Representation theory of alternating nil Hecke algebras

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Introduction

The nil Hecke algebras are a family $(\mathrm{NH_n})_{n\in\mathbb{N}_0}$ of graded algebras over a field k which is of huge importance in current research in representation theory. Originally, they were introduced by Kostant and Kumar in [KK86] in order to study the cohomology rings of flag varieties. A further field of representation theory in which they appear is the theory of categorification of quantum groups.

We now outline this in more detail. Nil Hecke algebras are special cases of Quiver Hecke algebras which are a family of graded k-algebras attached to a given quiver without loops whose construction is based on the famous Vershik-Okounkov approach in the representation theory of symmetric groups [OV96]. They were introduced independently by Khovanov and Lauda in [KL09, KL11] and Rouquier in [Rou08]. Thus, they are also called Khovanov–Lauda–Rouquier algebras.

One important aspect of quiver Hecke algebras is that their representation theory categorifies Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. The integral quantum group ${}_{\mathcal{A}}\mathbf{f}$ was introduced by Lusztig in [Lus93] and plays a fundamental role in the theory of quantum groups. The mentioned categorification theorem was proved by Khovanov and Lauda in [KL09, Theorem 1.1].

In the following, we elaborate on this result. For this, we fix some notation. Let Γ be a fixed quiver without loops and vertex set I. Let $Q^+ := \bigoplus_{i\in I} \mathbb{N}_0\alpha_i$ be the free commutative monoid on a basis with index set I. By definition, the family of quiver Hecke algebras is a family of algebras $(R(\nu))_{\nu\in Q^+}$ parameterized by Q^+. For each non-zero $\nu \in Q^+$, we have that $R(\nu)$ is an infinite dimensional graded k-algebra with finite dimensional graded components. Moreover, each $R(\nu)$ is bounded from below, i.e. there exists $d(\nu) \in \mathbb{Z}$ such that all homogeneous components of $R(\nu)$ of degree $< d(\nu)$ vanish.

Now, let $\mathrm{K}_0(R(\nu)\text{- pmod})$ be the split Grothendieck group of the additive category of finitely generated graded projective $R(\nu)$-modules and $\mathrm{G}_0(R(\nu)\text{- fmod})$ be the Grothendieck group of the category of finite dimensional graded $R(\nu)$-modules. Both Grothendieck groups admit the structure of an $\mathcal{A}$-module, where q acts via shifting degrees. We proceed with defining

$$\mathrm{K}_0(R) := \bigoplus_{\nu\in Q^+} \mathrm{K}_0(R(\nu)\text{- pmod}), \quad \mathrm{G}_0(R) := \bigoplus_{\nu\in Q^+} \mathrm{G}_0(R(\nu)\text{- fmod}).$$

Via induction and restriction functors, we obtain a multiplication and comultiplication on $\mathrm{K}_0(R)$ and $\mathrm{G}_0(R)$ turning them into Q^+-graded twisted bialgebras. Moreover, $\mathrm{G}_0(R)$ can be identified with the Q^+-graded dual of $\mathrm{K}_0(R)$. The categorification theorem of Khovanov–Lauda then states that there are isomorphisms of twisted bialgebras

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \to \mathrm{K}_0(R), \quad \gamma^* : \mathrm{G}_0(R) \to {}_{\mathcal{A}}\mathbf{f}^*,$$

where ${}_{\mathcal{A}}\mathbf{f}$ denotes Lusztig's integral quantum group corresponding to the unoriented underlying graph of Γ and ${}_{\mathcal{A}}\mathbf{f}^*$ denotes the graded dual of ${}_{\mathcal{A}}\mathbf{f}$.

In the special case, where Γ is the one-vertex quiver without arrows, the corresponding family of quiver Hecke algebras $(R(\nu))_{\nu\in Q^+}$ is by definition the family of nil Hecke algebras

$(\mathrm{NH_n})_{n\in\mathbb{N}_0}$. In this special case, we set $\mathrm{K}_0(\mathrm{NH}) := \mathrm{K}_0(R)$ and $\mathrm{G}_0(\mathrm{NH}) := \mathrm{G}_0(R)$. Moreover, we call $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ the nil Hecke Grothendieck groups.

This thesis is devoted to the study of an interesting family of graded subalgebras of nil Hecke algebras. Namely, the alternating nil Hecke algebras which we denote by $(\mathrm{ANH_n})_{n\in\mathbb{N}_0}$. In our studies, we focus in particular on the following interesting question:

(Q) Can we formulate an analogous version of the categorification theorem of Khovanov–Lauda for alternating nil Hecke algebras?

We now briefly outline the origins of alternating nil Hecke algebras. They are special cases of a more general family of algebras, namely the alternating quiver Hecke algebras which were introduced by Boys and Mathas in [Boy14, BM17]. The definition of alternating quiver Hecke algebras is based on the definition of the alternating group as fixed point subgroup of the symmetric group under the sign involution. Concretely, Boys and Mathas defined a sign involution

$$\mathrm{sgn} : R(\nu) \to R(\nu),$$

on the quiver Hecke algebras attached to quivers of type A_{2n+1} or $\tilde{A}_n$. They particularly used for the construction of the sign involution that these quivers admit isomorphisms to the opposite quivers which correspond to multiplication with -1 in a certain sense. This can be illustrated by the following pictures, where we choose $\{-n, -n+1, \dots, n\}$ as the vertices in type A_{2n+1} and $\mathbb{Z}/n$ as the vertices in type $\tilde{A}_n$.

$$\underset{-n}{\bullet} \longrightarrow \underset{1-n}{\bullet} \longrightarrow \dots \longrightarrow \underset{n}{\bullet} \qquad \xrightarrow[i\mapsto -i]{\sim} \qquad \underset{-n}{\bullet} \longleftarrow \underset{1-n}{\bullet} \longleftarrow \dots \longleftarrow \underset{n}{\bullet}$$

$$\begin{tikzcd}[column sep=small] & \underset{\overline{0}}{\bullet} \arrow[dl] & \\ \underset{\overline{1}}{\bullet} \arrow[r] & \dots \arrow[r] & \underset{\overline{n-1}}{\bullet} \arrow[ul] \end{tikzcd} \qquad \xrightarrow[\overline{i}\mapsto -\overline{i}]{\sim} \qquad \begin{tikzcd}[column sep=small] & \underset{\overline{0}}{\bullet} \arrow[dr] & \\ \underset{\overline{1}}{\bullet} \arrow[ur] & \dots \arrow[l] & \underset{\overline{n-1}}{\bullet} \arrow[l] \end{tikzcd}$$

By definition, the alternating nil Hecke algebras are the alternating quiver Hecke algebras corresponding to the one-vertex quiver without arrows.

In the following, we describe the results of this thesis. Overall, the thesis is divided into two parts.

Part 1. In the first two chapters, we discuss the representation theory of graded k-algebras which have finite dimensional graded components and are bounded from below. Following Kleshchev [Kle15a, Kle15b], we call these k-algebras Laurentian. As already mentioned above, quiver Hecke algebras and hence also alternating quiver Hecke algebras satisfy these finiteness conditions.

The crucial point why the representation theory of Laurentian k-algebras is worth to study is that firstly they form a huge class of graded k-algebras and secondly many pleasant results from the representation theory of finite dimensional k-algebras transfer to the setting of Laurentian k-algebras. We characterize this in detail in the first chapter.

One important fact is that a Laurentian k-algebra A is graded semiperfect. Hence, A admits only finitely many graded simple modules up to shift-isomorphism and each graded simple A-module admits a projective cover. In addition, the Laurentian property also implies that each graded simple A-module has finite dimension over k.

A further essential aspect of Laurentian k-algebras is the notion of graded composition multiplicities. If M is a finitely generated graded A-module, then M is not necessarily of finite length. In particular, M does in general not admit a composition series. However, in our setting, we have a satisfying alternative. Given a graded simple A-module L, then the graded composition multiplicity $[M : L]_q$ is a Laurent series with integer coefficients

$$\sum_{i \gg -\infty} a_i q^i \in \mathbb{Z}(\!(q)\!),$$

where for each $i \in \mathbb{Z}$ the coefficient a_i counts the multiplicity of the shifted module $L\langle i \rangle$ in a separated filtration

$$M = F_0 \supset F_1 \supset F_2 \supset \dots$$

with graded simple subquotients. The Laurentian property ensures that a_i is independent from the choice of the filtration. We will discuss this in detail in Section 1.4.

Altogether, the above properties imply that we have an interesting theory of Grothendieck groups of Laurentian k-algebras. In Chapter 2, we focus on working out important aspects of this theory.

In the following, let A-fmod be the Abelian category of finite dimensional graded A-modules and A-pmod additive category of finitely generated graded projective A-modules. We denote by $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{K}_0(A\text{-pmod})$ the corresponding Grothendieck groups. Moreover, let us assume that A is also graded left Noetherian and let A-mod denote the Abelian category of finitely generated A-modules. The corresponding Grothendieck group is denoted by $\mathrm{G}_0(A\text{-mod})$. We have that $\mathrm{G}_0(A\text{-fmod})$, $\mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-mod})$ are all $\mathcal{A}$-modules, where q acts by shifting degrees.

Using graded composition multiplicities we obtain a close connection between $\mathrm{G}_0(A\text{-mod})$ and $\mathrm{G}_0(A\text{-fmod})$ which is given by the graded character map

$$\mathrm{gch} : \mathrm{G}_0(A\text{-mod}) \to \mathbb{Z}(\!(q)\!) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}).$$

The graded character map can be viewed as mapping a class of a finitely generated graded A-module to the (possibly infinite) sum of the classes of its graded simple filtration quotients.

In our studies, we are particularly interested in characterizing relations between the graded characters of graded projective indecomposable and graded simple A-modules. For this, we transfer a further crucial aspect of the theory of Grothendieck groups of finite dimensional k-algebras to our setting. Namely, we define $\mathcal{A}$-bilinear Euler forms $\chi_{\mathrm{f}}, \chi_{\mathrm{p}}$ and χ_{m} on $\mathrm{G}_0(A\text{-fmod}), \mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-mod})$. Our motivation for this is to establish a duality relationship between the graded characters of graded projective indecomposable and graded simple A-modules.

To define these bilinear Euler forms, we make some assumptions on A. One assumption is that A has finite global dimension, so we have that all graded EXT-terms vanish in high enough degrees. A further assumption is that A admits a self-inverse anti-automorphism. This assumption provides that if M is a graded A-module, then $M^{\circledast} := \mathrm{HOM}_k(M, k)$ again admits a graded A-module structure. Here, $\mathrm{HOM}_k(M, k)$ is the graded k-vector space spanned by homogeneous k-homomorphisms between M and k. This fact is of significant importance in the definition of $\chi_{\mathrm{f}}, \chi_{\mathrm{p}}$ and χ_{m}, since it provides the $\mathcal{A}$-bilinearity condition.

Using the $\mathcal{A}$-bilinearity, we can then extend χ_{f} and χ_{p} to $\mathbb{Z}(\!(q)\!)$-bilinear forms $\hat{\chi}_{\mathrm{f}}$ and $\hat{\chi}_{\mathrm{p}}$ on $\mathbb{Z}(\!(q)\!) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ and $\mathbb{Z}(\!(q)\!) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$. The main result of the first part is then the following duality statement.

Theorem. *The following holds:*

(i) The Euler forms $\hat{\chi}_{\mathrm{p}}$ and $\hat{\chi}_{\mathrm{f}}$ are both non-degenerated.

(ii) If $L_1, \dots, L_r$ is a complete list of pairwise non-shift-isomorphic graded simple A-modules and $P_1, \dots, P_r$ are the corresponding projective covers, then

$$(\mathrm{gch}([L_1^{\circledast}]), \dots, \mathrm{gch}([L_r^{\circledast}])) \quad \textit{and} \quad (\mathrm{gch}([P_1]), \dots, \mathrm{gch}([P_r]))$$

are dual $\mathbb{Z}(\!(q)\!)$-bases of $\mathbb{Z}(\!(q)\!) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ with respect to $\hat{\chi}_{\mathrm{f}}$.

According to our choice of assumptions, this theorem can be applied to a large class of Laurentian k-algebras including (alternating) nil Hecke algebras, as we will outline in the second part.

Part 2. This part is devoted to the study of the representation theory of alternating nil Hecke algebras. Motivated by the definition of the nil Hecke Grothendieck groups, we define the alternating nil Hecke Grothendieck groups as

$$\mathrm{G}_0(\mathrm{ANH}) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}), \quad \mathrm{K}_0(\mathrm{ANH}) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}).$$

By construction, $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are $\mathbb{N}_0$-graded $\mathcal{A}$-modules. Moreover, in Section 4.3, we show that $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ both admit multiplicative and comultiplicative structures given by induction and restriction. In addition, $\mathrm{G}_0(\mathrm{ANH})$ is also the graded dual of $\mathrm{K}_0(\mathrm{ANH})$. However, in contrast to the nil Hecke Grothendieck groups, $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are no twisted bialgebras which we show in Proposition 4.6.2.

We now outline a further interesting aspect. For this, we first consider the following fact about $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ which is a consequence of the categorification theorem of Khovanov–Lauda:

(F) We have that $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ are not isomorphic as twisted algebras over $\mathcal{A}$. However, taking graded characters induces an isomorphism of twisted bialgebras over $\mathbb{Q}(q)$:

$$\phi : \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \to \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)},$$

where $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ are obtained from $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ via scalar extension to $\mathbb{Q}(q)$.

Using techniques from the first part, we show that this result generalizes to our setting as follows. Let $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ be the scalar extended versions of $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$. Then, in Theorem 4.4.4, we show that taking graded characters gives an isomorphism of $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-vector spaces between $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$, which is compatible with the multiplicative and comultiplicative structures. Moreover, in Proposition 4.6.1, we show that $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$ are neither isomorphic as $\mathbb{N}_0$-graded $\mathcal{A}$-algebras nor as $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras.

Hereafter, we come to a further application of the results from the first part. At first, we construct non-degenerated $\mathbb{Q}(q)$-bilinear Euler forms χ_{G} and χ_{K} on $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$. The categorification theorem implies that under the identification $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \cong \mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}$, we have that χ_K corresponds to Lusztig's symmetric form $(.,.)$. Here, ${}_{\mathcal{A}}\mathbf{f}$ denotes Lusztig's integral quantum group corresponding to the one-vertex graph without edges.

Motivated by these considerations, we also construct non-degenerated $\mathbb{Q}(q)$-bilinear Euler forms χ'_{G} and χ'_{K} on $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and show that they satisfies similar properties as $(.,.)$. In Theorem 4.4.4, we then apply the main theorem from the first part to determine explicit formulas for χ'_{G} and χ'_{K} with respect to appropriate bases.

Finally, we characterize the multiplicative and comultiplicative structures on $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. In view of the categorification theorem, we are particularly interested in relating these structures to ${}_{\mathcal{A}}\mathbf{f}$. For this, let $\mathcal{A}[\mathbb{Z}/2]$ be the group algebra of $\mathbb{Z}/2$ over $\mathcal{A}$ and let $\mathcal{A}[\mathbb{Z}/2]^*$ denote the graded dual coalgebra of $\mathcal{A}[\mathbb{Z}/2]$. In Theorem 4.5.9, we construct an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras

$$\psi : ({}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*)_{\geq 2} \to \mathrm{K}_0(\mathrm{ANH})_{\geq 2}.$$

Here, the subscript ≥ 2 means that we only have an isomorphism in degrees ≥ 2. This is due to the fact that ANH_0 and ANH_1 strongly differ from the alternating nil Hecke algebras $\mathrm{ANH}_{\mathrm{n}}$ for $n \geq 2$. Using the duality between $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$, we then obtain an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras

$$\psi^* : \mathrm{G}_0(\mathrm{ANH})_{\geq 2} \to ({}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2])_{\geq 2}.$$

In order to describe the comultiplication on $\mathrm{G}_0(\mathrm{ANH})$ and the multiplication on $\mathrm{K}_0(\mathrm{ANH})$, we define an $\mathbb{N}_0$-graded $\mathcal{A}$-algebra ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ which can be seen as a sign perturbated version of ${}_{\mathcal{A}}\mathbf{f}$. Let ${}_{\mathcal{A}}\tilde{\mathbf{f}}^*$ be the graded dual coalgebra of ${}_{\mathcal{A}}\tilde{\mathbf{f}}$. In Theorem 4.5.16, we show that the $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras $\mathrm{G}_0(\mathrm{ANH})_{\geq 2}$ and $({}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*)_{\geq 2}$ become isomorphic after extending the scalars to $\mathcal{A}' := \mathcal{A}[\frac{1}{2}]$. Likewise, we have that the $\mathbb{N}_0$-graded $\mathcal{A}$-algebras $\mathrm{K}_0(\mathrm{ANH})_{\geq 2}$ and the product algebra $({}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}})_{\geq 2}$ become isomorphic after scalar extension to $\mathcal{A}'$.

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1 Representation theory of Laurentian algebras

Convention. Throughout this chapter, we fix a ground field k.

Summary

This thesis is devoted to the study of the representation theory of alternating nil Hecke algebras which are interesting subalgebras of nil Hecke algebras. Both, the nil Hecke algebras and the alternating nil Hecke algebras, are graded k-algebras that satisfy the following conditions:

L1 All homogeneous components are of finite dimension over k.

L2 There exists $d \in \mathbb{Z}$ such that all homogeneous components of degree i vanish, for $i < d$.

Following Kleshchev [Kle15a, Kle15b], we call a graded k-algebra A *Laurentian*, if A satisfies the conditions L1 and L2. In this chapter, we discuss well-known and important properties of Laurentian k-algebras. The crucial point why the representation theory of Laurentian k-algebras is interesting is that Laurentian k-algebras satisfy many pleasant properties that are similar to the properties of finite dimensional k-algebras. We now formulate some of these properties. For this, let A be a Laurentian k-algebra:

1. We have that A is graded semiperfect. In particular, A admits only finitely many graded simple modules up to shift-isomorphism and each graded simple A-module admits a projective cover.
2. All graded simple A-modules are of finite dimension over k.
3. For finitely generated graded A-modules, we have a notion of graded composition multiplicities.

An important application of these properties is that there is an interesting theory of Grothendieck groups of Laurentian k-algebras. In Chapter 2, we will discuss this in sufficient detail.

In this chapter, we assume that the reader is familiar with the basics of graded algebras and graded modules, for a reference see e.g. [NvO04]. Furthermore, we assume basic knowledge about Abelian and additive categories and in particular the notion of projective covers and Krull–Schmidt categories, see for instance [Kra15]. Finally, we also assume some basic knowledge in homological algebra, for a reference see e.g. [Wei95].

1.1 Basic definitions and conventions

In this section, we recall important definitions and provide the general setup of this thesis. At first, we stress that by $\mathbb{N}$ we denote the natural numbers without 0 and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Now, we move to (graded) k-algebras.

Convention 1.1.1. Throughout the thesis, we adhere to the following conventions:

1. A *k-algebra A* is always assumed to be associative and unital. By a *module M* over A, we always mean a unital left A-module.

2. By a *graded k-algebra A*, we always mean an associative unital $\mathbb{Z}$-graded k-algebra. The homogeneous components of A are denoted by A_i, for $i \in \mathbb{Z}$. If $a \in A$ is homogeneous, then the degree of a is denoted by $|a|$, i.e. $|a| = i$ if $a \in A_i$. Whenever we write $|a|$, we always assume that a is homogenoues.

3. Let A be a graded k-algebra. By a *graded A-module M* over A, we always mean a unital graded left A-module. The homogeneous components of M are denoted by M_i, for $i \in \mathbb{Z}$. If $m \in M$ is an homogeneous element, then the degree of m is denoted by $|m|$. In particular, we have $|am| = |a||m|$ for $a \in A, m \in M$ homogeneous. Whenever we write $|m|$ for $m \in M$, we always assume that m is homogenoues.

We proceed with fixing the notation for the sets of homomorphisms between graded and ungraded modules.

Notation 1.1.2. Let A be a k-algebra and M, N be A-modules. Then the k-vector space of A-module homomorphisms between M and N is denoted by $\mathrm{Hom}_A(M, N)$.

Notation 1.1.3. Let A be a graded k-algebra and M, N be graded A-modules.

1. An A-linear map $f \in \mathrm{Hom}_A(M, N)$ is called a *homomorphism* of graded A-modules, if for all $i \in \mathbb{Z}$, we have $f(M_i) \subset N_i$. We denote by $\mathrm{hom}_A(M, N)$ the k-vector space of graded A-module homomorphisms between M and N. Moreover, we set $\mathrm{end}_A(M) := \mathrm{hom}_A(M, M)$. Note that $\mathrm{end}_A(M)$ is an ungraded k-algebra, where the multiplication is given by composition of functions.

2. Let $d \in \mathbb{Z}$. An A-linear map $f \in \mathrm{Hom}_A(M, N)$ is called a *homogeneous* of degree d, if for all $i \in \mathbb{Z}$, we have $f(M_i) \subset N_{i+d}$. We denote by $\mathrm{HOM}_A(M, N)_d \subset \mathrm{Hom}_A(M, N)$ the k-vector space of homogeneous A-linear maps of degree d. In particular, $\mathrm{hom}_A(M, N) = \mathrm{HOM}_A(M, N)_0$.

3. We set
$$\mathrm{HOM}_A(M, N) := \bigoplus_{d \in \mathbb{Z}} \mathrm{HOM}_A(M, N)_d \subset \mathrm{Hom}_A(M, N).$$
Then $\mathrm{HOM}_A(M, N)$ is a graded k-vector space. Moreover, we set
$$\mathrm{END}_A(M) := \mathrm{HOM}_A(M, M).$$
Then $\mathrm{END}_A(M)$ is a graded k-algebra with multiplication given by composition of functions. We call $\mathrm{END}_A(M)$ the *graded endomorphism algebra* of M.

In the following let A be a fixed graded k-algebra and M, N be graded A-modules. If M is a finitely generated graded A-module, then one can directly check that we have
$$\mathrm{HOM}_A(M, N) = \mathrm{Hom}_A(M, N).$$

We will use the following degree shifts of graded A-modules. For $d \in \mathbb{Z}$, let $M\langle d\rangle$ denote the graded A-module obtained from M by defining the homogeneous components

$$M\langle d\rangle_i := M_{i-d}, \quad \text{for all } i \in \mathbb{Z}.$$

It is a straightforward exercise to check that we have natural isomorphisms of k-vector spaces

$$\mathrm{HOM}_A(M,N)_d \cong \hom_A(M\langle d\rangle, N) \cong \hom_A(M, N\langle -d\rangle).$$

We call two graded A-modules M, N *shift-isomorphic* if there exists $d \in \mathbb{Z}$ such that $M \cong M\langle d\rangle$.
Next, we fix the notation of several categories of graded A-modules.

Notation 1.1.4. Let A be a graded k-algebra.

1. Let A-Mod denote the graded Abelian category of graded A-modules with morphism spaces $\hom_A(M,N)$.
2. Let A-mod $\subset$ A-Mod denote the full graded subcategory of finitely generated graded A-modules.
3. Let A-fmod $\subset$ A-Mod denote the full graded Abelian subcategory of graded A-modules that are of finite dimension over k.
4. Let A-pmod $\subset$ A-Mod denote the full graded additive subcategory of finitely generated graded projective A-modules.
5. Let $A\text{-Mod}^+ \subset A\text{-Mod}$ denote the full graded Abelian subcategory whose objects are the graded A-modules M that satisfy the following conditions:
   5.a. All homogeneous components of M are of finite dimension over k.
   5.b. There exists $d \in \mathbb{Z}$ (depending on M) such that $M_i = 0$, for $i < d$.
6. Let $A\text{-Mod}^- \subset A\text{-Mod}$ denote the full graded Abelian subcategory whose objects are the graded A-modules M that satisfy the following conditions:
   6.a. All homogeneous components of M are of finite dimension over k.
   6.b. There exists $d \in \mathbb{Z}$ (depending on M) such that $M_i = 0$, for $i > d$.

The categories $A\text{-Mod}^+$ and $A\text{-Mod}^-$ are of particular importance since we have a notion of graded dimensions on these categories. For this, we denote by $\mathbb{Z}(\!(q)\!)$ the ring of formal Laurent series with integer coefficients.

Definition 1.1.5. *Let $M \in A\text{-Mod}^+$. Then the graded dimension of M is defined as*

$$\mathrm{grdim}(M) := \sum_{i \gg -\infty} \dim_k(M_i) q^i \in \mathbb{Z}(\!(q)\!).$$

Likewise, for $N \in A\text{-Mod}^-$, we define the dual graded dimension by

$$\mathrm{dgrdim}(N) := \sum_{i \ll \infty} \dim_k(N_i) q^i \in \mathbb{Z}(\!(q^{-1})\!).$$

We now consider some basic examples of graded and dual graded dimensions. For this, we denote for a graded k-algebra A the regular A-module by ${}_A A$.

Example 1.1.6. (1) Let $A = k$ be the ground field. Then we have

$$\mathrm{grdim}({}_A A) = \mathrm{dgrdim}({}_A A) = 1.$$

(2) Let $A = k[x_1, \ldots, x_n]$, where each x_i is homogeneous of degree 1. Then the graded dimension of ${}_A A$ is

$$\mathrm{grdim}({}_A A) = \frac{1}{(1-q)^n}.$$

In particular, for $n = 1$ we have

$$\mathrm{grdim}({}_A A) = 1 + q + q^2 + q^3 + \ldots.$$

(3) Now, let $A = k[x_1, \ldots, x_n]$, where each x_i is homogeneous of degree -1. Then the dual graded dimension of ${}_A A$ is

$$\mathrm{dgrdim}({}_A A) = \frac{1}{(1-q^{-1})^n}.$$

In particular, for $n = 1$ we have

$$\mathrm{grdim}({}_A A) = 1 + q^{-1} + q^{-2} + q^{-3} + \ldots.$$

We end this section with recalling the notion of graded EXT-functors and Frobenius reciprocity. At first, we recall that it is a well-known fact that the category A-Mod admits enough projectives and enough injectives, see for instance [NvO04, Appendix]. In the following let vect denote the category of graded k-vector spaces. We have that the graded functor

$$\mathrm{HOM}_A(.,.) : A\text{-Mod} \times A\text{-Mod} \to \mathrm{vect}$$

is left-exact in both variables. For each $M \in A$-Mod, we denote by $\mathrm{EXT}^i_A(., M)$ the i-th right derived functor of $\mathrm{HOM}_A(., M)$, for $i \in \mathbb{N}_0$. Moreover, we denote by $\mathrm{R}^i(\mathrm{HOM}_A(M, .))$ the i-th right derived functor of $\mathrm{HOM}_A(M, .)$, for $i \in \mathbb{N}_0$. It is a well-known fact from homological algebra that for all $M, N \in A$-Mod, we have isomorphisms of graded k-vector spaces

$$\mathrm{EXT}^i_A(M, N) \cong \mathrm{R}^i(\mathrm{HOM}_A(M, .))(N).$$

A proof of this fact is given or instance in [Wei95, Theorem 2.7.6]. The reference only treats the ungraded case, but the arguments directly generalize to the graded setting.

Finally, we recall the notion of Frobenius reciprocity. For this, let A, B be graded k-algebras and $A \subset B$ a non-necessarily unital inclusion of graded k-algebras, i.e. we do not demand that the unity element 1_A of A is the unity element of B. The corresponding restriction functor is defined as

$$\mathrm{Res}^B_A : B\text{-Mod} \to A\text{-Mod}, \quad M \mapsto 1_A \cdot M.$$

We proceed with defining the corresponding induction functor. For this, note that if M is a graded right A-module and N is a graded left A-module, then $M \otimes_A N$ is a graded k-vector space, where the graded component $(M \otimes_A N)_i$ is spanned by the pure tensors $m \otimes n$, where $m \in M, n \in N$ are homogeneous with $|m| + |n| = i$. With this observation, we define the induction functor corresponding to $A \subset B$ by

$$\mathrm{Ind}^B_A : A\text{-Mod} \to B\text{-Mod}, \quad M \mapsto B \otimes_A M.$$

Note that $B \otimes_A M$ is a graded B-module with scalar multiplication

$$b(b' \otimes m) = bb' \otimes m, \quad \text{for all } b, b' \in B,\, m \in M.$$

Let $M \in A\text{-Mod}$ and $N \in B\text{-Mod}$. By the universal property of tensor products, we obtain a natural isomorphism of graded k-vector spaces

$$\mathrm{HOM}_B(\mathrm{Ind}_A^B(M), N) \cong \mathrm{HOM}_A(M, \mathrm{Res}_A^B(N)).$$

This isomorphism is called *Frobenius reciprocity.* Using the Grothendieck spectral sequence (see e.g. [Wei95, Theorem 5.8.3]) and the fact that Ind_A^B preserves graded projective modules, we obtain an isomorphism of the right-derived functors

$$\mathrm{EXT}^i_B(\mathrm{Ind}_A^B(M), N) \cong \mathrm{EXT}^i_A(M, \mathrm{Res}_A^B(N)), \quad \text{for all } i \in \mathbb{N}_0.$$

This isomorphism is called *generalized Frobenius reciprocity.*

We herewith end this section about general notions of graded k-algebras. In the upcoming section, we will come to crucial players of this thesis: the Laurentian k-algebras.

1.2 Laurentian algebras

In this section, we recall the definition of Laurentian k-algebras and discuss important well-known representation theoretic properties of these algebras. Our notation in this section is modeled on [Kle15a].

Definition 1.2.1. *Let A be a graded k-algebra. We say that A is Laurentian if A satisfies the following conditions:*

L1 For all $i \in \mathbb{Z}$, the homogeneous component A_i has finite dimension over k.

L2 There exists $d \in \mathbb{Z}$ such that $A_i = 0$, for all $i < d$.

Using the notation from the previous section, we have that a graded k-algebra A is Laurentian if and only if the regular A-module ${}_AA$ is contained in $A\text{-Mod}^+$.

Let A be a Laurentian k-algebra. By definition, we have $A\text{-mod} \subset A\text{-Mod}^+$. In particular, for every finitely generated graded A-module, the graded dimension is well-defined. As a direct consequence, we conclude that if M is a finitely generated graded A-module and $d \in \mathbb{Z}$, then $M \cong M\langle d\rangle$ if and only if $d = 0$.

Moreover, we define the graded dimension of A by

$$\mathrm{grdim}(A) := \sum_{i \gg -\infty} \dim_k(A_i) q^i \in \mathbb{Z}(\!(q)\!).$$

This observation motivates the name Laurentian, because the graded dimension of a Laurentian k-algebra is a Laurent series with integer coefficients.

We proceed with considering two important examples of Laurentian k-algebras.

Example 1.2.2. Let $A = k[x_1, \ldots, x_n]$ be a graded polynomial algebra, where all variables x_i are homogeneous of strictly positive degree. Then A is Laurentian.

The second example we consider are graded matrix algebras over Laurentian k-algebras. They give in particular examples for Laurentian k-algebras with non-vanishing negative homogeneous components.

Definition 1.2.3. *Let A be a graded k-algebra, $n \in \mathbb{N}$ and $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$. Let $\mathrm{M}_n(A)(\underline{d})$ be the k-vector space of $n \times n$ matrices over A with the grading such that for any homogeneous element $c \in A$ the matrix $cE_{i,j}$ is homogeneous of degree $|c| + d_j - d_i$. This means that the i-th graded component of $\mathrm{M}_n(A)(\underline{d})$ is given by*

$$\begin{pmatrix} A_i & A_{i+d_2-d_1} & \cdots & A_{i+d_n-d_1} \\ A_{i+d_1-d_2} & A_i & \cdots & A_{i+d_{n-1}-d_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i+d_1-d_n} & A_{i+d_2-d_{n-1}} & \cdots & A_i \end{pmatrix}.$$

One can verify directly that $\mathrm{M}_n(A)(\underline{d})$ with the usual matrix multiplication is a graded k-algebra. We call $\mathrm{M}_n(A)(\underline{d})$ the graded matrix algebra over A parameterized by $\underline{d}$.

If we assume that the graded k-algebra A is Laurentian, then it follows directly from the definition that also $\mathrm{M}_n(A)(\underline{d})$ is Laurentian. Moreover, the graded dimension of $\mathrm{M}_n(A)(\underline{d})$ is given by

$$\operatorname{grdim}(M_n(A)(\underline{d})) = \Big(\prod_{\substack{1 \le i \le n \\ 1 \le j \le n}} q^{d_i - d_j} \Big) \cdot \operatorname{grdim}(A).$$

In particular, in this way, we obtain examples of Laurentian k-algebras with non-vanishing components of strictly negative degree.

We proceed with recalling two fundamental properties of graded matrix algebras.

Proposition 1.2.4. *Let A be a graded k-algebra and A^{op} be the opposite k-algebra of A. Let M be a graded free module over A^{op} with homogeneous basis $(m_1, \dots, m_n)$ of degrees $\underline{d} := (d_1, \dots, d_n) \in \mathbb{Z}^n$. Then there is an isomorphism of graded k-algebras*

$$\mathrm{M}_n(A)(\underline{d}) \to \mathrm{END}_{A^{\mathrm{op}}}(M), \quad B \mapsto (\varphi_B : m_i \mapsto \sum_{j=1}^{n} B_{i,j} m_j).$$

Proof. This follows immediately from the definition of graded matrix algebras. □

Proposition 1.2.5. *Let A be a graded k-algebra, $n \in \mathbb{N}_0$ and $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$. Let $A(\underline{d})$ be the graded free A-right module with homogeneous basis $a_1, \dots, a_n$ and each a_i is homogeneous of degree d_i, for each $i \in \{1, \dots, n\}$. Via the usual matrix multiplication, we obtain a graded $(M_n(A)(\underline{d}), A)$-bimodule structure on $A(\underline{d})$. Then we have the following equivalence of graded categories*

$$A\text{-Mod} \to (M_n(A)(\underline{d}))\text{-Mod}, \quad M \mapsto A(\underline{d}) \otimes_A M$$

Proof. See e.g. [Haz16, Proposition 2.1.1]. □

Next, we describe some crucial representation theoretic properties of Laurentian k-algebras. Our first observation is that if A is a graded k-algebra that satisfies the condition L2, then for all finitely generated graded A-modules M, N, we have that $\hom_A(M, N)$ is a finite dimensional k-vector space. Thus, we immediately obtain the following proposition.

Proposition 1.2.6. *Let A be a graded k-algebra that satisfies the condition L2. Then the categories* A-mod *and* A-pmod *are both Krull–Schmidt categories, i.e. every object is isomorphic to a direct sum of objects having local endomorphism rings.*

The property that A-pmod is a Krull–Schmidt category can be reformulated as *graded semiperfectness* of A as follows.

Theorem 1.2.7. *Let A be a graded algebra over a field k. Then the following are equivalent:*

(i) *The additive category* A-pmod *is a Krull–Schmidt category.*

(ii) *Every object in* A-mod *admits a projective cover in* A-mod.

(iii) *Let $J^g(A)$ denote the graded Jacobson radical of A. Then the graded k-algebra $A/J^g(A)$ is graded semisimple and each homogeneous idempotent in $A/J^g(A)$ lifts to a homogeneous idempotent in A.*

We call A graded semiperfect, if A satisfies the above equivalent conditions.

Proof. See e.g. [AF12, Theorem 27.6]. This reference only treats the ungraded case, but the proof directly transfers to the graded case. □

Corollary 1.2.8. *Laurentian k-algebras are graded semiperfect.*

The graded semiperfectness property has many peasant consequences that we describe in the following. We begin with characterizing the graded radical of finitely generated graded modules over graded semiperfect algebras. For this, we recall the general definition of the graded radical.

Definition 1.2.9. *Let A be a graded k-algebra and M be a graded A-module.*

(i) *Let $N \subset M$ be a graded A-submodule. We call N graded superfluous if for any graded A-submodule $H \subset M$, we have $H + N = M$ if and only if $H = M$.*

(ii) *The graded radical* $\mathrm{rad}(M)$ *is defined to be the graded A-submodule of M generated by all graded superfluous graded A-submodules of M.*

If we assume that A is graded semiperfect, then the graded radical of finitely generated graded A-modules can be described in the following way.

Proposition 1.2.10. *Let A be a graded semiperfect k-algebra and M be a finitely generated graded A-module. Then we have* $\mathrm{rad}(M) = J^g(A)M$.

Proof. The inclusion $J^g(A)M \subset \mathrm{rad}(M)$ follows from the graded version of the Nakayama Lemma, see e.g. [NvO04, Corollary 2.9.2]. The other inclusion follows from the fact that $A/J^g(A)$ is graded semisimple. □

The Proposition 1.2.10 gives an adequate notion of the head of a finitely generated graded module over a graded semiperfect k-algebra.

Definition 1.2.11. *Let A be a graded semiperfect k-algebra and M be a finitely generated graded A-module. Then the head* $\mathrm{hd}(M)$ *of M is defined as*

$$\mathrm{hd}(M) := M/\operatorname{rad} M.$$

By Proposition 1.2.10, $\mathrm{hd}(M)$ *is the unique maximal graded semisimple quotient of M.*

We continue with a further useful consequence of the equivalent characterizations of the graded semiperfectness. Namely, by using standard arguments, we obtain a 1:1 correspondence between the graded simple A-modules and the finitely generated graded projective indecomposable A-modules which we describe in Theorem 1.2.13. In the theorem and also in the following, we use the following notational convention.

Convention 1.2.12. Given a graded k-algebra A, then we call the finitely generated graded projective indecomposable A-modules just graded projective indecomposable A-modules, so we omit the part 'finitely generated'.

Theorem 1.2.13. *Let A be a graded semiperfect k-algebra, then the following are true:*

(i) *Let P be graded projective indecomposable A-module P, then $\mathrm{hd}(P)$ is graded simple and P is the projective cover of $\mathrm{hd}(P)$.*

(ii) *A admits only finitely many graded simple A-modules up to shift-isomorphism.*

(iii) *Let $\mathcal{I}$ be the set of isomorphism classes of graded simple A-modules and $\mathcal{P}$ be the set of isomorphism classes of graded projective indecomposable A-modules. Then there is a bijection*

$$\mathcal{P} \xrightarrow{\sim} \mathcal{I}, \quad [P] \mapsto [\mathrm{hd}(P)].$$

The inverse map is given by assigning to a class $[L]$ the class $[P_L]$, where P_L denotes the projective cover of L.

After this discussion about the pleasant consequences of the graded semiperfectness property, we continue with describing a further crucial property of Laurentian k-algebras, namely that the graded simple modules over Laurentian k-algebras are always of finite dimension over k.

Theorem 1.2.14. *Let A be a Laurentian k-algebra and L be a graded simple A-module, then L has finite dimension over k.*

Proof. According to Theorem 1.2.7.(iii), we know that $A/J^g(A)$ is a graded semisimple Laurentian k-algebra. By the graded version of the Artin-Wedderburn theorem (see [Bla11, Corollary 9.4.5]), we obtain that $A/J^g(A)$ is isomorphic to a product of graded matrix algebras over graded k-division algebras. Recall at this point that a graded k-division algebra is a graded k-algebra with the property that every homogeneous element is invertible. In particular, a Laurentian k-division algebra has to be of finite dimension over k and to be concentrated in degree zero. From this observation, we can infer that $A/J^g(A)$ is of finite dimension over k which implies the assertion of Theorem 1.2.14 □

We end this section with a basic but important example, where we illustrate the results discussed in this section.

Example 1.2.15. Let $A = k[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ graded polynomial algebra, where all variables x_i are homogeneous of strictly positive degree. The regular A-module $P := {}_A A$ is the unique graded projective indecomposable A-module up to shift-isomorphism. Moreover, $J^g(A) = (x_1, \dots, x_n)$ and the head of P is given by $L := P/(x_1, \dots, x_n)P$. So L is one dimensional and the unique graded simple A-module up to shift-isomorphism.

Altogether, we conclude that the representation theory of $k[x_1, \dots, x_n]$ in the graded setting heavily differs from the ungraded setting. In the ungraded setting we have that the simple $k[x_1, \dots, x_n]$-modules correspond to maximal ideals in $k[x_1, \dots, x_n]$. However, in the graded setting, there exists only one graded simple module over $k[x_1, \dots, x_n]$ up to shift-isomorphism which we obtain by dividing out the ideal $(x_1, \dots, x_n)$.

1.3 Outer tensor products of Laurentian algebras

In this section, we discuss a useful application of the results of the previous section. Namely, we show that if A and B are graded Schurian Laurentian k-algebras, then the graded simple $(A \otimes_k B)$-modules are all given by outer tensor products $S \otimes_k T$, where S is a graded simple A-module and T is a graded simple B-module. From this, we also conclude the analogous statement for the graded projective indecomposable $(A \otimes_k B)$-modules. Again, this result is analogous to the corresponding statement for finite dimensional k-algebras, in particular for group algebras over finite groups.

At first, we recall the definition of graded Schurian k-algebras.

Definition 1.3.1. *Let A be a graded k-algebra. Then A is called graded Schurian, if for any graded simple A-module L, we have that* $\mathrm{end}(L)$ *is one dimensional over k.*

If A and B are graded k-algebras, then $A \otimes_k B$ is a graded k-vector space, where the graded component $(A \otimes_k B)_i$ is spanned by the pure tensors $a \otimes b$, where $a \in A, b \in B$ are homogeneous with $|a| + |b| = i$. Moreover, we have a multiplication on $A \otimes_k B$ given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb', \quad \text{for all } a, a' \in A, b, b' \in B.$$

In this way, $A \otimes_k B$ admits the structure of a graded k-algebra. We call $A \otimes_k B$ the *outer tensor product* of A and B.

Let M resp. N be a graded A- resp. B-module. Then as above, $M \otimes_k N$ is a graded k-vector space. For each $i \in \mathbb{Z}$, he graded component $(M \otimes_k N)_i$ is spanned by the pure tensors $m \otimes n$, where $m \in M, n \in N$ are homogeneous with $|m| + |n| = i$. We can endow $M \otimes_k N$ with a graded $(A \otimes_k B)$-module structure such that

$$(a \otimes b)(m \otimes n) = am \otimes bn,$$

for all $a \in A, b \in B, m \in M$ and $n \in N$. We call $M \otimes_k N$ the *outer tensor product* of M and N.

By the definition of the Laurentian property, we immediately obtain the following lemma.

Lemma 1.3.2. *If A and B are both Laurentian k-algebras, then $A \otimes_k B$ is also Laurentian k-algebra.*

Now, let us formulate the above mentioned classification result.

Theorem 1.3.3. *Let A, B be graded Schurian Laurentian k-algebras. Let $S_1, \ldots, S_r$ resp. $T_1, \ldots, T_s$ be a complete list of pairwise non-shift-isomorphic graded simple A- resp. B-modules. Then $S_i \otimes_k T_j$, for $i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}$ is a complete list of pairwise non-shift-isomorphic graded simple modules over $A \otimes_k B$.*

Proof. At first, note that the case where A and B are of finite dimension over k is well-known, see e.g. [EGH+11, Theorem 3.10.2]. The proof in the reference treats the ungraded case, but all arguments transfer directly to the graded setting.

Now, let us prove the general case. At first, note that $A/J^g(A)$ and $B/J^g(B)$ are graded semisimple. Moreover, as we assumed A and B to be graded Schurian, the graded version of the theorem of Artin-Wedderburn implies that $A/J^g(A) \otimes_k B/J^g(B)$ is graded semisimple and that $S_i \otimes T_j$ for $i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}$ is a complete list of pairwise non-shift-isomorphic graded simple $(A/J^g(A) \otimes_k B/J^g(B))$-modules. We conclude that $S_i \otimes T_j$ for $i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}$ is a list of pairwise non-shift-isomorphic graded simple $(A \otimes_k B)$-modules. So to conclude the theorem, it remains to show that the list is complete. For this, let M be a graded simple $(A \otimes_k B)$-module. According to Theorem 1.2.13, we know that M has finite dimenison over k. Now, the $(A \otimes_k B)$-module structure on M is given by a homomorphism of graded k-algebras

$$\rho : A \otimes_k B \to \mathrm{END}_k(M).$$

We set $A' := \rho(A \otimes_k k), B' := \rho(k \otimes_k B)$ and regard A' resp. B' as a finite dimensional graded quotient algebras of A resp. B. We have that $\rho(A \otimes_k B) = A' \otimes_k B'$ and M is a graded simple $(A' \otimes_k B')$-module. Since A' and B' are graded Schurian k-algebras of finite dimension, we conclude that M is isomorphic as $(A' \otimes_k B')$-module to an outer tensor product $S \otimes_k T$, where S resp. T is a graded simple A'- resp. B'-module. By inflation, it follows that S resp. T is also a graded simple A- resp. B-module and $M \cong S \otimes_k T$ as graded $(A \otimes_k B)$-module. As the $S_1, \ldots, S_r$ resp. $T_1, \ldots, T_s$ form a complete list of pairwise non-shift-isomorphic graded simple A- resp- B-modules, there exist unique $i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}$ such that S is shift-isomorphic to S_i and T is shift-isomorphic to T_j. Hence, M is shift-isomorphic to $S_i \otimes_k T_j$. This finishes the proof. □

The following are direct consequences of Theorem 1.3.3.

Corollary 1.3.4. *Let A, B be graded Schurian Laurentian k-algebras. Then the graded Jacobson radical of $A \otimes_k B$ is given by*

$$J^g(A \otimes_k B) = J^g(A) \otimes_k B + A \otimes_k J^g(B).$$

In particular, we obtain an isomorphism of graded k-algebras

$$A/J^g(A) \otimes_k B/J^g(B) \cong (A \otimes_k B)/J^g(A \otimes_k B).$$

Moreover, $A \otimes_k B$ is graded Schurian.

By applying Theorem 1.2.13 and Corollary 1.3.4, we get the following analogous description of the graded projective indecomposable modules over outer tensor products.

Corollary 1.3.5. *Let A, B be graded Schurian Laurentian k-algebras. Then the following holds:*

(i) Let S resp. T be a graded simple A- resp. B-module. Let P be the projective cover of S and Q be the projective cover of T. Then $P \otimes_k Q$ is the projective cover of $S \otimes_k T$.

(ii) Let $P_1, \ldots, P_r$ resp. $Q_1, \ldots, Q_s$ be a complete list of pairwise non-shift-isomorphic graded projective indecomposable A- resp. B-modules. Then $P_i \otimes_k Q_j$ for $i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}$ is a complete list of pairwise non-shift-isomorphic graded projective indecomposable modules over $A \otimes_k B$.

Proof. (i) At first, observe note that $P \otimes_k Q$ indeed is a finitely generated graded projective $(A \otimes_k B)$-module. Moreover, from Theorem 1.3.3, we directly conclude

$$\mathrm{hd}(P \otimes_k Q) = (P \otimes_k Q)/(J^g(A \otimes_k B)(P \otimes_k Q)) \cong P/J^g(A) \otimes_k Q/J^g(B) \cong S \otimes_k T.$$

Thus, $P \otimes_k Q$ is the projective cover of $S \otimes_k T$.

(ii) This assertion follows directly from the assertion (i) and Theorem 1.3.3, using the 1:1 correspondence between graded simple and graded projective indecomposable $(A \otimes_k B)$-modules from Theorem 1.2.13. □

1.4 Graded composition multiplicities

We now come to a further important aspect of the representation theory of Laurentian k-algebras. Namely, the notion of graded composition multiplicities. We now briefly describe the basic idea of graded composition multiplicities.

