DELIGNE CATEGORIES AND REPRESENTATIONS OF OSp(r | 2n)

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On the category of finite dimensional representations of OSp(r | 2n)*: Part II*

Abstract. We study Brauer and Deligne categories to describe the finite dimensional representation category $\mathcal{F}$ of the orthosymplectic supergroups OSp(r | 2n). On the way we show that the Deligne categories provide upper finite highest weight categories and categorify the Fock space of charge $\delta/2 - 1$ for the (quantum) symmetric pairs $(\mathfrak{g}^\theta, \mathfrak{gl}_{\mathbb{Z}+\delta/2})$ of type (AIII). The main result is an explicit description of the endomorphism ring of a projective generator of $\mathcal{F}$ in terms of a quotient of a type D Khovanov arc algebra. As an application we obtain graded versions of Brauer algebras and the Deligne category.

Contents

Introduction

Lie superalgebras and Lie supergroups have a rich representation theory. Already the finite dimensional representations provide interesting non-semisimple tensor categories. In this paper we will focus on the BCD series, that is the orthosymplectic Lie supergroups OSp(r | 2n) with Lie superalgebras $\mathfrak{osp}(r|2n)$, and study its representation theory.

Main Result: Endomorphism Theorem. The main result is Theorem 10.5, which gives a description of the endomorphism ring of a projective generating family of the category $\mathcal{F}$ of finite dimensional representations of OSp(r | 2n). The answer is explicit in terms of a diagrammatically defined algebra, which moreover comes with a natural grading, and thus

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induces a grading on the category. The main ingredient in the proof is Deligne's universal tensor category Rep_δ, i.e. the idempotent completion of the Brauer category $\mathrm{Br}(\delta)$.

Since the Brauer category is diagrammatic, but the idempotent completion is formal and not expressible in these diagrams, we have to solve on the way the fundamental problem

Find a diagrammatical definition of the Deligne category!

Despite tremendous recent developments in finite dimensional super representation theory, the BCD series is remarkably poorly understood. One reason for this is the lack of Kac modules, [Kac77] in $\mathcal{F}$, so that techniques from highest weight categories are not available. Motivated by the existing rather ad hoc replacements, see e.g. [GS10], [GS13], [ES17] we like to study more conceptually the category $\mathcal{F}$ using Deligne's category Rep_δ for $\delta = r - 2n$.

To describe some results in more detail, fix $\mathbb{C}$ as ground field and let $\delta \in \mathbb{C}$. Then Rep_δ is the idempotent completion of the free rigid symmetric monoidal category generated by one self-dual object $\star$ of categorical dimension δ, [Del07], see also [CW12b], [CH17], [Cou18].

(Graded) Highest weight categories. Let $\mathcal{D}(\delta)$ be the category of of representations of Rep_δ (Definition 2.1). Using the classification of indecomposable objects in Rep_δ (Theorem 1.15) we index the isomorphism classes of indecomposable projectives in $\mathcal{D}(\delta)$ by the set Λ of partitions (Lemma 2.3). We then show in Corollary 2.11 that $\mathcal{D}(\delta)$ is an upper finite highest weight category in the sense of [BS18] with Λ viewed as a poset with the reversed inclusion ordering. The categories $\mathcal{D}(\delta)$ are always highest weight, even though Brauer algebras might be only cellular, see e.g. [ES18, Theorem 5.13, Remark 5.14]. Tensoring with $\star$ followed by the projections onto prescribed generalized eigenspaces for the Jucys-Murphy elements defines *i-induction functors* (Section 2.3). We study these functors and describe their action on standard and projective objects (Lemma 2.16, Theorem 8.7). As an application we obtain a generalization of the Ariki-Grojnowski categorification theorem ([Ari96], [Gro99]) by using instead of symmetric groups the Deligne category to categorify a Fock space $\bigwedge^{\infty/2}\mathbb{V}_\delta$. This is now a type BCD Fock space, since the combinatorics is controlled by Weyl groups of types BCD.

Categorification Theorem. Since the most interesting cases, including the non-semisimple ones, appear for $\delta \in \mathbb{Z}$ let us assume this for this introduction. Let $\mathbb{V}_\delta$ be the vector space with basis v_i for $i \in \mathbb{Z} + \delta/2$ viewed as the vector representation of the infinite general linear Lie algebra $\mathfrak{g} = \mathfrak{gl}(\mathbb{V}_\delta)$ with Chevalley generators E_i, F_i, $i \in \mathbb{Z} + \frac{\delta+1}{2}$. Consider the classical Fock space $\bigwedge^{\infty/2}\mathbb{V}_\delta$ of semiinfinite wedges of $\mathbb{V}_\delta$ of charge $\frac{\delta-1}{2}$ with its standard basis vectors identified with Λ, see e.g. [Lec12], [KR87]. In Section 3 we identify $\bigwedge^{\infty/2}\mathbb{V}_\delta$ with the complexified Grothendieck group $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ of the exact subcategory $\mathcal{D}_\Delta(\delta)$ of $\mathcal{D}(\delta)$ given by all objects which admit a filtration with subquotients isomorphic to standard objects $\Delta(\lambda)$,

$$\mathrm{K}_0(\mathcal{D}_\Delta(\delta)) \cong \langle\, \Lambda \,\rangle \cong \bigwedge^{\infty/2} \mathbb{V}_\delta, \qquad [\Delta_\delta(\lambda)] \mapsto \lambda \;\mapsto v_\delta^\lambda. \tag{0.1}$$

We show in Theorem 3.5 that the exact i-induction functors define on $\mathcal{D}_\Delta(\delta)$ an action of the fixed point Lie subalgebra $\mathfrak{g}^\theta \subset \mathfrak{g}$ defined by the involution on $\mathfrak{g}$ sending E_i to F_{-i}. Using the classification of thick ideals in Rep_δ, we obtain a filtration on $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ by isotypical components for $\mathfrak{g}^\theta$ indexed by values of a combinatorial function κ. It turns out to be a special case of Lusztig's a-function, Remark 13.7, and later encodes the degree of atypicality, Corollary 13.6. This construction fits into the theory of categorified socle filtrations of limit Lie algebra, [HPS19].

Remarkably, the *canonical basis* of $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ given by the isomorphism classes of indecomposable projective objects can be described explicitly, Corollary 2.11, Remark 11.24).

The entries in the base change matrix D to the standard basis are parabolic Kazhdan-Lusztig-polynomials evaluated at 1 of type $(\mathrm{D}_N, \mathrm{A}_{N-1})$ or equivalently $(\mathrm{B}_{N-1}, \mathrm{A}_{N-2})$, [ES16b, Section 9.7]. Slightly different type BCD Fock spaces were defined in [LRS19] using affine Hecke algebras; those would appear if we passed from $\mathbb{C}$ to fields of characteristics $p > 2$; they would inherit an action of an analogous fixed point Lie subalgebra of affine $\mathfrak{sl}_p$.

The canonical basis differs from Lusztig's canonical basis on the $\mathfrak{g}$-module $\bigwedge^{\infty/2} \mathbb{V}_\delta$. Instead of Lusztig's bar involution on the quantised Fock space, one has to work with the bar involution from [ES18, Proposition 8.9]. It is compatible with the action of Letzter's quantum symmetric pair, [Let02], of type (AIII) attached to $(\mathfrak{g}^\theta, \mathfrak{g})$ instead of the quantum group for $\mathfrak{g}$. We refer to [Lec12] and to [ES18] for the two constructions. Bar involutions for quantum symmetric pairs were studied in general in [BK15], [BK19]. The notion of canonical bases for type (AIII) were independently introduced in [ES18] and, under the name *ι-canonical bases*, in [BW18] (with ι referring to the involution θ).

After having finished our paper we were informed about the independent result from [RS20]. Rui and Song proved the same categorification theorem and even generalisations thereof to higher levels. Our focus here is however different in the sense that we are interested in a generalisation to the quantum version by categorifying the Fock space for the corresponding quantum symmetric pair using graded enrichments of the involved categories. We therefore like to nevertheless give our independent proofs which have the advantage that they directly lift to the graded setting. We expect that all the results from [RS20] with a Lie theoretic origin generalise directly to a quantum setting using the Koszul grading on category $\mathcal{O}$ from [BGS96]. The case of interest in our paper is in some sense one of the most complicated one as we indicate below.

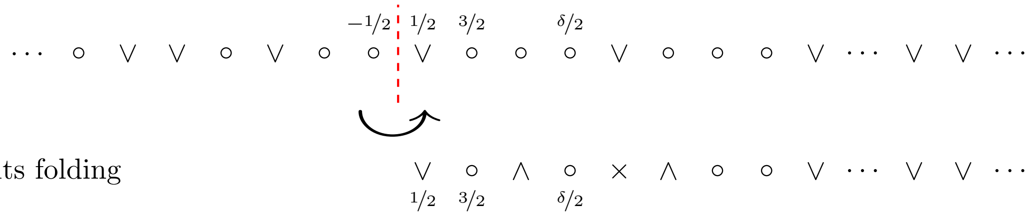
Higher structures. Section 4 deals with the higher structures of the categorification theorem, that is with natural transformations between induction functors. The main tool is the *affine VW-category* or *degenerate affine Brauer category* which was introduced in a super version in [BDE$^+$20] and then in the version relevant for our purposes in [RS19]. Important for us are two special cyclotomic quotients, the *Brauer quotient* of level 1, which we like to understand, and the *isotropic Grassmannian quotient* of level 2, which we connect via an equivalence of categories Theorem 4.7 to the well-understood parabolic category $\mathcal{O}$ of the above parabolic types. By pushing this equivalence further to categories of perverse sheaves on isotropic Grassmannians, geometric tools become available; in particular (Koszul) gradings appear naturally, [BGS96]. Via the equivalence in Theorem 4.8 we realise the first quotient as an idempotent truncation of the second and use this to define a graded version of the Deligne category, i.e. an enrichment in the category of graded vector spaces, Definition 10.2. We like to stress for the experts that this is not a trivial result at all in the sense, that the Brauer algebra is one of the cyclotomic quotients of the affine VW-category **??** which has not a direct Lie theoretic incarnation as a parabolic category $\mathcal{O}$ involving higher Schur-Weyl duality, since (nonsemisimple) parabolic category .$\mathcal{O}$ give rise to cyclotomic quotients of level strictly higher than 1. For a careful treatment of this connection with category $\mathcal{O}$ we refer to [RS19].

Finite dimensional representations of OSp. In Section 5 we finally pass to the Lie supergroup $G = \mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$ and the category $\mathcal{F}$. Let V be the vector representation of G. The connection to the Deligne category is given by the *First Fundamental Theorem* of invariant theory and the universal property of Rep_δ, [DLZ18], [Ser14]. Namely sending $\star$ to V defines a *full* tensor functor $\mathbb{F} = \mathbb{F}_{(r\,|\,2n)}$ from Rep_δ to the tensor category of representations of G generated by V, Theorem 5.8. For the fullness it is important to work with G and not just the special orthosymplectic supergroup or it Lie superalgebra $\mathfrak{osp}(r|2n)$, see [LZ15b] or [ES16d, Remark 5.35.].

Since any projective in $\mathcal{F}$ appears in some $V^{\otimes d}$, our strategy is to describe

- idempotents picking out the indecomposable summand (see Lemma 1.8, Remark 8.11),
- a characterisation of the projective summands among all summands (see Theorem 6.17, Proposition 7.6),
- *convenient* labelling sets depending first on δ and then also on r and n (Theorem 5.12),
- and finally a dictionary translating into highest weight modules with the choice of Borel as in [GS13] (Section 13).

The main tool hereby are the *weight, cup and circle diagrams* introduced in [ES16b]. To understand how they fit into the picture we go back to the Fock space with its standard basis of semiinfinite wedges. Every basis vector gives rise to a sequence $i_1, i_2, i_3, \ldots$ of indices $i_j \in \mathbb{Z} + \delta/2$ such that $i_j = -(\delta/2 + j - 1)$ for $j \gg 0$. We encode these by marking the corresponding positions i_j on the (half)-number line by $\vee$. To encode the action of the fixed point Lie algebra $\mathfrak{g}^\theta$, we *fold* the negative part of the line with the symbols swapped upside down onto the positive part (mimicking the involution θ). The result is what we call a *Deligne weight diagram* consisting out of symbols $\circ, \times, \wedge$ (finitely many) and $\vee$ (infinitely many). For $\delta = 7$ and basis vector $v_{-11/2} \wedge v_{-9/2} \wedge v_{-5/2} \wedge v_{1/2} \wedge v_{9/2} \wedge v_{17/2} \wedge \cdots$ we obtain



and its folding

To each Deligne weight diagram λ_δ we attach a dotted cup diagram $\underline{\lambda_\delta}$ via a simple rule, Definition 6.3. We have then bijections of sets (where the outer ones depend on δ)

$$\left\{\begin{matrix}\text{Indecomposable}\\ \text{objects } \mathtt{R}_\delta(\lambda) \text{ in}\\ \mathrm{Rep}_\delta\end{matrix}\right\} \longleftrightarrow \left\{\text{partitions } \lambda\right\} \longleftrightarrow \left\{\begin{matrix}\text{cup diagrams } \underline{\lambda_\delta}\\ \text{for Deligne}\\ \text{weight diagrams } \lambda_\delta\end{matrix}\right\}$$

The highest weight structure on $\mathcal{D}(\delta)$ implies that the Cartan matrix C of $\mathcal{D}(\delta)$ can be factorised as $C = DD^{\mathrm{t}}$. This allows us to represent homomorphisms between indecomposable objects in Rep_δ as linear combinations of *oriented circle diagrams*, Definition 6.7, i.e. pairs of cup diagrams with a compatible orientation. More precisely, we obtain an isomorphism of categories between the Deligne category and the category where objects are $\underline{\lambda_\delta}$, $\lambda \in \Lambda$ and morphism from $\underline{\lambda_\delta}$ to $\underline{\mu_\delta}$ is the vector space with basis all circle diagrams $(\underline{\mu_\delta}, \underline{\lambda_\delta}, \nu)$. The composition is given by the multiplication in the Khovanov arc algebras of type D, [ES16b]. This category is a solution to our fundamental problem.

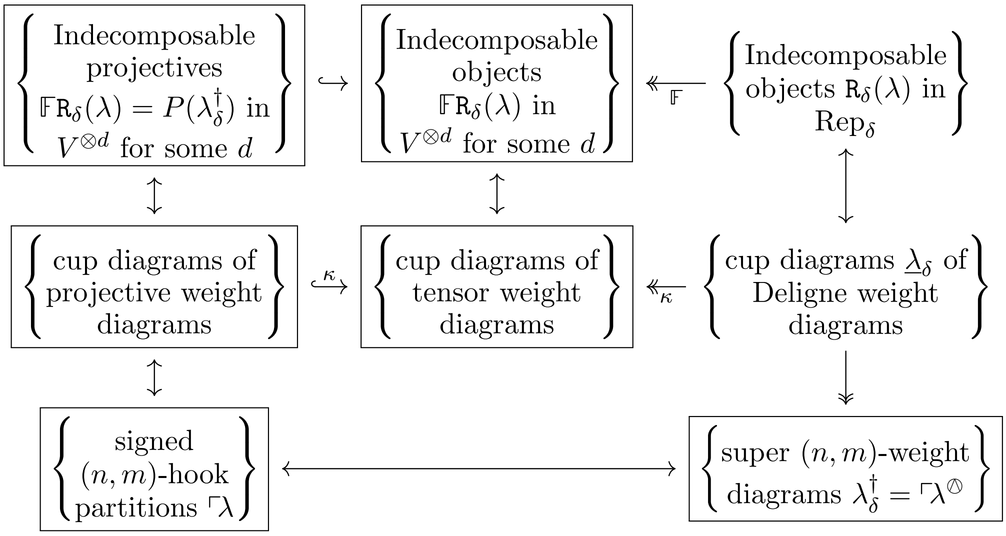
This diagrammatic description has moreover the following advantages:

- The morphism spaces can be equipped with a grading such that it is Koszul and the circle diagrams form a homogeneous basis (Definition 10.2).
- The kernel of the functor $\mathbb{F} = \mathbb{F}_{(r\,|\,2n)}$ has an easy description in terms of *nuclear* morphisms (see Definition 10.2, Lemma 10.4, Theorem 10.5). The nuclear circle diagrams form a homogeneous basis of the kernel.

These properties are used to finally prove the main theorem. A major combinatorial work is hereby required to translate between the combinatorics in the (two incarnations of the) Deligne category and the combinatorics of Lie theoretic weights.

In Section 12 we describe a few applications. This includes the structure of the indecomposable summands in $V^{\otimes d}$ and the minimal tensor power $V^{\otimes d}$ in which the determinant representation respectively super Pfaffian from [Ser01], [LZ15b] occurs in terms of the developed cup diagram combinatorics. We finish with a small section which explains the dictionary to Lie theoretic weights with the choice of Borel from [GS13].

The following chart illustrates the most important involved combinatorial sets. A frame indicates that a set depends on r and n, whereas the unframed sets only depend on δ.



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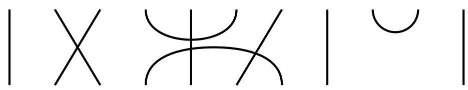
1. Brauer and Deligne categories

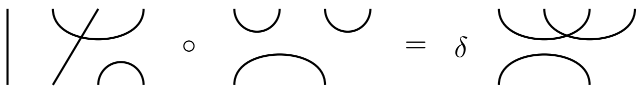
We fix as ground field the complex numbers $\mathbb{C}$. All vector spaces are over $\mathbb{C}$ and categories, functors etc. are assumed to be $\mathbb{C}$-linear. For any algebra A we denote by A-mod the category of finite dimensional A-modules. Our algebras are always associative and if not stated otherwise also unital, and algebra homomorphisms preserve units.

1.1. **Basics.** Let $\delta \in \mathbb{C}$. For fixed $r, s \in \mathbb{Z}_{\geq 0}$, a *Brauer diagram of type* (r, s) is a partitioning of the set $P := \{1, 2, \ldots, r\} \cup \{r+1, \ldots, r+s\}$ into subsets of cardinality two. We identify such diagrams with planar diagrams by identifying $p \in P$ with the point $(p, 0)$ in the plane if $1 \leq p \leq r$ and with $(p - r, 1)$ if $r + 1 \leq p \leq r + s$, and then connect the two points in each subset by an arc inside the rectangle $[1, \max\{r, s\}] \times [0, 1]$. Two Brauer diagrams of the same type are defined to be equivalent if the corresponding partitioning is the same. Here is an example of a Brauer diagram of type $(9, 11)$:



(1.2)

The *Brauer category* $\mathrm{Br}(\delta)$, see [LZ15a], has objects $r \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Hom}_{\mathrm{Br}(\delta)}(r, s)$ is the vector space with basis the equivalence classes of Brauer diagrams of type (r, s) (independent of δ) with the following diagrammatic composition (depending on δ). Given Brauer diagrams D_i of type (r_i, s_i) for $i = 1, 2$ with $s_1 = r_2$, we can stack D_2 on top of D_1 and obtain, after removing all internal closed components, again a Brauer diagram D. Then set $D_2 \circ D_1 = \delta^c D$, where c is the number of internal components removed, for example


(1.3)

The endomorphism algebra of the object d is the *Brauer algebra* $\mathrm{Br}_d(\delta)$ originally introduced diagrammatically in [Bra37]. For an algebraic presentation see e.g. [Naz96].

Lemma 1.1. *We have* $\dim \mathrm{Hom}_{\mathrm{Br}(\delta)}(r,s) = \dim \mathrm{Br}_{\frac{r+s}{2}}(\delta)$ *with* $\mathrm{Br}_{\frac{r+s}{2}} = \{0\}$ *if* $r+s$ *is odd.*

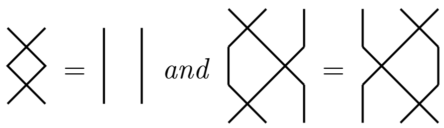
Proof. For an (r,s)-Brauer diagram to exist, $r+s$ must be even, and then the number of such diagrams only depends on $r+s$. □

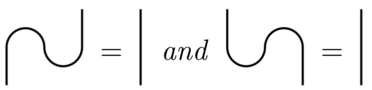
Note, that $\mathrm{Br}(\delta)$ is a rigid symmetric monoidal category, see [LZ15a]. The tensor product, denoted $\boxtimes$, on objects is $r \boxtimes s = r+s$, and $a \boxtimes b$ for basis morphisms a and b is obtained by placing the diagram for b to the right of the one for a. If we denote by id the identity morphism on the object 1, then $\mathrm{id}^{\boxtimes d}$ is the identity element in $\mathrm{Br}_d(\delta)$. Using this, $\mathrm{Br}(\delta)$ can be defined alternatively as follows (where $\star$ corresponds to the object 1, the diagrams to the Brauer diagrams displayed in the same way and the monoidal product $\otimes$ to $\boxtimes$):

Proposition 1.2. *The Brauer category* $\mathrm{Br}(\delta)$ *is the strict rigid monoidal category generated as monoidal category by a single object* $\star$ *and morphisms*

$s =$  $: \star \otimes \star \longrightarrow \star \otimes \star$, $\flat =$  $: \star \otimes \star \longrightarrow \mathbb{1}$, $\flat^* =$  $: \mathbb{1} \longrightarrow \star \otimes \star$

subject to the following relations

(1) *The braid relations:* 

(2) *The snake relations:* 

(3) *The untwisting relations:* 

(4) *The loop removing relation:*  $= \delta$

Proof. This follows directly from [LZ15a, Theorem 2.6]. □

Remark 1.3. Standard calculations imply that the mirrors of the untwisting relations hold as well, see e.g. [BDE+20, Lemmas 4-5, ignoring the signs coming from the super structure].

Definition 1.4. The *Deligne category* Rep_δ, or more precisely $\underline{\mathrm{Rep}}(\mathrm{O}_\delta)$, from [DLZ18], is the Karoubian envelope of the additive envelope of $\mathrm{Br}(\delta)$, i.e. it is obtained from $\mathrm{Br}(\delta)$ by adding finite direct sums and images of idempotent endomorphisms, see e.g. [CW12b] for more details on this construction.

Rep_δ is a rigid symmetric monoidal category with full subcategory $\mathrm{Br}(\delta)$. By definition, every idempotent e in $\mathrm{Br}_d(\delta) = \mathrm{Hom}_{\mathrm{Br}(\delta)}(d,d)$ has an image object $\mathrm{im}\,\mathrm{e}$ in Rep_δ and we identify $\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathrm{im}\,\mathrm{e}, \mathrm{im}\,\mathrm{e}')$ with $\mathrm{e}'\,\mathrm{Hom}_{\mathrm{Rep}_\delta}(d,d')\,\mathrm{e}$.

1.2. **Jucys-Murphy elements and semisimplicity.** Consider the Brauer algebra $\mathrm{Br}_d(\delta)$ for fixed d. Note that it contains as a subalgebra the group algebra $\mathbb{C}[S_d]$ of the symmetric group S_d (given by all permutation diagrams) with the standard generators $s_i = (i, i+1)$, $1 \leq i \leq d-1$. A general transposition $s_{i,j} = (i,j)$ for $i \neq j$ corresponds to the diagram, denoted by $s_{i,j}$ as well, which connects $(i,0)$ with $(j,1)$ and $(i,1)$ with $(j,0)$ and $(k,0)$ with $(k,1)$ for $1 \leq k \leq d$ otherwise. We define $\tau_{i,j} \in \mathrm{Br}_d(\delta)$, $1 \leq i \leq d-1$, to be the diagram which again connects the points $(k,0)$ with $(k,1)$ for any $1 \leq k \leq d$, $k \neq i,j$, but then $(i,0)$ with $(j,0)$ and $(i,1)$ with $(j,1)$. In particular. we have

$$\tau_i := \tau_{i,i+1} = (\mathrm{id}^{\boxtimes(i-1)} \boxtimes \cup \boxtimes \mathrm{id}^{\boxtimes d-i-1}) \circ (\mathrm{id}^{\boxtimes(i-1)} \boxtimes \cap \boxtimes \mathrm{id}^{\boxtimes d-i-1}),$$

where $\cup$ and $\cap$ are the unique $(0,2)$ respectively $(2,0)$ Brauer diagrams.

Following the approach of Okounkov and Vershik for the symmetric groups, Nazarov defined in [Naz96] the *Jucys-Murphy elements* $\xi_k \in \mathrm{Br}_d(\delta)$ for $1 \leq i \leq d$ as

$$\xi_k := \tfrac{\delta-1}{2} + \sum_{1 \leq i < k} (s_{i,k} - \tau_{i,k}). \tag{1.4}$$

Lemma 1.5. *The Jucys-Murphy elements generate a commutative subalgebra* $\mathrm{GZ}_d(\delta)$ *of* $\mathrm{Br}_d(\delta)$ *and the element* $\xi_1 + \xi_2 + \cdots + \xi_d$ *is central in* $\mathrm{Br}_d(\delta)$*.*

Proof. This is [Naz96, Corollaries 2.2 and 2.4]. In fact, the second claim follows from the equalities $s_k(\xi_k + \xi_{k+1}) = (\xi_k + \xi_{k+1})s_k$ and $\tau_k(\xi_k + \xi_{k+1}) = 0 = (\xi_k + \xi_{k+1})\tau_k$ (note the cancelling of terms involving δ) together with $s_k\xi_l = \xi_l s_k$ and $\tau_k\xi_l = \xi_l\tau_k$ for $|k-l| > 1$. □

In contrast to the symmetric group, the action of $\mathrm{GZ}_d(\delta)$ is not diagonalizable in general. The occurring Jordan blocks are however at most of size 2, as can be deduced for instance from [ES16c, Theorem B]. This is because the Brauer algebra is not semisimple for certain integral values of δ, as was first proved by Wenzl [Wen88] (using trace arguments), and then made explicit and generalized to any field in [Rui05] (using determinants of Cartan matrices). For an alternative proof using tilting modules for quantum groups see [AST17]. Over the complex numbers the result can be stated as follows:

Proposition 1.6. *Let* $\delta \in \mathbb{C}$*. Then* $\mathrm{Br}_d(\delta)$ *is semisimple if and only if* $\delta \notin \mathbb{Z}$ *or* $\delta = 0$ *and* $d \in \{1,3,5\}$ *or* $\delta \neq 0$ *and* $d \leq \delta + 1$*.*

The following consequence is [Del07, Theorem 9.7].

Corollary 1.7. Rep_δ *is abelian semisimple if and only if* $\delta \notin \mathbb{Z}$*.*

We are therefore mainly interested in the case $\delta \in \mathbb{Z}$ and $\boxed{\text{assume from now on } \delta \in \mathbb{Z}}$.

1.3. **Primitive idempotents.** We will need the following general well-known results.

Lemma 1.8. *Let* A *and* B *be finite dimensional algebras with a surjective algebra homomorphism* $f : A \twoheadrightarrow B$*. Let* $e \in B$ *be an idempotent, i.e.* $e^2 = e$*, then there exists an idempotent* $\hat{e} \in A$ *such that* $f(\hat{e}) = e$*. If* e *is primitive,* $\hat{e}$ *can be chosen to be primitive as well.*

Proof. This is the Idempotent Lifting Lemma, see e.g. [Lin18, Theorem 4.7.1.]. □

Lemma 1.9. *Let* $\mathcal{D}$ *be a (*$\mathbb{C}$*-linear) Krull-Schmidt category,* $\mathcal{D}'$ *a preadditive category and* $F : \mathcal{D} \to \mathcal{D}'$ *a full pre-additive functor. Then* $FX \in \mathcal{D}'$ *is indecomposable if* $X \in \mathcal{D}$ *is indecomposable; and for indecomposable objects* $X, Y \in \mathcal{D}$ *we have* $(FX \cong FY \Rightarrow X \cong Y$*.)*

Proof. A proof can be found in [CW12b, Proposition 2.7.4]. □

Definition 1.10. A *partition* of d is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers λ_i such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_i \lambda_i = d$. We call λ_i the *parts* of λ and set $|\lambda| = \sum_i \lambda_i$. Let $\varnothing$ be the *empty partition* with all parts being zero, and let Λ be the set of all partitions.

As usual we identify partitions with their Young diagram with λ_i boxes in row i and denote by λ^t the partition *transposed* to λ whose Young diagram is obtained by reflecting the one for λ in the middle diagonal such that λ^t has λ_i boxes in the ith column. For a partition λ of d fix the corresponding primitive Frobenius idempotent $z_\lambda \in \mathbb{C}[S_d]$, see e.g. [FH91, Section 4.2] for a construction.

Sending all diagrams to zero which are not permutation diagrams and sending permutation diagrams to itself defines an algebra homomorphism $\pi : \mathrm{Br}_d(\delta) \twoheadrightarrow \mathbb{C}[S_d]$. By Lemma 1.8 we can lift z_λ to a primitive idempotent $e_\lambda \in \mathrm{Br}_d(\delta)$ (which is uniquely defined up to conjugacy). To classify primitive idempotents in $\mathrm{Br}_d(\delta)$ define for $r, i \in \mathbb{Z}_{\geq 0}$ the $(r, r+2i)$-Brauer diagram $\psi_{r,i} = \mathrm{id}^{\boxtimes r} \boxtimes \cup^{\boxtimes i}$ and the $(r+2i, r)$-Brauer diagram $\varphi_{r,i} = \mathrm{id}^{\boxtimes r-1} \boxtimes \cap^{\boxtimes i} \boxtimes \mathrm{id}$ in case $r > 0$, and $\varphi_{0,i} = \frac{1}{\delta^i} \cap^{\boxtimes i}$ if $\delta \neq 0$. Define the following set of partitions

$$\Lambda_d(\delta) \quad = \quad \begin{cases} \{\lambda \mid |\lambda| = d - 2i, 0 \leq i \leq d/2\} & \text{if } \delta \neq 0, \text{ or } d \text{ odd, or } d = 0, \\ \{\lambda \mid |\lambda| = d - 2i, 0 \leq i < d/2\} & \text{if } \delta = 0 \text{ and } d > 0 \text{ even.} \end{cases}$$

For any $\lambda \in \Lambda_d(\delta)$, and $i \in \mathbb{Z}$ such that $d = |\lambda| + 2i$, the element $e_\lambda^{(i)} = \psi_{|\lambda|,i} e_\lambda \varphi_{|\lambda|,i} \in \mathrm{Br}_d(\delta)$ is an idempotent and the following holds, see e.g. [CH17, Theorem 3.4].

Proposition 1.11. *The elements $e_\lambda^{(i)}$, where $\lambda \in \Lambda_d(\delta)$, and $2i = d - |\lambda|$ form a complete set of pairwise orthogonal primitive idempotents in* $\mathrm{Br}_d(\delta)$.

Example 1.12. If $d = 2$, then we have the following complete sets of pairwise orthogonal primitive idempotents: $z_{(2)} = \frac{1+s_1}{2}$ and $z_{(1,1)} = \frac{1-s_1}{2}$ in $\mathbb{C}[S_2]$, and $e_{(2)} = \frac{1+s_1}{2} - \frac{1}{\delta}\tau_1$, $e_{(1,1)} = \frac{1-s_1}{2}$ and $e_\emptyset = \frac{1}{\delta}\tau_1$ in $\mathrm{Br}_d(\delta)$ if $\delta \neq 0$. In case $\delta = 0$ we only have $\frac{1+s_1}{2}$ and $\frac{1-s_1}{2}$ (note that e_1 is then nilpotent). In general, there is no explicit formula for the idempotents, for some explicit results see e.g. [DLS18], [MR13], [BB01].

It follows that in the category $\mathrm{Br}_d(\delta)$- mod of finite dimensional representations of $\mathrm{Br}_d(\delta)$, the isomorphism classes of irreducible representations are indexed by $\Lambda_d(\delta)$.

Definition 1.13. We denote by $L_{d,\delta}(\lambda)$ the irreducible module corresponding to $\lambda \in \Lambda_d(\delta)$ and by $P_{d,\delta}(\lambda)$ its indecomposable projective cover.

By definition of Karoubian closure, the idempotents $e_\lambda^{(i)}$ have an image, $\mathrm{im}\, e_\lambda^{(i)}$ in Rep_δ. By [CH17, Proposition 3.3] the $e_\lambda^{(i)}$, for fixed λ, are (up to isomorphism) independent of i.

Definition 1.14. For $\lambda \in \Lambda$ let $e_\lambda = e_\lambda^{(0)}$ and set $\mathtt{R}_\delta(\lambda) = \mathrm{im}\, e_\lambda$. This is an indecomposable object in Rep_δ.

There is the following classification theorem from [CH17, Theorem 3.5].

Theorem 1.15 (Indecomposables in Rep_δ)**.** *The assignment $\lambda \mapsto \mathtt{R}_\delta(\lambda)$ gives a bijection between the set Λ of all partitions and the set of isomorphism classes of nonzero indecomposable objects in* Rep_δ.

Remark 1.16. As in Lemma 1.1, $\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu)) = \{0\}$ if $|\lambda| \not\equiv |\mu| \bmod 2$; otherwise we have for any d such that $\lambda, \mu \in \Lambda_d(\delta)$ an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu)) \quad \cong \quad \mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\lambda), P_{d,\delta}(\mu)). \tag{1.5}$$

We fix such isomorphisms for every allowed d in a compatible way by fixing compatible isomorphisms between the idempotents $e_\lambda^{(i)}$ for any fixed λ.

1.4. **Cellularity of $\mathrm{Br}_d(\delta)$ and facts on cell modules.** For any $\delta \in \mathbb{C}$, $\mathrm{Br}_d(\delta)$ is a *cellular algebra* in the sense of Graham and Lehrer, [GL96]. This was first observed in [GL96]; but see also e.g. [Mar15], [CVM09], [AST18] for different approaches. Apart from the case $\delta = 0$ (see Example 1.12), $\mathrm{Br}_d(\delta)$ is always quasihereditary [CVM09, Corollary 2.3], that is $\mathrm{Br}_d(\delta)$- mod is a highest weight category. The only facts we need are summarized in the following proposition.

Proposition 1.17. *$\mathrm{Br}_d(\delta)$ is a cellular algebra with a collection of cell modules $\Delta_{d,\delta}(\lambda)$ labelled by $\lambda \in \Lambda_d^+(\delta) = \Lambda_d(\delta) \cup \{\varnothing\}$ and constructed as in Remark 1.18. Their irreducible quotients for $\lambda \in \Lambda_d(\delta)$ give a full set $L_{d,\delta}(\lambda)$ of pairwise non-isomorphic irreducible modules.*

Proof. These are Theorems (4.10) and (4.17) in [GL96] with [GL96, (3.4)]. □

Note that $\Lambda_d^+(\delta) = \Lambda_d(\delta)$ iff $\delta \neq 0$ or d odd, that is precisely if $\mathrm{Br}_d(\delta)$-mod is a highest weight category by [GL96, (3.10)].

Remark 1.18. To construct the cell module $\Delta_{d,\delta}(\lambda)$ explicitly set $t := {}^1\!/_2(d - |\lambda|)$ and consider the two-sided ideal J_d^t of $\mathrm{Br}_d(\delta)$ spanned by the Brauer diagrams with at least t horizontal arcs at the top (and bottom). The images of the Brauer diagrams with exactly t such horizontal arcs give a basis for the quotient J_d^t/J_d^{t+1}. Let I_d^t be the subspace of J_d^t/J_d^{t+1} spanned by the images of the diagrams with exactly t horizontal arcs at the bottom connecting the $(d-t-k+1)$th vertex to the $d-t+k$th vertex for each $k = 1, \ldots, t$. Identify $\mathbb{C}[S_{|\lambda|}]$ with the subalgebra of $\mathrm{Br}_d(\delta)$ generated by the s_i, $1 \leq i \leq |\lambda| - 1$ in the obvious way, I_d^t is invariant under right multiplication by elements of $\mathbb{C}[S_{|\lambda|}]$ and left multiplication by elements of $\mathrm{Br}_d(\delta)$, hence it is a $\big(\mathrm{Br}_d(\delta), \mathbb{C}[S_{|\lambda|}]\big)$-bimodule. Then we have by definition

$$\Delta_{d,\delta}(\lambda) := I_d^t \otimes_{\mathbb{C}[S_{|\lambda|}]} S(\lambda), \tag{1.6}$$

where $S(\lambda)$ is the ordinary Specht module for $\mathbb{C}[S_{|\lambda|}]$.

The bimodule I_d^t is free as right $\mathbb{C}[S_{|\lambda|}]$-module, with basis X_d^t given by the images of the Brauer diagrams with exactly t horizontal arcs at the bottom connecting the $(d-t-k+1)$th vertex to the $(d-t+k)$th vertex for each $k = 1, \ldots, t$ and in which no two vertical strands cross. Hence we can consider it as a vector space and compute its dimension

$$\Delta_{d,\delta}(\lambda) = \bigoplus_{\tau \in X_d^t} \tau \otimes S(\lambda), \qquad \dim \Delta_{d,\delta}(\lambda) = \tfrac{1}{(t+1)} \binom{2t}{t} \binom{d}{2t} \dim S(\lambda), \tag{1.7}$$

which is in particular independent of δ.

To study their behaviour under induction we recall from Lemma 1.5 the central element

$$\Omega_d = 2 \sum_{k=1}^{d-1} \xi_k \in \mathrm{Br}_d(\delta).$$

It defines an endomorphism of each irreducible module, thus by Schur's Lemma a scalar. To describe this scalar recall that the *content* of a box (i,j) in a partition λ is $j - i$.

