# Affine versions of Schur-Weyl duality 

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## Introduction

Many fundamental objects in representation theory such as Lie algebras, Weyl groups or Hecke algebras have an affine version. These affine algebras appear naturally in various places. For example, affine Hecke algebras and affine Weyl groups play an important role in the theory of $p$-adic groups (see for example [IM65) but they are also connected with various other objects in representation theory such as affine Lie algebras, affine quantum groups or diagram algebras. In this thesis we explore some of these connections by studying affine versions of Schur-Weyl duality.

The original version of Schur-Weyl duality goes back to Schur Sch01 and is concerned with the commuting actions of the general linear group $\mathrm{GL}_{n}=\mathrm{GL}_{n}(\mathbb{C})$ and the symmetric group $S_{r}$ on tensor space:

$$
\begin{equation*}
\mathrm{GL}_{n} \curvearrowright V^{\otimes r} \curvearrowleft S_{r} . \tag{1}
\end{equation*}
$$

Here $V=\mathbb{C}^{n}$ is the natural representation of $\mathrm{GL}_{n}$ and $S_{r}$ acts on $V^{\otimes r}$ by permuting the tensor factors. Usually this classical Schur Weyl duality also asserts that the two actions generate each other's centraliser. More categorically, this duality can be rephrased as an equivalence between certain subcategories of $\mathrm{GL}_{n}$ - $\bmod$ and $S_{r^{-}}$ mod. The action of $\mathrm{GL}_{n}$ can also be replaced by $\mathrm{SL}_{n}$ or the Lie algebras $\mathfrak{s l}_{n}$ and $\mathfrak{g l}_{n}$ and one can also pass to other fields than $\mathbb{C}$. This thesis looks at generalisations of Schur-Weyl duality in various directions that involve affine and quantum algebras of some kind. We collect some known results, sometimes with new proofs, but there will also be new examples. The following picture illustrates some of these known generalisations of Schur-Weyl duality which we take as the starting point of our investigation:


Let us explain these in a bit more detail. Affine Schur-Weyl duality, which appears in [CP96, can be obtained from classical Schur-Weyl duality by replacing $S_{r}$ with the affine Weyl group $S_{r}^{\text {aff }}=\mathbb{Z}^{r} \rtimes S_{r}$. At the same time one has to take an affine version of the Lie algebra $\mathfrak{s l}_{n}$ and an affine version of the natural representation $V$. Similarly, replacing $S_{r}$ by its Hecke algebra $\mathcal{H}_{r}$ and $\mathfrak{s l}_{n}$ by its quantum group $U_{q}\left(\mathfrak{s l}_{n}\right)$ yields quantum Schur-Weyl duality established by Jimbo in Jim86. This takes place over the field of rational functions $\mathbb{C}(q)$ in the indeterminate $q$. Fusing the two cases together, one obtains the quantum-affine case from [Gre97] and [P96] which involves the affine Hecke algebra $\mathcal{H}_{r}^{\text {aff }}$ associated to $S_{r}^{\text {aff }}$. All of these three dualities have a double centraliser property and an induced categorical correspondence which we will prove in this thesis.

These results are well-known but our exposition will sometimes deviate from the
standard literature in order to give more elementary arguments. In the classical and the quantum case the double centraliser property and the categorical correspondence is an easy consequence of the double centraliser theorem together with a few calculations. In the (quantum) affine case, however, the underlying vector space is infinite-dimensional so that we do not have the double centraliser theorem at our disposal. We will therefore need to use other techniques. This will lead us to look at the quantum and affine versions of the (classical) Schur algebra $S(n, r)=\operatorname{End}_{S_{r}}\left(V^{\otimes r}\right)$. Our proofs will make use of their structure and the quantum and affine versions of the functor $V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]}$.

Higher Schur-Weyl duality, going back to AS98 and then studied in detail in BK08, has a bit of a different flavour. It involves the so-called degenerate affine Hecke algebra $\mathcal{H}_{r}^{\text {deg }}{ }_{\text {introduced by Drinfeld [Dri86]. Roughly speaking, this algebra is con- }}$ structed by formally adding generators to the group algebra $\mathbb{C}\left[S_{r}\right]$ which behave very similar to the Jucys-Murphy elements of $\mathbb{C}\left[S_{r}\right]$. These are elements generating a maximal commutative subalgebra of $\mathbb{C}\left[S_{r}\right]$ which can be used to develop a weight theory for $S_{r}$ (similar to the highest weight theory for semisimple Lie algebras). As a consequence, the degenerate affine Hecke algebra contains a lot of information about the representation and weight theory of the symmetric group. At the same time, higher Schur-Weyl duality replaces tensor space by $M \otimes V^{\otimes r}$ where $M$ is an arbitrary (not necessarily finite-dimensional) $\mathfrak{g l}_{n}$-module and we will explain the construction of commuting actions $\mathfrak{g l}_{n} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\mathrm{deg}}$. This is more flexible than classical Schur-Weyl duality and makes it possible to connect the representation theory of $\mathcal{H}_{r}^{\text {deg }}$ or $S_{r}$ with infinite-dimensional $\mathfrak{g l}_{n}$-representations like Verma modules.

All of the dualities above can be associated to some kind of Lie algebra of type $A$. There are also generalisations of the duality from (1) for classical groups outside of type $A$ and also for the symmetric group. The respective actions on tensor space are given by restricting the $\mathrm{GL}_{n}$-action:


Here the centralising partner of the orthogonal group $\mathrm{O}_{n}$ is the Brauer algebra $B_{r}(n)$ for the parameter $n \in \mathbb{N}$ which was introduced by Brauer in [Bra37. If we replace the orthogonal group by the symplectic group $\mathrm{Sp}_{n}$, the centralising partner will be the Brauer algebra $B_{r}(-n)$. This type $B, C, D$ duality also has a higher version. To construct this, we will explain how the Jucys-Murphy elements of the Brauer algebra can be used to define a degenerate affine version of the Brauer algebra denoted by $\mathbb{W}(\Xi)$. This algebra was introduced in Naz96] and is nowadays called the affine VW-algebra (or sometimes also the Nazarov-Wenzl algebra). The associated higher Schur-Weyl duality was studied in [ES18] and we will explain part of this by constructing commuting actions $\mathfrak{g} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathbb{W}(\Xi)$ where $\mathfrak{g}$ is a simple Lie algebra of type $B, C$ or $D$ and $M$ is a highest weight module of $\mathfrak{g}$.

The duality at the bottom of (2) connects the symmetric group $S_{n}$ with the partition algebra $P_{r}(n)$ for the parameter $n \in \mathbb{N}$. This algebra was discovered independently
by Martin Mar91 and Jones Jon94 and studied in detail by Martin and others MS94, Mar96, Mar00, HR05]. Both the Brauer algebra and the partition algebra can be defined more generally for any parameter $\delta \in \mathbb{C}$ as vector spaces with a distinguished basis given by certain set partitions (usually drawn as diagrams) and a pictorial multiplication rule. There are many more algebras of this kind which are often referred to as diagram algebras. These diagram algebras will also play an important role in this thesis and we will review some important examples and the associated Schur-Weyl dualities. A more detailed overview of diagram algebras can be found in Koe08.

Our main new result:
As our main new result, we will construct an affine version of the duality between the symmetric group and the partition algebra. For this, we first have to define an action of the affine symmetric group $S_{n}^{\text {aff }}$ on $V^{\otimes r}$ extending the action on the bottom of (2). This will be the diagonal action induced along the group homomorphism $S_{n}^{\text {aff }} \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ which sends a permutation $\sigma \in S_{n}$ to the corresponding permutation matrix and $\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}^{n}$ to the diagonal matrix with diagonal entries $x^{a_{1}}, \ldots, x^{a_{n}}$ for a fixed $x \in \mathbb{C}^{\times}$. Since this extends the diagonal action of the symmetric group, the centralising partner of the $S_{n}^{\text {aff }}$-action should be contained in the partition algebra and we would to like understand this subalgebra. For this we will define a diagram algebra $P_{r}^{\text {bal }}$ which is spanned by partition diagrams that are balanced in a certain way and we call this the balanced partition algebra. The balanced partition algebra does not depend on a parameter $\delta$ but there is a canonical inclusion $P_{r}^{\text {bal }} \subset P_{r}(\delta)$ for all $\delta \in \mathbb{C}$. In particular, there is an induced right action of $P_{r}^{\text {bal }}$ on $V^{\otimes r}$. Our main result is then the following.

Theorem 3.41. The actions $S_{n}^{\text {aff }} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}^{\text {bal }}$ commute. If the multiplicative order of $x$ is bigger than $r$, the two actions generate each other's centraliser.

After we had proved this result, we found out that the balanced partition algebra was already defined in Har18, but for slightly different commuting actions. To be more precise, it is shown in Har18 that the $P_{r}^{\text {bal }}$-action on $V^{\otimes r}$ generates the algebra $\operatorname{End}_{\mathcal{M}_{n}}\left(V^{\otimes r}\right)^{\mathrm{op}}$ where $\mathcal{M}_{n} \subset \mathrm{GL}_{n}$ is the set of invertible monomial matrices acting diagonally on tensor space. We will recover this result by showing that $S_{n}^{\text {aff }}$ and $\mathcal{M}_{n}$ generate the same subalgebra of $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ if the multiplicative order of $x$ exceeds $r$. This also establishes the following double centraliser property which appears as a problem in Har18, p. 21].

Corollary 3.42. The commuting actions $\mathcal{M}_{n} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}^{\text {bal }}$ generate each other's centraliser.

We will also look at a few properties of the balanced partition algebra $P_{r}^{\text {bal }}$. In particular, we show that $P_{r}^{\text {bal }}$ is semisimple and we will parametrise the irreducible $P_{r}^{\text {bal }}$-representations by certain multipartitions. Moreover, we give a presentation of $P_{r}^{\text {bal }}$ by generators and relations.

An outlook on further generalisations:
It would also be desirable to have a higher version of the duality between $S_{n}$ and $P_{r}(n)$. Motivated by this, we will define the Jucys-Murphy elements of the partition algebra. These elements were already introduced by Halverson and Ram in HR05, but our construction will be slightly different. To be more precise, we will use SchurWeyl duality to define these elements for the partition algebras over large enough positive integers and then interpolate to arbitrary values $\delta \in \mathbb{C}$. One advantage
of this approach is that it makes computations with the Jucys-Murphy elements very straightforward. In fact, one can prove formulas involving the Jucys-Murphy elements first for $\delta \in \mathbb{N}$ by acting on tensor space and then for all $\delta \in \mathbb{C}$ by a basic interpolation argument. We will use this idea to verify a few formulas which were also obtained in Eny13 by other techniques.
These formulas will also make it clear that the relations between the standard generators and the Jucys-Murphy elements of the partition algebra are not local. This locality property is crucial in the definition of the degenerate affine Hecke algebra and the affine VW-algebra, which explains why defining a degenerate affine version of the partition algebra might be a more difficult task. In the Brauer algebra $B_{r}(\delta)$, for example, the Jucys-Murphy elements $\widehat{X}_{1}, \ldots, \widehat{X}_{r}$ satisfy the relations $s_{i} \widehat{X}_{i}-\widehat{X}_{i+1} s_{i}=e_{i}-1$ for $i=1, \ldots, r-1$ where $s_{i}$ and $e_{i}$ are standard generators of $B_{r}(\delta)$. These relations involve only neighbouring indices and, more importantly, they are stable under shifting indices. We will show that the analogous statement for the Jucys-Murphy elements of the partition algebra does not hold.

The locality properties above are closely related to the fact that our diagram algebras can be realised as endomorphism algebras in some monoidal diagram category. The definition of these categories can be motivated by the observation that it is actually unnatural to study Schur-Weyl duality on a fixed tensor space $V^{\otimes r}$ only. Instead, it can be fruitful to look at all tensor spaces and morphisms between any two of them at the same time. This categorical point of view has received a lot of attention in recent years. In the last section of this thesis we will explain how to generalise various diagram algebras into diagram categories and how to rephrase their respective Schur-Weyl dualities in this categorical setting. In particular, we will define a diagrammatic version of the balanced partition algebra and we give a presentation of this category by generators and relations.

Here is a short summary of each section of this thesis.

Section 1: We recall the definitions of (affine) Hecke algebras and Weyl groups and state a few basic properties. Proofs will be omitted.

Section 2: We give elementary proofs of quantum, affine and quantum affine SchurWeyl duality. We also explain how quantum Schur-Weyl duality can be used to link the bar involutions of quantum groups and Hecke algebras.

Section 3: The main purpose of this section is to prove the Schur-Weyl duality between the symmetric group and the partition algebra as well as an affine version for $S_{n}^{\text {aff }}$. This will lead us to defining the balanced partition algebra. We will also outline a few Schur-Weyl dualities for other diagram algebras and some basic techniques often used in the representation theory of the partition algebra and its diagram subalgebras.

Section 4: This section is mostly concerned with the construction of higher versions of Schur-Weyl dualities for simple Lie algebras. Our exposition will focus on the role of the Jucys-Murphy elements and we will briefly explain their applications to towers of semisimple algebras via the Okounkov-Vershik approach. We also give a simple construction of the Jucys-Murphy elements of the partition algebra using Schur-Weyl duality. We then use an interpolation argument to prove a few formu-
las involving the Jucys-Murphy elements of the partition algebra for any $\delta \in \mathbb{C}$ by checking them on tensor space.

Section 5: We introduce monoidal categories and explain how some of the diagram algebras and Schur-Weyl dualities from this thesis fit into this setting by defining their respective diagram categories. We also define the balanced partition category which generalises the balanced partition algebra from section 3 .

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## 1 Preliminaries on Weyl groups and Hecke algebras

In this section we will give a brief introduction to (affine) Weyl groups and Hecke algebras associated to simple Lie algebras. Many of the things we will talk about here are not strictly necessary to understand the succeeding sections since we will mostly be working with specific examples in type $A$. However, it is often useful to have the bigger picture in mind which is why we give a brief outline of the general set-up. For a self-contained introduction to Coxeter groups and Hecke algebras, we refer to [Hum90].

### 1.1 Affine Weyl groups

Let $\Phi$ be a (crystallographic) root system in a Euclidean space $E$ with inner product $(\cdot, \cdot)$. Let $\Phi^{\vee}$ be the dual root system in $E$ consisting of the coroots $\alpha^{\vee}=2 \frac{\alpha}{(\alpha, \alpha)}$. For any $\alpha \in \Phi$, let $s_{\alpha} \in \mathrm{GL}(E)$ be the reflection in the hyperplane perpendicular to $\alpha$. Then the (finite) Weyl group $W(\Phi)=W$ associated to $\Phi$ is the subgroup of $\mathrm{GL}(E)$ generated by the $s_{\alpha}(\alpha \in \Phi)$. This is the same as the Weyl group associated to $\Phi^{\vee}$. Let $\operatorname{Aff}(E)$ be the group of invertible affine transformations on $E$ (i.e. transformations of the form $x \mapsto A x+d$ with $A \in \mathrm{GL}(E)$ and $d \in E)$. Then there is a group homomorphism $E \rightarrow \operatorname{Aff}(E)$ sending any $d \in E$ to the corresponding affine translation map

$$
T(d): E \rightarrow E, \quad x \mapsto x+d
$$

Definition 1.1. The affine Weyl group $W^{\text {aff }}(\Phi)=W^{\text {aff }}$ associated to $\Phi$ is the subgroup of $\operatorname{Aff}(E)$ generated by $W$ and $T\left(\Phi^{\vee}\right)$.

Let $L\left(\Phi^{\vee}\right) \subset E$ be the coroot lattice, i.e. the abelian subgroup of $E$ generated by the coroots. We can identify this via the map $T$ with a subgroup of $W^{\text {aff }}$. Note that $T\left(L\left(\Phi^{\vee}\right)\right) \cap W=\left\{\operatorname{id}_{E}\right\}$. Moreover, the identity $s_{\alpha} T(d) s_{\alpha}=T\left(s_{\alpha}(d)\right)$ tells us that $T\left(L\left(\Phi^{\vee}\right)\right) \subset W^{\text {aff }}$ is a normal subgroup. This proves the following Proposition.

Proposition 1.2. The affine Weyl group is a semidirect product

$$
W^{\mathrm{aff}}=T\left(L\left(\Phi^{\vee}\right)\right) \rtimes W \cong \mathbb{Z}^{r} \rtimes W
$$

where $r$ is the rank of $\Phi$.
Example 1.3. Let $\Phi$ be the root system of type $A_{n-1}$. We realise this as the set of roots $\Phi=\left\{e_{i}-e_{j} \mid i \neq j\right\}$ in the Euclidean space $E=\left\{\sum_{i=1}^{n} a_{i} e_{i} \in \mathbb{R}^{n} \mid\right.$ $\left.\sum_{i=1}^{n} a_{i}=0\right\}$ with $\left(e_{i}, e_{j}\right)=\delta_{i, j}$. Then $\Phi=\Phi^{\vee}$ and the associated Weyl group $W=S_{n}$ acts on $E$ by permuting the coordinates. The coroot lattice is given by $L\left(\Phi^{\vee}\right)=\left\{\sum_{i=1}^{n} a_{i} e_{i} \in E \mid a_{i} \in \mathbb{Z}\right\}$. Hence, we can realise $W^{\text {aff }}$ as the subgroup of $\mathbb{Z}^{n} \rtimes S_{n}$ consisting of those elements whose lattice coordinates sum up to zero (where $S_{n}$ acts on $\mathbb{Z}^{n}$ by permuting lattice coordinates).

There also is a slightly different approach to affine Weyl groups through affine reflection groups. For any $k \in \mathbb{Z}$ let

$$
H_{\alpha, k}:=\{x \in E \mid(x, \alpha)=k\}
$$

and let $s_{\alpha, k} \in \operatorname{Aff}(E)$ be the reflection in the hyperplane $H_{\alpha, k}$. It is straightforward to check that $T\left(k \alpha^{\vee}\right) s_{\alpha}=s_{\alpha, k}$. In particular, $W^{\text {aff }}$ can also be described as the subgroup of $\operatorname{Aff}(E)$ generated by the affine reflections $s_{\alpha, k}$. Here is the analogue of the Weyl chambers in the affine situation.

Definition 1.4. The connected components of $E \backslash \bigcup_{\substack{\alpha \in \Phi \\ k \in \mathbb{Z}}} H_{\alpha, k}$ are called alcoves.
Now assume that $\Phi$ is an irreducible root system and let us choose a set of positive roots $\Phi^{+}$. Then there is a highest root $\alpha_{0} \in \Phi^{+}$uniquely determined by the property that $\alpha_{0}-\beta$ is a sum of simple roots for any $\beta \in \Phi^{+}$(including the empty sum for $\beta=\alpha_{0}$ ).

Example 1.5. As in Example 1.3, we realise the roots of $A_{n-1}$ as $\Phi=\left\{e_{i}-e_{j} \mid\right.$ $i \neq j\}$ in $E=\left\{\sum_{i=1}^{n} a_{i} e_{i} \in \mathbb{R}^{n} \mid \sum_{i=1}^{n} a_{i}=0\right\}$. If we choose the set of positive roots $\Phi^{+}=\left\{e_{i}-e_{j} \mid i<j\right\}$, then the simple roots are

$$
\alpha_{1}=e_{1}-e_{2}, \quad \alpha_{2}=e_{2}-e_{3}, \quad \ldots \quad \alpha_{n-1}=e_{n-1}-e_{n}
$$

In particular, $e_{i}-e_{j}=\alpha_{i}+\alpha_{i+1}+\ldots+\alpha_{j-1}$ for any $i<j$. It now follows that highest root is $\alpha_{0}=e_{1}-e_{n}=\alpha_{1}+\ldots+\alpha_{n-1}$.

Theorem 1.6. Let $\alpha_{1}, \ldots, \alpha_{r}$ be the simple roots of $\Phi$ and $\alpha_{0}$ the highest root. Then $S=\left\{s_{\alpha_{1}}, \ldots, s_{\alpha_{r}}, s_{\alpha_{0}, 1}\right\}$ is a generating system of $W^{\text {aff }}$. The affine Weyl group $W^{\text {aff }}$ acts simply-transitively on the set of alcoves. Moreover, $\left(W^{\text {aff }}, S\right)$ is a Coxeter system.

Proof. See [IM65, Prop. 1.2, Cor. 1.8, Cor. 1.16] or [Hum90, Section 4].
Next, we explain how to construct the Coxeter diagram of $W^{\text {aff }}$. Given two roots $\alpha, \beta \in \Phi$ meeting at an angle $\theta \neq 0, \pi$ the product $s_{\alpha, i} s_{\beta, j}$ is an affine rotation by $2 \theta$. If $(\alpha, \beta) \leq 0$, the 4 possible values of

$$
4 \cos (\theta)^{2}=4 \frac{(\alpha, \beta)^{2}}{\|\alpha\|^{2}\|\beta\|^{2}}=\left\langle\alpha, \beta^{\vee}\right\rangle\left\langle\beta, \alpha^{\vee}\right\rangle \in\{0,1,2,3\}
$$

correspond to the angles

$$
\theta=(1-1 / k) \pi \text { with } k \in\{2,3,4,6\}
$$

in this order. The order of $s_{\alpha, i} s_{\beta, j}$ is then $k$. Note that the Weyl group $W$ is generated by the simple reflections corresponding to the simple roots $\alpha_{i}(i=1, \ldots, r)$ which satisfy $\left(\alpha_{i}, \alpha_{j}\right) \leq 0$ for $i \neq j$. Thus, the Coxeter diagram of $\left(W,\left\{s_{\alpha_{1}}, \ldots, s_{\alpha_{r}}\right\}\right)$ is given by the Dynkin diagram of $\Phi$ with double and triple edges replaced by edges with label 4 and 6. A similar strategy works for the affine Weyl group: We have $\left(-\alpha_{0}, \alpha_{i}\right) \leq 0$ for $i=1, \ldots, r$ since otherwise $s_{\alpha_{i}}\left(\alpha_{0}\right)=\alpha_{0}-\left\langle\alpha_{0}, \alpha_{i}^{\vee}\right\rangle \alpha_{i}$ would be a higher root. Since $s_{\alpha_{0}, 1}=s_{-\alpha_{0},-1}$, the Coxeter diagram of $W^{\text {aff }}$ can thus be read off from the Dynkin diagram of $\Phi$ extended by the root $-\alpha_{0}$.

Definition 1.7. The extended Dynkin diagram associated to $\Phi$ is the diagram with vertices $-\alpha_{0}, \alpha_{1}, \ldots, \alpha_{r}$ and an edge between two vertices $\alpha$ and $\beta$ whenever $(\alpha, \beta) \neq 0$ and $\alpha \neq \beta$. The multiplicity of such an edge is $\left\langle\alpha, \beta^{\vee}\right\rangle\left\langle\beta, \alpha^{\vee}\right\rangle$ and an arrow points to the shorter root whenever this value is greater than 1.

Remark 1.8. Since $\left\langle-\alpha_{0}, \alpha_{i}^{\vee}\right\rangle\left\langle\alpha_{i},-\alpha_{0}^{\vee}\right\rangle=\left\langle\alpha_{0}, \alpha_{i}^{\vee}\right\rangle\left\langle\alpha_{i}, \alpha_{0}^{\vee}\right\rangle$ one can also construct the extended Dynkin diagram by adding the root $\alpha_{0}$ instead of $-\alpha_{0}$. However, it is more natural to work with $-\alpha_{0}$ so that $(\alpha, \beta) \leq 0$ for any distinct $\alpha, \beta \in$ $\left\{-\alpha_{0}, \alpha_{1}, \ldots, \alpha_{r}\right\}$.

Example 1.9. We determine the extended Dynkin diagram of the root system $\Phi$ of type $A_{n-1}$ for $n>2$. Keeping the notation from Example 1.5, the simple roots of $A_{n-1}$ are $\alpha_{1}=e_{1}-e_{2}, \ldots, \alpha_{n-1}=e_{n-1}-e_{n}$ with the associated Dynkin diagram

$$
A_{n-1}: \underset{\alpha_{1}}{\bullet} \quad \stackrel{\alpha_{2}}{\stackrel{\bullet}{\alpha_{n-2}} \quad \underset{\alpha_{n-1}}{\bullet}} .
$$

The highest root is $\alpha_{0}=e_{1}-e_{n}$ and

$$
\left\langle-\alpha_{0}, \alpha_{i}^{\vee}\right\rangle\left\langle\alpha_{i},-\alpha_{0}^{\vee}\right\rangle=\left\{\begin{array}{l}
1 \text { if } i=1, n-1 \\
0 \text { if } i=2, \ldots, n-2
\end{array}\right.
$$

The extended Dynkin diagram (and after relabelling of the vertices also the Coxeter diagram of $\left.\left(W^{\text {aff }}, S\right)\right)$ is thus given by


### 1.2 Hecke algebras

Let us recall the definition and some basic properties of Hecke algebras for arbitrary Coxeter groups. These algebras can be defined over the ring of Laurent polynomials $\mathbb{Z}\left[q, q^{-1}\right]$ but we are mostly interested in Hecke algebras over the ring of rational functions $\mathbb{C}(q)$.

