Geometric Realisations for Tensor Products of Representations of $U_q(\mathfrak{sl}_2)$

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1 Introduction

In the first part (section 3), Savage‘s result [21] on a geometric approach to realising the canonical basis of finite tensor products of integrable highest weight representations of $U_q(\mathfrak{sl}_2)$ is presented, but I have added some explicit examples and an explicit geometric description of the varieties used. Then, not following [21] anylonger, I find an alternative geometric realisation of finite tensor products of integrable highest weight representations of $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ and their bases with an analogue for type D (section 4 and 5). In the first part, a tensor product variety $\mathfrak{T}(\mathbf{d})$, a special form of Nakjima tensor product variety, is considered first over $\mathbb{C}$, then over the finite field $\mathbf{F}_{q^2}$ with q^2 elements (or its algebraic closure $\overline{\mathbf{F}}_{q^2}$). This allows me for example to count points and is used in one proof (Proposition 3.8). However, the combinatorics do not depend on the particular q, so q can be treated as a variable, which becomes the variable q in the quantum group. A tensor product variety associated to the tensor product of a finite number of integrable highest weight representations of a Lie algebra $\mathfrak{g}$ of type ADE was defined in [18] and [19], though over $\mathbb{C}$. For $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^k$, an $U_q(\mathfrak{sl}_2)$-action on the space of invariant functions, $T(\mathbf{d})$, (with respect to a natural group action) from $U_q(\mathfrak{sl}_2)$ into $\mathbb{C}(q)$ is presented. Two distinct subspaces of invariant functions, $T_0(\mathbf{d})$ and $T_c(\mathbf{d})$, isomorphic to $V_{d_1} \otimes \cdots \otimes V_{d_k}$, are introduced. I also present a natural basis for each of them: a basis B_e corresponding to the elementary basis, and a basis B_c corresponding to Lusztig's canonical basis [17]. B_c is characterized by its relation to the irreducible components of $\mathfrak{T}(\mathbf{d})$. These irreducible components, defined over $\mathbb{F}_{q^2}$, are defined as the $\mathbb{F}_{q^2}$-points of the irreducible components of the corresponding variety $\mathfrak{T}(\mathbf{d})'$ over the algebraic closure $\overline{\mathbb{F}}_{q^2}$. Distinct elements of B_c are supported on distinct irreducible components of $\mathfrak{T}(\mathbf{d})$ (where the supports are not necessarily disjoint) and are nonzero on the dense points of this irreducible component. The dense points are defined as $\mathbb{F}_{q^2}$-points of certain dense subsets of the irreducible components of $\mathfrak{T}(\mathbf{d})'$. The notation in the first part is mostly taken from [21]. The following conventions will be used throughout the thesis, unless otherwise stated. The topology used will always be the Zariski topology and a function on an algebraic variety will be a function into $\mathbb{C}(q)$, the field of rational functions in an indeterminate q. The span of a set of such will be their $\mathbb{C}(q)$ span. The support is defined as $\{x \mid f(x) \neq 0\}$. At several instances, the graphical calculus of intertwiners of $U_q(\mathfrak{sl}_2)$ will be used. This was introduced by Penrose, Kauffman and others, and is expanded in [6], see also [7]. In subsection 2.7, this is explained a little as well.

In the second part I return to $\mathbb{C}$ as ground field and define B^i as the set of functions from $W_i = S_d/(S_i \times S_{d-i})$ to $\mathbb{C}$ and C_{func} as the direct sum over all i of the sets of B^i-modules. My main result is Theroem 4.3: I define an isomorphism between $K_0(C_{func})$ and $V_1^{\otimes d}$, sending a natural basis of $K_0(C_{func})$, consisting of isomorphism classes of irreducible elements of C_{func}, to the elementary basis of $V_1^{\otimes d}$. This can be restricted to an isomorphism from a subcategory C'_{func} onto $V_{d_1} \otimes \cdots \otimes V_{d_k}$ (where it sends again a basis corresponding to simple modules in C'_{func} to the elementary basis) and can be defined both for $U(\mathfrak{sl}_2)$- and $U_q(\mathfrak{sl}_2)$-modules. An analogue for type D is presented as well.

I want to thank my supervisor, Prof. Stroppel, for suggesting this topic, and for her help in the development.

2 Finite Dimensional Representations of $U_q(\mathfrak{sl}_2)$

2.1 Some Definitions

Definition 2.1. Let $\mathbb{C}(q)$ denote the field of rational functions in an indeterminate q. Then define the quantum group $U_q(\mathfrak{sl}_2)$ (or U_q as a shorthand) as the associative algebra over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations

$$\begin{aligned} KK^{-1} = &\ K^{-1}K = 1 \\ KE = &\ q^2 EK \\ KF = &\ q^{-2} FK \\ EF - FE = &\ \tfrac{K-K^{-1}}{q-q^{-1}} \end{aligned} .$$

Remark 1. The quantum group can be defined more generally for any finite dimensional simple Lie algebra, see e.g. [14].

U_q has the structure of a Hopf algebra with the following comultiplication ([11])

$$\Delta: \begin{array}{rl} K^{\pm1} \mapsto & K^{\pm1} \otimes K^{\pm1} \\ E \mapsto & E \otimes 1 + K \otimes E \\ F \mapsto & F \otimes K^{-1} + 1 \otimes F. \end{array}$$

Hence tensor products of representations are again representations via

$$\Delta^{(k-1)}: \begin{array}{rl} K^{\pm1} \mapsto & K^{\pm1} \otimes \cdots \otimes K^{\pm1} \\ E \mapsto & \sum_{i=1}^{k} K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1 \\ F \mapsto & \sum_{i=1}^{k} 1 \otimes \cdots \otimes F \otimes K^{-1} \otimes \cdots \otimes K^{-1} \end{array}$$

(where E respectivly F in the second respecitvly third row are in the ith position).

Definition 2.2. Define an antiinvolution w, the Cartan antiinvolution, by

$$w(E) = F,\ w(F) = E,\ (K^{\pm1}) = K^{\pm1},\ w(q^{\pm1}) = q^{\pm1},\ w(xy) = w(y)w(x) \text{ for } x, y \in U_q.$$

Define also a second comultiplication $\overline{\Delta}$, using the so called "bar" involution σ. This will be used later to let the quantum group act on the dual space in a bilinear pairing. Set

$$\sigma(E) = E,\ \sigma(F) = F,\ \sigma(K^{\pm1}) = K^{\mp1},\ \sigma(q^{\pm1}) = q^{\mp1},\ \sigma(xy) = \sigma(x)\sigma(y) \text{ for } x, y \in U_q$$

and define

$$\overline{\Delta}(x) = (\sigma \otimes \sigma)\Delta(\sigma(x)), \text{ for } x \in U_q.$$

So

$$\overline{\Delta}: \begin{array}{rl} K^{\pm1} \mapsto & K^{\pm1} \otimes K^{\pm1} \\ E \mapsto & E \otimes 1 + K^{-1} \otimes E \\ F \mapsto & F \otimes K + 1 \otimes F \end{array}.$$

Recall from [14] that any finite $d+1$-dimensional irreducible U_q-module V is generated by a highest weight vector v_d of heighest weight ϵq^d, $\epsilon = \pm1$. In this thesis, only those of type I, i.e. with $\epsilon = 1$ are considered. Fixing ϵ, there is only one irreducible module in each dimension (up to isomorphism). Let V_d denote the $d+1$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-representation.

Definition 2.3. Define $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$, $[k]! = [k][k-1]\cdots[2][1]$ and $\begin{bmatrix} d \\ k \end{bmatrix} = \frac{[d]!}{[k]![d-k]!}$.

Set $v_{d-2k} = F^k v_d/[k]!$. Since $\dim(V_d) = d+1$ and V_d is irreducible, I have $v_{d-2k} = 0$ for $k > d$ and a basis of V_d is given by $\{v_d, v_{d-2}, \cdots, v_{-d}\}$. Then

$$\begin{array}{rl} K^{\pm1} v_m = & q^{\pm m} v_m \\ E v_m = & \left[\frac{d+m}{2} + 1\right] v_{m+2} \\ F v_m = & \left[\frac{d-m}{2} + 1\right] v_{m-2}. \end{array}$$

Define a bilinear symmetric pairing on V_d by $\langle v_{d-2k}, v_{d-2l}\rangle = \delta_{k,l}\begin{bmatrix} d \\ k \end{bmatrix}$. Then a straightforward calculation shows that this implies the conditions

$$\langle xu, v\rangle = \langle u, w(x)v\rangle,\ \langle v_d, v_d\rangle = 1 \quad \forall\, u, v \in V_d \text{ and } x \in U_q.$$

The dual basis with respect to the bilinear form is given by $v^{d-2k} = \begin{bmatrix} d \\ k \end{bmatrix}^{-1} v_{d-2k}$ with the action of U_q given by

$$\begin{aligned} K^{\pm 1}v^m &= q^{\pm m}v^m \\ Ev^m &= \left[\tfrac{d-m}{2}\right] v^{m+2} \\ Fv^m &= \left[\tfrac{d+m}{2}\right] v^{m-2} \end{aligned}.$$

Now consider tensor products of representations. Define a bilinear pairing

$$\langle -, -\rangle : V_{d_1} \otimes \cdots \otimes V_{d_k} \times V_{d_k} \otimes \cdots \otimes V_{d_1} \to \mathbb{C}$$

by

$$\langle v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1}\rangle = \delta_{i_1,l_1} \cdots \delta_{i_k,l_k}. \tag{1}$$

Note that this definition agrees with the earlier one for just one tensor factor. One can calculate that

$$\langle \Delta^{(k-1)}(x)v_{i_1} \otimes \cdots \otimes v_{i_k}, v^{l_k} \otimes \cdots \otimes v^{l_1}\rangle = \langle v_{i_1} \otimes \cdots \otimes v_{i_k}, \overline{\Delta}^{(k-1)}(w(x))v^{l_k} \otimes \cdots \otimes v^{l_1}\rangle.$$

Here the alternativ comultiplication is used.
When considering a tensor product of simple modules, the action of U_q on a standard basis vector of the form $v_{i_1} \otimes \cdots \otimes v_{i_k}$ does not in general give another standard basis vector, but rather a linear combination of several standard basis vectors. Therefore one wants to find some other basis on which U_q acts particularily nicely. This is called the canonical basis and denoted by $v_{i_1}\diamondsuit \cdots \diamondsuit v_{i_k}$ (see [17]). Denote its dual with respect to the bilinear pairing (1) by $v^{i_d}\heartsuit \cdots \heartsuit v^{i_1}$. The notion of a based module going back to Lusztig ([17]) makes it precise what it means that U_q "acts nicely" one a basis. In more detail, let A denote $\mathbb{Z}[q, q^{-1}]$ and consider finite-dimensional U_q-modules of type I. Any such module M has a decomposition $M = \bigoplus_{\lambda\in\mathbb{Z}} M^\lambda$ into weight spaces

$$M^\lambda = \{m \in M \mid Km = q^\lambda m\}.$$

Let B be a $\mathbb{C}(q)$-basis of M. Define an involution $\sigma_B : M \to M$ by

$$\sigma_B(fb) = \overline{f}b \quad \forall f \in \mathbb{C}(q),\, b \in B$$

(where $^-\colon \mathbb{C}(q) \to \mathbb{C}(q)$ such that $\overline{q^n} = q^{-n}$ for all n, is a $\mathbb{C}$-algebra involution). Then (M, B) is called a based module (with respect to the choice of generators $E, F, K^{\pm 1}$ of U_q) if the following conditions are satisfied:

1. $B \cap M^\lambda$ is a basis of M^λ, for any $\lambda \in \mathbb{Z}$ (so in particular all elements of B are weight vectors)

2. The $A-$submodule M_A generated by B is stable under $\frac{E^n}{[n]!}$ and $\frac{F^n}{[n]!}$;

3. The involution σ_B is compatible with the bar involution σ on U_q in the sense that $\sigma_B(xm) = \sigma(x)\sigma_B(m)$ for all $x \in U_q$, $m \in M$;

4. B is a crystal basis of M at ∞.

For the notion of a crystal basis, see [13] (e.g. the $\{v_i\}_{i=-d}^{d}$ are a crystal basis of V_d at ∞).

Lemma 2.4. *V_d is a based module with $\mathbb{C}(q)$-basis $B = \{v_d, v_{d-2}, \cdots, v_{-d}\}$ and involution σ_B as described above.*

Proof: As v_m is a basis of the weight space of V_d associated to the weight m, the first condition is satisfied. Moreover, by the definition of the action of E and F, the second condition is satisfied as well. Now, to see that the third condition is satisfied, consider

$$\sigma_B(xv_m) = \begin{cases} \overline{\left[\frac{d+m}{2}+1\right]} v_{m+2} & x = E \\ \overline{\left[\frac{d-m}{2}+1\right]} v_{m-2} & x = F \\ q^{\mp 1} v_m & x = K^{\pm 1} \end{cases} = \sigma(x)\sigma_B(v_m).$$

□

Lemma 2.5. *The direct sum of two based modules (M, B) and (M', B') is again a based module $(M \oplus M', B \oplus B')$*

Proof: As $x(m + m') = xm + xm'$ and $\sigma_{B\oplus B'}(m + m') := \sigma_B(m) + \sigma_{B'}(m')$, the first three conditions are satisfied. For the fourth condition, see [11]. □

Since all the representations considered here are semisimple, the above gives a description of tensor products of representations as based modules. However, one wants to have an intrinsic structure of based module for tensor products of representations, but the tensor product with the obvious basis $B \otimes B'$ does not in general satisfy property 3) of the definition. Lusztig introduces a modified basis $B \diamondsuit B'$ in the tensor product as follows:
Let $\Psi : M \otimes M' \to M \otimes M'$ be given by

$$\Psi(m \otimes m') = \Theta(\sigma_B(m) \otimes \sigma_{B'}(m')),$$

where

$$\Theta = \sum_{n\geq 0} (-1)^n q^{-n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!} F^n \otimes E^n \in \widehat{U_q \otimes U_q}, \text{ a completion of } U_q \otimes U_q.$$

A quick calculation shows that

$$\Psi^2 = 1$$

follows from $\Theta\overline{\Theta} = 1 \otimes 1$, which can be shown by a somewhat more lenghty and not entirely trivial calculation. Moreover, on can show

$$\Psi(x(m \otimes m')) = \sigma(x)\Psi(m \otimes m'), \; x \in U_q$$

(One has $\overline{\Delta}(x) = (\sigma \otimes \sigma)\Delta(\sigma(x))$, $\Theta\overline{\Delta} = \Delta\Theta$ and $\sigma_B(xm) = \sigma(x)\sigma_B(m)$, so

$$\begin{aligned}
\Psi(x(m \otimes m')) = \; & \Psi(\Delta(x)(m \otimes m')) = & \Theta(\sigma_B \otimes \sigma_{B'}(\Delta(x)(m \otimes m'))) \\
= \; & \Theta(\sigma \otimes \sigma(\Delta(x)(\sigma_B(m) \otimes \sigma_{B'}(m')))) = & \Theta(\overline{\Delta}(\sigma(x))(\sigma_B(m) \otimes \sigma_{B'}(m'))) \\
= \; & \Delta(\sigma(x))\Theta(\sigma_B(m) \otimes \sigma_{B'}(m')) = & \sigma(x)\Psi(m \otimes m')).
\end{aligned}$$

Set $\sigma_{B\diamondsuit B'} = \Psi$ and let $M \otimes M'_A$ (respectivly $M \otimes M'_{\mathbb{Z}[q^{-1}]}$) be the A- (resp. $\mathbb{Z}[q^{-1}]$-) submodule of $M \otimes M'$ generated by the basis $B \otimes B'$. The set $B \times B'$ has a partial ordering such that

$$(b_1, b_1') \geq (b_2, b_2') \Leftrightarrow \quad \begin{array}{c} b_i \in M^{\lambda_i},\ b_i' \in M'^{\lambda_i'} \text{ with} \\ \lambda_1 \geq \lambda_2,\ \lambda_1' \leq \lambda_2',\ \lambda_1 + \lambda_1' = \lambda_2 + \lambda_2'. \end{array}$$

Example 1. Let $M = M' = V_1$. Then $B = B' = \{v_1, v_{-1}\}$ and $v_1 \in (V_1)^1$, $v_{-1} \in (V_1)^{-1}$. So $(v_1, v_{-1}) \geq (v_{-1}, v_1)$, and of course the trivial relations $(v_1, v_1) \geq (v_1, v_1)$, $(v_{-1}, v_{-1}) \geq (v_{-1}, v_{-1})$ hold.

Then Lusztig proves the following result:

Theorem 2.6. *1. For any $(b_1, b_1') \in B \times B'$, there is a unique element $b_1 \diamondsuit b_1' \in M \otimes M'_{\mathbb{Z}[q^{-1}]}$ such that*

$$\Psi(b_1 \diamondsuit b_1') = b_1 \diamondsuit b_1'$$

and $b_1 \diamondsuit b_1' - b_1 \otimes b_1' \in q^{-1} M \otimes M'_{\mathbb{Z}[q^{-1}]}$.

2. *The element $b_1 \diamondsuit b_1'$ is equal to $b_1 \otimes b_1'$ plus a linear combination of elements $b_2 \otimes b_2'$ with $(b_2, b_2') \in B \times B'$, $(b_2, b_2') < (b_1, b_1')$ and coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.*

3. *These elements $b_1 \diamondsuit b_1'$ form a $\mathbb{C}(q)$-basis $B \diamondsuit B'$ of $M \otimes M'$, an A-basis of $M \otimes M'_A$, and a $\mathbb{Z}[q^{-1}]$-basis of $M \otimes M'_{\mathbb{Z}[q^{-1}]}$.*

4. *$(M \otimes M', B \diamondsuit B')$ is a based module with associated involution Ψ (so Ψ takes the role of σ_B).*

For more details, see [6], on which the preceeding paragraph, starting with Lusztig´s notion of based module, is based. An example for based modules are tensor products of irreducible representations with canonical bases (where the canonical basis is the basis defined in theorem 2.6 above).

Example 2. Again, consider $V_1 \otimes V_1$. The canonical basis is given by $\{v_{-1} \diamondsuit v_{-1} = v_{-1} \otimes v_{-1},\ v_{-1} \diamondsuit v_1 = v_{-1} \otimes v_1,\ v_1 \diamondsuit v_{-1} = v_1 \otimes v_{-1} + q^{-1} v_{-1} \otimes v_1,\ v_1 \diamondsuit v_1 = v_1 \otimes v_1\}$.

Write

$$\begin{aligned}
\otimes^{\mathbf{d}} v_{\mathbf{w}} = \; & v_{d_1 - 2w_1} \otimes \cdots \otimes v_{d_k - 2w_k} \\
\otimes^{\mathbf{d}} v^{\mathbf{w}} = \; & v^{d_1 - 2w_1} \otimes \cdots \otimes v^{d_k - 2w_k} \\
\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = \; & v_{d_1 - 2w_1} \diamondsuit \cdots \diamondsuit v_{d_k - 2w_k} \\
\heartsuit^{\mathbf{d}} v^{\mathbf{w}} = \; & v^{d_1 - 2w_1} \heartsuit \ldots \heartsuit v^{d_k - 2w_k}
\end{aligned}$$

where $\mathbf{d}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^k$. The bar involution σ can be extended to tensor products of irreducible representations in the following way [21]: Define $\hat{\sigma}$ by

$$\hat{\sigma}(\otimes^{\mathbf{d}} v_{\mathbf{w}}) = \otimes^{\mathbf{d}} v_{\mathbf{w}}$$

and extend it antilinearly via

$$\hat{\sigma}(f(q)(\otimes^{\mathbf{d}} v_{\mathbf{w}})) = f(q^{-1})(\otimes^{\mathbf{d}} v_{\mathbf{w}})$$

for any polynomial $f(q)$ in q, and extend by $\mathbb{C}$-linearity. Then $\hat{\sigma}$ is an isomorphism from $V_{d_1} \otimes \cdots \otimes V_{d_k}$ to itself and

$$\hat{\sigma}(\Delta^{(k-1)}(x)v) = ((\sigma \otimes \cdots \otimes \sigma)(\Delta^{(k-1)}x))(\hat{\sigma}v)$$

with $x \in U_q$, $v \in V_{d_1} \otimes \cdots \otimes V_{d_k}$ (so $\hat{\sigma}$ is the involution σ_B associated to σ as in the definition of based module above).

Now consider the space of intertwiners $\mathrm{Hom}_{U_q}(V_{d_1} \otimes \cdots \otimes V_{d_k}, V_{e_1} \otimes \cdots \otimes V_{e_l})$, consisting of intertwiners commuting with the U_q-action given by $(\Delta)^{(k-1)}$. A basis can be identified with the set of crossingless matchings $CM^{e_1,\cdots,e_l}_{d_1,\cdots,d_k}$ (for more details, see [6], [7]). However, the intertwiners used in [6] and [7] are commuting with the action of U_q given by $\overline{\Delta}^{(k-1)}$. For such an intertwiner $\tilde{\gamma}$, define $\gamma = \hat{\sigma}\tilde{\gamma}\hat{\sigma}$. Then γ is an intertwiner commuting with the action of U_q given by $\Delta^{(k-1)}$, as

$$\begin{aligned}
\gamma\Delta^{(k-1)}(x)(v) = & \ \hat{\sigma}\tilde{\gamma}\hat{\sigma}\Delta^{(k-1)}(x)(v) \\
= & \ \hat{\sigma}\tilde{\gamma}((\sigma \otimes \cdots \otimes \sigma)\Delta^{(k-1)}(x))(\hat{\sigma}v) \\
= & \ \hat{\sigma}\tilde{\gamma}\overline{\Delta}^{(k-1)}(\sigma x)(\hat{\sigma}v) \\
= & \ \hat{\sigma}\overline{\Delta}^{(k-1)}(\sigma x)\tilde{\gamma}(\hat{\sigma}v) \\
= & \ \hat{\sigma}((\sigma \otimes \cdots \otimes \sigma)\Delta^{(k-1)}(x))\hat{\sigma}\gamma(v) \\
= & \ \Delta^{(k-1)}(x)\gamma(v)
\end{aligned}$$

for $x \in U_q$ and $v \in V_{d_1} \otimes \cdots \otimes V_{d_k}$[21].

2.2 Diagrammatics of Intertwiners

The definitions of the crossingless matchings are taken from [21].

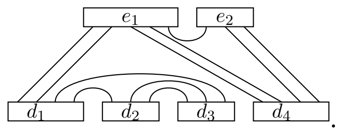
Definition 2.7. Depict V_d by a box with d vertices, marked with a d, and define the set of crossingless matchings $CM^{e_1,\cdots,e_l}_{d_1,\cdots,d_k}$ to be the set of non-intersecting curves in the plane (up to isotopy) connecting the vertices between a horizontal line consisting of the boxes depicting the V_{d_i} and another horizontal line above consisting of the boxes depicting the V_{e_i}, where the curves satisfy the following conditions:

1. Each curve connects exactly two vertices
2. Each vertex is endpoint of exactly one curve
3. No curve connects vertices of the same box

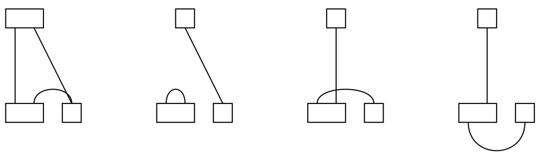
4. All curves lie inside the space bounded by the two horizontal lines and the vertical lines through the extreme left and right points.

The curves connecting two lower vertices are called *lower curves* or *caps*, those connecting upper vertices are called *upper curves* or *cups* and the remaining curves connecting an upper and a lower vertex are called *middle curves*.

Example 3. Let $\mathbf{d} = (4, 3, 3, 4)$ and $\mathbf{e} = (5, 3)$.
A crossingless matching:



The following are no crossingless matchings:



To see how a crossingless matching is associated to an intertwiner, see [7] and the following rough explanation: Fix maps $V_n \hookrightarrow V_1^{\otimes n}$, $V_1^{\otimes n} \twoheadrightarrow V_n$ and an identification between V_n and its dual. One has

$$V_{d_1} \otimes V_{d_2} \otimes V_{d_3} \otimes V_{d_4} \hookrightarrow V_1^{\otimes d_1} \otimes V_1^{\otimes d_2} \otimes V_1^{\otimes d_3} \otimes V_1^{\otimes d_4} = V_1^{\otimes \sum_{i=1}^4 d_i}$$

and similar

$$V_1^{\otimes e_1+e_2} \twoheadrightarrow V_{e_1} \otimes V_{e_2}.$$

Moreover, $V_1 \cong V_1^*$ canonically and there is a natural map $V_1 \otimes V_1 \cong V_1 \otimes V_1^* \to \mathbb{Q}(q)$, $v \otimes f \mapsto f(v)$, which is denoted by a cap. Similarly, a map $\mathbb{Q}(q) \to V_1 \otimes V_1$ can be defined, denoted by a cup. Then the crossingless match defines a map

$$V_{d_1} \otimes \cdots \otimes V_{d_4} \to V_{e_1} \otimes V_{e_2}$$

as composite of

$$V_{d_1} \otimes V_{d_2} \otimes V_{d_3} \otimes V_{d_4} \hookrightarrow V_1^{\otimes \sum_{i=1}^4 d_i} \text{ and } V_1^{\otimes e_1+e_2} \twoheadrightarrow V_{e_1} \otimes V_{e_2}$$

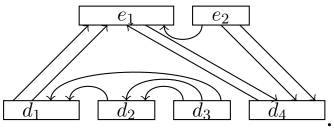
with

$$V_1^{\otimes \sum_{i=1}^4 d_i} \to V_1^{\otimes e_1+e_2}.$$

Middle curves map a V_1 in the tensor product $V_1^{\otimes \sum_{i=1}^4 d_i}$ to a V_1 in the tensor product $V_1^{\otimes e_1+e_2}$, and cups and caps act as described above.
Elements of the set of oriented crossingless matchings $OCM_{d_1,\cdots,d_k}^{e_1,\cdots,e_l}$ are given by crossingless matchings together with an orientation such that all upper and lower curves are oriented to

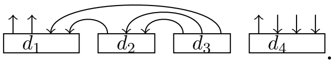
the left (i.e. if the curve connects vertices a and b and a is to the left of b, the curve must be oriented such that the arrow would point away from b if the arrow was placed at the right end of the curve) and the middle curves oriented upwards are to the left of the middle curves oriented downwards.

Example 4. An oriented crossingless matching:



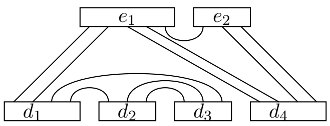
Furthermore define the set of lower crossingless matchings $LCM_{d_1,\cdots,d_k}$ and oriented lower crossingless matchings $OLCM_{d_1,\cdots,d_k}$. Elements are obtained by removing the upper boxes from elements of $CM^{e_1,\cdots,e_l}_{d_1,\cdots,d_k}$ respectivly $OCM^{e_1,\cdots,e_l}_{d_1,\cdots,d_k}$, converting middle curves to vertical rays, and keeping the orientation of the curves in the case of $OLCM_{d_1,\cdots,d_k}$. So in the case of $OLCM_{d_1,\cdots,d_k}$, the vertices oriented up must be to the left of those oriented down, as for the middle curves of $OCM^{e_1,\cdots,e_l}_{d_1,\cdots,d_k}$. Upper crossingless matchings are defined in an analogous way.