Lemma 1.19. *Ω_d acts on the irreducible module $L_{d,\delta}(\lambda)$ or cell module $\Delta_{d,\delta}(\lambda)$ for $\mathrm{Br}_d(\delta)$ by the scalar $c_\lambda^{\mathrm{Br}} := |\lambda|(\delta - 1) + 2\,\mathrm{cont}(\lambda)$, where $\mathrm{cont}(\lambda)$ is the sum of all contents of λ.*

Proof. It suffices to prove the second claim which is [Naz96, formula before (2.13)] or [DWH99, Theorem 3.2] using the definition (1.6) for cell modules. □

The Brauer algebra $\mathrm{Br}_d(\delta)$ embeds into the Brauer algebra $\mathrm{Br}_{d+1}(\delta)$ by adding to each diagram an additional strand to the right. We obtain the *induction functors*

$$\mathrm{ind}_d^{d+1} : \mathrm{Br}_d(\delta)\text{-mod} \to \mathrm{Br}_{d+1}(\delta)\text{- mod} \qquad M \mapsto \mathrm{Br}_{d+1}(\delta) \otimes_{\mathrm{Br}_d(\delta)} M. \tag{1.8}$$

The following *branching rules* are well-known, see e.g. [CVM09, Proposition 2.7].

Lemma 1.20. *Let $\lambda \in \Lambda_d^+(\delta)$. The module $\operatorname{ind}_d^{d+1} \Delta_{d,\delta}(\lambda)$ has a multiplicity free filtration with subquotients exactly of the form $\Delta_{d+1,\delta}(\mu)$ where μ runs through all partitions obtained from λ by adding or removing a box.*

Definition 1.21. An *up-down tableau of length d* is a sequence $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d)})$ of partitions such that $\lambda^{(0)} = \varnothing$ and $\lambda^{(i+1)}$ is obtained from $\lambda^{(i)}$ by adding or removing a single box. We call $\lambda^{(d)}$ the *shape of* $\boldsymbol{\lambda}$. Denote by 🚶_d the set [1]of all up-down tableaux of length d and by $\text{🚶}_d^{d'}$ the set of pairs $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \text{🚶}_d \times \text{🚶}_{d'}$ of the same shape.

Proposition 1.22. *We have* $\dim \operatorname{Hom}_{\operatorname{Br}(\delta)}(r,s) = \left| \text{🚶}_r^s \right|$.

Proof. By Lemma 1.1 the left hand side depends only on $r+s$ and is zero if this number is odd. On the other hand, the set 🚶_r^s can be identified with up-down-tableaux of shape $\varnothing$ and length $r+s$ by composing the pair (λ, μ) as $(\lambda^{(0)}, \ldots, \lambda^{(r-1)}, \lambda^{(r)} = \mu^{(s)}, \mu^{(s-1)} \ldots, \mu^{(0)})$. Hence also this side depends only on $r+s$, and 🚶_r^s is empty if $r+s$ is odd, since the parity of $|\lambda^{(r)}| = |\mu^{(s)}|$ agrees with the parity of r respectively s.

Thus we may assume $r = s$ and, since the left hand side is independent of δ, restrict to the semisimple case. But then the statement follows directly from Lemma 1.20 noting that induction sends the regular module to the regular module, $\operatorname{ind}_d^{d+1} \operatorname{Br}_d(\delta) \cong \operatorname{Br}_{d+1}(\delta)$. □

Definition 1.23. We fix the *reverse inclusion ordering* on Λ, i.e. $\lambda \geq \mu$ if the partition λ is contained in the partition μ, in formulas $\lambda \subset \mu$. In particular, $\varnothing$ is maximal.

By general properties of cellular algebras, [GL96, (2.9)(ii), (2.10)(i)], any $P_{d,\delta}(\lambda)$ has a filtration with subquotients of the form $\Delta_{d,\delta}(\mu)$; and any $\Delta_{d,\delta}(\lambda)$ has a Jordan-Hoelder series with subquotients $L_{d,\delta}(\mu)$'s where $\mu \leq \lambda$ and $L_{d,\delta}(\lambda)$ appears exactly once [GL96, 3.6].

Remark 1.24. It will be important for us that the cell module $\Delta_{d,\delta}(\lambda)$ for $\lambda \in \Lambda_d(\delta)$ is the maximal quotient of $P_{d,\delta}(\lambda)$ by the submodule generated by the images of all morphism from any $P_{d,\delta}(\mu)$ with $\mu > \lambda$, and all the endmorphisms in the maximal ideal of the local ring $\operatorname{End}_{\operatorname{Br}_d(\delta)}(P_{d,\delta}(\lambda))$. This follows from [GL96, (2.9), (3.7)].

Definition 1.25. We denote by $(P_{d,\delta}(\lambda) : \Delta_{d,\delta}(\mu)) =: \mathrm{d}_{\lambda,\mu} := [\Delta_{d,\delta}(\mu) : L_{d,\delta}(\lambda)]$ the cell- respectively Jordan-Hoelder multiplicities (which agree by [GL96, 3.7]). Note that then

$$\dim \operatorname{Hom}_{\operatorname{Br}_d(\delta)}(P_{d,\delta}(\lambda), P_{d,\delta}(\mu)) \quad = \quad \sum_{\nu} \mathrm{d}_{\lambda,\nu}\, \mathrm{d}_{\mu,\nu}\,. \tag{1.9}$$

Remark 1.26. Using the multiplicity formulas $\mathrm{d}_{\lambda,\mu}$ and Remark 1.18 one can inductively compute the dimensions of the projective and (in principle) of the irreducible modules.

An important observation is that $\mathrm{d}_{\lambda,\mu}$ are parabolic Kazhdan-Lusztig polynomials $\mathrm{d}_{\lambda,\mu}(q)$ of type $(\mathrm{D}_N, \mathrm{A}_{N-1})$ for large enough N evaluated at 1, see [CVM09]. We will recall in Lemma 6.20 a graphical interpretation of $\mathrm{d}_{\lambda,\mu}$ and $\mathrm{d}_{\lambda,\mu}(q)$. Important fur us is that these Kazhdan-Lusztig polynomials are always monomials, see [LS12] for a proof and for a general treatment. Hence all the multiplicities $\mathrm{d}_{\lambda,\mu}$ are always zero or one.

2. Abelianized Deligne category $\mathcal{D}(\delta)$ and highest weight structures

In this section we show that, for any $\delta \in \mathbb{C}$, the category of representations of $\operatorname{Rep}_\delta$ is an upper finite highest weight category in the sense of [BS18].

[1]We often think of these up-down-tableaux as describing paths of length d in a Bratelli type diagram. The symbol 🚶_d should indicate that for each element one walks d steps in this diagram.

2.1. Representations of Rep_δ.

Definition 2.1. A *representation* of Rep_δ is a contravariant functor from Rep_δ to Vect, the category of finite dimensional complex vector spaces. We denote the abelian category of all representations of Rep_δ by $\mathcal{D}(\delta)$ and call it the *abelianized Deligne category.*

Definition 2.2. For $\lambda \in \Lambda$ we define $\mathcal{P}_\delta(\lambda) = \mathrm{Hom}_{\mathrm{Rep}_\delta}(\_, \mathtt{R}_\delta(\lambda))$ in $\mathcal{D}(\delta)$, i.e. the functor that sends an object $M \in \mathrm{Rep}_\delta$ to $\mathrm{Hom}_{\mathrm{Rep}_\delta}(M, \mathtt{R}_\delta(\lambda))$ and a map $f : M \to M'$ to $\_ \circ f : \mathrm{Hom}_{\mathrm{Rep}_\delta}(M', \mathtt{R}_\delta(\lambda)) \to \mathrm{Hom}_{\mathrm{Rep}_\delta}(M, \mathtt{R}_\delta(\lambda))$.

Lemma 2.3. *Let $\lambda, \mu \in \Lambda$.*

(1) $\mathcal{P}_\delta(\lambda) \in \mathcal{D}(\delta)$ is an indecomposable projective object.
(2) $\mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\lambda), \mathcal{P}_\delta(\mu)) \cong \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu))$.
(3) For $\lambda \in \Lambda_d(\delta)$ we have $\dim \mathrm{Hom}_{\mathcal{D}(0)}(\mathcal{P}_0(\varnothing), \mathcal{P}_0(\lambda)) = (P_{d,0}(\lambda) : \Delta_{d,0}(\varnothing))$, and there is moreover an embedding of functors

$$\mathcal{P}_0(\varnothing)^{\oplus(P_{d,0}(\lambda):\Delta_{d,0}(\varnothing))} \quad \hookrightarrow \quad \mathcal{P}_\delta(\lambda). \tag{2.10}$$

Note that $\mathcal{D}(\delta)$ makes sense for $\delta \in \mathbb{C}$, but gives a semisimple category if $\delta \notin \mathbb{Z}$.

Proof. Statement (2) is just (the contravariant version of) the Yoneda lemma. Now $\mathcal{P}_\delta(\lambda)$ is indecomposable, since the only non-trivial idempotent of its endomorphism ring is the identity. For the projectivity assume we have $\mathrm{F}, \mathrm{G} \in \mathcal{D}(\delta)$ with $\alpha_1 \in \mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\lambda), \mathrm{G})$, and $\alpha_2 \in \mathrm{Hom}_{\mathcal{D}(\delta)}(\mathrm{F}, \mathrm{G})$ an epimorphism. We want to construct $\beta \in \mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\lambda), \mathrm{F})$ such that $\alpha_2 \circ \beta = \alpha_1$. For this take the identity map $e_\lambda \in \mathrm{End}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda)) = \mathcal{P}_\delta(\lambda)(\mathtt{R}_\delta(\lambda))$ and pick an element $b \in \alpha_2^{-1}(\alpha_1(e_\lambda)) \subseteq \mathrm{F}\,\mathtt{R}_\delta(\lambda)$. Then for $f \in \mathcal{P}_\delta(\lambda)(\mathcal{P}_\delta(\mu)) = \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\mu), \mathtt{R}_\delta(\lambda))$ set $\beta(f) = \mathrm{F}(f)(b)$ which is obviously a morphism in $\mathcal{D}(\delta)$. The definition of β and b imply

$$\alpha_2 \circ \beta(f) = \alpha_2(\mathrm{F}(f)(b)) = \mathrm{G}(f)(\alpha_2(b)) = \mathrm{G}(f)(\alpha_1(e_\lambda)) = \alpha_1(e_\lambda \circ f) = \alpha_1(f),$$

since the α_i's are natural transformations of functors. We use (2) and observe that $e_\varnothing$ is the identity morphism of 0. Thus, $\mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\varnothing), \mathcal{P}_\delta(\lambda)) \cong e_\lambda \mathrm{Hom}_{\mathrm{Rep}_\delta}(0, |\lambda|)$. But the latter is $e_\lambda \Delta_{d,\delta}(\varnothing)$ by the construction of cell modules (see (1.6)). It has dimension equal to $(\Delta_{d,\delta}(\varnothing) : L_{d,\delta}(\lambda)) = (P_{d,0}(\lambda) : \Delta_{d,0}(\varnothing))$. If we pick a basis $\{g_i\}$ for $\mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\varnothing), \mathcal{P}_\delta(\lambda)) = \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\varnothing), \mathtt{R}_\delta(\lambda))$ then $g = (\oplus_i g_i)\circ\_$ defines a map as required. (It is obviously injective, since diagrammatically it puts a certain fixed linear combination of $(0, |\lambda|)$-Brauer diagrams on top of $(|\mu|, 0)$-Brauer diagrams.) □

Definition 2.4. We denote by $\mathcal{L}_\delta(\lambda)$ be the unique irreducible quotient of $\mathcal{P}_\delta(\lambda)$ in $\mathcal{D}(\delta)$.

Remark 2.5. Observe that the category Rep_δ is a *finite dimensional category*, that is a small category with finite dimensional morphism spaces

$$e_\lambda A e_\mu := \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu)).$$

Equivalently, the (infinite dimensional and non-unital) algebra $A := \bigoplus_{\lambda,\mu} e_\lambda A e_\mu$ is a *locally finite dimensional locally unital algebra* in the sense of [BS18, Remark 2.3] and the category $\mathcal{D}(\delta)$ can be identified with the category of *locally finite dimensional* A-modules. Hereby $\mathcal{P}_\delta(\lambda)$ corresponds to the A-module Ae_λ.

Objects in $\mathcal{D}(\delta)$ are not of finite length in general. For objects M, L with L irreducible we define the *composition multiplicity* $[M : L]$ to be the supremum of the sizes of the sets $\{i = 1, \ldots, n \mid M_i/M_{i-1} \cong L\}$ taken over all finite filtrations $0 = M_0 < M_1 < \cdots < M_n = M$ by subobjects in $\mathcal{D}(\delta)$. This generalizes the Jordan-Hoelder multiplicities.

Lemma 2.6. *The composition multiplicities $[\mathcal{P}_\delta(\lambda) : \mathcal{L}_\delta(\mu)]$ are all finite.*

Proof. This holds in any finite dimensional category by [BS18, Section 2], in particular for Rep_δ by Remark 2.5. □

2.2. **Upper finite highest weight structure** Rep_δ**.** For $\lambda \in \Lambda$ we construct important objects in $\mathcal{D}(\delta)$ as follows. For each $\nu \geq \lambda$ pick a finite (which exists by (1.5)) generating set $G_{\nu,\lambda}$ (as vector space) of $\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\nu), \mathtt{R}_\delta(\lambda))$ in case $\nu > \lambda$ and of the unique maximal ideal $\mathfrak{m}_\lambda$ in $\mathrm{End}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda))$ in case $\nu = \lambda$. Note that $\mathfrak{m}_\lambda$ is indeed unique, since $\mathtt{R}_\delta(\lambda)$ is indecomposable, hence its endomorphism ring is local. Set $m_{\nu,\lambda} = |G_{\nu,\lambda}|$ and let

$$\mathfrak{f}^\lambda = \oplus_\nu \mathfrak{f}^\lambda_\nu : \bigoplus_{\nu \geq \lambda} \mathtt{R}_\delta(\nu)^{m_{\nu,\lambda}} \longrightarrow \mathtt{R}_\delta(\lambda) \quad \text{and} \quad \mathfrak{f}^\lambda_\nu := \oplus_{g_\nu \in G_{\nu,\lambda}} g_\nu. \tag{2.11}$$

Note that the direct sums are finite, the first by definition of the reversed inclusion ordering and the last by the definition of Rep_δ. Consider the Yoneda embedding functor

$$\mathcal{Y} := \mathrm{Hom}_{\mathrm{Rep}_\delta}(\_, ?) : \mathrm{Rep}_\delta \to \mathcal{D}(\delta), \qquad M \mapsto \mathrm{Hom}_{\mathrm{Rep}_\delta}(\_, M),$$

especially $\mathcal{Y}(\mathtt{R}_\delta(\mu)) = \mathcal{P}_\delta(\mu)$. It has image in the additive subcategory of $\mathcal{D}(\delta)$ generated by the projective objects $\mathcal{P}_\delta(\lambda)$'s. Applying $\mathcal{Y}$ to $\mathfrak{f}^\lambda$ from (2.11) yields a morphism in $\mathcal{D}(\delta)$

$$\mathcal{Y}(\mathfrak{f}^\lambda) : \quad \bigoplus_{\nu \geq \lambda} \mathcal{P}_\delta(\nu)^{m_{\nu,\lambda}} \longrightarrow \mathcal{P}_\delta(\lambda).$$

Since $\mathcal{D}(\delta)$ is abelian, $\mathcal{Y}(\mathfrak{f}^\lambda)$ has image, $\mathrm{im}(\mathcal{Y}(\mathfrak{f}^\lambda))$, and cokernel, $\mathrm{coker}(\mathcal{Y}(\mathfrak{f}^\lambda))$, in $\mathcal{D}(\delta)$.

Definition 2.7. For $\lambda \in \Lambda$ define the representation $\Delta_\delta(\lambda) := \mathrm{coker}(\mathcal{Y}(\mathfrak{f}^\lambda))$. We call it the *standard representation* of Rep_δ corresponding to λ.

Example 2.8. As special cases we obtain directly $\Delta_\delta(\varnothing) = \mathcal{P}_\delta(\varnothing)$ and $\Delta_\delta(\square) = \mathcal{P}_\delta(\square)$.

The standard representations are directly connected with the cell modules:

Lemma 2.9. *Let λ be a partition. Then $\Delta_\delta(\lambda)$ is isomorphic to the representation which sends $\mathtt{R}_\delta(\mu)$ to $\mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), \Delta_{d,\delta}(\lambda))$ for some (and then any) d satisfying $\lambda, \mu \in \Lambda_d(\delta)$, and a morphism $\alpha : \mathtt{R}_\delta(\nu) \to \mathtt{R}_\delta(\mu)$ to the precomposition $\_ \circ \alpha$ with α. Hereby we use the identifications from Remark 1.16.*

Proof. This is obvious from the definitions except in case $\lambda = \varnothing$. Note that $\mathcal{P}_\delta(\varnothing)(\mathtt{R}_\delta(\mu)) = \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\mu), \mathtt{R}_\delta(\varnothing)) = \mathrm{Hom}_{\mathrm{Rep}_\delta}(|\mu|, 0)e_\mu \cong (e_\mu \Delta_{d,\delta})^*$. But the latter can be identified with $\mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), \Delta_{d,\delta}(\varnothing))$ whenever $\mu, \varnothing \in \Lambda_d^+(\delta)$. □

Theorem 2.10 (Standard filtrations)**.** *For any partition λ, the representation $\mathcal{P}_\delta(\lambda)$ has a filtration with subquotients isomorphic to some $\Delta_\delta(\mu)$'s with $\mu \geq \lambda$, such that the following multiplicity formulas hold for $\lambda \in \Lambda_d(\delta)$, $\mu \in \Lambda_d(\delta) \cup \{\varnothing\}$,*

$$(\mathcal{P}_\delta(\lambda) : \Delta_\delta(\mu)) \quad = \quad (P_{d,\delta}(\lambda) : \Delta_{d,\delta}(\mu)). \tag{2.12}$$

Proof. Choose $\eta \in \Lambda$ maximal with $m_\eta := \dim \mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\eta), \mathcal{P}_\delta(\lambda)) \neq 0$. This exists, since $m_\lambda \neq 0$ and Λ has a unique maximal element. Pick a basis $\{g_i \mid 1 \leq i \leq m_\eta\}$ for this space and set $\gamma_\eta = \oplus_{i=1}^{m_\eta} g_i : \mathcal{P}_\delta(\eta)^{m_\eta} \to \mathcal{P}_\delta(\lambda)$. We claim that

$$\gamma_\eta \text{ factors through } {\Delta_\delta}^{m_\eta} \text{ and induces an isomorphism } \overline{\gamma}_\eta : \Delta_\delta(\eta)^{m_\eta} \cong \mathrm{im}\, \gamma_\eta.$$

This is clear in case $\eta = \varnothing$ by Example 2.8 and Lemma 2.3. Hence assume $\eta \neq \varnothing$.

To see that $\overline{\gamma}_\eta$ is well-defined we have to show that $\gamma_\eta \circ \mathcal{Y}(\mathfrak{f}^\eta) = 0$. Assume not, then $g_i \circ \mathcal{Y}(\mathfrak{f}^\eta_\nu) \neq 0$ for at least one pair (i, ν) with $\nu \geq \eta$. Now $\nu > \eta$ would give a contradiction to the maximality of η. If $\nu = \eta$, thus in particular $\nu \neq \varnothing$, then apply the functors to $\mathtt{R}_\delta(\mu)$ for arbitrary $\mu \in \Lambda$, with $\mu \neq \varnothing$ if $\delta = 0$. (The case $\mu = \varnothing$ is trivial, since $\Delta_\delta(\eta)^{m_\eta}(\mathtt{R}_\delta(\varnothing))$ is trivial by the assumption on η and thus $\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\varnothing), \mathtt{R}_\delta(\lambda)) = \mathrm{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\varnothing), \mathtt{R}_\delta(\lambda)) = \{0\} = \Delta_\delta(\eta)^{m_\eta}(\mathtt{R}_\delta(\varnothing))$.) Now consider the top square in the following diagram (with d

such that $\eta, \mu \in \Lambda_d(\delta)$). Hereby the top horizontal identification is obvious and induces via Lemma 2.9 an isomorphism α_2 as indicated.

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\mu), \mathtt{R}_\delta(\eta)^{m_\eta}) & = & \operatorname{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), P_{d,\delta}(\eta)^{m_\eta}) \\
\downarrow & & \downarrow \\
\Delta_\delta(\eta)^{m_\eta}(\mathtt{R}_\delta(\mu)) & \overset{\alpha_2}{\dashrightarrow} & \operatorname{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), \Delta_{d,\delta}(\eta)^{m_\eta}) \\
\downarrow \overline{\gamma}_{\eta|\mathtt{R}_\delta(\mu)} & & \\
\operatorname{im} \gamma_\eta(\mathtt{R}_\delta(\mu)) & & \downarrow \alpha_1 \\
\hookdownarrow & & \\
\operatorname{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\mu), \mathtt{R}_\delta(\lambda)) & = & \operatorname{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), P_{d,\delta}(\lambda)).
\end{array}
\tag{2.13}
$$

The bottom identification is also obvious and the outer bended arrows are induced by γ_η making the outer square commute. By definition of $\Delta_{d,\delta}(\eta)$ and the construction of a cell module filtration of $P_{d,\delta}(\lambda)$, the map on the right factors through $\operatorname{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\mu), \Delta_{d,\delta}(\eta)^{m_\eta})$ In fact, $\Delta_{d,\delta}(\eta)^{m_\eta}$ is a submodule of $P_{d,\delta}(\lambda)$ at the bottom of a cell module filtration, and hence the map on the left has to factor as well. Thus, $\overline{\gamma}_\eta$ is well-defined such that (2.13) commutes. Now α_1 must be injective, since $\Delta_{d,\delta}(\eta)^{m_\eta}$ is a submodule of $P_{d,\delta}(\lambda)$ and $P_{d,\delta}(\mu)$ is projective. Since α_2 is an isomorphism, a diagram chasing implies that $\overline{\gamma}_\eta$ is a monomorphism on any $\mathtt{R}_\delta(\mu)$. For $\mu = \varnothing$ we have $\operatorname{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\varnothing), \mathtt{R}_\delta(\lambda)) = \operatorname{Hom}_{\mathcal{D}(\delta)}(\mathcal{P}_\delta(\varnothing), \mathtt{R}_\delta(\lambda)) = \{0\} = \Delta_\delta(\eta)^{m_\eta}(\mathtt{R}_\delta(\varnothing))$ by assumption on η. Thus, $\overline{\gamma}_\eta$ is a monomorphism and therefore an isomorphism, since it was by definition an epimorphism. ☐

As a consequence, $\mathcal{D}(\delta)$ is a highest weight category (with an infinite poset):

Corollary 2.11. *For any $\delta \in \mathbb{C}$, the category $\mathcal{D}(\delta)$ is an upper finite highest weight category in the sense of* [BS18] *with standard objects $\Delta_\delta(\lambda)$, $\lambda \in \Lambda$.*

Proof. The poset Λ of all partitions with reversed inclusion ordering is an upper finite set (with maximal element $\varnothing$) which is in bijection with the irreducible modules in $\mathcal{D}(\delta)$. By Theorem 2.10, $\mathcal{D}(\delta)$ satisfies for $\delta \in \mathbb{Z}$ the property $(P\Delta_\epsilon)$ from [BS18] for ϵ the constant function $+$ with $\Delta_\delta(\lambda)$ defined as in [BS18, (1.3)]. For $\delta \notin \mathbb{Z}$ this is property is obvious. Hence $\Delta_\delta(\lambda)$ is an ϵ-stratified upper finite category in the sense of [BS18, Definition 3.32] by Remark 2.5. It is highest weight by [BS18, Lemma 3.6] since $(\Delta_\delta(\lambda) : \mathcal{L}_\delta(\lambda)) = 1$ for any $\lambda \in \Lambda$ by Lemma 2.9. ☐

We denote by $(\mathcal{P}_\delta(\mu) : \Delta_\delta(\lambda))$ the multiplicity of $\Delta_\delta(\lambda)$ in a (and then also any by [BS18, Lemma 3.38]) *standard flag*, i.e. a finite filtration of $\mathcal{P}_\delta(\mu)$ with subquotients isomorphic to standard representations (as given by Theorem 2.10).

Remark 2.12. We could also construct the standard representations diagrammatically using the concept of Borelic pairs as in [CZ19] or (weakly) triangular decompositions as in [BS18, Section 5] or in [MGS20]. Both constructions restrict to the description (1.6) of standard modules for $\mathrm{Br}_d(\delta)$ via Lemma 2.9. They all mimic the classical Lie theoretic constructions of Verma modules in a quite elegant way. We chose the different construction based on Lemma 2.9 and Remark 1.24 here, since it can directly be transferred to the graded setting in Section 11 without using an explicit isomorphism or explicit multiplication rules.

Lemma 2.13. *For any $\lambda, \mu \in \Lambda$ we have the following multiplicity formulas.*

(1) If $\lambda, \mu \in \Lambda_d(\delta)$ for some $d \in \mathbb{Z}_{\geq 0}$, then $[\mathcal{P}_\delta(\lambda) : \mathcal{L}_\delta(\mu)] = [P_{d,\delta}(\lambda) : L_{d,\delta}(\mu)]$ and $[\mathcal{P}_\delta(\lambda) : \mathcal{L}_\delta(\mu)] = 0$ otherwise.

(2) $[\Delta_\delta(\lambda) : \mathcal{L}_\delta(\mu)] = (\mathcal{P}_\delta(\mu) : \Delta_\delta(\lambda))$ for any $\lambda, \mu \in \Lambda$.

Proof. Part (1) follows directly from Lemma 2.3 and (1.5), cf. also [BS18, Theorem 3.37]. For the second observe first that $(\mathcal{P}_\delta(\mu) : \Delta_\delta(\lambda)) = [\nabla_\delta(\lambda) : \mathcal{L}_\delta(\mu)]$ for any $\lambda, \mu \in \Lambda$ by [BS18, Corollary 3.38], where $\nabla_\delta(\lambda)$ denote the costandard module in the sense of [BS18, (5.2)]. Flipping the diagrams representing morphisms in the Deligne category define an equivalence of categories between the Deligne category and its opposite. Together with the ordinary duality on Vect this defines a duality on $\mathcal{D}(\delta)$ which preserves the simple objects and sends $\Delta_\delta(\lambda)$ to $\nabla_\delta(\lambda)$. Therefore, $[\Delta_\delta(\lambda) : \mathcal{L}_\delta(\mu)] = [\nabla_\delta(\lambda) : \mathcal{L}_\delta(\mu)]$ and the claim follows. □

2.3. **Induction functors.** Consider the endofunctor ind $= \_ \boxtimes \mathtt{R}_\delta(\square)$ of Rep_δ given by tensoring with $\mathtt{R}_\delta(\square)$. Diagrammatically it adds to each basis morphism a strand to the right of each diagram. This functor induces an endofunctor ind $= \_ \boxtimes \mathtt{R}_\delta(\square)$ on $\mathcal{D}(\delta)$ by sending $M \in \mathcal{D}(\delta)$ to $F \boxtimes \mathtt{R}_\delta(\square)$ defined by $M \boxtimes \mathtt{R}_\delta(\square)(\mathtt{R}_\delta(\mu)) = M(\mathtt{R}_\delta(\mu) \boxtimes \mathtt{R}_\delta(\square))$ with the obvious assignment on morphisms. By definition, there are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\mu), \mathtt{R}_\delta(\lambda) \boxtimes \mathtt{R}_\delta(\square)) \quad \cong \quad \mathrm{Hom}_{\mathrm{Br}_{d+1}(\delta)} \left(P_{d,\delta}(\mu), \mathrm{ind}_d^{d+1} P_{d,\delta}(\lambda) \right) \qquad (2.14)$$

for $d = |\lambda| = |\mu| - 1$. We call these functors simply *induction functors.*

In case of $\mathcal{D}(\delta)$ or $\mathrm{Br}_d(\delta)$-mod they are right exact and preserve obviously the additive subcategories $\mathrm{Proj}(\mathcal{D}(\delta))$ respectively $\mathrm{Proj}(\mathrm{Br}_d(\delta)\text{-mod})$ generated by projective objects.

Definition 2.14. For $i \in \mathbb{Z} + \delta/2$ we define the i-induction functor

$$i\text{-ind} : \mathrm{Rep}_\delta \to \mathrm{Rep}_\delta \quad M \mapsto \mathrm{proj}_i \left(M \boxtimes \mathtt{R}_\delta(\square) \right),$$

where proj_i is the projection onto the generalized i-eigenspace of $\xi_{|\lambda|+1}$ viewed as an element in $\mathrm{End}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda) \boxtimes \mathtt{R}_\delta(\square))$ for an indecomposable object $\mathtt{R}_\delta(\lambda)$. This is extended to arbitrary objects M and is well-defined due to Lemma 2.15 below. We denote by i-ind also the induced endofunctor on $\mathcal{D}(\delta)$.

Lemma 2.15. *Let $\lambda, \mu \in \Lambda$ and $f \in \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu))$. Then*

$$(f \boxtimes \mathrm{id}) \circ \xi_{|\lambda|+1} = \xi_{|\mu|+1} \circ (f \boxtimes \mathrm{id}) \ \ \textit{in} \ \ \mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda) \boxtimes \mathtt{R}_\delta(\square), \mathtt{R}_\delta(\mu) \boxtimes \mathtt{R}_\delta(\square)).$$

Proof. Note that the formula makes sense, since $\xi_{|\lambda|+1}$ commutes with $e_\lambda \boxtimes \mathrm{id}$ and thus can be regarded as an element in $\mathrm{End}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda) \boxtimes 1)$. Since f is a linear combinations of elements of the form $e_\mu \circ \tilde{f} \circ e_\lambda$, for $\tilde{f}$ a $(|\lambda|, |\mu|)$-Brauer diagram, it is enough to check the claim for f a generating morphism for $\mathrm{Br}(\delta)$. For the crossing this is obvious. So it remains the case when f is a single cup or cap flanked by a number of identity strands. This can be checked by a quick calculation which we omit. □

Lemma 2.16. *Let $\lambda \in \Lambda$ and $i \in \mathbb{Z} + \delta/2$. Then $i\text{-ind}(\Delta_\delta(\lambda)) \in \mathcal{D}_\Delta(\delta)$ with a multiplicity free standard filtration with subquotients of the form $\Delta_\delta(\mu)$, where μ is obtained from λ by adding a box of content $i - \frac{\delta-1}{2}$ or removing a box of content $-(i - \frac{\delta-1}{2})$.*

Proof. The first claim follows directly from Lemma 2.9, Lemma 1.20 and (2.14). (Note that $i\text{-ind}(\Delta_\delta(\lambda))$ is obtained from $\mathrm{ind}(\Delta_\delta(\lambda))$ by projecting onto a certain generalized eigenspace of the *central* element $\Omega_{|\lambda|+1}$. Hence having a standard filtration is preserved under this projection.) The second statement follows with Lemma 2.9 and (2.14) from Lemma 1.19, since $\xi_{|\lambda|+1}$ acts on $\Delta_\delta(\mu)$ as $1/2 \left(\Omega_{|\lambda|+1} - \Omega_{|\lambda|} \right)$. □

Thus, i-induction behaves well on standard representations; moreover ind $= \oplus_{i \in \mathbb{Z}+\delta/2} i\text{-ind}$.

3. Application: The first categorification theorem

We like to establish an analogue of the famous Ariki-Grojnowski categorification result, [Ari96], [Gro99] for the category $\mathcal{D}(\delta)$. In this section we explain how the representation category $\mathcal{D}(\delta)$ of Rep_δ together with induction functors categorifies the Fock space of charge $\frac{\delta-1}{2}$ with an action of a fixed point subalgebra of $\mathfrak{gl}_{\mathbb{Z}+\delta/2}$ with respect to an involution.

For background and details on Fock spaces see e.g. [Lec12], [KR87].

Definition 3.1. Let $\mathcal{D}_\Delta(\delta)$ be the full subcategory of $\mathcal{D}(\delta)$ given by all objects which have a standard flag. Consider the free abelian group of isomorphism classes $[M]$ of objects M in $\mathcal{D}_\Delta(\delta)$ modulo the relation $[M]=[M_1]+[M_2]$ if there is a short exact sequence in $\mathcal{D}(\delta)$ of the form $M_1 \to M \to M_2$. We call this the *Grothendieck group of* $\mathcal{D}_\Delta(\delta)$ and denote by $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ its complexification. By abuse of notation we will always write $[M]$ instead of $[M]\otimes 1$ for the corresponding vector in the complexified Grothendieck group.

By Theorem 2.10, $\mathcal{P}_\delta(\lambda)$ is an object in $\mathcal{D}_\Delta(\delta)$ for any $\lambda \in \Lambda$. Obviously, the classes $[\Delta_\delta(\lambda)]$, $\lambda \in \Lambda$ form a basis of $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$. Starting from $\Delta_\delta(\varnothing)=\mathcal{P}_\delta(\varnothing)$ it follows from Theorem 2.10 by induction on the poset Λ that the $[\mathcal{P}_\delta(\lambda)]$, $\lambda\in\Lambda$, form a basis as well.

Definition 3.2. Let $\mathbb{V}_\delta$ be the vector space on basis $\{v_i \mid i \in \mathbb{Z}+\delta/2\}$. The corresponding *vacuum vector* is defined as the formal expression

$$v_\varnothing \quad = \quad v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\wedge v_{i_4}\wedge\cdots \tag{3.15}$$

of basis vectors from $\mathbb{V}_\delta$, such that the $i_j=-(\delta/2+j-1)$. The *Fock space of charge* $\frac{\delta-1}{2}$ is the vector space $\bigwedge^{\infty/2}\mathbb{V}_\delta$ with basis all formal semiinfinite wedges

$$v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\wedge v_{i_4}\wedge\cdots$$

where the indices are strictly decreasing and $i_j \neq -(\frac{\delta}{2}+j-1)$ for only finitely many $j\in\mathbb{Z}_{>0}$.

As usual, we can identify the basis vectors of $\bigwedge^{\infty/2}\mathbb{V}_\delta$ with partitions. More precisely, given a partition λ we attach the unique basis vector v_δ^λ with set of indices

$$X(\lambda) \quad = \quad \left\{\lambda_i^t - i + 1 - \delta/2 \,\middle|\, i \geq 1\right\} \subset \mathbb{Z}+\delta/2. \tag{3.16}$$

We get isomorphisms of vector spaces as in (10.46).

To describe the action of the induction functors on $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ we consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{gl}(\mathbb{V}_\delta)$, the Lie algebra of all complex matrices $(a_{i,j})$ where $i,j\in\mathbb{Z}+\delta/2$, with only finitely many nonzero entries. Then $\mathcal{U}(\mathfrak{g})$ has the usual Chevalley generators $E_{i+1/2,i-1/2}$, $E_{i-1/2,i+1/2}$ for $i\in\mathbb{Z}+\frac{\delta+1}{2}$ and $E_{i,i}$ for $i\in\mathbb{Z}+\delta/2$ subject to the type A Serre relations; it comes with the natural representation $\mathbb{V}_\delta$. Let θ be the involution on $\mathfrak{gl}(\mathbb{V}_\delta)$ given by $E_i\mapsto F_{-i}$, and $\mathfrak{gl}(\mathbb{V}_\delta)^\theta$ the Lie subalgebra of fixed points with universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$.