Definition 1.10. Let $(W, S)$ be a Coxeter system. The (generic) Hecke algebra $\mathcal{H}=\mathcal{H}(W)=\mathcal{H}(W, S)$ of $(W, S)$ is the $\mathbb{C}(q)$-algebra with generators $\left\{H_{s} \mid s \in S\right\}$ and relations
(H1) $H_{s}^{2}=1+\left(q^{-1}-q\right) H_{s} \quad$ for all $s \in S$
(H2) $\underbrace{H_{s} H_{t} H_{s} \ldots}_{m_{s t}}=\underbrace{H_{t} H_{s} H_{t} \ldots}_{m_{s t}} \quad$ for all $s, t \in S$ with $s \neq t$
where $m_{s t}$ is the order of $s t$.
Note that (H1) is equivalent to $H_{s}\left(H_{s}+q-q^{-1}\right)=1$ or $\left(H_{s}+q\right)\left(H_{s}-q^{-1}\right)=0$. In particular, $H_{s}$ is invertible with $H_{s}^{-1}=H_{s}+q-q^{-1}$. For any $w \in W$, choose a reduced expression $w=s_{1} \cdot \ldots \cdot s_{l(w)}$ and let

$$
H_{w}:=H_{s_{1}} \cdot \ldots \cdot H_{s_{l(w)}}
$$

Proposition 1.11. The element $H_{w}$ does not depend on the choice of a reduced expression of $w$ and $\left\{H_{w} \mid w \in W\right\}$ is a $\mathbb{C}(q)$-basis of $\mathcal{H}$. Moreover,

$$
\begin{array}{ll}
H_{w} H_{w^{\prime}}=H_{w w^{\prime}} & \text { if } l(w)+l\left(w^{\prime}\right)=l\left(w w^{\prime}\right) \\
H_{w} H_{s}=H_{w s}+\left(q^{-1}-q\right) H_{w} & \text { if } l(w)>l(w s) \tag{3}
\end{array}
$$

for any $w, w^{\prime} \in W$ and $s \in S$.
Proof. See [Hum90, Sections 7.1-7.4].
The Hecke algebra $\mathcal{H}(W, S)$ has the same generators as the group algebra $\mathbb{C}[W]$ but the quadractic relation $s^{2}=1$ is replaced by the twisted quadratic relation $H_{s}^{2}=1+\left(q^{-1}-q\right) H_{s}$. In fact, one recovers the quadratic relation of the group algebra by replacing the indeterminate $q$ by 1 and in light of that the Hecke algebra $\mathcal{H}(W, S)$ is sometimes called a deformation of the group algebra $\mathbb{C}[W]$. To be more precise, there is a specialisation homomorphism of $\mathbb{C}$-algebras

$$
\begin{aligned}
\varphi: \mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]} & \longrightarrow \mathbb{C}[W] \\
H_{x} & \longmapsto x \\
q & \longmapsto 1
\end{aligned}
$$

where $\mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}$ is the $\mathbb{C}\left[q, q^{-1}\right]$-subalgebra of $\mathcal{H}$ generated by the $H_{s}$ with $s \in S$. The homomorphism $\varphi$ is well-defined since it is compatible with the relations (H1) and (H2) which are defined over $\mathbb{C}\left[q, q^{-1}\right]$. It follows from Proposition 1.11 that the algebra $\mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}$ is also the $\mathbb{C}\left[q, q^{-1}\right]$-span of the basis $\left\{H_{w} \mid w \in W\right\}$ in $\mathcal{H}$. The specialisation homomorphism $\varphi$ can be used to prove some basic properties of the Hecke algebra like semisimplicity.

Proposition 1.12. If $W$ is a finite Coxeter group, the Hecke algebra $\mathcal{H}=\mathcal{H}(W)$ is semisimple.

Proof. See also [Mat99, Cor. 1.17]. If $\mathcal{H}$ is not semisimple, we can find $h \neq 0$ in the radical of $\mathcal{H}$. After multiplying with some element of $\mathbb{C}(q)^{\times}$and dividing by an appropriate power of $q-1$, we may assume that $h \in \mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}$ and $\varphi(h) \neq 0$. Since $\mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]} h \mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}$ is contained in the radical of $\mathcal{H}$, it is a nilpotent ideal. This shows that $\mathbb{C}[W] \varphi(h) \mathbb{C}[W]=\varphi\left(\mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]} h \mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}\right)$ is a non-zero nilpotent ideal in $\mathbb{C}[W]$ which contradicts the semisimplicity of $\mathbb{C}[W]$. Hence, $\mathcal{H}$ must be semisimple.

For $u \in\left\{-q, q^{-1}\right\}$ there is a ringhomomorphism

$$
\varphi_{u}: \mathcal{H} \rightarrow \mathbb{C}(q), \quad H_{s} \mapsto u
$$

This defines a (right) $\mathcal{H}$-module structure on $\mathbb{C}(q)$ which we denote by $\mathbb{C}(q)_{u}$. For $S_{\lambda} \subset S$, let $W_{\lambda}=\left\langle S_{\lambda}\right\rangle \subset W$ be the corresponding Coxeter group and let $\mathcal{H}_{\lambda}$ be the corresponding Hecke algebra. Note that there is a canonical inclusion $\mathcal{H}_{\lambda} \hookrightarrow \mathcal{H}$ with $H_{w} \mapsto H_{w}$.

Definition 1.13. The (right) parabolic Hecke modules corresponding to the parabolic subgroup $W_{\lambda} \subset W$ are the $\mathcal{H}$-modules

$$
\begin{aligned}
& \mathcal{M}^{\lambda}:=\mathbb{C}(q)_{q^{-1}} \otimes_{\mathcal{H}_{\lambda}} \mathcal{H} \\
& \mathcal{N}^{\lambda}:=\mathbb{C}(q)_{-q} \otimes_{\mathcal{H}_{\lambda}} \mathcal{H} .
\end{aligned}
$$

Let ${ }^{\lambda} W$ be the set of shortest right coset representatives of $W_{\lambda}$ in $W$. Recall that for any $w \in W$ there is a unique decomposition $w=w_{\lambda} \cdot{ }^{\lambda} w$ with $w_{\lambda} \in W_{\lambda}$ and ${ }^{\lambda} w \in{ }^{\lambda} W$. These elements satisfy $l(w)=l\left(w_{\lambda}\right)+l\left({ }^{\lambda} w\right)$. By Proposition 1.11, we also have $H_{w}=H_{w_{\lambda}} H_{\lambda_{w}}$. This implies that $\mathcal{H}$ is a free left $\mathcal{H}_{\lambda}$ module with basis $\left\{H_{w} \mid w \in{ }^{\lambda} W\right\}$. We thus obtain bases for parabolic Hecke modules.

Proposition 1.14. The parabolic Hecke module $\mathcal{M}^{\lambda}$ (resp. $\mathcal{N}^{\lambda}$ ) has the basis $\left\{M_{x}:=1 \otimes H_{x} \in \mathcal{M}^{\lambda} \mid x \in{ }^{\lambda} W\right\}$ (resp. $\left\{N_{x}:=1 \otimes H_{x} \in \mathcal{N}^{\lambda} \mid x \in{ }^{\lambda} W\right\}$ ). The generators of $\mathcal{H}$ act on these via

$$
1 \otimes H_{x} \cdot H_{s}= \begin{cases}1 \otimes H_{x s} & \text { if } x s \in^{\lambda} W, x s>x \\ 1 \otimes H_{x s}+\left(q^{-1}-q\right) H_{x} & \text { if } x s \in^{\lambda} W, x s<x \\ u \otimes H_{x} & \text { if } x s \not{ }^{\lambda} W\end{cases}
$$

where $u=q^{-1}$ (resp. $\left.u=-q\right)$ and where $>$ is the Bruhat order on $W$.
Proof. This follows from the multiplication formulas in (3) and the fact that $x s=r x$ for some $r \in S_{\lambda}$ if $x s \not{ }^{\lambda} W$ (see for example [GP00, Lemma 2.1.2]).

There is another way to construct the parabolic Hecke modules. Let

$$
x_{\lambda}:=\sum_{x \in W_{\lambda}} q^{-l(x)} H_{x} .
$$

For any $s \in S_{\lambda}$, we have

$$
\begin{aligned}
x_{\lambda} H_{s} & =\sum_{\substack{x \in W_{\lambda} \\
x s>x}} q^{-l(x)} H_{x s}+\sum_{\substack{x \in W_{\lambda} \\
x s<x}} q^{-l(x)}\left(H_{x s}+\left(q^{-1}-q\right) H_{x}\right) \\
& =\sum_{x \in W_{\lambda}} q^{-l(x)-1} H_{x}=q^{-1} x_{\lambda} .
\end{aligned}
$$

In particular,

$$
\begin{equation*}
x_{\lambda}^{2}=\left(\sum_{x \in W_{\lambda}} q^{-l(x)} H_{x}\right)^{2}=\left(\sum_{x \in W_{\lambda}} q^{-l(x)} H_{x}\right)\left(\sum_{x \in W_{\lambda}} q^{-2 l(x)}\right)=\left(\sum_{x \in W_{\lambda}} q^{-2 l(x)}\right) x_{\lambda} \tag{4}
\end{equation*}
$$

and $\mathbb{C}(q) x_{\lambda} \cong \mathbb{C}(q)_{q^{-1}}$ as $\mathcal{H}_{\lambda}$-modules. Using Frobenius reciprocity, we obtain an isomorphism of $\mathcal{H}$-modules

$$
\begin{align*}
\mathcal{M}^{\lambda} & \xrightarrow{\longrightarrow} x_{\lambda} \mathcal{H} \\
1 \otimes H_{x} & \longmapsto x_{\lambda} H_{x} . \tag{5}
\end{align*}
$$

In fact, this map is surjective by construction and for $y \in{ }^{\lambda} W$ we have $x_{\lambda} H_{y}=$ $\sum_{x \in W_{\lambda}} q^{-l(x)} H_{x y}$. Hence, the $x_{\lambda} H_{y}$ are linearly independent (for $y \in{ }^{\lambda} W$ ) which shows that the map above is injective.

Remark 1.15. Let $y_{\lambda}:=\sum_{x \in W_{\lambda}}(-q)^{l(x)} H_{x}$. A similar argument to the one above shows that $\mathcal{N}^{\lambda} \cong y_{\lambda} \mathcal{H}$.

Recall that a $\mathbb{C}$-linear map $\varphi: V \rightarrow W$ of $\mathbb{C}(q)$-vector spaces is called $\mathbb{C}(q)$ antilinear if $\varphi(f(q) v)=f\left(q^{-1}\right) \varphi(v)$ for all $f(q) \in \mathbb{C}(q)$ and $v \in V$. There is a unique $\mathbb{C}(q)$-antilinear ringhomomorphism

$$
{ }^{-}: \mathcal{H} \rightarrow \mathcal{H} \quad \text { with } \quad \overline{H_{s}}=H_{s}^{-1}=H_{s}+q-q^{-1}
$$

called the bar involution of $\mathcal{H}$. This also extends to $\mathbb{C}(q)$-antilinear bar involutions on the parabolic Hecke modules $\mathcal{M}^{\lambda}$ and $\mathcal{N}^{\lambda}$ via $a \otimes H \mapsto \bar{a} \otimes \bar{H}$. This is well-defined since

$$
\overline{\varphi_{u}\left(H_{s}\right)}=\bar{u}=u+q-q^{-1}=\varphi_{u}\left(H_{s}+q-q^{-1}\right)=\varphi_{u}\left(\overline{H_{s}}\right) .
$$

Theorem 1.16. For any $x \in{ }^{\lambda} W$, there are unique elements

$$
\begin{aligned}
& \underline{M}_{x} \in M_{x}+\sum_{y} q \mathbb{Z}[q] M_{y} \subset \mathcal{M}^{\lambda}, \quad \underline{\tilde{M}}_{x} \in M_{x}+\sum_{y} q^{-1} \mathbb{Z}\left[q^{-1}\right] M_{y} \subset \mathcal{M}^{\lambda}, \\
& \underline{N}_{x} \in N_{x}+\sum_{y} q \mathbb{Z}[q] N_{y} \subset \mathcal{N}^{\lambda}, \quad \underline{\underline{N}}_{x} \in N_{x}+\sum_{y} q^{-1} \mathbb{Z}\left[q^{-1}\right] N_{y} \subset \mathcal{N}^{\lambda}
\end{aligned}
$$

invariant under the bar involution. The elements $\underline{M}_{x}$ (resp. $\underline{\tilde{M}}_{x}, \underline{N}_{x}, \underline{\tilde{N}}_{x}$ ) for $x \in{ }^{\lambda} W$ form a basis of $\mathcal{M}^{\lambda}$ (resp. $\mathcal{N}^{\lambda}$ ) called the parabolic Kazhdan-Lusztig basis.

Proof. See [Soe97, Thm. 3.1, Thm. 3.5].
Remark 1.17. For $W_{\lambda}=\{e\}$ we have $\mathcal{M}^{\lambda}=\mathcal{N}^{\lambda}=\mathcal{H}$ and $\underline{M}_{x}=\underline{N}_{x}$ as well as $\underline{\tilde{M}}_{x}=\underline{\tilde{N}}_{x}$ for all $x \in{ }^{\lambda} W$. The $\underline{M}_{x}$ (resp. $\underline{\tilde{M}}_{x}$ ) then form a basis of $\mathcal{H}$ called the Kazhdan-Lusztig basis.

### 1.3 The $\mathfrak{g l}_{n}$ case

To any semisimple Lie algebra $\mathfrak{g}$ we can associate a root system $\Phi$. In the previous sections, we have defined the associated

- Weyl group $W$
- affine Weyl group $W^{\text {aff }}$
- finite Hecke algebra $\mathcal{H}(W)$
- affine Hecke algebra $\mathcal{H}\left(W^{\text {aff }}\right)$.

For our applications it will often be more natural to work with $\mathfrak{g l}_{n}$ instead of its semisimple cousin $\mathfrak{s l}_{n}$. In this section, we therefore define analogues of the above notions for $\mathfrak{g l}_{n}$. Note that $\mathfrak{g l}_{n}=\mathfrak{s l}_{n} \oplus \mathbb{C} \cdot \mathrm{I}_{n}$ is obtained from $\mathfrak{s l}_{n}$ by adding a one-dimensional center. This will be reflected in the definitions that follow.

Definition 1.18. The Weyl group associated to $\mathfrak{g l}_{n}$ is the same as the Weyl group $W=S_{n}$ associated to $\mathfrak{s l}_{n}$. The affine Weyl group associated to $\mathfrak{g l}_{n}$ is the affine symmetric group $S_{n}^{\text {aff }}:=\mathbb{Z}^{n} \rtimes S_{n}$ where $S_{n}$ acts on $\mathbb{Z}^{n}$ by permuting the lattice coordinates.

By Example 1.3 the affine Weyl group $W^{\text {aff }}$ associated to $\mathfrak{s l}_{n}$ is the subgroup of $S_{n}^{\text {aff }}$ consisting of those elements whose lattice coordinates sum up to 0 . By Theorem 1.6 and Example 1.9 the group $W^{\text {aff }}$ can also be described as the Coxeter group with Coxeter diagram


To obtain a similar description of the affine symmetric group $S_{n}^{\text {aff }}$ we have to extend the Coxeter group $W^{\text {aff }}$ by some diagram automorphisms. To be more precise, let $G$ be the group with generators $s_{0}, s_{1}, \ldots, s_{n-1}$ and $\tau$ with the relations from the Coxeter group $W^{\text {aff }}$ above together with $\tau s_{\bar{i}} \tau^{-1}=s_{\bar{i}-1}$ for $i \in \mathbb{Z}$ where $\bar{i} \in\{0, \ldots, n-1\}$ s.t. $\bar{i} \equiv i \bmod n$. Let $\epsilon_{1}, \ldots, \epsilon_{n}$ be the standard generators of the abelian group $\mathbb{Z}^{n}$.

Lemma 1.19. There is an isomorphism of groups

$$
\begin{aligned}
\varphi: G & \xrightarrow{\sim} S_{n}^{\text {aff }} \\
s_{i} & \mapsto s_{i} \text { for } i=1, \ldots, n-1 \\
s_{0} & \mapsto s_{n-1} \cdots s_{2} s_{1} s_{2} \cdots s_{n-1} \epsilon_{1} \epsilon_{n}^{-1} \\
\tau & \mapsto s_{n-1} \cdots s_{2} s_{1} \epsilon_{1} .
\end{aligned}
$$

Proof. See also MS19, Lemma 2.1]. Let

$$
\begin{aligned}
& \psi: S_{n}^{\text {aff }} \rightarrow G \\
& \rightarrow G \\
& s_{i} \mapsto s_{i} \text { for } i=1, \ldots, n-1 \\
& \epsilon_{1} \mapsto s_{1} s_{2} \cdots s_{n-1} \tau .
\end{aligned}
$$

One can show by direct computations that $\varphi$ and $\psi$ are well-defined mutually inverse group homomorphisms.

For any field $K$, the Weyl group $S_{n}$ of $\mathfrak{g l}_{n}$ can be embedded into $\mathrm{GL}_{n}(K)$ by identifying an element $\sigma \in S_{n}$ with the corresponding permutation matrix $P_{\sigma} \in$ $\mathrm{GL}_{n}(K)$. This can be extended to an embedding of the affine symmetric group as follows (at least if $K$ contains an element of infinite multiplicative order).

Lemma 1.20. For any $x \in K^{\times}$, there is a group homomorphism

$$
\begin{aligned}
\varphi: S_{n}^{\text {aff }} & \longrightarrow \mathrm{GL}_{n}(\mathbb{C}) \\
S_{n} \ni \sigma & \longmapsto P_{\sigma} \\
\mathbb{Z}^{n} \ni\left(a_{1}, \ldots, a_{n}\right) & \longmapsto\left(\begin{array}{cccc}
x^{a_{1}} & 0 & \cdots & 0 \\
0 & x^{a_{2}} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & x^{a_{n}}
\end{array}\right) .
\end{aligned}
$$

The map $\varphi$ is injective if and only if $x$ is of infinite multiplicative order.
Proof. The assignments above define group homomorphisms $S_{n} \hookrightarrow \mathrm{GL}_{n}(K)$ and $\mathbb{Z}^{n} \rightarrow \mathrm{GL}_{n}(K)$. The conjugation action of $S_{n} \subset \mathrm{GL}_{n}(K)$ on the set of diagonal matrices is the action that permutes the diagonal entries. This shows that $\varphi$ respects the semidirect product structure of $S_{n}^{\text {aff }}$ and hence $\varphi$ is a well-defined group homomorphism. Note that $\operatorname{ker}(\varphi)=\left\{\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}^{n} \mid x^{a_{i}}=1\right.$ for $\left.i=1, \ldots, n\right\}$. This proves the injectivity claim.

Remark 1.21. Using the embedding $S_{n} \hookrightarrow \mathrm{GL}_{n}(\mathbb{C})$, the well-known Bruhat decomposition for $\mathrm{GL}_{n}(\mathbb{C})$ states that

$$
\mathrm{GL}_{n}(\mathbb{C})=\bigsqcup_{w \in S_{n}} B w B
$$

where $B=\left\{A \in \mathrm{GL}_{n}(\mathbb{C}) \mid a_{i j}=0\right.$ for $\left.i>j\right\}$ is the standard Borel (details on this can be found in Hum12b, §28]). There is a similar result for the group homomorphism $S_{n}^{\text {aff }} \rightarrow \mathrm{GL}_{n}(K)$ from Lemma 1.20 . For this, let $K$ be a non-Archimedean local field and $x=\pi$ a uniformiser of the ring of integers $\mathcal{O} \subset K$. Let

$$
I:=\left\{A \in \mathrm{GL}_{n}(\mathcal{O}) \mid a_{i j} \in(\pi) \subset \mathcal{O} \text { for } i>j\right\}
$$

which is called an Iwahori subgroup of $\mathrm{GL}_{n}(K)$. Then there is an Iwahori-Bruhat decomposition

$$
\operatorname{GL}_{n}(K)=\bigsqcup_{w \in S_{n}^{\text {aff }}} I w I .
$$

This explains why $S_{n}^{\text {aff }}$ is the natural candidate for an affine Weyl group of $\mathrm{GL}_{n}$ or $\mathfrak{g l}_{n}$. For more details, see [IM65].

Next, we define the finite and the affine Hecke algebra associated $\mathfrak{g l}_{n}$. The finite Hecke algebra associated to $\mathfrak{g l}_{n}$ is the same as the finite Hecke algebra associated to $\mathfrak{s l}_{n}$, namely $\mathcal{H}\left(S_{n}\right)$. We construct the affine Hecke algebra of $\mathfrak{g l}_{n}$ by associating a Hecke algebra to the affine symmetric group $S_{n}^{\text {aff }}$. For this we extend the Hecke algebra of the affine Weyl group $W^{\text {aff }}$ of $\mathfrak{s l}_{n}$ by a diagram automorphism as in Lemma 1.19 ,

Definition 1.22. The (affine) Hecke algebra $\mathcal{H}\left(S_{n}^{\text {aff }}\right)$ associated to $S_{n}^{\text {aff }}$ is the $\mathbb{C}(q)$ algebra with generators $H_{s_{0}}, H_{s_{1}} \ldots, H_{s_{n-1}}$ and $H_{\tau}^{ \pm 1}$ subject to the relations
(AH1) $H_{s_{\bar{i}}}^{2}=1+\left(q^{-1}-q\right) H_{s_{\bar{i}}} \quad$ for $i \in \mathbb{Z}$
(AH2) $H_{s_{\bar{i}}} H_{s_{\overline{i+1}}} H_{s_{\bar{i}}}=H_{s_{\bar{i}+1}} H_{s_{\bar{i}}} H_{s_{\bar{i}+1}}$
(AH3) $H_{s_{\bar{i}}} H_{s_{\bar{j}}}=H_{s_{\bar{j}}} H_{s_{\bar{i}}}$
(AH4) $H_{\tau} H_{s_{\bar{i}}}=H_{s_{\bar{i}-1}} H_{\tau}$
for $i \in \mathbb{Z}$
for $i, j \in \mathbb{Z}$ with $\overline{i-j} \neq 0,1, n-1$ for $i \in \mathbb{Z}$
(AH5) $H_{\tau} H_{\tau}^{-1}=1=H_{\tau}^{-1} H_{\tau}$.
There is another presentation of $\mathcal{H}\left(S_{n}^{\text {aff }}\right)$ which is more similar to the presentation of $S_{n}^{\text {aff }}$ as a semidirect product $\mathbb{Z}^{n} \rtimes S_{n}$. Let $\mathcal{B}$ be the $\mathbb{C}(q)$-algebra with generators $H_{1}, \ldots, H_{n-1}$ and $X_{1}^{ \pm 1}, \ldots, X_{n}^{ \pm 1}$ subject to the relations
(BER1) (i) $H_{i}^{2}=1+\left(q^{-1}-q\right) H_{i} \quad$ for $i=1, \ldots, n-1$
(ii) $H_{i} H_{i+1} H_{i}=H_{i+1} H_{i} H_{i+1} \quad$ for $i=1, \ldots, n-2$
(iii) $H_{i} H_{j}=H_{j} H_{i} \quad$ for $|i-j|>1$
(BER2)
(i) $X_{i} X_{i}^{-1}=1=X_{i}^{-1} X_{i} \quad$ for $i=1, \ldots, n$
(ii) $X_{i} X_{j}=X_{j} X_{i}$
for $i, j=1, \ldots, n$
(BER3)
(i) $H_{i} X_{i} H_{i}=X_{i+1}$
for $i=1, \ldots, n-1$
(ii) $H_{i} X_{j}=X_{j} H_{i}$
for $j \neq i, i+1$.

Proposition 1.23. There is an isomorphism

$$
\begin{aligned}
\mathcal{H}\left(S_{n}^{\text {aff }}\right) & \stackrel{\sim}{\longrightarrow} \\
H_{s_{i}} & \longmapsto H_{i} \text { for } i=1, \ldots, n-1 \\
H_{s_{0}} & \longmapsto X_{1}^{-1} X_{n}\left(H_{n-1} \cdots H_{2} H_{1} H_{2} \cdots H_{n-1}\right)^{-1} \\
H_{\tau} & \longmapsto H_{n-1} \cdots H_{2} H_{1} X_{1} .
\end{aligned}
$$

Proof. This is a rescaled version of MS19, Lemma 3.2].
The defining presentation of $\mathcal{B}$ is often called the Bernstein presentation. There is a canonical algebra homomorphism

$$
\begin{aligned}
\mathcal{H}\left(S_{n}\right) \longrightarrow \mathcal{B} \cong \mathcal{H}\left(S_{n}^{\mathrm{aff}}\right) \\
H_{s_{i}} \longmapsto H_{i}
\end{aligned}
$$

Using this, we can talk about the elements $H_{w} \in \mathcal{H}\left(S_{n}^{\text {aff }}\right)$ for any $w \in S_{n}$. For any $\underline{a}=\left(a_{1}, . ., a_{n}\right) \in \mathbb{Z}^{n}$ let $X^{\underline{a}}=X_{1}^{a_{1}} \cdot \ldots \cdot X_{n}^{a_{n}}$. Moreover, let $\mathbb{Z}^{n} \curvearrowleft S_{n}$ act by permuting the coordinates. Then by [Lus89, Prop. 3.6], the following holds.