If M is a finitely generated graded module over a Laurentian k-algebra A, then M is in general not of finite length. So M does in general not admit a composition series. However, in our setting, we have a satisfying alternative to composition series. Namely, the Laurentain property of A implies that M admits a countable separated filtration $F = (M = F_0 \supset F_1 \supset \dots)$ with graded simple quotients. Now, let L be a graded simple A-module and $i \in \mathbb{Z}$. Let a_i be the multiplicity how often $L\langle i\rangle$ appears as filtration quotient of F. The Laurentian property of A ensures that a_i is a finite natural number and moreover, that there exists $d(M) \in \mathbb{Z}$ such that $a_i = 0$ if $i < d(M)$. In this way, we can assign to M a Laurent series

$$\sum_{i \gg -\infty} a_i q^i \in \mathbb{Z}(\!(q)\!). \tag{1.1}$$

Furthermore, we have that each coefficient a_i is independent of the choice of the filtration. By definition, the graded composition multiplicity of L in M is then the Laurent series (1.1).

Let us now translate this into practice. Our notation in this section is modeled on [Kle15a, Chapter 2]. Throughout this section let A be a fixed Laurentian k-algebra. At first, we recall the general definition of composition multiplicities for objects in A-Mod.

Definition 1.4.1. *Let L be a graded simple A-module.*

(i) Let $N \in A\text{-Mod}$ and $F = (N = F_0 \supset F_1 \supset \cdots \supset F_r = 0)$ be a finite filtration of N by graded A-modules. Then the composition multiplicity $[F : L]$ of L in F is defined as

$$[F : L] := |\{i = 0, \dots, r-1 | F_i/F_{i+1} \cong L\}|.$$

(ii) For any $M \in A\text{-Mod}$ the graded composition multiplicity $[M : L]$ of L in M is defined as

$$[M : L] := \sup\{[F : L] | F \textit{ is a finite filtration of } M\}.$$

Note that possibly $[M : L] = \infty$.

The following properties of composition multiplicities can easily be verified by using the results that were discussed in Section 1.2.

Lemma 1.4.2. *Let A be a Laurentian k-algebra and L be a graded simple A-module. Further let P be the projective cover of L and $M \in A\text{-mod}$. Then the following holds:*

(i) We have the equality

$$[M : L] = \frac{1}{\dim_k(\mathrm{end}_A(L))} \cdot \dim_k(\hom_A(P, M)).$$

In particular, $[M : L] < \infty$.

(ii) There exists $m_0 \in \mathbb{Z}$ such that $[M : L\langle i\rangle] = 0$ for all $i < m_0$.

Using Lemma 1.4.2, we now state the definition of graded composition multiplicities.

Definition 1.4.3. *Let A be a Laurentian k-algebra, L be a graded simple A-module and M be a finitely generated graded A-module. Then the graded composition multiplicity $[M : L]_q$ of L in M is defined as*

$$[M : L]_q := \sum_{i \gg -\infty} [M : L\langle i\rangle] q^i \in \mathbb{Z}((q)).$$

Note that Lemma 1.4.2.(i) implies

$$[M : L]_q = \frac{1}{\dim_k(\mathrm{end}_A(L))} \cdot \mathrm{grdim}(\mathrm{HOM}_A(P, M)), \tag{1.2}$$

where P is the projective cover of L.

Next, we explain the equivalent characterization of graded composition multiplicities that we already mentioned above.

Definition 1.4.4. *Let M be a graded A-module and let $I = \mathbb{N}_0$ or $I = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}_0$. Let $F = (F_i)_{i \in I}$ be a decreasing filtration of M by graded A-submodules, i.e. all F_i are graded A-submodules of M with $F_{i-1} \supset F_i$ for all $i \in I$ with $i \geq 1$ and $F_0 = M$.*

We call F a countable separated graded simple filtration of M if F satisfies the following properties:

1. *The filtration F is separated, i.e. we have*

$$\bigcap_{i \in I} F_i = 0.$$

2. *For each $i \in I$ with $i \geq 1$, we have that F_{i-1}/F_i is a graded simple A-module.*

Proposition 1.4.5. *Let M be a finitely generated graded A-module. Then the following holds:*

(i) M admits a countable separated graded simple filtration.

(ii) Let $F = (M = F_0 \supset F_1 \supset \dots)$ be a countable separated graded simple filtration of M and let L be a graded simple A-module. Then we have

$$[M : L] = |\{i \in \mathbb{N}_0 | F_i/F_{i+1} \cong L\}|.$$

In particular, we have

$$[M : L]_q = \sum_{i \gg -\infty} |\{j \in \mathbb{N}_0 | F_j/F_{j+1} \cong L\langle i\rangle\}| \cdot q^i.$$

Proof. (i) Let $m_1, \dots, m_r$ be homogeneous generators of M as graded A-module. Moreover, for $i \in \mathbb{N}_0$ let $\mathfrak{a}_i \subset A$ be the graded two-sided ideal generated by all homogeneous elements of degree $\geq i$. We further set $F_i := \mathfrak{a}_i M_i$. Then

$$M = F_0 \supset F_1 \supset F_2 \supset \dots$$

is a countable separated graded filtration of M such that all subquotients are of finite dimension over k. Since every graded finite dimensional A-module admits a composition series in A-fmod, we can refine the filtration $M = F_0 \supset F_1 \supset F_2 \supset \dots$ to a countable separated graded simple filtration of M.

(ii) If M is of finite dimension over k, the assertion is clear. So let us assume that M is of infinite dimension over k. In this case, we know that $F_i \neq 0$ for all $i \in \mathbb{N}_0$. Since M is finitely generated and L is of finite dimension over k, we know by the separateness of F that there exists $j \in \mathbb{N}_0$ such that

$$\max\{i \in \mathbb{Z} | L_i \neq 0\} < \min\{i \in \mathbb{Z} | (F_j)_i \neq 0\}.$$

This implies that

$$|\{i \in \mathbb{N}_0 | F_i/F_{i+1} \cong L\}| = |\{i \in \mathbb{N}_0 | F_i/F_{i+1} \cong L, i < j\}|.$$

Moreover, we also obtain $\hom_A(P, F_j) = 0$. Hence, by Lemma 1.4.2, we conclude

$$\begin{aligned}
[M : L] &= \frac{1}{\dim_k(\mathrm{end}(L))} \dim_k(\hom_A(P, M)) \\
&= \frac{1}{\dim_k(\mathrm{end}(L))} \dim_k(\hom_A(P, M/F_j)) \\
&= |\{i \in \mathbb{N}_0 | F_i/F_{i+1} \cong L, i < j\}| \\
&= |\{i \in \mathbb{N}_0 | F_i/F_{i+1} \cong L\}|.
\end{aligned}$$

This completes the proof. □

We end this section by explicitly computing some graded composition multiplicities.

Example 1.4.6. Let $A = k[x_1, \dots, x_n]$ with all $x_1, \dots, x_n$ homogeneous of strictly positive degree. Then $P := {}_AA$ be the regular A-module. So P is the unique graded projective indecomposable A-module and $L := P/(x_1, \dots, x_n)P$ is the graded simple A-module corresponding to P. It follows that

$$[P : L]_q = \mathrm{grdim}(\mathrm{HOM}_A(P, P)) = \mathrm{grdim}(P) = \prod_{i=1}^{n} \frac{1}{1 - q^{|x_i|}}.$$

For the next example, we first introduce the following notion.

Definition 1.4.7. *Let A be a graded k-algebra and $\phi : A \to A$ be a self-inverse graded automorphism of k-algebras. We define the graded semidirect product $A \rtimes \mathbb{Z}/2$ to be the k-algebra of formal sums*

$$A \rtimes \mathbb{Z}/2 = \{ae_1 + be_\tau | a, b \in A\},$$

where the addition is componentwise and the multiplication is given by

$$(ae_1 + be_\tau)(a'e_1 + b'e_\tau) = (aa' + b\phi(b'))e_1 + (ab' + b\phi(a'))e_\tau,$$

for all $a, a', b, b' \in A$. We endow $A \rtimes \mathbb{Z}/2$ with the unique grading such that $|e_1 a| = |a| = |e_\tau a|$ for all homogeneous $a \in A$.

Example 1.4.8. Let $A = k[x]$ with x homogeneous of degree 1. Let $\phi : A \to A$ be the isomorphism given by $x \mapsto -x$ and let $A \rtimes \mathbb{Z}/2$ be the graded semidirect product corresponding to ϕ. Now, set $e^+ := \frac{1}{2}(e_1 + e_\tau), e^- := \frac{1}{2}(e_1 - e_\tau) \in A$. By direct arguments, one can show that $P^+ := Ae^+$ and $P^- := Ae^-$ are non-shift-isomorphic graded projective indecomposable A-modules. Moreover, $L^+ := P^+/xP^+$ is the graded simple A-module corresponding to P^+ and $L^- := P^-/xP^-$ is the graded simple A-module corresponding to P^-. From the definition of e^+ and e^-, it follows that

$$\begin{aligned} \mathrm{HOM}_A(P^+, P^+) &\cong e^+Ae^+ = \mathrm{span}_k(x^i e^+ | i \in \mathbb{N}_0, 2|i), \\ \mathrm{HOM}_A(P^-, P^+) &\cong e^-Ae^+ = \mathrm{span}(x^i e^+ | i \in \mathbb{N}_0, 2 \nmid i). \end{aligned}$$

This implies $[P^+ : L^+]_q = (1-q^2)^{-1}$ and $[P^+ : L^-]_q = q(1-q^2)^{-1}$. Moreover, one can readily check that

$$P^+ \supset P^+/(xe^+)P^+ \supset P^+/(xe^+)^2P^+ \supset \dots$$

is a countable separated graded simple filtration of P^+. Similarly, with the same arguments, one can show that $[P^- : L^-]_q = (1-q^2)^{-1}$ and $[P^- : L^+]_q = q(1-q^2)^{-1}$ and

$$P^- \supset P^-/(xe^-)P^- \supset P^-/(xe^-)^2P^- \supset \dots$$

is a countable separated graded simple filtration of P^-.

1.5 Criterion for finiteness of global dimension

We end this chapter with considering a useful criterion to bound the global dimension of Laurentian k-algebras which are additionally also graded left Noetherian. Namely, the global dimension of a graded left Noetherian Laurentian k-algebra is controlled by the EXT-terms between the graded simple A-modules. This property is again analogous to the corresponding property of finite dimensional k-algebras. For a general reference for the notion of global dimension and its importance in homological algebra see [Wei95, Chapter 4].

At first, we recall the notions of projective dimensions. For this, we assume that $\mathcal{C}$ be an Abelian category with enough projectives. We also assume that $\mathcal{C}$ is not equivalent to the trivial category with only one object and one morphism.

Definition 1.5.1. *We define the following:*

(i) Let $C \in \mathcal{C}$ be a non-zero object and

$$\cdots \to P_2 \to P_1 \to P_0 \to C \to 0$$

be a projective resolution in $\mathcal{C}$ which we denote by $\mathcal{P}$. Then $l(\mathcal{P})$ is defined to be the minimal number n (if it exists) such that $P_n \neq 0$ and $P_i = 0$ for $i > n$. Otherwise, we set $l(\mathcal{P}) := \infty$. We call $l(\mathcal{P})$ the length of $\mathcal{P}$.

(ii) Let $C \in \mathcal{C}$ be a non-zero object. We define $\mathrm{pd}(C)$ to be the minimum number n (if it exists) such that there exists a projective resolution of C in $\mathcal{C}$ of length n. Otherwise, we set $\mathrm{pd}(C) = \infty$. We call $\mathrm{pd}(C)$ the projective dimension of M.

Next, we recall the definition of the (graded) global dimension.

Definition 1.5.2. *The global dimension of $\mathcal{C}$ is defined as*

$$\mathrm{gl}(\mathcal{C}) := \sup\{\mathrm{pd}(C) | C \in \mathcal{C},\ C \text{ non-zero}\}.$$

Let A is be a k-algebra. Then the global dimension of A is defined as

$$\mathrm{gl}(A) := \mathrm{gl}(\mathcal{D}),$$

where $\mathcal{D}$ is the category of A-modules. If A is a graded k-algebra, then the graded global dimension is defined as

$$\text{gr-gl}(A) := \mathrm{gl}(\mathcal{D}'),$$

where $\mathcal{D}'$ is the category of graded A-modules.

It is well-known that the finiteness condition L2 ensures that we do not have to distinguish between gl and gr-gl, i.e. the following holds.

Proposition 1.5.3. *Let A be a graded k-algebra that satisfies the condition L2. Then we have* $\mathrm{gl}(A) = \text{gr-gl}(A)$.

Proof. See e.g. [NVO79, Corollary 7.8]. □

Finally, we come to the mentioned criterion to bound the global dimension of graded left Noetherian Laurentian k-algebras. For this, we first briefly recall the corresponding criterion for finite dimensional k-algebras. So let A be a finite dimensional k-algebra, let $\mathcal{D}$ denote the category of A-modules and $\mathrm{Irr}(\mathcal{D})$ the set of simple objects in $\mathcal{D}$. From [Wei95, Theorem 4.1.2], we know that

$$\mathrm{gl}(A) = \sup\{\mathrm{pd}(M) |\, M \in \mathcal{D},\, \dim_k(M) < \infty\}.$$

Let us now assume that

$$m := \sup\{i \in \mathbb{N}_0 | \exists L, L' \in \mathrm{Irr}(\mathcal{D}) : \mathrm{Ext}^i_A(L, L') \neq 0\} < \infty.$$

Given a finite dimensional A-module M, then we can estimate the projective dimension of M as follows. At first, recall the well-known fact

$$\mathrm{pd}(M) = \max\{i \in \mathbb{N}_0 | \exists L \in \mathrm{Irr}(\mathcal{D}) : \mathrm{Ext}^i(M, L) \neq 0\}.$$

This fact id for instance proved in [Aus55, Proposition 3.7]. Now, using induction on the length of M and the long exact Ext-sequence, we obtain the estimate $\mathrm{pd}(M) \leq m$. Hence, we conclude that

$$\mathrm{gl}(A) = m.$$

The Laurentian and graded left Noetherian properties, allow us to adapt these ideas and translate the result to the setting of graded left Noetherian Laurentian k-algebras. This was for instance done by McNamara in [McN15, Lemma 4.11]. He proved the result only for finite type quiver Hecke algebras which are special examples of graded left Noetherian Laurentian k-algebras. However, the same arguments work also in general for graded left Noetherian Laurentian k-algebras.

Theorem 1.5.4. *Let A be a graded left Noetherian Laurentian k-algebra and let* $\operatorname{Irr}(A)$ *denote the set of graded simple A-modules. Suppose that*

$$m := \sup\{i \in \mathbb{N}_0 | \exists L, L' \in \operatorname{Irr}(A) : \operatorname{EXT}^i_A(L, L') \neq 0\} < \infty.$$

Then we have $\operatorname{gl}(A) = m$.

Next, we consider an example which illustrates Theorem 1.5.4.

Example 1.5.5. Let $A = k[x] \rtimes \mathbb{Z}/2$ be the semidirect product that was already considered in Example 1.4.8. With the notation from there, we have a short exact sequence

$$0 \to P^-\langle 1\rangle \xrightarrow{f_+} P^+ \to L^+ \to 0,$$

where f_+ is given by $\frac{1}{2}(e_1 - e_\tau) \mapsto \frac{x}{2}(e_1 + e_\tau)$ and the map $P^+ \to L^+$ is the projection to the head of P^+. Similarly, we also have a short exact sequence

$$0 \to P^+\langle 1\rangle \xrightarrow{f_-} P^- \to L^- \to 0,$$

where f_- is given by $\frac{1}{2}(e_1 + e_\tau) \mapsto \frac{x}{2}(e_1 - e_\tau)$ and $P^- \to L^-$ is the projection to the head of P^-. Using these short exact sequences, one can directly calculate that

$$\operatorname{EXT}^i_A(L^+, L^+) \cong \operatorname{EXT}^i_A(L^-, L^-) = \begin{cases} k & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

$$\operatorname{EXT}^i_A(L^+, L^-) \cong \operatorname{EXT}^i_A(L^-, L^+) = \begin{cases} k\langle -1\rangle & \text{if } i = 1, \\ 0 & \text{else.} \end{cases}$$

Hence, by Theorem 1.5.4, we conclude that $\operatorname{gl}(A) = 1$.

We herewith end this chapter about the representation theory of Laurentian k-algebras. In the next chapter, we will study Grothendieck groups of Laurentian k-algebras. Again, many results can be transferred from the finite dimensional k-algebras to the Laurentian k-algebras. However, there are also subtleties that we will work out.

2 Grothendieck groups of Laurentian algebras

Convention. Throughout this chapter let k be a fixed ground field and A, B be fixed Laurentian k-algebras. Moreover, we assume that A and B are graded left Noetherian and graded Schurian.

Summary

In this chapter, we study Grothendieck groups of module categories over Laurentian k-algebras. Our main focus lies on describing the Grothendieck groups $\mathrm{G}_0(A\text{-fmod})$, $\mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-mod})$. These Grothendieck groups are naturally modules over $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$, where q acts via degree shifting. In the first sections, we discuss well-known properties of these Grothendieck groups.

A crucial ingredient in our studies of these Grothendieck groups is the graded character map which connects the Grothendieck group $\mathrm{G}_0(A\text{-mod})$ with the Grothendieck group of finite dimensional graded A-modules but scalar extended to the ring of Laurent series $\hat{\mathcal{A}} := \mathbb{Z}(\!(q)\!)$. In this way, we obtain an $\mathcal{A}$-linear homomorphism

$$\mathrm{gch} : \mathrm{G}_0(A\text{-mod}) \to \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}), \quad [M] \mapsto \sum_{i=1}^{r} [M : L_i]_q \otimes [L_i],$$

where $L_1, \ldots, L_r$ is a complete list of pairwise non-shift-isomorphic graded simple A-modules and $[M : L_i]_q$ denotes the graded composition multiplicity of L_i in M. We discussed the notion of graded composition multiplicities in detail in Section 1.4. We like to warn the reader that the graded character map is in general not injective, see Example 2.2.6.

Now, via the graded character map, we obtain a homomorphism of $\hat{\mathcal{A}}$-modules

$$\hat{\phi} : \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod}) \to \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}).$$

If we additionally assume that A has finite global dimension, we then obtain by a standard argument that $\hat{\phi}$ is an isomorphism of $\hat{\mathcal{A}}$-modules.

In Section 2.5, we come to the main result of this chapter. Under the assumption that A is of finite global dimension and admits a self-inverse graded anti-automorphism $\mathfrak{T} : A \to A$, we define $\mathcal{A}$-bilinear Euler forms $\chi_{\mathrm{f}}, \chi_{\mathrm{p}}$ and χ_{m} on $\mathrm{G}_0(A\text{-fmod}), \mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-mod})$. In the definition of these bilinear Euler forms, we use that thanks to $\mathfrak{T}$, taking $\mathrm{HOM}_k(., k)$ gives contravariant equivalences

$$\circledast : A\text{-fmod} \to A\text{-fmod}, \quad \tilde{\circledast} : A\text{-Mod}^{+} \to A\text{-Mod}^{-},$$

These equivalences ensure the $\mathcal{A}$-bilinearity property of $\chi_{\mathrm{f}}, \chi_{\mathrm{p}}$ and χ_{m}. Now, via scalar extension, we obtain $\hat{\mathcal{A}}$-bilinear Euler forms $\hat{\chi}_{\mathrm{f}}$ and $\hat{\chi}_{\mathrm{p}}$ on $\hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ and $\hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$.

Our main result is then Theorem 2.5.14 which states that:

1. The isomorphism $\hat{\phi}$ is compatible with $\hat{\chi}_{\mathrm{p}}$ and $\hat{\chi}_{\mathrm{f}}$.

2. The Euler forms $\hat{\chi}_{\mathrm{p}}$ and $\hat{\chi}_{\mathrm{f}}$ are both non-degenerated.

3. If $P_1, \dots, P_r$ is a complete list of pairwise non-shift-isomorphic projective indecomposable graded A-modules and $L_1, \dots, L_r$ are the corresponding graded simple A-modules, then

$$(\hat{\phi}(1 \otimes [P_1]), \dots, \hat{\phi}(1 \otimes [P_r])) \quad \text{and} \quad (1 \otimes [L_1^{\circledast}], \dots, 1 \otimes [L_r^{\circledast}])$$

are dual $\hat{\mathcal{A}}$-bases of $\hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ with respect to $\hat{\chi}_{\mathrm{f}}$.

Altogether, we have that Theorem 2.5.14 holds for a large class of graded k-algebras. In particular, we can apply this theorem to (alternating) nil Hecke algebras as we will discuss in Section 3.3 and Section 4.4.

2.1 Definitions and fundamental properties

We start by introducing the several Grothendieck groups.

Definition 2.1.1. *We define the following:*

(i) Let $\mathcal{C}$ be a small Abelian category. Then the Grothendieck group $\mathrm{G}_0(\mathcal{C})$ of $\mathcal{C}$ is the Abelian group generated by the set $\{[C] | C \in \mathcal{C}\}$ of isomorphism classes of objects in $\mathcal{C}$, subject to the relation $[C] = [C'] + [C'']$, whenever there exists a short exact sequence

$$0 \to C' \to C \to C'' \to 0$$

in $\mathcal{C}$.

(ii) Let $\mathcal{P}$ be a small additive category. Then the split Grothendieck group $\mathrm{K}_0(\mathcal{P})$ of the category in $\mathcal{P}$ is the Abelian group generated by the set $\{[P] | P \in \mathcal{P}\}$ of isomorphism classes of objects in $\mathcal{P}$, subject to the relation $[P] = [P'] + [P'']$, whenever there exists an isomorphism $P \cong P' \oplus P''$ in $\mathcal{P}$.

Of particular interest in our studies are the Grothendieck groups $\mathrm{G}_0(A\text{-fmod})$, $\mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-mod})$. Note that the graded left Noetherian assumption on A implies that A-mod is an Abelian category.

We begin our studies with an important observation. Let $\mathcal{C}$ be a small Abelian subcategory of A-Mod. Then $\mathrm{G}_0(\mathcal{C})$ admits the structure of a $\mathbb{Z}[q, q^{-1}]$-module, where for every $i \in \mathbb{Z}$ the scalar multiplication with q^i is given by

$$q^i[M] = [M\langle i \rangle], \quad \text{for all } M \in \mathcal{C}.$$

Analogously, if $\mathcal{P}$ is a small additive subcategory of A-Mod, then $\mathrm{K}_0(\mathcal{P})$ is also a $\mathbb{Z}[q, q^{-1}]$-module, where for every $i \in \mathbb{Z}$ the scalar multiplication with q^i is given by

$$q^i[P] = [P\langle i \rangle], \quad \text{for all } P \in \mathcal{P}.$$

In particular, $\mathrm{G}_0(A\text{-fmod})$, $\mathrm{G}_0(A\text{-mod})$ and $\mathrm{K}_0(A\text{-pmod})$ are $\mathcal{A}$-modules.

Notation 2.1.2. Let $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$, $\hat{\mathcal{A}} := \mathbb{Z}((q))$ and $\hat{\mathcal{B}} := \mathbb{Z}((q^{-1}))$. Moreover, let $^- : \mathcal{A} \to \mathcal{A}$ be the unique additive map such that $\overline{q^i} = q^{-i}$ for all $i \in \mathbb{Z}$. Let M, N be $\mathcal{A}$-modules and $f : M \to N$ be an additive map. If f satisfies

$$f(am) = \bar{a}m, \quad \text{for all } a \in \mathcal{A},$$

then f is called $\mathcal{A}$-*anti-linear*. Moreover, let $^- : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ be the additive map defined as

$$\overline{\Big(\sum_{i \gg -\infty} a_i q^{-i} \Big)} := \sum_{i \gg -\infty} a_i q^i.$$

Finally, if we are given an element f contained in $\mathcal{A}, \hat{\mathcal{A}}$ or $\hat{\mathcal{B}}$, then f_i denotes the coefficient of q^i, for $i \in \mathbb{Z}$.

Remark 2.1.3. We like to stress that we never view $\mathcal{A}$, $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ as graded rings, but as ordinary ungraded rings.

In the following, we focus on the Grothendieck groups $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{K}_0(A\text{-pmod})$. Using that A-fmod is a finite length category and A-pmod is a Krull-Schmidt category, we obtain by well-known arguments the following basis theorem.

Theorem 2.1.4. *The following holds:*

(i) We have that $\mathrm{G}_0(A\text{-fmod})$ *is a free* $\mathcal{A}$*-module of finite rank with basis* $([L_1], \ldots, [L_r])$, *where* $L_1, \ldots, L_r$ *is a complete list of pairwise non-shift-isomorphic graded simple* A*-modules.*

(ii) We have that $\mathrm{K}_0(A\text{-pmod})$ *is a free* $\mathcal{A}$*-module of finite rank with basis* $([P_1], \ldots, [P_r])$, *where* $P_1, \ldots, P_r$ *is a complete list of pairwise non-shift-isomorphic graded projective indecomposable* A*-modules.*

By the graded semiperfectness of Laurentian k-algebras, we deduce that taking the projective cover gives an isomorphism of $\mathcal{A}$-modules between $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{K}_0(A\text{-pmod})$.

Proposition 2.1.5. *We have an isomorphism of* $\mathcal{A}$*-modules*

$$\mathrm{pc} : \mathrm{G}_0(A\text{-fmod}) \to \mathrm{K}_0(A\text{-pmod}),$$

given by assigning to a class $[M] \in \mathrm{G}_0(A\text{-fmod})$ *the class* $[P_M]$, *where* $M \in A$-fmod *and* P_M *is the projective cover of* M*. We call* pc *the projective cover map.*

Proof. At first, we observe that pc is well-defined. Indeed, this follows directly from the fact that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence in A-fmod and P' resp. P'' is the projective cover of M' resp. M'', then $P' \oplus P''$ is the projective cover of M. Hence, pc is a well-defined homomorphism of $\mathcal{A}$-modules. From Theorem 2.1.4 and the 1:1 correspondence between the graded simple and the graded projective indecomposable A-modules from Theorem 1.2.13, then directly follows that pc is an isomorphism of $\mathcal{A}$-modules. $\square$

Next we use the results discussed in Section 1.3 to describe the Grothendieck groups of outer tensor products.

Proposition 2.1.6. *The following holds:*

(i) There is an isomorphism of $\mathcal{A}$-modules

$$\Phi_{A,B} : \mathrm{G}_0(A\text{-fmod}) \otimes_{\mathcal{A}} \mathrm{G}_0(B\text{-fmod}) \to \mathrm{G}_0((A \otimes_k B)\text{-fmod}),$$

such that $\Phi_{A,B}([M] \otimes [N]) = [M \otimes_k N]$ *for all* $M, N \in A\text{-fmod}$.

(ii) There is an isomorphism of $\mathcal{A}$-modules

$$\Psi_{A,B} : \mathrm{K}_0(A\text{-pmod}) \otimes_{\mathcal{A}} \mathrm{K}_0(B\text{-pmod}) \to \mathrm{K}_0((A \otimes_k B)\text{-pmod}),$$

such that $\Psi_{A,B}([P] \otimes [Q]) = [P \otimes_k Q]$ *for all* $P \in A\text{-pmod}, Q \in B\text{-pmod}$.

(iii) These isomorphisms are compatible with taking projective covers, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathrm{G}_0(A\text{-fmod}) \otimes_{\mathcal{A}} \mathrm{G}_0(B\text{-fmod}) \arrow[r, "\Phi_{A,B}"] \arrow[d, "\mathrm{pc}_A \otimes \mathrm{pc}_B"'] & \mathrm{G}_0((A \otimes_k B)\text{-fmod}) \arrow[d, "\mathrm{pc}_{A\otimes_k B}"] \\
\mathrm{K}_0(A\text{-pmod}) \otimes_{\mathcal{A}} \mathrm{K}_0(B\text{-pmod}) \arrow[r, "\Psi_{A,B}"] & \mathrm{K}_0((A \otimes_k B)\text{-pmod})
\end{tikzcd}$$

Here, $\mathrm{pc}_A, \mathrm{pc}_B$ *and* $\mathrm{pc}_{A\otimes_k B}$ *are the respective projective cover maps.*

Proof. (i) The well-definedness of $\Phi_{A,B}$ is a consequence of the fact that taking tensor products over k preserves exactness. Now, from Theorem 2.1.4 and Theorem 1.3.3, it follows that $\Phi_{A,B}$ is an isomorphism of $\mathcal{A}$-modules.

(ii) If $P \in A\text{-pmod}, Q \in B\text{-pmod}$, then also $P \otimes_k Q \in (A \otimes_k B)\text{-pmod}$. Since, the tensor products over k preserves also direct sums, we deduce that $\Psi_{A,B}$ is well-defined. Using Theorem 2.1.4 and Corollary 1.3.5.(ii), we then deduce that $\Psi_{A,B}$ is an isomorphism of $\mathcal{A}$-modules.

(iii) This assertion is a direct consequence of Corollary 1.3.5.(i). □

After these considerations, we come in the following section to a crucial player in our study of Grothendieck groups: the graded character map.

2.2 The graded character map

The graded character map provides an important connection between the Grothendieck groups $\mathrm{G}_0(A\text{-mod})$ and $\mathrm{G}_0(A\text{-fmod})$. The target of the graded character map will be the following scalar extension of $\mathrm{G}_0(A\text{-fmod})$.

Definition 2.2.1. *We set*

$$\hat{G}_0(A\text{-fmod}) := \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}), \quad \hat{K}_0(A\text{-pmod}) := \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$$

and call $\hat{G}_0(A\text{-fmod})$ *the extended Grothendieck group of* $\mathrm{G}_0(A\text{-fmod})$ *and* $\hat{K}_0(A\text{-pmod})$ *the extended Grothendieck group of* $\mathrm{K}_0(A\text{-pmod})$.

For the definition of the graded character map, we use the notion of graded composition multiplicities that were discussed in Section 1.4.

Definition 2.2.2. *Let $L_1, \ldots, L_r$ be a complete list of pairwise non-shift-isomorphic graded simple A-modules. Then there exists a unique homomorphism of $\mathcal{A}$-modules*

$$\mathrm{gch} : \mathrm{G}_0(A\text{-mod}) \to \hat{G}_0(A\text{-fmod}),$$

such that

$$\mathrm{gch}([M]) = \sum_{i=1}^{r} [M : L_i]_q \otimes [L_i],$$

*for all $M \in A$-*mod*. We call* gch *the graded character map.*

The following lemma treats the well-definedness of the graded character map.

Lemma 2.2.3. *The graded character map* gch *is well-defined. In addition,* gch *is independent from the choice of complete list of pairwise non-shift-isomorphic graded simple A-modules.*

Proof. We first show that gch is well-defined. For this, let $L_1, \ldots, L_r$ be a complete list of pairwise non-shift-isomorphic graded simple A-modules and let P_i be the projective cover of L_i for $i \in \{1, \ldots, r\}$. At first, recall from the definition of graded composition multiplicities that $[M : L_i]_q$ is indeed contained in $\hat{\mathcal{A}}$. Thus, the element

$$\sum_{i=1}^{r} [M : L_i]_q \otimes [L_i]$$

is contained in $\hat{G}_0(A\text{-fmod})$. Now, let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence in A-mod. Then, by (1.2) and the graded Schurain property of A, we have the following equations

$$\begin{aligned} [M : L_i]_q &= \mathrm{grdim}(\mathrm{HOM}_A(P_i, M)) \\ &= \mathrm{grdim}(\mathrm{HOM}_A(P_i, M')) + \mathrm{grdim}(\mathrm{HOM}_A(P_i, M'')) \\ &= [M' : L_i]_q + [M'' : L_i]_q. \end{aligned}$$

Hence, we conclude that gch is well-defined. Next, we show the independence property. For this, let $L'_1, \ldots, L'_r$ be a second complete list of pairwise non-shift-isomorphic graded simple A-modules. Without loss of generality, we may assume that for each $i \in \{1, \ldots, r\}$ there exists $d_i \in \mathbb{Z}$ such that $L_i\langle d_i \rangle \cong L'_i$. Then we have

$$[M : L_i]_q \otimes [L_i] = (q^{-d_i}[M : L_i]_q) \otimes (q^{d_i}[L_i]) = [M : L'_i]_q \otimes [L'_i].$$

This implies that gch is independent from the choice of complete list of pairwise non-shift-isomorphic graded simple A-modules. □

Let us consider an example that illustrates the graded character map.

Example 2.2.4. Let $A = k[x_1, \ldots, x_n]$ be a graded polynomial algebra with all x_i homogeneous of degree 1. The regular A-module $P := {}_A A$ is the unique graded projective indecomposable A-module up to shift-isomorphism and $L := P/(x_1, \ldots, x_n)P$ is the corresponding graded simple A-module of P. Then the graded character of P is given by

$$\mathrm{gch}([P]) = \Big(\frac{1}{1-q}\Big)^n \otimes [L].$$

In particular, if $n = 1$, we have

$$\mathrm{gch}([P]) = \Big(\sum_{i=0}^{\infty} q^i\Big) \otimes [L].$$

Remarkably, the graded character map needs not to be injective as the following example shows. It relies on the following useful criterion.

Lemma 2.2.5. *Let $M, N \in A$-mod. Then, we have $[M] = [N]$ in $\mathrm{G}_0(A\text{-mod})$ if and only if there exists $E \in A$-mod such that E admits two finite filtrations in A-mod of the same length*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_s = E, \quad 0 = G_0 \subset G_1 \subset \cdots \subset G_s = E,$$

such that there exists $i, j \in \{1, \dots, s\}$ with

$$F_i/F_{i-1} \cong M, \quad G_j/G_{j-1} \cong N$$

and there exists a bijection $\pi : \{1, \dots, r\} \setminus \{i\} \to \{1, \dots, r\} \setminus \{j\}$ such that

$$F_l/F_{l-1} \cong G_{\pi(l)}/G_{\pi(l)-1}$$

for all $l \in \{1, \dots, r\} \setminus \{i\}$.

Proof. The lemma is a straightforward consequence of the definition of the Grothendieck group $\mathrm{G}_0(A\text{-mod})$. □

Example 2.2.6. Suppose that k is not of characteristic 2. Let $A = k[x, y]$ be the graded polynomial algebra with x and y homogeneous of degree 1. Again, let $P := {}_AA$ be the regular A-module, so P is the unique graded projective indecomposable A-module up to shift-isomorphism and $L := P/(x, y)P$ is the corresponding graded simple A-module.

Now, consider the graded A-modules

$$M_1 := A/(x + y)A, \quad M_2 := A/(x - y)A.$$

By definition, we have

$$\mathrm{grdim}(M_1) = \mathrm{grdim}(M_2) = (1 - q)^{-1}.$$

Thus, we conclude that the graded characters of $[M_1]$ and $[M_2]$ are given by

$$\mathrm{gch}([M_1]) = \mathrm{gch}([M_2]) = (1 - q)^{-1} \otimes [L].$$

In the remaining part of this example, we show that $[M_1] \neq [M_2]$ in $\mathrm{G}_0(A\text{-mod})$ which proves that gch is not injective.

For the sake of contradiction, we assume that $[M_1] = [M_2]$ and apply Lemma 2.2.5 to $M = M_1$ and $N = M_2$. Thus, there exists $E \in A$-mod and π as in Lemma 2.2.5. At first, we extend the map π to a bijection on $\{1, \dots, s\}$ by setting $\pi(i) = j$ and set $B := k[x, y, x^{-1}, y^{-1}]$. Then, we apply the functor $B \otimes_A (.)$ to the filtrations

$$0 = F_0 \subset F_1 \subset \cdots \subset F_s = E, \quad 0 = G_0 \subset G_1 \subset \cdots \subset G_s = E.$$

Since B is a graded flat A-module, we deduce that $B \otimes_A E$ is a finitely generated graded B-module with two finite filtrations

$$\begin{aligned} 0 = B \otimes_A F_0 \subset B \otimes_A F_1 \subset \cdots \subset B \otimes_A F_s = B \otimes_A E, \\ 0 = B \otimes_A G_0 \subset B \otimes_A G_1 \subset \cdots \subset B \otimes_A G_s = B \otimes_A E, \end{aligned}$$

such that

$$(B \otimes_A F_i)/(B \otimes_A F_{i-1}) \cong B \otimes_A M_1, \quad (B \otimes_A G_j)/(B \otimes_A G_{j-1}) \cong B \otimes_A M_2.$$

In addition, we have

$$(B \otimes_A F_l)/(B \otimes_A F_{l-1}) \cong (B \otimes_A G_{\pi(l)})/(B \otimes_A G_{\pi(l)-1}),$$

for all $l \in \{1, \ldots, r\} \backslash \{i\}$. Since $B \otimes_A E$ is a finitely generated B-module, all graded components of E are of finite dimension over k.

Now, we take a closer look at the graded B-modules $B \otimes_A M_1$ and $B \otimes_A M_2$. For each $l \in \mathbb{Z}$, the graded component $(B \otimes_A M_1)_l$ is of dimension 1 with generator $v_l := \overline{1} \otimes x^l$. The scalar multiplication of x and y on $B \otimes_A M_1$ is given by

$$xv_l = v_{l+1}, \quad yv_l = -v_{l+1}, \quad \text{for all } l \in \mathbb{Z}.$$

Similarly, each homogeneous component $(B \otimes_A M_2)_l$ is also of dimension 1 with generator $w_l := \overline{1} \otimes x^l$. The scalar multiplication of x and y on $B \otimes_A M_2$ is given by

$$xw_l = w_{l+1}, \quad yw_l = w_{l+1}, \quad \text{for all } l \in \mathbb{Z}.$$

From these considerations, we conclude that scalar multiplication with xy^{-1} is multiplication with -1 on $(B \otimes_A M_1)_0$ respectively multiplication with 1 on $(B \otimes_A M_2)_0$.

Now, consider the k-vector space filtrations

$$\begin{aligned} 0 = (B \otimes_A F_0)_0 \subset (B \otimes_A F_1)_0 \subset \cdots \subset (B \otimes_A F_s)_0 = (B \otimes_A E)_0, \\ 0 = (B \otimes_A G_0)_0 \subset (B \otimes_A G_1)_0 \subset \cdots \subset (B \otimes_A G_s)_0 = (B \otimes_A E)_0. \end{aligned}$$

Let $i_1, \ldots, i_r \in \{1, \ldots, s\}$ be those indices such that

$$(B \otimes_A F_{i_l})_0/(B \otimes_A F_{i_l-1})_0 \neq 0.$$

In particular, there is some $l_0 \in \{1, \ldots, r\}$ such that $i_{l_0} = i$ and hence

$$B \otimes_A F_{i_{l_0}}/B \otimes_A F_{i_{l_0}-1} \cong M_1.$$

Now, let

$$m : (B \otimes_A E)_0 \to (B \otimes_A E)_0,$$

and

$$m_l : \Big((B \otimes_A F_l)/(B \otimes_A F_{l-1})\Big)_0 \to \Big((B \otimes_A F_l)/(B \otimes_A F_{l-1})\Big)_0, \quad \text{for } l \in \{i_1, \ldots, i_r\},$$

as well as

$$m'_l : \Big((B \otimes_A G_l)/(B \otimes_A G_{l-1})\Big)_0 \to \Big((B \otimes_A G_l)/(B \otimes_A G_{l-1})\Big)_0, \quad \text{for } l \in \{\pi(i_1), \ldots, \pi(i_r)\},$$

denote the k-linear automorphisms given by scalar multiplication with xy^{-1}, respectively. These are automorphisms of finite dimensional k-vector spaces, hence they have non-zero determinant. Moreover, they satisfy

$$\prod_{l \in \{i_l, \ldots, i_r\}} \det(m_l) = \det(m) = \prod_{l \in \{\pi(i_1), \ldots, \pi(i_r)\}} \det(m'_l) \tag{2.1}$$

By our assumptions on π, we also have the equality

$$\prod_{l \in \{i_l, \ldots, i_r\} \setminus \{i\}} \det(m_l) = \prod_{l \in \{\pi(i_1), \ldots, \pi(i_r)\} \setminus \{j\}} \det(m'_l). \tag{2.2}$$

The equations (2.1) and (2.2) imply $\det(m_i) = \det(m_j)$ which contradicts our calculations above. So we have finally proved that $[M_1] \neq [M_2]$ in $\mathrm{G}_0(A\text{-mod})$.

We proceed with a crucial property of the graded character map. Namely, if we assume that A is of finite global dimension, then the graded character map induces an isomorphism of $\hat{\mathcal{A}}$-modules

$$\hat{\phi} : \hat{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod}).$$

So beside the projective cover map, we obtain a second isomorphism between $\hat{K}_0(A\text{-pmod})$ and $\hat{G}_0(A\text{-fmod})$. The great advantage of the isomorphism $\hat{\phi}$ is that it satisfies neat compatibility conditions as we will discuss in Section 2.3 and Section 2.5.

In order to put this into practice, we first show the following general proposition.

Proposition 2.2.7. *Given $M \in A\text{-mod}$, then M admits a minimal projective resolution in the category $A\text{-mod}$:*

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0.$$

This means that each P_i is a finitely generated graded projective A-module and the homomorphisms $P_i \to \ker(\partial_i)$ are projective covers.

Proof. Recall from Corollary 1.2.8 that A is graded semiperfect. Thus, M admits a projective cover in A-mod. We choose $\partial_0 : P_0 \to M$ to be the projective cover of M in A-mod. By assumption, A is graded left Noetherian. Hence, $\ker(\partial_0)$ is also a finitely generated graded A-module. According to the graded semiperfectness of A there exists a projective cover of $\ker(\partial_0)$ in A-mod. Let $p_1 : P_1 \to \ker(\partial_0)$ be the projective cover in A-mod and let $\iota_0 : \ker(\partial_0) \to P_0$ be the inclusion. Then, we define $\partial_1 : P_1 \to P_0$ as $\partial_1 := \iota_0 \circ p_1$. By continuing this procedure, we obtain a minimal resolution of M in the category A-mod. ☐

The following properties of minimal projective resolutions are well-known and can be shown directly by using the definition of projective covers.

Lemma 2.2.8. *Let $M \in A\text{-mod}$ and let $\mathcal{P}$ be a minimal projective resolution of M in $A\text{-mod}$. Let $\mathcal{Q}$ be a projective resolution of M in $A\text{-mod}$. Then $\mathcal{P}$ is a direct summand of $\mathcal{Q}$. In particular, if M is non-zero, then we have $\mathrm{pd(M)} = l(\mathcal{P})$.*

We now come to the promised isomorphism theorem. For this, let $\phi : \mathrm{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod})$ be the unique homomorphism of $\mathcal{A}$-modules such that

$$[P] \mapsto \mathrm{gch}([P]), \quad \textit{for all } P \in A\text{-pmod}.$$

Note that $[P]$ denotes on the left hand side a class in $\mathrm{K}_0(A\text{-pmod})$ and on the right hand side a class in $\mathrm{G}_0(A\text{-mod})$. Let

$$\hat{\phi} : \hat{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod}) \tag{2.3}$$

be the $\hat{\mathcal{A}}$-linear extension of ϕ.

Theorem 2.2.9. *Suppose* $\mathrm{gl}(A) < \infty$. *Then* $\hat{\phi}$ *is an isomorphism of* $\hat{\mathcal{A}}$*-modules.*

Proof. According to Proposition 2.1.4, we know that $\hat{K}_0(A\text{-pmod})$ and $\hat{G}_0(A\text{-fmod})$ are free $\hat{\mathcal{A}}$-modules of finite rank and the rank of $\hat{K}_0(A\text{-pmod})$ and $\hat{G}_0(A\text{-fmod})$ coincides. Thus, it suffices to show that $\hat{\phi}$ is surjective. Let $M \in A\text{-fmod}$ be non-zero. Then by Lemma 2.2.8 and Proposition 2.2.7, we know that M admits a projective resolution in A-mod of length $\leq n$. Let

$$\ldots \to Q_2 \to Q_1 \to Q_0 \to M \to 0$$

be such a projective resolution. From this, we deduce the following equality in $\mathrm{G}_0(A\text{-mod})$:

$$\sum_{i=0}^{n} (-1)^i [Q_i] = [M].$$

Thus, we obtain the following equality in $\hat{G}_0(A\text{-fmod})$:

$$\sum_{i=0}^{n} (-1)^i \,\mathrm{gch}([Q_i]) = 1 \otimes [M].$$

This implies

$$\hat{\phi}\Big(\sum_{i=0}^{n} (-1)^i \otimes [Q_i]\Big) = 1 \otimes [M].$$

Thus, we proved that $\hat{\phi}$ is surjective and hence, $\hat{\phi}$ is an isomorphism of $\hat{\mathcal{A}}$-modules. $\square$

We end this section, with showing that the graded character map is compatible with outer tensor products. For this, we use the following well-known fact.

Lemma 2.2.10. *Let* $M \in A\text{-mod}, N \in B\text{-mod}$. *Then for all graded* A*-modules* M' *and all graded* B*-modules* N' *the canonical map*

$$\mathrm{HOM}_A(M, M') \otimes_k \mathrm{HOM}_A(N, N') \to \mathrm{HOM}_{A\otimes_k B}(M \otimes_k N, M' \otimes_k N') \tag{2.4}$$

is an isomorphism of graded k*-vector spaces.*

Proof. See e.g. [Del90, Corollary 5.4]. The reference only treats the ungraded case, but the argument generalizes directly to the graded setting. $\square$

This fact implies that the graded composition multiplicities are compatible with outer tensor products.

Proposition 2.2.11. *Let* $M \in A\text{-mod}, N \in B\text{-mod}$ *and let* S *resp.* T *be graded simple* A*-resp.* B*-modules. Then we have*

$$[M \otimes_k N : S \otimes_k T]_q = [M : S]_q \cdot [N : T]_q.$$

Proof. At first note that by Corollary 1.3.4, we have that $A \otimes_k B$ is graded Schurian. Let P resp. Q be the projective cover of S resp. T. By Corollary 1.3.5, we know that $P \otimes_k Q$ is the projective cover of $S \otimes_k T$. Hence, we obtain

$$\begin{aligned}
[M \otimes_k N : S \otimes_k T]_q &= \operatorname{grdim}(\mathrm{HOM}_{A\otimes_k B}(P \otimes_k Q, M \otimes_k N)) \\
&= \operatorname{grdim}(\mathrm{HOM}_A(P, M) \otimes_k \mathrm{HOM}_B(Q, N)) \\
&= \operatorname{grdim}(\mathrm{HOM}_A(P, M)) \operatorname{grdim}(\mathrm{HOM}_B(Q, N)) \\
&= [M : S]_q \cdot [N : T]_q,
\end{aligned}$$

where in the first and last equality, we used the graded Schurian property and in the second equality, we used Lemma 2.2.10. $\square$

From Proposition 2.2.11, we directly obtain a compatibility statement for the graded character map with outer tensor products. For this, note that by Proposition 2.1.6, we have an isomorphism of $\hat{\mathcal{A}}$-modules

$$\hat{\Phi}_{A,B} : \hat{G}_0(A\text{-fmod}) \otimes_{\hat{\mathcal{A}}} \hat{\mathrm{G}}_0(B\text{-fmod}) \to \hat{\mathrm{G}}_0((A \otimes_k B)\text{-fmod}),$$

such that

$$(f \otimes [M]) \otimes (g \otimes [N]) \mapsto fg \otimes [M \otimes_k N],$$

for all $f, g \in \hat{\mathcal{A}}, M \in A\text{-fmod}, N \in B\text{-fmod}$. Moreover, we also have an isomorphism of $\hat{\mathcal{A}}$-modules

$$\hat{\Psi}_{A,B} : \hat{K}_0(A\text{-pmod}) \otimes_{\hat{\mathcal{A}}} \hat{\mathrm{K}}_0(B\text{-pmod}) \to \hat{\mathrm{K}}_0((A \otimes_k B)\text{-pmod}),$$

such that

$$(f \otimes [P]) \otimes (g \otimes [Q]) \mapsto fg \otimes [P \otimes_k Q],$$

for all $f, g \in \hat{\mathcal{A}}, P \in A\text{-pmod}, Q \in B\text{-pmod}$.