Lemma 3.3. *If δ is odd, then $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ is isomorphic the algebra $\mathcal{H}^\jmath_{\mathrm{class}}$ with generators*

$$\{B_i \mid i\in\mathbb{Z}\}\cup\{H_i \mid i\in\mathbb{Z}_{\geq 0}+1/2\},$$

subject to the following defining relations

1. *The B_i, B_{-i}, for $i\in\mathbb{Z}_{\geq 1}$, and H_j, for $j\in\mathbb{Z}_{\geq 0}+1/2$ satisfy the usual defining relations of $\mathcal{U}(\mathfrak{gl}_\mathbb{N}(\mathbb{C}))$ with B_i and B_{-i} taking the place of the generator E_i and F_i respectively.*
2. *The generator B_0 commutes with all generators except for B_1 and B_{-1}, for which we have the relations*

$$B_1^2B_0-2B_1B_0B_1+B_0B_1^2=0, \quad B_0^2B_1-2B_0B_1B_0+B_1B_0^2=B_1,$$

$$B_{-1}^2B_0-2B_{-1}B_0B_{-1}+B_0B_{-1}^2=0, \quad B_0^2B_{-1}-2B_0B_{-1}B_0+B_{-1}B_0^2=B_{-1}.$$

Lemma 3.4. *If δ is even, then $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ is isomorphic the algebra $\mathcal{H}_{\mathrm{class}}$ with generators*

$$\{B_i \mid i\in\mathbb{Z}+1/2\}\cup\{H_i \mid i\in\mathbb{Z}_{\geq 0}\},$$

subject to the following relations

1. *The B_i, B_{-i}, for $i\in\mathbb{Z}_{\geq 1}+1/2$, and H_j, for $j\in\mathbb{Z}_{\geq 1}$ satisfy the usual defining relations for $\mathcal{U}(\mathfrak{gl}_\mathbb{N}(\mathbb{C}))$ with B_i taking the place of the generator E_i and B_{-i} the place of F_i.*

(2) *The generator* H_0 *commutes with all* $B_{\pm i}$, *for* $i \neq 1/2$ *and* H_i, *and both* $B_{\pm 1/2}$ *commute with all generators apart from* H_0, H_1, *and* $B_{\pm 3/2}$, *for which we have the following relations*

(a) *Commutativity relations*

$$[H_0, B_{1/2}] = -B_{1/2}, \quad [H_0, B_{-1/2}] = B_{-1/2}, \quad [H_1, B_{-1/2}] = -B_{-1/2}, \quad [H_1, B_{1/2}] = B_{1/2}.$$

(b) *Serre relations*

$$B_{3/2}^2 B_{1/2} - 2B_{3/2}B_{1/2}B_{3/2} + B_{1/2}B_{3/2}^2 = 0, \quad B_{-3/2}^2 B_{-1/2} - 2B_{-3/2}B_{-1/2}B_{-3/2} + B_{-1/2}B_{-3/2}^2 = 0,$$
$$B_{1/2}^2 B_{3/2} - 2B_{1/2}B_{3/2}B_{1/2} + B_{3/2}B_{1/2}^2 = 0, \quad B_{-1/2}^2 B_{-3/2} - 2B_{-1/2}B_{-3/2}B_{-1/2} + B_{-3/2}B_{-1/2}^2 = 0.$$

(c) *Modified Serre relations*

$$B_{1/2}^2 B_{-1/2} - 2B_{1/2}B_{-1/2}B_{1/2} + B_{-1/2}B_{1/2}^2 = -4B_{1/2},$$
$$B_{-1/2}^2 B_{1/2} - 2B_{-1/2}B_{1/2}B_{-1/2} + B_{1/2}B_{-1/2}^2 = -4B_{-1/2}.$$

Proofs of Lemma 3.3 and Lemma 3.4. The assignments

$$B_i \mapsto E_{i-1/2,i+1/2} + E_{-i+1/2,-i-1/2} \quad \text{and} \quad H_j \mapsto E_{j,j} + E_{-j,-j} \tag{3.17}$$

preserve the relations in both, $\mathcal{H}_{\text{class}}^{\lrcorner}$ and $\mathcal{H}_{\text{class}}$, and thus define an algebra homomorphism $\mathcal{H}_{\text{class}}^{\lrcorner} \to \mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ respectively $\mathcal{H}_{\text{class}} \to \mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ depending on whether δ is odd or even. Now $(\mathfrak{gl}(\mathbb{V}_\delta), \mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ is the limit $n \to \infty$ of the symmetric pair $(\mathfrak{gl}_{2n}, \mathfrak{gl}_n \oplus \mathfrak{gl}_n)$ of type (AIII) in the odd case (with a similar limit version of $(\mathfrak{gl}_{2n+1}, \mathfrak{gl}_{n+1} \oplus \mathfrak{gl}_n)$ in the even case), see [Hel01, Table V] for the classification and [ES18, Section 9] for an explicit isomorphism. Then our map is an isomorphism by [Let03, Section 7] with [ES18, Section 9.2]. □

Now $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ is a Hopf subalgebra of $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta))$ which acts on $\bigwedge^{\infty/2} \mathbb{V}_\delta$ by restricting the natural $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta))$-action. Via the isomorphisms (3.17) the following holds.

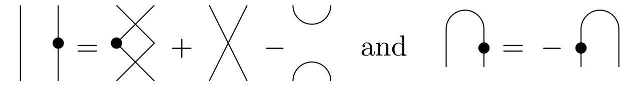
Theorem 3.5 (First categorification Theorem)**.** *The isomorphism* (10.46) *of vector spaces is an isomorphism of* $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$*-modules, where the action of* $\mathcal{U}(\mathfrak{gl}(\mathbb{V}_\delta)^\theta)$ *on* $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ *is given by sending* B_i *to* $[i\text{-ind}]$.

Proof. This is clear by Lemma 2.16, since $E_{-i+1/2,-i-1/2}$ and $E_{i-1/2,i+1/2}$ acts on $\bigwedge^{\infty/2} \mathbb{V}_\delta \cong \langle \Lambda \rangle$ by adding respectively removing a box of content i, see e.g. [Lec12, Exercise 1]. □

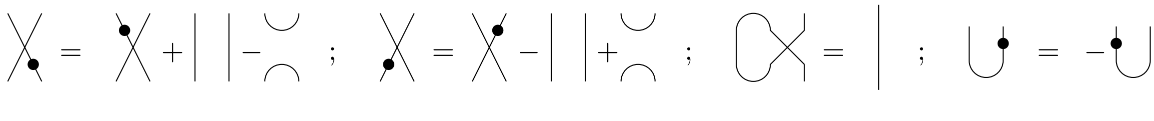
4. The affine VW-category and (cyclotomic) VW-algebras

The importance of Jucys-Murphy elements illustrated in the previous sections motivates us to define the affine VW-algebra or degenerate affine Brauer category. This will allow us to connect Brauer categories to geometry and classical Lie theory and in particular allows to define a graded version of the Deligne cateory.

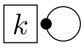
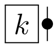
Definition 4.1. The *(affine) VW-category* is the $\mathbb{C}$-linear strict rigid monoidal category $\mathbb{W}$ generated as monoidal category by a single object $\star$, morphisms s, $\flat$, $\flat^*$ as in Proposition 1.2 and an additional generator $y = $ (dotted strand) $: \star \longrightarrow \star$ subject to the relations (1)–(3) from Proposition 1.2 and

the *dot sliding relations*:  (4.18)

Remark 4.2. As in [BDE[+]20] one can deduce the additional relations



Note that the first two relations are exactly the relations satisfied by the Jucys-Murpys elements in the Brauer algebras.

The elements $\mathrm{id}^{\otimes(k-1)} \otimes y \otimes \mathrm{id}^{\otimes(d-k)} \in \mathrm{End}_{\mathbb{W}}(d)$ are abbreviated as y_k. We omit hereby the dependence on d in the notation, since it will be clear from the context. Also abbreviate $\flat_k = \mathrm{id}^{\otimes(k-1)} \otimes \flat \otimes \mathrm{id}^{\otimes(d-k-1)}$ and similarly $\flat_k^*$ and s_k. For any $k \in \mathbb{N}$ we have endomorphisms $\omega_k =$  $\in \mathrm{End}_{\mathbb{W}}(0)$, where  denotes the k-fold composition y^k of y.

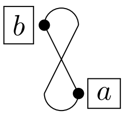
The above category was studied the first time in [RS19] following ideas of [BCDR17]. A similar category s$\mathbb{W}$ was already defined in [BDE+20] in the super setting in connection with the periplectic Lie superalgebras. The category $\mathbb{W}$ is however much richer because of the existence of the nontrivial elements ω_k (cf. [BDE+20, Lemma 5]).

Remark 4.3. Our terminology *affine VW-category* follows [ES18] and [BDE+20], (and differs from [RS19]). We like to emphasize that we work with a *degenerate* affine version, i.e. endomorphism algebras are *degenerate* affine BMW algebras that is affine VW-algebras.

Lemma 4.4. *The following relation holds in* $\mathrm{End}_{\mathbb{W}}(0)$.

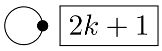
$$\omega_{2a+1} = \frac{1}{2}\left(-\omega_{2a} + \sum_{j=0}^{2a}(-1)^j \omega_{2a-j}\omega_j\right). \tag{4.19}$$

Proof. Using the relations in $\mathbb{W}$ one easily shows by induction on a that the following holds



$$= \begin{cases} \omega_{a+b} - \sum_{j=0}^{a-1}(-1)^j \omega_{a+b-j-1}\omega_j & \text{if } a \text{ is even,} \\ \omega_{a+b} + \omega_{a+b-1} - \sum_{j=0}^{a-1}(-1)^j \omega_{a+b-j-1}\omega_j & \text{if } a \text{ is odd.} \end{cases}$$

for any $a \in \mathbb{N}$ and $b \in \mathbb{Z}_{\geq 0}$. Together with the dot sliding relations we obtain

$-\omega_{2k+1} =$  $= \omega_{2k+1} - \left(\sum_{j=0}^{2k-1}(-1)^j \omega_{2k-j}\omega_j\right) + \omega_{2a} - \omega_a\omega_{2a}.$ □

In particular, the ω_k for k odd can be expressed in terms of Δ_k's for k even. It is known, see [RS19, Theorem B], that the ω_k for k even are algebraically independent and generate the endomorphism algebra of the unit object, thus $\mathrm{End}(0) \cong \mathbb{C}[\omega_{2k} \mid k \in \mathbb{Z}_{\geq 0}]$. For a more detailed description of the morphism spaces as left End(0)-module we refer to [RS19].

4.1. Cyclotomic quotients.

Definition 4.5. Recall that a (right) *tensor ideal* in a $\mathbb{C}$-linear monoidal category $\mathcal{C}$ is a collection of vector subspaces $I(d, d')$ for any pair of objects d, d' such that

$$f \otimes \mathrm{id} \in I(d \otimes b, d' \otimes b) \quad \text{and} \quad g \circ f \circ h \in I(b, c) \tag{4.20}$$

for any $f \in I(d, d')$ and objects b, c whenever $g \in \mathrm{Hom}_{\mathcal{C}}(d', c)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(b, d)$.

Definition 4.6. Given a monic polynomial $\mathtt{f}(t) = \prod_{i=1}^{\ell}(t - a_i) \in \mathbb{C}[t]$ and a set $\overline{\omega}$ of complex numbers $\overline{\omega}_k$, $k \geq 0$ satisfying the equality (4.19) (with ω_k replaced by $\overline{\omega}_k$) let $I_{\mathtt{f},\overline{\omega}}$ be the tensor ideal generated by $\mathtt{f}(y)$ and $\omega_k - \overline{\omega}_k$, $k \geq 0$. We call it the *cyclotomic ideal* attached to $\mathtt{f}$ and $\overline{\omega}$. The corresponding *cyclotomic quotient* is the quotient category $\mathbb{W}_{\mathtt{f},\overline{\omega}} = \mathbb{W}/I_{\mathtt{f},\overline{\omega}}$ which is the tensor category with the same objects as $\mathbb{W}$, but morphism spaces $\mathrm{Hom}_{\mathbb{W}_{\mathtt{f},\overline{\omega}}}(d, d') = \mathrm{Hom}_{\mathbb{W}}(d, d')/I_{\mathtt{f},\overline{\omega}}(d, d')$

Note that the cyclotomic quotient is not a monoidal category itself, but rather a right module category over $\mathbb{W}$. We call ℓ the *level* of the quotient. General cyclotomic quotients were studied in [RS19], see also [MGS20]. We need two important special cases (depending on our fixed $\delta \in \mathbb{Z}$) which were already considered in [ES18, Section 5].

Brauer quotient: This is the level 1 cyclotomic quotient defined by $\overline{\omega}_k = \delta(\frac{\delta-1}{2})^k$ and $\mathtt{f}(t) = (t - \frac{\delta-1}{2})$. The corresponding cyclotomic quotient of $\mathbb{W}$ is then isomorphic to the Brauer category $\mathrm{Br}(\delta)$ (which is in fact a monoidal category). Under this identification, the elements y_i are hereby sent to the Jucys-Murphy elements.

Isotropic Grassmannian quotient: This level 2 cyclotomic quotient, denoted $\mathbb{W}(\alpha, \beta, N)$, is defined as follows. Let $N = 2n$ be an even natural number and $\mathtt{f}(t) = (t - \alpha)(t - \beta)$ with $\alpha = 1/2(1 - \delta)$ and $\alpha + \beta = n$. Define $\overline{\omega}_k$ by the recursion $\overline{\omega}_0 = N$, $\overline{\omega}_1 = N\frac{N-1}{2}$ and $\overline{\omega}_k = (\alpha + \beta)\overline{\omega}_{k-1} - \alpha\beta\overline{\omega}_{k-2}$ for $k \geq 0$. Let $\mathbb{W}_\delta(N)$ be the corresponding cyclotomic quotient of level 2.

We are ultimately interested in a better understanding of the Brauer category. In [ES18] we connected $\mathbb{W}_\delta(\alpha, \beta, N)$ with a certain parabolic category $\mathcal{O}$ for the semisimple complex Lie algebra of type $\mathrm{D}_{\frac{N}{2}}$ and could understand this cyclotomic quotient in this way and connect it with the category of perverse sheaves on isotropic Grassmannians, [ES16b]. This is why we call this quotient the *Isotropic Grassmannian quotient.* Via an idempotent truncation, and not (!) by taking a further quotient, we will now pass from the Isotropic Grassmannian quotient (of level two) to the Brauer quotient (of level one).

For this pick a (large) even natural number N and consider the complex Lie algebra $\mathfrak{so}(N)$ of type $\mathrm{D}_{N/2}$ with a fixed maximal proper parabolic subalgebra $\mathfrak{p}$ of type $\mathrm{A}_{N/2-1}$. Let $\underline{\delta}$ be $\delta\omega$ where ω is the fundamental weight orthogonal to all simple roots for $\mathrm{A}_{N/2-1}$ and let $M(\underline{\delta})$ be the parabolic Verma module of highest weight $\underline{\delta}$. For any $d \in \mathbb{N}_0$ let $M_\delta^d = M(\underline{\delta}) \otimes (\mathbb{C}^N)^{\otimes d}$, where $\mathbb{C}^N$ is the vector representation of $\mathfrak{so}(N)$. Given N let $\mathbb{W}_{\delta,\leq}(N)$ be the full subcategory of $\mathbb{W}_\delta(N)$ given by all objects d with $N \geq 4d$.

Theorem 4.7. *There is an equivalence of categories between $\mathbb{W}_{\delta,\leq}(N)$ and the category with objects M_δ^d for $N \geq 4d$ and morphisms all $\mathfrak{so}(N)$ module homomorphisms.*

This equivalence sends the object d to M_δ^d, and the generating morphisms s, b, b^* to the flip map, the counit and unit acting on the corresponding tensor factors in M_δ^d. where the unit is given by the bilinear form on $\mathbb{C}^N \otimes \mathbb{C}^N$ defining $\mathfrak{so}(N)$. The identity element with a dot on the first strand acts by the Casimir element in $\mathfrak{so}(N) \otimes \mathfrak{so}(N)$ on the factor $M(\underline{\delta}) \otimes V$.

Proof. This is a reformulation of the isomomorphism theorem [ES18, Theorem 3.1] using the adjunctions respectively snake relations. □

The morphism spaces $\mathrm{Hom}_{\mathbb{W}_\delta(N)}(d, d')$ are finite dimensional $\big(\mathrm{End}_{\mathbb{W}_\delta(N)}(d'), \mathrm{End}_{\mathbb{W}_\delta(N)}(d)\big)$-bimodules. In particular they are bimodules over the respective subalgebras generated by the pairwise commuting elements y_i. The occurring eigenvalues are all in $\mathbb{Z} + 1/2(1 - \delta)$, [ES18, Section 3]. We denote by $\mathbf{f}$ (suppressing again the dependence on the object) the idempotent endomorphisms projecting onto the simultaneous generalized eigenspaces for all y_i's with eigenvalues ξ such that $|\xi| < \beta$. We denote by $\mathbf{f}\mathbb{W}_{\delta,\leq}(N)\mathbf{f}$ the idempotent truncation of $\mathbb{W}_{\delta,\leq}(N)$, by which we mean the category with the same objects as $\mathbb{W}_{\delta,\leq}(N)$, but with the morphism spaces $\mathrm{Hom}_{\mathbf{f}\mathbb{W}_{\delta,\leq}(N)\mathbf{f}}(d, d') = \mathbf{f}\,\mathrm{Hom}_{\mathbb{W}_{\delta,\leq}(N)}(d, d')\mathbf{f}$.

The following crucial result provides now an interesting passage from the loop parameter N to the loop parameter δ.

Theorem 4.8. *There is an equivalence of categories between the full subcategory of the Brauer category* $\mathrm{Br}(\delta)$ *given by the objects d with $N \geq 4d$ and the category $\mathbf{f}\mathbb{W}_{\delta,\leq}(N)\mathbf{f}$.*

Proof. Consider the elements $b_k := \beta + y_k$ and $b'_k := \beta - y_k$ in $\mathrm{End}_{\mathbb{W}_\delta(N)}(d)$ for any object d and $1 \leq k \leq d$, and set (for a choice of square root) $Q_k = \sqrt{\frac{b_{k+1}}{b_k}}\mathbf{f}$. For the existence and well-definedness of this element and of similar elements used in the following calculations we refer to [ES16c, Section 4]. We claim that the assignments $d \mapsto d$ on objects and

$$s_k \mapsto -\mathbf{f}Q_k s_k Q_k \mathbf{f} + \mathbf{f}\tfrac{1}{b_k}\mathbf{f}, \quad \flat_k \mapsto \mathbf{f}\flat_k Q_k \mathbf{f}, \quad \flat_k^* \mapsto \mathbf{f}Q_k \flat_k^* \mathbf{f} \tag{4.21}$$

on generating morphisms defines the desired equivalence. We need to show the well-definedness which is done in Section 15. The faithfullness, that is the surjectivity and injectivity on morphism spaces (via (4.21)), is then a direct consequence of the Isomorphism Theorem [ES16c, Theorem 4.3] using adjunctions together with the snake relations. □

5. Representation Theory of the orthosymplectic supergroup OSp(r | 2n)

In this section we pass now to the representation theory of the orthosymplectic supergroup OSp(r | 2n). The main result is a classification of the indecomposable summands in the tensor products of the natural representation for OSp(r | 2n) using Deligne categories.

5.1. **The orthosymplectic supergroup** OSp(r | 2n)**.** By a *(vector) superspace* we always mean a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $V = V_0 \oplus V_1$. For any homogeneous element $v \in V$ we denote by $|v| \in \{0,1\}$ its parity. The integer $\dim V_0 - \dim V_1$ is called the *superdimension* of V. Given a superspace V let $\mathfrak{gl}(V)$ be the corresponding *general Lie superalgebra*, i.e. the superspace $\mathrm{End}_{\mathbb{C}}(V)$ of all endomorphism with the superbracket defined on homogeneous elements by $[X,Y] = X \circ Y - (-1)^{|X|\cdot|Y|} Y \circ X$.

Assume that V is additionally equipped with a non-degenerate even super-symmetric bilinear form $\langle -,-\rangle$, i.e. a bilinear form $V \times V \to \mathbb{C}$ which is symmetric when restricted to $V_0 \times V_0$, skew-symmetric on $V_1 \times V_1$ and zero on pairs with different parities. Then the *orthosymplectic Lie superalgebra* $\mathfrak{osp}(V)$ is the Lie supersubalgebra of $\mathfrak{gl}(V)$ consisting of all endomorphisms which respect this form. Explicitly, a homogeneous element $X \in \mathfrak{osp}(V)$ has to satisfy for any homogeneous $v \in V$

$$\langle Xv, w\rangle + (-1)^{|X|\cdot|v|}\langle v, Xw\rangle = 0. \tag{5.22}$$

For more details on Lie superalgebras see for instance [Mus12], [Ser14].

From now on fix $r = \dim V_0$ and $2n = \dim V_1$ and let $m = \lfloor r/2 \rfloor$ and $\delta = r - 2n$.

5.2. **Concrete realisation.** It is sometimes convenient to work with a concrete realization of $\mathfrak{g} = \mathfrak{osp}(V)$ in terms of endomorphism of V which we identify with $\mathbb{C}^{r|n}$ fixing a homogeneous basis v_i, $1 \leq i \leq r + 2n$ of V with the first r vectors being a basis of V_0. We pick the supersymmetric bilinear form given by the (skew)symmetric matrices

$$J^{\mathrm{sym}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_m \\ 0 & \mathbf{1}_m & 0 \end{pmatrix} \qquad \text{and} \qquad J^{\mathrm{skew}} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

where $\mathbf{1}_k$ denotes the respective identity matrix and r is equal to $2m+1$ or equal to $2m$, in the latter case the first column and row of J^{sym} are removed. Then $\mathfrak{g}$ can be realized as the Lie super subalgebra of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathfrak{gl}(r \mid 2n)$ where $A^t J^{\mathrm{sym}} + J^{\mathrm{sym}} A = B^t J^{\mathrm{sym}} - J^{\mathrm{skew}} C = D^t J^{\mathrm{skew}} + J^{\mathrm{skew}} D = 0$. The even part $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) is the subset of all such matrices with $B = C = 0$ (resp. $A = D = 0$). In particular, $\mathfrak{g}_0 \cong \mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$ with its standard Cartan $\mathfrak{h} = \mathfrak{h}_0$ of all diagonal matrices. Let

$$X = X(\mathfrak{g}) = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^{n} \mathbb{Z}\delta_j \tag{5.23}$$

be the integral weight lattice whose elements we call just *weights*. Here the ε_i's and δ_j's are the standard basis vectors of $\mathfrak{h}^*$ picking out the i-th respectively $(r+j)$-th diagonal matrix entry. Let the *parity map* be 0 on the ϵ's and 1 on the δ's, and extend this (uniquely) to a map of abelian groups from the whole weight lattice to $\mathbb{Z}/2\mathbb{Z}$. We fix the symmetric bilinear pairing given on the basis vectors by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$ and $(\delta_i, \delta_j) = -\delta_{i,j}$ and zero otherwise. We will often denote weights as $(m+n)$-tuples $(a_1, a_2, \ldots, a_m \mid b_1, b_2, \ldots, b_n)$, with the ε-coefficients to the left and the δ-coefficients to the right of the vertical line. We chose a Borel as in [GS13], [ES17, Definition 2.2], so that the dominance condition looks similar to the one for semisimple Lie algebras, see Definition 13.2 for the precise condition.

5.3. **Finite dimensional representations.** Let $G_0 = \mathrm{O}(r) \times \mathrm{Sp}(2n)$ be the algebraic group corresponding the (ordinary) Lie algebra $\mathfrak{g} = \mathfrak{osp}(V)_0 \cong \mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$ with its adjoint action a on $\mathfrak{g}$, and let $G = \mathrm{OSp}(V) = \mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$ be the algebraic supergroup given by the Harish-Chandra pair $(\mathfrak{g}, G_0, \mathrm{a})$ in the sense of [Ser11, Section 3], see [ES17, 1.2] especially for the orthosymplectic case.

We denote by $\mathcal{C}_{(r\,|\,2n)}$ the category of finite dimensional representations of G, or equivalently the category of finite dimensional Harish-Chandra modules for $(\mathfrak{g}, G_0, \mathrm{a})$.

Then $\mathcal{C}_{(r\,|\,2n)}$ decomposes into a direct sum of two equivalent categories $\mathcal{F}_{(r\,|\,2n)} \oplus \pi\mathcal{F}_{(r\,|\,2n)}$, where π denotes the parity shift, and $\mathcal{F}_{(r\,|\,2n)}$ contains all objects where the parity of any weight space agrees with the parity of the corresponding weight. In the following we restrict our study to the summand $\mathcal{F} = \mathcal{F}_{(r\,|\,2n)}$.

We first state the classification of the irreducible representation in $\mathcal{F}$, see e.g. [CW12a] for the general theory and [ES17] for our special case.

Definition 5.1. An (n,m)-*hook partition* is a partition λ such that $\lambda_{n+1} \leq m$. We call such a hook partition *special* if $\lambda_{n+1} = m$. Let $\Gamma(n|m)$ be the set of (n,m)-hook partition.

Proposition 5.2. *Let $r = 2m$ or $r = 2m+1$ and consider $G = \mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$.*

(1) If r odd, the irreducible objects in $\mathcal{F}$ are in bijection with $s\Gamma_\delta(n|m) := \Gamma(n|m) \times \{\pm\}$.

(2) If r even, the irreducible objects in $\mathcal{F}$ are in bijection with the equivalence classes $s\Gamma_\delta(n|m)$ of $\Gamma(n|m) \times \{\pm\}$, where two pairs (λ, ϵ) and (λ', ϵ') are equivalent if $\lambda = \lambda'$ and they are special, or if $\lambda = \lambda'$ and additionally $\epsilon = \epsilon'$ if they are not special.

In any case we call the elements in $s\Gamma_\delta(n|m)$ *signed hook partitions.*

Proof. This is [ES17, Propositions 2.6 and 2.13] together with [ES17, Lemma 2.21]. $\square$

Remark 5.3. Note that in case of odd r, any irreducible object in $\mathcal{F}$ is in fact an irreducible representation of $\mathfrak{g}$ together with an action of the central element $-\mathbb{1} \in G$ acting by $\epsilon\,\mathrm{id}$. In particular, V corresponds to $(\square, -)$.

Definition 5.4. In part (2) of Proposition 5.2 we denote the equivalence class of (λ, ϵ) usually just by (λ, ϵ). We say this equivalence class *has no sign* if (λ, ϵ) and (λ, ϵ') are equivalent; in which case we denote the class also by $(\lambda, \pm)$.

Remark 5.5. The bijections in Proposition 5.2 between highest weights and hook partitions are explicit, but do depend on our choice of a Borel which we do not want to recall here in detail. Up to a shift by ρ, the coefficient of ϵ_i respectively δ_i for the attached highest weight is determined by the lengths of the columns respectively rows of the hook partition λ, see [ES17, Definition 2.19] for the precise relation.

We fix now bijections as in Proposition 5.2 and denote by $L(\lambda, \epsilon)$ the irreducible object in $\mathcal{F}$ corresponding to $(\lambda, \epsilon) \in \Gamma(n|m) \times \{\pm\}$. We will use the explicit bijection only in one proof, otherwise the abstract existence statement is sufficient for our purposes. The category $\mathcal{F}$ has enough projective objects which are in fact also injective, see [BKN11, Proposition 2.2.2]. We denote the projective cover of $L(\lambda, \epsilon)$ by $P(\lambda, \epsilon)$.

Example 5.6. Consider the case $G = \mathrm{OSp}(3\,|\,2)$. It this case the irreducibles $\mathcal{L}(\lambda, \epsilon)$ are labelled by a sign and and an element $\lambda \in \Gamma(1|1)$ which are the hook partitions $[a; b] := (b+1, 1^a)$ of arbitrary arm length $b+1$ and leg length a. Then $[a; b]$ corresponds to the irreducible $\mathfrak{g}$-module with highest weight $a\epsilon_1 + \max\{b, 0\}\delta_1$. In particular a single box corresponds to the natural representation[2] V of G.

5.4. **Universal property and tensor space.** The category $\mathcal{F}$ is (by definition) a tensor category, which we want to study using the universal property of Deligne categories and Schur-Weyl duality. For this consider the natural representation $V = \mathbb{C}^{\mathrm{r}\,|\,2\mathrm{n}} \in \mathcal{F}$ for G.

Lemma 5.7. *Let $d, d' \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{Hom}_{\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})}(V^{\otimes d}, V^{\otimes d'}) \neq 0$ implies $d \equiv d' \bmod 2$.*

Proof. In case r is odd, this is easy, since $-\mathbb{1} \in G$ is central and acts by $(-1)^d$ respectively $(-1)^{d'}$ on the source respectively target. In case r is is not necessarily odd, $V^{\otimes d}$ has weights λ of the form $\sum_{i=1}^{d} \nu_i$, where ν_i is of the form $\pm\epsilon_j$, $\pm\delta_j$ for some standard basis vector ϵ_j in the integral weight lattice. In particular, although the signs can cause cancellations, the sum of the coefficients when writing λ in the standard basis, has the same parity as d. □

The vector representation V of $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$ has superdimension $\delta = r - 2n$. By the universal property of Deligne's category Rep_δ, see e.g. [Ost04], [DLZ18], [Cou18], there is a unique monoidal functor $\mathbb{F} = \mathbb{F}_{(r\,|\,2n)}$ from Rep_δ to $\mathcal{F}$ sending the object 1 to V. Then for any fixed $d \in \mathbb{Z}_{>0}$ we have $\mathbb{F}(d) = V^{\otimes d}$ and there is an action $\Upsilon_{d,\delta}$ of $\mathrm{Br}_d(\delta)$ on $V^{\otimes d}$ commuting with the action of G. For explicit formulas in our normalization see [ES16d].

Theorem 5.8 (First Fundamental Theorem)**.** *The functor $\mathbb{F} = \mathbb{F}_{(r\,|\,2n)}$ is full. This implies that, for $\delta = r - 2n$, the action map $\Upsilon_{d,\delta}$ is a surjective algebra homomorphism*

$$\Upsilon_{d,\delta} : \mathrm{Br}_d(\delta) \twoheadrightarrow \mathrm{End}_{\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})}(V^{\otimes d}). \tag{5.24}$$

Proof. This is [DLZ18, Section 3.13], see also [Ser14]. □

Corollary 5.9. *Let $M \subset V^{\otimes d}$ be a direct summand as an $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$-module. Then there is a natural isomorphism of $\mathrm{Br}_{d+1}(\delta)$-modules*

$$\mathrm{ind}_d^{d+1} \mathrm{Hom}_{\mathcal{F}}(V^{\otimes d}, M) \cong \mathrm{Hom}_{\mathcal{F}}(V^{\otimes d+1}, M \otimes V).$$

Proof. By adjunction, constructing a $\mathrm{Br}_{d+1}(\delta)$-morphism

$$\tau_M : \mathrm{Br}_{d+1}(\delta) \otimes_{\mathrm{Br}_d(\delta)} \mathrm{Hom}_{\mathcal{F}}(V^{\otimes d}, M) \to \mathrm{Hom}_{\mathcal{F}}(V^{\otimes d+1}, M \otimes V) := Y$$

is equivalent to a $\mathrm{Br}_d(\delta)$-morphism $\check{\tau}_M : \mathrm{Hom}_{\mathcal{F}}(V^{\otimes d}, M) \to Y$. If we take $\check{\tau}_M(f) = f \otimes \mathrm{id}$, then $\tau_M(b \otimes f) = b(f \otimes \mathrm{id})$. This is by Theorem 5.8 an isomorphism if $M = V^{\otimes d}$ and then also an isomorphism for any summand. □

5.5. **Direct summands in $V^{\otimes d}$.** We refine the fact that V is a tensor generator of $\mathcal{F}$:

Proposition 5.10. *Let $J \subset \Lambda(\mathcal{B})$ be a finite subset of weights such that all $P(\lambda)$ are in the same block $\mathcal{B}$ of $\mathcal{F}$. Then $\oplus_{\lambda \in J} P(\lambda)$ is isomorphic to some direct summand P' of $V^{\otimes d}$ for some large enough d.*

Proof. By [CH17, Lemma 7.5] every $P(\lambda)$ appears in $V^{\otimes d}$ for some large enough d. The bilinear form β defines a surjective morphism $\mathrm{id}_{\mathrm{P}(\lambda)} \otimes \beta : P(\lambda) \otimes V \otimes V \to P(\lambda) \otimes \mathbb{C}$ which splits, since $P(\lambda)$ is projective. Hence if $P(\lambda)$ appears as a summand in $V^{\otimes d}$ then also in $V^{\otimes d'}$ for any $d' \geq d$ with the same parity. Since by Lemma 5.7 there are only morphisms between tensor powers of V of the same parity, the statement follows by realizing any $P(\lambda)$, $\lambda \in J$ in some tensor power of V and choosing then for d the maximal occurring power. □

[2]We want to warn the reader that there is a typo in [ES17, Remark 2.7]: the conditions $m > n$, $m \leq n$ should be replaced by $m \geq n$, $m < n$ respectively.

Theorem 5.11. *Let J and $P' = \oplus_{\lambda \in J} P(\lambda)$ be as in Proposition 5.10. Then there is an idempotent $e = e_{d,\delta}$ in $\mathrm{Br}_d(\delta)$ such that $\Upsilon_{d,\delta}$ induces a surjective algebra homomorphism*

$$\Upsilon_{d,\delta}:\ e\mathrm{Br}_d(\delta)e \twoheadrightarrow \mathrm{End}_{\mathcal{F}}(P') \tag{5.25}$$

identifying the primitive idempotents in both algebras. Thus, idempotents in $\mathrm{End}_{\mathcal{F}}(P')$ lift.

Proof. Assume R is an indecomposable summand in $V^{\otimes d}$. It corresponds to an idempotent $e_R \in \mathrm{End}_{\mathcal{F}}(V^{\otimes d})$. Since $\Upsilon_{d,\delta}$ is surjective by Theorem 5.8, e_R can be lifted to an idempotent $\overline{e}_R \in \mathrm{Br}_d(\delta)$ by Lemma 1.8. If $\mathrm{im}(\overline{e}_R)$ decomposes into several indecomposable summands in Rep_δ, then all of them except of one, say X, are sent to zero under the functor $\mathbb{F}$, since $\mathbb{F}(\mathrm{im}(\overline{e}_R))$ is indecomposable. Hence $\mathbb{F}(X) = R$ for some indecomposable object $X \in \mathrm{Rep}_\delta$. And this is unique up to isomorphism by Lemma 1.9. Then id_X is a primitive idempotent in Rep_δ, since Rep_δ is idempotent complete. By construction, id_X is mapped to e_R. Applying this to any summand in P' and summing them up defines an idempotent e as required. □

Our goal is an explicit description of the algebra $\mathrm{End}_{\mathcal{F}}(P')$. As a first step we recall the classification from [CH17, Theorem 7.3, Lemma 7.16] of indecomposable summands in $V^{\otimes d}$ in terms of a certain set $\Lambda(d,r,n)$ of partitions, depending on d,m,n. This set will be specified in the next section using the combinatorics of dotted cup diagrams.

Theorem 5.12 (Indecomposables in $V^{\otimes d}$)**.** *The assignment $\lambda \mapsto \mathbb{F}\mathrm{R}_\delta(\lambda)$ gives a bijection between the set $\Lambda(d,r,n) := \{\lambda \in \Lambda_d(\delta) \mid \mathbb{F}\mathrm{R}_\delta(\lambda) \neq 0\}$ and any set of representatives for the isomorphism classes of nonzero indecomposable summands of $V^{\otimes d}$.*

6. Dotted cup diagrams and the graded basic Brauer algebra $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$

We review now some of the combinatorics of dotted cup diagrams introduced in [ES16b], motivated by [BS11], [BS12b], which allows us to describe the decomposition numbers $\mathrm{d}_{\lambda,\mu}$ from (1.9) as well as the classification of indecomposable summands in tensor space $V^{\otimes d}$ by making explicit the set $\Lambda(d,r,n)$ from Theorem 5.12. As an application we define the idempotented version $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ of the basic version of the Brauer algebra $\mathrm{Br}_d(\delta)$ and equip it with a grading.

Remark 6.1. Although the dotted cup diagrams look very similar to Brauer diagrams, they should be considered as a quite different constructions and instead be viewed as half type BCD Temperley-Lieb diagrams, see e.g. [LS12]. Diagrams similar to our dotted cup diagrams were introduced independently as curl diagrams in [CVM09].

6.1. **(Infinite) weight, cup and circle diagrams.** We consider the nonnegative (half) integer line $\mathrm{L} = \mathbb{Z} + {}^{\delta}\!/_2 \cap \mathbb{R}_{\geq 0}$ and call its elements *vertices*. A *weight diagram* is a map μ from L to the set $\{\wedge, \vee, \times, \circ, \diamond\}$ such that $\diamond$ can only occur as the image of 0 and conversely 0 can only be mapped to $\circ$ or $\diamond$. Hence a weight diagram assigns to each vertex a unique label of the form $\wedge$, $\vee$, $\times$, $\circ$ or $\diamond$ called *up*, *down*, *nought* or *diamond* respectively. We usually draw it as a sequence of symbols indicating the lowest number of L, e.g.

$$\overset{0}{\diamond} \circ \wedge \vee \circ \vee \times \vee \cdots \quad \text{respectively} \quad \overset{1/2}{\wedge} \circ \wedge \vee \circ \vee \times \vee \cdots .$$

Definition 6.2. A weight diagram μ is called *admissible* if $\mu^{-1}(\{\circ,\times,\wedge\})$ is finite, and it is called *flipped* if $\mu^{-1}(\{\circ,\times,\vee\})$ is finite. For such a weight diagram and $? \in \{\circ\times,\wedge,\vee\}$ we denote by $\#?(\mu) \in \mathbb{N}_0 \cap \{\infty\}$ the number of ?'s appearing in μ.

Obviously, turning each symbol upside down gives a bijection between admissible weight diagrams and flipped weight diagrams. Given an admissible or a flipped weight diagram μ we like to assign a *cup diagram* $\underline{\mu}$. For this we say that two vertices in a weight diagram are *neighboured* if they are only separated by vertices labelled from the set $\{\circ, \times\}$ and proceed as follows:

Definition 6.3. The *cup diagram* $\underline{\mu}$ associated with an admissible or a flipped weight diagram μ is obtained by applying the following steps in order.

(C-1) If the weight diagram μ contains a $\diamond$ we change it into an $\wedge$ or $\vee$ such that the resulting number of $\wedge$'s becomes odd in case this number is finite and into an $\wedge$ in case this number is infinite.
(C-2) First connect neighboured vertices labelled $\vee\wedge$ successively by a cup, i.e. an arc forming a cup below the labels, ignoring already joint vertices as long as possible. (The result is independent of the order in which the connections are made).
(C-3) Attach to each remaining $\vee$ a vertical ray.
(C-4) Connect from left to right pairs of two neighboured $\wedge$'s by cups. In case only $\wedge$'s are left over we we might have to attach infinitely many cups in this step.[3]
(C-5) If a single $\wedge$ remains, attach a vertical ray.
(C-6) Put a marker $\bullet$ on each cup created in (C-4) and each ray created in (C-5).
(C-7) Finally delete all labels at vertices.

The arcs for the connections should always be drawn without intersections. If a cup has a marker $\bullet$, it is called a *dotted cup*, otherwise an *undotted cup*. Two cup diagrams are considered the same if there is a bijection between the set of arcs respecting the connected vertices and the property whether they are dotted or undotted.

Remark 6.4. The admissibility condition makes sure that the algorithm producing the cup diagram stops after finitely many steps creating finitely many cups only. In case of a flipped weight diagram, the algorithm ends as well thanks to the rule (C-4). It produced infinitely many dotted cups.