Lemma 1.24. For any $\underline{a} \in \mathbb{Z}^{n}$ we have

$$
H_{i} X^{\underline{a}}-X^{\underline{a} \cdot s_{i}} H_{i} \in \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{n}^{ \pm 1}\right]
$$

Proof. For $\underline{a}, \underline{b} \in \mathbb{Z}^{n}$, we have

$$
H_{i} X^{\underline{a}+\underline{b}}-X^{(\underline{a}+\underline{b}) \cdot s_{i}} H_{i}=\left(H_{i} X^{\underline{a}}-X^{\underline{a} \cdot s_{i}} H_{i}\right) X^{\underline{b}}+X^{\underline{a} \cdot s_{i}}\left(H_{i} X^{\underline{b}}-X^{\underline{b} \cdot s_{i}} H_{i}\right)
$$

Hence, the claim follows by induction if we can show the claim for $X^{\underline{a}}=X_{j}^{ \pm 1}$. For $j \neq i, i+1$ the elements $H_{i}$ and $X_{j}$ commute and the claim is obvious. For $X^{\underline{a}}=X_{i}$ this follows from the Bernstein relations:

$$
\begin{aligned}
H_{i} X_{i}-X_{i+1} H_{i} & =X_{i+1} H_{i}^{-1}-X_{i+1} H_{i} \\
& =X_{i+1}\left(H_{i}+\left(q-q^{-1}\right)\right)-H_{i} X_{i+1} \\
& =\left(q-q^{-1}\right) X_{i+1}
\end{aligned}
$$

The $X^{\underline{a}}=X_{i}^{-1}$ case also follows from this. In fact, multiplying with $X_{i+1}^{-1}$ on the left and with $-X_{i}^{-1}$ on the right in the equation above yields $H_{i} X_{i}^{-1}-X_{i+1}^{-1} H_{i}=$ $\left(q^{-1}-q\right) X_{i}^{-1}$. Now using

$$
\begin{aligned}
H_{i} X_{i+1}-X_{i} H_{i} & =H_{i} X_{i+1}-H_{i}^{-1} X_{i+1} \\
& =H_{i} X_{i+1}-\left(H_{i}+\left(q-q^{-1}\right)\right) X_{i+1} \\
& =\left(q^{-1}-q\right) X_{i+1}
\end{aligned}
$$

the $X^{\underline{a}}=X_{i+1}^{ \pm 1}$ case follows similarly.
Lemma 1.24 implies that the set $\left\{H_{w} X^{\underline{a}} \mid w \in S_{n}, \underline{a} \in \mathbb{Z}^{n}\right\}$ spans $\mathcal{H}\left(S_{n}^{\text {aff }}\right)$ as a $\mathbb{C}(q)$-vector space. In fact, this is even a basis.

Proposition 1.25. The set $\left\{H_{w} X^{\underline{a}} \mid w \in S_{n}, \underline{a} \in \mathbb{Z}^{n}\right\}$ is a $\mathbb{C}(q)$-basis of $\mathcal{H}\left(S_{n}^{\text {aff }}\right)$.
Proof. This is also shown in Lus89, Prop. 3.7]. Assume there is a linear dependence

$$
\sum_{w \in S_{n}, \underline{a} \in \mathbb{Z}^{n}} p_{w, \underline{a}}(q) \cdot H_{w} X^{\underline{a}}=0
$$

with $p_{w, \underline{a}}(q) \in \mathbb{C}(q)^{\times}$not all 0 . After multiplying with some element of $\mathbb{C}(q)^{\times}$and dividing by some power of $q-1$, we may assume that $p_{w, \underline{a}}(q) \in \mathbb{C}\left[q, q^{-1}\right]$ and $p_{w, \underline{a}}(1)$ not all 0 . Let $\mathcal{B}_{\mathbb{C}\left[q, q^{-1}\right]}$ be the $\mathbb{C}\left[q, q^{-1}\right]$-subalgebra of $\mathcal{B}$ generated by the $H_{i}$ and the $X_{i}$. Let $\varphi: \mathcal{B}_{\mathbb{C}\left[q, q^{-1}\right]} \rightarrow \mathbb{C}\left[S_{n}^{\text {aff }}\right]$ be the evaluation at $q=1$ with $\varphi\left(H_{i}\right)=s_{i}$ and $\varphi\left(X_{i}\right)=\epsilon_{i}$. The $\mathbb{C}$-algebra homomorphism $\varphi$ is well-defined since it is compatible with the relations (BER1)-(BER3) which are defined over $\mathbb{C}\left[q, q^{-1}\right]$. We get

$$
0=\varphi\left(\sum_{w \in S_{n}, \underline{a} \in \mathbb{Z}^{n}} p_{w, \underline{a}}(q) \cdot H_{w} X^{\underline{a}}\right)=\sum_{w \in S_{n}, \underline{a} \in \mathbb{Z}^{n}} p_{w, \underline{a}}(1) \cdot w \epsilon^{\underline{a}} .
$$

This is a contradiction since the $w \epsilon^{\underline{a}}$ form a basis of $\mathbb{C}\left[S_{n}^{\text {aff }}\right]$. Hence, the $H_{w} X^{\underline{a}}$ are linearly independent.

## 2 Quantum and affine versions of Schur-Weyl duality

In this section we want to look at affine and quantum generalisations of classical type A Schur-Weyl duality. Schur-Weyl dualities are concerned with commuting algebra actions $A \curvearrowright W \curvearrowleft B$ on a vector space $W$ that generate each other's centraliser (meaning that the canonical algebra homomorphisms $A \rightarrow \operatorname{End}_{B}(W)$ and $B \rightarrow \operatorname{End}_{A}(W)^{\mathrm{op}}$ are surjective). Very often this will induce an equivalence between certain subcategories of $A$-mod and $B$-mod. We will explain four such dualities in this section (classical, quantum, affine and quantum affine). The strategy will be the same in all four cases:

Step 1: Construct commuting actions $A \curvearrowright W \curvearrowleft B$.
Step 2: Show that $\operatorname{End}_{B}(W) \curvearrowright W \curvearrowleft B$ generate each other's centraliser.
Step 3: Show that the induced map $A \rightarrow \operatorname{End}_{B}(W)$ is surjective.
Together this implies a double centraliser property for the actions $A \curvearrowright W \curvearrowleft B$. Surprisingly, Step 2 is the easiest and follows from a general argument which exploits the structure of the algebras $\operatorname{End}_{B}(W)$ called Schur algebras. Step 2 also follows
from the double centraliser theorem if $B$ and $W$ are finite-dimensional and $B$ is semisimple but this will not always be the case. The real work has to be done in Step 1 and Step 3. This will usually boil down to a few explicit calculations.
The Schur-Weyl dualities from this section go back to [Jim86] in the quantum case and to CP96 and Gre97 in the (quantum) affine case. However, some of our proofs will be new (or at least a modification of the arguments in the sources above) since we want to avoid using more complicated machinery. In particular, all our proofs are self-contained and mostly rely on a few elementary calculations.

### 2.1 Classical Schur-Weyl duality

Before we look at the quantum or affine setting, let us briefly recall classical SchurWeyl duality. For $V=\mathbb{C}^{n}$, we can equip $V^{\otimes r}$ with the structure of a left $\mathfrak{s l}_{n}$-module and the structure of a right $S_{r}$-module (by permuting the tensor factors). SchurWeyl duality then states the following.

Theorem 2.1. The commuting actions $\mathfrak{s l}_{n} \curvearrowright V^{\otimes r} \curvearrowleft S_{r}$ generate each other's centraliser.

Here the action of $\mathfrak{s l} n$ can also be replaced by $\mathrm{SL}_{n}(\mathbb{C}), \mathrm{GL}_{n}(\mathbb{C})$ or $\mathfrak{g l}_{n}$ and the statement still holds. Let us briefly outline the standard proof of classical SchurWeyl duality. Details can be found in [EGH ${ }^{+11}$, Section 5.18]. The main ingredient is the double centraliser theorem.

Theorem 2.2 (Double Centraliser Theorem). Let $K$ be a field and let $E$ be a finite-dimensional $K$-vector space. For any semisimple $K$-algebra $A \subset \operatorname{End}_{K}(E)$ and $B:=\operatorname{End}_{A}(E)$, the following hold:

1. The algebra $B$ is semisimple;
2. $\operatorname{End}_{B}(E)=A$;
3. $E$ decomposes as a $(A, B)$-bimodule into

$$
E \cong \bigoplus V_{i} \otimes W_{i}
$$

where the $V_{i}$ are the irreducible left $A$-modules and the $W_{i}$ are the irreducible right $B$-modules.

In light of this, it seems reasonable to look at $\operatorname{End}_{S_{r}}\left(V^{\otimes r}\right)$ in more detail.
Definition 2.3. The algebra $S(n, r):=\operatorname{End}_{S_{r}}\left(V^{\otimes r}\right)$ is called the Schur algebra.
Remark 2.4. We consider the Schur algebra $S(n, r)$ as an endomorphism algebra but it can also be constructed by dualising the coalgebra of degree $r$ homogeneous polynomials in $n$ variables (see [Gre06, Section 2]). For a more complete description of the Schur algebra which arises from viewing $\mathrm{GL}_{n}(\mathbb{C})$ as an algebraic group, we refer to Don86.

Since $\mathbb{C}\left[S_{n}\right]$ is semisimple, the double centraliser theorem tells us that the commuting actions

$$
S(n, r) \curvearrowright V^{\otimes r} \curvearrowleft S_{r}
$$

generate each other's centraliser. Classical Schur-Weyl duality then follows if one can show that the algebra homomorphism $U\left(\mathfrak{s l}_{n}\right) \rightarrow S(n, r)$ induced by the $\mathfrak{s l}_{n}$ action is surjective. This can be done by direct computation. The double centraliser theorem also establishes a link between the representation theories of $\mathfrak{s l}_{n}$ and $S_{r}$.

Corollary 2.5. There is a bijection
$\left\{\begin{array}{c}\text { iso. classes of } \\ \text { irreducible } \mathfrak{s l}_{n} \text {-representations } \\ \text { appearing in } V^{\otimes r}\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}\text { iso. classes of } \\ \text { irreducible } S_{r} \text {-representations } \\ \text { appearing in } V^{\otimes r}\end{array}\right\}$.
For the rest of this section, we look at a slightly different approach to Schur-Weyl duality which relies on the structure of the Schur algebra $S(n, r)$ but avoids finitedimensionality and semisimplicity arguments as in the double centraliser theorem. This will be useful when we prove generalisations of classical Schur-Weyl duality.

Definition 2.6. For $n, r \in \mathbb{N}$ a composition of $r$ with $n$ parts is a sequence $\lambda=$ $\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right) \in \mathbb{N}_{0}^{n}$ such that $\lambda_{1}+\lambda_{2}+\ldots+\lambda_{n}=r$. We denote the set of all such sequences by $\Lambda(n, r)$ and $l(\lambda):=(\underbrace{1, \ldots, 1}_{\lambda_{1}}, \ldots, \underbrace{n, \ldots, n}_{\lambda_{n}}) \in\{1, \ldots, n\}^{r}$ for any $\lambda \in \Lambda(n, r)$.

For $\lambda \in \Lambda(n, r)$ and $V=\mathbb{C}^{n}$ (with standard basis $\left.v_{1}, \ldots, v_{n}\right)$ let

$$
\left(V^{\otimes r}\right)_{\lambda}:=\operatorname{Span}_{\mathbb{C}}\left\{v_{\underline{i}}=v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \mid \#\left\{l \mid i_{l}=k\right\}=\lambda_{k}\right\}
$$

This is the $\left(\lambda_{1}-\lambda_{2}\right) h_{1}^{*}+\left(\lambda_{2}-\lambda_{3}\right) h_{2}^{*}+\ldots+\left(\lambda_{n-1}-\lambda_{n}\right) h_{n-1}^{*}$-weight space for the $\mathfrak{s l}_{n}(\mathbb{C})$-action where $h_{1}, \ldots, h_{n-1}$ are the standard basis elements of the diagonal Cartan subalgebra of $\mathfrak{s l}_{n}$. In other words,

$$
\left(V^{\otimes r}\right)_{\lambda}=\left\{v \in V^{\otimes r} \mid h_{i} \cdot v=\left(\lambda_{i}-\lambda_{i+1}\right) \cdot v\right\}
$$

and we have the weight space decomposition $V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)}\left(V^{\otimes r}\right)_{\lambda}$. Since the $\mathfrak{s l}_{n^{-}}$ action and the $S_{r}$-action on $V^{\otimes r}$ commute, the weight spaces are preserved by the $S_{r}$-action. Hence, $\left(V^{\otimes r}\right)_{\lambda}$ is a (right) $S_{r}$-submodule of $V^{\otimes r}$. There is a distinguished element

$$
v_{l(\lambda)}:=v_{1}^{\otimes \lambda_{1}} \otimes v_{2}^{\otimes \lambda_{2}} \otimes \ldots \otimes v_{n}^{\otimes \lambda_{n}} \in\left(V^{\otimes r}\right)_{\lambda}
$$

The element $v_{l(\lambda)}$ is stabilised by the Young subgroup

$$
S_{\lambda}:=S_{\lambda_{1}} \times S_{\lambda_{2}} \times \ldots \times S_{\lambda_{r}}
$$

and $v_{l(\lambda)}$ generates the weight space $\left(V^{\otimes r}\right)_{\lambda}$ as an $S_{r}$-module. By Frobenius reciprocity we get a surjective $S_{r}$-homomorphism

$$
\begin{aligned}
\mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}} & \longrightarrow\left(V^{\otimes r}\right)_{\lambda} \\
1 \otimes 1 & \longmapsto v_{l(\lambda)}
\end{aligned}
$$

where $\mathbf{1}_{\lambda}$ is the trivial $S_{\lambda}$ representation and $\mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}}=\mathbf{1}_{\lambda} \otimes_{\mathbb{C}\left[S_{\lambda}\right]} \mathbb{C}\left[S_{r}\right]$ denotes induction. In fact, the map above is an isomorphism since both vector spaces have dimension $\frac{\left|S_{r}\right|}{\left|S_{\lambda}\right|}$. Hence, there is an isomorphism of $S_{r}$-representations

$$
\begin{gather*}
\bigoplus_{\lambda \in \Lambda(n, r)} \mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}} \xrightarrow{\sim} V^{\otimes r}  \tag{6}\\
\mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}} \ni 1 \otimes \sigma \longmapsto v_{l(\lambda) \cdot \sigma} .
\end{gather*}
$$

There is another way we can realise a weight space as an $S_{r}$-module. Let

$$
e_{\lambda}:=\sum_{\sigma \in S_{\lambda}} \sigma
$$

Then $\mathbb{C} e_{\lambda} \cong \mathbf{1}_{\lambda}$ as $S_{\lambda}$-representations. By Frobenius reciprocity, there is an induced homomorphism of $S_{r}$-representations

$$
\begin{equation*}
\left(V^{\otimes r}\right)_{\lambda} \cong \mathbf{1}_{\lambda} \uparrow \uparrow_{S_{\lambda}}^{S_{r}} \xrightarrow{\sim} e_{\lambda} \mathbb{C}\left[S_{r}\right] \tag{7}
\end{equation*}
$$

and a dimension count shows that this in fact an isomorphism. Note that

$$
e_{\lambda}^{2}=\left(\sum_{\sigma \in S_{\lambda}} \sigma\right)^{2}=\sum_{\sigma, \tau \in S_{\lambda}} \sigma \tau=\left|S_{\lambda}\right| e_{\lambda}
$$

Lemma 2.7. Let $K$ be field, $A$ a $K$-algebra, $M$ a right $A$-module and $N$ a left $A$ module. Let $e \in A$ such that $e^{2}=c e$ for some $c \in K^{\times}$. Then there are isomorphisms of $K$-vector spaces

$$
\begin{array}{ll}
M e \xrightarrow{\sim} \operatorname{Hom}_{A}(e A, M) & e N \xrightarrow{\sim} \operatorname{Hom}_{A}(A e, N) \\
m e \longmapsto(e a \mapsto(m e) \cdot(e a)) & e n \longmapsto(a e \mapsto(a e) \cdot(e n)) .
\end{array}
$$

Proof. The assignment $\varphi \mapsto \frac{1}{c} \varphi(e)$ is an inverse to the maps above.
By Lemma 2.7 there is an isomorphism of vector spaces

$$
\begin{align*}
S(n, r) & \stackrel{(77}{=} \operatorname{End}_{S_{r}}\left(\bigoplus_{\lambda \in \Lambda(n, r)} e_{\lambda} \mathbb{C}\left[S_{r}\right]\right) \\
& \cong \bigoplus_{\lambda, \mu \in \Lambda(n, r)} \operatorname{Hom}_{S_{r}}\left(e_{\lambda} \mathbb{C}\left[S_{r}\right], e_{\mu} \mathbb{C}\left[S_{r}\right]\right)  \tag{8}\\
& \cong \bigoplus_{\lambda, \mu \in \Lambda(n, r)} e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}
\end{align*}
$$

For any $\sigma \in S_{r}$ we have

$$
e_{\mu} \sigma e_{\lambda}=e_{\mu} \cdot \sum_{\tau \in \sigma S_{\lambda}} \tau=\left|\sigma S_{\lambda} \sigma^{-1} \cap S_{\mu}\right| \cdot\left(\sum_{\tau \in S_{\mu} \sigma S_{\lambda}} \tau\right)
$$

We see that $\left\{e_{\mu} y e_{\lambda} \mid y \in{ }^{\mu} S_{r}^{\lambda}\right\}$ is a basis of $e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}$ where ${ }^{\mu} S_{r}^{\lambda}$ is the set of (shortest) double coset representatives of $S_{\mu} \backslash S_{r} / S_{\lambda}$. We can use this to count the dimension of the Schur algebra. In fact,

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}}(S(n, r))=\sum_{\lambda, \mu \in \Lambda(n, r)} \operatorname{dim}_{\mathbb{C}}\left(e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}\right)=\sum_{\lambda, \mu \in \Lambda(n, r)}\left|S_{\mu} \backslash S_{r} / S_{\lambda}\right| \tag{9}
\end{equation*}
$$

Let us study the Schur algebra and its idempotents in more detail using the isomorphism from (8). For $x \in e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}$, let $\phi_{\mu, \lambda}^{x} \in \operatorname{Hom}_{S_{r}}\left(e_{\lambda} \mathbb{C}\left[S_{r}\right], e_{\mu} \mathbb{C}\left[S_{r}\right]\right)$ be the homomorphism induced by left multiplication with $x$. We can consider this as an element of $S(n, r)$ via the isomorphism $V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} e_{\lambda} \mathbb{C}\left[S_{r}\right]$ (so that $\phi_{\mu, \lambda}^{x}$ acts by 0 on $e_{\lambda^{\prime}} \mathbb{C}\left[S_{r}\right]$ for $\left.\lambda^{\prime} \neq \lambda\right)$. Then $\phi_{\mu, \lambda}^{x}$ is the element corresponding to $x$ under the isomorphism from (8). Note that

$$
\phi_{\mu, \lambda}^{x} \cdot \phi_{\mu^{\prime}, \lambda^{\prime}}^{x^{\prime}}= \begin{cases}\phi_{\mu, \lambda^{\prime}}^{x x^{\prime}} & \text { if } \lambda=\mu^{\prime}  \tag{10}\\ 0 & \text { otherwise }\end{cases}
$$

In particular, $\frac{1}{\left|S_{\lambda}\right|} \phi_{\lambda, \lambda}^{e_{\lambda}}=\phi_{\lambda, \lambda}^{\frac{1}{S_{\lambda}} e_{\lambda}} \in S(n, r)$ an idempotent which is the projection onto the weight space $e_{\lambda} \mathbb{C}\left[S_{r}\right] \cong\left(V^{\otimes r}\right)_{\lambda}$ and $1=\sum_{\lambda \in \Lambda(n, r)} \frac{1}{\left|S_{\lambda}\right|} \phi_{\lambda, \lambda}^{e_{\lambda}}$. Moreover,

$$
\phi_{\mu, \mu}^{e_{\mu}} S(n, r) \phi_{\lambda, \lambda}^{e_{\lambda}}=\operatorname{Hom}_{S_{r}}\left(\left(V^{\otimes r}\right)_{\lambda},\left(V^{\otimes r}\right)_{\mu}\right) \stackrel{[2.7}{=}\left\{\phi_{\mu, \lambda}^{x} \mid x \in e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}\right\} .
$$

For $n \geq r$ let

$$
\omega:=(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r}) \in \Lambda(n, r) .
$$

Then $S_{\omega} \subset S_{r}$ is the trivial subgroup and $\phi_{\omega, \omega}^{1} \in S(n, r)$ is an idempotent which is the projection onto $\left(V^{\otimes r}\right)_{\omega}$. By 10 there is an algebra isomorphism

$$
\begin{aligned}
\phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega}^{1}=\left\{\phi_{\omega, \omega}^{x} \mid x \in \mathbb{C}\left[S_{r}\right]\right\} & \xrightarrow{\sim} \mathbb{C}\left[S_{r}\right] \\
\phi_{\omega, \omega}^{x} & \longmapsto x .
\end{aligned}
$$

In particular, we can consider $V^{\otimes r}$ as a right $\phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega}^{1}$-module.
Lemma 2.8. We have $V^{\otimes r} \cong S(n, r) \phi_{\omega, \omega}^{1}$ as $\left(S(n, r), \phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega}^{1}\right)$-bimodules.
Proof. The isomorphism is given by

$$
\begin{aligned}
V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} e_{\lambda} \mathbb{C}\left[S_{r}\right] & \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda(n, r)} \phi_{\lambda, \lambda}^{e_{\lambda}} S(n, r) \phi_{\omega, \omega}^{1} \cong S(n, r) \phi_{\omega, \omega}^{1} \\
e_{\lambda} x & \longmapsto \phi_{\lambda, \omega}^{e_{\lambda} x} .
\end{aligned}
$$

The formulas from 10 show that this is both $S(n, r)$-linear and $\phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega^{-}}^{1}$ linear, i.e. an isomorphism of bimodules.

This can be used to give a proof of the double centraliser property without using semisimplicity.

Corollary 2.9. For $n \geq r$ the commuting action $S(n, r) \curvearrowright V^{\otimes r} \curvearrowleft S_{r}$ generate each other's centraliser.

Proof. By definition, we have $S(n, r)=\operatorname{End}_{S_{r}}\left(V^{\otimes r}\right)$. We need to show that the homomorphism $\mathbb{C}\left[S_{r}\right] \rightarrow \operatorname{End}_{S(n, r)}\left(V^{\otimes r}\right)^{\text {op }}$ is surjective. By Lemma 2.8 this is equivalent to showing that the homomorphism $\phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega}^{1} \rightarrow \operatorname{End}_{S(n, r)}\left(S(n, r) \phi_{\omega, \omega}^{1}\right)^{\text {op }}$ is surjective. Lemma 2.7 actually shows that this map is an isomorphism.

The representation theoretic correspondence from Corollary 2.5 can also be proved using the structure of the Schur algebra. Let us first explain this approach in a more general framework.

Let $K$ be a field and let $A$ be a $K$-algebra which is not necessarily finite-dimensional (but unital).

Definition 2.10. We denote by $A$-Mod the category of (left) $A$-modules and by $A$-mod the full subcategory consisting of the finite-dimensional (left) $A$-modules.

For any idempotent $e \in A$ we can consider the algebra $e A e$ with unit element $e$. There are functors

$$
\begin{align*}
F: A-\operatorname{Mod} & \longrightarrow e A e-\operatorname{Mod} \\
M & \longmapsto e M \\
G: e A e-\operatorname{Mod} & \longrightarrow A \text {-Mod }  \tag{11}\\
N & \longmapsto A e \otimes_{e A e} N .
\end{align*}
$$

There is an isomorphism of $e A e$-modules $e A e \otimes_{e A e} N \cong N$ which is natural in $N$ and hence $F \circ G \cong \mathrm{id}_{e A e-\mathrm{Mod}}$. On the other hand, there is a natural transformation $\eta: G \circ F \rightarrow \operatorname{id}_{A \text {-Mod }}$ with

$$
\begin{aligned}
\eta_{M}: A e \otimes_{e A e} e M & \longrightarrow M \\
a e \otimes e m & \longmapsto \text { aem. }
\end{aligned}
$$

Note, however, that $\eta_{M}$ is not an isomorphism in general (for instance, $\eta_{M}=0$ if $e M=0)$. Still, we can make the following observation.

Lemma 2.11. We have $e \cdot \operatorname{ker}\left(\eta_{M}\right)=0$. Moreover, $\eta_{M}$ is surjective if $A e M=M$.
Proof. Since $\eta_{M}$ restricts to an isomorphism of $e A e$-modules $e \cdot\left(A e \otimes_{e A e} e M\right) \xrightarrow{\sim} e M$, we get $e \cdot \operatorname{ker}\left(\eta_{M}\right)=0$. Moreover, $\operatorname{im}\left(\eta_{M}\right)=A e \cdot e M=A e M$ which implies the surjectivity claim.

Corollary 2.12. If $A e A=A$, then $\eta_{M}$ is an isomorphism for all $A$-modules $M$. In particular, $G \circ F \cong \mathrm{id}_{A-\mathrm{Mod}}$ and $G$ and $F$ are equivalences of categories. Moreover, if $G$ sends finite-dimensional $e A e$-modules to finite-dimensional $A$-modules, this descends to an equivalence $A-\bmod \cong e A e-\bmod$.