Example 5. An oriented lower crossingless matching:

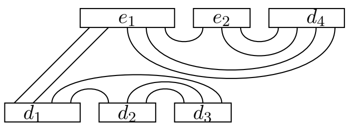


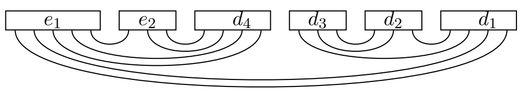
Remark 2. This is not taken from [21]. Using $V_1 \cong V_1^*$ and the canonical isomorphisms $\mathrm{Hom}_{U_q}(V \otimes W, X) \cong \mathrm{Hom}_{U_q}(V, \mathrm{Hom}(W, X)) \cong \mathrm{Hom}_{U_q}(V, \mathrm{Hom}(W, X^*)) \cong \mathrm{Hom}_{U_q}(V, (W \otimes X)^*) \cong \mathrm{Hom}_{U_q}(V, X \otimes W)$, one obtains (The isomorphisms correspond to the operations on the matchings, see [23, Chapter VI 3.2])

$$\mathrm{Hom}_{U_q}(V_{d_1} \otimes \cdots \otimes V_{d_k}, V_{e_1} \otimes \cdots \otimes V_{e_l})$$



$$\cong \mathrm{Hom}_{U_q}(V_{d_1} \otimes \cdots \otimes V_{d_{k-1}}, V_{e_1} \otimes \cdots \otimes V_{e_l} \otimes V_{d_k})$$



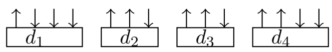


$$\cong \mathrm{Hom}_{U_q}(\mathbb{Q}(q), V_{e_1} \otimes \cdots \otimes V_{e_l} \otimes V_{d_k} \otimes \cdots \otimes V_{d_1})$$
$$\cong (V_{e_1} \otimes \cdots \otimes V_{e_l} \otimes V_{d_k} \otimes \cdots \otimes V_{d_1})^{Inv}$$

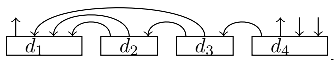
since a map $f : \mathbb{Q}(q) \to V_{e_1} \otimes \cdots \otimes V_{e_l} \otimes V_{d_k} \otimes \cdots \otimes V_{d_1}$ is given by $f(1)$ and E, F act trivially on $\mathbb{Q}(q)$. This illustrates a relation between upper or lower crossingless matchings without vertical rays and general crossingless matchings and gives an easy way of obtaining the elements of a tensor product of representations invariant under the action of U_q.

Given $\mathbf{a}$ and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^k$ with $a_i \leq d_i \,\forall\, i$, a lower oriented crossingless matching $M(\mathbf{d}, \mathbf{a}) \in OLCM_{d_1,\cdots,d_k}$ can be associated to it as follows [21]:
For each i, place downwards oriented arrows on the rightmost a_i vertices of the box representing V_{d_i}, and upwards oriented arrows on the remaining $d_i - a_i$ vertices. There is a unique way to connect the vertices such that $M(\mathbf{d}, \mathbf{a})$ forms a lower oriented crossingless matching, respecting the orientation of the arrows on the vertices. Starting from the right, connect each down arrow to the first up arrow to its right not already connected, if there is any (as the up arrows of each box are the the left of the down arrows in the same box, the resulting curves do not connect vertices of the same box). This produces a lower oriented crossingless matching with all unmatched downwards oriented arrows to the right of all unmatched upwards oriented arrows, as required.

Example 6. Let $\mathbf{d} = (4, 3, 3, 4)$.
Orientation of arrows for $\mathbf{d} = (4, 3, 3, 4)$ and $\mathbf{a} = (3, 1, 1, 2)$:



and resulting $M(\mathbf{d}, \mathbf{a})$:



.

Lemma 2.8. *The correspondence between $OLCM_{d_1,\cdots,d_k}$ and $\{\mathbf{a} \in \mathbb{Z}_{\geq 0}^k \mid a_i \leq d_i \,\forall i\}$ is one to one.*

Proof: From the definition it becomes clear that any element of $OLCM_{d_1,\cdots,d_k}$ can be associated to precisely one such $\mathbf{a}$. a_i denotes the number of down arrows of the ith box d_i and by fixing the order in which arrows are connected, only one lower oriented crossingless matching is associated to an $\mathbf{a}$. $\square$

Definition 2.9. [21] A partial order on the sets $CM_{d_1,\cdots,d_k}^{e_1,\cdots,e_l}$, $OCM_{d_1,\cdots,d_k}^{e_1,\cdots,e_l}$, $LCM_{d_1,\cdots,d_k}$ and $OLCM_{d_1,\cdots,d_k}$ can be defined by setting $S_1 \leq S_2$ if the set of lower curves of S_1 is a subset of the set of lower curves of S_2, for any two elements S_1, S_2 of one of these sets.

3 A Geometric Realisation of the Canonical Basis

I want to realise the canonical basis of a representation of $U_q(\mathfrak{sl}_2)$ geometrically.

3.1 The Tensor Product Variety

Let $D = \mathbb{C}^d$ and let $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^k$ such that $\sum_{i=1}^k d_i = d$.

Definition 3.1. Let k be an arbitrary field and set $GL_k(d) = \{f \in \mathrm{End}(k^d) \mid f \text{ invertible}\}$ and $\mathfrak{gl}_k(d) = \mathrm{End}(k^d)$. If the ground field is clear, I will write $GL(d)$ respectivly $\mathfrak{gl}(d)$ instead of $GL_k(d)$ respectivly $\mathfrak{gl}_k(d)$.

I now assume $k = \mathbb{C}$.

Definition 3.2. Fix $\mathbf{d}$. Define the variety $Fl(\mathbf{d})$ of flags of type $\mathbf{d}$ via

$$Fl(\mathbf{d}) = \{\mathbf{D} = \{D_i\}_{i=0}^{k} \mid 0 = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_{k-1} \subseteq D_k = D, \dim(D_i/D_{i-1}) = d_i\}.$$

Note that this makes sense over an arbitrary field.

Remark 3. The variety of partial flags $Fl(\mathbf{d})$ can be identified with the set of parabolic subalgebras of $\mathfrak{gl}(d)$ of type $\mathbf{d}$ (see [10]) via

$$\begin{array}{rcl} Fl(\mathbf{d}) & \xrightarrow{\sim} & \{\text{parabolic subalgebras of } \mathfrak{gl}(d) \text{ of type } \mathbf{d}\} \\ \mathbf{D} = D_1 \subset \cdots \subset D_k & \longmapsto & \{x \mid xD_i \subset D_i \forall i\} = p(\mathbf{D}) = \text{“stabilizer”of } \mathbf{D}. \end{array}$$

All parabolics of type $\mathbf{d}$ are $GL(d)$-conjugate to the standard parabolic of type $\mathbf{d}$,

$$p_{\mathbf{d}} = \left\{ \begin{array}{c} d_1 \\ d_2 \\ \vdots \\ d_k \end{array} \begin{pmatrix} * & & & * \\ & * & & \\ & 0 & \ddots & \\ & & & * \end{pmatrix} \right\}$$

(and all the subalgebras conjugate to $p_{\mathbf{d}}$ are parabolics of type $\mathbf{d}$). Now fix the “standard”flag

$$\mathbf{D}_{st} = < e_1, \cdots, e_{d_1} > \subset < e_1, \cdots, e_{d_1+d_2} > \subset \cdots \subset < e_1, \cdots, e_d >;$$

then its stabilizer is $p_{\mathbf{d}}$. An arbitrary element $\mathbf{D} \in Fl(\mathbf{d})$ is therefore of the form $g\mathbf{D}_{st}$ for some $g \in GL(d)$. Then

$$xD_i \subset D_i \Leftrightarrow xg(\mathbf{D}_{st})_i \subset g(\mathbf{D}_{st})_i \Leftrightarrow g^{-1}xg(\mathbf{D}_{st})_i \subset (\mathbf{D}_{st})_i,$$

so the stabilizer of $\mathbf{D}$ is the conjugate by g of the stabilizer $p_{\mathbf{d}}$ of $\mathbf{D}_{st}$. Note that here one needs the ground field to be $\mathbb{C}$.

Remark 4. One has furthermore $Fl(\mathbf{d}) \hat{=} Gl(d)/p$ for a parabolic subgroup p of the correct type $\mathbf{d}$ (identify a flag with the coset of matricies sending the standard basis to a basis compatible with the flag). $Gl(d)/p$ is a subvariety of the product of projective spaces $G(d_1, d) \times \ldots \times G(d_k, d)$ (see [10, section 1.8]), where $G(l, d)$ gets a projective structure in the following way:
Consider the exterior algebra $\bigwedge D$. $\bigwedge^d D$ is 1-dimensional. If V is a subspace, then $\bigwedge^l W$ may be identified canonically with a subspace of $\bigwedge^l D$. Thus there is a map $G(l, d) \to \mathbb{P}(\bigwedge^l D)$ sending a subspace V to the corresponding point in projective space belonging to $\bigwedge^l V$. Moreover, the cartesian product of projective varieties can be viewed again as a projective variety.
$Gl(d)/p$ is projective because one can embed it into the product of projective spaces (or either because it is a homogeneous space). The projective space $\mathbb{P}(D)$ is a special example of some G/p, namely the one where p has 2 blocks of size 1 and $d - 1$.

Definition 3.3. Define the tensor product variety

$$\mathfrak{T}(\mathbf{d}) = \{(\mathbf{D} = \{D_i\}_{i=0}^{k}, W, t) \mid \mathbf{D} \in Fl(\mathbf{d}), W \subseteq D, t \in \mathrm{End}(D), t(D_i) \subseteq D_{i-1}, \mathrm{im}(t) \subseteq W \subseteq \ker(t)\}$$

with subvariety

$$\mathfrak{T}_0(\mathbf{d}) = \{(\mathbf{D}, W, 0) \in \mathfrak{T}(\mathbf{d})\} = Fl(\mathbf{d}) \times \coprod_{i=0}^{d} G(i, d),$$

where $G(i, d)$ denotes a Grassmannian of subspaces of dimension i.

Remark 5. Using remark 3, the tensor product variety can be described as follows:
Denote the standard parabolics of type $\mathbf{d}$ respectivly $(i, d-i)$ by $p_{\mathbf{d}}$ respectivly $p_{(i,d-i)}$. Then these contain the standard Levi subalgebras $l_{\mathbf{d}}$ respectivly $l_{(i,d-i)}$ given by the elements with zeros outside the blockmatricies on the diagonal. Moreover, there are the unipotent subalgebras $u_{\mathbf{d}}$ respectivly $u_{(i,d-i)}$ consisting of the matricies with zeros in the blockmatricies and underneath, such that $p_{\mathbf{d}} = l_{\mathbf{d}} + u_{\mathbf{d}}$ and $p_{(i,d-i)} = l_{(i,d-i)} + u_{(i,d-i)}$. Define orthogonal projections $\pi : p_{\mathbf{d}} \to u_{\mathbf{d}}$ and $\pi' : p_{(i,d-i)} \to u_{(i,d-i)}$ and extend these to all parabolics of type $\mathbf{d}$ respectivly $(i, d-i)$ as follows:
Let p be a parabolic of type $\mathbf{d}$ and $F \in p$. Then there is $g \in GL(d)$ such that $gpg^{-1} = p_{\mathbf{d}}$. Set $\pi(F) := g^{-1}\pi(gFg^{-1})g$ and analogously for parabolics of type $(i, d-i)$. So

$$\begin{aligned} Fl(\mathbf{d}) &\longleftrightarrow \text{ parabolics of type } \mathbf{d} \\ \text{Grassmannian } G(i,d) &\longleftrightarrow \text{ parabolics of type } (i, d-i) \\ \text{endomorphisms } t &\longrightarrow \text{ nilpotent elements in } \mathfrak{gl}(d) \text{ which square to zero} \end{aligned}$$

The condition $t(D_i) \subseteq D_{i-1}$ then implies $t \in \pi(p(\mathbf{D}))$. Similarly, the condition $\mathrm{im}(t) \subseteq W \subseteq \ker(t)$ can be reformulated as $t(F_i) \subseteq F_{i-1}$ for $\mathbf{F} = \{0\} \subseteq W \subseteq D$ and thus, $t \in \pi'(p(\mathbf{F})) = \pi'(p(W \subseteq D))$, where $p(W \subseteq D)$ is the parabolic associated to $W \subseteq D$. Thus, for fixed dimension w of W, one obtains the following variety of triples:

$$\begin{aligned} &\{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \dim W = w\} \\ &\cong \left\{ (x, p_1, p_2) \;\middle|\; \begin{array}{c} p_1 \text{ parabolic of type } \mathbf{d},\ p_2 \text{ parabolic of type } (w, d-w), \\ x \in \pi(p_1) \cap \pi'(p_2) \text{ with } x^2 = 0 \end{array} \right\} = St(\mathbf{d}, w). \end{aligned}$$

This is called the Steinberg variety (see [2, section 3.3]). Therefore $\mathfrak{T}(\mathbf{d}) = \bigcup_{w=0}^{d} St(\mathbf{d}, w)$.
$GL(D)$ acts on $\mathfrak{T}(\mathbf{d})$ via $g.(\mathbf{D}, W, t) = (\{g.D_i\}_{i=0}^{k}, g.W, gtg^{-1})$.
The same definition of $\mathfrak{T}(\mathbf{d})$ makes sense when substituting a finite field $\mathbb{K} = \mathbb{F}_{q^2}$ for $\mathbb{C}$ (where q of course has to be chosen as a power of a prime number instead of an invariant), so from now on, let D be a d-dimensional vector space over $\mathbb{F}_{q^2}$.
An example for the tensor product variety follows.

3.2 Explicit Examples

In the following I describe some small examples of these varieties explicitly.

Example 7. Let $d = 2$, thus $\mathbf{d} \in \{(2), (1, 1)\}$ (ignore zeros in the vector, e.g. $(2, 0) = \mathbf{d}$). I describe these two cases explicitly.

- $\mathfrak{T}(2) = \{(\mathbf{D} = \{D_i\}_{i=0}^1, W, t) \mid 0 = D_0 \subseteq D_1 = D, \dim(D_1/D_0) = d, W \subseteq D, t \in \mathrm{End}(D), t(D) \subseteq D_0 = 0, 0 = \mathrm{im}(t) \subseteq W \subseteq \ker(t) = D\}$,
  hence
  $Fl(2) = \{0 \subset D\}$ and $\mathfrak{T}(2) = \{(0 \subseteq D, W, 0)\} = \mathfrak{T}_0(2) = \bigcup_{w=0}^2 Fl(2) \times G(w, 2)$
  is a union of Grassmannian varieties and each Grassmannian is an orbit for the action of $GL(d)$. Thus $\mathfrak{T}(2)$ has 3 orbits.
  If k is a finite field then the variety contains only finietly many points, for instance over the field $\mathbb{F}_{q^2}$ with q^2 elements I have the following:
  Since $G(0,2) = \{0\}$, $G(1,2) = \{span\{e_2\}, span\{e_1 + \lambda e_2\}\}_{\lambda \in \mathbb{F}_{q^2}}$, and $G(2,2) = \{D\}$, it follows that $|G(0,2)| = 1$, $|G(1,2)| = (q^2+1)$, and $|G(2,2)| = 1$ and so $\mathfrak{T}(2)$ has $1 + (q^2+1) + 1$ points.

- Consider $\mathfrak{T}(1,1) = \{(\mathbf{D} = \{D_i\}_{i=0}^2, W, t) \mid 0 = D_0 \subseteq D_1 \subseteq D_2 = D, \dim(D_i/D_{i-1}) = d_i, W \subseteq D, t \in \mathrm{End}(D), t(D_i) \subseteq D_{i-1}, \mathrm{im}(t) \subseteq W \subseteq \ker(t)\}$
  $= \mathfrak{T}_0(1,1) \cup \{(0 \subset < v > \subset D, < v >, t \neq 0) \mid D = < v > \oplus < u >, t = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \lambda \in \mathbb{F}_{q^2}^\times$
  for some u completing v to a basis of D and t as a matrix in this basis $(v, u)\}$
  $\cong \mathfrak{T}_0(1,1) \cup \mathbb{P}^1 \times \mathbb{F}_{q^2}^x$ ($W = < v >$ in the second set as $t(D) \subset D_1 = < v >$, $\mathrm{im}(t) \subset W$ and $t \neq 0$).
  To calculate the cardinality of $\mathfrak{T}(1,1)$, note that the number of different flags of type $(1,1)$ equals the number of different one-dimensional subspaces of $(\mathbb{F}_{q^2})^2$, which is q^2+1. Therefore $|\mathfrak{T}_0(1,1)| = q^2+1+(q^2+1)^2+q^2+1 = q^4+4q^2+3$ and $|\mathfrak{T}(1,1)| = |\mathfrak{T}_0(1,1)| + (q^2+1)(q^2-1)$ (as there are q^2+1 possibilities for the flag, which also fixes W, and for each flag $q^2 - 1 = |\mathbb{F}_{q^2}^x|$ possibilities for the endomorphism t for the elements of $\mathfrak{T}(1,1)$ with nonzero endomorphism).

Example 8. Now let $d = 3$, so $\mathbf{d} \in \{(3), (1,2), (2,1), (1,1,1)\}$.

- Then

$$\begin{array}{lllllll} \mathfrak{T}(3) = \mathfrak{T}_0(3) & & & & & & \\ \cong Fl(3) & \cup & Fl(3) \times \mathbb{P}^2 & \cup & Fl(3) \times G(2,3) & \cup & Fl(3) \\ \cong \{(D,0,0)\} & \cup & \{(D, < v >, 0)\}_{v \neq 0} & \cup & \{(D, < v, w >, 0)\}_{\substack{v,w\neq 0 \\ v \neq \lambda w}} & \cup & \{(D, D, 0)\} \end{array}$$

  (As seen before, $\mathfrak{T}_0(\mathbf{d})$ is generally of the form $\coprod_{i=0}^d Fl(\mathbf{d}) \times G(i,d)$. However, for example $Fl(3) \times \mathbb{P}^2$ divides into several $GL(D)$-orbits, depending on how W lies in $\mathbf{D}$.).

- $\mathfrak{T}(2,1) = \mathfrak{T}_0(2,1) \cup R$ with

$$R = \left\{ ((D_1 \subset D), W, t) \;\middle|\; \begin{array}{c} W \subset D, \dim(D_1) = 2, t \neq 0 \\ \mathrm{im}(t) \subset D_1 \subset \ker(t) \\ \mathrm{im}(t) \subset W \subset \ker(t) \end{array} \right\}.$$

One has $((D_1 \subset D), W, t) \in R \Rightarrow \dim(\ker(t)) = 2$ and $\dim(\mathrm{im}(t)) = 1$ as $t \neq 0$, $\mathrm{im}(t) \subset \ker(t)$ and $\dim(\mathrm{im}(t)) + \dim(\ker(t)) = 3$. So $D_1 = \ker(t)$ and $W = \mathrm{im}(t)$ or $W = \ker(t)$. It follows

$$R = \left\{ (\mathbf{D}, \mathrm{im}(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 2, \\ \mathrm{im}(t) \subset D_1 = \ker(t) \end{array} \right\} \dot{\cup} \left\{ (\mathbf{D}, \ker(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 2, \\ \mathrm{im}(t) \subset D_1 = \ker(t) \end{array} \right\}$$

with $\mathbf{D} = (D_1 \subset D)$. So it divides into the $\mathfrak{T}_0(2,1)$-part and a union of Spaltenstein-varieties.

- $\mathfrak{T}(1,2) = \mathfrak{T}_0(1,2) \cup R$ with

$$R = \left\{ ((D_1 \subset D), W, t) \;\middle|\; \begin{array}{c} W \subset D,\ \dim(D_1) = 1,\ t \neq 0 \\ \mathrm{im}(t) \subset D_1 \subset \ker(t) \\ \mathrm{im}(t) \subset W \subset \ker(t) \end{array} \right\}.$$

As before, one has $((D_1 \subset D), W, t) \in R \Rightarrow \dim(\mathrm{im}(t)) = 1$, $\dim(\ker(t)) = 2$ and $W = \mathrm{im}(t)$ or $W = \ker(t)$. But in this case $\mathrm{im}(t) = D_1$. So it follows

$$R = \left\{ (\mathbf{D}, \mathrm{im}(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 1, \\ \mathrm{im}(t) = D_1 \subset \ker(t) \end{array} \right\} \dot{\cup} \left\{ (\mathbf{D}, \ker(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 1, \\ \mathrm{im}(t) = D_1 \subset \ker(t) \end{array} \right\}$$

with $\mathbf{D} = (D_1 \subset D)$.

- $\mathfrak{T}(1,1,1) = \mathfrak{T}_0(1,1,1) \cup R$ with

$$R = \left\{ ((D_1 \subset D_2 \subset D), W, t) \;\middle|\; \begin{array}{c} W \subset D,\ \dim(D_1) = 1,\ \dim(D_2) = 2,\ t \neq 0 \\ \mathrm{im}(t) \subset D_2,\ D_1 \subset \ker(t),\ t(D_2) \subset D_1 \\ \mathrm{im}(t) \subset W \subset \ker(t) \end{array} \right\}.$$

Again, $((D_1 \subset D_2 \subset D), W, t) \in R \Rightarrow \dim(\ker(t)) = 2$, $\dim(\mathrm{im}(t)) = 1$ and $W = \mathrm{im}(t)$ or $W = \ker(t)$. It follows

$$R = \left\{ (\mathbf{D}, \mathrm{im}(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 1, \\ \dim(D_2) = 2, \\ \mathrm{im}(t) \subset D_2, \\ D_1 \subset \ker(t), \\ t(D_2) \subset D_1 \end{array} \right\} \dot{\cup} \left\{ (\mathbf{D}, \ker(t), t) \;\middle|\; \begin{array}{c} t \neq 0 \\ \dim(D_1) = 1, \\ \dim(D_2) = 2, \\ \mathrm{im}(t) \subset D_2, \\ D_1 \subset \ker(t), \\ t(D_2) \subset D_1 \end{array} \right\}$$

with $\mathbf{D} = (D_1 \subset D_2 \subset D)$.

3.3 Relative Positions of Subspaces

In the following I introduce a function α which describes the relative position of a subspace $V \subseteq D$ with respect to the flag $\mathbf{D}$.

Definition 3.4. Define $\alpha : (V, \mathbf{D}) \mapsto \alpha(V, \mathbf{D}) \in (\mathbb{Z}_{\geq 0})^k$, $\alpha(V, \mathbf{D})_i = \dim((V \cap D_i)/(V \cap D_{i-1}))$ and denote the following unions of orbits of $\mathfrak{T}(\mathbf{d})$ under the action of $GL(D)$ by $A_{\mathbf{w},\mathbf{r},\mathbf{n}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) | \alpha(W, \mathbf{D}) = \mathbf{w},\ \alpha(\text{im } t, \mathbf{D}) = \mathbf{r},\ \alpha(\ker t, \mathbf{D}) = \mathbf{n}\}$ for fixed $\mathbf{w}, \mathbf{r}, \mathbf{n} \in (\mathbb{Z}_{\geq 0})^k$.

The $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ will be used to characterize the canonical basis later on.

Remark 6. As $\text{im}(t) \subseteq W \subseteq \ker(t)$, $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is empty unless $\sum_{i=1}^j r_i \leq \sum_{i=1}^j w_i \leq \sum_{i=1}^j n_i$ for all j and $\sum_{i=1}^k r_i + n_i = d$(as $\dim(\text{im}(t)) + \dim(\ker(t)) = d$).

Example 9. This example illustrates the counting of points over finite fields of cardinality q^2. The results will always depend on a polynomial of q. This allows me later to treat q as a formal variable and connect it with the modules over the quantum group U_q. Let $d = 3$ and $\mathbf{d} = (1, 2)$. Then $\mathbb{F}_{q^2}^3$ has $(q^2)^3$ different elements of which all but one are nonzero. The Grassmannian $G(1, 3)$ has $q^2[3]$ points because leaving out linear multiples, one obtains $\frac{q^6-1}{q^2-1} = \frac{q^3(q^3-q^{-3})}{q(q-q^{-1})} = q^2[3]$ different "lines ", i.e pairwise linear independent vectors. So there are $q^2[3]$ different flags of type $(1, 2)$, since a flag of type $\mathbf{d} = (1, 2)$ is of the form $(< v > \subset D)$. In general, $G(1, d) = \mathbb{P}^{d-1}$ has $(q^2)^n - 1$ elements and $\frac{q^{2n}-1}{q^2-1}$ points. Using the following easy identities $q^2[3] - 1 = \frac{q^6-1}{q^2-1} - 1 = \frac{q^6-1-q^2+1}{q^2-1} = q^2\frac{q^4-1}{q^2-1} = q^3[2]$ and $q^3[2] = q^2\frac{q^4-1}{q^2-1}$, so $q^3[2](q^2-1) = q^2(q^4-1)$ and $q^2[3](q^4 - 1) = q^3[3][2](q^2 - 1) = q^3[3]!(q^2 - 1)$, I can now determine the number of points (not elements!) in A. The result is given in the following tables:
First let $\mathbf{r} = 0$, $\mathbf{n} = (1, 2)$, so $t = 0$.

$\mathbf{w}$	$\lvert A_{\mathbf{w},\mathbf{r},\mathbf{n}}\rvert$	Explanatory Remarks
$(1,0)$	$q^2[3]$	W has to be equal to D_1
$(1,1)$	$q^5[3]!$	$W =< v, v' >$, $< v >= D_1$ and v' has to be linear independent of v
$(1,2)$	$q^2[3]$	$W = D = \mathbb{F}_{q^2}^3$
$(0,0)$	$q^2[3]$	$W = 0$
$(0,1)$	$q^5[3]!$	$W =< v' >$, $< v >= D_1$ and v' has to be linear independant of v
$(0,2)$	$\frac{1}{2}q^7[3]!$	$q^2[3]$ possibilities for $< v >= D_1$ and $W =< v', v'' >$, so $q^2[3] - 1$ possibilities for v' and then only q^2 for v'' to obtain different W, but the order in which v', v'' are choosen does not matter.

.

Now let $\mathbf{r} = (1, 0)$, $\mathbf{n} = (1, 1)$, so $t \neq 0$ and $\mathbf{w}$ is of the form $(1, |\mathbf{w}| - 1)$:

$\mathbf{w}$	$\lvert A_{\mathbf{w},\mathbf{r},\mathbf{n}}\rvert$	Explanatory Remarks
$(1,0)$	$q^3[3]!(q^2 - 1)$	$W = D_1$ and $t = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ in a basis compatible with the flag, so there are $q^4 - 1 = (q^2 + 1)(q^2 - 1)$ possibilities for t
$(1,1)$	$q^5[3]!(q^2 - 1)$	As in the $t = 0$-case, there are $q^5[3]!$ possibilities for the tupel $(\mathbf{D}, W)$ and since $t(W) = 0$, $t \neq 0$, $q^2 - 1$ possibilities for t

.

