Combinatorial models for cohomology theories of Grassmannians

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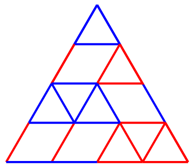
Introduction

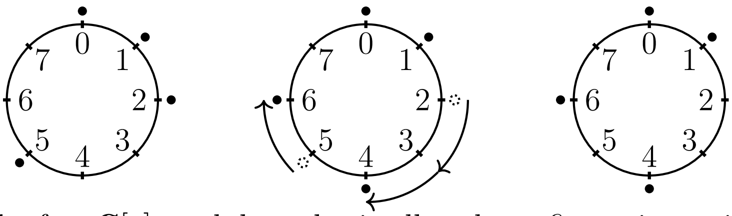
The Grassmannian $Gr(k, N)$ is the manifold whose points correspond to k-dimensional subspaces of $\mathbb{C}^N$. These are geometric objects where geometry, representation theory and combinatorics intertwine. For geometers, Grassmannians are among the first examples of moduli spaces, which are equipped with God-given vector bundles or families of geometric objects. Furthermore, Chern classes and vector bundles play a fundamental role in intersection theory, and the infinite Grassmannian (in the analytic topology) is a classifying space for complex vector bundles, hence all Chern classes are pullbacks of its cohomology classes, see [MSSU74] for details. For combinatorists, the Schubert classes are symmetric polynomials known as Schur polynomials, which are labeled by partitions and carry much combinatorial information. And for representation theorists, the Schur functions encode for example characters of the symmetric groups and of the general linear group. Furthermore, they form a $\mathbb{Z}$-basis of the ring of symmetric functions, which in turn carries many algebraic structures. Thus, the classical theory of the cohomology ring of Grassmannians is already of much interest.

From other areas of mathematics arise various enhancements of the cohomology ring of nice spaces, among which we are interested in *torus equivariant cohomology*. denoted by H_T^* and *quantum cohomology*, denoted by QH^*. The former arises from topology, and the reason why we focus on tori is that a general Lie group has the same homotopy type as its maximal reductive quotient, hence they yield the same equivariant cohomology theory. Then the equivariant cohomology for a reductive group is essentially the Weyl group invariants of the equivariant cohomology of its maximal torus. See [Bri98a] for details. This is of much interest for representation theorists because e.g. they provide a geometric construction for Soergel bimodules. On the other hand, quantum cohomology arises from physics and enumerative geometry, and plays a central role e.g. in mirror symmetry. The general theories are rich and giving a substantially general treatment is far beyond the scope of this thesis. We will rather give a glimpse of the beauty of the theory by focusing on combinatorial models for both $H_T^*Gr(k, N)$ and $QH^*Gr(k, N)$, and discusses some connections between them.

More specifically, we consider the Knutson-Tao puzzle rule for equivariant cohomology of Grassmannians as in [KT+03] and the fermionic model for the quantum cohomology of Grassmannians as in [KS10]. The former described the following puzzle game: take an equilateral triangle with side length N, where the sides have certain 2-colored patterns, and try to fill it with certain puzzle pieces into a puzzle of the following form:



And Knutson-Tao assert that the product structure of the Grassmannian is characterized by the number of such puzzles. On the other hand, [KS10] considers particle configurations on a circle with N slots, where each slot allows at most one particle. The following picture presents an example with $N = 7$.



Let $\mathcal{F}_k[q]$ be the free $\mathbb{C}[q]$-module on basis all such configurations with k particles. Then there are natural particle hopping operators acting on $\mathcal{F}_k[q]$. As a $\mathbb{C}[q]$-module, $QH^*Gr(k, N)$ is trivially isomorphic to $\mathcal{F}_k[q]$. Now multiplication by a Schubert class in $QH^*Gr(k, N)$ can be viewed as a $\mathbb{C}[q]$-linear operator on $\mathcal{F}_k[q]$, which can be expressed in terms of the particle hopping operators.

Our first main result (see §3.2.6) is the connection between the fermion model and the *boson-fermion correspondence* in [KRR13]. The latter is an isomorphism (as certain Lie algebra representations) between a space of infinite wedges $F^{(k)}$ and the polynomial ring $B^{(k)}$ with infinitely many variables.

Theorem. There is a commutative diagram:

$$\begin{tikzcd} F^{(k)}[q] \arrow[r] \arrow[d, two heads] & B^{(k)}[q] \arrow[d, two heads] \\ \mathcal{F}_k[q] \arrow[r] & QH^*Gr(k, N) \end{tikzcd}$$

Our second main result is presented in §3.3, where we interpret certain puzzles pieces as particle hopping operators. From this point of view we deduce an algorithm producing the puzzles for $H^*Gr(k, N)$.

Theorem. Algorithm 1 in §3.3 produces all Knutson-Tao puzzles of a given boundary condition.

Our conjecture is that, this interpretation could lead to a similar puzzle rule for the quantum cohomology of Grassmannians.

The structure of this thesis is as follows: in the first chapter we present a brief introduction to the construction of torus equivariant cohomology. Next, in the second chapter we review basic properties of Grassmannians and the Knutson-Tao puzzle rule. In the third chapter we define the quantum cohomology of Grassmannians and explain the construction of [KS10]. Then we establish the main results mentioned above.

Chapter 1

Equivariant cohomology

The aim of the first chapter is to provide the reader with knowledge from first courses on manifolds and topology or on algebraic geometry a quick introduction to the basic properties of equivariant cohomology of torus actions. In §1 We shall quickly introduce the topological backgrounds used in the theory, then give a quick overview of the main results in §2. In the last section a proof of the main Theorem 1.3.7 of this chapter is presented. In this thesis we are only interested in Grassmannians, which we discuss in more details in the next chapter.

Conventions. Throughout this thesis, a space always means a topological space with homotopy type of a CW complex, e.g. a compact manWefold is homotopy equivalent to a CW complex via Morse theory. A map between spaces is always assumed to be continuous. All varieties are over $\mathbb{C}$, and dimension for varieties always means complex dimension. Let N be a fixed positive integer, denote the N-fold product of a space X by $X^N = X \times X \times \ldots \times X$. All group actions are left actions.

1.1 Prerequisites

Here we collect the topological ingredients for equivariant cohomology.

1.1.1 Tori

An *(algebraic) torus* $T = T^N$ of rank N is a product of N copies of $\mathbb{C}^*$ (the multiplicative group of complex numbers). It is a Lie group under the analytic topology, i.e. a manifold with a group structure such that the multiplication and the inversion are *smooth* maps.

The set of all left invariant vector fields on T is a Lie algebra with the usual commutator of vector fields on manifolds as the Lie bracket. This Lie algebra is identified with the tangent space at the identity element via restriction. For a torus the Lie bracket is identically 0, since the multiplication of T is commutative.

Remark 1.1.1. A torus is also an algebraic group under the Zariski topology. The reader can always think of either topology or algebraic geometry, but in this chapter We will mostly stick to the former.

Our basic setting is a continuous action of T on a space X, i.e. an action

$$\alpha : T \times X \longrightarrow X$$

that is a continuous map under the product topology of $T \times X$.

Example 1.1.2. The action of $T = (\mathbb{C}^*)^2$ on $\mathbb{C}^2$ given by

$$(x, y) \cdot (z, w) = (xz, yw) \qquad \forall (x, y) \in T, (z, w) \in \mathbb{C}^2$$

is continuous. In other words, T acts via 2×2 diagonal matrices in $GL_2(\mathbb{C})$ and hence brings linear subspaces to linear subspaces. Therefore this descends to an action of T^2 on $\mathbb{P}^1$.

Definition 1.1.3. A *character* of T is an algebraic group homomorphism $T \to \mathbb{C}^*$. The set $X(T)$ of all characters of T has a group structure induced by the multiplication of $\mathbb{C}^*$.

Lemma 1.1.4. *A character $\mathbb{C}^* \to \mathbb{C}^*$ is of the form $z \mapsto z^n$. More generally, $X(T)$ is a free abelian groups of rank N if T is a rank N torus.*

Proof. Given a character $\chi : \mathbb{C}^* \to \mathbb{C}^*$, consider the associated algebra homomorphism of coordinate rings $\chi^\sharp : \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$. This is determined by the image of z, and we want to show that $\chi^\sharp(z) = z^n$ for some $n \in \mathbb{Z}$. Since z is invertible, so is $\chi^\sharp(z)$. The unit group of $\mathbb{C}[z, z^{-1}]$ is isomorphic to $\mathbb{C}^* \times \mathbb{Z}$ via $\lambda z^n \mapsto (\lambda, n)$ for $\lambda \in \mathbb{C}^*$, hence $\chi^\sharp(z) = \lambda z^n$ for $\lambda \in \mathbb{C}^*$ and $n \in \mathbb{Z}$. Finally, $\chi(1) = 1$ implies that $\chi^\sharp(z - 1) = \lambda z^n - 1$ is divisible by $z - 1$, or $\lambda \cdot 1^n - 1 = 0$, which gives $\lambda = 1$.

Now consider a character $\chi : T \to \mathbb{C}^*$ of a rank N torus T. Choose a splitting $T \cong (\mathbb{C}^*)^N$, compose the inclusion of the i-th factor with χ $(1 \leq i \leq N)$, we get a character $\mathbb{C}^* \hookrightarrow T \to \mathbb{C}^*$, which is of the form $z \mapsto z^{n_i}$ by the previous paragraph. Hence the assignment

$$\chi \longmapsto (n_1, \dots, n_N)$$

defines a group isomorphism $X(T) \cong \mathbb{Z}^{\oplus N}$, and via this identification the projections $T \cong (\mathbb{C}^*)^N \to \mathbb{C}^*$ form a basis of $X(T)$. $\square$

Example 1.1.5. Given two characters $\chi_1, \chi_2 : T \to \mathbb{C}^*$ of a torus T, we can define a T-action on $\mathbb{P}^1$ via the above example:

$$t \cdot [x : y] = [\chi_1(t)x : \chi_2(t)y] \qquad \forall t \in T, [x : y] \in \mathbb{P}^1$$

For later use we notice that if $\chi_1\chi_2^{-1}$ is not the trivial character sending everything to $1 \in \mathbb{C}^*$, then the set of fixed points of this action is

$$(\mathbb{P}^1)^T = \{\ 0 = [0 : 1],\ \infty = [1 : 0]\ \}$$

because if we view $\mathbb{P}^1$ as lines in $\mathbb{C}^2$, then the two lines corresponding to these two points are exactly the common eigenspaces of the corresponding T-action $\mathbb{C}^2$. We will see below (Proposition 1.2.16) that for "good" T-varieties all one dimensional orbit closures are isomorphic to $\mathbb{P}^1$ with 2 fixed points.

Given a continuous torus action, one might want to define equivariant cohomology as the singular cohomology of the orbit space X/T. The problems is that when the action is not free (i.e. all stabilizers are trivial), it is not clear whether or not X/T is still a space in our sense. The idea of the *Borel construction* is to replace X by a space with the same homotopy type, but with a free action. For this we need the notion of *principal bundles.*

1.1.2 Principal bundles

Here we present only a minimal review of the theory of principal bundles and refer to [Hus13] and [Ste99] for a more thorough discussion.

Definition 1.1.6. A fibre bundle with fibre F is a is a continuous map $\pi : E \to B$, such that there is an open cover $\{U_i\}_i$ of B with homeomorphisms

$$\pi^{-1}(U_i) \cong U_i \times F$$

B is called the *base* and E the *total space.*

A *principal T-bundle* is a fibre bundle $\pi : E \to B$ with fibre T, with a free T-action on E that preserves the fibres. In what follows we always assume that the base space is path connected.

Remark 1.1.7. For every principal T-bundle $\pi : E \to B$, one can equip B with the trivial T-action, then π becomes a T-equivariant map, since by assumption

$$t\pi(e) = \pi(e) = \pi(te) \qquad \forall\ e \in E, t \in T$$

and then one can check that the orbit space $E/T \cong B$. When the group is clear from context or is not emphasized we will also simply say principal bundles instead of principal T-bundles.

Just like the case of vector bundles, there are pullback principal bundles. Given a principal T-bundle $E \to B$ and a map $f : X \to B$, taking fiber products in the category of topological spaces gives the pullback bundle $f^*E \to X$.

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

This is still a principal T-bundle. To see this, take a trivialization over an open set $U \subset B$, consider the following diagram where the front, back, bottom and right facets are Cartesian, then one checks that so is the left one, hence the fiber of $f^*E \to X$ over $f^{-1}(U)$ is $f^{-1}(U) \times T$.

$$\begin{tikzcd}
& f^*E \arrow[rr] \arrow[dd] & & E \arrow[dd] \\
f^{-1}(U) \times T \arrow[ur] \arrow[rr] \arrow[dd] & & U \times T \arrow[ur] \arrow[dd] & \\
& X \arrow[rr, "f"] & & B \\
f^{-1}(U) \arrow[ur] \arrow[rr] & & U \arrow[ur] &
\end{tikzcd}$$

Lemma 1.1.8 (Homotopy invariance of pullback bundles)**.** *Let $E \to B$ be a principal T-bundle and $f, g : X \to B$ homotopic maps, then $f^*E \cong g^*E$ as principal T-bundles.*

The miracle is that there is a (unique up to homotopy) universal bundle such that all principal T-bundles are obtained from pulling back this bundle along some map.

Definition 1.1.9. A *classifying bundle* for T, or a *universal* principal T-bundle, is a principal T-bundle $\mathbb{E}T \to \mathbb{B}T$ satisfying:

> For any principal bundle $E \to B$, there is a unique continuous map $f : B \to \mathbb{B}T$, up to homotopy, such that $E \cong f^*B$.

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & \mathbb{E}T \arrow[d] \\
B \arrow[r, "f"] & \mathbb{B}T
\end{tikzcd}$$

Remark 1.1.10. The base space $\mathbb{B}T$ of the universal bundle, if exists, is unique up to homotopy equivalence. This can be seen, for example, from the fact that the assignment sending each space to the set of isomorphism classes of principal T-bundles on it defines a contravariant functor from the homotopy category of topological spaces to the category of sets. The existence of the universal bundle is equivalent to the representability of this functor by the universal base $\mathbb{B}T$.

Theorem 1.1.11 (Milnor)**.** *For any topological group G with homotopy type of a CW complex, there is a universal principal G-bundle $\mathbb{E}G \to \mathbb{B}G$.*

Proof. See [Hus13, chapter 4 §11]. □

Remark 1.1.12. In what follows we are only concerned with the case $G = T$. In fact any principal bundle with *weakly contractible* total space is universal, see [Ste99, §19]. (A space E is weakly contractible means that for any $n \in \mathbb{N}$, any continuous map $S^n \to E$ is null-homotopic.) We next use this to construct a universal principal T-bundle.

Example 1.1.13 (Universal principal $\mathbb{C}^*$-bundle)**.** Consider the space

$$\mathbb{C}^\infty - \{0\} = \varinjlim_{m\in\mathbb{N}} \mathbb{C}^m - \{0\}$$

obtained as follows: identify $\mathbb{C}$ as the line $x_1 = 0$ on the (x_1, x_2)-coordinate plane $\mathbb{C}^2$ minus the origin, then identify $\mathbb{C}^2$ as the plane $x_2 = 0$ inside the (x_1, x_2, x_3)-coordinate space $\mathbb{C}^3$ minus the origin, etc. Similarly, we can form the infinite projective space $\mathbb{CP}^\infty = \varinjlim_{m\in\mathbb{N}} \mathbb{CP}^m$.

Composing the projection $\mathbb{C}^{m+1} - \{0\} \to \mathbb{CP}^m$ with $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^\infty$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^{m+1} - \{0\} \arrow[r] \arrow[d, hook] & \mathbb{CP}^m \arrow[d, hook] \\
\mathbb{C}^{m+2} - \{0\} \arrow[r] & \mathbb{CP}^{m+1}
\end{tikzcd}$$

Hence by the universal property of $\mathbb{C}^\infty - \{0\}$ we get a map

$$\pi : \mathbb{C}^\infty - \{0\} \longrightarrow \mathbb{CP}^\infty$$

We claim that π is a universal principal $\mathbb{C}^*$-bundle.

To see that the map π is a principal bundle, note that by construction each $\{x_i \neq 0\} \subset \mathbb{CP}^\infty$ intersects $\mathbb{CP}^m \subset \mathbb{CP}^\infty$ at the standard open set with $x_i \neq 0$, provided that $m > i$, and does not intersect $\mathbb{CP}^m$ otherwise. This means that $\{x_i \neq 0\} \subset \mathbb{CP}^\infty$ is open under the weak/direct limit topology. On the other hand, each standard projection $\mathbb{C}^{m+1} - \{0\} \to \mathbb{CP}^m$ is a trivial principal bundle on $\{x_i \neq 0\} \subset \mathbb{CP}^m$. Since such open sets cover $\mathbb{CP}^\infty$, π is indeed a principal $\mathbb{C}^*$-bundle.

Now we claim that the total space $\mathbb{C}^\infty - \{0\}$ is weakly contractible, so that in view of Remark 1.1.12, π is indeed a universal bundle. To prove the claim, notice that $\mathbb{C}^m - \{0\}$ is homotopy equivalent to the sphere S^{2m-1}, hence for any $i < 2m-1$, any amp $S^i \to \mathbb{C}^m - \{0\}$ is null-homotopic. Now take any map $f : S^i \to \mathbb{C}^\infty - \{0\}$, since S^i is compact, its image is contained in some $\mathbb{C}^m - \{0\}$, and we can assume that $2m - 1 > i$, therefore f is contractible.

Example 1.1.14 (Universal principal T-bundle)**.** Now from the above example we can construct a universal bundle for a torus of arbitrary rank N. Take the product of N copies of $\mathbb{P}^\infty$ as $\mathbb{B}T$, and the product of N copies of $\mathbb{C}^\infty - \{0\}$ as $\mathbb{E}T$. Since $\mathbb{C}^\infty - \{0\}$ is weakly contractible, so is its N-fold product. This concludes that we have found the classifying bundle for T.

1.1.3 Chern classes

Here is a quick overview of Chern classes. This is closely related to the classifying bundles, but we will only need some elementary properties of Chern classes. For a more thorough introduction of Chern classes the reader is referred to [BT13], [EH16] or [MSSU74]. Denote by H^* singular cohomology with complex coefficients, $\cup$ the cup product and by $\cap$ the cap product with homology.

Definition 1.1.15. Consider a complex vector bundle $\pi : E \to B$. Its Chern classes are the cohomology classes

$$c_i(E) \in H^{2i}(B)$$

characterized by the following axioms:

- $c_0(E) = 1 \in H^0(B)$ and $c_i(E) = 0$ for $i > \operatorname{rank} E$.
- (Normalization). For the *tautological line bundle* $\mathcal{O}(-1)$ on $\mathbb{P}^1$,

  $$c_1(\mathcal{O}(-1)) \cap [\mathbb{P}^1] = -[\mathrm{pt}]$$

  where $[\mathrm{pt}] \in H_0(\mathbb{P}^1)$ is the class of a point, and $[\mathbb{P}^1] \in H_1(\mathbb{P}^1)$ is the fundamental class.
- (Naturality). For any $f : B' \to B$ and vector bundle ξ over B,

  $$f^* c_i(\xi) = c_i(f^*\xi)$$

- (Whitney sum formula). Given a short exact sequence

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

of vector bundles over B, we have

$$c(\xi) = c(\xi')c(\xi'')$$

where $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \ldots = \sum_{0 \leq 2i \leq \mathrm{rank}\, \xi} c_i(\xi) \in H^*(B)$ is the *total Chern class.*

Example 1.1.16. Let me elaborate a little more about the normalization axiom for those not familiar with the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$. This bundle is defined as follows: view the points $\mathbb{P}^n$ as lines in $\mathbb{C}^{n+1}$, and consider the subspace of the trivial bundle

$$\{(L, x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in L\}$$

then the projection to the first factor defines a line bundle over $\mathbb{P}^n$, with fiber over $L \in \mathbb{P}^n$ equal to $L \subset \mathbb{C}^{n+1}$.

Remark 1.1.17. As a matter of fact, naturality and normalization axioms determine c_1 for all line bundles, Then one reaches the higher Chern classes via the Whitney sum formula plus the following *splitting principle.*

Lemma 1.1.18 (Splitting principle)**.** *For any complex vector bundle $E \to B$ there exists a space $B' \to B$ such that the pullback $H^*(B) \to H^*(B')$ is injective and the pullback bundle on B' has a filtration with line bundle subquotients.*

Remark 1.1.19. The top Chern classes are also the Euler classes of the complex vector bundles. The difference is that the Euler class is defined for any oriented real vector bundle. A complex vector bundle carries a canonical orientation therefore has an Euler class. Some authors use Euler classes in what follows instead.

1.2 Torus Equivariant Cohomology

1.2.1 Definition and first examples

Fix a torus T of rank N and a universal bundle $\mathbb{E}T = (\mathbb{C}^\infty - \{0\})^N \to \mathbb{B}T = (\mathbb{CP}^\infty)^N$ as in Example 1.1.14. Recall that T acts on $\mathbb{E}T$ on the left.

Definition 1.2.1. Given a T-space X, the *Borel construction* is defined to be the space

$$\mathbb{E}T \times_T X := \mathbb{E}T \times X/\sim$$

where $\sim$ is defined by the diagonal action

$$(e, x) \sim (te, tx) \qquad \forall e \in \mathbb{E}T, x \in X, t \in T$$

Note that the T-action on $\mathbb{E}T \times X$ is free, and it is homotopy equivalent to X.

Lemma 1.2.2. *The Borel construction $\mathbb{E}T \times_T X$ has the homotopy type of a CW complex.*

Proof. The total spaces $\mathbb{E}T$ of the universal bundles in Example 1.1.14 are actually T-CW complexes, i.e. they are inductively constructed from pushouts of the form

$$\begin{array}{ccc} S^n \times T & \hookrightarrow & D^n \times T \\ \downarrow & & \downarrow \\ Y_n & \hookrightarrow & Y_{n+1} \end{array}$$

where $Y_n \hookrightarrow \mathbb{E}T$ is the n-skeleton of $\mathbb{E}T$. Therefore $\mathbb{E}T \times_T X$ is constructed inductively via pushouts of the form

$$\begin{array}{ccc} S^n \times X & \hookrightarrow & D^n \times X \\ \downarrow & & \downarrow \\ Z_n & \hookrightarrow & Z_{n+1} \end{array}$$

where $Z_n \subset \mathbb{E}T \times_T X$ is the n-skeleton of the Borel construction. Now X itself by our convention has the homotopy type of a CW complex, so all maps are homotopic to cellular maps, see [Hat02]. Replacing all maps by these cellular maps, we get a CW complex which is homotopy equivalent to Z_{n+1}, see [Hat02].

□

Remark 1.2.3. In fact when the T-action on X is free, one can show that the Borel construction is homotopy equivalent to X/T. See [Aud12] for a proof when X is a manifold. See [Jan87, part I chapter 5] for a similar construction using schemes.

Definition 1.2.4. The *T-equivariant cohomology* of a T-space X with coefficient ring Λ is by definition the ring

$$H^i_T(X, \Lambda) = H^i(\mathbb{E}T \times_T X, \Lambda)$$

where $H^*(-, \Lambda)$ is the singular cohomology with coefficient ring Λ and multiplication given by the usual cup product. In what follows we abbreviate $H^*_T(-) = H^*_T(-, \mathbb{C})$.

Lemma 1.2.5. *A T-*equivariant *map $X \to X'$ induces an algebra homomorphism*

$$H^*(X', \Lambda) \longrightarrow H^*(X, \Lambda)$$

In this way equivariant cohomology becomes a contravariant functor from the category of T-spaces to that of abelian groups. Moreover, taking $X' = \mathrm{pt}$ yields a canonical ring homomorphism

$$H^*_T(\mathrm{pt}, \Lambda) \longrightarrow H^*_T(X, \Lambda)$$

thus all equivariant cohomology rings are $H^*_T(\mathrm{pt}, \Lambda)$-algebras.

Example 1.2.6. If X is a point with a $T = \mathbb{C}^*$-action, then $\mathbb{E}T \times_T X = \mathbb{E}T/T = \mathbb{B}T$, therefore

$$H^*_T(\mathrm{pt}) = H^*(\mathbb{B}T)$$

Recall Example 1.1.14, we have chosen $\mathbb{B}T = \mathbb{CP}^\infty$ and have an isomorphism

$$H^*_T(\mathrm{pt}) = \mathbb{C}[X]$$

$$c_1(\mathcal{O}(-1)) \longmapsto X$$

where $\mathbb{C}[X]$ is a polynomial ring with $\deg X = 2$. We use the Chern class as the generator so that there is no ambiguity in the generator on the right hand side.

Lemma 1.2.7. *If rank $T = N$, we have $H^*_T(\mathrm{pt}) \cong \mathrm{Sym}^\bullet \mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the linear space dual of the Lie algebra of T.*

Proof. By the Künneth formula and Example 1.2.6 we get

$$H^*_T(\mathrm{pt}) \cong \mathbb{C}[X_1, X_2, \dots X_N]$$

where each X_i can be taken to be the first Chern class of the pull back of $\mathcal{O}(-1)$ on $\mathbb{P}^\infty$ via the i-th projection $(\mathbb{P}^\infty)^N \to \mathbb{P}^\infty$. This however depends on the choice of a splitting $T \cong (\mathbb{C}^*)^N$. Our goal is thus a *natural* description of $H^*_T(\mathrm{pt})$.

We make use of the character group $X(T)$ of T. Take a character $\lambda : T \to \mathbb{C}^* = GL_1(\mathbb{C})$ and the corresponding one dimensional representation $\mathbb{C}_\lambda$, form the line bundle $L = \mathbb{E}T \times_T \mathbb{C}_\lambda \to \mathbb{B}T$, then take the first Chern class $c_1(L) \in H^2(\mathbb{B}T)$. In view of Whitney sum formula, this gives a linear map

$$\mathbb{C} \otimes_{\mathbb{Z}} X(T) \longrightarrow H^2(\mathbb{B}T)$$

We claim that this is bijective. Choose a splitting $T \cong (\mathbb{C}^*)^N$, the basis elements of $X(T)$ as in Lemma 1.1.4 are brought to the pullbacks of $c_1(\mathcal{O}(1))$ via the projections $\mathbb{B}T \cong (\mathbb{P}^\infty)^N \to \mathbb{P}^\infty$, which are known to form a basis of $H^2(\mathbb{B}T)$ from our non-natural description. Thus we have

$$\mathrm{Sym}^\bullet (\mathbb{C} \otimes_{\mathbb{Z}} X(T)) \longrightarrow H^*(\mathbb{B}T) = H^*_T(\mathrm{pt})$$

Here $\mathrm{Sym}^\bullet$ denotes the symmetric algebra over $\mathbb{C}$. Now we cite the fact that by taking the tangent map, $\mathbb{C} \otimes_{\mathbb{Z}} X(T)$ is naturally identified with $\mathbf{t}^*$, the dual of the Lie algebra of T. Then we obtain a canonical isomorphism

$$\mathrm{Sym}^\bullet \mathbf{t}^* \longrightarrow H^*(\mathbb{B}T) = H_T^*(\mathrm{pt})$$

In what follows we shall often use this description of $H_T^*(\mathrm{pt})$. $\square$

Example 1.2.8. Consider $\mathbb{P}^1$ with the standard action (Example 1.1.2), we can also determine its equivariant cohomology by hand. From $\{\infty\} \hookrightarrow \mathbb{P}^1$ we obtain a closed embedding

$$\mathbb{E}T \times_T \{\infty\} \longhookrightarrow \mathbb{E}T \times_T \mathbb{P}^1$$

Denote $E_0 = \mathbb{E}T \times_T \mathbb{P}^1 - \mathbb{E}T \times_T \{\infty\}$, consider the relative cohomology sequence of the pair $(\mathbb{E}T \times_T \mathbb{P}^1, E_0)$:

$$\dots \longrightarrow H^i(\mathbb{E}T \times_T \mathbb{P}^1) \longrightarrow H^i(E_0) \longrightarrow H^{i+1}(\mathbb{E}T \times_T \mathbb{P}^1, E_0) \longrightarrow \dots$$

Note that in some CW structure all these spaces have no cells of odd real dimensions, this sequence breaks down to the following short exact sequences:

$$0 \longrightarrow H^{2i}(\mathbb{E}T \times_T \mathbb{P}^1, E_0) \longrightarrow H^{2i}(\mathbb{E}T \times_T \mathbb{P}^1) \longrightarrow H^{2i}(E_0) \longrightarrow 0$$

The fibers of the line bundle $E_0 \to \mathbb{B}T$ can be contracted to the zero section, which makes the base $\mathbb{B}T$ a deformation retract of E_0. This gives $H^*(E_0) \cong H^*(\mathbb{B}T) = H_T^*(\mathrm{pt})$. In particular, the following short exact sequence of $H_T^*(\mathrm{pt})$-modules splits.

$$0 \longrightarrow H^*(\mathbb{E}T \times_T \mathbb{P}^1, E_0) \longrightarrow H^*(\mathbb{E}T \times_T \mathbb{P}^1) \longrightarrow H^*(E_0) \longrightarrow 0$$

On the other hand, by excision we have

$$H^{2i}(\mathbb{E}T \times_T \mathbb{P}^1, E_0) \cong H^{2i}(E_0, E_0 - \text{zero section})$$

then the Thom isomorphism (see [BT13] or [MSSU74]) applied to the line bundle $E_0 \to \mathbb{B}T$ yields

$$H^{2i+2}(E_0, E_0 - \text{zero section}) \cong H^{2i}(\mathbb{B}T) = H_T^{2i}(\mathrm{pt})$$

In other words $H^*(E_0, E_0 - \text{zero section}) \cong H_T^*(\mathbb{P}^1) \cong H_T^*(\mathrm{pt})\langle 2\rangle$ as a graded $H_T^*(\mathbb{P}^1) \cong H_T^*(\mathrm{pt})$-module, where the i-th graded piece of $H_T^*(\mathrm{pt})\langle 2\rangle$ is $H_T^{i-2}(\mathrm{pt})$. Thus we get $H_T^*(\mathbb{P}^1) \cong H_T^*(\mathrm{pt})\langle 2\rangle \oplus H_T^*(\mathrm{pt})$ as a graded module over $H_T^*(\mathrm{pt})$.

Remark 1.2.9. The above arguments requires some knowledge in algebraic topology. We will see that the main Theorem 1.3.7 in this chapter reduces everything to elementary combinatorics which also gives the ring structure of $H_T^*(\mathbb{P}^1)$.

The following properties follow immediately from the standard properties of singular cohomology.

Proposition 1.2.10. *Let X be a T-space.*

- *An* equivariant *closed embedding $Y \hookrightarrow X$ gives a long exact sequence*

$$\ldots \longrightarrow H_T^i(X) \longrightarrow H_T^i(Y) \longrightarrow H_T^{i+1}(X,Y) \longrightarrow \ldots$$

  where $H_T^*(X,Y) := H^*(\mathbb{E}T \times_T X, \mathbb{E}T \times_T Y)$.

- *(Mayer-Vietoris sequence) Let $U, V \hookrightarrow X$ be* equivariant *open embeddings such that $X = U \cup V$, then there is a long exact sequence*

$$\ldots \longrightarrow H_T^i(X) \longrightarrow H_T^i(U) \oplus H_T^i(V) \longrightarrow H_T^i(U \cap V) \longrightarrow \ldots$$

From examples 1.2.6, 1.2.7 we see that, in contrast to the non-equivariant case, points actually carry nontrivial information for the equivariant cohomology. In general one can define principal bundles for arbitrary topological groups and obtain the corresponding equivariant cohomology theory. Its value on a point contains information about the group, even though its action on a point is trivial. e.g. $H_{GL_n}^*(\mathrm{pt}) = (\mathrm{Sym}^\bullet \mathfrak{t}^*)^{S_n}$, the symmetric group invariants in $\mathrm{Sym}^\bullet \mathfrak{t}^*$, see [Bri98a]. In what follows we will see that often one can describe $H_T^*(X)$ in terms of the fixed point set.

The slogans

Given a T-space X, let X^T be the set of fixed points. Then the inclusion $\iota : X^T \to X$ is equivariant, hence induces a map

$$\iota^* : H_T^*(X) \longrightarrow H_T^*(X^T)$$

Our slogans are:

- Equivariant cohomology is determined by that of the fixed locus: in nice situations

  $$\iota^* : H_T^*(X) \longrightarrow H_T^*(X^T) \text{ is } \textit{injective.}$$

  and the image can be determined explicitly.

- Equivariant cohomology determines the usual one: under nice circumstances we want to have a canonical *surjective* map $H_T^*(X) \twoheadrightarrow H^*(X)$, whose kernel can be described explicitly.

Remark 1.2.11. Both slogans can miserably fail in general. Consider $T = X = \mathbb{C}^*$, acting on itself via left multiplication, then there are no fixed points, so the second statement obviously fails. On the other hand, $\mathbb{E}T \times_T T = \mathbb{E}T$ has vanishing cohomology, hence $H_T^*(T) = H^*(\mathbb{E}T) = 0$ whereas the singular cohomology $H^1(T) \cong \mathbb{C}$, consequently the first statement also fails.

1.2.2 Setting the stage

Next, we will formulate a suitable setting, and work exclusively under these reasonable assumptions thereafter.

Definition 1.2.12. A topological space with T-action satisfying the following condition (EF) is called *equivariantly formal.*

> $H_T^*(X)$ is a free $H_T^*(\mathrm{pt})$-module, and has a $H_T^*(\mathrm{pt})$-basis that restricts to a $\mathbb{Z}$-basis for $H^*(X)$.

To understand (EF), from topology (or for an algebraic geometry version [Jan87, §5.14]) we know that the Borel construction is a fibre bundle $\mathbb{E}T \times_T X \to \mathbb{B}T$ with fibre X, (EF) considers the restriction of cohomology to a fibre $H^*(\mathbb{E}T \times_T X) \to H^*(X)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{E}T \times_T X \arrow[d] \\ pt \arrow[r] & \mathbb{B}G \end{tikzcd}$$

Theorem 1.2.13 (Leray-Hirsch)**.** *Given a fibre bundle $E \to B$ with fibre F, assume that there are classes $\alpha_1, \dots \alpha_r \in H^*(E)$ whose restriction to each fibre form a $\mathbb{C}$-basis of $H^*(F)$, then α_i also form a $H^*(B)$-basis for $H^*(E)$. In particular, there is an isomorphism of $H^*(B)$-modules*

$$H^*(B) \otimes_{\mathbb{C}} H^*(F) \cong H^*(E)$$

So (EF) amounts to the setting of the Leray-Hirsch theorem. Although this assumption already gives the $\mathbb{Z}$-module structure of the equivariant cohomology, the ring structure is our major concern. The following lemma specifies a convenient class of equivariantly formal spaces.

Lemma 1.2.14. *Suppose X is a smooth projective variety with an algebraic action, then X is equivariantly formal.*

This comes in handy when we want to check equivariant formality. For a proof see [GKM98, Theorem 14.1 (6)].

From now on we consider a projective T-variety X satisfying the following conditions.

- Equivariantly formal, i.e. (EF) holds.
- There are finitely many fixed points, i.e. $|X^T| < \infty$.
- There are finitely many one dimensional orbits.

In section §1.4.3 we will see that this class of T-spaces yield the desired picture in our slogans.

Example 1.2.15. Consider the T action on $\mathbb{P}^1$ defined in Example 1.1.5. By the above lemma $\mathbb{P}^1$ is equivariantly formal: it is a smooth projective variety, and there are two fixed points $0 = [0:1], \infty = [1:0] \in \mathbb{P}^1$. This also checks the second assumption above. Finally $\mathbb{P}^1 - \{0, \infty\}$ is the only one dimensional orbit, because the whole space is one dimensional. This turns out to be the basic building block of the theory for T-varieties satisfying the above conditions.