Proof. We have

$$
A e M=A e A M=A M=M
$$

so $\eta_{M}$ is surjective by Lemma 2.11. Note that

$$
\operatorname{ker}\left(\eta_{M}\right)=A \operatorname{ker}\left(\eta_{M}\right)=A e A \operatorname{ker}\left(\eta_{M}\right)=A e \operatorname{ker}\left(\eta_{M}\right)=0
$$

Hence, $\eta_{M}$ is an isomorphism. This shows $G \circ F \cong \mathrm{id}_{A \text {-Mod }}$. We have already seen that $F \circ G \cong \mathrm{id}_{e A e-M o d}$ proving that $F$ and $G$ are equivalences of categories. $F$ clearly preserves finite-dimensionality, so the equivalence descends to $A$ - mod $\cong e A e-\bmod$ if $G$ preserves finite-dimensionality.

We want to apply Corollary 2.12 to the Schur algebra. For this, we claim that $S(n, r) \phi_{\omega, \omega}^{1} S(n, r)=S(n, r)$. In fact, this follows from $1=\sum_{\lambda \in \Lambda(n, r)} \frac{1}{\left|S_{\lambda}\right|} \phi_{\lambda, \lambda}^{e_{\lambda}}$ and

$$
\begin{equation*}
\left|S_{\lambda}\right| \phi_{\lambda, \lambda}^{e_{\lambda}}=\left(\phi_{\lambda, \lambda}^{e_{\lambda}}\right)^{2}=\phi_{\lambda, \omega}^{e_{\lambda}} \phi_{\omega, \omega}^{1} \phi_{\omega, \lambda}^{e_{\lambda}} \in S(n, r) \phi_{\omega, \omega}^{1} S(n, r) \tag{12}
\end{equation*}
$$

for any $\lambda \in \Lambda(n, r)$. Moreover, we have

$$
V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]}(-) \cong S(n, r) \phi_{\omega, \omega}^{1} \otimes_{\phi_{\omega, \omega}^{1} S(n, r) \phi_{\omega, \omega}^{1}}(-)
$$

as functors from $\mathbb{C}\left[S_{r}\right]$-mod to $S(n, r)$-mod by Lemma 2.8 . Applying Corollary 2.12 yields the following.

Proposition 2.13. The functor

$$
\begin{aligned}
\mathbb{C}\left[S_{r}\right]-\bmod & \longrightarrow S(n, r)-\bmod \\
M & V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]} M
\end{aligned}
$$

is an equivalence of categories.

### 2.2 Jimbo's quantum Schur-Weyl duality

We have already encountered the Hecke algebra as a deformation of the group algebra. There is a similar construction for the universal enveloping algebra of a semisimple Lie algebra (but recovering the original algebra in this case is more involved than just specialising the parameter $q \rightarrow 1$ ). These are called quantum groups. In this section we define quantum $\mathfrak{s l}_{n}$ and explain its connection with the Hecke algebra. Recall the Serre presentation of the semisimple Lie algebra $\mathfrak{s l}_{n}$ (see for example Hum12a, Thm. 18.3] for a proof).

Theorem 2.14. The Lie algebra $\mathfrak{s l}_{n}$ is isomorphic to the Lie algebra with generators $e_{i}, f_{i}, h_{i} \quad(1 \leq i \leq n-1)$ and relations
(S1) $\left[h_{i}, h_{j}\right]=0$
(S2) $\left[h_{i}, e_{j}\right]=a_{i j} e_{j}$ and $\left[h_{i}, f_{j}\right]=-a_{i j} f_{j}$
(S3) $\left[e_{i}, f_{j}\right]=\delta_{i, j} h_{i}$
(S4) $\operatorname{ad}\left(e_{i}\right)^{1-a_{i j}}\left(e_{j}\right)=0$ and $\operatorname{ad}\left(f_{i}\right)^{1-a_{i j}}\left(f_{j}\right)=0$ for $i \neq j$
where $A=\left(a_{i j}\right)_{1 \leq i, j \leq n-1}=\left(\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle\right)_{1 \leq i, j \leq n-1}$ is the Cartan matrix of $\mathfrak{s l}_{n}$. The relations above are called the Serre relations.

The defining relations of quantum $\mathfrak{s l}_{n}$ are inspired by these relations.
Definition 2.15. The quantum group $U_{q}\left(\mathfrak{s l}_{n}\right)$ is the $\mathbb{C}(q)$-algebra with generators

$$
E_{i}, F_{i}, K_{i}^{ \pm 1} \quad(1 \leq i \leq n-1)
$$

and relations (whenever they make sense)
(i) $K_{i} K_{i}^{-1}=1=K_{i}^{-1} K_{i}$
(ii) $K_{i} K_{j}=K_{j} K_{i}$
(UQ2) (i) $K_{i} E_{j}=q^{a_{i j}} E_{j} K_{i}$
(ii) $K_{i} F_{j}=q^{-a_{i j}} F_{j} K_{i}$

$$
\begin{equation*}
\left[E_{i}, F_{j}\right]=\delta_{i, j} \frac{K_{i}-K_{i}^{-1}}{q-q^{-1}} \tag{UQ3}
\end{equation*}
$$

(i) $E_{i}^{2} E_{i \pm 1}-\left(q+q^{-1}\right) E_{i} E_{i \pm 1} E_{i}+E_{i \pm 1} E_{i}^{2}=0$
(ii) $F_{i}^{2} F_{i \pm 1}-\left(q+q^{-1}\right) F_{i} F_{i \pm 1} F_{i}+F_{i \pm 1} F_{i}^{2}=0$
(iii) $E_{i} E_{j}=E_{j} E_{i}$ if $|i-j|>1$
(iv) $F_{i} F_{j}=F_{j} F_{i}$ if $|i-j|>1$
where $A=\left(a_{i j}\right)_{1 \leq i, j \leq n-1}$ is the Cartan matrix of $\mathfrak{s l}_{n}$.
The quantum group $U_{q}\left(\mathfrak{s l}_{n}\right)$ is a Hopf algebra with compultiplication

$$
\begin{aligned}
\Delta: U_{q}\left(\mathfrak{s l}_{n}\right) & \rightarrow U_{q}\left(\mathfrak{s l}_{n}\right) \otimes_{\mathbb{C}(q)} U_{q}\left(\mathfrak{s l}_{n}\right) \\
E_{i} & \mapsto E_{i} \otimes K_{i}^{-1}+1 \otimes E_{i} \\
F_{i} & \mapsto F_{i} \otimes 1+K_{i} \otimes F_{i} \\
K_{i} & \mapsto K_{i} \otimes K_{i} .
\end{aligned}
$$

Remark 2.16. There are various other normalisations of the comultiplication $\Delta$. We have chosen this specific normalisation since it will behave well with respect to the bar involutions we study in the next section. The other normalisations, which are often used in the literature, can be obtained from our normalisation by passing to the opposite comultiplication or by twisting by an (anti-)automorphism. For example, our comultiplication is obtained from the comultiplictation in Jan96 by twisting with the bar involution introduced in the next section (see also Jan96, (4)]) and then passing to the opposite comultiplication.

Of course, one has to check that $\Delta$ is a well-defined algebra homomorphism (i.e. that this is compatible with (UQ1)-(UQ4)). These are some straightforward calculations which will be omitted here. The comultiplication is also coassociative which is easily checked on the generators. The comultiplication can be used to define tensor products of $U_{q}\left(\mathfrak{s l}_{n}\right)$-representations. There also is a counit and an antipode for $U_{q}\left(\mathfrak{s l}_{n}\right)$ but we do not write these down since we will not use them explicitly. For more details on the construction of quantum groups and their Hopf algebra structure, we refer to Jan96 and Lus10.

Proposition 2.17. The quantum group $U_{q}\left(\mathfrak{s l}_{n}\right)$ has the structure of a Hopf-algebra with comultiplication $\Delta$ defined above.

Proof. This can be found in [Jan96, Prop. 4.11] (but one has to use the twist from Remark 2.16 to get the result in our normalisation).

There also is the notion of a natural representation. Consider the $n$-dimensional vector space $V_{q}=\mathbb{C}(q)^{n}$ with standard basis $v_{1}, \ldots, v_{n}$. Then $V_{q}$ can be given the structure of a $U_{q}\left(\mathfrak{s l}_{n}\right)$-module via

$$
\begin{aligned}
E_{i} \cdot v_{j} & =\delta_{i+1, j} \cdot v_{j-1} \\
F_{i} \cdot v_{j} & =\delta_{i, j} \cdot v_{j+1} \\
K_{i} \cdot v_{j} & =q^{\delta_{i, j}-\delta_{i+1, j}} \cdot v_{j} .
\end{aligned}
$$

Again, one can check by hand that this is compatible with (UQ1)-(UQ4). Using the comultiplication $\Delta$ we can thus act with $U_{q}\left(\mathfrak{s l}_{n}\right)$ on tensor powers of $V_{q}$. We can now ask what the centralising partner of this action is

$$
U_{q}\left(\mathfrak{s l}_{n}\right) \curvearrowright V_{q}^{\otimes r} \curvearrowleft ? .
$$

Since we have replaced the action of $U\left(\mathfrak{s l}_{n}\right)$ used in the classical Schur-Weyl duality by its quantum version $U_{q}\left(\mathfrak{s l}_{n}\right)$, it is natural to take the quantum version of $\mathbb{C}\left[S_{r}\right]$ on the other side as well, that is the Hecke algebra

$$
\mathcal{H}_{r}:=\mathcal{H}\left(S_{r}\right)
$$

To make this work, we need to define a right $\mathcal{H}_{r}$-module structure on $V_{q}^{\otimes r}$. This will be done by 'deforming' the permutation action of $S_{r}$ on tensor space. In the classical case, we were able to decompose our tensor space into weight spaces and hence into a direct sum of induced representation $V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} \mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}}$. Let us investigate what the Hecke analogue of this construction would be. For $V_{q}=\mathbb{C}(q)^{n}$, the space

$$
\left(V_{q}^{\otimes r}\right)_{\lambda}:=\operatorname{Span}_{\mathbb{C}(q)}\left\{v_{\underline{i}}=v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \mid \#\left\{l \mid i_{l}=k\right\}=\lambda_{k}\right\}
$$

is the simultaneous eigenspace for the action of $K_{1}, \ldots, K_{n-1}$ where the eigenvalue of $K_{i}$ is $q^{\lambda_{i}-\lambda_{i+1}}$. We call $\left(V_{q}^{\otimes r}\right)_{\lambda}$ a weight space (of the $U_{q}\left(\mathfrak{s l}_{n}\right)$-module $\left.V_{q}^{\otimes r}\right)$. The
analogue of the trivial $S_{\lambda}$-representation $\mathbf{1}_{\lambda}$ for the algebra $\mathcal{H}_{\lambda}:=\mathcal{H}\left(S_{\lambda}\right)$ is the module $\mathbb{C}(q)_{q^{-1}}$ and when inducing this up to $\mathcal{H}_{r}$, we obtain the parabolic Hecke module

$$
\mathcal{M}^{\lambda}=\mathbb{C}(q)_{q^{-1}} \otimes_{\mathcal{H}_{\lambda}} \mathcal{H}_{r}
$$

Recall from Proposition 1.14 that $\mathcal{M}^{\lambda}$ has the basis $\left\{1 \otimes H_{x} \mid x \in{ }^{\lambda}\left(S_{r}\right)\right\}$ where ${ }^{\lambda}\left(S_{r}\right)$ is the set of shortest right coset representatives of $S_{\lambda}$ in $S_{r}$. So by identifying a basis vector $1 \otimes H_{x} \in \mathcal{M}^{\lambda}$ for $x \in{ }^{\lambda}\left(S_{r}\right)$ with $v_{l(\lambda) \cdot x} \in\left(V_{q}^{\otimes r}\right)_{\lambda}$ we obtain an isomorphism of vector spaces

$$
\begin{equation*}
\Psi_{r}=\oplus_{\lambda \in \Lambda(n, r)} \Psi_{r}^{(\lambda)}: \bigoplus_{\lambda \in \Lambda(n, r)} \mathcal{M}^{\lambda} \xrightarrow{\sim} V_{q}^{\otimes r} \tag{13}
\end{equation*}
$$

Remark 2.18. Note that the identification $\bigoplus_{\lambda \in \Lambda(n, r)} \mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}} \cong V^{\otimes r}$ from (6) takes $1 \otimes \sigma, 1 \otimes \tau \in \mathbf{1}_{\lambda} \uparrow_{S_{\lambda}}^{S_{r}}$ to the same basis vector $v_{l(\lambda) \cdot \sigma}=v_{l(\lambda) \cdot \tau}$ in tensor space if $\sigma$ and $\tau$ lie in the same coset of $S_{\lambda} \backslash S_{r}$. However, for $x \in{ }^{\lambda}\left(S_{r}\right)$ and $y \in S_{\lambda}$, we get

$$
\begin{gathered}
1 \otimes H_{x} \stackrel{\Psi_{r}^{(\lambda)}}{\mapsto} v_{l(\lambda) \cdot x} \\
1 \otimes H_{y x}=q^{-l(y)} \otimes H_{x} \stackrel{\stackrel{\Psi^{(\lambda)}}{\mapsto}}{\mapsto} q^{-l(y)} \cdot v_{l(\lambda) \cdot x}
\end{gathered}
$$

under the identification from (13). This means that in the quantum setting it does actually matter what coset representatives we work with.

There is a (right) $\mathcal{H}_{r}$-module structure on $V_{q}^{\otimes r}$ induced along the identification from (13). By construction, the $H_{r}$-action preserves weight spaces and hence the action commutes with the elements $K_{1}, \ldots, K_{n-1}$ on $V_{q}^{\otimes r}$. However, it is not obvious from the construction how the elements $E_{i}, F_{i} \in U_{q}\left(\mathfrak{s l}_{n}\right)$ interact with the Hecke algebra. To understand this better, we derive explicit formulas for the $H_{r}$-action on tensor space.

Lemma 2.19. For any $k=1, \ldots, r-1$ and $\underline{i}=\left(i_{1}, i_{2}, \ldots, i_{r}\right) \in\{1, \ldots, n\}^{r}$ we have

$$
v_{\underline{i}} \cdot H_{s_{k}}= \begin{cases}v_{\underline{i} \cdot s_{k}} & \text { if } i_{k}<i_{k+1} \\ v_{\underline{i} \cdot s_{k}}+\left(q^{-1}-q\right) v_{\underline{i}} & \text { if } i_{k}>i_{k+1} \\ q^{-1} \cdot v_{\underline{i}} & \text { if } i_{k}=i_{k+1}\end{cases}
$$

Proof. Let $v_{\underline{i}}=v_{l(\lambda) \cdot x}=\Psi_{r}^{(\lambda)}\left(1 \otimes H_{x}\right)$ for some $\lambda \in \Lambda(n, r)$ and $x \in{ }^{\lambda}\left(S_{r}\right)$. Then using the multiplication formulas for parabolic Hecke modules from Proposition 1.14, we obtain

$$
\begin{aligned}
v_{\underline{i}} \cdot H_{s_{k}} & =\Psi_{r}^{(\lambda)}\left(1 \otimes H_{x} H_{s_{k}}\right) \\
& = \begin{cases}\Psi_{r}^{(\lambda)}\left(1 \otimes H_{x s_{k}}\right) & \text { if } x s_{k} \in^{\lambda}\left(S_{r}\right), x s_{k}>x \\
\Psi_{r}^{(\lambda)}\left(1 \otimes H_{x s_{k}}\right)+\left(q^{-1}-q\right) \Psi_{r}^{(\lambda)}\left(1 \otimes H_{x}\right) & \text { if } x s_{k} \in^{\lambda}\left(S_{r}\right), x s_{k}<x \\
q^{-1} \cdot \Psi_{r}^{(\lambda)}\left(1 \otimes H_{x}\right) & \text { if } x s_{k} \nexists^{\lambda}\left(S_{r}\right)\end{cases} \\
& = \begin{cases}v_{\underline{i}} \cdot s_{k} & \text { if } x s_{k} \in^{\lambda}\left(S_{r}\right), x s_{k}>x \\
v_{\underline{i}} \cdot s_{k}+\left(q^{-1}-q\right) v_{\underline{i}} & \text { if } x s_{k} \in^{\lambda}\left(S_{r}\right), x s_{k}<x \\
q^{-1} \cdot v_{\underline{i}} & \text { if } x s_{k} \nexists^{\lambda}\left(S_{r}\right) .\end{cases}
\end{aligned}
$$

The rest of the proof follows from some well-known facts about the (Coxeter) group $S_{n}$ : We have

$$
i_{k}=i_{k+1} \Leftrightarrow l(\lambda) \cdot x s_{k}=l(\lambda) \cdot x \Leftrightarrow S_{\lambda} x s_{k}=S_{\lambda} x \Leftrightarrow x s_{k} \not{ }^{\lambda}\left(S_{r}\right)
$$

where the last equivalence uses that $x s_{k} \nexists^{\lambda}\left(S_{r}\right)$ implies $x s_{k}=r x$ for some $r \in S_{\lambda}$ (see for example [GP00, Lemma 2.1.2]). Note that

$$
l\left(x s_{k}\right)= \begin{cases}l(x)+1 & \text { if } k \cdot x^{-1}<(k+1) \cdot x^{-1} \\ l(x)-1 & \text { if } k \cdot x^{-1}>(k+1) \cdot x^{-1}\end{cases}
$$

(c.f. Hum90, Thm. 5.4]). The entries of $l(\lambda)$ are weakly increasing. Since $i_{k}$ and $i_{k+1}$ are the $k$-th (resp. $(k+1)$-st) entry of $l(\lambda) \cdot x=\left(l(\lambda)_{1 \cdot x^{-1}}, l(\lambda)_{2 \cdot x^{-1}}, \ldots, l(\lambda)_{r \cdot x^{-1}}\right)$ we get $i_{k} \leq i_{k+1}$ if $x s_{k}>x$ and $i_{k} \geq i_{k+1}$ if $x s_{k}<x$. This finishes the proof.

Proposition 2.20. The actions $U_{q}\left(\mathfrak{s l}_{n}\right) \curvearrowright V_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}$ commute.
Proof. We first check the claim for $r=2$. We have $\mathcal{H}_{2}=\operatorname{Span}_{\mathbb{C}(q)}\left\{H_{e}, H_{s}\right\}$. As noted before, $H_{s}$ preserves the weight spaces of $V \otimes V$ and hence $H_{s}$ commutes with $K_{1}, \ldots, K_{n-1}$. To prove that $H_{s}$ commutes with the $E_{i}$ and the $F_{i}$, we decompose $V \otimes V$ into eigenspaces for the endomorphism $H_{s}$ and show that these eigenspaces are preserved by the $E_{i}$ and the $F_{i}$. For any $1 \leq j<k \leq n$ the formulas from Lemma 2.19 tell us that

$$
\begin{aligned}
v_{j} \otimes v_{j} \cdot H_{s} & =q^{-1} \cdot v_{j} \otimes v_{j} \\
\left(q v_{j} \otimes v_{k}+v_{k} \otimes v_{j}\right) \cdot H_{s} & =q v_{k} \otimes v_{j}+v_{j} \otimes v_{k}+\left(q^{-1}-q\right) v_{k} \otimes v_{j} \\
& =q^{-1} \cdot\left(q v_{j} \otimes v_{k}+v_{k} \otimes v_{j}\right) \\
\left(q^{-1} v_{j} \otimes v_{k}-v_{k} \otimes v_{j}\right) \cdot H_{s} & =q^{-1} v_{k} \otimes v_{j}-v_{j} \otimes v_{k}-\left(q^{-1}-q\right) v_{k} \otimes v_{j} \\
& =(-q) \cdot\left(q^{-1} v_{j} \otimes v_{k}-v_{k} \otimes v_{j}\right) .
\end{aligned}
$$

Thus, we have the eigenspace decomposition

$$
\begin{aligned}
V \otimes V= & \operatorname{Span}_{\mathbb{C}(q)}\left(\left\{v_{j} \otimes v_{j} \mid 1 \leq j \leq n\right\} \cup\left\{q v_{j} \otimes v_{k}+v_{k} \otimes v_{j} \mid 1 \leq j<k \leq n\right\}\right) \\
& \oplus \operatorname{Span}_{\mathbb{C}(q)}\left\{q^{-1} v_{j} \otimes v_{k}-v_{k} \otimes v_{j} \mid 1 \leq j<k \leq n\right\} .
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\Delta\left(E_{i}\right) & =E_{i} \otimes K_{i}^{-1}+1 \otimes E_{i} \\
\Delta\left(F_{i}\right) & =F_{i} \otimes 1+K_{i} \otimes F_{i} .
\end{aligned}
$$

Hence, the $E_{i}$ and the $F_{i}$ act on the bases of the eigenspaces found above as follows (where $1 \leq j<k \leq n$ ):

$$
\begin{aligned}
& v_{j} \otimes v_{j} \stackrel{E_{i}}{\longleftrightarrow} \begin{cases}q v_{i} \otimes v_{i+1}+v_{i+1} \otimes v_{i} & \text { if } j=i+1 \\
0 & \text { otherwise }\end{cases} \\
& q v_{j} \otimes v_{k}+v_{k} \otimes v_{j} \stackrel{E_{i}}{\longleftrightarrow} \begin{cases}q v_{i} \otimes v_{k}+v_{k} \otimes v_{i} & \text { if } j=i+1 \\
q v_{j} \otimes v_{i}+v_{i} \otimes v_{j} & \text { if } j \neq i, k=i+1 \\
\left(q+q^{-1}\right) v_{i} \otimes v_{i} & \text { if } j=i, k=i+1 \\
0 & \text { otherwise }\end{cases} \\
& q^{-1} v_{j} \otimes v_{k}-v_{k} \otimes v_{j} \stackrel{E_{i}}{\longleftrightarrow} \begin{cases}q^{-1} v_{i} \otimes v_{k}-v_{k} \otimes v_{i} & \text { if } j=i+1 \\
q^{-1} v_{j} \otimes v_{i}-v_{i} \otimes v_{j} & \text { if } j \neq i, k=i+1 \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

$$
\begin{gathered}
v_{j} \otimes v_{j} \stackrel{F_{i}}{\longmapsto} \begin{cases}q v_{i} \otimes v_{i+1}+v_{i+1} \otimes v_{i} & \text { if } j=i \\
0 & \text { otherwise }\end{cases} \\
q v_{j} \otimes v_{k}+v_{k} \otimes v_{j} \stackrel{F_{i}}{\longmapsto} \begin{cases}q v_{j} \otimes v_{i+1}+v_{i+1} \otimes v_{j} & \text { if } k=i \\
q v_{i+1} \otimes v_{k}+v_{k} \otimes v_{i+1} & \text { if } j=i, k \neq i+1 \\
\left(q+q^{-1}\right) v_{i+1} \otimes v_{i+1} & \text { if } j=i, k=i+1 \\
0 & \text { otherwise }\end{cases} \\
q^{-1} v_{j} \otimes v_{k}-v_{k} \otimes v_{j} \stackrel{F_{i}}{\longmapsto} \begin{cases}q^{-1} v_{j} \otimes v_{i+1}-v_{i+1} \otimes v_{j} & \text { if } k=i \\
q^{-1} v_{i+1} \otimes v_{k}-v_{k} \otimes v_{i+1} & \text { if } j=i, k \neq i+1 \\
0 & \text { otherwise. }\end{cases}
\end{gathered}
$$

We see that the $E_{i}$ and the $F_{i}$ preserve the eigenspaces of the $H_{s}$-action. This shows the claim of the proposition for $r=2$. For the general case, consider $H_{s_{k}} \in \mathcal{H}_{r}$ with $k \in\{1, \ldots, r-1\}$. By Lemma 2.19 this acts on $V_{q}^{\otimes r}$ via

$$
\mathrm{id}^{\otimes k-1} \otimes H_{s} \otimes \mathrm{id}^{r-k-1}
$$

Since the comultiplication $\Delta$ is coassociative, any $x \in U_{q}\left(\mathfrak{s l}_{n}\right)$ acts on $V_{q}^{\otimes r}$ as

$$
\left(\mathrm{id}^{\otimes k-1} \otimes \Delta \otimes \mathrm{id}^{\otimes r-k-1}\right)(y) \in U_{q}\left(\mathfrak{s l}_{n}\right)^{\otimes r}
$$

for some $y \in U_{q}\left(\mathfrak{s l}_{n}\right)^{\otimes r-1}$. By the $r=2$ case, $H_{s}$ commutes with $\Delta(z)$ on $V \otimes V$ for any $z \in U_{q}\left(\mathfrak{s l}_{n}\right)$ and hence $H_{s_{k}}$ commutes with $x$ on $V_{q}^{\otimes r}$. We have thus shown the proposition for arbitrary $r$.

Remark 2.21. Our construction of the $\mathcal{H}_{r}$-action is a straightforward 'by hand' deformation of the classical permutation action on tensor space. However, the fact that this commutes with the $U_{q}\left(\mathfrak{s l}_{n}\right)$-action is a bit mysterious when just checked by brute force calculations as above. There is a more conceptual approach to the Hecke algebra action on tensor space via the so-called universal $R$-matrix (we refer to JJan96, Section 7] for more details on R-matrices). Using this $R$-matrix the two actions will commute pretty much by construction (see [Jim86]). We have chosen a more computational approach instead since it highlights the analogy with classical Schur-Weyl duality and requires less machinery.

