f_\bullet)_j = f_{j+1} \circ d_j^A + (-1)^{i+1} d^B_{i+j} \circ f_j.$$

Now we claim the following statement:

(a) For $A_\bullet, B_\bullet \in \mathcal{C}(\mathcal{A})$ and $n \in \mathbb{Z}$ there is an isomorphism of abelian groups

$$\zeta \colon H^n(\mathrm{Hom}^\bullet_{\mathcal{C}(\mathcal{A})}(A_\bullet, B_\bullet)) \to \mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}}(\mathcal{A})}(A_\bullet, B_\bullet[n]).$$

Proof of (a). We define $\zeta(\bar{f_\bullet}) = f_\bullet$ and check that this is well-defined: So at first we have to check that $f_\bullet$ is indeed a morphism of complexes $A_\bullet$ and $B_\bullet[n]$ if $\partial^n(f_\bullet) = 0$. But this condition is exactly that the following diagram commutes for all $j \in \mathbb{Z}$:

$$\begin{tikzcd} A_j \arrow[r, "d_j^A"] \arrow[d, "f_j"'] & A_{j+1} \arrow[d, "f_{j+1}"] \\ B_{j+n} \arrow[r, "(-1)^n d^B_{j+n}"'] & B_{j+n+1} \end{tikzcd} \tag{29}$$

and this is exactly the condition that $f_\bullet$ is a morphism from $A_\bullet$ to $B_\bullet[n]$. Now we have to check that $f_\bullet = \partial^{n-1}(h_\bullet)$ is null-homotopic in $\mathcal{C}(\mathcal{A})$. But the condition that

$$f_j = \partial^{n-1}(h_\bullet)_j = h_{j+1} \circ d_j^A + (-1)^n d^B_{n+j-1} \circ h_j$$

is exactly the condition that we have the following diagrams for all $j \in \mathbb{Z}$:

$$\begin{tikzcd}
A_{j-1} \arrow[r, "d^A_{j-1}"] \arrow[d, "f_{j-1}"'] & A_j \arrow[r, "d^A_j"] \arrow[d, "f_j"] \arrow[dl, "h_j"'] & A_{j+1} \arrow[d, "f_{j+1}"] \arrow[dl, "h_{j+1}"'] \\
B_{j+n-1} \arrow[r, "(-1)^n d^B_{j+n-1}"'] & B_{j+n} \arrow[r, "(-1)^n d^B_{j+n}"'] & B_{j+n+1}
\end{tikzcd} \tag{30}$$

giving a null-homotopy of $f_\bullet$ via $h_\bullet$ as a morphism from $A_\bullet$ to $B_\bullet[n]$. So this shows that ζ is well-defined. But since (29) holds exactly if $f_\bullet$ is in the kernel of ∂^n and we have a diagram (30) yielding a null-homotopy exactly if $f_\bullet$ is in the image of ∂^{n-1}, this already shows surjectivity and injectivity.

(b) Now we are ready to prove (28): Choose a finite projective resolution

$$\ldots \to P_{-2} \to P_{-1} \to P_0 \to X \to 0$$

of X and obtain the complex $\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Y)$:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P_0, Y) \to \mathrm{Hom}_{\mathcal{A}}(P_1, Y) \to \mathrm{Hom}_{\mathcal{A}}(P_2, Y) \to \ldots$$

Note that up to the sign of the differential this complex is isomorphic to $\mathrm{Hom}^\bullet_{\mathcal{C}(\mathcal{A})}(P_\bullet, Y_\bullet)$ and since sign changes do not affect cohomology we have that $H^n(\mathrm{Hom}^\bullet_{\mathcal{C}(\mathcal{A})}(P_\bullet, Y_\bullet)) \cong H^n(\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Y))$. So finally by taking (a) and A 6.1(c) into account we optain

$$\begin{aligned}
\mathrm{Ext}^i_{\mathcal{A}}(X, Y) &= H^i(\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Y)) \\
&\cong H^i(\mathrm{Hom}^\bullet_{\mathcal{C}(\mathcal{A})}(P_\bullet, Y_\bullet)) \\
&\cong \mathrm{Hom}_{\mathcal{H}o_{\mathbb{Z}}(\mathcal{A})}(P_\bullet, Y_\bullet[i]) \\
&\cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P_\bullet, Y_\bullet[i]) \\
&\cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X_\bullet, Y_\bullet[i])
\end{aligned}$$

where the last step follows because $X_\bullet$ is quasi-isomorphic to $P_\bullet$. $\square$