Proposition 1.2.16. *Let X be an equivariantly formal projective T-variety, with finitely many fixed points and one dimensional orbits. Suppose the tangent weights at each fixed point are linearly independent. Then*

- *The closure of each one dimensional orbit of T in X is isomorphic to $\mathbb{P}^1$.*
- *There are exactly two fixed points in the closure of each one dimensional orbit.*
- *The torus acts on each one dimensional orbit via a character.*

Proof. Take any one dimensional orbit O, and $x \in O$. Then O is the image of T under $\cdot x : T \times \{x\} \to X$ sending $(t, x) \mapsto t \cdot x$, hence is isomorphic to T/T_x where $T_x \subset T$ denotes the stabilizer of x.

Claim 1. O is isomorphic to $\mathbb{C}^*$ as a variety.

Consider first the case rank $T = 1$: the only possible one dimensional quotient of $T = \mathbb{C}^*$ by a closed subgroup is $\mathbb{C}^*$, because its proper closed subgroups are zero dimensional, hence finite, hence cyclic groups generated by a root of unity (see [Mor96]).

Now take a general torus, we do induction on the rank. Take a splitting $T \cong (\mathbb{C}^*)^N$, and consider the first $N-1$ factors $T' = (\mathbb{C}^*)^{N-1} \subset T$. The closure of its image in O under the map $\cdot x$ is either 1 dimensional or 0 dimensional, and in the former case the claim follows from induction hypothesis. In the latter case, since T' is connected, so is its image, which implies $T'x = \{x\}$, and $T' \subset T_x$. Then O is a quotient of $T/T' \cong \mathbb{C}^*$, hence isomorphic to $\mathbb{C}^*$.

Claim 2. There is a character χ of T which defines a T-action on $\mathbb{C}^*$, such that the above isomorphism $O \cong \mathbb{C}^*$ is T-equivariant.

Observe that $T/T_x \cong \mathbb{C}^*$ as algebraic groups, because T is abelian. The above action map $\cdot x : T \to O$ then defines a character $\chi : T \to T/T_x \cong \mathbb{C}^*$. Then T acts on $\mathbb{C}^*$ via χ, and the above isomorphism $O \cong \mathbb{C}^*$ sends $t \cdot x \mapsto \chi(t)$, hence is T-equivariant.

Claim 3. The closure $\overline{O}$ of O in X is isomorphic to $\mathbb{P}^1$.

Now since X is proper, the inclusion $\mathbb{C}^* \cong O \hookrightarrow X$ extends to a map $\mathbb{P}^1 \to X$, whose image is closed, and $\overline{O} - O$ consists of at most 2 fixed points. This map is either injective, in which case $\overline{O} \cong \mathbb{P}^1$ as desired, or sends two poles of $\mathbb{P}^1$ to the same point. In the latter case, suppose T acts on $\mathbb{P}^1$ via the character χ, then the tangent weights at the two poles of $\mathbb{P}^1$ are $\chi, -\chi$ resp. Since the poles are mapped to the same fixed point $p \in X$, this pair of linearly dependent weights occur as tangent weights of p, a contradiction. $\square$

Remark 1.2.17. The condition on the tangent weights in Proposition 1.2.16 is in fact superfluous. Using the nontrivial theorem of Sumihiro-Kambayashi, there is an equivariant embedding $X \hookrightarrow \mathbb{P}(V)$ for some finite dimensional representation V, see [CG10, Theorem 5.1.25] . Then the argument of [Mil17, online version, Theorem 14.47] proves that the one dimensional orbits are isomorphic to $\mathbb{P}^1$. Our treatment is more elementary and it is easy to check this condition for our main examples.

Example 1.2.18. Proposition 1.2.16 says that under nice circumstances, if we consider the restriction of the T-action to any one dimensional orbit, we always have the following setting: $\mathbb{P}^1$ with T-action via a character χ as in Example 1.1.5. Let us examine this basic example first.

The fixed points and the unique one dimensional orbit of this T-action on $\mathbb{P}^1$ can be presented by a graph

$$[1:0] \;\text{———}\; [0:1]$$

One might want to label the line with χ to indicate the T-action. However, to be consistent with Example 1.2.7, we take the tangent map of χ at the identity element $D\chi|_1 \in \mathfrak{t}^*$.

We make the following important observation: consider the tangent space of a fixed point $0 \in \mathbb{P}^1$, which is a representation of T via

$$t \cdot v = D(\chi(t)\cdot)|_0 v \in T_0\mathbb{P}^1 \qquad \forall t \in T,\ v \in T_0\mathbb{P}^1$$

where $\chi(t)\cdot : \mathbb{P}^1 \to \mathbb{P}^1$ denotes multiplication by $\chi(t)$. Explicitly, take a standard coordinate chart $z : \mathbb{C} = \mathbb{P}^1 - \{\infty\} \to \mathbb{C}$

$$D(\chi(t)\cdot)|_0 (\partial_z) = \chi(t)\partial_z$$

Thus we can read off the torus character defining the action on $\mathbb{P}^1$ from this representation of T.

Notation. From now on we denote by $\epsilon_i = dt_i$ $(1 \leq i \leq N)$ the i-th standard cotangent vector of $(\mathbb{C}^*)^N$ at the origin, where $(t_1, \ldots, t_N)$ are the standard coordinates on the torus. So this will give a (non-canonical) basis of the dual of the Lie algebra $\mathfrak{t}^*$, as well as $H_T^2(\mathrm{pt})$. Denote by $S^\vee$ the dual of a vector bundle S.

Example 1.2.19. Consider the $T = (\mathbb{C}^*)^2$ action on $\mathbb{P}^2$ given by the following T-action on $\mathbb{C}^3$:

$$(t_1, t_2)(x_0, x_1, x_2) = (x_0, t_1 x_1, t_2 x_2)$$

We can find out all the fixed points and one dimensional orbits by hand. The fixed points are exactly the lines in $\mathbb{C}^3$ invariant under the T action, which are spanned by common eigenvectors of T. In the standard basis the matrices of elements in T are already simultaneously diagonal, so the fixed points are

$$p_1 = [1, 0, 0], \qquad p_2 = [0, 1, 0], \qquad p_3 = [0, 0, 1]$$

Similarly, the one dimensional orbits are 2 dimensional subspaces in $\mathbb{C}^3$ that are stable under T. Given such a 2 dimensional subspace W, we restrict the T-action to W and again find linearly independent common eigenvectors of T, which are also common eigenvectors of T in $\mathbb{C}^3$, hence are simply two of the standard basis elements. Consequently the one dimensional orbits are

$$O_1 = \{[0 : * : *]\}, \qquad O_2 = \{[* : 0 : *]\} \qquad O_3 = \{[* : * : 0]\}$$

where $*$ denotes arbitrary nonzero complex numbers. Then in view of Example 1.1.5, the T-action here on O_2 is given by the character $(t_1, t_2) \mapsto t_2$ if we set $\infty = [1 : 0 : 0]$.

Now we also want to represent these data by a graph. We take the vertices to be fixed points and edges the 1-dimensional orbits connecting two fixed points. Then we want to label the edges that tell tales about the torus action.

$$\begin{tikzcd} & p_3 \arrow[dl, dash, "\epsilon_2"'] \arrow[dr, dash, "\epsilon_1 - \epsilon_2"] & \\ p_1 \arrow[rr, dash, "\epsilon_1"'] & & p_2 \end{tikzcd}$$

In view of the above Example 1.2.18, one might want to calculate the torus action on the orbits and differentiate to get the labels. There is a more convenient way: consider the tangent spaces at the fixed points. If O is a 1-dimensional orbit, then we have an equivariant inclusion

$$T_p\overline{O} \hookrightarrow T_p\mathbb{P}^2$$

In other words, O gives a subrepresentation of $T_p\mathbb{P}^2$. Any finite dimensional T-representation splits into a direct sum of 1-dimensional representations, which are called *weight spaces*, and the character through which T acts on each 1-dimensional representation is called the *weight*. In view of Example 1.2.7 we also view the weight as a cotangent vector in $\mathfrak{t}^*$. This is exactly the tangent map we described in Example 1.2.18.

Next we have to compute these weights. Recall the standard description of the tangent bundle of $\mathbb{P}^n$: the tautological bundle $\mathcal{O}(-1)$ is by definition a subbundle of the trivial bundle $E = \mathbb{P}^n \times \mathbb{C}^{n+1}$, and the tangent bundle is $\mathcal{O}(1)^\vee \otimes E/\mathcal{O}(-1)$. We postpone the proof that this identification is compatible with the torus actions to Lemma 2.1.4. In particular, at the point corresponding to a line $L \subset \mathbb{C}^3$ the tangent space is naturally identified with the quotient space $L^\vee \otimes \mathbb{C}^3/L$ as a representation of T. Now let e_1, e_2, e_3 be the standard basis of $\mathbb{C}^3$ as a T-representation and e^1, e^2, e^3 the dual basis of $(\mathbb{C}^3)^*$. Denote by $\bar{e}_i$ the image of e_i in the quotient space $\mathbb{C}^3/L$. Combining the above facts, the tangent space at p_1 is spanned by

$$e^1 \otimes \bar{e}_2 \qquad e^1 \otimes \bar{e}_3 \in (\mathbb{C}e_1)^\vee \otimes \mathbb{C}^3/\mathbb{C}e_1$$

The corresponding torus characters are projections to the first resp. the second factor of $T = (C^*)^2$. Thus $\epsilon_1, \epsilon_2 \in \mathfrak{t}^*$ label the edges corresponding to O_3, O_2 resp. Similarly, for p_2 the corresponding $\mathfrak{t}^*$ elements are $-\epsilon_1, \epsilon_2 - \epsilon_1$. From this we see that there is a sign ambiguity in the labeling of the edges. This does not harm for our purposes.

1.3 The GKM theorem

1.3.1 GKM conditions

In this subsection we extract combinatorial information from a T-variety. In the above examples we have seen that for certain T-varieties there is a graph encoding some information about the T-action. We make the following definition.

Definition 1.3.1. Under the setting of Proposition 1.2.16, consider a graph defined as follows:

- the vertices are the fixed points;
- the edges correspond to 1-dimensional orbits;
- two vertices are connected by an edge iff the corresponding one dimensional orbit closure contains the two corresponding fixed points;
- let p be a fixed point in a one dimensional orbit closure O. The corresponding edge by the tangent weight of $T_p\overline{O}$, up to a sign.

This graph is called the *GKM graph* (Goresky-Kottwitz-MacPherson).

Next, since there are finitely many fixed points in the T-variety X, then $H_T^*(X^T)$ is a direct sum of $|X^T|$ copies of $H_T^*(\mathrm{pt}) \cong \mathrm{Sym}^\bullet \mathfrak{t}^*$, with product ring structure. This can also be identified with the set $\mathrm{Maps}(X^T, \mathrm{Sym}^\bullet \mathfrak{t}^*)$ of maps of sets $X^T \to \mathrm{Sym}^\bullet \mathfrak{t}^*$ with the obvious ring structure.

Definition 1.3.2. Given the GKM graph for a T-variety X, for any one dimensional orbit O denote by $\alpha(O) \in \mathfrak{t}^*$ the label of the edge in the GKM graph corresponding to the orbit O. Consider the condition on $\mathrm{Maps}(X^T, \mathrm{Sym}^\bullet \mathfrak{t}^*)$

$$f(p) \equiv f(q) \mod \alpha(O) \qquad \forall\, p, q \in X^T \cap \overline{O} \tag{GKM}$$

A map in $\mathrm{Maps}(X^T, \mathrm{Sym}^\bullet \mathfrak{t}^*)$ satisfying (GKM) is called a *GKM class*. Observe that the set of GKM classes is a $\mathrm{Sym}^\bullet \mathfrak{t}^*$-submodule of $\mathrm{Maps}(X^T, \mathrm{Sym}^\bullet \mathfrak{t}^*)$.

Remark 1.3.3 (Here be dragons). We do not want to get into whether or not there are several edges between two given fixed points in general, but rather contend ourselves with the fact that this does not happen in all examples interesting to us.

1.3.2 Examples of GKM classes

According to our slogan, the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective. Then we want to determine its image. On the other hand, we have the submodule of GKM classes inside $H_T^*(X^T)$. Is there a good reason to expect a connection between the geometric object $H_T^*(X)$ and the combinatorial GKM classes? Let us examine the examples at hand.

Example 1.3.4. Consider the standard action of $T = (\mathbb{C}^*)^2$ on $\mathbb{P}^1$. On the local coordinate chart near $[0, 1]$ this is

$$(a, b) \cdot z = ab^{-1}z$$

hence T acts via the character $(a, b) \mapsto ab^{-1}$. The corresponding Lie algebra element is $\epsilon_1 - \epsilon_2 \in \mathfrak{t}^*$. The GKM classes are

$$H = \{(f, g) \mid f - g \equiv 0 \mod \epsilon_1 - \epsilon_2\} \subset \mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*$$

In fact H is a rank 2 free module over $\mathrm{Sym}^\bullet \mathfrak{t}^*$. To see this, take any $(f, g) \in H$, write

$$(f, g) = (0, g - f) + (f, f) = (0, g - f) + f(1, 1)$$

This makes sense because $(1, 1) \in H$ automatically. Since $g - f$ is divisible by $\epsilon_1 - \epsilon_2$, this proves that $(1, 1)$, $(0, \epsilon_1 - \epsilon_2)$ span H. They are linearly independent over $\mathrm{Sym}^\bullet \mathfrak{t}^*$ because looking at the first component, $r(1, 1)+s(0, \epsilon_1-\epsilon_2) = 0$ implies that $r = 0$, and then s also has to be 0. This proves that $H \cong \mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*\langle 2\rangle$ as a graded $\mathrm{Sym}^\bullet \mathfrak{t}^*$-module.

The careful reader would recall from Example 1.2.8, that $H_T^*(\mathbb{P}^1)$ in this case is also isomorphic to $\mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*\langle 2\rangle$ as a graded $\mathrm{Sym}^\bullet \mathfrak{t}^*$-module. We strongly suspect that they coincide, and Theorem 1.3.7 below confirms this.

We also make the following observation, parallel to our slogan: there is a short exact sequence of $\mathrm{Sym}^\bullet \mathfrak{t}^*$-modules

$$0 \longrightarrow H \longrightarrow \mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^* \longrightarrow \mathrm{Sym}^\bullet \mathfrak{t}^*/(\epsilon_1 - \epsilon_2) \longrightarrow 0$$

Notice that the cokernel is torsion, and after localization (in the sense of basic commutative algebra) with respect to the multiplicative set $\{(\epsilon_1-\epsilon_2)^n\} \subset \mathrm{Sym}^\bullet \mathfrak{t}^*$, this term vanishes and the inclusion $H \hookrightarrow \mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*$ becomes an isomorphism.

Example 1.3.5. More generally, let T act on $\mathbb{P}^1$ via a character χ (see Example 1.1.5). The GKM classes are

$$\{(f, g) \mid f - g \equiv 0 \mod \chi\}$$

where by abuse of notation we also denote by $\chi \in \mathfrak{t}^*$ the Lie algebra element corresponding to the character.

Example 1.3.6. Consider the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2$ from Example 1.2.19. We determine the GKM classes in $(\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 3}$. First of all we have multiples $f(1,1,1)$ of $(1,1,1)$. To obtain others we write a GKM class (f,g,h) as

$$(f,g,h) = f(1,1,1) + (0, \epsilon_1 g', \epsilon_2 h'), \qquad g'\epsilon_1 = g - f,\ h'\epsilon_2 = h - f,$$

By the same spirit, one might want to write $(0,\epsilon_1 g', \epsilon_2 h') = g'\epsilon_1(0,1,1) + (0,0,\epsilon_2 h' - \epsilon_1 g')$. At a second thought, this does not work because $(0,1,1)$ is not a GKM class. If we want to put a zero at the first component, we have to take a multiple of ϵ_1 at the second component. Let us try the easiest guess: take $(0,\epsilon_1,\epsilon_2)$. Write

$$(0,\epsilon_1 g', \epsilon_2 h') = g'(0,\epsilon_1,\epsilon_2) + (0,0,\epsilon_2(h'-g'))$$

At this point we are not comfortable with the second summand, which is not a priori a GKM class if we pick random g', h'. On the other hand, we do know that it IS a GKM class because all other terms in this equation are GKM classes, and the set of GKM classes is a submodule of $(\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 3}$. What makes it work? It is the GKM condition on g, h in the original tuple (f,g,h), which implies that $g'\epsilon_1 - h'\epsilon_2$ is divisible by $(\epsilon_1 - \epsilon_2)$. Then

$$(g'-h')\epsilon_2 \equiv g'(\epsilon_2 - \epsilon_1) + g'\epsilon_1 - h'\epsilon_2 \equiv 0 \mod (\epsilon_2 - \epsilon_1)$$

therefore $h' - g' = h''(\epsilon_1 - \epsilon_2)$, because ϵ_2 and $\epsilon_1 - \epsilon_2$ are coprime in the polynomial ring $\mathrm{Sym}^\bullet \mathfrak{t}^*$. As a result, any GKM class is a $\mathrm{Sym}^\bullet \mathfrak{t}^*$-linear combination of the form

$$(f,g,h) = f(1,1,1) + g'(0,\epsilon_1,\epsilon_2) + h''(0,0,\epsilon_2(\epsilon_1 - \epsilon_2))$$

where f, g', h'' can take any value in $\mathrm{Sym}^\bullet \mathfrak{t}^*$. Now $(0,0,\epsilon_2(\epsilon_1-\epsilon_2))$ is indeed a GKM class. Again the set of GKM classes is a free $\mathrm{Sym}^\bullet \mathfrak{t}^*$-module of rank 3 (the number of fixed points), compare with Leray-Hirsch theorem.

From the above example we observe that in general the combinatorics in the computations can be rather complicated. What we used here implicitly to make thing easier is a partial ordering on the set of vertices. This will be crucial when we work with Grassmannians, which are the most interesting examples for us.

1.3.3 The GKM theorem

Here we formulate the main theorems in this chapter, which will be established later. Recall the equivariant closed embedding $X^T \to X$, which induces the restriction map

$$H_T^*(X) \longrightarrow H_T^*(X^T).$$

Theorem 1.3.7 (Goresky-Kottwitz-MacPherson)**.** *Let X be an equivariantly formal T-variety, with finitely many fixed points and finitely many one dimensional orbits. Then the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective, with image exactly the GKM classes.*

Example 1.3.8. Consider the standard rank 2 torus action on $\mathbb{P}^1$, the theorem says that the restriction map

$$H_T^*(\mathbb{P}^1) \hookrightarrow \mathrm{Sym}^\bullet \mathfrak{t}^* \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*$$

is injective with image $\mathrm{Sym}^\bullet \mathfrak{t}^*(1,1) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, \epsilon_1 - \epsilon_2)$, by Example 1.3.4. This gives an isomorphism of graded algebras $H_T^*(\mathbb{P}^1) \cong \mathrm{Sym}^\bullet \mathfrak{t}^*(1,1) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, \epsilon_1 - \epsilon_2)$, which is exactly what we expected from the direct computations. Furthermore, the ring structure is give by

$$(0, \epsilon_1 - \epsilon_2)^2 = (0, (\epsilon_1 - \epsilon_2)^2) = (\epsilon_1 - \epsilon_2) \cdot (0, \epsilon_1 - \epsilon_2) \in \mathrm{Sym}^\bullet \mathfrak{t}^*(1,1) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, \epsilon_1 - \epsilon_2)$$

Now with the help of Theorem 1.3.7 we can also deal with $\mathbb{P}^2$. Consider the T-action on $\mathbb{P}^2$ as in Examples 1.2.19, 1.3.6. The image of the restriction map $H_T^*(\mathbb{P}^2) \to (\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 3}$ is

$$\mathrm{Sym}^\bullet \mathfrak{t}^*(1,1,1) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, \epsilon_1, \epsilon_2) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, 0, \epsilon_2(\epsilon_1 - \epsilon_2)) \subset (\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 3}$$

This again describes $H_T^*(\mathbb{P}^2)$ as a graded $\mathbb{C}$-algebra, where for example

$$(0, \epsilon_1, \epsilon_2)^2 = \epsilon_1 \cdot (0, \epsilon_1, \epsilon_2) - (0, 0, \epsilon_2(\epsilon_1 - \epsilon_2))$$

Note that the computation of H_T^* via GKM theorem is totally combinatorial, and gives more than Leray-Hirsch theorem by also describing the ring structure.

1.4 The slogans revisited

In this section we introduce some convenient geometric gadgets: equivariant Chern classes and Gysin pushforward, then we establish the slogans. The GKM theorem is not necessary here, however, we introduced the GKM theorem before this section because it is recommended to keep in mind these examples before entering the somewhat dazzling world of geometry.

1.4.1 Equivariant Chern classes

There is also an equivariant version of vector bundles for general nice algebraic groups or Lie groups. We are only concerned with closed subgroups of $GL_N(\mathbb{C})$.

Definition 1.4.1. Let H be a closed subgroup of $GL_N(\mathbb{C})$, an H-equivariant vector bundle is a vector bundle $E \to B$ such that E carries an H-action that preserves the fibres.

Observe that for an H-equivariant vector bundle $E \to B$ of rank r, there is an induced map

$$\mathbb{E}T \times_T E \longrightarrow \mathbb{E}T \times_T B$$

which is also a vector bundle of rank r. Then we can take its Chern classes.

Definition 1.4.2. Let $E \to B$ be a T-equivariant vector bundle. Its equivariant Chern classes is by definition

$$c_i^T := c_i\left(\mathbb{E}T \times_T E \to \mathbb{E}T \times_T B\right) \in H_T^{2i}(B)$$

From the Axioms of ordinary Chern class we deduce the following properties:

Lemma 1.4.3. *Let $\xi = (E \to B)$ be a T-equivariant vector bundle.*

- $c_0^T(E) = 1 \in H^0(B)$ *and* $c_i^T(E) = 0$ *for* $i > \operatorname{rank} E$.
- *(Naturality). For any T-equivariant vector bundle $\xi' = (E' \to B')$, and any T-equivariant bundle map $f : \xi' \to \xi$,*

$$f^* c_i^T(\xi) = c_i^T(f^*\xi)$$

- *(Whitney sum formula). Given a short exact sequence*

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

of T-equivariant vector bundles over B and T-equivariant bundle maps

$$c^T(\xi) = c^T(\xi')c^T(\xi'')$$

where $c^T(\xi) = 1 + c_1^T(\xi) + c_2^T(\xi) + \ldots = \sum_{0 \le 2i \le \operatorname{rank}\xi} c_i^T(\xi) \in H_T^*(B)$ is the total equivariant Chern class.

Example 1.4.4. For later applications, the only case we will need is when B is a point. In this case an equivariant vector bundle $E \to B$ is just a representation of T. Since T is abelian, this representation splits as a direct sum of one dimensional representations given by characters $\chi_i : T \to \mathbb{C}^* = GL_1(\mathbb{C})$. Then these weights χ_i can be used to represent the first equivariant Chern classes of the corresponding one dimensional representations. To be more precise, recall Example 1.2.7, taking equivariant Chern classes defines an abelian group isomorphism

$$\mathrm{ch} : X(T) \longrightarrow H^2(\mathbb{B}T, \mathbb{Z})$$

So the image of the weight χ_i in $H^2(\mathbb{B}T, \mathbb{Z})$ are the first equivariant Chern class of the corresponding one dimensional representation. Under the identification $X(T) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^2(\mathbb{B}T, \mathbb{C}) \cong \mathfrak{t}^*$ we will take the corresponding element in $\mathfrak{t}^*$ to represent the equivariant Chern class. The Whitney sum formula gives the total equivariant Chern class of E as

$$1 + c_1^T(E) + c_2^T(E) + \cdots = \prod (1 + \operatorname{ch} \chi_i)$$

so $c_j^T(E)$ is the j-th elementary symmetric polynomial in all $\operatorname{ch} \chi_i$.

Example 1.4.5. Consider the standard T-action on $\mathbb{P}^1$ in Example 1.1.2. The restrictions of $c_1^T(T^*\mathbb{P}^1)$ to the fixed points can be calculated as follows: let p be a fixed point, consider the following pullback diagram viewed as an equivariant map of equivariant line bundles.

$$\begin{tikzcd} T_p^*\mathbb{P}^1 \arrow[r] \arrow[d] & T^*\mathbb{P}^1 \arrow[d] \\ p \arrow[r, "\iota_p"] & \mathbb{P}^1 \end{tikzcd}$$

By naturality $\iota_p^* c_1^T(\mathbb{P}^1) = c_1^T(T_p^*\mathbb{P}^1)$, which is just the weight of the cotangent space at p. Let $0 = [0:1], \infty = [1:0]$, we have

$$c_1^T(T_0^*\mathbb{P}^1) = \epsilon_2 - \epsilon_1, \qquad c_1^T(T_\infty^*\mathbb{P}^1) = -\epsilon_2 + \epsilon_1 \tag{1.1}$$

where we identified $H^2(\mathbb{B}T) \cong \mathfrak{t}^*$ as usual and ϵ_i are the standard basis elements.

Remark 1.4.6. The labels of the edges in a GKM graph are in fact equivariant Chern classes. To see this, take a one dimensional orbit, it suffices to discuss the $\mathbb{P}^1$-case. Notice that the tangent bundle $T\mathbb{P}^1$ is an equivariant line bundle. Let p be a fixed point, the fibre diagram

$$\begin{tikzcd} T_p\mathbb{P}^1 \arrow[r, hook] \arrow[d] & T\mathbb{P}^1 \arrow[d] \\ p \arrow[r, hook, "\iota"] & \mathbb{P}^1 \end{tikzcd}$$

is a map of equivariant vector bundles, therefore we can form the pullback

$$\iota^* c_1^T(T\mathbb{P}^1) = c_1^T(T_p\mathbb{P}^1) \in \operatorname{Sym}^\bullet \mathfrak{t}^*$$

This is exactly the label on the edge corresponding to this orbit, up to a sign.

Remark 1.4.7. Using the equivariant Chern classes, in the setting of example 1.4.5 one can show that

$$H_T^*(\mathrm{pt})[\zeta]/\left((\zeta - c_1^T(T_0\mathbb{P}^1))(\zeta - c_1^T(T_\infty\mathbb{P}^1))\right) \cong H_T^*(\mathbb{P}^1)$$

where the isomorphism is given by $\zeta \mapsto c_1^T\mathcal{O}(-1)$, the equivariant Chern class of the tautological line bundle on $\mathbb{P}^1$.

1.4.2 Gysin maps

The next geometric ingredient is a push forward map. To give some motivation, consider a fibre bundle $f : X \to Y$ where both the fibre F and the base Y are compact smooth manifolds. One can define a push-forward map of de Rham cohomology $f_* : H^*_{dR}(X) \to H^*_{dR}(Y)$ via the formula

$$\alpha \longmapsto \int_F \alpha \in H^*_{dR}(Y)$$

see [BT13] for its interaction with characteristic classes. When Y is a point this is just the usual integration of differential forms.

With more work one can construct a Gysin pushforward for the equivariant cohomology for proper maps. see [Bot99]. We are only concerned with two cases:

1. Equivariant fundamental classes. Let $\iota : Y \to X$ be a T equivariant codimension d closed embedding, where Y is irreducible. Then Y has an *equivariant fundamental class* in X

$$[Y]^T \in H_T^{2d}(X)$$

see [Bri98b]. Furthermore, there is a push forward map

$$\iota_* : H_T^*(Y) \longrightarrow H_T^{*+2d}(X)$$

satisfying $\iota_*(1) = \iota_*[Y]^T = [Y]^T \in H_T^{2d}(X)$. We are only using this to define the equivariant Schubert classes for Grassmannians, and then work with the combinatorics thereafter.

Example 1.4.8. For $\mathbb{P}^1$ with the standard action of the rank 2 torus, only the two fixed points $0, \infty$ and $\mathbb{P}^1$ itself give equivariant fundamental classes. We will see that the classes of the two fixed points are also not equal.

2. Integrals. When X is compact and nonsingular of dimension n, the pushforward along the constant map $X \to \mathrm{pt}$ gives an integration map

$$\int_X : H_T^*(X) \longrightarrow H_T^{*-2n}(\mathrm{pt})$$

Next, we turn to the interaction of Gysin maps with equivariant Chern classes. Given an n-dimensional T-variety X and the inclusion of a fixed point $\iota : p \to X$, recall that we want to consider the restriction map $H_T^*(X) \to H_T^*(p)$. The following proposition is basically due to one of the constructions of Chern classes.

Proposition 1.4.9 (Self-intersection formula)**.** *Given an n-dimensional smooth T-variety X and the inclusion $\iota : p \hookrightarrow X$ of a fixed point to X, then*

$$\iota^* \iota_* \alpha = c_n^T(T_p X) \cdot \alpha.$$

More generally, if $Y \subset X$ is any closed subvariety (not necessarily smooth) and $p \in Y$ is a smooth fixed point, then

$$\iota^* [Y]^T = c_{\text{top}}^T \left(N_p Y \right),$$

where $N_p Y$ denotes the normal space to Y at p.

1.4.3 The localization theorems and the slogans

Let X be a T-variety. For each fixed point $p \in X^T$ denote the inclusion by

$$\iota_p : p \longrightarrow X$$

Theorem 1.4.10 (Algebraic localization theorem)**.** *Let X be an n-dimensional smooth projective T-variety with finitely many 1-dimensional orbits and finitely many fixed points. Let $S \subset H_T^*(\mathrm{pt})$ be a multiplicative set containing the element*

$$c := \prod_{p \in X^T} c_n^T \left(T_p X \right)$$

Then the localization of the restriction map with respect to S is an isomorphism

$$S^{-1}\iota : S^{-1} H_T^*(X) \cong S^{-1} H_T^*(X^T)$$

Proof. We first prove that $S^{-1}\iota$ is surjective. Consider also the localization of the pullback map $S^{-1}\iota^* : S^{-1}H_T^*(X) \to S^{-1}H_T^*(X^T)$, we take the composition

$$S^{-1}\iota^* \circ S^{-1}\iota_* = S^{-1} \left(\iota^* \iota_* \right) : S^{-1} H_T^*(X^T) \to S^{-1} H_T^*(X^T)$$

Note that $S^{-1}H_T^*(X^T)$ is a free $H_T^*(\mathrm{pt})$-module of rank $|X^T|$. By the self intersection formula 1.4.9, for each $p \in X^T$ the map $S^{-1}\iota^* S^{-1}\iota_*$ sends the basis element 1_p corresponding to the identity in $S^{-1}H_T^*(p)$ to $c_n^T(T_pX)1_p$, and since we allow the inverse of $c_n^T(T_pX)$, the collection $\{c_n^T(T_pX)1_p \mid p \in X^T\}$ again form a basis of $S^{-1}H_T^*(X^T)$. Conclusion: $S^{-1}\iota^* \circ S^{-1}\iota_*$ is surjective, hence so is $S^{-1}\iota^*$.

To show that $S^{-1}\iota^*$ is injective, we use the fact that $S^{-1}H^*_T(\mathrm{pt})$ is a noetherian ring. By equivariant formality $H^*_T(X)$ is a free module, and by the Białynicki-Birula decomposition (see [ByB73]), of rank at most $|X^T|$. Hence the same is true for $S^{-1}H^*_T(X)$. By surjectivity of $S^{-1}\iota^*$, the rank has to be equal to $|X^T|$ (tensor with the fraction field to see this). Then the conclusion follows from the next Lemma 1.4.11. $\square$

Lemma 1.4.11. *Let $f : M \to N$ be a surjective map of finite free modules of the same rank over a noetherian ring A. Then f is an isomorphism.*

Proof. Take any prime ideal $\mathfrak{p} \subset A$. It suffices to show that $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective. Let $K = \ker f$, we have the following short exact sequence

$$0 \longrightarrow K \otimes k(\mathfrak{p}) \longrightarrow M \otimes k(\mathfrak{p}) \longrightarrow N \otimes k(\mathfrak{p}) \longrightarrow 0$$

where $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. But $M \otimes k(\mathfrak{p}), N \otimes k(\mathfrak{p})$ are vector spaces of the same (finite) dimension, so we have $K_\mathfrak{p} \otimes k(\mathfrak{p}) = K \otimes k(\mathfrak{p}) = 0$, or $K_\mathfrak{p} = \mathfrak{p}K_\mathfrak{p}$. By Nakayama's lemma this implies $K_\mathfrak{p} = 0$, and we have $\ker f_\mathfrak{p} = K_\mathfrak{p}$. $\square$

Corollary 1.4.12. *Let X be an n dimensional smooth projective T-variety with finitely many one dimensional orbits and finitely many fixed points. Then the restriction map*

$$\iota^* H^*_T(X) \longrightarrow H^*_T\left(X^T\right)$$

is injective.

Proof. Let $m \in \ker \iota^*$, then $m \in S^{-1}\iota^*$ for S as in Theorem 1.4.10. This means that $m = 0$ in $S^{-1}H^*_T(X)$. On the other hand, $H^*_T(X)$ is a free module over $H^*_T(\mathrm{pt})$, which is a polynomial ring by Example 1.2.7. Therefore, $H^*_T(\mathrm{pt})$ has no torsion and $m = 0$. $\square$

At this point we have established the first slogan: under our assumptions the localization map $\iota^* : H^*_T(X) \to H^*_T(X^T)$ is injective. The image is described by the GKM theorem, which we prove later.

The second slogan also follows immediately. View $\mathbb{E}T \times_T X$ as a fibre bundle over $\mathbb{B}T$ with fibre X, restriction to the fibre gives a natural map

$$H^*_T(X) \longrightarrow H^*(X)$$

Then equivariant formality says that this is surjective. In particular, when X is a point, we get a short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow H^*_T(\mathrm{pt}) \longrightarrow H^*(\mathrm{pt}) \longrightarrow 0$$

Lemma 1.4.13. *Let X be an n dimensional smooth projective T-variety with finitely many one dimensional orbits and finitely many fixed points. The kernel of the natural map $H_T^*(X) \to H^*(X)$ is $\mathfrak{m}H_T^*(X)$.*

Proof. By Leray-Hirsch theorem, $H_T^*(X) \cong H_T^*(\mathrm{pt}) \otimes_{\mathbb{C}} H^*(X)$ as a $H_T^*(\mathrm{pt})$ module, where the $H_T^*(\mathrm{pt}) = H^*(\mathbb{B}T)$ factor comes from the base of the fibre bundle $\mathbb{E}T \times_T X \to \mathbb{B}T$ and $H^*(X)$ comes from the fibre, therefore restriction to the fibre is given by the map $H_T^*(\mathrm{pt}) \otimes_{\mathbb{C}} H^*(X) \longrightarrow (H_T^*(\mathrm{pt})/\mathfrak{m}H_T^*(\mathrm{pt})) \otimes_{\mathbb{C}} H^*(X)$. □

Theorem 1.4.14. *Let X be an n-dimensional smooth projective T-variety with finitely many fixed points, then for any $\alpha \in H_T^* X$,*

$$\alpha = \sum_{p \in X^T} \frac{(\iota_p)_* \iota_p^* \alpha}{c_n^T(T_p X)}$$

in a localization of $H_T^*(X)$ such that all the Chern classes are invertible.

Proof. By Theorem 1.4.10 it suffices to check that

$$\iota_q^* \alpha = \iota_q^* \sum_{p \in X^T} \frac{(\iota_p)_* \iota_p^* \alpha}{c_n^T(T_p X)}$$

for all fixed point q. By self intersection formula 1.4.9

$$\iota_q^* \sum_{p \in X^T} \frac{(\iota_p)_* \iota_p^* \alpha}{c_n^T(T_p X)} = \iota_q \frac{(\iota_q)_* \iota_q^* \alpha}{c_n^T(T_p X)} = c_n^T(T_p X) \frac{\iota_q^* \alpha}{c_n^T(T_p X)} = \iota_q^* \alpha$$

□

Example 1.4.15. Consider $\mathbb{P}^1$ with standard T action, we have calculated the cotangent weights of the fixed points in Example 1.4.5. The tangent spaces are the dual representations of the cotangent spaces, hence the tangent weights are obtained by multiplying the cotangent weights by -1:

$$c_1^T(T_0 \mathbb{P}^1) = -\epsilon_2 + \epsilon_1, \qquad c_1^T(T_\infty \mathbb{P}^1) = \epsilon_2 - \epsilon_1$$

Identify the localization map $H_T^*(\mathbb{P}^1) \hookrightarrow H_T^*(\{0, \infty\})$ with the inclusion $\mathrm{Sym}^\bullet \mathfrak{t}^*(1,1) \oplus \mathrm{Sym}^\bullet \mathfrak{t}^*(0, \epsilon_1 - \epsilon_2) \hookrightarrow (\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 2}$, the self-intersection formula says that the equivariant fundamental class $[0]^T$ is sent to

$$(-\epsilon_2 + \epsilon_1) \cdot (1, 0) = (-\epsilon_2 + \epsilon_1) \cdot (1, 1) - (0, \epsilon_1 - \epsilon_2) \in (\mathrm{Sym}^\bullet \mathfrak{t}^*)^{\oplus 2}$$

The following formula is one of the main computational tools of equivariant cohomology.