Corollary 2.2.12. *Let $M \in A\text{-mod}, N \in B\text{-mod}$. Then we have*

$$(\hat{\Phi}_{A,B} \circ \operatorname{gch}_{A\otimes_k B})([M \otimes_k N]) = \operatorname{gch}_A([M]) \otimes \operatorname{gch}_B([N]),$$

where $\operatorname{gch}_{A\otimes_k B}$, gch_A and gch_B denote the respective graded character maps.

Proof. Let $S_1, \dots, S_r$ resp. $T_1, \dots, T_s$ be a complete list of pairwise non-shift-isomorphic A- resp. B-modules. Using Proposition 2.2.11, we immediately obtain

$$\begin{aligned}
(\hat{\Phi}_{A,B} \circ \operatorname{gch}_{A\otimes_k B})([M \otimes_k N]) &= \hat{\Phi}_{A,B}\Big(\sum_{\substack{1\le i\le r \\ 1\le j\le s}} [M \otimes_k N : S_i \otimes_k T_j]_q \otimes [S_i \otimes_k T_j] \Big) \\
&= \hat{\Phi}_{A,B}\Big(\sum_{\substack{1\le i\le r \\ 1\le j\le s}} ([M : S_i]_q [N : T_j]_q) \otimes [S_i \otimes_k T_j] \Big) \\
&= \sum_{\substack{1\le i\le r \\ 1\le j\le s}} ([M : S_i]_q \otimes [S_i]) \otimes ([N : T_j]_q \otimes T_j]) \\
&= \operatorname{gch}_A([M]) \otimes \operatorname{gch}_B([N]).
\end{aligned}$$

This finishes the proof. $\square$

Corollary 2.2.13. *We have the following commutative diagram:*

$$\begin{tikzcd}
\hat{\mathrm{K}}_0(A\text{-pmod}) \otimes_{\hat{\mathcal{A}}} \hat{\mathrm{K}}_0(B\text{-pmod}) \arrow[r, "\hat{\Psi}_{A,B}"] \arrow[d, "\hat{\phi}_A \otimes \hat{\phi}_B"'] & \hat{\mathrm{K}}_0((A \otimes_k B)\text{-pmod}) \arrow[d, "\hat{\phi}_{A\otimes_k B}"] \\
\hat{\mathrm{G}}_0(A\text{-fmod}) \otimes_{\hat{\mathcal{A}}} \hat{\mathrm{G}}_0(B\text{-fmod}) \arrow[r, "\hat{\Phi}_{A,B}"] & \hat{\mathrm{G}}_0((A \otimes_k B)\text{-fmod})
\end{tikzcd}$$

Here, $\hat{\phi}_{A\otimes_k B}, \hat{\phi}_A$ and $\hat{\phi}_B$ are the respective homomorphisms of $\hat{\mathcal{A}}$-modules from (2.3)*. If we assume that A and B are of finite global dimension, then $\hat{\phi}_A$ and $\hat{\phi}_B$ are isomorphism of $\hat{\mathcal{A}}$-modules. Hence, we deduce that in this case also $\hat{\phi}_{A\otimes_k B}$ is an isomorphism of $\hat{\mathcal{A}}$-modules.*

2.3 Induction and restriction

In the previous sections, we studied the projective cover map and the graded character map. We now discuss how these homomorphisms behave under induction and restriction. In Example 2.3.3, we will see that pc is in general not compatible with induction and restriction. However, in Proposition 2.3.4, we show that the graded composition multiplicities satisfy compatibility relations with induction and restriction. From this, it follows that the graded character map is compatible with induction and restriction.

At first, we discuss some basic facts about induction and restriction functors. For this, let $A \subset B$ be a non-necessary unital inclusion of graded k-algebras and let

$$\begin{aligned}
\mathrm{Ind}_A^B &: A\text{-Mod} \to B\text{-Mod}, \quad M \mapsto B \otimes_A M, \\
\mathrm{Res}_A^B &: B\text{-Mod} \to A\text{-Mod}, \quad M \mapsto 1_A \cdot M,
\end{aligned}$$

denote the corresponding induction and restriction functors. In our studies, we are mostly interested in the case where these functors give well-defined $\mathcal{A}$-linear homomorphisms between Grothendieck groups. Since Res_A^B is exact and preserves finite dimensionality, we conclude that

$$\mathrm{G}_0(B\text{-fmod}) \to \mathrm{G}_0(A\text{-fmod}), \quad [M] \mapsto [\mathrm{Res}_A^B(M)],$$

is a well-defined $\mathcal{A}$-linear homomorphism. It is also always true that Ind_A^B preserves finitely generated graded projective modules, hence

$$\mathrm{K}_0(A\text{-pmod}) \to \mathrm{K}_0(B\text{-pmod}), \quad [P] \mapsto [\mathrm{Ind}_A^B(P)],$$

is a well-defined $\mathcal{A}$-linear homomorphism. However, in general the following holds:

1. In general, the functor Ind_A^B neither gives an $\mathcal{A}$-homomorphism between $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{G}_0(B\text{-fmod})$ nor between $\mathrm{G}_0(A\text{-mod})$ and $\mathrm{G}_0(B\text{-mod})$. This is due to the fact that in general, Ind_A^B neither preserves finite dimensionality nor is exact.

2. In general, Res_A^B neither gives a well-defined $\mathcal{A}$-homomorphism between $\mathrm{G}_0(B\text{-fmod})$ and $\mathrm{G}_0(A\text{-fmod})$ nor between $\mathrm{G}_0(B\text{-mod})$ and $\mathrm{G}_0(A\text{-mod})$. This is due to the fact that in general, Res_A^B neither preserves finitely generated modules nor preserves graded projective modules.

However, we will mostly consider inclusions $A \subset B$ such that the induction and restriction functors satisfy the following conditions:

G1 For all $P \in B$-pmod, we have $\mathrm{Res}_A^B(P) \in A$-pmod.

G2 For all $M \in A$-fmod, we have $\mathrm{Ind}_A^B(M) \in B$-fmod.

G3 The functor Ind_A^B is exact.

These conditions are for example satisfied if B is a graded free right A-module of finite rank, or, more generally, if B is a finitely generated graded projective right A-module.

Proposition 2.3.1. *Suppose that the inclusion $A \subset B$ satisfies the properties G1, G2 and G3. Then we have homomorphisms of $\mathcal{A}$-modules*

$$\mathrm{R_f} : \mathrm{G_0}(B\text{-fmod}) \to \mathrm{G_0}(A\text{-fmod}), \quad \mathrm{R_m} : \mathrm{G_0}(B\text{-mod}) \to \mathrm{G_0}(A\text{-mod}),$$
$$\mathrm{R_p} : \mathrm{K_0}(B\text{-pmod}) \to \mathrm{K_0}(A\text{-pmod}),$$

each given by assigning to a class $[M]$ the class $[\mathrm{Res}_A^B(M)]$ in the respective Grothendieck group. Likewise, we have also homomorphisms of $\mathcal{A}$-modules

$$\mathrm{I_f} : \mathrm{G_0}(A\text{-fmod}) \to \mathrm{G_0}(B\text{-fmod}), \quad \mathrm{I_m} : \mathrm{G_0}(A\text{-mod}) \to \mathrm{G_0}(B\text{-mod}),$$
$$\mathrm{I_p} : \mathrm{K_0}(A\text{-pmod}) \to \mathrm{K_0}(B\text{-pmod}),$$

each given by assigning to a class $[M]$ the class $[\mathrm{Ind}_A^B(M)]$ in the respective Grothendieck group.

Proof. This follows directly from the definitions and the assumptions G1, G2 and G3. □

Convention 2.3.2. Throughout this chapter, we assume that the conditions G1, G2 and G3 are always satisfied whenever we consider induction and restriction functors.

The following example shows that the projective cover map is in general not compatible with induction and restriction.

Example 2.3.3. Let $B := k[x]$ be the polynomial algebra with x homogeneous of degree 1 and let $A := k[x^2] \subset B$. Let

$$\mathrm{pc}_A : \mathrm{G_0}(A\text{-fmod}) \to \mathrm{K_0}(A\text{-pmod}), \quad \mathrm{pc}_B : \mathrm{G_0}(B\text{-fmod}) \to \mathrm{K_0}(B\text{-pmod})$$

be the respective projective cover maps. Let $P := {}_B B$ resp. $Q := {}_A A$ be the regular A- resp. B-module. Then P resp. Q is the unique graded projective indecomposable A- resp. B-module up to shift-isomorphism. We denote the corresponding graded simple quotients by $S := P/xP$ and $T := Q/x^2Q$. Readily, we have $\mathrm{Res}_A^B(S) \cong T$. However, $\mathrm{Res}_A^B(P) \cong Q \oplus Q\langle 1\rangle$. Hence, it follows that

$$\mathrm{pc}_A(\mathrm{R_f}([S])) = [Q] \neq (1+q)[Q] = \mathrm{R_p}(\mathrm{pc}_B([S])).$$

Similarly, $\mathrm{Ind}_A^B(T) \cong S \oplus S\langle 1\rangle$, whereas $\mathrm{Ind}_A^B(Q) \cong P$. Thus, we can infer that

$$\mathrm{pc}_B(\mathrm{I_f}([T])) = (1+q)[P] \neq [P] = \mathrm{I_p}(\mathrm{pc}_A([T])).$$

Hence, we observe that in this example, the projective cover map is neither compatible with induction nor with restriction.

In contrast to the projective cover map, the graded character map is compatible with induction and restriction. This is thanks to the following transitivity formulas of graded composition multiplicities.

Proposition 2.3.4. *Let $L_1, \dots, L_r$ be a complete list of non-shift-isomorphic graded simple A-modules and $S_1, \dots, S_s$ be a complete list of non-shift-isomorphic graded simple B-modules. Then the following holds:*

(i) Let S be a graded simple B-module and $M \in A\text{-mod}$. Then we have

$$[\operatorname{Ind}_A^B(M) : S]_q = \sum_{i=1}^{r} [M : L_i]_q [\operatorname{Ind}_A^B(L_i) : S]_q.$$

(ii) Let L be a graded simple A-module and $N \in B\text{-mod}$. Then we have

$$[\operatorname{Res}_A^B(N) : L]_q = \sum_{i=1}^{s} [M : S_i]_q [\operatorname{Res}_A^B(S_i) : L]_q.$$

Proof. We only prove (i) since (ii) can be shown in the same way. Let

$$F = (M \supset F_0 \supset F_1 \supset \dots)$$

be a countable separated graded simple filtration of M. Using the Laurentian property and the assumption that Ind_A^B preserves finite dimensionality, we conclude that

$$\operatorname{Ind}_A^B(M) \supset \operatorname{Ind}_A^B(F_0) \supset \operatorname{Ind}_A^B(F_1) \supset \dots$$

is a countable separated graded filtration of $\operatorname{Ind}_A^B(M)$ with finite dimensional subquotients. We denote this filtration by $\operatorname{Ind}_A^B(F)$. As we assumed that Ind_A^B is exact, we also have

$$\operatorname{Ind}_A^B(F_i)/\operatorname{Ind}_A^B(F_{i+1}) \cong \operatorname{Ind}_A^B(F_i/F_{i+1}), \quad \text{for all } i \in \mathbb{N}_0.$$

Given $N \in B\text{-fmod}$, then we define

$$[\operatorname{Ind}_A^B(F) : N]_q := \Big(\sum_{i >> -\infty} |\{i \geq 1 | \operatorname{Ind}_A^B(F_i)/\operatorname{Ind}_A^B(F_{i+1}) \cong N\}| \cdot q^i \Big) \in \mathbb{Z}(\!(q)\!).$$

Using the Laurentian property of B, one can easily check that $[\operatorname{Ind}_A^B(F) : N]_q$ is indeed well-defined. Next, we introduce the equivalence relation $\sim$ on the set $\{L_1, \dots, L_r\}$, where $L_i \sim L_j$ if and only if $\operatorname{Ind}_A^B(L_i) \cong \operatorname{Ind}_A^B(L_j)$. Let $L_{i_1}, \dots, L_{i_t}$ be a system of representatives for $\sim$. Then we conclude the following equalities

$$\begin{aligned}
[\operatorname{Ind}_A^B(M) : S]_q &= \sum_{j=1}^{t} [\operatorname{Ind}_A^B(F) : \operatorname{Ind}_A^B(L_{i_j})]_q [\operatorname{Ind}_A^B(L_{i_j}) : S]_q \\
&= \sum_{j=1}^{t} \Big(\sum_{\substack{1 \leq i \leq r \\ L_i \sim L_{i_j}}} [M : L_i]_q [\operatorname{Ind}_A^B(L_{i_j}) : S]_q \Big) \\
&= \sum_{i=1}^{r} [M : L_i]_q [\operatorname{Ind}_A^B(L_{i_j}) : S]_q.
\end{aligned}$$

This finishes the proof. □

As a direct consequence, we can infer that the graded character map is compatible with induction and restriction. It follows that also the $\hat{\mathcal{A}}$-homomorphism $\hat{\phi}$ from (2.3) is compatible with induction and restricition.

Corollary 2.3.5. *In the setting of Proposition 2.3.4 let*

$$\mathrm{gch}_A : \mathrm{G}_0(A\text{-mod}) \to \hat{G}_0(A\text{-fmod}), \quad \mathrm{gch}_B : \mathrm{G}_0(B\text{-mod}) \to \hat{\mathrm{G}}_0(B\text{-fmod})$$

denote the respective graded character maps and let

$$\hat{\mathrm{I}}_{\mathrm{f}} : \hat{G}_0(A\text{-fmod}) \to \hat{\mathrm{G}}_0(B\text{-fmod}), \quad \hat{\mathrm{R}}_{\mathrm{f}} : \hat{\mathrm{G}}_0(B\text{-fmod}) \to \hat{G}_0(A\text{-fmod})$$

denote the $\hat{\mathcal{A}}$-linear maps obtained via scalar extension from I_{f} *and* R_{f}. *Then we have*

$$\hat{\mathrm{I}}_{\mathrm{f}} \circ \mathrm{gch}_A = \mathrm{gch}_B \circ \mathrm{I}_{\mathrm{m}} \quad \textit{and} \quad \hat{\mathrm{R}}_{\mathrm{f}} \circ \mathrm{gch}_B = \mathrm{gch}_A \circ \mathrm{R}_{\mathrm{m}}\,.$$

Corollary 2.3.6. *In the setting of Corollary 2.3.5, let*

$$\hat{\phi}_A : \hat{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod}), \quad \hat{\phi}_B : \hat{\mathrm{K}}_0(B\text{-pmod}) \to \hat{\mathrm{G}}_0(B\text{-fmod})$$

denote the respective homomorphisms of $\hat{\mathcal{A}}$-modules from (2.3) *and let*

$$\hat{\mathrm{I}}_{\mathrm{p}} : \hat{K}_0(A\text{-pmod}) \to \hat{\mathrm{K}}_0(B\text{-pmod}), \quad \hat{\mathrm{R}}_{\mathrm{p}} : \hat{\mathrm{K}}_0(B\text{-pmod}) \to \hat{K}_0(A\text{-pmod})$$

denote the $\hat{\mathcal{A}}$-linear maps obtained via scalar extension from I_{p} *and* R_{p}. *Then we have*

$$\hat{\mathrm{I}}_{\mathrm{f}} \circ \hat{\phi}_A = \hat{\phi}_B \circ \hat{\mathrm{I}}_{\mathrm{p}} \quad \textit{and} \quad \hat{\mathrm{R}}_{\mathrm{f}} \circ \hat{\phi}_B = \hat{\phi}_A \circ \hat{\mathrm{R}}_{\mathrm{p}}.$$

2.4 HOM-pairings

In this section, we discuss HOM-pairings between $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{K}_0(A\text{-pmod})$. Via these pairings we get a connection between $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{K}_0(A\text{-pmod})$ which is also compatible with induction and restriction thanks to Frobenius reciprocity. Our notation in this section is modeled on the notation in [KL09, Section 2.5].

We begin with defining the semi-linear HOM-pairing between the Grothendieck groups $\mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-fmod})$.

Definition 2.4.1. *There exists a unique $\mathcal{A}$-semi-linear pairing*

$$\langle .,. \rangle : \mathrm{K}_0(A\text{-pmod}) \times \mathrm{G}_0(A\text{-fmod}) \to \mathcal{A},$$

such that for all $P \in A\text{-pmod}, M \in A\text{-fmod}$, we have

$$\langle [P], [M] \rangle = \mathrm{grdim}(\mathrm{HOM}_A(P, M)).$$

We call $\langle .,. \rangle$ the semi-linear HOM*-pairing between the Grothendieck groups* $\mathrm{K}_0(A\text{-pmod})$ *and* $\mathrm{G}_0(A\text{-fmod})$. *Here, $\mathcal{A}$-semi-linear means, that $\langle .,. \rangle$ is $\mathcal{A}$-anti-linear in the first variable and $\mathcal{A}$-linear in the second variable.*

Let P be a graded projective indecomposable A-module and let L be corresponding graded simple A-module and recall that we assumed A to be graded Schurian. Hence, from (1.2) follows

$$\langle P, M\rangle = [M : L]_q, \quad \text{for all } M \in A\text{-fmod}.$$

Thus, we deduce the following duality result.

Proposition 2.4.2. *Let $P_1, \ldots, P_r$ be a complete list of pairwise non-shift-isomorphic graded projective indecomposable A-modules and let $L_1, \ldots, L_r$ be the corresponding graded simple A-modules, then we have*

$$\langle P_i, L_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i, j \in \{1, \ldots, r\}$.

Moreover, by Frobenius reciprocity, it follows that the semi-linear HOM-pairing satisfies the following compatibility relations with induction and restriction. Let $A \subset B$ be a non-necessary unital graded inclusion. Then we have

$$\langle [\mathrm{Ind}_A^B(P)], [M]\rangle = \langle [P], [\mathrm{Res}_A^B(M)]\rangle,$$

for all $P \in A$-pmod, $M \in B$-fmod.

Next, we describe a possibility how we can obtain an $\mathcal{A}$ bilinear pairing from $\langle ., .\rangle$. In general, bilinear pairings have the advantage that we can extend them to ring extensions of $\mathcal{A}$ via scalar extension.

Now, in order to obtain a bilinear pairing from $\langle ., .\rangle$, we use certain dualities on the categories A-fmod and A-pmod. To define these dualities we make the following assumption on A.

Convention 2.4.3. For the rest of this section, we assume that A that admits a self-inverse graded anti-isomorphism $\mathfrak{T} : A \to A$. This means, $\mathfrak{T}$ is an automorphism of graded k-vector spaces such that $\mathfrak{T}(ab) = \mathfrak{T}(b)\mathfrak{T}(a)$ for all $a, b \in A$.

As we will show in the following chapters, we have that (alternating) nil Hecke algebras satisfy this assumption. So the results that we discuss in the following can be applied to them.

The crucial step to obtain a bilinear pairing from $\langle ., .\rangle$ is to observe that we have the following dualities on the categories A-fmod and A-pmod.

Definition 2.4.4. *We define the following:*

(i) Let $\circledast : A\text{-fmod} \to A\text{-fmod}$ be the duality given by

$$M \mapsto \mathrm{HOM}_k(M, k),$$

where the graded A-module structure on $\mathrm{HOM}_k(M, k)$ is defined as

$$(af)(m) = f(\mathfrak{T}(a)m),$$

for all $m \in M, f \in \mathrm{HOM}_k(M, k)$ and $a \in A$.

(ii) Let $\# : A\text{-pmod} \to A\text{-pmod}$ *be the duality given by*

$$P \mapsto \mathrm{HOM}_A(P, A),$$

where the graded A-module structure on $\mathrm{HOM}_A(P, A)$ *is defined as*

$$(af)(p) = f(p)\mathfrak{T}(a),$$

for all $p \in P, f \in \mathrm{HOM}_A(P, A)$ *and* $a \in A$.

Note that if $e \in A$ *is a homogeneous idempotent, then* $(Ae)^{\#} \cong A\mathfrak{T}(e)$.

In the following lemma, we list important properties of these dualities. These properties are formulated for instance in [KL09, Section 2.5] in the context of quiver Hecke algebras and transfer directly to our more general setting.

Lemma 2.4.5. *Let* $P \in A\text{-pmod}$ *and* $M \in A\text{-fmod}$. *Then we have*

$$\langle [P^{\#}], [M] \rangle = \mathrm{grdim}(P^{\mathfrak{T}} \otimes_A M) = \overline{\langle [P], [M^{\circledast}] \rangle},$$

where $P^{\mathfrak{T}}$ *denotes the graded right A-module obtained from P via the map* $\mathfrak{T}$. *Moreover,* $^{-}$ *is the involution on* $\mathcal{A}$ *from Notation 2.1.2.*

Proof. It suffices to prove the equalities in the case $P = Ae$, where $e \in A$ is a homogeneous idempotent. The equality

$$\mathrm{grdim}(\mathrm{HOM}_A(P^{\#}, M)) = \mathrm{grdim}(P^{\mathfrak{T}} \otimes_A M)$$

follows from the fact that both $\mathrm{HOM}_A(P^{\#}, M)$ and $P^{\mathfrak{T}} \otimes_A M$ are isomorphic as graded k-vector space to $\mathfrak{T}(e)M$. Now, let us prove the second equality. For this, we have to show

$$\mathrm{grdim}(P^{\mathfrak{T}} \otimes_A M) = \overline{\mathrm{grdim}(\mathrm{HOM}_A(P, M^{\circledast}))}.$$

Since $P^{\mathfrak{T}} \otimes_A M$ is isomorphic as graded k-vector space to $\mathfrak{T}(e)M$ it suffices to show

$$\dim_k(\mathfrak{T}(e)M)_i = \dim_k(\mathrm{HOM}_A(P, M^{\circledast})_{-i}), \quad \text{for each } i \in \mathbb{Z}.$$

Let us fix $i \in \mathbb{Z}$. At first, note that $\mathrm{HOM}_A(P, M^{\circledast})$ is isomorphic as graded k-vector space to $eM^{\circledast}$. Let $m_{\mathfrak{T}(e)} : M_i \to M_i$ denote k-linear homomorphism given by left multiplication with $\mathfrak{T}(e)$. By the definition of $M^{\circledast}$, we have that $(eM^{\circledast})_{-i}$ and $\mathrm{Hom}_k(m_{\mathfrak{T}(e)}(M_i), k)$ are isomorphic as k-vector spaces. Since $(\mathfrak{T}(e)M)_i = m_{\mathfrak{T}(e)}(M_i)$, we can infer that $(\mathfrak{T}(e)M)_i$ and $(eM^{\circledast})_{-i}$ have the same dimension over k. $\square$

As a direct consequence we obtain the following result.

Corollary 2.4.6. *Let L be a graded simple A-module with projective cover P. Then* $P^{\#}$ *is the projective cover of* $L^{\circledast}$.

After these considerations, we now define the bilinear HOM-pairing.

Definition 2.4.7. *There exists a unique $\mathcal{A}$-bilinear pairing*

$$(.,.) : \mathrm{K}_0(A\text{-pmod}) \times \mathrm{G}_0(A\text{-fmod}) \to \mathcal{A},$$

such that for all $P \in A\text{-pmod}, M \in A\text{-fmod}$, we have

$$([P],[M]) = \langle [P^{\#}],[M] \rangle.$$

We call $(.,.)$ the $\mathcal{A}$-bilinear HOM*-pairing between* $\mathrm{K}_0(A\text{-pmod})$ *and* $\mathrm{G}_0(A\text{-fmod})$.

Let P be a graded projective indecomposable A-module and let L be corresponding graded simple A-module. According to Corollary 2.4.6, we have

$$(P,M) = [M : L^{\circledast}]_q,$$

for all $M \in A\text{-fmod}$. Thus, we obtain a slightly different duality result for the bilinear HOM-pairing.

Proposition 2.4.8. *Let $P_1,\dots,P_r$ be a complete list of pairwise non-shift-isomorphic graded projective indecomposable A-modules and let $L_1,\dots,L_r$ be the corresponding graded simple A-modules. Then we have*

$$(P_i, L_j^{\circledast}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i,j \in \{1,\dots,r\}$. In particular, $(.,.)$ is non-degenerate.

We end this section with compatibility statements of $(.,.)$ with induction, restriction and outer tensor products. For this, we assume that also B admits a self-inverse graded anti-automorphism $\mathfrak{H} : B \to B$.

Proposition 2.4.9. *Let $A \subset B$ be a non-necessary unital graded inclusion such that $\mathfrak{H}_{|A} = \mathfrak{T}$. Then the following holds:*

(i) For all $P \in A\text{-pmod}, M \in B\text{-fmod}$, we have

$$(\mathrm{Ind}_A^B[P],[M]) = ([P], \mathrm{Res}_A^B[M]),.$$

(ii) For all $P \in B\text{-pmod}, M \in A\text{-fmod}$, we have

$$(\mathrm{Res}_A^B[P],[M]) = ([P], \mathrm{Ind}_A^B[M]).$$

Proof. This is a direct application of Lemma 2.4.5. For details, see e.g. [KL09, Proposition 3.3]. The reference only treats quiver Hecke algebras, but the arguments directly generalize to our more general setting. □

We continue with stating a compatibility relation of the bilinear HOM-pairing with respect to outer tensor products. For this, note that

$$\mathfrak{T} \otimes \mathfrak{H} : A \otimes_k B \to A \otimes_k B$$

is a self-inverse graded anti-automorphism of $A \otimes_k B$.

Proposition 2.4.10. *Let* $M \in A\text{-fmod}, N \in B\text{-fmod}, P \in A\text{-pmod}, Q \in B\text{-pmod}$. *Then we have*

$$(P \otimes_k Q, M \otimes_k N) = (P, M)(Q, N).$$

Proof. By Lemma 2.2.10, we have $(P \otimes_k Q)^{\#} \cong P^{\#} \otimes_k Q^{\#}$. Thus, we conclude

$$\begin{aligned}
(P \otimes_k Q, M \otimes_k N) &= \operatorname{grdim}(\operatorname{HOM}_{A \otimes_k B}((P \otimes_k Q)^{\#}, M \otimes_k N)) \\
&= \operatorname{grdim}(\operatorname{HOM}_{A \otimes_k B}(P^{\#} \otimes_k Q^{\#}, M \otimes_k N)) \\
&= \operatorname{grdim}(\operatorname{HOM}_A(P^{\#}, M) \ \otimes_k \ \operatorname{HOM}_B(Q^{\#}, N)) \\
&= \operatorname{grdim}(\operatorname{HOM}_A(P^{\#}, M)) \cdot \operatorname{grdim}(\operatorname{HOM}_B(Q^{\#}, N)) \\
&= (P, M)(Q, N),
\end{aligned}$$

where in the third equation we also used Lemma 2.2.10. □

2.5 Euler Forms

In this section, we define bilinear Euler forms on $\mathrm{G}_0(A\text{-fmod}), \mathrm{G}_0(A\text{-mod})$ and $\mathrm{K}_0(A\text{-pmod})$ and prove the results that were outlined in the summary of this chapter.

Convention 2.5.1. Recall that we assume in this chapter that A is a graded Schurian, graded left Noetherian and Laurentian k-algebra. Throughout this section, we further make the following assumptions on A:

1. A has finite global dimension.
2. A admits a self-inverse graded anti-automorphism $\mathfrak{T} : A \to A$.

We set $n := \operatorname{gl}(A)$ and denote the associated dualities to $\mathfrak{T}$ by $\circledast$ and $\#$.

In the following, we define the $\mathcal{A}$-bilinear Euler forms χ_{f} and χ_{m} on the Grothendieck groups $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{G}_0(A\text{-mod})$. For this, we extend the functor $\circledast$ appropriately. Recall at this point the definition of the categories $A\text{-Mod}^+$ and $A\text{-Mod}^-$ from Section 1.1. Then we have a contravariant equivalence of categories

$$\tilde{\circledast} : A\text{-Mod}^+ \to A\text{-Mod}^-, \quad M \mapsto \operatorname{HOM}_A(M, k),$$

where the graded A-module structure on $\operatorname{HOM}_k(M, k)$ is defined as

$$(af)(m) = f(\mathfrak{T}(a)m),$$

for all $m \in M, f \in \operatorname{HOM}_k(M, k)$ and $a \in A$. By definition, $\tilde{\circledast}$ maps the subcategory A-fmod to itself and coincides with $\circledast$ on A-fmod.

We proceed with introducing some notation. Let vect denote the category of graded k-vector spaces and homomorphisms between graded k-vector spaces and let $\mathrm{vect}^- \subset \mathrm{vect}$ be the full graded subcategory whose objects are the graded k-vector spaces V that satisfy the following conditions:

1. All homogeneous components of V are of finite dimension over k.
2. There exists $d \in \mathbb{Z}$ (depending on V) such that $V_i = 0$, for $i > d$.

Recall from Definition 1.1.5 that for every $V \in \mathrm{vect}^-$ the dual graded dimension is defined as

$$\mathrm{dgrdim}(V) := \sum_{i \ll \infty} \dim_k(V_i) q^i \in \hat{\mathcal{B}}.$$

Before we define the $\mathcal{A}$-bilinear Euler forms on $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{G}_0(A\text{-mod})$, we list some useful facts in the following lemma. They can be proved directly by standard arguments using Proposition 2.2.7 and Lemma 2.2.8.

Lemma 2.5.2. *The following holds:*

(i) Let L be a graded simple A-module with projective cover P. Then $L^{\circledast}$ is graded simple and $P^{\tilde{\circledast}}$ is the injective hull of $L^{\circledast}$.

(ii) Let $M, N \in A\text{-mod}$. Then $\mathrm{EXT}^i_A(M, N^{\tilde{\circledast}}) \in \mathrm{vect}^-$, for all $i \in \mathbb{N}_0$.

(iii) Let $M, N \in A\text{-mod}$. Then we have an isomorphism of graded k-vector spaces

$$\mathrm{EXT}^i_A(M, N^{\tilde{\circledast}}) \cong \mathrm{EXT}^i_A(N, M^{\tilde{\circledast}}),$$

for all $i \in \mathbb{N}_0$.

Equipped with Lemma 2.5.2, we now state the definition of the bilinear Euler forms on $\mathrm{G}_0(A\text{-fmod})$ and $\mathrm{G}_0(A\text{-mod})$.

Definition 2.5.3. *We define the following:*

(i) Let $\chi_{\mathrm{f}} : \mathrm{G}_0(A\text{-fmod}) \times \mathrm{G}_0(A\text{-fmod}) \to \mathcal{A}$ be the unique $\mathcal{A}$-bilinear form such that

$$\chi_{\mathrm{f}}([M], [N]) = \sum_{i=0}^{n} (-1)^i \, \overline{\mathrm{grdim}(\mathrm{EXT}^i_A(M, N^{\circledast}))},$$

for all $M, N \in A\text{-fmod}$. Here, $^{-} : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ denotes the additive isomorphism from Notation 2.1.2. We call χ_{f} the bilinear Euler form on $\mathrm{G}_0(A\text{-fmod})$.

(ii) Let $\hat{\chi}_{\mathrm{f}} : \hat{G}_0(A\text{-fmod}) \times \hat{G}_0(A\text{-fmod}) \to \hat{\mathcal{A}}$ be the unique $\hat{\mathcal{A}}$-bilinear form obtained from χ_{f} by scalar extension. We call $\hat{\chi}_{\mathrm{f}}$ the bilinear Euler form on $\hat{G}_0(A\text{-fmod})$.

(iii) Let $\chi_{\mathrm{m}} : \mathrm{G}_0(A\text{-mod}) \times \mathrm{G}_0(A\text{-mod}) \to \hat{\mathcal{A}}$ be the unique $\mathcal{A}$-bilinear form such that

$$\chi_{\mathrm{m}}([M], [N]) = \sum_{i=0}^{n} (-1)^i \, \overline{\mathrm{dgrdim}(\mathrm{EXT}^i_A(M, N^{\tilde{\circledast}}))},$$

for all $M, N \in A\text{-mod}$. Here, $^{-} : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ denotes the additive isomorphism from Notation 2.1.2. We call χ_{m} the bilinear Euler form on $\mathrm{G}_0(A\text{-mod})$.

From Lemma 2.5.2.(iii), we immediately obtain the following consequence.

Corollary 2.5.4. *The bilinear Euler forms $\chi_{\mathrm{f}}, \hat{\chi}_{\mathrm{f}}$ and χ_{m} are symmetric.*

Next, we prove the first crucial result of this section. Namely, we show that the graded character map is compatible with the bilinear Euler forms χ_{m} and $\hat{\chi}_{\mathrm{f}}$.

Theorem 2.5.5. *Let $M, N \in A$-mod. Then we have*

$$\chi_{\mathrm{m}}([M],[N]) = \hat{\chi}_{\mathrm{f}}(\mathrm{gch}([M]),\mathrm{gch}([N])). \tag{2.5}$$

As preparation for the proof of Theorem 2.5.5, we first prove two lemmata. The first lemma allows us to control the graded dimension of minimal projective resolutions of finitely generated graded A-modules. This type of argument is commonly used in the context of Laurentian and graded left Noetherian k-algebras. For example, McNamara used a similar technique in his proof of the theorem that the global dimension of finite type quiver Hecke algebras is finite, see [McN15, Theorem 4.7]. To formulate the first lemma, we define

$$\mathrm{b}(M) := \min\{i \in \mathbb{Z} : M_i \neq 0\},$$

for any non-zero $M \in A$-mod

Lemma 2.5.6. *There exists $m_0 \in \mathbb{N}_0$, only depending on the algebra A, such that the following holds. Let $M \in A$-Mod^+, P be a graded projective indecomposable A-module and $f : P \to M$ be a non-zero homomorphism of graded A-modules. Then we have*

$$\mathrm{b}(P) \geq \mathrm{b}(M) - m_0.$$

Proof. Let $P_1, \ldots, P_r$ be a complete list of pairwise non-shift-isomorphic graded projective indecomposable A-modules. Let $L_1, \ldots, L_r$ be the corresponding graded simple A-modules. We choose

$$m_0 := \max\{\mathrm{b}(L_i) - \mathrm{b}(P_i) | i = 1, \ldots, r\}.$$

Note that m_0 is contained in $\mathbb{N}_0$ and does not depend on our choice of $P_1, \ldots, P_r$. Since $L_1, \ldots, L_r$ are all of finite dimension, we can assume without loss of generality that for all $i \in \{1, \ldots, r\}$, we have $\mathrm{b}(L_i) = 0$. By this assumption, we obtain that for all $i \in \{1, \ldots, r\}$, the Laurent series $[M : L_i]_q$ vanishes in all degrees strictly less than $\mathrm{b}(M)$. Finally, let $d \in \mathbb{Z}$ and $j \in \{1, \ldots, r\}$ such that $P_j\langle d\rangle \cong P$. By our choice of m_0, we have $\mathrm{b}(P) \geq d - m_0$. As f was supposed to be non-zero, it follows that $[M : L_j]_q$ has a non-zero coefficient of degree d. Hence, we have $d \geq \mathrm{b}(M)$. We conclude that $\mathrm{b}(P) \geq \mathrm{b}(M) - m_0$ which finishes the proof. ☐

The following lemma will be crucial in the proof of Theorem 2.5.5.

Lemma 2.5.7. *Let $M, N \in A$-mod. Then there exists a graded A-submodule $G \subset N$ such that the following three conditions are satisfied:*

(i) *N/G is of finite dimension over k,*

(ii) $\mathrm{EXT}^i_A(M, G^{\tilde{\circledast}})_0 = 0$ *for all $i \in \mathbb{N}_0$,*

(iii) $\hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M], \mathrm{gch}[G])_0 = 0$.

Note that (ii) implies $\chi_{\mathrm{m}}(M, G)_0 = 0$.

Proof. Clearly, we can assume that M, N are non-zero. In order to construct the graded A-submodule $G \subset N$, we first record two general facts:

(a) There exists a natural number $n_1 \in \mathbb{N}_0$, only depending on M, such that for all non-zero $M' \in A$-mod with $\mathrm{b}(M') \geq n_1$, we have $\hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M], \mathrm{gch}[M'])_0 = 0$.

(b) According to Proposition 2.2.7, we have that M admits a minimal projective resolution in A-mod:

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

We denote the resolution by $\mathcal{P}$. By Lemma 2.2.8, we know that the length of $\mathcal{P}$ is bounded by n, i.e. $P_i = 0$ for $i > n$. Thus, by Lemma 2.5.6 there exists $m_1 \in \mathbb{N}_0$, only depending on the algebra A, such that all non-zero P_i satisfy

$$\mathrm{b}(P_i) \geq \mathrm{b}(M) - m_1.$$

According to (a) and (b), it is hence sufficient to construct a graded A-submodule $G \subset N$, such that N/G is of finite dimension over k and we have

$$\mathrm{b}(G) > \max\{n_1, m_1 - \mathrm{b}(M)\}.$$

Indeed, by (a), we immediately get (iii). Now, let $r := l(\mathcal{P})$. Then $r \leq n$ by (b). Moreover, from $\mathrm{b}(G) > m_1 - \mathrm{b}(M)$, we conclude

$$\max\{j \in \mathbb{Z} | (G^{\tilde{\circledast}})_j \neq 0\} = -\mathrm{b}(G) < \mathrm{b}(M) - m_1 \leq \mathrm{b}(P_i),$$

for all $i \in \{0, \ldots, r\}$. This implies

$$\mathrm{HOM}_A(P_i, G^{\tilde{\circledast}})_0 = 0,$$

for all $i \in \{0, \ldots, r\}$ which gives (ii). Now, we define $G \subset N$ to be the graded A-submodule generated by all homogeneous elements of degree strictly greater than

$$\max\{n_1, m_1 - \mathrm{b}(M)\} + \mathrm{b}({}_A A),$$

where ${}_A A$ denotes the regular A-module. According to our discussion above, it follows that G satisfies all the desired properties. □

Finally, we prove Theorem 2.5.5.

Proof of Theorem 2.5.5. At first, note that (2.5) is satisfied for $M, N \in A$-fmod. Moreover, recall that χ_{m} and $\hat{\chi}_{\mathrm{f}}$ are symmetric by Corollary 2.5.4. Now, we complete the proof of Theorem 2.5.5 in the following two steps.

Step 1. We prove (2.5) for $M \in A$-fmod and $N \in A$-mod. At first, note that it suffices to prove that for all $M \in A$-mod and $N \in A$-fmod the degree 0 coefficients of $\chi_{\mathrm{m}}([M], [N])$ and $\hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M], \mathrm{gch}[N])$ coincide. Now, choose $G \subset M$ as in Lemma 2.5.7. Then we have the following equalities

$$\begin{aligned} \chi_{\mathrm{m}}([M], [N])_0 &= \chi_{\mathrm{m}}([M], [N/G])_0 \\ &= \hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M], \mathrm{gch}[N/G])_0 \\ &= \hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M], \mathrm{gch}[N])_0, \end{aligned}$$

where the first equality follows from Lemma 2.5.7.(ii), the second from the fact that that (2.5) holds for finite dimensional modules and Lemma 2.5.7.(i) and the third from Lemma 2.5.7.(iii).

Step 2. Now, we prove (2.5) for $M, N \in A$-mod. Again, we choose $G \subset N$ as in Lemma 2.5.7. Using Step 1 and the properties (i), (ii), and (iii) from Lemma 2.5.7, we obtain by the same arguments as in Step 1, the following equalities

$$\begin{aligned} \chi_{\mathrm{m}}([M],[N])_0 &= \chi_{\mathrm{m}}([M],[N/G])_0 \\ &= \hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M],\mathrm{gch}[N/G])_0 \\ &= \hat{\chi}_{\mathrm{f}}(\mathrm{gch}[M],\mathrm{gch}[N])_0. \end{aligned}$$

This finishes the proof of Step 2 and also the proof of Theorem 2.5.5. □

As a short reality check, we consider an elementary but insightful example illustrating Theorem 2.5.5.

Example 2.5.8. Let $A = k[x]$ be a polynomial algebra with x homogeneous of degree 1. Since A is commutative, we can choose $\mathfrak{T} = \mathrm{id}_A$. Moreover, since A is a graded principal ideal domain, A has global dimension 1. Let $P := {}_AA$ be the regular A-module, so P is the unique graded projective indecomposable A-module up to shift-isomorphism. Let $S := P/xP$ be the graded simple A-module corresponding to P. In the following, we directly calculate $\chi_{\mathrm{m}}([P],[P])$ and $\hat{\chi}_{\mathrm{f}}(\mathrm{gch}([P]),\mathrm{gch}([P]))$. At first, we observe that

$$\begin{aligned} \chi_{\mathrm{m}}([P],[P]) &= \overline{\mathrm{dgrdim}(\mathrm{HOM}_A(P, P^{\tilde{\circledast}}))} \\ &= \overline{\mathrm{dgrdim}(P^{\tilde{\circledast}})} \\ &= \overline{((1-q^{-1})^{-1})} \\ &= (1-q)^{-1}. \end{aligned}$$

Next, we calculate $\hat{\chi}_{\mathrm{f}}(\mathrm{gch}([P]),\mathrm{gch}([P]))$. At first, note that $\mathrm{gch}([P]) = (1-q)^{-1} \otimes [S]$. Moreover, we have

$$\mathrm{EXT}^0_A(S,S) \cong k, \quad \mathrm{EXT}^1_A(S,S) = k\langle -1\rangle.$$

Here, k denotes the graded one-dimensional k-vector space concentrated in degree 0. Thus, we obtain

$$\hat{\chi}_{\mathrm{f}}(\mathrm{gch}([P]),\mathrm{gch}([P])) = (1-q)^{-2}\hat{\chi}_{\mathrm{f}}(1 \otimes [S], 1 \otimes [S]) = (1-q)^{-1}.$$

This gives $\chi_{\mathrm{m}}([P],[P]) = \hat{\chi}_{\mathrm{f}}(\mathrm{gch}([P]),\mathrm{gch}([P]))$.

We proceed with defining the $\mathcal{A}$-bilinear Euler form on $\mathrm{K}_0(A\text{-pmod})$. In our approach, we are particularly motivated by Lemma 2.4.5 which transfers to our setting as follows.

Lemma 2.5.9. *Let $P, Q \in A$-pmod. Then we have the following equalities*

$$\mathrm{grdim}(\mathrm{HOM}_A(P^{\#}, Q)) = P^{\mathfrak{T}} \otimes_A Q = \overline{\mathrm{dgrdim}\,\mathrm{HOM}_A(P, Q^{\tilde{\circledast}})}.$$

Proof. The proof is a straightforward adaption of the proof of Lemma 2.4.5. □

Definition 2.5.10. *Let χ_{p} be the unique $\mathcal{A}$-bilinear pairing*

$$\mathrm{K}_0(A\text{-pmod}) \times \mathrm{K}_0(A\text{-pmod}) \to \hat{\mathcal{A}},$$

such that

$$\chi_{\mathrm{p}}([P],[Q]) = \mathrm{grdim}(\mathrm{HOM}_A(P^{\#}, Q)),$$

for all $P, Q \in A$-pmod. We call χ_{p} the bilinear Euler form on $\mathrm{K}_0(A\text{-pmod})$. *Furthermore, let*

$$\hat{\chi}_{\mathrm{p}} : \hat{K}_0(A\text{-pmod}) \times \hat{K}_0(A\text{-pmod}) \to \hat{\mathcal{A}}$$

be the $\hat{\mathcal{A}}$-bilinear extension of χ_{p}. We call $\hat{\chi}_{\mathrm{p}}$ the bilinear Euler form on $\hat{K}_0(A\text{-pmod})$.

Again, we immediately conclude from Lemma 2.5.2 the following consequence.

Corollary 2.5.11. *We have that χ_{p} and $\hat{\chi}_{\mathrm{p}}$ are symmetric.*

Let P be a graded projective indecomposable A-module and L is the corresponding graded simple A-module. Then we have

$$\chi_{\mathrm{p}}([P], [Q]) = [Q^{\#} : L]_q,$$

for all $Q \in A$-pmod. So, χ_{p} is given by taking graded composition multiplicities.

Remark 2.5.12. At this point, we stress that in the definition of χ_{p} and $\hat{\chi}_{\mathrm{p}}$, we do not need the assumption on A to be of finite global dimension. So χ_{p} and $\hat{\chi}_{\mathrm{p}}$ are also well-defined if A is not of finite global dimension. Moreover, the equivalent descriptions of Lemma 2.5.9 and the symmetry of χ_{p} and $\hat{\chi}_{\mathrm{p}}$ also remain true in the case where A is not of finite global dimension.

Finally, we come to the main result of this section. For this, we first fix the following notation.

Notation 2.5.13. Let $(.,.) : \mathrm{K}_0(A\text{-pmod}) \times \mathrm{G}_0(A\text{-fmod}) \to \mathcal{A}$ be the $\mathcal{A}$-bilinear HOM-pairing between $\mathrm{K}_0(A\text{-pmod})$ and $\mathrm{G}_0(A\text{-fmod})$ from Definition 2.4.7. Let

$$(.,.) : \hat{K}_0(A\text{-pmod}) \times \hat{G}_0(A\text{-fmod}) \to \hat{\mathcal{A}}$$

bet the $\hat{\mathcal{A}}$-bilinear pairing obtained from the $\mathcal{A}$-bilinear HOM-pairing via scalar extension. We call $(.,.)$ the *extended bilinear* HOM-*pairing.*

Theorem 2.5.14. *Let $\hat{\phi} : \hat{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod})$ be the isomorphism of $\hat{\mathcal{A}}$-modules from Theorem 2.2.9. Then the following holds:*

(i) For all $P, Q \in A$-pmod, we have

$$\hat{\chi}_{\mathrm{p}}(1 \otimes [P], 1 \otimes [Q]) = \hat{\chi}_{\mathrm{f}}(\hat{\phi}(1 \otimes [P]), \hat{\phi}(1 \otimes [Q])).$$

(ii) The bilinear Euler forms $\hat{\chi}_{\mathrm{p}}$ and $\hat{\chi}_{\mathrm{f}}$ are both non-degenerated.

(iii) The $\hat{\mathcal{A}}$-bilinear pairings $\hat{\chi}_{\mathrm{f}}(\hat{\phi}(.), .)$ and $(.,.)$ coincide.

(iv) If $P_1, \ldots, P_r$ is a complete list of pairwise non-shift-isomorphic projective indecomposable graded A-modules and $L_1, \ldots, L_r$ are the corresponding graded simple A-modules, then

$$(\hat{\phi}(1 \otimes [P_1]), \ldots, \hat{\phi}(1 \otimes [P_r])) \quad \textit{and} \quad (1 \otimes [L_1^{\circledast}], \ldots, 1 \otimes [L_r^{\circledast}])$$

are dual $\hat{\mathcal{A}}$-bases of $\hat{G}_0(A\text{-fmod})$ with respect to $\hat{\chi}_{\mathrm{f}}$.