Definition 6.5. For a weight diagram μ we denote by $\mathrm{def}(\mu) = \mathrm{def}(\underline{\mu})$ its *defect*, which is the total number of cups (dotted or undotted) in $\underline{\mu}$ and by $\mathrm{rk}(\mu) = \min(\#\circ(\mu), \#\times(\mu))$ its *rank*. Define $\kappa(\mu) := \mathrm{def}(\underline{\mu}) + \mathrm{rk}(\underline{\mu})$. We call this the *layer number* of μ.

Admissibility implies that these numbers are finite. For a flipped weight diagram the rank is still finite, whereas the defect is not.

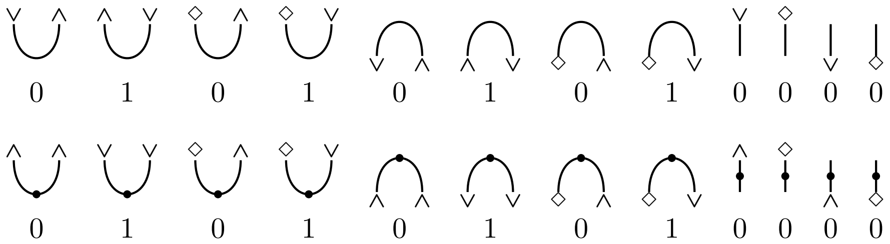
Definition 6.6. (Weight dictionary) Let $S \subseteq \mathbb{Z}+{}^{\delta}/_{2}$. Then the *weight diagram* λ_S associated with S is defined as follows (where $i \in \mathrm{L}$)

$$\lambda_S(i) \quad = \quad \begin{cases} \diamond & \text{if } i = 0 \in S, \text{ and otherwise} \\ \wedge & \text{if } i \in X(\lambda) \text{ but } -i \notin X(\lambda), \\ \vee & \text{if } -i \in X(\lambda) \text{ but } i \notin X(\lambda), \\ \times & \text{if } i \in X(\lambda) \text{ and } -i \in X(\lambda), \\ \circ & \text{if } i \notin X(\lambda) \text{ and } -i \notin X(\lambda). \end{cases} \tag{6.26}$$

6.2. **Circle diagrams.** A pair of a cup diagram $\underline{\lambda}$ and a weight diagram ν is an *oriented* cup diagram, denoted $\underline{\lambda}\nu$, if the following holds. If we put ν on top of $\underline{\lambda}$, then the symbols $\circ$ and $\times$ match and each undotted ray carries the label $\vee$, each dotted ray the label $\wedge$, each undotted cup one label $\vee$ and one label $\wedge$, and a dotted cup either two $\vee$'s or two $\wedge$'s. For the purpose of being oriented, the symbol $\diamond$ can be regarded as an $\wedge$ or a $\vee$.

Definition 6.7. A *circle diagram* is a pair $\underline{\lambda}\nu$, $\underline{\mu}\nu$ of two oriented cup diagrams with the same weight diagram ν. Diagrammatically a circle diagram is displayed by putting $\underline{\mu}$ upside down on top of $\underline{\lambda}\nu$, and also denoted $\underline{\lambda}\nu\overline{\mu}$. The orientation means then that it is built from pieces which look as follows (the numbers below will only be used later):

[3]The additional rule in (Cup-(C-4)) in [ES17] is incorrect and should be omitted.



(6.27)

Note that for cups and caps that do not involve the zero position, an orientation is the naive idea of an orientation of the line segment, with the extra condition that the dot should be thought of as an orientation reversal point.

The connected components of a circle diagram $\underline{\lambda}\nu\overline{\mu}$ are called *circles* if they are closed (i.e. a circle embedded into the plane), and are called *lines* otherwise.

6.3. **Deligne weight diagrams.** Let $\lambda \in \Lambda$ and recall the set $X(\lambda)$ from (3.16).

Definition 6.8. For $\lambda \in \Lambda$ we abbreviate the associated weight diagram $\lambda_\delta = \lambda_{X(\lambda)}$ and call it the *(Deligne) weight diagram* attached to the partition λ.

Example 6.9. The partition $\varnothing$ corresponds in case of odd δ to

$$\varnothing_\delta = \begin{cases} \overset{1/2}{\circ} \;\cdots\; \circ \;\overset{\delta/2}{\vee}\; \vee \;\cdots\; \vee \;\vee\; \vee\; \vee\; \vee\; \cdots & \text{if } \delta > 0, \\ \overset{1/2}{\times} \;\cdots\; \times \;\overset{-\delta/2}{\times}\; \vee \;\cdots\; \vee \;\vee\; \vee\; \vee\; \vee\; \cdots & \text{if } \delta < 0. \end{cases} \tag{6.28}$$

and in case of even δ to

$$\varnothing_\delta = \begin{cases} \overset{0}{\circ} \;\cdots\; \circ \;\overset{\delta/2}{\vee}\; \vee \;\cdots\; \vee \;\vee\; \vee\; \vee\; \vee\; \cdots & \text{if } \delta > 0, \\ \overset{0}{\diamond} \;\times\; \cdots \;\overset{-\delta/2}{\times}\; \vee \;\cdots\; \vee \;\vee\; \vee\; \vee\; \vee\; \cdots & \text{if } \delta \leq 0. \end{cases} \tag{6.29}$$

Starting from $\varnothing_\delta$, the Deligne weight diagram λ_δ for $\lambda \in \Lambda$ is obtained by moving the ith symbol λ_i^t steps to the left until it reaches the first position, where it gets flipped upside down and afterwards moved further to the right. The definitions directly imply the following.

Lemma 6.10. *Adding a box to a partition λ corresponds to applying diagrammatically to λ_δ one of the following* local moves *(which should be read from bottom to top):*

$$\begin{array}{r||c|c|c|c|c|c|c|c|c|c|c|c|c} \lambda + \square & \vee\circ & \times\wedge & \circ\wedge & \vee\times & \vee\wedge & \vee\wedge & \times\circ & \circ\times & \diamond\circ & \circ\times & \circ\wedge & \diamond\wedge & \wedge \\ \lambda & \circ\vee & \wedge\times & \wedge\circ & \times\vee & \circ\times & \times\circ & \wedge\vee & \wedge\vee & \circ\vee & \diamond\vee & \diamond\circ & \circ\times & \vee \end{array}, \tag{6.30}$$

with the last scenario only possible at position $1/2$.

Note that λ_δ is always admissible; thus the corresponding cup diagram $\underline{\lambda_\delta}$ is defined.

Remark 6.11. (1) If we consider the symbol $\times$ as a union of $\wedge$ and $\vee$, the local moves move a symbol $\vee$ one step to the left or a symbol $\wedge$ one step to the right, or turn $\vee$ into $\wedge$ at position $1/2$.

(2) We observe that each local move (6.30) preserves the difference of the number of $\circ$ and $\times$. In particular, for fixed δ, the rank of a Deligne weight diagrams is nothing else than the nunber of $\times$ in case $\delta \geq 0$ and the number of $\circ$ in case $\delta < 0$.

For δ odd, the parity of $|\lambda|$ for $\lambda \in \Lambda$ can be read off from λ_δ as follows.

Definition 6.12. Let ν be an admissible weight diagram. For each $\circ$ or $\times$ in ν count the total number of $\wedge$ and $\vee$ in $\underline{\nu}$ to the left of the symbol. Sum all of these up and add the number of $\wedge$ in $\underline{\nu}$. The parity of the result is called the *parity* $\mathrm{par}(\nu)$ of ν.

Lemma 6.13. *Let $\lambda \in \Lambda$. The parity of $|\lambda|$ equals $\mathrm{par}(\lambda_\delta)$.*

Proof. We prove this by induction on the number of boxes of λ. In case of the empty partition, the corresponding weight diagram is given in (6.28) with obviously $\mathrm{par}(\varnothing_\delta) = 0$. Let now μ be a partition obtained from λ by adding a box and assume the claim holds for λ. Comparing $\mathrm{par}(\lambda_\delta)$ and $\mathrm{par}(\mu_\delta)$ in each local move from (6.30), one checks easily that they always differ by $\pm$. We exemplify this for two of the local moves:

$\lambda + \square$ λ	$\vee\circ$ $\circ\vee$	In this case the number of $\wedge$'s does not change, but the $\circ$ got moved and we now count one more symbol to its left, while all other numbers involved are kept. Hence altogether the parity changes.
$\lambda + \square$ λ	$\vee\wedge$ $\times\circ$	In this case the contributions given by $\wedge$ and $\vee$ on the left of $\times$, respectively $\circ$ cancelled each other, since $\times$ does not change the parity. Any $\circ$ or $\times$ to the right of the vertices involved in the local move will count two more symbols after the move, also not changing the parity. We however created one new $\wedge$. Altogether the parity changes.

For the other moves the arguments are similar and therefore omitted. □

6.4. **Central characters.** We descibe now diagrammatically how the scalar c_μ^{Br} by which Ω_d acts on cell modules (cf. *Lemma* 1.19) behaves under i-induction.

Lemma 6.14. *Let $\lambda, \mu \in \Lambda$ such that μ_δ is obtained from λ_δ by a local move involving the positions $p \pm 1/2$. Then $c_\mu^{\mathrm{Br}} = c_\lambda^{\mathrm{Br}} \pm 2p$, where we add (or remove) $2p$ if a $\vee$ is moved from right to left (respectively an $\wedge$ is moved from left to right). A $\diamondsuit$ is interpreted as an $\wedge$ or $\vee$.*

Proof. We computing the change $c_\lambda^{\mathrm{Br}} \rightsquigarrow c_\mu^{\mathrm{Br}}$

- *Assume a $\vee$ is moved from the right to the left:* Enumerating all symbols in λ by first counting all $\wedge$ from right to left and then all $\vee$ from left to right let i be the index of the symbol moved. Then to obtain λ from the ground state $\varnothing$ the symbol was moved a total of $\delta/2 + i - 1 - (p + 1/2)$ steps. Thus in the language of partitions a new box is added in row $\frac{\delta+1}{2} + i - 1 - p$ and column i. Thus we have by Lemma 1.19

$$\begin{aligned} c_\mu^{\mathrm{Br}} &= |\mu|(\delta-1) + 2c(\mu) = |\lambda|(\delta-1) + 2c(\lambda) + (\delta-1) + 2\left(i - \left(\tfrac{\delta+1}{2} + i - 1 - p\right)\right) \\ &= c_\lambda^{\mathrm{Br}} + 2p. \end{aligned}$$

- *Assume an $\wedge$ is moved from the left to the right:* The only change to above is that the new box in the partition is now added in row $\frac{\delta-1}{2} + i + p$, which results in $c_\mu^{\mathrm{Br}} = c_\lambda^{\mathrm{Br}} - 2p$. □

Denote by $\mathrm{C}_{r\,|\,2n}$ the Casimir element [Mus12, Lemma 8.5.1] in the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{osp}(\mathrm{r}\,|\,2\mathrm{n}))$. Let mult_Ω denote the map given by the multiplication with Ω_d on $V^{\otimes d}$ and by mult_C the map given by acting on tensor space with $\mathrm{C}_{r\,|\,2n}$.

Proposition 6.15. *Let $M \subset V^{\otimes d}$ be an $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$-submodule. Then mult_Ω and mult_C restrict to M and agree thereon.*

Proof. The action of $\mathrm{C}_{r\,|\,2n}$ is well-defined as a central element of $\mathcal{U}(\mathfrak{osp}(\mathrm{r}\,|\,2\mathrm{n}))$. The action of Ω_d is well-defined, since M is given by an idempotent in $\mathrm{End}_\mathcal{F}(V^{\otimes d})$ which is the image of the Brauer algebra by Theorem 5.8 and hence commutes with the action of Ω_d by Lemma 1.5. To see that both maps agree, note that $\Omega_d = d\,(\delta-1) + 2\sum_{1\le i<k\le d}(s_{i,k} - \tau_{i,k})$. Multiplication

with Ω_d thus agrees with the action of

$$\begin{aligned}\mathrm{mult}_\Omega &= \left(\Delta^{d-1}(\mathrm{C}_{r\,|\,2n}) - \sum_{i=1}^{d} \mathrm{id}_V^{i-1} \otimes C \otimes \mathrm{id}_V^{d-i}\right) + d\,(\Delta - 1)\,\mathrm{id}_{V^{\otimes d}} \\ &= \left(\Delta^{d-1}(\mathrm{C}_{r\,|\,2n}) - (\Delta-1)\mathrm{id}_{V^{\otimes d}}\right) + d\,(\Delta - 1)\,\mathrm{id}_{V^{\otimes d}} = \Delta^{d-1}(\mathrm{C}_{r\,|\,2n}) = \mathrm{mult}_{\mathrm{C}},\end{aligned}$$

where Δ denotes the comultiplication of $\mathcal{U}(\mathfrak{osp}(\mathrm{r}\,|\,2\mathrm{n}))$, given on an element $x \in \mathfrak{osp}(\mathrm{r}\,|\,2\mathrm{n})$ by $\Delta(x) = x\otimes 1 + 1\otimes x$. Note further that the action of $\mathrm{C}_{r\,|\,2n}$ on V is given by multiplication with $\delta - 1$, see [Mus12, Lemma 8.5.3] using the explicit formula for ρ from [ES17, 2.1] and the fact that V is irreducible with highest weight ϵ_1 if $m \geq n$ and δ_1 if $m < n$. □

Corollary 6.16. *Let $\lambda \in \Lambda(d, r, n)$. Then $\mathrm{Hom}_{\mathcal{F}}(V^{\otimes d}, \mathbb{FR}_\delta(\lambda))$ is contained in a single generalized eigenspace for the right action of Ω_d and for the precomposition with* $\mathrm{mult}_{\mathrm{C}}$ *respectively. The two eigenvalues agree.*

Proof. Since $\mathbb{FR}_\delta(\lambda)$ is an indecomposable summand of $V^{\otimes d}$ by Theorem 5.12, the module $\mathrm{Hom}_{\mathcal{F}}(V^{\otimes d}, \mathbb{FR}_\delta(\lambda))$ it is contained in a single generalized eigenspace for Ω, since Ω induces a central element in $\mathrm{End}_{\mathcal{F}}(V^{\otimes d})$. But then all statements follow from Proposition 6.15. □

We obtain an explicit description of the set $\Lambda(d, r, n)$ from Theorem 5.12 via certain admissible weight diagrams, called therefore *tensor weight diagrams*.

Theorem 6.17 (Tensor weight diagrams)**.** *Recall the notation from Definition 6.5.*

(1) There is an equality of sets $\Lambda(d, r, n) = \{\lambda \in \Lambda_d(\delta) \mid \kappa(\lambda_\delta) \leq \min(m, n)\}$.
(2) The summand $\mathbb{FR}_\delta(\lambda)$ is projective, if and only if $\kappa(\lambda_\delta) = \min(m, n)$. In which case we call λ a projective weight diagram.

Proof. This is [CH17, Corollary 7.14 with Lemma 7.16]. □

Example 6.18. The following table displays in the first two columns examples of partitions and their corresponding Deligne weight diagrams and cup diagrams for $\delta = 1$. The third column checks which of them correspond to projective modules for $\mathrm{OSp}(3\,|\,2)$ via Theorem 6.17. None of them is projective for $\mathrm{OSp}(2n+1\,|\,2n)$ for $n > 1$.

λ	λ_δ / $\underline{\lambda_\delta}$	proj. ?	$\ulcorner\lambda^\infty$ / $\underline{\ulcorner\lambda^\infty}$	ϵ	$\lambda_\epsilon^\oslash$ / $\underline{\lambda_\epsilon^\oslash}$
$\varnothing$	∨ ∨ ∨ ∨ ∨ ∨ ⋯	no	∧ ∧ ∧ ∧ ∧ ∧ ⋯	$+$	∧ ∧ ∧ ∨ ∨ ∨ ⋯
□	∧ ∨ ∨ ∨ ∨ ∨ ⋯	no	∨ ∧ ∧ ∧ ∧ ∧ ⋯	$-$	∨ ∧ ∧ ∨ ∨ ∨ ⋯
(1,1)	○ × ∨ ∨ ∨ ∨ ⋯	✓	○ × ∧ ∧ ∧ ∧ ⋯	$+$	○ × ∧ ∨ ∨ ∨ ⋯
(2)	× ○ ∨ ∨ ∨ ∨ ⋯	✓	× ○ ∧ ∧ ∧ ∧ ⋯	$+$	× ○ ∧ ∨ ∨ ∨ ⋯
(1,1,1)	○ ∨ × ∨ ∨ ∨ ⋯	✓	○ ∧ × ∧ ∧ ∧ ⋯	$-$	○ ∧ × ∨ ∨ ∨ ⋯
(2,1)	∨ ∧ ∨ ∨ ∨ ∨ ⋯	✓	∨ ∧ ∧ ∧ ∧ ∧ ⋯	$-$	∨ ∧ ∧ ∨ ∨ ∨ ⋯
(3)	× ∨ ○ ∨ ∨ ∨ ⋯	✓	× ∧ ○ ∧ ∧ ∧ ⋯	$-$	× ∧ ○ ∨ ∨ ∨ ⋯

Here the labelling set of the indecomposable objects $\mathtt{R}_\delta(\lambda)$ in terms of cup diagrams is easy. We have either one cup or no cup and then one $\circ$ and one $\times$ somewhere. We do not know any reasonable description of the corresponding set of partitions even in this small case.

Question 6.19. Is there an intrinsic description of the involved sets of partitions, namely for $\Lambda(d,r,n)$ or the subset corresponding to projective summands $\mathbb{F}\mathtt{R}_\delta(\lambda)$)?

6.5. **Decomposition numbers and idempotented basic Brauer algebras.** Oriented cup diagrams encode the decomposition number $d_{\lambda,\mu}$ and dimensions in (1.9):

Lemma 6.20. *For* $\mathrm{Br}_d(\delta)$ *and* $\lambda,\nu\in\Lambda_d(\delta)$*, the multiplicity* $(P_{d,\delta}(\lambda):\Delta_{d,\delta}(\nu))$ *is equal to* 1 *if* $\underline{\lambda}_\delta$
nu$_\delta$ *is oriented and it is zero otherwise.*

Proof. This is [CVM09, Theorem 4.11] adapted to our setup as in [ES18, Proposition 4.4]. □

Definition 6.21. For $\lambda,\mu\in\Lambda$, we denote by $\mathbb{B}_\delta(\lambda,\mu)$ the vector space with basis consisting of all oriented circle diagrams $\underline{\lambda}_\delta\nu\overline{\mu}_\delta$ for ν a weight diagram. We set

$$\mathbb{D}_d^{\mathrm{bsc}}(\delta):=\bigoplus_{\lambda,\mu\in\Lambda_d(\delta)}\mathbb{B}_\delta(\lambda,\mu)\quad\text{and}\quad\mathbb{D}^{\mathrm{bsc}}(\delta):=\bigoplus_{\lambda,\mu\in\Lambda}\mathbb{B}_\delta(\lambda,\mu).\tag{6.31}$$

As a consequence of Lemma 6.20, respectively from Definition 1.25 and Lemma 2.3 we obtain isomorphisms of vector spaces

$$\mathbb{B}_\delta(\lambda,\mu)\;\cong\;\mathrm{Hom}_{\mathrm{Rep}_\delta}(\mathtt{R}_\delta(\lambda),\mathtt{R}_\delta(\mu))\cong\mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\lambda),P_{d,\delta}(\mu))\tag{6.32}$$

for any $\lambda,\mu\in\Lambda$ and d such that $|\lambda|,|\mu|\leq d$.

Now $\mathrm{Br}_d^{\mathrm{bsc}}(\delta)=\bigoplus_{\lambda,\mu\in\Lambda_d(\delta)}\mathrm{Hom}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\lambda),P_{d,\delta}(\mu))$ is an algebra by viewing it as a subalgebra of the Brauer algebra $\mathrm{Br}_d(\delta)$. We call it the *basic Brauer algebra*, since it is by construction a basic algebra Morita equivalent to $\mathrm{Br}_d(\delta)$.

In [ES16b] we defined a diagrammatic multiplication generalizing the construction of Khovanov's arc algebra from [Kho00], [BS11] to a situation where we allow dotted cup diagrams which we apply now to $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ and $\mathbb{D}^{\mathrm{bsc}}(\delta)$. Up to possible signs, the multiplication rules are exactly as in [BS11]. We refer the reader for details to [ES16b].

Theorem 6.22 (Basic Brauer algebra)**.**
The isomorphisms (6.32) *can be chosen such that the following holds.*

(1) They define an isomorphism of algebras

$$\mathrm{Br}_d^{\mathrm{bsc}}(\delta)\cong\mathbb{D}_d^{\mathrm{bsc}}(\delta).$$

(2) Under this isomorphism the primitive idempotent corresponding to $P_{d,\delta}(\lambda)$ *is mapped to the circle diagram of the form* $\underline{\lambda}_\delta\lambda_\delta\overline{\lambda}_\delta$.

(3) The primitive idempotents in $\mathbb{D}^{\mathrm{bsc}}(\delta)$ *are the elements* $\underline{\lambda}_\delta\lambda_\delta\overline{\lambda}_\delta$*, for* $\lambda\in\Lambda$.

Proof. The last statement is [ES16b, Theorem 6.2]. For the first statement, recall that Theorem 4.8 identifies the Brauer algebra $\mathrm{Br}_d(\delta)$ with the endomorphism algebra of d in the idempotent truncation $\mathbf{f}\mathbb{W}(N,\alpha,\beta)\mathbf{f}$ of $\mathbb{W}(N,\alpha,\beta)$. On the other hand, Theorem 4.7 identifies $\mathrm{End}_{\mathbb{W}(N,\alpha,\beta)}(d)$ with the endomorphism ring of $M(\underline{\delta})\otimes V^{\otimes d}$. The identification with the algebra $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ is then given by [ES16b, Theorem 9.1]. The second statement holds again by [ES16b, Theorem 6.2]. □

Circle diagrams give a quite different diagrammatic description of the basic version $\mathrm{Br}_d^{\mathrm{bsc}}(\delta)$ of the Brauer algebra with the advantage of providing a distinguished basis involving primitive idempotents and an explicit multiplication rule for basis elements, [ES16b, Theorem 6.2]. This approach will be extended to the Brauer algebra $\mathrm{Br}_d(\delta)$ in Theorem 11.20.

It is worth to examine this distinguished basis a bit further. For this consider $\mathbb{D}^{\mathrm{bsc}}(\delta) = \bigoplus_{\lambda,\mu\in\Lambda} \mathbb{B}_\delta(\lambda,\mu)$. It is a locally unital locally finite dimensional algebra with a distinguished basis given by oriented circle diagrams, that is pairs of oriented cup diagrams $\underline{\lambda}_\delta\nu_\delta$, $\underline{\mu}_\delta\nu_\delta$ with the same weight diagram ν_δ. For $\nu\in\Lambda$ define the set $Y(\nu)$ (and $X(\nu)$) to be the set of all oriented cup diagrams (respectively cap diagrams) with weight ν_δ. We have $Y(\nu) = \bigcup_{\lambda\in\Lambda} Y(\lambda,\nu)$, where $Y(\lambda,\nu)$ contains the unique oriented cup diagram $\underline{\lambda}_\delta\nu_\delta$ in case this exists and it is empty otherwise; similarly we have $X(\nu) = \bigcup_{\lambda\in\Lambda} X(\nu,\lambda)$ by fixing the cap diagram. We view elements $x\in X(\nu,\lambda)$ and $y\in Y(\lambda,\nu)$ as basis vectors, i.e. as oriented circle diagrams, by putting the cup diagram $\underline{\nu}_\delta$ on the bottom, respectively the cap diagram $\overline{\nu}_\delta$ on the top. By definition of the multiplication, see [ES16b, Merge Case in (4.20)], the product xy of basis vectors equals exactly the oriented circle diagram (x,y). By construction we have $X(\nu,\lambda)\subset\mathbb{B}_\delta(\nu,\lambda)$ and $Y(\lambda,\nu)\subset\mathbb{B}_\delta(\lambda,\nu)$. Now the following holds:

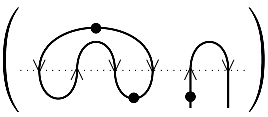
Theorem 6.23 (Based quasi-hereditary)**.**

1. *Consider the algebra $\mathbb{D}^{\mathrm{bsc}}(\delta) = \bigoplus_{\lambda,\mu\in\Lambda}\mathbb{B}_\delta(\lambda,\mu)$ with the set of primitive idempotents $\underline{\lambda}_\delta\lambda_\delta\overline{\lambda}_\delta$ for $\lambda\in\Lambda$. Set $I:=\Lambda$ and $B:=\Lambda$ with the reverse inclusion ordering on partitions. This data together with the sets $X(\nu,\lambda)$ and $Y(\lambda,\nu)$ equip $\mathbb{D}^{\mathrm{bsc}}(\delta)$ with the structure of an upper finite based quasi-hereditary algebra in the sense of* [BS18, Definition 5.1] *(and in the notation from there).*
2. *If we define $Y_d(\nu) = \bigcup_{\lambda\in\Lambda} Y_d(\lambda,\nu)$, $X_d(\nu) = \bigcup_{\lambda\in\Lambda} Y_d(\nu,\lambda)$ by restricting λ,ν to elements in $I:=B:=\Lambda_d(\delta)$, then this equips $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ with the structure of an upper finite based quasi-hereditary algebra.*

Proof. We only need to check the conditions (QH4)-(QH6) from [BS18, Definition 5.1], since all the other assumptions are obvious. By definition of the multiplication rules, the elements yx with $(y,x)\in\bigcup_{\nu\in\Lambda} Y(\nu)\times X(\nu)$ form exactly our distinguished basis, hence (QH4) holds. (QH6) is obvious. To see (QH5) we observe that by [ES16b, Lemma 3.17] $X(\nu,\lambda)\neq\emptyset$ or $Y(\nu,\lambda)\neq\emptyset$ only if ν_δ is obtained from λ_δ by a finite sequence of swaps of symbols turning $\vee\wedge$ into $\wedge\vee$ or $\wedge\wedge$ into $\vee\vee$. From Lemma 6.10 we see that this means $\nu\geq\lambda$. ☐

6.6. **Gradings.** Recall from Definition 6.21 the vector space $\mathbb{B}_\delta(\lambda,\mu)$ with basis the space of oriented circle diagrams of the form $\underline{\lambda}_\delta\nu\overline{\mu}_\delta$. Following [ES16b] we equip this vector space, as well as $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ and $\mathbb{D}^{\mathrm{bsc}}(\delta)$ with a grading by viewing each basis vector as built from oriented cups, oriented caps and oriented rays.

Definition 6.24. The *degree* $\deg(\underline{\lambda}\nu\overline{\mu})$ of an oriented circle diagram $\underline{\lambda}\nu\overline{\mu}$ is defined to be the total sum of the degrees of all oriented cups, caps and rays; where the degrees are defined as indicated by the numbers in (6.27), e.g.

deg () = 4

Definition 6.25. The *graded basic Deligne category* $\mathrm{gRep}_\delta^{\mathrm{bsc}}$ is the additive category enriched in graded vector spaces with objects $\lambda\in\Lambda$, and morphism spaces $\mathrm{Hom}_{\mathrm{gRep}_\delta}(\lambda,\mu)=\mathbb{B}_\delta(\lambda,\mu)$ if there is some d such that $\lambda,\mu\in\Lambda_d(\delta)$ and $\mathrm{Hom}_{\mathrm{gRep}_\delta}(\lambda,\mu)=\{0\}$ otherwise. The grading on morphisms spaces is defined by Definition 6.24. The composition of morphisms is given by the diagrammatical multiplication rules from [ES16b, Section 6.2].

Remark 6.26. The definition makes sense, since $\mathbb{B}_\delta(\lambda,\mu)$ is independent of d as long as $\lambda,\mu\in\Lambda_d(\delta)$ for some d. It is the same space for varying d, hence does not require identifications as in Remark 1.16.

The graded basic Deligne category is indeed a graded lift of the basic version of the additive category ${\mathrm{Rep}_\delta}^{\mathrm{bsc}}$ of Rep_δ (where by ${\mathrm{Rep}_\delta}^{\mathrm{bsc}}$ we mean the full additive subcategory of Rep_δ

given by the chosen representatives $\mathtt{R}_\delta(\lambda)$ for the isomorphism classes of indecomposable objects $\mathtt{R}_\delta(\lambda)$ from Theorem 1.15):

Lemma 6.27. *When forgetting the grading,* $\mathrm{gRep}_\delta^{\mathrm{bsc}}$ *is isomorphic to* $\mathrm{Rep}_\delta^{\mathrm{bsc}}$.

Proof. We have an isomorphism on morphism spaces by (6.32) and (1.5). Then the claim follows with the arguments as in the proof of Theorem 6.22 using [ES16b, Theorem 9.1] and also the second part of Theorem 6.22. □

Definition 6.28. A *representation* of $\mathrm{gRep}_\delta^{\mathrm{bsc}}$ is a contravariant functor G from $\mathrm{gRep}_\delta^{\mathrm{bsc}}$ to the category gVect of finite dimensional graded vector spaces. Let $\mathbb{D}^{\mathrm{gr}}(\delta)$ be the category of such representations.

By Remark 2.5, $\mathbb{D}^{\mathrm{gr}}(\delta)$ is equivalent to the category of locally finite dimensional graded $\mathbb{D}^{\mathrm{bsc}}(\delta)$-modules, where we view $\mathbb{D}^{\mathrm{bsc}}(\delta)$ as a locally finite dimensional graded algebra. We call an upper finite based quasi-hereditary algebra which is additionally a graded algebra such that the sets defining the based quasi-hereditary structure consist of homogeneous elements a *graded upper finite based quasi-hereditary algebra.* By definition we then havex

Theorem 6.29. *The graded algebras* $\mathbb{D}^{\mathrm{bsc}}(\delta)$ *and* $\mathbb{D}_d^{\mathrm{bsc}}(\delta)$ *for* $d \in \mathbb{N}$, *are with the data from Theorem 6.23 graded upper finite based quasi-hereditary algebra.*

Remark 6.30. Any upper finite based quasi-hereditary algebra comes with a collection of standard modules by [BS18, (5.2)]. In the case of $\mathbb{D}^{\mathrm{bsc}}(\delta)$, these are modules $\Delta(\lambda)$, $\lambda \in \Lambda$ which can be realised as a subquotient of $\mathbb{D}^{\mathrm{bsc}}(\delta)$. The underlying vector space has basis y, $y \in Y(\lambda)$. Under the equivalence of categories with $\mathcal{D}(\delta)$, they correspond to the standard modules from Theorem 2.10. By construction the analogous statements hold in the graded setting as well.

7. Classification of indecomposable summands in $V^{\otimes d}$

In this section we establish first the important bijections mentioned in the introduction and then show that the indecomposable summands in tensor space $V^{\otimes d}$ have always irreducible heads. With the dictionaries given in Section 13 this allows to determine the highest weight of the irreducible head.

7.1. **Combinatorial bijections: partitions versus weight diagrams.** We start by characterizing Deligne weight diagrams (depending on δ) among all admissible weight diagrams.

Lemma 7.1. *Fix* $\delta \in \mathbb{Z}$. *The assignment* $\lambda \mapsto \lambda_\delta$ *gives a bijection*

$$W_\delta : \quad \{\textit{partitions}\} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{matrix} \textit{admissible weight diagrams } \mu \textit{ such that} \\ \#\circ(\mu) - \#\times(\mu) = \lfloor \delta/2 \rfloor \end{matrix} \right\}. \qquad (7.33)$$

Proof. Let X be the set on the right hand side. Then $\varnothing_\delta \in X$ by definition. Now, the image of our map is indeed contained in X, since adding a box to a partition λ corresponds on λ_δ to a local move of the form (6.30) which does not change the difference $\#\circ(\mu) - \#\times(\mu)$. The assignment is obviously injective. On the other hand let $\mu \in X$. By admissibility, it has far to right only $\vee$'s. Take the rightmost symbol of the form $\wedge$ or $\times$. Assume it is at position j_1. Then set $\lambda_1 = \lfloor \delta/2 \rfloor + \lfloor j_1 - 1/2 \rfloor + 1$. Continue defining $\lambda_i = \lfloor \delta/2 \rfloor + \lfloor j_i - 1/2 \rfloor + i$ using the next rightmost symbol $\wedge$ or $\times$ at position j_i as long as possible. Then continue with the symbols $\vee$, $\times$ and $\diamond$'s and record (read from left to right) the number of $\wedge$ or $\circ$ to the right of each such symbol. By admissibility this sequence will be identically zero after finitely many steps and thus defines together with the λ_i from above a partition λ such that $\lambda_\delta = \mu$. This shows the surjectivity. □

The same argument, but with the roles of $\wedge$'s and $\vee$'s swapped, implies the following, where λ^∞ is the weight diagram λ_δ but with all symbols turned upside down.

Lemma 7.2. *Fix $\delta \in \mathbb{Z}$. Sending λ to the weight diagram λ^∞ gives a bijection*

$$W_\delta^\infty: \quad \{\textit{partitions}\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{flipped weight diagrams } \mu \textit{ such that} \\ \#\circ(\mu) - \#\times(\mu) = \lfloor \delta/2 \rfloor \end{array} \right\}. \tag{7.34}$$

Remark 7.3. In fact, λ^∞ equals the weight diagram associated with the set

$$S^\infty(\lambda) := \{-\lambda_i + i - 1 + \delta/2 \mid i \geq 1\} \subset \mathbb{Z} + \delta/2. \tag{7.35}$$

To see this, note first that the partition $\varnothing$ corresponds to the weight diagram $\varnothing^\infty$ obtained from $\varnothing_\delta$ in (3.16) by turning all symbols upside down (respectively by swapping $\wedge$'s and $\vee$'s. Then compare with λ_δ by adding successively boxes to the partition.

We characterize flipped weight diagrams for (n,m)-hook partitions.

Lemma 7.4. *(1) Let $\ulcorner\lambda$ be an (n,m)-hook partition. Then the associated flipped weight diagram $\ulcorner\lambda^\infty$ has at most $\min(m,n) - \mathrm{rk}(\mu)$ many $\vee$'s, and $\underline{\ulcorner\lambda}^\infty$ has at most that many undotted cups.*

(2) The bijection W_δ^∞ from (7.34) restricts to a bijection

$$\left\{ \begin{array}{c} \textit{(n,m)-hook} \\ \textit{partitions} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{flipped weight diagrams } \mu \textit{ such that} \\ \#\circ(\mu) - \#\times(\mu) = \lfloor \delta/2 \rfloor \textit{ and} \\ \#\vee(\mu) \leq \min(m,n) - \mathrm{rk}(\mu) \end{array} \right\} =: \Gamma_\delta(n,m). \tag{7.36}$$

Elements in the set $\Gamma_\delta(n,m)$ are called *hook weight diagrams.*

Proof. The second statement in (1) follows directly from the first since each undotted cup requires a $\vee$. If $\delta \geq 0$, then $m \geq n$. The first n rows in $\ulcorner\lambda$ could possibly each create a $\vee$ or $\times$ in $\ulcorner\lambda^\infty$, hence a total of at most n of such, let us call them, *bad symbols.* The $n+j$th symbol (with $j \geq 1$) has to get moved at least $\delta/2 + n + j$ to create an additional bad symbol. But $\delta/2 + n + j > m$. This is however impossible, since the $n+j$th row in $\ulcorner\lambda$ has at most m boxes. Thus we have at most $n = \min(m,n)$ bad symbols, that means at most $\min(m,n) - \mathrm{rk}(\mu)$ many $\vee$'s by Lemma 6.10. Conversely, if μ does not come from a (n,m)-hook partition, then the $(n+1)$th $\wedge$ has been turned into an $\vee$, possible as a part of a $\times$ which means (2) holds as well.

If $\delta < 0$, then $m < n$ and there are $n - m$ crosses in the weight diagram attached to $\varnothing$. The first n rows of the hook partition move the $\vee$'s which are part of the crosses and then the leftmost m from $\diamond$'s and $\wedge$'s (including those contained in crosses). This can create a total of at most m *bad symbols* by which we mean now $\vee$'s (not inside crosses) and $\circ$'s. To create more bad symbols the $m+1$th from the $\diamond$'s and $\wedge$'s in the original weight must be moved at least $m+1$ steps which is impossible, since the partition had at most m boxes in the $(n+1)$th row. Again, (2) follows as well. □

Definition 7.5. By identifying $\Gamma_\delta(n,m)$ with $\Gamma(n|m)$ from Definition 5.1 we can transfer the equivalence relation on $\Gamma(n|m) \times \{\pm\}$ defined in Proposition 5.2 to $\Gamma_\delta(n,m) \times \{\pm\}$. We denote by $s\Gamma_\delta(n|m)$ the corresponding set of equivalence classes. Its elements are called *signed hook weight diagram* and denoted by the same notation as in Proposition 5.2. We also label the irreducible $\mathrm{OSp}(\mathrm{r} \mid 2\mathrm{n})$-modules by signed hook weight diagrams, $L(\ulcorner\lambda^\infty, \epsilon) = L(\ulcorner\lambda, \epsilon)$.

We use this setup now to describe the summands $\mathtt{R}_\delta(\lambda)$ appearing in tensor space $V^{\otimes d}$.