Corollary 1.4.16 (Atiyah-Bott integration formula). *Let X be an n-dimensional smooth projective T-variety with finitely many fixed points, then for any $\alpha \in H_T^* X$,*

$$\int_X \alpha = \sum_{p \in X^T} \frac{\iota_p^* \alpha}{c_n^T(T_p X)} \in H_T^*(\mathrm{pt})$$

Example 1.4.17. Consider the standard torus action on $\mathbb{P}^1$. We integrate the class of a fixed point $[\infty]^T \in H_T^*(\mathbb{P}^1)$. The pullback of this class to the other fixed point is 0, hence

$$\int_{\mathbb{P}^1} [\infty]^T = \frac{\iota^*[\infty]^T}{c_1^T(T_\infty \mathbb{P}^1)} = 1$$

because the self-intersection formula says

$$\iota^*[\infty]^T = c_1^T(T_\infty \mathbb{P}^1) \in H_T^*(\mathrm{pt})$$

Note that the verbatim repetition of this calculation gives that the integral of any fixed point in any smooth projective variety is 1.

Remark 1.4.18. We close this section with a few remarks for the die-hard algebraic people. One can also construct equivariant cohomology using only algebraic geometry, however, the major drawback is the need of stacks for a universal principal bundle. On the other hand, the algebraic setting is best suited for our purposes, hence we present the (for us) important aspects of the theory on the topology side, but always keep in our minds an algebraic setting.

1.5 The proof of the GKM theorem

In this section we present a proof of Theorem 1.3.7 following [GKM98]. Recall the identification 1.2.7 $H_T^*(\mathrm{pt}) \cong \mathrm{Sym}^\bullet \mathfrak{t}^*$. Recall that under the assumptions of the theorem, by equivariant formality, $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt})$, and by Proposition 1.2.16 and Definition 1.3.1, we can make sense of the GKM condition.

1.5.1 Equivariant cohomology gives GKM classes

Lemma 1.5.1. *Under the assumptions of Theorem 1.3.7, the image of the restriction map $H_T^*(X) \to H^T(X^T)$ is contained in the set of GKM classes.*

Proof. Pick a one dimensional orbit O with fixed points N, S in its closure, consider the following diagram:

$$\begin{tikzcd} H_T^*(X) \arrow[rr] \arrow[dr] & & H_T^*(N) \oplus H_T^*(S) \\ & H_T^*(\overline{O}) \arrow[ur] & \end{tikzcd}$$

In view of Proposition 1.2.16, T acts on O via a character χ. Take an element $(f, g) \in H_T^*(N) \oplus H_T^*(S)$. We write down the GKM equation for the edge O, and compare it with that of $\overline{O} \cong \mathbb{P}^1$ with the inherited action. The former condition says that $f \equiv g \mod \chi$, on the other hand, recall Example 1.2.18, this is also the GKM condition on $\mathbb{P}^1$. Next, by functoriality and by the above commutative diagram, it suffices to show that any class living in $H_T^*(\mathbb{P}^1)$ gives a GKM class in $H_T^*(N) \oplus H_T^*(S)$. For this use the following commutative diagram

$$\begin{tikzcd} H_T^*(\mathbb{P}^1) \arrow[r] \arrow[d] & H_T^*(\mathbb{P}^1 - N) \arrow[d] \\ H_T^*(\mathbb{P}^1 - S) \arrow[r] & H_T^*(\mathbb{P}^1 - \{N, S\}) \end{tikzcd}$$

We have the identifications

$$H_T^*(O \cup N) \cong H_T^*(N) \qquad H_T^*(O \cup S) \cong H_T^*(S)$$

Commutativity of the diagram says that for any class $\alpha \in H_T^*(\mathbb{P}^1)$, we have $\alpha|_N, \alpha|_S$ goes to the same class in $H_T^*(\mathbb{P}^1 - \{N, S\}) \cong H_T^*(\mathrm{pt})/(\chi)$ (Definition 1.2.4). The restriction map $H_T^*(\mathbb{P}^1 - S) \to H_T^*(\mathbb{P}^1 - \{N, S\})$ is the natural quotient map, and similarly for N. Hence we get $\alpha|_N \equiv \alpha|_S \mod \chi$ as desired. □

1.5.2 Topological localization

The other ingredient is the topological localization, which describes $H_T^*(X)$ as a kernel. Let X_1 be the union of the orbits of dimension ≤ 1. This is the "equivariant 1-skeleton" of X. We are going to recover $H_T^*(X)$ from the relative equivariant cohomology of the pair (X_1, X^T).

Theorem 1.5.2. *Under the assumptions of Theorem 1.3.7, then there is an exact sequence*

$$0 \longrightarrow H_T^*(X) \xrightarrow{\iota^*} H_T^*(X^T) \xrightarrow{\delta} H_T^*(X_1, X^T)$$

Proof. We have shown that the restriction map $H_T^*(X) \to H_T^*(X^T)$is injective. Consider the long exact sequences 1.2.10 of the equivariant pair (X, X^T) as well as (X_1, X^T), we have the following commutative diagram

$$\begin{tikzcd} H_T^*(X) \arrow[r, "\iota^*"] \arrow[d] & H_T^*(X^T) \arrow[r, "\delta'"] \arrow[d, equal] & H_T^{*+1}(X, X^T) \arrow[d] \\ H_T^*(X_1) \arrow[r] & H_T^*(X^T) \arrow[r, "\delta"] & H_T^{*+1}(X_1, X^T) \end{tikzcd}$$

Therefore the image of $H_T^*(X)$ in $H_T^*(X^T)$ is contained in $\ker \delta$. To show the reversed inclusion, let $\xi \in H_T^*(X^T)$ with $\delta(\xi) = 0$, we want to show that $\delta'(\xi) = 0$. Suppose this is not the case, then the annihilator $\mathrm{ann}(\delta'(\xi)) \subset H_T^*(\mathrm{pt})$ is a proper ideal. We now introduce another exact sequence and use this to study $\mathrm{ann}(\delta'(\xi))$.

For each $x \in X - X^T$ we can consider the Lie algebra of (the identity component of) its stabilizer in T, viewed as a Lie subalgebra of $\mathfrak{t}$. Let P be the set of all such Lie subalgebras. For each $\mathfrak{l} \in P$ we have again an exact sequence of relative cohomology

$$H_T^*(X^{\mathfrak{l}}) \xrightarrow{\gamma} H_T^*(X^{\mathfrak{l}} \cap X^T) \xrightarrow{\delta^{\mathfrak{l}}} H_T^*(X^{\mathfrak{l}}, X^T)$$

By the Lemma 1.5.3 below, we have

$$\mathrm{Supp}\ \mathrm{ann}(\delta'(\xi)) \subset \bigcup_{\substack{\mathfrak{l} \in P \\ \delta^{\mathfrak{l}}(\xi) \neq 0}} \mathfrak{l} \subset \mathrm{Spec}\ H_T^*(\mathrm{pt})$$

Since any nonzero element in $\mathrm{ann}\, \delta'(\xi)$ is a nonzero divisor, some $\mathfrak{l} \in P$ has codimension 1 inside $\mathfrak{t}$ (Krull's Hauptidealsatz). For this $\mathfrak{l}$ we have $X^{\mathfrak{l}} \subset X_1$, hence the following diagram:

$$\begin{tikzcd} H_T^*(X) \arrow[r, "\iota^*"] \arrow[d] & H_T^*(X^T) \arrow[r, "\delta'"] \arrow[d, equal] & H_T^{*+1}(X, X^T) \arrow[d] \\ H_T^*(X_1) \arrow[r] \arrow[d] & H_T^*(X^T) \arrow[r, "\delta"] \arrow[d] & H_T^{*+1}(X_1, X^T) \arrow[d] \\ H_T^*(X^{\mathfrak{l}}) \arrow[r, "\gamma"] & H_T^*(X^T \cap X^{\mathfrak{l}}) \arrow[r, "\delta^{\mathfrak{l}}"] & H_T^{*+1}(X^{\mathfrak{l}}, X^T \cap X^{\mathfrak{l}}) \end{tikzcd} \tag{1.2}$$

but then $\delta^{\mathfrak{l}}(\xi) \neq 0$ contradicts $\delta(\xi) = 0$ in view of the bottom right square. $\square$

Below is a technical support of the proof of the topological localization theorem. All notations as above.

Lemma 1.5.3. *Suppose $\delta'(\xi) \neq 0$, then the support of $\operatorname{ann}(\delta'(\xi))$ satisfies*

$$\mathrm{Supp}\ \operatorname{ann}(\delta'(\xi)) \subset \bigcup_{\substack{\mathfrak{l}\in P \\ \delta^{\mathfrak{l}}(\xi)\neq 0}} \mathfrak{l} \subset \mathrm{Spec}\ H_T^*(\mathrm{pt})$$

Proof. See [GKM98, §15.9]. □

1.5.3 Proof of the GKM theorem

Now we have shown that the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective, and the image is contained in the subset of GKM classes. To prove the GKM theorem it remains to show that any GKM class glue to an equivariant cohomology class. We first prove a technical lemma.

Lemma 1.5.4. *Under the assumption of Theorem 1.3.7, take a one dimensional orbit O, with fixed points $N, S \in \overline{O}$. Then there is an isomorphism $H_T^{i+1}(\overline{O}, \{N, S\}) \cong H_T^i(O)$ that fits into the following commutative diagram*

$$\begin{tikzcd}
H_T^i(U) \oplus H_T^i(V) \arrow[r, "\beta"] \arrow[d, "\simeq"] & H_T^i(O) \arrow[d, "\simeq"] \\
H_T^i(N) \oplus H_T^i(S) \arrow[r, "\beta'"] & H_T^{i+1}(\overline{O}, N \cup S)
\end{tikzcd}$$

Proof. Consider the long exact sequence 1.2.10 of the equivariant closed embedding $N \cup S \hookrightarrow \overline{O}$.

$$\longrightarrow H_T^i(\overline{O}) \longrightarrow H_T^i(N) \oplus H_T^i(S) \xrightarrow{\beta} H_T^{i+1}(\overline{O}, \{N, S\}) \longrightarrow 0$$

Here right exactness follows from the fact that $\overline{O}$ has no odd cohomology. To deal with this relative cohomology we apply the MV sequence to the open equivariant cover by $U = \overline{O} - S$ and $V = \overline{O} - N$. Since all $H_T^{2i+1}(\bar{O}) \cong H_T^{2i+1}(\mathbb{P}^1), H_T^{2i+1}(O) \cong H_T^{2i+1}(\mathbb{B}T)$ vanish, the MV sequence breaks into short exact sequences. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H_T^i(\overline{O}) \arrow[r] \arrow[d, equal] & H_T^i(U) \oplus H_T^i(V) \arrow[r, "\beta"] \arrow[d, "\simeq"] & H_T^i(O) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H_T^i(\overline{O}) \arrow[r, "\mathrm{restr}"] & H_T^i(N) \oplus H_T^i(S) \arrow[r, "\beta'"] & H_T^{i+1}(\overline{O}, \{N, S\}) \arrow[r] & 0
\end{tikzcd} \tag{†}$$

From injectivity of the top row we deduce injectivity of the bottom left arrow, hence the right vertical arrow is an isomorphism. □

Proposition 1.5.5. *In the setting of Theorem 1.5.2, any GKM class lies in* $\ker \delta$.

Proof. By Theorem 1.5.2 it suffices to show that the GKM classes are contained in the skernel of the map $\delta : H_T^*(X^T) \to H_T^*(X_1, X^T)$. For this we take any one dimensional orbit $O \subset X_1$ with fixed points $N, S \subset \overline{O}$. Consider the following commutative diagram:

$$\begin{tikzcd} H_T^*(X^T) \arrow[r, "\delta"] \arrow[d] & H_T^{*+1}(X_1, X^T) \arrow[d] \\ H_T^*(N) \oplus H_T^*(S) \arrow[r, "\beta'"] & H_T^{*+1}(\overline{O}, \{N, S\}) \end{tikzcd}$$

we want to show that β' is given by $(f, g) \mapsto \bar{f} - \bar{g} \in \mathrm{Sym}^\bullet \mathfrak{t}^*/\chi$, where χ is the character through which T acts on $\overline{O}$. By Lemma 1.5.4 this is true, hence the set of GKM classes in $H_T^*(X^T)$ is the same as $\cap \ker \beta'$.

Now it remains to show that this is in $\ker \delta$. For this we consider the following modification of diagram 1.2:

$$\begin{tikzcd} H_T^*(X) \arrow[r, "\iota^*"] \arrow[d] & H_T^*(X^T) \arrow[r, "\delta'"] \arrow[d, equal] & H_T^{*+1}(X, X^T) \arrow[d] \\ H_T^*(X_1) \arrow[r] \arrow[d] & H_T^*(X^T) \arrow[r, "\delta"] \arrow[d] & H_T^{*+1}(X_1, X^T) \arrow[d] \\ \oplus H_T^*(X^{\mathfrak{l}}) \arrow[r] & \oplus H_T^*(X^T \cap X^{\mathfrak{l}}) \arrow[r, "\oplus\delta^{\mathfrak{l}}"] & \oplus H_T^{*+1}(X^{\mathfrak{l}}, X^T \cap X^{\mathfrak{l}}) \end{tikzcd}$$

where the direct sum is over codimension 1 $\mathfrak{l}$'s. The bottom middle vertical arrow is injective, hence by the algebraic localization theorem so is the bottom left vertical arrow. Then by five lemma so is the bottom right vertical arrow. Now observe that the GKM classes are in the kernel of $\oplus\delta^{\mathfrak{l}}$, hence in $\ker \delta$. □

Chapter 2

Grassmannians

In this chapter we collect useful facts about Grassmannians. The Grassmannian with the standard torus action is the most interesting example for us. Recall that the set $Gr(k,N)$ of all k-dimensional linear subspaces of $\mathbb{C}^N$ embeds via the Plücker embedding as a closed subset of some projective space. Thus $Gr(k,N)$ admits the structure of a projective variety, known as the *Grassmannian.*

Notations. Fix positive integers $0 \leq k \leq N$. Consider $GL_N(\mathbb{C}) \subset B \subset T$ where T is the standard torus and B is the *opposite* standard Borel subgroup. Let $P_k \subset GL_N(\mathbb{C})$ be the opposite standard parabolic subgroup containing B. Explicitly, B is the subgroup of invertible upper triangular matrices, and T is the subgroup of invertible diagonal matrices, and P_k is the set of invertible matrices of the following form:

$$\begin{pmatrix} C & 0 \\ * & D \end{pmatrix} \qquad C \in GL_k,\ D \in GL_{N-k}$$

Usually the opposite groups are convenient when we want to connect geometry to combinatorics. The Grassmannian is also realized as a Chevalley quotient $GL_N(\mathbb{C})/P_k$ (see [Spr09] chapter 5 §5).

The T-action on $GL_N(\mathbb{C})$ by left multiplication descends to a T-action on $Gr(k,N)$, which we refer to as the *standard action.* Note that $\mathbb{P}^N = Gr(1, N+1)$, the reader may find it helpful to take this as a running example.

2.1 Basic structures

2.1.1 Vector bundles

Let us first describe some natural vector bundles on the Grassmannian, starting with the *tautological bundle*.

Definition 2.1.1. View $Gr(k,N)$ as the set of k-dimensional subspaces of $\mathbb{C}^N$. The tautological bundle $\mathscr{S}$ on $Gr(k,N)$ is by definition

$$\mathscr{S} = \{(V,v) \in Gr(k,N) \times \mathbb{C}^N \mid v \in V\} \xrightarrow{\text{proj}} Gr(k,N)$$

There is a short exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow Gr(k,N) \times \mathbb{C}^N \longrightarrow \mathscr{Q} \longrightarrow 0 \tag{2.1}$$

Intuitively, $\mathscr{S}$ is a rank k vector bundle whose fiber over $V \subset Gr(k,N)$ is V itself, and thus the name. Since $Gr(k,N)$ carries a natural GL_N-action, we can view all these natural vector bundles $\mathscr{S}, \mathscr{Q}, TGr(k,N)$ as GL_N-equivariant vector bundles. e.g. the GL_N-action on $\mathscr{S}$, is given by the restriction of the diagonal action on $Gr(k,N) \times \mathbb{C}^N$. Then intuitively, we can use the GL_N-action to translate the information of a point to anywhere else.

Theorem 2.1.2. *There is an equivalence between the category of GL_N-equivariant vector bundles on $Gr(k,N) \cong GL_N/P_k$ and finite dimensional representations of P_k, given by taking the fibre at the coset $P_k \in GL_N/P_k$. Furthermore, this equivalence preserves tensor products and duals.*

Proof. A quasi-inverse of the functor of taking the fibre at P_k is given as follows: given a representation V of P_k, define $\sim$ on $GL_N \times V$ by

$$(g,v) \sim (gp^{-1}, pv) \qquad \forall\, g \in GL_N, v \in V, p \in P_k$$

the projection $(g,v) \mapsto gP_k$ defines a vector bundle $GL_N \times_{P_k} V := GL_N \times V/\sim \to GL_N/P_k$. See [Jan87, part I §5.14] for details. □

Let $\mathfrak{p}_k$ be the Lie algebra of P_k. Explicitly, $\mathfrak{p}_k$ consists of matrices of the form

$$\begin{pmatrix} C & 0 \\ * & D \end{pmatrix} \qquad C \in \mathfrak{gl}_k,\ D \in \mathfrak{gl}_{N-k}$$

This is naturally a representation of P_k via differentiating the conjugation action of P_k on itself. We have a short exact sequence of P_k-modules

$$0 \longrightarrow \mathfrak{p}_k \longrightarrow \mathfrak{gl}_N \longrightarrow \mathfrak{gl}_N/\mathfrak{p}_k \longrightarrow 0$$

Intuitively, the quotient map $GL_N \to GL_N/P_k$ is a submersion, and the tangent vectors to P_k at the identity element is killed, so the quotient space $\mathfrak{gl}_N/\mathfrak{p}_k$ should be the tangent space of GL_N/P_k at the coset P_k.

Proposition 2.1.3. *The tangent bundle of GL_N/P_k is isomorphic to the bundle $GL_N \times_{P_k} (\mathfrak{gl}_N/\mathfrak{p}_k)$ as GL_N-equivariant vector bundles.*

Proof. See [Jan87, part II, §4.2]. □

In what follows we will denote the dual of a vector bundle $\mathscr{F}$ by $\mathscr{F}^\vee$.

Lemma 2.1.4. *Let $\mathscr{S}, \mathscr{Q}$ be as in (2.1). The tangent bundle of $Gr(k, N)$ is isomorphic to $\mathscr{S}^\vee \otimes \mathscr{Q}$ as GL_N-equivariant vector bundles.*

Proof. In view of Theorem 2.1.2 it suffices to compare the fibres over the coset $P_k \in G/P_k$ as P_k representations. The fibre of $S^\vee \otimes \mathscr{Q}$ is spanned by basis vectors

$$\{e^i \otimes \bar{e}_j \mid 1 \leq i \leq k, k+1 \leq j \leq N\}$$

Take any $p \in P_k$, we calculate

$$pe^i \otimes \bar{e}_j = p^{-1}e^i \otimes p\bar{e}_j$$

where $\bar{e}_i$ denote the congruent class of the standard basis in the fibre of $\mathscr{Q}$, and e^i denote the dual basis of the standard basis in the fibre of $\mathscr{S}$. On the other hand, the action of P_k on $\mathfrak{gl}_N/\mathfrak{p}_k$ is given by conjugation

$$p\bar{E}_{ji} = p\bar{E}_{ji}p^{-1} \qquad 1 \leq i \leq k,\ k+1 \leq j \leq N$$

where $\bar{E}_{ji} \in \mathfrak{gl}_N/\mathfrak{p}_k$ denotes the image of the matrix unit $E_{ji}e_l = \delta_{li}e_j$. Define a linear map from the fibre of $\mathscr{S}^\vee \otimes \mathscr{Q}$ to $\mathfrak{gl}_N/\mathfrak{p}_k$ by

$$e^i \otimes \bar{e}_j \longmapsto \bar{E}_{ji}$$

this is bijective, and compatible with the P_k-action, as desired. □

2.1.2 Fixed points

Notations. We denote by $\Lambda(k, N)$ the set of 01-strings $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \{0,1\}^N$ such that $\sum_{i=1}^{N} \lambda_i = k$.

Lemma 2.1.5. *The fixed points of the standard torus action on $Gr(k, N)$ are precisely the coordinate subspaces, i.e. the linear subspaces spanned by k of the standard basis elements*

$$e_i = (0, 0, \ldots 0, \underset{i\text{-}th}{1}, 0, \ldots, 0) \in \mathbb{C}^N$$

Proof. Note that a torus element $t \in T$ simply sends a subspace $W \subset \mathbb{C}^N$ to the subspace $tW \subset \mathbb{C}^N$, thus W is a fixed point iff $W = tW$ for all $t \in T$. It is then a linear algebra exercise to show that this is equivalent to W being spanned by common eigenvectors of T. $\square$

Corollary 2.1.6. *The set of 01-strings $\Lambda(k, N)$ corresponds bijectively to the torus fixed points in $Gr(k, N)$ via*

$$\lambda = \lambda_1 \lambda_2 \ldots \lambda_n \longmapsto \sum \mathbb{C}\lambda_i e_i \subset \mathbb{C}^N$$

Notation. By abuse of notation, denote by λ the fixed point corresponding to λ. Denote the corresponding linear subspace of $\mathbb{C}^N$ by $\mathbb{C}^\lambda$.

Now by Lemma 2.1.4 we can also read off the T-action on the tangent bundle of $Gr(k, N)$ via that of $\mathscr{S}^\vee \otimes \mathscr{Q}$, whose fibre over any fixed point $W \subset Gr(k, N)$ is naturally identified with $W^\vee \otimes \mathbb{C}^N/W$ as a T-module. Since T is abelian, the tangent space at any fixed point splits into a direct sum of common eigenspaces of T called *weight spaces*. On each summand T acts via a character called the *weight*. Recall example 1.2.7 that we identify weights as certain Lie algebra elements.

Example 2.1.7. Consider $Gr(2, 4)$ with the standard T-action. By Lemma 2.1.5 the torus fixed points are the coordinate subspaces

$$\text{span}\{e_i, e_j\}, \qquad 1 \leq i < j \leq 4$$

Therefore by the argument above the tangent space at $0110 \in \Lambda(2, 4)$ splits as a T-module:

$$T_{0110}Gr(2, 4) = \mathbb{C}e^2 \otimes \bar{e}_1 \oplus \mathbb{C}e^3 \otimes \bar{e}_1 \oplus \mathbb{C}e^2 \otimes \bar{e}_4 \oplus \mathbb{C}e^3 \otimes \bar{e}_4$$

For a general Grassmannian we can do the same thing. By Example 1.2.7 we have the identification $H_T^*(\text{pt}) \cong \text{Sym}^\bullet \mathbf{t}^*$. For our standard torus $T \subset GL_N(\mathbb{C})$ we have a standard choice of a basis of $\mathfrak{t}^*$ given by the tangent map ϵ_i of the coordinate function

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_N \end{pmatrix} \longmapsto t_i \in \mathbb{C}^*$$

Denote them by ϵ_i, then we have an identification

$$H_T^*(\mathrm{pt}, \mathbb{Z}) \cong \mathrm{Sym}_{\mathbb{Z}}^{\bullet} \mathfrak{t}^* \cong \mathbb{Z}[\epsilon_1 \ldots . \epsilon_N] \tag{2.2}$$

where $\deg \epsilon_i = 2$.

Proposition 2.1.8. *In $Gr(k, N)$ the top T-equivariant Chern class of the tangent space at the fixed point λ is*

$$c_{\mathrm{top}}^T(T_\lambda Gr(k, N)) = \prod_{\lambda_i > \lambda_j} (\epsilon_j - \epsilon_i) \in H_T^{2k(N-k)} Gr(k, N)$$

Proof. The tangent space of $Gr(k, N)$ at a fixed point λ is identified with $(\mathbb{C}^\lambda)^* \otimes \mathbb{C}^N / \mathbb{C}^\lambda$ as a T-module, which has an common eigenbasis for the T-action

$$\{e^i \otimes \bar{e}_j \mid \lambda_j = 0, \lambda_i = 1\}$$

The T-weights are given by

$$(t_1, \ldots t_N) \cdot e^i \otimes \bar{e}_j = t_j t_i^{-1} e^i \otimes \bar{e}_j$$

under the identification (2.2) these weights are identified with $\epsilon_i - \epsilon_j \in \mathfrak{t}^*$ for $\lambda_i = 1$ and $\lambda_j = 0$. □

2.1.3 Schubert cells and Schubert varieties

To study the geometry of $Gr(k, N)$ one usually start with a cellular decomposition associated to the fixed points.

Definition 2.1.9. The *Schubert cells* are the B-orbits of the torus fixed points in $Gr(k, N)$. The Schubert varieties are the closures of the Schubert cells.

These cells are studied via the following combinatorial information of the fixed points.

Definition 2.1.10. Define a map $I : \Lambda(k, N) \to \mathbb{N}$ by

$$I(\lambda) = \sharp\{(i, j) \mid i < j, \lambda_i < \lambda_j\}$$

Define a partial order $\geq$ on $\Lambda(k, N)$ by

$$\lambda' \geq \lambda \iff \sum_{i=1}^{n} \lambda'_i \geq \sum_{i=1}^{n} \lambda_i \quad \forall\, 1 \leq n \leq N$$

Proposition 2.1.11. *Let* $0 = V_0 \subset V_1 \subset \dots V_N = \mathbb{C}^N$ *be the* opposite standard flag, *where* V_i *is the span of the last* i *standard basis vectors* $e_{N-i+1}, \dots e_N$*.*

- $Gr(k, N)$ *is the disjoint union Schubert cells.*
- *For each torus fixed point* $\lambda \in Gr(k, N)$ *the unique Schubert cell* C_λ *containing* λ *is isomorphic to the affine space* $\mathbb{C}^{k(N-k)-I(\lambda)}$*.*
- $C_\lambda = \{W \subset \mathbb{C}^N \mid \dim W \cap V_{N-i+1}/W \cap V_{N-i} = \lambda_i\}$.
- C_λ *is dense and open in its closure* $\Omega_\lambda := \overline{C}_\lambda$*.*
- $\Omega_\lambda = \coprod_{\mu \leq \lambda} C_\mu$
- $\Omega_\lambda = \{W \subset \mathbb{C}^N \mid \dim W \cap V_i \geq \sum_{j=N-i+1}^{N} \lambda_j\}$

For the proof the reader is referred to [EH16], chapter 4.

Example 2.1.12. Take $\mathbb{P}^{N-1} = Gr(1, N)$ as an example. Then

$$\Lambda(1, N) = \{\lambda_r = 00 \dots \underset{r\text{-th}}{1}0 \dots 00 \mid 1 \leq r \leq N\}$$

The Schubert cell $C_r := C_{\lambda_r}$ is by definition the B-orbit of the fixed point λ, hence in homogeneous coordinates $[x_1 : x_2 \dots : x_N]$ on $\mathbb{P}^{N-1}$ we have

$$C_r = \{[0 : \dots : 0 : 1 : x_{r+1} : x_{r+2} \dots : x_N] \mid x_i \in \mathbb{C}\} \cong \mathbb{C}^{(N-1)-(r-1)}$$

Note that $I(\lambda_r) = r - 1$, as in the proposition. It is also clear that the disjoint union of Schubert cells is $Gr(1, N)$. On the other hand, C_r consists of lines spanned by some $v \in Be_r$ consisting of vectors satisfying

$$v \equiv ce_r \mod \operatorname{span}\{e_{r+1} \dots e_N\} \qquad c \in \mathbb{C}^*$$

One can directly check that this is equivalent to

$$\dim \mathbb{C}v \cap V_i/(\mathbb{C}v \cap V_{i-1}) = \begin{cases} 1 & \text{if } i = r \\ 0 & \text{otherwise.} \end{cases}$$

Next, one can directly check that

$$\Omega_r = \overline{C}_r = \{[0 : \cdots : 0 : x_r : x_{r+1} : x_{r+2} \cdots : x_N] \in \mathbb{P}^{N-1}\} \cong \mathbb{P}^{r-1}$$

and C_r is dense open in Ω_r. This is the set of lines L satisfying

$$\dim L \cap V_{N-r+1} \geq 1$$

and one can see that $\Omega_r = \cup_{i \geq r} C_i$. This cellular structure is exactly the standard CW structure for the complex projective space in topology.

Tangent and normal spaces of Schubert varieties

Next we write down the tangent and normal spaces of the Schubert variety Ω_λ at the corresponding torus fixed point λ.

Lemma 2.1.13. *The tangent space to Ω_λ at the torus fixed point λ is*

$$\{\bar{e}_i \otimes de_j \mid \lambda_i = 0, \lambda_j = 1, i > j\} \subset T_\lambda Gr(k, N)$$

This identification is compatible with the T-action.

Proof. By Proposition 2.1.11, C_λ is dense open in Ω_λ and contains λ, hence $T_\lambda \Omega_\lambda = T_\lambda C_\lambda$. To determine the latter, we produce some tangent vectors as follows: consider the one-parameter subgroup

$$\phi_{ij} : t \mapsto I + tE_{ij} \in B \qquad \lambda_i = 0, \lambda_j = 1, i > j$$

where E_{ij} is the matrix defined by $E_{ij} e_l = \delta_{lj} e_i$. Note that the stabilizer of λ of the left GL_N-action consists of matrices preserving the subspace $\mathbb{C}^\lambda = \sum \mathbb{C}\{e_i \mid \lambda_i = 1\} \subset \mathbb{C}^N$, therefore $\phi_{ij}(t)\lambda$ is a curve on $Gr(k, N)$. Taking the tangent vector of this curve at $t = 0$ gives $\bar{e}_i \otimes de_j$. The collection of these tangent vectors is linearly independent. Notice that the torus fixed point λ is always in the Schubert cell and thus a smooth point. Then by dimension reasons we see that all such vectors already span $T_\lambda \Omega_\lambda$. □

The cotangent space is then identified with the dual representation of $T_\lambda \Omega_\lambda$, i.e. the T-weights are exactly the negative of the weights of the tangent space. Recall the identification (2.2) $H^*_T(\mathrm{pt}, \mathbb{Z}) \cong \mathbb{Z}[\epsilon_1 \ldots \epsilon_N]$.

Corollary 2.1.14. *For any $\lambda \in \Lambda(k, N)$ the top T-equivariant Chern class of the normal space of the Schubert variety Ω_λ at the fixed point λ is*

$$c^T_{\mathrm{top}}(N_\lambda \Omega_\lambda) = \prod_{\substack{\lambda_i < \lambda_j \\ i < j}} (\epsilon_i - \epsilon_j)$$

The top T-equivariant Chern class of the conormal space of the Schubert variety Ω_λ at the fixed point λ is

$$c^T_{\text{top}}(N^*_\lambda \Omega_\lambda) = \prod_{\substack{\lambda_i < \lambda_j \\ i<j}} (-\epsilon_i + \epsilon_j)$$

2.1.4 Cohomology

Now we briefly recap the (integral) cohomology or Chow ring of the Grassmannian $Gr(k, N)$ (they are isomorphic as $\mathbb{Z}$-algebras in this case). For the proofs and a more thorough discussion see [EH16] or [Ful98].

The Schubert variety Ω_λ gives a fundamental class in homology (or more precisely, Borel-Moore homology, but for compact spaces they are the same, see [CG10] §2.6), whose Poincaré dual defines a cohomology class $S_\lambda \in H^*(Gr(k, N), \mathbb{Z})$. These classes are known as the *Schubert classes.*

Theorem 2.1.15. *The Schubert classes S_λ form a $\mathbb{Z}$-basis of $H^*(Gr(k, N), \mathbb{Z})$, with each S_λ sitting in degree $2I(\lambda)$, where $I(\lambda)$ is defined in 2.1.10.*

For the proof the reader is again referred to [EH16] §4.2.2.

Remark 2.1.16. Note that there is a unique Schubert class Div sitting in degree 2, corresponding to the string $\lambda = 11 \dots 10100 \dots 0$ and $I(\lambda) = 1$. This is called the *divisor class.* The the Theorem 2.1.15 implies that

$$\operatorname{rank} H^2(Gr(k, N), \mathbb{Z}) = 1$$

This divisor defines a line bundle $\mathcal{O}(1)$ satisfying $c_1(\mathcal{O}(1)) = \text{Div}$. In fact $\mathcal{O}(1) = \bigwedge^k \mathscr{S}^\vee$, where $\mathscr{S}$ is the tautological bundle (2.1). Furthermore, $\mathcal{O}(1)$ has $\binom{N}{k}$ linearly independent sections, corresponding to $e_{i_1} \wedge \dots e_{i_k}$, hence yields the Plücker embedding.

Dually, there is a unique Schubert class in degree $2k(N-k)-2$, corresponding to the string $00 \dots 01011 \dots 1$. The corresponding Schubert variety is a T-invariant $\mathbb{P}^1$, hence this Schubert class is often called *the class of a line.* We again deduce

$$\operatorname{rank} H^{2k(N-k)-2}(Gr(k, N), \mathbb{Z}) = 1$$

Example 2.1.17. Consider $\mathbb{P}^{N-1} = Gr(1, N)$, recall example 2.1.12 that the Schubert varieties are linear subspaces $\mathbb{P}^r \subset \mathbb{P}^{N-1}$, i.e. the vanishing loci of the first $r-1$ coordinates $x_1, \dots x_{r-1}$ on $\mathbb{P}^N$. Since this is the standard CW structure for the projective space, this illustrates Theorem 2.1.15.

The intersection pairing

Recall that the Grassmannian is connected, hence we have an isomorphism

$$\phi : H_0 Gr(k,N) \cong \mathbb{Z}$$

There is an *intersection pairing* on the cohomology give by

$$\int_{Gr(k,N)} \alpha \cup \beta := \phi\left(\alpha \cup \beta \cap [Gr(k,N)]\right) \in \mathbb{Z}$$

where $[Gr(k,N)] \in H_{k(N-k)} Gr(k,N)$ denotes the fundamental class. This number is also called the intersection number. By definition

$$\int_{Gr(k,N)} \alpha \cup \beta \neq 0 \implies \deg \alpha + \deg \beta = 2 \dim Gr(k,N) = 2k(N-k)$$

This pairing is nondegenerate by (a version of) Poincaré duality, see [CG10, §2.6] for details.

Remark 2.1.18. When $\deg \alpha + \deg \beta = 2k(N-k)$, the integer $\int_{Gr(k,N)} \alpha \cup \beta$ is called the intersection number because cohomology classes represent (certain equivalence classes) of closed submanifolds, and under nice circumstances (transversal intersection) it counts the number of intersection points of the two submanifolds. e.g. the fact that two "generic" lines on $\mathbb{P}^2$ intersect at exactly one point is reflected by

$$\int_{\mathbb{P}^2} S_{010} \cup S_{010} = 1$$

because the Schubert variety Ω_{010} is a line in view of example 2.1.12, and on any projective space the fundamental class of any linear $\mathbb{P}^1$ is the Schubert class of a line. The pairing is written as an integral because using de Rham cohomology this is indeed the honest integral of differential forms.

Example 2.1.19.