Proof. We begin with proving (i). From Theorem 2.5.5 and Lemma 2.5.9 follows

$$\begin{aligned}\hat{\chi}_{\mathrm{p}}(1\otimes[P],1\otimes[Q]) &= \overline{\operatorname{dgrdim}(\operatorname{HOM}_A(P,Q^{\tilde{\circledast}}))}\\ &= \chi_{\mathrm{m}}([P],[Q])\\ &= \hat{\chi}_{\mathrm{f}}(\operatorname{gch}([P]),\operatorname{gch}([Q]))\\ &= \hat{\chi}_{\mathrm{f}}(\hat{\phi}(1\otimes[P]),\hat{\phi}(1\otimes[Q])),\end{aligned}$$

for all $P,Q\in A$-pmod. Hence, we proved (i). Next, we show the assertion (iv). For this, recall from Theorem 2.1.4 that $(1\otimes[P_1],\dots,1\otimes[P_r])$ is an $\hat{\mathcal{A}}$-basis of $\hat{K}_0(A\text{-pmod})$ and likewise, $(1\otimes[L_1^{\circledast}],\dots,1\otimes[L_r^{\circledast}])$ is an $\hat{\mathcal{A}}$-basis of $\hat{G}_0(A\text{-fmod})$. Now, from Theorem 2.5.5 we conclude

$$\begin{aligned}\hat{\chi}_{\mathrm{f}}(\hat{\phi}(1\otimes[P_i]),1\otimes[L_j^{\circledast}]) &= \chi_{\mathrm{m}}([P_i],[L_j^{\circledast}])\\ &= \overline{\operatorname{dgrdim}(\operatorname{HOM}_A(P_i,L_j))}\\ &= \begin{cases}1 & \text{if } i=j,\\ 0 & \text{else,}\end{cases}\end{aligned}$$

for all $i,j\in\{1,\dots,r\}$. Hence, it follows that

$$(\hat{\phi}(1\otimes[P_1]),\dots,\hat{\phi}(1\otimes[P_r]))\quad\text{and}\quad(1\otimes[L_1^{\circledast}],\dots,1\otimes[L_r^{\circledast}])$$

are dual $\hat{\mathcal{A}}$-bases of $\hat{G}_0(A\text{-fmod})$ with respect to $\hat{\chi}_{\mathrm{f}}$ which proves (iv). We also immediately conclude that $\hat{\chi}_{\mathrm{f}}$ is non-degenerate. From (i), it directly follows that also $\hat{\chi}_{\mathrm{p}}$ is non-degenerated which gives (ii). Finally, note that by Proposition 2.4.8, the extended HOM-pairing is uniquely determined by

$$(1\otimes[P_i],1\otimes[L_j^{\circledast}])=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{else,}\end{cases}$$

for all $i,j\in\{1,\dots,r\}$. Thus, from (iv), we directly deduce that $\hat{\chi}_{\mathrm{f}}(\hat{\phi}(.),.)=(.,.)$. Hence, we proved (iii). ☐

As a direct consequence of Theorem 2.5.14, we conclude the following degeneracy result for χ_{f}.

Corollary 2.5.15. *Suppose that A is not of finite dimension, then χ_{f} is a degenerate $\mathcal{A}$-bilinear form on* $\mathrm{G}_0(A\text{-fmod})$.

Remark 2.5.16. Theorem 2.5.14 and Corollary 2.5.15 demonstrate that if the algebra A is of infinite dimension over k, then we prefer to work with the bilinear Euler forms on $\hat{G}_0(A\text{-fmod})$ and $\hat{K}_0(A\text{-pmod})$ because here, the bilinear Euler forms are non-degenerate and we have the duality between the graded projective indecomposable A-modules and graded simple A-modules as described in Theorem 2.5.14.(iii).

We proceed with considering a compatibility relation of the bilinear Euler forms with induction and restriction. For this, let B be a second graded k-algebra that satisfies the conditions formulated in Convention 2.5.1. We denote the self-inverse graded anti-automorphism of B by $\mathfrak{H}$. In addition, we denote the corresponding Euler forms on the graded B-module categories also by $\chi_{\mathrm{f}},\chi_{\mathrm{m}}$ and χ_{p}. From the context, it will be always clear, where the forms are defined.

Proposition 2.5.17. *Suppose $A \subset B$ such that $\mathfrak{H}_{|A} = \mathfrak{T}$. Let*

$$(\chi, \mathcal{C}, \mathcal{D}) \in \{(\chi_{\mathrm{f}}, A\text{-fmod}, B\text{-fmod}), (\chi_{\mathrm{m}}, A\text{-mod}, B\text{-mod}), (\chi_{\mathrm{p}}, A\text{-pmod}, B\text{-pmod})\}.$$

Then for all $M \in \mathcal{C}, N \in \mathcal{D}$, we have

$$\chi([\mathrm{Ind}_A^B(M)], [N]) = \chi([M], [\mathrm{Res}_A^B(N)]).$$

Proof. The assertion directly follows from generalized Frobenius reciprocity and the fact that the dualities $\circledast$ and $\tilde{\circledast}$ commute with the functor Res_A^B. $\square$

We end this section with considering a further interesting aspect. Namely, we discuss under which conditions the statement of Theorem 2.5.14 also holds for the Grothendieck groups

$$\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod}), \quad \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}).$$

An important reason why one might prefer the scalar extension to $\mathbb{Q}(q)$ to the scalar extension to $\hat{\mathcal{A}}$ is the following. The duality $\circledast$ gives an $\mathcal{A}$-anti-linear involution

$$\mathrm{G}_0(A\text{-fmod}) \to \mathrm{G}_0(A\text{-fmod}), \quad [M] \mapsto [M^{\circledast}].$$

Likewise, we also obtain an $\mathcal{A}$-anti-linear involution

$$\mathrm{K}_0(A\text{-pmod}) \to \mathrm{K}_0(A\text{-pmod}), \quad [P] \mapsto [P^{\#}].$$

These involutions are further useful structures on these Grothendieck groups. Now, it is possible to extend these involutions to $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ and $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$. However, these involutions can not be extended to the Grothendieck groups $\hat{G}_0(A\text{-fmod})$ and $\hat{K}_0(A\text{-pmod})$.

In order to translate Theorem 2.5.14 to the rational setting, we assume in the following that the finitely generated graded projective A-modules have only rational graded composition multiplicities. This means that

$$[P : L]_q \in \hat{\mathcal{A}} \cap \mathbb{Q}(q),$$

for every finitely generated graded projective A-module P and every graded simple A-module L.

Definition 2.5.18. *We define the following:*

(i) Let $\chi_{\mathrm{f},\mathbb{Q}(q)}$ be the $\mathbb{Q}(q)$-bilinear extension of χ_{f} to $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$. We call $\chi_{\mathrm{f},\mathbb{Q}(q)}$ the rational blinear Euler form on $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$.

(ii) Let $\chi_{\mathrm{p},\mathbb{Q}(q)}$ be the $\mathbb{Q}(q)$-bilinear extension of χ_{p} to $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$. This is well-defined since, by assumption, the image of χ_{p} is contained in $\hat{\mathcal{A}} \cap \mathbb{Q}(q)$. We call $\chi_{\mathrm{p},\mathbb{Q}(q)}$ the rational blinear Euler form on $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$.

(iii) Let

$$(.,.) : (\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})) \times (\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})) \to \mathbb{Q}(q)$$

be the $\mathbb{Q}(q)$-bilinear extension of the $\mathcal{A}$-bilinear HOM*-pairing between* $\mathrm{K}_0(A\text{-pmod})$ *and* $\mathrm{G}_0(A\text{-fmod})$*. We call $(.,.)$ the rational* HOM*-pairing between* $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod})$ *and* $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$*.*

In addition, let $\mathcal{A}' := \hat{\mathcal{A}} \cap \mathbb{Q}(q)$. Then by our assumption, we have that the image of

$$\phi : \mathrm{K}_0(A\text{-pmod}) \to \hat{G}_0(A\text{-fmod}), \quad [P] \mapsto \mathrm{gch}([P])$$

is contained in $\mathcal{A}' \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$. Thus, we can extend ϕ to a $\mathbb{Q}(q)$-linear map

$$\phi_{\mathbb{Q}(q)} : \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(A\text{-pmod}) \to \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod}).$$

Now, using exactly the same arguments as in the proof of Theorem 2.5.14, one can directly show that Theorem 2.5.14 literally also holds in the rational setting.

Theorem 2.5.19. *In the above setting, we have that $\phi_{\mathbb{Q}(q)}$ is an isomorphism of $\mathbb{Q}(q)$-vector spaces. In addition, the following holds:*

(i) *For all $P, Q \in A\text{-pmod}$, we have*

$$\chi_{\mathrm{p},\mathbb{Q}(q)}(1 \otimes [P], 1 \otimes [Q]) = \chi_{\mathrm{f},\mathbb{Q}(q)}(\phi_{\mathbb{Q}(q)}(1 \otimes [P]), \phi_{\mathbb{Q}(q)}(1 \otimes [Q])).$$

(ii) *The bilinear Euler forms $\chi_{\mathrm{p},\mathbb{Q}(q)}$ and $\chi_{\mathrm{f},\mathbb{Q}(q)}$ are both non-degenerated.*

(iii) *The $\mathbb{Q}(q)$-bilinear pairings $\chi_{\mathrm{f},\mathbb{Q}(q)}(\phi_{\mathbb{Q}(q)}(.), .)$ and $(., .)$ coincide.*

(iv) *If $P_1, \ldots, P_r$ is a complete list of pairwise non-shift-isomorphic projective indecomposable graded A-modules and $L_1, \ldots, L_r$ are the corresponding graded simple A-modules. Then*

$$(\phi_{\mathbb{Q}(q)}(1 \otimes [P_1]), \ldots, \phi_{\mathbb{Q}(q)}(1 \otimes [P_r])) \quad \textit{and} \quad (1 \otimes [L_1^{\circledast}], \ldots, 1 \otimes [L_r^{\circledast}])$$

are dual $\mathbb{Q}(q)$-bases of $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(A\text{-fmod})$ with respect to $\chi_{\mathrm{f},\mathbb{Q}(q)}$.

Proof. By using the same argument as in Theorem 2.2.9, we deduce that $\phi_{\mathbb{Q}(q)}$ is an isomorphism of $\mathbb{Q}(q)$-vector spaces. Now, the assertions (i)-(iv) can be shown exactly in the same way as the assertions (i)-(iv) of Theorem 2.5.14. □

We herewith end this chapter. In the next two chapters, we will study concrete applications of Theorem 2.5.14. In particular, we explicitly calculate the bilinear Euler forms χ_{p} and χ_{f} for Grothendieck groups of (alternating) nil Hecke algebras in Section 3.3 and in Section 4.4.

3 Nil Hecke algebras

Convention. Throughout this chapter we fix a ground field k.

Summary

Our aim in the next two chapters is to study the representation theory of alternating nil Hecke algebras. For this, we recall in this chapter important representation theoretic properties of nil Hecke algebras. In the following chapter, we will then use these results to describe the representation theory of alternating nil Hecke algebras.

The nil Hecke algebras are a family of graded k-algebras $(\mathrm{NH_n})_{n\in\mathbb{N}_0}$ which were introduced by Kostant and Kumar in [KK86] to study the cohomology rings of flag varieties. In our discussion in this chapter, we focus on the following two aspects.

At first, we recall the faithful operation of the nil Hecke algebra $\mathrm{NH_n}$ on the polynomial algebra $P_n := k[x_1, \ldots, x_n]$ which is defined via Demazure operators. For this, let $\mathrm{Sym}_n := P_n^{S_n}$ denote the symmetric functions in P_n. The Demazure operators are a family of Sym_n-linear operators $\partial_1, \ldots, \partial_{n-1}$ on P_n which were introduced independently by Bernstein, Gelfand and Gelfand in [BGG73] and Demazure in [Dem74]. Like the nil Hecke algebras, they were also introduced in a geometrical context. Namely, in order to study properties of Schubert classes in cohomology rings of flag varieties.

Now, using Demazure operators, we obtain a faithful operation of $\mathrm{NH_n}$ on P_n which is given by an isomorphism of graded k-algebras

$$\Phi : \mathrm{NH_n} \to \mathrm{END}_{\mathrm{Sym}_n}(P_n).$$

This isomorphism implies many pleasant properties of $\mathrm{NH_n}$. In particular, P_n is the unique graded projective indecomposable $\mathrm{NH_n}$-module up to shift-isomorphism, see Theorem 3.2.4. In Chapter 4, we will then use this result to establish a classification of the graded projective indecomposable modules over alternating nil Hecke algebras.

The second aspect is the description of the categorification theorem of Khovanov–Lauda ([KL09, Theorem 1.1]) in the special case of nil Hecke algebras. We now give a brief characterization of this theorem. Let ${}_{\mathcal{A}}\mathbf{f}$ denote Lusztig's integral quantum group corresponding to the one-vertex graph without edges, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. For a general introduction to Lusztig's integral quantum groups, see [Lus93].

We set

$$\mathrm{G}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}), \quad \mathrm{K}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$$

and call $\mathrm{G}_0(\mathrm{NH})$ the *nil Hecke Grothendieck group* and $\mathrm{K}_0(\mathrm{NH})$ the *split nil Hecke Grothendieck group.* By construction, $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ are both $\mathbb{N}_0$-graded free $\mathcal{A}$-modules. Via induction and restriction, we obtain multiplicative and comultiplicative structures on $\mathrm{G}_0(\mathrm{NH})$

and $K_0(\mathrm{NH})$ turning them into $\mathbb{N}_0$-graded twisted bialgebras. Using the HOM-pairing from Section 2.4, we deduce that $G_0(\mathrm{NH})$ is the $\mathbb{N}_0$-graded dual of $K_0(\mathrm{NH})$.

Finally, the categorification theorem of Khovanov–Lauda states that the $\mathbb{N}_0$-graded twisted bialgebra $K_0(\mathrm{NH})$ is isomorphic to ${}_{\mathcal{A}}\mathbf{f}$. By duality, we deduce that $G_0(\mathrm{NH})$ is isomorphic to the $\mathbb{N}_0$-graded dual ${}_{\mathcal{A}}\mathbf{f}^*$ of ${}_{\mathcal{A}}\mathbf{f}$.

In the context of our studies of the representation theory of alternating nil Hecke algebras in Chapter 4, this theorem is an important motivation. Modeled on the definition of $G_0(\mathrm{NH})$ and $K_0(\mathrm{NH})$, we will define alternating nil Hecke Grothendieck groups $G_0(\mathrm{ANH})$ and $K_0(\mathrm{ANH})$ and study their algebraic properties as well as their relations to $G_0(\mathrm{NH})$ and $K_0(\mathrm{NH})$.

3.1 Demazure operators

In this section, we give an overview to well-known properties of Demazure operators. As we will discuss in the following section, there is a close connection between Demazure operators and nil Hecke algebras with which the representation theory of the nil Hecke algebras can be described in an adequate way. Our notation in this section is modeled on [Man01, Section 2.3].

At first, we fix some notation.

Notation 3.1.1. For a given $n \in \mathbb{N}_0$, let $P_n := k[x_1, \ldots, x_n]$ be the polynomial algebra in n variables, where each x_i is homogeneous of degree 2. Let S_n denote the symmetric group. Then S_n acts on P_n by permuting the variables. We set $\mathrm{Sym}_n := P_n^{S_n}$. Moreover, for each $i \in \{1, \ldots, n-1\}$, let $s_i \in S_n$ denote the *simple transposition* $s_i := (i, i+1)$.

Recall at this point that by the fundamental theorem of symmetric polynomials, Sym_n is a graded polynomial algebra over k which is generated by the elementary symmetric functions $e_1, \ldots, e_n \in \mathrm{Sym}_n$, where

$$e_i := \sum_{1 \le j_1 < \ldots < j_i \le n} x_{j_1} \ldots x_{j_i}.$$

So in particular, the graded dimension of Sym_n is given by

$$\mathrm{grdim}(\mathrm{Sym}_n) = \prod_{i=1}^{n} \frac{1}{1 - q^{2i}}.$$

We now come to the definition of Demazure operators.

Definition 3.1.2. *Let $n \in \mathbb{N}$ with $n \ge 2$. Then the Demazure operators $\partial_1, \ldots, \partial_{n-1}$ are the graded k-linear operators on P_n of degree -2 given by*

$$\partial_i : P_n \to P_n, \quad \partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}, \quad \textit{for all } f \in P_n.$$

In the following lemma, we list of important properties of Demazure operators that follow directly from the definition.

Lemma 3.1.3. *Let $n \in \mathbb{N}$ with $n \ge 2$. Then the following holds:*

(i) Let $f \in P_n$ such that $s_i(f) = f$. Then $\partial_i(f) = 0$.

(ii) Each Demazure operator ∂_i is a twisted derivation, i.e. for all $f, g \in P_n$, we have

$$\partial_i(fg) = \partial_i(f)g + (s_i f)\partial_i(g).$$

If $s_i(f) = f$, then $\partial_i(fg) = f\partial_i(g)$. In particular, ∂_i is Sym_n-linear.

(iii) The Demazure operators satisfy the following relations

$$\partial_i^2 = 0, \tag{3.1}$$

$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}, \tag{3.2}$$

$$\partial_i\partial_j = \partial_j\partial_i \quad \text{if } |i-j| > 1, \tag{3.3}$$

for all admissible i, j.

Through the relations (3.1), (3.2) and (3.3), we get a close connection between Demazure operators and elements of the symmetric group S_n. In order to characterize this connection, we first recall important well-known facts about symmetric groups. For this, we follow [BB06, Chapter 2 and 3].

The symmetric group S_n is generated by the simple transpositions $s_1, \ldots, s_{n-1} \in S_n$. We denote the set of simple transpositions by $\mathcal{S} \subset S_n$. The defining relations of the symmetric group with respect to the generators $s_1, \ldots, s_{n-1}$ are

$$s_i^2 = e \tag{3.4}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{3.5}$$

$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \tag{3.6}$$

for all admissible i, j. Here, $e \in S_n$ denotes the neutral element. The relation (3.4) is called the *quadratic relation.* The relations (3.5) and (3.6) are called *braid relations.* An important aspect of the theory of symmetric groups is the *length function* $l : S_n \to \mathbb{N}_0$ which is defined as

$$l(w) := \min\{m \in \mathbb{N}_0 | \exists s_{i_1}, \ldots, s_{i_m} \in \mathcal{S} : w = s_{i_1} \ldots s_{i_m}\}.$$

If we are given $w \in S_n$ and an expression $w = s_{i_1} \ldots s_{i_m}$ with $s_{i_1}, \ldots, s_{i_m} \in \mathcal{S}$ and $m = l(w)$, then we call $s_{i_1} \ldots s_{i_m}$ a *reduced expression of* w. Otherwise, we call $s_{i_1} \ldots s_{i_m}$ an *unreduced expression of* w. The symmetric group S_n admits a unique element $w_{0,n}$ of maximal length given by the permutation

$$w_{0,n} = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}.$$

We have that $l(w_{0,n}) = \frac{1}{2}n(n-1)$.

In the following, we use these facts to assign to each $w \in S_n$ an operator ∂_w in terms of the Demazure operators $\partial_1, \ldots, \partial_{n-1}$. The key input for this assignment is the theorem of Matsumoto–Tits. For this, we recall the notion of nil-moves and braid-moves. Let $w \in S_n$ and $w = s_{i_1} \ldots s_{i_m}$ and $w = s_{j_1} \ldots s_{j_r}$ be expressions of w. If $s_{j_1} \ldots s_{j_r}$ is obtained from $s_{i_1} \ldots s_{i_m}$ by deleting a factor of the form ss, we say that the expressions $s_{j_1} \ldots s_{j_r}$ and $s_{i_1} \ldots s_{i_m}$ are linked by a *nil-move.* Likewise, if $s_{j_1} \ldots s_{j_r}$ is obtained from $s_{i_1} \ldots s_{i_m}$ by applying once the a braid relation (3.5) or (3.6), we say that $s_{j_1} \ldots s_{j_r}$ and $s_{i_1} \ldots s_{i_m}$ are linked by a *braid move.*

Theorem 3.1.4 (Matsumoto–Tits)**.** *Let $w \in S_n$. Then the following holds:*

(i) Any expression $w = s_{i_1} \dots s_{i_m}$ with $s_{i_1}, \dots, s_{i_m}$ can be transformed into a reduced expression of w by a sequence of nil-moves and braid-moves.

(ii) Every two reduced expressions of w can be transformed into each other by a sequence of braid-moves.

Proof. See e.g. [BB06, Theorem 3.3.1]. □

From Theorem 3.1.4 and the relations (3.1), (3.2) and (3.3), we immediately obtain the following consequence.

Corollary 3.1.5. *Let $w \in S_n$. For a reduced expression $w = s_{i_1} \dots s_{i_m}$, we define the* Sym_n*-linear operator*

$$\partial_w : P_n \to P_n, \quad \partial_w := \partial_{i_1} \circ \dots \circ \partial_{i_m}.$$

Then ∂_w is independent of the choice of reduced expression. We have that ∂_w is homogeneous of degree $-2l(w)$. By convention, we set $\partial_e := \mathrm{id}_{P_n}$, where $e \in S_n$ is the neutral element.

By Theorem 3.1.4, we directly deduce that if $s_{i_1} \dots s_{i_m}$ is an unreduced expression of an element in S_n, then we have

$$\partial_{i_1} \dots \partial_{i_m} = 0.$$

Thus, it follows that for all $v, w \in S_n$, we have

$$\partial_v \partial_w = \begin{cases} \partial_{vw} & \textit{if } l(vw) = l(v) + l(w), \\ 0 & \textit{else.} \end{cases}$$

Next, we discuss an important basis theorem for the graded k-algebra $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$ which involves the Demazure operators. For this, we first recall the definition of Schubert polynomials.

Definition 3.1.6. *For each $n \in \mathbb{N}$, we set $x^{\rho_n} := x_1^{n-1} x_2^{n-2} \dots x_{n-1}$. Then for $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w \in P_n$ is defined as $\mathfrak{S}_w := \partial_{w^{-1} w_{0,n}}(x^{\rho_n})$.*

Note that $\mathfrak{S}_w$ is homogeneous of degree $n(n-1) - 2l(w)$. Moreover, we have by definition

$$\partial_v \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wv^{-1}} & \text{if } l(vw) = l(v) + l(w), \\ 0 & \text{else.} \end{cases} \tag{3.7}$$

We now describe some Schubert polynomials explicitly. From the definition, it follows that $\mathfrak{S}_{w_{0,n}} = x^{\rho_n}$. The other extreme case is the following.

Lemma 3.1.7. *For each $n \in \mathbb{N}$, we have $\mathfrak{S}_e = 1$.*

Proof. We proof this assertion by induction on n. The case $n = 1$ is clear by definition. For the induction step, note that $w_{0,n} = s_1 \dots s_{n-1} w_{0,n-1}$. In the following equations, we use in the third equality the induction hypothesis. The other equations follow from Lemma 3.1.3.(ii), since $x_1 \dots x_j \in P_n$ is s_i-invariant for each $i < j$.

$$\begin{aligned}\partial_{w_{0,n}}(x^{\rho_n}) &= \partial_1 \dots \partial_{n-1} \partial_{w_{0,n-1}}(x_1 \dots x_{n-1} x^{\rho_{n-1}}) \\ &= \partial_1 \dots \partial_{n-1}(x_1 \dots x_{n-1} \partial_{w_{0,n-1}}(x^{\rho_{n-1}})) \\ &= \partial_1 \dots \partial_{n-1}(x_1 \dots x_{n-1}) \\ &= \partial_1 \dots \partial_{n-2}(x_1 \dots x_{n-2} \partial_{n-1}(x_{n-1})) \\ &= \partial_1 \dots \partial_{n-2}(x_1 \dots x_{n-2}) \\ &= \dots \\ &= 1.\end{aligned}$$

This completes the induction step. □

It is a well-known property of the Schubert polynomials that they form a homogeneous Sym_n-basis of P_n, i.e. we have the following important theorem.

Theorem 3.1.8. *We have that P_n is a graded free Sym_n-module and $(\mathfrak{S}_w)_{w \in S_n}$ is a homogeneous basis of P_n.*

Proof. See e.g. [Man01, Proposition 2.5.2]. □

A crucial consequence of Theorem 3.1.8 is the following basis theorem for the graded k-algebra $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$. It can be proved by considering the action of the Demazure operators on the Schubert polynomials and using the relations from Corollary 3.1.5.

Theorem 3.1.9. *Let $n \in \mathbb{N}$, then the elements $x_1^{m_1} \dots x_n^{m_n} \partial_w$ for $m_1, \dots, m_n \in \mathbb{N}_0^n, w \in S_n$ form a homogeneous k-basis of $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$.*

Proof. See e.g. [Lau10, Proposition 3.5]. □

As we will explain in the following section, this theorem proves to be very useful to describe the representation theory of nil Hecke algebras.

We end this section with an explicit example that illustrates Theorem 3.1.9.

Example 3.1.10. In the following let $n = 2$. We write $S_2 = \{e, s\}$, where e is the neutral element and s is the transposition $(1, 2)$. Then we have the Schubert polynomials

$$\mathfrak{S}_e = 1, \quad \mathfrak{S}_s = x_1.$$

By Theorem 3.1.8, we know that P_2 is a graded free Sym_2-module and $(\mathfrak{S}_e, \mathfrak{S}_s)$ is a homogeneous basis. Now, for $a, b \in S_2$, we denote by $E_{a,b} \in \mathrm{END}_{\mathrm{Sym}_2}(P_2)$ the Sym_2-linear operator given by

$$E_{a,b}(\mathfrak{S}_c) = \begin{cases} \mathfrak{S}_a & \text{if } b = c, \\ 0 & \text{else,} \end{cases}$$

for all $c \in S_2$. By Proposition 1.2.4, we know that $\mathrm{END}_{\mathrm{Sym}_2}(P_2)$ is isomorphic as graded Sym_2-algebra to the graded matrix algebra $\mathrm{M}_2(\mathrm{Sym}_2)(\underline{d})$, where $\underline{d} = (0, 2) \in \mathbb{Z}^2$. Let $E_{1,1}, E_{1,2}, E_{2,1}$ and $E_{2,2}$ denote the elementary matrices in $\mathrm{M}_2(\mathrm{Sym}_2)(\underline{d})$. Then an explicit isomorphism of graded Sym_2-algebras is given by

$$\mathrm{M}_2(\mathrm{Sym}_2)(\underline{d}) \to \mathrm{END}_{\mathrm{Sym}_2}(P_2),$$

where

$$E_{1,1} \mapsto E_{e,e}, \quad E_{1,2} \mapsto E_{e,s}, \quad E_{2,1} \mapsto E_{s,e}, \quad E_{2,2} \mapsto E_{s,s}.$$

So the operators $E_{e,e}, E_{e,s}, E_{s,e}$ and $E_{s,s}$ correspond to the diagonal matrices in $\mathrm{M}_2(\mathrm{Sym}_2)(\underline{d})$ under this isomorphism.

According to Theorem 3.1.9, the elements $x_1^{m_1} x_2^{m_2} \partial_w$ for $m_1, m_2 \in \mathbb{N}_0$ and $w \in S_2$ form a homogeneous k-basis of $\mathrm{END}_{\mathrm{Sym}_2}(P_2)$. In the following, we describe the operators $E_{e,e}, E_{e,s}, E_{s,e}$ and $E_{s,s}$ as k-linear combination with respect to this basis. By definition, we have $\partial_e = \mathrm{id}_{P_2}$ and by (3.7), we have $\partial_s = E_{e,s}$. From this, we directly deduce

$$E_{s,s} = x_1 \partial_s, \quad E_{e,e} = \partial_e - x_1 \partial_s.$$

So finally, we consider the operator $E_{s,e}$. At first, note that the operator $x_1 \cdot \mathrm{id}_{P_2}$ maps $\mathfrak{S}_e$ to $\mathfrak{S}_s$. Moreover, we have

$$x_1 \mathfrak{S}_s = x_1^2 = (x_1 + x_2)\mathfrak{S}_s - x_1 x_2 \mathfrak{S}_e.$$

Hence, we deduce the equality

$$x_1 \cdot \mathrm{id}_{P_2} = E_{s,e} - x_1 x_2 E_{e,s} + (x_1 + x_2) E_{s,s}.$$

Inserting the above formulas for $\mathrm{id}_{P_2}, E_{e,s}, E_{s,s}$ then yields

$$E_{s,e} = x_1 \partial_e - x_1^2 \partial_s.$$

Thus, we described the operators $E_{e,e}, E_{e,s}, E_{s,e}$ and $E_{s,s}$ as k-linear combinations with respect to the k-basis from Theorem 3.1.9.

3.2 Nil Hecke algebras

The Demazure operators together with the polynomial algebra define an algebra called the nil Hecke algebra. In this section, we consider well-known important representation theoretic properties of these algebras. All the results, we discuss are well-known. We model this section on [Bru13, Chapter 2]. Our main focus is on the description of the graded simple and the graded projective indecomposable modules over nil Hecke algebras.

Definition 3.2.1. *Let $n \in \mathbb{N}_0$, then the nil Hecke algebra NH_n is the k-algebra with generators*

$$\{\tau_1, \dots, \tau_{n-1}\} \cup \{y_1, \dots, y_n\}$$

subject to the relations

$$\tau_i^2 = 0, \tag{3.8}$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \tag{3.9}$$
$$\tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i-j| > 1, \tag{3.10}$$
$$y_i y_j = y_j y_i, \tag{3.11}$$
$$\tau_i y_j = y_j \tau_i \quad \text{if } |i-j| > 1, \tag{3.12}$$
$$\tau_i y_i = 1 + y_{i+1} \tau_i, \tag{3.13}$$
$$y_i \tau_i = 1 + \tau_i y_{i+1}, \tag{3.14}$$

for all admissible i, j. There is a well-defined grading on $\mathrm{NH_n}$ such that each y_i is homogeneous of degree 2 and each τ_i is homogeneous of degree -2.

By definition, we have that $\mathrm{NH}_0 = k$ and $\mathrm{NH}_1 = k[y_1]$ with y_1 homogeneous of degree 2. In the following let $n \in \mathbb{N}_0$ be fixed. The relations (3.8), (3.9) and (3.10) coincide with the relations (3.1), (3.2) and (3.3) for Demazure operators. Thus, we can again apply Theorem 3.1.4 in the following way.

Corollary 3.2.2. *For a reduced expression $w = s_{i_1} \dots s_{i_m}$ we define the element*

$$\tau_w := \tau_{i_1} \dots \tau_{i_m}.$$

Then τ_w is independent of the choice of reduced expression. By convention, we set $\tau_e = 1$, where $e \in S_n$ is the neutral element. We have that τ_w is homogeneous of degree $-2l(w)$.

Moreover, just as for the Demazure operators, Theorem 3.1.4 also implies that if $s_{i_1} \dots s_{i_m}$ is an unreduced expression of an element in S_n, then we have

$$\tau_{i_1} \dots \tau_{i_m} = 0.$$

Hence, we conclude that for all $v, w \in S_n$, we have

$$\tau_v \tau_w = \begin{cases} \tau_{vw} & \text{if } l(vw) = l(v) + l(w), \\ 0 & \text{else,} \end{cases}$$

In the following, we apply the results from the previous section to describe the representation theory of nil Hecke algebras. For this, we continue to use the notation that was used in the previous section. The central result is that via Demazure operators, we obtain the following faithful operation of $\mathrm{NH_n}$ on P_n.

Theorem 3.2.3. *The following holds:*

(i) We have that $\mathcal{B} := (y_1^{m_1} \dots y_n^{m_n} \tau_w \,|\, m_1, \dots, m_n \in \mathbb{N}_0^n, w \in S_n)$ is a homogeneous k-basis of NH_n.

(ii) There is a isomorphism of graded k-algebras

$$\Phi : \mathrm{NH}_n \to \mathrm{END}_{\mathrm{Sym}_n}(P_n)$$

given by $\tau_i \mapsto \partial_i$ and $y_i \mapsto x_i$ for all admissible i.

Proof. Using the defining relations of $\mathrm{NH_n}$, one can directly check that Φ is a well-defined homomorphism of graded k-algebras. Moreover, using the relations (3.13) and (3.14), we obtain that the elements of $\mathcal{B}$ generate NH_n. Finally, we have that the elements of $\mathcal{B}$ are linear independent because by Theorem 3.1.9, they are mapped via Φ to a basis of $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$. From this, the assertions (i) and (ii) directly follow. □

From now on, we identify $\mathrm{NH_n}$ and $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$ via the isomorphism Φ. Moreover, we view P_n as $\mathrm{NH_n}$-module, where the action is given via the isomorphism Φ and we consider Sym_n as graded k-subalgebra of $\mathrm{NH_n}$.

As a direct consequence of Theorem 3.2.3, we obtain that $\mathrm{NH_n}$ is a Laurentian k-algebra and hence all the results outlined in the first chapter can be applied to $\mathrm{NH_n}$. However, Theorem 3.2.3 implies a more profound description of the representation theory of $\mathrm{NH_n}$. By Theorem 3.1.8, we know that P_n is a graded free Sym_n-module with homogeneous basis given by the Schubert polynomials. Thus, by Proposition 1.2.4, we have that $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$ is isomorphic as graded k-algebra to a graded matrix algebra over Sym_n. From these observations, we immediately obtain the following classification of the graded simple and graded projective indecomposable $\mathrm{NH_n}$-modules.

Theorem 3.2.4. *The following holds:*

(i) P_n *is the unique graded projective indecomposable* $\mathrm{NH_n}$*-module up to shift-isomorphism.*

(ii) The graded Jacobson radical of $\mathrm{NH_n}$ *is* $J^g(\mathrm{NH_n}) = (\mathrm{Sym}_n^+)\,\mathrm{NH_n}$*, where*

$$\mathrm{Sym}_n^+ := \bigoplus_{i\geq 1} (\mathrm{Sym}_n)_i$$

is the augmentation ideal of Sym_n*.*

(iii) $L_n := P_n/(\mathrm{Sym}_n^+)P_n$ *is the unique graded simple* $\mathrm{NH_n}$*-module up to shift-isomorphism. As graded* k*-vector space,* L_n *has a homogeneous basis given by the residue classes of Schubert polynomials* $(\overline{\mathfrak{S}_w})_{w\in S_n}$ *and* $\mathrm{NH_n}$ *acts on* L_n *through the epimorphism*

$$\mathrm{NH_n} \twoheadrightarrow \mathrm{END}_{\mathrm{Sym}_n}(P_n)/((\mathrm{Sym}_n^+)\,\mathrm{END}_{\mathrm{Sym}_n}(P_n)) \cong \mathrm{END}_k(L_n).$$

For $v, w \in S_n$, let $E_{v,w} \in \mathrm{NH_n}$ denote the elementary matrix given by

$$E_{v,w}(\mathfrak{S}_z) = \begin{cases} \mathfrak{S}_v & \text{if } z = w, \\ 0 & \text{else,} \end{cases}$$

for all $z \in S_n$. Note that $E_{v,w}$ is homogeneous of degree $2(l(v) - l(w))$. Evidently, the elementary matrices $E_{w,w}$ for $w \in S_n$ form a complete set of primitive orthogonal homogeneous idempotents in $\mathrm{NH_n}$. We now describe the corresponding graded projective indecomposable $\mathrm{NH_n}$-module of the idempotent $E_{w,w}$. For this, note that we have an explicit isomorphism of graded $\mathrm{NH_n}$-modules

$$P_n\langle -2l(w)\rangle \xrightarrow{\sim} \mathrm{NH_n}\, E_{w,w}, \quad f\mathfrak{S}_v \mapsto fE_{v,w}, \quad \text{for all } f \in \mathrm{Sym}_n, v \in S_n.$$

Our next goal is to describe the graded dimension of P_n and L_n and compute the graded composition multiplicity of L_n in P_n. In order to formulate these quantities appropriately, we use the notion of quantum numbers. Our notation is modeled on [Kas12, Chapter IV].

Definition 3.2.5. *For a natural number* $n \in \mathbb{N}$*, the quantum number* $(n)_q \in \mathbb{Z}[q]$ *is defined as*

$$(n)_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \in \mathbb{Z}[q].$$

The quantum factorial $(n)_q!$ *is defined as*

$$(n)_q! := (1)_q(2)_q \dots (n)_q = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n} \in \mathbb{Z}[q].$$

Let $m, n \in \mathbb{N}_0$ *with* $m \leq n$*. Then we define the corresponding quantum binomial coefficient as*

$$\binom{n}{m}_q := \frac{(n)_q!}{(m)_q!}(n-m)_q! \in \mathbb{Z}[q].$$

For the fact that $\binom{n}{m}_q$ *is indeed contained in* $\mathbb{Z}[q]$*, see e.g. [Kas12, Proposition IV.2.1].*

We also have the notion of symmetric quantum numbers.

Definition 3.2.6. *The symmetric quantum number $[n]_q$ is defined as*

$$[n]_q = q^{-(n-1)}(n)_q = \frac{q^{-n} - q^n}{q^{-1} - q} = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1} \in \mathbb{Z}[q, q^{-1}].$$

Similarly, the symmetric quantum factorial is defined as

$$[n]_q! := [1]_q[2]_q \dots [n]_q = q^{-\frac{1}{2}n(n-1)}(n)_{q^2}! \in \mathbb{Z}[q, q^{-1}].$$

Let $m, n \in \mathbb{N}_0$ with $m \leq n$. Then we define the corresponding symmetric quantum binomial coefficient as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[m]_q![n-m]_q!} \in \mathbb{Z}[q, q^{-1}].$$

It is a well-known fact that we have the equality

$$\sum_{w \in S_n} q^{l(w)} = (n)_q!. \tag{3.15}$$

One can for example prove (3.15) by induction on n. Now, from equation (3.15), we can infer the following useful consequence. Let $m, n \in \mathbb{N}_0$ and let $W_{m,n} \subset S_{m+n}$ denote the set of shortest left coset representatives of $S_{m+n}/(S_m \times S_n)$. Then we have

$$\sum_{w \in W_{m,n}} q^{l(w)} = \binom{m+n}{m}_q. \tag{3.16}$$

Indeed, given $w \in S_{m+n}$, then exist unique $x \in S_m, y \in S_n$ and $z \in W_{m,n}$ such that $w = z(x \times y)$ and $l(w) = l(x) + l(y) + l(z)$. Hence, by (3.15), we obtain

$$\Big(\sum_{w \in W_{m,n}} q^{l(w)}\Big)(m)_q!(n)_q! = (m+n)_q!.$$

By definition, this is equivalent to (3.16).

We proceed with determining the graded dimension of P_n and L_n. Using equation (3.15) and the fundamental theorem of symmetric polynomials, we immediately obtain the following formulas

$$\operatorname{grdim}(P_n) = (n)_{q^2}! \prod_{i=1}^{n} \frac{1}{1 - q^{2i}}, \quad \operatorname{grdim}(L_n) = (n)_{q^2}!. \tag{3.17}$$

Since L_n is the unique graded simple $\mathrm{NH_n}$-module up to shift-isomorphism, we also directly obtain a formula for the graded composition multiplicity of L_n in P_n. Namely, we have

$$[P_n : L_n]_q = \operatorname{grdim}(P_n) \cdot (\operatorname{grdim}(L_n))^{-1} = \prod_{i=1}^{n} \frac{1}{1 - q^{2i}}. \tag{3.18}$$

We end this section with discussing some duality properties of P_n and L_n. For this, recall that in Definition 2.4.4, we defined dualities $\circledast$ and $\#$ on the categories A-fmod and A-pmod, where A is a Laurentian k algebra that admits a self-inverse graded anti-automorphism. In order to apply this to $\mathrm{NH_n}$, we use that $\mathrm{NH_n}$ admits a self-inverse graded anti-automorphism $\mathfrak{T} : \mathrm{NH}_n \to \mathrm{NH}_n$ given by

$$\tau_i \mapsto \tau_i, \quad y_i \mapsto y_i,$$

for all admissible i. Thus, by Definition 2.4.4, we have dualities

$$\begin{aligned} \circledast : &\mathrm{NH}_n\text{-fmod} \to \mathrm{NH}_n\text{-fmod}, \quad M^{\circledast} := \mathrm{HOM}_k(M, k), \\ \# : &\mathrm{NH}_n\text{-pmod} \to \mathrm{NH}_n\text{-pmod}, \quad P^{\#} := \mathrm{HOM}_{\mathrm{NH}_n}(P, \mathrm{NH}_n), \end{aligned}$$

defined with respect to $\mathfrak{T}$.

Next, we investigate how $\circledast$ and $\#$ act on NH_n-fmod and NH_n-pmod. Since we have

$$\mathrm{grdim}(L_n^{\circledast}) = (n)_{q^{-2}}! = q^{-n(n-1)}(n)_{q^2}! = q^{-n(n-1)}\,\mathrm{grdim}(L_n),$$

we conclude that $L_n^{\circledast} \cong L_n\langle -n(n-1)\rangle$ because L_n is the unique graded simple $\mathrm{NH_n}$-module up to shift-isomorphism. From Corollary 2.4.6, it then follows that $P_n^{\#} \cong P_n\langle -n(n-1)\rangle$. Now, we set

$$T_n := L_n\langle -\frac{1}{2}n(n-1)\rangle, \quad Q_n := P_n\langle -\frac{1}{2}n(n-1)\rangle.$$

Then we have that, up to isomorphism, T_n is the unique graded simple $\mathrm{NH_n}$-module such that $T_n^{\circledast} \cong T_n$. Likewise, up to isomorphism, we have that Q_n is the unique graded projective indecomposable $\mathrm{NH_n}$-module such that $Q_n^{\#} \cong Q_n$. Moreover, Q_n is the projective cover of T_n.

3.3 Nil Hecke Grothendieck groups

In the following sections we study Grothendieck groups of nil Hecke algebras. Our main goal is to explicitly describe the categorification theorem of Khovanov–Lauda ([KL09, Theorem 1.1]) in the special case of nil Hecke algebras. In Chapter 4, we then discuss how we can generalize this theorem for alternating nil Hecke algebras. Now, in this section, we follow [KL09, Chapter 3] to define the nil Hecke Grothendieck groups $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ and characterize their algebraic properties.

We use the notation that was introduced in Chapter 2. In particular $\mathcal{A}$ denotes the ring $\mathbb{Z}[q, q^{-1}]$. In the following proposition, we list basic facts about the Grothendieck groups $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ and $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ that are direct consequences of the results from the previous section.

Proposition 3.3.1. *Let $n \in \mathbb{N}_0$. Then the following holds:*

(i) The Grothendieck group $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ *admits a self-inverse $\mathcal{A}$-anti-linear automorphism*

$$\overline{} : \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}) \to \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}), \quad [M] \mapsto [M^{\circledast}],$$

for all $M \in \mathrm{NH}_n$-fmod. *We call* $\overline{}$ *the bar involution on* $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$.

(ii) The Grothendieck group $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ *admits a self-inverse $\mathcal{A}$-anti-linear automorphism*

$$\overline{} : \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}) \to \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}), \quad [P] \mapsto [P^{\#}],$$

for all $P \in \mathrm{NH}_n$-pmod. *We call* $\overline{}$ *the bar involution on* $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$.

(iii) The $\mathcal{A}$-modules $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ *and* $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ *are free of rank one. We have that* $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ *is generated by the class* $[T_n]$ *and* $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ *is generated by the class* $[Q_n]$.

(iv) Let $(.,.)_n : \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}) \times \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}) \to \mathcal{A}$*, be the* $\mathcal{A}$*-bilinear* HOM*-pairing from Definition 2.4.7. So* $(.,.)_n$ *is the unique* $\mathcal{A}$*-bilinear pairing such that*

$$([P],[M])_n = \mathrm{grdim}(\mathrm{HOM}_{\mathrm{NH_n}}(P^{\#}, M)),$$

for all $P \in \mathrm{NH}_n\text{-pmod}, M \in \mathrm{NH}_n\text{-fmod}$*. Then we have* $([Q_n],[T_n])_n = 1$*.*

Proof. (i) Since $\circledast$ is a duality, we have that the bar involution on $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ is a well-defined and self-inverse additive map. The $\mathcal{A}$-anti-linearity follows from

$$(M\langle d\rangle)^{\circledast} \cong \mathrm{HOM}_k(M\langle d\rangle, k) \cong (\mathrm{HOM}_k(M,k))\langle -d\rangle \cong (M^{\circledast})\langle -d\rangle,$$

for all $M \in \mathrm{NH}_n\text{-fmod}, d \in \mathbb{Z}$.

(ii) We can use the same argument as in (i). As $\#$ is a duality, we obtain that the bar involution on $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ is a well-defined and self-inverse additive map. The anti-linearity follows from

$$(P\langle d\rangle)^{\#} \cong \mathrm{HOM}_{\mathrm{NH_n}}(P\langle d\rangle, \mathrm{NH_n}) \cong (\mathrm{HOM}_{\mathrm{NH_n}}(P, \mathrm{NH_n}))\langle -d\rangle \cong (P^{\#})\langle -d\rangle,$$

for all $P \in \mathrm{NH}_n\text{-pmod}, d \in \mathbb{Z}$.

(iii) This follows directly from Theorem 2.1.4 and the classification of the graded simple and graded projective indecomposable $\mathrm{NH_n}$-modules from Theorem 3.2.4.

(iv) At first, recall from Theorem 3.2.3 that $\mathrm{NH_n}$ is graded Schurian. Moreover, Q_n is the projective cover of T_n and $Q_n^{\#} \cong Q_n$. Thus, using (1.2), we conclude

$$([Q_n],[T_n])_n = \mathrm{grdim}(\mathrm{HOM}_{\mathrm{NH_n}}(Q_n, T_n)) = [T_n : T_n]_q = 1.$$

This finishes the proof. □

We choose the class $[T_n]$ as standard generator of $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ since $[T_n]$ is the unique generator of $\mathrm{G}_0(\mathrm{NH})$ that is invariant under the bar involution. With the same motivation, we choose the class $[Q_n]$ as standard generator of $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$.

Definition 3.3.2. *The nil Hecke Grothendieck group* $\mathrm{G}_0(\mathrm{NH})$ *and the split nil Hecke Grothendieck group* $\mathrm{K}_0(\mathrm{NH})$ *are defined as*

$$\mathrm{G}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}), \quad \mathrm{K}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}).$$

They are in a natural way $\mathbb{N}_0$*-graded* $\mathcal{A}$*-modules.*

The bar involutions on $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ and $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ give $\mathcal{A}$-anti-linear self-inverse automorphisms

$$\overline{} : \mathrm{G}_0(\mathrm{NH}) \to \mathrm{G}_0(\mathrm{NH}), \quad \overline{} : \mathrm{K}_0(\mathrm{NH}) \to \mathrm{K}_0(\mathrm{NH}),$$

which we call the *bar involutions* on $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$. Moreover, the $\mathcal{A}$-bilinear HOM-pairings between $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ and $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ give a pairing

$$(.,.) : \mathrm{K}_0(\mathrm{NH}) \times \mathrm{G}_0(\mathrm{NH}) \to \mathcal{A},$$

such that

$$([P],[M]) = \begin{cases} ([P],[M])_n & \text{if } P \in \mathrm{NH}_n\text{-pmod}, M \in \mathrm{NH}_n\text{-fmod}, \\ 0 & \text{else,} \end{cases}$$

for all $P \in \mathrm{NH}_n$-pmod, $M \in \mathrm{NH}_m$-fmod. We call $(.,.)$ the $\mathcal{A}$-*bilinear* HOM-*pairing* between $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$.

Our next aim is to show that $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ admit both an algebra and a coalgebra structure which is defined via induction and restriction functors. For this, we introduce the following inclusions.