More concretely, take e.g. $q = 2$, $\mathbb{F}_{q^2} = \mathbb{F}_4 \cong \{0, 1, e^{\frac{2\pi i}{3}} = x, e^{\frac{4\pi i}{3}} = y\}$, so x, y are third roots of unity. Then the one-dimensional subspaces of $\mathbb{F}_4^3$ are given by the spans of the vectors in

$$V = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\x\\0\end{pmatrix}, \begin{pmatrix}1\\y\\0\end{pmatrix}, \begin{pmatrix}1\\0\\x\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1\\0\\y\end{pmatrix}, \begin{pmatrix}0\\1\\x\end{pmatrix}, \begin{pmatrix}0\\1\\y\end{pmatrix}, \begin{pmatrix}1\\x\\x\end{pmatrix}, \begin{pmatrix}1\\y\\y\end{pmatrix}, \begin{pmatrix}1\\x\\1\end{pmatrix}, \begin{pmatrix}1\\y\\1\end{pmatrix}, \begin{pmatrix}1\\1\\x\end{pmatrix}, \begin{pmatrix}1\\1\\y\end{pmatrix}, \begin{pmatrix}1\\x\\y\end{pmatrix}, \begin{pmatrix}1\\y\\x\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

and

$$\begin{aligned}
\mathfrak{T}_0(1,2) = \ & \{(< v >\subset D, 0, 0) \mid v \in V\} \cup \{(< v >\subset D, < v >, 0) \mid v \in V\} \\
& \cup\{(< v >\subset D, < v' >, 0) \mid v \neq v' \in V\} \cup \{(< v >\subset D, < v, v' >, 0) \mid v \neq v' \in V\} \\
& \cup\{(< v >\subset D, < v', v'' >, 0) \mid v \neq v', v'' \in V \text{ such that } < v', v'' > \cap < v >= 0\} \\
& \cup\{(< v >\subset D, D, 0) \mid v \in V\} \\
= \ & A_{0,0,\mathbf{d}} \cup A_{(1,0),0,\mathbf{d}} \\
& \cup A_{(0,1),0,\mathbf{d}} \cup A_{(1,1),0,\mathbf{d}} \\
& \cup A_{(0,2),0,\mathbf{d}} \\
& \cup A_{(1,2),0,\mathbf{d}}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\{\mathbf{D} \in Fl(1,2) | tD_i \subset D_{i-1}\} \times \{\mathbf{D} \in Fl(1,2) | tD_i \subset D_{i-1}\} \\
&\quad = \{(< v >\subset D, < v >, t = \left(\begin{smallmatrix} 0 & \lambda & \mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)) \mid v \in V,\ \lambda, \mu \in \mathbb{F}_4 \text{ not both zero}\} = A_{(1,0),(1,0),(1,1)}
\end{aligned}$$

and

$$\begin{aligned}
&\{\mathbf{D} \in Fl(1,2) | tD_i \subset D_{i-1}\} \times \{\mathbf{D} \in Fl(2,1) | tD_i \subset D_{i-1}\} \\
&= \{(< v >\subset D, < v, v' >, t = \left(\begin{smallmatrix} 0 & \lambda & \mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)) \mid v \neq v' \in V,\ t(v') = 0,\ \lambda, \mu \in \mathbb{F}_4 \text{ not both zero}\} \\
&= A_{(1,1),(1,0),(1,1)}
\end{aligned}$$

(t as a matrix for a basis compatible with the flag).
The cardinalities are

$$\begin{aligned}
|A_{0,0,\mathbf{d}}| &= q^2[3] &&= 21 \\
|A_{(1,0),0,\mathbf{d}}| &= q^2[3] &&= 21 \\
|A_{(0,1),0,\mathbf{d}}| &= q^5[3]! &&= 420 \\
|A_{(1,1),0,\mathbf{d}}| &= q^5[3]! &&= 420 \\
|A_{(0,2),0,\mathbf{d}}| &= \tfrac{1}{2}q^7[3]! &&= 840 \\
|A_{(1,2),0,\mathbf{d}}| &= q^2[3] &&= 21 \\
|A_{(1,0),(1,0),(1,1)}| &= q^3[3]!(q^2-1) &&= 315 \\
|A_{(1,1),(1,0),(1,1)}| &= q^5[3]!(q^2-1) &&= 1260.
\end{aligned}$$

The following corrects a claim made in [21]:

Lemma 3.5. *The varieties $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are unions of orbits. In general, they are not single orbits (in contrast to the claim made in [21]).*

Proof: The first claim is clear by definition, since the condition is $GL(D)$-equivariant. For the second claim, I refer to the following example. $\square$

Example 10. Let $\mathbf{d} = (1,1,1,1)$ and let $A_{(1,1,0,0,),(1,1,0,0),(1,1,0,0,)} = \{(\mathbf{D}, W, t) | W = D_2 = \ker t = \operatorname{im} t\}$. Then, for $\mathbf{D} = (< e_1 > \subset < e_1, e_2 > \subset < e_1, e_2, e_3 > \subset < e_1, e_2, e_3, e_4 >= D)$, the two elements

$$(\mathbf{D}, < e_1, e_2 >, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) \text{ and } (\mathbf{D}, < e_1, e_2 >, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})$$

are in this set. Assuming both were in the same orbit, there should exist a $g \in GL(D)$ such that

$$g(\mathbf{D}, < e_1, e_2 >, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = (\mathbf{D}, < e_1, e_2 >, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}).$$

Since $g\mathbf{D} = \mathbf{D}$, $g = (g_{i,j})$, $g_{i,i} \neq 0$ has to be an upper triangular matrix. Thus

$$g \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & 0 & g_{1,2} & g_{1,1} \\ 0 & 0 & g_{2,2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & 0 & g_{1,2}g_{3,3}^{-1} & g_{1,1}g_{4,4}^{-1} - g_{1,2}g_{3,4}g_{4,4}^{-1}g_{3,3}^{-1} \\ 0 & 0 & g_{2,2}g_{3,3}^{-1} & -g_{2,2}g_{3,4}g_{4,4}^{-1}g_{3,3}^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But $g_{2,2}g_{3,3}^{-1} \neq 0$, so the two elements can not be in the same orbit. So the $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are merely a union of orbits. However, the $A_{\mathbf{w},\mathbf{0},\mathbf{d}}$ always are orbits and the projection of $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ onto the first two components (sending $(\mathbf{D}, W, t)$ to $(\mathbf{D}, W)$) is an orbit under the $GL(D)$-action as well.

Remark 7. To identify the orbits in general, consider $(\mathbf{D}, W, t), (\mathbf{D}', W', t') \in A_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Without loss of generality, I can assume $\mathbf{D} = \mathbf{D}'$, $W = W'$. Choose a basis $(u_i)_{i=1}^d$ of D such that

$$D_i = span\{u_i\}_{i=1}^{d_i}$$

and

$$W \cap D_j = span \bigcup_{l=0}^{j} \{u_i\}_{i=(\sum_{s=1}^{l-1} d_s)+1}^{(\sum_{s=1}^{l-1} d_s)+w_l}.$$

Then t, t' have to fulfill the conditions posed by $\mathbf{r}, \mathbf{n}$. If $g.(\mathbf{D}, W, t) = (\mathbf{D}, W, t')$, then g is an upper triangular matrix, so if $t(u_i) \in D_j$, then $t'(u_i) \in D_j$. However, if t' fulfills this, then there also exists g such that $gtg^{-1} = g'$. This describes the orbits.

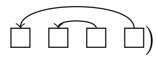
It remains to see which $\mathbf{r}, \mathbf{n}$ allow more than one orbit. I need $u_{i_1} \in D_{j_1}, u_{i_2} \in D_{j_2}$, $j_1 < j_2$ with $n_{j_l} \neq d_{j_l}$, $l = 1, 2$ (so $t(D_{j_l}) \neq 0$) and $\sum_{i=j_1}^k r_i < \sum_{i=j_1+1}^k x_i$, with $d_i = n_i + x_i$ i.e u_{i_1} and u_{i_2} can be mapped to D_{j_1-1} and not to zero. Moreover, there must be $l_1 \neq l_2$ with $r_{l_i} \neq 0$ and $l_i < j_1$

such that t may map u_{i_1}, u_{i_2} to D_{l_1}, D_{l_2} or D_{l_2}, D_{l_1}. Then t, t' with $t(u_{i_1}) \in D_{l_1}$, $t(u_{i_2}) \in D_{l_2}$ and $t'(u_{i_1}) \in D_{l_2}$, $t'(u_{i_2}) \in D_{l_1}$ are not in the same orbit. E.g.

$$\mathbf{n} = (*, \cdots, *, d_{j_1} - x_{j_1}, \cdots, d_{j_2} - x_{j_2}, *, \cdots, *),\ x_i \neq 0$$

and

$$\mathbf{r} = (*, \cdots, *, \overset{l_{i_1}}{1}, *, \cdots, *, \overset{l_{i_2}}{1}, *, \cdots, *, \overset{j_1}{0}, \cdots, \overset{j_2}{0}, *, \cdots, *).$$

(if looking at the corresponding crossingless matching, it must have at least two arrows running above one another, e.g ).

3.4 The Spaces $T(\mathbf{d})$ and $T_0(\mathbf{d})$

Definition 3.6. A function $f : \mathfrak{T}(\mathbf{d}) \to \mathbb{C}$ such that $(g.f)(x) := f(g^{-1}x) = f(x)\, \forall g \in GL(D)$ is called invariant. Let $T(\mathbf{d})$ denote the space of invariant functions on $\mathfrak{T}(\mathbf{d})$.
Define $1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$ the indicator function. Set

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\sum_{i<j} r_i w_j + w_i n_j - w_i w_j}$$

a constant and define

$$f_{\mathbf{w},\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} 1_{A_{\mathbf{w},\mathbf{r},\mathbf{n}}}.$$

Define $T_0(\mathbf{d}) = span\{f_{\mathbf{w},\mathbf{0},\mathbf{d}}\}_{\mathbf{w}}$, the set of invariant functions on $\mathfrak{T}_0(\mathbf{d})$ (Recall that $A_{\mathbf{w},\mathbf{0},\mathbf{d}}$ is a single orbit).

Remark 8. Then $T(\mathbf{d}) \supset span\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w},\mathbf{r},\mathbf{n}}$, but in general not equal (the inclusion is in general strict, e.g. consider f the indicator function of some orbit strictly contained in an $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$. Recall that $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is not necessarily an orbit, see example 10).
Only finitly many of the $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are nonzero, more precisely $f_{\mathbf{w},\mathbf{r},\mathbf{n}} = 0$ unless $|\mathbf{r}| + |\mathbf{n}| = |\mathbf{d}| = d$ (as $|\mathbf{r}| = \dim(\text{im } t)$, $|\mathbf{n}| = \dim(\ker t)$ and $\dim(D) = d = \dim(\ker t) + \dim(\text{im } t)$) and $\mathbf{r} \leq \mathbf{w} \leq \mathbf{n}$ (as $\text{im } t \subseteq W \subseteq \ker t$) where $|\mathbf{a}| := \sum_{i=1}^{k} a_i$ and $\mathbf{a} \leq \mathbf{b} \Leftrightarrow \sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} b_i\, \forall 1 \leq j \leq k$, $\mathbf{a} < \mathbf{b} \Leftrightarrow \mathbf{a} \leq \mathbf{b}, \mathbf{a} \neq \mathbf{b}$, for $\mathbf{a}, \mathbf{b} \in (\mathbb{Z}_{\geq 0})^k$.

Example 11.

- $\mathbf{d} = (2)$: $T(2) = span\{f_{0,0,2}, f_{1,0,2}, f_{2,0,2}\}$ (Recall that $\mathfrak{T}(\mathbf{d}) = \mathfrak{T}_0(\mathbf{d})$, so $f_{\mathbf{w},\mathbf{r},\mathbf{n}} = 0$ unless $\mathbf{r} = 0$, $\mathbf{n} = \mathbf{d}$). So the span equals $T(2)$ and is not just a subset.

- $\mathbf{d} = (1, 1)$: Recall $\mathfrak{T}(1,1) = \mathfrak{T}_0(1,1) \cup \{(0 \subset < v > \subset D, < v >, t \neq 0) | D = < v > \oplus < v >^{\perp}, t = \left(\begin{smallmatrix} 0 & \lambda \\ 0 & 0 \end{smallmatrix}\right)\}$, so $A_{\mathbf{w},\mathbf{r},\mathbf{n}} \neq \emptyset$ if and only if $\mathbf{w} = \mathbf{r} = \mathbf{n} = (1, 0)$ or $\mathbf{r} = 0$, $\mathbf{n} = \mathbf{d}$ and $\mathbf{w} \in \{(0,0), (1,0), (0,1), (1,1)\}$. So $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \neq 0$ if and only if $\mathbf{w} = \mathbf{r} = \mathbf{n} = (1, 0)$ or $\mathbf{r} = 0$, $\mathbf{n} = \mathbf{d}$ and $\mathbf{w} \in \{(0,0), (1,0), (0,1), (1,1)\}$. Again, $T(1,1) = span\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}$, since in the case of $\mathbf{d} = (1, 1)$, all the $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ are single orbits.

One wants to equip $T(\mathbf{d})$ with a U_q-module action such that there is a module-isomorphism $T_0(\mathbf{d}) \overset{\eta_{0,\mathbf{d}}}{\to} V_{d_1} \otimes \cdots \otimes V_{d_k}$ sending $f_{\mathbf{w},\mathbf{0},\mathbf{d}}$ to $v_{d_1 - 2w_1} \otimes \cdots \otimes v_{d_k - 2w_k} =: \otimes^{\mathbf{d}} v_{\mathbf{w}}$, the elementary basis element corresponding to $\mathbf{w}$.

Set

$$\mathfrak{T}(w;\mathbf{d}) = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}) \mid \dim W = w\}$$

and

$$\mathfrak{T}(w, w+1;\mathbf{d}) = \{(\mathbf{D}, U, W, t) \mid (\mathbf{D}, W, t), (\mathbf{D}, U, t) \in \mathfrak{T}(\mathbf{d}),\ \dim U = w,\ \dim W = w+1\}.$$

Then there is a correspondence

$$\mathfrak{T}(\mathbf{d}) \overset{\pi_1}{\leftarrow} \bigcup_w \mathfrak{T}(w, w+1;\mathbf{d}) \overset{\pi_2}{\rightarrow} \mathfrak{T}(\mathbf{d})$$

with $\pi_1((\mathbf{D}, U, W, t)) = (\mathbf{D}, U, t)$ and $\pi_2((\mathbf{D}, U, W, t)) = (\mathbf{D}, W, t)$. Define $\pi_!(f)(x) := \sum_{y\in\pi^{-1}(x)} f(y)$ (recall that I am working over a finite field) and $\pi^* f(x) = f(\pi(x))$.

Remark 9. The correspondence can be defined over $\mathbb{C}$ as well.

Definition 3.7. [21, Theorem 2.2.1]
$T(\mathbf{d})$ becomes a $U_q(sl_2)$ module via the following action of $E, F, K^{\pm 1}$: Set

$$Ef = q^{-\dim(\pi_1^{-1}(-))}(\pi_1)_!\pi_2^* f,$$

$$Ff = q^{-\dim(\pi_2^{-1}(-))}(\pi_2)_!\pi_1^* f$$

and

$$K^{\pm 1} f = q^{\pm(d-2\ \dim(-))} f.$$

So

$$\begin{aligned} Ef(\mathbf{D}, U, t) = q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))}(\pi_1)_!\pi_2^* f(\mathbf{D}, U, t) \\ = q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d})} f(\mathbf{D}, W, t), \end{aligned}$$

$$\begin{aligned} Ff(\mathbf{D}, W, t) = q^{-\dim(\pi_2^{-1}(\mathbf{D},W,t))}(\pi_2)_!\pi_1^* f(\mathbf{D}, W, t) \\ = q^{-\dim(\pi_2^{-1}(\mathbf{D},W,t))} \sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d})} f(\mathbf{D}, U, t) \end{aligned}$$

and

$$K^{\pm 1} f(\mathbf{D}, W, t) = q^{\pm(d-2\ \dim W)} f(\mathbf{D}, W, t).$$

Remark 10. π_1, π_2 are in general not surjective, e.g. consider $A_{\mathbf{w},\mathbf{r},\mathbf{n}} = A_{(1,1,0,0),(1,1,0,0),(1,1,0,0)}$.

$$\begin{aligned} & \mathfrak{T}(w, w+1;\mathbf{d}) \\ = & \{(\mathbf{D}, U, W, t) \mid (\mathbf{D}, W, t), \mathbf{D}, U, t) \in \mathfrak{T}(\mathbf{d}),\ \dim U = w,\ \dim W = w+1\} \\ = & \{(\mathbf{D}, U, W, t) \mid \mathrm{im}\ t \subset U \subsetneq W \subset \ker t,\ (\mathbf{D}, W, t), \mathbf{D}, U, t) \in \mathfrak{T}(\mathbf{d}),\ \dim U = w,\ \dim W = w+1\}, \end{aligned}$$

so if $\ker t = \operatorname{im} t$, no $(\mathbf{D}, U, W, t) \in \mathfrak{T}(w, w+1; \mathbf{d})$ for any $w, \mathbf{d}$. Thus

$$\begin{aligned}
& Ef_{(1,1,0,0),(1,1,0,0),(1,1,0,0)}(\mathbf{D}, U, t) \\
= \ & q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \textstyle\sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d})} f(\mathbf{D}, W, t) \\
= \ & q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \textstyle\sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d}),\,(\mathbf{D},W,t)\in A_{\mathbf{w},\mathbf{r},\mathbf{n}}} f(\mathbf{D}, W, t) \\
= \ & q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \textstyle\sum_{(\mathbf{D},U,W,t)\in\emptyset} f(\mathbf{D}, W, t) \\
= \ & 0 \quad \forall (\mathbf{D}, U, t),
\end{aligned}$$

similarly for F.

Proposition 3.8. *Applying the action of $E, F, K^{\pm 1}$ to the vectors $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$, one obtains*

$$K^{\pm 1} f_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\pm(d-2|\mathbf{w}|)} f_{\mathbf{w},\mathbf{r},\mathbf{n}},$$

$$E f_{\mathbf{w},\mathbf{r},\mathbf{n}} = \sum_{j=1}^{k} q^{\sum_{i=1}^{j-1} \mathbf{n}_i - \mathbf{r}_i - 2(\mathbf{w}_i - \mathbf{r}_i)} [\mathbf{n}_j - \mathbf{w}_j + 1] f_{\mathbf{w}-\delta^j,\mathbf{r},\mathbf{n}}$$

and

$$F f_{\mathbf{w},\mathbf{r},\mathbf{n}} = \sum_{j=1}^{k} q^{-\sum_{i=j+1}^{k} \mathbf{n}_i - \mathbf{r}_i - 2(\mathbf{w}_i - \mathbf{r}_i)} [\mathbf{w}_j - \mathbf{r}_j + 1] f_{\mathbf{w}+\delta^j,\mathbf{r},\mathbf{n}}$$

(where $\delta^j \in (\mathbb{Z}_{\geq 0})^k$ is the element such that $\delta^j_i = 0 \,\forall\, i \neq j$, $\delta^j_j = 1$).

Proof: Let $(\mathbf{D}, U, t) \in \mathfrak{T}(\mathbf{d})$ be fixed. It is clear that $E f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, U, t) = 0$ unless $\alpha(U, \mathbf{D}) = \mathbf{w} - \delta^j$ for some j since

$$E f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, U, t) = q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d})} f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t),$$

so there must exist a W such that $(\mathbf{D}, W, t) \in A_{\mathbf{w},\mathbf{r},\mathbf{n}}$.
Then

$$\begin{aligned}
E f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, U, t) = \ & q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \textstyle\sum_{(\mathbf{D},U,W,t)\in\bigcup_w \mathfrak{T}(w,w+1;\mathbf{d})} f_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D}, W, t) \\
= \ & k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\dim(\pi_1^{-1}(\mathbf{D},U,t))} \chi_q(\pi_1^{-1}(\mathbf{D}, U, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}})).
\end{aligned}$$

($\chi_q(A)$ is the Euler characteristic, i.e. the number of points in A, which is finite since $k = \mathbb{F}_{q^2}$ and $\chi_q(\pi_1^{-1}(\mathbf{D}, U, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}}))$ is the number of W such that $(\mathbf{D}, W, t) \in A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ and $(\mathbf{D}, U, W, t) \in \bigcup_w \mathfrak{T}(w, w+1; \mathbf{d})$)
Now,

$$\begin{aligned}
\pi_1^{-1}(\mathbf{D}, U, t) \cong \ & \{W | U \subset W \subset \ker t,\ \dim W = \dim U + 1\} \\
\cong \ & \{W | W \subset \ker t/U,\ \dim W = 1\} \\
\cong \ & \mathbb{P}^{\dim(\ker t) - \dim U - 1} \\
= \ & \mathbb{P}^{|\mathbf{n}| - (|\mathbf{w}|-1) - 1} \\
= \ & \mathbb{P}^{|\mathbf{n}| - |\mathbf{w}|}
\end{aligned}$$

and thus $\dim(\pi_1^{-1}(\mathbf{D}, W, t)) = |\mathbf{n}| - |\mathbf{w}|$ (remember $\alpha(U, \mathbf{D}) = \mathbf{w} - \delta^j$, so $\dim(U) = |\mathbf{w}| - 1$). Moreover,

$$\begin{aligned}
& \pi_1^{-1}(\mathbf{D}, U, t) \cap \pi_2^{-1}(A_{\mathbf{w},\mathbf{r},\mathbf{n}}) \\
\cong \ & \{W | U \subset W \subset \ker t,\ \alpha(W, \mathbf{D}) = \mathbf{w}\} \\
\cong \ & \{W | (U \cap D_j) \subset W \subset (\ker t \cap D_j),\ \dim(W \cap D_{j-1}) = \mathbf{w}^{(1,j-1)},\ \dim W = \mathbf{w}^{(1,j)}\} \\
\cong \ & \{W | W \subset (\ker t \cap D_j)/(U \cap D_j)\,, W \nsubseteq (\ker t \cap D_{j-1})/(U \cap D_{j-1}),\ \dim W = 1\},
\end{aligned}$$

(where $\mathbf{w}^{(i,j)} = \sum_{l=i}^{j} w_l$), so the dimension equals

$$\begin{aligned} & \dim \mathbb{P}^{\dim(\ker\, t\cap D_j)/(U\cap D_j)-1} - \dim \mathbb{P}^{\dim(\ker\, t\cap D_{j-1})/(U\cap D_{j-1})-1} \\ = \;& \dim \mathbb{P}^{\mathbf{n}^{(1,j)}-(\mathbf{w}-\delta^j)^{(1,j)}-1} - \dim \mathbb{P}^{\mathbf{n}^{(1,j-1)}-(\mathbf{w}-\delta^j)^{(1,j-1)}-1} \\ = \;& \dim \mathbb{P}^{\mathbf{n}^{(1,j)}-\mathbf{w}^{(1,j)}} - \dim \mathbb{P}^{\mathbf{n}^{(1,j-1)}-\mathbf{w}^{(1,j-1)}-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D},U,t) = \;& k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-(|\mathbf{n}|-|\mathbf{w}|)} \left(\sum_{i=0}^{\mathbf{n}^{(1,j)}-\mathbf{w}^{(1,j)}} q^{2i} - \sum_{i=0}^{\mathbf{n}^{(1,j-1)}-\mathbf{w}^{(1,j-1)}-1} q^{2i} \right) \\ = \;& k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}|-|\mathbf{n}|} \sum_{i=\mathbf{n}^{(1,j-1)}-\mathbf{w}^{(1,j-1)}}^{\mathbf{n}^{(1,j)}-\mathbf{w}^{(1,j)}} q^{2i} \\ = \;& k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{|\mathbf{w}|-|\mathbf{n}|+2(\mathbf{n}^{(1,j-1)}-\mathbf{w}^{(1,j-1)})} \sum_{i=0}^{n_j-w_j} q^{2i} \\ = \;& k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{\mathbf{w}^{j+1,k}-\mathbf{w}^{1,j-1}+\mathbf{n}^{1,j-1}-\mathbf{n}^{j+1,k}} [n_j - w_j + 1]. \end{aligned}$$

Using $k_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}} = k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{-\mathbf{r}^{1,j-1}-\mathbf{n}^{j+1,k}+\mathbf{w}^{1,j-1}+\mathbf{w}^{j+1,k}}$, one obtains

$$k_{\mathbf{w},\mathbf{r},\mathbf{n}} q^{\mathbf{w}^{j+1,k}-\mathbf{w}^{1,j-1}+\mathbf{n}^{1,j-1}-\mathbf{n}^{j+1,k}} = k_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{1,j-1}-2\mathbf{w}^{1,j-1}}.$$

Inserting this gives $Ef_{\mathbf{w},\mathbf{r},\mathbf{n}}(\mathbf{D},U,t) = k_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{1,j-1}-2\mathbf{w}^{1,j-1}} [n_j - w_j + 1]$. Thus

$$\begin{aligned} Ef_{\mathbf{w},\mathbf{r},\mathbf{n}} = \;& \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{1,j-1}-2\mathbf{w}^{1,j-1}} [n_j - w_j + 1] k_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}} 1_{A_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}}} \\ = \;& \sum_{j=1}^{k} q^{\mathbf{r}^{(1,j-1)}+\mathbf{n}^{1,j-1}-2\mathbf{w}^{1,j-1}} [n_j - w_j + 1] f_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}} \\ = \;& \sum_{j=1}^{k} q^{\sum_{i=1}^{j-1}(n_i - r_i - 2(w_i - r_i))} [n_j - w_j + 1] f_{\mathbf{w}-\delta \mathbf{j},\mathbf{r},\mathbf{n}}. \end{aligned}$$

Similarly,

$$Ff_{\mathbf{w},\mathbf{r},\mathbf{n}} = \sum_{j=1}^{k} q^{-\sum_{i=j+1}^{k}(n_i - r_i - 2(w_i - r_i))} [w_j - r_j + 1] f_{\mathbf{w}+\delta \mathbf{j},\mathbf{r},\mathbf{n}}.$$

It follows from the definition that

$$K^{\pm 1} f_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\pm(d-2|\mathbf{w}|)} f_{\mathbf{w},\mathbf{r},\mathbf{n}} = q^{\pm \sum_{i=1}^{k}(n_i - r_i - 2(w_i - r_i))} f_{\mathbf{w},\mathbf{r},\mathbf{n}}$$

as $|\mathbf{r}| + |\mathbf{n}| = |\mathbf{d}| = d$. □

3.5 Relation between $T_0(\mathbf{d})$ and $V_{d_1} \otimes \cdots \otimes V_{d_k}$

Definition 3.9. Define $\eta_{\mathbf{r},\mathbf{n}} : span\{f_{\mathbf{w},\mathbf{r},\mathbf{n}}\}_{\mathbf{w}} \to V_{n_1-r_1} \otimes \cdots \otimes V_{n_k-r_k}$ by $f_{\mathbf{w},\mathbf{r},\mathbf{n}} \mapsto \otimes^{\mathbf{n}-\mathbf{r}} v_{\mathbf{w}-\mathbf{r}}$, extended by linearity.

Proposition 3.10. *η is a U_q-module isomorphism.*

Proof: The action of $x \in U_q$ on $V_{d_1} \otimes \cdots \otimes V_{d_k}$ was defined as $\Delta^{k-1}(x)$, so

$$\Delta^{k-1} E = \sum_{i=1}^{k} K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1$$

$$\Delta^{k-1} F = \sum_{i=1}^{k} 1 \otimes \cdots \otimes 1 \otimes F \otimes K^{-1} \otimes \cdots \otimes K^{-1}$$

$$\Delta^{k-1} K^{\pm} = K^{\pm} \otimes \cdots \otimes K^{\pm}$$

where E and F appear in the i^{th} position in the first two equations. Comparing this with the action on $T(\mathbf{d})$, the claim follows. □

Definition 3.11. Denote by $h^{\mathbf{d}}_{\mathbf{w}}$ the preimage under $\eta_{\mathbf{0},\mathbf{d}}$ of $v_{d_1-2w_1}\diamondsuit\cdots\diamondsuit v_{d_k-2w_k} =: \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}$, the canonical basis element corresponding to $\mathbf{w}$. So the canonical basis can be interpreted as certain invariant functions on a subvariety of $\mathfrak{T}(\mathbf{d})$.