Now by non-degeneracy of the intersection pairing, the matrix with entries $\int S_\lambda \cup S_\mu$ is invertible. Denote its inverse matrix by $(g^{\lambda\mu})$, we can write any cohomology class as

$$\alpha = \left(\int_{Gr(k,N)} \alpha \cup S_\lambda\right) g^{\lambda\mu} S_\mu$$

In particular,

$$S_\lambda \cup S_\mu = \left(\int_{Gr(k,N)} S_\lambda \cup S_\mu \cup S_\nu\right) g^{\nu\xi} S_\xi \tag{2.3}$$

The coefficients $\int_{Gr(k,N)} S_\lambda \cup S_\mu \cup S_\nu$ are sometimes called *Schubert intersection numbers.* Thus to determine the cup product structure it suffices to compute the Schubert intersection numbers and the matrix $(g^{\lambda\mu})$.

Lemma 2.1.20. *For any $\lambda \in \Lambda(k,N)$ we have*

$$\int_{Gr(k,N)} S_\lambda \cup S_\mu = \begin{cases} 1 & \mu = w_0\lambda \\ 0 & \text{otherwise.} \end{cases}$$

where $w_0\lambda$ means the string λ read backwards.

Proof. See [EH16, §4.2.2]. □

Now we can rewrite equation 2.3 as

$$S_\lambda \cup S_\mu = \left(\int_{Gr(k,N)} S_\lambda \cup S_\mu \cup S_{w_0\nu} \right) S_\nu \tag{2.4}$$

2.2 Schubert calculus and puzzle games

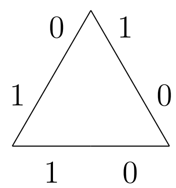
The ring structure of the cohomology of the Grassmannian is determined by the equation 2.4. Abbreviate the Schubert intersection numbers

$$c^\nu_{\lambda\mu} := \int_{Gr(k,N)} S_\lambda \cup S_\mu \cup S_{w_0\nu} \in \mathbb{Z}$$

In general, intersection numbers can be negative. However, as we will see, for Grassmannians all $c^\nu_{\lambda\mu}$ turn out to be nonnegative. They have interesting combinatorial interpretations, one of which is that they count the puzzles of [KT$^+$03].

Definition 2.2.1. A labeled equilateral triangle is an upward pointing equilateral triangle of some integral side length n, with the 3n unit edges labeled by elements in $\{0.1\}$.

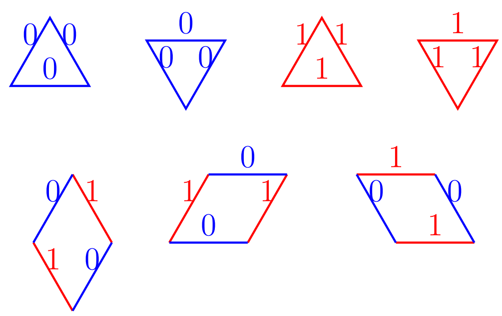
Example 2.2.2. In the previous example the boundary of the puzzle is the following labeled equilateral triangle of side length 2.



We call the left side of such a triangle the NW side, the right one the NE side, and the bottom one the S side. To relate this to the Schubert calculus, for any three strings $\lambda, \mu, \nu \in \Lambda(k, N)$, let $\Delta_{\lambda,\mu,\nu}$ be the labeled equilateral triangle with the NW side λ, NE side μ and S side ν, all read clockwise. Next, like actual puzzle games, a Knutson-Tao puzzle is obtained by filling such a labeled equilateral triangle by basic puzzle pieces.

Definition 2.2.3. An ordinary puzzle piece is one of the following 3 plane figures with labeled edges

- a unit triangle with all edges labeled 0,
- a unit triangle with all edges labeled 1,
- a unit rhombus, the two edges clockwise of acute vertices labeled 0, the others labeled 1.



Note that this set of puzzle pieces is invariant under 120° rotation. Our labeling is opposite to the ones in [KT+03]: what we label 1 would be 0 for them, and vice versa. This is because we want to align with our conventions on Grassmannians.

Definition 2.2.4. A puzzle is a decomposition of an equilateral triangle with side length $N \in \mathbb{N}$ into triangles and rhombi with all edges labeled 0 or 1, so that each region is a puzzle piece.

Example 2.2.5. In the following picture the left figure is a puzzle and the right one is not.

Theorem 2.2.6 (Puzzles compute Schubert calculus)**.** *Let $0 \le k \le N$ and $\lambda, \mu, \nu \in \Lambda(k, N)$. The number of puzzles P with boundary $\Delta_{\lambda,\mu,\nu}$ is the intersection number*

$$\int_{Gr(k,N)} S_\lambda \cup S_\mu \cup S_\nu$$

In particular, the structure constant $c^\nu_{\lambda\mu}$ is equal to the number of puzzles P with boundary $\Delta_{\lambda,\mu,w_0\nu}$;

Remark 2.2.7. In the Theorem 2.2.6, the symmetries of the intersection numbers

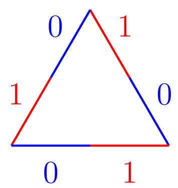
$$\int S_\lambda \cup S_\mu \cup S_\nu = \int S_\mu \cup S_\nu \cup S_\lambda = \int S_\nu \cup S_\lambda \cup S_\mu$$

corresponds to our earlier observation that the 120° rotation preserves the set of puzzle pieces and hence takes a puzzle to a puzzle.

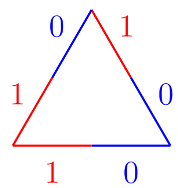
Example 2.2.8. Consider $\mathbb{P}^1 = Gr(1, 2)$, we haves seen that the Schubert cells are $C_{01} = \{\mathrm{pt}\}$, $C_{10} = \mathbb{P}^1 - \{\mathrm{pt}\}$ The Schubert classes are $S_{01} = [\mathrm{pt}]$ and $S_{10} = [\mathbb{P}^1]$. As a reality check, we examine all puzzles in the multiplication table of $H^*(\mathbb{P}^1)$.

	S_{10}	S_{01}
S_{10}	S_{10}	
S_{01}	S_{01}	0

Start with $S_{10}S_{10} = S_{10}$. In the following puzzles we color the 1-edges by red and 0-edges by blue and thus omit the 0's and 1's. Now we count the corresponding puzzles.



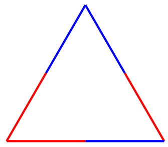
$$\int_{\mathbb{P}^1} S_{10} \cup S_{10} \cup S_{10}$$

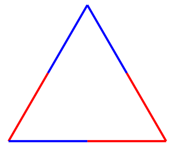


$$\int_{\mathbb{P}^1} S_{10} \cup S_{10} \cup S_{01}$$

Of course the first intersection number is 0 for degree reasons. Let us check that there can be no such puzzles: the upper acute angle has edges of different colors, which only fits in the rhombus puzzle piece. But then this is also the case for any other angle, and there is simply not enough room for so many rhombi in this triangle. For the other integral, by the same observation we have to put a vertical rhombus at the top, and this gives the unique puzzle given in example 2.2.5.

Next we consider $S_{10}S_{01} = S_{01}$. We count the puzzles with the following boundary conditions.





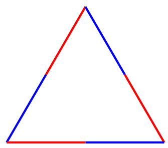
$$\int_{\mathbb{P}^1} S_{10} \cup S_{01} \cup S_{01} \qquad\qquad \int_{\mathbb{P}^1} S_{10} \cup S_{01} \cup S_{10}$$

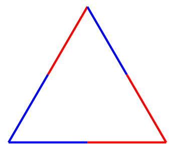
Notice that the second puzzle problem has be solved before, if we rotate it 120° clockwise. This gives the correct answer to the second integral. The first one has no solution because we have to put a red triangle at the top and a blue one at the bottom left corner, leaving nothing to fill in the remaining space.

Next, write $S_{01} \cup S_{01} = 0$ as

$$\int_{\mathbb{P}^1} S_{01} \cup S_{01} \cup S_{01} = 0 \qquad \int_{\mathbb{P}^1} S_{01} \cup S_{01} \cup S_{10} = 0$$

We count the puzzles with the following boundary conditions.





$$\int_{\mathbb{P}^1} S_{01} \cup S_{01} \cup S_{01} \qquad\qquad \int_{\mathbb{P}^1} S_{01} \cup S_{01} \cup S_{10}$$

These boundaries have no solution because the upper acute angle does not fit into any puzzle pieces.

Example 2.2.9. There are Littlewood-Richardson coefficients greater than 1. For example in $Gr(6, 10)$ we have the following puzzles, where for clarity we left out all triangles and only marked the rhombi pieces.

Remark 2.2.10. Theorem 2.2.6 used integral coefficients for the cohomology, but the results also apply to complex coefficients.

2.3 Equivariant Schubert calculus

Now let us discuss the equivariant cohomology of $Gr(k, N)$. In what follows we upgrade the puzzle rule to the equivariant setting following [KT+03]. In fact one

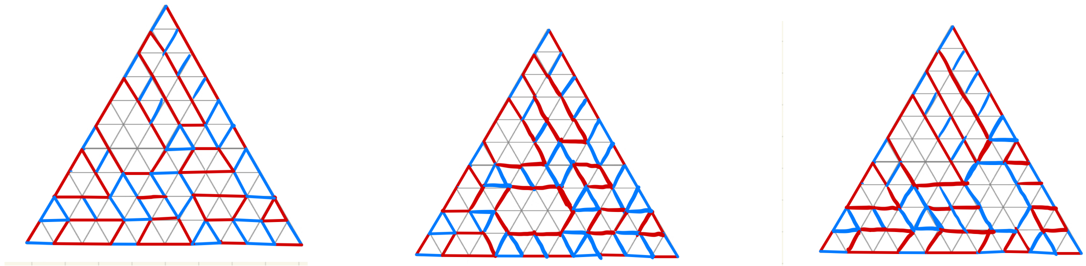


Figure 2.1: All $\Delta^{0110110101}_{1011011010,1111010010}$ puzzles

can prove the equivariant case directly without appealing to the non equivariant geometry, then obtain the non equivariant version as a corollary.

2.3.1 The GKM graph of Grassmannians

Recall that we can describe the equivariant cohomology in terms of GKM classes. For this we need to consider the GKM graph of Grassmannians. Recall that a transposition is a symmetric group element of the form $(ij) \in S_N$.

Lemma 2.3.1. *For any $Gr(k,N)$ let $\lambda \neq \lambda'$ be two torus fixed points.*

- *If there is a transposition $\sigma \in S_N$ such that $\lambda_i = \lambda'_{\sigma i}$ for all i, then there is a unique one dimensional orbit whose closure contains λ, λ'.*
- *Otherwise there are no one dimensional orbits whose closure contains λ, λ'.*

Proof. By Proposition 1.2.16 each one dimensional orbit O is isomorphic to $\mathbb{P}^1 - \{0,\infty\}$ and its closure $\overline{O}$ isomorphic to $\mathbb{P}^1$ containing exactly two fixed points $\lambda, \lambda' \in Gr(k,N)$ identified with $0, \infty \in \mathbb{P}^1$ resp. As in example 1.2.19, via the equivariant inclusion

$$T_0\mathbb{P}^1 \hookrightarrow T_\lambda Gr(k,N)$$

$$T_\infty\mathbb{P}^1 \hookrightarrow T_{\lambda'} Gr(k,N)$$

$\bar{O}$ specifies a weight space of the torus in each of the tangent spaces of $Gr(k,N)$ at these two fixed points. In view of Proposition 2.1.8, this weight has to be the form $\epsilon_i - \epsilon_j$ at λ, with $\lambda_i = 1, \lambda_j = 0$. Recall example 1.2.18, we deduce that the torus T acts on $O \cong \mathbb{P}^1$ via the character $t_i t_j^{-1}$. Hence the torus elements with 1's at the i-th and j-th diagonal fixes O pointwise. Take $V \in O$ and a nonzero vector $v \in V$. Write $v = \sum_i a_i e_i$ and assume that $a_l \neq 0$ for some $l \neq i, j$. Take the T element with the l-th diagonal entry equal to $2a_i$ and the other diagonal entries equal to 1.

$$t = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 2a_i & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Then $tV = V \implies tv \in V$, and consequently $e_l = a_i^{-1}(tv - v) \in V$. In this way we conclude that V is the span of $k-1$ standard basis vectors, together with some $ae_i + be_j$, and both $a, b \neq 0$ because V is itself not a fixed point. Then one can check that

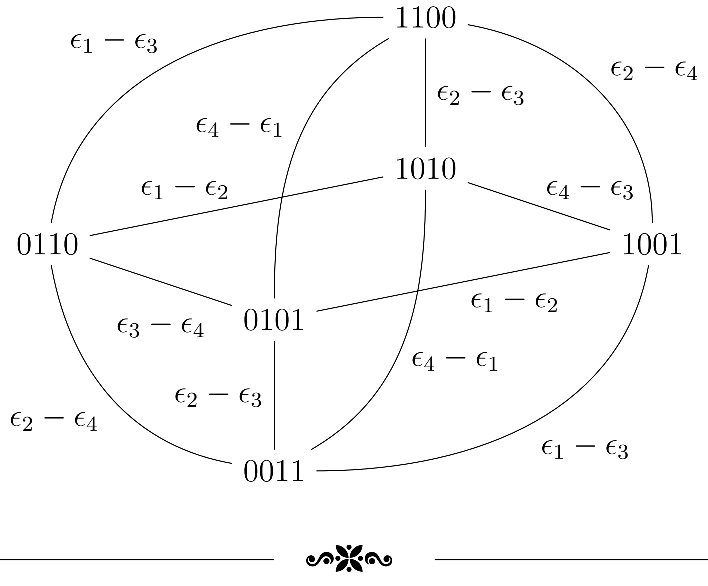
$$\overline{O} = \left\{ W \subset \mathbb{C}^N \mid W \subset \sum_{\lambda_l = 1} \mathbb{C}e_l + \mathbb{C}e_j \right\} \tag{2.5}$$

Then observe that the only torus fixed points in RHS is λ and λ' obtained from λ via the transposition (ij). This proves the second assertion, together with the uniqueness in the first one. To prove existence, just define the RHS in the above equation 2.5 and check that it is a one dimensional orbit closure. $\square$

Example 2.3.2. Consider the standard T^4-action on $Gr(2,4)$. The torus fixed points are the coordinate subspaces

$$W_{ij} = \text{span}\{e_i, e_j\}, \qquad 1 \le i < j \le 4$$

Recall the identification (2.2), $H_T^*(\text{pt}, \mathbb{Z}) \cong \mathbb{Z}[\epsilon_1, \ldots, \epsilon_N]$, these weights at W_{ij} are $\epsilon_i - \epsilon_l, \epsilon_j - \epsilon_l$ for $l \neq i, j$. The GKM graph is thus depicted as follows:



One can explicitly check that λ, λ' are connected by an edge iff there is a transposition $\sigma \in S_4$ such that $\lambda_i = \lambda'_{\sigma i}$ for all i.

In general the GKM graph can have so many edges that it could be hardly helpful to draw it explicitly. We shall see in what follows that the partial order on the set of vertices is sufficient for our purposes.

2.3.2 Equivariant Schubert classes

We have to first define the equivariant version of the Schubert classes. Following Knutson and Tao's treatment, we give a combinatorial definition and then construct these classes via geometry.

Recall that we identify the set of fixed points with the set $\Lambda(k, N)$. Recall the partial order 2.1.10 on it. Recall Definition 1.3.2 that a GKM class is a map from the set of torus fixed points to $\mathrm{Sym}^\bullet \mathfrak{t}^* \cong H^*_T(\mathrm{pt})$ subject to the GKM conditions.

Definition 2.3.3. Let $\lambda \in \Lambda(k, N)$. The *support* Supp f of is the subset of $\lambda \in \Lambda(k, N)$ such that $f(\lambda) \neq 0$ in $H^*_T Gr(k, N)$. We say that f is *supported below* λ if for all $\lambda' \in$ Supp f we have $\lambda' \leq \lambda$.

Recall example 1.3.6, where we found a basis for the equivariant cohomology consisting of classes supported below some λ. To generalize this, observe that for each of such pairs (i, j), we swap λ_i and λ_j to obtain another vertex $\lambda' > \lambda$ in the GKM graph, which is connected to λ by an edge thanks to Lemma 2.3.1. Consequently the GKM condition requires

$$\alpha(\lambda) \equiv 0 \mod (\epsilon_i - \epsilon_j)$$

Then the fact that $\mathrm{Sym}^\bullet \mathfrak{t}^*$ is a factorial ring concludes that a GKM class f supported below λ satisfies

$$f(\lambda) \equiv 0 \mod \prod_{\substack{\lambda_i < \lambda_j \\ i<j}} (\epsilon_i - \epsilon_j) \in \mathrm{Sym}^\bullet \mathfrak{t}^*$$

Definition 2.3.4. The *equivariant Schubert classes* are GKM classes characterized by the following properties:

- $\tilde{S}_\lambda$ is supported below λ;
- $\tilde{S}_\lambda(\lambda) = \prod(\epsilon_i - \epsilon_j)$, where the product is over (i, j) with $\lambda_i < \lambda_j$ and $i < j$;
- $\tilde{S}_\lambda(\mu)$ is homogeneous of degree $2I(\lambda)$.

Lemma 2.3.5. *Suppose the equivariant Schubert class $\tilde{S}_\lambda$ exists for all λ, then they are uniquely characterized by the Definition 2.3.4, and form a* $\operatorname{Sym}^\bullet \mathfrak{t}^*$*-basis of the GKM classes of $Gr(k, N)$.*

Proof. Suppose the equivariant Schubert classes exist. They form a $\operatorname{Sym}^\bullet \mathfrak{t}^*$-basis due to the "upper triangular" shape provided by its first defining property. More precisely, to show linear independence, we take any linear combination

$$\sum a_\lambda \tilde{S}_\lambda = 0$$

consider a minimal μ with $a_\mu \neq 0$. We want to evaluate the above equation at μ. Suppose $a_\nu \tilde{S}_\nu(\mu) \neq 0$, then μ is in the support of ν, i.e. $\nu \geq \mu$, and also $a_\nu \neq 0$, hence contradicts the minimality of μ. Then we have

$$0 = \sum a_\lambda \tilde{S}_\lambda(\mu) = a_\mu \tilde{S}_\mu(\mu) = a_\mu \prod (\epsilon_i - \epsilon_j)$$

Consequently $a_\mu = 0$, a contradiction. This shows that we cannot have any nonzero a_λ at all. To show that any $x \in H_T^* Gr(k, N)$ is in $\operatorname{Sym}^\bullet \mathfrak{t}^* \tilde{S}_\lambda$, use induction on $\min\{I(\lambda) \mid \lambda \in \operatorname{Supp} x\}$. The case $\operatorname{Supp} x = \{00\ldots011\ldots1\}$ is clear: since

$$\tilde{S}_{00\ldots011\ldots1}(\lambda) = \begin{cases} \prod_{\substack{1 \leq i \leq N-k \\ N-k+1 \leq j \leq N}} (\epsilon_i - \epsilon_j) & \lambda = 00\ldots011\ldots1 \\ 0 & \text{otherwise.} \end{cases}$$

x is automatically divisible by $\tilde{S}_{00\ldots011\ldots1}$. Now for general x, suppose $\mu_1, \ldots \mu_s \in \operatorname{Supp} x$ have minimal $I(\mu_i)$, then each $x(\mu_i)$ is divisible by $\tilde{S}_{\mu_i}(\mu_i)$, hence

$$x' := x - \sum_i \frac{x(\mu_i)}{\tilde{S}_{\mu_i}(\mu_i)} \tilde{S}_{\mu_i} \in H_T^* Gr(k, N)$$

and $\min\{I(\lambda) \mid \lambda \in \operatorname{Supp} x'\} > \min\{I(\lambda) \mid \lambda \in \operatorname{Supp} x\}$. Induction hypothesis thus yields $x' \in \sum \operatorname{Sym}^\bullet \mathfrak{t}^* \tilde{S}_\lambda$.

To show uniqueness, suppose $\tilde{S}'_\lambda$ is another set of equivariant Schubert classes, then we can write $\tilde{S}'_\lambda = \sum a_\mu \tilde{S}_\mu$. Take a maximal $\nu \not\leq \lambda$ such that $a_\nu \neq 0$, then by definition we have $\nu \notin \operatorname{Supp} \tilde{S}'_\lambda$ and evaluation at ν gives

$$0 = \tilde{S}'_\lambda(\nu) = \sum_{\mu \geq \nu} a_\mu \tilde{S}_\mu(\nu) = a_\nu \prod_{\substack{\nu_i < \nu_j \\ i<j}} (\epsilon_i - \epsilon_j)$$

because $\tilde{S}_\mu(\nu) \neq 0$ only if $\mu \geq \nu$, and such $\mu \in \operatorname{Supp} \lambda$ implies that $\lambda \geq \mu \geq \nu$, a contradiction, therefore $\mu \not\leq \lambda$. By maximality of μ we get $\mu = \nu$. This implies that $a_\nu = 0$, hence $\tilde{S}'_\lambda = a_\lambda \tilde{S}_\lambda + \sum_{\mu<\lambda} a_\mu \tilde{S}_\mu$. Again evaluation at λ gives $a_\lambda = 1$. Finally all lower terms vanish because $\tilde{S}'_\lambda$ is by definition homogeneous, and for $\mu < \lambda$ we have $\deg \tilde{S}_\mu > \deg \tilde{S}_\lambda = \deg \tilde{S}'_\lambda$. $\square$

2.3.3 Equivariant Schubert classes exist

Next we construct the equivariant Schubert classes from geometry. As a motivating observation, recall the computation of equivariant Chern classes 2.1.14 of the normal space of the Schubert variety Ω_λ at the tours fixed point λ, recall the equivariant fundamental classes in section 1.4.2 and the self-intersection formula 1.4.9. Comparing with the combinatorial Definition 2.3.4 of the equivariant Schubert classes we get

$$\tilde{S}_\lambda(\lambda) = c^T_{\mathrm{top}}(N_\lambda \Omega_\lambda) = \iota^* \iota_* [\Omega_\lambda]^T$$

where $\iota : \lambda \hookrightarrow Gr(k, N)$ is the inclusion of the fixed point.

Lemma 2.3.6. *The equivariant fundamental class of the Schubert varieties satisfy the defining properties 2.3.4 of the equivariant Schubert classes.*

Proof. That the equivariant fundamental classes $[\Omega_\lambda]^T$ give rise to GKM classes is a part of our main Theorem 1.3.7. Next, by Proposition 2.1.11, the equivariant fundamental class $[\Omega_\lambda]^T$ is homogeneous of degree $2I(\lambda)$, and Ω_λ is the union of C_μ for $\mu \leq \lambda$. Hence for $\mu \not\leq \lambda$ the torus fixed point $\mu \notin \Omega_\lambda$ and consequently $[\Omega_\lambda]^T$ restricts to 0 at the torus fixed point μ. In other words, the equivariant fundamental class $[\Omega_\lambda]^T$ is supported below λ. Finally the last axiom is our motivating observation. $\square$

Before we proceed let us see a few examples to keep in mind.

Example 2.3.7. For $\mathbb{P}^1 = Gr(1, 2)$ the GKM graph is given in example 1.2.18. The equivariant Schubert classes are

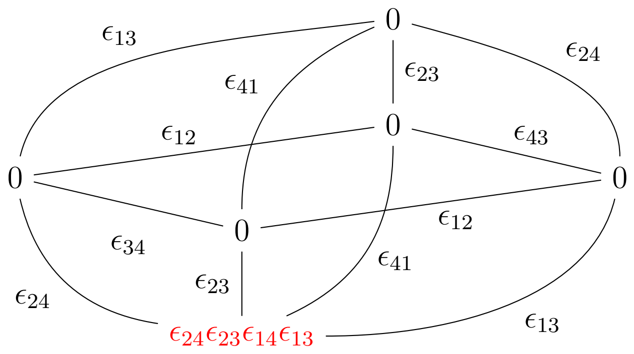
$$\tilde{S}_{01} = \begin{array}{c} 0 \\ \Big| \\ \epsilon_1 - \epsilon_2 \end{array} \qquad \qquad \tilde{S}_{10} = \begin{array}{c} 1 \\ \Big| \\ 1 \end{array}$$

Since the localization map $H^*_T(\mathbb{P}^1) \to \mathrm{Sym}^\bullet \mathfrak{t}^* \times \mathrm{Sym}^\bullet \mathfrak{t}^*$is a ring homomorphism, one could also explicitly describe the multiplication of the equivariant Schubert classes in this case. But this becomes harder for larger Grassmannians.

Example 2.3.8. Consider $Gr(2, 4)$. Recall the GKM graph 2.3.2, the equivariant Schubert classes are listed below, where we abbreviate the labels $\epsilon_{ij} := \epsilon_i - \epsilon_j$.

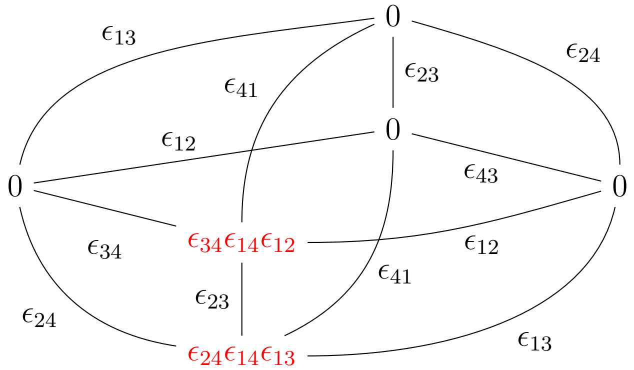
The class of a fixed point:

$\tilde{S}_{1100} =$



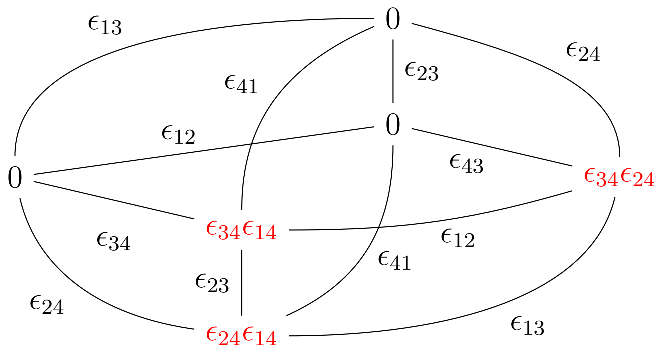
The class of a T-stable line:

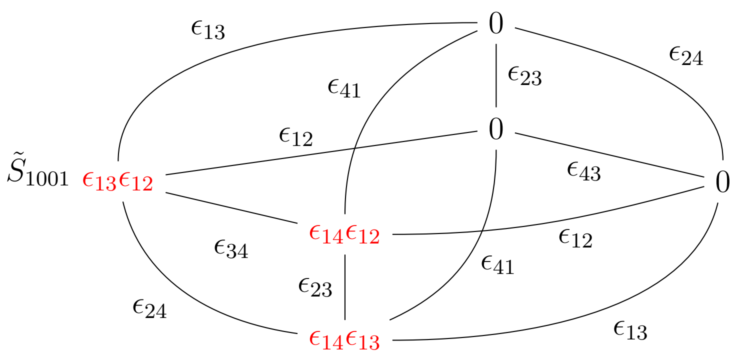
$\tilde{S}_{1010} =$



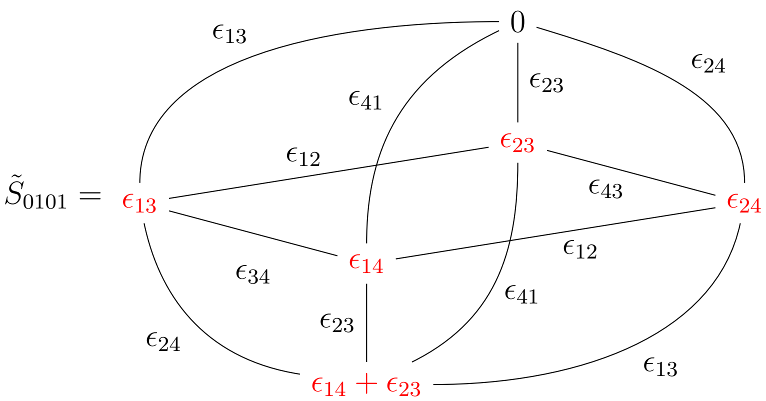
Two codimension 2 Schubert varieties.

$\tilde{S}_{0110} =$

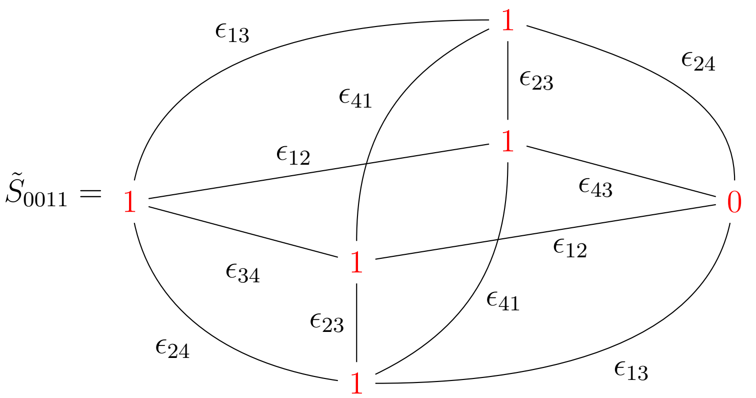




The divisor class, also the class of the unique singular Schubert variety in $Gr(2,4)$.



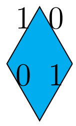
And the identity element is the fundamental class of the total space $Gr(2,4)$.



2.3.4 Equivariant Schubert calculus of Grassmannians

In view of Lemma 2.3.5, the multiplicative structure of $H_T^*Gr(k,N)$ is again determined by the product formula for the equivariant Schubert classes. In this section we state the combinatorial rule in [KT+03] computing the product of two equivariant Schubert classes.

Write $\tilde{S}_\lambda \tilde{S}_\mu = \sum \tilde{c}^\nu_{\lambda\mu} \tilde{S}_\nu$. Recall the identification $H^*_T(\mathrm{pt}, \mathbb{Z}) \cong \mathbb{Z}[\epsilon_1 \ldots . \epsilon_N]$ (2.2). Here the coefficient $\tilde{c}^\nu_{\lambda\mu} \in \mathbb{Z}[\epsilon_1, \ldots, \epsilon_N]$ is homogeneous of degree $2I(\lambda) + 2I(\mu) - 2I(\nu)$. Recall the map $H^*_T Gr(k, N) \to H^* Gr(k, N)$ sending all equivariant parameters $\epsilon_i \mapsto 0$, by construction this map takes each equivariant Schubert class $\tilde{S}_\lambda$ to the Schubert class S_λ, hence $\tilde{c}^\nu_{\lambda\mu}$ give the same coefficient in the usual cohomology if $2I(\lambda)+2I(\mu) = 2I(\nu)$. In particular, they count the puzzles as in Theorem 2.2.6. In order to encode other equivariant coefficients, consider the following *equivariant puzzle piece*: this is the vertical rhombus puzzle with 1 and 0 interchanged.



Warning. Unlike the usual puzzle pieces, this special piece can NOT be rotated. We also don't have an intersection number formula in the equivariant setting.

It turns out that allowing this extra piece in our puzzles yields these general equivariant coefficients $\tilde{c}^\nu_{\lambda\mu}$. But there is one small problem: in the $\mathbb{P}^1$ example 2.3.7 we have

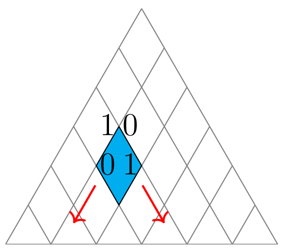
$$\tilde{S}^2_{01} = (\epsilon_1 - \epsilon_2)\tilde{S}_{01}$$

So we have to also produce these extra polynomial coefficients in the puzzles.

Definition 2.3.9. Consider the same notion of puzzles and labeled equilateral triangles, except that we allow this new puzzle piece. To each equivariant puzzle piece p in a puzzle P we associate a weight $wt(p) \in \mathrm{Sym}^\bullet \mathbf{t}^*$ by dropping the rhombi in the SW resp. SE direction until it pokes out the i-th resp. j-th segment on the south side and set $wt(p) := \epsilon_i - \epsilon_j$.

We illustrate this by the following example.

Example 2.3.10. The following equivariant piece has weight $\epsilon_2 - \epsilon_4$.



Definition 2.3.11. To each puzzle P we associate its weight $\prod_p wt(p)$ where p runs over the equivariant puzzle pieces of P. By convention an empty product is 1. Note that by definition the weights are products of $\epsilon_i - \epsilon_j$ with $j > i$.

Theorem 2.3.12 (Puzzle computes equivariant Schubert calculus)**.** *Let $\lambda, \mu, \nu \in \Lambda(k, N)$ and corresponding to $\tilde{S}_\lambda, \tilde{S}_\mu, \tilde{S}_\nu$, then*

$$\tilde{S}_\lambda \tilde{S}_\mu = \sum_{P_{NW}=\lambda, P_{NE}=\mu} wt(P)\tilde{S}_{w_0 P_S}$$

In other words, the structure constant $\tilde{c}^\nu_{\lambda\mu}$ is equal to the sum of all weights of P with $\partial P = \Delta_{\lambda,\mu,w_0\nu}$. In particular,

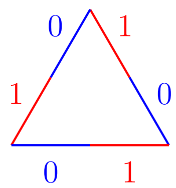
$$\tilde{c}^\nu_{\lambda\mu} \in \mathbb{N}[\epsilon_1 - \epsilon_2, \dots, \epsilon_{N-1} - \epsilon_N]$$

This implies Theorem 2.2.6.

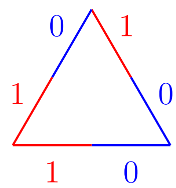
Example 2.3.13. As a reality check, we examine all puzzles in the multiplication table of $H^*(\mathbb{P}^1)$. The reader might want to compare this with example 2.2.8. The multiplication table is computed using example 2.3.7.

	$\tilde{S}_{10}$	$\tilde{S}_{01}$
$\tilde{S}_{10}$	$\tilde{S}_{10}$	
$\tilde{S}_{01}$	$\tilde{S}_{01}$	$(\epsilon_1 - \epsilon_2)\tilde{S}_{01}$

In the puzzles we again color the 1-edges by red and 0-edges by blue and thus omit the 0's and 1's. Start with $\tilde{S}_{10}\tilde{S}_{10} = \tilde{S}_{10}$.

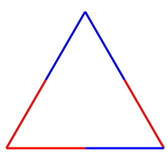


$\tilde{c}_{10,10,01}$

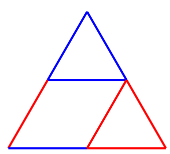


$\tilde{c}_{10,10,10}$

The solutions to these puzzles are the same as before. Next consider the product $\tilde{S}_{10}\tilde{S}_{01} = \tilde{S}_{01}$. We try to complete the following into puzzles.

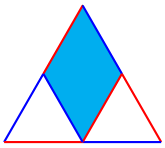


$\tilde{c}_{01,10,10}$

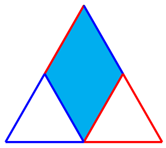


$\tilde{c}_{10,01,01}$

The equivariant piece also does not increase the number of puzzles in this case. Next, for $\tilde{S}_{01} \cup \tilde{S}_{01} = (\epsilon_1 - \epsilon_2)\tilde{S}_{01}$, we count the puzzles with the following boundary conditions.



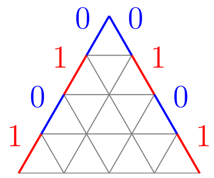
$\tilde{c}_{01,01,10}$



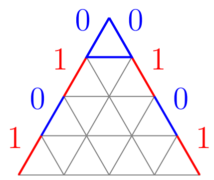
$\tilde{c}_{01,01,01}$

Either case we have to put the equivariant rhombus at the top, then the first puzzle has no solution, and the second has a unique one, with weight $\epsilon_1 - \epsilon_2$. Thus we have checked the multiplication table of $H_T^*(\mathbb{P}^1)$. Compare this with example 1.2.8, we have substantially reduced the abstractness of the matter!

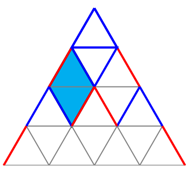
Example 2.3.14. Consider $Gr(2, 4)$, we compute the product $\tilde{S}_{1010}\tilde{S}_{0101}$ by trying to complete the following into a puzzle.