Proposition 7.6. *There is a bijection*

$$\begin{array}{rcl} \dagger: \left\{ \begin{array}{c} \textit{projective} \\ \textit{weight diagrams} \end{array} \right\} & \overset{1:1}{\leftrightarrow} & \left\{ \begin{array}{c} \textit{signed } (n,m)\textit{-hook} \\ \textit{weight diagrams} \end{array} \right\} = s\Gamma_\delta(n|m) \\ \lambda_\delta & \mapsto & \lambda_\delta^\dagger := (\Phi(\lambda), \epsilon), \end{array}$$

where $\Phi(\lambda)$ *is the weight diagram obtained from* λ *by swapping all symbols corresponding to rays in* $\underline{\lambda_\delta}$ *from* $\vee$ *to* $\wedge$*; and*

- *in case* δ *is odd:* $\epsilon \in \{+,-\}$ *is* $+$ *respectively* $-$ *if the parity of the partition corresponding to* λ_δ *via Lemma 7.1 is even respectively odd.*
- *in case* δ *is even:* $\epsilon \in \{+,-,\pm\}$ *is* $+$ *(or* $-$*) if the leftmost ray in* $\underline{\lambda_\delta}$ *is undotted (respectively dotted) and not at position zero;* $\epsilon = \pm$ *if the leftmost ray is at position zero.*

Proof. The map is well-defined, since $\underline{\lambda_\delta}$ has at most $\min(m,n) - \mathrm{rk}(\lambda)$ many undotted cups, thus $\Phi(\lambda)$ has at most that many $\vee$'s, and therefore corresponds to a hook partition by (7.36). It is obviously injective, since two different projective weight diagrams give only the same hook weight diagram if their cup diagrams agree up to a dot on the leftmost ray in which case the signs are different. For surjectivity, consider a hook weight diagram ν and its cup diagram $\underline{\nu}$ with infinitely many dotted cups. Now, starting from the left, keep as many dotted cups as required for the cup diagram corresponding to a projective, and replace all others by undotted rays. The corresponding weight diagram λ is then projective and $\lambda_\delta^\dagger = (\nu, \epsilon)$ for some ϵ. Attaching a dot to the leftmost ray in $\underline{\lambda}$ defines a projective weight μ such that $\mu_\delta^\dagger = (\nu, -\epsilon)$. □

We extend now the above map $\lambda_\delta \mapsto \lambda^\dagger$ to all tensor weight diagrams.

Definition 7.7. The *signed hook weight diagram* $\lambda_\delta^\dagger = (\Phi(\lambda), \epsilon)$ attached to a non-projective tensor weight diagram λ_δ, is the pair of a weight diagram $\ulcorner\lambda^\infty$ obtained from λ_δ by flipping *all* symbols attached to rays in $\underline{\lambda_\delta}$ upside down, and a sign ϵ which is defined as above in case δ is odd and is always $+$ in case δ is even.

We defined therefore a surjection (extending the bijection from Proposition 7.6),

$$\begin{array}{rcl} \left\{ \begin{array}{c} \text{tensor} \\ \text{weight diagrams} \end{array} \right\} & \twoheadrightarrow & s\Gamma_\delta(n|m) \\ \lambda_\delta & \mapsto & \lambda_\delta^\dagger := (\Phi(\lambda), \epsilon). \end{array}$$

Via Theorem 6.17 we also obtain a map $\lambda \mapsto \lambda_\delta^\dagger$ on the corresponding set of partitions, i.e. on $\{\lambda \in \Lambda_d(\delta) \mid \kappa(\lambda) \leq \min(m,n)\}$; see Example 6.18 for concrete values of this map.

7.2. **The classification theorem.** We can now determine the irreducible quotients of the summands $\mathbb{F}\mathtt{R}_\delta(\lambda)$ in $V^{\otimes d}$ (following the notation from Section 5.3).

Theorem 7.8 (Classification Theorem). *Let* $\lambda \in \Lambda(d,r,n)$*:*

(1) The indecomposable summand $\mathbb{F}\mathtt{R}_\delta(\lambda)$ *of the* OSp(r | 2n)*-module* $V^{\otimes d}$ *has irreducible head isomorphic to* $L(\lambda_\delta^\dagger)$*.*
(2) In particular, if $\mathbb{F}\mathtt{R}_\delta(\lambda)$ *is projective, then* $\mathbb{F}\mathtt{R}_\delta(\lambda) \cong P(\lambda_\delta^\dagger)$*.*
(3) Any indecomposable projective in $\mathcal{F}$ *is obtained in this way for some* λ *and* d*.*

Proof of Theorem 7.8. The second statement is a direct consequence of the first. The last statement follows again from the first with Theorem 5.12 and Proposition 5.10. Hence it remains to show the first statement. This will be done in Section 9. □

7.3. **Central characters - again.** The missing part in the proof of Theorem 7.8 will use translation functors and in particular a good understanding of central characters. Therefore we add here a description of the change in the value of the Casimir action on highest weight modules if the corresponding weight diagrams are related by a local move (which up to signs corresponds to adding or removing a box in the hook partitions). This should be compared with Lemma 6.14.

In this section, a symbol $\times$ will be considered as a union of two symbols, one $\wedge$ and one $\vee$, so that a local move either moves an $\wedge$ or $\vee$ to a neighboured position or swaps one of them upside down at position $1/2$, see Remark 6.11.

Consider again the Casimir element $\mathrm{C}_{r\,|\,2n} \in Z(\mathcal{U}(\mathfrak{osp}(\mathrm{r}\,|\,2\mathrm{n})))$, [Mus12, Lemma 8.5.1]. For a signed hook weight diagram $\lambda_\delta^\dagger$ denote by $c_{\lambda_\delta^\dagger}$ the value by which $\mathrm{C}_{r\,|\,2n}$ acts on the corresponding irreducible $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$-module $L(\lambda_\delta^\dagger)$. If λ_{Lie} denotes the highest weight in the Lie theoretic sense, then we have the standard formula $c_{\lambda_\delta^\dagger} = (\lambda_{\mathrm{Lie}} + 2\rho, \lambda_{\mathrm{Lie}})$, see e.g. [Mus12, Lemma 8.5.3], where we will use the choice for ρ from Section 13.

Lemma 7.9. *Let $\lambda_\delta^\dagger = (\ulcorner\lambda^\infty, \epsilon)$, $\mu_\delta^\dagger = (\ulcorner\mu^\infty, \epsilon')$ be signed hook weight diagrams such that $\mu_\delta^\dagger$ is obtained from $\lambda_\delta^\dagger$ by a local move* (6.30) *involving the positions $p \pm 1/2$. Then*

$$c_{\mu_\delta^\dagger} = c_{\lambda_\delta^\dagger} \pm 2p. \tag{7.37}$$

Hereby we add $2p$ if a symbol $\vee$ or $\wedge$ is moved from right to left, and subtract $2p$ if a symbol $\vee$ or $\wedge$ is moved from left to right.[4] *Changing the sign ϵ does not change the value $c_{\lambda_\delta^\dagger}$.*

Proof. Attached to $\lambda_\delta^\dagger$ we have the irreducible module $L(\lambda_\delta^\dagger)$. Let $\lambda_{\mathrm{Lie}} = \sum_{i=1^m} a_i\epsilon_i + \sum_{i=1^n} b_i - \rho$ be its highest weight as in Section 13; similar for μ. The a_i and b_i are determined by the lengths of the first m columns respectively first n rows in $\ulcorner\lambda$. Let first $p \neq 0$.

- *A $\vee$ moves to the left:* In this case $\epsilon' = \epsilon$. Assume that the ℓth symbol gets moved, where we enumerate the symbols according to the rows in $\ulcorner\lambda$, i.e., first counting the $\vee$ from right to left followed by the $\wedge$ from left to right. Then by assumption in the lemma $\mathcal{S}(\lambda)_\ell = -(p + 1/2)$. By the definition of $\mathcal{S}(\lambda)$ it follows that $\ulcorner\lambda_\ell = \delta/2 + \ell + p - 1/2$ and $\ulcorner\mu_\ell = \ulcorner\lambda_\ell - 1$. Since a $\vee$ is moved the coefficient b_ℓ is changed from $b_\ell = p + 1/2$ to $p - 1/2$. This implies using the biliner form on (5.23) (defined there) that

$$\begin{aligned} c_{\mu_\delta^\dagger} &= (\mu + 2\rho, \mu) = (\lambda - \delta_\ell + 2\rho, \lambda - \delta_\ell) = (\lambda + 2\rho, \lambda) + (\delta_\ell, \delta_\ell) - 2(\lambda + 2\rho, \delta_\ell) \\ &= c_{\lambda_\delta^\dagger} - 1 + 2b_\ell = c_{\lambda_\delta^\dagger} + 2p. \end{aligned}$$

- An $\wedge$ moves to the left: Again we have $\epsilon' = \epsilon$. With the notation as above we have now $\mathcal{S}(\lambda)_\ell = p + 1/2$. Which implies for the hook partition $\ulcorner\lambda_\ell = \delta/2 + \ell - (p + 1/2) - 1$ and $\ulcorner\mu_\ell = \ulcorner\lambda_\ell + 1$. Since an $\wedge$ is moved, the coefficient a_k gets changed from $p - 1/2$ to $p + 1/2$, where k is given by $\delta/2 + \ell - (p + 1/2)$. We obtain

$$\begin{aligned} c_{\mu_\delta^\dagger} &= (\mu + 2\rho, \mu) = (\lambda + \epsilon_k + 2\rho, \lambda + \epsilon_k) = (\lambda + 2\rho, \lambda) + (\epsilon_k, \epsilon_k) - 2(\lambda + 2\rho, \epsilon_k) \\ &= c_{\lambda_\delta^\dagger} + 1 + 2a_k = c_{\lambda_\delta^\dagger} + 2p. \end{aligned}$$

- *An $\wedge$ or a $\vee$ moves to the right: In both cases, independent whether an $\wedge$ or a $\vee$ is moved, a similar calculation as the two previous cases shows that $c_{\mu_\delta^\dagger} = c_{\lambda_\delta^\dagger} - 2p$.*

If $p = 0$ then only the sign gets changed, the hook partition is preserved. Hence the value of the $\mathrm{C}_{r\,|\,2n}$-action is preserved. (The underlying SOSp-modules are isomorphic). ☐

8. Translation functors on $\mathcal{F}$

Instead of working with the set $s\Gamma_\delta(n|m)$ of equivalence classes of weight diagrams with signs we prefer to work with weight diagrams only. Which set of weight diagrams we obtain will depend on r and n and not only on δ, and its elements are obtained from the $\lambda_\delta^\dagger$ by an easy procedure. That means we will rewrite each equivalence class of signed weight diagrams as a certain new weight diagram which we call a super weight diagram.

[4] A $\diamond$ is counted as $\wedge$ or $\vee$.

These super weight diagrams will by construction be admissible, and so the corresponding cup diagrams have finitely many cups. We will later see that the number of cups is related to the atypicality of the irreducible OSp(r | 2n)-module corresponding to the signed hook partition we started with, Corollary 13.6. These cup diagrams will be used later to give a basis for the morphism space of indecomposable projective objects in $\mathcal{F}$ as well as of $\mathrm{Hom}_{\mathcal{F}}(\mathtt{R}_\delta(\lambda), \mathtt{R}_\delta(\mu))$.

The super weight diagrams are also particularly useful in understanding the behaviour of $\mathbb{F}\mathtt{R}_\delta(\lambda)$ under the action of translation functors. This will be used crucially in the proof of the remaining part of Theorem 7.8.

8.1. **The super weight diagram attached to an irreducible** OSp(r | 2n)**-module.** We assign now a certain admissible weight diagram to each irreducible OSp(r | 2n)-module.

Definition 8.1. Let λ_δ be a projective weight diagram, then the corresponding *super weight diagram* $\lambda_\epsilon^{\oslash}$ is obtained as follows: Consider the signed (n, m)-hook weight diagram $\lambda_\delta^\dagger = (\Phi(\lambda), \epsilon)$ associated to λ_δ via Proposition 7.6 and its cup diagram $\underline{\Phi(\lambda)}$. Then $\lambda_\epsilon^{\oslash}$ is defined as the unique admissible weight diagram μ with $\mathrm{def}(\mu) + \mathrm{rk}(\mu) = \min(m, n)$ such that

- $\underline{\mu}$ is obtained from $\underline{\Phi(\lambda)}$ by replacing (infinitely many) dotted cups with two undotted vertical rays each,
- and possibly a dot on the resulting leftmost ray depending on the sign of $\lambda_\delta^\dagger$ according to the following *sign rule.*

Sign rule: If δ is even, then we put a dot on the first ray if $\epsilon = +$ and no dot if $\epsilon = -$. If δ is odd, then we put a dot if the leftmost ray in $\underline{\lambda_\delta}$ is undotted and we do not put a dot if the leftmost ray in $\underline{\lambda_\delta}$ is dotted. The weight diagram $\lambda_\epsilon^{\oslash}$ is then called the *super weight diagram attached to* $W_\delta^{\infty\,-1}(\Phi(\lambda), \epsilon)$. Let $s\Gamma_\delta(n|m)$ be the set of all super weight diagrams.

The super weight diagram is well-defined thanks to Lemma 7.4. When passing from $\Phi(\lambda)$ to $\lambda_\epsilon^{\oslash}$ all undotted cups and then the required amount of leftmost dotted cups are kept. The dotted cups which got removed in Definition 8.1 were called *fake cups* in [ES17, Definition 4.1] and the vertices attached to them *frozen*. Some examples are given in Section 12.3. The frozen vertices are indicated by small circles around the symbols.

Remark 8.2. In the case of odd δ, we can also formulate the sign rule directly in terms of the signed (n, m)-hook weight diagram $(\ulcorner\lambda^\infty, \epsilon)$ without knowledge of the corresponding projective weight diagram λ_δ using Lemma 6.13 and Proposition 8.4 below[5]. Namely for each symbol $\circ$ or $\times$ in $\ulcorner\lambda^\infty$ we count the number of endpoints of rays and cups in $\underline{\ulcorner\lambda^\infty}$ to the left of this symbol, and take their sum plus the total number of undotted cups in $\underline{\ulcorner\lambda^\infty}$. Let this be s. In case s is even, we put a dot on the first ray if $\epsilon = +$ and no dot if $\epsilon = -$. In case s is odd, we put a dot on the first ray if $\epsilon = -$, and no dot if $\epsilon = +$. In other words, we choose β such that, with $\mu := \ulcorner\lambda^\infty$, the number $u + \sum_{i \in \mu^{-1}(\{\circ,\times\})} |\{j < i \mid \mu(j) \in \{\wedge, \vee\}\} + \beta$ is odd in case $\epsilon = +$ and even in case $\epsilon = -$.

Definition 8.3. The cup diagram for a signed hook partition $\lambda_\delta^\dagger$ is defined as $\underline{\lambda_\delta^\dagger} := \underline{\ulcorner\lambda_\epsilon^{\oslash}}$. Via the map $\lambda \mapsto \underline{\lambda_\delta^\dagger}$ we can also talk about the *super cup diagram attached to* $\mathtt{R}_\delta(\lambda)$.

Here in an easy, but important observation:

Proposition 8.4. *If* $\mathbb{F}\mathtt{R}_\delta(\lambda)$ *is projective, then* $\underline{\lambda_\delta^\dagger}$ *agrees with* $\underline{\lambda_\delta}$ *up to a dot on the leftmost ray, and additionally a dot on the cup attached to* $\Diamond$ *in case there is such a cup.*

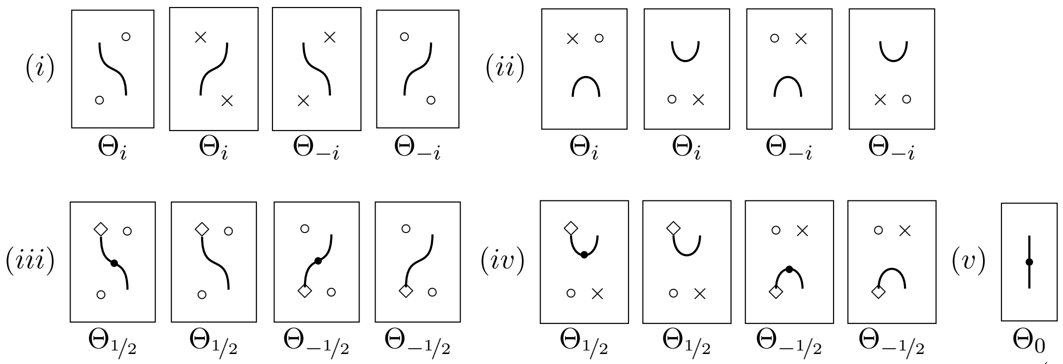
Proof. The first part is clear by construction, see Definition 8.1, since passing from λ_δ to $\lambda_\delta^\dagger$ means we label all vertices corresponding to rays in $\underline{\lambda_\delta}$ with $\wedge$. By projectivity they are all frozen and hence get again turned into $\vee$'s in $\lambda_\epsilon^{\oslash}$ apart from maybe the leftmost one. Hence,

[5] Note that this rule is just a reformulation of the rule in [ES17, Definition 4.6].

the two cup diagrams in question agree up to possibly a dot on the first ray. That they are in fact different follows then from Lemma 6.13 together with (C-1) from Definition 6.3 in case of a cup attached to $\diamond$. ☐

8.2. ***i*-induction and *i*-translation functors.** We will now introduce translation functors with the nice property that they send an indecomposable projective module either to an indecomposable projective module, to zero, or to the direct sum of two copies of an indecomposable projective module.

Definition 8.5. Let $i \in \mathbb{Z} + \delta/2$. Given a cup diagram $\underline{\lambda}$ for an admissible weight diagram λ consider the following *i-translation pictures* involving the vertices $|i| - 1/2$ and $|i| + 1/2$.



(8.38)

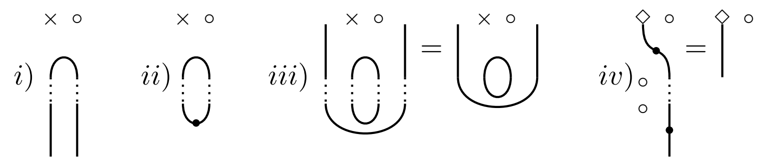
By *applying i-translation* to $\underline{\lambda}$ we mean

(T-1) in case $i \neq 0$, to put a translation picture Θ_i from i)-iv) on top of $\underline{\lambda}$ identifying the vertices at the bottom of the translation picture with two neighboured vertices $|i| \pm 1/2$ at the top of $\underline{\lambda}$, and afterwards remove pairs of two dots on the same connected component so that each component has at most one dot. Hereby the involved $\circ$'s, $\times$'s and $\diamond$'s in the translation picture have to match the labels at vertices $|i| \pm 1/2$ in λ, and a cap cannot be put on top of two undotted rays. The resulting diagram is (topologically) either again a cup diagram $\underline{\mu}$ or a cup diagram $\underline{\mu}$ with an internal circle; and

(T-2) in case $i = 0$ to put the picture v) on top of $\underline{\lambda}$ at position $1/2$, if the position $1/2$ exists and does not carry a $\circ$ or $\times$. Again, we obtain a cup diagram $\underline{\mu}$.

In either case just described we call the cup diagram $\underline{\mu}$ a *translated cup diagram* or more precisely a *cup diagram obtained from $\underline{\lambda}$ by an i-translation.*

Example 8.6. A number of example of translation pictures applied to cup diagrams is given. Here the first two are not defined, while the last two give the indicated cup diagram, with the third one having an internal circle.



The translation pictures describe i-induction on projective objects in $\mathcal{D}(\delta)$.

Theorem 8.7. *Let $\lambda \in \Lambda$ and $i \in \mathbb{Z} + \delta/2$. Then*

$$i\text{-ind}(\mathtt{R}_\delta(\lambda)) \cong \bigoplus_\mu \mathtt{R}_\delta(\mu)^{\oplus s_\mu}, \quad i\text{-ind}(\mathcal{P}_\delta(\lambda)) \cong \bigoplus_\mu \mathcal{P}_\delta(\mu)^{\oplus s_\mu}, \text{ and}$$

$$i\text{-ind}(P_{d,\delta}(\lambda)) \cong \bigoplus_\mu P_{d+1,\delta}(\mu)^{\oplus s_\mu},$$

where μ runs through all partitions, such that $\underline{\mu}_\delta$ can be obtained by an i-translation applied to $\underline{\lambda}_\delta$ not creating a circle with an odd number of dots. The multiplicity s_μ is $s_\mu = 2$ if hereby a circle (with an even number of dots) occurred, and $s_\mu = 1$ otherwise.

Proof. This is [ES18, Theorem 4.5]. □

The following easy observation will be crucial in proving the categorification statements.

Proposition 8.8. *Let λ, $\mu \in \Lambda$ then $\kappa(\mu_\delta) \geq \kappa(\lambda_\delta)$ if $\underline{\mu}_\delta$ is obtained by a translation from $\underline{\lambda}_\delta$. Moreover, $\kappa(\mu_\delta) > \kappa(\lambda_\delta)$ only if the translation picture is a cap put on top of a dotted and an undotted ray in $\underline{\lambda}$.*

Proof. By applying a translation picture to $\underline{\lambda}_\delta$ we never destroy cups except when we create (using a translation picture with a cap) a small circle consisting of one cup and one cap which then gets removed. But in this case we create additionally one $\circ$ and one $\times$. We can create a new cups, but only by destroying one $\circ$ and one $\times$. Conversely we only can destroy $\circ$ and $\times$ in pairs and at the same time create a cup. We can only create $\circ$ and $\times$'s in pairs by either creating a circle as in the beginning of the proof or by putting a cap on top of a dotted and an undotted ray, see (8.40). In this case we obviously have $\kappa(\mu_\delta) > \kappa(\lambda_\delta)+1$. □

Let $\lambda_\delta^\dagger = (\Phi(\lambda), \epsilon)$ and $\mu_\delta^\dagger = (\Phi(\mu), \epsilon')$ be signed hook partitions. By [ES17, Propositions 6.2 and 6.3], $L(\lambda_\delta^\dagger)$ and $L(\mu_\delta^\dagger)$ are in the same block of $\mathcal{F}$ if the positions of the $\circ$ and $\times$ coincide in $\ulcorner\lambda^\infty$ and $\ulcorner\mu^\infty$ and $\epsilon = \epsilon'$.

Definition 8.9. The endofunctor $\_ \otimes V$ of $\mathcal{F}$ decomposes as $\_ \otimes V = \oplus_{i \in \mathbb{Z}+\delta/2} \Theta_i$, where Θ_i is the direct summand which changes the generalized eigenvalue of the $\mathrm{C}_{r\,|\,2n}$-action by $2i$. We call Θ_i the *i-translation functor*.

By Lemma 7.9, Θ_i can be non-zero only on blocks whose weight diagrams look locally as at the bottom of a picture in (8.38) subtitled Θ_i (with image in the block at the top). We finish this section by the following very important application of Theorem 7.8.

Theorem 8.10. *The functor $\mathbb{F}_{(r\,|\,2n)}$ interwiners i-induction with i-translation, that means*

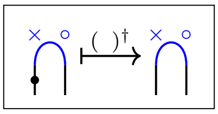
$$\mathbb{F}_{(r\,|\,2n)} \circ i\text{-ind} \cong \Theta_i \mathbb{F}_{(r\,|\,2n)} \tag{8.39}$$

for any $i \in \mathbb{Z} + \delta/2$. Moreover, $\Theta_i(P(\lambda_\delta^\dagger) \cong \bigoplus_{\mu_\delta^\dagger} P(\lambda_\delta^\dagger)^{\otimes s}$ where $\mu_\delta^\dagger$ runs through all signed hook partitions, such that $\underline{\mu}_\delta^{\oslash}$ can be obtained by an i-translation applied to $\underline{\lambda}_\delta^{\oslash}$ not creating a circle with an odd number of dots. The results are zero in case the sum is empty. The multiplicity is $s = 2$ if hereby a circle occurred, and $s = 1$ otherwise.

Remark 8.11. Note that Theorem 8.7 and Theorem 8.10 imply that i-induction for $i \neq -1/2$ sends the indecomposable objects $\mathtt{R}_\delta(\lambda)$, $P_{d,\delta}(\lambda)$ to indecomposable objects, to zero or to two copies of an indecomposable object. Thus Θ_i sends indecomposable projectives to indecomposable projectives to zero or to two copies of an indecomposable projective. This fact was already observed in [GS13].

Proof. The statement (8.39) is clear from the definitions and Proposition 6.15. For the second part we assume Theorem 7.8. Let $P(\lambda_\delta^\dagger)$. Clearly, $\Theta_i(P)$ is projective, since $\_ \otimes V$ is exact and selfadjoint. By Theorem 7.8 $P \cong \mathbb{F}_{(r\,|\,2n)}(\mathtt{R}_\delta(\lambda))$, and thus $\Theta_i(P) \cong \mathbb{F}_{(r\,|\,2n)}(i\text{-ind}\mathtt{R}_\delta(\lambda))$. By Theorem 8.7 the indecomposable summands in $i\text{-ind}\mathtt{R}_\delta(\lambda)$ are of the form $\mathtt{R}_\delta(\mu)$ where $\underline{\mu}_\delta$ is obtained from $\underline{\lambda}_\delta$ by applying i-translation. Then by Theorem 5.12 and Theorem 6.17, the direct summands in $\Theta_i(P)$ are precisely the corresponding $P(\mu_\delta^\dagger)$ (with multiplicities) for $\kappa(\mu_\delta) = \min(m, n)$. We have to match it now with the diagrammatics.

In case $\kappa(\mu_\delta) > \min(m,n)$, we are by Proposition 8.8 in a local situation as follows:



(8.40)

with $\mathbb{F}_{(r\,|\,2n)} \circ i\text{-ind}\mathtt{R}_\delta(\lambda) = \{0\}$, hence $\Theta_i(P) = \{0\}$. The dotted ray here must be the leftmost ray. By assumption $\kappa(\lambda_\delta) = \min(m,n)$, and so Proposition 8.4 applies. Therefore, $\underline{\ulcorner\lambda}^{\circledcirc}_{\epsilon}$ is the same cup diagram as $\underline{\lambda}$ except that the leftmost ray is now undotted, see (8.40). Then no i-translation picture can be applied and hence the diagrammatical formula is true.

If $\kappa(\mu_\delta) = \min(m,n)$ the diagrammatical formula follows from Proposition 8.4. (This is indeed obvious except that one has to keep track of dots on the leftmost ray. We leave this check to the reader, but in fact one can see it also by following through the proof of Proposition 9.1). □

Remark 8.12. Note that Θ_i, $i \neq -1/2$ is a classical translation functor in Lie theory, since restricted to a block it is the functor $-\otimes V$ followed by projecting onto some block. Note however that $\Theta_{-1/2}$ has image in two different blocks (see the two pictures in iv)), hence this functor decomposes further if we project onto one of the two blocks.

We finish with an application in direction of our main categorification theorem.

Definition 8.13. For any $k \in \mathbb{Z}_{\geq 0}$ let $I_k = \{\lambda \in \Lambda \mid \kappa(\lambda_\delta) \geq k\}$ and denote by $\mathcal{I}_k$ the full additive subcategory of Rep_δ generated by the $\{\mathtt{R}_\delta(\lambda) \mid \kappa(\lambda_\delta) \geq k\}$.

This can be used to define a filtration on the Fock space considered in Theorem 3.5.

Corollary 8.14 (Layerfiltration)**.** *The isomorphism classes $[\mathcal{P}_\delta(\lambda)]$, $\lambda \in I_k$ generate a vector subspace $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))_{>k}$ of $\mathrm{K}_0(\mathcal{D}_\Delta(\delta))$ which is stable under the action of $\mathcal{U}(\mathfrak{g}^\theta)$.*

Proof. The action of the generators B_i in Theorem 3.5 is given by $[i\text{-ind}]$. Then the claim follows from Theorem 8.7 and Proposition 8.8. □

Definition 8.15. We call the subquotient $\mathrm{K}_0\left(\mathcal{D}_\Delta(\delta)\right)\big)_k := \mathrm{K}_0^{\oplus}(\mathcal{D}_\Delta(\delta))_{>k}/\,\mathrm{K}_0(\mathcal{D}_\Delta(\delta))_{>(k+1)}$ the kth *layer* of the filtration. It has basis $\{[\mathcal{P}_\delta(\lambda)] \mid \kappa(\lambda_\delta) = k\}$.

9. Proof of Theorem 7.8 using the Eigenvalue Comparison Lemma

It remains to show that the indecomposable summand $\mathbb{F}\mathtt{R}_\delta(\lambda)$ of the $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$-module $V^{\otimes d}$ has irreducible head isomorphic to $L(\lambda^\dagger)$ to complete the proof of Theorem 7.8. The proof contains two main ideas: one is the matching of the combinatorics of i-induction and i-translation functors and the second is the Eigenvalue Comparison Lemma justifying that the two values i correspond under the functor $\mathbb{F}$.

Proof of Theorem 7.8. As before we abbreviate $\mathbb{F} = \mathbb{F}_{(r\,|\,2n)}$. If $\lambda = \varnothing$, then $\mathbb{F}\mathtt{R}_\delta(\lambda)$ is the trivial $\mathrm{OSp}(\mathrm{r}\,|\,2\mathrm{n})$-module, i.e. $\mathbb{F}\mathtt{R}_\delta(\lambda) \cong L((0,+))$, which via Proposition 5.2 corresponds to the empty hook partition. Now $\ulcorner\varnothing^\infty$ is equal to $\varnothing_\delta$ with all symbols flipped upside down. On the other hand, $\underline{\varnothing}_\delta$ has no cups, and therefore $\lambda^\dagger_\delta = (\ulcorner\varnothing^\infty, +)$ and $\underline{\ulcorner\varnothing}^{\circledcirc}_{+}$ has precisely $\min(m,n)$ dotted cups and a dotted ray. The claim follows in this case.

Assume now that $\mu \in \Lambda$ is obtained from $\lambda \in \Lambda$ by adding one box and that the claim is proved for $\mathbb{F}\mathtt{R}_\delta(\lambda)$ with $\lambda^\dagger_\delta = (\ulcorner\lambda^\infty, \epsilon)$. Then μ_δ differs from λ_δ by exactly one local move from (6.30), say at position $p \pm 1/2$. Assume first that we do not have the situation that $\Diamond\vee$ turns into $\circ\wedge$.

By Lemma 6.14 and Remark 8.11, $\mathtt{R}_\delta(\mu)$ is (up to isomorphism) the unique indecomposable summand in $i\text{-ind}(\mathtt{R}_\delta(\lambda))$ where $i = \pm p$ with the sign depending on whether a symbol was moved to the left or to the right. Therefore, $\mathbb{F}(\mathtt{R}_\delta(\mu))$ is the unique indecomposable summand in $\mathbb{F}(\mathtt{R}_\delta(\lambda) \boxtimes \mathtt{R}_\delta(\square)) = \mathbb{F}(\mathtt{R}_\delta(\lambda)) \otimes V$ contained in the generalized eigenspace for Ω_d (with

$d = |\mu|$) with eigenvalue $c_\mu^{\mathrm{Br}} = c_\lambda^{\mathrm{Br}} \pm 2p$, which is also the generalized eigenvalue for the $\mathrm{C}_{r\,|\,2n}$-action by Proposition 6.15.

By induction hypothesis we have a surjection $P(\lambda_\delta^\dagger) \twoheadrightarrow \mathbb{F}\mathtt{R}_\delta(\lambda)$, hence also $P(\lambda_\delta^\dagger) \otimes V \twoheadrightarrow \mathbb{F}\mathtt{R}_\delta(\lambda) \otimes V \cong \mathbb{F}(\mathtt{R}_\delta(\lambda) \boxtimes \mathtt{R}_\delta(\varnothing))$. In fact, already the direct summand contained in the generalized $2p$-eigenspace surjects onto $\mathbb{F}\mathtt{R}_\delta(\mu)$. Thus by Lemma 7.9 we obtain that $P(\nu_\delta^\dagger)$ must surject onto $\mathbb{F}\mathtt{R}_\delta(\mu)$ where $\nu_\delta^\dagger$ is obtained from $\ulcorner\lambda_d e^\dagger$ by a local move of a symbol at positions $p \pm 1/2$ and by Lemma 6.14 such that this symbol is moved in the same direction as when passing from λ_δ to μ_δ.

In case $\diamond\vee$ turns into $\circ\wedge$, the the summand $P(\nu_\delta^\dagger)$ is not unique. Instead, we have to work with a direct sum $P(\nu_\delta^\dagger) \oplus P(\eta_\delta^\dagger)$ where $\nu_\delta^{\oslash}$ and $\eta_\delta^{\oslash}$ only differ by a dot on the leftmost ray. However for one of the summands we know the claim already by induction, say $\mathbb{F}\mathtt{R}_\delta(\eta) \cong P(\eta_\delta^\dagger)$. Thus by Lemma 7.9 we obtain that $P(\nu_\delta^\dagger)$ must surject onto $\mathbb{F}\mathtt{R}_\delta(\mu)$.

It remains to show $\mu_\delta^\dagger = \nu_\delta^\dagger$, since then $\mathbb{F}\mathtt{R}_\delta(\mu)$ has irreducible head $L(\mu_\delta^\dagger)$. By the same induction on $|\lambda|$, this is Proposition 9.1 below, since we have established the case $\lambda = \varnothing$. □

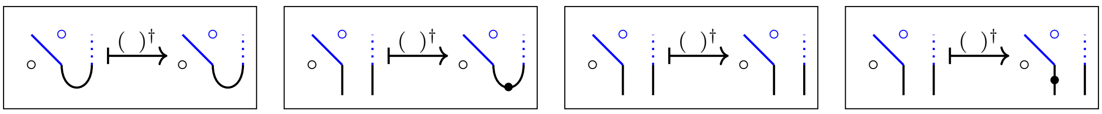
Proposition 9.1 (Eigenvalue Comparison Lemma). *Let $\lambda, \mu \in \Lambda$ and $\lambda_\delta^\dagger = (\ulcorner\lambda^\infty, \epsilon)$ and $\mu_\delta^\dagger = (\ulcorner\mu^\infty, \epsilon')$. If μ is obtained from λ by adding one extra box, then*

$$c_\lambda^{\mathrm{Br}} = c_{\lambda_\delta^\dagger} \quad \Longrightarrow \quad c_\mu^{\mathrm{Br}} = c_{\mu_\delta^\dagger}. \tag{9.41}$$

Moreover, $\underline{\mu}_\delta$ is obtained from $\underline{\lambda}_\delta$ and $\underline{\ulcorner\mu_{\epsilon'}^{\oslash}}$ from $\underline{\ulcorner\lambda_\epsilon^{\oslash}}$ by some i-translation for the same i. The corresponding translation pictures (8.38) *agree except of an extra dot on on the cup or ray attached to position zero if a diamond gets moved out of position zero (that is if $i = -1/2$).*

Proof. Since the partitions differ by a box, the Deligne weight μ_δ is obtained from λ_δ by a local move from (6.30). We start with the case where δ is odd.

- Assume first μ_δ is obtained from λ_δ by one of the first four "easy" moves in (6.30). We explain the argument for the first of these moves only, since the others are similar. The $\vee$ involved in the move could correspond to a cup or a ray in $\underline{\lambda}_\delta$, as in one of the following local situation [6]

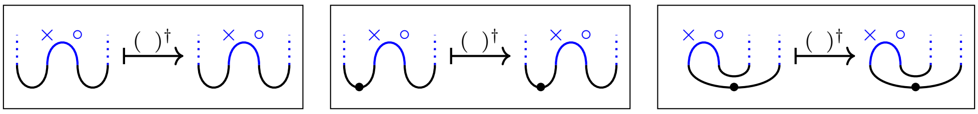


  The first picture illustrates the situation, when we have a cup, the others when we have a ray (which must be undotted). We indicated in each case the change when passing from $\underline{\lambda}_\delta$ to $\underline{\mu}_\delta$. In the second pictures we have $\kappa(\lambda_\delta) < \min(m, n)$, whereas in the last $\kappa(\lambda_\delta) = \min(m, n)$ and the move happens at the leftmost ray. Applying $(\ )^\dagger$ gives $\mu_\delta^\dagger$ respectively $\lambda_\delta^\dagger$ with the cup diagrams $\underline{\lambda}_\delta^{\oslash}$ respectively $\underline{\mu}_\delta^{\oslash}$ as indicated. Indeed, in the first picture the cup exists in $\underline{\lambda}_\delta$ and $\underline{\mu}_\delta$ and must stay when we apply the map. In the second picture $\kappa(\lambda_\delta) < \min(m, n)$ and then also $\kappa(\mu_\delta) < \min(m, n)$ by Proposition 8.8 and a dotted cup gets created at the position involved in the move when passing to $\underline{\lambda}_\delta^{\oslash}$, but then the same happens for $\underline{\mu}_\delta^{\oslash}$. In the third picture $\kappa(\lambda_\delta) \leq \min(m, n)$ and none of the possibly newly created cups is attached to the position involved in the move (if this holds for λ_δ then also for μ_δ). In the last picture we have $\kappa(\lambda_\delta) = \min(m, n) = \kappa\mu_\delta$ and therefore applying the map $(\ )^\dagger$ keeps the cup diagrams except of maybe an extra dot on the leftmost ray, see Proposition 8.4. To check the signs note that adding a box of course changes the parity of the number of boxes in a partition. On the other hand par is changed locally at the two vertices involved in the move. Hence the leftmost ray in

[6]Note that here and in the following illustrations the indicated cups could be much bigger than drawn and could have other cups nested inside. Only the endpoints at the vertices involved in the move are fixed.