3.6 Examples

Example 12. Action of U_q on $T_0(\mathbf{d})$ (I will abbreviate $f_{\mathbf{w},\mathbf{0},\mathbf{d}}$ by $f_{\mathbf{w}}$).
$\mathbf{d} = (1,1,1)$:

$$\begin{array}{llllllll}
Ef_{(0,0,0)} = & 0 & Ef_{(0,0,1)} = & q^2 f_{(0,0,0)} & Ef_{(0,1,1)} = & q f_{(0,0,1)} & Ef_{(1,1,1)} = & f_{(0,1,1)} \\
 & & & & & +f_{(0,1,0)} & & +q^{-1} f_{(1,0,1)} \\
 & & & & & & & +q^{-2} f_{(1,1,0)} \\
 & & Ef_{(0,1,0)} = & q f_{(0,0,0)} & Ef_{(1,1,0)} = & f_{(0,1,0)} & & \\
 & & & & & +q^{-1} f_{(1,0,0)} & & \\
 & & Ef_{(1,0,0)} = & f_{(0,0,0)} & Ef_{(1,0,1)} = & f_{(0,0,1)} & & \\
 & & & & & +f_{(1,0,0)} & & ,
\end{array}$$

$$\begin{array}{llllllll}
Ff_{(0,0,0)} = & f_{(0,0,1)} & Ff_{(0,0,1)} = & q f_{(0,1,1)} & Ff_{(0,1,1)} = & q^2 f_{(1,1,1)} & Ff_{(1,1,1)} = & 0 \\
 & +q^{-1} f_{(0,1,0)} & & +f_{(1,0,1)} & & & & \\
 & +q^{-2} f_{(1,0,0)} & & & & & & \\
 & & Ff_{(0,1,0)} = & f_{(0,1,1)} & Ff_{(1,1,0)} = & f_{(1,1,1)} & & \\
 & & & +f_{(1,1,0)} & & & & \\
 & & Ff_{(1,0,0)} = & f_{(1,0,1)} & Ff_{(1,0,1)} = & q f_{(1,1,1)} & & \\
 & & & +q^{-1} f_{(1,1,0)} & & & &
\end{array}$$

and

$$\begin{array}{llllllll}
Kf_{(0,0,0)} = & q^3 f_{(0,0,0)} & Kf_{(0,0,1)} = & q f_{(0,0,1)} & Kf_{(0,1,1)} = & q^{-1} f_{(0,1,1)} & Kf_{(1,1,1)} = & q^{-3} f_{(1,1,1)} \\
 & & Kf_{(0,1,0)} = & q f_{(0,1,0)} & Kf_{(1,1,0)} = & q^{-1} f_{(1,1,0)} & & \\
 & & Kf_{(1,0,0)} = & q f_{(1,0,0)} & Kf_{(1,0,1)} = & q^{-1} f_{(1,0,1)} & &
\end{array}$$

One sees that there are four weightspaces determined by the absolute value of $\mathbf{w}$. Compare this to the action of U_q on $V_{d_1} \otimes \cdots \otimes V_{d_k}$ ($\otimes^{\mathbf{d}} v_{\mathbf{w}} = v_{d_1-2w_1} \otimes \cdots \otimes v_{d_k-2w_k}$ and recall the action of U_q (see the proof of Proposition 3.10)):
$\mathbf{d} = (1,1,1)$:

$$\begin{array}{llll}
E\otimes^{\mathbf{d}} v_{(0,0,0)} & E\otimes^{\mathbf{d}} v_{(0,0,1)} & E\otimes^{\mathbf{d}} v_{(0,1,1)} & E\otimes^{\mathbf{d}} v_{(1,1,1)} \\
=0 & =q^2\otimes^{\mathbf{d}} v_{(0,0,0)} & =q\otimes^{\mathbf{d}} v_{(0,0,1)} & =\otimes^{\mathbf{d}} v_{(0,1,1)} \\
 & & +\otimes^{\mathbf{d}} v_{(0,1,0)} & +q^{-1}\otimes^{\mathbf{d}} v_{(1,0,1)} \\
 & & & +q^{-2}\otimes^{\mathbf{d}} v_{(1,1,0)}
\end{array}$$

$$\begin{array}{ll}
E\otimes^{\mathbf{d}} v_{(0,1,0)} & E\otimes^{\mathbf{d}} v_{(1,1,0)} \\
=q\otimes^{\mathbf{d}} v_{(0,0,0)} & =\otimes^{\mathbf{d}} v_{(0,1,0)} \\
 & +q^{-1}\otimes^{\mathbf{d}} v_{(1,0,0)}
\end{array}$$

$$\begin{array}{ll}
E\otimes^{\mathbf{d}} v_{(1,0,0)} & E\otimes^{\mathbf{d}} v_{(1,0,1)} \\
=\otimes^{\mathbf{d}} v_{(0,0,0)} & =\otimes^{\mathbf{d}} v_{(0,0,1)} \\
 & +\otimes^{\mathbf{d}} v_{(1,0,0)}
\end{array},$$

$$\begin{array}{llll}
F\otimes^{\mathbf{d}} v_{(0,0,0)} & F\otimes^{\mathbf{d}} v_{(0,0,1)} & F\otimes^{\mathbf{d}} v_{(0,1,1)} & F\otimes^{\mathbf{d}} v_{(1,1,1)} \\
=\otimes^{\mathbf{d}} v_{(0,0,1)} & =q\otimes^{\mathbf{d}} v_{(0,1,1)} & =q^2\otimes^{\mathbf{d}} v_{(1,1,1)} & =0 \\
+q^{-1}\otimes^{\mathbf{d}} v_{(0,1,0)} & +\otimes^{\mathbf{d}} v_{(1,0,1)} & & \\
+q^{-2}\otimes^{\mathbf{d}} v_{(1,0,0)} & & &
\end{array}$$

$$\begin{array}{ll}
F\otimes^{\mathbf{d}} v_{(0,1,0)} & F\otimes^{\mathbf{d}} v_{(1,1,0)} \\
=\otimes^{\mathbf{d}} v_{(0,1,1)} & =\otimes^{\mathbf{d}} v_{(1,1,1)} \\
+\otimes^{\mathbf{d}} v_{(1,1,0)} &
\end{array}$$

$$\begin{array}{ll}
F\otimes^{\mathbf{d}} v_{(1,0,0)} & F\otimes^{\mathbf{d}} v_{(1,0,1)} \\
=\otimes^{\mathbf{d}} v_{(1,0,1)} & =q\otimes^{\mathbf{d}} v_{(1,1,1)} \\
+q^{-1}\otimes^{\mathbf{d}} v_{(1,1,0)} &
\end{array}$$

and

$$\begin{array}{llll}
K\otimes^{\mathbf{d}} v_{(0,0,0)} & K\otimes^{\mathbf{d}} v_{(0,0,1)} & K\otimes^{\mathbf{d}} v_{(0,1,1)} & K\otimes^{\mathbf{d}} v_{(1,1,1)} \\
=q^3\otimes^{\mathbf{d}} v_{(0,0,0)} & =q\otimes^{\mathbf{d}} v_{(0,0,1)} & =q^{-1}\otimes^{\mathbf{d}} v_{(0,1,1)} & =q^{-3}\otimes^{\mathbf{d}} v_{(1,1,1)}
\end{array}$$

$$\begin{array}{ll}
K\otimes^{\mathbf{d}} v_{(0,1,0)} & K\otimes^{\mathbf{d}} v_{(1,1,0)} \\
=q\otimes^{\mathbf{d}} v_{(0,1,0)} & =q^{-1}\otimes^{\mathbf{d}} v_{(1,1,0)}
\end{array}$$

$$\begin{array}{ll}
K\otimes^{\mathbf{d}} v_{(1,0,0)} & K\otimes^{\mathbf{d}} v_{(1,0,1)} \\
=q\otimes^{\mathbf{d}} v_{(1,0,0)} & =q^{-1}\otimes^{\mathbf{d}} v_{(1,0,1)}
\end{array}$$

Example 13. $\mathbf{d} = (1,1,1)$:
The coefficients of the $f_{\mathbf{w}',\mathbf{0},\mathbf{d}}$ occuring in the representation of $h^{\mathbf{d}}_{\mathbf{w}}$ in this basis are calculated using the Kazhdan-Lusztig polynomials $p_{\mathbf{w}',\mathbf{w}}$, with q^{-1} inserted (see [1, section 5]) (Again, I

will abbreviate $f_{\mathbf{w},\mathbf{0},\mathbf{d}}$ by $f_{\mathbf{w}}$).

$$
\begin{array}{ll|ll|ll|ll}
h^{\mathbf{d}}_{(0,0,0)} = & f_{(0,0,0)} & h^{\mathbf{d}}_{(0,0,1)} = & f_{(0,0,1)} & h^{\mathbf{d}}_{(0,1,1)} = & f_{(0,1,1)} & h^{\mathbf{d}}_{(1,1,1)} = & f_{(1,1,1)} \\
 & & & +q^{-1}f_{(0,1,0)} & & +q^{-1}f_{(1,0,1)} & & \\
 & & & +q^{-2}f_{(1,0,0)} & & +q^{-2}f_{(1,1,0)} & & \\
 & & h^{\mathbf{d}}_{(0,1,0)} = & f_{(0,1,0)} & h^{\mathbf{d}}_{(1,1,0)} = & f_{(1,1,0)} & & \\
 & & & +q^{-1}f_{(1,0,0)} & & & & \\
 & & h^{\mathbf{d}}_{(1,0,0)} = & f_{(1,0,0)} & h^{\mathbf{d}}_{(1,0,1)} = & f_{(1,0,1)} & & \\
 & & & & & +q^{-1}f_{(1,1,0)} & &
\end{array}
$$

Operation of $U_q(sl_2)$ on $T_0(\mathbf{d}) \cong V_1^{\otimes 3}$ with basis $h^{\mathbf{d}}_{\mathbf{w}}$:

$$
\begin{array}{ll|ll|ll|ll}
Eh^{\mathbf{d}}_{(0,0,0)} = & 0 & Eh^{\mathbf{d}}_{(0,0,1)} = & [3]h^{\mathbf{d}}_{(0,0,0)} & Eh^{\mathbf{d}}_{(0,1,1)} = & [2]h^{\mathbf{d}}_{(0,0,1)} & Eh^{\mathbf{d}}_{(1,1,1)} = & h^{\mathbf{d}}_{(0,1,1)} \\
 & & Eh^{\mathbf{d}}_{(0,1,0)} = & [2]h^{\mathbf{d}}_{(0,0,0)} & Eh^{\mathbf{d}}_{(1,1,0)} = & h^{\mathbf{d}}_{(0,1,0)} & & \\
 & & Eh^{\mathbf{d}}_{(1,0,0)} = & h^{\mathbf{d}}_{(0,0,0)} & Eh^{\mathbf{d}}_{(1,0,1)} = & h^{\mathbf{d}}_{(1,0,0)} & & \\
 & & & & & +h^{\mathbf{d}}_{(0,0,1)} & & \\
Fh^{\mathbf{d}}_{(0,0,0)} = & h^{\mathbf{d}}_{(0,0,1)} & Fh^{\mathbf{d}}_{(0,0,1)} = & [2]h^{\mathbf{d}}_{(0,1,1)} & Fh^{\mathbf{d}}_{(0,1,1)} = & [3]h^{\mathbf{d}}_{(1,1,1)} & Fh^{\mathbf{d}}_{(1,1,1)} = & 0 \\
 & & Fh^{\mathbf{d}}_{(0,1,0)} = & h^{\mathbf{d}}_{(1,1,0)} & Fh^{\mathbf{d}}_{(1,1,0)} = & h^{\mathbf{d}}_{(1,1,1)} & & \\
 & & & +h^{\mathbf{d}}_{(0,1,1)} & & & & \\
 & & Fh^{\mathbf{d}}_{(1,0,0)} = & h^{\mathbf{d}}_{(1,0,1)} & Fh^{\mathbf{d}}_{(1,0,1)} = & [2]h^{\mathbf{d}}_{(1,1,1)} & &
\end{array}
$$

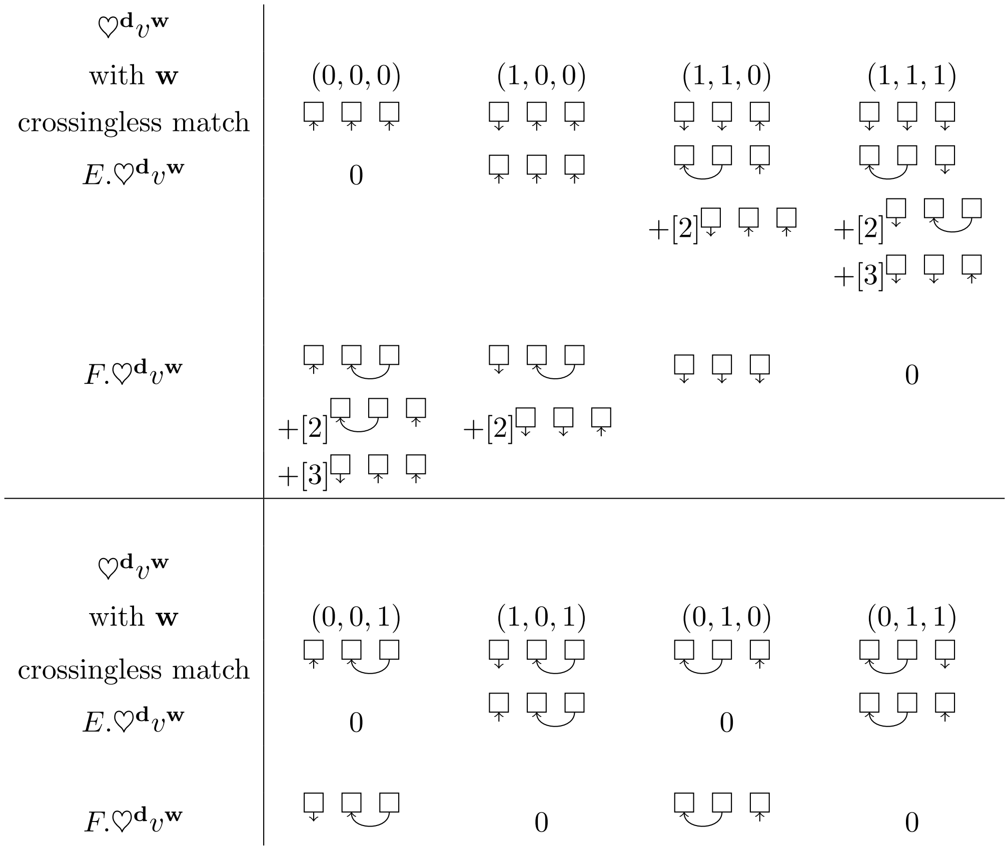
(use $q + q^{-1} = [2]$ and $q^2 + 1 + q^{-2} = [3]$) and

$$
\begin{array}{lll}
K^{\pm 1}h^{\mathbf{d}}_{(0,0,0)} = q^{\pm 3}h^{\mathbf{d}}_{(0,0,0)} & & \\
K^{\pm 1}h^{\mathbf{d}}_{(0,0,1)} = q^{\pm 1}h^{\mathbf{d}}_{(0,0,1)} & K^{\pm 1}h^{\mathbf{d}}_{(0,1,0)} = q^{\pm 1}h^{\mathbf{d}}_{(0,1,0)} & K^{\pm 1}h^{\mathbf{d}}_{(1,0,0)} = q^{\pm 1}h^{\mathbf{d}}_{(1,0,0)} \\
K^{\pm 1}h^{\mathbf{d}}_{(0,1,1)} = q^{\pm -1}h^{\mathbf{d}}_{(0,1,1)} & K^{\pm 1}h^{\mathbf{d}}_{(1,1,0)} = q^{\pm -1}h^{\mathbf{d}}_{(1,1,0)} & K^{\pm 1}h^{\mathbf{d}}_{(1,0,1)} = q^{\pm -1}h^{\mathbf{d}}_{(1,0,1)} \\
K^{\pm 1}h^{\mathbf{d}}_{(1,1,1)} = q^{\pm -3}h^{\mathbf{d}}_{(1,1,1)} & &
\end{array}.
$$

Notice that all coefficients are positive for the action of U_q on the $h^{\mathbf{d}}_{\mathbf{w}}$.

Remark 11. Compare this to the action of U_q on the dual basis given by [6]. The dual basis can be denoted by (upper) crossingless matchings where an arc is drawn between to arrows if the one oriented up is to the left of the one oriented down and as usual w_i denotes the number of down verticies in the box corresponding to V_{d_i} (a down vertex is associated to v_{-1} and an up vertex to v_1, so e.g. $v_1 \otimes v_{-1}$ is associated to

). To determine the action of E, numerate all down-oriented verticies not connected to some other vertex with an arc, starting from the left, by $(1, 2, \cdots, l_{down})$. E acts on $\heartsuit^{\mathbf{d}} v^{\mathbf{w}}$ by $\sum_i [i] E_{(i)} \heartsuit^{\mathbf{d}} v^{\mathbf{w}}$, where $E_{(i)}$ reverses the i^{th} down arrow not connected with an arc to an up arrow and draws an arc if possible (i.e. if there is a neighboring down arrow to the right of the up arrow). Similarly, F reverses the up-arrows, starting from the right. For $\mathbf{d} = (1, 1, 1)$, one gets:



and one has $\langle u, \overline{\Delta}^{(k-1)}(w(x))v^* \rangle = \langle \Delta^{(k-1)}(x)u, v^* \rangle$ for u in usual basis and v^* in dual basis, so e.g. $\langle \diamondsuit^{\mathbf{d}} u_{\mathbf{w}}, \overline{\Delta}^{(k-1)}(E) \heartsuit^{\mathbf{d}} v^{\mathbf{w}'} \rangle = \langle \Delta^{(k-1)}(F) \diamondsuit^{\mathbf{d}} u_{\mathbf{w}}, \heartsuit^{\mathbf{d}} v^{\mathbf{w}'} \rangle$. This gives a further way of checking that the results on the canonical basis calculated before in the example are indeed correct. Again, all the coefficients occuring are positive.

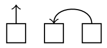
3.7 The Space $T_c(\mathbf{d})$ and a Canonical Basis of It

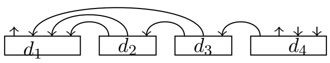
One can find an extension $e : T_0(\mathbf{d}) \to T(\mathbf{d})$ (module homomorphism, isomorphism onto its image) extending invariant functions on $\mathfrak{T}_0(\mathbf{d})$ to invariant functions on $\mathfrak{T}(\mathbf{d})$ with larger support. By this, one wishes to obtain from the $h^{\mathbf{d}}_{\mathbf{w}}$ a basis of invariant functions on $\mathfrak{T}(\mathbf{d})$ with a nice geometric interpretation. So the aim is to find an extension that will yield such a nice basis and that is an isomorphism onto its image, such that the new basis can again be identified with the canonical basis via e and $\eta_{\mathbf{0},\mathbf{d}}$.

Definition 3.12. Define an extension e extending a function $f \in T_0(\mathbf{d})$ to a function $f^e \in T(\mathbf{d})$ by

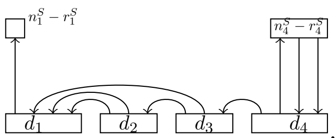
$$f^e = \sum_{\mathbf{r},\mathbf{n}} (\eta_{\mathbf{r},\mathbf{n}})^{-1} \circ \gamma_{\mathbf{r},\mathbf{n}} \circ \eta_{\mathbf{0},\mathbf{d}}(f)$$

(Recall the definition 3.9 of η), where the $\gamma_{\mathbf{r},\mathbf{n}}$ in the set of intertwiners $\{\gamma_{\mathbf{r},\mathbf{n}} : V_{d_1} \otimes \cdots \otimes V_{d_k} \to V_{n_1-r_1} \otimes \cdots \otimes V_{n_k-r_k}\}_{\mathbf{r},\mathbf{n}}$ are defined below.

To define the $\gamma_{\mathbf{r},\mathbf{n}}$, some preparation is needed. It is known from [6], that a basis of the space of intertwiners (though linear endomorphisms commuting with the alternative comultiplication) between two tensor product representations of U_q is given by the corresponding crossingless matchings. For a lower crossingless matching S, define $\mathbf{r}^S$ by setting r_i^S equal to the number of left endpoints of lower curves contained in V_{d_i} and define $\mathbf{n}^S$ by setting n_i^S equal to d_i minus the number of right endpoints of lower curves contained in V_{d_i}. One can associate to any lower crossingless matching an endomorphism t sending a vector of $D_i \backslash D_{i-1}$ to a vector of $D_j \backslash D_{j-1}$, $j < i$, for any curve connecting V_{d_i} and V_{d_j} (choose a basis of D compatible with the flag $\mathbf{D}$ and define the matrix of t in this basis by $(C_t)_{i,j} = 1$ if $i < j$ and S has a curve connecting the i^{th} and j^{th} vertices and equal to zero otherwise). E.g. let S be the crossingless matching  , so $\mathbf{n^S} = (1,1,0)$ and $\mathbf{r^S} = (0,1,0)$, and let $\mathbf{D}$ be the standard flag $< e_1 > \subset < e_1, e_2 > \subset < e_1, e_2, e_3 >$. Then let the matrix of t in the standard basis be $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.
So one actually obtains $\mathbf{r^S} = \alpha(\text{im } t, \mathbf{D})$ and $\mathbf{n^S} = \alpha(\ker t, \mathbf{D})$. This S can be completed to a crossingless matching to $V_{n_1^S - r_1^S} \otimes \cdots \otimes V_{n_k^S - r_k^S}$ in a unique way as $n_i^S - r_i^S$ is the number of unconnected vertices of the ith box.
E.g. the lower crossingless matching S



with $\mathbf{r^S} = (3,1,1,0)$ and $\mathbf{n^S} = (4,1,1,3)$ can be completed to a crossingless match to $V_1 \otimes V_0 \otimes V_0 \otimes V_2 = V_{n_1^S - r_1^S} \otimes \cdots \otimes V_{n_4^S - r_4^S}$,



Then let $\tilde{\gamma}_{\mathbf{r^S},\mathbf{n^S}}$ be the corresponding intertwiner commuting with the action of U_q given by $\overline{\Delta}^{(k-1)}$. This is welldefined as $S \mapsto (\mathbf{r^S}, \mathbf{n^S})$ is injective. Define $\gamma_{\mathbf{r^S},\mathbf{n^S}} = \hat{\sigma}\tilde{\gamma}_{\mathbf{r^S},\mathbf{n^S}}\hat{\sigma}$; this is an intertwiner commuting with the action of U_q given by $\Delta^{(k-1)}$. If $(\mathbf{r}, \mathbf{n})$ is not of the form $(\mathbf{r^S}, \mathbf{n^S})$ for any crossingless matching S, set $\gamma_{\mathbf{r},\mathbf{n}} = 0$.

Proposition 3.13. *e is an isomorphism onto its image and $f^e|_{T_0(\mathbf{d})} = f$.*

Proof: Follows from Proposition 3.10 and the way the intertwiner associated to a crossingless matching is defined. □
Let $T_c(\mathbf{d}) := span\{f^e_{\mathbf{w},\mathbf{0},\mathbf{d}}\}$. Now one wants to show that the distinguished basis $g^{\mathbf{d}}_{\mathbf{w}} = (h^{\mathbf{d}}_{\mathbf{w}})^e$ of this space corresponds to the irreducible components of $\mathfrak{T}(\mathbf{d})$ and to the canonical basis of $V_{d_1} \otimes \cdots \otimes V_{d_k}$, thus getting a geometric interpretation of the canonical basis.

In order to do this, it is necessary to work over the algebraic closure of the field for some time. Let $\mathfrak{T}(\mathbf{d})'$ denote the variety over the closure of the field defined in the same fashion as $\mathfrak{T}(\mathbf{d})$ and set $Z'_{\mathbf{w}} = \{(\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d})' | \alpha(W, \mathbf{D}) = \mathbf{w}\} = \bigcup_{\mathbf{r},\mathbf{n}} A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ (where $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is defined in analogy to $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$).

Proposition 3.14. *The $\overline{Z'_w}$ are the irreducible components of $\mathfrak{T}(\mathbf{d})'$.*

Remark 12. An analogous statement for the Steinberg varieties introduced in section 3.1 is well-known ([2], [5]).

Proof: Clearly $\dot{\bigcup}_{\mathbf{w}} Z'_{\mathbf{w}} = \mathfrak{T}(\mathbf{d})'$. Moreover, the connected components of $\mathfrak{T}(\mathbf{d})'$ are given by fixing the dimension of W, i.e. by $\bigcup_{\substack{\mathbf{w}', \\ |\mathbf{w}'|=|\mathbf{w}|}} A'_{\mathbf{w}',\mathbf{r},\mathbf{n}}$. Thus it sufficies to show that the $Z'_{\mathbf{w}}$ are irreducible and locally closed and that their dimension is independent of $\mathbf{w}$ for fixed $|\mathbf{w}|$ (so the closures (the sets themselves are disjoint), are not contained in one another). In order to do so, consider the maps

$$\begin{tikzcd} & Z'_{\mathbf{w}} \arrow[dl, "p_1"] & \\ Z'^1_{\mathbf{w}} \arrow[rr, "p_2"] & & Z'^2_{\mathbf{w}} \end{tikzcd}$$

given by $p_1(\mathbf{D}, W, t) = (\mathbf{D}, W)$ and $p_2(\mathbf{D}, W) = \mathbf{D}$ with

$$Z'^1_{\mathbf{w}} = \{(\mathbf{D}, W) | (\mathbf{D}, W, t) \in Z'_{\mathbf{w}} \text{ for some } t\} \hat{=} A_{\mathbf{w},\mathbf{0},\mathbf{d}}$$

and

$$Z'^2_{\mathbf{w}} = \{\mathbf{D} | (\mathbf{D}, W) \in Z'^1_{\mathbf{w}} \text{ for some } W\} = Fl(\mathbf{d})$$

a flag manifold. p_1 and p_2 are locally trivial fibrations, i.e. for each point $(\mathbf{D}, W, t)$ ($(\mathbf{D}, W)$ respectivly) there is an open neighborhood U of $(\mathbf{D}, W)$ ($\mathbf{D}$ respectivly) such that $p_1^{-1}(U) \cong (\mathbf{D}, W) \times \{t \in \mathrm{End}(D) \mid t(D_i) \subset D_{i-1}, \mathrm{im}(t) \subset W \subset \ker(t)\}$ ($p_2^{-1}(U) \cong \mathbf{D} \times \{W \subset D \mid \alpha(W, \mathbf{D}) = \mathbf{w}\}$ respectivly). $GL(D)$ acts transitive on $Z'^1_{\mathbf{w}} = A_{\mathbf{w},\mathbf{0},\mathbf{d}}$ with stabilizer

$$G_1 = \left\{ \begin{pmatrix} M_1 & * & * & * & * & \cdots & \cdots & * \\ 0 & N_1 & 0 & * & 0 & \cdots & \cdots & * \\ 0 & 0 & M_2 & * & * & & & * \\ 0 & 0 & 0 & N_2 & 0 & & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & & * \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & M_k & * \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & N_k \end{pmatrix} \middle| M_i \in GL(w_i), N_i \in GL(d_i - w_i) \right\}.$$

Thus $\dim(A_{\mathbf{w},\mathbf{0},\mathbf{d}}) = \dim GL(D) - \dim G_1 = \sum_{i<j} d_i d_j + \sum_{i \leq j} w_j(d_i - w_i)$. The fiber of p_1 over a point $(\mathbf{D}, W) \in Z'^1_w$ is

$$F_1 = \{t \in \mathrm{End}(D) \mid t(D_i) \subset D_{i-1}, \text{ im } t \subset W \subset \ker t\}.$$

In order to describe the dimension of this fiber, pick a basis $\{u_i\}_{i=1}^{d}$ of D such that $\{u_i\}_{i=1}^{d_1+\ldots+d_j}$ is a basis of D_j and such that $\bigcup_{l=0}^{j}\{u_i\}_{i=\mathbf{d}^{(1,l-1)}+1}^{\mathbf{d}^{(1,l-1)}+w_l}$ is a basis for $W \cap D_j$ (where $\mathbf{d}^{(i,j)} = \sum_{l=i}^{j} d_l$). By considering the matrices of t in this basis

$$t = \begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad\qquad}^{d_1} & \overbrace{\qquad\qquad}^{d_2} & \cdots & \overbrace{\qquad\qquad}^{d_k} \end{array} \\
\left(\begin{array}{cc:cc:c:cc}
\overbrace{0}^{w_1} & 0 & \overbrace{0}^{w_2} & * & \cdots & \overbrace{0}^{w_k} & * \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hdashline
 & & 0 & 0 & \cdots & 0 & * \\
 & & 0 & 0 & \cdots & 0 & 0 \\
\hdashline
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\hdashline
 & & & & \cdots & 0 & 0 \\
 & & & & \cdots & 0 & 0
\end{array}\right)
\end{array}$$

one sees that F_1 is an affine space of dimension $\sum_{i>j} w_j(d_i - w_i)$. Finally, one obtains

$$\dim Z'_{\mathbf{w}} = \dim Z'^{1}_{\mathbf{w}} + \dim F_1 = \sum_{i<j} d_i d_j + \sum_{i,j=1}^{k} w_j(d_i - w_i) = \sum_{i<j} d_i d_j + |\mathbf{w}|(d - |\mathbf{w}|),$$

which is independent of $\mathbf{w}$ for fixed $|\mathbf{w}|$. The spaces $Z'^{2}_{\mathbf{w}}, F_1$ and $F_2 = \{W \subset D \mid \alpha(W, \mathbf{D}) = \mathbf{w}\}$, the fiber of p_2 over a point $\mathbf{D} \in Z'^{2}_{\mathbf{w}}$, are all smooth and connected, hence irreducible. Furthermore, $Z'^{2}_{\mathbf{w}}$ and F_1 are closed and F_2 is locally closed since F_2 is equal to the closed set $\{W \subset D \mid \alpha(W, \mathbf{D}) \geq \mathbf{w}\}$ minus the finite collection of closed sets $\{W \subset D \mid \alpha(W, \mathbf{D}) \geq \mathbf{a}\}_{\mathbf{a}>\mathbf{w}}$. Thus $Z'_{\mathbf{w}}$ is irreducible and locally closed. $\square$

Remark 13. In a similar fashion as in the case of $A'_{\mathbf{w},\mathbf{0},\mathbf{d}}$, one can calculate the dimensions of the other orbits. Let $W \in F_2$ for some $\mathbf{D}$ and $A'_{\mathbf{w},\mathbf{r},\mathbf{n}} \neq \emptyset$. Then one can define a t such that $(\mathbf{D}, W, t) \in A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ as follows. $\mathbf{r}$ and $\mathbf{n}$ tell me how many basis elements of D_i have to be sent to 0 and onto how many basis vectors of D_j the rest may be sent (use the same basis as in the proof of proposition 3.14). Since $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ is not empty, these vectors can be chosen from the basis of W. Thus $p_1|_{A'_{\mathbf{w},\mathbf{r},\mathbf{n}}} = Z'^{1}_{\mathbf{w}}$. The difference in dimension therefore can only occur in the fiber F'_1 of $p_1|_{A'_{\mathbf{w},\mathbf{r},\mathbf{n}}}$ over some point $(\mathbf{D}, W)$. Consider $t \in F'_1$ in the same basis as in the proof of proposition 3.14:

$$t = \begin{array}{c}
\begin{array}{ccccccc} \overbrace{\qquad\qquad}^{d_1} & \qquad & \overbrace{\qquad\qquad}^{d_2} & \qquad & \cdots & \qquad & \overbrace{\qquad\qquad}^{d_k} \end{array} \\
\left(\begin{array}{cccccccccc}
\overbrace{0}^{w_1} & 0 & \vdots & \overbrace{0}^{w_2} & * & \vdots & \cdots & \vdots & \overbrace{0}^{w_k} & * \\
0 & 0 & \vdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 \\
\multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
 & & \vdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & * \\
 & & \vdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 \\
\multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
 & & \vdots & & & \vdots & \cdots & \vdots & 0 & 0 \\
 & & \vdots & & & \vdots & \cdots & \vdots & 0 & 0
\end{array}\right) \\
\begin{array}{ccccccc} \underbrace{\qquad\qquad}_{rank\,0} & \qquad & \underbrace{\qquad\qquad}_{rank\,d_2-n_2} & \qquad & \cdots & \qquad & \underbrace{\qquad\qquad}_{rank\,d_k-n_k} \end{array}
\end{array}$$

In addition, to fulfill the condition posed by $\mathbf{r}$, there must by r_1 linear independant columns with zero entries outside the first d_1 rows (so that im $t \cap D_1$ has dimension r_1), r_2 linear independant collumns with zero entries in rows below the first $d_1 + d_2$ rows and not all entries zero in the rows below the first d_1 rows and so on. Clearly, as $\mathbf{r}$ increases and subsequently $\mathbf{n}$ decreases, the number of possibilities and thus the dimension of $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$ increases.