Our next goal is a double centraliser property for the quantum case.
Definition 2.22. The algebra $S_{q}(n, r):=\operatorname{End}_{\mathcal{H}_{r}}\left(V_{q}^{\otimes r}\right)$ is called the quantum Schur algebra or short the $q$-Schur algebra.

By Proposition 1.12, the Hecke algebra $\mathcal{H}_{r}$ is semisimple and by the double centraliser theorem (Theorem 2.2) we get that

$$
S_{q}(n, r) \curvearrowright V_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}
$$

generate each other's centraliser. The $U_{q}\left(\mathfrak{s l}_{n}\right)$-action on $V_{q}^{\otimes r}$ induces an algebra homomorphism $U_{q}\left(\mathfrak{s l}_{n}\right) \rightarrow S_{q}(n, r)$ so proving a quantum Schur-Weyl duality boils down to showing that this map is surjective. For this, we observe that the structure of the $q$-Schur algebra is in many ways similar to that of the ordinary Schur algebra. In fact, the isomorphism of $\mathcal{H}_{r}$-modules

$$
\begin{equation*}
V_{q}^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} \mathcal{M}^{\lambda} \stackrel{\stackrel{(5)}{=}}{=} \bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \mathcal{H}_{r} \tag{14}
\end{equation*}
$$

together with Lemma 2.7 yields the isomorphism of vector spaces

$$
S_{q}(n, r) \cong \operatorname{End}_{\mathcal{H}_{r}}\left(\bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \mathcal{H}_{r}\right) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, r)} x_{\mu} \mathcal{H}_{r} x_{\lambda} .
$$

Lemma 2.23. The set $\left\{x_{\mu} H_{y} x_{\lambda} \mid y \in{ }^{\mu} S_{r}^{\lambda}\right\}$ is a $\mathbb{C}(q)$-basis of $x_{\mu} \mathcal{H}_{r} x_{\lambda}$. In particular, $\operatorname{dim}_{\mathbb{C}(q)}\left(S_{q}(n, r)\right)=\operatorname{dim}_{\mathbb{C}}(S(n, r))$.

Proof. Any $x \in S_{r}$ is of the form $x=a y b$ for some $a \in S_{\mu}, b \in S_{\lambda}$ and $y \in{ }^{\mu} S_{r}^{\lambda}$ with $l(x)=l(a)+l(y)+l(b)$ (see for example [GP00, Prop. 2.1.7]). Then $H_{x}=H_{a} H_{y} H_{b}$ by Proposition 1.11 and $x_{\mu} H_{x} x_{\lambda}=q^{-l(a)-l(b)} x_{\mu} H_{y} x_{\lambda}$. This implies that the $x_{\mu} H_{y} x_{\lambda}$ with $y \in{ }^{\mu} S_{r}^{\lambda}$ span $x_{\mu} \mathcal{H}_{r} x_{\lambda}$. We claim that they are also linearly independent. If not, there is a linear dependence

$$
\sum_{y \in^{\mu} S_{r}^{\lambda}} p_{y}(q) \cdot x_{\mu} H_{y} x_{\lambda}=0
$$

for some $p_{y}(q) \in \mathbb{C}(q)$ not all 0 . After multiplying with an element of $\mathbb{C}(q)^{\times}$and dividing by some power of $q-1$, we may assume that $p_{y}(q) \in \mathbb{C}\left[q, q^{-1}\right]$ and $p_{y}(1)$ not all 0 . Let $\varphi: \mathcal{H}_{\mathbb{C}\left[q, q^{-1}\right]}\left(S_{r}\right) \rightarrow \mathbb{C}\left[S_{r}\right]$ be the specialisation homomorphism at $q \rightarrow 1$. Then $\varphi\left(x_{\mu} H_{y} x_{\lambda}\right)=e_{\mu} y e_{\lambda}$ and

$$
0=\varphi\left(\sum_{y \not{ }^{\mu} S_{r}^{\lambda}} p_{y}(q) \cdot x_{\mu} H_{y} x_{\lambda}\right)=\sum_{y \in^{\mu} S_{r}^{\lambda}} p_{y}(1) \cdot e_{\mu} y e_{\lambda} .
$$

This contradicts the linear independence of the $e_{\mu} y e_{\lambda}$ in $e_{\mu} \mathbb{C}\left[S_{r}\right] e_{\lambda}$ and the claim follows. In particular, $\operatorname{dim}_{\mathbb{C}(q)} S_{q}(n, r)=\sum_{\lambda, \mu \in \Lambda(n, r)}\left|S_{\mu} \backslash S_{r} / S_{\lambda}\right| \stackrel{99}{=} \operatorname{dim}_{\mathbb{C}} S(n, r)$.

Now, we are ready prove quantum Schur-Weyl duality which was first proved by Jimbo Jim86.

Theorem 2.24. The commuting actions $U_{q}\left(\mathfrak{s l}_{n}\right) \curvearrowright V_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}$ generate each other's centraliser.

Proof. Our proof is an adaptation of the arguments from [KS12, Section 8.6.3]. However, we avoid using facts about the representation theory of $U_{q}\left(\mathfrak{s l}_{n}\right)$. The actions $\mathfrak{s l}_{n} \curvearrowright V^{\otimes r}$ and $U_{q}\left(\mathfrak{s l}_{n}\right) \curvearrowright V_{q}^{\otimes r}$ induce algebra homomorphisms

$$
\begin{gathered}
\psi: U\left(\mathfrak{s l}_{n}\right) \longrightarrow S(n, r) \subset \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right) \\
\psi_{q}: U_{q}\left(\mathfrak{s l}_{n}\right) \longrightarrow S_{q}(n, r) \subset \operatorname{End}_{\mathbb{C}(q)}\left(V_{q}^{\otimes r}\right) .
\end{gathered}
$$

We know that $\psi$ is surjective and we need to show that $\psi_{q}$ is surjective. The space $\operatorname{End}_{\mathbb{C}(q)}\left(V_{q}^{\otimes r}\right)$ has the basis $\left\{E_{i_{1}, j_{1}} \otimes \ldots \otimes E_{i_{r}, j_{r}} \mid i_{l}, j_{l} \in\{1, \ldots, n\}\right\}$. Let $A_{\mathbb{C}\left[q, q^{-1}\right]}$ be the $\mathbb{C}\left[q, q^{-1}\right]$-span of this basis which is a $\mathbb{C}\left[q, q^{-1}\right]$-algebra. Let

$$
\varphi: A_{\mathbb{C}\left[q, q^{-1}\right]} \longrightarrow \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)
$$

be the specialisation at $q \rightarrow 1$. Clearly, $\psi_{q}\left(E_{i}\right), \psi_{q}\left(F_{i}\right) \in A_{\mathbb{C}\left[q, q^{-1}\right]}$ and

$$
\begin{aligned}
\varphi\left(\psi_{q}\left(E_{i}\right)\right) & =\psi\left(e_{i}\right) \\
\varphi\left(\psi_{q}\left(F_{i}\right)\right) & =\psi\left(f_{i}\right) .
\end{aligned}
$$

This shows that $S(n, r)=\psi\left(U\left(\mathfrak{s l}_{n}\right)\right) \subset \varphi \circ \psi_{q}\left(U_{\mathbb{C}\left[q, q^{-1}\right]}\right)$ where $U_{\mathbb{C}\left[q, q^{-1}\right]}$ is the $\mathbb{C}\left[q, q^{-1}\right]$-subalgebra of $U_{q}\left(\mathfrak{s l}_{n}\right)$ generated by the $E_{i}, F_{i}$. In particular, we can find $X_{1}, \ldots, X_{m} \in U_{\mathbb{C}\left[q, q^{-1}\right]}$ that map to a basis of $S(n, r)$ under $\varphi \circ \psi_{q}$. We claim that $\psi_{q}\left(X_{1}\right), \ldots, \psi_{q}\left(X_{m}\right)$ are $\mathbb{C}(q)$-linearly independent in $S_{q}(n, r)$. If not, there is a linear dependence

$$
\sum_{i=1}^{m} p_{i}(q) \cdot \psi_{q}\left(X_{i}\right)=0
$$

where the $p_{i}(q) \in \mathbb{C}(q)$ are not all 0 . By our standard division technique, we may assume $p_{i}(q) \in \mathbb{C}\left[q, q^{-1}\right]$ for all $i$ with $p_{i}(1)$ not all 0 . Then

$$
0=\varphi\left(\sum_{i=1}^{m} p_{i}(q) \cdot \psi_{q}\left(X_{i}\right)\right)=\sum_{i=1}^{m} p_{i}(1) \cdot \varphi\left(\psi_{q}\left(X_{i}\right)\right)
$$

contradicting the linear independence of $\varphi\left(\psi_{q}\left(X_{1}\right)\right), \ldots, \varphi\left(\psi_{q}\left(X_{m}\right)\right)$. This shows that $\psi_{q}\left(X_{1}\right), \ldots, \psi_{q}\left(X_{m}\right)$ are linearly independent. The surjectivity of $\psi_{q}$ now follows since

$$
m=\operatorname{dim}_{\mathbb{C}}(S(n, r)) \stackrel{\text { Lemma }}{=} \xlongequal{[2.23} \operatorname{dim}_{\mathbb{C}(q)}\left(S_{q}(n, r)\right) .
$$

This finishes the proof.
Since $\mathcal{H}_{r}$ is semisimple, we can apply the double centraliser theorem to obtain the following duality on the level of representations.

Corollary 2.25. There is a bijection

$$
\left\{\begin{array}{c}
\text { iso. classes of } \\
\text { simple } U_{q}\left(\mathfrak{s l}_{n}\right) \text {-modules } \\
\text { appearing in } V_{q}^{\otimes r}
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { iso. classes of } \\
\text { simple } \mathcal{H}_{r} \text {-modules } \\
\text { appearing in } V_{q}^{\otimes r}
\end{array}\right\} .
$$

Remark 2.26. One can show that any finite-dimensional $U_{q}\left(\mathfrak{s l}_{n}\right)$-representations is completely reducible (see [Jan96, Thm. 5.17]). Hence, by Corollary 2.25 there is an equivalence between the subcategories of $U_{q}\left(\mathfrak{s l}_{n}\right)$ - $\bmod$ and $\bmod -\mathcal{H}_{r}$ which consist of those finite-dimensional representations whose simple constituents appear in $V_{q}^{\otimes r}$. Alternatively, there is a straightforward quantum version of the arguments we used to establish Proposition 2.13. In particular, these arguments show that for $n \geq r$ the functor $V_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}}(-)$ induces an equivalence of categories between $\mathcal{H}_{r}$-mod and the subcategory of $U_{q}\left(\mathfrak{s l}_{n}\right)$-mod consisting of those finite-dimensional representations that are annihilated by $\operatorname{ker}\left(U_{q}\left(\mathfrak{s l}_{n}\right) \rightarrow S_{q}(n, r)\right)$.

### 2.3 Bar involutions

Recall the bar involution of the Hecke algebra which is the unique $\mathbb{C}(q)$-antilinear ring endomorphism of $\mathcal{H}_{r}$ with the property that $\overline{H_{s}}=H_{s}^{-1}$. There also is a bar involution on $U_{q}\left(\mathfrak{s l}_{n}\right)$ and its representations $V_{q}^{\otimes r}$. In [FKK98] a connection between the bar involution of the Hecke algebra and the quantum group is established (see also [FK97]). The involution on $V_{q}^{\otimes r}$ is usually constructed using the so-called quasi- $R$-matrix which is closely related to the $R$-matrix from Remark 2.21. The general construction of the quasi- $R$-matrix is non-trivial and can be found in Lus10, Chapter 4]. We will focus on the $n=2$ case so that we can keep things explicit. In other words, we consider the quantum group $U_{q}\left(\mathfrak{s l}_{2}\right)$ with generators $E, F, K^{ \pm 1}$ and relations

$$
\begin{equation*}
K K^{-1}=1=K^{-1} K, \quad K E=q^{2} E K, \quad K F=q^{-2} F K, \quad[E, F]=\frac{K-K^{-1}}{q-q^{-1}} . \tag{15}
\end{equation*}
$$

In this section we will also use the notation

$$
[m]:=\frac{q^{m}-q^{-m}}{q-q^{-1}}=q^{m-1}+q^{m-3}+\ldots+q^{1-m} \in \mathbb{C}\left[q, q^{-1}\right]
$$

for any $m \in \mathbb{Z}$. This is sometimes called a quantum integer since one recovers the integer $m \in \mathbb{Z}$ for $q \rightarrow 1$. Similarly, we define the quantum factorial and the quantum binomial coefficient

$$
[m]!:=[m] \cdot[m-1] \cdot \ldots \cdot[1], \quad\left[\begin{array}{l}
n \\
k
\end{array}\right]:=\frac{[n]!}{[k]![n-k]]} .
$$

Definition 2.27. The bar involution of $U_{q}\left(\mathfrak{s l}_{2}\right)$ is the $\mathbb{C}(q)$-antilinear ring endomorphism

$$
\begin{aligned}
{ }^{-}: U_{q}\left(\mathfrak{s l}_{2}\right) & \longrightarrow U_{q}\left(\mathfrak{s l}_{2}\right) \\
E & \longmapsto E \\
F & \longmapsto F \\
K & \longmapsto K^{-1} .
\end{aligned}
$$

It is easy to check that this is compatible with the relations from (15) and it is clear that this is an involution. We also define an involution $\sigma_{V_{q}}$ on the natural representation $V_{q}=\mathbb{C}(q)^{2}$ of $U_{q}\left(\mathfrak{s l}_{2}\right)$. This is the unique $\mathbb{C}(q)$-antilinear map that fixes the standard basis, i.e.

$$
\sigma_{V_{q}}\left(f_{1}(q) v_{1}+f_{2}(q) v_{2}\right)=f_{1}\left(q^{-1}\right) v_{1}+f_{2}\left(q^{-1}\right) v_{2} .
$$

This is $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear in the following sense.
Definition 2.28. Let $M$ be a $U_{q}\left(\mathfrak{s l}_{2}\right)$-module and $\sigma_{M} \in \operatorname{End}_{\mathbb{C}}(M)$. We say that $\sigma_{M}$ is $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear if $\sigma_{M}(x \cdot m)=\bar{x} \cdot m$ for all $x \in U_{q}\left(\mathfrak{s l}_{2}\right)$ and $m \in M$.

We would like to construct a $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear involution on $V_{q}^{\otimes r}$. The obvious candidate is the $\mathbb{C}(q)$-antilinear involution $\sigma_{V_{q}}^{\otimes r}$ which fixes the standard basis. However, this is not $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear.
Example 2.29. We have on $V_{q} \otimes V_{q}$

$$
\begin{aligned}
& \bar{E} \cdot \sigma_{V_{q}} \otimes \sigma_{V_{q}}\left(v_{2} \otimes v_{1}\right)=E \cdot v_{2} \otimes v_{1}=\left(E \otimes K^{-1}+1 \otimes E\right) \cdot v_{2} \otimes v_{1}=q^{-1} v_{1} \otimes v_{1} \\
& \sigma_{V_{q}} \otimes \sigma_{V_{q}}\left(E \cdot v_{2} \otimes v_{1}\right)=\sigma_{V_{q}} \otimes \sigma_{V_{q}}\left(q^{-1} v_{1} \otimes v_{1}\right)=q v_{1} \otimes v_{1} .
\end{aligned}
$$

This shows that $\sigma_{V_{q}} \otimes \sigma_{V_{q}}$ is not $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear.
To solve this problem, we will twist the involution $\sigma_{V_{q}} \otimes \sigma_{V_{q}}$ by an endomorphism of $V_{q} \otimes V_{q}$. For this, we extend the bar involution of $U_{q}\left(\mathfrak{s l}_{2}\right)$ to $U_{q}\left(\mathfrak{s l}_{2}\right) \otimes U_{q}\left(\mathfrak{s l}_{2}\right)$ via $\overline{x_{1} \otimes x_{2}}=\overline{x_{1}} \otimes \overline{x_{2}}$. Then, we can consider the map

$$
\bar{\Delta}: U_{q}\left(\mathfrak{s l}_{2}\right) \rightarrow U_{q}\left(\mathfrak{s l}_{2}\right) \otimes U_{q}\left(\mathfrak{s l}_{2}\right), \quad x \mapsto \overline{\Delta(\bar{x})} .
$$

This is a coassociative comultiplication which is $\mathbb{C}(q)$-linear. Specifically, we have

$$
\bar{\Delta}(E)=E \otimes K+1 \otimes E, \quad \bar{\Delta}(F)=F \otimes 1+K^{-1} \otimes F, \quad \bar{\Delta}(K)=K \otimes K .
$$

Now let $M, N$ be $U_{q}\left(\mathfrak{s l}_{2}\right)$-modules equipped with $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear involutions $\sigma_{M} \in$ $\operatorname{End}_{\mathbb{C}}(M), \sigma_{N} \in \operatorname{End}_{\mathbb{C}}(N)$. For any $\varphi \in \operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$ define the bar conjugate

$$
\bar{\varphi}=\left(\sigma_{M} \otimes \sigma_{N}\right) \circ \varphi \circ\left(\sigma_{M} \otimes \sigma_{N}\right) \in \operatorname{End}_{\mathbb{C}(q)}(M \otimes N)
$$

Since $\sigma_{M}$ and $\sigma_{N}$ are $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear, the bar involution of $U_{q}\left(\mathfrak{s l}_{2}\right) \otimes U_{q}\left(\mathfrak{s l}_{2}\right)$ gets identified with involution of $\operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$ under the map $U_{q}\left(\mathfrak{s l}_{2}\right) \otimes U_{q}\left(\mathfrak{s l}_{2}\right) \rightarrow$ $\operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$. We can now give a general recipe to construct a $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear involution $\sigma_{M \otimes N}$ for certain $M$ and $N$.

Proposition 2.30. Let $\Theta \in \operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$ such that

$$
\Delta(x) \circ \Theta=\Theta \circ \bar{\Delta}(x) \quad \text { and } \quad \Theta \circ \bar{\Theta}=1
$$

in $\operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$ for all $x \in U_{q}\left(\mathfrak{s l}_{2}\right)$. Then

$$
\sigma_{M \otimes N}:=\Theta \circ\left(\sigma_{M} \otimes \sigma_{N}\right) \in \operatorname{End}_{\mathbb{C}}(M \otimes N)
$$

is a $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear involution.
Proof. We have

$$
\sigma_{M \otimes N}^{2}=\Theta \circ\left(\sigma_{M} \otimes \sigma_{N}\right) \circ \Theta \circ\left(\sigma_{M} \otimes \sigma_{N}\right)=\Theta \circ \bar{\Theta}=1
$$

proving that $\sigma_{M \otimes N}$ is an involution. Moreover, we have

$$
\begin{aligned}
\sigma_{M \otimes N} \circ \Delta(x) & =\Theta \circ\left(\sigma_{M} \otimes \sigma_{N}\right) \circ \Delta(x) \\
& =\Theta \circ \overline{\Delta(x)} \circ \sigma_{M} \otimes \sigma_{N} \\
& =\Delta(\bar{x}) \circ \Theta \circ \sigma_{M} \otimes \sigma_{N} \\
& =\Delta(\bar{x}) \circ \sigma_{M \otimes N}
\end{aligned}
$$

in $\operatorname{End}_{\mathbb{C}}(M \otimes N)$ for any $x \in U_{q}\left(\mathfrak{s l}_{2}\right)$. This implies that $\sigma_{M \otimes N}$ is $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear.

The hard part is the construction of an element $\Theta$ as above. This will make use of the following observation.

Lemma 2.31. For any $m \geq 0$ we have

1) $\Delta(E) \cdot E^{m} \otimes F^{m}-E^{m} \otimes F^{m} \cdot \bar{\Delta}(E)=E^{m} \otimes\left[E, F^{m}\right]+p_{m}(q) E^{m+1} \otimes\left[E, F^{m+1}\right]$
2) $\Delta(F) \cdot E^{m} \otimes F^{m}-E^{m} \otimes F^{m} \cdot \bar{\Delta}(F)=\left[F, E^{m}\right] \otimes F^{m}+p_{m}(q)\left[F, E^{m+1}\right] \otimes F^{m+1}$
3) $\Delta(K) \cdot E^{m} \otimes F^{m}-E^{m} \otimes F^{m} \cdot \bar{\Delta}(K)=0$
where $p_{m}(q)=\frac{-q^{m}\left(q-q^{-1}\right)}{[m+1]}$.
Proof. Let us prove 1) (the other cases are similar). We have

$$
\begin{aligned}
& \Delta(E) \cdot E^{m} \otimes F^{m}-E^{m} \otimes F^{m} \cdot \bar{\Delta}(E) \\
= & E^{m+1} \otimes K^{-1} F^{m}+E^{m} \otimes E F^{m}-E^{m+1} \otimes F^{m} K-E^{m} \otimes F^{m} E \\
= & E^{m+1} \otimes\left(K^{-1} F^{m}-F^{m} K\right)+E^{m} \otimes\left[E, F^{m}\right] .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
p_{m}(q)\left[E, F^{m+1}\right] & =\sum_{k=0}^{m} p_{m}(q) F^{k}[E, F] F^{m-k} \\
& =\sum_{k=0}^{m} \frac{-q^{m}}{[m+1]} F^{k}\left(K-K^{-1}\right) F^{m-k} \\
& =\sum_{k=0}^{m} \frac{-q^{m}}{[m+1]}\left(q^{-2(m-k)} F^{m} K-q^{-2 k} K^{-1} F^{m}\right) \\
& =\frac{-q^{m}}{[m+1]}\left(q^{-m}[m+1] F^{m} K-q^{-m}[m+1] K^{-1} F^{m}\right) \\
& =K^{-1} F^{m}-F^{m} K .
\end{aligned}
$$

Together, these two calculations imply 1).

We use a telescope sum argument and Lemma 2.31 to construct an element $\Theta \in \operatorname{End}_{\mathbb{C}(q)}(M \otimes N)$ as it appears in Proposition 2.30. For this, let

$$
\Theta^{(k)}:=\sum_{m=0}^{k}(-1)^{m} \hat{p}_{m}(q) E^{m} \otimes F^{m} \in U_{q}\left(\mathfrak{s l}_{2}\right) \otimes U_{q}\left(\mathfrak{s l}_{2}\right)
$$

with $\hat{p}_{m}(q)=\prod_{i=0}^{m-1} p_{i}(q)$. By Lemma 2.31 we get

$$
\begin{aligned}
\Delta(E) \Theta^{(k)}-\Theta^{(k)} \bar{\Delta}(E)= & \sum_{m=0}^{k}(-1)^{m} \hat{p}_{m}(q)\left(E^{m} \otimes\left[E, F^{m}\right]+p_{m}(q) E^{m+1} \otimes\left[E, F^{m+1}\right]\right) \\
= & \sum_{m=0}^{k}(-1)^{m} \hat{p}_{m}(q) E^{m} \otimes\left[E, F^{m}\right] \\
& +\sum_{m=0}^{k}(-1)^{m} \hat{p}_{m+1}(q) E^{m+1} \otimes\left[E, F^{m+1}\right] \\
= & (-1)^{k} \hat{p}_{k+1}(q) E^{k+1} \otimes\left[E, F^{k+1}\right] \\
\Delta(F) \Theta^{(k)}-\Theta^{(k)} \bar{\Delta}(F)= & \ldots=(-1)^{k} \hat{p}_{k+1}(q)\left[F, E^{k+1}\right] \otimes F^{k+1}, \\
\Delta(K) \Theta^{(k)}-\Theta^{(k)} \bar{\Delta}(K)= & 0 .
\end{aligned}
$$

Now assume that $E^{m} \cdot M=0$ and $F^{m} \cdot N=0$ for $m \gg 0$. Then for $k \gg 0$ the element $\Theta^{(k)}$ defines a unique endomorphism (independent of $k$ )

$$
\begin{equation*}
\Theta:=\sum_{m \geq 0}(-1)^{m} \hat{p}_{m}(q) E^{m} \otimes F^{m} \in \operatorname{End}_{\mathbb{C}(q)}(M \otimes N) \tag{16}
\end{equation*}
$$

and $\Delta(x) \circ \Theta=\Theta \circ \bar{\Delta}(x)$.
Lemma 2.32. We have $\Theta \circ \bar{\Theta}=1$.
Proof.

$$
\begin{aligned}
\Theta \circ \bar{\Theta} & =\left(\sum_{m \geq 0}(-1)^{m} \hat{p}_{m}(q) E^{m} \otimes F^{m}\right) \circ\left(\sum_{m \geq 0}(-1)^{m} \hat{p}_{m}\left(q^{-1}\right) E^{m} \otimes F^{m}\right) \\
& =\sum_{m \geq 0}(-1)^{m}\left(\sum_{k+s=m} \hat{p}_{k}(q) \hat{p}_{s}\left(q^{-1}\right)\right) E^{m} \otimes F^{m} .
\end{aligned}
$$

Now $\hat{p}_{m}(q)=\prod_{i=0}^{m-1} p_{i}(q)=\prod_{i=0}^{m-1} \frac{-q^{i}\left(q-q^{-1}\right)}{[i+1]}=(-1)^{m} q^{m(m-1) / 2} \frac{\left(q-q^{-1}\right)^{m}}{[m]!}$ for $m>0$.
Note that if $k+s=m$ then

$$
k(k-1)-s(s-1)=k^{2}-k-s^{2}+s=(k+s-1)(k-s)=(m-1)(2 k-m)
$$

and hence

$$
\begin{aligned}
\sum_{k+s=m} \hat{p}_{k}(q) \hat{p}_{s}\left(q^{-1}\right) & =\sum_{k+s=m}(-1)^{k+s} q^{\frac{1}{2}(k(k-1)-s(s-1))} \frac{\left(q-q^{-1}\right)^{k}\left(q^{-1}-q\right)^{s}}{[k]![s]!} \\
& =\sum_{k+s=m}(-1)^{k} q^{\frac{1}{2}(m-1)(2 k-m)} \frac{\left(q-q^{-1}\right)^{k+s}}{[k]![s]!} \\
& =q^{-m(m-1) / 2} \frac{\left(q-q^{-1}\right)^{m}}{[m]!} \sum_{k=0}^{m}(-1)^{k} q^{(m-1) k}\left[\begin{array}{c}
m \\
k
\end{array}\right] .
\end{aligned}
$$

The quantum binomial theorem implies that $\sum_{k=0}^{m}(-1)^{k} q^{(m-1) k}\left[\begin{array}{c}m \\ k\end{array}\right]=0$ for $m>0$ (see Lus10, 1.3.4]). Hence $\Theta \circ \bar{\Theta}=\hat{p}_{0}(q) \hat{p}_{0}\left(q^{-1}\right)=1$.