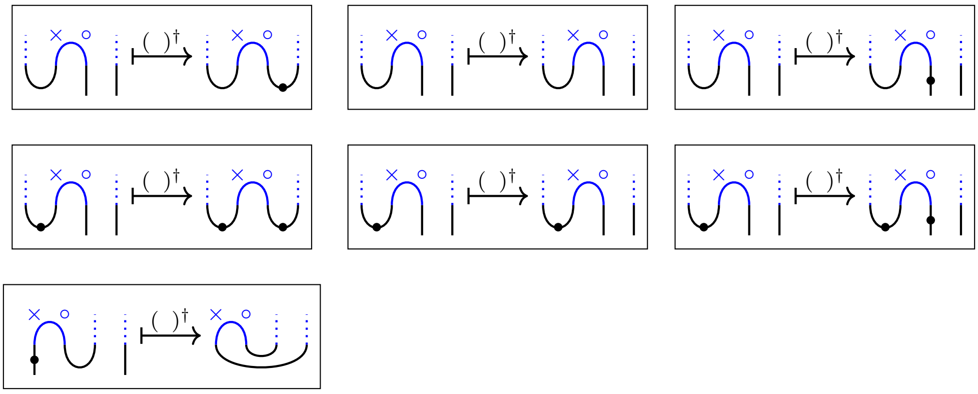
$\underline{\mu}_\delta^{\oslash}$ has a dot if and only if the leftmost ray in $\underline{\lambda}_\delta^{\oslash}$ has a dot. Therefore $\underline{\mu}_\delta^{\oslash}$ differs from $\underline{\mu}_\delta^{\oslash}$ exactly by the same i-translation as $\underline{\mu}_\delta$ does from $\underline{\lambda}_\delta$. The changes in the Ω and the $\mathrm{C}_{r\,|\,2n}$-value agree by Lemma 7.9 and Lemma 6.14.

- For the fifth and sixth move in (6.30) the claim is obviously true.
- We argue now for the seventh move in (6.30), but omit the eighth, since only the involved $\circ$ and $\times$ are switched. If $\underline{\lambda}_\delta$ has cups attached to both vertices involved in the move, we have local pictures as follows



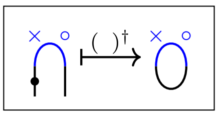
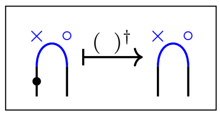
(Note that λ_δ has the configuration $\wedge,\vee$ at the two vertices in question.) Since in each move par is changed the leftmost ray is kept dotted respectively undotted in the corresponding move on the super side. Again, the changes in the Ω and the $\mathrm{C}_{r\,|\,2n}$-value agree by Lemma 7.9 and Lemma 6.14.

If $\underline{\lambda}_\delta$ has a cup attached to one of the vertices involved in the move, we have locally



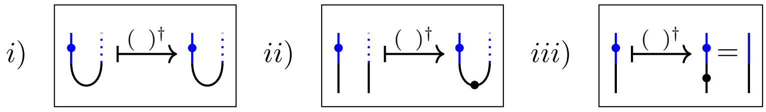
In the leftmost picture in the first two rows the ray becomes part of a (dotted) cup on the super side (assuming λ_δ is not a projectiv weight diagram). Here and in the following we use Proposition 8.8 without mentioning it explicitly. The second and third pictures from these rows show the cases where the ray stays a ray. It might turn into a dotted one or from a dotted into an undotted when we apply $(\ )^\dagger$, but this happens in the same way for λ and μ. This is because locally at the two vertices (and then also globally) par changes. In the last case we create a new undotted cup out of the dotted ray. Obviously the same i-translation is applied on each side of map. Again, the changes in the Ω and the $\mathrm{C}_{r\,|\,2n}$-value agree by Lemma 7.9 and Lemma 6.14.

If we have two rays attached to the two involved vertices, the left one must be dotted and the right one undotted. In case $\kappa(\lambda_\delta) < \min(m,n)$ a new undotted cup gets created

$\kappa(\lambda_\delta)<\min(m,n)$:  $\kappa(\lambda_\delta)=\min(m,n)$:  (9.42)

and, again, the changes in the Ω and the $\mathrm{C}_{r\,|\,2n}$-value agree by Lemma 7.9 and Lemma 6.14. In the second case, we create an additional $\circ$, $\times$ pair in μ_δ which means $\mathbb{F}\mathtt{R}_\delta(\mu) = \{0\}$. But on the other hand, the translation on the super side is not defined.

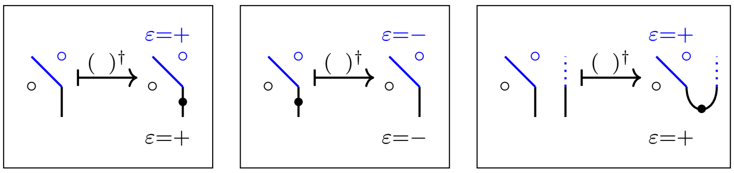
- Finally we have to consider the case, where μ_δ differs from λ_δ by a flip of a $\vee$ into an $\wedge$ at $1/2$. Now $\underline{\lambda}_\delta$ could have a cup or a ray at position $1/2$. If there is a (necessarily undotted) cup, then it stays a cup when we apply $(\ )^\dagger$. Moving to $\underline{\mu}_\delta$ creates a dotted cup which again stays when applying $(\ )^\dagger$. Since par is changed in the diagram during the move, the leftmost ray will be unchanged after we apply $(\ )^\dagger$, see picture i),



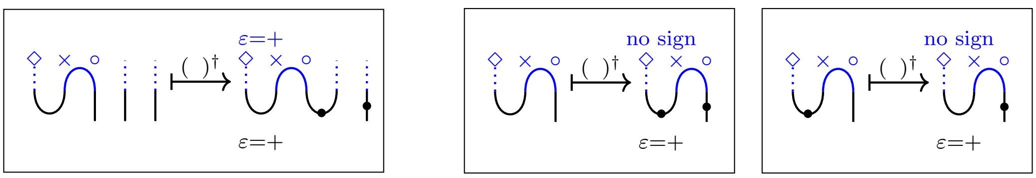
Otherwise there must be a ray (necessarily undotted) at position $1/2$. If λ is a projective weight diagram, then the claim follows from Proposition 8.4, see iii). If λ_δ is not a projective weight diagram, then $\vee$ at position $1/2$ corresponding to a ray in $\underline{\lambda_\delta}$ becomes an $\wedge$ when we apply the $\dagger$-map and hence creates a dotted cup in the corresponding cup diagram. Then μ_δ has an $\wedge$ instead which turns into a $\vee$ when we apply the $\dagger$-map and hence creates an undotted cup. In all these cases neither the Ω- nor the $\mathrm{C}_{r\,|\,2n}$-value change by Lemma 7.9 and Lemma 6.14. (In fact $L(\lambda_\delta^\dagger)$ and $L(\mu_\delta^\dagger)$ are isomorphic as $\mathfrak{osp}$-modules, but different as OSp(r | 2n)-modules.)

This proves all claims in the proposition in case δ is odd. Now assume δ is even. Let ϵ and ϵ' be the signs attached to λ respectively μ.

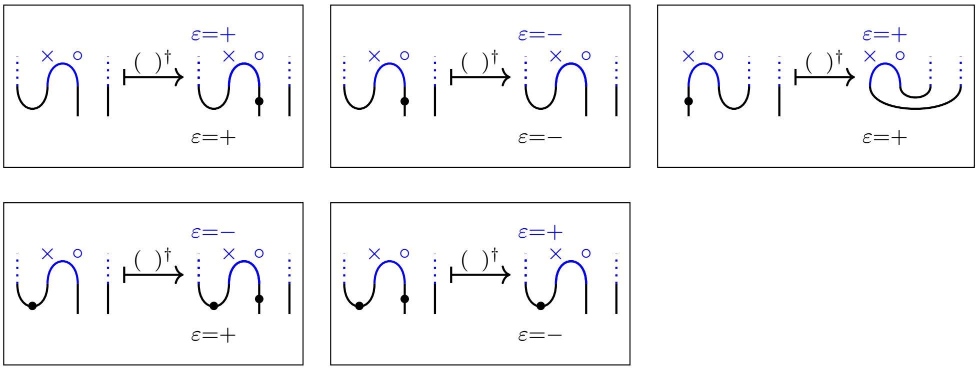
- For the first four "easy" moves from (6.30) the arguments are exactly as in the odd case, as long as the vertices involved in the move are not connected with the leftmost ray. We explain the argument for the first of these moves assuming the leftmost ray is connected to the involved vertex. Then the situation looks locally as follows:



  In the first two pictures we have the situation $\kappa(\lambda_\delta) = \min(m, n)$, whereas in the third $\kappa(\lambda_\delta) < \min(m, n)$ and hence we create an extra dotted cup on the super side. Since $\kappa(\lambda_\delta) = \kappa(\mu_\delta)$ and also $\epsilon = \epsilon'$ the claims follow.
- For the fifth and sixth move in (6.30) the claims are again obviously true.
- We argue for the seventh move in (6.30), and omit the eighth.
  - We have the special situation that the first involved vertex is connected in $\underline{\lambda_\delta}$ with the zero position. (This cannot happen with the second involved vertex, since it is labelled by a $\vee$.) If $\kappa(\lambda_\delta) < \min(m, n)$, then we have the first situation here, otherwise the second or third.



    The dotted ray is always the leftmost ray in the diagram. Again all claims follow.
  - The remaining arguments for the seventh move are the same as for odd δ as long as the leftmost ray in $\underline{\lambda_\delta}$ is not attached to the involved vertices.
    * If there is exactly one ray attached to the involved vertices. then we have one of the following five situations

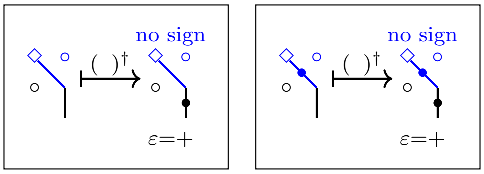


In the first two columns $\kappa(\lambda_\delta) = \min(m, n)$. This is preserved when passing to μ_δ, but also the sign since we destroy a cup but create additionally a pair $\times\circ$, the leftmost ray is kept dotted respectively undotted and by then passing to the super side, the leftmost ray becomes undotted respectively dotted. In the rightmost picture $\kappa(\lambda_\delta) < \min(m, n)$, hence a cup gets created on the super side. This cup is undotted, since the symbol attached to the dotted ray on the Brauer/Deligne side is turned into a $\vee$ on the super side. Again $\kappa(\mu_\delta) = \kappa(\lambda_\delta)$, and the sign is preserved.

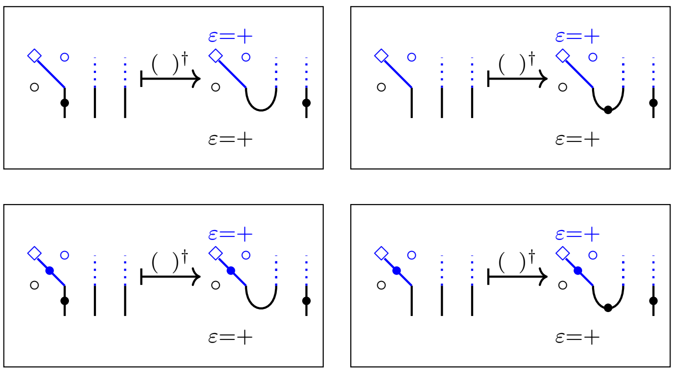
∗ The case that each of the involved vertices is attached to a ray in $\underline{\lambda}_\delta$ is (8.40).

This settles the first eight (and of course the last) moves from (6.30)

- Now consider the ninth move. We have the following local situations if $\kappa(\lambda_\delta) = \min(m, n)$

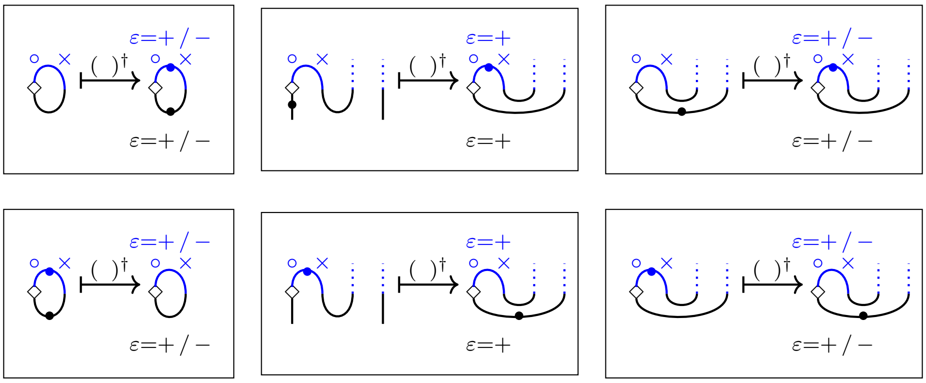


where by (6.30) we always start with an undotted ray, hence the sign is $+$. Clearly, $\kappa(\mu_\delta) = \min(m, n)$ as well. By the rule how to interpret $\diamond$ it has to be opposite before to after applying the map $(\ )^\dagger$. If $\kappa(\mu_\delta) < \min(m, n)$ then we create (possibly many) additional cups, but one of them at position zero:

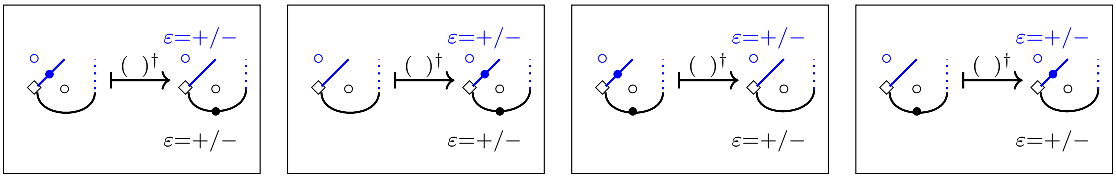


Here the sign is is $+$ and so the leftmost ray on the super side must be dotted (as indicated). Outside the displayed parts λ_δ and μ_δ respectively $\underline{\lambda}_\delta$ and $\underline{\mu}_\delta$ do not differ.

- For the tenth move we argue as for the seventh. The possible local situations are

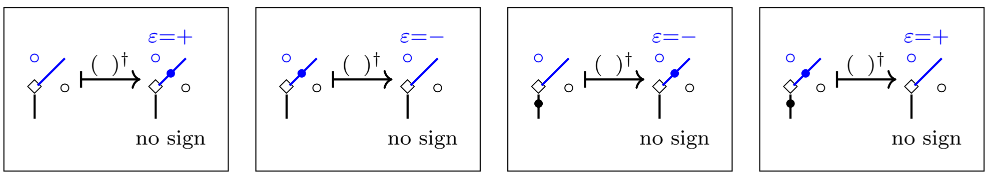


Note that in all situations, the left most ray on the super side is swapped from undotted to dotted or vice versa compared to the Deligne side. This holds for both $\kappa(\lambda_\delta) = \min(m,n)$ and $\kappa(\lambda_\delta) < \min(m,n)$ and swaps the role of $\diamond$ from one side to the other.
- For the eleventh move we first assume that $\underline{\lambda}_\mu$ has a cup connected with position zero:

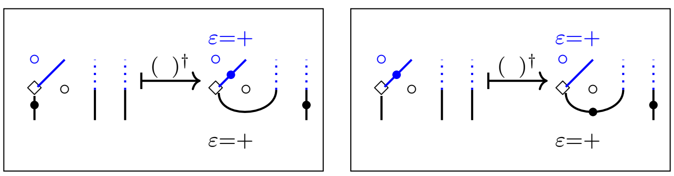


If λ_δ is a projective weight diagram, the sign is $+$ or $-$, depending on the leftmost ray, and it is $+$ otherwise. In particular it stays the same for μ_δ. This leftmost ray gets changed and an additional dot appears on the cup when passing to the super side, see Proposition 8.4. The translation pictures on the two sides differ by a dot as claimed.

Now assume there is not a cup, hence there must be a ray, at position zero. If $\kappa(\lambda_\delta) = \min(m,n)$ then the possible scenarios are as follows:

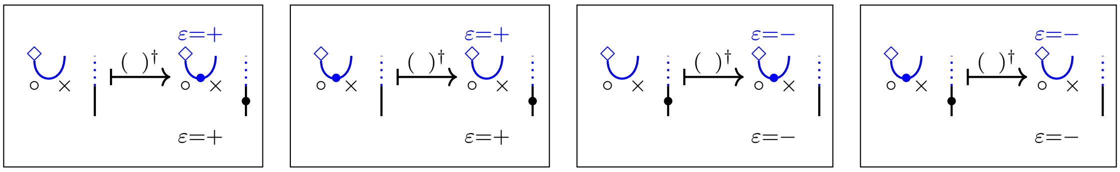


If $\kappa(\lambda_\delta) < \min(m,n)$ then we create a new cup at zero on the super side. Whether it is dotted or not depends on how the $\diamond$ has to be interpreted.



We display here for the Brauer/Deligne side the first two rays and then the first ray which does not turn into a cup when we apply the map $(\ )^\dagger$. Again we see that the translation pictures on the two sides differ by a dot. The remaining claims are obvious.

- For the 12th move we obtain the following local situations



Note that μ_δ has an $\wedge$ at position 1 creating a cup at zero in $\underline{\mu}_\delta$. Whether there is a dot on the cup depends on how the $\diamond$ gets interpreted, but the leftmost ray should not change when passing from $\underline{\lambda}_\delta$ to $\underline{\mu}_\delta$, in particular the sign is preserved. The depicted situations are in case λ_δ is projective, where by Proposition 8.4 the super side adds additional dots on the leftmost ray and on the cup at zero. The case of non-projective λ and μ are the same as the first two depicted situations with the difference that the left most ray on the super side comes from some undotted ray on the Deligne side, not necessarily the left most. Hence the sign is always $+$ and again the cup at zero must get an additional dot by the rules how to interpret the $\diamond$. We see that the translation pictures on the two sides differ by a dot.

This finished finally all the moves from (6.30). The proposition follows. □

10. The category $\mathcal{F}$ as a graded module category

We now prepare the proof of the main theorem, an explicit description of the category $\mathcal{F}_{(r\,|\,2n)}$ as the category of finite dimensional modules over the algebra $A_{(r\,|\,2n)}$.

10.1. The algebra $A_{(r\,|\,2n)}$.

Definition 10.1. The *algebra describing* $\mathcal{F}_{(r\,|\,2n)}$, denoted $A_{(r\,|\,2n)}$, is defined as

$$A_{(r\,|\,2n)} \quad = \quad \left(\bigoplus_{(\lambda,\epsilon),(\mu,\epsilon')\in\Gamma_\delta(n,m)\times\{\pm\}} \mathrm{Hom}_{\mathcal{F}_{(r\,|\,2n)}}(P(\lambda,\epsilon),P(\mu,\epsilon')) \right)^{\mathrm{op}}. \tag{10.43}$$

This is a locally finite dimensional (not unital, but) locally unital algebra with primitive idempotents labelled by signed hook partitions or equivalently their signed hook weight diagrams. By general theory, see e.g. [Mit72],

$$\bigoplus_{(\lambda,\epsilon)\in\Gamma_\delta(n,m)\times\{\pm\}} \mathrm{Hom}_{\mathcal{F}_{(r\,|\,2n)}}(P(\lambda,\epsilon),\_\,):\ \mathcal{F}_{(r\,|\,2n)} \ \longrightarrow\ A_{(r\,|\,2n)}\text{-}\mathrm{mod} \tag{10.44}$$

is an equivalence of categories. To describe $A_{(r\,|\,2n)}$ we first express the category Rep_δ (or rather a basic version) in terms of circle diagrams, then connect Rep_δ with $\mathcal{F}_{(r\,|\,2n)}$ and describe the algebra $A_{(r\,|\,2n)}$. The construction moreover allows to enrich morphism spaces in graded vector spaces (where by graded we always mean $\mathbb{Z}$-graded). We start by defining this graded enrichment.

10.2. The basic graded Deligne category $\mathrm{gRep}_\delta^{\mathrm{bsc}}$ and the nuclear ideal.

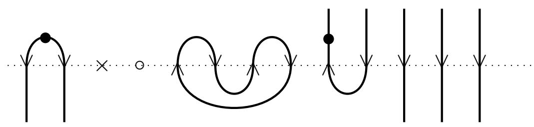
Definition 10.2. Let $k\in\mathbb{Z}_{\geq 0}$ and I_k as in Definition 8.13.

(1) Define the (locally finite dimensional locally unital) algebra $\mathbf{1}_kA\mathbf{1}_k=\oplus_{\lambda,\mu\in I_k}e_\lambda Ae_\mu$ with its distinguished basis given by $\underline{\lambda}\nu\overline{\mu}\in\mathbb{B}_{d,\delta}(\lambda,\mu)$ where $\lambda,\mu\in I_k$.

(2) Denote by $\mathbf{1}_kA\mathbf{1}_{k+1}A\mathbf{1}_k$ the twosided ideal of $\mathbf{1}_kA\mathbf{1}_k$ generated by the idempotents $e_\lambda=\underline{\lambda}\lambda\overline{\lambda}$ with $\lambda\in I_{k+1}$. For $\lambda\in I_k$ we denote by $\overline{e}_\lambda\in\mathbf{1}_kA\mathbf{1}_k/\mathbf{1}_kA\mathbf{1}_{k+1}A\mathbf{1}_k$ the idempotent which obtained as the image of $e_\lambda=\underline{\lambda}\lambda\overline{\lambda}$ under the canonical projection.

(3) A basis vector $\underline{\lambda}\nu\overline{\mu}\in\mathbf{1}_kA\mathbf{1}_k$ is *nuclear* if it contains at least one non-propagating line. We denote by $\mathbb{I}_k\subseteq\mathbf{1}_kA\mathbf{1}_k$ the span of such nuclear basis vectors and call it the *nuclear ideal* in $\mathbf{1}_kA\mathbf{1}_k$.

Example 10.3. An example for a nuclear basis vector, in this case for $k = 4$, is



It was shown in [ES17, Proposition 5.3] that $\mathbb{I}_k \subseteq \mathbf{1}_k A \mathbf{1}_k$ is indead an ideal. The algebra $\mathbf{1}_k A \mathbf{1}_k$, the nuclear ideal $\mathbb{I}_k$, and thus also the quotient algebra $\mathbf{1}_k A \mathbf{1}_k / \mathbb{I}_k$ inherit a $\mathbb{Z}_{\geq 0}$-grading from gRep_δ. We give now a categorical characterisation using Definition 8.13.

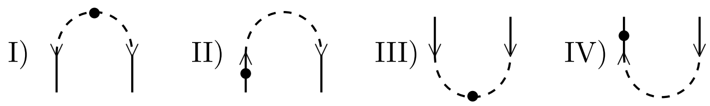
Lemma 10.4. *Let $k \in \mathbb{Z}_{\geq 0}$, $\lambda, \mu \in I_k$ and $f \in \mathbb{B}_{d',\delta}(\lambda, \mu)$ with $\lambda, \mu \in \Lambda_d(\delta)$. Then*

f is nuclear if and only if it factors through an object $\nu \in I_{k+1}$

(i.e. there exists $d \in \mathbb{Z}_{\geq 0}$, $\nu \in I_{k+1}$ and $f_1 \in \mathbb{B}_{d,\delta}(\nu, \mu)$, $f_2 \in \mathbb{B}_{d,\delta}(\lambda, \nu)$ such that $f_1 \circ f_2 = f$).

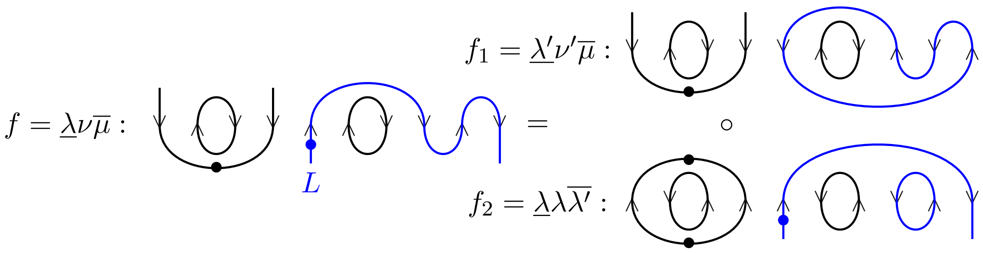
Proof. $\Rightarrow$:) It obviously suffices to check this for basis morphisms. So let f be given by a circle diagram, $f = \underline{\lambda}\nu\overline{\mu}$.

Each non-propagating line L in D must end with two rays at the same side of the circle diagram. Hence, this line looks roughly as follows,



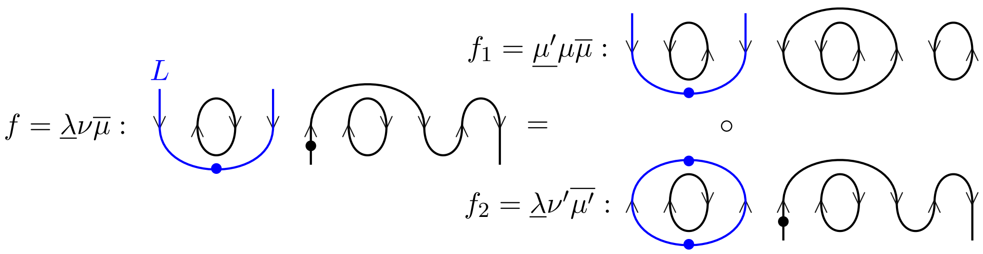
where the dashed part can be a combination of an arbitrary number of cups and caps. In case I) and III) there are an odd number of dots (indicated by a single dot) on the dashed part, whereas in case II) and IV) there is an even number of dots on the dashed part (indicated by putting no dot). These are all possible configurations and thus non-propagating lines cannot be nested one inside the other, since this would contradict the rule that dots must be visible from the left in the sense of [ES16b, Definition 3.5]. In particular this implies that there are no further rays between the two, indicated in the cases, at the ends of L.

Pick the left-most non-propagating lines L in D ending at the bottom. This is always possible by [ES17, Proof of Proposition 5.3] since the number of non-propagating lines starting at top and bottom agree. For $f = \underline{\lambda}\nu\overline{\mu}$ we now define a few modified weights. First let λ' be the weight diagram that is obtained from λ by swapping both symbols at the end of the two rays involved in L. For $\underline{\lambda'}$ this implies that it agrees with $\underline{\lambda}$ except that the two rays are replaced by a single cup, which is dotted or undotted according two whether the two rays where undotted or one was dotted. One can replace the two rays also by a dotted cup since we choose the leftmost non-propagating line. Now let ν' be the weight diagram obtained from ν by swapping all symbols that lie on the line L, not just the two for the rays as in λ'. We can then form the two oriented circle diagrams $f_1 = \underline{\lambda'}\nu'\overline{\mu}$ and $f_2 = \underline{\lambda}\lambda\overline{\lambda'}$. Note that the f_1 only differs from f by having the two rays involved in L replaced by a cup and the newly created circle being oriented anticlockwise (with the λ it would have been oriented clockwise), while f_2 only consists of small anticlockwise circles, i.e. circles consisting of a single cup and cap, except for a single non-propagating line starting at the bottom consisting of the two original rays and a single cup. Thus f_2 is an element of degree 1 while f_1 has degree exactly one less than f and we are done by induction. We illustrate this in two examples



for the case of the non-propagating line starting as indicated above. The composition $f_1 \circ f_2$ is well-defined and checking the multiplication rule [ES16b, Remark 5.13] one obtains that the product only consists of merges and only the merge with the non-propagating line in f_2 is non-trivial, which exactly creates the line L in the result. Hence $f = f_1 \circ f_2$ and by construction it factors through an object in layer $k+1$, since we increased the defect from λ to λ' by one and have the same rank, thus the layer is increased by 1.

Analogously one could have taken the non-propagating line starting at the top and split off a degree 1 element. In this case we obtain:



$\Leftarrow$:) Assume that f factors through the layer $k+1$. Hence f is a linear combination of elements of the form $f = f_1 \circ f_2$ for some circle diagrams f_i, such that the there are $k+1$ cups at the bottom of f_1 (and thus also $k+1$ caps at the top of f_2). Using the kink removal argument as in [ES16b, (4.18)] it is easy to see that each closed circle and each propagating line in a circle diagram must contain as many cups as caps. Hence there is at least one non-progating line in f_1 as well as in f_2, and so they are nuclear and so is their composition. Therefore f is nuclear. $\square$

Theorem 10.5 (Main Endomorphism Theorem)**.** *There is an isomorphism of algebras*

$$\Psi : \quad A_{(r\,|\,2n)} \quad \cong \quad \mathbf{1}_k A \mathbf{1}_k / \mathbf{1}_k A \mathbf{1}_{k+1} A \mathbf{1}_k$$

which identifies the primitive idempotent corresponding to $P(\lambda) \in \mathcal{F}_{(r\,|\,2n)}$ *with the idempotent* $\overline{e}_{\Psi(\lambda)}$ *such that* $\underline{\Psi(\lambda)} = \underline{\lambda}_{\epsilon}^{\circledcirc}$. *Moreover,* $A_{(r\,|\,2n)}$ *inherits a positive grading from* A.

Proof. We claim that there is a surjective algebra homomorphism $\mathbf{1}_k A \mathbf{1}_k \to A_{(r\,|\,2n)}$. To see this let $f \in A_{(r\,|\,2n)}$. By definition of $A_{(r\,|\,2n)}$, it can be identified with some $f \in \mathrm{End}_{\mathcal{F}_{(r\,|\,2n)}}(P, P)$ for some projective module P which is the direct sum of finitely many pairwise non-isomorphic indecomposable projective objects in $\mathcal{F}_{(r\,|\,2n)}$. We can assume that they are in the same block. By Proposition 5.10 there exists some $V^{\otimes d}$ which contains P as a summand and hence f can be identified with an endomorphism of $V^{\otimes d}$. It can be lifted to an endomorphism in $\mathrm{Br}_d(\delta)$ by Theorem 5.8, and by construction and Theorem 6.17 actually to an element in A. To make these choices canonical we realize each indecomposable projective as a summand in some $V^{\otimes d}$ and consider the embeddings $\mathrm{End}_{\mathcal{F}_{(r\,|\,2n)}}(V^{\otimes d}) \hookrightarrow \mathrm{End}_{\mathcal{F}_{(r\,|\,2n)}}(V^{\otimes d+1})$, $f \mapsto f \otimes 1$. We then can embed $A_{(r\,|\,2n)}$ as a subalgebra in $\varinjlim \mathrm{End}_{\mathcal{F}_{(r\,|\,2n)}}(V^{\otimes d})$. By Theorem 6.17 and Theorem 5.8 there is a surjective algebra homomorphism $\Psi : \mathbf{1}_k A \mathbf{1}_k \to A_{(r\,|\,2n)}$. By Theorem 7.8, it identifies the primitive idempotent corresponding to $P(\lambda) \in \mathcal{F}_{(r\,|\,2n)}$ with the idempotent $\overline{e}_{\Psi(\lambda)}$ such that $\underline{\Psi(\lambda)} = \underline{\lambda}^{\oslash}_{\epsilon}$. By [ES17, Theorem 5.1], the map Ψ factors through the nuclear ideal $\mathbb{I}_k$ inducing an isomorphism $\mathbf{1}_k A \mathbf{1}_k / \mathbb{I}_k \cong A_{(r\,|\,2n)}$. In particular, $A_{(r\,|\,2n)}$ inherits a positive grading from A. By Lemma 10.4 we obtain also an isomorphism $\mathbf{1}_k A \mathbf{1}_k / \mathbf{1}_k A \mathbf{1}_{k+1} A \mathbf{1}_k \cong A_{(r\,|\,2n)}$. □

10.3. **Categorification of the layers.** Let $k \in \mathbb{Z}_{\geq 0}$. Then the full additive subcategory $\mathcal{I}_k$ generated by the $\mathtt{R}_\delta(\lambda)$, with $\lambda \in I_k$ is a thick ideal in the sense of [CH17, 1.2]. To see this note that it is stable under $\_ \boxtimes \mathtt{R}_\delta(\square)$, and hence under $\_ \boxtimes \mathtt{R}_\delta(\mu)$ thanks to Theorem 8.7. It was proved in [CH17, Theorem 6.11], that these are precisely the thick ideals in Rep_δ. If $\langle \mathcal{I}_{k+1} \rangle$ defines the tensor ideal in $\mathcal{I}_k$ generated by $\mathcal{I}_{k+1}$, then Theorem 10.5 can be reformulated (using the notion from from Section 5.3) as an equivalence of additive categories

$$\mathcal{I}_k / \langle \mathcal{I}_{k+1} \rangle \quad \cong \quad \mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)}),$$

where $\mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)})$ denotes the full subcategory of $\mathcal{F}_{(r\,|\,2n)}$ given by projective objects.

This equivalence was in fact proved already in [Cou18, Theorem 7.1.1]. Our approach has the advantage that we can lift it to a graded setting which we will do in the next section.

For a thick ideal $\mathcal{I}_k$, the quotient $\mathcal{I}_k / \mathcal{I}_{k+1}$ is defined as the quotient category of $\mathcal{I}_k$ with respect to all morphisms that factor through an objects in $\mathcal{I}_{k+1}$.

Proposition 10.6. *The functor $\mathbb{F} : \mathrm{Rep}_\delta \to \mathcal{F}$ restricts to a tensor functor $\mathbb{F} : \mathcal{I}_{\min\{m,n\}} \to \mathcal{F}$ and factors through a functor*

$$\mathcal{I}_{\min\{m,n\}} / \mathcal{I}_{\min\{m,n\}+1} \to \mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)}), \quad \mathtt{R}_\delta(\lambda) \mapsto \mathbb{F}\mathtt{R}_\delta(\lambda) \cong P(\lambda^\dagger_\delta) \tag{10.45}$$

which is a bijection on objects and full on morphisms.

Proof. The first claim is clear, and the functor (10.45) is well-defined by Theorem 6.17, surjective on objects by Proposition 5.10 and bijective by Theorem 6.17. It is full by Theorem 5.8. □

The layers from Definition 8.15 describe the combinatorics of translation functors on $\mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)})$, where $\mathrm{K}_0^{\oplus}(\mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)}))$ denotes the split Grothendieck group of the additive category $\mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)}))$

Corollary 10.7 (Categorification of Layers)**.** *There is a canonical isomorphism of $\mathcal{U}(\mathfrak{g}^\theta)$-modules*

$$\mathrm{K}_0\left(\mathcal{D}_\Delta(\delta)\right)_k \cong \mathrm{K}_0^{\oplus}(\mathrm{Proj}(\mathcal{F}_{(r\,|\,2n)}))$$

Remark 10.8. That the functor induced by $\mathbb{F}$ in (10.45) is actually an equivalence was also shown in [Cou18, Theorem 7.1.1]. In the notation therein, it is the bijectivity of Ob.

10.4. **Additional remarks.**

Graded Categorification Theorem. Note that the The First Categorification Theorem and the Categorification of the Layers has naturally a graded version, since the nuclear ideal is homogeneous. Including the grading corresponds to a quantized version of the involved algebras, where $\mathcal{U}(\mathfrak{g})$ is replaced by its usual quantum analogue $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}^\theta)$ is replaced by a subalgebra $\mathcal{H}^{\lrcorner}$ respectively $\mathcal{H}$ of $\mathcal{U}_q(\mathfrak{g})$ defined analogously to the classical case. For their definition we refer to [ES18, Proposition 7.18. Proposition 7.17]. (The notation is chosen such that the generators above correspond to the generator in the quantum version denoted by the same letters.) To state the theorem consider the graded category $\mathbb{D}^{\mathrm{gr}}(\delta)$ of graded representations of gRep_δ as defined in **??** with its subcategory $g\mathcal{D}_\Delta(\delta)$ of graded representations which have a filtration with subquotients graded standard representations, see Remark 6.30, $DeltaD(\lambda)\langle i \rangle$. For $i \in \mathbb{Z}$ we denote here by $\Delta_\delta(\lambda)\langle i \rangle$ the unique graded lift of $\Delta_\delta(\lambda)$ normalised such that the head is in degree $i \in \mathbb{Z}$.

Theorem 10.9. *There is an isomorphism of*

$$\mathrm{K}_0(\mathcal{D}_\Delta(\delta)) \cong \langle\, \Lambda \,\rangle \cong \bigwedge^{\infty/2} \mathbb{V}_\delta, \qquad [\Delta_\delta(\lambda)] \mapsto \lambda \,\mapsto v_\delta^\lambda. \tag{10.46}$$

In this case however the fixed point subalgebras are only coideal subalgebras, not Hopf subalgebras. The appearing coideal subalgebras are the type (AIII) examples in the general family of *quantum symmetric pairs* which were classified and studied by Letzter in [Let02] [Let03] and further studied in [Kol14]. The lift to the graded setup is straight-forward using the results from [ES18]. To get however *explicit* character formulas of simple modules one has to develop the Fock space combinatorics further. This will be done in the sequel to this paper.

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The same coideal subalgebras (but without interpretation of the quantum parameter as a grading shift) where used to obtain multiplicity formulas for category $\mathcal{O}$ for the orthosymplectic Lie superalgebra. In this setup also Fock spaces appeared and the arguments were based on the so-called super duality connecting the combinatorics of the Lie superalgebras with the combinatorics of classical category $\mathcal{O}$. In contrast to our result for finite dimensional representations, there is however no concrete description of the endomorphism ring of a projective generator or of the morphism spaces between projective objects available (and might in fact be far out of reach).