Proposition 3.15. *Setting $M = M(\mathbf{d}, \mathbf{w})$, the crossingless matching corresponding to $\mathbf{w}$, then $A'_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ is open and dense in $\overline{Z'_{\mathbf{w}}}$.*

Proof: It is obvious that $A'_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}} \subset \overline{Z'_{\mathbf{w}}}$, so it only remains to show that $A'_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ is dense in Z'_w (Recall that in the Zariski-topology, a subset of an irreducible variety is dense if it is open and not empty). As seen in the remark, the projection of $A'_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ onto Z'^1_w is all of Z'^1_w. Thus it suffices to show that $A'_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ is dense in each fiber. For fixed $(\mathbf{D}, W) \in Z'^1_w$, the intersection F'_1 of F_1 with $(p_1|_{A'_{\mathbf{w},\mathbf{r},\mathbf{n}}})^{-1}(\mathbf{D}, W)$ is given by

$$F'_1 = \{t \in \mathrm{End}(D) | t(D_i) \subset D_{i-1},\ \mathrm{im}\ t \subset W \subset \ker\ t,\ \alpha(\mathrm{im}\ t, \mathbf{D}) = \mathbf{r^M},\ \alpha(\ker\ t, \mathbf{D}) = \mathbf{n^M}\}.$$

Choose a basis as in the proof of proposition 3.14. Since im $t \subset W \subset$ ker t, t can be factored through D/W and viewed as map into W. Then t is uniquely determined by the correspponding $\bar{t} \in \mathrm{End}(D/W, W)$. Then (see proof of proposition 3.14) the matrix C_t of $\bar{t}$ is of the form

$$C_t = \begin{pmatrix} 0 & A_{1,2} & A_{1,3} & \dots & A_{1,k} \\ \vdots & 0 & A_{2,3} & \dots & A_{2,k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & A_{k-1,k} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A_{i,j}$ a $(w_i)\times(d_j-w_j)$-matrix (corresponding to the $*$ in the matrix in the proof of proposition 3.14). I claim that $t \in F_1'$ if and only if each submatrix

$$C_t^{i,j} = \begin{pmatrix} A_{i,i+1} & A_{i,i+2} & \dots & A_{i,j+1} \\ 0 & A_{i+1,i+2} & \dots & A_{i+1,j+1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & A_{j,j+1} \end{pmatrix}$$

for $1 \leq i \leq j \leq k-1$ has maximal rank. That is, $\alpha(\text{im } t, \mathbf{D})_l$ and $d_l - \alpha(\ker t, \mathbf{D})_l$ are maximal for all l. Now consider a diagram M' of non-crossing oriented curves connecting the V_{d_l} associated to a $t \in F_1$, i.e. the number of downward vertices among those associated to V_{d_l} is given by w_l and the number of left and right endpoints of curves of M' in V_{d_l} is given by $\alpha(\text{im } t, \mathbf{D})_l$ and $d_l - \alpha(\ker t, \mathbf{D})_l$ respectively (So M' illustrates how t maps the basis vectors of D_l/D_{l-1} to those of D_m/D_{m-1} in a certain especially nice basis). A priori, this need not be an oriented lower crossingless matching, as for example the unmatched vertices might not be arranged such that those oriented down are to the right of those oriented up. However, requiring the rank of $C_t^{i,j}$ to be maximal is equivalent to M' having the maximal number of curves connecting V_{d_i}, $V_{d_{i+1}}, \dots$ and $V_{d_{j+1}}$. Comparing this to the definition of $M(\mathbf{d}, w)$ in definition 2.7, one sees that $C_t^{i,j}$ having maximal rank is equivalent to $M' = M$ and thus to $\mathbf{r^M} = \mathbf{r^{M'}}$ and $\mathbf{n^M} = \mathbf{n^{M'}}$, therefore to $t \in F_1'$. This prooves the claim. This argument shows once more, that $F_1' \neq \emptyset$, since one can define $t \in F_1'$ by $(C_t)_{(i,j)} = 1$ if $i < j$ and M has a curve connecting the i^{th} and j^{th} vertices, and $(C_t)_{(i,j)} = 0$ otherwise. To be more precise, as seen above, any $t \in F_1'$ has a matrix of this form for a basis chosen accordingly.

Now one still has to see that the set F_1' is open and dense. Being non-empty, it is clear that F_1' is dense if it is open.

I claim that $N_{m,n} = \{A \in M_{m,n} | A \text{ has maximal rank}\} \subset M_{m,n} = m \times n -$ matrices is open in $M_{m,n}$. To see this, let $r = min(m, n)$.

Then $N_{m,n} = \{A \in M_{m,n} |$ at least one $r \times r$ submatrix of A has rank r$\}$, which is a union of open subsets of $M_{m,n}$ since a $r \times r$ matrix has rank r if and only if it has a nonzero determinant, thus open. Since $N_{m,n}$ is open, it is given by the non-vanishing of a finite collection of polynomials in the matrix elements of $M_{m,n}$ (since I am working over the Zariski-topology). Applying this to the $C_t^{i,j}$, requiring $C_t^{i,j}$ has maximal rank is equivalent to the non-vanishing of a finite number of polynomials in the matrix elements of $C_t^{i,j}$, and thus of C_t. Therefore F_1' is the intersection of a finite number of open subsets of F_1, and hence open.

$\square$

3.8 Examples for the $M(\mathbf{d}, \mathbf{w})$ and Corresponding $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$

In this section I describe explicitly the spaces $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ assigned to crossingless matchings in some small examples.

Example 14. $M(\mathbf{d}, \mathbf{w})$ and the corresponding $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ for $d = 3, 4$:
Recalling the inclusion $V_{d_1} \otimes \cdots \otimes V_{d_k} \to V_1^{\otimes \sum_{i=1}^k d_i}$, one can reduce the case where $\mathbf{d'} \neq (1, \dots, 1)$

to the case $\mathbf{d} = (1, \ldots, 1)$, where $|\mathbf{d}'| =$ number of 1s in $(1, \ldots, 1) = |\mathbf{d}|$, by "merging" the boxes in the diagram according to $\mathbf{d}'$ (so that each box contains the correct number of vertices) and regarding only those lower oriented crossingless matchings which do not have a lower curves among vertices of a single box. For fixed $\mathbf{d}$, all lower oriented crossingless matchings can be parametrised by the $\mathbf{w}$, yielding the $M(\mathbf{d}, \mathbf{w})$.

$d = 3$

$\mathbf{w}$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,1,1)$
$M((1,1,1),\mathbf{w})$				
$A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$	$A_{\mathbf{w},(0,0,0)(1,1,1)}$	$A_{\mathbf{w},(0,0,0)(1,1,1)}$	$A_{\mathbf{w},(0,0,0)(1,1,1)}$	$A_{\mathbf{w},(0,0,0)(1,1,1)}$
element of orbit	$(\mathbf{D}, 0, 0)$	$(\mathbf{D}, < e_3 >, 0)$	$(\mathbf{D}, < e_2, e_3 >, 0)$	$(\mathbf{D}, D, 0)$

$\mathbf{w}$	$(0,1,0)$	$(1,1,0)$
$M((1,1,1),\mathbf{w})$		
$A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$	$A_{\mathbf{w},(0,1,0)(1,1,0)}$	$A_{\mathbf{w},(0,1,0)(1,1,0)}$
element of orbit	$(\mathbf{D}, < e_2 >, \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix})$	$(\mathbf{D}, < e_1, e_2 >, \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix})$

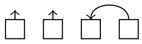
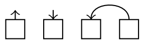
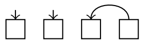
$\mathbf{w}$	$(1,0,0)$	$(1,0,1)$
$M((1,1,1),\mathbf{w})$		
$A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$	$A_{\mathbf{w},(1,0,0)(1,0,1)}$	$A_{\mathbf{w},(1,0,0)(1,0,1)}$
element of orbit	$(\mathbf{D}, < e_1 >, \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix})$	$(\mathbf{D}, < e_1, e_3 >, \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix})$

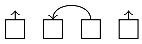
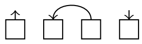
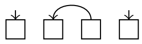
($\mathbf{D} = (0 \subset < e_1 > \subset < e_1, e_2 > \subset < e_1, e_2, e_3 > = D)$, $< e_i >$ the standard basis vectors of D). Now, for $\mathbf{d} = (3)$, only the first four $M(\mathbf{d}, \mathbf{w})$ are admitted, for $\mathbf{d} = (2, 1)$, the first six, for $\mathbf{d} = (1, 2)$ the first four and the last two, and all for $\mathbf{d} = (1, 1, 1)$.

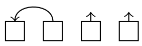
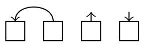
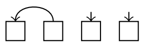
$d = 4$

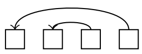
$\mathbf{w}$	$(0,0,0,0)$	$(0,0,0,1)$	$(0,0,1,1)$	$(0,1,1,1)$
$M((1,1,1,1),\mathbf{w})$				
$A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$	$A_{\mathbf{w},(0,0,0,0)(1,1,1,1)}$	$A_{\mathbf{w},(0,0,0,0)(1,1,1,1)}$	$A_{\mathbf{w},(0,0,0,0)(1,1,1,1)}$	$A_{\mathbf{w},(0,0,0,0)(1,1,1,1)}$
element of orbit	$(\mathbf{D}, 0, 0)$	$(\mathbf{D}, < e_4 >, 0)$	$(\mathbf{D}, < e_3, e_4 >, 0)$	$(\mathbf{D}, < e_2, e_3, e_4 >, 0)$

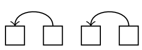
$\mathbf{w}$	$(1,1,1,1)$
$M((1,1,1,1),\mathbf{w})$	
$A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$	$A_{\mathbf{w},(0,0,0,0)(1,1,1,1)}$
element of orbit	$(\mathbf{D}, D, 0)$

$\mathbf{w}$	$(0,0,1,0)$	$(0,1,1,0)$	$(1,1,1,0)$
$M((1,1,1,1),\mathbf{w})$			
$A_{\mathbf{w},\mathbf{r}^{\mathrm{M}},\mathbf{n}^{\mathrm{M}}}$	$A_{\mathbf{w},(0,0,1,0)(1,1,1,0)}$	$A_{\mathbf{w},(0,0,1,0)(1,1,1,0)}$	$A_{\mathbf{w},(0,0,1,0)(1,1,1,0)}$
W in element	$< e_3 >$	$< e_2, e_3 >$	$< e_1, e_2, e_3 >$
$(\mathbf{D}, W, t)$ of set			
with $t = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$			

$\mathbf{w}$	$(0,1,0,0)$	$(0,1,0,1)$	$(1,1,0,1)$
$M((1,1,1,1),\mathbf{w})$			
$A_{\mathbf{w},\mathbf{r}^{\mathrm{M}},\mathbf{n}^{\mathrm{M}}}$	$A_{\mathbf{w},(0,1,0,0)(1,1,0,1)}$	$A_{\mathbf{w},(0,1,0,0)(1,1,0,1)}$	$A_{\mathbf{w},(0,1,0,0)(1,1,0,1)}$
W in element	$< e_3 >$	$< e_2, e_3 >$	$< e_1, e_2, e_3 >$
$(\mathbf{D}, W, t)$ of set			
with $t = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$			

$\mathbf{w}$	$(1,0,0,0)$	$(1,0,0,1)$	$(1,0,1,1)$
$M((1,1,1,1),\mathbf{w})$			
$A_{\mathbf{w},\mathbf{r}^{\mathrm{M}},\mathbf{n}^{\mathrm{M}}}$	$A_{\mathbf{w},(1,0,0,0)(1,0,1,1)}$	$A_{\mathbf{w},(1,0,0,0)(1,0,1,1)}$	$A_{\mathbf{w},(1,0,0,0)(1,0,1,1)}$
W in element	$< e_3 >$	$< e_2, e_3 >$	$< e_1, e_2, e_3 >$
$(\mathbf{D}, W, t)$ of set			
with $t = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$			

$\mathbf{w}$	$(1,1,0,0)$
$M((1,1,1,1),\mathbf{w})$	
$A_{\mathbf{w},\mathbf{r}^{\mathrm{M}},\mathbf{n}^{\mathrm{M}}}$	$A_{\mathbf{w},(1,1,0,0)(1,1,0,0)}$
W in element	$< e_1, e_2 >$
$(\mathbf{D}, W, t)$ of set	
with $t = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$	

$\mathbf{w}$	$(1,0,1,0)$
$M((1,1,1,1),\mathbf{w})$	
$A_{\mathbf{w},\mathbf{r}^{\mathrm{M}},\mathbf{n}^{\mathrm{M}}}$	$A_{\mathbf{w},(1,0,1,0)(1,0,1,0)}$
W in element	$< e_1, e_3 >$
$(\mathbf{D}, W, t)$ of set	
with $t = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$	

3.9 A more Detailed Description of the Irreducible Components

Proposition 3.16. $\overline{A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}}} \subset \overline{Z'_w}$ *for all* $S \leq M$, $\mathbf{a} \geq \mathbf{w}$, $|\mathbf{a}| = |\mathbf{w}|$.

Proof: It is sufficient to show $A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}} \subset \overline{Z'_w}$. Since the connected components of $\mathfrak{T}(\mathbf{d})'$ are given by fixing the dimension of W, i.e. by

$$\bigcup_{\substack{\mathbf{w}',\\ |\mathbf{w}'|=|\mathbf{w}|}} A'_{\mathbf{w}',\mathbf{r},\mathbf{n}},$$

$|\mathbf{a}| = |\mathbf{w}|$ is clear as well. First consider $A'_{\mathbf{a},\mathbf{0},\mathbf{d}}$. I want to show that

$$A'_{\mathbf{a},\mathbf{0},\mathbf{d}} \subset \overline{A'_{\mathbf{w},\mathbf{0},\mathbf{d}}} \text{ if and only if } |\mathbf{a}| = |\mathbf{w}| \text{ and } \mathbf{a} \geq \mathbf{w}.$$

Since $Z'^1_{\mathbf{w}} = A'_{\mathbf{w},\mathbf{0},\mathbf{d}}$, it follows that $p_1(A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}}) \subset \overline{Z'^1_{\mathbf{w}}}$ if and only if $|\mathbf{a}| = |\mathbf{w}|$ and $\mathbf{a} \geq \mathbf{w}$, so $A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}} \subset \overline{Z'_{\mathbf{w}}}$ only if $|\mathbf{a}| = |\mathbf{w}|$ and $\mathbf{a} \geq \mathbf{w}$.

One has $A'_{\mathbf{a},\mathbf{0},\mathbf{d}} \subset Fl(\mathbf{d}) \times G(|\mathbf{a}|, d)$, which is projective (see remark 4). Let $I(X)$ denote the ideal of homogenous polynomials vanishing on X, then $I(A'_{\mathbf{a},\mathbf{0},\mathbf{d}}) \supset I(A'_{\mathbf{w},\mathbf{0},\mathbf{d}})$ for $\mathbf{a} \geq \mathbf{w}$ (Since for fixed $\mathbf{D}$, consider $(\mathbf{D}, W, 0) \in A'_{\mathbf{a},\mathbf{0},\mathbf{d}}$ and let $(u_l)_l = \bigcup_{i=1}^k (u_l^i)_l$ denote a basis compatible with $\mathbf{D}$ and W as in the proof of proposition 3.14, where $\bigcup_{i=1}^j (u_l^i)_l$ denotes the basis of D_j. Then one can define a $(\mathbf{D}, W', 0) \in A'_{\mathbf{w},\mathbf{0},\mathbf{d}}$: for each i such that $a_i \geq w_i$, $a_i = w_i + l$, choose $j_1, \ldots, j_l$ with $a_{j_k} \leq w_{j_k}$ such that $(u_l^i + \lambda u_m^{j_k})_{i,l}$ for some λ's forms a basis of a W' as required. Then a polynomial vanishing on $(\mathbf{D}, W', 0)$ for all λ, already has to vanish on $(\mathbf{D}, W, 0)$. Hence the inclusion follows.).

Let $V(I)$ denote the vanishing set of an ideal, then $V(I(A'_{\mathbf{a},\mathbf{0},\mathbf{d}})) \subset V(I(A'_{\mathbf{w},\mathbf{0},\mathbf{d}})) = \overline{A'_{\mathbf{w},\mathbf{0},\mathbf{d}}}$.

Next, consider the fiber of the projection p_1 over a point $(\mathbf{D}, W)$ given by

$$\{(\mathbf{D}, W, t) \in A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}} \mid p_1(\mathbf{D}, W, t) = (\mathbf{D}, W)\}.$$

So the first two entries are fixed and the fiber can be identified with

$$\{t | (\mathbf{D}, W, t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^S,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^S\}.$$

This is in the closure of

$$\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M\}$$

(since $S \leq M$, one has $r_i^S \leq r_i^M$, $n_i^S \geq n_i^M$, and therefore $A'_{\mathbf{w},\mathbf{r^S},\mathbf{n^S}} \in Z'_{\mathbf{w}}$, so for each t with $(\mathbf{D}, W, t) \in A'_{\mathbf{a},\mathbf{r^S},\mathbf{n^S}}$, there is W' with $(\mathbf{D}, W', t) \in A'_{\mathbf{w},\mathbf{r^S},\mathbf{n^S}}$. But then

$$(\mathbf{D}, W', t) \in \overline{\{(\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}) \mid \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M,\ p_1(\mathbf{D}, W', t) = (\mathbf{D}, W')\}},$$

which can be identified with the closure of

$$\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M\}.$$

So all t lie in the closure of

$$\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M,\ \alpha(W', \mathbf{D}) = \mathbf{w}\}.)$$

Now,

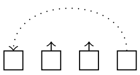
$$\{t|(\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M,\ \alpha(W', \mathbf{D}) = \mathbf{w}\} \in \overline{Z'_{\mathbf{w}}}$$

by proposition 3.15, thus

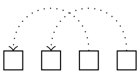
$$\overline{\{t|(\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\ \alpha(\ker t, \mathbf{D}) = n^M,\ \alpha(\operatorname{im} t, \mathbf{D}) = r^M,\ \alpha(W', \mathbf{D}) = \mathbf{w}\}} \in \overline{Z'_{\mathbf{w}}}.$$

So it is proven that $A'_{\mathbf{a},\mathbf{r}^{\mathbf{S}},\mathbf{n}^{\mathbf{S}}} \subset \overline{Z'_{\mathbf{w}}}$. □

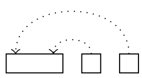
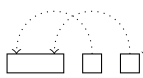
Given an algebraic group G acting on an algebraic variety X, the closure of an orbit O of G is of course again G-invariant, hence a union of G-orbits. In fact, see [10, 8.3], $\overline{O} - O$ is a union of orbits of strictly smaller dimension than O. This applies in particular to the situation here, and I am interested in describing the induced partial ordering on orbits given by $O' < O$ if O' is contained in the closure of O in more detail.

Remark 14. In a similar manner as in the proof, one can describe more generally some of the $A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'}$ lying in the closure of $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$. For one thing, $|\mathbf{a}| = |\mathbf{w}|$ and $\mathbf{a} \geq \mathbf{w}$ needs to be satisfied. It remains to consider the fiber. To each orbit I can associate a "generalised" lower oriented (crossingless) matching by arranging the vertices and up and down arrows as usual and drawing caps from the d_ith to the d_jth vertices for each basis vector of D_i/D_{i-1} mapped to D_j/D_{j-1} by some t belonging to an element of this orbit (these matchings are no longer necessarily crossingless, e.g.  corresponding to

$$t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with standard basis and standard flag or  corresponding to

$$t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

but I can let them have as few crossings as possible, e.g.  rather than ).

Proposition 3.17. $A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'} \subset \overline{A'_{\mathbf{w},\mathbf{r},\mathbf{n}}}$ *if all the diagramms corresponding to the orbits of* $A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'}$ *are* $<$ *some diagramm corresponding to an orbit of* $A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$*, i.e. if* $r'_i \leq r_i$*,* $n'_i \geq n_i\, \forall i$*, and* $|\mathbf{a}| = |\mathbf{w}|$*,* $\mathbf{a} \geq \mathbf{w}$*.*

In particular, the $<$ ordering on diagrams is a refinement of the partial ordering on orbits, i.e. all the diagramms corresponding to the orbits of $A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'}$ are $<$ some diagramm corresponding to an orbit of $A'_{\mathbf{w},\mathbf{r},\mathbf{n}} \Rightarrow A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'} < A'_{\mathbf{w},\mathbf{r},\mathbf{n}}$. *Proof:* One has that if $r'_i \leq r_i$, $n'_i \geq n_i$, then $A'_{\mathbf{w},\mathbf{r}',\mathbf{n}'} \in Z'_{\mathbf{w}}$, so for each t' with $(\mathbf{D}, W, t') \in A'_{\mathbf{a},\mathbf{r}',\mathbf{n}'}$, there is $W_{t'}$ with $(\mathbf{D}, W_{t'}, t') \in A'_{\mathbf{w},\mathbf{r}',\mathbf{n}'}$. But then

$$(\mathbf{D}, W_{t'}, t') \in \overline{\{(\mathbf{D}, W_{t'}, t') \in \mathfrak{T}(\mathbf{d}) \mid \alpha(\ker t, \mathbf{D}) = n,\, \alpha(\operatorname{im} t, \mathbf{D}) = r,\, p_1(\mathbf{D}, W_{t'}, t') = (\mathbf{D}, W_{t'})\}},$$

which can be identified with the closure of

$$\{t | (\mathbf{D}, W_{t'}, t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = n,\, \alpha(\operatorname{im} t, \mathbf{D}) = r\}.$$

So all t' lie in the closure of

$$\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = n,\, \alpha(\operatorname{im} t, \mathbf{D}) = r,\, \alpha(W', \mathbf{D}) = \mathbf{w}\}.$$

Then again,

$$\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = n,\, \alpha(\operatorname{im} t, \mathbf{D}) = r,\, \alpha(W', \mathbf{D}) = \mathbf{w}\} \in \overline{Z'_{\mathbf{w}}},$$

thus

$$\overline{\{t | (\mathbf{D}, W', t) \in \mathfrak{T}(\mathbf{d}),\, \alpha(\ker t, \mathbf{D}) = n,\, \alpha(\operatorname{im} t, \mathbf{D}) = r,\, \alpha(W', \mathbf{D}) = \mathbf{w}\}} \in \overline{Z'_{\mathbf{w}}}.$$

□

Define the irreducible components of $\mathfrak{T}(\mathbf{d})$ to be the k-points of $\overline{Z'_{\mathbf{w}}}$ and denote them by $\overline{Z_{\mathbf{w}}}$. Moreover, defining the dense points of $\overline{Z_{\mathbf{w}}}$ to be the k-points of the dense subset $A'_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$ of $\overline{Z'_{\mathbf{w}}}$, these dense points are exactly the elements of $A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$.

Example 15. Consider the irreducible components:
The following tables illustrate the decomposition of the $Z_{\mathbf{w}}$ into the $A_{\mathbf{w},\mathbf{r},\mathbf{n}}$ and the $A_{\mathbf{w}',\mathbf{r}',\mathbf{n}'}$ contained in $\overline{Z_{\mathbf{w}}} - Z_{\mathbf{w}}$, which the propositions above yield. $\mathbf{d} = (1, 1, 1)$, and the $A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$ are colored blue and underlined.

$\mathbf{w}$	$Z_{\mathbf{w}}$	the part of $\overline{Z_{\mathbf{w}}}$ the propositions yield
$(0,0,0)$	$\underline{A}_{\mathbf{0},\mathbf{0},\mathbf{d}}$	$Z_{\mathbf{w}}$
$(0,0,1)$	$\underline{A}_{(\mathbf{0},\mathbf{0},\mathbf{1}),\mathbf{0},\mathbf{d}}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{0},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}} \cup A_{(\mathbf{1},\mathbf{0},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(0,1,0)$	$A_{(\mathbf{0},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}} \cup \underline{A}_{(\mathbf{0},\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0})}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{0},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,0,0)$	$A_{(\mathbf{1},\mathbf{0},\mathbf{0}),\mathbf{0},\mathbf{d}} \cup A_{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0})}$ $\cup \underline{A}_{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{1})}$	$Z_{\mathbf{w}}$
$(0,1,1)$	$\underline{A}_{(\mathbf{0},\mathbf{1},\mathbf{1}),\mathbf{0},\mathbf{d}}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{0},\mathbf{1}),\mathbf{0},\mathbf{d}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,0,1)$	$A_{(\mathbf{1},\mathbf{0},\mathbf{1}),\mathbf{0},\mathbf{d}} \cup \underline{A}_{(\mathbf{1},\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{0},\mathbf{1})}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,1,0)$	$A_{(\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0})}$ $\cup \underline{A}_{(\mathbf{1},\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0})}$	$Z_{\mathbf{w}}$
$(1,1,1)$	$\underline{A}_{(\mathbf{1},\mathbf{1},\mathbf{1}),\mathbf{0},\mathbf{d}}$	$Z_{\mathbf{w}}$

$\mathbf{d} = (1,1,1,1)$, and again, the $A_{\mathbf{w},\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}$ are colored blue and underlined.