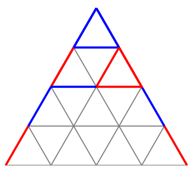


There is only one puzzle piece we can put on the upper most position:

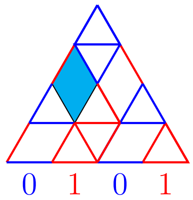


Then for the next 1 on the NE side there are two possibilities:

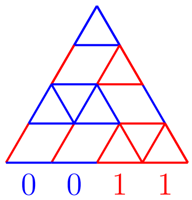


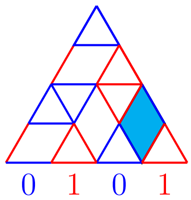


In either case the puzzle piece to the left of our rhombus is determined. Next, in the first case we have only one possible completed puzzle:



In this manner one can check that the following are the other possible puzzles.





Next, the weights of the two equivariant puzzles are $\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_4$ resp. This concludes

$$\tilde{S}_{1010}\tilde{S}_{0101} = \tilde{S}_{0011} + (\epsilon_1 - \epsilon_4)\tilde{S}_{0101}$$

Finally set all equivariant parameters to 0, we get the usual product $S_{1010}S_{0101} = S_{0011}$ is the class of a point. At this point it is desirable to have an algorithm giving the puzzles, instead of really having to play an individual puzzle game each time. Fortunately there is a source provided at `https://doc.sagemath.org/html/en/reference/combinat/sage/combinat/knutson_tao_puzzles.html`. This sage program uses the Knutson-Tao convention about the 01-strings, but their coloring agrees with us.

Chapter 3

Quantum cohomology

To give some motivation, we take a smooth projective variety X (We will only consider $X = Gr(k, N)$) and describe the classical intersection product on the Chow ring A^*X as follows. Consider the diagonal embedding Δ:

$$\begin{tikzcd} X & & \\ & X \times X \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & X \arrow[l, hook', "\Delta"'] \\ X & & \end{tikzcd}$$

Then for any $\alpha, \beta \in A^*X$, we have

$$\Delta^*(\pi_1^{-1}\alpha \cup \pi_2\beta) = \alpha \cup \beta$$

see e.g. [Ful98] §5.2. The "quantum" input is to replace the diagonal embedding by certain moduli spaces $\overline{M}_{0,n}(X, \beta)$ with natural maps

$$\overline{M}_{0,n}(X, \beta) \xrightarrow{\text{ev}} X^n$$

which depends on certain $\beta \in A^{\dim X - 1}$, then pullback n classes $\gamma_1, \ldots \gamma_n$ on X and form the *n-point genus 0 Gromov-Witten invariants*

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \text{ev}^* (\gamma_1 \times \ldots \gamma_n)$$

We shall produce a new product structure on A^*X involving certain I_β's. For an introduction to the moduli space $\overline{M}_{g,n}(X, \beta)$ and the Gromov-Witten invariants we refer to [FP96] and [KV07].

3.1 Quantum cohomology of G/P

In this section we review the construction of the (small) quantum cohomology. Let G be a reductive algebraic group and $P \subset G$ a parabolic subgroup. We define the small quantum cohomology for the smooth variety $X = G/P$, which is projective by e.g. [Spr09] §6.2.

There is a cycle map $A^*X \to H^*(X, \mathbb{Z})$ sending each subvariety to its fundamental class in the Borel-Moore homology and doubling the degree, see [Ful98] chapter 19. Similar to theorem 2.1.15, there are general Schubert cells which give a basis for both sides, we get the following theorem.

Theorem 3.1.1. *The a cycle map $A^iX \to H^{2i}(X, \mathbb{Z})$ is an isomorphism.*

For details we refer to [Kum02], [LG01] or [Bri05].

3.1.1 Definition

We write down again the product formula for the Schubert classes. Denote all the Schubert classes by $U_0, U_1, U_2, \dots U_m \in A^*X$. Denote the intersection numbers

$$g_{ij} = \int_X U_i \cup U_j \in \mathbb{Z}$$

The matrix (g^{ij}) is invertible by Poincaré duality. Denote (g^{ij}) the inverse matrix, then

$$U_i \cup U_j = \sum_{e,f} \left(\int_X U_i \cup U_j \cup U_e \right) g^{ef} U_f$$

One of the original motivations for quantum cohomology is counting the number of certain curves on the projective spaces.

Definition 3.1.2. An *effective curve class* $\beta \in A_1X$ is an $\mathbb{N}$-linear combination of the Schubert classes in $A^{\dim X - 1}X$.

Following [FP96], we shall add the "quantum deformation" terms $I_\beta(U_i \cdot U_j \cdot U_e)$ on the right hand side to get a new composition law of the form

$$U_i * U_j = \sum_{e,f} \left(\int_X U_i \cup U_j \cup U_e \right) g^{ef} U_f + \sum_{\beta > 0} q^\beta I_0(U_i, U_j, U_e) g^{ef} U_f$$

where the sum is over *effective curve classes* β.

The quantum potential

Let $U_0 = 1 \in A^0X$ and let $U_1, \dots U_p$ be the divisor Schubert classes (i.e. the Schubert classes in A^1X). Let $U_0, U_1, U_2, \dots U_m \in A^*X$ be all the Schubert classes. For a class $\gamma \in A^*X$, consider the "potential function"

$$\Phi(\gamma) = \sum_{n\geq 3} \sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n) \tag{3.1}$$

where γ^n is the n-fold cup product of γ. Here $n \geq 3$ because the Gromov-Witten invariants I_β vanish for $n \leq 3$. The next lemma allows to set γ as a formal linear combination of the Schubert classes.

Lemma 3.1.3. *Given any integer n, there can be only finitely many $\beta \in A_1X$ such that $I_\beta(\gamma^n)$ is nonzero for some γ.*

Proof. Let β_i be the Schubert classes in $A^{\dim X-1}$. We need the following two facts, see e.g. [FP96].

- $\int_{\beta_i} c_1(TX) \geq 2$.
- $\dim \overline{M}_{0,n}(X,\beta) = \dim X + \int_\beta c_1(TX) + n - 3$.

Hence $I_\beta(\gamma^n) \neq 0$ only if

$$\int_\beta c_1(TX) \leq \dim X + \int_\beta c_1(TX) + n - 3 = n \deg\gamma \leq n \dim X$$

write $\beta = \sum m_i\beta_i$, and note that the sum 3.1 is over $n \geq 3$, we get

$$\sum 2m_i \leq \sum m_i \int_{\beta_i} c_1(TX) = \int_\beta c_1(TX) \leq n \dim X$$

Since each $m_i \in \mathbb{N}$, there are only finitely many such linear combinations.

□

Next we introduce formal parameters $y_0, \dots y_m$ and replace $\gamma \in A^*X$ above by $\gamma = \sum_{i=0}^m y_iU_i$. Thanks to Lemma 3.1.3, we can consider the following formal power series in $\mathbb{Q}[[y_0, \dots y_m]]$,

$$\Phi(y_0, y_1, \dots, y_m) = \sum_{n_0+\dots n_m\geq 3} \sum_\beta I_\beta(U_0^{n_0}, \dots U_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

We take the formal partial derivatives of the potential function $\Phi(y)$. Denote

$$\Phi_{ijk} = \frac{\partial^3}{\partial y_i \partial y_j \partial y_k}\Phi = \sum_{n_0+\dots n_m\geq 0} \sum_\beta I_\beta(U_0^{n_0}, \dots, U_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

The quantum product

If we proceed with Φ_{ijk} we end up with the *big quantum cohomology.* However, we are interested in the *small quantum cohomology*, i.e. we consider only the formal variables $y_1, \ldots y_p$ corresponding to the divisor Schubert classes.

$$\begin{aligned}\overline{\Phi}_{ijk} := \Phi_{ijk}(y_0, y_1, \ldots y_p, 0, \ldots, 0) = & \sum_{n\geq 0}\sum_{\beta} I_\beta(\gamma^n \cdot U_i \cdot U_j \cdot U_k) \\ = & \sum_{n\geq 0} I_0(\gamma^n \cdot U_i \cdot U_j \cdot U_k) + \overline{\Gamma}_{ijk} \qquad (3.2)\end{aligned}$$

where $\gamma = y_0U_0 + y_1U_1 + \ldots y_pU_p$, and $I_\beta(\gamma^n \cdot U_i \cdot U_j \cdot U_k)$ are $n+3$ pointed Gromov-Witten invariants, and

$$\overline{\Gamma}_{ijk} = \sum_{n\geq 0}\sum_{\beta>0} I_\beta(\gamma^n \cdot U_i \cdot U_j \cdot U_k)$$

is the sum over nonzero β's. We cite the following standard facts.

Theorem 3.1.4. *Let $X = G/P$.*

- *If $n = 3$ then*
$$I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$$
otherwise $I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = 0$ for all γ_i.
- *If $D \in A^1X$, then*
$$I_\beta(D \cdot \gamma_1 \ldots \gamma_n) = \left(\int_\beta D\right) I_0(\gamma_1 \ldots \gamma_n)$$
- *For all $\gamma_i \in A^*X$ we have $I_0(1 \cdot \gamma_1 \ldots \gamma_n) = 0$.*

From these properties we write the two summands of equation 3.2 as

$$\sum_{n\geq 0} I_0(\gamma^n \cdot U_i \cdot U_j \cdot U_k) = \int_X U_i \cup U_j \cup U_k$$

$$\begin{aligned}\overline{\Gamma}_{ijk} &= \sum_{n_i\geq 0}\sum_{\beta>0} I_\beta(U_1^{n_1} \ldots U_p^{n_p} \cdot U_i \cdot U_j \cdot U_k)\frac{y_1^{n_1}}{n_1!} \cdots \frac{y_p^{n_p}}{n_p!} \\ &= \sum_{n_i\geq 0}\sum_{\beta>0} \left(\int_\beta U_1^{n_1} \cdots \int_\beta U_p^{n_p}\right) I_\beta(U_i \cdot U_j \cdot U_k)\frac{y_1^{n_1}}{n_1!} \cdots \frac{y_p^{n_p}}{n_p!} \\ &= \sum_{\beta>0} I_\beta(U_i \cdot U_j \cdot U_k) e^{\int_\beta U_1 y_1} \ldots e^{\int_\beta U_p y_p}\end{aligned}$$

Replace the formal variable $q_i = e^{y_i}$, we get

$$\overline{\Gamma}_{ijk} = \sum_{\beta>0} I_\beta(U_i \cdot U_j \cdot U_k) q_1^{\int_\beta U_1} \dots q_p^{\int_\beta U_p}$$

So the small quantum cohomology essentially only involves the 3-pointed Gromov-Witten invariants. Again by Lemma 3.1.3 applied to $n = 3$, $\overline{\Gamma}_{ijk}$ is a finite sum, hence the above $\overline{\Phi}_{ijk} \in \mathbb{Z}[q]$.

Definition 3.1.5. The small quantum cohomology of a Grassmannian X is the free $\mathbb{Z}[q_1 \dots q_p]$-module $QH^*X := A^*X \otimes_{\mathbb{Z}} \mathbb{Z}[q_1 \dots q_p]$ with multiplication

$$U_i * U_j = \sum_{e,l} \overline{\Phi}_{ije} g^{el} U_l \tag{3.3}$$

Theorem 3.1.6. *Let G be a reductive group and $P \subset G$ a parabolic subgroup. Let $X = G/P$. The small quantum cohomology QH^*X is commutative, associative $\mathbb{Z}[q_1, \dots q_p]$-algebra with unit U_0. Furthermore, it becomes a graded algebra with*

$$\deg q_i = 2 \int_{\beta_i} c_1(TX)$$

where TX is the tangent bundle of X and $\beta_1, \dots \beta_p$ is the dual Schubert class of the divisor Schubert class $U_1, \dots U_p$.

By construction, setting all $q_i = 0$ recovers the original multiplication in A^*X.

Remark 3.1.7. Commutativity follows directly from the symmetry of Gromov-Witten invariants. The difficulty lies completely in the associativity, which gives an easy solution to a highly nontrivial enumerative geometry problem!

Remark 3.1.8. The construction of quantum cohomology is independent of the choice of the basis U_i in the following sense: for another choice of basis of $A^*(X)$ one can define a new product on $A^*(X)[q]$ analogous to 3.3. This again makes $A^*(X)[q]$ a commutative associative algebra, which is canonically isomorphic to the one we have. One could also consider $A^*X \otimes_{\mathbb{Z}} \mathbb{C}$ and obtain the quantum cohomology "with complex coefficients".

Example 3.1.9. Recall 2.1.16, for $Gr(k, N)$ there is a unique Schubert class of codimension $\dim Gr(k, N) - 1$, which is an effective curve class. Thus we abbreviate the Gromov-Witten invariants as

$$I_d(\gamma_1 \cdot \dots \gamma_n) := I_{d[\text{line}]}(\gamma_1 \cdot \dots \gamma_n)$$

There is also only one divisor Schubert class, hence the quantum cohomology is a $\mathbb{Z}[q]$-algebra. To determine the degree of q, Recall Lemma 2.1.4 that $TGr(k,N) \cong \mathscr{S}^\vee \otimes Q$. Hence we have

$$\begin{aligned}\deg q &= 2\int_{[\text{line}]} c_1(TGr(k,N)) \\ &= 2\int_{[\text{line}]} c_1(\mathscr{S}^\vee) + c_1(Q) \\ &= 2N.\end{aligned}$$

Here we used the fact that $c_1(\bigwedge^{N-k} Q) = -c_1(\bigwedge^k S)$ is the divisor Schubert class, see Remark 2.1.16. We also get

$$\dim \overline{M}_{0,3}(X, d[\text{line}]) = \dim Gr(k,N) + Nd. \tag{3.4}$$

In order to evaluate the Gromov-Witten invariants, we often use the following theorem. See [LM11] for a proof.

Theorem 3.1.10. *Let $X = Gr(k,N)$, let $\Gamma_1, \Gamma_2, \Gamma_3$ be Schubert varieties satisfying*

$$\sum \operatorname{codim} \Gamma_i = \dim \overline{M}_{0,n}(X,d) = \dim X + dN$$

then the Gromov-Witten invariant $I_\beta([\Gamma_1]\cdot[\Gamma_2]\cdot[\Gamma_3])$ is equal to the number of smooth *degree d genus 0 curves on X incident to general translates of the Γ_i.*

Remark 3.1.11. To prove properties of the Gromov-Witten invariants we have to allow certain singular curves so as to compactify the moduli space. However, this theorem tells us that to evaluate the Gromov-Witten invariants, it suffices to consider smooth curves.

Example 3.1.12. We use Theorem 3.1.10 to determine the small quantum cohomology of projective spaces. Recall that $A^*\mathbb{P}^{N-1} \cong \mathbb{Z}[U]/(U^N)$, where $U = c_1(\mathcal{O}(1))$ is the unique divisor Schubert class of $\mathbb{P}^1$, and U^i is the class of a codimension i linear subspace. By 3.4 the dimension of the 3-pointed moduli space is

$$\dim \overline{M}_{0,3}(\mathbb{P}^{N-1}, d) = N - 1 + Nd$$

Then $I_d(U^i \cdot U^j \cdot U^k) \neq 0$ only if $i+j+k = N-1+Nd$, in which case we must have $d = 0,1$ because $i,j,k \leq N-1 < N$. Thus we only need to compute $I_1(U^i \cdot U^j \cdot U^{2N-1-i-j})$. For this we take a generic codimension i, j and $2N-1-i-j$ linear subspaces U,V,W resp. which are disjoint for dimension reasons. The fundamental classes of these subspaces are exactly $U^i, U^j, U^{2N-1-i-j}$ resp. On the other hand, generic degree 1 curve is a one dimensional linear subspace, and given one point on each of the U, V, W, there is a unique line through these points. Thus by theorem 3.1.10,

$$I_1(U^i \cdot U^j \cdot U^{2N-1-i-j}) = 1$$

This gives the multiplication law

$$U^i * U^j = \begin{cases} U^{i+j}, & i+j < N \\ qU^{2N-1-i-j}, & i+j \geq N \end{cases}$$

Note that the above multiplication formula indeed respects the grading. Thus we get the presentation

$$QH^*\mathbb{P}^{N-1} = \mathbb{Z}[q, U]/(U^N - q)$$

In either form it can be seen that if we set $q = 0$, then this gives the usual cohomology of $\mathbb{P}^{N-1}$.

3.1.2 A presentation of QH^*X

For general Grassmannians there is also a presentation for the small quantum cohomology due to Siebert-Tian [ST97]. In fact any presentation of A^*X gives one for QH^*X. In this subsection we describe this general story.

Take the polynomial ring $\mathbb{Z}[Z_1, \ldots, Z_r]$ with $\deg Z_i = \deg z_i$. Define the surjective graded algebra homomorphism

$$\phi : \mathbb{Z}[Z_1, \ldots, Z_r] \longrightarrow A^*X$$

by $Z_i \mapsto z_i$. Then $\ker \phi$ is a homogeneous ideal. Thanks to theorem 3.1.6, we can also define an homomorphism of graded algebras

$$\phi' : \mathbb{Z}[q_1, \ldots q_p, Z_1, \ldots, Z_r] \longrightarrow QH^*X$$

by sending $Z_i \mapsto z_i$.

Lemma 3.1.13. *The elements $\phi'(Z_1), \ldots \phi'(Z_r)$ generate QH^*X as a $\mathbb{Z}[q_1, \ldots, q_p]$-algebra.*

Proof. For any multi-index $\underline{i} = (i_1, \ldots . i_r) \in \mathbb{N}^r$ define

$$z^{*\underline{i}} := \underbrace{z_1 * \ldots z_1}_{i_1} * \underbrace{z_2 * \ldots z_2}_{i_2} * \cdots * \underbrace{z_r * \ldots z_r}_{i_r}$$

$$z^{\underline{i}} := z_1^{i_1} \ldots z_r^{i_r}$$

Notice that by definition $z_i * z_j \equiv z_i z_j \mod (q_1, \ldots, q_p)$, thus we have $z^{*\underline{i}} \equiv z^{\underline{i}} \mod (q_1, \ldots, q_p)$. For convenience we will abbreviate $0 = (0, \ldots, 0) \in \mathbb{N}^r$. Let

$S \subset QH^*X$ be the subalgebra generated by z_i. Write any homogeneous element $f \in QH^*X$ as $f = \sum f_{\underline{i}} q^{\underline{i}}$ where $f_{\underline{i}} \in A^*X$. Thanks to the grading, it suffices to show that

$$\exists f' \in S \qquad \deg f' \leq \deg f, \qquad f' - f \equiv 0 \mod (q_1, \dots q_r)^{\deg f} \qquad (\star)$$

This can be proven via induction. There is a polynomial $f'_0 \in \mathbb{Z}[Z_1, \dots Z_r]$ such that

- $\deg f'_0 \leq \deg f$.
- $\phi'(f'_0) - f \equiv \phi'(f'(0)) - f_0 \equiv 0 \mod (q_1, \dots, q_p)$

Next, we can choose $f'_{(1,0,\dots 0)}, \dots f_{(0,\dots,0,1)} \in \mathbb{Z}[Z_1, \dots Z_r]$ of degree $\leq \deg f$, such that

$$\phi'(f'_0 + f'_{(1,0,\dots 0)} q_1 + \dots f_{(0,\dots,0,1)} q_p) - f \equiv 0 \mod (q_1, \dots, q_p)^2$$

etc. This proves $(\star)$. □

Proposition 3.1.14. *Let $X = G/P$. Choose a graded surjective homomorphism*

$$\phi : \mathbb{Z}[Z_1, \dots, Z_r] \cong A^*X$$

and homogeneous generators $f_1, \dots, f_s$ of $\ker \phi$. Let $f'_1 \dots f'_s$ be polynomials in $\mathbb{Z}[q_1, \dots, q_p, Z_1, \dots, Z_r]$ satisfying

- *$f'(0, \dots 0, Z_1, \dots, Z_r) \mapsto 0$ under ϕ;*
- *$f'(q_1, \dots q_p, Z_1, \dots, Z_r) = 0$ in QH^*X.*

Then

$$QH^*X = \mathbb{Z}[q_1, \dots, q_p, Z_1, \dots, Z_r]/(f'_1, \dots f'_s)$$

Proof. Use the arguments in Lemma 3.1.13, mutatis mutandis.

□

Finally we construct the f'_i satisfying the conditions of Proposition 3.1.14. Take any $f \in \ker \phi \subset \mathbb{Z}[Z_1, \dots Z_r]$, viewed as an element in $\mathbb{Z}[q_1, \dots q_r, Z_1, \dots, Z_r]$. Thanks to Lemma 3.1.13, we can fix a $\mathbb{Z}[q_1, \dots, q_p]$-basis $\{z^{*\underline{i}} \mid \underline{i} \in \mathcal{S}\}$ for QH^*X. Then we can write

$$\phi'(f) = \sum_{I \in \mathcal{S}} \xi_{\underline{i}} z^{*\underline{i}} \in QH^*X$$

for uniquely determined polynomials $\xi_{\underline{i}} \in \mathbb{Z}[q_1, \dots q_p]$. Then the polynomial

$$f' = f(Z_1, \dots Z_r) - \sum_{I \in \mathcal{S}} \xi_{\underline{i}} Z^{\underline{i}} \in \ker \phi'$$

and $f' = f$ if we set all $q_i = 0$. So given any presentation of A^*X we can always find a presentation of QH^*X.

3.1.3 Cohomology of Grassmannian Revisited

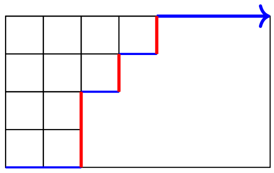
Now we need a presentation of $H^*(Gr(k, N), \mathbb{Z})$, which is also classical.

Partitions and 01-strings

Consider the following set of partitions

$$\mathfrak{P}(k, N) := \{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k \mid N - k \geq \lambda_1 \geq \lambda_2 \dots \geq \lambda_k \geq 0\} \tag{3.5}$$

Homo sapiens find it more convenient to work with diagrams rather than formulae. A partition $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathfrak{P}(k, N)$ is represented by a Young diagram with λ_1 boxes in the top row and λ_2 in the second, etc. Thus this Young diagram is contained in a rectangle of width $N - k$ and height k.



e.g. the above Young diagram represents the partition $(4, 3, 2, 2)$ in $\mathfrak{P}(7, 11)$.

Lemma 3.1.15. *There is a bijection* $F : \mathfrak{P}(k, N) \cong \Lambda(k, N)$ *such that*

$$\mathrm{Inv}(F(\lambda)) = |\lambda| := \sum_{i=1}^{k} \lambda_i \qquad \forall \lambda \in \mathfrak{P}(k, N)$$

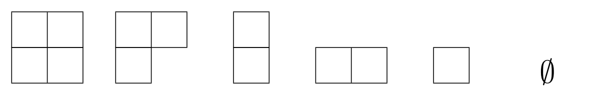
where Inv *is the function defined in 2.1.10.*

Proof. For each partition $\lambda \in \mathfrak{P}(k, N)$ we take the boundary path of the Young diagram, starting from the bottom left corner of the $k \times (N - k)$ rectangle, and we interpret each horizontal line segment as 0 and each vertical segment as 1. This produces a 01-string $F(\lambda)$. Conversely, given a 01-string we can form the boundary path and thus obtain the corresponding partition. It is easy to see that these two operations are mutually inverse. □

By Lemma 3.1.15, the Schubert classes of $Gr(k,N)$ are also labeled by $\mathfrak{P}(k,N)$, and for each partition λ the total number of boxes in its Young diagram is the codimension of the corresponding Schubert class S_λ.

Example 3.1.16. Consider $X = Gr(2,4)$. We list the correspondence between the above two different labels.

0011	0101	0110	1001	1010	1100
(2,2)	(2,1)	(1,1)	(2,0)	(1,0)	(0,0)



Here we omitted the ambient 2×2 square for the Young diagrams.

From this point of view, it is more natural to notice partitions of the form

$$(1^p) := (\underbrace{1,1,\dots,1}_{p},0,\dots 0) \in \mathfrak{P}(k,N)$$

In other words, the corresponding Young diagram has only one column of p boxes. We can also consider the partitions with a single row

$$(p) := (p,\underbrace{0,\dots 0}_{k-1}) \in \mathfrak{P}(k,N)$$

We will see that the Schubert classes labeled by (1^p) generate the Chow ring and the same is true for (p).

Schur polynomials

Partitions are natural labels for Schubert classes, one explanation being the explicit formulae for Schubert classes in terms of *Schur polynomials.* Define the i-th elementary symmetric polynomial $e_i(x_1,\dots x_k)$ in k variables $x_1,\dots x_k$ via the generating series

$$\mathcal{E}(X) := \prod_{i=1}^{k}(1+x_iX) = \sum_{n\geq 0} e_nX^n \in \mathbb{Z}[x_1,\dots x_k][[X]] \tag{3.6}$$

Define the i-th complete symmetric polynomial $h_i(x_1,\dots x_k)$ in k variables by

$$\mathcal{H}(X) := \prod_{i=1}^{k}(1-x_iX)^{-1} = \sum_{n\geq 0} h_nX^n \in \mathbb{Z}[x_1,\dots x_k][[X]] \tag{3.7}$$

Definition 3.1.17. For any partition λ, denote by λ^t the transposed partition, which is by definition

$$\lambda_i^t := \sharp\{j \mid \lambda_j \geq i\}$$

The Young diagram of λ^t is obtained from that of λ by "matrix transposition".

Definition 3.1.18. Define the Schur polynomial via the *Jacobi-Trudi* formula

$$s_\lambda = s_\lambda(x_1, \dots x_k) = \det(e_{\lambda_j^t+i-j}) = \det(h_{\lambda_i-i+j}) \in \mathbb{Z}[x_1, \dots x_k]$$

Observe that $s_{(1^p)} = e_p$, and $s_{(p)} = h_p$.

Of course one has to show that the above two determinants are equal. For this we refer to [Pra19] or [Ful97] chapters 2 and 6.

Now we connect the combinatorics to geometry. Recall the tautological sequence (2.1) on $Gr(k, N)$:

$$0 \longrightarrow \mathscr{S} \longrightarrow Gr(k, N) \times \mathbb{C}^N \longrightarrow \mathscr{Q} \longrightarrow 0$$

Lemma 3.1.19. *There is a ring extension $f^* : H^*(Gr(k, N), \mathbb{Z}) \hookrightarrow A'$ and classes $x_1, \dots x_k \in A'$ such that*

$$f^*c_i(S^\vee) = e_i(x_1, \dots x_k) \qquad f^*c_i(\mathscr{Q}) = h_i(x_1, \dots, x_k) \in A'$$

Proof. By the splitting principle 1.1.18, there is a $f : B' \to Gr(k, N)$ such that the pullback $H^*(Gr(k, N), \mathbb{Z}) \to A^*B'$ is injective and the pullback bundle $f^*S^\vee$ on B' has a filtration with line bundle subquotients. Let $x_1, \dots x_k \in A^*B'$ be the first Chern classes of these subquotients, known as the *Chern roots* of the tautological bundle. By the Whitney sum formula 1.1.15 we get $c_i(S^\vee) = e_i(x_1, \dots x_k)$. To compute the Chern classes of $\mathscr{Q}$, by Whitney sum formula $c(S^\vee)c(\mathscr{Q}) = 1$, and observe that the generating series 3.6 and 3.7 satisfies $\mathcal{E}(X)\mathcal{H}(-X) = 1$. $\square$

Remark 3.1.20. Here we use the dual tautological bundle because we want to avoid signs when we express them in terms of the Schubert classes. e.g. for $\mathbb{P}^{N-1}$ the negative of the divisor Schubert class is $c_1(\mathcal{O}(-1))$.

Now it remains to connect these Chern classes to the Schubert classes.

Proposition 3.1.21. *In $H^*(Gr(k, N), \mathbb{Z})$ hold the following identities:*

$$S_{(i)} = c_i(\mathscr{Q}) \qquad S_{(1^i)} = c_i(S^\vee)$$

Proof. Fix the flag $0 \subset F_0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N$ as in Proposition 2.1.11, the Schubert variety labeled by $\lambda = (\lambda_1, \ldots, \lambda_{N-k}) \in \mathfrak{P}(k, N)$ can be described as

$$\Omega_\lambda = \{V \subset \mathbb{C}^N \mid \dim V \cap F_{N-k-\lambda_i+i} \geq i, \ \forall\, 1 \leq i \leq k\} \tag{3.8}$$

In particular, for $\lambda = (p)$ we have

$$\Omega_{(p)} = \{V \subset \mathbb{C}^N \mid \dim V \cap F_{N-k-p+1} \geq 1\} \tag{3.9}$$

For convenience denote $E = Gr(k, N) \times \mathbb{C}^N$. Consider the projective bundles $\mathbb{P}(\mathscr{S})$ and $\mathbb{P}(E) = \mathbb{P}^{N-1} \times Gr(k, N)$ of the vector bundles $\mathscr{S}, E$ resp. The embedding $\mathscr{S} \hookrightarrow E$ induces an embedding $\eta : \mathbb{P}(\mathscr{S}) \hookrightarrow \mathbb{P}(E)$.

$$\mathbb{P}(\mathscr{S}) \xhookrightarrow{\ \eta\ } \mathbb{P}(E) = \mathbb{P}^{N-1} \times Gr(k, N) \xrightarrow{\ \pi\ } Gr(k, N)$$

By the description 3.9 and the "tautological" property of $\mathscr{S}$, Ω_p is the image of $\mathbb{P}(\mathscr{S}) \cap \mathbb{P}(F_{k-p+1})$ under π. Then we compute

$$S_{(p)} = [\Omega_p] = \pi_* \eta^* [\mathbb{P}(F_{k+1-p})] = \pi_* \eta^* c_1(\mathcal{O}_E(1))^{k-1+p} \tag{3.10}$$

where $\mathcal{O}_E(1)$ is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^{N-1}$ via the projection $\mathbb{P}^{N-1} \times X \to \mathbb{P}^{N-1}$. Next let $\mathcal{O}_{\mathscr{S}}(1) = \eta^* \mathcal{O}_E(1)$, by naturality of the Chern class we rewrite 3.10 as

$$\pi_* \eta^* c_1(\mathcal{O}_E(1))^{k-1+p} = \pi_* c_1(\mathcal{O}_{\mathscr{S}}(1))^{k-1+p} =: s_p(\mathscr{S})$$

where $s_p(\mathscr{S})$ is known as the p-th Segre class of S, satisfying

$$(1 + s_1(\mathscr{S})X + s_2(\mathscr{S})X^2 + \ldots)(1 + c_1(\mathscr{S})X + c_2(\mathscr{S})X^2 + \ldots) = 1$$

where the identity is in $H^*(Gr(k, N), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[X]]$, see [Ful98] §3.2. By the Whitney sum formula we get $c(\mathscr{Q}) = s(\mathscr{S})$, hence we conclude

$$S_{(p)} = [\Omega_p] = s_p(\mathscr{S}) = c_p(\mathscr{Q}), \qquad S_{(1^p)} = c_p(S^\vee)$$

□

Corollary 3.1.22. *Let $x_1, \ldots, x_k$ be the Chern roots of $S^\vee$. For any $\lambda \in \mathfrak{P}(k, N)$ the Schubert class $S_\lambda = s_\lambda(x_1, \ldots x_k)$. In particular, as a $\mathbb{Z}$-algebra $H^*(Gr(k, N), \mathbb{Z})$ is generated by $S_{(1^p)}$ for $1 \leq p \leq k$.*

Proof. In view of our Definition 3.1.18, it suffices to cite the following result (the Giambelli formula)

$$S_\lambda = \det\left(c_{\lambda_i - i + j}(\mathscr{Q})\right)_{i,j} = \det\left(c_{\lambda_i^t + i - j}(S^\vee)\right)_{i,j}$$

which is a corollary of the Kempf-Larksov formula for degeneracy loci by considering the quotient bundle $\mathscr{Q}$ and the trivial bundle $Gr(k,N) \times \mathbb{C}^N$ with the filtration $Gr(k,N) \times F_i$ for the flag $0 \subset F_0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N$ as in Proposition 2.1.11. For more details we refer to [Ful98] chapter 14, or to [And11] for an equivariant treatment. $\square$

Remark 3.1.23. There is an asymmetry in the Schur polynomials. Note that by Definition 3.1.18 the Schur polynomials in k formal variables lie in the ring of symmetric polynomials $\mathbb{Z}[e_1, \ldots, e_k]$. The elementary symmetric polynomial $e_r = 0$ for $r > k$, but h_r is nonzero for all r. Thus sometimes it is convenient to pass to the ring of symmetric functions

$$\mathbb{Z}[e_1, \ldots, e_k] \longrightarrow \varinjlim_n \mathbb{Z}[e_1, \ldots, e_n]$$

The canonical image of s_λ in the ring of symmetric functions is called the *Schur function* corresponding to λ.

Theorem 3.1.24 (Pieri rule for Schur functions). *For any partition λ let s_λ be the corresponding Schur function. We have*

$$s_{(1^p)} s_\lambda = \sum s_\mu \in \varinjlim_n \mathbb{Z}[e_1, \ldots, e_n]$$

where the sum is over partitions μ whose Young diagram is obtained from adding p boxes to that of λ, with no two added to the same row. Dually, for any partition λ we have

$$s_{(p)} s_\lambda = \sum s_\mu$$

where the sum is over partitions μ whose Young diagram is obtained from adding p boxes to that of λ, with no two added to the same column.

Corollary 3.1.25 (Pieri rule for Grassmannians). *For any $\lambda \in \mathfrak{P}(k,N)$ the product of the Schubert classes is computed as follows:*

$$S_{(1^p)} S_\lambda = \sum s_\mu$$

where the sum is over $\mu \in \mathfrak{P}(k,N)$ whose Young diagram is obtained from adding p boxes to that of λ, with no two added to the same row.

Remark 3.1.26. The Pieri rule for Grassmannians can be obtained from the *equivariant Pieri rule*, which can be proved using the *divided difference operators* thanks to the localization theorems. See [KT+03]. For a combinatorial proof, we refer to [Ful97] chapter 6 or [Pra19].

A presentation of $H^*(Gr(k,N),\mathbb{Z})$

Now recall that we need a presentation of $H^*(Gr(k,N),\mathbb{Z})$. Since rank $\mathscr{Q} = N-k$, we have the natural vanishing

$$S_{(r)} = 0 \in H^*(Gr(k,N),\mathbb{Z}) \qquad N-k+1 \leq r \leq N$$

Let $e_1,\dots e_k$ be symmetric polynomials in k variables, with $\deg e_i = 2i$. By a slight abuse of language we also write

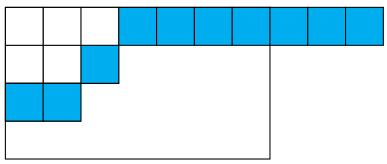
$$s_\lambda(e_1,\dots e_k) := \det(e_{\lambda_i^t - i+j}) \in \mathbb{Z}[e_1,\dots e_k]$$

Recall that the complete symmetric polynomial $h_i = s_{(i)} \in \mathbb{Z}[e_1,\dots e_k]$.

Theorem 3.1.27. *There is a canonical isomorphism of graded $\mathbb{Z}$-algebras*

$$\Phi : \mathbb{Z}[e_1,\dots e_k]/(h_{N-k+1},\dots,h_N) \cong A^*Gr(k,N)$$
$$s_\lambda(e_1,\dots e_k) \longmapsto S_\lambda$$

Proof. This map is clearly well defined and surjective in view of corollary 3.1.22. To see injectivity, view $\mathbb{Z}[e_1,\dots e_{N-k}]$ as the ring of symmetric polynomials in k variables, then it has a basis given by the Schur polynomials labeled by partitions $(\lambda_1,\dots\lambda_k) \in \mathbb{N}^k$, see [Ful97].