We can now define the involutions $\sigma_{V_{q}{ }^{\otimes r}}$ as follows. We have already constructed $\sigma_{V_{q}}$. Assume we have constructed $\sigma_{V_{q}^{\otimes r^{\prime}}}$ for all $r^{\prime}<r$. Choose $r_{1}, r_{2}>0$ such that $r=r_{1}+r_{2}$. It is easy to check that $E^{m} \cdot V_{q}^{\otimes k}=F^{m} \cdot V_{q}^{\otimes k}=0$ for all $k$ and $m \gg 0$. Hence, we can define $\sigma_{V_{q}^{\otimes r}}$ by applying Proposition 2.30 with $M=V_{q}^{\otimes r_{1}}, N=V_{q}^{\otimes r_{2}}$ and $\Theta$ from (16).

Example 2.33. Let us look at $V_{q} \otimes V_{q}$. We compute the action of the involution $\sigma_{V_{q} \otimes V_{q}}=\Theta \circ \sigma_{V_{q}} \otimes \sigma_{V_{q}}$ on the standard basis. Since $\sigma_{V_{q}} \otimes \sigma_{V_{q}}$ fixes the standard basis pointwise, this is the same as acting with $\Theta=1+\left(q-q^{-1}\right) E \otimes F+\ldots$. Note that $F^{2} \cdot V_{q}=0=E^{2} \cdot V_{q}$. We have

$$
\begin{aligned}
& \Theta\left(v_{1} \otimes v_{1}\right)=v_{1} \otimes v_{1}+\left(q-q^{-1}\right) E v_{1} \otimes F v_{1}=v_{1} \otimes v_{1} \\
& \Theta\left(v_{1} \otimes v_{2}\right)=v_{1} \otimes v_{2}+\left(q-q^{-1}\right) E v_{1} \otimes F v_{2}=v_{1} \otimes v_{2} \\
& \Theta\left(v_{2} \otimes v_{1}\right)=v_{2} \otimes v_{1}+\left(q-q^{-1}\right) E v_{2} \otimes F v_{1}=v_{2} \otimes v_{1}+\left(q-q^{-1}\right) v_{1} \otimes v_{2} \\
& \Theta\left(v_{2} \otimes v_{2}\right)=v_{1} \otimes v_{1}+\left(q-q^{-1}\right) E v_{2} \otimes F v_{2}=v_{2} \otimes v_{2} .
\end{aligned}
$$

This looks very similar to the bar involution of the Hecke algebra. In fact, the isomorphism $\oplus_{\lambda \in \Lambda(2,2)} \mathcal{M}^{\lambda} \xrightarrow{\oplus_{\lambda} \Psi^{(\lambda)}} V_{q} \otimes V_{q}$ of right $\mathcal{H}_{r}$-modules from $\sqrt{13}$ induces a bar involution on $V_{q} \otimes V_{q}$ and we have

$$
\begin{aligned}
\overline{v_{1} \otimes v_{1}}=\Psi_{2}^{(2,0)}\left(\overline{1 \otimes H_{e}}\right) & =\Psi_{2}^{(2,0)}\left(1 \otimes H_{e}\right)=v_{1} \otimes v_{1} \\
\overline{v_{1} \otimes v_{2}}=\Psi_{2}^{(1,1)}\left(\overline{1 \otimes H_{e}}\right) & =\Psi_{2}^{(1,1)}\left(1 \otimes H_{e}\right)=v_{1} \otimes v_{2} \\
\overline{v_{2} \otimes v_{1}}=\Psi_{2}^{(1,1)}\left(\overline{1 \otimes H_{s}}\right) & =\Psi_{2}^{(1,1)}\left(1 \otimes\left(H_{s}+\left(q-q^{-1}\right) H_{e}\right)\right. \\
& =v_{2} \otimes v_{1}+\left(q-q^{-1}\right) v_{1} \otimes v_{2} \\
\overline{v_{2} \otimes v_{2}}=\Psi_{2}^{(0,2)}\left(\overline{1 \otimes H_{e}}\right) & =\Psi_{2}^{(0,2)}\left(1 \otimes H_{e}\right)=v_{2} \otimes v_{2} .
\end{aligned}
$$

We see that the involution $\sigma_{V_{q} \otimes V_{q}}$ and the bar involution on $V_{q} \otimes V_{q}$ are the same.
There is one more technical problem we need to address. In fact, our construction of $\sigma_{V_{q}}{ }^{\otimes r}$ depends on how we write $r$ as a sum $r=r_{1}+r_{2}$ or equivalently on how we bracket $V_{q}^{\otimes r}$. It can be shown that the involution $\sigma_{V_{q}^{\otimes r}}$ is independent of this choice.

Lemma 2.34. Let $M, M^{\prime}, M^{\prime \prime}$ be $U_{q}\left(\mathfrak{s l}_{2}\right)$-modules with $U_{q}\left(\mathfrak{s l}_{2}\right)$-antilinear involutions $\sigma_{M}, \sigma_{M^{\prime}}, \sigma_{M^{\prime \prime}}$ and $E^{m} \cdot M=0, E^{m} \cdot M^{\prime}=F^{m} \cdot M^{\prime}=0$ and $F^{m} \cdot M^{\prime \prime}=0$ for $m \gg 0$. Then $\sigma_{M \otimes\left(M^{\prime} \otimes M^{\prime \prime}\right)}=\sigma_{\left(M \otimes M^{\prime}\right) \otimes M^{\prime \prime}}$ for $\sigma_{M \otimes\left(M^{\prime} \otimes M^{\prime \prime}\right)}$ and $\sigma_{\left(M \otimes M^{\prime}\right) \otimes M^{\prime \prime}}$ constructed as in Proposition 2.30 with the $\Theta$ from (16).

Proof. See Lus10, 27.3.6].
We can now extend the observations from Example 2.33 to arbitrary tensor powers.

Proposition 2.35. The bar involution on $V_{q}^{\otimes r}$ induced by $\bigoplus_{\lambda \in \Lambda(n, r)} \mathcal{M}^{\lambda} \xrightarrow{\Psi_{r}} V_{q}^{\otimes r}$ is the same as the involution $\sigma_{V_{q}^{\otimes r}}$.
Proof. The $r=1$ case is obvious. The $r=2$ is proved in Example 2.33. We proceed by induction. Assume we have shown the claim for all $r^{\prime}<r$. The bar involution on $V_{q}^{\otimes r}$ is the $\mathbb{C}(q)$-antilinear map uniquely determined by the properties

$$
\overline{v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}}=v_{1}^{\otimes k} \otimes v_{2}^{\otimes s} \quad \text { and } \quad \overline{x \cdot H_{s_{i}}}=\bar{x} \cdot \overline{H_{s_{i}}}
$$

whenever $k+s=r, x \in V_{q}^{\otimes r}$ and $i \in\{1, \ldots, r-1\}$. We have $\Theta\left(v_{1} \otimes x\right)=v_{1} \otimes x$ for any $x \in V_{q}^{\otimes r-1}$ since $E \cdot v_{1}=0$. Similarly, we get $\Theta\left(x \otimes v_{2}\right)=x \otimes v_{2}$ since $F \cdot v_{2}=0$ and by the induction hypothesis we have

$$
\sigma_{V_{q}} \otimes \sigma_{V_{q}^{\otimes r-1}}\left(v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}\right)=v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}=\sigma_{V_{q}}^{\otimes r-1} \otimes \sigma_{V_{q}}\left(v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}\right) .
$$

Since $\sigma_{V_{q}^{\otimes r}}=\Theta \circ \sigma_{V_{q}} \otimes \sigma_{V_{q}^{\otimes r-1}}=\Theta \circ \sigma_{V_{q}^{\otimes r-1}} \otimes \sigma_{V_{q}}$ we get $\sigma_{V_{q}^{\otimes r}}\left(v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}\right)=v_{1}^{\otimes k} \otimes v_{2}^{\otimes s}$. It remains to show that $\sigma_{V_{q}{ }^{\otimes r}}\left(x \cdot H_{i}\right)=\sigma_{V_{q}{ }^{\otimes r}}(x) \cdot \overline{H_{i}}$. For $i>1$ we have

$$
\begin{aligned}
\sigma_{V_{q}^{\otimes r}}\left(x \cdot H_{i}\right) & =\Theta \circ \sigma_{V_{q}} \otimes \sigma_{V_{q}^{\otimes r-1}}\left(x \cdot\left(1 \otimes H_{i-1}\right)\right) \\
& \stackrel{(\text { a) }}{=} \Theta\left(\left(\sigma_{V_{q}} \otimes \sigma_{V_{q}^{\otimes r-1}}(x)\right) \cdot\left(1 \otimes \overline{H_{i-1}}\right)\right) \\
& \stackrel{(\mathrm{b})}{=}\left(\Theta \circ \sigma_{V_{q}} \otimes \sigma_{V_{q}^{\otimes r-1}}(x)\right) \cdot\left(1 \otimes \overline{H_{i-1}}\right) \\
& =\left(\sigma_{V_{q}^{\otimes r}}(x)\right) \cdot \overline{H_{i}}
\end{aligned}
$$

using the induction hypothesis in (a) and that $\overline{H_{i-1}}$ commutes with the $U_{q}\left(\mathfrak{s l}_{2}\right)$ action on $V_{q}^{\otimes r-1}$ in (b). The $i=1$ case follows from a similar argument using that $\sigma_{V_{q}^{\otimes r}}=\Theta \circ\left(\sigma_{V_{q}^{\otimes r-1}} \otimes \sigma_{V_{q}}\right)$.

By Theorem 1.16 the isomorphism $\bigoplus_{\lambda \in \Lambda(n, r)} \mathcal{M}^{\lambda} \xrightarrow{\Psi_{r}} V^{\otimes r}$ also provides us with bases $\left\{b_{\underline{i}} \mid \underline{i} \in\{1,2\}^{r}\right\}$ and $\left\{b^{\underline{i}} \mid \underline{i} \in\{1,2\}^{r}\right\}$ of $V_{q}^{\otimes r}$ uniquely determined by the properties

$$
\begin{aligned}
& \sigma_{V_{q}^{\otimes r}}\left(b_{\underline{i}}\right)=b_{\underline{i}} \quad \text { and } \quad b_{\underline{i}}=v_{\underline{i}}+\sum_{\underline{j}} q \mathbb{Z}[q] v_{\underline{j}} \\
& \text { resp. } \sigma_{V_{q} \otimes r}\left(b^{\underline{i}}\right)=b^{\underline{i}} \quad \text { and } \quad b^{\underline{i}}=v_{\underline{i}}+\sum_{\underline{j}} q^{-1} \mathbb{Z}\left[q^{-1}\right] v_{\underline{j}}
\end{aligned}
$$

where the sums run over all $\underline{j} \in\{1,2\}^{r}$ such that $v_{\underline{i}}$ and $v_{\underline{j}}$ lie in the same weight space. These are called the canonical, respectively dual canonical basis of $V_{q}^{\otimes r}$. The existence and uniqueness of the (dual) canonical bases can also be shown independently (i.e. not using the Hecke algebra), see [Lus10, Thm. 27.3.2] and [FKK98, Prop. 2.3']. As an application of Proposition [2.35, we compute some parabolic Kazhdan-Lusztig basis elements by computing (dual) canonical basis elements.

Example 2.36. We use the notation $\left(i_{1}, \ldots, i_{r}\right) \mid\left(j_{1}, \ldots, j_{s}\right)=\left(i_{1}, \ldots, i_{r}, j_{1}, \ldots, j_{s}\right)$. If $E \cdot b^{i}=0$ or $F \cdot b^{\underline{j}}=0$, then

$$
\sigma_{V_{q}^{r+s}}\left(b^{\underline{i}} \otimes b_{\underline{\underline{j}}}\right)=\Theta\left(b^{\underline{i}} \otimes b^{\underline{j}}\right)=b^{\underline{i}} \otimes b^{\underline{j}}
$$

and hence $b^{\underline{i} \underline{j}}=b^{\underline{i}} \otimes b^{\underline{j}}$. We immediately see that

$$
b^{1 \mid \underline{i}}=v_{1} \otimes b^{\underline{i}} \quad \text { and } \quad b^{i \mid 2}=b^{\underline{i}} \otimes v_{2}
$$

Moreover, it follows from Example 2.29 that $b^{(2,1)}=v_{(2,1)}-q^{-1} v_{(1,2)}$. A direct computation shows that $E \cdot b^{(2,1)}=\stackrel{F \cdot b^{(2,1)}}{ }=0$ and hence

$$
b^{(2,1) \mid \underline{i}}=b^{(2,1)} \otimes b^{\underline{i}} \quad \text { and } \quad b^{i}\left((2,1)=b^{\underline{i}} \otimes b^{(2,1)} .\right.
$$

For example, we get

$$
\begin{aligned}
b^{(2,1,2,1)}=b^{(2,1)} \otimes b^{(2,1)} & =\left(v_{(2,1)}-q^{-1} v_{(1,2)}\right) \otimes\left(v_{(2,1)}-q^{-1} v_{(1,2)}\right) \\
& =v_{(2,1,2,1)}-q^{-1} v_{(1,2,2,1)}-q^{-1} v_{(2,1,1,2)}+q^{-2} v_{(1,2,1,2)} .
\end{aligned}
$$

On the parabolic Hecke modules side, this tells us that

$$
\underline{\tilde{M}}_{s_{2} s_{1} s_{3}}=M_{s_{2} s_{1} s_{3}}-q^{-1} M_{s_{2} s_{3}}-q^{-1} M_{s_{2} s_{1}}+q^{-2} M_{s_{2}}
$$

in $\mathcal{M}^{(2,2)}$.
Remark 2.37. There are recursive formulas that compute the (dual) canonical basis in FKK98. These formulas are used in [FKK98] to derive recursive expressions for Kazhdan-Lusztig polynomials corresponding to maximal parabolic subgroups which were originally obtained in LS81 using other techniques. There also is a graphical calculus for the (dual) canonical basis introduced in [FK97 (see also BS11, (5.12)] and [BS10, (5.3)]) and the action of the elements $E$ and $F$ on the (dual) canonical basis can be described explicitly in this graphical interpretation.

### 2.4 Affine Schur-Weyl duality

In this section we want to look at an affine version of classical Schur-Weyl duality. The natural replacement for the $S_{r}$-action on tensor space will be an action of the affine symmetric group $S_{r}^{\text {aff }}=\mathbb{Z}^{r} \rtimes S_{r}$. The replacement for the Lie algebra $\mathfrak{s l}_{n}$ will be the affine Lie algebra $\widehat{\mathfrak{s l}}_{n}$ which we will define now. Recall that $\mathfrak{s l}_{n}$ can be presented by taking generators $e_{i}, f_{i}$ and $h_{i}$ subject to the Serre relations from Theorem 2.14. The structure of these relations is encoded in the Cartan matrix of $\mathfrak{s l}_{n}$ or equivalently in its Dynkin diagram. We have already encountered the Dynkin diagram of affine type $A$ in Example 1.9. This is the diagram


We can associate a (generalised) Cartan matrix $\tilde{A}=\left(a_{i, j}\right)_{0 \leq i, j \leq n-1}$ to this with entries

$$
\begin{align*}
a_{i j} & =\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle \\
a_{0, j} & =\left\langle-\alpha_{0}, \alpha_{j}^{\vee}\right\rangle \\
a_{i, 0} & =\left\langle\alpha_{i},-\alpha_{0}^{\vee}\right\rangle  \tag{17}\\
a_{0,0} & =\left\langle-\alpha_{0},-\alpha_{0}^{\vee}\right\rangle
\end{align*}
$$

for $1 \leq i, j \leq n-1$.
Definition 2.38. The affine Lie algebra $\widehat{\mathfrak{s l}}_{n}$ is the Lie algebra with generators $e_{i}, f_{i}, h_{i}(0 \leq i \leq n-1)$ subject to the Serre relations from Theorem 2.14 with the $a_{i j}$ being the entries from the generalised Cartan matrix $\tilde{A}$ defined above.

Note that there is a natural inclusion $\mathfrak{s l}_{n} \hookrightarrow \widehat{\mathfrak{s l}}_{n}$ with $e_{i} \mapsto e_{i}$ and $f_{i} \mapsto f_{i}$. The affine Lie algebra $\widehat{\mathfrak{s l}}_{n}$ can also be constructed explicitly as follows. For a Lie algebra $\mathfrak{g}$, the loop algebra of $\mathfrak{g}$ is the (complex) Lie algebra

$$
\mathcal{L}(\mathfrak{g}):=\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}\left[t, t^{-1}\right]
$$

with Lie bracket

$$
[x \otimes p(t), y \otimes q(t)]=[x, y] \otimes p(t) q(t)
$$

We can extend the loop algebra of $\mathfrak{s l}_{n}$ by a one-dimensional centre

$$
\widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right):=\mathcal{L}\left(\mathfrak{s l}_{n}\right) \oplus \mathbb{C} c
$$

with Lie bracket

$$
[x \otimes p(t)+\alpha c, y \otimes q(t)+\beta c]=[x, y] \otimes p(t) q(t)+\kappa(x, y) n \delta_{n,-m} c
$$

where $\kappa(\cdot, \cdot)$ is the Killing form of $\mathfrak{s l}_{n}$.
Theorem 2.39. There is an isomorphism of Lie algebras $\widehat{\mathfrak{s}}_{n} \cong \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)$ which identifies $x \in \mathfrak{s l}_{n} \subset \widehat{\mathfrak{s}}_{n}$ with $x \otimes 1 \in \mathcal{L}\left(\mathfrak{s l}_{n}\right) \subset \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)$.
Proof. This can be found in Kac90, §0.3, §9.11].
Note that there is a natural representation of the loop algebra $\mathcal{L}\left(\mathfrak{s l}_{n}\right)$ (or in fact even $\mathcal{L}\left(\mathfrak{g r}_{n}\right)$ ) given by

$$
\widehat{V}:=V \otimes_{\mathbb{C}} \mathbb{C}\left[t, t^{-1}\right]
$$

with $V=\mathbb{C}^{n}$ and $(x \otimes p(t)) \cdot(v \otimes q(t))=x v \otimes p(t) q(t)$ for $x \in \mathfrak{s l}_{n}, v \in V$ and $p(t), q(t) \in \mathbb{C}\left[t, t^{-1}\right]$. This extends to a representation of $\widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)$ by letting $c$ act by 0 . The vector space $\widehat{V}$ has the standard basis $\left\{v_{i} \mid i \in \mathbb{Z}\right\}$ where (by abuse of notation)

$$
v_{i+k n}:=v_{i} \otimes t^{-k}
$$

for any $i \in\{1, \ldots, n\}$ and $k \in \mathbb{Z}$.
Remark 2.40. The action $\widehat{\mathfrak{s}}_{n} \cong \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right) \curvearrowright \widehat{V}$ can also be constructed using the generators $e_{i}, f_{i}$ and $h_{i}$ from Definition 2.38. In fact, one checks that

$$
\begin{aligned}
e_{i} \cdot v_{j} & =\delta_{\overline{i+1}, \bar{j}} \cdot v_{j-1} \\
f_{i} \cdot v_{j} & =\delta_{i, \bar{j}} \cdot v_{j+1} \\
h_{i} \cdot v_{j} & =\left(\delta_{\bar{i}, \bar{j}}-\delta_{\overline{i+1, j}}\right) \cdot v_{j} .
\end{aligned}
$$

(where $\bar{i} \in\{0, \ldots, n-1\}$ such that $\bar{i} \equiv i \bmod n$ ) is compatible with the Serre relations and hence defines an action $\widehat{\mathfrak{s}}_{n} \curvearrowright \widehat{V}$. Note that acting with $e_{0}$ (resp. $f_{0}$ ) on $\widehat{V}$ is the same as acting with $E_{n, 1} \otimes t \in \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)$ (resp. $\left.E_{1, n} \otimes t^{-1} \in \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)\right)$. Conversely, the action of $e_{i} \otimes t^{k}, f_{i} \otimes t^{k} \in \widehat{\mathcal{L}}\left(\mathfrak{s l}_{n}\right)$ for $1 \leq i \leq n$ and $k \in \mathbb{Z}$ can easily be constructed by acting with the $e_{j}, f_{j} \in \widehat{\mathfrak{s l}}_{n}$ for $0 \leq j \leq n$. This shows that one can use both constructions of the $\widehat{\mathfrak{s l}}_{n}$-action interchangeably. In practice, we will only be working with the loop algebra realisation of this action, though.

There is a natural (right) $S_{r}$-action on $\widehat{V}^{\otimes r}$ (with tensor products taken over $\mathbb{C}$ ) given by permuting the tensor factors. We can also act with $\mathbb{Z}^{r}$ on $\widehat{V}^{\otimes r}$ by letting the $k$-th standard basis vector $\epsilon_{k} \in \mathbb{Z}^{r}$ act by multiplying with $t$ in the $k$-th factor of the tensor product, i.e.

$$
v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \cdot \epsilon_{k}=v_{i_{1}} \otimes \ldots \otimes v_{i_{k}-n} \otimes \ldots \otimes v_{i_{r}} .
$$

One checks on the standard basis of $\widehat{V}^{\otimes r}$ that $s_{k} \circ \epsilon_{k} \circ s_{k}=\epsilon_{k+1}$ in $\operatorname{End}_{\mathbb{C}}\left(\widehat{V}^{\otimes r}\right)$. Hence, we obtain an $S_{r}^{\text {aff }}$-action on $\hat{V}^{\otimes r}$. By construction, the actions

$$
\widehat{\mathfrak{s}}_{n} \curvearrowright \widehat{V}^{\otimes r} \curvearrowleft S_{r}^{\text {aff }}
$$

commute. There is another way we can think about $\widehat{V}^{\otimes r}$.
Lemma 2.41. The action map

$$
\begin{equation*}
V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\text {aff }}\right] \longrightarrow \widehat{V}^{\otimes r} \tag{18}
\end{equation*}
$$

is an isomorphism of $\left(U\left(\mathfrak{s l}_{n}\right), \mathbb{C}\left[S_{r}^{\text {aff }}\right]\right)$-bimodules. Similarly, the action map

$$
\begin{equation*}
V^{\otimes r} \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right] \longrightarrow \widehat{V}^{\otimes r} \tag{19}
\end{equation*}
$$

is an isomorphism of $\left(U\left(\mathfrak{s l}_{n}\right), \mathbb{C}\left[\mathbb{Z}^{r}\right]\right)$-bimodules.

Proof. It is clear that the two maps above are bimodule homomorphisms. The map from 19$]$ is an isomorphism since it identifies the standard basis of $V^{\otimes r} \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right]$ with the standard basis of $\widehat{V}^{\otimes r}$. Note that the multiplication map

$$
\mathbb{C}\left[S_{r}\right] \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right] \longrightarrow \mathbb{C}\left[S_{r}^{\text {aff }}\right]
$$

is an isomorphism of $\left(\mathbb{C}\left[S_{r}\right], \mathbb{C}\left[\mathbb{Z}^{r}\right]\right)$-bimodules. Then the composition

$$
V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right] \cong V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}\right] \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right] \cong V^{\otimes r} \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right] \xrightarrow{\sim} \widehat{V}^{\otimes r}
$$

is the action map from (18). This shows that (18) is also an isomorphism.
Definition 2.42. The algebra $\widehat{S}(n, r):=\operatorname{End}_{S_{r}^{\text {aff }}}\left(\widehat{V}^{\otimes r}\right)$ is called the affine Schur algebra.