$\mathbf{w}$	$Z_{\mathbf{w}}$	the part of $\overline{Z_{\mathbf{w}}}$ the propositions yield
$(0,0,0,0)$	$\underline{A_{\mathbf{w,0,d}}}$	$Z_{\mathbf{w}}$
$(0,0,0,1)$	$\underline{A_{\mathbf{w,0,d}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(0,0,1,0),0,d}}$ $\cup A_{\mathbf{(0,1,0,0),0,d}} \cup A_{\mathbf{(1,0,0,0),0,d}}$
$(0,0,1,0)$	$A_{\mathbf{w,0,d}} \cup \underline{A_{\mathbf{w,(0,0,1,0),(1,1,1,0)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(0,1,0,0),0,d}}$ $\cup A_{\mathbf{(1,0,0,0),0,d}}$
$(0,1,0,0)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(0,1,0,0),(1,1,1,0)}}$ $\cup \underline{A_{\mathbf{w,(0,1,0,0),(1,1,0,1)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(1,0,0,0),0,d}}$
$(1,0,0,0)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(1,0,0,0),(1,1,1,0)}}$ $\cup A_{\mathbf{w,(1,0,0,0),(1,1,0,1)}} \cup \underline{A_{\mathbf{w,(1,0,0,0),(1,0,1,1)}}}$	$Z_{\mathbf{w}}$
$(0,0,1,1)$	$\underline{A_{\mathbf{w,0,d}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(0,1,0,1),0,d}}$ $\cup A_{\mathbf{(0,1,1,0),0,d}} \cup A_{\mathbf{(1,0,0,1),0,d}}$ $\cup A_{\mathbf{(1,0,1,0),0,d}} \cup A_{\mathbf{(1,1,0,0),0,d}}$
$(0,1,0,1)$	$A_{\mathbf{w,0,d}} \cup \underline{A_{\mathbf{w,(0,1,0,0),(1,1,0,1)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(0,1,1,0),0,d}}$ $\cup A_{\mathbf{(1,0,0,1),0,d}} \cup A_{\mathbf{(1,0,1,0),0,d}}$ $\cup A_{\mathbf{(1,1,0,0),0,d}} \cup A_{(1,1,0,0),(0,1,0,0),(1,1,0,1)}$
$(0,1,1,0)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(0,1,0,0),(1,1,1,0)}}$ $\cup \underline{A_{\mathbf{w,(0,0,1,0),(1,1,1,0)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(1,0,1,0),0,d}}$ $\cup A_{\mathbf{(1,1,0,0),0,d}} \cup A_{(1,1,0,0),(0,1,0,0),(1,1,1,0)}$
$(1,0,0,1)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(1,0,0,0),(1,1,0,1)}}$ $\cup \underline{A_{\mathbf{w,(1,0,0,0),(1,0,1,1)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(1,0,1,0),0,d}}$ $\cup A_{\mathbf{(1,1,0,0),0,d}} \cup A_{(1,0,1,0),(1,0,0,0),(1,0,1,1)}$ $\cup A_{(1,1,0,0),(1,0,0,0),(1,1,0,1)}$
$(1,0,1,0)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(0,0,1,0),(1,1,1,0)}}$ $\cup A_{\mathbf{w,(1,0,0,0),(1,1,1,0)}} \cup A_{\mathbf{w,(1,0,0,0),(1,0,1,1)}}$ $\cup \underline{A_{\mathbf{w,(1,0,1,0),(1,0,1,0)}}}$	$Z_{\mathbf{w}} \cup A_{\mathbf{(1,1,0,0),0,d}}$ $\cup A_{(1,1,0,0),(1,0,0,0),(1,1,1,0)}$
$(1,1,0,0)$	$A_{\mathbf{w,0,d}} \cup A_{\mathbf{w,(0,1,0,0),(1,1,1,0)}}$ $\cup A_{\mathbf{w,(1,0,0,0),(1,1,1,0)}} \cup A_{\mathbf{w,(0,1,0,0),(1,1,0,1)}}$ $\cup A_{\mathbf{w,(1,0,0,0),(1,1,0,1)}} \cup \underline{A_{\mathbf{w,(1,1,0,0),(1,1,0,0)}}}$	$Z_{\mathbf{w}}$

$(0,1,1,1)$	$\underline{A}_{\mathbf{w},\mathbf{0},\mathbf{d}}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1}),\mathbf{0},\mathbf{d}}$ $\cup A_{(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}),\mathbf{0},\mathbf{d}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,0,1,1)$	$A_{\mathbf{w},\mathbf{0},\mathbf{d}} \cup \underline{A}_{\mathbf{w},(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1})}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}),\mathbf{0},\mathbf{d}}$ $\cup A_{(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,1,0,1)$	$A_{\mathbf{w},\mathbf{0},\mathbf{d}} \cup A_{\mathbf{w},(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1})}$ $\cup \underline{A}_{\mathbf{w},(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1})}$	$Z_{\mathbf{w}} \cup A_{(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0}),\mathbf{0},\mathbf{d}}$
$(1,1,1,0)$	$A_{\mathbf{w},\mathbf{0},\mathbf{d}} \cup A_{\mathbf{w},(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0})}$ $\cup A_{\mathbf{w},(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0})} \cup \underline{A}_{\mathbf{w},(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0})}$	$Z_{\mathbf{w}}$

Remark 15. Compare the part of $\overline{Z_{\mathbf{w}}}$ provided by the propositions to the $h^{\mathbf{d}}_{\mathbf{w}}$. For the case $d = 3$, one can see from example 13 that the $A_{\mathbf{w}',\mathbf{0},\mathbf{d}}$ added to $Z(\mathbf{w})$ to obtain this part of the closure correspond precisly to the $f_{\mathbf{w}',\mathbf{0},\mathbf{d}}$ added to $f_{\mathbf{w},\mathbf{0},\mathbf{d}}$ to obtain $h^{\mathbf{d}}_{\mathbf{w}}$.

3.10 A Geometric Interpretation of the Canonical Basis Elements

In the following one wants to define a basis of $T_c(\mathbf{d})$ related to the irreducible components of $\mathfrak{T}(\mathbf{d})$ and the the canonical basis, thus obtaining a geometric interpretation of the canonical basis.

Definition 3.18. Define $g^{\mathbf{d}}_{\mathbf{w}} = (h^{\mathbf{d}}_{\mathbf{w}})^e$.

Proposition 3.19. *$g^{\mathbf{d}}_{\mathbf{w}}$ can be written as*

$$g^{\mathbf{d}}_{\mathbf{w}} = \sum_{S \leq M(\mathbf{d},\mathbf{w})} (\eta_{\mathbf{r}^S,\mathbf{n}^S})^{-1}(\diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S})$$

(recall definitions 3.9, 3.12).

Proof: $g^{\mathbf{d}}_{\mathbf{w}} = (h^{\mathbf{d}}_{\mathbf{w}})^e = \sum_S (\eta_{\mathbf{r}^S,\mathbf{n}^S})^{-1}(\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\eta_{\mathbf{0},\mathbf{d}}(h^{\mathbf{d}}_{\mathbf{w}}))) = \sum_S (\eta_{\mathbf{r}^S,\mathbf{n}^S})^{-1}(\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}))$. I claim that

$$\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}}_{\mathbf{w}}) = \begin{cases} \diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S} & \text{if } S \leq M(\mathbf{d},\mathbf{w}) \\ 0 & \text{otherwise} \end{cases}.$$

From the graphical calculus in [6], it follows that if $S \leq M(\mathbf{d},\mathbf{w})$, then

$$(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^{\dagger}((\heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r) = (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r$$

and other dual canonical basis elements $(\heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{a}})^r$, $\mathbf{a} \neq \mathbf{w} - \mathbf{r}^S$ are sent to elements of the form $(\heartsuit^{\mathbf{d}} v^{\mathbf{a}'})^r$, $\mathbf{a}' \neq \mathbf{w}$. This yields

$$\langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}), (\heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r \rangle = \langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, (\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^{\dagger}((\heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r) \rangle = \langle \diamondsuit^{\mathbf{d}} v_{\mathbf{w}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r \rangle = 1$$

and

$$\langle \gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}), (\heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{a}})^r \rangle = 0 \text{ for all } \mathbf{a} \neq \mathbf{w} - \mathbf{r}^S.$$

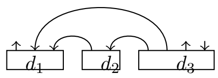
Therefore

$$\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) = \diamondsuit^{\mathbf{n}^S - \mathbf{r}^S} v_{\mathbf{w}-\mathbf{r}^S}.$$

Similarly, one can see that

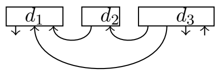
$$\gamma_{\mathbf{r}^S,\mathbf{n}^S}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) = 0 \text{ if } S \not\leq M(\mathbf{d}, \mathbf{w})$$

as in this case the image of $(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger$ is spanned by $\heartsuit^{\mathbf{d}} v^{\mathbf{a}}$ with $\mathbf{a} \neq \mathbf{w}$.
Let me illustrate the graphical calculus used above in an example:
Let $\mathbf{d} = (3, 2, 4), \mathbf{w} = (2, 1, 1)$, so

$M(\mathbf{d}, \mathbf{w}) =$ 

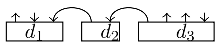
.

Then

$\heartsuit^{\mathbf{d}} v^{\mathbf{w}} =$ 

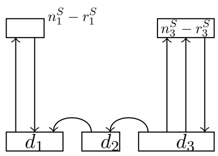
.

Let

$S =$ 

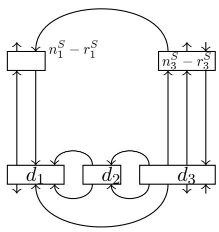
,

then

$\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S} =$ 

.

Therefore,

$\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S}(\heartsuit^{\mathbf{d}} v^{\mathbf{w}}) =$ 

$= \heartsuit^{\mathbf{n}^S - \mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S}$.

So $(\tilde{\gamma}_{\mathbf{r}^S,\mathbf{n}^S})^\dagger((\heartsuit^{\mathbf{n}^S-\mathbf{r}^S} v^{\mathbf{w}-\mathbf{r}^S})^r) = (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r$ and similarly, the other claims on the graphical calculus follow. □

Proposition 3.20. *Writing $\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}$ as a linear combination of elementary basis elements, it equals $\otimes^{\mathbf{d}} v_{\mathbf{w}}$ plus a linear combination of elements $\otimes^{\mathbf{d}} v_{\mathbf{a}}$ with $\mathbf{a} > \mathbf{w}$ and $|\mathbf{a}| = |\mathbf{w}|$ with coefficients in $q^{-1}\mathbb{N}[q^{-1}]$.*

Proof: This follows from [6, section 1.5,1.6]. □

Example 16. Using Proposition 3.19 and example 14, I can compute the $g^{\mathbf{d}}_{\mathbf{w}}$ for $d = 3, 4$.
$d = 3$: For $\mathbf{w}$ in the first row in the table of example 14, only the S in the first row are $S \leq M(\mathbf{d}, \mathbf{w})$, so

$$g^{\mathbf{d}}_{\mathbf{w}} = 4\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}).$$

If $\mathbf{w}$ lies in the second row in example 14, all S from the first two rows are $S \leq M(\mathbf{d},\mathbf{w})$. Therefore

$$g^{\mathbf{d}}_{\mathbf{w}} = 4\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) + 2\eta^{-1}_{(0,1,0),(1,1,0)}(\diamondsuit^{(1,0,0)} v_{\mathbf{w}-(0,1,0)}).$$

Similarly, if $\mathbf{w}$ lies in the third row, $S \leq M(\mathbf{d},\mathbf{w})$ for all S from the first and third row, and

$$g^{\mathbf{d}}_{\mathbf{w}} = 4\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) + 2\eta^{-1}_{(1,0,0),(1,0,1)}(\diamondsuit^{(0,0,1)} v_{\mathbf{w}-(1,0,0)}).$$

Again as in example 14, if $\mathbf{d} \neq (1,1,1)$, only those $\mathbf{w}$ and $M(\mathbf{d},\mathbf{w})$, that are admitted, are used (so, e.g. for $\mathbf{d} = (2,1)$, the $g^{\mathbf{d}}_{\mathbf{w}}$ from the third row are left aside). Using the expansion of the canonical basis in the standard basis calculated in example 13 (since $h^{\mathbf{d}}_{\mathbf{w}} = \eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}})$ and the canonical basis of e.g. $V_1 \otimes V_0 \otimes V_0$ is simply the standard basis), one thus obtains for $g^{\mathbf{d}}_{\mathbf{w}}$ (with $\mathbf{d} = (1,1,1)$):

$$\begin{aligned}
g^{\mathbf{d}}_{(0,0,0)} &= 4f_{(0,0,0),0,\mathbf{d}}\\
g^{\mathbf{d}}_{(0,0,1)} &= 4(f_{(0,0,1),0,\mathbf{d}} + q^{-1}f_{(0,1,0),0,\mathbf{d}} + q^{-2}f_{(1,0,0),0,\mathbf{d}})\\
g^{\mathbf{d}}_{(0,1,1)} &= 4(f_{(0,1,1),0,\mathbf{d}} + q^{-1}f_{(1,0,1),0,\mathbf{d}} + q^{-2}f_{(1,1,0),0,\mathbf{d}})\\
g^{\mathbf{d}}_{(1,1,1)} &= 4f_{(1,1,1),0,\mathbf{d}}\\
g^{\mathbf{d}}_{(0,1,0)} &= 4(f_{(0,1,0),0,\mathbf{d}} + q^{-1}f_{(1,0,0),0,\mathbf{d}}) + 2f_{(0,1,0),(0,1,0),(1,1,0)}\\
g^{\mathbf{d}}_{(1,1,0)} &= 4f_{(1,1,0),0,\mathbf{d}} + 2f_{(1,1,0),(0,1,0),(1,1,0)}\\
g^{\mathbf{d}}_{(1,0,0)} &= 4f_{(1,0,0),0,\mathbf{d}} + 2f_{(1,0,0),(1,0,0),(1,0,1)}\\
g^{\mathbf{d}}_{(1,0,1)} &= 4(f_{(1,0,1),0,\mathbf{d}} + q^{-1}f_{(1,1,0),0,\mathbf{d}}) + 2f_{(1,0,1),(1,0,0),(1,0,1)}
\end{aligned}$$

$d = 4$: Similarly as in the case $d = 3$, one obtains that if $\mathbf{w}$ lies in the first row of the $d = 4$-part of example 14, then

$$g^{\mathbf{d}}_{\mathbf{w}} = 5\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}),$$

if $\mathbf{w}$ lies in the second, third or fourth row, then

$$g^{\mathbf{d}}_{\mathbf{w}} = 5\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) + 3\eta^{-1}_{\mathbf{r}^{\mathbf{M}},\mathbf{n}^{\mathbf{M}}}(\diamondsuit^{\mathbf{n}^{\mathbf{M}}-\mathbf{r}^{\mathbf{M}}} v_{\mathbf{w}-\mathbf{r}^{\mathbf{M}}})$$

(where M is from the second, third, or fourth row, respectivly). For $\mathbf{w}$ in the fifth row, one obtains

$$g^{\mathbf{d}}_{\mathbf{w}} = 5\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) + 3\eta^{-1}_{(0,1,0,0),(1,1,0,1)}(\diamondsuit^{(1,0,0,1)} v_{\mathbf{w}-(0,1,0,0)}) + \eta^{-1}_{(1,1,0,0),(1,1,0,0)}(\diamondsuit^{(0)} v_{(0)})$$

and for $\mathbf{w}$ in the sixth row,

$$\begin{aligned}
g^{\mathbf{d}}_{\mathbf{w}} = 5\eta^{-1}_{\mathbf{0},\mathbf{d}}(\diamondsuit^{\mathbf{d}} v_{\mathbf{w}}) &+ 3\eta^{-1}_{(0,0,1,0),(1,1,1,0)}(\diamondsuit^{(1,1,0,0)} v_{\mathbf{w}-(0,0,1,0)})\\
&+ 3\eta^{-1}_{(1,0,0,0),(1,0,1,1)}(\diamondsuit^{(0,0,1,1)} v_{\mathbf{w}-(1,0,0,0)}) + \eta^{-1}_{(1,0,1,0),(1,0,1,0)}(\diamondsuit^{(0)} v_{(0)})
\end{aligned}$$

(if one labels the rows by $i), ii), \ldots, vi)$), then

$$\begin{aligned}
&i) \leq ii) \leq vi)\\
&i) \leq iv) \leq vi)\\
&i) \leq iii) \leq v)
\end{aligned}$$

for the crossingless matchings, which gives, together with the number of $\mathbf{w}$ in each row, the η and their coefficients)

Theorem 3.21. *$g^{\mathbf{d}}_{\mathbf{w}}$ is, up to a multiplicativ constant, the unique element of $T_c(\mathbf{d})$ satisfying*

1. *$g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a non-zero constant on the set $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ of dense points of the irreducible component $\overline{Z_{\mathbf{w}}}$.*

2. *The support of $g^{\mathbf{d}}_{\mathbf{w}}$ lies in $\overline{Z_{\mathbf{w}}}$.*

Moreover, the $g^{\mathbf{d}}_{\mathbf{w}}$ form a basis of $T_c(\mathbf{d})$ and

$$\diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto g^{\mathbf{d}}_{\mathbf{w}},$$

extended by linearity, is a $U_q(sl_2)$-module isomorphism $V_{d_1} \otimes \cdots \otimes V_{d_k} \to T_c(\mathbf{d})$.

Proof: Since

$$V_{d_1} \otimes \cdots \otimes V_{d_k} \overset{\eta_{\mathbf{0},\mathbf{d}}}{\leftarrow} T_0(\mathbf{d}) \overset{e}{\to} T_c(\mathbf{d}),\ \diamondsuit^{\mathbf{d}} v_{\mathbf{w}} \mapsto h^{\mathbf{d}}_{\mathbf{w}} \mapsto g^{\mathbf{d}_{\mathbf{w}}}$$

are U_q-module isomorphisms, it is clear that the $g^{\mathbf{d}}_{\mathbf{w}}$ form a basis and that the map given in the theorem is an U_q-module isomorphism.
It remains to prove the first part of the theorem. Surpressing the isomorphism $\eta_{\mathbf{r},\mathbf{n}}$ in order to simplify notation, I may identify $f_{\mathbf{w},\mathbf{r},\mathbf{n}}$ with $\otimes^{\mathbf{d}} v_{\mathbf{w}}$. To show uniqueness, consider a $\hat{g}^{\mathbf{d}}_{\mathbf{w}}$ satisfying the conditions of the theorem and let $\hat{h}^{\mathbf{d}}_{\mathbf{w}} \in T_0(\mathbf{d})$ be such that

$$\hat{g}^{\mathbf{d}}_{\mathbf{w}} = (\hat{h}^{\mathbf{d}}_{\mathbf{w}})^e = \sum_{\mathbf{r},\mathbf{n}} \gamma_{\mathbf{r},\mathbf{n}}(\hat{h}^{\mathbf{d}}_{\mathbf{w}}).$$

Then

$$\hat{g}^{\mathbf{d}}_{\mathbf{w}} = \sum a_{\mathbf{w},\mathbf{r},\mathbf{n}} f_{\mathbf{w},\mathbf{r},\mathbf{n}}.$$

Therefore, the value of $\hat{g}^{\mathbf{d}}_{\mathbf{w}}$ on $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ is given by $a_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}} k_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$. One has

$$a_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}} = \langle \gamma_{\mathbf{r^M},\mathbf{n^M}}(\hat{h}^{\mathbf{d}}_{\mathbf{w}}), (\otimes^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r \rangle$$

(where r stands for reversed, i.e. $(\otimes^{\mathbf{d}} v^{\mathbf{w}})^r = v^{d_k - 2w_k} \otimes \cdots \otimes v^{d_1 - 2w_1}$) since

$$f_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}} \hat{=} \otimes^{\mathbf{n^M}-\mathbf{r^M}} v_{\mathbf{w}}$$

(In more detail: $\hat{g}^{\mathbf{d}}_{\mathbf{w}} = (\hat{h}^{\mathbf{d}}_{\mathbf{w}})^e = \sum_{\mathbf{r},\mathbf{n}} \gamma_{\mathbf{r},\mathbf{n}}(\hat{h}^{\mathbf{d}}_{\mathbf{w}})$, so the coefficient of

$$V_{n_1^M - r_1^M} \otimes \cdots \otimes V_{n_k^M - r_k^M} \ni \otimes^{\mathbf{n^M}-\mathbf{r^M}} v_{\mathbf{w}-\mathbf{r^M}} = \eta_{\mathbf{r^M},\mathbf{n^M}}(f_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}})$$

is given by inserting $\gamma_{\mathbf{r^M},\mathbf{n^M}}(\hat{h}^{\mathbf{d}}_{\mathbf{w}})$ into the scalar product (where $\gamma_{\mathbf{r},\mathbf{n}} : V_{d_1} \otimes \cdots \otimes V_{d_k} \to V_{n_1 - r_1} \otimes \cdots \otimes V_{n_k - r_k}$)).
$k_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ being nonzero, the first condition in the theorem is equivalent to

$$\langle \gamma_{\mathbf{r^M},\mathbf{n^M}}(\hat{h}^{\mathbf{d}}_{\mathbf{w}}), (\otimes^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r \rangle \neq 0$$

which is equivalent to

$$\langle \hat{h}^{\mathbf{d}}_{\mathbf{w}}, (\tilde{\gamma}_{\mathbf{r^M},\mathbf{n^M}})^{\dagger}((\otimes^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r) \rangle \neq 0.$$

Since $M = M(\mathbf{d},\mathbf{w})$, it follows that $M(\mathbf{n^M} - \mathbf{r^M}, \mathbf{w} - \mathbf{r^M})$ has no curves and all down arrows are to the right of all up arrows (it would have curves otherwise). After rotating this diagramm by 180°, but keeping the original orientation (such that arrows oriented up remain oriented up, but all arrows are "below" the boxes), all down arrows are to the left of all up arrows. Then by [6, section 2.3],

$$(\otimes^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r = (\heartsuit^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r$$

and by the graphical calculus in [6] it follows that

$$(\tilde{\gamma}_{\mathbf{r^M},\mathbf{n^M}})^\dagger((\heartsuit^{\mathbf{n^M}-\mathbf{r^M}} v^{\mathbf{w}-\mathbf{r^M}})^r) = (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r.$$

So condition 1 (stating that $g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a nonzero constant on $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$) is equivalent to

$$\langle \hat{h}^{\mathbf{d}}_{\mathbf{w}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}})^r \rangle \neq 0.$$

To satisfy the second condition, $\hat{g}^{\mathbf{d}}_{\mathbf{w}}$ must be equal to zero on $A_{\mathbf{w}',\mathbf{r}^{M'},\mathbf{n}^{M'}}$ for all $\mathbf{w} \neq \mathbf{w}'$ and $M' = M(\mathbf{d}, \mathbf{w}')$. In the same way as above, one shows that this condition is equivalent to

$$\langle \hat{h}^{\mathbf{d}}_{\mathbf{w}}, (\heartsuit^{\mathbf{d}} v^{\mathbf{w}'})^r \rangle = 0$$

for all $\mathbf{w} \neq \mathbf{w}'$. But this shows that

$$\hat{h}^{\mathbf{d}}_{\mathbf{w}} = c^{\mathbf{d}}_{\mathbf{w}} \cdot \diamondsuit^{\mathbf{d}} v_{\mathbf{w}} = c^{\mathbf{d}}_{\mathbf{w}} \cdot h^{\mathbf{d}}_{\mathbf{w}}$$

for some constant $c^{\mathbf{d}}_{\mathbf{w}} \neq 0$. Therefore $g^{\mathbf{d}}_{\mathbf{w}}$ is indeed unique up to a multiplicative constant. It only remains to show that $g^{\mathbf{d}}_{\mathbf{w}}$ fulfills the conditions. By Proposition 3.19, the value of $g^{\mathbf{d}}_{\mathbf{w}}$ on $A_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ is equal to $k_{\mathbf{w},\mathbf{r^M},\mathbf{n^M}}$ times the coefficient of $\otimes^{\mathbf{n^M}-\mathbf{r^M}} v_{\mathbf{w}-\mathbf{r^M}}$ in the expression of $\diamondsuit^{\mathbf{n^M}-\mathbf{r^M}} v_{\mathbf{w}-\mathbf{r^M}}$ as a linear combination of elementary basis elements, and by [6, section 1.5,1.6] (or proposition 3.20), this coefficient is 1. Thus $g^{\mathbf{d}}_{\mathbf{w}}$ meets condition 1. Propositions 3.19 and 3.20 furthermore show that $g^{\mathbf{d}}_{\mathbf{w}}$ is equal to a linear combination of of functions of the form

$$f_{\mathbf{a}+\mathbf{r^S},\mathbf{r^S},\mathbf{n^S}} = (\eta_{\mathbf{r^S},\mathbf{n^S}})^{-1}(\otimes^{\mathbf{n^S}-\mathbf{r^S}} v_{\mathbf{a}})$$

with $S \leq M$ and $|\mathbf{a}| = |\mathbf{w} - \mathbf{r^S}| (\Rightarrow |\mathbf{a} + \mathbf{r^S}| = |\mathbf{w}|)$, $\mathbf{a} \geq \mathbf{w} - \mathbf{r^S} (\Rightarrow \mathbf{a} + \mathbf{r^S} \geq \mathbf{w})$. Then Proposition 3.16 shows that the support of $g^{\mathbf{d}}_{\mathbf{w}}$ lies in $\overline{Z_{\mathbf{w}}}$. this proves the theorem. ☐

Remark 16. Using 3.8 and $\overline{Z_{\mathbf{w}}} \subset \bigcup_{\mathbf{w}',\, |\mathbf{w}'|=|\mathbf{w}|} A_{\mathbf{w}',\mathbf{r},\mathbf{n}}$, it is clear that $g^{\mathbf{d}}_{\mathbf{w}}$ lies in the weight space corresponding to the weight $d - 2|\mathbf{w}|$. Since e is a module homomorphism, the action of U_q can also be calculated on the $h^{\mathbf{d}}_{\mathbf{w}}$ to obtain the action on the $g^{\mathbf{d}}_{\mathbf{w}}$ (see example 13).

Definition 3.22. There exists a scalar product on $T(\mathbf{d})$ in respect to which the standard basis is orthogonal, i.e. $< f_{\mathbf{w},\mathbf{r},\mathbf{n}}, f_{\mathbf{w}',\mathbf{r}',\mathbf{n}'} >= \delta_{(\mathbf{w},\mathbf{r},\mathbf{n}),(\mathbf{w}',\mathbf{r}',\mathbf{n}')}$.

Remark 17. Which Orbits are contained in $\overline{Z_{\mathbf{w}}} \cap \overline{Z_{\mathbf{w}'}}$? At the least all those whose diagram is included in some diagram of an orbit in $\overline{Z_{\mathbf{w}}}$ and a diagram of an orbit in $\overline{Z_{\mathbf{w}'}}$.
In which relation do diagrams (or their cups) and the $h^{\mathbf{d}}_{\mathbf{w}}$ stand? $M(\mathbf{d}, \mathbf{w})$ corresponds to $h^{\mathbf{d}}_{\mathbf{w}}$.
Can intersection of closures be calculated using scalar products of the $h_{\mathbf{w}}$ (and how do these scalar products look like?)? I do not think so.
How do dimension of orbits depend on number of cups in diagrams? If there are more cups,

more diagramms can be included in the corresponding diagram. As seen in remark 14, inclusions of closures of orbits correspond to inclusions of diagrams.
The number of different types of diagrams (ordered by their number of cups) of $V_1^{\otimes d}$ corresponds to the number of different irreducible submodules occuring in the decomposition into irreducible submodules, e.g. $V_1 \otimes V_1 = V_0 \oplus V_2$ and there are the diagramms without cups and one diagram with one cup, similar $V_1 \otimes V_1 \otimes V_1 = V_1 \oplus V_1 \oplus V_3$ and again there are the diagramms without cups as well as the diagrams with one cup. For $V_1^{\otimes 4} = V_0^2 \oplus V_2^3 \oplus V_4$, there are the diagrams with no cups, with one cup, and with two cups. To see this, consider the canonical basis. The $d+1$ canonical basis elements corresponding to the diagrams without cups form the irreducible submodule of largest dimension (i.e V_d) and linear combinations with the other canonical basis vectors form the irreducible components of smaller dimension (e.g. for $d = 3$, $< \diamondsuit^{\mathbf{d}} v_{(0,1,0)} - \frac{[2]}{[3]} \diamondsuit^{\mathbf{d}} v_{(0,0,1)}, \diamondsuit^{\mathbf{d}} v_{(1,1,0)} - \frac{1}{[3]} \diamondsuit^{\mathbf{d}} v_{(0,1,1)} > \cong V_1$ and $< \diamondsuit^{\mathbf{d}} v_{(1,0,0)} - \frac{1}{[3]} \diamondsuit^{\mathbf{d}} v_{(0,0,1)}, \diamondsuit^{\mathbf{d}} v_{(1,0,1)} - \frac{[2]}{[3]} \diamondsuit^{\mathbf{d}} v_{(0,1,1)} > \cong V_1$). The graphical way to describe the decomposition into irreducible modules was used in [7, p.43] in the context of categorification of tensor products of irreducble $\mathfrak{sl}_2$-modules.