Thus it suffices to show that s_λ is in the ideal generated by $h_{N-k+1},\dots h_N$ for $\lambda \notin \mathfrak{P}(k,N)$. In this case $\lambda_1 > N-k$, and by Pieri's rule we can write $s_\lambda = s_\mu h_{\lambda_1}$, as illustrated by the above picture. Then it suffice to prove this for $\lambda = (r)$, $r > N$. This can be done using induction and the Pieri rule for multiplying by e_1. Finally the degree of the Schur polynomials can be calculated from the Jacobi-Trudi formula 3.1.18 as follows:

$$\begin{aligned} s_\lambda &= \det(h_{\lambda_i - i+j}) \\ &= \sum_{\pi\in\mathfrak{S}_k} \mathrm{sgn}\pi \prod h_{\lambda_i - i+\pi(i)} \end{aligned}$$

here $\mathfrak{S}_k$ denotes the symmetric group. Since $\deg h_i = 2i$,

$$\deg \prod h_{\lambda_i - i + \pi(i)} = 2\left(\sum_{i=1}^{k} \lambda_i - i + \pi(i)\right) = 2|\lambda|$$

which is exactly the degree of the Schubert class S_λ. $\square$

3.1.4 A presentation of $QH^*Gr(k, N)$

Now thanks to Proposition 3.1.14 and theorem 3.1.27, we have a surjective homomorphism of graded $\mathbb{Z}$-algebras

$$\Phi' : \mathbb{Z}[e_1, \dots e_k][q] \longrightarrow QH^*Gr(k, N) \tag{3.11}$$

where $\mathbb{Z}[e_1, \dots e_k]$ is the ring of symmetric polynomials in k variables.

Lemma 3.1.28. *Suppose $N - k < r < N$. Then*

$$\Phi' h_r = 0 \in QH^*Gr(k, N)$$

Proof. Let $\lambda = (r)$ for some $N-k < r < N$. Since Φ' is an algebra homomorphism, we have $\Phi' s_\lambda = \Phi' h_r$. Thus it suffices to show that

$$\begin{aligned}\det(\Phi' h_{\lambda_i - i + j}) &= \sum_{\pi \in \mathfrak{S}_k} (\operatorname{sgn}\pi) \Phi' h_{\lambda_1 - 1 + \pi(1)} * \Phi' h_{\lambda_2 - 2 + \pi(2)} * \cdots * \Phi' h_{\lambda_k - k + \pi(k)} \\ &= 0\end{aligned}$$

Recall that $\deg q = 2N$ in $QH^*Gr(k, N)$, and the quantum product respects the grading, hence $I_d(S_\lambda \cdot S_\mu \cdot S_\nu) = 0$ for $d > 0$, unless

$$\deg S_\lambda + \deg S_\mu = 2|\lambda| + 2|\mu| \geq 2N$$

On the other hand, $\mathbb{Z}[e_1, \dots e_k][q]$ is also graded, with $\deg s_\lambda = 2|\lambda|$. The homomorphism Φ' respects this grading, hence for $r < N$ and $\lambda = (r)$,

$$\deg \Phi' h_{\lambda_1 - 1 + \pi(1)} * \Phi' h_{\lambda_2 - 2 + \pi(2)} * \cdots * \Phi' h_{\lambda_k - k + \pi(k)} = 2|\lambda| = 2r < 2N$$

Which means that all quantum products must not produce any q's, otherwise the degree wouldn't match. So we conclude that

$$\Phi'(s_{(r)}) = \Phi(h_r) = 0 \qquad N - k < r < N$$

where Φ is as in 3.1.27. $\square$

Lemma 3.1.29. *In $QH^*Gr(k,N)$ we have $\Phi'(h_N)=q$.*

Proof. Use the generating functions 3.6 3.7 to get

$$\begin{aligned}\Phi'(h_N) &= \Phi'(e_1h_{N-1}-e_2h_{N-2}+\cdots+(-1)^{k-1}h_{N-k}e_k)\\ &= (-1)^{k-1}\Phi'(h_{N-k}e_k)\\ &= (-1)^{k-1}S_{(N-k)}*S_{(1^k)}\end{aligned}$$

thanks to the vanishing of $\Phi'(h_{N-k+1}),\dots,\Phi'(h_{N-1})$. So we only need to compute $S_{N-k}*S_{(1^k)}$. By Pieri's rule the classical product $S_{(N-k)}S_{(1^k)}=0$. On the other hand, $\deg S_{(N-k)}*S_{(1^k)}=2N=\deg q$, we must have

$$S_{(N-k)}*S_{(1^k)}=\sum_\mu I_1(S_{N-k}\cdot S_{(1^k)}\cdot S_\mu)q$$

Recall 3.4 that the dimension of the moduli space is

$$\dim \overline{M}_{0,3}(Gr(k,N),1)=\dim Gr(k,N)+N$$

Hence the Gromov-Witten invariant $I_1(S_{N-k}\cdot S_{(1^k)}\cdot S_\mu)$ is 0 unless $|\mu|=\dim Gr(k,N)$. Thus we have

$$S_{(N-k)}*S_{(1^k)}=\sum_\mu I_1(S_{N-k}\cdot S_{(1^k)}\cdot[\mathrm{pt}])q$$

Now we have to evaluate the Gromov-Witten invariant. Thanks to theorem 3.1.10, we count the number of lines in $Gr(k,N)$ that meets generic translates of the Schubert varieties $\Omega_{(1^k)},\Omega_{(N-k)}$ and a point. Recall equation 3.8, these Schubert varieties are

$$\begin{aligned}\Omega_{(N-k)} &= \{V\subset\mathbb{C}^N \mid V\supset F_1\}\\ \Omega_{(1^k)} &= \{V\subset\mathbb{C}^N \mid V\subset F_{N-1}\}\end{aligned}$$

So a generic translation of $\Omega_{(1^k)}$ is

$$g'\Omega_{(1^k)}=\{V\subset\mathbb{C}^N \mid V\subset B\}$$

where B is a $N-1$ dimensional subspace not containing F_1, which we may take to be the span of $e_1,\dots e_{N-1}$; finally a generic point is a k dimensional subspace not in B and not containing F_1, which we may take to be the span of $e_2,\dots e_k,e_1+e_N$.

One final ingredient is Lemma 3.1.30 below. Note that there is a unique line meeting $\Omega_{(N-k)},g'\Omega_{(1^k)}$ and a generic point, namely

$$L=\{V\subset\mathbb{C}^N \mid e_2,\dots e_k\in V\subset\operatorname{span}\{e_1,\dots e_k,e_N\}\}\cong\mathbb{P}^1$$

Thus we conclude that

$$I_1(S_{N-k}\cdot S_\lambda\cdot[\mathrm{pt}])=1$$

therefore $S_{N-k}*S_k=I_1(S_{N-k}\cdot S_\lambda\cdot[\mathrm{pt}])q=q$. $\square$

Lemma 3.1.30. *A degree 1 embedding $L : \mathbb{P}^1 \to Gr(k, N)$ is a line*

$$L = \{V \subset \mathbb{C}^N \mid U \subset V \subset W\} \cong \mathbb{P}^1$$

where $U, W \subset \mathbb{C}^N$ are linear subspaces of dimension $k-1, k+1$ resp.

Proof. Consider the Plücker embedding

$$\begin{aligned} Gr(k,N) &\xhookrightarrow{\mathrm{Pl}} \mathbb{P}^m = \mathbb{P}\left(\bigwedge^k \mathbb{C}^N\right) \\ \operatorname{span}\{v_i \mid 1 \leq i \leq k\} &\longmapsto \mathbb{C}v_1 \wedge \ldots v_k \end{aligned}$$

where $m = \binom{N}{k} - 1$. Denote the standard basis $\mathbb{P}^m$ by

$$\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \mid I = \{i_1, \ldots i_k\} \subset \{1, 2, \ldots N\},\ |I| = k\}$$

The corresponding homogeneous coordinates $\{x_I\}$ are viewed as the global sections of $\mathcal{O}_{\mathbb{P}^m}(1)$. Recall Remark 2.1.16, $\bigwedge^k \mathscr{S}^k$ is the pullback of $\mathcal{O}_{\mathbb{P}^m}(1)$ to $Gr(k, N)$ along Pl, and since f has degree 1, the pullback of $\mathcal{O}_{\mathbb{P}^m}(1)$ to $\mathbb{P}^1$ via $\mathrm{Pl} \circ L$ is again $\mathcal{O}(1)$. This implies that the image of $\mathbb{P}^1$ under $\mathrm{Pl} \circ L$ is a linear subspace in $\mathbb{P}^m$.

Now, up to a linear transformation on $\mathbb{C}^N$, we may assume that the subspace spanned by coordinate vectors $e_1, \ldots e_k$ viewed as a point on the Grassmannian is in the image of L. Then $\mathrm{Pl} \circ L(\mathbb{P}^1)$ contains the point $p_0 := \mathbb{C}e_1 \wedge \cdots \wedge e_k$ whose only nonzero coordinate is $x_{I_0} = 1$. Such a line on $\mathbb{P}^m$ has the form

$$\{[\underset{I_0}{b} : 0 : \cdots : 0 : ac_{I_1} : \cdots : ac_{I_r} : 0 : \ldots] \mid [a : b] \in \mathbb{P}^1, c_{I_j} \in \mathbb{C} \text{ are fixed}\}$$

The tangent space of this line at p_0, identified as a subspace of $T_{p_0}\mathbb{P}^m$, is spanned by the tangent vector

$$e^{I_0} \otimes (c_{I_1} e_{I_1} + \ldots c_{I_r} e_{I_r}) \in T_{p_0}\mathbb{P}^m \tag{3.12}$$

where the convention is as in 2.1.4. On the other hand, the tangent space of $T_{p_0}Gr(k, N) \subset T_{p_0}\mathbb{P}^m$ is spanned by

$$\{e^{I_0} \otimes e_{J_{ij}} \in T_{p_0}\mathbb{P}^m \mid i \in I_0,\ j \notin I_0 \quad J_{ij} = I_0 - \{i\} + \{j\}\}$$

which implies that all I_j in 3.12 are of the form J_{ij}. Now consider the vectors $v_i = \alpha_{il} e_j$ with coefficient matrix

$$(\alpha_{il}) = \begin{pmatrix} b & \ldots & 0 & \ldots & ac_{J_{1,k+1}} & \ldots & ac_{J_{1,N}} \\ & 1 & \ldots & 0 & 0 & \ldots & \\ & & \ddots & \vdots & 0 & \ldots & \\ & & & 1 & 0 & \ldots & \end{pmatrix}$$

For each $[a : b] \in \mathbb{P}^1$, we get a point in $Gr(k, N)$ whose Plücker coordinates are the $k \times k$ minors of (α_{il}). Thus we obtain an embedded line L' in $Gr(k, N)$ with $\deg L' = 1$ because it meets the divisor Schubert variety

$$\Omega_{(1)} = \{V \in Gr(k, N) \mid e_1 \in V\}$$

only at p_0, which is a transversal intersection. Thus $\mathrm{Pl}\, L'$ is a line on $\mathbb{P}^m$ such that

$$p_0, [0 : \dots : 0 : c_{I_1} : \dots : c_{I_r} : 0 : \dots] \in \mathrm{Pl}\, L' \cap \mathrm{Pl}\, L$$

This proves that $L' = L$ because Euclid said that there is only one line through two given points.

Finally, it is easy to see that L' has the desired form with $U = \mathrm{span}\{e_2, \dots e_k\}$ and

$$W = \mathrm{span}\{e_1, \dots, e_k, c_{I_1} e_{I_1} + \dots c_{I_r} e_{I_r}\} \qquad \square$$

Assembling lemmata 3.1.28, 3.1.29 together with Proposition 3.1.14, we get the following theorem.

Theorem 3.1.31 (Siebert-Tian)**.** *Let $e_1, \dots e_k$ be elementary symmetric polynomials in k variables. There is a graded $\mathbb{Z}[q]$-algebra isomorphism*

$$Z[q, e_1, \dots e_k]/(h_{N-k+1}, \dots h_{N-1}, h_N + (-1)^k q) \cong QH^* Gr(k, N)$$

$$e_i \longmapsto S_{(1^k)}$$

where $h_r(e_1, \dots e_{N-k}) = \det(e_{1+i-j})_{1 \le i,j \le r}$ *and* $\deg e_i = 2i, \deg q = 2N$.

3.2 A fermion model of quantum cohomology

We next give a combinatorial description of the quantum cohomology of the Grassmannian following [KS10], providing some detailed proof.

3.2.1 Fermions on a circle

Inspired by physics, we consider k "fermions" on a circle with N slots, i.e. in each slot there is at most one particle. We also refer to such particle configurations as states. Number the slots by integers $1, 2, \dots, N$ clockwise, then the set of states with k particles is identified with the following sets:

$$\begin{aligned} \Lambda(k, N) &\xrightarrow{\sim} \mathfrak{P}(k, N) \\ w &\longmapsto \lambda(w) \\ w(\lambda) &\longleftarrow\!\shortmid \lambda \end{aligned} \tag{3.13}$$

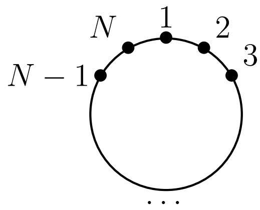


Figure 3.1: N slots on a circle, each can be occupied by at most one particle.

see Lemma 3.1.15 for this identification. In what follows we will always use Latin letters for 01-strings and reserve the Greek letters for partitions.

Let $\mathcal{F}_k := \mathbb{C}\Lambda(k,N) \cong \bigwedge^k \mathbb{C}^N \cong \mathbb{C}\mathfrak{P}(k,N)$, where the first isomorphism is given by

$$w = 00\ldots \underset{i_1}{1} \ldots \underset{i_k}{1} \ldots 00 \longmapsto e_{i_k} \wedge \ldots e_{i_1}$$

We often denote the empty wedge by $\emptyset_0 \in \mathcal{F}$, which corresponding to the string $00\ldots 0$. Consider the *state space*

$$\mathcal{F} := \bigoplus_{k=0}^{N} \mathcal{F}_k \cong \bigwedge^k \mathbb{C}^N$$

Definition 3.2.1. Let $e^i, \ldots, e^N$ the dual basis, extended to operators $\bigwedge^k \mathbb{C}^N \to \bigwedge^{k-1} \mathbb{C}^N$. Consider the following natural linear operators on $\mathcal{F}$:

$$\psi_i^*(w) = e_i \wedge w$$

$$\psi_i(w) := e^i(w)$$

Remark 3.2.2. In terms of physics, the empty wedge $\emptyset_0$ is called the vacuum vector, i.e. the state with no particles. Up to a sign, ψ^*, ψ are the creation resp. annihilation operators of the i-th particle, and the notation $\mathcal{F}$ comes from the *Fock space*, which we introduce later.

Proposition 3.2.3 (Clifford algebra)**.** *Consider the subalgebra $\mathfrak{C} \subset \mathrm{End}_{\mathbb{C}}\mathcal{F}$ generated by ψ_i, ψ_i^*. In $\mathrm{End}_{\mathbb{C}}(\mathcal{F})$ hold the following relations*

$$\psi_i\psi_j = -\psi_j\psi_i \qquad \psi_i^*\psi_j^* = -\psi_j^*\psi_i^*$$

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}$$

for $1 \le i,j \le N$. Furthermore, $\mathfrak{C}$ is isomorphic to the 2^{2N} dimensional Clifford algebra $\mathfrak{C}'$.

Proof. To check the relations, we calculate

$$e_i \wedge e_j \wedge v = -e_j \wedge e_i \wedge v$$

for any $v \in \bigwedge^k \mathbb{C}^N$. This gives $\psi_i^* \psi_j^* = -\psi_j^* \psi_i^*$. Next, if $i = j$ then we have $\psi_i^2 = 0$ so the relation is trivial. If $i \neq j$ we may assume that $i < j$, then

$$e^i e^j \left(e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N}\right) = \begin{cases} (-1)^{s-1+t-1} e_{i_1} \wedge \dots \hat{e}_{i_s} \wedge \dots \hat{e}_{i_t} \wedge \dots e_{i_N} & i_s = i, i_t = j \\ 0 & \text{otherwise} \end{cases}$$

$$e^j e^i \left(e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N}\right) = \begin{cases} (-1)^{s-1+t-2} e_{i_1} \wedge \dots \hat{e}_{i_s} \wedge \dots \hat{e}_{i_t} \wedge \dots e_{i_N} & i_s = i, i_t = j \\ 0 & \text{otherwise} \end{cases}$$

where $\hat{e}_i$ means no such term in the wedge. So we get $\psi_i \psi_j = -\psi_j \psi_i$. Next, we have

$$e_i \wedge e^i \left(e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N}\right) = \begin{cases} e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N} & \text{if some } i_l = i \\ 0 & \text{otherwise} \end{cases}$$

$$e^i \left(e_i \wedge \left(e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N}\right)\right) = \begin{cases} 0 & \text{if some } i_l = i \\ e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N} & \text{otherwise} \end{cases}$$

hence $\psi_i \psi_i^* + \psi_i^* \psi_i = 1$. This checks the Clifford relations, therefore there is a surjective $\mathbb{C}$-algebra homomorphism $\mathfrak{C}' \to \mathfrak{C}$ from the Clifford algebra $\mathfrak{C}'$ to $\mathfrak{C}$. To see that this is injective, note that there is a PBW type basis of $\mathfrak{C}'$ given by

$$\{\psi_1^{\epsilon_1} \dots \psi_N^{\epsilon_N} (\psi_1^*)^{\epsilon_1'} \dots (\psi_N^*)^{\epsilon_N'} \mid \epsilon_i, \epsilon_i' \in \{0, 1\}\}$$

We abbreviate

$$\psi^\varepsilon = \psi_1^{\epsilon_1} \dots \psi_N^{\epsilon_N} (\psi_1^*)^{\epsilon_1'} \dots (\psi_N^*)^{\epsilon_N'} \qquad \forall\, \varepsilon = (\epsilon_1, \dots \epsilon_N, \epsilon_1', \dots \epsilon_N')$$

Suppose in $\mathfrak{C} \subset \mathrm{End}(\mathcal{F})$ holds

$$\sum a_\varepsilon \psi^\varepsilon = 0 \qquad a_\varepsilon \in \mathbb{C}$$

For each $1 \leq i \leq N$ we compute its action on the wedge

$$e_1 \wedge \dots \hat{e}_i \wedge \dots e_N := e_1 \wedge \dots e_{i-1} \wedge e_{i+1} \wedge \dots e_N$$

$$\begin{aligned} \sum a_\varepsilon \psi^\varepsilon e_1 \wedge \dots \hat{e}_i \wedge \dots e_N &= \sum a_\varepsilon (e^1)^{\epsilon_1} \dots (e^N)^{\epsilon_N} e_1 \wedge \dots \wedge e_N \\ &= \sum \pm a_\varepsilon e_1^{1-\epsilon_1} \wedge \dots \wedge e_N^{1-\epsilon_N} = 0 \end{aligned}$$

where the sums are over all $\varepsilon = (\epsilon_1, \dots \epsilon_N, \epsilon'_1, \dots \epsilon'_N)$ such that $\epsilon'_i = 1, \epsilon'_j = 0$ for all $j \neq i$. Two such ε is equal iff all their first N entries coincide. Thus for different ε the wedges $e_1^{1-\epsilon_1} \wedge \dots \wedge e_N^{1-\epsilon_N}$ are distinct, and thus linearly independent in $\mathcal{F}$. This implies that all $a_\varepsilon = 0$ for such ε. Then we calculate the action of $\sum a_\varepsilon \psi^\varepsilon$ on the wedge $e_1 \wedge \dots \hat{e}_i \wedge \dots \hat{e}_j \wedge \dots \wedge e_N$.

$$\begin{aligned}\sum a_\varepsilon \psi^\varepsilon e_1 \wedge \dots \hat{e}_i \wedge \dots \hat{e}_j \wedge \dots \wedge e_N &= \sum a_\varepsilon (e^1)^{\epsilon_1} \dots (e^N)^{\epsilon_N} e_1 \wedge \dots \wedge e_N \\ &= \sum \pm a_\varepsilon e_1^{1-\epsilon_1} \wedge \dots \wedge e_N^{1-\epsilon_N} = 0\end{aligned}$$

where the sums are over all $\varepsilon = (\epsilon_1, \dots \epsilon_N, \epsilon'_1, \dots \epsilon'_N)$ such that $\epsilon'_i = \epsilon'_j = 1, \epsilon'_l = 0$ for all $l \neq i, j$. This is because we have shown that all $a_\varepsilon = 0$ for ε having only one nonzero ϵ'_i, hence no $N-1$ wedges appear in the first sum. Verbatim repetition of the argument for the previous case gives all such $a_\varepsilon = 0$. Inductively, we can show that all $a_\varepsilon = 0$. This concludes injectivity. □

Remark 3.2.4. Notice that $\dim \mathcal{F} = 2^N$ hence $\dim \operatorname{End}(\mathcal{F}) = 2^{2N}$ is exactly the dimension of the Clifford algebra, hence the creation and annihilation operators actually generate $\operatorname{End}(\mathcal{F})$, and thus we get an identification $\mathfrak{C} \cong \operatorname{End}(\mathcal{F})$. Since $\mathcal{F}$ is an irreducible module over $\operatorname{End}(\mathcal{F})$, it follows that $\mathcal{F}$ is an irreducible representation of the Clifford algebra.

Lemma 3.2.5. *Consider the scalar product*

$$\langle \alpha w, \beta w \rangle = \bar{\alpha}\beta\delta_{w,w'}$$

where bar denotes complex conjugation. Then

$$\langle \psi_i w, w' \rangle = \langle w, \psi_i^* w' \rangle$$

for all $w, w' \in \mathcal{F}$.

Proof. By definition $\langle w, \psi_i^* w' \rangle \neq 0$ only if $w = \pm\psi_i^* w' \neq 0$. Suppose this is the case, then $\langle w, \psi_i^* w' \rangle = \pm 1$, and we check that

$$\begin{aligned}\langle \psi_i w, w' \rangle &= \langle \pm\psi_i\psi_i^* w', w' \rangle \\ &= \langle \pm w', w' \rangle = \pm 1 \\ &= \langle w, \psi_i^* w' \rangle\end{aligned}$$

where the second line is because $\psi_i^* w' \neq 0 \implies \psi_i w' = 0$, then the Clifford relation gives $w' = (\psi_i\psi_i^* + \psi_i^*\psi_i)w' = \psi_i\psi_i^* w'$.

There is another possibility: if $\psi_i^* w' = e_i \wedge \psi_i \neq \pm w$, then we also have $\psi_i w \neq \pm w'$, hence $\langle w, \psi_i^* w' \rangle = \langle \psi_i w, w' \rangle = 0$. □

Apparently $\mathcal{F}$ is generated over $\mathfrak{C}$ by the vacuum vector $\emptyset_0$. In fact we have a triangular decomposition $\mathfrak{C} = \mathfrak{C}_- \otimes \mathbb{C}\,\mathrm{id} \otimes \mathfrak{C}_+$, where $\mathfrak{C}_+$ is the subalgebra generated by ψ_i^*, and $\mathfrak{C}_-$ is the subalgebra generated by ψ_i. Then clearly $\mathfrak{C}_-$ annihilates $\emptyset_0 \in \mathcal{F}$, in other words, $\mathbb{C}\emptyset_0$ is a $\mathfrak{C}_-$-module.

Lemma 3.2.6. *$\mathcal{F}$ is a free $\mathfrak{C}_+$-module of rank 1 generated by $\emptyset_0 \in \mathcal{F}$. Thus it is obtained from the one dimensional C_--module $\mathbb{C}\emptyset_0$ via induction:$\mathcal{F} = \mathfrak{C} \otimes_{\mathfrak{C}_-} \mathbb{C}\emptyset_0$.*

3.2.2 Particle hopping

Consider the following "particle hopping" operators

$$u_i := \psi_{i+1}^* \psi_i \qquad 1 \le i \le N-1 \tag{3.14}$$

We also add in the quantum parameter q. Take the $\mathbb{C}[q]$-module $\mathcal{F}[q]$, we still have the ψ_i, ψ_i^* and thus u_i action on $\mathcal{F}[q]$ extended $\mathbb{C}[q]$-linearly. In order to capture the "quantum behaviour", define an "affine" operator

$$u_N := (-1)^{k-1} q \psi_1^* \psi_N \text{ on } \mathcal{F}_k[q] \tag{3.15}$$

The algebra generated by the u_i's is not new to representation theory.

Definition 3.2.7. The *nil-Temperley-Lieb algebra* $0\text{-}TL_N$ is an associative $\mathbb{C}$-algebra with generators $\hat{u}_i$ for $1 \le i \le N-1$ and relations

$$\hat{u}_i^2 = \hat{u}_i \hat{u}_{i+1} \hat{u}_i = \hat{u}_{i+1} \hat{u}_i \hat{u}_{i+1} = 0 \qquad \hat{u}_i \hat{u}_j = \hat{u}_j \hat{u}_i \text{ for } |i-j| > 1 \tag{3.16}$$

The *affine nil-Temperley-Lieb algebra* $0\text{-}\widehat{TL}_N$ is an associative $\mathbb{C}$-algebra with generators $\hat{u}_i$ for $1 \le i \le N$ and relations

$$\hat{u}_i^2 = \hat{u}_i \hat{u}_{i+1} \hat{u}_i = \hat{u}_{i+1} \hat{u}_i \hat{u}_{i+1} = 0 \qquad \hat{u}_i \hat{u}_j = \hat{u}_j \hat{u}_i \text{ for } i-j \not\equiv \pm 1 \mod N \tag{3.17}$$

where all indices are understood modulo N.

Proposition 3.2.8. *Let $N \ge 2$, the $\mathbb{C}$-subalgebra generated by $\{u_i \mid 1 \le i \le N-1\}$ of $\mathrm{End}(\mathcal{F})$ is isomorphic to the nil-Temperley-Lieb algebra $0\text{-}TL_N$.*

Proof. **Step 1.** We check that the obvious assignment $u_i \mapsto \hat{u}_i$ defines an algebra homomorphism $0\text{-}TL_N \to \mathrm{End}(\mathcal{F})$. First write the Clifford relations 3.2.3 as

$$\psi_i^2 = (\psi_i^*))^2 = 0 \qquad \psi_i \psi_j^* = -\psi_j \psi_i^* \ \forall\, i \ne j$$

Then this is straightforward computation:

$$u_i^2 = \psi_{i+1}^* \psi_i \psi_{i+1}^* \psi_i = -\psi_{i+1}^* \psi_{i+1}^* \psi_i \psi_i = 0$$

$$u_i u_{i+1} u_i = \psi^*_{i+1}\psi_i\psi^*_{i+2}\psi_{i+1}\psi^*_{i+1}\psi_i = -\psi^*_{i+1}\psi_i\psi_i\psi^*_{i+2}\psi_{i+1}\psi^*_{i+1} = 0$$
$$u_{i+1} u_i u_{i+1} = \psi^*_{i+2}\psi_{i+1}\psi^*_{i+1}\psi_i\psi^*_{i+2}\psi_{i+1} = -\psi^*_{i+1}\psi_i\psi_i\psi^*_{i+2}\psi_{i+1}\psi^*_{i+1} = 0$$

And for $|i-j| > 1$, we have

$$\begin{aligned} u_i u_j &= \psi^*_{i+1}\psi_i\psi^*_{j+1}\psi_j \\ &= (-1)^2\psi^*_{j+1}\psi^*_{i+1}\psi_i\psi_j \\ &= (-1)^4\psi^*_{j+1}\psi_j\psi^*_{i+1}\psi_i = u_j u_i \end{aligned}$$

Now we get an algebra homomorphism $0\text{-}TL_N \to \mathrm{End}(\mathcal{F})$, we want to show that this is injective, or equivalently, $\mathcal{F}$ is a faithful representation of $0\text{-}TL_N$.

Step 2. For any monomial $m'(u_i) \neq 0 \in 0\text{-}TL_N$ we take a lexicographically minimal monomial $m(u_i)$ such that $m(u_i) = m'(u_i) \in 0\text{-}TL_N$. These monomials apparently span $0\text{-}TL_N$. To describe these monomials we define the descending strings

$$\mathrm{dcnd}(i,r) = u_{i+r}u_{i+r-1}\dots u_i \qquad 1 \le i < i+r \le N$$

Given a monomial $m(u_i) = u_{i_1}\dots u_{i_l}$, we take the maximal descending string $u_{i_1}\dots u_{i_{l_1}}$ starting with u_{i_1}, then the maximal descending string $u_{i_{l_1+1}}\dots u_{i_{l_2}}$, etc. In this way any monomial is written as

$$m(u_i) = u_{i_1}\dots u_{i_l} = \mathrm{dcnd}(i_{l_1}, l_1)\,\mathrm{dcnd}(i_{l_2}, l_2 - l_1)\dots\mathrm{dcnd}(i_{l_r}, l_r - l_{r-1}) \tag{3.18}$$

where $l_r = l$. Observe that for our minimal monomials, the initial index of each maximal descending string is strictly increasing:

$$i_{l_{j+1}} + l_{j+1} - l_j > i_{l_j} + l_j - l_{j-1} \qquad \forall j$$

Denote $s_{j+1} := i_{l_{j+1}} + l_{j+1} - l_j$. There are two cases:

- $s_{j+1} < l_j$ for some j, we would have in the middle of the monomial a product

  $$\begin{aligned} \mathrm{dcnd}(i_{l_j}, l_j - l_{j-1})\,\mathrm{dcnd}(i_{l_{j+1}}, l_{j+1} - l_j) &= u_{s_j}\dots u_{l_j}u_{s_{j+1}}\dots u_{l_{j+1}} \\ &= u_{s_{j+1}}\dots u_{l_{j+1}}u_{s_j}\dots u_{l_j} \end{aligned}$$

  which contradicts the minimality.

- If $l_j \le s_{j+1} \le s_j$, then $\mathrm{dcnd}(i_{l_j}, l_j - l_{j-1})\,\mathrm{dcnd}(i_{l_{j+1}}, l_{j+1} - l_j) = 0$. We only prove this for$s_{j+1} = s_j$,

  $$\mathrm{dcnd}(i_{l_j}, l_j - l_{j-1})\,\mathrm{dcnd}(i_{l_{j+1}}, l_{j+1} - l_j) = u_{s_j}u_{s_j}\dots u_{l_j}u_{s_j+1}\dots u_{l_{j+1}} = 0$$

  but the original monomial is by construction nonzero. In general the proof is similar.

Step 3. On the other hand, the action of $\mathrm{dcnd}(i,r)$ on $\mathcal{F}$ is equal to $\psi^*_{i+r}\psi_i$. So the monomial $m(u_i)$ in 3.18 acts as

$$e_{i_{j_1}} \wedge \cdots \wedge e_{i_{j_r}} \longmapsto \begin{cases} \pm e_{i_{l_1}+l_1} \wedge \cdots \wedge e_{i_{l_r}+l_r-l_{r-1}} & j_s = l_s,\ \forall s \\ 0 & \text{otherwise} \end{cases}$$

Suppose a linear combination of minimal monomials is 0. Write this as

$$\sum_j a_j \,\mathrm{dcnd}(i_{l_{1,j}}, l_{1,j}) \ldots \mathrm{dcnd}(i_{l_{r_j,j}}, l_{r_j} - l_{r_j-1}) \qquad a_j \in \mathbb{C}$$

acts by 0 on $\mathcal{F}$. If the data $(l_{1,j}, \ldots l_{r_j}, i_{l_{1,j}}, \ldots, i_{r_j,j})$ are all distinct, then we can find a wedge which only the summand for $j = 1$ acts by a nonzero operator, hence we conclude that $a_1 = 0$, etc. Thus conclude that all $a_j = 0$. Now we have to show that the data $(l_{1,j}, \ldots l_{r_j}, i_{l_{1,j}}, \ldots, i_{r_j,j})$ are all distinct. By the observation in step 2, a minimal monomial is uniquely determined by these data, so we are done. $\square$

Remark 3.2.9. We have also shown that the minimal monomials in u_i are linearly independent, hence yield a monomial basis for $0\text{-}TL_N$, and in view of the canonical form of these monomials are uniquely determined by the input and output 01-strings.

We cite following analogue of Proposition 3.2.8 from [BM16].

Lemma 3.2.10. *For $N \geq 2$, the state space $\mathcal{F}[q]$ is a faithful representation of the affine nil-Temperley-Lieb algebra $0\text{-}\widehat{TL}_N$.*

Remark 3.2.11. A canonical form of nonzero monomials in $0\text{-}\widehat{TL}_N$ is also given in [BM16] for $N \geq 3$. Similar to the $0\text{-}TL_N$ case, these monomials are uniquely determined by the input 01string with minimal k, the output 01string, and the power of q it raises.

As a free $\mathbb{C}[q]$-module, $\mathcal{F}_k[q]$ has the rank equal to that of $QH^*Gr(k,N)$, and we next specifying a collection of $\mathbb{C}[q]$-linear operators on $\mathcal{F}_k[q]$ indexed by $\Lambda(k,N)$, which turns out to be the quantum multiplication by the corresponding Schubert classes on $QH^*Gr(k,N)$.

3.2.3 The noncommutative Schur polynomials

Recall the Siebert-Tian presentation 3.1.31 of the quantum cohomology

$$QH^*Gr(k,N) = \mathbb{Z}[e_1, \ldots e_k]/(h_{N-k+1}, \ldots h_{N-1}, h_N + (-1)^k q)$$

Postnikov [Pos05] found the appropriate counterparts of e_i, h_i in 0-$\widehat{TL}_N$, the "symmetric polynomials" in the non-commuting generators u_i. Here we must be careful about the order of the product. Recall that we label the slots on the circle clockwise, see figure 3.1. Note that all that matters is the order of consecutive generators, and they are either in clockwise or counterclockwise order in figure 3.1. For a set of indices $I = \{i_1, i_2, \dots i_r\}$ where the consecutive ones are in clockwise order, define

$$\prod_{i\in I}^{\circlearrowright} u_i := u_{i_1}u_{i_2}\dots u_{i_r} \qquad \prod_{i\in I}^{\circlearrowleft} u_i := u_{i_r}u_{i_{r-1}}\dots u_{i_1}$$

Since the case $N = 2$ is completely understood, from now on we assume $N \geq 3$ unless otherwise specified.

Definition 3.2.12. For $1 \leq r \leq N-1$, define the noncommutative elementary symmetric polynomial

$$e_r(\mathcal{U}) := \sum_{|I|=r} \prod_{i\in I}^{\circlearrowright} u_i$$

similarly, define

$$h_r(\mathcal{U}) := \sum_{|I|=r} \prod_{i\in I}^{\circlearrowleft} u_i$$

where the sum runs through subsets of $\{1, 2, \dots, N\}$. For $r = N$ we define

$$h_N(\mathcal{U}) = (-1)^{k-1}q \text{ on } \mathcal{F}_k[q]$$

$$e_N(\mathcal{U}) = \begin{cases} (-1)^N q & \text{restriced on } \mathcal{F}_N[q] \\ 0 & \text{restriced on } \sum_{i=1}^{N-1} \mathcal{F}_i[q] \end{cases}$$

Example 3.2.13. If $N = 3$, then

$$e_2(\mathcal{U}) = u_1u_2 + u_2u_3 + u_3u_1 \qquad h_2(\mathcal{U}) = u_3u_2 + u_2u_1 + u_1u_3$$

and we write down their action on $\mathcal{F}_2[q] = \mathbb{C}[q]\Lambda(2, 3)$:

	110	101	011
$e_2(\mathcal{U})$	011	$-q \cdot 110$	$-q{\cdot}101$
$h_2(\mathcal{U})$	0	0	0

One can see that $e_2(\mathcal{U})$ moves every "particle" clockwise by one slot, and on the contrary, $h_2(\mathcal{U})$ wants to move particles more than one step, hence ends up killing everything.

Remark 3.2.14. We defined $e_N(\mathcal{U})$ and $h_N(\mathcal{U})$ specifically to produce the "quantum cohomology relation" $h_N = (-1)^{k-1}q$. If we set $u_N, e_N(\mathcal{U}), h_N(\mathcal{U}) = 0$ in what follows we will obtain a description of the usual cohomology in terms of $0\text{-}TL_N$.

Definition 3.2.15. For any partition $\lambda \in \mathfrak{P}(k,N)$ define the *noncommutative Schur polynomials* by the Jacobi-Trudi formula

$$s_\lambda(\mathcal{U}) = \det(e_{\lambda^t_j+i-j}(\mathcal{U})) = \det(h_{\lambda_i-i+j}(\mathcal{U})) \in 0\text{-}\widehat{TL}_N \tag{3.19}$$

where λ^t is the transposed partition and $1 \leq i,j \leq N$.