We define

$$
\begin{aligned}
\left(\widehat{V}^{\otimes r}\right)_{\lambda} & :=\operatorname{Span}_{\mathbb{C}}\left\{v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \mid \#\left\{l \mid i_{l} \equiv k \quad \bmod n\right\}=\lambda_{k}\right\} \\
& \cong\left(V^{\otimes r}\right)_{\lambda} \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right] \cong\left(V^{\otimes r}\right)_{\lambda} \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\text {aff }}\right]
\end{aligned}
$$

for any composition $\lambda \in \Lambda(n, r)$. We can also interpret $\left(\widehat{V}^{\otimes r}\right)_{\lambda}$ as a permutation module via

$$
\left(\widehat{V}^{\otimes r}\right)_{\lambda} \cong\left(V^{\otimes r}\right)_{\lambda} \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right] \stackrel{(7)}{=} e_{\lambda} \mathbb{C}\left[S_{r}\right] \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right] \cong e_{\lambda} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right]
$$

Most of our arguments about the structure of the Schur algebra can be applied word by word to the affine Schur algebra. In fact, by Lemma 2.7 there is an isomorphism of vector spaces

$$
\widehat{S}(n, r)=\operatorname{End}_{S_{r}^{\mathrm{aff}}}\left(\widehat{V}^{\otimes r}\right) \cong \operatorname{End}_{S_{r}^{\mathrm{aff}}}\left(\bigoplus_{\lambda \in \Lambda(n, r)} e_{\lambda} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right]\right) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, r)} e_{\mu} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right] e_{\lambda}
$$

and for any $x \in e_{\mu} \mathbb{C}\left[S_{r}^{\text {aff }}\right] e_{\lambda}$ there is a corresponding element $\phi_{\mu, \lambda}^{x} \in \widehat{S}(n, r)$. These multiply as in 10 . For $n \geq r$ we can consider

$$
\omega:=(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r}) \in \Lambda(n, r) .
$$

The same arguments as in the classical case show that there is an algebra isomorphism

$$
\phi_{\omega, \omega}^{1} \widehat{S}(n, r) \phi_{\omega, \omega}^{1}=\operatorname{End}_{S_{r}^{\text {aff }}}\left(\left(\widehat{V}^{\otimes r}\right)_{\omega}\right)=\left\{\phi_{\omega, \omega}^{x} \mid x \in \mathbb{C}\left[S_{r}^{\text {aff }}\right]\right\} \cong \mathbb{C}\left[S_{r}^{\text {aff }}\right]
$$

and a corresponding isomorphism

$$
\begin{equation*}
\widehat{V}^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} e_{\lambda} \mathbb{C}\left[S_{r}^{\mathrm{aff}}\right] \cong \bigoplus_{\lambda \in \Lambda(n, r)} \phi_{\lambda, \lambda}^{e_{\lambda}} \widehat{S}(n, r) \phi_{\omega, \omega}^{1} \cong \widehat{S}(n, r) \phi_{\omega, \omega}^{1} \tag{20}
\end{equation*}
$$

of $\left(\widehat{S}(n, r), \phi_{\omega, \omega}^{1} \widehat{S}(n, r) \phi_{\omega, \omega}^{1}\right)$-bimodules.
Proposition 2.43. For $n \geq r$, the commuting actions $\widehat{S}(n, r) \curvearrowright \widehat{V}^{\otimes r} \curvearrowleft S_{r}^{\text {aff }}$ generate each other's centraliser.

Proof. The proof works exactly as in Corollary 2.9. In fact, by the definition of the affine Schur algebra, we have $\widehat{S}(n, r)=\operatorname{End}_{S_{r}^{\text {af }}}\left(\widehat{V}^{\otimes r}\right)$. Moreover, there is an isomorphism $\phi_{\omega, \omega}^{1} \widehat{S}(n, r) \phi_{\omega, \omega}^{1} \xrightarrow{\sim} \operatorname{End}_{\widehat{S}(n, r)}\left(\widehat{S}(n, r) \phi_{\omega, \omega}^{1}\right)^{\text {op }}$ by Lemma 2.7 . It follows from 20 that $S_{r}^{\text {aff }}$ generates $\operatorname{End}_{\widehat{S}(n, r)}\left(\widehat{V}^{\otimes r}\right)^{\text {op }}$ and the proof is complete.

This duality can also be made into a categorical equivalence.
Proposition 2.44. For $n \geq r$, the functor

$$
\begin{aligned}
\mathbb{C}\left[S_{r}^{\mathrm{aff}}\right]-\mathrm{Mod} & \longrightarrow \widehat{S}(n, r) \text {-Mod } \\
M & \longmapsto \widehat{V}^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}^{\text {aff }]}\right.} M
\end{aligned}
$$

is an equivalence of categories. Moreover, this descends to an equivalence of categories $\mathbb{C}\left[S_{r}^{\text {aff }}\right]$ - $\bmod \cong \widehat{S}(n, r)$-mod.

Proof. The proof is the same as the proof of Proposition 2.13. In fact, we have $\widehat{S}(n, r) \phi_{\omega, \omega}^{1} \widehat{S}(n, r)=\widehat{S}(n, r)$ which follows from $1=\sum_{\lambda \in \Lambda(n, r)} \frac{1}{\left|S_{\lambda}\right|} \phi_{\lambda, \lambda}^{e} \lambda$ together with $\phi_{\lambda, \lambda}^{e_{\lambda}} \in \widehat{S}(n, r) \phi_{\omega, \omega}^{1} \widehat{S}(n, r)$ (see $\sqrt{12}$ ). It then follows from Corollary 2.12 that $\widehat{V}^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}^{\text {aff }}\right]}(-)$ is an equivalence of categories since

$$
\widehat{V}^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}^{\text {aff }}\right]}(-) \cong \widehat{S}(n, r) \phi_{\omega, \omega}^{1} \otimes_{\phi_{\omega, \omega}^{1}} \widehat{S}(n, r) \phi_{\omega, \omega}^{1}(-)
$$

as functors from $\mathbb{C}\left[S_{r}^{\text {aff }}\right]-$ Mod to $\widehat{S}(n, r)$-Mod by 20). The equivalence of categories $\mathbb{C}\left[S_{r}^{\text {aff }}\right]-\bmod \cong \widehat{S}(n, r)$-mod also follows from Corollary 2.12 using that the functor

$$
\widehat{V}^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}^{\mathrm{aff}}\right]}(-) \cong V \otimes_{\mathbb{C}\left[S_{r}\right]} \mathbb{C}\left[S_{r}^{\text {aff }}\right] \otimes_{\mathbb{C}\left[S_{r}^{\mathrm{aff}]}\right]}(-) \cong V^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}\right]}(-)
$$

preserves finite-dimensionality since $V^{\otimes r}$ is finite-dimensional.
We see that proving an affine Schur-Weyl duality involving the affine Lie algebra $\widehat{\mathfrak{s}}_{n}$ boils down to showing that the map $U\left(\widehat{\mathfrak{s}}_{n}\right) \longrightarrow \widehat{S}(n, r)$ is surjective. Let us first look at the $\mathbb{Z}^{r}$-endomorphisms on $\widehat{V}^{\otimes r}$ in more detail.

Lemma 2.45. The natural algebra homomorphism $\operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V})^{\otimes r} \rightarrow \operatorname{End}_{\mathbb{Z}^{r}}\left(\widehat{V}^{\otimes r}\right)$ is an isomorphism.

Proof. There is a commutative diagram


The two vertical maps are isomorphisms of vector spaces since $\widehat{V}=V \otimes_{\mathbb{C}} \mathbb{C}\left[t, t^{-1}\right]$ and $\widehat{V}^{\otimes r} \cong V^{\otimes r} \otimes_{\mathbb{C}} \mathbb{C}\left[\mathbb{Z}^{r}\right]$. The horizontal map in the bottom row is an isomorphism of vector spaces since $\left\{E_{i_{1}, j_{1}} \otimes \ldots \otimes E_{i_{r}, j_{r}} \mid j_{l} \in\{1, \ldots, n\}\right.$ and $\left.i_{l} \in \mathbb{Z}\right\}$ is a basis of both $\operatorname{Hom}_{\mathbb{C}}(V, \widehat{V})^{\otimes r}$ and $\operatorname{Hom}_{\mathbb{C}}\left(V^{\otimes r}, \widehat{V}^{\otimes r}\right)$. Hence, the horizontal map in the top row is an isomorphism.

We can use this to compute the $S_{r}^{\text {aff }}$-endomorphisms of $\widehat{V}^{\otimes r}$. In fact,

$$
\begin{equation*}
\operatorname{End}_{S_{r}^{\text {aff }}}\left(\widehat{V}^{\otimes r}\right)=\left(\operatorname{End}_{\mathbb{Z}^{r}}\left(\widehat{V}^{\otimes r}\right)\right)^{S_{r}} \cong\left(\operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V})^{\otimes r}\right)^{S_{r}} \tag{21}
\end{equation*}
$$

where $S_{r}$ acts on $\operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V})^{\otimes r}$ by permuting the tensor factors.

Lemma 2.46. Let $A$ be a $\mathbb{C}$-algebra (not necessarily finite-dimensional). Then $\left(A^{\otimes r}\right)^{S_{r}}$ is generated as an algebra by the elements

$$
\Delta_{r}(a)=a \otimes 1^{\otimes r-1}+1 \otimes a \otimes 1^{\otimes r-2} \ldots+1^{\otimes r-1} \otimes a
$$

where $a \in A$.
Proof. This is well-known for $A$ finite-dimensional but generalises to $A$ infinitedimensional without problem. For example, the proof in [EGH ${ }^{+} 11,5.18 .3$ ] uses finite-dimensionality only to show that for a finite-dimensional $\mathbb{C}$-vector space $W$ the $S_{r}$-invariants of $W^{\otimes r}$ are spanned by the $w \otimes \ldots \otimes w$ for $w \in W$. This extends to $W$ infinite-dimensional since any $x \in W^{\otimes r}$ is contained in $\tilde{W}^{\otimes r}$ for some finitedimensional subspace $\tilde{W} \subset W$.

Using Lemma 2.46 and 21 , we see that $\widehat{S}(n, r)=\operatorname{End}_{S_{r}^{\text {aff }}}\left(\widehat{V}^{\otimes r}\right)$ is generated by the action of the Lie algebra $\operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V})$ (with the commutator bracket) on $\widehat{V}^{\otimes r}$.
Lemma 2.47. The action $\mathcal{L}\left(\mathfrak{g l}_{n}\right) \curvearrowright \widehat{V}$ induces an isomorphism of Lie algebras $\mathcal{L}\left(\mathfrak{g l}_{n}\right) \xrightarrow{\sim} \operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V}) \subset \operatorname{End}_{\mathbb{C}}(\widehat{V})$.
Proof. We have

$$
\operatorname{End}_{\mathbb{C}\left[t, t^{-1}\right]}(\widehat{V}) \cong \operatorname{Hom}_{\mathbb{C}}\left(V, V \otimes_{\mathbb{C}} \mathbb{C}\left[t, t^{-1}\right]\right) \cong \operatorname{End}_{\mathbb{C}}(V) \otimes_{\mathbb{C}} \mathbb{C}\left[t, t^{-1}\right]=\mathcal{L}\left(\mathfrak{g l}_{n}\right)
$$

We have thus shown the following.
Corollary 2.48. The commuting actions $\mathcal{L}\left(\mathfrak{g l}_{n}\right) \curvearrowright \widehat{V}^{\otimes r} \curvearrowleft S_{r}^{\text {aff }}$ generate each other's centraliser.

To be consistent with previous dualities we have seen, we would like to replace $\mathcal{L}\left(\mathfrak{g l}_{n}\right)$ by $\mathcal{L}\left(\mathfrak{s l}_{n}\right)$. This is not quite as obvious as in the classical case since $\mathcal{L}\left(\mathfrak{g l}_{n}\right)$ also contains the elements $I_{n} \otimes t^{k} \in \mathcal{L}\left(\mathfrak{g l}_{n}\right)$ for $k \neq 0$. In contrast to $I_{n} \in \mathfrak{g l}_{n}$, these do not act by multiplying with a scalar.
Lemma 2.49. For $n>r$, the actions of the Lie algebras $\mathcal{L}\left(\mathfrak{s l}_{n}\right)$ and $\mathcal{L}\left(\mathfrak{g l}_{n}\right)$ generate the same subalgebra of $\operatorname{End}_{\mathbb{C}}\left(\widehat{V}^{\otimes r}\right)$.
Proof. Let $\lambda \in \Lambda(n, r)$. By the classical Schur-Weyl duality, we can find an element $X_{\lambda} \in U\left(\mathfrak{s l}_{n}\right)$ such that $X_{\lambda}$ acts as $\mathrm{pr}_{\left(V^{\otimes r}\right)_{\lambda}}$ on $V^{\otimes r}$. By Lemma $2.41, X_{\lambda}$ will then act as $\operatorname{pr}_{\left(\hat{V}^{\otimes r}\right)_{\lambda}}$ on $\widehat{V}^{\otimes r}$. Now let $x \in \mathfrak{g l}_{n}$ and $p(t) \in \mathbb{C}\left[t, t^{-1}\right]$. Since $n>r$, we can find some $k \in\{1, \ldots, n\}$ such that $\lambda_{k}=0$. Then $x \otimes p(t)$ induces the same action on $\left(\widehat{V}^{\otimes r}\right)_{\lambda}$ as $\left(x-\operatorname{Tr}(x) E_{k, k}\right) \otimes p(t)$. Since $x-\operatorname{Tr}(x) E_{k, k} \in \mathfrak{s l}_{n}$, we have shown that $x \otimes p(t) \circ \operatorname{pr}_{(\hat{V} \otimes r)_{\lambda}}$ is induced by the $\mathcal{L}\left(\mathfrak{s l}_{n}\right)$-action. We can do this for any $\lambda \in \Lambda(n, r)$ and $\sum_{\lambda \in \Lambda(n, r)} \operatorname{pr}_{\left(\hat{V}^{\otimes r}\right)_{\lambda}}=1$. Hence the action of $x \otimes p(t)$ can be constructed from the action of $\mathcal{L}\left(\mathfrak{s l}_{n}\right)$. This proves the claim.

We have thus shown the following.
Corollary 2.50. For $n>r$, the commuting actions $\widehat{\mathfrak{s l}}_{n} \curvearrowright \widehat{V}^{\otimes r} \curvearrowleft S_{r}^{\text {aff }}$ generate each other's centraliser.
Proof. This follows from Lemma 2.49 and Corollary 2.48 .
Using Proposition 2.44 this double centraliser property can also be rephrased as the following categorical equivalence which also appears in [CP96, Thm. 4.9].
Corollary 2.51. For $n>r$, the functor $\widehat{V}^{\otimes r} \otimes_{\mathbb{C}\left[S_{r}^{\mathrm{ff}}\right]}(-)$ induces a an equivalence of categories between $\mathbb{C}\left[S_{r}^{\text {aff }}\right]$-mod and the category of finite-dimensional $\widehat{\mathfrak{s l}}_{n}$-modules annihilated by $\operatorname{ker}\left(U\left(\widehat{\mathfrak{s}}_{n}\right) \rightarrow \widehat{S}(n, r)\right)$.

### 2.5 Quantum affine Schur-Weyl duality

In the previous sections, we have seen a quantum and an affine version of classical Schur-Weyl duality. In this section we want to combine the two and describe a quantum affine version of Schur-Weyl duality. The quantum affine replacement of the group $S_{r}$ from the classical setting will be the affine Hecke algebra $\mathcal{H}_{r}^{\text {aff }}:=\mathcal{H}\left(S_{r}^{\text {aff }}\right)$. We still need to define quantum affine $\mathfrak{s l}_{n}$. For any $i \in \mathbb{Z}$, let $\bar{i} \in\{0,1, \ldots, n-1\}$ such that $\bar{i} \equiv i \bmod n$.

Definition 2.52. The affine quantum group $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$ is the $\mathbb{C}(q)$-algebra with generators

$$
E_{0}, E_{1}, \ldots, E_{n-1}, \quad F_{0}, F_{1}, \ldots, F_{n-1}, \quad K_{0}^{ \pm 1}, K_{1}^{ \pm 1}, \ldots, K_{n-1}^{ \pm 1}
$$

and relations (for all $i, j \in \mathbb{Z}$ )
( $\widehat{\mathrm{UQ}} 1$ ) (i) $K_{\bar{i}} K_{\bar{i}}^{-1}=1=K_{\bar{i}}^{-1} K_{\bar{i}}$
(ii) $K_{\bar{i}} K_{\bar{j}}=K_{\bar{j}} K_{\bar{i}}$
( $\widehat{\mathrm{UQ}} 2$ )
(i) $K_{\bar{i}} E_{\bar{j}}=q^{a_{\bar{i}, \bar{j}}} E_{\bar{j}} K_{\bar{i}}$
(ii) $K_{\bar{i}} F_{\bar{j}}=q^{-a_{\bar{i}, \bar{j}}} F_{\bar{j}} K_{\bar{i}}$

$$
\begin{equation*}
\left[E_{\bar{i}}, F_{\bar{j}}\right]=\delta_{\bar{i}, \bar{j}} \frac{K_{\overline{\bar{i}}}-K_{\bar{i}}^{-1}}{q-q^{-1}} \tag{UQ}
\end{equation*}
$$

$(\widehat{\mathrm{UQ}} 4)$
(i) $E_{\bar{i}}^{2} E_{\overline{i \pm 1}}-\left(q+q^{-1}\right) E_{\bar{i}} E_{\overline{i \pm 1}} E_{\bar{i}}+E_{\overline{i \pm 1}} E_{\bar{i}}^{2}=0$
(ii) $F_{\bar{i}}^{2} F_{\overline{i \pm 1}}-\left(q+q^{-1}\right) F_{\bar{i}} F_{\overline{i \pm 1}} F_{\bar{i}}+F_{\overline{i \pm 1}} F_{\bar{i}}^{2}=0$
(iii) $E_{\bar{i}} E_{\bar{j}}=E_{\bar{j}} E_{\bar{i}}$ if $\overline{i-j} \neq 0,1, n-1$
(iv) $F_{\bar{i}} F_{\bar{j}}=F_{\bar{j}} F_{\bar{i}}$ if $\overline{i-j} \neq 0,1, n-1$.
where $A=\left(a_{i, j}\right)_{0 \leq i, j \leq n-1}$ is the (generalised) Cartan matrix of $\widehat{\mathfrak{s l}}_{n}$ as in 17 .
Note that there is a natural algebra homomorphism $U_{q}\left(\mathfrak{s l}_{n}\right) \rightarrow U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$ with $E_{i} \mapsto E_{i}, F_{i} \mapsto F_{i}$ and $K_{i} \mapsto K_{i}$. As for the quantum group $U_{q}\left(\mathfrak{s l}_{n}\right)$, one can check that

$$
\begin{aligned}
\Delta: U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right) & \rightarrow U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right) \otimes_{\mathbb{C}(q)} U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right) \\
E_{i} & \mapsto E_{i} \otimes K_{i}^{-1}+1 \otimes E_{i} \\
F_{i} & \mapsto F_{i} \otimes 1+K_{i} \otimes F_{i} \\
K_{i} & \mapsto K_{i} \otimes K_{i} .
\end{aligned}
$$

defines a coassociative comultiplication on $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$. There also is a natural representation $\widehat{V}_{q}$ of $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$. This is the $\mathbb{C}(q)$-vector space with basis $\left\{v_{i} \mid i \in \mathbb{Z}\right\}$ and action

$$
\begin{aligned}
E_{i} \cdot v_{j} & =\delta_{\overline{i+1}, \bar{j}} \cdot v_{j-1} \\
F_{i} \cdot v_{j} & =\delta_{\bar{i}, \bar{j}} \cdot v_{j+1} \\
K_{i} \cdot v_{j} & =q^{\delta_{\bar{i}, \bar{j}}-\delta_{\overline{i+1}, \bar{j}}} \cdot v_{j} .
\end{aligned}
$$

Hence, we have an action $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right) \curvearrowright \widehat{V}_{q}^{\otimes r}$. We define a (right) $\mathcal{H}_{r}^{\text {aff }}$-action on $\widehat{V}_{q}^{\otimes r}$ by mimicking the identifications from Lemma 2.41. More precisely, we start with the isomorphism of vector spaces

$$
\begin{aligned}
\widehat{V}_{q}^{\otimes r} & \cong V_{q}^{\otimes r} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right] \\
v_{i_{1}-k_{1} n} \otimes \ldots \otimes v_{i_{r}-k_{r} n} & \leftarrow\left(v_{i_{1}} \otimes \ldots \otimes v_{i_{r}}\right) \otimes X_{1}^{k_{1}} \cdot \ldots \cdot X_{r}^{k_{r}} .
\end{aligned}
$$

By Proposition 1.25 , the multiplication map

$$
\mathcal{H}_{r} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right] \longrightarrow \mathcal{H}_{r}^{\text {aff }}
$$

is an isomorphism of $\left(\mathcal{H}_{r}, \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right]\right)$-bimodules and we get an $\mathcal{H}_{r}^{\text {aff }}$-action on $\widehat{V}_{q}^{\otimes r}$ induced along the identifications

$$
\begin{aligned}
\widehat{V}_{q}^{\otimes r} & \cong V_{q}^{\otimes r} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right] \\
& \cong V_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}} \mathcal{H}_{r} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right] \\
& \cong V_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}} \mathcal{H}_{r}^{\text {aff }}
\end{aligned}
$$

Note that by construction, $X_{i}$ acts on $\widehat{V}_{q}^{\otimes r}$ as the shift $v_{k} \mapsto v_{k-n}$ in the $i$-th coordinate. Moreover, the action of $\mathcal{H}_{r} \subset \mathcal{H}_{r}^{\text {aff }}$ on $V_{q}^{\otimes r} \subset \widehat{V}_{q}^{\otimes r}$ is just the usual action from quantum Schur-Weyl duality (i.e. given by the the formulas in Lemma 2.19). However, the action of $\mathcal{H}_{r}$ on the whole space $\widehat{V}_{q}^{\otimes r}$ is more complicated than this. To gain a better understanding of the $\mathcal{H}_{r}^{\text {aff }}$-module $\widehat{V}_{q}^{\otimes r}$, let us first show that the $\mathcal{H}_{r}^{\text {aff }}$-action is local in the following sense.

Lemma 2.53. For $k \in\{1, \ldots, r-1\}$ consider the algebra homomorphism

$$
\iota_{k}: \mathcal{H}_{2}^{\mathrm{aff}} \rightarrow \mathcal{H}_{r}^{\mathrm{aff}}, \quad \iota_{k}\left(X_{1}\right)=X_{k}, \quad \iota_{k}\left(X_{2}\right)=X_{k+1}, \quad \iota_{k}\left(H_{1}\right)=H_{k}
$$

Then for any $x \in \mathcal{H}_{2}^{\text {aff }}$, we have $\iota_{k}(x)=\mathrm{id}^{\otimes k-1} \otimes x \otimes \mathrm{id}^{r-k-1}$ as operators on $\widehat{V}_{q}^{\otimes r}$.
Proof. Let $A:=\left\{x \in \mathcal{H}_{2}^{\text {aff }} \mid \iota_{k}(x)=\mathrm{id}^{\otimes k-1} \otimes x \otimes \mathrm{id}^{r-k-1}\right.$ on $\left.\widehat{V}_{q}^{\otimes r}\right\}$. This is a subalgebra of $\mathcal{H}_{2}^{\text {aff }}$ and we need to show that $A=\mathcal{H}_{2}^{\text {aff }}$. Since $X_{i}$ acts on $\widehat{V}_{q}^{\otimes r}$ as the shift $v_{k} \mapsto v_{k-n}$ in the $i$-th coordinate, we have $X_{1}^{ \pm 1}, X_{2}^{ \pm 1} \in A$ and hence $\mathbb{C}(q)\left[X_{1}^{ \pm 1}, X_{2}^{ \pm 1}\right] \subset A$. Next, we show that $H_{1} \in A$. The space $\widehat{V}_{q}^{\otimes r}$ is spanned by the elements of the form $v \cdot X^{\underline{a}}$ where $v \in V_{q}^{\otimes r}$ and $\underline{a} \in \mathbb{Z}^{r}$ so it suffices to show that $\iota_{k}\left(H_{1}\right)$ acts as $\mathrm{id}^{\otimes k-1} \otimes H_{1} \otimes \mathrm{id}^{r-k-1}$ on $v \cdot X^{\underline{a}}$. Note that the elements $\iota_{k}\left(H_{1}\right)=H_{k}$ and $\mathrm{id}^{\otimes k-1} \otimes H_{1} \otimes \mathrm{id}^{r-k-1}$ both commute with the action of $X_{j}$ for $j \neq k, k+1$. Hence we may assume $X^{\underline{a}}=X_{k}^{m_{1}} X_{k+1}^{m_{2}}=\iota_{k}\left(X_{1}^{m_{1}} X_{2}^{m_{2}}\right)=\iota_{k}\left(X^{\left(m_{1}, m_{2}\right)}\right)$ for some $m_{1}, m_{2} \in \mathbb{Z}$. Then

$$
\begin{aligned}
& v \cdot X^{\underline{a}} \cdot \iota_{k}\left(H_{1}\right)= v \cdot \iota_{k}\left(X^{\left(m_{1}, m_{2}\right)} H_{1}\right) \\
&= v \cdot\left(\iota_{k}\left(H_{1} X^{\left(m_{2}, m_{1}\right)}\right)+\iota_{k}\left(X^{\left(m_{1}, m_{2}\right)} H_{1}-H_{1} X^{\left(m_{2}, m_{1}\right)}\right)\right) \\
& \stackrel{(*)}{=} v \cdot \mathrm{id}^{\otimes k-1} \otimes H_{1} X^{\left(m_{2}, m_{1}\right)} \otimes \mathrm{id}^{r-k-1} \\
&+v \cdot \mathrm{id}^{\otimes k-1} \otimes\left(X^{\left(m_{1}, m_{2}\right)} H_{1}-H_{1} X^{\left(m_{2}, m_{1}\right)}\right) \otimes \mathrm{id}^{r-k-1} \\
&= v \cdot \mathrm{id}^{\otimes k-1} \otimes X^{\left(m_{1}, m_{2}\right)} H_{1} \otimes \mathrm{id}^{r-k-1} \\
&= v \cdot X^{\underline{a}} \cdot \mathrm{id}^{\otimes k-1} \otimes H_{1} \otimes \mathrm{id}^{r-k-1}
\end{aligned}
$$

Here ( $*$ ) uses that $X^{\left(m_{1}, m_{2}\right)} H_{1}-H_{1} X^{\left(m_{2}, m_{1}\right)} \in \mathbb{C}(q)\left[X_{1}^{ \pm 1}, X_{2}^{ \pm 1}\right] \subset A$ by Lemma 1.24 and that $H_{k}$ acts as id ${ }^{\otimes k-1} \otimes H_{1} \otimes \mathrm{id}^{r-k-1}$ on $v \in V_{q}^{\otimes r}$ by the formulas in Lemma 2.19. This shows that $H_{1} \in A$ and hence $A=\mathcal{H}_{2}^{\text {aff }}$ which completes the proof.