Proposition 3.3.3. *Let $m, n \in \mathbb{N}_0$. Then we have inclusions of graded k-algebras*

$$i_{m,n} : \mathrm{NH}_m \hookrightarrow \mathrm{NH}_{m+n}, \quad y_i \mapsto y_i, \quad \tau_i \mapsto \tau_i$$

and

$$j_{m,n} : \mathrm{NH}_n \hookrightarrow \mathrm{NH}_{m+n}, \quad y_i \mapsto y_{i+m}, \quad \tau_i \mapsto \tau_{i+m}$$

for all admissible i. In particular, we obtain an inclusion of graded k-algebras

$$\iota_{m,n} := i_{m,n} \otimes j_{m,n} : \mathrm{NH}_m \otimes_k \mathrm{NH}_n \hookrightarrow \mathrm{NH}_{m+n}\,.$$

Proof. It follows directly from the defining relations of nil Hecke algebras that $i_{m,n}$ and $j_{m,n}$ are well-defined homomorphisms of graded k-algebras. The injectivity of $i_{m,n}$ and $j_{m,n}$ follows from Theorem 3.2.3.(i). Thus, by definition we have that $\iota_{m,n}$ is also an inclusion of k-algebras. □

From now on, we consider $\iota_{m,n}$ as standard inclusion $\mathrm{NH}_m \otimes_k \mathrm{NH}_n \hookrightarrow \mathrm{NH}_{m+n}$. By Theorem 3.2.3.(i), we directly obtain the following important result.

Lemma 3.3.4. *Let $m, n \in \mathbb{N}_0$. Then the following holds:*

(i) The graded left $(\mathrm{NH}_m \otimes_k \mathrm{NH}_n)$-module NH_{m+n} is graded free and a homogeneous basis is given by $(\tau_w)_{w \in W_{m,n}}$, where $W_{m,n}$ is the set of shortest right coset representatives of $(S_m \times S_n)\backslash S_{m+n}$.

(ii) The graded right $(\mathrm{NH}_m \otimes_k \mathrm{NH}_n)$-module NH_{m+n} is graded free and a homogeneous basis is given by $(\tau_w)_{w \in W'_{m,n}}$, where $W'_{m,n}$ is the set of shortest left coset representatives of $S_{m+n}/(S_m \times S_n)$.

We denote the induction and restriction functors corresponding to $\iota_{m,n}$ by

$$\mathrm{Ind}_{m,n}^{m+n} : (\mathrm{NH}_m \otimes_k \mathrm{NH}_n)\text{-Mod} \to \mathrm{NH}_{m+n}\text{-Mod},$$
$$\mathrm{Res}_{m,n}^{m+n} : \mathrm{NH}_{m+n}\text{-Mod} \to (\mathrm{NH}_m \otimes_k \mathrm{NH}_n)\text{-Mod}.$$

In the following theorem, we describe the multiplicative and comultiplicative structure on $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ via these functors. For this, we will in the following implicitly use the identifications

$$\mathrm{G}_0(\mathrm{NH}_m\text{-fmod}) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}) \cong \mathrm{G}_0((\mathrm{NH}_m \otimes_k \mathrm{NH}_n)\text{-fmod}),$$
$$\mathrm{K}_0(\mathrm{NH}_m\text{-pmod}) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}) \cong \mathrm{K}_0((\mathrm{NH}_m \otimes_k \mathrm{NH}_n)\text{-pmod}),$$

given by the $\mathcal{A}$-module isomorphisms from Proposition 2.1.6.

Theorem 3.3.5. *The following holds:*

(i) We have that $\mathrm{G}_0(\mathrm{NH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebra with unit* $[T_0]$ *and multiplication*

$$[M]\cdot[N]=[\mathrm{Ind}_{m,n}^{m+n}(M\otimes_k N)],$$

for all $m,n\in\mathbb{N}_0, M\in \mathrm{NH}_m\text{-fmod}, N\in\mathrm{NH}_n\text{-fmod},$. *Similarly,* $\mathrm{K}_0(\mathrm{NH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebra with unit* $[Q_0]$ *and multiplication*

$$[P]\cdot[Q]=[\mathrm{Ind}_{m,n}^{m+n}(P\otimes_k Q)],$$

for all $m,n\in\mathbb{N}_0, P\in\mathrm{NH}_m\text{-pmod}, Q\in\mathrm{NH}_n\text{-pmod}.$

(ii) We have that $\mathrm{G}_0(\mathrm{NH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebra, where the counit is the projection to* $\mathrm{G}_0(\mathrm{NH}_0\text{-fmod})$ *and the comultiplication is*

$$\Delta_{\mathrm{G}}([M])=\sum_{r=0}^{n}[\mathrm{Res}_{r,n-r}^{n}(M)],\quad \textit{for all } n\in\mathbb{N}_0, M\in\mathrm{NH}_n\text{-fmod}.$$

Analogously, $\mathrm{K}_0(\mathrm{NH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebra, where counit is the projection to* $\mathrm{K}_0(\mathrm{NH}_0\text{-pmod})$ *and the comultiplication is*

$$\Delta_{\mathrm{K}}([P])=\sum_{r=0}^{n}[\mathrm{Res}_{r,n-r}^{n}(P)],\quad \textit{for all } n\in\mathbb{N}_0, P\in\mathrm{NH}_n\text{-pmod}.$$

Proof. From Lemma 3.3.4 and Proposition 2.1.6, we obtain that the above multiplications and comultiplications are well-defined. The associativity and coassociativity is a direct consequence of the associativity of induction, restriction and outer tensor products. ☐

We proceed with discussing an important duality relationship between $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ which is given by the bilinear HOM-pairing. For this, we first recall some general notions.

Notation 3.3.6. Let R be commutative unital ring and A be an $\mathbb{N}_0$-graded free R-module such that all graded components of A are of finite rank. We set

$$A^*:=\bigoplus_{n\in\mathbb{N}_0}\mathrm{Hom}_R(A_n,R).$$

By definition, A^* is an $\mathbb{N}_0$-graded R-module. For each $i\in\mathbb{Z}$, let $(a_{i,1},\dots,a_{i,m_i})$ be an R-basis of A_i and let $(a^*_{i,1},\dots,a^*_{i,m_i})$ be the corresponding dual basis of $\mathrm{Hom}_R(A_i,R)$. Then $(a^*_{i,j}|i\in\mathbb{N}_0, j\in\{1,\dots m_i\})$ is a homogeneous R-basis of A^* which we call the *dual basis* of $(a_{i,j}|i\in\mathbb{N}_0, j\in\{1,\dots m_i\})$.

Definition 3.3.7. *Let* R *be commutative unital ring.*

(i) Let A *be an* $\mathbb{N}_0$*-graded free* R*-algebra such that all graded components of* A *are of finite rank. Let* $\eta:R\to A$ *be the unit and let*

$$\mu:A\otimes_R A\to A,\quad a\otimes b\mapsto ab,\quad \textit{for all } a,b\in A.$$

For $i,j\in\mathbb{N}_0$*, let* $\mu_{i,j}:A_i\otimes_R A_j\to A_{i+j}$ *be the restriction of* μ*. Then the* $\mathbb{N}_0$*-graded dual* R*-coalgebra* A^* *of* A *is defined to be the* $\mathbb{N}_0$*-graded* R*-coalgebra with underlying* $\mathbb{N}_0$*-graded* R*-module*

$$A^*:=\bigoplus_{n\in\mathbb{N}_0}\mathrm{Hom}_R(A_n,R)$$

and counit

$$A^* \xrightarrow{q_0} \mathrm{Hom}_R(A_0, R) \xrightarrow{\eta^*} R,$$

where q_0 *is the projection to the zeroth component* $A_0^* = \mathrm{Hom}_R(A_0, R)$ *and* η^* *is the adjoint map of* η*. The comultiplication on* A^* *given by*

$$f \mapsto \sum_{i+j=|f|} \mu_{i,j}^*(f) \in \bigoplus_{i+j=|f|} A_i^* \otimes_R A_j^*,$$

where $f \in A^*$ *is homogeneous and* $\mu_{i,j}^* : A_{i+j}^* \to A_i^* \otimes_R A_j^*$ *is the adjoint map of* $\mu_{i,j}$*. Here, we identify* $A_i^* \otimes_R A_j^*$ *with* $\mathrm{Hom}_R(A_i \otimes_R A_j, R)$ *via the canonical* R*-linear isomorphism.*

(ii) Let C *be an* $\mathbb{N}_0$*-graded free* R*-coalgebra with comultiplication* Δ *and counit* ε*. For each* $i \in \mathbb{N}_0$*, let* $p_i : C \to C_i$ *denote the projection. We set*

$$\Delta_{i,j} : C_{i+j} \to C_i \otimes_R C_j, \quad \Delta_{i,j} := (p_i \otimes p_j) \circ \Delta.$$

Then the $\mathbb{N}_0$*-graded dual* R*-algebra* C^* *of* C *is defined to be the* $\mathbb{N}_0$*-graded* R*-algebra with underlying* $\mathbb{N}_0$*-graded* R*-module*

$$C^* := \bigoplus_{n \in \mathbb{N}_0} \mathrm{Hom}_R(C_n, R)$$

and unit $\varepsilon^* : R \to C^*$*, where* ε^* *is the adjoint of* ε*. The multiplication on* C^* *is given by*

$$C_i^* \otimes_R C_j^* \xrightarrow{\iota_{i,j}} \mathrm{Hom}_R(C_i \otimes_R C_j, R) \xrightarrow{\Delta_{i,j}^*} C_{i+j}^*.$$

for all $i, j \in \mathbb{N}_0$*. Here,* $\Delta_{i,j}^*$ *denotes the adjoint map of* $\Delta_{i,j}$ *and* $\iota_{i,j}$ *is the canonical* R*-linear inclusion.*

Let A *be an* $\mathbb{N}_0$*-graded free* R*-algebra such that all graded components of* A *are of finite rank which also admits an* $\mathbb{N}_0$*-graded coalgebra structure. Then the* $\mathbb{N}_0$*-graded dual* A^* *of* A *is defined to be the* $\mathbb{N}_0$*-graded* R*-module*

$$A^* := \bigoplus_{n \in \mathbb{N}_0} \mathrm{Hom}_R(A_n, R),$$

endowed with the $\mathbb{N}_0$*-graded* R*-algebra structure from (i) and with the* $\mathbb{N}_0$*-graded* R*-coalgebra from (ii).*

We now come to the duality relationship between $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$. For this, we define the $\mathcal{A}$-bilinear pairing

$$(.,.) : (\mathrm{K}_0(\mathrm{NH}) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{NH})) \times (\mathrm{G}_0(\mathrm{NH}) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{NH})) \to \mathcal{A}, \tag{3.19}$$

via $(a \otimes b, c \otimes d) = (a, c)(b, d)$, for $a, b \in \mathrm{K}_0(\mathrm{NH}), c, d \in \mathrm{G}_0(\mathrm{NH})$.

Theorem 3.3.8. *The following holds:*

(i) For each $n \in \mathbb{N}_0$*, let*

$$f_n : \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{NH}_n\text{-pmod}), \mathcal{A})$$

be the homomorphism of $\mathcal{A}$*-modules given by* $[M] \mapsto (., [M])_n$*, for all* $M \in \mathrm{NH_n}\text{-fmod}$*. Then*

$$f := \bigoplus_{n \in \mathbb{N}_0} f_n : \mathrm{G}_0(\mathrm{NH}) \to \mathrm{K}_0(\mathrm{NH})^*$$

is an isomorphism of $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebras and* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebras.*

(ii) For each $n \in \mathbb{N}_0$, let

$$g_n : \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}_0(\mathrm{NH}_n\text{-fmod}), \mathcal{A})$$

be the isomorphism of $\mathcal{A}$-modules given by $[P] \mapsto ([P], .)_n$, for all $P \in \mathrm{NH_n}\text{-pmod}$. Then

$$g := \bigoplus_{n \in \mathbb{N}_0} g_n : \mathrm{K}_0(\mathrm{NH}) \to \mathrm{G}_0(\mathrm{NH})^*$$

is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras.

Proof. We only show (i) since the proof for (ii) is analogous. By construction, f is a homomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-modules. Next, we show that f is a homomorphism of algebras. Let $m, n \in \mathbb{N}_0$ and let $M \in \mathrm{NH}_m\text{-fmod}, N \in \mathrm{NH}_n\text{-fmod}$ and $P \in \mathrm{ANH}_{m+n}\text{-pmod}$. Moreover, let

$$P_1', \ldots, P_r' \in \mathrm{NH}_m\text{-pmod}, \quad \text{and} \quad P_1'' \ldots, P_r'' \in \mathrm{NH}_n\text{-pmod}$$

such that

$$[\mathrm{Res}_{m,n}^{m+n}(P)] = \sum_{i=1}^{r} [P_i' \otimes_k P_i''].$$

Then we have the following equalities, where the second equality follows from Proposition 2.4.9 and the fourth equality follows from Proposition 2.4.10.

$$\begin{aligned}
(f([M] \cdot [N]))([P]) &= ([P], [\mathrm{Ind}_{m,n}^{m+n}(M \otimes_k N)]) \\
&= ([\mathrm{Res}_{m,n}^{m+n}(P)], [M \otimes_k N]) \\
&= \sum_{i=1}^{r} ([P_i' \otimes_k P_i''], [M \otimes_k N]) \\
&= \sum_{i=1}^{r} ([P_i'], [M])([P_i''], [N]) \\
&= (f([M]) \otimes f([N]))(\Delta_{\mathrm{K}}([P])) \\
&= (f([M]) \cdot f([N]))([P]).
\end{aligned}$$

Thus, we conclude that f is a homomorphism of algebras. By a similar argument, one can also show that f is a homomorphism of coalgebras. Hence, it is only left to show that f is bijective. By Proposition 3.3.1, we have that $(., [T_n])_n$ is an $\mathcal{A}$-module generator of $\mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{NH}_n\text{-pmod}), \mathcal{A})$ for all $n \in \mathbb{N}_0$. Hence, f maps an $\mathcal{A}$-basis of $\mathrm{G}_0(\mathrm{NH})$ to an $\mathcal{A}$-basis of $\mathrm{K}_0(\mathrm{NH})^*$. Thus, we conclude that f is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras. □

We end this section by applying the results of Section 2.5 to the nil Hecke Grothendieck groups. In this way, we obtain further interesting algebraic structures. In addition, this gives us a favorable opportunity to see some explicit examples of the results of Section 2.5. At first, we make two preparatory observations.

Firstly, in order to apply the results from Section 2.5, we have to ensure that $\mathrm{NH_n}$ satisfies the conditions from Convention 2.5.1. According to Theorem 3.2.3, $\mathrm{NH_n}$ is a graded matrix algebra over Sym_n. Furthermore, in Section 3.2, we defined a self-inverse anti-automorphism $\mathfrak{T} : \mathrm{NH_n} \to \mathrm{NH_n}$. Hence, it follows that the conditions from Convention 2.5.1 are satisfied. For the finiteness of the global dimension, note that by Proposition 1.2.5, $\mathrm{NH_n}$ is graded Morita equivalent to Sym_n. Now, Sym_n is a graded polynomial algebra over k in n variables and hence has global dimension n, see for instance [Wei95, Theorem 4.3.7]. Thus, we obtain

$$\mathrm{gl}(\mathrm{NH_n}) = n. \tag{3.20}$$

Secondly, by (3.18), we have that the graded composition multiplicities of finitely generated graded projective $\mathrm{NH_n}$-modules are contained in $\mathbb{Q}(q)$. Thus, we can apply the rational versions of the results of Section 2.5. For this, we introduce the rational nil Hecke Grothendieck groups.

Definition 3.3.9. *The rational nil Hecke Grothendieck group* $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *and the rational split nil Hecke Grothendieck group* $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *are defined as*

$$\begin{aligned}\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)} &:= \bigoplus_{n\in\mathbb{N}_0} \mathbb{Q}(q)\otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{NH}_n\text{-fmod}),\\ \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} &:= \bigoplus_{n\in\mathbb{N}_0} \mathbb{Q}(q)\otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{NH}_n\text{-pmod}).\end{aligned}$$

By Theorem 3.3.5, we have that both $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ are $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-algebras and $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-coalgebras. In addition, the involution $^-$ on $\mathcal{A}$ extends uniquely to $\mathbb{Q}(q)$. Thus, we have that the bar involutions on $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ naturally extend to bar involutions on $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$. Moreover, the $\mathcal{A}$-bilinear HOM-pairing $(.,.)$ between $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ extends to a $\mathbb{Q}(q)$-bilinear HOM-pairing $(.,.)$ between $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

Let us now apply the results from Section 2.5 to $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$. We begin with defining $\mathbb{Q}(q)$-bilinear Euler forms on $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

Definition 3.3.10. *We define the following:*

(i) For $n\in\mathbb{N}_0$*, let* $\chi_{\mathrm{K},n}$ *be the rational bilinear Euler on* $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ *from Definition 2.5.18.(i). So* $\chi_{\mathrm{K},n}$ *is the unique* $\mathbb{Q}(q)$*-bilinear form such that*

$$\chi_{\mathrm{K},n}(f\otimes[M], g\otimes[N]) = fg\sum_{i=0}^{n}(-1)^i\,\overline{\mathrm{grdim}(\mathrm{EXT}^i_{\mathrm{NH_n}}(M, N^{\circledast}))},$$

for all $f,g\in\mathbb{Q}(q)$ *and* $M,N\in\mathrm{NH}_n\text{-fmod}$.

(ii) Let χ_{G} *be the unique* $\mathbb{Q}(q)$*-bilinear form on* $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *such that*

$$\chi_{\mathrm{G}}(f\otimes[M], g\otimes[N]) = \begin{cases}\chi_{\mathrm{K},n}(f\otimes[M], g\otimes[N]) & \text{if } m=n,\\ 0 & \text{if } m\neq n,\end{cases}$$

holds for all $f,g\in\mathbb{Q}(q), M\in\mathrm{NH}_m\text{-fmod}$ *and* $N\in\mathrm{NH}_n\text{-fmod}$*. We call* χ_{G} *the* $\mathbb{Q}(q)$*-bilinear Euler form on* $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

(iii) For $n\in\mathbb{N}_0$*, let* $\chi_{\mathrm{K},n}$ *be the rational bilinear Euler on* $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ *from Definition 2.5.18.(ii). So* $\chi_{\mathrm{K},n}$ *is the unique* $\mathbb{Q}(q)$*-bilinear form such that*

$$\chi_{\mathrm{K},n}(f\otimes[P], g\otimes[Q]) = fg\ \mathrm{grdim}(\mathrm{HOM}_{\mathrm{NH_n}}(P^{\#}, Q)),$$

for all $f,g\in\mathbb{Q}(q)$ *and* $P,Q\in\mathrm{NH}_n\text{-pmod}$.

(iv) *Let* χ_{K} *be the unique* $\mathbb{Q}(q)$*-bilinear form on* $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *such that*

$$\chi_{\mathrm{K}}(f \otimes [P], g \otimes [Q]) = \begin{cases} \chi_{\mathrm{K},n}(f \otimes [P], g \otimes [Q]) & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

holds for all $f, g \in \mathbb{Q}(q)$, $P \in \mathrm{NH}_m$-pmod *and* $Q \in \mathrm{NH}_n$-pmod. *We call* χ_{K} *the* $\mathbb{Q}(q)$*-bilinear Euler form on* $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

Corollary 3.3.11. *The bilinear Euler forms* χ_{G} *and* χ_{K} *are symmetric.*

Proof. This follows directly from Corollary 2.5.4 and Corollary 2.5.11. □

Next, we apply Theorem 2.5.19 to $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$. For this, note that by (3.18), the graded character of $[Q_n]$ is

$$\mathrm{gch}([Q_n]) = \Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) \otimes [T_n], \quad \text{for each } n \in \mathbb{N}_0.$$

Thus, we obtain the following result.

Theorem 3.3.12. *We have an isomorphism of* $\mathbb{N}_0$*-graded* $\mathbb{Q}(q)$*-algebras and* $\mathbb{N}_0$*-graded* $\mathbb{Q}(q)$*-coalgebras*

$$\phi : \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \to \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)},$$

such that

$$[Q_n] \mapsto \Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) \otimes [T_n], \quad \text{for all } n \in \mathbb{N}_0.$$

Furthermore, the following holds:

(i) *We have* $\chi_{\mathrm{K}}(x, y) = \chi_{\mathrm{G}}(\phi(x), \phi(y))$ *for all* $x, y \in \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

(ii) *The* $\mathbb{Q}(q)$*-bilinear Euler forms* χ_{G} *and* χ_{K} *are non-degenerate and* $\chi_{\mathrm{G}}(\phi(.), .) = (., .)$. *In particular,* $(\phi(1 \otimes [Q_n]))_{n \in \mathbb{N}_0}$ *and* $(1 \otimes [T_n])_{n \in \mathbb{N}_0}$ *are dual bases of* $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *with respect to* χ_{G}.

Proof. The compatibility of the graded character map with induction, restriction and outer tensor products from Corollary 2.3.6 and Corollary 2.2.13 implies that ϕ is a homomorphism of $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-algebras and $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-coalgebras. The remaining assertions immediately follow from Theorem 2.5.19. □

We now conclude useful properties of χ_{G} and χ_{K} from Theorem 3.3.12.

Proposition 3.3.13. *The following holds:*

(i) *For all* $x, y, z \in \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$, *we have*

$$\chi_{\mathrm{G}}(x \cdot y, z) = \chi_{\mathrm{G}}(x \otimes y, \Delta_{\mathrm{G}}(z)),$$

where the $\mathbb{Q}(q)$*-bilinear form* χ_{G} *on* $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *is given by*

$$\chi_{\mathrm{G}}(a \otimes b, c \otimes d) = \chi_{\mathrm{G}}(a, c)\chi_{\mathrm{G}}(b, d),$$

for all $a, b, c, d \in \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

(ii) Likewise, for all $x, y, z \in \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$, we have

$$\chi_{\mathrm{K}}(x \cdot y, z) = \chi_{\mathrm{K}}(x \otimes y, \Delta_{\mathrm{K}}(z)),$$

where the $\mathbb{Q}(q)$-bilinear form χ_{K} on $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ is given by

$$\chi_{\mathrm{K}}(a \otimes b, c \otimes d) = \chi_{\mathrm{K}}(a, c)\chi_{\mathrm{K}}(b, d),$$

for all $a, b, c, d \in \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

Proof. Note that (i) and (ii) are equivalent by Theorem 3.3.12. Hence, we only have to prove (i). According to Theorem 3.3.12, it suffices to show the following. Let $m, n \in \mathbb{N}_0$ and let $P \in \mathrm{NH}_n\text{-pmod}, Q \in \mathrm{NH}_n\text{-pmod}$ and $M \in \mathrm{NH}_{m+n}\text{-fmod}$ be arbitrary. Moreover, let

$$M'_1, \ldots, M'_r \in \mathrm{NH}_m\text{-fmod} \quad \text{and} \quad M''_1, \ldots, M''_r \in \mathrm{NH}_n\text{-fmod}$$

such that $[\mathrm{Res}^{m+n}_{m,n}(M)] = \sum_{i=1}^r [M'_i \otimes_k M''_i]$. Then we have

$$\begin{aligned}\chi_{\mathrm{G}}(\phi(1 \otimes [\mathrm{Ind}^{m+n}_{m,n}(P \otimes_k Q)]), 1 \otimes [M]) \\ = \sum_{i=1}^r \chi_{\mathrm{G}}(\phi(1 \otimes [P]), 1 \otimes [M'_i])\chi_{\mathrm{G}}(\phi(1 \otimes [Q]), 1 \otimes [M''_i]).\end{aligned} \tag{3.21}$$

We prove (3.21) by using the compatibility properties of the bilinear HOM-pairing. For this, let

$$(.,.) : (\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}) \times (\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}) \to \mathbb{Q}(q),$$

be the $\mathbb{Q}(q)$-bilinear extension of the pairing $(.,.)$ from (3.19). Then we have the following equalities, where in the second equality we use Proposition 2.4.9 and in the third equality we use Proposition 2.4.10.

$$\begin{aligned}\chi_{\mathrm{G}}(\phi(1 \otimes [\mathrm{Ind}^{m+n}_{m,n}(P \otimes_k Q)]), 1 \otimes [M]) &= (1 \otimes [\mathrm{Ind}^{m+n}_{m,n}(P \otimes_k Q)], 1 \otimes [M]) \\ &= \Big((1 \otimes [P]) \otimes (1 \otimes [Q]), \sum_{i=1}^r (1 \otimes [M'_i]) \otimes (1 \otimes [M''_i])\Big) \\ &= \sum_{i=1}^r (1 \otimes [P], 1 \otimes [M'_i])(1 \otimes [Q], 1 \otimes [M''_i]) \\ &= \sum_{i=1}^r \chi_{\mathrm{G}}(\phi(1 \otimes [P]), 1 \otimes [M'_i])\chi_{\mathrm{G}}(\phi(1 \otimes [Q]), 1 \otimes [M''_i]).\end{aligned}$$

Thus, we proved (3.21) and hence we obtain (i). □

We end this section with explicitly describing the $\mathbb{Q}(q)$-bilinear Euler forms χ_{K} and χ_{G} with respect to the bases $(1 \otimes [Q_n])_{n \in \mathbb{N}_0}$ and $(1 \otimes [T_n])_{n \in \mathbb{N}_0}$. By the definition of χ_{K} and χ_{G}, we only have to calculate the products $\chi_{\mathrm{K}}(1 \otimes [Q_n], 1 \otimes [Q_n])$ and $\chi_{\mathrm{G}}(1 \otimes [T_n], 1 \otimes [T_n])$ since the other products vanish. Now, from (3.18), we obtain

$$\chi_{\mathrm{K}}(1 \otimes Q_n], 1 \otimes [Q_n]) = [Q_n : T_n]_q = \prod_{i=1}^n \frac{1}{1 - q^{2i}}, \quad \text{for all } n \in \mathbb{N}_0. \tag{3.22}$$

By Theorem 3.3.12, we then conclude

$$\chi_{\mathrm{G}}(1 \otimes [T_n], 1 \otimes [T_n]) = \prod_{i=1}^{n}(1-q^{2i})\chi_{\mathrm{G}}(\phi(1 \otimes [Q_n]), 1 \otimes [T_n]) = \prod_{i=1}^{n}(1-q^{2i}).$$

This provides an explicit description of the bilinear Euler forms χ_{K} and χ_{G} with respect to the bases $(1 \otimes [Q_n])_{n\in\mathbb{N}_0}$ and $(1 \otimes [T_n])_{n\in\mathbb{N}_0}$.

3.4 Classification of nil Hecke Grothendieck groups

After the preparations made in the previous section, we now finally discuss the categorification theorem of Khovanov–Laudain the special case of nil Hecke algebras. In this case, the categorification theorem states that $\mathrm{K}_0(\mathrm{NH})$ is an $\mathbb{N}_0$-graded twisted bialgebra over $\mathcal{A}$ and isomorphic to Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$ corresponding to the one-vertex graph without edges. Using the duality between $\mathrm{K}_0(\mathrm{NH})$ and $\mathrm{G}_0(\mathrm{NH})$ from Theorem 3.3.8, we can infer that $\mathrm{G}_0(\mathrm{NH})$ is also an $\mathbb{N}_0$-graded twisted bialgebra over $\mathcal{A}$ and isomorphic to the $\mathbb{N}_0$-graded dual of ${}_{\mathcal{A}}\mathbf{f}$. To formulate this theorem, we first recall the definition of Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$. For this, we follow [Lus93, Chapter 1].

At first, we fix the notion of $\mathbb{N}_0$-graded twisted bialgebras.

Definition 3.4.1. *Let $R = \mathcal{A}$ or $R = \mathbb{Q}(q)$. Let A be an $\mathbb{N}_0$-graded R-algebra. We call A an $\mathbb{N}_0$-graded twisted R-bialgebra if A admits a coalgebra structure (A, ε, Δ) such that $\varepsilon : A \to R$ and $\Delta : A \to A \otimes_R A$ are homomorphisms of $\mathbb{N}_0$-graded R-algebras, where $A \otimes_R A$ is endowed with the twisted multiplication given by*

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = q^{-|y_1||x_2|} x_1x_2 \otimes y_1y_2,$$

where $x_1, x_2, y_1, y_2 \in A$ are homogeneous elements.

We proceed with recalling the definition of Lusztig's quantum group $\mathbf{f}$ and Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$. In general, the definition of $\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{f}$ depends on a choice of Cartan datum or a choice of finite unoriented graph.We are only interested in the very special case where the graph consists of a single vertex without edges (and the associated Cartan type A_1). In this case, *Lusztig's quantum group* $\mathbf{f}$ is defined to be the $\mathbb{N}_0$-graded polynomial algebra $\mathbf{f} = \mathbb{Q}(q)[\theta]$ with θ homogeneous of degree 1. We have that $\mathbf{f}$ is a $\mathbb{N}_0$-graded twisted $\mathbb{Q}(q)$-bialgebra, where the counit is the projection to the zeroth component and the comultiplication is the unique algebra homomorphism

$$r : \mathbf{f} \to \mathbf{f} \otimes_{\mathbb{Q}(q)} \mathbf{f}, \quad \theta \mapsto \theta \otimes 1 + 1 \otimes \theta.$$

Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$ is defined to be the $\mathbb{N}_0$-graded submodule of $\mathbf{f}$ generated by the elements $\theta^{(n)} := \theta^n/[n]_q!$, for all $n \in \mathbb{N}_0$. We have that ${}_{\mathcal{A}}\mathbf{f}$ inherits the structure of an $\mathbb{N}_0$-graded twisted $\mathcal{A}$-bialgebra. Moreover, the canonical inclusion ${}_{\mathcal{A}}\mathbf{f} \hookrightarrow \mathbf{f}$ extends to an isomorphism of $\mathbb{N}_0$-graded twisted $\mathbb{Q}(q)$-bialgebras $\mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f} \cong \mathbf{f}$.

By construction, one can directly check that the multiplication on ${}_{\mathcal{A}}\mathbf{f}$ is given by

$$\theta^{(m)} \cdot \theta^{(n)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q \theta^{(m+n)}, \quad \text{for all } m, n \in \mathbb{N}_0.$$

Moreover, the comultiplication on ${}_{\mathcal{A}}\mathbf{f}$ is given by

$$\theta^{(n)} \mapsto \sum_{r=0}^{n} q^{-r(n-r)}\theta^{(r)} \otimes \theta^{(n-r)}, \quad \text{for all } n \in \mathbb{N}_0.$$

Additionally, we have the following algebraic structures on $\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{f}$:

1. There is a symmetric non-degenerate $\mathbb{Q}(q)$-bilinear form $(.,.)$ on $\mathbf{f}$ given by

   $$(\theta^{(m)}, \theta^{(n)}) = \delta_{m,n} \prod_{i=1}^{m} \frac{1}{1-q^{2i}},$$

   for all $m, n \in \mathbb{N}_0$. Here, $\delta_{m,n}$ denotes the Kronecker symbol. The form $(.,.)$ is called *Lusztig's symmetric form.* In addition, $(.,.)$ satisfies

   $$(xy, z) = (x \otimes y, r(z)),$$

   for all $x, y, z \in \mathbf{f}$, where the $\mathbb{Q}(q)$-bilinear form $(.,.)$ on $\mathbf{f} \otimes_{\mathbb{Q}(q)} \mathbf{f}$ is given by

   $$(a \otimes b, c \otimes d) = (a, c)(b, d), \quad \text{for all } a, b, c, d \in \mathbf{f}.$$

2. There exists a multiplicative self-inverse $\mathbb{Q}(q)$-anti-linear automorphism $\mathrm{b} : \mathbf{f} \to \mathbf{f}$ such that $\mathrm{b}(\theta) = \theta$. We call b the *bar involution* on $\mathbf{f}$.

3. We have that the bar involution on $\mathbf{f}$ restricts to a self-inverse $\mathcal{A}$-anti-linear automorphism $\mathrm{b} : {}_{\mathcal{A}}\mathbf{f} \to {}_{\mathcal{A}}\mathbf{f}$ that we call the *bar involution* on ${}_{\mathcal{A}}\mathbf{f}$.

Let ${}_{\mathcal{A}}\mathbf{f}^*$ denote the $\mathbb{N}_0$-graded dual of ${}_{\mathcal{A}}\mathbf{f}$, see Defintion 3.3.7. The fact that ${}_{\mathcal{A}}\mathbf{f}$ is an $\mathbb{N}_0$-graded twisted bialgebra over $\mathcal{A}$ implies that also ${}_{\mathcal{A}}\mathbf{f}^*$ is an $\mathbb{N}_0$-graded twisted bialgebra over $\mathcal{A}$. Let $(\theta^{(n)*})_{n\in\mathbb{N}_0}$ be the dual basis of $(\theta^{(n)})_{n\in\mathbb{N}_0}$. Then by definition, one can directly check that the multiplication on ${}_{\mathcal{A}}\mathbf{f}^*$ is given by

$$\theta^{(m)*} \cdot \theta^{(n)*} = q^{-mn}\theta^{(m+n)*}, \quad \text{for all } m, n \in \mathbb{N}_0.$$

Furthermore, the comultiplication is given by

$$\theta^{(n)*} \mapsto \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q \theta^{(r)*} \otimes \theta^{(n-r)*}, \quad \text{for all } n \in \mathbb{N}_0.$$

The dependence of the structure constants on $m, n \in \mathbb{N}_0$ implies that ${}_{\mathcal{A}}\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{f}^*$ are not isomorphic as $\mathbb{N}_0$-twisted bialgebras over $\mathcal{A}$. However, the following holds.

Lemma 3.4.2. *We have an inclusion of $\mathbb{N}_0$-graded $\mathcal{A}$-modules*

$$\iota : {}_{\mathcal{A}}\mathbf{f}^* \hookrightarrow \mathbf{f}, \quad \theta^{(n)*} \mapsto \Big(\prod_{i=1}^{n}(1-q^{2i})\Big) \cdot \theta^{(n)}.$$

that is compatible with the multiplication and comultiplication. In addition, let

$$\tilde{\iota} : \mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^* \to \mathbf{f}$$

be the map obtained via scalar extension. Then $\tilde{\iota}$ is an isomorphism of $\mathbb{N}_0$-graded twisted bialgebras over $\mathbb{Q}(q)$.

Proof. At first note that by construction, we have $\theta^{(n)*} = (\iota(\theta^{(n)*}), .)$. Now, using the compatibility of $(.,.)$ with respect to the multiplication and comultiplication on $\mathbf{f}$, we obtain that ι is multiplicative and comultiplicative. Finally, we observe that $\tilde{\iota}$ is an isomorphism of $\mathbb{N}_0$-graded twisted bialgebras over $\mathbb{Q}(q)$, since it maps a homogeneous basis of $\mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^*$ to a homogeneous basis of $\mathbf{f}$. $\square$

Lemma 3.4.2 directly implies that ${}_{\mathcal{A}}\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{f}^*$ become isomorphic as $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-twisted bialgebras after extending scalars to $\mathbb{Q}(q)$.

From now on, we view ${}_{\mathcal{A}}\mathbf{f}^*$ as embedded into $\mathbf{f}$ via the inclusion from Lemma 3.4.2. Let $\mathrm{b}^* : \mathbf{f} \to \mathbf{f}$ be the $\mathcal{A}$-anti-linear adjoint map of b, i.e. b^* is uniquely determined by

$$(\mathrm{b}(x), y) = \overline{(x, \mathrm{b}^*(y))}, \quad \text{for all } x, y \in \mathbf{f}.$$

One can directly check that we have

$$\mathrm{b}^*(\theta^{(n)}) = \prod_{i=1}^{n} \frac{1}{1 - q^{-2i}}, \quad \text{for all } n \in \mathbb{N}_0.$$

It follows that b^* is self-inverse and restricts to an $\mathcal{A}$-anti-linear automorphism $\mathrm{b}^* : {}_{\mathcal{A}}\mathbf{f}^* \to {}_{\mathcal{A}}\mathbf{f}^*$, which we call the *bar involution* on ${}_{\mathcal{A}}\mathbf{f}^*$. According to the above description of b^*, we have $\mathrm{b}^*(\theta^{(n)*}) = \theta^{(n)*}$, for all $n \in \mathbb{N}_0$.

Finally, after these preparations, we state the categorification theorem of Khovanov–Lauda for nil Hecke Grothendieck groups.

Theorem 3.4.3. *The split nil Hecke Grothendieck group* $\mathrm{K}_0(\mathrm{NH})$ *is an* $\mathbb{N}_0$*-graded twisted bialgebra over* $\mathcal{A}$*. In addition, we have an isomorphism of* $\mathbb{N}_0$*-graded twisted* $\mathcal{A}$*-bialgebras*

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \to \mathrm{K}_0(\mathrm{NH}), \quad \theta^{(n)} \mapsto [Q_n], \quad \textit{for all } n \in \mathbb{N}_0.$$

Moreover, the following holds:

(i) For all $x \in {}_{\mathcal{A}}\mathbf{f}$*, we have* $\gamma(\mathrm{b}(x)) = \overline{\gamma(x)}$*.*

(ii) Let $\gamma_{\mathbb{Q}(q)} : \mathbf{f} \to \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ *be the scalar extension of* γ*. Then for all* $x, y \in \mathbf{f}$*, we have*

$$(x, y) = \chi_{\mathrm{K}}(\gamma_{\mathbb{Q}(q)}(x), \gamma_{\mathbb{Q}(q)}(y)).$$

Proof. According to Proposition 3.3.1 and equation (3.22), the only assertion that is left to show is that γ is multiplicative and comultiplicative. Both properties can be proved in the same way. Hence, we here only prove that γ is multiplicative. For this, we have to show that

$$[\mathrm{Ind}_{m,n}^{m+n}(Q_m \otimes_k Q_n)] = \begin{bmatrix} m+n \\ m \end{bmatrix}_q [Q_{m+n}]$$

holds for all $m, n \in \mathbb{N}_0$. Since both NH_{m+n} and $\mathrm{NH}_m \otimes_k \mathrm{NH}_\mathrm{n}$ admit only one graded projective indecomposable module up to shift-isomorphism, it suffices to show

$$\mathrm{grdim}(\mathrm{Ind}_{m,n}^{m+n}(Q_m \otimes_k Q_n)) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q \cdot \mathrm{grdim}(Q_{m+n}).$$

This is however an immediate consequence of Lemma 3.3.4. $\square$

By the duality between $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ from Theorem 3.3.8, we direclty obtain the corresponding result for $\mathrm{G}_0(\mathrm{NH})$.

Theorem 3.4.4. *The nil Hecke Grothendieck group* $\mathrm{G}_0(\mathrm{NH})$ *is an* $\mathbb{N}_0$*-graded twisted bialgebra over* $\mathcal{A}$*. In addition, let* $\gamma^* : \mathrm{G}_0(\mathrm{NH}) \to {}_{\mathcal{A}}\mathbf{f}^*$ *be the adjoint map of* γ*. Then* γ^* *is an isomorphism of* $\mathbb{N}_0$*-graded twisted* $\mathcal{A}$*-bialgebras and we have*

$$\gamma^*([T_n]) = \theta^{(n)*}, \qquad \text{for all } n \in \mathbb{N}_0.$$

Moreover, the following holds:

(i) For all $x \in \mathrm{G}_0(\mathrm{NH})$*, we have* $\gamma^*(\overline{x}) = \mathrm{b}^*(\gamma^*(x))$*.*

(ii) Let $\gamma^*_{\mathbb{Q}(q)} : \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)} \to \mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^* \cong \mathbf{f}$ *be the scalar extension of* γ^**. Then for all* $x, y \in \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$*, we have*

$$\chi_{\mathrm{G}}(x,y) = (\gamma^*_{\mathbb{Q}(q)}(x), \gamma^*_{\mathbb{Q}(q)}(y)).$$

Finally, we investigate how the isomorphism $\phi : \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \to \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ from Theorem 3.3.12 fits into the picture. For this, note that under the identifications $\mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f} \cong \mathbf{f}$ and $\mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^* \cong \mathbf{f}$, we have the following commuting diagram

$$\begin{tikzcd} \mathbf{f} \arrow[r, "\gamma_{\mathbb{Q}(q)}"] \arrow[d, "\mathrm{id}_{\mathbf{f}}"] & \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)} \arrow[d, "\phi"] \\ \mathbf{f} \arrow[r, "(\gamma^*_{\mathbb{Q}(q)})^{-1}"] & \mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)} \end{tikzcd}$$

Hence, we observe that ϕ corresponds under these identifications to the isomorphism

$$\mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f} \cong \mathbb{Q}(q) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^*,$$

that we described above.

With this observation, we end this chapter about the representation theory of nil Hecke algebras. In the following chapter, we establish similar results for the alternating nil Hecke algebras.

4 Alternating nil Hecke algebras

Convention. Throughout this chapter, let k be a fixed field with $\mathrm{char}(k) \neq 2$.

Summary

This chapter is devoted to the study of the representation theory of the alternating nil Hecke algebras. These are special cases of a more general family of algebras called alternating quiver Hecke algberas which were introduced by Boys and Mathas in [Boy14, BM17]. Our main focus in our studies lies on the description of their Grothendieck groups.

By definition, the alternating nil Hecke algebras are a family of graded k-algebras denoted by $(\mathrm{ANH_n})_{n\in\mathbb{N}_0}$, where $\mathrm{ANH_n}$ is the fixed point subalgebra of the nil Hecke algebra $\mathrm{NH_n}$ under the sign involution on $\mathrm{NH_n}$. The definition of the sign involution on $\mathrm{NH_n}$ is based on the definition of the sign involution on symmetric groups.

In the first section, we discuss fundamental algebraic properties of alternating nil Hecke algebras. Then, in Section 4.2, we give a classification of their graded simple and graded projective indecomposable modules. Since the (graded) representation theory of $\mathrm{ANH_0} = k$ and $\mathrm{ANH_1} = k[y_1]$ with y_1 homogeneous of degree 4 is well-understood, let us now assume that $n \geq 2$. Let P_n be the polynomial representation of $\mathrm{NH_n}$ from Theorem 3.2.3. Via restriction P_n becomes an $\mathrm{ANH_n}$-module. Then we observe that it decomposes into a direct sum

$$\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(P_n) = P_n^{\mathrm{e}} \oplus P_n^{\mathrm{o}},$$

where P_n^{e} is the 'even' and P_n^{o} is the 'odd' part of P_n. We specify this in detail in Section 4.2. In Theorem 4.2.5, we then show that $P_n^{\mathrm{e}}, P_n^{\mathrm{o}}$ is a complete list of pairwise non-shift-isomorphic graded projective indecomposable $\mathrm{ANH_n}$-modules. From this, we obtain an analogous classification of the graded simple $\mathrm{ANH_n}$-modules.

Hereafter, we define the alternating nil Hecke Grothendieck groups $\mathrm{G_0(ANH)}$ and $\mathrm{K_0(ANH)}$ and study their algebraic structures. We define $\mathrm{G_0(ANH)}$ and $\mathrm{K_0(ANH)}$ analogously to the nil Hecke Grothendieck groups $\mathrm{G_0(NH)}$ and $\mathrm{K_0(NH)}$. Just as $\mathrm{G_0(NH)}$ and $\mathrm{K_0(NH)}$, we have that $\mathrm{G_0(ANH)}$ and $\mathrm{K_0(ANH)}$ admit both a multiplication and comultiplication which is given by induction and restriction functors. Moreover, we also have that $\mathrm{G_0(ANH)}$ is the $\mathbb{N}_0$-graded dual of $\mathrm{K_0(ANH)}$ as $\mathbb{N}_0$-graded algebra and $\mathbb{N}_0$-graded coalgebra over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. However, in contrast to $\mathrm{G_0(NH)}$ and $\mathrm{K_0(NH)}$, we show that $\mathrm{G_0(ANH)}$ and $\mathrm{K_0(ANH)}$ are no twisted bialgebras, see Propotsition 4.6.2.

Then, in Section 4.4, we use the techniques from Section 2.5 to show the following crucial results:

1. There is a non-degenerated $\mathbb{Q}(q)$-bilinear Euler forms χ_{K} resp. χ_{G} on $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K_0(ANH)}$ resp. $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G_0(ANH)}$, and

2. the graded character map induces an isomorphism of $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-vector spaces

$$\phi : \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K_0(ANH)} \to \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G_0(ANH)},$$

which is compatible with the multiplication and comultiplication. In addition, ϕ is also compatible with χ_{K} and χ_{G}.

Furthermore, we explicitly calculate the isomorphism ϕ and its inverse ϕ^{-1} in Theorem 4.4.9 and also provide formulas for the Euler forms χ_{K} and χ_{G} in Theorem 4.4.10.

Finally, in Section 4.5, we study in detail the multiplicative and comultiplicative structure of $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. At first, we give explicit formulas on appropriate bases. Then we relate these structures to Lusztig's integral quantum group ${}_{\mathcal{A}}\mathbf{f}$ corresponding to the one-vertex graph without edges.

In Theorem 4.5.9, we establish an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras between $\mathrm{K}_0(\mathrm{ANH})$ and ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*$ in degrees ≥ 2. Here, $\mathcal{A}[\mathbb{Z}/2]^*$ is the dual coalgebra of the group algebra $\mathcal{A}[\mathbb{Z}/2]$. By using the duality between $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$, we then construct an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras between $\mathrm{G}_0(\mathrm{ANH})$ and ${}_{\mathcal{A}}\mathbf{f}^* \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]$ in degrees ≥ 2, where ${}_{\mathcal{A}}\mathbf{f}^*$ is the $\mathbb{N}_0$-graded dual of ${}_{\mathcal{A}}\mathbf{f}$.

In order to describe the comultiplication on $\mathrm{G}_0(\mathrm{ANH})$ and the multiplication on $\mathrm{K}_0(\mathrm{ANH})$, we extend the scalars to $\mathcal{A}' := \mathcal{A}[\frac{1}{2}]$. Moreover, we introduce an $\mathbb{N}_0$-graded $\mathcal{A}$-algebra ${}_{\mathcal{A}}\tilde{\mathbf{f}}$, which is in a certain sense a sign perturbated version of ${}_{\mathcal{A}}\mathbf{f}$. Let ${}_{\mathcal{A}}\tilde{\mathbf{f}}^*$ denote the $\mathbb{N}_0$-graded dual $\mathcal{A}$-coalgebra of ${}_{\mathcal{A}}\tilde{\mathbf{f}}$. Then, in Theorem 4.5.16, we establish an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}'$-coalgebras between $\mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*)$ and $\mathcal{A}' \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH})$ in degrees ≥ 2. Again, using a duality argument, we conclude in Corollary 4.5.17 that we have an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}'$-algebras between $\mathcal{A}' \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH})$ and the direct product $\mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}})$ in degrees ≥ 2.

4.1 Alternating nil Hecke algebras

In this section, we describe important algebraic properties of alternating nil Hecke algebras. For this, we mostly follow [Boy14, Chapter 5]. The algebras are defined via the following involution.

Definition 4.1.1. *Let $n \in \mathbb{N}_0$. Then there exists a unique self-inverse graded k-algebra automorphism* $\mathrm{sgn} : \mathrm{NH_n} \to \mathrm{NH_n}$*, such that*

$$\tau_i \mapsto -\tau_i, \qquad y_j \mapsto -y_j,$$

for all $1 \leq i \leq n-1$, $1 \leq j \leq n$. The automorphism sgn *is called the sign involution on* $\mathrm{NH_n}$*.*

Using the description of the nil Hecke algebra $\mathrm{NH_n}$ by generators and relations, one can verify directly that sgn is well defined and also uniquely determined by the above properties.