4 Another Construction for a U_q-Module

I want to introduce a more naive construction of a U_q-module $\cong V_{d_1} \otimes \cdots \otimes V_{d_k}$ using functions on finite sets.

Definition 4.1. Let $W = S_d$ be the symmetric group and $S_{d_1} \times \cdots \times S_{d_k} = S_{\mathbf{d}} \subseteq W$ the (Young) subgroup generated by $\{s_1, \cdots, s_{d_1-1}, s_{d_1+1} \cdots, s_{d_1+d_2-1}, \cdots\}$ (where S_d is generated by the $d-1$ generators $\{s_1, \cdots, s_{d-1}\}$). E.g. $< s_1, s_2 > \times < s_4 > \times < s_6 > \cong S_3 \times S_2 \times S_2$.
Now let

$$B^i = \{ \text{ complex valued functions on } S_d/(S_i \times S_{d-i})\}$$

and

$$B^{i,i+1} = \{ \text{ complex valued functions on } S_d/(S_i \times S_1 \times S_{d-i-1})\}.$$

Then a basis of B^i (resp. $B^{i,i+1}$) is given by the set of indicator functions on $S_d \diagup (S_i \times Sd - i)$ (resp. $S_d \diagup (S_i \times S_1 \times S_{d-i-1})$). All these sets of functions are algebras (isomorphic to copies of $\mathbb{C}$).
There are natural surjections

$$\begin{tikzcd} W_{i,i+1} := W/(S_i \times S_1 \times S_{d_i-1}) \arrow[r, two heads, "\pi_{i+1}"] \arrow[dr, two heads, "\pi_i"] & W/(S_{i+1} \times S_{d-i-1}) =: W_{i+1} \\ & W/(S_i \times S_{d-i}) =: W_i \end{tikzcd}$$

For $g \in B^j$, $j \in \{i, i+1\}$ and $f \in B^{i,i+1}$, define $g.f(x) := g(\pi_j(x))$; this turns $B^{i,i+1}$ into a B^j-module. All rings being commutative, $B^{i,i+1}$ thus turns into a $B^i - B^{i+1}$-bimodule as well as a $B^{i+1} - B^i$-bimodule.

Lemma 4.2. *$B^{i,i+1}$ is a free B^i-module of rank* $|(W/W_i)/(W/W_{i,i+1})| = |W_{i,i+1}/W_i|$.

Proof: As a B^i-module, $B^{i,i+1} = \bigoplus_{i=1}^{l} f_{w_i} B^i$ for a complete transversal $(w_1, \cdots, w_l)$ of W_i in $W_{i,i+1}$ and f_w the indicator function of w (since $g_w.f = \sum_{i=1}^{l} f_{w_i w}$). □

4.1 A Construction for a U-Module

Now let $C_{func} = \bigoplus_{i=0}^{d} B^i - mod$ and set $E := \bigoplus_{i=0}^{d} E_i$ and $F := \bigoplus_{i=0}^{d} F_i$, where

$$E_i : B^i - mod \to B^{i+1} - mod,\ M \mapsto B^{i,i+1} \otimes_{B^i} M$$

for all $i < d$ and zero otherwise, and

$$F_i : B^i - mod \to B^{i-1} - mod,\ M \mapsto B^{i-1,i} \otimes_{B^i} M$$

for $i > 0$ and zero otherwise.

Theorem 4.3. *$K_0(C_{func}) \cong V_1^{\otimes d}$ as $U(\mathfrak{sl}_2)$-module, where K_0 = (free abelian group of isomorphism classes $[M]$ of objects) modulo $[B] = [A] + [C]$ if $A \hookrightarrow B \twoheadrightarrow C$ is a short exact sequence, so in this case it is enough to say if $B = A \oplus C$.*

A proof will follow later.

Remark 18. $K_0(C_{func})$ is a Grothendieck group. Actually, I consider the group algebra of $K_0(C_{func})$ over $\mathbb{C}$.

Claim: this generalises to $V_{d_1} \otimes \cdots \otimes V_{d_k}$ by taking functions on $\bigcup_{i=0}^{d} (S_{\mathbf{d}} \diagdown W_i)$.

Definition 4.4. Set

$$B^{i\,\prime} = \{\mathbb{C}\text{-valued functions on } S_{\mathbf{d}} \diagdown S_d \diagup (S_i \times S_{d-i})\},$$

similar $B^{i,i+1\prime}$. Then $B^{i\prime} \hookrightarrow B^i$, as

$$B^{i\prime} \hat{=} \{\mathbb{C}\text{-valued functions on } S_d \diagup (S_i \times S_{d-i}) \text{ that are constant on left } S_{\mathbf{d}}\text{-cosets}\}.$$

A basis of B^i is given by $\{f_w\}_{w \in W_i}$, where f_w is the indicator function of the coset w, i.e. $f_w(x) = \delta_{w,x}$. Then $B^{i\prime}$ has a basis corresponding to $\{\sum_{a \in S_{\mathbf{d}} w} f_a\}_{w \in \{\text{system of representatives of cosets of } S_{\mathbf{d}} \text{ in } W_i\}}$.
I get

$$\begin{tikzcd}
W_{i,i+1} \arrow[r, "\pi"] & S_{\mathbf{d}} \diagdown W_{i,i+1} \arrow[r, "\pi'_{i+1}"] \arrow[dr, "\pi'_i"] & S_{\mathbf{d}} \diagdown W_{i+1} \\
 & & S_{\mathbf{d}} \diagdown W_i
\end{tikzcd}$$

and as before, I can make $B^{i,i+1}$ and $B^{i,i+1\prime}$ into a $B^{i\prime}$-module. Setting $C'_{func} = \bigotimes_{i=0}^{d} B^{i\prime} - mod$, I can define the action of E and F as before. Any B^i-module is also a $B^{i\prime}$-module with the restricted action. Vice versa, for a $B^{i\prime}$-module M, I can let $f_w \in B^i$ act as $\frac{1}{|S_{\mathbf{d}} w|} \sum_{a \in S_{\mathbf{d}} w} f_a$. So

$$B^i - mod \overset{\rightarrow}{\hookleftarrow} B^{i\prime} - mod.$$

Using $B^{i,i+1}$ for the action of E, F, the correspondance between B^i-mod and $B^{i\prime}$-mod commutes with the action of E, F, thus yielding

$$C_{func} \overset{\rightarrow}{\hookleftarrow} C'_{func}$$

and

$$K_0(C_{func}) \overset{\twoheadrightarrow}{\hookleftarrow} K_0(C'_{func}).$$

On the other hand,

Definition 4.5. For $\mathbf{a} = (a_1, \cdots, a_n) \in \{0,1\}^n$, let $v_{\mathbf{a}} = v_{a_1} \otimes \cdots \otimes v_{a_n} \in V_1^{\otimes n}$ be the corresponding basis vector.
Then define

$$\pi_n : V_1^{\otimes n} \to V_n$$

by

$$\pi_n(v_{\mathbf{a}}) = v_{|\mathbf{a}|}.$$

This gives the projection $\pi_{d_1} \otimes \cdots \otimes \pi_{d_k} : V_1^{(d_1+\cdots+d_k)} \to V_{d_1} \otimes \cdots \otimes V_{d_k}$.
Furthermore define

$$\iota_n : V_n \to V_1^{\otimes n}$$

by

$$\iota_n(v_k) = \sum_{|\mathbf{a}|=k} v_{\mathbf{a}}.$$

Thus one obtaines the inclusion $\iota_{d_1} \otimes \cdots \otimes \iota_{d_k} : V_{d_1} \otimes \cdots \otimes V_{d_k} \to V_1^{(d_1+\cdots+d_k)}$.
The composition $\iota_n \circ \pi_n = p_n$ is the Jones-Wenzl projector.

This yields

$$\begin{array}{ccc} K_0(C'_{func}) & \overset{\twoheadleftarrow}{\hookrightarrow} & K_0(C_{func}) \\ \overset{?}{\cong} & & \overset{\phi}{\cong} \\ V_{d_1} \otimes \cdots \otimes V_{d_k} & \overset{\twoheadleftarrow}{\hookrightarrow} & V_1^{\otimes d} \end{array} . \tag{2}$$

So how is the isomorphism ϕ on the right defined? Can it be restricted to an isomorphism on the left?
To answer these questions, I first need to define bases of $K_0(C_{func})$ and $V_1^{\otimes d}$.

A basis of $K_0(C_{func})$ is given by the isomorphism classes of simple modules. Since $C_{func} = \bigoplus_{i=1}^d B^i - mod$, these are simple modules over the B^i. Addition and multiplication being defined pointwise in B^i, I have $f_w f_{w'} = \delta_{w,w'} f_w$ and $\sum_w f_w$ is the identity element of the multiplication, and thus for some simple B^i-module V and $v \in V$,

$$f_w^n.(v) = f_w.(v),\ f_w(f_{w'}.v) = \delta_{w,w'} f_w.v \text{ and } v = \sum_w f_w.v.$$

So, V being simple, $V =< \{v, f_w v \mid w \in W_i\} >_{\mathbb{C}}$.
Therefore $\dim V \leq |W_i|$. However, $f_w.v$ would span a $1-$dimensional subspace of V, so V must have been 1-dimensional from the beginning. Two 1-dimensional irreducible modules are not isomorphic in general as B^i-modules, e.g. take $V = \{\mathbb{C}v \mid f_w.v \neq 0\}$, $V' = \{\mathbb{C}v' \mid f_{w'}.v' \neq 0\}$, $w \neq w'$. Then $f_w.V \neq 0 = f_w.V'$. So the isomorphism classes of simple modules are given as the 1-dim. modules where one of the f_w acts nontrivialy, and they can thus be parametrised by

the f_w. So write V^i_w for the simple module corresponding to $f_w \in B^i$ (a 1-dimensional module where two different $f_w, f_{w'}$ act nontriavially cannot occur, as then $f_w.v = \lambda v$, $f_{w'}.v = \mu v$, but $f_w f_{w'} = 0$).
Similarly, a basis of $K_0(C'_{func})$ is given by the isomorphism classes of simple $B^{i\prime}$-modules. Then, for some $v \in V$ and $w \in W_i$, $V^i_{S_{\mathbf{d}}w} = \{\mathbb{C}v \mid \{(\sum_{a\in S_{\mathbf{d}}w} f_a).v \neq 0\}$. Then V^i_a is mapped to this $V^i_{S_{\mathbf{d}}w}$ under the correspondance explained above (4.5), for all $a \in S_{\mathbf{d}}w$, and $V^i_{S_{\mathbf{d}}w}$ is mapped to $\oplus_{a\in S_{\mathbf{d}}w} V^i_a$. It remains to find a nice basis for $V_1^{\otimes d}$.

Remark 19. [4] Recall the Schur-Weyl duality between GL_n and S_d: Let V be a n-dimensional vector space, then GL_n acts on $V^{\otimes d}$ by $g.(v_1 \otimes \cdots \otimes v_d) = g.v_1 \otimes \cdots \otimes g.v_d$ and S_d by permuting the entries. Schur observed that the centralizer algebra of each of the two actions equals the image of the other action in $\mathrm{End}(V^{\otimes d})$ in characteristic zero. Schur and Weyl used this to obtain information about representations of GL_n from information about representations of S_d. A similar correspondance has been found between $\mathfrak{sl}_n$ and S_d and both actions commute.

S_d acts on $V_1^{\otimes d}$ by permuting the entries, i.e

$$v_{i_1} \otimes \cdots \otimes v_{i_d}.\pi = v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(d)}},$$

and thus $V_1^{\otimes d}$ can be decomposed into

$$V_1^{\otimes d} = \bigoplus_{j=0}^{d} < \bigcup_{\sum_{l=1}^d i_l = j} v_{i_1} \otimes \cdots \otimes v_{i_d} >_{\mathbb{C}} = \bigoplus_{j=0}^{d} (V_1^{\otimes d})_{2j-d}$$

($i_l \in \{0,1\}$, (v_0, v_1) is a basis of V_1 and $(V_1^{\otimes d})_{2j-d}$ is the weight space of $V_1^{\otimes n}$ associated to the weight $\mu = 2j - d$, as K acts as 1 on v_1 and as -1 on v_0) and

$$(V_1^{\otimes n})_{2j-d} = < \bigcup_{\sum_{l=1}^d i_l = j} v_{i_1} \otimes \cdots \otimes v_{i_d} >_{\mathbb{C}} \cong 1 \uparrow_{S_j \times S_{d-j}}^{S_d} .$$

Recall the definition of induced action:

Definition 4.6. Let G be group with subgroup H, and $\{t_1, \cdots, t_l\}$ a fixed transversal for the cosets of H, i.e. $G = \bigcup_i t_i H$. Then for a representation Y of H, the induced representation $Y \uparrow_H^G$ is given by $Y \uparrow_H^G (g) = (Y(t_i^{-1} g t_j))_{i,j}$ (as a matrix in the basis given by the transversal), with $Y(g) = 0$ for $g \notin H$.

In this particular case, $1 \uparrow_{S_i \times S_{d-i}}^{S_d}$ is a right S_d-module with a basis given by the cosets $\{\overline{t_1}, \cdots, \overline{t_l}\}$ for a fixed transversal $\{t_1, \cdots, t_l\}$ for the cosets of $S_d / (S_i \times S_{d-i})$. Then $\overline{t_i}.s = \overline{t_i s}$ for some $s \in S_d$.
The induced representation $1 \uparrow_{S_i \times S_{d-i}}^{S_d}$ is isomorphic to the representation

$$V^{(i,d-i)} = \mathbb{C}\{S_i \times S_{d-i}\pi_1, \cdots, S_i \times S_{d-i}\pi_l\},$$

where $\{\pi_1, \cdots, \pi_l\}$ is a transversal of $S_i \times S_{d-i}$ in S_d, and S_d acts on the basis elements $S_i \times S_{d-i}\pi_i$ by multiplication from the right ([20, Prop. 1.12.3]).

Then I can identify

$$< \bigcup_{\sum_{l=1}^{d} i_l = j} v_{i_1} \otimes \cdots \otimes v_{i_d} >_{\mathbb{C}}$$

with $V^{(j,d-j)}$. $V^{(i,d-i)}$ is cyclic ([20]), so I can just choose one $(0,1)$-sequence to correspond to $(1)S_i \times S_{d-i}$, for example let

$$v_{i_1} \otimes \cdots \otimes v_{i_n} = v_1 \otimes \cdots \otimes v_1 \otimes v_0 \otimes \cdots \otimes v_0 \hat{=} (1, \cdots, 1, 0 \cdots, 0)$$

correpond to $(1)S_i \times S_{d-i}$. Then

$$v_{i_1} \otimes \cdots \otimes v_{i_d}.\pi = v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(d)}}$$

correponds to $S_i \times S_{d-i}\pi$.

So as a decomposition, one gets precisely the induced trivial modules for the Young subgroups $S_i \times S_{d-i}$.
This action can be generalised to the Hecke algebra:

Definition 4.7. [22] The Hecke algebra $\mathcal{H}_d$ over $\mathbb{Z}[v, v^{-1}]$ (with v generic) associated to S_d is the associative algebra with generators $\{T_\pi \mid \pi \in S_d\}$ and relations $T_{\pi\sigma} = T_\pi T_\sigma$ if $l(\pi)+l(\sigma) = l(\pi\sigma)$ (where l is the usual length function given by a shortest representation as a product of simple reflections $(i, i+1)$) and $T_s^2 = v^{-2}T_e + (v^{-2} - 1)T_s$ for all simple reflections $s \in S_d$ ($v^{-2} = q$ yields the version of the definition of Kazhdan and Lusztig).
Define $H_s = vT_s$, then $H_s^2 = 1 + (v^{-1} - v)H_s$ (where $1 = T_e$) and $H_s^{-1} = H_s + (v - v^{-1})$, and the H_s generate $\mathcal{H}_d$ as well.

Remark 20. It follws from Lusztig's version of Tits' deformation theorem ([16, Theorem 3.1]), that the group algebra of S_d over $\mathbb{Q}(q^{\frac{1}{2}})$ may be embedded in the Hecke algebra $\mathcal{H}_d(q)$ of S_d (with $q \in \mathbf{C}$) and $\pi_\in S_d$ may be written as linear combination of the T_v, $v \in S_d$. Since $\{\pi \in S_d\}$ forms a basis of S_d and $\{T_v \mid v \in S_d\}$ a basis of $\mathcal{H}_d(q)$ as $\mathbb{Q}(q^{\frac{1}{2}})$-vector space, one can invert this and write the T_v as linear combination of the $\pi \in S_d$. Then the action of S_d on $V_1^{\otimes d}$ can be extended to an action of the Hecke algebra. However, this isomorphism is not useful for explicit calculations.

For some $S_\lambda \subset S_d$ (e.g. $\lambda = (i, d-i)$), define the subalgebra $\mathcal{H}(S_\lambda)$ of $\mathcal{H}_d$ generated by the T_s, $s \in S_\lambda$. Since $H_s^2 = 1 + (v^{-1} - v)H_s \Leftrightarrow (H_s + v)(H_s - v^{-1}) = 0$, there is a surjective $\mathbb{Z}[v, v^{-1}]$-algebra morphism $\mathcal{H}(S_\lambda) \to \mathbb{Z}[v, v^{-1}]$, $H_s \mapsto v^{-1}$ for $s \in S_\lambda$ a simple reflection. This turns $\mathbb{Z}[v, v^{-1}]$ into an $\mathcal{H}(S_\lambda)$-bimodule where H_s acts as v^{-1}. This can be induced to a right $\mathcal{H}_d$-module $\mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}(S_\lambda)} \mathcal{H}_d$ with basis given by $\{1 \otimes H_{t_i}\}$ for a fixed transversal $\mathbf{t} = \{t_1, \cdots, t_l\}$ for the cosets of $S_\lambda \diagdown S_d$ (where $H_\pi := v^{l(\pi)}T_\pi$). Choose the transversal such that the t_i have minimal length. Then the action of $\mathcal{H}_d$ is given by

$$(1 \otimes H_\pi).H_s = \begin{cases} 1 \otimes H_{\pi s} & \pi s \in \mathbf{t},\ \pi s > \pi \\ 1 \otimes H_{\pi s} + (v^{-1} - v)H_\pi & \pi s \in \mathbf{t},\ \pi s < \pi \\ v^{-1}(1 \otimes H_\pi) & \pi s \notin \mathbf{t} \end{cases}$$

(from $\pi s \notin \mathbf{t}$, it follows that $\pi s = r\pi$ for some simple reflection $r \in S_\lambda$, and so πs is in the same coset as π) [22, chapter 3]. Notice that for $v = 1$, one obtains the action of the group algebra on $1 \uparrow_{S_\lambda}^{S_d}$ again (when identifying H_s with s) and $\mathbb{Z}[v, v^{-1}]$ corresponds to the trivial representation for S_λ.

Remark 21. [4] Jimbo [12] and (independantly) Dipper and James [3] observed that there is a q-analogue of $V^{\otimes d}$ and the mutually centralizing actions of GL_n and S_d on $V^{\otimes d}$ become mutually centralizing actions of $U_q(\mathfrak{gl}_n)$ and the Iwahori-Hecke algebra $\mathcal{H}_d(q)$.

Example 17. Consider

$$V_1 \otimes V_1 \hat{=} <(0,0),(0,1),(1,0),(1,1)> = <(1,1)> \oplus <(1,0),(0,1)> \oplus <(0,0)>$$

as S_2-module ($(x,y)\hat{=}v_x \otimes v_y$, (v_0, v_1) being a basis of V_1). Then $1 \uparrow_{S_1\times S_1}^{S_2}$ is a right S_2-module and 2-dimensional as $\mathbb{C}$-vector space with basis given by the transversal $\{\overline{1}, \overline{s} = (1,2)\}$ and the operation of $(1,2)$ in this basis is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so $(1,0).(1,2) = 0(1,0) + Id(0,1) = (0,1)$ and vice versa. Thus the S_2-module $<(1,0),(0,1)>$ corresponds to $1 \uparrow_{S_1\times S_1}^{S_2}$ (which is a sum of sign representations for S_2, namely $<(0,1)+(1,0)> \oplus <(0,1)-(1,0)>$). As $S_2 \times S_0 \cong S_2$ has transversal $\{(1)\}$ in S_2, $1 \uparrow_{S_2\times S_0}^{S_2} ((1,2)) = 1(1,2) = Id.$ and similar for $S_0 \times S_2$, so $<(0,0)>$ and $<(1,1)>$ correspond to the induced trivial representations of $S_2 \times S_0$ and $S_0 \times S_2$. The operation of H_{s_1} on the $\mathcal{H}_d$-module induced from the trivial representation of $\mathcal{H}(S_1 \times S_1)$ in the basis $\{1 \otimes H_e, 1 \otimes H_{(1,2)}\}$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & v^{-1} - v \end{pmatrix}.$$

So a nice basis (the standard basis) for the induced modules is given by the $S_i \times S_{d-i}\pi_j$. I have precisly

$$W_i = S_d/(S_i \times S_{d-i}) = \{\pi_1, \cdots, \pi_l\}$$

($V^{(i,d-i)} = \mathbb{C}\{S_i \times S_{d-i}\pi_1, \cdots, S_i \times S_{d-i}\pi_l\}$, where $\{\pi_1, \cdots, \pi_l\}$ is a transversal of $S_i \times S_{d-i}$ in S_d) and the isomorphism ϕ in (2) is defined as sending V_w to $S_i \times S_{d-i}w$.

Proposition 4.8. *The map defined thus is indeed an isomorphism.*

Proof: $E.V_w^i = B^{i,i+1} \otimes_{B^i} V_w^i$ is a B^{i+1}-module and can thus be decomposed into a direct sum of simple B^{i+1}-modules. Then $V_{w'}^{i+1}$ is a summand if and only if $f_{w'}^{i+1} \in B^{i+1}$ acts nontrivially on $E.V_w^i$. Since the action of B^{i+1} on $B^{i,i+1}$ is defined as $f^{i+1}.g(x) = f^{i+1}(\pi_{i+1}(x))$ for $f^{i+1} \in B^{i+1}$, $g \in B^{i,i+1}$ and $x \in W_{i,i+1}$, it follows that $f_{w'}^{i+1} B^{i,i+1} \otimes_{B^i} V_w^i = f_{\pi_{i+1}^{-1}(w')} \otimes_{B^i} V_w^i = 1 \otimes_{B^i} f_{\pi_i \circ \pi_{i+1}^{-1}(w')}$. I have $E.V_w^i \stackrel{!}{=} \sum_{\tau=(i+1,j), j\geq i+1} V_{w\tau}^{i+1} = \Phi^{-1}(E.S_i \times S_{d-i}.w)$, so all the $f_{w\tau}^{i+1}$ and no other $f_{w'}^{i+1}$ should act nontrivially on $E.V_w^i$. Therefore one w_j in $f_{\pi_i \circ \pi_{i+1}^{-1}(w\tau)} = \sum_j f_{(w_j)}^i$ should equal w and none for $w' \neq w\tau \, \forall \tau = (i+1, j)$, $j \geq i+1$.

Now $W_i \hat{=} \{\wedge, \vee\}$-sequences of length d with i-times $\wedge$ and $d-i$-times $\vee$ (identify $e \hat{=} \wedge \cdots \wedge \vee \cdots \vee$ and let W_i act by permuting elements of the sequence). Then $W_i \overset{\pi_{i+1}\circ\pi_i^{-1}}{\rightarrow} W_{i+1}$, where a $\{\wedge, \vee\}$-sequence of length d with i-times $\wedge$ and $d-i$-times $\vee$ is mapped to the sum of all $\{\wedge, \vee\}$-sequences of length d with $i+1$-times $\wedge$ and $d-i-1$-times $\vee$ obtained from the original sequence by converting one $\vee$ to a $\wedge$. Similarly for $W_{i+1} \overset{\pi_i\circ\pi_{i+1}^{-1}}{\rightarrow} W_i$. Then $\sum_{\tau=(i+1,j), j\geq i+1} f^{i+1}_{w\tau} = f_{\pi_{i+1}\circ\pi_i^{-1}(w)}$ and $\sum_{\nu=(i+1,j), j\leq i+1} f^i_{w\nu} = f_{\pi_i\circ\pi_{i+1}^{-1}(w)}$. The desired result follows for the action of E (since $f_{\pi_i\circ\pi_{i+1}^{-1}(w\tau)} = \sum_{\nu=(i+1,j),j\leq i+1} f^i_{w\tau\nu} = \sum_{\nu=(i+1,j),j<i+1} f^i_{w\tau\nu} + f^i_{w\tau}$ and $f^i_{w\tau} = f^i_w$, and for $w\prime \neq w\tau$ in W_{i+1} with τ as before, $w'\nu \neq w$ in W_i) and the analogous result for F follows using $\sum_{\tau=(i+1,j),j\geq i+1} f^{i+1}_{w\tau} = f_{\pi_{i+1}\circ\pi_i^{-1}(w)}$. □

Example 18. Let $d = 3$ and $i = 1$.
Then

$$\begin{aligned} W_1 = S_3/(S_1 \times S_2) \hat{=} &\{1S_1 \times S_2, (1,2)S_1 \times S_2, (1,3)S_1 \times S_2\} \\ &= \{\{1, (2,3)\}, \{(1,2), (1,2)(2,3)\}, \{(1,3), (1,3)(2,3) = (2,3)(1,2)\}\} \end{aligned}$$

and

$$\begin{aligned} W_2 \hat{=} &\{1S_2 \times S_1, (2,3)S_2 \times S_1, (1,3)S_2 \times S_1\} \\ &= \{\{1, (1,2)\}, \{(2,3), (2,3)(1,2)\}, \{(1,3), (1,3)(1,2) = (1,2)(2,3)\}\}, \end{aligned}$$

and $W_{1,2} = S_3$.
Let $w = (1,2)$. Then $f_{\pi_i\circ\pi_{i+1}^{-1}(w)} = f^i_w + f^i_1$ and $f_{\pi_i\circ\pi_{i+1}^{-1}(w(2,3))} = f^i_{w(2,3)} + f^i_{(1,3)} = f^i_w + f^i_{(1,3)}$ and indeed $E.V^i_w = \sum_{\tau=(i+1,j),j\geq i+1} V^{i+1}_{w\tau} = V^{i+1}_w + V^{i+1}_{w(2,3)}$.

In order to restrict the isomorphism in (2) to $V_{d_1} \otimes \cdots \otimes V_{d_k}$ and $K_0(C'_{func})$, I need to check that the images of the projection maps on both sides correspond. Since the inclusion maps are injective, it is enough to show that the isomorphism commutes with the composition, i.e.

$$\phi \circ \iota \circ \pi = \iota_{d_1} \otimes \cdots \otimes \iota_{d_k} \circ \pi_{d_1} \otimes \cdots \otimes \pi_{d_k} \circ \phi.$$

Let $v_{a_1} \otimes \cdots \otimes v_{a_d} \hat{=} S_i \times S_{d-i} w$, i.e $\mathbf{a} = (1, \cdots, 1, 0, \cdots, 0)w$ and $v_{a_1} \otimes \cdots \otimes v_{a_d} = (v_1 \otimes \cdots \otimes v_1 \otimes v_0 \otimes \cdots \otimes v_0)w$. Then

$$\pi_{d_1} \otimes \cdots \otimes \pi_{d_k}(v_{a_1} \otimes \cdots \otimes v_{a_d}) = \bigotimes_{i=1}^k v_{|\mathbf{a}^i|}$$

and

$$\begin{aligned} &\iota_{d_1} \otimes \cdots \otimes \iota_{d_k}(\textstyle\bigotimes_{i=1}^k v_{|\mathbf{a}^i|}) \\ = &\textstyle\bigotimes_{i=1}^k \sum_{|\hat{\mathbf{a}}^i| = |\mathbf{a}^i|} v_{\hat{\mathbf{a}}^i} \\ = &\textstyle\sum_{\hat{\mathbf{a}}=\sigma(\mathbf{a}),\, \sigma\in S_{\mathbf{d}}} \bigotimes_{i=1}^k v_{\hat{\mathbf{a}}^i} \end{aligned}$$

(Set $a_{(i,j)} = (a_i, \cdots, a_j)$ and $\mathbf{a}^i = a_{(d_1+\cdots+d_{i-1}+1, d_1+\cdots+d_i)}$), as $\sigma \in S_{\mathbf{d}}$ precisely means that $\sigma(\mathbf{a}) = \hat{\mathbf{a}}$ in

$$\iota_{d_1} \otimes \cdots \otimes \iota_{d_k}(\bigotimes_{i=1}^k v_{|\mathbf{a}^i|}) = \bigotimes_{i=1}^k \sum_{|\hat{\mathbf{a}}^i| = |\mathbf{a}^i|} v_{\hat{\mathbf{a}}^i}.$$

Furthermore, $\pi(V^i_w) = V^i_{S_{\mathbf{d}}w}$ with $v \in V^i_w$ and $\iota(V^i_{S_{\mathbf{d}}w}) = \oplus_{\sigma\in S_{\mathbf{d}}w} V^i_\sigma$.