Highest weight structures. Recall the algebra A from Remark 2.5. By [BS18, Theorem 5.7] Corollary 2.11 can be reformulated by saying that A can be equipped with the structure of an *upper finite based quasi-hereditary algebra* in the sense of [BS18, Definition 5.1]. In contrast, the algebra $A_{(r\,|\,2n)}$ is an *essentially finite* algebra in the sense of [BS18], that is a locally unital algebra $A_{(r\,|\,2n)} = \oplus_{e_{\lambda_\delta^\dagger} \in s\Gamma_\delta(n|m)} e_{\lambda_\delta^\dagger} A e_{\mu_\delta^\dagger}$ with infinitely many primitive idempotents $e_{\lambda_\delta^\dagger}$, but finite dimensional pieces $e_{\lambda_\delta^\dagger} A e_{\mu_\delta^\dagger}$, and all indecomposable projectives have finite length. Note that $I_k \subset \Lambda$ is not an upper set in the sense of [BS18, 3.1]. Hence $A_{(r\,|\,2n)}$ does not inherit a nice stratification from the quasi-hereditary structure of A. However, the standard filtrations of the $\mathcal{P}_\delta(\lambda)$, $\lambda \in \Lambda$ from Corollary 2.11 nevertheless induces a filtration on any $P(\lambda) \in \mathcal{F}_{(r\,|\,2n)}$ where the subquotients are quotients of the standard representations $\Delta_\delta(\mu)$, but the quotients we obtain could vary depending on $P(\lambda)$. The combinatorics of these filtrations is described in [ES17, Proposition 7.3]. The fact that the nuclear ideal is not compatible with the poset Λ is one of the facts which make the category $\mathcal{F}$ hard to understand.

Koszulity. The notion of Koszulity in [MOS09] generalizes (in natural way) the well-known notion of Koszulity for finite dimensional algebras from [BGS96].

Conjecture 10.10. *We conjecture that the algebra $A_{(r\,|\,2n)}$ equipped with the grading from Theorem 10.5 is a locally finite dimensional Koszul algebra in the sense of* [MOS09].

The conjecture is true in case $\min(m, n) \leq 2$, see [ES16a]. In these cases we obtain infinite zigzag algebras which are Koszul, see e.g. [ET19].

11. The graded Brauer algebras

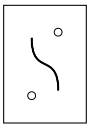
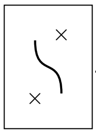
In this section we introduce a graded version of the Brauer algebras, not just the basic versions. We start by introducing graded versions of the up-down tableaux from Definition 1.21, the so-called oriented stretched cup diagrams. In contrast to ordinary up-down tableaux they depend on δ.

11.1. **Grading on up-down tableaux.** Recall the i-translations from Equation (8.38).

Definition 11.1. Let $i \in \mathbb{Z} + \delta/2$ and let $\Theta = \Theta_{\pm i}$ be a translation move at position i.

(1) Given a weight diagram λ we say that λ is *compatible* with the bottom respectively top of Θ if the symbols $\circ$ and $\times$ in the bottom respectively top row of Θ match the symbols of λ at positions $|i| \pm 1/2$.
(2) Given two weight diagrams λ and μ λ is compatible with the bottom of Θ and μ is compatible with the top of Θ. Then we say that $\lambda\Theta\mu$ is *oriented* if λ and μ agree outside of the positions $|i| \pm 1/2$ and at positions $|i| \pm 1/2$ putting the entries of λ below Θ and the ones from μ above Θ gives an oriented diagram in the sense of (6.27).

Example 11.2. Let $\lambda = \circ \vee \times \wedge \vee \vee \vee \cdots$, $\mu_1 = \vee \circ \times \wedge \vee \vee \vee \cdots$, $\mu_2 = \vee \circ \wedge \times \vee \vee \vee \cdots$ and

$\Theta = \Theta_1 =$  and $\Theta' = \Theta_{-2} =$ .

We see that λ is compatible with the bottom of both Θ and Θ', since it has $\circ$ at position $1/2$ and $\times$ at position $5/2$, but not with the top of either one. On the other hand μ_1 is only compatible with the top of Θ and the bottom of Θ', while μ_2 is only compatible with the top of both Θ and Θ'. For the orientability we have that $\lambda\Theta\mu_1$ and $\mu_1\Theta'\mu_2$ are oriented, while $\lambda\Theta\mu_2$ and $\lambda\Theta'\mu_2$ are not oriented.

Definition 11.3. (1) A *stretched cup diagram* (for δ) is a sequence $\mathbf{c} = (c_0, \ldots, c_d)$ of cup diagrams such that $c_0 = \underline{\varnothing}_\delta$ and c_r is obtained from c_{r-1} by some i_r-translation Θ_{i_r} for $1 \leq r \leq d$. We call $(i_1, \ldots, i_d)$ the *type* of $\mathbf{c}$ and d its *length*.
(2) An *orientation* of $\mathbf{c}$ is a sequence $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_d)$ of weight diagrams such that $c_i\lambda_i$ is oriented in the sense of (6.27) for any $0 \leq i \leq d$. Furthermore the diagrams $\lambda_{r-1}\Theta_{i_r}\lambda_r$ must all be oriented.
(3) We call λ_d the *final* weight diagram. The pair $(\mathbf{c}, \boldsymbol{\lambda})$ is called an *oriented stretched cup diagram* and will in general be denoted by $\mathbf{c}\boldsymbol{\lambda}$.

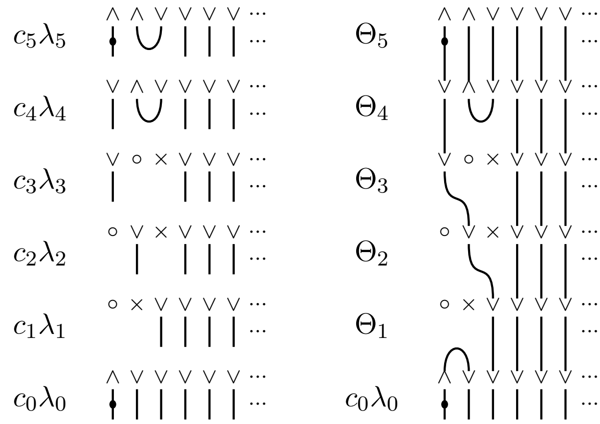
We will denote by $\mathcal{X}_d$ the set[7] of oriented stretched cup diagrams of length d. For a fixed stretched cup diagram $\mathbf{c}$ (of length d) we will denote by $\mathbf{c}\mathcal{X}_d$ the set of all oriented stretched cup diagrams of type $\mathbf{c}$.

Remark 11.4. We identify an oriented stretched cup diagram with the picture obtained by drawing the sequence $\boldsymbol{\lambda}$ of weights in order from bottom to top, putting $\underline{\varnothing}_\delta$ below and the appropriate translation pictures $\Theta_{i_1}, \ldots, \Theta_{i_r}$ (from (8.38)) between the weight diagrams with additional vertical connections for the parts of the cup diagrams which do not get changed under the translation. Note that the requirement of being oriented in each step makes this into an oriented picture in the naive sense with dots interpreted as orientation reversing points. For more details about stretched cup diagrams (but without dots) we refer to [BS12a].

Example 11.5. Let us illustrate the two ways of viewing stretched cup diagrams from Remark 11.4in an example. Here $\delta = 1$ and the stretched cup diagram $\mathbf{c}\boldsymbol{\lambda}$ has length 5. On the left is the sequence of cup diagrams with corresponding orientations, while on the right

[7]Note the same notation for this space as for Verma paths in [ES18, Section 2.2], which is not a coincidence.

shows the translations used in each step as well as the orientations.



Definition 11.6. The reduction red($\mathbf{c}$) of a stretched cup diagram $\mathbf{c}$ of length d is the cup diagram c_d. Similarly red($\mathbf{c}\boldsymbol{\lambda}$) $= c_d\lambda_d$ for an oriented diagram.

As for ordinary cup diagrams, we can use the stretched version to obtain circle diagrams.

Definition 11.7. An *oriented stretched circle diagram*, or *osc-diagram* for short, is a pair $(\mathbf{c}\boldsymbol{\lambda}, \mathbf{c}'\boldsymbol{\lambda}')$ of oriented stretched cup diagrams with the same final weight diagram. We will call it a (d, d')-osc-diagram if we want to emphasize that $\mathbf{c}$ has length d and $\mathbf{c}'$ has length d'.

Similar to stretched cup diagrams, we write $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$ for $(\mathbf{c}\boldsymbol{\lambda}, \mathbf{c}'\boldsymbol{\lambda}')$ and we call $(\mathbf{c}, \mathbf{c}')$ the *type* of $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$.

We depict an osc-diagram by reflecting the diagram $\mathbf{c}'$ vertically, but keeping the same orientations, just in reversed order, and putting it on top of $\mathbf{c}\boldsymbol{\lambda}$ in analogy to [BS12a, Section 6].

Definition 11.8. In analogy to the cup diagram case we denote by $\mathbb{X}_d^{d'}$ the set of all (d, d')-osc-diagram and by $\mathbf{c}\mathbb{X}_d^{d'}\mathbf{c}'$ the sets of (d, d')-osc-diagram of type $(\mathbf{c}, \mathbf{c}')$. The circles in an osc-diagram that are already contained in either the stretched cup respectively cap diagram part will be called *internal circles* in the following.

For an up-down tableaux $\boldsymbol{\lambda}$ denote by $\boldsymbol{\lambda}_\delta$ the corresponding sequence of Deligne weight diagrams obtained via Definition 6.8.

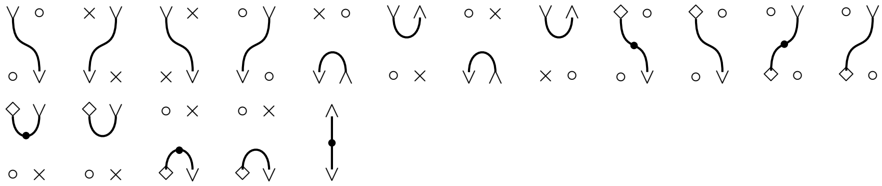
Proposition 11.9. *(1) There is a canonical bijection*

$$\mathbb{T}_d^{d'} \overset{1:1}{\longleftrightarrow} \mathbb{X}_d^{d'} \tag{11.47}$$

sending $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *to* $\mathbf{c}\boldsymbol{\lambda}_\delta\boldsymbol{\mu}_\delta\mathbf{c}'$. *Here* $\mathbf{c}$, *respectively* $\mathbf{c}'$, *is the unique stretched cup diagram of type* $(i_1, \ldots, i_d)$, *respectively* $(i'_1, \ldots, i'_{d'})$, *where the* $|i_j| \pm 1/2$, *respectively* $|i_j|' \pm 1/2$, *are the involved vertices in the local moves* (6.30) *connecting* $\lambda_\delta^{(j)}$ *with* $\lambda_\delta^{(j-1)}$, *respectively* $\mu_\delta^{(j)}$ *with* $\mu_\delta^{(j-1)}$.

(2) In particular, the number of (r, s)*-osc diagrams is equal to* $\dim \mathrm{Hom}_{\mathrm{Br}(\delta)}(r, s)$.

Proof. Consider the translation pictures from (8.38). One easily verifies that the orientations must look as follows



(11.48)

or the role of $\vee$ and $\wedge$ swapped. Further they are all allowed. But this corresponds exactly to the local moves (6.30) read from top or from the bottom. Therefore (11.47) follows. Statement (2) is then clear from Proposition 1.22. $\square$

Remark 11.10. Note that by Proposition 11.9 part (1) there is a natural bijection between $\text{🧍}_d^{d'}$ and $\text{🚶}_d^{d'}$. Hence one should think of $\text{🧍}_d^{d'}$ as the labelling set that is independent of δ, i.e. it originates from the invariant theoretic side of the Brauer algebra, while the the set $\text{🚶}_d^{d'}$ depends on δ and originates in the version of the Brauer algebra obtained via Verma paths and category $\mathcal{O}$ as described in Theorem 4.8.

As seen in the Proof of Proposition 11.9, internal circles of stretched cup diagrams encode multiplicities of indecomposable projective modules $P_{d,\delta}(\lambda)$ in $\mathrm{Br}_d(\delta)$ for $\lambda \in \Lambda_d(\delta)$. To link this directly to osc-diagrams, let $\nu \in \Lambda_d(\delta)$ and $\mu \in \Lambda_{d'}$ and denote by $\underline{\nu}_\delta \text{🚶}_d^{d'} \underline{\mu}_\delta$ the set of (d, d')-osc diagrams $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$ such that $\mathrm{red}(\mathbf{c}) = \underline{\nu}_\delta$ and $\mathrm{red}(\mathbf{c}') = \underline{\mu}_\delta$. Then we have

Proposition 11.11. *For the Brauer algebras* $\mathrm{Br}_d(\delta)$ *the following holds*

1. *Let* $\mathrm{Br}_d(\delta) \cong \bigoplus_{\lambda \in \Lambda_d(\delta)} P_{d,\delta}(\lambda)^{\oplus \mathrm{m}_{d,\lambda}}$ *as a* $\mathrm{Br}_d(\delta)$*-module. Then* $\mathrm{m}_{d,\lambda} = \sum_{\mathbf{c}} 2^{\mathrm{m}_{\mathbf{c}}}$, *where* $\mathbf{c}$ *runs over all stretched cup diagrams of length* d *with* $\mathrm{red}(\mathbf{c}) = \underline{\lambda}_\delta$ *and* $\mathrm{m}_{\mathbf{c}}$ *is the number of internal circles in* $\mathbf{c}$.
2. *For* $\nu \in \Lambda_d(\delta)$ *and* $\mu \in \Lambda_{d'}$, *we have the following equalities*

$$\mathrm{m}_{d,\nu}\, \mathrm{m}_{d',\mu} \dim e_\mu \operatorname{Hom}_{\mathrm{Br}(\delta)}(d, d') e_\nu = \left| \underline{\mu}_\delta \text{🚶}_d^{d'} \underline{\nu}_\delta \right|,$$

in particular $\dim \operatorname{End}_{\mathrm{Br}_d(\delta)}(P_{d,\delta}(\lambda)^{\oplus m_{d,\lambda}}) = \left| \underline{\lambda}_\delta \text{🚶}_d^{d} \underline{\lambda}_\delta \right|$.

Proof. By Theorem 8.7 the multiplicity of $P_{d,\delta}(\lambda)$ in $\mathrm{Br}_d(\delta)$ equals the number of stretched cup diagrams $\mathbf{c}$ of length d ending in $\underline{\lambda}_\delta$ counted with multiplicities 2^{m}, where m is the number of internal circles in $\mathbf{c}$. By definition, the connected components in a stretched circle diagram $\mathbf{c}\boldsymbol{\lambda}$ that are removed for the reduction $\mathrm{red}(\mathbf{c}\boldsymbol{\lambda})$ either have exactly one orientation, if they are lines, or exactly two orientations, if they are internal circles. Then the claims follow from (6.32) which determines $e_\mu \operatorname{Hom}_{\mathrm{Br}(\delta)}(d, d') e_\nu$ in terms of ordinary oriented circle diagrams up to the multiplicity 2^m above. $\square$

Definition 11.12. The *degree* of an oriented stretched cup diagram $\mathbf{c}\boldsymbol{\lambda}$ is the degree of $c_0\lambda_0$ plus the sum of the degrees of the translation pieces $\lambda_{r-1}\Theta_{i_r}\lambda_r$ in the sense of (6.27) minus the total number of caps appearing in the translation pieces.

The *degree* of an osc-diagram $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$ is the sum of the degrees of $\mathbf{c}\boldsymbol{\lambda}$ and $\mathbf{c}'\boldsymbol{\lambda}'$.

Example 11.13. The osc-diagram $\mathbf{c}\boldsymbol{\lambda}$ from Example 11.5 has degree 1, coming from $c_0\lambda_0$ having degree 0 plus 2 from a clockwise oriented cap in $\lambda_0\Theta_1\lambda_2$ and a clockwise oriented cup in $\lambda_3\Theta_3\lambda_4$, minus 1 from the cap in Θ_1.

To define an algebra using stretched circle diagrams we introduce the following space of diagrams with a fixed form.

Definition 11.14. Let $\mathbb{D}_{d,d'}(\delta)$ be the vector space with basis $\text{🚶}_d^{d'}$ and by $\mathbf{c}\mathbb{D}_{d,d'}(\delta)\mathbf{c}'$ the subspace with basis $\mathbf{c}\text{🚶}_d^{d'}\mathbf{c}'$.

Given two osc-diagrams $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$ and $\mathbf{b}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{b}'$ their product is defined via the following *extended surgery procedure*: Place $\mathbf{b}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{b}'$ on top of $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$

- If $\mathbf{c}' \neq \mathbf{b}$, then the result is zero.
- If $\mathbf{c}' = \mathbf{b}$, but there is a circle in $\mathbf{c}'$ that has the same orientation in $\mathbf{c}'\boldsymbol{\lambda}'$ and in $\mathbf{c}'\boldsymbol{\mu}$, then the result is zero as well.

- Otherwise, i.e. if $\mathbf{c}' = \mathbf{b}$ and all mirror symmetric pairs of circles in the middle part have opposite orientations, we replace $\mathbf{c}'\boldsymbol{\lambda}'$ and $\mathbf{b}\boldsymbol{\mu}$ by their reductions and apply the surgery procedure from [ES16b], as in Theorem 6.22, to this reduced part.

The result is a linear combination of osc-diagrams of the type $(\mathbf{c}, \mathbf{b}')$.

Theorem 11.15. *The extended surgery procedure defines a well defined associative multiplication on the vector space $\mathbb{D}(\delta) = \bigoplus_{d,d'} \mathbb{D}_{d,d'}(\delta)$, where the sum runs over all pairs non-negative integers. When equipping this vector space with the grading induced by the grading degree of the basis vectors $\mathbb{D}(\delta)$ turns into a graded algebra.*

Proof. The associativity and well-definedness follows directly from [ES16b, Theorem 6.2]. The compatibility of the grading follows then from [ES16b, Theorem 6.2] and the definition of the degrees with Lemma 11.16 below. Note that, due to Lemma 11.16, pairs of internal circles that are oriented in opposite ways cancel each others' degree contributions in the middle part, while lines eliminated in the reduction give no contribution at all. □

Lemma 11.16. *Let $\mathbf{c}\boldsymbol{\lambda}$ be an oriented stretched cup diagram.*

(1) Let C be an internal circle in $\mathbf{c}$, then its contribution to the degree of $\mathbf{c}\boldsymbol{\lambda}$ is ± 1, with -1 in the anti-clockwise case and $+1$ in the clockwise case.
(2) Let L be a line in $\mathbf{c}$ containing two rays from c_0 then its contribution to the degree of $\mathbf{c}\boldsymbol{\lambda}$ is zero.

Proof. Part (1): Assume that C is a small circle, i.e. containing exactly one cup and one cap, the claim is immediate since the orientation of the circle either adds zero or 2 to the degree and the single cap contained in it will subtract 1 from this.

If C is not a small circle, one can successively eliminate kinks from the circle, i.e. a pair of a cup and a cap with opposite orientations. By construction each kink has degree zero and after removing finitely many one obtains a small circle. For more details on this, especially how to handle possible dotted cups and caps, see [ES16b, (4.18) & (4.19)].

Part (2): Note that by assumption L contains at least one cap. Assume first that it contains exactly one. Since L is a line and connects two rays in c_0 it follows immediately that the cap is oriented clockwise, i.e, its orientation contributes 1 to the degree, but being a cap it subtracts one again. Thus the total contribution is zero. For an arbitrary line one now argues as in case (1). □

Remark 11.17. Note that an osc-diagram $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$ is an idempotent if and only if $\mathbf{c}\boldsymbol{\lambda}$ and $\mathbf{c}'\boldsymbol{\lambda}'$ agree except that all internal circles in $\mathbf{c}\boldsymbol{\lambda}$ are oriented opposite to the ones in $\mathbf{c}'\boldsymbol{\lambda}'$, and all other circles are oriented anti-clockwise. This again follows directly from the definition of the multiplication and from [ES16b, Theorem 6.2].

We have now a graded algebra

$$\mathbb{D}(\delta) = \bigoplus_{d,d'} \mathbb{D}_{d,d'}(\delta)$$

Alternatively we can also formulate this as a category directly and obtain the following graded versions of the Deligne and Brauer categories.

Definition 11.18 (Graded Deligne and Brauer categories)**.**

(1) The *graded Brauer category* $\mathrm{gBr}(\delta)$ is the category enriched in graded vector spaces with set of objects $\mathbb{Z}_{\geq 0}$, morphism spaces $\mathrm{Hom}_{\mathrm{gBr}(\delta)}(d, d') = \mathbb{D}_{d,d'}(\delta)$, and composition given my multiplication of osc-diagrams.
(2) The *graded Deligne category* gRep_δ is the additive closure of the idempotent completion of $\mathrm{gBr}(\delta)$.

Remark 11.19. Note that $\mathbb{D}_d(\delta) := \mathrm{End}_{\mathrm{gBr}(\delta)}(d) = \mathbb{D}_{d,d}(\delta)$. This is a graded algebra with degree zero part having a basis of primitive, pairwise orthogonal idempotents $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}$ as described in Remark 11.17, i.e. internal mirror symmetric circles oriented in opposite ways and all other circles oriented anti-clockwise. Thus an object in gRep_δ is the formal direct sum of objects of the form $(d, \mathbf{e})$ with $\mathbf{e}$ an idempotent in $\mathbb{D}_d(\delta)$ which is a sum of such primitive idempotents. Thus morphisms in gRep_δ are just matrices of osc-diagrams with prescribed stretched cup and cap diagram part and orientations for internal circles.

Theorem 11.20 (Isomorphism Theorem II)**.**
After forgetting the grading there are an equivalences of categories

$$\mathrm{gRep}_\delta \cong \mathrm{Rep}_\delta \text{ and } \mathrm{gBr}(\delta) \cong \mathrm{Br}(\delta),$$

such that one has isomorphisms of ungraded algebras

$$\mathbb{D}_d(\delta) \cong \mathrm{Br}_d(\delta).$$

We call $\mathbb{D}_d(\delta)$ with the grading from gRep_δ the graded Brauer algebra.

Proof. First note that the equivalence on the Deligne categories follows from the equivalence on the Brauer categories. Since $d \in \mathbb{Z}_{\geq 0}$ is already an object of $\mathrm{Br}(\delta)$ the statement on the endomorphism ring also follows from this.

We first enumerate certain osc-diagrams. Let $d \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Lambda_d(\delta)$. For a weight diagram ν such that $\underline{\lambda}_\delta\nu$ is oriented, the set of oriented stretched cup diagrams $\mathbf{c}\boldsymbol{\nu}$ such that $\mathrm{red}(\mathbf{c}) = \underline{\lambda}_\delta$ and $\nu_d = \nu$ has cardinality $\mathrm{m}_{d,\lambda}$, where $\mathrm{m}_{d,\lambda}$ is given as in Proposition 11.11 as the sum of $2^{\mathrm{m}_{\mathbf{c}}}$ with $\mathrm{m}_{\mathbf{c}}$ being the number of internal circles in $\mathbf{c}$. We write $\underline{\boldsymbol{\lambda}}_i\nu$ with $1 \leq i \leq \mathrm{m}_{d,\lambda}$ for them. We make this enumeration consistently for all such ν, i.e. $\underline{\boldsymbol{\lambda}}_i\nu$ and $\underline{\boldsymbol{\lambda}}_i\nu'$ have the same type and orientation on internal circles. Finally we fix an involution $i \mapsto i'$ on the set $\{1, \ldots, \mathrm{m}_{d,\lambda}\}$ such that for any ν, $\underline{\boldsymbol{\lambda}}_{i'}\nu$ is obtained from $\underline{\boldsymbol{\lambda}}_i\nu$ by reversing the orientation of all internal circles, especially the underlying stretched cup diagrams agree.

Now let $d, d' \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Lambda_d(\delta)$ and $\mu \in \Lambda_{d'}(\delta)$. For a weight diagram ν such that $\underline{\lambda}_\delta\nu$ and $\underline{\mu}_\delta\nu$ are oriented we thus have a collection of stretched oriented circle diagrams

$$\left\{\underline{\boldsymbol{\lambda}}_i\nu\overline{\boldsymbol{\mu}}_{j'} \mid 1 \leq i \leq \mathrm{m}_{d,\lambda} \text{ and } 1 \leq j \leq \mathrm{m}_{d',\mu}\right\}.$$

Note that all orientations of internal circles are fixed by the enumeration, while orientations for components not connected to the final diagram of either $\underline{\boldsymbol{\lambda}}_i$ or $\underline{\boldsymbol{\mu}}_i$ have fixed orientations since they are lines. For each i and j, this gives a basis element of $e_{\mu,j}\,\mathrm{Hom}_{\mathrm{gRep}_\delta}(d, d')e_{\lambda,i}$, for idempotents $e_{\lambda,i} = \underline{\boldsymbol{\lambda}}_i\lambda_\delta\overline{\boldsymbol{\lambda}}_{i'}$ and $e_{\mu,j} = \underline{\boldsymbol{\mu}}_j\mu_\delta\overline{\boldsymbol{\mu}}_{j'}$. Furthermore the for each i and j, the space $e_{\mu,j}\,\mathrm{Hom}_{\mathrm{gRep}_\delta}(d, d')e_{\lambda,i}$ can be identified with $\mathrm{Hom}_{\mathrm{gRep}^{\mathrm{bsc}}_\delta}(\lambda, \mu)$ by sending $\underline{\boldsymbol{\lambda}}_i\nu\overline{\boldsymbol{\mu}}_{j'}$ to $\underline{\lambda}_\delta\nu\overline{\mu}_\delta$. Taking these identifications together we obtain an isomorphism

$$\Phi_{(d,\lambda),(d',\mu)} : \bigoplus_{i,j} e_{\mu,j}\,\mathrm{Hom}_{\mathrm{gRep}_\delta}(d, d')e_{\lambda,i} \longrightarrow \mathbb{M}_{(d,\lambda),(d',\mu)},$$

where $\mathbb{M}_{(d,\lambda),(d',\mu)} = \mathbb{M}_{m_{d',\mu},m_{d,\lambda}}(\mathrm{Hom}_{\mathrm{gRep}^{\mathrm{bsc}}_\delta}(\lambda, \mu))$ are matrices with entries from the morphism space $\mathrm{Hom}_{\mathrm{gRep}^{\mathrm{bsc}}_\delta}(\lambda, \mu)$ and $\Phi_{(d,\lambda),(d',\mu)}$ sends $\underline{\boldsymbol{\lambda}}_i\nu\overline{\boldsymbol{\mu}}_{j'}$ to the matrix with $\underline{\lambda}_\delta\nu\overline{\mu}_\delta$ at position (j, i) and zero otherwise. We define a composition on

$$\mathbb{M} = \bigoplus_{d,d'} \bigoplus_{\lambda \in \Lambda_d(\delta),\, \mu \in \Lambda_{d'}(\delta)} \mathbb{M}_{(d,\lambda),(d',\mu)},$$

by declaring for two summand that $AB = 0$ unless $A \in \mathbb{M}_{(d,\lambda),(d',\mu)}$ and $B \in \mathbb{M}_{(d',\mu),(d'',\nu)}$ for some (d, λ), (d', μ), and (d'', ν), in which case we use matrix multiplication with the entries being multiplied as usual for oriented circle diagrams. Then $\bigoplus_{(d,\lambda),(d',\mu)} \Phi_{(d,\lambda),(d',\mu)}$ intertwines the multiplication rules on both sides, by definition of the surgery procedure for osc-diagrams.

In case the parity of d, d', and d'' are not equal one of the elements is necessarily zero. Hence we can assume that the parity of d, d', and d'' is equal, in which case we set $\hat{d} = \max(d, d', d'')$. Then by Proposition 11.11 and Remark 1.16 we have

$$\mathbb{M}_{(d,\lambda),(d',\mu)} \cong \mathrm{Hom}_{\mathrm{Br}_{\hat{d}}(\delta)}\left(P_{\hat{d},\delta}(\lambda)^{\oplus m_{d,\lambda}}, P_{\hat{d},\delta}(\mu)^{\oplus m_{d',\mu}}\right)$$

and similarly on (d', μ) and (d'', ν) and matrix multiplication is intertwined with composition of maps on the right hand side. Thus summing up over all λ for a fixed d we see that there is an equivalence on the level of the graded and ungraded Brauer categories. This extends then also to the Deligne categories. □

Remark 11.21. The connection between the Deligne category and its basic version is easy to describe. Let $d, d' \in \mathbb{Z}_{\geq 0}$, $\lambda \in \Lambda_d(\delta)$ and $\mu \in \Lambda_{d'}(\delta)$ and set $e_\lambda = \sum_i e_{\lambda,i} \in \mathrm{End}_{\mathrm{gRep}_\delta}(d)$ and $e_\mu = \sum_i e_{\mu,i} \in \mathrm{End}_{\mathrm{gRep}_\delta}(d')$. Then we have

$$e_\mu \mathrm{Hom}_{\mathrm{gRep}_\delta}(d, d') e_\lambda \;\cong\; \bigoplus_{\mathbf{c},\mathbf{c}'} N^{\otimes m_{\mathbf{c}}} \otimes \mathbb{B}_\delta(\lambda, \mu) \otimes N^{\otimes m_{\mathbf{c}'}}, \tag{11.49}$$

where the sum runs over all stretched cup diagrams $\mathbf{c}$ of length d with $\mathrm{red}(\mathbf{c}) = \underline{\lambda}_\delta$ and $\mathbf{c}'$ of length d' with $\mathrm{red}(\mathbf{c}') = \underline{\mu}_\delta$ and the space N is a graded vector space with graded dimension $q + q^{-1}$. This follows directly from Lemma 11.16.

Now we can consider graded representation theory:

Definition 11.22. A *representation* of gRep_δ is a contravariant functor from gRep_δ to gVect, the category of graded finite dimensional complex vector spaces. We denote the abelian category of all representations of gRep_δ by $\mathrm{g}\mathcal{D}(\delta)$ and call it the *abelianized graded Deligne category.*

We can now directly mimic the construction of Definition 2.7 in this graded setting and define completely analogously projective objects $\mathcal{P}_\delta(\lambda, i) \in \mathrm{g}\mathcal{D}(\delta)$ and standard objects $\Delta(\lambda, i)$ for $(\lambda, i) \in \Lambda \times \mathbb{Z}$. These are graded lifts of the representations $\Delta(\lambda)$ defined in Definition 2.7 such that the idempotent e_λ in the vector space associated to the object λ is concentrated in degree i. By construction we have the following:

Proposition 11.23. *In the category $\mathrm{g}\mathcal{D}(\delta)$ the following holds:*

(1) $\mathcal{P}_\delta(\lambda, i)$ has a filtration with subquotients isomorphic to $\Delta(\mu, j)$ with $\Delta(\lambda, i)$ at the top and some $\Delta(\mu, i)$ for $\mu > \lambda$ and $j > i$.

(2) The multiplicities satisfy $\sum_j (\mathcal{P}_\delta(\lambda, i) : \Delta_\delta(\mu, j)) = (\mathcal{P}_\delta(\lambda) : \Delta_\delta(\mu))$ for any $o \in \mathbb{Z}$.

Proof. The first claim follows directly from the construction in the non-graded situation noting that all steps have a graded analogue. The second statement is then also clear by forgetting the grading. □

Remark 11.24. By following carefully the construction, one can also deduce with [ES16b, Lemma 8.6] that the graded multiplicities $(\mathcal{P}_\delta(\lambda, i) : \Delta_\delta(\mu, j))$ are given by the coefficient of q^{j-i} in some parabolic Kazdhan-Lusztig polyomial of type $(\mathrm{D}_N, \mathbf{A}_{N-1})$ for large N. To make the precise translation to [ES16b, Lemma 8.6] take λ_δ and μ_δ. These are two diagrammatical weights which have far to the left only $\vee$'s. Pick a vertex such that no cups involve vertices to the right right of it. Then remove all the vertices to the right to create finite weight and cup diagrams. To these resulting weight diagrams we can apply [ES16b, Lemma 8.6].

Associated with the graded Deligne category we have the locally unital graded algebra

$$\mathbb{D}(\delta) = \bigoplus_{d,d'} \mathbb{D}_{d,d'}(\delta)$$

with its distinguished basis of stretched circle diagrams $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}'$. We now play a similar game as in Theorem 6.23.

Denote by I the set of pairwise orthogonal idempotents $\mathbf{c}\boldsymbol{\lambda}\boldsymbol{\lambda}'\mathbf{c}$ of $\mathbb{D}(\delta)$ as described in Remark 11.17, i.e. internal mirror symmetric circles oriented in opposite ways and all other circles oriented anti-clockwise. Recall that by definition stretched circle diagrams are pairs $(\mathbf{c}\boldsymbol{\lambda}, \mathbf{c}'\boldsymbol{\lambda}')$ of oriented stretched cup diagrams with the same final weight diagram. We define for $\nu \in \Lambda$ and $i, j \in I$, the sets $Y(i, \nu)$ (and $X(\nu, j)$) as the set of all oriented stretched cup diagrams (respectively oriented stretched cap diagrams) of shape i respectively j with final weight ν_δ. Let $Y(\nu) = \bigcup_{i \in I} Y(i, \nu)$ and $X(\nu) = \bigcup_{j \in I} X(\nu, j)$ by fixing the cap diagram. We view elements $x \in X(\nu, j)$ and $y \in Y(i, \nu)$ as basis vectors, i.e. as oriented stretched circle diagrams, by putting the cup diagram $\underline{\nu}_\delta$ on the bottom, respectively the cap diagram $\overline{\nu}_\delta$ on the top. By definition of the multiplication via the extended surgery rule defined right before Theorem 11.15, the product xy of basis vectors equals exactly the oriented stretched circle diagram (x, y). By construction we have $X(\nu, j) \subset e_\nu \mathbb{D}(\delta) j$ and $Y(i, \nu) \subset i\mathbb{D}(\delta) e_\nu$. Now the following holds:

Theorem 11.25. *Consider the algebra $\mathbb{D}(\delta)$ with the set I of primitive idempotents. Let $B := \Lambda$ with the reverse inclusion ordering on partitions. This data together with the sets $Y(i, \nu)$ and $X(\nu, j)$ for $\nu \in \Lambda$, $i, j \in I$ defined as above, equip $\mathbb{D}(\delta)$ with the structure of an upper finite based quasi-hereditary algebra in the sense of* [BS18, Definition 5.1].

Proof. The proof is totally analogous to the proof of Theorem 6.23, except that we have now that B is a proper subset of I which is justified by Remark 11.21. □

Since $\mathbb{D}(\delta)$ is additionally graded and all data is homogeneous, it is also an upper finite based quasi-hereditary graded algebra.

12. Applications and examples

We finish this paper by stating a few applications and examples about the representation theory of OSp(r | 2n) which we found interesting on its own.

12.1. **Self-duality.** Recall from [Mus12, 13.7] the usual duality on $\mathcal{F}$ preserving irreducible modules. Then the following holds:

Theorem 12.1. *The $\mathbb{F}\mathtt{R}_\delta(\lambda)$ are self-dual. In particular they have irreducible socle isomorphic to $L(\lambda_\delta^\dagger)$.*

Proof. The claim is obviously true for $\lambda = \varnothing$. Assume now that it holds for $\mathbb{F}\mathtt{R}_\delta(\lambda)$, then it also holds for $\mathbb{F}\mathtt{R}_\delta(\lambda) \otimes V \cong \mathbb{F}(\mathtt{R}_\delta(\lambda) \otimes \mathtt{R}_\delta(\square))$ by [Mus12, 13.7.2] and thus also for any $\mathbb{F}\mathtt{R}_\delta(\mu)$ where $\mathtt{R}_\delta(\mu)$ is obtained from $\mathtt{R}_\delta(\lambda)$ by some i-induction thanks to Theorem 8.10 and [Mus12, 13.7.1] using Remark 8.12. Thus the claim follows inductively. □

12.2. **Determinant representation and super Pfaffian.** In [Ser01], Sergeev established the existence of an even Pfaffian, an analogue of a determinant representation for OSp(r | 2n) for r even. It was revisited and reformulated in terms of a subrepresentation of $V^{\otimes d}$ by Lehrer and Zhang, see [LZ15b, Theorem 3.5]. In fact, this 1-dimensional representation was constructed explicitly as a pseudo-invariant polynomial function.

In our notation, this representation is precisely $L(0, -)$. The following gives an alternative approach for the computation of the degree of the super Pfaffian, with the second and third part being a refinement of the known results.

Theorem 12.2. *(1) The minimal $d = d_{\min}$ such that $L(0, -)$ appears as a submodule in $V^{\otimes d}$ is $d_{\min} = r(2n + 1)$.*

(2) The minimal $d = d_{\min}$ such that $L(0, -)$ appears as a quotient of $V^{\otimes d}$ is $d_{\min} = r(2n + 1)$.

(3) Moreover, $P(0,-)$ *is a summand of* $V^{\otimes d_{\min}}$.

Proof. By Theorem 12.1 the first and second statement are equivalent. By Proposition 7.6 and Definition 8.3 the super cup diagram associated to the signed hook partition $(\varnothing,-)$ equals

(a)  if $\delta > 0$ and odd,

(b)  if $\delta \leq 0$ and odd,

(c)  if $\delta > 0$ and even, and

(d)  if $\delta \leq 0$ and even.

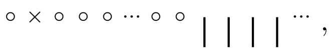
We have $n-m$ leading symbols $\circ$ followed by n dotted cups if $\delta > 0$, while we have $m-n$ leading symbols $\times$ followed by m dotted cups if $\delta > 0$ and odd, while in the even case we have $m-1$ dotted cups instead and an additional undotted cup. In each case we have the corresponding Deligne weight diagram λ_δ via Theorem 7.8, whose cup diagram $\underline{\lambda}_\delta$ differs from the diagrams in $(a)-(d)$ by having a dotted leftmost ray in each case. We call the corresponding diagram $(a')-(d')$ correspondingly. To prove the first statement we need to determine the minimal number $d_{\min}$ needed to obtain the diagrams $(a')-(d')$ from $\underline{\varnothing}_\delta$ via translations.