Definition 4.1.2. *Let $n \in \mathbb{N}_0$. Then the alternating nil Hecke algebra* $\mathrm{ANH_n}$ *is defined to be the fixed point k-subalgebra* $\mathrm{ANH_n} := \mathrm{NH}_n^{\mathrm{sgn}} \subset \mathrm{NH_n}$.

Remark 4.1.3. Since sgn is a graded k-algebra automorphism, we have that $\mathrm{ANH_n}$ inherits a grading from $\mathrm{NH_n}$ that turns $\mathrm{ANH_n}$ into a graded k-algebra.

Now, let us consider basic examples of alternating nil Hecke algebras.

Example 4.1.4. (1) If $n = 0$, then $\mathrm{NH}_0 = k$ and by definition, $\mathrm{sgn} = \mathrm{id}_k$. So we also have $\mathrm{ANH}_0 = k$.

(2) If $n = 1$, then $\mathrm{NH}_1 = k[y_1]$ with y_1 homogeneous of degree 2. The sign involution is given by $y_1^{\mathrm{sgn}} = -y_1$, hence $\mathrm{ANH}_1 = k[y_1^2] \subset \mathrm{NH}_1$.

In the following let $n \in \mathbb{N}_0$ be fixed. In order to obtain first properties of the alternating nil Hecke algebra $\mathrm{ANH_n}$, we first describe the sign involution in more detail. For this, we fix some notation.

Notation 4.1.5. Let A be a graded k-algebra such that the graded components A_i vanish for all $i \in 1 + \mathbb{Z}$. We call a non-zero homogeneous element $a \in A$ of degree d *even* if $4|d$ and we call a *odd* if $4 \nmid d$. Moreover, we denote by $A^{\mathrm{e}} \subset A$ the *even part* of A, i.e. A^{e} is the graded k-subalgebra whose graded components are given by

$$A_i^{\mathrm{e}} = \begin{cases} A_i & \text{if } i \equiv 0 \mod 4, \\ 0 & \text{else.} \end{cases}$$

Furthermore, let $A^{\mathrm{o}} \subset A$ be *odd part* of A, i.e. A^{o} is the graded $(A^{\mathrm{e}}, A^{\mathrm{e}})$-bimodule whose graded components are given by

$$A_i^{\mathrm{o}} = \begin{cases} A_i & \text{if } i \equiv 2 \mod 4, \\ 0 & \text{else.} \end{cases}$$

Using this notation, the sign involution on $\mathrm{NH_n}$ can be described in the following way.

Proposition 4.1.6. *Let $a \in \mathrm{NH}_n$ be a homogeneous element. Then we have*

$$a^{\mathrm{sgn}} = \begin{cases} a & \text{if } a \text{ is even,} \\ -a & \text{if } a \text{ is odd.} \end{cases}$$

Proof. The proposition follows directly from the definition of the sign involution and the fact that the generators $\tau_1, \dots, \tau_{n-1}, y_1, \dots, y_n \in \mathrm{NH}_n$ are all odd. □

Hence, one might view $\mathrm{ANH_n}$ as thinning of $\mathrm{NH_n}$.

Proposition 4.1.7. *We have that* $\mathrm{ANH_n} = (\mathrm{NH_n})^{\mathrm{e}}$.

Next, we discuss a further useful interpretation of $\mathrm{ANH_n}$. For this, recall the isomorphism of graded k-algebras

$$\Phi : \mathrm{NH_n} \to \mathrm{END}_{\mathrm{Sym}_n}(P_n),$$

from Theorem 3.2.3. As in Theorem 3.2.3, we denote by P_n the graded polynomial algebra $k[x_1, \dots, x_n]$ with $x_1, \dots, x_n$ homogeneous of degree 2 and by Sym_n the graded k-subalgebra $P_n^{S_n} \subset P_n$. By Theorem 3.1.8, we know that P_n is a graded free Sym_n-module with a basis given by the Schubert polynomials $(\mathfrak{S}_w)_{w \in S_n}$. Hence, we can view $\mathrm{END}_{\mathrm{Sym}_n}(P_n)$ as a graded matrix algebra. For $v, w \in S_n$ let $E_{v,w} \in \mathrm{END}_{\mathrm{Sym}_n}(P_n)$ denote the elementary matrix given by

$$E_{v,w}(\mathfrak{S}_{w'}) = \begin{cases} \mathfrak{S}_v & \text{if } w = w', \\ 0 & \textit{else}, \end{cases}$$

for all $w' \in S_n$. Note that $E_{v,w}$ is homogeneous of degree $2(l(v) - l(w))$, where l denotes the length function on S_n. Now, using the description of $\mathrm{ANH_n}$ as a thinning of $\mathrm{NH_n}$, we immediately get a description of the image $\Phi_n(\mathrm{ANH_n})$.

Proposition 4.1.8. *Using the notation from above, we have that the underlying graded k-vector space of the graded k-subalgebra $\Phi_n(\mathrm{ANH_n}) \subset \mathrm{END}_{\mathrm{Sym}_n}(P_n)$ is spanned by the following elements:*

(i) $fE_{v,w}$*, for* $f \in \mathrm{Sym}_n^{\mathrm{e}}$ *and* $v, w \in S_n$ *such that* $2|(l(v) - l(w))$*,*

and the elements:

(ii) $fE_{v,w}$*, for* $f \in \mathrm{Sym}_n^{\mathrm{o}}$ *and* $v, w \in S_n$ *such that* $2 \nmid (l(v) - l(w))$*.*

In the following example, we illustrate the statement of Proposition 4.1.8 in the case $n = 2$.

Example 4.1.9. We consider the special case $\Phi_2(\mathrm{ANH}_2) \subset \mathrm{END}_{\mathrm{Sym}_2}(P_2)$. Let $S_2 = \{e, s\}$, where e is the neutral element and s is the transposition $(1, 2)$. We pick $(\mathfrak{S}_e, \mathfrak{S}_s)$ as ordered homogeneous Sym_2-basis of P_2 view $\mathrm{END}_{\mathrm{Sym}_2}(P_2)$ as graded matrix algebra with respect to this choice of basis. We then have by Proposition 4.1.8 that the elements of $\Phi_2(\mathrm{ANH}_2)$ are exactly the matrices of the following form

$$\Phi_2(\mathrm{ANH}_2) = \begin{pmatrix} \mathrm{Sym}_2^{\mathrm{e}} & \mathrm{Sym}_2^{\mathrm{o}} \\ \mathrm{Sym}_2^{\mathrm{o}} & \mathrm{Sym}_2^{\mathrm{e}} \end{pmatrix} \subset \mathrm{END}_{\mathrm{Sym}_2}(P_2).$$

We will usually identify algebras $\mathrm{ANH_n}$ and $\Phi_n(\mathrm{ANH_n})$. It turns out that the description of $\Phi_n(\mathrm{ANH_n})$ from Proposition 4.1.8 is very useful to describe the representation theory of $\mathrm{ANH_n}$. We will discuss this in detail in the following section.

We end this section by showing that $\mathrm{ANH_n}$ inherits some properties from $\mathrm{NH_n}$.

Proposition 4.1.10. *The following holds:*

(i) $\mathrm{ANH_n}$ *is Laurentian.*

(ii) $\mathrm{ANH_n}$ *has the homogeneous* k*-basis*

$$(\tau_w y_1^{m_1} \dots y_n^{m_n} | \, w \in S_n, m_1, \dots, m_n \in \mathbb{N}_0, 2|(l(w) + m_1 + \dots + m_n)).$$

(iii) Let $Z(\mathrm{ANH_n})$ *resp.* $Z(\mathrm{NH_n})$ *denote the centre of* $\mathrm{ANH_n}$ *resp.* $\mathrm{NH_n}$*. Recall that we can identify* $Z(\mathrm{NH_n})$ *with* Sym_n *by Theorem 3.2.3. Then we have*

$$Z(\mathrm{ANH_n}) = Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} = \mathrm{Sym}_n^{\mathrm{e}}.$$

Proof. (i) We know from Theorem 3.2.3 that $\mathrm{NH_n}$ is Laurentian. Since $\mathrm{ANH_n}$ is a graded k-subalgebra of $\mathrm{NH_n}$, it follows that also $\mathrm{ANH_n}$ is Laurentian.

(ii) This assertion follows immediately from Theorem 3.2.3 and the description of $\mathrm{ANH_n}$ as thinning of $\mathrm{NH_n}$.

(iii) The assertion is trivial for $n = 0$. So let us assume that $n \geq 1$. We clearly have $Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} \subset Z(\mathrm{ANH_n})$. For the converse inclusion let $z \in Z(\mathrm{ANH_n})$. At first, we show that z commutes with every odd element $x \in \mathrm{NH_n}$. For this, let $e_1 \in \mathrm{Sym}_n$ be the first elementary symmetric polynomial. Since $e_1 \in Z(\mathrm{NH_n})$ and $e_1 x \in \mathrm{ANH_n}$, we have the equality $e_1 xz = e_1 zx$. As $\mathrm{NH_n}$ is a graded free Sym_n module, the map

$$\mathrm{NH_n} \to \mathrm{NH_n}, \quad y \mapsto e_1 y, \quad \text{for all } y \in \mathrm{NH_n},$$

is injective. Hence, $e_1 xz = e_1 zx$ implies $xz = zx$. Thus, we proved that z commutes with every odd element in $\mathrm{NH_n}$. Since $z \in Z(\mathrm{ANH_n})$, we also have that z commutes with every even element of $\mathrm{NH_n}$. Thus, z also commutes with arbitrary sums of odd and even elements, which implies that z commutes with every element in $\mathrm{NH_n}$. This gives $z \in Z(\mathrm{NH_n})$ and hence the inclusion $Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} \subset Z(\mathrm{ANH_n})$. So, we proved $Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} = Z(\mathrm{ANH_n})$.

Finally, the equality $Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} = \mathrm{Sym}_n^{\mathrm{e}}$ follows directly from

$$\mathrm{Sym}_n^{\mathrm{e}} = \mathrm{Sym}_n \cap (\mathrm{NH_n})^{\mathrm{e}} = \mathrm{Sym} \cap \mathrm{ANH_n} = Z(\mathrm{NH_n}) \cap \mathrm{ANH_n} = Z(\mathrm{ANH_n}).$$

This completes the proof. □

We now describe an important property of the left and right ideals of $\mathrm{ANH_n}$. By the description of $\mathrm{ANH_n}$ as thinning of $\mathrm{NH_n}$, it follows that they are always contracted ideals from $\mathrm{NH_n}$. By this we mean the following.

Proposition 4.1.11. *Let I be a possibly ungraded left ideal of* $\mathrm{ANH_n}$. *Let $J \subset \mathrm{NH_n}$ be the left ideal generated by I. Then $I = J \cap \mathrm{ANH_n}$. The assertion remains true if we replace left ideal by right ideal.*

Proof. The inclusion $I \subset J \cap \mathrm{ANH_n}$ is clear. So let us show $J \cap \mathrm{ANH_n} \subset I$. Let $x \in J \cap \mathrm{ANH_n}$. We can write x as

$$x = \sum_{i=1}^{r} (a_i + b_i) y_i, \quad \text{where } a_i \in \mathrm{ANH_n}, b_i \in (\mathrm{NH_n})^{\mathrm{o}}, y_i \in I.$$

We have that

$$\Big(\sum_{i=1}^{r} a_i y_i\Big) \in \mathrm{ANH_n}, \quad \Big(\sum_{i=1}^{r} b_i y_i\Big) \in (\mathrm{NH_n})^{\mathrm{o}}.$$

Since $\mathrm{NH_n} = \mathrm{ANH_n} \oplus (\mathrm{NH_n})^{\mathrm{o}}$, the assumption $x \in \mathrm{ANH_n}$ implies $\sum_{i=1}^{r} b_i y_i = 0$. Thus, we have

$$x = \Big(\sum_{i=1}^{r} a_i y_i\Big) \in \mathrm{ANH_n}.$$

This proves the inclusion $J \cap \mathrm{ANH_n} \subset I$. Thus, we have $I = J \cap \mathrm{ANH_n}$. The same argument works also for right ideals. □

Corollary 4.1.12. $\mathrm{ANH_n}$ *is Noetherian as ungraded k-algebra.*

Proof. The assertion follows directly from the fact that $\mathrm{NH_n}$ is Noetherian as ungraded k-algebra and Proposition 4.1.11. □

In particular, it follows that $\mathrm{ANH_n}$ a graded Noetherian k-algebra.

Remark 4.1.13. The statement of Corollary 4.1.12 could also be deduced from general invariant theory. Let R be a ring and G be a finite group that acts by ring automorphisms on R. Let $R^G \subset R$ denote the fixed subring of G. Assume that $|G|$ is invertible in R. If R is left (or right) Noetherian, then so is R^G. For a proof of this result, see for instance [Mon80, Corollary 1.12].

Altogether, in this section, we have considered several algebraic properties of the alternating nil Hecke algebras. In the next section, we will use them to study their representation theory.

4.2 Graded modules over alternating nil Hecke algebras

As the alternating nil Hecke algebra $\mathrm{ANH_n}$ is Laurentian, we know by the results treated in the first chapter, that $\mathrm{ANH_n}$ admits many useful properties. In particular, $\mathrm{ANH_n}$ is graded semiperfect. So $\mathrm{ANH_n}$ admits only finitely many graded simple modules up to shift-isomorphism and every graded simple $\mathrm{ANH_n}$-module admits a projective cover. Moreover, all graded simple $\mathrm{ANH_n}$-modules have finite dimension over k. In this section, we classify the graded simple and graded projective indecompoable $\mathrm{ANH_n}$-modules. In addition, we outline interesting properties properties of these modules.

The cases $n = 0$ and $n = 1$ are clear. Hence, we make the following assumption.

Convention 4.2.1. Throughout this section, we fix $n \in \mathbb{N}_0$ with $n \geq 2$.

Using the notation from the previous section, recall from Theorem 3.2.4 that P_n is the unique graded projective indecomposable $\mathrm{NH_n}$-module up to shift-isomorphism. The underlying graded k-vector space of the unique (up to shift-isomorphism) graded simple $\mathrm{NH_n}$-module $L_n := P_n/(\mathrm{Sym}_n^+)P_n$ has a homogeneous basis given by the residue classes of Schubert polynomials $(\overline{\mathfrak{S}_w})_{w \in S_n}$ and $\mathrm{NH_n}$ acts on L_n via the epimorphism

$$\mathrm{NH_n} \twoheadrightarrow \mathrm{END}_{\mathrm{Sym}_n}(P_n)/((\mathrm{Sym}_n^+)\,\mathrm{END}_{\mathrm{Sym}_n}(P_n)) \cong \mathrm{END}_k(L_n).$$

Now, we pass to the alternating nil Hecke algebras.

Notation 4.2.2. Let M be a graded $\mathrm{ANH_n}$-module such that the graded components M_i vanish for all $i \in 1 + \mathbb{Z}$. We define the $M^{\mathrm{e}} \subset M$ to be the *even part* of M, i.e. M^{e} is the graded $\mathrm{ANH_n}$-module whose graded components are

$$M_i^{\mathrm{e}} = \begin{cases} M_i & \text{if } i \equiv 0 \mod 4, \\ 0 & \text{else.} \end{cases}$$

Likewise, we define $M^{\mathrm{o}} \subset M$ to be the *odd part* of M, i.e. M^{o} is the graded $\mathrm{ANH_n}$-module whose graded components are

$$M_i^{\mathrm{o}} = \begin{cases} M_i & \text{if } i \equiv 2 \mod 4, \\ 0 & \text{else.} \end{cases}$$

There is a decomposition of $\mathrm{ANH_n}$-modules $M = M^{\mathrm{e}} \oplus M^{\mathrm{o}}$.

If N is a graded $\mathrm{NH_n}$-module such that the graded components N_i vanish for all $i \in 1 + \mathbb{Z}$, then we have a decomposition of $\mathrm{ANH_n}$-modules

$$\mathrm{Res}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(N) = N^{\mathrm{e}} \oplus N^{\mathrm{o}},$$

where $N^{\mathrm{e}} := (\mathrm{Res}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(N))^{\mathrm{e}}$ and $N^{\mathrm{o}} := (\mathrm{Res}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(N))^{\mathrm{o}}$. As above, we call N^{e} the *even part* and N^{o} the *odd part* of N.

At this point, we warn the reader that this notation is not compatible with Notation 4.1.5.

The following lemma will be of great use in our study of $\mathrm{ANH_n}$.

Lemma 4.2.3. *Let M be a graded* $\mathrm{ANH_n}$*-module such that the graded components M_i vanish for all $i \in 1+\mathbb{Z}$. Let $N \subset M$ be a graded* $\mathrm{ANH_n}$*-submodule. Then we have an isomorphism of graded* $\mathrm{ANH_n}$*-modules*

$$F : M^{\mathrm{e}}/N^{\mathrm{e}} \to (M/N)^{\mathrm{e}}, \quad \overline{m} \mapsto \overline{m}, \quad \text{for all } \overline{m} \in M^{\mathrm{e}}/N^{\mathrm{e}}.$$

Similarly, we also have an isomorphism of graded $\mathrm{ANH_n}$*-modules*

$$G : M^{\mathrm{o}}/N^{\mathrm{o}} \to (M/N)^{\mathrm{o}}, \quad \overline{m} \mapsto \overline{m}, \quad \text{for all } \overline{m} \in M^{\mathrm{o}}/N^{\mathrm{o}}.$$

Proof. At first, note that

$$(M/N)_i = \{\overline{m} \in M/N | m \in M_i\}, \quad \text{for all } i \in \mathbb{Z}. \tag{4.1}$$

Hence, we conclude that F is indeed a well-defined homomorphism of graded $\mathrm{ANH_n}$-modules. From (4.1), we also deduce that F is surjective. In order to show the injectivity, let $m \in M^{\mathrm{e}}$ be homogeneous with $F(m) = \overline{m} = \overline{0} \in (M/N)^{\mathrm{e}}$. Hence, we have $m \in N$. This implies $m \in N \cap M^{\mathrm{e}} = N^{\mathrm{e}}$. So $\overline{m} = \overline{0} \in M^{\mathrm{e}}/N^{\mathrm{e}}$ and hence, F is injective.

The second assertion can be shown in exactly the same way. □

Using Notation 4.2.2, we have the following decompositions of $\mathrm{ANH_n}$-modules.

Lemma 4.2.4. *We have that* $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(P_n) = P_n^{\mathrm{e}} \oplus P_n^{\mathrm{o}}$ *and* $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(L_n) = L_n^{\mathrm{e}} \oplus L_n^{\mathrm{o}}$.

Proof. Recall from (3.17) that

$$\mathrm{grdim}(P_n) = (n)_{q^2}! \prod_{i=1}^{n} \frac{1}{1-q^{2i}}, \quad \mathrm{grdim}(L_n) = (n)_{q^2}!.$$

Here, we use the notion of quantum numbers from Definition 3.2.5. These formulas imply that $(P_n)_i = 0$ and $(L_n)_i = 0$ for $i \in 1+\mathbb{Z}$. Thus, we can take the even and the odd part of $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(P_n)$ and $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(L_n)$ as described in Notation 4.2.2. □

One can easily check that L_n^{e} and L_n^{o} are graded simple $\mathrm{ANH_n}$-modules. In fact, we will prove in the following theorem that P_n^{e} and P_n^{o} are exactly the unique graded projective indecomposable $\mathrm{ANH_n}$-modules up to shift-isomorphism. To see this, recall the notion of the elementary matrices $E_{v,w} \in \mathrm{END}_{\mathrm{Sym}_n}(P_n)$ for $v, w \in S_n$ from the previous section. In particular, note that for each $w \in S_n$, we have that $E_{w,w}$ is contained in $\mathrm{ANH_n}$.

Theorem 4.2.5. *The following assertions are true:*

(i) The graded $\mathrm{ANH_n}$*-modules P_n^{e} and P_n^{o} are graded projective indecomposable and non-shift-isomorphic.*

(ii) We have that $\mathrm{hd}(P^{\mathrm{e}}) \cong L^{\mathrm{e}}$ *and* $\mathrm{hd}(P^{\mathrm{o}}) \cong L^{\mathrm{o}}$.

(iii) Any graded projective indecomposable $\mathrm{ANH_n}$*-module is shift-isomorphic to P_n^{e} or P_n^{o}.*

(iv) Let $w \in S_n$, then $E_{w,w}$ is a homogeneous primitive idempotent in $\mathrm{ANH_n}$*. Moreover, we have*

$$\mathrm{ANH_n}\, E_{w,w} \cong \begin{cases} P_n^{\mathrm{e}}\langle -2l(w)\rangle & \text{if } 2|l(w), \\ P_n^{\mathrm{o}}\langle -2l(w)\rangle & \text{if } 2 \nmid l(w). \end{cases} \tag{4.2}$$

Proof. We begin with proving (iv). Since $E_{w,w}$ is a homogeneous primitive idempotent in $\mathrm{NH_n}$, it is also a homogeneous primitive idempotent in $\mathrm{ANH_n}$. Now, we prove the relation (4.2). For this, we only consider the case $2|l(w)$. The other case can be proved in the same way. In Section 3.2, we showed that there is an isomorphism $\mathrm{NH_n}\, E_{w,w} \cong P_n\langle -2l(w)\rangle$ of graded $\mathrm{NH_n}$-modules. Thus, we conclude

$$\mathrm{ANH_n}\, E_{w,w} \cong (\mathrm{NH_n}\, E_{w,w})^{\mathrm{e}} \cong (P_n\langle -2l(w)\rangle)^{\mathrm{e}} = P_n^{\mathrm{e}}\langle -2l(w)\rangle.$$

This gives (iv). Next, we show the remaining assertions. Since $\{E_{w,w}|\ w \in S_n\}$ is a complete set of homogeneous pairwise orthogonal primitive idempotents in $\mathrm{ANH_n}$, we obtain by (iv) that P_n^{e} and P_n^{o} are graded projective indecomposable and any graded projective indecomposable $\mathrm{ANH_n}$-module is shift-isomorphic to P_n^{e} or P_n^{o}. Hence, we obtain (iii). Now, since we have $\mathrm{hd}(P_n) \cong L_n$, Lemma 4.2.3 implies that L_n^{e} is a quotient of P_n^{e}. Since L_n^{e} is graded simple and P_n^{e} is graded projective indecomposable, we deduce that $\mathrm{hd}(P_n^{\mathrm{e}}) \cong L_n^{\mathrm{e}}$. The same argument gives $\mathrm{hd}(P_n^{\mathrm{o}}) \cong L_n^{\mathrm{o}}$. Thus, we proved (ii). To conclude (i), it is left to show that P_n^{e} and P_n^{o} are non-shift-isomorphic. For this, it suffices to show that L_n^{e} and L_n^{o} are non-shift-isomorphic. Now, note that L_n^{e} has a homogeneous k-basis given by the residue classes of Schubert polynomials $\overline{\mathfrak{S}_w}$, for $w \in S_n$ with $2|l(w)$. Similarly, L_n^{o} has a homogeneous k-basis given by the $\overline{\mathfrak{S}_w}$, for $w \in S_n$ with $2 \nmid l(w)$. Thus, it follows that for each $v \in S_n$ with $2|l(v)$, we have that $E_{v,v}$ operates on L_n^{e} by a non-zero homomorphism. However, $E_{v,v}L_n^{\mathrm{o}} = 0$, so L_n^{e} and L_n^{o} are non-shift-isomorphic. $\square$

As a direct consequence, we obtain a classification of the graded simple $\mathrm{ANH_n}$-modules.

Corollary 4.2.6. *The following assertions are true:*

(i) The graded $\mathrm{ANH_n}$*-modules* L_n^{e} *and* L_n^{o} *are graded simple and non-shift-isomorphic.*

(ii) Any graded simple $\mathrm{ANH_n}$*-module is shift-isomorphic to* L_n^{e} *or* L_n^{o}*.*

(iii) $\mathrm{ANH_n}$ *is graded Schurian.*

Proof. The assertions (i) and (ii) directly follow from Theorem 4.2.5. So let us prove (iii). We have to show that $\mathrm{END}_{\mathrm{ANH_n}}(L_n^{\mathrm{e}}) \cong k$ and $\mathrm{END}_{\mathrm{ANH_n}}(L_n^{\mathrm{o}}) \cong k$. In the following, we just prove the assertion for L_n^{e}. The assertion for L_n^{o} can be shown in the similarly. By construction, L_n^{e} has a homogeneous k-basis given by the residue classes of Schubert polynomials $\overline{\mathfrak{S}_w}$, for $w \in S_n$ with $2|l(w)$. Let $e \in S_n$ be the neutral element. Then by Theorem 4.2.5, we have $P_n^{\mathrm{e}} \cong \mathrm{ANH_n}\, E_{e,e}$ and hence

$$\mathrm{HOM}_A(P_n^{\mathrm{e}}, L_n^{\mathrm{e}}) \cong E_{e,e}L_n^{\mathrm{e}} = \mathrm{span}_k(\overline{\mathfrak{S}_e}).$$

Since P_n^{e} is the projective cover of L_n^{e}, we obtain $\mathrm{END}_{\mathrm{ANH_n}}(L_n^{\mathrm{e}}) \cong k$. $\square$

Corollary 4.2.7. *For each* $n \in \mathbb{N}_0$*, we have that* $\mathrm{NH_n}$ *is a finitely generated graded projective* $\mathrm{ANH_n}$*-module.*

In the following proposition, we consider the induction behavior of the graded projective indecomposable and the graded simple $\mathrm{ANH_n}$-modules. This behavior is analogous to the Clifford theory of group algebras of finite groups, see e.g. [FH13, Proposition 5.1].

Proposition 4.2.8. *There are isomorphisms of graded* $\mathrm{NH_n}$*-modules*

$$\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(P_n^{\mathrm{e}}) \cong \mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(P_n^{\mathrm{o}}) \cong P_n$$

and also

$$\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{e}}) \cong \mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{o}}) \cong L_n.$$

Proof. Let $e \in S_n$ be the neutral element. Since $P_n^{\mathrm{e}} \cong \mathrm{ANH_n}\, E_{e,e}$ we obtain

$$\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(P_n^{\mathrm{e}}) \cong \mathrm{NH_n}\, E_{e,e} \cong P_n.$$

The assertion $\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(P_n^{\mathrm{o}}) \cong P_n$ can be proved similarly. Next, we show $\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{e}}) \cong L_n$. To this end note that there is a homomorphism of graded $\mathrm{NH_n}$-modules

$$f : \mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{e}}) \to L_n, \quad \xi \otimes a \mapsto \xi a, \quad \text{for all } \xi \in \mathrm{NH_n}, a \in L_n^{\mathrm{e}}.$$

In order show that f is an isomorphism, we construct an inverse. For this, we fix an arbitrary simple transposition $s \in S_n$ and define the following homomorphism of graded k-vector spaces

$$g : L_n \to \mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{e}}), \quad \overline{\mathfrak{S}_w} \mapsto \begin{cases} 1 \otimes \overline{\mathfrak{S}_w} & \text{if } 2|l(w), \\ E_{w,ws} \otimes \overline{\mathfrak{S}_{ws}} & \text{if } 2 \nmid l(w), \end{cases}$$

for all $w \in S_n$. At first sight g is just a homomorphism of graded k-vector spaces. However, it is a straightforward exercise to check that g is the inverse of f. Thus, f and g are isomorphisms of graded $\mathrm{NH_n}$-modules. The fact $\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(L_n^{\mathrm{o}}) \cong L_n$ can be shown in the same way. □

Our next aim is to compute the graded dimension of the $\mathrm{ANH_n}$-modules $P_n^{\mathrm{e}}, P_n^{\mathrm{o}}, L_n^{\mathrm{e}}$ and L_n^{o}. In addition, we also determine the graded composition multiplicities of P_n^{e} and P_n^{o}. For this, we use that we already know the corresponding formulas for P_n and L_n, see (3.17) and (3.18).

Notation 4.2.9. Let $f = \sum_{i \gg -\infty} a_i q^i$ be a Laurent series with integer coefficients such that $a_i = 0$ for all $i \in 1 + 2\mathbb{Z}i$. Then we define the Laurent series

$$\mathrm{Even}(f) := \sum_{i \gg -\infty} a_{4i} q^{4i}, \quad \mathrm{Odd}(f) := \sum_{i \gg -\infty} a_{4i+2} q^{4i+2}.$$

Note that if f is a rational function, i.e. $f = g(q^2)$ for some $g(q) \in \mathbb{Q}(q)$, then also $\mathrm{Even}(f)$ and $\mathrm{Odd}(f)$ are contained in $\mathbb{Q}(q)$. This follows from the equations

$$\mathrm{Even}(f) = \frac{g(q^2) + g(-q^2)}{2}, \quad \mathrm{Odd}(f) = \frac{g(q^2) - g(-q^2)}{2}.$$

Now, recall from (3.17), that we have

$$\mathrm{grdim}(L_n) = (n)_{q^2}!, \quad \mathrm{grdim}(P_n) = (n)_{q^2}! \prod_{i=1}^{n} \frac{1}{1 - q^{2i}}.$$

By the construction of the $\mathrm{ANH_n}$-modules $P_n^{\mathrm{e}}, P_n^{\mathrm{o}}, L_n^{\mathrm{e}}$ and L_n^{o}, we hence obtain the following formulas for their graded dimensions.

Proposition 4.2.10. *We have*

$$\mathrm{grdim}(L_n^{\mathrm{e}}) = \mathrm{Even}((n)_{q^2}!), \quad \mathrm{grdim}(L_n^{\mathrm{o}}) = \mathrm{Odd}((n)_{q^2}!)$$

and

$$\mathrm{grdim}(P_n^{\mathrm{e}}) = \mathrm{Even}\Big((n)_{q^2}!\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big), \quad \mathrm{grdim}(P_n^{\mathrm{o}}) = \mathrm{Odd}\Big((n)_{q^2}!\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big).$$

Proof. Note that for each $i \in 1+2\mathbb{Z}$, the i-coefficients $(n)_{q^2}!$ and $(n)_{q^2}!\prod_{i=1}^{n}(1-q^{2i})^{-1}$ vanishes. Thus, we can apply Even and Odd to them. Now, by definition, the homogeneous components of L_n^{e} are given by

$$(L_n^{\mathrm{e}})_i = \begin{cases}(L_n)_i & \text{if } i \equiv 0 \mod 4,\\ 0 & \text{else,}\end{cases}$$

for all $i \in \mathbb{Z}$. Thus, $\mathrm{grdim}(L_n^{\mathrm{e}}) = \mathrm{Even}((n)_{q^2}!)$. The other graded dimensions can be determined in exactly the same way. $\square$

Proposition 4.2.11. *The graded composition multiplicities of P_n^{e} and P_n^{o} are given by*

$$[P_n^{\mathrm{e}} : L_n^{\mathrm{e}}]_q = \mathrm{Even}\Big(\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big), \quad [P_n^{\mathrm{e}} : L_n^{\mathrm{o}}]_q = \mathrm{Odd}\Big(\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big),$$

and

$$[P_n^{\mathrm{o}} : L_n^{\mathrm{e}}]_q = \mathrm{Odd}\Big(\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big), \quad [P_n^{\mathrm{o}} : L_n^{\mathrm{o}}]_q = \mathrm{Even}\Big(\prod_{i=1}^{n}\frac{1}{1-q^{2i}}\Big).$$

Note that all the above graded composition multiplicities are contained in $\mathbb{Q}(q)$.

Proof. We only determine graded composition multiplicities of P_n^{e}, since the graded composition multiplicities of P_n^{o} can be determined in the same way. At first, recall from (3.18) that

$$[P_n : L_n]_q = \prod_{i=1}^{n}\frac{1}{1-q^{2i}}.$$

Let $P_n = F_0 \supset F_1 \supset F_2 \supset \dots$ be a countable separated graded simple filtration of P_n with $F_i/F_{i+1} \cong L_n\langle d_i\rangle$ for some $d_i \in 2\mathbb{N}_0$. Now, recall from Lemma 4.2.3 that $F_i^{\mathrm{e}}/F_{i+1}^{\mathrm{e}} \cong (L_n\langle d_i\rangle)^{\mathrm{e}}$ for all $i \in \mathbb{N}_0$. Hence, we deduce that $P_n^{\mathrm{e}} = F_0^{\mathrm{e}} \supset F_1^{\mathrm{e}} \supset F_2^{\mathrm{e}} \supset \dots$ is a countable separated graded simple filtration of P_n^{e} and we have

$$F_i^{\mathrm{e}}/F_{i+1}^{\mathrm{e}} \cong (L_n\langle d_i\rangle)^{\mathrm{e}} \cong \begin{cases}L_n^{\mathrm{e}}\langle d_i\rangle & \text{if } i \equiv 0 \mod 4,\\ L_n^{\mathrm{o}}\langle d_i\rangle & \text{if } i \equiv 2 \mod 4.\end{cases}$$

This directly implies the stated formulas for the composition multiplicities of P_n^{e}. $\square$

We end this section with studying duality properties of the modules $P_n^{\mathrm{e}}, P_n^{\mathrm{o}}, L_n^{\mathrm{e}}$ and L_n^{o}. Our procedure is analogous as for the nil Hecke algebras in Section 3.2. At first, recall that in Definition 2.4.4, we defined dualities $\circledast$ and $\#$ on the categories A-fmod and A-pmod, where A is a Laurentian k algebra that admits a self-inverse graded anti-automorphism. Moreover, recall that $\mathrm{NH_n}$ admits a self-inverse graded anti-automorphism $\mathfrak{T} : \mathrm{NH}_n \to \mathrm{NH}_n$ given by

$$\tau_i \mapsto \tau_i, \quad y_i \mapsto y_i,$$

for all admissible i. Since $\mathfrak{T}$ is graded, we know that $\mathfrak{T}$ maps $\mathrm{ANH_n}$ onto $\mathrm{ANH_n}$. Hence, $\mathfrak{T}$ restricts to a self-inverse graded anti-automorphism of $\mathrm{ANH_n}$. Now, by applying Definition 2.4.4, we also obtain dualities

$$\circledast : \mathrm{ANH}_n\text{-fmod} \to \mathrm{ANH}_n\text{-fmod}, \qquad M^{\circledast} := \mathrm{HOM}_k(M, k),$$
$$\# : \mathrm{ANH}_n\text{-pmod} \to \mathrm{ANH}_n\text{-pmod}, \qquad P^{\#} := \mathrm{HOM}_{\mathrm{ANH}_n}(P, \mathrm{ANH}_n),$$

which are defined with respect to $\mathfrak{T}$.

In the following proposition, we describe how the duality $\circledast$ acts on the graded simple $\mathrm{ANH_n}$-modules and, equivalently, how $\#$ acts on the graded projective indecomposable $\mathrm{ANH_n}$-modules.

Proposition 4.2.12. *If* $4|(n(n-1))$*, then we have*

$$(L_n^{\mathrm{e}})^{\circledast} \cong L_n^{\mathrm{e}}\langle -n(n-1)\rangle, \qquad (L_n^{\mathrm{o}})^{\circledast} \cong L_n^{\mathrm{o}}\langle -n(n-1)\rangle,$$
$$(P_n^{\mathrm{e}})^{\#} \cong P_n^{\mathrm{e}}\langle -n(n-1)\rangle, \qquad (P_n^{\mathrm{o}})^{\#} \cong P_n^{\mathrm{o}}\langle -n(n-1)\rangle.$$

However, if $4 \nmid (n(n-1))$*, then we have*

$$(L_n^{\mathrm{e}})^{\circledast} \cong L_n^{\mathrm{o}}\langle -n(n-1)\rangle, \qquad (L_n^{\mathrm{e}})^{\circledast} \cong L_n^{\mathrm{o}}\langle -n(n-1)\rangle,$$
$$(P_n^{\mathrm{e}})^{\#} \cong P_n^{\mathrm{e}}\langle -n(n-1)\rangle, \qquad (P_n^{\mathrm{o}})^{\#} \cong P_n^{\mathrm{o}}\langle -n(n-1)\rangle.$$

Proof. We only prove the assertion for L_n^{e}. The assertion for L_n^{o} can be shown in the same way. The assertions for P_n^{e} and P_n^{o} then follow from Corollary 2.4.6. Now, note that we have a natural isomorphism of $\mathrm{ANH_n}$-modules

$$\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(L_n^{\circledast}) \cong (\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(L_n))^{\circledast}.$$

This implies in particular that $(L_n^{\circledast})^{\mathrm{e}} \cong (L_n^{\mathrm{e}})^{\circledast}$. In Section 3.2, we showed that

$$L_n^{\circledast} \cong L_n\langle -n(n-1)\rangle.$$

Hence, we deduce

$$(L_n^{\mathrm{e}})^{\circledast} \cong (L_n\langle -n(n-1)\rangle)^{\mathrm{e}} \cong \begin{cases} L_n^{\mathrm{e}}\langle -n(n-1)\rangle & \text{if } 4|(n(n-1)), \\ L_n^{\mathrm{o}}\langle -n(n-1)\rangle & \text{if } 4 \nmid (n(n-1)). \end{cases}$$

This proves the assertion for L_n^{e}. □

Similar as for the nil Hecke algebra, we can symmetrize the graded simple and graded projective indecomposable $\mathrm{ANH_n}$-modules, i.e. we set

$$T_n^{\mathrm{e}} := L_n^{\mathrm{e}}\langle -\frac{1}{2}n(n-1)\rangle, \quad T_n^{\mathrm{o}} := L_n^{\mathrm{o}}\langle -\frac{1}{2}n(n-1)\rangle$$

and

$$Q_n^{\mathrm{e}} := P_n^{\mathrm{e}}\langle -\frac{1}{2}n(n-1)\rangle, \quad Q_n^{\mathrm{o}} := P_n^{\mathrm{o}}\langle -\frac{1}{2}n(n-1)\rangle.$$

By Proposition 4.2.12, we have that if $4|(n(n-1))$, the duality $\circledast$ fixes T_n^{e} and T_n^{o}. Equivalently, the duality $\#$ fixes Q_n^{e} and Q_n^{o}. So, we have

$$(T_n^{\mathrm{e}})^{\circledast} \cong T_n^{\mathrm{e}}, \quad (T_n^{\mathrm{o}})^{\circledast} \cong T_n^{\mathrm{o}}, \quad (Q_n^{\mathrm{e}})^{\#} \cong Q_n^{\mathrm{e}}, \quad (Q_n^{\mathrm{o}})^{\#} \cong Q_n^{\mathrm{o}}. \tag{4.3}$$

However, if $4 \nmid (n(n-1))$, then we have

$$(T_n^{\mathrm{e}})^{\circledast} \cong T_n^{\mathrm{o}}, \quad (T_n^{\mathrm{o}})^{\circledast} \cong T_n^{\mathrm{e}}, \quad (Q_n^{\mathrm{e}})^{\#} \cong Q_n^{\mathrm{o}}, \quad (Q_n^{\mathrm{o}})^{\#} \cong Q_n^{\mathrm{e}}. \tag{4.4}$$

So in this case, the duality $\circledast$ switches T_n^{e} and T_n^{o}. Likewise, $\#$ switches Q_n^{e} and Q_n^{o}.

4.3 Alternating nil Hecke Grothendieck groups

The remaining sections of this chapter are devoted to the study of the alternating nil Hecke Grothendieck groups $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. The definition of $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ is modeled on the definition of the nil Hecke Grothendieck groups $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$ which were discussed in Section 3.3. In this section, we consider fundamental properties of $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ that are analogous to the properties $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$.

We start with some preparations. For $n \in \{0,1\}$, we have that $\mathrm{ANH_n}$ admits a unique (up to isomorphism) graded simple module that is concentrated in degree zero which we denote T^{e}_n. Let Q^{e}_n be the projective cover of T^{e}_n. Furthermore, let the dualities $\circledast$ and $\#$ be defined as in the case $n \geq 2$. One can then directly check that $(T^{\mathrm{e}}_n)^{\circledast} \cong T^{\mathrm{e}}_n$ and $(Q^{\mathrm{e}}_n)^{\#} \cong Q^{\mathrm{e}}_n$.

In the following, we use the notation that was introduced in Chapter 2. In particular, $\mathcal{A}$ denotes the ring $\mathbb{Z}[q,q^{-1}]$. In the next proposition, we list some important properties of the Grothendieck groups of alternating nil Hecke algebras.

Proposition 4.3.1. *The following assertions are true:*

(i) *We have that* $\mathrm{G}_0(\mathrm{ANH}_0\text{-fmod})$ *and* $\mathrm{G}_0(\mathrm{ANH}_1\text{-fmod})$ *are free $\mathcal{A}$-modules of rank* 1 *with generator* $[T^{\mathrm{e}}_0]$ *resp.* $[T^{\mathrm{e}}_1]$. *For $n \in \mathbb{N}_0$ with $n \geq 2$, we have that* $\mathrm{G}_0(\mathrm{ANH}_n\text{-fmod})$ *is a free $\mathcal{A}$-modules of rank* 2 *with generators* $[T^{\mathrm{e}}_n]$ *and* $[T^{\mathrm{o}}_n]$.

(ii) *We have that* $\mathrm{K}_0(\mathrm{ANH}_0\text{-pmod})$ *and* $\mathrm{K}_0(\mathrm{ANH}_1\text{-pmod})$ *are free $\mathcal{A}$-modules of rank* 1 *with generator* $[Q^{\mathrm{e}}_0]$ *resp.* $[Q^{\mathrm{e}}_1]$. *For $n \in \mathbb{N}_0$ with $n \geq 2$, we have that* $\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod})$ *is a free $\mathcal{A}$-modules of rank* 2 *with generators* $[Q^{\mathrm{e}}_n]$ *and* $[Q^{\mathrm{o}}_n]$.

(iii) *There exists a unique self-inverse $\mathcal{A}$-anti-linear automorphism*

$$\overline{}: \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}) \to \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}), \quad [M] \mapsto [M^{\circledast}],$$

for all $M \in \mathrm{ANH}_n\text{-fmod}$. *We call* $\overline{}$ *the bar involution on* $\mathrm{G}_0(\mathrm{ANH}_n\text{-fmod})$.

(iv) *There exists a unique self-inverse $\mathcal{A}$-anti-linear automorphism*

$$\overline{}: \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}) \to \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}), \quad [P] \mapsto [P^{\#}],$$

for all $P \in \mathrm{ANH}_n\text{-pmod}$. *We call* $\overline{}$ *the bar involution on* $\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod})$.

Proof. (i) This follows directly from Theorem 2.1.4.(i) and the classification of the graded simple $\mathrm{ANH_n}$-modules from Corollary 4.2.6.

(ii) Like in (i), the assertion (ii) follows from Theorem 2.1.4.(ii) and the classification of the graded projective indecomposable $\mathrm{ANH_n}$-modules from Theorem 4.2.5.

(iii) Since $\circledast$ is a duality on the category $\mathrm{ANH}_n\text{-fmod}$, we obtain that the bar involution on $\mathrm{G}_0(\mathrm{ANH}_n\text{-fmod})$ is a well-defined self-inverse additive map. The $\mathcal{A}$-anti-linearity then follows from

$$(M\langle d\rangle)^{\circledast} = \mathrm{HOM}_k(M\langle d\rangle, k) \cong (\mathrm{HOM}_k(M,k))\langle -d\rangle = (M^{\circledast})\langle -d\rangle,$$

for all $M \in \mathrm{ANH}_n\text{-fmod}, d \in \mathbb{Z}$.

(iv) We can use the same argument as in (ii). Since $\#$ is a duality on the category $\mathrm{ANH}_n\text{-pmod}$, the bar involution on $\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod})$ is a well-defined self-inverse additive map. The $\mathcal{A}$-anti-linearity is a consequence of

$$(P\langle d\rangle)^{\#} = \mathrm{HOM}_{\mathrm{ANH_n}}(P\langle d\rangle, \mathrm{ANH_n}) \cong (\mathrm{HOM}_{\mathrm{ANH_n}}(P, \mathrm{ANH_n}))\langle -d\rangle = (P^{\#})\langle -d\rangle,$$

for all $P \in \mathrm{ANH}_n\text{-pmod}, d \in \mathbb{Z}$. $\square$

Now, let $(.,.)_n : \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}) \times \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}) \to \mathcal{A}$ be the $\mathcal{A}$-bilinear HOM-pairing that was from Definition 2.4.7. So $(.,.)_n$ is given by

$$([P],[M])_n = \mathrm{grdim}(\mathrm{HOM}_{\mathrm{ANH_n}}(P^{\#}, M)),$$

for all $P \in \mathrm{ANH}_n\text{-pmod}$, $M \in \mathrm{ANH}_n\text{-fmod}$. Using the description of the action of the involution $\#$ on the graded projective indecomposable $\mathrm{ANH_n}$-modules from Proposition 4.2.12, we obtain an explicit description of $(.,.)_n$.

Proposition 4.3.2. *The following holds:*

(i) We have $([Q_0^{\mathrm{e}}],[T_0^{\mathrm{e}}])_0 = 1$.

(ii) We have $([Q_1^{\mathrm{e}}],[T_1^{\mathrm{e}}])_1 = 1$.

(iii) Let $n \in \mathbb{N}_0$ *with* $n \geq 2$ *and* $4|n(n-1)$. *Then we have*

$$([Q_n^{\mathrm{e}}],[T_n^{\mathrm{e}}])_n = 1, \qquad ([Q_n^{\mathrm{e}}],[T_n^{\mathrm{o}}])_n = 0,$$
$$([Q_n^{\mathrm{o}}],[T_n^{\mathrm{e}}])_n = 0, \qquad ([Q_n^{\mathrm{o}}],[T_n^{\mathrm{o}}])_n = 1.$$

(iv) Let $n \in \mathbb{N}_0$ *with* $n \geq 2$ *and* $4 \nmid n(n-1)$. *Then we have*

$$([Q_n^{\mathrm{e}}],[T_n^{\mathrm{e}}])_n = 0, \qquad ([Q_n^{\mathrm{e}}],[T_n^{\mathrm{o}}])_n = 1,$$
$$([Q_n^{\mathrm{o}}],[T_n^{\mathrm{e}}])_n = 1, \qquad ([Q_n^{\mathrm{o}}],[T_n^{\mathrm{o}}])_n = 0.$$

Proof. The assertions (i) and (ii) are clear. So let $n \in \mathbb{N}_0$ with $n \geq 2$. At first, we recall that $\mathrm{ANH_n}$ is graded Schurian by Corollary 4.2.6. Now, if $4|n(n-1)$, then we know by (4.3) that $(Q_n^{\mathrm{e}})^{\#} \cong Q_n^{\mathrm{e}}$ and $(Q_n^{\mathrm{o}})^{\#} \cong Q_n^{\mathrm{o}}$. Moreover, Q_n^{e} is the projective cover of T_n^{e} and Q_n^{o} is the projective cover of T_n^{o}. Thus, by (1.2), we obtain

$$([Q_n^{\mathrm{e}}],[T_n^{\mathrm{e}}])_n = \mathrm{grdim}\,\mathrm{HOM}_{\mathrm{ANH_n}}(Q_n^{\mathrm{e}}, T_n^{\mathrm{e}}) = [T_n^{\mathrm{e}} : T_n^{\mathrm{e}}]_q = 1.$$

The other formulas from (iii) can be shown in the same way. Likewise, if $4 \nmid n(n-1)$, then we know from (4.4) that $(Q_n^{\mathrm{e}})^{\#} \cong Q_n^{\mathrm{o}}$ and $(Q_n^{\mathrm{o}})^{\#} \cong Q_n^{\mathrm{e}}$. With the same argument as in the previous case, we conclude the formulas stated in (iv). ☐

Definition 4.3.3. *The alternating nil Hecke Grothendieck group* $\mathrm{G}_0(\mathrm{ANH})$ *and the split alternating nil Hecke Grothendieck group* $\mathrm{K}_0(\mathrm{ANH})$ *are defined as*

$$\mathrm{G}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}), \quad \mathrm{K}_0(\mathrm{NH}) := \bigoplus_{n\in\mathbb{N}_0} \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}).$$

By definition, $\mathrm{G}_0(\mathrm{ANH})$ *and* $\mathrm{K}_0(\mathrm{ANH})$ *both admit an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-module structure.*

The bar involutions on $\mathrm{G}_0(\mathrm{NH}_n\text{-fmod})$ and $\mathrm{K}_0(\mathrm{NH}_n\text{-pmod})$ give in a natural way an $\mathcal{A}$-anti-linear self-inverse automorphisms

$$^{-} : \mathrm{G}_0(\mathrm{ANH}) \to \mathrm{G}_0(\mathrm{ANH}), \quad ^{-} : \mathrm{K}_0(\mathrm{ANH}) \to \mathrm{K}_0(\mathrm{ANH}),$$

which we call the *bar involutions* on $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$. In addition, we can also extend the bilinear HOM-pairing to the alternating nil Hecke Grothendieck groups as follows. Let

$$(.,.) : \mathrm{K}_0(\mathrm{ANH}) \times \mathrm{G}_0(\mathrm{ANH}) \to \mathcal{A}$$

be the unique $\mathcal{A}$-bilinear pairing such that

$$([P],[M]) = \begin{cases} ([P],[M])_m & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

holds for all $P \in \mathrm{ANH}_m$-pmod, $M \in \mathrm{ANH}_n$-fmod. We call $(.,.)$ the *bilinear* HOM-*pairing* between $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$.