The definition of the noncommutative symmetric polynomials makes sense only if we can show that the e_r's resp. h_r's pairwise commute, and the two determinants in 3.19 are equal. For this we mimic the *Bethe Ansatz*, which is known in physics as the problem of simultaneously diagonalizing certain linear operators, see e.g. [KBI97] or [Bet97]. (An Ansatz is a particular speculation of solutions.)

Construction of common eigenvectors

Extend the scalar to $\mathbb{C}(q^{\frac{1}{N}})$. For $z \in \mathbb{C}(q^{\frac{1}{N}})$ define the operator

$$\hat{\psi}^*(z) = \sum_{j=1}^{N} z^{-j}\psi_j^*$$

For a k-tuple $y = (y_1, \dots y_k)$ of pairwise distinct elements in $\mathbb{C}(q^{\frac{1}{N}})$, consider the "Bethe vector"

$$b(y) := \eta(y)\hat{\psi}^*(y_1)\dots\hat{\psi}^*(y_k)\emptyset_0 \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(q^{\frac{1}{N}}) \tag{3.20}$$

The normalization factor is by definition

$$\eta(y) = \frac{(-1)^{\frac{k(k-1)}{2}} y_1 \dots y_k}{\mathrm{Van}(y)}$$

where $\mathrm{Van}(y) = \prod_{i<j}(y_i^{-1} - y_j^{-1})$ is the Vandermonde determinant. This factor is introduced in order to have the following equality. Recall that $\mathfrak{P}(k,N)$ is the set of partitions in a $k \times (N-k)$ rectangle.

Lemma 3.2.16. *Identify $\mathcal{F}_k[q] \cong \mathbb{C}[q]\mathfrak{P}(k,N)$. For any y_i we have*

$$b(y) = \sum_{\lambda \in \mathfrak{P}(k,N)} s_\lambda(y_1^{-1}, \dots y_k^{-1})\lambda$$

where s_λ denote the usual Schur polynomial.

Proof. By direct computation:

$$\begin{aligned}
\hat{\psi}^*(y_1)\dots\hat{\psi}^*(y_k)\emptyset_0 &= \sum_{j_1,\dots j_k} y_1^{-j_1}\psi^*_{j_1}\dots y_k^{-j_1}\psi^*_{j_k}\emptyset_0 \\
&= \sum_{\pi\in\mathfrak{S}_k}\sum_{j_1<\dots<j_k}(-1)^\pi y_{\pi(1)}^{-j_1}\dots y_{\pi(k)}^{-j_k}\psi^*_{j_1}\dots\psi^*_{j_k}\emptyset_0 \\
&= \sum_{\lambda\in\mathfrak{P}(k,N)}\sum_{\pi\in\mathfrak{S}_k}(-1)^\pi y_{\pi(1)}^{-\lambda_k-1}\dots y_{\pi(k)}^{-\lambda_1-k}\lambda
\end{aligned}$$

where $\mathfrak{S}_k$ is the symmetric group. Now we multiply both sides by $y_1\dots y_k$

$$\begin{aligned}
y_1\dots y_k\hat{\psi}^*(y_1)\dots\hat{\psi}^*(y_k)\emptyset_0 &= y_1\dots y_k\sum_{\lambda\in\mathfrak{P}(k,N)}\sum_{\pi\in\mathfrak{S}_k}(-1)^\pi y_{\pi(1)}^{-\lambda_k-1}\dots y_{\pi(k)}^{-\lambda_1-k}\lambda \\
&= \sum_{\lambda\in\mathfrak{P}(k,N)} y_1\dots y_k \det\begin{pmatrix} y_1^{-\lambda_k} & \dots & y_k^{-\lambda_k} \\ y_1^{-\lambda_{k-1}-1} & \ddots & y_k^{-\lambda_{k-1}-1} \\ \vdots & \ddots & \vdots \\ y_1^{-\lambda_1-k+1} & \dots & y_k^{-\lambda_1-k+1}\end{pmatrix}\lambda
\end{aligned}$$

Permute the rows of the matrix on RHS by $(k\dots 21)$, then compare with Cauchy's bialternant formula (see e.g. [Pra19]) we recognize that

$$\text{RHS} = \sum_{\lambda\in\mathfrak{P}(k,N)}(-1)^{\frac{k(k-1)}{2}}\operatorname{Van}(y_i)s_\lambda(y_1^{-1},\dots y_k^{-1})\lambda \qquad \square$$

Next we derive equations for $b(y)$ to be a common eigenvector of the noncommutative symmetric polynomials.

Lemma 3.2.17. *Suppose a k-tuple $y = (y_1,\dots y_k)$ of pairwise distinct y_i's is a solution for the equations*

$$y_1^N = \dots = y_k^N = (-1)^{k-1}q \tag{3.21}$$

then the corresponding vector $b(y)$ 3.20 is a common eigenvector for the noncommutative symmetric polynomials $e_r(\mathcal{U})$ and $h_r(\mathcal{U})$ 3.2.12, and we have

$$e_r(\mathcal{U})b(y) = e_r(y)b(y) \qquad h_r(\mathcal{U})b(y) = h_r(y)b(y) \tag{3.22}$$

for $0\le r\le N$.

Proof. For convenience we some a notations

$$v_{j_1,\dots i_k} := \psi^*_{j_1}\dots\psi^*_{j_k}\emptyset_0 \qquad 1 \le j_1 \le j_2 \dots \le i_k \le N$$

Note that $v_{j_1,\dots i_k} = 0$ unless $1 \le j_1 < j_2 \dots < i_k \le N$. By Clifford relations we get

$$\begin{aligned} u_r v_{j_1,\dots i_k} &= \psi^*_{r+1}\psi_r\psi^*_{j_1}\dots\psi^*_{j_k}\emptyset_0 \\ &= \psi^*_{j_1}\dots\psi^*_{r+1}\psi_r\psi^*_{j_s}\dots\psi^*_{j_k}\emptyset_0 \\ &= \begin{cases} v_{j_1,\dots,j_s+1,\dots,j_k} & \text{if some } j_s = r \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} u_N v_{j_1,\dots i_k} &= (-1)^{k-1} q\psi^*_1\psi_N\psi^*_{j_1}\dots\psi^*_{j_k}\emptyset_0 \\ &= q\delta_{j_k,N}\psi^*_1\psi^*_{j_1}\dots\psi^*_{j_k}\emptyset_0 \\ &= q\delta_{j_k,N} v_{1,j_1,\dots j_{k-1}} \end{aligned}$$

Hence to unify the above expressions we define

$$v_{j_1,\dots j_{k-1},N+1} := q v_{1,j_1,\dots j_{k-1}}$$

Then we compute the action of $e_i(\mathcal{U})$ on the Bethe vectors.

$$\begin{aligned} e_r(\mathcal{U})b(y) &= \left(\sum_{|I|=r}\prod_{i\in I}^{\circlearrowleft} u_i\right)\eta(y)\sum_{\pi\in\mathfrak{S}_k}\sum_{j_1<\dots<j_k}(-1)^\pi y_{\pi(1)}^{-j_1}\dots y_{\pi(k)}^{-j_k} v_{j_1\dots j_k} \\ &= \eta(y)\sum_{\pi\in\mathfrak{S}_k}\sum_{j_1<\dots<j_k}(-1)^\pi y_{\pi(1)}^{-j_1}\dots y_{\pi(k)}^{-j_k}\sum_{1\le i_1<\dots<i_r\le k} u_{j_{i_1}}\dots u_{j_{i_r}} v_{j_1\dots j_k} \end{aligned}$$

This is 0 if $r > k$, and otherwise

$$u_{j_{i_1}}\dots u_{j_{i_r}} v_{j_1,\dots j_r} = v_{j_1,\dots j_{i_1}+1,\dots j_{i_r}+1,\dots j_k}$$

Denote the anti-symmetrization operator $A_k = \sum_{\pi\in\mathfrak{S}_k}(-1)^\pi\pi$, where π permutes the subscripts of y_i. We have for $r \le k$

$$\begin{aligned} e_r(\mathcal{U})b(y) &= \eta(y)A_k\sum_{1\le j_1<\dots<j_k\le N+1} y_1^{-j_1}\dots y_k^{-j_k}\sum_{1\le i_1<\dots<i_r\le k} v_{j_1,\dots j_{i_1}+1,\dots j_{i_r}+1,\dots j_k} \\ &= \eta(y)A_k\sum_{1\le j_1<\dots<j_k\le N+1} y_1^{-j_1}\dots y_k^{-j_k}\sum_{1\le i_1<\dots<i_r<k} y_{\pi(j_{i_1})}\dots y_{\pi(j_{i_r})} v_{j_1\dots j_k} \\ &= \eta(y)A_k\sum_{1\le j_1<\dots<j_k\le N+1} y_1^{-j_1}\dots y_k^{-j_k} e_r(y_1,\dots y_k) v_{j_1\dots j_k} \\ &= e_r(y)b(y) \end{aligned}$$

where the second equality is because if $i_r = k, j_k = N$, then using the assumption that $y_i^N = (-1)^{k-1}q$, we calculate

$$\begin{aligned}
&A_k y_1^{-j_1} \dots y_k^{-N} v_{j_1,\dots j_{i_1}+1,\dots j_{i_r}+1,\dots j_{k-1},N+1} \\
&\qquad = A_k y_1^{-j_1} \dots y_{k-1}^{-j_{k-1}} (-1)^{k-1} q^{-1} q v_{1,j_1,\dots j_{i_1}+1,\dots j_{i_r}+1,\dots j_{k-1}} \\
&\qquad = A_k (-1)^{k-1} y_{j_1} \dots y_{j_{r-1}} y_k y_1^{-j_1} \dots y_{k-1}^{-j_{k-1}} y_k^{-1} v_{1,j_1,\dots j_{k-1}} \\
&\qquad\qquad = A_k y_1 y_{j_1} \dots y_{j_{r-1}} y_1^{-1} y_2^{-j_1} \dots y_{k-1}^{-j_{k-1}} v_{1,j_1,\dots j_{k-1}}
\end{aligned}$$

This proves our claim for $0 \leq r \leq k$ and $r < N$. For $r = N$, recall Definition 3.2.12 that $e_N(\mathcal{U}) = (-1)^N q$ only on $\mathcal{F}_k[q]$, so the eigenvalue formula 3.22 boils down to

$$e_N(y_1, \dots y_N) = \prod_{i=1}^{N} y_i = (-1)^N q$$

which is automatic if the y_i's are distinct roots of $X^N + (-1)^N q$ viewed as a polynomial in X.

Similarly, we compute the action of $h_r(\mathcal{U})$:

$$\begin{aligned}
h_r(\mathcal{U}) b(y) &= \left(\sum_{|I|=r} \prod_{i \in I}^{\circlearrowleft} u_i \right) \eta(y) A_k \sum_{j_1 < \dots < j_k} y_1^{-j_1} \dots y_k^{-j_k} v_{j_1 \dots j_k} \\
&= \eta(y) A_k \sum_{\substack{l_1 + \dots l_s = r \\ 1 \leq i_1 < \dots i_s + l_s}} \operatorname{dcnd}(i_1, l_1) \dots \operatorname{dcnd}(i_s, l_s) \sum_{j_1 < \dots < j_k} y_1^{-j_1} \dots y_k^{-j_k} v_{j_1 \dots j_k} \\
&= \eta(y) A_k \sum_{\substack{l_1 + \dots l_s = r \\ 1 \leq i_1 < \dots i_s + l_s}} \sum_{j_1 < \dots < j_k} y_1^{-j_1} \dots y_k^{-j_k} v_{j_1 \dots i_1 + l_1, \dots i_r + l_r, \dots j_k}
\end{aligned}$$

$$\begin{aligned}
&= \eta(y) A_k \sum_{\substack{l_1 + \dots l_s = r \\ 1 \leq i_1 < \dots i_s + l_s}} \sum_{j_1 < \dots < j_k} y_{i_1}^{l_1} \dots y_{i_r}^{l_r} y_1^{-j_1} \dots y_k^{-j_k} v_{j_1,\dots j_k} \\
&= h_r(y) b(y) \qquad \square
\end{aligned}$$

Let $\widehat{\mathrm{Sol}}$ the set of solutions to equations 3.21 such that the y_i's are distinct. Then clearly there is a bijection

$$\begin{aligned}
\widehat{\mathrm{Sol}} &\longrightarrow \Lambda(k, N) \\
(-1)^{k-1}(\zeta^{i_1} q^{\frac{1}{N}}, \dots, \zeta^{i_k} q^{\frac{1}{N}}) &\longmapsto e_{i_1} \wedge \dots e_{i_k}
\end{aligned}$$

where $\zeta = \exp(\frac{2\pi i}{N})$ is a primitive N-th root of unity. For

$$w = 00\ldots\underset{i_1}{1}\ldots\underset{i_k}{1}\ldots 00 \in \Lambda(k,N)$$

denote the corresponding solution tuple $y_w := (-1)^{k-1}(\zeta^{i_1}q^{\frac{1}{N}},\ldots,\zeta^{i_k}q^{\frac{1}{N}})$.

Theorem 3.2.18. *The Bethe vectors $b(y_w)$ for $y_w \in \widehat{\mathrm{Sol}}$ are pairwise orthogonal, and form a common eigenbasis of $\mathcal{F}\otimes\mathbb{C}(q^{\frac{1}{N}})$ for the action of the operators $e_r(\mathcal{U})$ and $h_s(\mathcal{U})$. We have the eigenvalue formula*

$$e_r(\mathcal{U})b(y_w) = \begin{cases} e_r(y_w)b(y_w) & r \leq k \\ 0 & r > k \end{cases} \tag{3.23}$$

$$h_r(\mathcal{U})b(y_w) = \begin{cases} h_r(y_w)b(y_w) & r \neq N \\ (-1)^{k-1}q\cdot b(y_w) & r = N \end{cases} \tag{3.24}$$

And the norm of the Bethe vectors is given by

$$\langle b(y_w), b(y_w)\rangle = \frac{n(n+k)^n}{\mathrm{Van}(y_w)} \tag{3.25}$$

Proof. We have shown that the Bethe vectors are common eigenvectors for $e_r(\mathcal{U})$ and $h_s(\mathcal{U})$. Suppose for the moment that we have shown that they are pairwise orthogonal, then they are linearly independent, and there are $|\widehat{\mathrm{Sol}}| = \dim\mathcal{F}\otimes\mathbb{C}(q^{\frac{1}{N}})$. To show orthogonality, by Lemma 3.2.16 and the definition of the bilinear form 3.2.5

$$\begin{aligned}\langle b(y), b(y')\rangle &= \left\langle \sum_\lambda s_\lambda(y_1^{-1},\ldots y_k^{-1})\lambda, \sum_\lambda s_\lambda((y_1')^{-1},\ldots(y_k')^{-1})\lambda \right\rangle \\ &= \sum_\lambda \overline{s_\lambda(y_1^{-1},\ldots y_k^{-1})}s_\lambda\left((y_1')^{-1},\ldots(y_k')^{-1}\right)\end{aligned}$$

Then we refer to [Rie01] proposition 4.3 to conclude that $RHS = 0$ if $(y_1,\ldots y_k) \neq (y_1',\ldots y_k')$ and is equal to 3.25 otherwise. □

Corollary 3.2.19. *The operators $e_r(\mathcal{U})$ and $h_s(\mathcal{U})$ pairwise commute. Thus the noncommutative Schur polynomials are well-defined. Moreover,*

$$s_\lambda(\mathcal{U})b(y_w) = s_\lambda(y_w)b(y_w) \qquad \forall\ \lambda \in \mathfrak{P}(k,N),\ w \in \Lambda(k,N)$$

hence by the usual Jacobi-Trudi formula 3.1.18 we have

$$s_\lambda(\mathcal{U}) = \det(e_{\lambda_i^t-i+j}(\mathcal{U})) = \det(h_{\lambda_i-i+j}(\mathcal{U})) \tag{3.26}$$

3.2.4 The combinatorial quantum cohomology

Motivated by the description of the quantum cohomology of the Grassmannian in terms of Schur polynomials, we make the following definition. Fix $k \in \mathbb{N}$, recall the bijection $\Lambda(k,N) \cong \mathfrak{P}(k,N)$ 3.13, consider

$$(w,u) \longmapsto w \star u := s_{\lambda(w)}(\mathcal{U})u$$

for $w, u \in \Lambda(k, N-k)$. This defines an composition on $\mathcal{F}_k[q]$.

Theorem 3.2.20. *The composition $\star$ is commutative, associative and unital. Thus $\mathcal{F}_k[q]$ becomes a $\mathbb{C}[q]$-algebra QH^*_{comb}.*

Proof. We again extend the scalars to $\mathbb{C}(q^{\frac{1}{N}})$. By definition the empty partition acts as the identity operator. For $v \in \Lambda(k,N)$ write

$$v = \sum \frac{\langle b(y_w), v\rangle}{\langle b(y_w), b(y_w)\rangle} b(y_w)$$

Then we compute

$$\begin{aligned}
\langle b(y_w), u \star v\rangle &= \langle b(y_w), s_{\lambda(u)}(\mathcal{U})v\rangle \\
&= \left\langle b(y_w), \sum_{w'} s_{\lambda(u)}(\mathcal{U}) \frac{\langle b(y_{w'}), v\rangle}{\langle b(y_{w'}), b(y_{w'})\rangle} b(y_{w'}) \right\rangle \\
&= \left\langle b(y_w), \sum_{w'} \frac{\langle b(y_{w'}), v\rangle}{\langle b(y_{w'}), b(y_{w'})\rangle} s_{\lambda(u)}(y_{w'}) b(y_{w'}) \right\rangle \\
&= \langle b(y_w), v\rangle s_{\lambda(u)}(y_w)
\end{aligned}$$

We first show commutativity, which amounts to $\langle x \star v, b(y_w)\rangle = \langle v \star x, b(y_w)\rangle$ by orthogonality of the Bethe vectors. Equivalently, we want to have

$$s_{\lambda(u)}(y_w)\langle b(y_w), v\rangle = s_{\lambda(v)}(y_w)\langle b(y_w), u\rangle$$

This is true, because by Lemma 3.2.16,

$$b(y_w) = \sum_{\mu} s_\mu(y_w) w(\mu)$$

thus we compute

$$\begin{aligned}
s_{\lambda(u)}(y_w)\langle b(y_w), v\rangle &= s_{\lambda(u)}(y_w) \left\langle \sum_{\mu} s_\mu(y_w) w(\mu), v \right\rangle \\
&= s_{\lambda(u)}(y_w) s_{\lambda(v)}(y_w)
\end{aligned}$$

for all $u, v, w \in \Lambda(k, N)$. The second equality is because $\lambda(v) = \mu \iff w(\mu) = v$. Interchanging u, v, we get

$$s_{\lambda(v)}(y_u)\langle b(y_w), u\rangle = s_{\lambda(v)}(y_w)s_{\lambda(u)}(y_w)$$

as we desired.

Next, using commutativity, we rewrite associativity as

$$\begin{aligned} & w \star (v \star u) = (w \star v) \star u = u \star (v \star w) \\ \iff & s_{\lambda(w)}(\mathcal{U})s_{\lambda(v)}(\mathcal{U})u = s_{\lambda(u)}(\mathcal{U})s_{\lambda(v)}(\mathcal{U})w \end{aligned}$$

Hence we calculate

$$\begin{aligned} \left\langle b(y_{w'}), s_{\lambda(w)}(\mathcal{U})s_{\lambda(v)}(\mathcal{U})u\right\rangle &= \left\langle b(y_{w'}), \sum_{u'} \frac{\langle b(y_{u'}), u\rangle}{\langle b(y_{u'}), b(y_{u'})\rangle} s_{\lambda(w)}(y_{u'})s_{\lambda(v)}(y_{u'})b(y_{u'})\right\rangle \\ &= \langle b(y_{w'}), u\rangle s_{\lambda(w)}(y_{w'})s_{\lambda(v)}(y_{w'}) \\ &= \left\langle \sum_{\mu} s_\mu(y_{w'})w(\mu), u\right\rangle s_{\lambda(w)}(y_{w'})s_{\lambda(v)}(y_{w'}) \\ &= s_{\lambda(u)}(y_{w'})s_{\lambda(w)}(y_{w'})s_{\lambda(v)}(y_{w'}) \end{aligned}$$

Interchanging u, w, we get

$$\langle b(y_{w'}), s_{\lambda(u)}(\mathcal{U})s_{\lambda(v)}(\mathcal{U})w\rangle = s_{\lambda(w)}(y_{w'})s_{\lambda(u)}(y_{w'})s_{\lambda(v)}(y_{w'})$$

This proves associativity. $\square$

Recall that for any $u \in \Lambda(k, N)$, the string $w_0u \in \Lambda(k, N)$ is obtained from reading u backwards.

Theorem 3.2.21. *There is an isomorphism of $\mathbb{Z}[q]$-algebras.*

$$\mathbb{Z}[e_1, \ldots e_k]/(h_{N-k+1}, \ldots h_{N-1}, h_N + (-1)^k q) \xrightarrow{\sim} QH^*_{\text{comb}}$$

$$e_i \longmapsto e_i(\mathcal{U})$$

In particular, the Gromov-Witten invariants are give by

$$q^d I_d(S_u, S_w, S_v) = \langle w_0 v, u \star w\rangle$$

where $d = N^{-1}(I(u) + I(w) - I(v))$ and I is the function defined in 2.1.10.

Proof. First we check the relations. For any $w, w' \in \Lambda(k, N)$

$$\begin{aligned}\langle b(y_w), h_i(\mathcal{U})b(y_{w'})\rangle &= \langle b(y_w), h_i(y_{w'})b(y_{w'})\rangle \\ &= \delta_{w,w'}h_i(y_{w'})\langle b(y_{w'}), b(y_{w'})\rangle\end{aligned}$$

We claim that $h_i(y_{w'}) = 0$ for $N-k < i < N$ and all $w' \in \Lambda(k, N)$. To see this, recall that $y_{w'} = (y_1, \dots y_k)$ is a tuple of distinct N-th roots of q. Let $\zeta_1, \dots \zeta_{N-k}$ be those N-th roots of q distinct from y_i. Recall 3.7, we have the expansion

$$(1-qX^N)^{-1}\prod_j(1-\zeta_j X) = \prod(1-y_iX)^{-1} = \sum h_n(y_{w'})X^n \in \mathbb{C}(q^{\frac{1}{N}})[[X]]$$

On the other hand

$$(1-qX^N)^{-1}\prod_j(1-\zeta_j X) = \sum c_n X^{Nn}\prod_j(1-\zeta_j X)$$

where $c_i \in \mathbb{C}(q^{\frac{1}{N}})$. The X^i term is 0 for $N-k < i < N$, and compare with the previous equation we have $h_i(y') = 0$ for such i. Since the Bethe vectors form an orthogonal basis, we conclude that $h_i(\mathcal{U})$ acts on $\mathcal{F}_k[q]$ by 0 for $N-k < i < N$.

Next, for $i = N$ by Definition 3.2.12

$$h_N(\mathcal{U})b(y') = h_N(y')b(y') = (-1)^k q b(y')$$

By definition QH^*_{comb} is generated as an algebra by the 01-string's corresponding to (1^p) for $1 \le p \le k$, hence the assignment $e_i \to e_i(\mathcal{U})$ defines a surjective algebra homomorphism. Since both $QH^*Gr(k,N)$ and QH^*_{comb} are by definition free $\mathbb{Z}[q]$-modules of rank $\binom{N}{k}$, this is an isomorphism. The Gromov-Witten invariants are given by

$$s_\lambda * s_\mu = \sum_d q^d I_d(S_\lambda, S_\mu, S_{w_0\nu})\nu$$

hence in QH^*_{comb} holds

$$w(\lambda) \star w(\mu) = \sum_d q^d I_d(S_\lambda, S_\mu, S_{w_0\nu})w(\nu)$$

taking inner product gives the desired formula. $\square$

In [KS10] a commutator relation is used to derive a recursion formula for the Gromov-Witten invariants. In fact, they also used this relation to derived a combinatorial description of the $\star$ product.

Proposition 3.2.22. *The following relation holds in the endomorphism ring of $\mathcal{F}[q]$.*

$$s_\lambda(\mathcal{U})\psi_i^* = \sum_{r=0}^{\lambda_1} \psi_{i+r}^* \sum_{\lambda/\mu=(r)} \hat{s}_\mu(\mathcal{U}) \tag{3.27}$$

where $\hat{s}_\mu(\mathcal{U})$ denotes the noncommutative Schur polynomial (3.19) with q replaced by $-q$ and we impose the quasi-periodic boundary condition

$$\psi_{i+N}^* = (-1)^{k-1} q \psi_i^*. \tag{3.28}$$

Proof. Here it is convenient to define the particle number operator $K : \mathcal{F}[q] \to \mathbb{C}$ sending $\mathcal{F}_k[q]$ to k. Then we can uniformly write $u_N = -(-1)^K q\psi_1^*\psi_N$ as an operator. Note that we have the following commutator relation:

$$-(-1)^K q \circ \psi_j^* = \psi_j^* \circ (-1)^K q.$$

By Clifford relations 3.2.3 we have for $1 \leq i < N$

$$\begin{aligned} u_i\psi_j^* &= \psi_{i+1}^*\psi_i\psi_j^* \\ &= \psi_{i+1}^*(\delta_{ij} - \psi_j^*\psi_i) \\ &= \delta_{ij}\psi_{i+1}^* + \psi_j^* u_i \end{aligned}$$

Now for u_N we calculate

$$\begin{aligned} u_N\psi_j^* &= -(-1)^K q\psi_1^*\psi_N\psi_j^* \\ &= -(-1)^K q\psi_1^*(\delta_{jN} - \psi_j^*\psi_N) \\ &= \delta_{jN}\psi_{N+1}^* + \psi_j^*\hat{u}_N \end{aligned}$$

Write $\hat{u}_i = u_i$ for $1 \leq i < N$ and $\hat{u}_N = -u_N$, the above two formulae can be compressed into one:

$$u_i\psi_j^* = \delta_{ij}\psi_{i+1}^* + \psi_j^*\hat{u}_i \tag{3.29}$$

Next, for indices $i_1, \ldots, i_r$ in the clockwise order, we use induction to show that

$$u_{i_1} \ldots u_{i_r}\psi_j^* = \psi_j^*\hat{u}_{i_1} \ldots \hat{u}_{i_r} + \psi_{j+1}^* \sum_{s=1}^{r} \delta_{i_s,j}\hat{u}_{i_1} \ldots \cancel{\hat{u}_{i_s}} \ldots \hat{u}_{i_r} \tag{3.30}$$

where $\cancel{\hat{u}_{i_s}}$ means omitting this generator in the product. Then 3.29 is exactly the case $r = 1$. We use induction to prove 3.30: suppose this is true for some $r < N-1$, then for $l, i_1, \dots i_r$ in the clockwise order we compute:

$$\begin{aligned} u_l u_{i_1} \dots u_{i_r} \psi_j^* &= u_l \left(\psi_j^* \hat{u}_{i_1} \dots \hat{u}_{i_r} + \psi_{j+1}^* \sum_{s=1}^{r} \delta_{i_s,j} \hat{u}_{i_1} \dots \cancel{\hat{u}_{i_s}} \dots \hat{u}_{i_r} \right) \\ &= (\delta_{lj} \psi_{j+1}^* + \psi_j^* \hat{u}_l) \hat{u}_{i_1} \dots \hat{u}_{i_r} + \hat{u}_l \psi_{j+1}^* \sum_{s=1}^{r} \delta_{i_s,j} \hat{u}_{i_1} \dots \cancel{\hat{u}_{i_s}} \dots \hat{u}_{i_r} \end{aligned}$$

Note that the second summand is 0 unless $j = i_s$ for some s, in which case $j+1 \neq l$ because $r < N-1$. Thus we have $\hat{u}_l \psi_{j+1}^* = \psi_{j+1}^* \hat{u}_l$, and compare with 3.30, there is only one missing summand, namely $\delta_{lj} \psi_{l+1}^* \hat{u}_{i_1} \dots \hat{u}_{i_r}$. Notice that if $j = l$, then $\psi_{j+1}^* u_j = \psi_{j+1}^* \psi_{j+1}^* \psi_j = 0$, hence we are fine.

As a consequence, 3.30 immediately yields

$$e_r(\mathcal{U}) \psi_j^* = \psi_j^* \hat{e}_r(\mathcal{U}) + \psi_{j+1}^* \hat{e}_{r-1}(\mathcal{U}) \tag{3.31}$$

which is the special case of our aim 3.2.22 for $\lambda = (1^r)$. Finally for partition $\lambda^t = (\lambda_1, \dots, \lambda_r) \in \mathfrak{P}(k, N)$, we do induction on r, the case $r = 1$ is exactly 3.31. Suppose the case this is true for r, and $\mu^t = (\lambda_0, \lambda_1, \dots, \lambda_r) \in \mathfrak{P}(k, N)$, we calculate

$$\begin{aligned} s_\mu(\mathcal{U}) \psi_j^* &= \det\left(e_{\mu_i - i + j}(\mathcal{U})\right) \\ &= \sum_{l=0}^{r} (-1)^l e_{\lambda_l - l} s_{\mu(l)^t}(\mathcal{U}) \psi_j^* \end{aligned}$$

where the second line is expansion with respect to the first column, and $\mu(l) := (\lambda_0 + 1, \dots \lambda_{l-1} + 1, \lambda_{l+1}, \dots, \lambda_r)$. Apply the induction hypothesis gives

$$\begin{aligned} s_\mu(\mathcal{U}) \psi_j^* &= \sum_{l=0}^{r} (-1)^l e_{\lambda_l - l} s_{\mu(l)^t}(\mathcal{U}) \psi_j^* \\ &= \sum_{l=0}^{r} (-1)^l e_{\lambda_l - l} \sum_{s=0}^{r} \psi_{j+s}^* \sum_{\mu(l)^t / \nu^t = (s)} \hat{s}_{\nu^t}(\mathcal{U}) \\ &= \sum_{l=0}^{r} \sum_{s=0}^{r} (-1)^l (\psi_{j+s}^* \hat{e}_{\lambda_l - l} + \psi_{j+s+1}^* \hat{e}_{\lambda_l - l - 1}) \sum_{\mu(l)^t / \nu^t = (s)} \hat{s}_{\nu^t}(\mathcal{U}) \\ &= \sum_{s=0}^{r} \psi_{j+s}^* \sum_{l=0}^{r} (-1)^l \hat{e}_{\lambda_l - l} \sum_{\mu(l)^t / \nu^t = (s)} \hat{s}_{\nu^t}(\mathcal{U}) \\ &+ \sum_{s=0}^{r} \psi_{j+s+1}^* \sum_{l=0}^{r} (-1)^l \hat{e}_{\lambda_l - l - 1} \sum_{\mu(l)^t / \nu^t = (s)} \hat{s}_{\nu^t}(\mathcal{U}) \end{aligned}$$

Now we would like to interchange the last two summations: for a fixed l, the first summand contributes the partitions obtained from μ^t by deleting a horizontal strip but not in the l-th column, and the second contributes those deleting one box from the l-th column, therefore we indeed get the right hand side of 3.2.22. $\square$

Remark 3.2.23. Using the commutator relation 3.2.22 one can derive a recursion formula and many symmetries for the Gromov-Witten invariants, see [KS10].

Corollary 3.2.24. *For partitions λ, μ, impose the quasi-periodic boundary condition 3.28, we have*

$$\lambda \star \mu := \sum_T \psi^*_{l_n(\mu)+\alpha_n} \psi^*_{l_{n-1}(\mu)+\alpha_{n-1}} \psi^*_{l_{n-2}(\mu)+\alpha_{n-2}} \cdots \psi^*_{l_1(\mu)+\alpha_1} \ \emptyset \tag{3.32}$$

where the sum runs over all semi-standard tableaux T of shape λ with fillings from $[1, n]$, and weight *$\alpha = (\alpha_1, \ldots, \alpha_k)$, i.e. α_i is the number of times i occurs in T.*

Proof. Write $w \in \Lambda(k, N)$ as $w = \psi^*_{l_k} \cdots \psi^*_{l_2} \psi^*_{l_1} \emptyset$ with $1 \leq l_k < l_{k-1} < \cdots < l_1 \leq N$. By the commutation relation (3.27),

$$\begin{aligned} v \star w = s_{\lambda(v)}(\mathcal{U}) w \ &= \ s_{\lambda(v)}(\mathcal{U}) \psi^*_{l_k(\mu)} \cdots \psi^*_{l_2(\mu)} \psi^*_{l_1(\mu)} \emptyset \\ &= \sum_{\rho_{k-1}} \psi^*_{l_k(\mu)+|\lambda/\rho_{k-1}|} \hat{s}_{\rho_{k-1}}(\mathcal{U}) \psi^*_{l_{k-1}(\mu)} \cdots \psi^*_{l_1(\mu)} \emptyset \\ &= \sum_{\rho_{k-2}, \rho_{k-1}} \psi^*_{l_{k-1}(\mu)+|\lambda/\rho_{k-1}|} \psi^*_{l_{k-1}(\mu)+|\rho_{k-1}/\rho_{k-2}|} s_{\rho_{k-2}}(\mathcal{U}) \psi^*_{l_{k-2}(\mu)} \cdots \psi^*_{l_1(\mu)} \emptyset \\ &\ \ \vdots \\ &= \sum_{(\rho_{k-1}, \ldots, \rho_1)} \psi^*_{l_k(\mu)+|\lambda/\rho_{k-1}|} \psi^*_{l_{k-1}(\mu)+|\rho_{k-1}/\rho_{k-2}|} \psi^*_{l_{k-2}(\mu)+|\rho_{k-2}/\rho_{k-3}|} \cdots \emptyset, \end{aligned}$$

where eventually we arrive at $\rho_0 = \emptyset$ because $s_\rho \emptyset$ nonzero implies $\rho = \emptyset$, hence the sums run over partitions ρ_i such that $\rho_k = \lambda$, $\rho_0 = \emptyset$ and ρ_{k+1-i}/ρ_{k-i} is a horizontal strip. Such a sequence of partitions $\rho_0, \ldots \rho_{k-1}$ is equivalent to a semi-standard Young tableau T, where ρ_i is obtained by deleting all boxes with entries exceeding i from T. $\square$

Example 3.2.25. Consider $Gr(2, 4)$, take $\lambda = \mu = (2, 1)$, then the corresponding 01-string is 0101, and all possible semi-standard Young tableaux with entries from $\{1, 2\}$ are

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array}$$

hence we compute in $QH^*Gr(2,4)$

$$S_{(2,1)} * S_{(2,1)} = qS_{1001} = qS_{(2)}$$

From this example we see that in general there will be cancellations in this tableaux description, so we cannot just count tableaux to get the Gromov-Witten invariants. A natural question is that can one find a manifestly positive combinatorial formula for them? We will discuss this in the next section.

3.2.5 The boson-fermion correspondence

In this subsection we connect the boson-fermion correspondence to the fermionic model. Following [KRR13], the boson-fermion correspondence asserts that a "fermi -onic" construction and a "bosonic" one both give the same irreducible representation of the infinite dimensional *Heisenberg Lie algebra.*

Definition 3.2.26. The infinite Heisenberg algebra $\mathbf{H}$ is the Lie algebra on basis a_n, $n \in \mathbb{Z}$ and C where C is central and $[a_n, a_l] = l\delta_{l,-n}C$.

The "bosonic construction" is a natural representation of $\mathbf{H}$ on the polynomial ring in countably many variables $B = \mathbb{C}[z^{\pm 1}, p_1, p_2, \ldots]$. Denote by ∂_{p_l} the operator of formal derivatives and $p_l\cdot$ multiplication by p_l, we have the commutator relation

$$[\partial_{p_l}, p_n\cdot] = \delta_{l,n}\,\mathrm{id}\,.$$

The $\mathbf{H}$-action is given by letting the central element C act as the identity and the basis vectors a_l act by

$$\begin{cases} l\partial_{p_l} & \text{if } l > 0 \\ p_{-l}\cdot & \text{if } l < 0 \\ z\partial_z & \text{if } l = 0. \end{cases}$$

Since higher powers of p_i's are allowed in B, we think of them as states of bosons, thus for $m > 0$ we call a_m resp. a_{-l} the annihilation resp. creation operators for bosons.