We first consider case (a), i.e. the case where

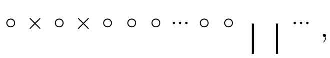
$\underline{\varnothing}_\delta =$ 

with $m-n$ leading $\circ$ symbols followed by rays. To obtain the diagram (a') we need to put a dot on the $2n+1$st ray. For this we first need to eliminate the $2n$ rays to the left of it. Looking at the moves from (8.38), we see that we first move the rays to the very left of the diagram and then apply a dot on the first ray and the a cup. Thus one first obtains

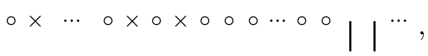
by applying $(8.38)(i)$ a total of $m-n$ times to each of the two left-most rays. Afterwards we apply $(8.38)(v)$ followed by the third from $(8.38)(ii)$, leaving us with



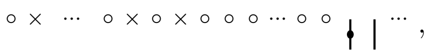
after a total of $2(m-n+1)$ translation pictures. We proceed by using the first and third from $(8.38)(i)$ and move the next two rays to the front. This is followed by $(8.38)(v)$ and the third in $(8.38)(ii)$ to obtain



after a total of $2(m-n+3)$ more translation pictures. This we continue for the first $2n$ rays in total to obtain

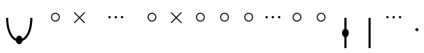


where we have a total of n pairs of $\circ\times$ at the front, followed by $m-n$ times the symbol $\circ$ and then an infinite number of rays. In total this will be $2\sum_{k=1}^{n}(m-n+2k-1)$ translation pictures. By using another $2(m-n+2n)+1$ translation pictures we obtain

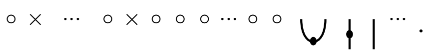


by just moving the ray to the front, applying $(8.38)(v)$ and moving it back into position. To create now a dotted cup we apply the second picture from $(8.38)(ii)$ for $i=1$, followed by

(8.38)(v) to obtain



Using now the second and fourth picture from (8.38)(i) we move the cup to the right, giving us



Creating and moving this dotted cup will take a total of $2(m-n+2n-1)$ translation pictures. This we repeat with the rest of the $n-1$ pairs of $\circ\times$ and end up with the cup diagram (a') for a total number of

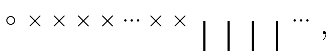
$$\begin{aligned}
&2\sum_{k=1}^{n}(m-n+2k-1)+2(m+n)+1+2\sum_{k=1}^{n}(m+n-2k+1)\\
=&2\sum_{k=1}^{n}2m+2(m+n)+1 \qquad (12.50)\\
=&4mn+2m+2n+1=(2m+1)(2n+1).
\end{aligned}$$

We leave it to the reader to check that this is the minimum , i.e. $d_{\text{min}}=(2m+1)(2n+1)$ for this case. In case of diagram (b') the arguments are nearly identical, with the roles of m and n interchanged.

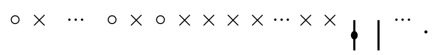
In case (c') is nearly identical as well, except that instead of using a combination of (8.38)(v) and (8.38)(ii) to create the dotted cups we instead use a single move from (8.38)(iv) instead. Thus the summand in (12.50) are all reduced by 1 and we obtain

$$d_{\text{min}}=(2m+1)(2n+1)-(2n+1)=2m(2n+1)$$

in this case as well. For the final case (d'), we first need $(n-m+1)$ translations to obtain



with one $\circ\times$ pair in the front followed by $n-m$ times the symbol $\times$. We then proceed analogous to case (a') and



with a total of $2m$ pairs of $\circ\times$, followed by $n-m$ times the symbol $\times$. Creating and moving the dotted cups to the correct place is the same as for (a'), followed by creating the single undotted cup and moving it so that it encloses the $\times$ symbols. We leave it to the reader to check that the total number of translations is again $d_{\text{min}}=2m(2n+1)$. By construction and Theorem 6.17, we have $\mathbb{F}\mathtt{R}_\delta(\lambda_\delta)$ is projective for the constructed Deligne weight diagram λ_δ in each of the cases $(a')-(d')$. Hence $\mathbb{F}\mathtt{R}_\delta(\lambda)\cong P(0,-)$. Thus, we have shown part (3) with $d_{\text{min}}=r(2n+1)$.

Hence $L(0,-)$ appears as a quotient of $V^{\otimes d_{\text{min}}}$. It is left to show that d_{min} is the minimal power where this occurs.

Since the diagrams $(a')-(d')$ were constructed with a minimal number of steps, we only have to check that there are no non-projective summands for smaller powers. Thus we assume that there exists μ constructed in less than d_{min} steps such that $\mathbb{F}\mathtt{R}_\delta(\mu)$ surjects onto $L(0,-)$ but $\mathbb{F}\mathtt{R}_\delta(\mu)\not\cong P(0,-)$, i.e. $\mu_\delta^\dagger$ is equal to the diagram $(a)-(d)$ in each case respectively. In case δ even, i.e. (c) and (d), the assertion follows directly since all non-projective weight diagrams μ give $\mu_\delta^\dagger$ with sign $+$. Thus there exists no such μ as required above. In case δ odd and $\delta>0$, i.e. (a) we obtain that μ_δ is of the form

$$\mu_\delta=\circ\ \circ\cdots\circ\ \wedge\ \wedge\cdots\wedge\ \vee\,\vee\cdots$$

by Theorem 7.8 with $m-n$ times the symbol $\circ$ follows by $2s$ times the symbol $\wedge$ for $s<n$ followed by an infinite sequence of $\vee$. This holds since otherwise $\mu_\delta^\dagger$ would not have the

correct form. In this case we have that $|\mu|$ is even, see Lemma 6.13, which implies that when passing to $\mu_\delta^\dagger$ the sign is + contradicting the assumption on μ. Hence the assertion follows in this case as well. The case that $\delta < 0$ is completely analogous, just with the roles of n and m reversed. □

12.3. **Explicit examples for frozen symbols.** We just briefly give some examples for frozen symbols as introduced in [ES17].

Example 12.3. Consider the case of OSp$(6\,|\,4)$ and the hook partition $\ulcorner\lambda = (4, 2, 1)$, then

$$\ulcorner\lambda^\infty : \ \diamond \circ \wedge \vee \wedge \underline{\circledcirc \circledcirc \circledcirc \circledcirc \cdots} \quad \rightsquigarrow \quad \ulcorner\lambda^{\circledcirc} : \ \diamond \circ \wedge \vee \wedge \underline{\vee \vee \vee \vee \cdots}$$

where we indicated the relevant positions by a horizontal line. For $\ulcorner\lambda = (4, 1, 1)$ on the other hand we obtain

$$\ulcorner\lambda^\infty : \ \circ \wedge \wedge \vee \wedge \underline{\circledcirc \circledcirc \circledcirc \circledcirc \cdots} \quad \rightsquigarrow \quad \ulcorner\lambda^{\circledcirc} : \ \circ \wedge \wedge \vee \wedge \underline{\vee \vee \vee \vee \cdots}$$

12.4. **Explicit tensor product decompositions in small examples.** We illustrate the the decomposition of $V^{\otimes d}$ into indecomposable in case $G = \mathrm{OSp}(3\,|\,2)$ for small d. In this case $\delta = 1/2$ and indecomposable summands in the tensor powers of $V^{\otimes d}$ are given by tensor weight diagrams, i.e. $\{\lambda \in \Lambda_d(1/2) \mid \kappa(\lambda_\delta) \leq 1\}$. The summand is projective if there is equality, hence $\kappa(\lambda_\delta) = 1$.

Case $d = 0$: In this case $\varnothing_\delta = \ \vee \vee \vee \vee \vee \vee \cdots$ is the only relevant Deligne weight diagram and $\underline{\varnothing}_\delta = |\ |\ |\ |\ |\ |\ \cdots$. Applying $\mathbb{F}$ to $\mathtt{R}_\delta(\varnothing)$ gives a quotient of $P(\cup\ \downarrow\ |\ |\ |\ \cdots)$ that is self-dual and not projective, which in this case is the trivial representation.

Case $d = 1$: There is again only a single Deligne weight diagram corresponding to a single box in the partition, with corresponding cup diagram $\downarrow\ |\ |\ |\ |\ |\ \cdots$. Applying $\mathbb{F}$ to the associated indecomposable object in Rep_δ gives a non-projective quotient of $P(\cup\ \downarrow\ |\ |\ |\ \cdots)$, which in this case is the representation V itself.

Case $d = 2$: At this point there are three possible Deligne weight diagrams indexing summands of $V^{\otimes 3}$. We list the cup diagram indexing the $\mathtt{R}_\delta(\lambda)$ in Rep_δ below, together with the cup diagram indexing the projective module mapping onto $\mathbb{F}\mathtt{R}_\delta(\lambda)$.

$\underline{\lambda_\delta}$	$\underline{\lambda_\delta^\dagger}$	simple/projective?
$\circ\ \times\ |\ |\ |\ |\ \cdots$	$\circ\ \times\ \downarrow\ |\ |\ |\ \cdots$	simple and projective
$\times\ \circ\ |\ |\ |\ |\ \cdots$	$\times\ \circ\ \downarrow\ |\ |\ |\ \cdots$	simple and projective
$|\ |\ |\ |\ |\ |\ \cdots$	$\cup\ \downarrow\ |\ |\ |\ \cdots$	simple, but not projective

That the first two summands are projective follows by looking at their κ values, which are both 1. They are simple, since there is a unique orientation that fits on top of the cup diagram. The last one is the same one that already appears in the case $d = 0$ and is the trivial representation. Especially it holds

$$V^{\otimes 2} \cong P(\times\ \circ\ \downarrow\ |\ |\ |\ \ldots) \oplus P(\circ\ \times\ \downarrow\ |\ |\ |\ \ldots) \oplus L(\cup\ \downarrow\ |\ |\ |\ \cdots)$$

Case $d = 3$: The total number of summands up to isomorphism is 4 in this case, but starting from the $d = 3$ case and using the translation pictures, we see that one summand appears twice. We list again the summands and the projective modules mapping onto them as in the previous case.

$\underline{\lambda_\delta}$	$\underline{\lambda_\delta^\dagger}$	simple/projective?
$\circ \mid \times \mid \mid \mid \cdots$	$\circ \downarrow \times \mid \mid \mid \cdots$	simple and projective
$\times \mid \circ \mid \mid \mid \cdots$	$\times \downarrow \circ \mid \mid \mid \cdots$	simple and projective
$\downarrow \mid \mid \mid \mid \mid \cdots$	$\cup \downarrow \mid \mid \mid \cdots$	simple, but not projective
$\cup \mid \mid \mid \mid \cdots$	$\cup \downarrow \mid \mid \mid \cdots$	projective, but not simple

The first three occur when applying the translation to the three cup diagrams from the case $d = 3$ in the same order as they are listed. While the last one occurs in the first two cases from $d = 3$, hence it appears twice in decomposition of $V^{\otimes 3}$. Note that the third and forth have the same projective cover, in one case the summand is the simple module corresponding to $\cup \downarrow \mid \mid \mid \cdots$, in the other it is the projective cover. In total we get as a decomposition of the tensor power

$$V^{\otimes 2} \cong P(\circ \downarrow \times \mid \mid \mid \ldots) \oplus P(\times \downarrow \circ \mid \mid \mid \ldots) \oplus \big(L(\cup \downarrow \mid \mid \mid \cdots) \oplus P(\cup \downarrow \mid \mid \mid \cdots)^{\oplus 2}\big)$$

Note that the first two summands lie in different, semisimple, blocks, while three summands lie in the same, non semisimple, block.

13. Dictionary to the Gruson-Serganova labelling of highest weights

In this section we give an explicit dictionary which translates between the Gruson-Serganova labelling of irreducible modules from [GS13] and our labelling with weight diagrams respectively corresponding cup diagram.

13.1. **Dominant weights.** We consider $\mathfrak{g} = \mathfrak{osp}(r|2n)$ and pick the Borel subalgebra and simple roots as in [GS13], [ES17]. With this choice the sum of all the positive even roots minus the sum of the positive odd roots equals 2ρ, where ρ is given as follows.

For $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ we have $\delta/2 = m - n + 1/2$ and

$$\rho \quad = \quad \begin{cases} (\delta/2 - 1, \delta/2 - 2, \ldots, 1/2, -1/2, \ldots, -1/2 \,|\, 1/2, \ldots, 1/2) & \text{if } m \geq n, \\ (-1/2, \ldots, -1/2 \,|\, -\delta/2, -\delta/2 - 1, \ldots, 1/2, \ldots, 1/2) & \text{if } m < n. \end{cases}$$

For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ we have $\delta/2 = m - n$ and

$$\rho \quad = \quad \begin{cases} (\delta/2 - 1, \delta/2 - 2, \ldots, 1, 0, \ldots, 0 \,|\, 0, \ldots, 0) & \text{if } m > n, \\ (0, \ldots, 0 \,|\, -\delta/2, -\delta/2 - 1, \ldots, 1, 0, \ldots, 0) & \text{if } m \leq n. \end{cases}$$

Remark 13.1. Note that $n = 0$ gives $\rho = (m-1, m-2, \ldots, 0)$ for m even and $\rho = (m - 1/2, m - 3/2, \ldots, 1/2)$ for m odd; and $\rho = (n, n-1, \ldots, 1)$ in case $m = 0$. These are the values for ρ for the semisimple Lie algebras of type D_m, B_m, C_n.

Definition 13.2. A weight $\lambda \in X(\mathfrak{g})$ is *dominant (integral)* if

$$\lambda + \rho \quad = \quad \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j \tag{13.51}$$

satisfies the following dominance condition, see [GS10].

For $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$:

(1) either $a_1 > a_2 > \cdots > a_m \geq 1/2$ and $b_1 > b_2 > \cdots > b_n \geq 1/2$,
(2) or $a_1 > a_2 > \cdots > a_{m-l-1} > a_{m-l} = \cdots = a_m = -\frac{1}{2}$ and
$b_1 > b_2 > \cdots > b_{n-l-1} \geq b_{n-l} = \cdots = b_n = \frac{1}{2}$,

For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$:

(1) either $a_1 > a_2 > \cdots > a_{m-1} > |a_m|$ and $b_1 > b_2 > \cdots > b_n > 0$,
(2) or $a_1 > a_2 > \cdots > a_{m-l-1} \geq a_{m-l} = \cdots = a_m = 0$ and
$b_1 > b_2 > \cdots > b_{n-l-1} > b_{n-l} = \cdots = b_n = 0$.

The set of dominant weights is denoted $X^+(\mathfrak{g})$. Note that

$$X^+(\mathfrak{osp}(2m+1|2n)) \subset (\mathbb{Z} + 1/2)^{m+n} \text{ and } X^+(\mathfrak{osp}(2m|2n)) \subset \mathbb{Z}^{m+n}.$$

Definition 13.3. Weights satisfying (i) are called *tailless* and the number $l+1$ from Definition 13.2 is the *tail length,* $\mathrm{tail}(\lambda)$, *of* λ.

Consider first the case $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$. Given a weight λ such that $\lambda + \rho$ is dominant in the sense of Definition 13.2, there are precisely two finite dimensional irreducible representations $L(\lambda, +)$ and $L(\lambda, -)$ of $\mathrm{OSp}(2m+1|2n)$ which are irreducible highest weight representations of highest weight λ for $\mathfrak{g}$. Note that $\mathrm{tail}(\lambda)$ in this case is multiplicity how often $-1/2$ appears amongst the a_i's.

Theorem 13.4. *(1) The indecomposable projective module $P(\lambda, \epsilon)$ for $\mathrm{OSp}(2m+1|2n)$ with λ as in Definition 13.2 and $\epsilon \in \{+,-\}$ correspond to the super cup diagram constructed as follows. Define $A(\lambda) = \{a_1, a_2, \ldots, a_m\}$ and $B(\lambda) = \{b_1, b_2, \ldots, b_n\}$. Then put at position $i \geq 1/2$ the symbol*

$$\begin{cases} \vee & \text{if } i \in A(\lambda) \cap B(\lambda), \\ \circ & \text{if } i \in A(\lambda), \notin B(\lambda), \\ \times & \text{if } i \notin A(\lambda) \in B(\lambda), \\ \wedge & \text{otherwise} \end{cases}$$

and construct the cup diagram following Definition 6.3, except that we stop in Step (C-4) after having created $\mathrm{tail}(\lambda)$ *many dotted cups. Instead of creating further cups we attach rays to these remaining vertices with the leftmost ray being dotted in case $\epsilon = +$ and undotted in case $\epsilon = -$.*

(2) Conversely, given a cup diagram, the corresponding indecomposable projective $P(\lambda, \epsilon)$ for $\mathrm{OSp}(2m+1|2n)$ has $\epsilon = +$ if the leftmost ray is dotted and $\epsilon = -$ if the leftmost ray is undotted. The positions of $\circ$ and left endpoints of cups determine the coefficients $a_i \geq 1/2$, the positions of $\times$ and left endpoints of cups determine the coefficients $b_i \geq 1/2$; and the number of dotted cups equals the tail length $\mathrm{tail}(\lambda)$.

Consider now the case $\mathfrak{g} = \mathfrak{osp}(2m|2n)$. Given a weight λ such that $\lambda+\rho$ is dominant in the sense of Definition 13.2, there are precisely two finite dimensional irreducible representations $L(\lambda, +)$ and $L(\lambda, -)$ of $\mathrm{OSp}(2m|2n)$ which are irreducible highest weight representations of highest weight λ for $\mathfrak{g}$ in case $a_m > 0$ and one irreducible represntation $L(\lambda, \pm)$ in case $a_m = 0$. Note that now $\mathrm{tail}(\lambda)$ is the multiplicity how often 0 appears amongst the b_i's.

Theorem 13.5. *The indecomposable projective module $P(\lambda, \epsilon)$ for $\mathrm{OSp}(2m|2n)$ with λ as in Definition 13.2 and $\epsilon \in \{+, -, \pm\}$ correspond to the super cup diagram constructed as follows. Define $A(\lambda) = \{a_1, a_2, \ldots, a_m\}$ and $B(\lambda) = \{b_1, b_2, \ldots, b_n\}$. Then put at position $i > 0$ the symbol*

$$\begin{cases} \vee & \text{if } i \in A(\lambda) \cap B(\lambda), \\ \circ & \text{if } i \in A(\lambda), \notin B(\lambda), \\ \times & \text{if } i \notin A(\lambda) \in B(\lambda), \\ \wedge & \text{otherwise} \end{cases}$$

and construct the corresponding cup diagram following Definition 6.3, except that we stop in Step $(C-4)$ after having created $\mathrm{tail}(\lambda)$ *dotted cups and instead of creating cups attach rays to these remaining vertices with the leftmost ray being dotted in case $\epsilon = +$ and undotted in case $\epsilon = -$.*

As an application of the main theorem we obtain an interpretation the atypicality of a block in the sense of [GS13] in terms of the defect from Definition 6.5.

Corollary 13.6. *If $P(\lambda)$ and $P(\mu)$ are are indecomposable projectives in the same block $\mathcal{B}$ of $\mathcal{F}$ then* $\operatorname{def}(\underline{\Psi(\lambda)}) = \operatorname{def}(\underline{\Psi(\lambda)})$. *In particular, one can talk about the defect* $\operatorname{def}(\mathcal{B})$ *of a block $\mathcal{B}$ of $\mathcal{F}$. Moreover this defect is the atypicality of $\mathcal{B}$ and the Loewy length of any indecomposable projective in $\mathcal{B}$ equals* $2\operatorname{def}(\mathcal{B}) + 1$.

Proof. The first statement is [ES17, Corollary 6.5] which now holds by our main Theorem 10.5. The second is just a translation from [GS13], see [ES17, Corollary 6.6]. Consider now an indecomposable projective in $\mathcal{B}$ and let P be the corresponding module over the diagrammatically defined algebra from Theorem 10.5. We first claim that the top degree of P equals $2\operatorname{def}(\mathcal{B})$. It comes from the unique basis vector b of P' which is the oriented circle diagram where bottom and top are both given by the cup diagram corresponding to P and where the orientation is such that all circles are clockwise (i.e. of degree 2). Indeed, one can directly see that this circle diagram can provide as maximal degree $2\operatorname{def}(\mathcal{B})$. By an easy induction which successively removes kinks as in [ES17, proof of Proposition 4.9] one can show that other basis vectors would have strictly smaller degree, thus the claim holds.

By construction, the projective module corresponding via Theorem 10.5 to P on the diagrammatical side is a quotient of an indecomposable projective P'' from the algebras studied in [ES16b]. In particular the Loewy length of P is at most that of P'. Now by [ES17, Corollary 9.3] P' can be chosen to be projective-injective and so has in particular simple head and simple socle. By [ES17][Theorem 6.10] P' is a module over a graded algebra which is semisimple in degree zero and generated in degree 1. Since it has simple head and socle, [BGS96, Proposition 2.4.1.] implies that the grading filtration of P' agrees with both the socle and the radical filtration of P'. Hence the subspace of maximal degree of P' is one-dimensional (i.e. the socle) and again of degree $2\operatorname{def}(\mathcal{B})$. The distinguished basis vector is sent to $b \in P$ under the quotient map. Thus the Loewy length of P cannot be smaller than of P' and thus equals $2\operatorname{def}(\mathcal{B}) + 1$. □

Remark 13.7. The combinatorics of the translation functors as well as the multiplicity formulas are controlled by the combinatorics of a type D Weyl group or Hecke algebra, in fact of the Weyl group of type D_∞, see [CVM09], [ES16b], [ES18]. One can easily verify that the categorification of the layers from and [] gives can be seen as a categorification of a filtration by isotypical components. Each layer is isomorphic to a direct sum of right cell modules with a fixed a-function in the sense of Lusztig. This can be seen as an infinite and type D generalisations of the filtrations from [FKS06, Section 11].

14. Algorithm to compute the highest weight of the head of an indecomposable summand in $V^{\otimes d}$

In this section we like to give a float-chart which describes how one can compute the Lie theoretic highest weight of an indecomposable tensor module, i.e. an indecomposable summand in $V^{\otimes d}$ from the Classification Theorem 7.8.

We have fixed m, n, δ and given the indecomposable summand $\mathbb{F}\mathtt{R}_\delta(\lambda)$ labelled by the partition λ. To compute the labelling $\delta^\dagger = (\Phi(\lambda), \epsilon)$ of the head of $\mathbb{F}\mathtt{R}_\delta(\lambda)$ proceed as follows:

1. Compute the Deligne weight diagram λ_δ according to λ by applying the dictionary (6.26) to the sequence $X(\lambda)$ from (3.16).
2. Construct the corresponding cup diagram $\underline{\lambda_\delta}$ using the rules from Definition 6.3.
3. Check whether the result is projective via the criterion from Theorem 6.17, namely whether $\kappa(\lambda_\delta) = \min(m, n)$, with κ as defined in Definition 6.5. Continue depending on the answer as follows:

- (yes): swap in λ_δ all $\vee$'s corresponding to rays in $\underline{\lambda_\delta}$ into $\wedge$'s and define the corresponding cup diagram $\underline{\lambda}_\delta^\dagger$ via Definition 6.3. Also define $\epsilon \in \{+.-, \mp\}$ as in Proposition 7.6.
- (no) swap in λ_δ all symbols $\vee$ or $\wedge$ corresponding to rays in $\underline{\lambda_\delta}$ and define the corresponding cup diagram $\underline{\lambda}_\delta^\dagger$ via Definition 6.3. Also define $\epsilon \in \{+.-, \mp\}$ as in Proposition 7.6.

Now one can read off the weight diagram $\Phi(\lambda)$ from the cup diagram $\underline{\lambda}_\delta^\dagger$. This give us the signed (n, m)-hook weight diagram $\delta^\dagger = (\Phi(\lambda), \epsilon)$ from Theorem 7.8.

15. Proof of Theorem 4.8

We now finish the proof of Theorem 4.8. We abbreviate $e_k = \flat_k \flat_k^*$ and observe that the following formulas hold

$$\tfrac{1}{b_k} s_k = s_k \tfrac{1}{b_{k+1}} + \tfrac{1}{b_k b_{k+1}} - \tfrac{1}{b_k} e_k \tfrac{1}{b_{k+1}} \qquad s_k \tfrac{1}{b_k} = \tfrac{1}{b_{k+1}} s_k + \tfrac{1}{b_k} \tfrac{1}{b_{k+1}} - \tfrac{1}{b_{k+1}} e_k \tfrac{1}{b_k}. \tag{15.52}$$

thanks to Remark 4.2, whenever the quotients are defined. In the following we will not argue why the expessions are defined, but instead refer to [ES16c] for a detailed treatment of these issues. We also refer to [ES16c] for the justification that the idempotent on the left side of the expressions in (4.21) can be omitted. In the following we will in fact quite often omit some idempotents $\mathbf{f}$ in the middle and the left of expressions, since they are not necessary thanks to the appearance of $\mathbf{f}$ on the right. We refer to [ES16c] for a detailed treatment of this phenomenon.

Lemma 15.1. *We claim that* $\mathbf{f}\flat_k \frac{1}{b_k} \flat_k^* \mathbf{f} = (1 + \frac{1}{2\beta})\mathbf{f}$ *for any* $k \geq 1$.

Proof. Let first $k = 1$. Then by definition of the cyclotomic quotient

$$\begin{aligned}
\mathbf{f}\flat_1 \tfrac{1}{b_1} \flat_1^* \mathbf{f} &= \mathbf{f}\flat_1 \left(\tfrac{1}{\alpha+\beta} \tfrac{y_1-\beta}{\alpha-\beta} + \tfrac{1}{2\beta} \tfrac{y_1-\alpha}{\beta-\alpha} \right) \flat_1^* \mathbf{f} \\
&= \tfrac{1}{(\alpha-\beta)(\alpha+\beta)2\beta} \flat_1 \left((\beta-\alpha) y_1 + (\alpha-\beta)(\alpha+2\beta) \right) \flat_1^* \mathbf{f} \\
&= \tfrac{1}{(\alpha+\beta)2\beta} \flat_1 \left(-y_1 + (\alpha+2\beta) \right) \flat_1^* \mathbf{f} \\
&= \tfrac{1}{(\alpha+\beta)2\beta} (-\overline{\omega}_1 + (\alpha+2\beta)N)\mathbf{f} \\
&= \tfrac{1}{\beta} \left(\tfrac{1-N}{N} + \tfrac{N}{2} + \beta \right) \mathbf{f} = \left(\tfrac{1}{2\beta} + 1 \right) \mathbf{f}
\end{aligned}$$

Now assume the lemma holds for k and consider $A := \mathbf{f}\flat_{k+1} \frac{1}{b_{k+1}} \flat_k^* \mathbf{f} = \mathbf{f}\flat_{k+1} s_k s_k \flat_{k+1}^* \mathbf{f}$. Then

$$A \;=\; \mathbf{f}\flat_{k+1} s_k \tfrac{1}{b_k} s_k \flat_{k+1}^* \mathbf{f} - \mathbf{f}\flat_{k+1} s_k \tfrac{1}{b_k b_{k+1}} \flat_{k+1}^* \mathbf{f} + \mathbf{f}\flat_{k+1} s_k \tfrac{1}{b_k} e_k \tfrac{1}{b_{k+1}} \flat_{k+1}^* \mathbf{f} \tag{15.53}$$

Now the first term in (15.53) equals

$$\mathbf{f}\flat_{k+1} s_k \tfrac{1}{b_k} s_k \flat_{k+1}^* \mathbf{f} = \flat_{k+1} e_k \tfrac{1}{b_k} e_k \flat_{k+1}^* \mathbf{f} = \left(1 + \tfrac{1}{2\beta}\right) \mathbf{f}$$

by induction. The second term in (15.53) equals, using (15.52),

$$\begin{aligned}
\mathbf{f}\flat_{k+1} s_k \tfrac{1}{b_{k+1}} \flat_{k+1}^* \tfrac{1}{b_k} \mathbf{f} &= \mathbf{f}\flat_{k+1} \tfrac{1}{b_k} s_k \flat_{k+1}^* \tfrac{1}{b_k} \mathbf{f} - \mathbf{f}\flat_{k+1} \tfrac{1}{b_k b_{k+1}} \flat_{k+1}^* \tfrac{1}{b_k} \mathbf{f} + \mathbf{f}\flat_{k+1} \tfrac{1}{b_k} e_k \tfrac{1}{b_{k+1}} \flat_{k+1}^* \tfrac{1}{b_k} \mathbf{f} \\
&= \mathbf{f} \tfrac{1}{b_k^2} \flat_{k+1} s_k \flat_{k+1}^* \mathbf{f} - \mathbf{f} \tfrac{1}{b_k^2} A \flat_{k+1} e_k \flat_{k+1}^* \mathbf{f} + \mathbf{f} \tfrac{1}{b_k^2 b_k'} \mathbf{f} \\
&= \mathbf{f} \tfrac{1}{b_k^2 b_k'} (b_k' + 1) \mathbf{f} - \mathbf{f} \tfrac{1}{b_k^2} A \mathbf{f}.
\end{aligned}$$

The third term in (15.53) equals, using (15.52),

$$\begin{aligned}
&\mathbf{f}\flat_{k+1} \tfrac{1}{b_{k+1}} s_k \flat_{k+1}^* \mathbf{f} + \flat_{k+1} \tfrac{1}{b_k} \tfrac{1}{b_{k+1}} \flat_{k+1}^* \mathbf{f} - \mathbf{f}\flat_{k+1} \tfrac{1}{b_{k+1}} e_k \tfrac{1}{b_k} \flat_{k+1}^* \mathbf{f} \\
= \;& \mathbf{f} \tfrac{1}{b_k'^2} \flat_{k+1} e_k \flat_{k+1}^* \mathbf{f} + \mathbf{f} \tfrac{1}{b_k b_k'^2} \flat_{k+1} e_k \flat_{k+1}^* \mathbf{f} - \mathbf{f} \tfrac{1}{b_k'^2} \left(1 + \tfrac{1}{2\beta}\right) \flat_{k+1} e_k \flat_{k+1}^* \mathbf{f} \\
= \;& \mathbf{f} \tfrac{1}{b_k'^2} \left(\tfrac{1}{b_k} - \tfrac{1}{\beta} \right) \mathbf{f}.
\end{aligned}$$

Altogether we obtain

$$\mathbf{f} \left(1 - \tfrac{1}{b_k^2}\right) A \mathbf{f} \;=\; \mathbf{f} \left(\left(1 + \tfrac{1}{2\beta}\right) - \tfrac{1}{b_k^2} - \tfrac{1}{b_k^2 b_k'} + \tfrac{1}{b_k b_k'^2} - \tfrac{1}{2\beta b_k'^2} \right) \mathbf{f}.$$

This is equivalent to

$$\mathbf{f}\left(b_k^2 b_k'^2 - b_k'^2\right) A\mathbf{f} \quad = \quad \mathbf{f}\left(\left(b_k^2 b_k'^2 - b_k'^2\right)\left(1 + \tfrac{1}{2\beta}\right)\right)\mathbf{f} + \mathbf{f}\left(b_k - b_k' + \tfrac{1}{2\beta}(b_k'^2 - b_k^2)\right)\mathbf{f}.$$

By definition the second summand here equals $\mathbf{f}(2y_k - \frac{1}{2\beta}(4\beta y_k))\mathbf{f} = 0$. Now the lemma is proved, since $\left(b_k^2 b_k'^2 - b_k'^2\right)$ has a left inverse. □

Proposition 15.2. *The assignments for the morphisms in the proof of Theorem 4.8 define a well-defined algebra homomorphism.*

Proof. We have to verify the defining relations (1)-(4) from Proposition 1.2. Let us start with the most interesting one, the loop removing relation. The loop is sent under our assignments to

$$\mathbf{f}\flat_k Q\mathbf{f}Q_k\flat_k^*\mathbf{t} = \flat_k \tfrac{b_{k+1}}{b_k}\flat_k^*\mathbf{f} = \flat_k \tfrac{b_k'}{b_k}\flat_k^*\mathbf{f} = 2\beta\flat_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} - \flat_k\flat_k^*\mathbf{f} = (2\beta + 1 - N)\mathbf{f} = \delta\mathbf{f}.$$

Here, we first used the definitions and the fact that $\mathbf{f}$ can be removed, then applied the dot sliding relation, invoked the definition $b_k' = 2\beta - b_k$, and finally we applied Lemma 15.1. Since $(2\beta + 1 - N) = \delta$ the dot removal relation holds.

Consider the first untwisting relation. The left hand side of the relation is mapped to

$$\begin{aligned} &-Q_{k+1}s_{k+1}Q_{k+1}Q_k\flat_k^* + \tfrac{1}{b_{k+1}}Q_k\flat_k^*\mathbf{f} = -\tfrac{Q_{k+1}}{\sqrt{b_k}}s_{k+1}\sqrt{b_{k+2}}\flat_k^*\mathbf{f} + \tfrac{1}{\sqrt{b_{k+1}b_k}}\flat_k^*\mathbf{f} \\ = \quad &-\tfrac{Q_{k+1}}{\sqrt{b_k}}s_{k+1}\flat_k^*\sqrt{b_k}\mathbf{f} + \tfrac{1}{\sqrt{b_{k+1}b_k}}\flat_k^*\mathbf{f}. \end{aligned} \tag{15.54}$$

The right hand side is, using (15.52), mapped to $A := -Q_k s_k Q_k Q_{k+1}\flat_{k+1}^*\mathbf{f} + \frac{1}{b_k}Q_{k+1}Q_k\mathbf{f}$. Now its first summand equals

$$-Q_k\sqrt{b_{k+2}}\tfrac{1}{b_{k+1}}s_k\flat_{k+1}^*\sqrt{b_k}\mathbf{f} - -Q_k\sqrt{b_{k+2}}\tfrac{1}{b_k b_{k+1}}\flat_{k+1}^*\sqrt{b_k}\mathbf{f} + -Q_k\sqrt{b_{k+2}}\tfrac{1}{b_{k+1}}e_k\frac{1}{b_k}\flat_{k+1}^*\sqrt{b_k}\mathbf{f}$$

here, the second summand equals $-\frac{\sqrt{b_{k+2}}}{\sqrt{b_k b_{k+1}}}\flat_{k+1}^*\mathbf{f}$ which cancels with the second summand in A we have ignored thus far. Hence we are left with

$$Q_{k+1}\tfrac{1}{\sqrt{b_k}}s_{k+1}\flat_k^*\sqrt{b_k}\mathbf{f} + Q_k\tfrac{\sqrt{b_{k+2}}}{b_{k+1}}\flat_k^*\tfrac{1}{\sqrt{b_k}}\mathbf{f}$$

Since the second summand here is equal to $Q_k\frac{1}{b_{k+1}}\flat_k^*\sqrt{b_k}\frac{1}{\sqrt{b_k}}\mathbf{f}$ which equals the second summand in (15.54) and so the desired relation is satisfied.

Consider the second untwisting relation. The left hand side is mapped to

$$\begin{aligned} -\mathbf{f}Q_k s_k\tfrac{b_{k+1}}{b_k}\flat_k^*\mathbf{f} + \mathbf{f}\tfrac{1}{b_k}Q_k\flat_k^*\mathbf{f} \quad &= \quad -\mathbf{f}Q_k b_k s_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} + \mathbf{f}Q_k e_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} \\ &= \quad -\mathbf{f}Q_k b_k s_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} + \mathbf{f}\left(1 + \tfrac{1}{2\beta}\right)Q_k\flat_k^*\mathbf{f} \end{aligned}$$

where we used Remark 4.2 and Lemma 15.1. Now the desired untwisting relation holds if we can show $\mathbf{f}s_k\frac{1}{b_k}\flat_k^*\mathbf{f} = \mathbf{f}\frac{1}{2\beta}\frac{1}{b_k}\flat_k^*\mathbf{f}$. This follows from [ES16c, Proposition 6.2], but we give (apart from the cancellation of the $\mathbf{f}$) the arguments here. We have

$$\begin{aligned} \mathbf{f}s_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} \quad &= \quad \mathbf{f}\tfrac{1}{b_{k+1}}\flat_k^*\mathbf{f} + \mathbf{f}\tfrac{1}{b_k b_{k+1}}\flat_k^*\mathbf{f} - \mathbf{f}\tfrac{1}{b_{k+1}}e_k\tfrac{1}{b_k}\flat_k^*\mathbf{f} = \mathbf{f}\tfrac{1}{b_{k+1}}\left(1 + \tfrac{1}{b_k} - (1 + \tfrac{1}{2\beta})\right)\flat_k^*\mathbf{f} \\ &= \quad \mathbf{f}\tfrac{1}{b_{k+1}}\left(\tfrac{1}{b_k} - \tfrac{1}{2\beta}\right)\flat_k^*\mathbf{f} = \mathbf{f}\tfrac{1}{2\beta}\tfrac{1}{b_{k+1}}\left(\tfrac{2\beta - b_k}{b_k}\right)\flat_k^*\mathbf{f} = \mathbf{f}\tfrac{1}{2\beta}\tfrac{1}{b_{k+1}}\flat_k^*\mathbf{f} \end{aligned}$$

The arguments for the snake relations are easy and therefore omitted. The braid relations are involved, but were established in [ES16c, Lemma 6.5 and Proposition 6.6]. Thus the proposition follows. □

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