Next, we show that the alternating nil Hecke Grothendieck groups $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ both admit multiplicative and comultiplicative structure. For this, we proceed as for the nil Hecke Grothendieck groups. At first, recall the inclusions

$$\iota_{m,n} : \mathrm{NH}_m \otimes_k \mathrm{NH_n} \hookrightarrow \mathrm{NH}_m \otimes_k \mathrm{NH}_n$$

from Proposition 3.3.3. Since $\iota_{m,n}$ is a graded inclusion, we deduce

$$\iota_{m,n}(\mathrm{ANH}_m \otimes_k \mathrm{ANH}_n) \subset \mathrm{ANH}_{m+n}\,.$$

Thus, via $\iota_{m,n}$, we obtain inclusions of graded algebras $\mathrm{ANH}_m \otimes_k \mathrm{ANH_n} \hookrightarrow \mathrm{ANH}_{m+n}$. In the following let

$$\mathrm{Ind}_{m,n}^{m+n} : (\mathrm{ANH}_m \otimes_k \mathrm{ANH}_n)\text{-Mod} \to \mathrm{ANH}_{m+n}\text{-Mod},$$
$$\mathrm{Res}_{m,n}^{m+n} : \mathrm{ANH}_{m+n}\text{-Mod} \to (\mathrm{ANH}_m \otimes_k \mathrm{ANH}_n)\text{-Mod}$$

denote the induction and restriction functors corresponding to this inclusion. To see that these functors give well-defined maps on the alternating nil Hecke Grothendieck groups, we use the following general proposition.

Proposition 4.3.4. *Let $m, n \in \mathbb{N}_0$. Then the following holds:*

(i) $\mathrm{NH_n}$ *is a finitely generated graded projective left* $\mathrm{ANH_n}$*-module,*

(ii) $\mathrm{NH_n}$ *is a finitely generated graded projective right* $\mathrm{ANH_n}$*-module,*

(iii) ANH_{m+n} *is a finitely generated graded projective left* $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$*-module,*

(iv) ANH_{m+n} *is a finitely generated graded projective right* $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$*-module.*

Proof. The assertion (i) was already shown in Corollary 4.2.7. We now use this result to prove (ii). For this, let N be $\mathrm{NH_n}$ viewed as left $\mathrm{ANH_n}$-module and N' be $\mathrm{NH_n}$ viewed as right $\mathrm{ANH_n}$-module. Let $N^{\mathfrak{T}}$ be the right $\mathrm{ANH_n}$-module obtained from N by twisting with $\mathfrak{T}$. Then (i) implies that $N^{\mathfrak{T}}$ is a finitely generated graded projective $\mathrm{ANH_n}$ right-module. Moreover, we have an isomorphism of right $\mathrm{ANH_n}$-modules

$$N^{\mathfrak{T}} \to N', \quad x \mapsto \mathfrak{T}(x), \quad \text{for all } x \in N^{\mathfrak{T}}.$$

Thus, N' is a finitely generated graded projective right $\mathrm{ANH_n}$-module.

Next we prove (iii). According to (i), $\mathrm{NH}_m \otimes_k \mathrm{NH}_n$ is a finitely generated graded projective left $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$-module. By Lemma 3.3.4.(i), we know that NH_{m+n} is a graded free left $(\mathrm{NH}_m \otimes_k \mathrm{NH}_n)$-module of finite rank. Hence, NH_{m+n} is a finitely generated graded projective left $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$-module. Finally, since ANH_{m+n} is a direct summand of NH_{m+n} as left $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$-module, we obtain that ANH_{m+n} is a finitely generated graded projective left $(\mathrm{ANH}_m \otimes_k \mathrm{ANH_n})$-module.

The proof of (iv) is completely analogous to the proof of (iii). □

Just like in the case of the nil Hecke Grothendieck groups which we considered in Theorem 3.3.5, we obtain via the above induction and restriction functors multiplicative and comultiplicative structures on $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. For this, recall from Proposition 2.1.6 the isomorphisms of $\mathcal{A}$-modules

$$\mathrm{G}_0(\mathrm{NH}_m\text{-fmod}) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}) \cong \mathrm{G}_0((\mathrm{ANH}_m \otimes_k \mathrm{ANH}_n)\text{-fmod}),$$
$$\mathrm{K}_0(\mathrm{NH}_m\text{-pmod}) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}) \cong \mathrm{K}_0((\mathrm{ANH}_m \otimes_k \mathrm{ANH}_n)\text{-pmod}).$$

In the following, we will always identify these Grothendieck groups via the isomorphisms from Proposition 2.1.6.

Theorem 4.3.5. *The following assertions are true:*

(i) We have that $\mathrm{G}_0(\mathrm{ANH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebra, with unit* $[T_0^{\mathrm{e}}]$ *and multiplication*

$$[M]\cdot[N] = [\mathrm{Ind}_{m,n}^{m+n}(M \otimes_k N)],$$

for all $m, n \in \mathbb{N}_0, M \in \mathrm{ANH}_m\text{-fmod}, N \in \mathrm{ANH}_n\text{-fmod}$*. Similarly,* $\mathrm{K}_0(\mathrm{ANH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebra, with unit* $[Q_0^{\mathrm{e}}]$ *and multiplication*

$$[P]\cdot[Q] = [\mathrm{Ind}_{m,n}^{m+n}(P \otimes_k Q)],$$

for all $m, n \in \mathbb{N}_0, P \in \mathrm{ANH}_m\text{-pmod}, Q \in \mathrm{ANH}_n\text{-pmod}$*.*

(ii) We have that $\mathrm{G}_0(\mathrm{ANH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebra, where the counit is the projection to* $\mathrm{G}_0(\mathrm{ANH}_0\text{-fmod})$ *and the comultiplication is*

$$\Delta_{\mathrm{G}}([M]) = \sum_{r=0}^{n} [\mathrm{Res}_{n,n-r}^{n}(M)], \quad \textit{for all } n \in \mathbb{N}_0, M \in \mathrm{ANH}_n\text{-fmod}.$$

Analogously, $\mathrm{K}_0(\mathrm{ANH})$ *admits the structure of an* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebra, where counit is the projection to* $\mathrm{K}_0(\mathrm{ANH}_0\text{-pmod})$ *and the comultiplication is*

$$\Delta_{\mathrm{K}}([P]) = \sum_{r=0}^{n} [\mathrm{Res}_{r,n-r}^{n}(P)], \quad \textit{for all } n \in \mathbb{N}_0, P \in \mathrm{ANH}_n\text{-pmod}.$$

Proof. From Proposition 4.3.4.(iv) and Proposition 2.1.6, we obtain that the above multiplication and comultiplication maps are well-defined. The associativity and coassociativity follows from the associativity of induction, restriction and outer tensor products. □

In the following let $\mathrm{K}_0(\mathrm{ANH})^*$ be the $\mathbb{N}_0$-graded dual of $\mathrm{K}_0(\mathrm{ANH})$, see Definition 3.3.7. So $\mathrm{K}_0(\mathrm{ANH})^*$ is the $\mathbb{N}_0$-graded $\mathcal{A}$-module

$$\mathrm{K}_0(\mathrm{ANH})^* = \bigoplus_{n\in\mathbb{N}_0} \mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}), \mathcal{A})$$

with $\mathbb{N}_0$-graded $\mathcal{A}$-algebra and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebra structure as defined in Definition 3.3.7. Likewise, let $\mathrm{G}_0(\mathrm{ANH})^*$ be the $\mathbb{N}_0$-graded dual of $\mathrm{G}_0(\mathrm{ANH})$. In Theorem 3.3.8, we described a duality between $\mathrm{G}_0(\mathrm{NH})$ and $\mathrm{K}_0(\mathrm{NH})$. Using the same argument as in Theorem 3.3.8, we deduce the following duality $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$.

Theorem 4.3.6. *The following assertions are true:*

(i) For each $n \in \mathbb{N}_0$, let

$$f_n : \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}), \mathcal{A})$$

be the homomorphism of $\mathcal{A}$-modules given by $[M] \mapsto (.,[M])_n$, for all $M \in \mathrm{ANH_n}\text{-fmod}$. Then

$$f := \bigoplus_{n\in\mathbb{N}_0} f_n : \mathrm{G}_0(\mathrm{ANH}) \to \mathrm{K}_0(\mathrm{ANH})^*$$

is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras.

(ii) For each $n \in \mathbb{N}_0$, let

$$g_n : \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}), \mathcal{A})$$

be the homomorphism of $\mathcal{A}$-modules given by $[P] \mapsto ([P],.)_n$, for all $P \in \mathrm{ANH_n}\text{-pmod}$. Then

$$g := \bigoplus_{n\in\mathbb{N}_0} g_n : \mathrm{K}_0(\mathrm{ANH}) \to \mathrm{G}_0(\mathrm{ANH})^*$$

is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras.

Proof. We only show (i), since (ii) can be shown in exactly the same way. With exactly the same argument as in Theorem 3.3.8, one can deduce from the compatibility of the HOM-pairing with induction, restriction and outer tensor products, that f is a homomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras. By Proposition 4.3.2, we know that $((.,[T_n^{\mathrm{e}}])_n, (.,T_n^{\mathrm{o}})_n)$ is an $\mathcal{A}$-basis of $\mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}), \mathcal{A})$ for each $n \geq 2$. If $n \in \{0,1\}$, then $\mathrm{Hom}_{\mathcal{A}}(\mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}), \mathcal{A})$ is free of rank 1 with generator $(.,[T_n^{\mathrm{e}}])_n$. Thus, from Proposition 4.3.1, it follows that f maps an $\mathcal{A}$-basis of $\mathrm{G}_0(\mathrm{ANH})$ to an $\mathcal{A}$-basis of $\mathrm{K}_0(\mathrm{ANH})^*$. This implies that f is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras. □

We end this section by observing that induction resp. restriction give algebra resp. coalgebra homomorphisms between the alternating nil Hecke Grothendieck groups and the nil Hecke Grothendieck groups.

Theorem 4.3.7. *The following assertions are true:*

(i) There exists a unique homomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras

$$\mathrm{I_G} : \mathrm{G}_0(\mathrm{ANH}) \to \mathrm{G}_0(\mathrm{NH}),$$

such that

$$[M] \mapsto [\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(M)],$$

for all $n \in \mathbb{N}_0, M \in \mathrm{ANH}_n\text{-fmod}$. Similarly, there exists a unique homomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras

$$\mathrm{I_K} : \mathrm{K}_0(\mathrm{ANH}) \to \mathrm{K}_0(\mathrm{NH}),$$

such that

$$[P] \mapsto [\mathrm{Ind}_{\mathrm{ANH_n}}^{\mathrm{NH_n}}(M)],$$

for all $n \in \mathbb{N}_0, P \in \mathrm{ANH}_n\text{-pmod}$.

(ii) There exists a unique homomorphism of $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebras*

$$\mathrm{R_G} : \mathrm{G}_0(\mathrm{NH}) \to \mathrm{G}_0(\mathrm{ANH}),$$

such that

$$[M] \mapsto [\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(M)],$$

for all $n \in \mathbb{N}_0, M \in \mathrm{NH}_n$ -fmod. *Analogously, there exists a unique homomorphism of* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebras*

$$\mathrm{R_K} : \mathrm{K}_0(\mathrm{NH}) \to \mathrm{K}_0(\mathrm{ANH}),$$

such that

$$[P] \mapsto [\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(P)],$$

for all $n \in \mathbb{N}_0, P \in \mathrm{NH}_n$ -pmod.

Moreover, under the identifications

$$\mathrm{G}_0(\mathrm{ANH}) \cong \mathrm{K}_0(\mathrm{ANH})^*, \qquad \mathrm{K}_0(\mathrm{ANH}) \cong \mathrm{G}_0(\mathrm{ANH})^*,$$
$$\mathrm{G}_0(\mathrm{NH}) \cong \mathrm{K}_0(\mathrm{NH})^*, \qquad \mathrm{K}_0(\mathrm{NH}) \cong \mathrm{G}_0(\mathrm{NH})^*,$$

from Theorem 4.3.6 and Theorem 3.3.8, we have that $\mathrm{R_G}$ *is the adjoint map of* $\mathrm{I_K}$*, vice versa,* $\mathrm{I_K}$ *is the adjoint of* $\mathrm{R_G}$*. Moreover,* $\mathrm{I_G}$ *is the adjoint map of* $\mathrm{R_K}$ *and vice versa,* $\mathrm{R_K}$ *is the adjoint of* $\mathrm{I_G}$*.*

Proof. By Proposition 4.3.4, we know that $\mathrm{NH_n}$ is a finitely generated graded projective right $\mathrm{ANH_n}$-module for each $n \in \mathbb{N}_0$. This implies that the maps $\mathrm{I_G}$, $\mathrm{I_K}$, $\mathrm{R_G}$ and $\mathrm{R_K}$ are well-defined. From the transitivity of induction, we obtain that $\mathrm{I_G}$ and $\mathrm{I_K}$ are compatible with the multiplicative structures. Likewise, the transitivity of restriction implies that $\mathrm{R_G}$ and $\mathrm{R_K}$ are compatible with the comultiplicative structures. Next, we show that $\mathrm{R_G}$ is the adjoint map of $\mathrm{I_K}$. From Proposition 2.4.9, we know that

$$([\mathrm{Ind}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(P)], [M]) = ([P], [\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH_n}}(M)]),$$

for all $P \in \mathrm{ANH}_n$ -pmod, $M \in \mathrm{NH}_n$ -fmod. Hence, we obtain

$$(., [M]) \circ \mathrm{I_K} = (., \mathrm{R_G}([M])).$$

According to the identifications $\mathrm{G}_0(\mathrm{ANH}) \cong \mathrm{K}_0(\mathrm{ANH})^*$ and $\mathrm{G}_0(\mathrm{NH}) \cong \mathrm{K}_0(\mathrm{NH})^*$ from Theorem 4.3.6 and Theorem 3.3.8, we conclude that $\mathrm{R_G}$ is the adjoint map of $\mathrm{I_K}$. The remaining adjunctions can be shown in exactly the same way. □

4.4 Rational alternating nil Hecke Grothendieck groups

In this section, we study the rational alternating nil Hecke Grothendieck groups $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ which we obtain by scalar extension to $\mathbb{Q}(q)$. In particular, we apply the results from Section 2.5 to $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

Definition 4.4.1. *The rational alternating nil Hecke Grothendieck group* $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ *and the rational split alternating nil Hecke Grothendieck group* $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ *are defined as*

$$\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)} := \bigoplus_{n \in \mathbb{N}_0} \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod}),$$
$$\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)} := \bigoplus_{n \in \mathbb{N}_0} \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}).$$

By definition, $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ are $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-vector spaces. In addition, Theorem 4.3.5 implies that both $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ are $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-algebras and $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-coalgebras. We also have that the bar involutions extend to $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$. Likewise, the $\mathcal{A}$-bilinear HOM-pairing between $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$ extends to a $\mathbb{Q}(q)$-bilinear pairing $(.,.)$ between $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

In order to apply the results from Section 2.5 to $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$, we have to ensure that the conditions that were formulated in Convention 2.5.1 are satisfied. According to our discussion from Section 4.1 and Section 4.2, it is only left to show that $\mathrm{ANH_n}$ has finite global dimension for all $n \in \mathbb{N}_0$. This follows from the following proposition.

Proposition 4.4.2. *For each $n \in \mathbb{N}_0$, we have* $\mathrm{gl}(\mathrm{ANH_n}) = n$.

Proof. The assertion is clear for $n = 0, 1$. So in the following, we assume that $n \geq 2$. From (3.20), we know that $\mathrm{gl}(\mathrm{NH_n}) = n$. By Theorem 1.5.4, we can infer that

$$\max\{i \in \mathbb{N}_0 |\, \mathrm{EXT}^i_{\mathrm{NH_n}}(L_n, L_n) \neq 0\} = n.$$

Using generalized Frobenius reciprocity and Proposition 4.2.8, we hence obtain

$$\mathrm{EXT}^i_{\mathrm{ANH_n}}(L^{\mathrm{e}}_n, L^{\mathrm{e}}_n \oplus L^{\mathrm{o}}_n) \cong \mathrm{EXT}^i_{\mathrm{ANH_n}}(L^{\mathrm{o}}_n, L^{\mathrm{e}}_n \oplus L^{\mathrm{o}}_n) \cong \mathrm{EXT}^i_{\mathrm{NH_n}}(L_n, L_n) = 0,$$

for all $i \geq n+1$. Since $L^{\mathrm{e}}_n, L^{\mathrm{o}}_n$ are the unique graded simple $\mathrm{ANH_n}$-modules up to shift-isomorphism, we deduce

$$\max\{i \in \mathbb{N}_0 |\, \mathrm{EXT}^i_{\mathrm{ANH_n}}(S, T) \neq 0, \text{ where } S, T \text{ are graded simple } \mathrm{ANH_n}\text{-modules}\} \leq n.$$

Hence, by Theorem 1.5.4, we conclude that $\mathrm{gl}(\mathrm{ANH_n}) \leq n$. Finally, from

$$\mathrm{EXT}^n_{\mathrm{ANH_n}}(L^{\mathrm{e}}_n, L^{\mathrm{e}}_n) \oplus \mathrm{EXT}^n_{\mathrm{ANH_n}}(L^{\mathrm{e}}_n, L^{\mathrm{o}}_n) \cong \mathrm{EXT}^n_{\mathrm{ANH_n}}(L^{\mathrm{e}}_n, L^{\mathrm{e}}_n \oplus L^{\mathrm{o}}_n) \cong \mathrm{EXT}^n_{\mathrm{NH_n}}(L_n, L_n) \neq 0,$$

we obtain that $\mathrm{gl}(\mathrm{ANH_n}) = n$. □

Next, we note that by Proposition 4.2.11, all graded composition multiplicities of finitely generated graded projective $\mathrm{ANH_n}$-modules are contained in $\mathbb{Q}(q)$. Hence, we can apply the rational versions of the results from Section 2.5.

We begin with defining $\mathbb{Q}(q)$-bilinear Euler forms on $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$. The definition is completely analogous to the corresponding definition of the $\mathbb{Q}(q)$-bilinear Euler forms on $\mathrm{G}_0(\mathrm{NH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$, see Definition 3.3.10.

Definition 4.4.3. *We define the following:*

(i) For $n \in \mathbb{N}_0$ let $\chi_{\mathrm{K},n}$ be the rational bilinear Euler form on $\mathrm{G}_0(\mathrm{ANH}_n\text{-fmod})$ *from Definition 2.5.18.(i). So $\chi_{\mathrm{K},n}$ is the unique $\mathbb{Q}(q)$-bilinear form such that*

$$\chi_{\mathrm{K},n}(f \otimes [M], g \otimes [N]) = fg \sum_{i=0}^{n} (-1)^i\, \overline{\mathrm{grdim}(\mathrm{EXT}^i_{\mathrm{ANH_n}}(M, N^{\circledast}))},$$

for all $f, g \in \mathbb{Q}(q)$ and $M, N \in \mathrm{ANH}_n\text{-fmod}$.

(ii) Let χ_{G} be the unique $\mathbb{Q}(q)$-bilinear form on $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ *such that*

$$\chi_{\mathrm{G}}(f \otimes [M], g \otimes [N]) = \begin{cases} \chi_{\mathrm{K},n}(f \otimes [M], g \otimes [N]) & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

holds for all $f, g \in \mathbb{Q}(q), M \in \mathrm{ANH}_m$-fmod and $N \in \mathrm{ANH}_n$-fmod. We call χ_{G} the $\mathbb{Q}(q)$-bilinear Euler form on $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

(iii) For $n \in \mathbb{N}_0$ let $\chi_{\mathrm{K},n}$ be the rational bilinear Euler form on $\mathrm{K}_0(\mathrm{ANH}_n$-pmod$)$ from Definition 2.5.18.(ii). So $\chi_{\mathrm{K},n}$ is the unique $\mathbb{Q}(q)$-bilinear form such that

$$\chi_{\mathrm{p},n}(f \otimes [P], g \otimes [Q]) = fg \ \operatorname{grdim}(\mathrm{HOM}_{\mathrm{ANH_n}}(P^{\#}, Q)),$$

for all $f, g \in \mathbb{Q}(q)$ and $P, Q \in \mathrm{ANH}_n$-pmod.

(iv) Let χ_{K} be the unique $\mathbb{Q}(q)$-bilinear form on $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ such that

$$\chi_{\mathrm{K}}(f \otimes [P], g \otimes [Q]) = \begin{cases} \chi_{\mathrm{K},n}(f \otimes [P], g \otimes [Q]) & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

holds for all $f, g \in \mathbb{Q}(q), P \in \mathrm{ANH}_m$-pmod and $Q \in \mathrm{ANH}_n$-pmod. We call χ_{K} the $\mathbb{Q}(q)$-bilinear Euler form on $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

We proceed with applying Theorem 2.5.19 to $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ and $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

Theorem 4.4.4. *For each $n \in \mathbb{N}_0$, let*

$$\phi_n : \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH}_n\text{-pmod}) \to \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH}_n\text{-fmod})$$

be the unique $\mathbb{Q}(q)$-linear map such that

$$f \otimes [P] \mapsto f \cdot \operatorname{gch}([P]), \quad \text{for all } f \in \mathbb{Q}(q), P \in \mathrm{ANH}_n\text{-pmod}.$$

Here, gch *denotes the graded character map from Definition 2.2.2. Let*

$$\phi := \bigoplus_{n \in \mathbb{N}_0} \phi_n : \mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)} \to \mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}.$$

Then ϕ is an isomorphism of $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-algebras and $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-coalgebras. Furthermore, the following holds:

(i) We have $\chi_{\mathrm{K}}(x, y) = \chi_{\mathrm{G}}(\phi(x), \phi(y))$, for all $x, y \in \mathrm{K}_0(\mathrm{NH})_{\mathbb{Q}(q)}$.

(ii) The $\mathbb{Q}(q)$-bilinear Euler forms χ_{G} and χ_{K} are non-degenerate and $\chi_{\mathrm{G}}(\phi(.), .) = (., .)$.

Proof. The fact that ϕ is a homomorphism of $\mathbb{N}_0$-graded $\mathbb{Q}(q)$-algebras and $\mathbb{N}_0$-graded coalgebras follows directly from the compatibility of the graded character map with induction, restriction and outer tensor products, see Corollary 2.3.6 and Corollary 2.2.13. The remaining assertions follow immediately from Theorem 2.5.19. □

From Theorem 4.4.4.(ii), we directly obtain the following consequence.

Corollary 4.4.5. *We have that*

$$\{1 \otimes [T_0^{\mathrm{e}}], 1 \otimes [T_1^{\mathrm{e}}]\} \cup \{1 \otimes [T_n^{\mathrm{e}}], 1 \otimes [T_n^{\mathrm{o}}] | n \in \mathbb{N}_0, n \geq 2\}$$

and

$$\{\phi(1 \otimes [Q_0^{\mathrm{e}}]), \phi(1 \otimes [Q_1^{\mathrm{e}}])\} \cup \{\phi(1 \otimes [(Q_n^{\mathrm{e}})^{\#}]), \phi(1 \otimes [(Q_n^{\mathrm{o}})^{\#}]) | n \in \mathbb{N}_0, n \geq 2\}$$

are dual homogeneous $\mathbb{Q}(q)$-bases of $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ with respect to χ_{G}.

Moreover, using exactly the same arguments as in Corollary 3.3.11 and Proposition 3.3.13, we deduce the following properties of χ_{G} and χ_{K}.

Corollary 4.4.6. *The following holds:*

(i) The Euler forms χ_{G} and χ_{K} are symmetric.

(ii) For all $x, y, z \in \mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$, we have

$$\chi_{\mathrm{G}}(x \cdot y, z) = \chi_{\mathrm{G}}(x \otimes y, \Delta_{\mathrm{G}}(z)),$$

where the $\mathbb{Q}(q)$-bilinear form χ_{G} on $\mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ is given by

$$\chi_{\mathrm{G}}(a \otimes b, c \otimes d) = \chi_{\mathrm{G}}(a, c)\chi_{\mathrm{G}}(b, d),$$

for all $a, b, c, d \in \mathrm{G}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

(iii) Likewise, for all $x, y, z \in \mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$, we have

$$\chi_{\mathrm{K}}(x \cdot y, z) = \chi_{\mathrm{K}}(x \otimes y, \Delta_{\mathrm{K}}(z)),$$

where the $\mathbb{Q}(q)$-bilinear form χ_{K} on $\mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$ is given by

$$\chi_{\mathrm{K}}(a \otimes b, c \otimes d) = \chi_{\mathrm{K}}(a, c)\chi_{\mathrm{K}}(b, d),$$

for all $a, b, c, d \in \mathrm{K}_0(\mathrm{ANH})_{\mathbb{Q}(q)}$.

In the following, we work out an explicit description of ϕ and the inverse map ϕ^{-1}. Furthermore, we also determine explicit formulas for the bilinear Euler forms χ_{G} and χ_{K}.

Notation 4.4.7. Let $n \in \mathbb{N}_0$. Then we define the following elements of $\mathbb{Q}(q)$:

$$H_n(q) := \prod_{i=1}^{n} \frac{1}{1 - q^{2i}}, \quad F_n(q) := \prod_{i=1}^{n} (1 - q^{2i}) = H^{-1}(q),$$
$$H_{n,\mathrm{e}}(q) := \mathrm{Even}(H_n(q)), \quad H_{n,\mathrm{o}}(q) := \mathrm{Odd}(H_n(q)),$$
$$F_{n,\mathrm{e}}(q) := \mathrm{Even}(F_n(q)), \quad F_{n,\mathrm{o}}(q) := \mathrm{Odd}(F_n(q)).$$

We have the following relations between these rational functions.

Lemma 4.4.8. *We have the equalities*

$$F_{n,\mathrm{e}}(q) = \frac{H_{n,\mathrm{e}}(q)}{H_{n,\mathrm{e}}(q)^2 - H_{n,\mathrm{o}}(q)^2}, \quad F_{n,\mathrm{o}}(q) = \frac{-H_{n,\mathrm{o}}(q)}{H_{n,\mathrm{e}}(q)^2 - H_{n,\mathrm{o}}(q)^2}.$$

Proof. We set $G_n(q) := \prod_{i=1}^{n}(1-q^i)^{-1} \in \mathbb{Q}(q)$. Then we have

$$H_{n,\mathrm{e}}(q) = \mathrm{Even}(H_n(q)) = \mathrm{Even}(G_n(q^2)) = \frac{1}{2}(G_n(q^2) + G_n(-q^2)).$$

Similarly, we also have the equalities

$$\begin{aligned} H_{n,\mathrm{o}}(q) &= \frac{1}{2}(G_n(q^2) - G_n(-q^2)), \\ F_{n,\mathrm{e}}(q) &= \frac{1}{2}(G_n^{-1}(q^2) + G_n^{-1}(-q^2)), \\ F_{n,\mathrm{o}}(q) &= \frac{1}{2}(G_n(q^2)^{-1} - G_n(-q^2)^{-1}). \end{aligned}$$

Using these equalities, we conclude

$$\begin{aligned} \frac{H_{n,\mathrm{e}}(q)}{H_{n,\mathrm{e}}(q)^2 - H_{n,\mathrm{o}}(q)^2} &= \frac{G_n(q^2) + G_n(-q^2)}{2G_n(q^2)G_n(-q^2)} \\ &= \frac{G_n^{-1}(-q^2) + G_n^{-1}(q^2)}{2} \\ &= F_{n,\mathrm{e}}(q). \end{aligned}$$

Thus, we proved the first equation. The second equation can be shown in the same way. □

Using these preparations, we now give an explicit description of ϕ and ϕ^{-1}.

Theorem 4.4.9. *Let $\phi = \bigoplus_{n\in\mathbb{N}_0} \phi_n$ be as in Theorem 4.4.4. Then the following holds:*

(i.a) We have $\phi_0(1 \otimes [Q_0^{\mathrm{e}}]) = 1 \otimes [T_0^{\mathrm{e}}]$.

(i.b) We have $\phi_1(1 \otimes [Q_1^{\mathrm{e}}]) = (1-q^4)^{-1} \otimes [T_1^{\mathrm{e}}]$.

(i.c) For $n \in \mathbb{N}_0$ with $n \geq 2$, we have

$$\begin{aligned} \phi_n(1 \otimes [Q_n^{\mathrm{e}}]) &= H_{n,\mathrm{e}}(q) \otimes [T_n^{\mathrm{e}}] + H_{n,\mathrm{o}}(q) \otimes [T_n^{\mathrm{e}}], \\ \phi_n(1 \otimes [Q_n^{\mathrm{o}}]) &= H_{n,\mathrm{o}}(q) \otimes [T_n^{\mathrm{e}}] + H_{n,\mathrm{e}}(q) \otimes [T_n^{\mathrm{o}}]. \end{aligned}$$

Write $\phi^{-1} = \bigoplus_{n\in\mathbb{N}_0} \phi_n{}^{-1}$. Then the following holds:

(ii.a) We have $\phi_0^{-1}(1 \otimes [T_0^{\mathrm{e}}]) = 1 \otimes [Q_0^{\mathrm{e}}]$.

(ii.b) We have $\phi_1^{-1}(1 \otimes [T_1^{\mathrm{e}}]) = (1-q^4) \otimes [Q_1^{\mathrm{e}}]$.

(ii.c) For $n \in \mathbb{N}_0$ with $n \geq 2$, we have

$$\begin{aligned} \phi_n{}^{-1}(1 \otimes [T_n^{\mathrm{e}}]) &= F_{n,\mathrm{e}}(q) \otimes [Q_n^{\mathrm{e}}] + F_{n,\mathrm{o}}(q) \otimes [Q_n^{\mathrm{o}}], \\ \phi_n{}^{-1}(1 \otimes [T_n^{\mathrm{o}}]) &= F_{n,\mathrm{o}}(q) \otimes [Q_n^{\mathrm{e}}] + F_{n,\mathrm{e}}(q) \otimes [Q_n^{\mathrm{o}}]. \end{aligned}$$

Proof. The exceptional cases (i.a), (i.b), (ii.a) and (ii.b) can be shown by direct calculations. So in the following, we only prove (i.c) and (ii.c). At first, recall from Proposition 4.2.11 that we have

$$[Q_n^{\mathrm{e}} : T_n^{\mathrm{e}}]_q = \mathrm{Even}\Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) = H_{n,\mathrm{e}}, \quad [Q_n^{\mathrm{e}} : T_n^{\mathrm{o}}]_q = \mathrm{Odd}\Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) = H_{n,\mathrm{o}},$$
$$[Q_n^{\mathrm{o}} : T_n^{\mathrm{e}}]_q = \mathrm{Odd}\Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) = H_{n,\mathrm{o}}, \quad [Q_n^{\mathrm{o}} : T_n^{\mathrm{o}}]_q = \mathrm{Even}\Big(\prod_{i=1}^{n} \frac{1}{1-q^{2i}}\Big) = H_{n,\mathrm{e}}.$$

Hence, the graded characters of Q_n^{e} and Q_n^{o} are given by

$$\mathrm{gch}([Q_n^{\mathrm{e}}]) = H_{n,\mathrm{e}} \otimes [T_n^{\mathrm{e}}] + H_{n,\mathrm{o}} \otimes [T_n^{\mathrm{o}}], \quad \mathrm{gch}([Q_n^{\mathrm{o}}]) = H_{n,\mathrm{o}} \otimes [T_n^{\mathrm{e}}] + H_{n,\mathrm{e}} \otimes [T_n^{\mathrm{o}}].$$

By the definition of ϕ, we hence immediately obtain (i.c). Now, in order to compute ϕ_n^{-1}, we have to compute the inverse of the matrix

$$\begin{pmatrix} H_{n,\mathrm{e}} & H_{n,\mathrm{o}} \\ H_{n,\mathrm{o}} & H_{n,\mathrm{e}} \end{pmatrix} \in \mathrm{M}_2(\mathbb{Q}(q)).$$

For this, we use Lemma 4.4.8 as follows

$$\begin{pmatrix} H_{n,\mathrm{e}} & H_{n,\mathrm{o}} \\ H_{n,\mathrm{o}} & H_{n,\mathrm{e}} \end{pmatrix}^{-1} = \frac{1}{H_{n,\mathrm{e}}^2 - H_{n,\mathrm{o}}^2} \begin{pmatrix} H_{n,\mathrm{e}} & -H_{n,\mathrm{o}} \\ -H_{n,\mathrm{o}} & H_{n,\mathrm{e}} \end{pmatrix} = \begin{pmatrix} F_{n,\mathrm{e}} & F_{n,\mathrm{o}} \\ F_{n,\mathrm{o}} & F_{n,\mathrm{e}} \end{pmatrix}. \tag{4.5}$$

This implies (ii.c). □

From these formulas for ϕ and the inverse ϕ^{-1}, we derive explicit formulas for the bilinear Euler forms χ_{K} and χ_{G} with respect to the bases given by graded projective indecomposable resp. graded simple modules.

Theorem 4.4.10. *The following assertions are true:*

(i) We have $\chi_{K,0}(1 \otimes [Q_0^{\mathrm{e}}], 1 \otimes [Q_0^{\mathrm{e}}]) = 1$. Moreover, we have $\chi_{G,0}(1 \otimes [T_0^{\mathrm{e}}], 1 \otimes [T_0^{\mathrm{e}}]) = 1$.

(ii) We have $\chi_{K,1}(1 \otimes [Q_1^{\mathrm{e}}], 1 \otimes [Q_1^{\mathrm{e}}]) = (1-q^4)^{-1}$. Moreover, we have

$$\chi_{G,1}(1 \otimes [T_1^{\mathrm{e}}], 1 \otimes [T_1^{\mathrm{e}}]) = 1 - q^4.$$

(iii) Let $n \in \mathbb{N}_0$ with $n \geq 2$ and $4 | n(n-1)$. With respect to the basis $(1 \otimes [Q_n^{\mathrm{e}}], 1 \otimes [Q_n^{\mathrm{o}}])$, we have that $\chi_{K,n}$ is given by the matrix

$$\begin{pmatrix} H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \\ H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \end{pmatrix}.$$

Moreover, with respect to the basis $(1 \otimes [T_n^{\mathrm{e}}], 1 \otimes [T_n^{\mathrm{o}}])$, we have that $\chi_{\mathrm{K},n}$ is given by the matrix

$$\begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix}.$$

(iv) *Let $n \in \mathbb{N}_0$ with $n \geq 2$ and $4 \nmid n(n-1)$. Then, with respect to the basis $(1\otimes[Q_n^{\mathrm{e}}], 1\otimes[Q_n^{\mathrm{o}}])$, we have that $\chi_{K,n}$ is given by the matrix*

$$\begin{pmatrix} H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \\ H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \end{pmatrix}$$

Moreover, with respect to the basis $(1 \otimes [T_n^{\mathrm{e}}], 1 \otimes [T_n^{\mathrm{o}}])$, we have that $\chi_{\mathrm{K},n}$ is given by the matrix

$$\begin{pmatrix} F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \\ F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \end{pmatrix}.$$

Proof. Again, the exceptional cases (i) and (ii) can be shown by straightforward computations. Hence, in the following, we only show (iii) and (iv).

(iii) By (4.3), we know that $(Q_n^{\mathrm{e}})^\# \cong Q_n^{\mathrm{e}}$ and $(Q_n^{\mathrm{o}})^\# \cong Q_n^{\mathrm{o}}$. Thus, from Proposition 4.2.11, we directly obtain

$$\chi_{K,n}(1 \otimes [Q_n^{\mathrm{e}}], 1 \otimes [Q_n^{\mathrm{e}}]) = [Q_n^{\mathrm{e}} : T_n^{\mathrm{e}}]_q = H_{n,\mathrm{e}}, \quad \chi_{K,n}(1 \otimes [Q_n^{\mathrm{e}}], 1 \otimes [Q_n^{\mathrm{o}}]) = [Q_n^{\mathrm{e}} : T_n^{\mathrm{o}}]_q = H_{n,\mathrm{o}},$$
$$\chi_{K,n}(1 \otimes [Q_n^{\mathrm{o}}], 1 \otimes [Q_n^{\mathrm{e}}]) = [Q_n^{\mathrm{o}} : T_n^{\mathrm{e}}]_q = H_{n,\mathrm{o}}, \quad \chi_{K,n}(1 \otimes [Q_n^{\mathrm{o}}], 1 \otimes [Q_n^{\mathrm{o}}]) = [Q_n^{\mathrm{o}} : T_n^{\mathrm{o}}]_q = H_{n,\mathrm{o}}.$$

Thus, we have that with respect to the basis $(1 \otimes [Q_n^{\mathrm{e}}], 1 \otimes [Q_n^{\mathrm{o}}])$, the bilinear form $\chi_{\mathrm{K},n}$ is given by the following matrix

$$\begin{pmatrix} H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \\ H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \end{pmatrix}$$

From Theorem 4.4.9.(ii.c), it then follows that with respect to the basis $(1\otimes[T_n^{\mathrm{e}}], 1\otimes[T_n^{\mathrm{o}}])$, we have that $\chi_{\mathrm{K},n}$ is given by the following product of matrices

$$\begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix} \underbrace{\begin{pmatrix} H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \\ H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \end{pmatrix} \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix}}_{=\mathrm{id}} = \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix}.$$

Here, we used the equation (4.5).

(iv) By (4.4), we have $(Q_n^{\mathrm{e}})^\# \cong Q_n^{\mathrm{o}}$ and $(Q_n^{\mathrm{o}})^\# \cong Q_n^{\mathrm{e}}$. With the same argument as in (iii), we conclude that with respect to the basis $(1 \otimes [Q_n^{\mathrm{e}}], 1 \otimes [Q_n^{\mathrm{o}}])$, the bilinear form $\chi_{\mathrm{K},n}$ is given by the matrix

$$\begin{pmatrix} H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \\ H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \end{pmatrix}.$$

Again applying Theorem 4.4.9.(ii.c), we deduce that with respect to the basis $(1\otimes[T_n^{\mathrm{e}}], 1\otimes[T_n^{\mathrm{o}}])$, we have that $\chi_{\mathrm{K},n}$ is given by the following product of matrices

$$F := \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix} \begin{pmatrix} H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \\ H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \end{pmatrix} \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix}.$$

Using (4.5), we determine F as follows

$$\begin{aligned} F &= \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_{n,\mathrm{e}}(q) & H_{n,\mathrm{o}}(q) \\ H_{n,\mathrm{o}}(q) & H_{n,\mathrm{e}}(q) \end{pmatrix} \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix} \\ &= \begin{pmatrix} F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \\ F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{n,\mathrm{o}}(q) & F_{n,\mathrm{e}}(q) \\ F_{n,\mathrm{e}}(q) & F_{n,\mathrm{o}}(q) \end{pmatrix}. \end{aligned}$$

Thus, we proved (iv). □

4.5 Multiplicative and comultiplicative structures

In this section, we characterize the multiplicative and the comultipliative structure on the alternating nil Hecke Grothendieck groups $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. For this, we identify $\mathrm{G}_0(\mathrm{ANH}) \cong \mathrm{K}_0(\mathrm{ANH})^*$ and $\mathrm{K}_0(\mathrm{ANH}) \cong \mathrm{G}_0(\mathrm{ANH})^*$ by the isomorphisms of $\mathbb{N}_0$-graded $\mathcal{A}$-algebras and $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras from Theorem 4.3.6. In addition, we use the following notation which is analogous to Notation 4.2.2.

Notation 4.5.1. Let $m, n \in \mathbb{N}_0$. Let M be a graded $(\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n})$-module such that the graded components M_i vanish for all $i \in 1 + \mathbb{Z}$. We define $M^\mathrm{e} \subset M$ to be the *even part* of M, i.e. M^e is the graded $(\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n})$-module whose graded components are

$$M_i^\mathrm{e} = \begin{cases} M_i & \text{if } i \equiv 0 \mod 4, \\ 0 & \text{else.} \end{cases}$$

Moreover, we define $M^\mathrm{o} \subset M$ to be the *odd part* of M, i.e. M^o is the graded $(\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n})$-module whose graded components are

$$M_i^\mathrm{o} = \begin{cases} M_i & \text{if } i \equiv 2 \mod 4, \\ 0 & \text{else.} \end{cases}$$

We have the following decomposition of $(\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n})$-modules $M = M^\mathrm{e} \oplus M^\mathrm{o}$.

If N is a graded $(\mathrm{NH}_m \otimes_k \mathrm{NH}_\mathrm{n})$-module such that the graded components N_i vanish for all $i \in 1 + \mathbb{Z}$, then we have a decomposition of $\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n}$-modules

$$\mathrm{Res}^{\mathrm{NH}_m \otimes_k \mathrm{NH}_\mathrm{n}}_{\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n}}(N) = N^\mathrm{e} \oplus N^\mathrm{o},$$

where $N^\mathrm{e} := (\mathrm{Res}^{\mathrm{NH}_m \otimes_k \mathrm{NH}_\mathrm{n}}_{\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n}}(N))^\mathrm{e}$ and $N^\mathrm{o} := (\mathrm{Res}^{\mathrm{NH}_m \otimes_k \mathrm{NH}_\mathrm{n}}_{\mathrm{ANH}_m \otimes_k \mathrm{ANH}_\mathrm{n}}(N))^\mathrm{o}$. We call N^e the *even part* and N^o the *odd part* of N.

We begin with determining explicit formulas for the comultiplication on $\mathrm{K}_0(\mathrm{ANH})$.

Notation 4.5.2. Let R be a commutative unital ring and let C be an $\mathbb{N}_0$-graded R-module. For $n \in \mathbb{N}_0$ let $p_n : C \to C_n$ be the projection to the n-th homogeneous component. Let $c \in C \otimes_R C$ be a homogeneous element and $r, s \in \mathbb{N}_0$ with $r + s = |c|$. Then we set

$$c_{r,s} := (p_r \otimes p_s)(c) \in C_r \otimes_R C_s.$$

Theorem 4.5.3. *Let $r, s \in \mathbb{N}_0$ with $r, s \geq 2$ and set $n := r + s$. Then we have*

$$\Delta_\mathrm{K}([Q_n^\mathrm{e}])_{r,s} = q^{-rs}([Q_r^\mathrm{e}] \otimes [Q_s^\mathrm{e}] + [Q_r^\mathrm{o}] \otimes [Q_s^\mathrm{o}]),$$
$$\Delta_\mathrm{K}([Q_n^\mathrm{o}])_{r,s} = q^{-rs}([Q_r^\mathrm{e}] \otimes [Q_s^\mathrm{o}] + [Q_r^\mathrm{o}] \otimes [Q_s^\mathrm{e}]).$$

Proof. We only show the formula for $[Q_n^\mathrm{e}]$. The same argument can be used to show the formula for $[Q_n^\mathrm{o}]$. Now, by definition, we have

$$Q_n^\mathrm{e} = P_n^\mathrm{e}\langle \frac{1}{2} n(n-1) \rangle.$$

Now, according to Theorem 3.4.3, we know that $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{NH}_r \otimes_k \mathrm{NH}_s}(P_n) \cong P_r \otimes_k P_s$. From this, we conclude

$$\begin{aligned}\mathrm{Res}^{\mathrm{ANH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(P_n^{\mathrm{e}}) &\cong \left(\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(P_n)\right)^{\mathrm{e}} \\ &\cong \left(\mathrm{Res}^{\mathrm{NH}_r \otimes_k \mathrm{NH}_s}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(P_r \otimes_k P_s)\right)^{\mathrm{e}} \\ &\cong (P_r^{\mathrm{e}} \otimes_k P_s^{\mathrm{e}}) \oplus (P_r^{\mathrm{o}} \otimes_k P_s^{\mathrm{o}}).\end{aligned}$$

Hence, we obtain

$$\mathrm{Res}^{\mathrm{ANH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(Q_n^{\mathrm{e}}) \cong (Q_r^{\mathrm{e}} \otimes_k Q_s^{\mathrm{e}})\langle -rs\rangle \oplus (Q_r^{\mathrm{o}} \otimes_k Q_s^{\mathrm{o}})\langle -rs\rangle,$$

which gives the stated formula for $[Q_n^{\mathrm{e}}]$. □

We further have the following exceptional cases. They can be computed with same argument as the ordinary cases from Theorem 4.5.3.

Remark 4.5.4. Let $m \in \mathbb{N}_0$. Then we have

$$\begin{aligned}\Delta_{\mathrm{K}}([Q^{\mathrm{e}}_{m+1}])_{1,m} &= q^{-m}([Q_1^{\mathrm{e}}] \otimes [Q_m^{\mathrm{e}}]) + q^{2-m}([Q_1^{\mathrm{e}}] \otimes [Q_m^{\mathrm{o}}]), \\ \Delta_{\mathrm{K}}([Q^{\mathrm{e}}_{m+1}])_{m,1} &= q^{-m}([Q_m^{\mathrm{e}}] \otimes [Q_1^{\mathrm{e}}]) + q^{2-m}([Q_m^{\mathrm{o}}] \otimes [Q_1^{\mathrm{e}}]), \\ \Delta_{\mathrm{K}}([Q^{\mathrm{o}}_{m+1}])_{1,m} &= q^{-m}([Q_1^{\mathrm{e}}] \otimes [Q_m^{\mathrm{o}}]) + q^{2-m}([Q_1^{\mathrm{e}}] \otimes [Q_m^{\mathrm{e}}]), \\ \Delta_{\mathrm{K}}([Q^{\mathrm{o}}_{m+1}])_{m,1} &= q^{-m}([Q_m^{\mathrm{o}}] \otimes [Q_1^{\mathrm{e}}]) + q^{2-m}([Q_m^{\mathrm{e}}] \otimes [Q_1^{\mathrm{e}}]).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\Delta_{\mathrm{K}}([Q_2^{\mathrm{e}}])_{1,1} &= (q^{-1} + q^3)[Q_1^{\mathrm{e}}] \otimes [Q_1^{\mathrm{e}}], \\ \Delta_{\mathrm{K}}([Q_2^{\mathrm{o}}])_{1,1} &= 2q[Q_1^{\mathrm{e}}] \otimes [Q_1^{\mathrm{e}}].\end{aligned}$$

Using the duality between $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$, we derive from Theorem 4.5.3 explicit formulas for the multiplication on $\mathrm{G}_0(\mathrm{ANH})$. In order to determine these formulas, we have to take the duality relations (4.3) and (4.4) into account. For this, we use the following lemma. It can be proved by straightforward computations.