For $m \in \mathbb{Z}$ write $B^{(m)} = z^m\mathbb{C}[p_1, p_2 \ldots]$, then a_0 acts as multiplication by m on $B^{(m)}$, hence $B^{(m)}$ is a subrepresentation, known as the *bosonic Fock space of central charge m.* In fact, one can get any monomial in $B^{(m)}$ by applying the creation operators to the *vacuum vector* $z^m \in B^{(m)}$, hence it is a highest weight representation generated by z^m, see [KRR13]. We define a grading on $B^{(m)}$ by $\deg p_i = i$.

Remark 3.2.27. There is a surjective algebra homomorphism from the ring of symmetric functions to the invariant ring $\mathbb{C}[x_1, \ldots x_N]^{S_N}$

$$p_n \longmapsto \frac{1}{n}(x_1^n + \ldots x_N^n)$$

Recall the ring $\Lambda = \varinjlim \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ of symmetric functions (see Remark 3.1.23). We can identify $\mathbb{C}[p_1, p_2, \ldots]$ with Λ as graded algebras. One can define Schur functions in terms of these normalized power sums, which are the preimages of the usual Schur functions under this identification, see [KRR13, Lecture 6] for details.

Next, we define the infinite dimensional *fermionic Fock space*. Let $V = \bigoplus_{j \in \mathbb{Z}} e_j$ be a complex vector space on basis $\{e_j,\ j \in \mathbb{Z}\}$.

Definition 3.2.28. A *semi-infinite wedge* is a formal expression of the form

$$e_{\mathbf{i}} := e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots$$

where $\mathbf{i} = (i_j)_{j \in \mathbb{N}_0}$ is an infinite tuple of integers satisfying $i_0 > i_1 > i_2 > i_3 > \ldots$ and $i_n = i_{n-1} - 1$ for $n \gg 0$. Let F be the complex vector space with basis given by all semi-infinite infinite wedges.

The infinite dimensional space F becomes an $\mathbf{H}$-module as follows: we define the wedge and contraction operators:

$$\psi_j^*(e_{\mathbf{i}}) = e_j \wedge e_{\mathbf{i}} := \begin{cases} (-1)^{k+1} e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge e_j \wedge e_{i_{k+1}} \wedge \ldots & \text{if } i_k > j > i_{k+1} \\ 0 & \text{if } i = i_k \text{ for some } k. \end{cases}$$

$$\psi_j(e_{\mathbf{i}}) = \begin{cases} (-1)^k e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge e_{i_{k+1}} \wedge \ldots & \text{if } j = i_k \\ 0 & \text{if } j \neq i_k \text{ for all } k \in \mathbb{Z}_{>0}. \end{cases}$$

Then we let C act as the identity, and a_n act as the following infinite sums

$$\begin{cases} \sum_{i \in \mathbb{Z}} {\psi_i}^* \psi_{i+n} & \text{if } n \in \mathbb{Z} - \{0\}, \\ \sum_{i>0} {\psi_i}^* \psi_i - \sum_{i \leq 0} \psi_i \psi_i^* & \text{if } i = 0. \end{cases}$$

Obviously $\psi_i \psi_i^* e_{\mathbf{i}} = 0$ for $i \gg 0$, and there are only finitely many positive indices in $\mathbf{i}$, hence $\sum_{i>0} {\psi_i}^* \psi_i - \sum_{i \leq 0} \psi_i \psi_i^* e_{\mathbf{i}}$ is always a finite sum. For other operators, note that for any $e_{\mathbf{i}} \in F^{(m)}$ we have ${\psi_i}^* \psi_{i+n} e_{\mathbf{i}} = 0$ for $i \gg 0$ because we would replace a factor e_i in the semi-infinite wedge $e_{\mathbf{i}}$ by e_{i+n} which already appears in $e_{\mathbf{i}}$ given that $i \gg 0$. Therefore the above infinite sums are well-defined operators on F. One can directly check the commutator relations of $\mathbf{H}$, so that these operators indeed define an $\mathbf{H}$-action on F.

Similar to the bosonic picture, for $m \in \mathbb{Z}$ we consider the subspace $F^{(m)} \subset F$ of infinite wedges $e_{\mathbf{i}}$ with $i_n = m - n$ for $n \gg 0$. Here the vacuum vector is given by $\emptyset_m = e_m \wedge e_{m-1} \wedge \dots$, and we can again apply a_n for $n < 0$ to get all semi-infinite wedges in $F^{(m)}$. From this one deduces that a_0 acts as multiplication by m on $F^{(m)}$. Again $F^{(m)}$ is an irreducible highest weight representation of $\mathbf{H}$, known as the *fermionic Fock space of central charge m*. For physicists, $\emptyset_0 \in F^{(0)}$ is known as the *perfect vacuum.*

We can also define a grading on F as follows: for each semi-infinite wedge $e_{\mathbf{i}} \in F^{(m)}$ we assign the infinite tuple $(i_0 - m, i_1 - m + 1, i_2 - m + 2, \dots)$, then only finitely many entires concentrated at the beginning are nonzero, and $i_0 - m \geq i_1 - m + 1 \geq i_2 - m + 2 \geq \dots \geq 0$, hence we actually get a partition. Clearly this is a bijection between semi-infinite wedges in $F^{(m)}$ and the set of all partition. We define the degree of the semi-infinite wedge to be the number of boxes of the corresponding partition. We refer to [KRR13, Lecture 6] for the following theorem.

Theorem 3.2.29 (Boson-fermion correspondence)**.** *There is a unique isomorphism of graded* $\mathbf{H}$*-modules* $\sigma : F^{(m)} \cong B^{(m)}$ *such that* $\sigma(\emptyset_0) = 1$*. Furthermore, under the identification* $B^{(m)} \cong \Lambda$ *in Remark 3.2.27,* σ *sends a semi-infinite wedge to* z^m *times the Schur polynomial labeled by its partition.*

3.2.6 Connection to the dual rim hook algorithm

Now recall the Siebert-Tian presentation 3.1.31, which gives for any k, N a canonical quotient map of algebras $\Delta : \Lambda[q] \to QH^*Gr(k, N)$, where Λ denotes the ring of symmetric functions (see Remark 3.1.23). Now we will translate the map Δ to the fermionic side via the boson-fermion correspondence. For this we consider the Fock space of central charge k.

Definition 3.2.30. Consider wedges of the following form

$$e = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \wedge e_0 \wedge e_{-1} \wedge e_{-2} \dots \tag{3.33}$$

Assign to such a wedge a number $a \in \mathbb{Z}$ as follows: write $i_j = a_j N + \bar{i}_j$ for $1 \leq j \leq k$, such that $1 \leq \bar{i}_j \leq N$ and set $a := \sum a_i$.

Define a $\mathbb{C}[q]$-linear map $q_{k,N} : F^{(k)} \to \mathcal{F}_k[q]$ by sending wedges of the form 3.33 to

$$(-1)^{a(k-1)} q^a e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \in \mathcal{F}_k[q],$$

and other semi-infinite wedges to 0.

To connect this map with the bosonic picture, we need a more explicit description of the quotient map Δ. For this we cite the dual rim hook algorithm in [BCFF99].

Recall that Schur polynomials can be defined for arbitrary integer tuples $v \in \mathbb{Z}^n$ by passing from an integer tuple to a partition using the following straightening rules (see [Pra19]):

$$s_{(\dots,a,b,\dots)} = -s_{(\dots,b-1,a+1,\dots)} \quad \text{and} \quad s_{(\dots,a,a+1,\dots)} = 0. \tag{3.34}$$

Definition 3.2.31 (Dual rim hook algorithm)**.** Given two partitions $\lambda, \mu \in \mathfrak{P}(k, N)$, first perform the standard Littlewood-Richardson algorithm in Λ to obtain the expansion

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

Then one imposes the quotient condition by discarding all terms s_ν with partitions ν which contain more than k nonzero parts and by replacing all the remaining s_ν with the polynomials $(-1)^{d(k-1)} q^d s_{v(\nu)}$ where $v(\nu)$ is the unique set of integers such that

$$v_i(\nu) = \nu_i \mod N \qquad \text{and} \qquad i - k \le v_i(\nu) \le i + N - k - 1\ . \tag{3.35}$$

and $d = N^{-1}(|\nu| - |v(\nu)|) \in \mathbb{Z}$. This gives the product in $QH^*Gr(k, N)$.

Now define a $\mathbb{C}[q]$-linear map $\Delta' : \Lambda[q] \to QH^*Gr(k, N)$ by

$$\Delta'(s_\lambda) = \begin{cases} 0, & \text{if } \lambda \text{ contains more than } k \text{ parts} \\ (-1)^{d(k-1)} q^d s_{\overline{\lambda}} & \text{otherwise.} \end{cases}$$

where $\overline{\lambda}$ is the composition obtained from λ by taking all parts modulo N satisfying the condition 3.35. In particular, $\Delta(s_\lambda) = s_\lambda$ if $\lambda \in \mathfrak{P}(k, N)$. Then the dual rim hook algorithm asserts that this map is a ring homomorphism, hence is equal to the canonical quotient map $\Delta : \Lambda[q] \to QH^*(k, N)$.

Theorem 3.2.32. *The following diagram commutes:*

$$\begin{tikzcd} F^{(k)}[q] \arrow[r, "\sigma"] \arrow[d, "q_{k,N}"] & B[q] \arrow[rr, "\sim"] & & \Lambda[q] \arrow[d, "\Delta"] \\ \mathcal{F}_k[q] \arrow[rrr, "\sim"] & & & \mathbb{C}[e_1, \dots, e_k]/(h_{N-k+1} \dots h_N + (-1)^k q) \end{tikzcd} \tag{3.36}$$

Proof. All we need to check is that the reduction 3.35 is given by $q_{k,N}$. First of all Δ sends all partitions with more than k parts to 0, which corresponds to the definition that $q_{k,N}$ kills the corresponding wedges. Thus it suffices to consider semi-infinite wedges of the form $e_{i_0} \wedge e_{i_1} \wedge \dots e_{i_{k-1}} \wedge e_0 \wedge e_{-1} \wedge \dots \in F^{(k)}$. Let λ be the corresponding partition, then we have

$$\lambda_j = i_{j-1} - (k - j + 1). \tag{3.37}$$

On the one hand, $q_{k,N}e_{\mathbf{i}} = (-1)^{d(k-1)}q^d e_{\bar{\mathbf{i}}}$ where $\bar{\mathbf{i}}$ is obtained from $\mathbf{i}$ by taking residue within range $1 \le \bar{i}_j \le N$ modulo N. The sequence $\mathbf{i}$ translates to the integer tuple $\overline{\lambda}$ with at most k parts satisfying

$$N - k + j - 1 \ge \overline{\lambda}_j \ge 1 - k + j - 1 = j - k \qquad 1 \le j \le k$$

Compare with the reduction condition 3.35, we find that this is the image of s_λ under Δ is equivalent to $d = a$ where a is as in Definition 3.2.30. In view of the relation 3.37 and the reduction condition 3.35, this indeed holds. □

Remark 3.2.33. The straightening rule 3.34 also translates nicely to the fermionic side nicely: if $\overline{\lambda} = (\dots, a, a+1, \dots)$ in the tuple $\overline{\lambda}$, then we have repeated indices in $e_{\mathbf{i}}$, hence it is zero; and if $\overline{\lambda}(\dots, a-1, b+1, \dots)$, suppose $a-1$ appears at the jth slot, then $i_{j-1} = a + k - j$ and $i_j = b + k - j + 1$. Interchanging these two indices produces a minus sign on the semi-infinite wedge, and turns $\overline{\lambda}$ into $(\dots, b, a, \dots)$.

Example 3.2.34. Set $k = 3, N = 7$ and consider the partitions $\lambda = (3, 1, 0)$ and $\mu = (3, 2, 0)$. The Littlewood-Richardson rule says that in the ring of symmetric functions, the nonzero summands s_ν of $s_\lambda s_\mu$ are given by

$$\begin{aligned}\nu = {} & (6,3,0), (6,2,1), (5,4,0), (5,3,1), (5,3,1), (5,2,2), (4,4,1), \\ & (4,3,2), (4,3,2), (3,3,3), (5,2,1,1), (4,3,1,1), (4,2,2,1), (3,3,2,1). \end{aligned} \tag{3.38}$$

Then we discard the last four as they have more than three parts. As for the rest, take $\nu = (5, 3, 1)$ as an example. Reduce the entries modulo N yields $v(\nu) = (-2, 3, 1)$, and following the dual rim hook algorithm $s_{\overline{\nu}} = qs_{(-2,3,1)} = -qs_{(2,-1,1)} = qs_{(2,0,0)}$. On the other hand, the semi-infinite wedge corresponding to the partition $(5, 3, 1)$ is $e_8 \wedge e_5 \wedge e_2 \wedge e_0 \wedge e_{-1} \wedge \dots$. Apply $q_{3,7}$, we obtain $qe_1 \wedge e_5 \wedge e_2 = e_5 \wedge e_2 \wedge e_1$ which corresponds to the partition $(2, 0, 0)$, see Lemma 3.1.15.

There is another partition $\nu = (6, 3, 0)$ with $s_{\overline{\nu}} = -qs_{(2,0,0)}$. The corresponding semi-infinite wedge is $e_9 \wedge e_5 \wedge e_1 \wedge e_0 \wedge e_{-1} \wedge \dots$, which goes to $e_9 \wedge e_5 \wedge e_1$ via $q_{k,N}$, and we normalize it into $qe_2 \wedge e_5 \wedge e_1 = -qe_5 \wedge e_2 \wedge e_1$ to get the partition $(2, 0, 0)$. These are all $(2, 0, 0)$ terms, hence the corresponding Gromov-Witten invariant is $2 - 1 = 1$. In this way one can write down the product expansion

$$\ydiagram{3,1} \star \ydiagram{3,2} = q\,\ydiagram{2} + q\,\ydiagram{1,1} + \ydiagram{4,4,1} + 2\,\ydiagram{4,3,2} + \ydiagram{3,3,3}\,.$$

One may also use the tableaux description (3.32). In this case we have 15 tableaux from which we can discard 5. Amongst the remaining ones there are two pairs which cancel each other which leaves the 6 summands from above.

3.3 Puzzles revisited

With the fermion model in mind, let us consider the Knutson-Tao puzzles for usual cohomology $H^*Gr(k, N)$ again. We first make some observations.

Lemma 3.3.1. *For $\mathbb{P}^{N-1} = Gr(1, N)$ the only puzzles are of the following form.*

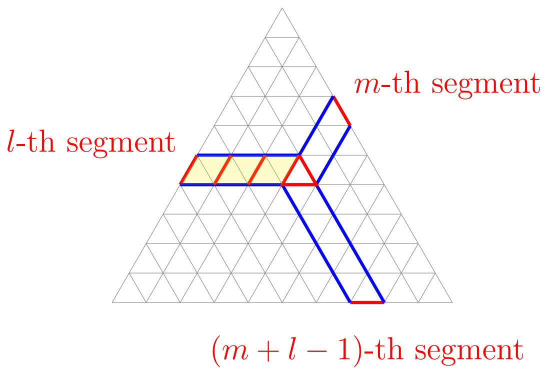


Figure 3.2: A $\mathbb{P}^9$ puzzle with $N = 10, m = 4, l = 5$.

Here the parallelogram regions with thick red and blue boundary lines are filled with rhombi, the remaining regions are filled with triangles, and we count all red segments on the boundaries from left to right.

Proof. Notice that these are indeed puzzles, and by Theorem 2.2.6 and the fact that $H^*(\mathbb{P}^n) \cong \mathbb{C}[X]/(X^{n+1})$, these are the only possible puzzles. □

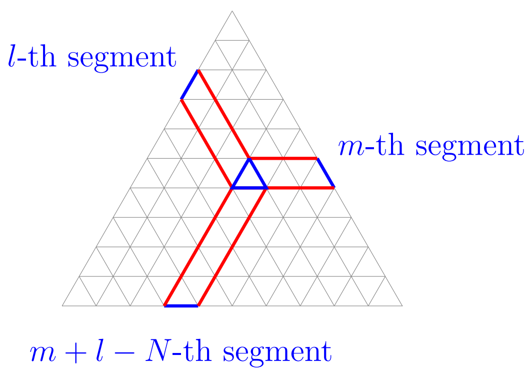


Figure 3.3: $Gr(N - 1, N)$ puzzles

Remark 3.3.2. Dually, by the same argument, all puzzles for $Gr(N-1, N)$ are of the form shown in figure 3.3, where the marked parallelograms are filled with rhombi and the remaining regions are filled with red triangles.

Observe that to determine a puzzle for a general Grassmannian it is also sufficient to determine all the rhombi. Now we want to reconsider these puzzles from the viewpoint of the fermion model. For 01-strings $x, v, w \in \Lambda(k, N)$, consider in any $\Delta_{x,v,w}$ puzzle the SW-NE rhombi, i.e. those whose edges are parallel to the x and w edges. For example, in figure 3.2 we consider the light yellow shaded rhombi. Recall 3.13, the partition corresponding to a 01-string x is denoted by $\lambda(x)$. On the other hand, on the fermion side there is a grading on 0-TL_N by setting $\deg u_i = 1$.

Lemma 3.3.3. *In any $\Delta_{x,v,w}$ puzzle, the number of SW-NE rhombi is equal to* $\deg s_{\lambda(v)}$. *Similarly, the number of the NW-SE rhombi is equal to* $\deg s_{\lambda(x)}$.

Proof. In [KT+03] it is proven that in any $\Delta_{x,v,w}$ puzzle the number of SW-NE rhombi is equal to $I(w) - I(x)$ where I is defined in Definition 2.1.10. Then for non equivariant puzzles we get $\deg s_{\lambda(v)} = I(v) = I(w) - I(x)$ is the number of SW-NE rhombi. $\square$

Lemma 3.3.3 suggests that SW-NE rhombi could be interpreted in terms of the 0-TL_N action on the state space $\mathcal{F}$.

Definition 3.3.4. We divide any puzzle into NW-SE strips which we call *lanes*, as illustrated by figure 3.4. We number the lanes from left to right.

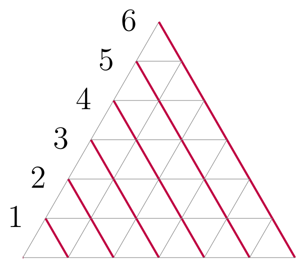


Figure 3.4: Lanes, counted from left to right

Similarly we define *opposite lanes* to be SW-NE strips.

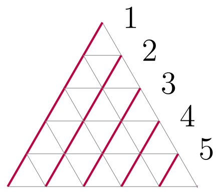


Figure 3.5: Opposite lanes, counted from left to right

Example 3.3.5. Let us first consider the projective space. In figure 3.2, consider the edges in the i-th lane that are parallel to the south or NW boundary of the puzzle, and are boundary edged of the puzzle pieces. For example, in the 7th lane we don't consider all but the short diagonals of the NW-SE rhombi. Observe that the color of the edges do not change in the first 4 and the last 2 lanes. On the other hand, there are no SW-NE rhombi in these lanes. Consider the 4,5,6th lanes, it is easy to see that the rhombi transports the red edge on the NW side to the 5th, then the 6th and finally the 7th lane. This corresponds to the 0-TL_N action $u_7u_6u_5$0000100000.

3.3.1 SW-NE rhombi as 0-TL_N generators

The observation in example 3.3.5 generalizes. Given any puzzle, we can track the color of the edges as we travel from the top of each lane downwards. However, it is possible to find us in the following situation: in figure 3.6 the yellow shaded lane has a NW-SE rhombus at the bottom, whose short diagonal is not colored and should be ignored.

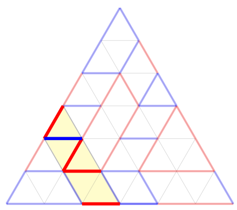


Figure 3.6: A $Gr(3,6)$ puzzle

Therefore we make the following definition: for any puzzle, let $L(i)$ be the set of the colored edges satisfying

- parallel to the south or NW boundary of the puzzle
- contained in the i-th lane.

Example 3.3.6. Consider the puzzle in figure 3.6, where we again only depicted the edges of the rhombi. The set $L(3)$ consists of the highlighted edges. The red edge at the top of $L(3)$ is first turned blue by the SW-NE rhombus adjacent to it, and then turned red again by the SW-NE rhombus below. On the other hand, this puzzle computes the product $S_{011100}S_{010110}$, which corresponds to the 0-TL_N-action

$$s_{(2,2,1)}(\mathcal{U})011100 = u_4u_5u_2u_3u_4 = 011100 = 001011 \tag{3.39}$$

In the fermion model picture, the 3rd position of the string turns into 0 by u_4 and then 1 by u_2. This coincides with the color change of the edges in $L(3)$.

Lemma 3.3.7. *Given any puzzle, let $E(i)$ be the sequence of the colors of edges in $L(i)$ read from top to bottom. Then*

- *Blue followed by red in the sequence $E(i)$ is given by an upward pointing triangle part of a SW-NE rhombus;*
- *Red followed by blue in the sequence $E(i)$ is given by an downward pointing triangle part of a SW-NE rhombus.*
- *Another puzzle piece gives rise to the same color twice consecutively in $E(i)$.*

Similar statements hold for NW-SE rhombi and opposite lanes.

Proof. Try every puzzle piece 2.2.3. □

Note that a rhombus in fact interchanges the colors in two *neighboring lanes*. For example, in figure 3.6 the top SW-NE rhombus in the yellow shaded lane interchanges the red-blue pattern on the NW side of the puzzle into blue-red in the 3rd and 4th lanes. Recall our convention that red translates into 1 and blue into 0 in the corresponding 01-sequence, we propose the following interpretation of any puzzle:

Definition 3.3.8 (Puzzle to 0-TL_N-monomial)**.** For any puzzle we define a monomial in 0-TL_N which we call the *rhombi monomial*, as follows: travel from the top downwards along the leftmost opposite lane, and when we meet an SW-NE rhombi whose south edge is in the i-th lane, we write down the 0-TL_N generator u_i, and for the next rhombus corresponding to u_j, write u_j to the *left* of u_i, etc. until we reach the bottom of the current opposite lane, then we go to the remaining leftmost lane and repeat this process.

For example, the puzzle in figure 3.6 yields the rhombi monomial $u_4u_5u_2u_3u_4$, which coincides with what we had in equation 3.39.

Lemma 3.3.9. *Up to a scalar multiple, the rhombi monomial of a $\Delta_{x,v,w}$ puzzle is a summand of $s_{\lambda(v)}(\mathcal{U}) \in 0\text{-}TL_N$.*

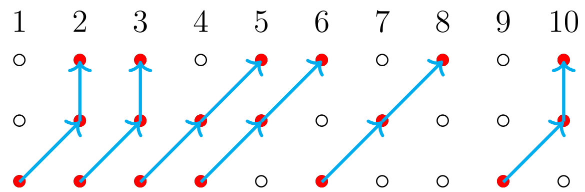
Proof. The specializing $q = 0$ in theorem 3.2.21 plus theorem 2.2.6 imply that in a $\Delta_{x,v,w}$ puzzle the above monomial acting on the 01-string x on the NW side is a summand of $s_{\lambda(v)}(\mathcal{U})x$. Next we notice that for any monomial $m, m' \in 0\text{-}TL_N$ if $mv = m'v$ for some 01-string v, then $m = m'$. For this we write m, m' in the canonical form as in the proof of Lemma 3.2.8 and find that their canonical forms must agree, see also remark 3.2.9. □

We have the following result refining Lemma 3.3.3. Here we switch to the NW-SE rhombi because the author tried to formulate it in terms of the good old SW-NE rhombi but ended up this way.

Lemma 3.3.10. *Given a $\Delta_{x,v,w}$ puzzle, suppose $1 \leq i_1 < i_2 \dots i_k \leq N$ and $1 \leq j_1 < j_2 \dots j_k \leq N$ are the indices such that $v_{i_l} = w_{j_l} = 1$. Then the number of downward pointing triangle parts that belong to a NW-SE rhombus in the i-th opposite lane is equal to $\#\{i_l \leq i \mid j_l > i\}$.*

A similar statement holds for SW-NE rhombi.

Proof. For clarity we use some diagrammatics. By the fermion model, we put the particle configurations corresponding to v at the bottom of that of w, and mark the particle hopping by arrows. For example, the puzzle in figure 2.1 yields



Note that no crossings occur in such diagrams, hence the path starting at i_l ends at j_l for $1 \leq l \leq k$. Thus, for $1 \leq i \leq N$ the number $\#\{i_l \leq i \mid j_l > i\}$ is the number of diagonal arrows starting at some level in the i-th column. On the other hand, Interpreting the NW-SE rhombi as in Lemma 3.3.7, we get that this number is equal to the number of NW-SE rhombi in the i-th opposite lane. □

Remark 3.3.11. From this point of view, we can give an explanation why triangular puzzles work: because the red color at the first segment of the NE side can be transfered by $0\text{-}TL_N$ at most $(N-1)$ times to the last one, hence in the last lane we can only allow $N-1$ NW-SE rhombi. In fact, this upper bound is assumed by $\mathbb{P}^{N-1}$. By the same reason in the second last lane admits only $(N-2)$ NW-SE rhombi, etc. and the last opposite lane admits no such rhombi at all.

3.3.2 A crystallization algorithm

Based on the above observations, we propose an algorithm that produces puzzles for the usual cohomology of the Grassmannians. Here we divide any input equilateral triangle into unit triangular grids. And the input $\underline{a}$ is the integer tuple computed in Lemma 3.3.10.

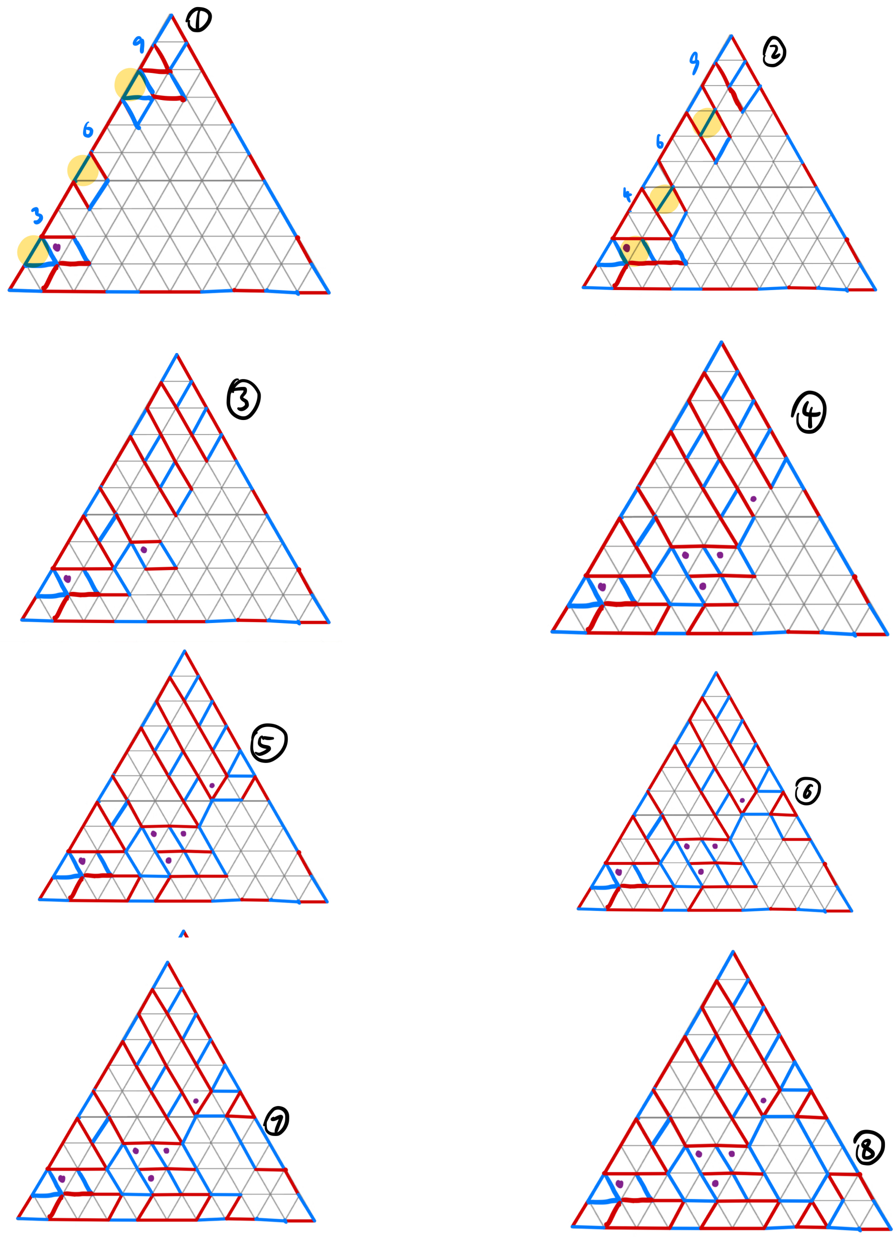
Algorithm 1: Crystallization algorithm for $H^*Gr(k,N)$ puzzles

---

Input: A labeled equilateral triangle $\Delta_{x,v,w}$
Result: All $\Delta_{x,v,w}$ puzzles
Function `Crystallize(`*a partially filled puzzle,* $\underline{a} \in \mathbb{Z}^N_{\geq 0}$`):`
  if *the puzzle is completed* **then**
    Return(the current puzzle);
  else
    Let the i-th opposite lane be the lefmost among the ones that is not completely filled;
    Let $J \subset \{2 \leq j \leq N\}$ such that
    $\star$ the j-th upward pointing triangle in the current opposite lane (count from the top) with its NW edge being either red or the short diagonal of an NW-SE rhombus;
    $\star$ the NW edge of the $(j-1)$-th upward pointing triangle is either blue or the short diagonal of an NW-SE rhombus;
    for *choices of* a_i *elements* $j_1 \dots j_{a_i} \in J$ **do**
      Put a NW-SE rhombus at the j_l-th downward pointing unit triangle (do nothing if $a_i = 0$);
      For the other upward pointing triangles in the i-th opposite lane, put a SW-NE or vertical rhombus whenever possible, satisfying the following rule:
      The color of all NE and south edges of the unit triangles between any two NW-SE rhombi or one such rhombus and the boundary of the puzzle are the same;
      if *cannot fill the current opposite lane* **then**
        Clear the current opposite lane;
      end
      `Crystallize` (the current puzzle);
    end
  end
End function;
$\underline{a} = (a_1, \dots, a_N)$=the number of NW-SE rhombi in each opposite lane;
`Crystallize` ($\Delta_{x,v,w}$, $\underline{a}$);

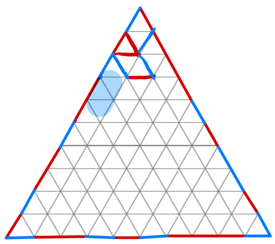
Before proving algorithm 1, let us first examine an example.

Example 3.3.12. The following is an example of (a branch of) the crystallization algorithm that gives the leftmost puzzle in figure 2.1.



We numbered the pictures by the opposite lane to be filled. For example, in the first picture we start from the first lane and find the blue edges on the NW side to get $J = \{3, 6, 9\}$, because no edges on the boundary of the puzzle can be the short diagonal of any rhombi. Now since the first red edge of the NE side is moved only once to get the bottom side, we have $a_1 = 1$. Here we picked $j_1 = 2$ and put a NW-SE rhombus at the 3rd lane. Then in the upper region of this rhombus we can only color any NE or south grid edge red, and when they meet a blue edge we get a vertical rhombus, as shown in the picture. Similarly, below this rhombus we have a SW-NE rhombus. These are all rhombus pieces in the first opposite lane.

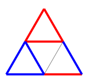
Next, we turn to the second opposite lane, then we have $a_2 = 1$ and $J = \{4, 6, 9\}$ and we picked $j_1 = 4$, and thus the second picture.



On the other hand, if in the first opposite lane we pick $j_1 = 9$, we must then have a vertical rhombus at the top of the puzzle, then we soon find that no puzzle piece fits into the blue highlighted region in the above picture. Hence we delete this SW-NE rhombus and try the next j_l, but there isn't a next j_l, the algorithm will in fact end the recursion here.

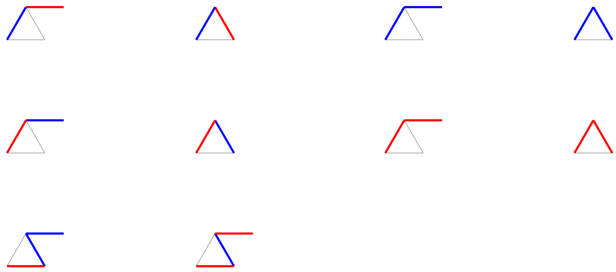
Theorem 3.3.13. *For 01-strings $x, v, w \in \Lambda(k, N)$, the crystallization algorithm 1 yields all puzzles with a given boundary condition $\Delta_{x,v,w}$.*

Proof. Given any puzzle, we use induction on the label of opposite lanes to show that it is given by Algorithm 1. Starting from a labeled equilateral triangle, if there is a NW-SE rhombus in the first opposite lane at the i-th position, then the 01-string x on the NW side satisfies $x_i = 0, x_{i+1} = 1$. In other words, we have the following local picture, which is included in the conditions defining the set J in Algorithm 1.



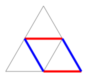
Now suppose the rhombi in the first i opposite lanes are determined. We again divide the entire puzzle into unit triangular grids.

- If there are no NW-SE rhombi, in which case the NE and south edges in the current opposite lane are all blue or all red, and a NW edge is either blue or red or the short diagonal of a NW-SE rhombus. Start from the top, the following are all the possible patterns of the upward pointing unit triangle in the current opposite lane: in the first row the NW edge is blue and in the second row red, and in the last row it is the short diagonal of a NW-SE rhombus.



In each case there is at mots one combination of puzzle pieces having the pattern as its upper boundary, where all the rhombi pieces are described by the algorithm.

- If there is a NW-SE rhombus as shown in the following picture, consider the upward pointing triangle to its left, whose NW edge is either the short diagonal of another NW-SE rhombus, or it has to be blue.

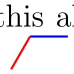
  Similarly, the NW edge of the upward pointing unit triangle above it has to be either red or the short diagonal of another NW-SE rhombus. Therefore the conditions in Algorithm 1 captures all possible occurrences of NW-SE rhombi.

  Finally, by lemma 3.3.7 the NE and south edges between the NW-SE rhombi are all of the same color, and we can use the arguments in the case where there are no SW-NE rhombi to determine all other rhombi.

This proves that our algorithm exhausts all possible puzzles. □

Remark 3.3.14. I call this "crystallization" because it resembles how crystals grow on surfaces. One can grow the "crystals" (i.e. the rhombi) simultaneously on all three edges of the puzzle to get a faster algorithm.

The `Crystallize` function defined in algorithm 1 can be also used to define an algorithm with input only the NW and NE sides of the to be determined puzzle and output all possible puzzles with these prescribed sides. Namely one just tries all south edges, or only the edges with the correct I-function value (Definition 2.1.10) or only all possible south edges determined e.g. via the state picture.

Finally, if we allow the equivariant rhombus, one can also modify this algorithm to produce equivariant puzzles. Note that in this case the pattern  fits into either a SW-NE rhombus or an equivariant rhombus, both increasing the degree of the NW side by 2, which reflects the fact that the multiplication respects the grading on $H_T^*Gr(k,N)$. However, I don't have a finer explanation of equivariant puzzles.

Conjecture. In this section we have only used the $q=0$ specialization of the fermion model. By the same spirit we can allow N rhombi in each lane, and interpret the affine generator also as a rhombus, thus we expect a puzzle rule for the quantum cohomology, possibly by tiling the puzzle pieces on a cylinder. If we can find such a puzzle rule, then for example we might be able to provide a combinatorial proof of the commutativity of the noncommutative elementary and complete symmetric polynomials.

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