Example 19. $d = 3, \mathbf{d} = (2,1)$.

$$\begin{aligned} \pi_2 \otimes \pi_1(v_1 \otimes v_0 \otimes v_0) = & \bigotimes_{i=1}^{2} v_{|\mathbf{a}^i|} \\ = & v_1 \otimes v_0 \end{aligned} .$$

Then

$$\begin{aligned} \iota_2 \otimes \iota_1(v_1 \otimes v_0) = & \sum_{|\hat{\mathbf{a}}^i=1} v_{\hat{\mathbf{a}}^i} \otimes \sum_{|\hat{\mathbf{a}}^i|=0} v_{\hat{\mathbf{a}}^i} \\ = & (v_{(1,0)} + v_{(0,1)}) \otimes v_{(0)} \\ = & v_1 \otimes v_0 \otimes v_0 + v_0 \otimes v_1 \otimes v_0 \end{aligned} .$$

Example 20. Let $d = 3, \mathbf{d} = (2,1)$.
Then $S_{(2,1)} = \{(1,2) \times (3), e\}$ and $\iota(\pi(V_e^i)) = V_e^i + V_{(1,2)\times(3)}^i$.

So both maps correspond to one another up to constants and since ι is injective on both sides, the images of the projections must already correspond to one another. Therefore the isomorphism from (2) can be restricted as claimed.

4.2 A Similar Construction for U_q

Again, one can define bases of $V_1^{\otimes d}$ and $K_0(C_{func})$ as well as their subspaces $V_{d_1} \otimes \cdots \otimes V_{d_k}$ and $K_0(C'_{func})$ as before. To pay reference to the modified action induced by the q in U_q, the projection and inclusion maps however are changed sligthly.

Definition 4.9. For $\mathbf{a} = (a_1, \cdots, a_n) \in \{0,1\}^n$, let $v_{\mathbf{a}} = v_{a_1} \otimes \cdots \otimes v_{a_n} \in V_1^{\otimes n}$ be the corresponding basis vector.
Then define

$$\pi_n : V_1^{\otimes n} \to V_n$$

by

$$\pi_n(v_{\mathbf{a}}) = q^{-l(\mathbf{a})} \frac{1}{\begin{bmatrix} n \\ |\mathbf{a}| \end{bmatrix}} v_{|\mathbf{a}|} = q^{-l(\mathbf{a})} v^{|\mathbf{a}|}$$

where $l(\mathbf{a})$ is equal to the number of pairs $i < j$ with $a_i < a_j$. This gives the projection

$$\pi_{d_1} \otimes \cdots \otimes \pi_{d_k} : V_1^{(d_1+\cdots+d_k)} \to V_{d_1} \otimes \cdots \otimes V_{d_k}.$$

Furthermore define

$$\iota_n : V_n \to V_1^{\otimes n}$$

by

$$\iota_n(v_k) = \sum_{|\mathbf{a}|=k} q^{b(\mathbf{a})} v_{\mathbf{a}}$$

where $b(\mathbf{a}) = |\mathbf{a}|(n - |\mathbf{a}|) - l(\mathbf{a})$, i.e. the number of pairs $i < j$ with $a_i > a_j$. Thus one obtaines the inclusion

$$\iota_{d_1} \otimes \cdots \otimes \iota_{d_k} : V_{d_1} \otimes \cdots \otimes V_{d_k} \to V_1^{(d_1+\cdots+d_k)}.$$

The composition $\iota_n \circ \pi_n = p_n$ is the Jones-Wenzl projector.

Similarly, map the class of the B^i-module V^i_w to the class of the $B^{i,i+1}$-module $V^i_{S_{\mathbf{d}}w}$ multiplied by the same constant as $\bigotimes_{j=1}^k v_{|\mathbf{a}^j|}$ in the case of $\pi_{d_1}\otimes\cdots\otimes\pi_{d_k}(v_{\mathbf{a}})$, where w should be choosen as representative of minimal lenght of the coset in W_i and $\mathbf{a}=w(1,\cdots,1,0,\cdots,0)$ and map $V^i_{S_{\mathbf{d}}w}$ to $\oplus_{a\in S_{\mathbf{d}}w}V^i_a$ (with some constants λ_a corresponding to the constants in the case of $\iota_{d_1}\otimes\cdots\otimes\iota_{d_k}$).

Example 21. $d=3, \mathbf{d}=(2,1)$.

$$\begin{aligned}\pi_2\otimes\pi_1(v_1\otimes v_0\otimes v_0) = &\quad \textstyle\bigotimes_{i=1}^2 q^{-l(\mathbf{a}^i)}\frac{1}{\left[\begin{smallmatrix}d_i\\|\mathbf{a}^i|\end{smallmatrix}\right]}v_{|\mathbf{a}^i|}\\ = &\quad q^{-2}\frac{1}{\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]}v_1\otimes q^{-1}\frac{1}{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}v_0\\ = &\quad q^{-3}\frac{1}{[2]}v_1\otimes v_0\\ = &\quad q^{-3}\frac{q-q^{-1}}{q^2-q^{-2}}v_1\otimes v_0\end{aligned}.$$

Then

$$\begin{aligned}\iota_2\otimes\iota_1(q^{-3}\tfrac{1}{[2]}v_1\otimes v_0) = &\quad q^{-3}\tfrac{1}{[2]}\textstyle\sum_{|\hat{\mathbf{a}}^i|=1}q^{b(\hat{\mathbf{a}}^i)}v_{\hat{\mathbf{a}}^i}\otimes\sum_{|\hat{\mathbf{a}}^i|=0}q^{b(\hat{\mathbf{a}}^i)}v_{\hat{\mathbf{a}}^i}\\ = &\quad q^{-3}\tfrac{1}{[2]}(q^{b(1,0)}v_{(1,0)}+q^{b(0,1)}v_{(0,1)})\otimes q^{b(0)}v_{(0)}\\ = &\quad q^{-3}\tfrac{1}{[2]}(qv_{(1,0)}+v_{(0,1)})\otimes v_{(0)}\\ = &\quad q^{-2}\tfrac{1}{[2]}v_1\otimes v_0\otimes v_0+q^{-3}\tfrac{1}{[2]}v_0\otimes v_1\otimes v_0\end{aligned}.$$

How should the action of U_q be defined such that so $V^i_w\mapsto(S_i\times S_{d-i})w$ remains an isomorphism? $E.(S_i\times S_{d-i})w := \Delta^{(d-1)}E.(v_{a_1}\otimes\cdots\otimes v_{a_d}) = \sum_{j=1}^d Kv_{a_1}\otimes\cdots\otimes Kv_{a_{j-1}}\otimes Ev_{a_j}\otimes v_{a_{j+1}}\otimes\cdots\otimes v_{a_d} = \sum_{j=1}^d q^{\alpha_1}v_{a_1}\otimes\cdots\otimes q^{\alpha_{j-1}}v_{a_{j-1}}\otimes\left[\frac{1+\alpha_j}{2}+1\right]v_{a_j+1}\otimes v_{a_{j+1}}\otimes\cdots\otimes v_{a_d}$ (Let $\alpha_j=a_j-\delta_{0,a_j}$, so $\alpha_j\in\{1,-1\}$).
Then

$$\begin{aligned}E.V^i_{Id} = &\quad \textstyle\sum_{(i+1,j)=\tau\in S_{d-i}}q^{i-(j-1-i)}\left[\frac{1+\alpha_j=0}{2}+1\right]V^{i+1}_\tau\\ \hat{=} &\quad \textstyle\sum_{j=i+1}^d q^{i-(j-1-i)}v_1\otimes\cdots\otimes v_1\otimes v_0\otimes\cdots\otimes v_{a_{j-1}=0}\\ &\quad \otimes v_{a_j+1=1}\otimes v_{a_{j+1}=0}\otimes\cdots\otimes v_{a_d=0}\\ = &\quad \textstyle\sum_{j=1}^d q^{a_1}v_{a_1}\otimes\cdots\otimes q^{a_{j-1}}v_{a_{j-1}}\otimes v_{a_j+1}\otimes v_{a_{j+1}}\otimes\cdots\otimes v_{a_d}\end{aligned}$$

($\mathbf{a}=(1,\cdots,1,0,\cdots,0)$, $|\mathbf{a}|=i$)
and $E.V^i_w$ should be

$$\begin{aligned}E.V^i_w = &\quad \textstyle\sum_{(w^{-1}(i+1),w^{-1}(j))=\sigma,(i+1,j)\in S_{d-i}}q^{i-(j-1-i-2b_{i+1,j}(w))}\left[\frac{1+\alpha_{w(w^{-1}(j))}=0}{2}+1\right]V^{i+1}_{\sigma w}\\ = &\quad \textstyle\sum_{\tau=(i+1,j)\in S_{d-i}}q^{i-(j-1-i-2b_{i+1,j}(w))}V^{i+1}_{w\tau}\\ \hat{=} &\quad \textstyle\sum_{(w^{-1}(i+1),w^{-1}(j))=\sigma,(i+1,j)\in S_{d-i}}q^{\alpha_{w(1)}}v_{a_{w(1)}}\otimes\cdots\otimes q^{\alpha_{w(w^{-1}(j)-1)}}v_{a_{w(w^{-1}(j)-1)}}\\ &\quad \otimes v_{a_{w(w^{-1}(j))}+1=1}\otimes v_{a_{w(w^{-1}(j)+1)}}\otimes\cdots\otimes v_{a_{w(d)}}\\ = &\quad \textstyle\sum_{j=1}^d q^{\alpha_{w(1)}}v_{a_{w(1)}}\otimes\cdots\otimes q^{\alpha_{w(j-1)}}v_{a_{w(j-1)}}\otimes\left[\frac{1+\alpha_{w(j)}}{2}+1\right]v_{a_{w(j)}+1}\otimes v_{a_{w(j+1)}}\otimes\cdots\otimes v_{a_{w(d)}}\end{aligned}$$

($\mathbf{a}=(1,\cdots,1,0,\cdots,0)$, $|\mathbf{a}|=i$).
(let $b_{i,j}(w)$ denote the number of $l<i$ such that $w(l)>j$, and $\sigma w=w\tau$).
This shows how the action ought to be defined in order for (3) to be a commutative diagram and ϕ an isomorphism. It remains to interprete this action in some natural way. By adapting the action of B^i, B^{i+1} from left and right on $B^{i,i+1}$, the action of E, F on B^i, B^{i+1} can be deformed such that E can again act as $E.V^i_w = B^{i,i+1}\otimes_{B^i}V^i_w$ (let $f^{i+1}_w.g := q^{i+1}f_{\pi^{-1}_{i+1}(w)}$,

$g.f^i_w := q^{1+2b_{i+1,i+1}(w)} f_{\pi_i^{-1}}$ to obtain the action of E, and use an analogous approach for the action of F; I need $f_{\pi_i\circ\pi_{i+1}^{-1}(w\tau)} = q^{2i+1-j-2b_{i+1,j}(w)} f^i_w + \sum_{l<i+1,\nu=(l,i+1)} q^{\lambda_l} f^i_{w\tau\nu}$ for $w=(i+1,j)$, and use $b_{i+1,j}(w) = b_{i+1,i+1}(w\tau)$ for $\tau=(i+1,j)$).

Remark 22. (this was used in the calculations above)

$$\begin{aligned}
\mathbf{a} = \;& (a_1 = 1, \cdots, \overset{w^{-1}(j)^{th}}{a_{w^{-1}(j)}}, \cdots, a_i = 1, \overset{i+1^{th}}{a_{i+1} = 0}, \cdots, \overset{j^{th}}{a_j = 0}, \cdots, \overset{w^{-1}(i+1)^{th}}{a_{w^{-1}(i+1)}}, \cdots, a_d = 0)\\
w\mathbf{a} = \;& (a_{w(1)}, \cdots, \overset{w^{-1}(j)^{th}}{a_j = 0}, \cdots, a_{w(i)}, \overset{i+1^{th}}{a_{w(i+1)}}, \cdots, \overset{j^{th}}{a_{w(j)}}, \cdots, \overset{w^{-1}(i+1)^{th}}{a_{i+1} = 0}, \cdots, a_{w(d)})\\
\sigma w\mathbf{a} = \;& (a_{w(1)}, \cdots, \overset{w^{-1}(j)^{th}}{a_{i+1} = 0}, \cdots, a_{w(i)}, \overset{i+1^{th}}{a_{w(i+1)}}, \cdots, \overset{j^{th}}{a_{w(j)}}, \cdots, \overset{w^{-1}(i+1)^{th}}{a_j = 0}, \cdots, a_{w(d)})\\
E.v_{w\mathbf{a}} = \;& \textstyle\sum_{(w^{-1}(i+1),w^{-1}(j))=\sigma,(i+1,j)\in S_{d-i}} K.v_{a_{w(1)}} \otimes \cdots \otimes \overset{w^{-1}(j)^{th}}{E.v_{a_j=0}} \otimes \cdots \otimes v_{a_{w(i)}}\\
& \otimes \overset{i+1^{th}}{v_{a_{w(i+1)}}} \otimes \cdots \otimes \overset{j^{th}}{v_{a_{w(j)}}} \otimes \cdots \otimes \overset{w^{-1}(i+1)^{th}}{v_{a_{i+1}=0}} \otimes \cdots \otimes v_{a_{w(d)}}\\
= \;& \textstyle\sum_{(w^{-1}(i+1),w^{-1}(j))=\sigma,(i+1,j)\in S_{d-i}} K.v_{a_{w(1)}} \otimes \cdots \otimes \overset{w^{-1}(j)^{th}}{E.v_{a_{\sigma(w(w^{-1}(i+1)))=j}=0}} \otimes \cdots \otimes v_{a_{w(i)}}\\
& \otimes \overset{i+1^{th}}{v_{a_{w(i+1)}}} \otimes \cdots \otimes \overset{j^{th}}{v_{a_{w(j)}}} \otimes \cdots \otimes \overset{w^{-1}(i+1)^{th}}{v_{a_{\sigma(w(w^{-1}(j)))=i+1}=0}} \otimes \cdots \otimes v_{a_{w(d)}}
\end{aligned}$$

(of course the position of $w^{-1}(j)$ will vary and may e.g. lie to the right of the i^{th} position... $(w^{-1}(i+1), w^{-1}(j)) = \sigma, (i+1,j) \in S_{d-i}$ precisely means that the action of E on the $w^{-1}(j)^{th}$ position is not zero, i.e. the basis vector in this position is v_0 and not v_1).

So I have

$$\begin{array}{ccc}
K_0(C'_{func}) & \overset{\twoheadleftarrow}{\hookrightarrow} & K_0(C_{func})\\
\overset{?}{\cong} & & \overset{\phi}{\cong}\\
V_{d_1} \otimes \cdots \otimes V_{d_k} & \overset{\twoheadleftarrow}{\hookrightarrow} & V_1^{\otimes d}
\end{array} . \tag{3}$$

In order to restrict the isomorphism in (3) to $V_{d_1} \otimes \cdots \otimes V_{d_k}$ and $K_0(C'_{func})$, I need to check that the images of the projection maps on both sides correspond. Since the inclusion maps are injective, it is enough to show that the isomorphism commutes with the composition, i.e.

$$\phi \circ \iota \circ \pi = \iota_{d_1} \otimes \cdots \otimes \iota_{d_k} \circ \pi_{d_1} \otimes \cdots \otimes \pi_{d_k} \circ \phi.$$

Let $v_{a_1} \otimes \cdots \otimes v_{a_d} \hat{=} wS_i \times S_{d-i}$, i.e $\mathbf{a} = w(1, \cdots, 1, 0, \cdots, 0)$ and $v_{a_1} \otimes \cdots \otimes v_{a_d} = w(v_1 \otimes \cdots \otimes v_1 \otimes v_0 \otimes \cdots \otimes v_0)$. Then

$$\pi_{d_1} \otimes \cdots \otimes \pi_{d_k}(v_{a_1} \otimes \cdots \otimes v_{a_d}) = \bigotimes_{i=1}^{k} q^{-l(\mathbf{a}^i)} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} v_{|\mathbf{a}^i|}$$

and

$$\begin{aligned}
& \iota_{d_1} \otimes \cdots \otimes \iota_{d_k} \Big(\textstyle\bigotimes_{i=1}^{k} q^{-l(\mathbf{a}^i)} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} v_{|\mathbf{a}^i|}\Big)\\
= \;& \textstyle\bigotimes_{i=1}^{k} q^{-l(\mathbf{a}^i)} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} \sum_{|\hat{\mathbf{a}}^i| = |\mathbf{a}^i|} q^{b(\hat{\mathbf{a}}^i)} v_{\hat{\mathbf{a}}^i}\\
= \;& \textstyle\sum_{\hat{a}=\sigma(a),\, \sigma\in S_{\mathbf{d}}} \bigotimes_{i=1}^{k} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} q^{b(\hat{\mathbf{a}}^i) - l(\mathbf{a}^i)} v_{\hat{\mathbf{a}}^i}
\end{aligned}$$

(Set $a_{(i,j)} = (a_i, \cdots, a_j)$ and $\mathbf{a}^i = a_{(d_1+\cdots+d_{i-1}+1, d_1+\cdots+d_i)})$, as $\sigma \in S_{\mathbf{d}}$ precisely means that $\sigma(a) = \hat{a}$ in

$$\iota_{d_1} \otimes \cdots \otimes \iota_{d_k}\Big(\bigotimes_{i=1}^{k} q^{-l(\mathbf{a}^i)} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} v_{|\mathbf{a}^i|}\Big) = \bigotimes_{i=1}^{k} q^{-l(\mathbf{a}^i)} \frac{1}{\begin{bmatrix} d_i \\ |\mathbf{a}^i| \end{bmatrix}} \sum_{|\hat{\mathbf{a}}^i| = |\mathbf{a}^i|} q^{b(\hat{\mathbf{a}}^i)} v_{\hat{\mathbf{a}}^i}.$$

Furthermore, $\pi(V_w^i) = V_{S_{\mathbf{d}}w}^i$ with $v \in V_w^i$ and $\iota(V_{S_{\mathbf{d}}w}^i) = \oplus_{\sigma \in S_{\mathbf{d}}w} V_\sigma^i$.
So both maps correspond to one another as for the U-case, and since ι is injective on both sides, the images of the projections must already correspond to one another (in fact, the projection and inclusion map for $K_0(C'_{func})$ was chosen precisely so that it would correspond). Therefore the isomorphism from (3) can be restricted as claimed.

5 A Construction for a $U_q(\mathfrak{so}_{2n})$-Module

A similar construction is possible for type D.

Definition 5.1. The (even) special orthogonal Lie algebra $\mathfrak{so}_{2n}$, the finite dimensional simple Lie algebra of type D_n ($n \geq 4$), is defined as

$$\mathfrak{so}_{2n} = \left\{ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n \times 2n}(\mathbb{C}) \mid A, B, C, D \in M_{n \times n}(\mathbb{C}),\ A^t = -D,\ B^t = -B,\ C^t = -C \right\}.$$

The associated quantum group, the quantum special orthogonal algebra $U_q(\mathfrak{so}_{2n})$, is defined as the quotient of the $k = \mathbb{C}(q)$-algebra with unit generated by E_a, F_a, K_a, K_a^{-1}, $a \in I = \{i, k, j_1 \ldots, j_{n-2}\}$ with relations

$$\begin{array}{ccc} K_a K_a^{-1} = 1 & K_a E_b = q^{C_{ab}} E_b K_a & E_a F_b - F_b E_a = \delta_{ab} \frac{K_a - K_a^{-1}}{q - q^{-1}} \\ K_a K_b = K_b K_a & K_a F_b = q^{-C_{ab}} F_b K_a & \forall a, b \in I \end{array}$$

by the ideal generated by

$$\begin{array}{ccc} E_a^2 E_b - (q - q^{-1}) E_a E_b E_a + E_b E_a^2 & F_a^2 F_b - (q - q^{-1}) F_a F_b F_a + F_b F_a^2 & C_{ab} = 1 \\ E_a E_b - E_b E_a & F_a F_b - F_b F_a & C_{ab} = 0. \end{array}$$

$C_{aa} = 2$ and $C_{ab} = -1$ if there is an edge between a and b in the Dynkin diagramm of type D_n, else $C_{ab} = 0$.

Remark 23. The Dynkin diagramm of type D_n is given by

$$\begin{array}{lllllll} i = j_n & & & & & & \\ & \diagdown & & & & & \\ & & j_{n-2} & — & j_{n-3} & — \cdots — & j_1 \\ & \diagup & & & & & \\ k = j_{n-1} & & & & & & \end{array}$$

Now define the vector representation for $U_q(\mathfrak{so}_{2n})$:

Definition 5.2. [9] Let $V = \left(\bigoplus_{j=1}^{n} kv_i\right) \oplus \left(\bigoplus_{j=1}^{n} kv_{\overline{i}}\right)$ be a $2n$-dimensional vector space. Introduce a linear ordering on the index set by

$$1 \prec 2 \prec \cdots \prec \overset{n}{\overline{n}} \prec \cdots \prec \overline{2} \prec \overline{1}$$

(Notice that the order between n and $\overline{n}$ is not defined). The $U_q(\mathfrak{so}_{2n})$-module action is defined as follows:

$$K_a v_j = \begin{cases} qv_j & \text{if } j = a \\ q^{-1}v_j & \text{if } j = a+1 \\ qv_j & \text{if } j = n-1, a = n \\ q^{-1}v_j & \text{if } j = \overline{a} \\ qv_j & \text{if } j = \overline{a+1} \\ q^{-1}v_j & \text{if } j = \overline{n-1}, a = n \\ v_j & \text{else} \end{cases}$$

$$E_a v_j = \begin{cases} v_a & \text{if } j = a+1, a \neq n \\ v_{\overline{a+1}} & \text{if } j = \overline{a}, a \neq n \\ v_n & \text{if } j = \overline{n-1}, a = n \\ v_{n-1} & \text{if } j = \overline{n}, a = n \\ 0 & \text{else} \end{cases}$$

$$F_a v_j = \begin{cases} v_{a+1} & \text{if } j = a, a \neq n \\ v_{\overline{a}} & \text{if } j = \overline{a+1}, a \neq n \\ v_{\overline{n-1}} & \text{if } j = n, a = n \\ v_{\overline{n}} & \text{if } j = n-1, a = n \\ 0 & \text{else} \end{cases}$$

(so $E_aF_av_j = v_j = F_aE_av_j$ or zero and $K_av_j = q^{\langle h_a, wt(v_j)\rangle}$ for $wt(v_j) = \epsilon_j$, $wt(v_{\overline{j}}) = -\epsilon_j$ where $\epsilon_i(A) = a_{ii}$ for a $2n \times 2n$-matrix A and h_a the ath diagonal generator of $\mathfrak{so}_{2n}$).

Then a basis of $V^{\otimes d}$ is given by the $v_{\mathbf{a}} = v_{a_i} \otimes \cdots \otimes v_{a_d}$, with $a_i \in \{1 \prec 2 \prec \cdots \prec \overset{n}{\overline{n}} \prec \cdots \prec \overline{2} \prec \overline{1}\}$. Again, S_d can act by permuting the indicies and if $\mathbf{x} = (|\{a_i = 1\}|, \ldots, |\{a_i = \overline{1}\}|)$ denotes the type of $\mathbf{a}$, then $V^{\otimes d} = \bigoplus_{\mathbf{x}} \{v_{\mathbf{a}} \mid \mathit{type}(\mathbf{a}) = \mathbf{x}\}_k$ is a decomposition into S_d-submodules. Such a submodule $\{v_{\mathbf{a}} \mid \mathit{type}(\mathbf{a}) = \mathbf{x}\}_k$ is isomorphic to $1 \uparrow_{S_{\mathbf{x}}}^{S_d}$ ($S_{\mathbf{x}} = S_{x_1} \times \cdots \times S_{x_{2n}}$), as in the case of $\mathfrak{sl}_2$. Furthermore, I can again identify a basis element $wS_{\mathbf{x}}$ with a simple $B^{\mathbf{x}}$-module $V_w^{\mathbf{x}}$, where $B^{\mathbf{x}}$ is the space of maps $S_d/S_{\mathbf{x}} \to \mathbb{C}$, as before. Furthermore, I can define $\pi_{\mathbf{x}} : S_d/S_{\mathbf{x}} \cap S_{\mathbf{x}'} = W_{\mathbf{x},\mathbf{x}'} \to W_{\mathbf{x}}$.
How do the elements of U_q act on these modules? The comultiplication is given by

$$\Delta : \begin{array}{rcl} K_a^{\pm 1} & \mapsto & K_a^{\pm 1} \otimes K_a^{\pm 1} \\ E_a & \mapsto & K_a \otimes E_a + E_a \otimes 1 \\ F_a & \mapsto & 1 \otimes F_a + F_a \otimes K_a^{-1} \end{array}$$

and so

$$\Delta^{(d-1)}(E_a) = \sum_{j=1}^{d-1} K_a \otimes \cdots \otimes K_a \otimes E_a \otimes 1 \otimes \cdots \otimes 1$$

and similar for the other generators, as in the $U_q(\mathfrak{sl}_2)$-case. So $v_{\mathbf{a}}$ of type $\mathbf{x}$ is mapped by E_a to a sum of $v_{\mathbf{a}'}$ of type $\mathbf{x}' = (x_1, \ldots, x_a + (1 - \delta_{0,x_{a+1}}), max\{x_{a+1} - 1, 0\}, \ldots, x_{2n-a} + (1 - \delta_{0,x_{2n-a+1}}), max\{x_{2n-a+1} - 1, 0\}, \ldots, x_{2n})$, or to zero if $x_{a+1} = 0 = x_{2n-a+1}$, if $a \neq n$, and of type $\mathbf{x}' = (x_1, \ldots, x_{n-1} + (1 - \delta_{0,x_{n+1}}), x_n + (1 - \delta_{0,x_{n+2}}), max\{x_{n+1} - 1, 0\}, max\{x_{n+2} - 1, 0\}, \ldots, x_{2n})$, or to zero if $x_{n+1} = 0 = x_{n+2}$, if $a = n$. Then a submodule $\{v_{\mathbf{a}} \mid type(\mathbf{a}) = \mathbf{x}\}_k \cong 1 \uparrow_{S_{\mathbf{x}}}^{S_d}$ is a weight space and the type $\mathbf{x}$ again determines the weight.
Similarly, $B^{\mathbf{x}}$-modules can be mapped by E_a to $B^{\mathbf{x}'}$-modules as in the $U_q(\mathfrak{sl}_2)$-case, but of course this ought to be interpreted in some fashion perhaps similar to the case of $\mathfrak{sl}_2$ (If I consider the case of $U(\mathfrak{so}_m)$ rather than $U_q(\mathfrak{so}_m)$, E_i acts as $E_i.V_{\mathbf{x}} = V_{\mathbf{x}'} = B^{\mathbf{x}\mathbf{x}'} \otimes_{B^{\mathbf{x}}} V_{\mathbf{x}}$; one must of course choose the $\mathbf{x}'$ accordingly.).

Remark 24. In the case of $\mathfrak{so}_m$, the Schur-Weyl duality becomes a duality between the Lie algebra and the Brauer algebra (instead of the group algebra of S_d or the corresponding Weyl group for type D) [8, section 10.1]. The Brauer algebra is slightly larger than the group algebra of the symmetric group. For the quantum case, it is an open question how the problem may be solved in general.

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