Lemma 4.5.5. *Let $m, n \in \mathbb{N}_0$. Then we have:*

(i) Suppose that $4|(m+n)(m+n-1)$. Then $2|mn$ if and only if either $4|m(m-1), 4|n(n-1)$ or $4 \nmid m(m-1), 4 \nmid n(n-1)$.

(ii) Suppose that $4 \nmid (m+n)(m+n-1)$. Then $2|mn$ if and only if either $4|m(m-1), 4 \nmid n(n-1)$ or $4 \nmid m(m-1), 4|n(n-1)$.

Theorem 4.5.6. *Let $m, n \in \mathbb{N}_0$ with $m, n \geq 2$. Then we have*

$$[T_m^{\mathrm{e}}] \cdot [T_n^{\mathrm{e}}] = [T_m^{\mathrm{o}}] \cdot [T_n^{\mathrm{o}}] = \begin{cases} q^{-mn}[T^{\mathrm{e}}_{m+n}] & \text{if } 2|mn, \\ q^{-mn}[T^{\mathrm{o}}_{m+n}] & \text{if } 2 \nmid mn,\end{cases}$$

$$[T_m^{\mathrm{e}}] \cdot [T_n^{\mathrm{o}}] = [T_m^{\mathrm{o}}] \cdot [T_n^{\mathrm{e}}] = \begin{cases} q^{-mn}[T^{\mathrm{o}}_{m+n}] & \text{if } 2|mn, \\ q^{-mn}[T^{\mathrm{e}}_{m+n}] & \text{if } 2 \nmid mn.\end{cases}$$

Proof. We only show the formula for $[T^{\mathrm{e}}_m] \cdot [T^{\mathrm{e}}_n]$. The other stated formulas can be shown in exactly the same way. Now, we know that

$$[T^{\mathrm{e}}_m] \cdot [T^{\mathrm{e}}_n] = a[T^{\mathrm{e}}_{m+n}] + b[T^{\mathrm{o}}_{m+n}], \tag{4.6}$$

for unique $a, b \in \mathcal{A}$. So, we have to determine the coefficients a and b. For this, we first define the $\mathcal{A}$-bilinear pairing

$$(.,.) : (\mathrm{K}_0(\mathrm{ANH}) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH})) \times (\mathrm{G}_0(\mathrm{ANH}) \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH})) \to \mathcal{A},$$

via $(a \otimes b, c \otimes d) = (a, c)(b, d)$, for $a, b \in \mathrm{K}_0(\mathrm{ANH}), c, d \in \mathrm{G}_0(\mathrm{ANH})$.

Now, assume that $4|(m+n)(m+n-1)$. According to Proposition 4.3.2, we can determine the coefficient a by applying $(., [Q^{\mathrm{e}}_{m+n}])$ to (4.6). Using the compatibility of the HOM-pairing with induction and restriction from Proposition 2.4.9 and the explicit formulas for $\Delta_{\mathrm{K}}([Q^{\mathrm{e}}_{m+n}]$ from Theorem 3.4.3, we then deduce the following equalities

$$\begin{aligned} a &= ([Q^{\mathrm{e}}_{m+n}], [T^{\mathrm{e}}_m] \cdot [T^{\mathrm{e}}_n]) \\ &= (\Delta_{\mathrm{K}}([Q^{\mathrm{e}}_{m+n}]), [T^{\mathrm{e}}_m] \otimes [T^{\mathrm{e}}_n]) \\ &= q^{-mn}([Q^{\mathrm{e}}_m] \otimes [Q^{\mathrm{e}}_n] + [Q^{\mathrm{o}}_m] \otimes [Q^{\mathrm{o}}_n], [T^{\mathrm{e}}_m] \otimes [T^{\mathrm{e}}_n]) \\ &= q^{-mn}([Q^{\mathrm{e}}_m], [T^{\mathrm{e}}_m])([Q^{\mathrm{e}}_n], [T^{\mathrm{e}}_n]) + q^{-mn}([Q^{\mathrm{o}}_m], [T^{\mathrm{e}}_m])([Q^{\mathrm{o}}_n], [T^{\mathrm{e}}_n]). \end{aligned} \tag{4.7}$$

By applying Proposition 4.3.2, we directly obtain the following possibilities

$$(4.7) = \begin{cases} q^{-mn} & \textit{if } 4|m(m-1), 4|n(n-1), \\ q^{-mn} & \textit{if } 4 \nmid m(m-1), 4 \nmid n(n-1), \\ 0 & \textit{if } 4|m(m-1), 4 \nmid n(n-1), \\ 0 & \textit{if } 4 \nmid m(m-1), 4|n(n-1). \end{cases}$$

Finally, Lemma 4.5.5 implies

$$a = \begin{cases} q^{-mn} & \textit{if } 2|mn, \\ 0 & \textit{if } 2 \nmid mn. \end{cases}$$

Now, by applying $(., [Q^{\mathrm{o}}_{m+n}])$ to (4.6) and using exactly the same argument as above, we obtain

$$b = \begin{cases} 0 & \textit{if } 2|mn, \\ q^{-mn} & \textit{if } 2 \nmid mn. \end{cases}$$

Thus, we showed the stated formula for $[T^{\mathrm{e}}_m] \cdot [T^{\mathrm{e}}_n]$ in the case $4|(m+n)(m+n-1)$. The case $4 \nmid (m+n)(m+n-1)$ can be proved in the same way. □

Again, we list the exceptional cases in the following remark. They can be verified with the same argument as in Theorem 4.5.6.

Remark 4.5.7. Let $n \in \mathbb{N}_0$ with $n \geq 2$. If $2|n$, then we have

$$\begin{aligned} [T^{\mathrm{e}}_1] \cdot [T^{\mathrm{e}}_n] = [T^{\mathrm{e}}_n] \cdot [T^{\mathrm{e}}_1] = q^{-n}[T^{\mathrm{e}}_{n+1}] + q^{2-n}[T^{\mathrm{o}}_{n+1}], \\ [T^{\mathrm{e}}_1] \cdot [T^{\mathrm{o}}_n] = [T^{\mathrm{o}}_n] \cdot [T^{\mathrm{e}}_1] = q^{2-n}[T^{\mathrm{e}}_{n+1}] + q^{-n}[T^{\mathrm{o}}_{n+1}]. \end{aligned}$$

If $2 \nmid n$, then we have

$$[T_1^{\mathrm{e}}] \cdot [T_n^{\mathrm{e}}] = [T_n^{\mathrm{e}}] \cdot [T_1^{\mathrm{e}}] = q^{2-n}[T_{n+1}^{\mathrm{e}}] + q^{-n}[T_{n+1}^{\mathrm{o}}],$$
$$[T_1^{\mathrm{e}}] \cdot [T_n^{\mathrm{o}}] = [T_n^{\mathrm{o}}] \cdot [T_1^{\mathrm{e}}] = q^{-n}[T_{n+1}^{\mathrm{e}}] + q^{2-n}[T_{n+1}^{\mathrm{o}}].$$

Finally, we also have

$$[T_1^{\mathrm{e}}] \cdot [T_1^{\mathrm{e}}] = 2q[T_2^{\mathrm{e}}] + (q^{-1} + q^3)[T_2^{\mathrm{o}}].$$

Next, we evaluate the results from Theorem 4.5.3 and Theorem 4.5.6. At first, we decide that we only consider the comultiplication on $\mathrm{K}_0(\mathrm{ANH})$ and the multiplication on $\mathrm{G}_0(\mathrm{ANH})$ in degrees ≥ 2. So in the following, we ignore the exceptional cases.

Notation 4.5.8. Let R be a ring.

1. Let A be an $\mathbb{N}_0$-graded R-algebra. For $n \geq 1$, we denote by $A_{\geq n}$ the two-sided ideal $A_{\geq n} := \bigoplus_{i \geq n} A_i$. We consider $A_{\geq 2}$ as non-unital $\mathbb{N}_0$-graded R-algebra.

2. Let C be an $\mathbb{N}_0$-graded R-coalgebra with comultiplication Δ. For $m \in \mathbb{N}_0$, we denote by $p_m : C \to C_m$ the projection to the m-th homogeneous component. For $n \geq 1$, we denote by $C_{\geq n}$ the non-counital $\mathbb{N}_0$-graded R-coalgebra $C_{\geq n} := \bigoplus_{i \geq n} C_i$, with comultiplication

$$c \mapsto \Big(\sum_{r=n}^{|c|} (p_r \otimes p_{|c|-r})\Delta(c) \Big) \in \bigoplus_{r=n}^{|c|} C_r \otimes_R C_{|c|-r}.$$

   for any homogeneous $c \in C_{\geq n}$.

In the following, let ${}_{\mathcal{A}}\mathbf{f}$ denote Lusztig's integral quantum group corresponding to the one-vertex graph without edges. We already discussed ${}_{\mathcal{A}}\mathbf{f}$ in Section 3.4 and continue to use the notation that we used there. In particular, ${}_{\mathcal{A}}\mathbf{f}^*$ denotes the $\mathbb{N}_0$-graded dual of ${}_{\mathcal{A}}\mathbf{f}$. Furthermore, let

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \to \mathrm{K}_0(\mathrm{NH}), \quad \gamma^* : \mathrm{G}_0(\mathrm{NH}) \to {}_{\mathcal{A}}\mathbf{f}^*$$

be the isomorphisms of $\mathbb{N}_0$-graded twisted $\mathcal{A}$-bialgebras from Theorem 3.4.3 and Theorem 3.4.4.

Let $\mathcal{A}[\mathbb{Z}/2]$ be the group algebra of $\mathbb{Z}/2$ over $\mathcal{A}$, i.e. $\mathcal{A}[\mathbb{Z}/2]$ is the $\mathcal{A}$-algebra of formal sums

$$\mathcal{A}[\mathbb{Z}/2] = \{ae_1 + be_\tau | a, b \in \mathcal{A}\},$$

where the addition is componentwise and the multiplication is given by

$$(ae_1 + be_\tau)(a'e_1 + b'e_\tau) = (aa' + bb')e_1 + (ab' + ba')e_\tau,$$

for all $a, a', b, b' \in \mathcal{A}$. We have that $\mathcal{A}[\mathbb{Z}/2]$ admits a unique grading, where e_1, e_τ are homogeneous of degree 0. Let $\mathcal{A}[\mathbb{Z}/2]^*$ be the $\mathbb{N}_0$-graded dual $\mathcal{A}$-coalgebra of $\mathcal{A}[\mathbb{Z}/2]$, see Definition 3.3.7. By definition, the comultiplication Δ on $\mathcal{A}[\mathbb{Z}/2]^*$ is given as follows. Let (e_1^*, e_τ^*) be the dual basis of (e_1, e_τ). Then we have

$$\Delta(e_1^*) = e_1^* \otimes e_1^* + e_\tau^* \otimes e_\tau^*, \quad \Delta(e_\tau^*) = e_1^* \otimes e_\tau^* + e_\tau^* \otimes e_1^*.$$

Next, we consider the tensor product ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*$. It can be equipped with the tensor product coalgebra structure that turns ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*$ into an $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebra, where the comultiplication is given by

$$\theta^{(n)} \otimes e_1 \mapsto \sum_{r=0}^{n} (q^{-r(n-r)}(\theta^{(r)} \otimes e_1^*) \otimes (\theta^{(n-r)} \otimes e_1^*) + q^{-r(n-r)}(\theta^{(r)} \otimes e_\tau^*) \otimes (\theta^{(n-r)} \otimes e_\tau^*)),$$
$$\theta^{(n)} \otimes e_\tau \mapsto \sum_{r=0}^{n} (q^{-r(n-r)}(\theta^{(r)} \otimes e_1^*) \otimes (\theta^{(n-r)} \otimes e_\tau^*) + q^{-r(n-r)}(\theta^{(r)} \otimes e_\tau^*) \otimes (\theta^{(n-r)} \otimes e_1^*)),$$

for all $n \in \mathbb{N}_0$. Finally, we note that we have a canonical embedding of $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras

$$\iota : {}_{\mathcal{A}}\mathbf{f} \to {}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*, \quad \theta^{(n)} \mapsto \theta^{(n)} \otimes (e_1^* + e_\tau^*), \quad \text{for all } n \in \mathbb{N}_0.$$

Summarizing the above considerations, we obtain the following description of the comultilicative structure on $\mathrm{K}_0(\mathrm{ANH})$.

Theorem 4.5.9. *There is an isomorphism of non-counital $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras*

$$\varphi : ({}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*)_{\geq 2} \to \mathrm{K}_0(\mathrm{ANH})_{\geq 2},$$

such that

$$\theta^{(n)} \otimes e_1^* \mapsto [Q_n^{\mathrm{e}}], \quad \theta^{(n)} \otimes e_\tau^* \mapsto [Q_n^{\mathrm{o}}], \quad \textit{for all } n \in \mathbb{N}_0 \textit{ with } n \geq 2.$$

Moreover, we have a commuting diagram:

$$\begin{tikzcd}
({}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*)_{\geq 2} \arrow[r, "\varphi"] & \mathrm{K}_0(\mathrm{ANH})_{\geq 2} \\
{}_{\mathcal{A}}\mathbf{f}_{\geq 2} \arrow[u, "\iota"] \arrow[r, "\gamma"] & \mathrm{K}_0(\mathrm{NH})_{\geq 2} \arrow[u, "\mathrm{R_K}"']
\end{tikzcd}$$

Here, $\mathrm{R_K}$ *is the restriction homomorphism from Theorem 4.3.7.*

Proof. According to Proposition 4.3.1, φ is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}$-modules. So, it is left to show that φ is compatible with the comultiplication. For this, let Δ' denote the comultiplication on ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*$. Let $r, s \in \mathbb{N}_0$ with $r, s \geq 2$ and $n := r + s$. Then we conclude the following equalities from Theorem 4.5.3.

$$\begin{aligned}
(\varphi \otimes \varphi)(\Delta'(\theta^{(n)} \otimes e_1^*)_{r,s}) &= q^{-rs}((\varphi \otimes \varphi)(\theta^{(r)} \otimes e_1^* \otimes \theta^{(s)} \otimes e_1^*)) \\
&\quad + q^{-rs}((\varphi \otimes \varphi)(\theta^{(r)} \otimes e_\tau^* \otimes \theta^{(s)} \otimes e_\tau^*)) \\
&= q^{-rs}([Q_r^{\mathrm{e}}] \otimes [Q_s^{\mathrm{e}}] + [Q_r^{\mathrm{o}}] \otimes [Q_s^{\mathrm{o}}]) \\
&= \Delta_{\mathrm{K}}([Q_n^{\mathrm{e}}]) \\
&= \Delta_{\mathrm{K}}(\varphi(\theta^{(n)} \otimes e_1^*))_{r,s}.
\end{aligned}$$

In addition, we also have

$$\begin{aligned}
(\varphi \otimes \varphi)(\Delta'(\theta^{(n)} \otimes e_\tau^*)_{r,s}) &= q^{-rs}((\varphi \otimes \varphi)(\theta^{(r)} \otimes e_1^* \otimes \theta^{(s)} \otimes e_\tau^*)) \\
&\quad + q^{-rs}((\varphi \otimes \varphi)(\theta^{(r)} \otimes e_\tau^* \otimes \theta^{(s)} \otimes e_1^*)) \\
&= q^{-rs}([Q_r^{\mathrm{e}}] \otimes [Q_s^{\mathrm{o}}] + [Q_r^{\mathrm{o}}] \otimes [Q_s^{\mathrm{e}}]) \\
&= \Delta_{\mathrm{K}}([Q_n^{\mathrm{o}}]) \\
&= \Delta_{\mathrm{K}}(\varphi(\theta^{(n)} \otimes e_\tau^*))_{r,s}.
\end{aligned}$$

This implies that φ is compatible with the comultiplication and hence, φ is an isomorphism of $\mathbb{N}_0$-graded non-counital $\mathcal{A}$-coalgebras. The commutativity of the stated diagram follows directly from

$$\mathrm{R_K}(\gamma(\theta^{(n)})) = \mathrm{R_K}([Q_n]) = [Q_n^{\mathrm{e}}] + [Q_n^{\mathrm{o}}] = \varphi(\theta^{(n)} \otimes e_1^* + \theta^{(n)} \otimes e_\tau^*) = \varphi(\iota(\theta^{(n)})),$$

for $n \in \mathbb{N}_0$ with $n \geq 2$. This completes the proof. □

Using the duality between $\mathrm{K}_0(\mathrm{ANH})$ and $\mathrm{G}_0(\mathrm{ANH})$, we directly obtain from Theorem 4.5.9 a characterization of the multiplication on $\mathrm{G}_0(\mathrm{ANH})$. For this, note that the $\mathbb{N}_0$-graded dual $\mathcal{A}$-algebra $({}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]^*)^*$ can be naturally identified with ${}_{\mathcal{A}}\mathbf{f}^* \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2]$. The adjoint map $\iota^* : {}_{\mathcal{A}}\mathbf{f}^* \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2] \to {}_{\mathcal{A}}\mathbf{f}^*$ of ι is then given by

$$(\theta^{(n)*} \otimes e_1) \mapsto \theta^{(n)*}, \quad (\theta^{(n)*} \otimes e_\tau) \mapsto \theta^{(n)*},$$

for all $n \in \mathbb{N}_0$.

Corollary 4.5.10. *Let φ be as in Theorem 4.5.3 and let*

$$\varphi^* : \mathrm{G}_0(\mathrm{ANH})_{\geq 2} \to ({}_{\mathcal{A}}\mathbf{f}^* \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2])_{\geq 2}$$

be the adjoint map. Then φ^* is an isomorphism of non-unital $\mathbb{N}_0$-graded $\mathcal{A}$-algebras. Moreover, for each $n \in \mathbb{N}_0$ with $n \geq 2$, we have

$$\varphi^*([T_n^{\mathrm{e}}]) = \begin{cases} \theta^{(n)*} \otimes e_1 & \text{if } 4 | n(n-1), \\ \theta^{(n)*} \otimes e_\tau & \text{if } 4 \nmid n(n-1), \end{cases} \qquad \varphi^*([T_n^{\mathrm{o}}]) = \begin{cases} \theta^{(n)*} \otimes e_\tau & \text{if } 4 | n(n-1), \\ \theta^{(n)*} \otimes e_1 & \text{if } 4 \nmid n(n-1). \end{cases}$$

In addition, the following diagram commutes:

$$\begin{tikzcd} \mathrm{G}_0(\mathrm{ANH})_{\geq 2} \arrow[r, "\varphi^*"] \arrow[d, "\mathrm{I_G}"] & ({}_{\mathcal{A}}\mathbf{f}^* \otimes_{\mathcal{A}} \mathcal{A}[\mathbb{Z}/2])_{\geq 2} \arrow[d, "\iota^*"] \\ \mathrm{G}_0(\mathrm{NH})_{\geq 2} \arrow[r, "\gamma^*"] & {}_{\mathcal{A}}\mathbf{f}^*_{\geq 2} \end{tikzcd}$$

Here, $\mathrm{I_G}$ is the induction homomorphism from Theorem 4.3.7.

We proceed with characterizing the comultiplicative structure on $\mathrm{G}_0(\mathrm{ANH})$ and the multiplicative structure on $\mathrm{K}_0(\mathrm{ANH})$.

Convention 4.5.11. In the following, we use in our formulas the non-symmetric quantum binomial coefficients from Definition 3.2.5. This is due to our definition of the operators Even and Odd. Namely, we have that

$$\mathrm{Even}\left(\binom{n}{m}_{q^2}\right), \quad \mathrm{Odd}\left(\binom{n}{m}_{q^2}\right),$$

are both well-defined for all $m, n \in \mathbb{N}_0$ with $0 \leq m \leq n$. In general, this is not true for the symmetrized quantum binomial coefficients.

Theorem 4.5.12. *Let $r, s \in \mathbb{N}_0$ with $r, s \geq 2$ and set $n := r + s$. Then we have*

$$\begin{aligned}\Delta_{\mathrm{G}}([T_n^{\mathrm{e}}])_{r,s} &= q^{-rs}\,\mathrm{Even}\left(\binom{n}{r}_{q^2}\right)([T_r^{\mathrm{e}}] \otimes [T_s^{\mathrm{e}}] + [T_r^{\mathrm{o}}] \otimes [T_s^{\mathrm{o}}]) \\ &\quad + q^{-rs}\,\mathrm{Odd}\left(\binom{n}{r}_{q^2}\right)([T_r^{\mathrm{e}}] \otimes [T_s^{\mathrm{o}}] + [T_r^{\mathrm{o}}] \otimes [T_s^{\mathrm{e}}]), \\ \Delta_{\mathrm{G}}([T_n^{\mathrm{o}}])_{r,s} &= q^{-rs}\,\mathrm{Odd}\left(\binom{n}{r}_{q^2}\right)([T_r^{\mathrm{e}}] \otimes [T_s^{\mathrm{e}}] + [T_r^{\mathrm{o}}] \otimes [T_s^{\mathrm{o}}]) \\ &\quad + q^{-rs}\,\mathrm{Even}\left(\binom{n}{r}_{q^2}\right)([T_r^{\mathrm{e}}] \otimes [T_s^{\mathrm{o}}] + [T_r^{\mathrm{o}}] \otimes [T_s^{\mathrm{e}}]).\end{aligned}$$

Proof. We only prove the formula for $[T_n^{\mathrm{e}}]$. The formula for $[T_n^{\mathrm{o}}]$ can be shown in the same way. Recall that by definition $T_n^{\mathrm{e}} = L_n^{\mathrm{e}}\langle \frac{1}{2}n(n-1)\rangle$. Now, by Theorem 3.4.4, we have

$$[\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{NH}_r \otimes_k \mathrm{NH}_s}(L_n)] = \binom{n}{r}_{q^2}[L_r \otimes_k L_s].$$

This means that $\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{NH}_r \otimes_k \mathrm{NH}_s} L_n$ admits a finite filtration

$$\mathrm{Res}^{\mathrm{NH_n}}_{\mathrm{NH}_r \otimes_k \mathrm{NH}_s}(L_n) = F_0 \supset F_1 \supset \ldots \supset F_N,$$

such that for each $i \in \{0, \ldots, N-1\}$, there exists $d_i \in 2\mathbb{N}_0$ such that $F_i/F_{i+1} \cong L_n\langle d_i\rangle$. Moreover, we have $\sum_{i=0}^{N-1} q^{d_i} = \binom{n}{r}_{q^2}$. Now, note that

$$\mathrm{Res}^{\mathrm{ANH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(L_n^{\mathrm{e}}) = F_0^{\mathrm{e}} \supset F_1^{\mathrm{e}} \supset \ldots \supset F_N^{\mathrm{e}}$$

is a finite filtration of $\mathrm{Res}^{\mathrm{ANH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(L_n^{\mathrm{e}})$. Furthermore, by using the same argument as in Lemma 4.2.3, we obtain

$$F_i^{\mathrm{e}}/F_{i+1}^{\mathrm{e}} \cong (F_i/F_{i+1})^{\mathrm{e}}, \quad \text{for } i \in \{0, \ldots, N-1\}.$$

Thus, we deduce

$$F_i^{\mathrm{e}}/F_{i+1}^{\mathrm{e}} \cong \begin{cases} (L_r^{\mathrm{e}} \otimes_k L_s^{\mathrm{e}})\langle d_i\rangle \oplus (L_r^{\mathrm{o}} \otimes_k L_s^{\mathrm{o}})\langle d_i\rangle & \text{if } 4|d_i, \\ (L_r^{\mathrm{e}} \otimes_k L_s^{\mathrm{o}})\langle d_i\rangle \oplus (L_r^{\mathrm{o}} \otimes_k L_s^{\mathrm{e}})\langle d_i\rangle & \text{if } 4 \nmid d_i. \end{cases}$$

Hence, we directly obtain the following equality

$$\begin{aligned}[\mathrm{Res}^{\mathrm{ANH_n}}_{\mathrm{ANH}_r \otimes_k \mathrm{ANH}_s}(L_n^{\mathrm{e}})] &= \mathrm{Even}\left(\binom{n}{r}_{q^2}\right)([L_r^{\mathrm{e}} \otimes_k L_s^{\mathrm{e}}] + [L_r^{\mathrm{o}} \otimes_k L_s^{\mathrm{o}}]) \\ &\quad + \mathrm{Odd}\left(\binom{n}{r}_{q^2}\right)([L_r^{\mathrm{e}} \otimes_k L_s^{\mathrm{o}}] + [L_r^{\mathrm{o}} \otimes_k L_s^{\mathrm{e}}]).\end{aligned}$$

From this, the stated formula for $[T_n^{\mathrm{e}}]$ directly follows. □

Again, we list the exceptional cases in the following remark. They can be shown in exactly the same way as the formulas from Theorem 4.5.12.

Remark 4.5.13. Let $m \in \mathbb{N}_0$ with $m \geq 2$. Then we have

$$\begin{aligned}
\Delta_{\mathrm{G}}([T^{\mathrm{e}}_{m+1}])_{1,m} &= q^{-m}\operatorname{Even}((n+1)_{q^2}[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{e}}_m] + q^{-m}\operatorname{Odd}((n+1)_{q^2})[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{o}}_m],\\
\Delta_{\mathrm{G}}([T^{\mathrm{e}}_{m+1}])_{m,1} &= q^{-m}\operatorname{Even}((n+1)_{q^2})[T^{\mathrm{e}}_m] \otimes [T^{\mathrm{e}}_1] + q^{-m}\operatorname{Odd}((n+1)_{q^2})[T^{\mathrm{o}}_m] \otimes [T^{\mathrm{e}}_1],\\
\Delta_{\mathrm{G}}([T^{\mathrm{o}}_{m+1}])_{1,m} &= q^{-m}\operatorname{Odd}((n+1)_{q^2})[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{e}}_m] + q^{-m}\operatorname{Even}((n+1)_{q^2})[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{o}}_m],\\
\Delta_{\mathrm{G}}([T^{\mathrm{o}}_{m+1}])_{m,1} &= q^{-m}\operatorname{Odd}((n+1)_{q^2})[T^{\mathrm{e}}_m] \otimes [T^{\mathrm{e}}_1] + q^{-m}\operatorname{Even}((n+1)_{q^2})[T^{\mathrm{o}}_m] \otimes [T^{\mathrm{e}}_1]
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\Delta_{\mathrm{G}}([T^{\mathrm{e}}_2])_{1,1} &= q^{-1}[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{e}}_1],\\
\Delta_{\mathrm{G}}([T^{\mathrm{o}}_2])_{1,1} &= q[T^{\mathrm{e}}_1] \otimes [T^{\mathrm{e}}_1].
\end{aligned}$$

By using the same duality argument as in Theorem 4.5.6, we obtain the corresponding formulas for the multiplication on $\mathrm{K}_0(\mathrm{ANH})$. We omit the proof, since the argument is completely analogous.

Theorem 4.5.14. *Let $m, n \in \mathbb{N}_0$ with $m, n \geq 2$. Then we have*

$$\begin{aligned}
&[Q^{\mathrm{e}}_m] \cdot [Q^{\mathrm{e}}_n] = [Q^{\mathrm{o}}_m] \cdot [Q^{\mathrm{o}}_n]\\
&\qquad = \begin{cases} q^{-mn}\operatorname{Even}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{e}}_{m+n}] + q^{-mn}\operatorname{Odd}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{o}}_{m+n}] & \text{if } 2|mn,\\ q^{-mn}\operatorname{Odd}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{e}}_{m+n}] + q^{-mn}\operatorname{Even}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{o}}_{m+n}] & \text{if } 2\nmid mn,\end{cases}\\
&[Q^{\mathrm{e}}_m] \cdot [Q^{\mathrm{o}}_n] = [Q^{\mathrm{o}}_m] \cdot [Q^{\mathrm{e}}_n]\\
&\qquad = \begin{cases} q^{-mn}\operatorname{Odd}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{e}}_{m+n}] + q^{-mn}\operatorname{Even}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{o}}_{m+n}] & \text{if } 2|mn,\\ q^{-mn}\operatorname{Even}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{e}}_{m+n}] + q^{-mn}\operatorname{Odd}(\binom{m+n}{n}_{q^2})[Q^{\mathrm{o}}_{m+n}] & \text{if } 2\nmid mn.\end{cases}
\end{aligned}$$

Moreover, we have the following exceptional cases.

Remark 4.5.15. Let $n \in \mathbb{N}_0$ with $n \geq 2$ such that $2|n$. Then we have

$$\begin{aligned}
[Q^{\mathrm{e}}_1] \cdot [Q^{\mathrm{e}}_n] = [Q^{\mathrm{e}}_n] \cdot [Q^{\mathrm{e}}_1] &= q^{-n}\operatorname{Even}((n+1)_{q^2})[Q^{\mathrm{e}}_{n+1}] + q^{-n}\operatorname{Odd}((n+1)_{q^2})[Q^{\mathrm{o}}_{n+1}],\\
[Q^{\mathrm{e}}_1] \cdot [Q^{\mathrm{o}}_n] = [Q^{\mathrm{o}}_n] \cdot [Q^{\mathrm{e}}_1] &= q^{-n}\operatorname{Odd}((n+1)_{q^2})[Q^{\mathrm{e}}_{n+1}] + q^{-n}\operatorname{Even}((n+1)_{q^2})[Q^{\mathrm{o}}_{n+1}].
\end{aligned}$$

If $2 \nmid n$, then we have

$$\begin{aligned}
[Q^{\mathrm{e}}_1] \cdot [Q^{\mathrm{e}}_n] = [Q^{\mathrm{e}}_n] \cdot [Q^{\mathrm{e}}_1] &= q^{-n}\operatorname{Odd}((n+1)_{q^2})[Q^{\mathrm{e}}_{n+1}] + q^{-n}\operatorname{Even}((n+1)_{q^2})[Q^{\mathrm{o}}_{n+1}],\\
[Q^{\mathrm{e}}_1] \cdot [Q^{\mathrm{o}}_n] = [Q^{\mathrm{o}}_n] \cdot [Q^{\mathrm{e}}_1] &= q^{-n}\operatorname{Even}((n+1)_{q^2})[Q^{\mathrm{e}}_{n+1}] + q^{-n}\operatorname{Odd}((n+1)_{q^2})[Q^{\mathrm{o}}_{n+1}].
\end{aligned}$$

Finally, we also have

$$[Q^{\mathrm{e}}_1] \cdot [Q^{\mathrm{e}}_1] = q[Q^{\mathrm{e}}_2] + q^{-1}[Q^{\mathrm{o}}_2].$$

In order to describe the comultiplication on $\mathrm{G}_0(\mathrm{ANH})$ and the multiplication on $\mathrm{K}_0(\mathrm{ANH})$, we define the $\mathbb{N}_0$-graded $\mathcal{A}$-algebra ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ as follows. For each $n \in \mathbb{N}_0$, let ${}_{\mathcal{A}}\tilde{\mathbf{f}}_n$ be the free $\mathcal{A}$-module of rank 1, with generator $\tilde{\theta}^{(n)}$. Then we set

$${}_{\mathcal{A}}\tilde{\mathbf{f}} := \bigoplus_{n \in \mathbb{N}_0} {}_{\mathcal{A}}\tilde{\mathbf{f}}_n$$

and endow ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ with the multiplication

$$\tilde{\theta}^{(m)} \cdot \tilde{\theta}^{(n)} = q^{-mn} \binom{m+n}{m}_{-q^2}, \quad \text{for } m, n \in \mathbb{N}_0.$$

Finally, we equip ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ with the obvious $\mathbb{N}_0$-grading. By construction, we can view ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ as a sign perturbated version of ${}_{\mathcal{A}}\mathbf{f}$. Now, let ${}_{\mathcal{A}}\tilde{\mathbf{f}}^*$ be the $\mathbb{N}_0$-graded dual $\mathcal{A}$-coalgebra of ${}_{\mathcal{A}}\tilde{\mathbf{f}}$. Then the comultiplication on ${}_{\mathcal{A}}\tilde{\mathbf{f}}^*$ can be characterized as follows. Let $(\tilde{\theta}^{(n)*})_{n\in\mathbb{N}_0}$ be the dual basis of $(\tilde{\theta}^{(n)})_{n\in\mathbb{N}_0}$. Then we have

$$\tilde{\theta}^{(n)*} \mapsto \sum_{r=0}^{n} q^{-r(n-r)} \binom{n}{r}_{-q^2} \tilde{\theta}^{(r)*} \otimes \tilde{\theta}^{(n-r)*}, \quad \text{for all } n \in \mathbb{N}_0.$$

According to the construction of ${}_{\mathcal{A}}\tilde{\mathbf{f}}$ and the formulas from Theorem 4.5.12, we then obtain the following description of the comultiplication on $\mathrm{K}_0(\mathrm{ANH})$.

Theorem 4.5.16. *Let $\mathcal{A}' := \mathcal{A}[\frac{1}{2}]$. There is an isomorphism of non-counital $\mathbb{N}_0$-graded $\mathcal{A}'$-coalgebras*

$$\psi : \mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*)_{\geq 2} \to \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH})_{\geq 2}$$

such that

$$1 \otimes \theta^{(n)*} \mapsto 1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}], \quad 1 \otimes \tilde{\theta}^{(n)*} \mapsto 1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}],$$

for all $n \in \mathbb{N}_0$ with $n \geq 2$. Moreover, we have a commuting diagram:

$$\begin{array}{ccc} \mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*)_{\geq 2} & \xrightarrow{\psi} & \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH})_{\geq 2} \\ {\scriptstyle\mathrm{inc}}\big\uparrow & & {\scriptstyle\mathrm{R_G}}\big\uparrow \\ \mathcal{A}' \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}^*_{\geq 2} & \xrightarrow{(\gamma^*)^{-1}} & \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{NH})_{\geq 2} \end{array}$$

Here, inc *is the canonical inclusion and* $\mathrm{R_K}$ *is the restriction homomorphism from Theorem 4.3.7.*

Proof. At first note that $(1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}], 1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}] \,|\, n \in \mathbb{N}_0, n \geq 2)$ is an $\mathcal{A}'$-basis of $\mathcal{A}' \otimes_{\mathcal{A}} \mathrm{G}_0(\mathrm{ANH})_{\geq 2}$. Thus, we conclude that ψ is an isomorphism of $\mathbb{N}_0$-graded $\mathcal{A}'$-modules. So it is left to show that ψ is compatible with the comultiplication. Let $r, s \in \mathbb{N}_0$ with $r, s \geq 2$ and we set $n := r + s$. Using the equations

$$\binom{n}{r}_{q^2} = \mathrm{Even}\left(\binom{n}{r}_{q^2}\right) + \mathrm{Odd}\left(\binom{n}{r}_{q^2}\right),$$
$$\binom{n}{r}_{-q^2} = \mathrm{Even}\left(\binom{n}{r}_{q^2}\right) - \mathrm{Odd}\left(\binom{n}{r}_{q^2}\right),$$

we directly conclude from Theorem 4.5.12 that

$$\begin{aligned} &\Delta_{\mathrm{G}}(1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}]) \\ &\qquad = q^{-r(n-r)} \binom{n}{r}_{q^2} ((1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}]) \otimes (1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}])), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\Delta_{\mathrm{G}}(1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}]) \\ &\qquad = q^{-r(n-r)} \binom{n}{r}_{-q^2} ((1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}]) \otimes (1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}])). \end{aligned} \tag{4.9}$$

Now, let Δ' denote the comultiplication on $\mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*)_{\geq 2}$. Then we have the following equalities

$$\begin{aligned}(\psi \otimes \psi)((\Delta'(1 \otimes \theta^{(n)*}))_{r,s}) &= q^{-r(n-r)}(\psi \otimes \psi)\left(\binom{n}{r}_{q^2}(1 \otimes \theta^{(n)*}) \otimes (1 \otimes \theta^{(n)*})\right) \\ &= q^{-r(n-r)}\binom{n}{r}_{q^2}((1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}]) \otimes (1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}])) \\ &\overset{(4.8)}{=} \Delta_{\mathrm{G}}(1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}]) \\ &= \Delta_{\mathrm{G}}(\psi(1 \otimes \theta^{(n)*})).\end{aligned}$$

Moreover, we have

$$\begin{aligned}(\psi \otimes \psi)((\Delta'(1 \otimes \tilde{\theta}^{(n)*}))_{r,s}) &= q^{-r(n-r)}(\psi \otimes \psi)\left(\binom{n}{r}_{-q^2}(1 \otimes \tilde{\theta}^{(n)*}) \otimes (1 \otimes \tilde{\theta}^{(n)*})\right) \\ &= q^{-r(n-r)}\binom{n}{r}_{-q^2}((1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}]) \otimes (1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}])) \\ &\overset{(4.9)}{=} \Delta_{\mathrm{G}}(1 \otimes [T_n^{\mathrm{e}}] - 1 \otimes [T_n^{\mathrm{o}}]) \\ &= \Delta_{\mathrm{G}}(\psi(1 \otimes \tilde{\theta}^{(n)*})),\end{aligned}$$

Thus, we proved that ψ is an isomorphism of non-counital $\mathbb{N}_0$-graded $\mathcal{A}'$-coalgebras. The commutativity of the stated diagram follows from

$$\mathrm{R}_{\mathrm{G}}((\gamma^*)^{-1}(1 \otimes \theta^{(n)*})) = \mathrm{R}_{\mathrm{G}}(1 \otimes [T_n]) = 1 \otimes [T_n^{\mathrm{e}}] + 1 \otimes [T_n^{\mathrm{o}}] = \psi(\mathrm{inc}(1 \otimes \theta^{(n)*})),$$

for all $n \in \mathbb{N}_0$ with $n \geq 2$. This finishes the proof. □

Again, by the duality between $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$, we obtain the corresponding description of the multilication on $\mathrm{K}_0(\mathrm{ANH})$. For this, note that the $\mathbb{N}_0$-graded dual $\mathcal{A}$-algebra of ${}_{\mathcal{A}}\mathbf{f}^* \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}^*$ is naturally isomorphic to the direct product algebra ${}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}}$. In the following corollary, we identify the underlying $\mathbb{N}_0$-graded $\mathcal{A}$-module of ${}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}}$ with the direct sum ${}_{\mathcal{A}}\mathbf{f} \oplus {}_{\mathcal{A}}\tilde{\mathbf{f}}$.

Corollary 4.5.17. *Let ψ be as in Theorem 4.5.16 and let*

$$\psi^* : \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH}) \to \mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}})$$

be the adjoint map. Then ψ^* is an isomorphism of non-unital $\mathbb{N}_0$-graded $\mathcal{A}'$-algebras. For each $n \in \mathbb{N}_0$, we have

$$\psi^*(1 \otimes [Q_n^{\mathrm{e}}]) = \begin{cases} \frac{1}{2}(1 \otimes \theta^{(n)} + 1 \otimes \tilde{\theta}^{(n)}) & \text{if } 4 | n(n-1), \\ \frac{1}{2}(1 \otimes \theta^{(n)} - 1 \otimes \tilde{\theta}^{(n)}) & \text{if } 4 \nmid n(n-1), \end{cases}$$

and

$$\psi^*(1 \otimes [Q_n^{\mathrm{o}}]) = \begin{cases} \frac{1}{2}(1 \otimes \theta^{(n)} - 1 \otimes \tilde{\theta}^{(n)}) & \text{if } 4 | n(n-1), \\ \frac{1}{2}(1 \otimes \theta^{(n)} + 1 \otimes \tilde{\theta}^{(n)}) & \text{if } 4 \nmid n(n-1). \end{cases}$$

In addition, we have a commuting diagram:

$$\begin{tikzcd} \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH})_{\geq 2} \arrow[r, "\psi^*"] \arrow[d, "\mathrm{I}_{\mathrm{K}}"] & \mathcal{A}' \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\tilde{\mathbf{f}})_{\geq 2} \arrow[d, "\mathrm{pr}"] \\ \mathcal{A}' \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{NH})_{\geq 2} \arrow[r, "\gamma^{-1}"] & {}_{\mathcal{A}}\mathbf{f}_{\geq 2} \end{tikzcd}$$

Here, pr *denotes the canonical projection and* I_{K} *is the induction homomorphism from Theorem 4.3.7.*

4.6 Applications

We now come to two applications of our characterization of the multiplicative and comultiplicative structures on $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$. At first, we show that $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are neither isomorphic as $\mathbb{N}_0$-graded $\mathcal{A}$-algebras nor as $\mathbb{N}_0$-graded $\mathcal{A}$-coalgebras. So this behavior is analogous to the corresponding property of the nil Hecke Grothendieck groups, as we discussed in Section 3.4.

Proposition 4.6.1. *We have that* $\mathrm{G}_0(\mathrm{ANH})$ *and* $\mathrm{K}_0(\mathrm{ANH})$ *are neither isomorphic as* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-algebras nor as as* $\mathbb{N}_0$*-graded* $\mathcal{A}$*-coalgebras.*

Proof. According to the duality between $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$, it suffices to show that $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are not isomorphic as $\mathbb{N}_0$-graded $\mathcal{A}$-algebras. For this, let $m, n \in \mathbb{N}_0$ with $m, n \geq 2$. Now, Theorem 4.5.6 implies that the multiplication map

$$\mathrm{G}_0(\mathrm{ANH})_m \times \mathrm{G}_0(\mathrm{ANH})_n \to \mathrm{G}_0(\mathrm{ANH})_{m+n}, \quad (x, y) \mapsto x \cdot y$$

is surjective. However, by Theorem 4.5.14, the image of the multiplication map

$$\mathrm{K}_0(\mathrm{ANH})_m \times \mathrm{K}_0(\mathrm{ANH})_n \to \mathrm{K}_0(\mathrm{ANH})_{m+n}, \quad (x, y) \mapsto x \cdot y,$$

is the $\mathcal{A}$-linear span of the elements

$$\xi := \mathrm{Even}\left(\binom{m+n}{n}_{q^2}\right)[Q^{\mathrm{e}}_{m+n}] + \mathrm{Odd}\left(\binom{m+n}{n}_{q^2}\right)[Q^{\mathrm{o}}_{m+n}]$$

and

$$\eta := \mathrm{Odd}\left(\binom{m+n}{n}_{q^2}\right)[Q^{\mathrm{e}}_{m+n}] + \mathrm{Even}\left(\binom{m+n}{n}_{q^2}\right)[Q^{\mathrm{o}}_{m+n}].$$

Finally, we observe that the $\mathcal{A}$-linear span of ξ and η is unequal to $\mathrm{K}_0(\mathrm{ANH})_{m+n}$. For this, note that

$$\det\begin{pmatrix}\mathrm{Even}(\binom{m+n}{n}_{q^2}) & \mathrm{Odd}(\binom{m+n}{n}_{q^2})\\ \mathrm{Odd}(\binom{m+n}{n}_{q^2}) & \mathrm{Even}(\binom{m+n}{n}_{q^2})\end{pmatrix} = \mathrm{Even}\left(\binom{m+n}{n}_{q^2}\right)^2 - \mathrm{Odd}\left(\binom{m+n}{n}_{q^2}\right)^2$$
$$= \left(\mathrm{Even}\left(\binom{m+n}{n}_{q^2}\right) + \mathrm{Odd}\left(\binom{m+n}{n}_{q^2}\right)\right)$$
$$\cdot \left(\mathrm{Even}\left(\binom{m+n}{n}_{q^2}\right) - \mathrm{Odd}\left(\binom{m+n}{n}_{q^2}\right)\right)$$
$$= \binom{m+n}{n}_{q^2}\binom{m+n}{n}_{-q^2}.$$

Hence, the above determinant is not a unit in $\mathcal{A}$. Indeed, we have that the units of $\mathcal{A}$ are $\{aq^i | a \in \{1,-1\}, i \in \mathbb{Z}\}$. However, according to (3.16), $\binom{m+n}{n}_{q^2}$ and $\binom{m+n}{n}_{-q^2}$ are polynomials in $\mathbb{Z}[q]$ of degree $2mn$ with absolute coefficient equal to 1. Thus, the product $\binom{m+n}{n}_{q^2}\binom{m+n}{n}_{-q^2}$ is not a unit in $\mathcal{A}$. Altogether, we conclude that the $\mathcal{A}$-linear span of ξ and η is a proper subset of $\mathrm{K}_0(\mathrm{ANH})_{m+n}$. It follows that $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are not isomorphic as $\mathbb{N}_0$-graded $\mathcal{A}$-algebras. □

Finally, we show that $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ are both no $\mathbb{N}_0$-graded twisted $\mathcal{A}$-bialgebras. So this points out a difference between the alternating nil Hecke Grothendieck groups and the nil Hecke Grothendieck groups.

Proposition 4.6.2. *Neither* $\mathrm{G}_0(\mathrm{ANH})$ *nor* $\mathrm{K}_0(\mathrm{ANH})$ *is an* $\mathbb{N}_0$*-graded twisted* $\mathcal{A}$*-bialgebra.*

Proof. By the duality between $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$, it suffices to show that only one of $\mathrm{G}_0(\mathrm{ANH})$ and $\mathrm{K}_0(\mathrm{ANH})$ is not an $\mathbb{N}_0$-graded twisted $\mathcal{A}$-bialgebra. We choose $\mathrm{K}_0(\mathrm{ANH})$. Recall that we endow $\mathrm{K}_0(\mathrm{ANH}) \otimes_{\mathcal{A}} \mathrm{K}_0(\mathrm{ANH})$ with the twisted multiplication as described in Definition 3.4.1. Now, by the formulas from Remark 4.5.4 and Remark 4.5.15, we have

$$\begin{aligned}\Delta_{\mathrm{K}}([Q_1^{\mathrm{e}}]\cdot[Q_1^{\mathrm{e}}])_{1,1} &= \Delta_{\mathrm{K}}(q[Q_2^{\mathrm{e}}])_{1,1} + \Delta_{\mathrm{K}}(q^{-1}[Q_2^{\mathrm{o}}])_{1,1}\\ &= (1+q^4)[Q_1^{\mathrm{e}}]\otimes[Q_1^{\mathrm{e}}] + 2[Q_1^{\mathrm{e}}]\otimes[Q_1^{\mathrm{e}}]\\ &= (3+q^4)([Q_1^{\mathrm{e}}]\otimes[Q_1^{\mathrm{e}}]).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(\Delta_{\mathrm{K}}([Q_1^{\mathrm{e}}])\cdot\Delta_{\mathrm{K}}([Q_1^{\mathrm{e}}]))_{1,1} &= ((1\otimes[Q_1^{\mathrm{e}}] + [Q_1^{\mathrm{e}}]\otimes 1)\cdot(1\otimes[Q_1^{\mathrm{e}}] + [Q_1^{\mathrm{e}}]\otimes 1))_{1,1}\\ &= (1+q^{-2})([Q_1^{\mathrm{e}}]\otimes[Q_1^{\mathrm{e}}]).\end{aligned}$$

Thus, $\mathrm{K}_0(\mathrm{ANH})$ is not an $\mathbb{N}_0$-graded twisted $\mathcal{A}$-bialgebra. □

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