By construction, the $\mathcal{H}_{r}^{\text {aff }}$-action on $\widehat{V}_{q}^{\otimes r} \cong V_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}} \mathcal{H}_{r}^{\text {aff }}$ commutes with the $U_{q}\left(\mathfrak{s l}_{n}\right)$-action. However, it is not obvious from the definition that the $\mathcal{H}_{r}^{\text {aff }}$-action commutes with the action of the whole affine quantum group $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$. To prove this, consider the endomorphism

$$
L: \widehat{V}_{q} \longrightarrow \widehat{V}_{q}, \quad v_{i} \mapsto v_{i+1} \quad \forall i \in \mathbb{Z}
$$

Lemma 2.54. $L^{\otimes r}$ commutes with the $\mathcal{H}_{r}^{\text {aff }}$ action on $\widehat{V}_{q}^{\otimes r}$, i.e. $L^{\otimes r} \in \operatorname{End}_{\mathcal{H}_{r}^{\text {aff }}}\left(\widehat{V}_{q}^{\otimes r}\right)$.
Proof. It is clear that $L^{\otimes r}$ commutes with the action of $\mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right]$ on $\widehat{V}_{q}^{\otimes r}$. Consider the vector space

$$
W:=\left\{w \in \widehat{V}_{q}^{\otimes r} \mid L^{\otimes r} \cdot\left(w \cdot H_{i}\right)=\left(L^{\otimes r} \cdot w\right) \cdot H_{i} \text { for } i=1, \ldots, r-1\right\} \subset \widehat{V}_{q}^{\otimes r}
$$

The lemma follows if we can show that $W=\widehat{V}_{q}^{\otimes r}$.
Claim 1: $V_{q}^{\otimes r} \subset W$.
By Lemma 2.53 the element $H_{k} \in \mathcal{H}_{r}^{\text {aff }}$ acts as $\mathrm{id}^{\otimes k-1} \otimes H_{1} \otimes \mathrm{id}^{r-k-1}$ on $V_{q}^{\otimes r}$. Hence, $H_{k}$ commutes with $L^{\otimes r}$ on $V_{q}^{\otimes r}$ if $H_{1}$ commutes with $L \otimes L$ on $V_{q} \otimes V_{q}$. In other words, we may assume $r=2$. We show that $v_{i} \otimes v_{j} \in W$ for any $1 \leq i, j \leq n$. For $1 \leq i, j<n$, the claim follows directly from the formulas in Lemma 2.19. The remaining cases can be checked by direct calculation. In fact, we have

$$
\begin{aligned}
L^{\otimes 2} \cdot\left(v_{i} \otimes v_{n} \cdot H_{s}\right) & =L^{\otimes 2} \cdot v_{n} \otimes v_{i} \\
& =v_{n+1} \otimes v_{i+1} \\
& =v_{1} \otimes v_{i+1} \cdot X_{1}^{-1} \\
& \stackrel{(\mathrm{BER} 3)}{=} v_{1} \otimes v_{i+1} \cdot H_{s} X_{2}^{-1} H_{s} \\
& =v_{i+1} \otimes v_{1} \cdot X_{2}^{-1} H_{s} \\
& =v_{i+1} \otimes v_{n+1} \cdot H_{s} \\
& =\left(L^{\otimes 2} \cdot v_{i} \otimes v_{n}\right) \cdot H_{s}
\end{aligned}
$$

for $1 \leq i<n$, as well as

$$
\begin{aligned}
L^{\otimes 2} \cdot\left(v_{n} \otimes v_{i} \cdot H_{s}\right) & =L^{\otimes 2} \cdot\left(v_{i} \otimes v_{n}+\left(q^{-1}-q\right) v_{n} \otimes v_{i}\right) \\
& =v_{i+1} \otimes v_{n+1}+\left(q^{-1}-q\right) v_{n+1} \otimes v_{i+1} \\
& =v_{i+1} \cdot v_{1} X_{2}^{-1}+\left(q^{-1}-q\right) v_{1} \otimes v_{i+1} X_{1}^{-1} \\
& =v_{1} \otimes v_{i+1} \cdot\left(H_{s} X_{2}^{-1}+\left(q^{-1}-q\right) X_{1}^{-1}\right) \\
& \stackrel{(\text { BER3) }}{=} v_{1} \otimes v_{i+1} \cdot\left(X_{1}^{-1} H_{s}^{-1}+\left(q^{-1}-q\right) X_{1}^{-1}\right) \\
& =v_{1} \otimes v_{i+1} \cdot X_{1}^{-1} H_{s} \\
& =v_{n+1} \otimes v_{i+1} \cdot H_{s} \\
& =\left(L^{\otimes 2} \cdot v_{n} \otimes v_{i}\right) \cdot H_{s}
\end{aligned}
$$

and finally

$$
\begin{aligned}
L^{\otimes 2} \cdot\left(v_{n} \otimes v_{n} \cdot H_{s}\right) & =q^{-1} L^{\otimes 2} \cdot v_{n} \otimes v_{n} \\
& =q^{-1} v_{n+1} \otimes v_{n+1} \\
& =v_{1} \otimes v_{1} \cdot H_{s} X_{2}^{-1} X_{1}^{-1} \\
& \stackrel{(\text { BER3 })}{=} v_{1} \otimes v_{1} \cdot H_{s} X_{2}^{-1} H_{s}\left(H_{s} X_{1} H_{s}\right)^{-1} H_{s} \\
& =v_{1} \otimes v_{1} \cdot X_{1}^{-1} X_{2}^{-1} H_{s} \\
& =v_{n+1} \otimes v_{n+1} \cdot H_{s} \\
& =\left(L^{\otimes 2} \cdot v_{n} \otimes v_{n}\right) \cdot H_{s}
\end{aligned}
$$

This shows Claim 1.
Claim 2: If $w \in W$, then $w \cdot X^{\underline{a}} \in W$ for any $a \in \mathbb{Z}^{r}$.
$L^{\otimes r}$ commutes with the $\mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm r}\right]$-action on $\widehat{V}_{q}^{\otimes r}$ and $X^{\underline{a}} H_{i}-H_{i} X^{\underline{a}} \in$ $\mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right]$ by Lemma 1.24 . This implies

$$
\begin{aligned}
L^{\otimes r} \cdot\left(w \cdot X^{\underline{a}} H_{i}\right) & =L^{\otimes r} \cdot\left(w \cdot H_{i} X^{\underline{a} \cdot s_{i}}\right)+L^{\otimes r} \cdot\left(w \cdot\left(X^{\underline{a}} H_{i}-H_{i} X^{\underline{a} \cdot s_{i}}\right)\right) \\
& =\left(L^{\otimes r} \cdot w\right) \cdot H_{i} X^{\underline{a} \cdot s_{i}}+\left(L^{\otimes r} \cdot w\right) \cdot\left(X^{\underline{a}} H_{i}-H_{i} X^{\underline{a} \cdot s_{i}}\right) \\
& =\left(L^{\otimes r} \cdot w\right) \cdot X^{\underline{a}} H_{i} \\
& =\left(L^{\otimes r} \cdot\left(w \cdot X^{\underline{a}}\right)\right) \cdot H_{i}
\end{aligned}
$$

proving Claim 2.
Combining Claim 1 and Claim 2, we get $\widehat{V}_{q}^{\otimes r}=V_{q}^{\otimes r} \cdot \mathbb{C}\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm r}\right] \subset W$ and hence $W=\widehat{V}_{q}^{\otimes r}$. This finishes the proof.

Corollary 2.55. The actions $U_{q}\left(\widehat{\mathfrak{s}}_{n}\right) \curvearrowright \widehat{V}_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\text {aff }}$ commute.
Proof. Since the $U_{q}\left(\mathfrak{s l}_{n}\right)$-action and the $\mathcal{H}_{r}^{\text {aff }}$-action on $\widehat{V}_{q}^{\otimes r}$ commute, we only need to show that the $\mathcal{H}_{r}^{\text {aff }}$-action commutes with $E_{0}, F_{0}$ and $K_{0}$. To see this, we observe that

$$
L^{-1} E_{1} L \cdot v_{i}=L^{-1} E_{1} v_{i+1}=\delta_{2, \overline{i+1}} L^{-1} v_{i}=\delta_{1, \bar{i}} v_{i-1}=E_{0} \cdot v_{i}
$$

and by a similar argument $L^{-1} F_{1} L \cdot v_{i}=F_{0} \cdot v_{i}$ and $L^{-1} K_{1} L \cdot v_{i}=K_{0} \cdot v_{i}$. It follows that $\left(L^{\otimes r}\right)^{-1} \circ E_{1} \circ L^{\otimes r}=E_{0},\left(L^{\otimes r}\right)^{-1} \circ F_{1} \circ L^{\otimes r}=F_{0}$ and $\left(L^{\otimes r}\right)^{-1} \circ K_{1} \circ L^{\otimes r}=K_{0}$ in $\operatorname{End}_{\mathbb{C}(q)}\left(\widehat{V}_{q}^{\otimes r}\right)$. Since $L^{\otimes r}$ and $E_{1}, F_{1}, K_{1}$ commute with the $\mathcal{H}_{r}^{\text {aff }}$-action, $E_{0}, F_{0}, K_{0}$ also commutes with the $\mathcal{H}_{r}^{\text {aff }}$ action. This proves the claim.

Our next goal is establishing a double centraliser property.
Definition 2.56. The algebra $\widehat{S}_{q}(n, r):=\operatorname{End}_{\mathcal{H}_{r}^{\text {aff }}}\left(\widehat{V}_{q}^{\otimes r}\right)$ is called the affine quantum Schur algebra or short the affine $q$-Schur algebra.

We define

$$
\begin{aligned}
\left(\hat{V}_{q}^{\otimes r}\right)_{\lambda} & :=\operatorname{Span}_{\mathbb{C}}\left\{v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \mid \#\left\{l \mid i_{l} \equiv k \quad \bmod n\right\}=\lambda_{k}\right\} \\
& \cong\left(V_{q}^{\otimes r}\right)_{\lambda} \otimes_{\mathbb{C}(q)} \mathbb{C}(q)\left[X_{1}^{ \pm 1}, \ldots, X_{r}^{ \pm 1}\right] \cong\left(V_{q}^{\otimes r}\right)_{\lambda} \otimes \mathcal{H}_{r} \mathcal{H}_{r}^{\text {aff }}
\end{aligned}
$$

for any $\lambda \in \Lambda(n, r)$. Then

$$
\left(\widehat{V}_{q}^{\otimes r}\right)_{\lambda} \cong\left(V_{q}^{\otimes r}\right)_{\lambda} \otimes_{\mathcal{H}_{r}} \mathcal{H}_{r}^{\mathrm{aff}} \stackrel{[14]}{\cong} x_{\lambda} \mathcal{H}_{r} \otimes_{\mathcal{H}_{r}} \mathcal{H}_{r}^{\text {aff }} \cong x_{\lambda} \mathcal{H}_{r}^{\text {aff }}
$$

and therefore
by Lemma 2.7. For $y \in x_{\mu} \mathcal{H}_{r}^{\text {aff }} x_{\lambda}$ there is a corresponding element

$$
\begin{equation*}
\phi_{\mu, \lambda}^{y} \in \operatorname{Hom}_{\mathcal{H}_{r}^{\text {aff }}}\left(x_{\lambda} \mathcal{H}_{r}^{\text {aff }}, x_{\mu} \mathcal{H}_{r}^{\text {aff }}\right) \subset \widehat{S}_{q}(n, r) \tag{23}
\end{equation*}
$$

under the isomorphism from (22) which acts on $x_{\lambda} \mathcal{H}_{r}^{\text {aff }}$ by multiplying with $y$ on the left (and by 0 on $x_{\lambda^{\prime}} \mathcal{H}_{r}^{\text {aff }}$ for $\lambda^{\prime} \neq \lambda$ ). The $\phi_{\mu, \lambda}^{y}$ multiply as in 10. Note that $\operatorname{pr}_{\left(\hat{V}_{q}^{\otimes r}\right)_{\lambda}}=\frac{1}{\sum_{x \in W_{\lambda}} q^{-2 l(x)}} \phi_{\lambda, \lambda}^{x_{\lambda}}$ by (4). The double centraliser property for the affine $q$-Schur algebra now follows from the same argument as for the (affine) Schur algebra.

Proposition 2.57. For $n \geq r$, the commuting actions $\widehat{S}_{q}(n, r) \curvearrowright \widehat{V}_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\text {aff }}$ generate each other's centraliser.

Proof. See also [Gre97, Thm 2.3.3]. The proof works exactly as the proof of Corollary 2.9 and Proposition 2.43 . In fact, using the identification $\mathcal{H}_{r}^{\text {aff }} \cong \phi_{\omega, \omega}^{1} \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1}$ with $y \mapsto \phi_{\omega, \omega}^{y}$ we get the isomorphism

$$
\widehat{V}_{q}^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \mathcal{H}_{r}^{\mathrm{aff}} \cong \bigoplus_{\lambda \in \Lambda(n, r)} \phi_{\lambda, \lambda}^{x_{\lambda}} \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1} \cong \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1}
$$

of $\left(\widehat{S}_{q}(n, r), \phi_{\omega, \omega}^{1} \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1}\right)$-bimodules. Now Lemma 2.7 implies that the canonical map $\mathcal{H}_{r}^{\text {aff }} \rightarrow \operatorname{End}_{\widehat{S}_{q}(n, r)}\left(\widehat{V}_{q}^{\otimes r}\right)^{\text {op }}$ is an isomorphism.

Similar to the affine situation, this duality can also be made into a categorical equivalence.

Proposition 2.58. For $n \geq r$, the functor

$$
\begin{aligned}
\mathcal{H}_{r}^{\text {aff }}-\mathrm{Mod} & \longrightarrow \widehat{S}_{q}(n, r)-\mathrm{Mod} \\
M & \longmapsto \widehat{V}_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}^{\text {aff }}} M
\end{aligned}
$$

is an equivalence of categories. This also descends to an equivalence of categories $\mathcal{H}_{r}^{\text {aff }}-\bmod \cong \widehat{S}_{q}(n, r)-\bmod$.

Proof. This follows from Corollary 2.12 using $\widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1} \widehat{S}_{q}(n, r)=\widehat{S}_{q}(n, r)$ and the isomorphism of $\left(\widehat{S}_{q}(n, r), \phi_{\omega, \omega}^{1} \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1}\right)$-bimodules $\widehat{V}_{q}^{\otimes r} \cong \widehat{S}_{q}(n, r) \phi_{\omega, \omega}^{1}$. The details work exactly as in the proofs of Proposition 2.44 and Proposition 2.13.

We see that proving a Schur-Weyl duality involving $U_{q}\left(\widehat{\mathfrak{s}}_{n}\right)$ boils down to showing that the homomorphism $U_{q}\left(\widehat{\mathfrak{s}}_{n}\right) \longrightarrow \widehat{S}_{q}(n, r)$ is surjective. This will use the structure of the affine $q$-Schur algebra.

Lemma 2.59. For $n \geq r$, the affine $q$-Schur algebra $\widehat{S}_{q}(n, r)$ is generated by the $\phi_{\mu, \lambda}^{x_{\mu} x_{\lambda}}$ and the $\phi_{\omega, \omega}^{y}$ from 23 with $y \in \mathcal{H}_{r}^{\text {aff }}$ and $\lambda, \mu \in \Lambda(n, r)$.

Proof. By Lemma 2.7 we have $\operatorname{Hom}_{\mathcal{H}_{r}^{\text {aff }}}\left(x_{\lambda} \mathcal{H}_{r}^{\text {aff }}, x_{\mu} \mathcal{H}_{r}^{\text {aff }}\right)=\left\{\phi_{\mu, \lambda}^{y} \mid y \in x_{\mu} \mathcal{H}_{r}^{\text {aff }} x_{\lambda}\right\}$. This shows that $\widehat{S}_{q}(n, r)$ is spanned by the $\phi_{\mu, \lambda}^{y}$ with $y \in x_{\mu} \mathcal{H}_{r}^{\text {aff }} x_{\lambda}$ and $\lambda, \mu \in \Lambda(n, r)$. This implies the lemma using that $\phi_{\mu, \omega}^{x_{\mu}} \circ \phi_{\omega, \omega}^{y} \circ \phi_{\omega, \lambda}^{x_{\lambda}}=\phi_{\mu, \lambda}^{x_{\mu} y_{\lambda}}=p(q) \phi_{\mu, \lambda}^{y}$ with $p(q)=\left(\sum_{x \in W_{\mu}} q^{-2 l(x)}\right)\left(\sum_{x \in W_{\lambda}} q^{-2 l(x)}\right) \in \mathbb{C}(q)^{\times}$by 4. .

Now, we are ready to prove quantum affine Schur-Weyl duality.
Theorem 2.60. For $n>r$, the commuting actions $U_{q}\left(\widehat{\mathfrak{s}}_{n}\right) \curvearrowright \widehat{V}_{q}^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\text {aff }}$ generate each other's centraliser.

Proof. We use ideas from [Gre97, Section 3.3]. By the quantum Schur-Weyl duality we can find $X \in U_{q}\left(\mathfrak{s l}_{n}\right)$ such that $X$ acts as $\phi_{\mu, \lambda}^{x_{\mu} x_{\lambda}}$ on $V_{q}^{\otimes r}$ for any $\lambda, \mu \in \Lambda(n, r)$. This already uniquely determines $X$ as an element of $\widehat{S}_{q}(n, r)$ and hence $X$ acts as $\phi_{\mu, \lambda}^{x_{\mu} x_{\lambda}}$ on $\widehat{V}_{q}^{\otimes r}$. The claim now follows from Lemma 2.59 if we can show that the $\phi_{\omega, \omega}^{y}$ are induced by the $U_{q}\left(\widehat{\mathfrak{s}}_{n}\right)$-action where $y \in \mathcal{H}_{r}^{\text {aff }}$. For $y \in \mathcal{H}_{r}$ this follows again from
the quantum Schur-Weyl duality. Note that $x_{\omega} \in x_{\omega} \mathcal{H}_{r}^{\text {aff }} \cong\left(\widehat{V}_{q}^{\otimes r}\right)_{\omega}$ corresponds to the basis vector $v_{l(\omega)}:=v_{1} \otimes v_{2} \otimes \ldots \otimes v_{r}$. We then have

$$
\begin{aligned}
E_{r} \cdot \ldots \cdot E_{n-1} E_{r-1} \cdot \ldots \cdot E_{0} \cdot v_{l(\omega)} & =E_{r} \cdot \ldots \cdot E_{n-1} \cdot v_{0} \otimes v_{1} \otimes \ldots \otimes v_{r-1} \\
& =v_{r-n} \otimes v_{1} \otimes \ldots \otimes v_{r-1} \\
& =v_{r} \otimes v_{1} \otimes \ldots \otimes v_{r-1} \cdot X_{1} \\
& =v_{l(\omega)} \cdot H_{r-1} \cdot \ldots \cdot H_{1} X_{1}
\end{aligned}
$$

and hence

$$
E_{r} \cdot \ldots \cdot E_{n-1} E_{r-1} \cdot \ldots \cdot E_{0} \circ \phi_{\omega, \omega}^{1}=\phi_{\omega, \omega}^{H_{r-1} \cdot \ldots \cdot H_{1} X_{1}}
$$

on $\widehat{V}_{q}^{\otimes r}$. Similarly, we have

$$
\begin{aligned}
F_{0} F_{n-1} \cdot \ldots \cdot F_{r+1} F_{1} \cdot \ldots \cdot F_{r} \cdot v_{l(\omega)} & =F_{0} F_{n-1} \cdot \ldots \cdot F_{r+1} \cdot v_{2} \otimes v_{3} \otimes \ldots \otimes v_{r+1} \\
& =v_{2} \otimes \ldots \otimes v_{r} \otimes v_{n+1} \\
& =v_{2} \otimes \ldots \otimes v_{r} \otimes v_{1} \cdot X_{r}^{-1} \\
& =v_{l(\omega)} \cdot H_{1} \cdot \ldots \cdot H_{r-1} X_{r}^{-1}
\end{aligned}
$$

and hence

$$
F_{0} F_{n-1} \cdot \ldots \cdot F_{r+1} F_{1} \cdot \ldots \cdot F_{r} \circ \phi_{\omega, \omega}^{1}=\phi_{\omega, \omega}^{H_{1} \cdot \ldots \cdot H_{r-1} X_{r}^{-1}}
$$

Note that $\operatorname{End}_{\mathcal{H}_{r}^{\text {aff }}}\left(x_{\omega} \mathcal{H}_{r}^{\text {aff }}\right) \rightarrow \mathcal{H}_{r}^{\text {aff }}, \phi_{\omega, \omega}^{y} \mapsto y$ is an algebra isomorphism. Moreover, the Bernstein relations imply that $\mathcal{H}_{r}^{\text {aff }}$ is generated as an algebra by $\mathcal{H}_{r}, X_{1}$ and $X_{r}^{-1}$. In particular, $\operatorname{End}_{\mathcal{H}_{r}^{\text {aff }}}\left(x_{\omega} \mathcal{H}_{r}^{\text {aff }}\right)$ is generated by $\phi_{\omega, \omega}^{H_{r-1} \cdots \ldots \cdot H_{1} X_{1}}, \phi_{\omega, \omega}^{H_{1} \cdots \cdot H_{r-1} X_{r}^{-1}}$ and the $\phi_{\omega, \omega}^{y}$ with $y \in \mathcal{H}_{r}$. We see that any $\phi_{\omega, \omega}^{y} \in \operatorname{End}_{\mathcal{H}_{r}^{\text {aff }}}\left(x_{\omega} \mathcal{H}_{r}^{\text {aff }}\right)$ is induced by the action $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$ for any $y \in \mathcal{H}_{r}^{\text {aff }}$. This finishes the proof.

Using Proposition 2.58, this double centraliser property can also be rephrased as the following categorical equivalence which also appears in [CP96, Thm. 4.2].

Corollary 2.61. For $n>r$, the functor $\widehat{V}_{q}^{\otimes r} \otimes_{\mathcal{H}_{r}^{\text {aff }}}(-)$ induces an equivalence of categories between $\mathcal{H}_{r}^{\text {aff }}-\bmod$ and the category of finite-dimensional $U_{q}\left(\widehat{\mathfrak{s l}}_{n}\right)$-modules annihilated by $\operatorname{ker}\left(U_{q}\left(\widehat{\mathfrak{s}}_{n}\right) \rightarrow \widehat{S}_{q}(n, r)\right)$.

## 3 Diagram algebras

In this section we look at more generalisations of Schur-Weyl duality which are still fairly classical in flavour. To this end, we will introduce several algebras with a distinguished basis given by certain partition diagrams. In particular, we will introduce the partition algebra which was defined and studied in detail by Martin Mar91, MS94, Mar96, Mar00]. We will also explain a known Schur-Weyl duality between the partition algebra and the symmetric group. Motivated by this, we will prove a new Schur-Weyl duality in Section 3.3 involving the diagonal action of the affine symmetric group $S_{n}^{\text {aff }}$ on tensor space. For this we will introduce a diagram subalgebra of the partition algebra which we will refer to as the balanced partition algebra. The balanced partition algebra was also defined in Har18 in a slightly different context but we will look at the structure of this algebra in more detail. In particular, we will show that the balanced partition algebra is semisimple and we will parametrise its irreducible representations. We will also give a presentation of the balanced partition algebra by generators and relations.

### 3.1 A Schur-Weyl duality for $S_{n}$

Recall that classical Schur-Weyl duality states that for $V=\mathbb{C}^{n}$, the commuting actions

$$
\mathrm{GL}_{n}(\mathbb{C}) \curvearrowright V^{\otimes r} \curvearrowleft S_{r}
$$

generate each other's centraliser. One can also consider the symmetric group as a subgroup $S_{n} \subset \mathrm{GL}_{n}(\mathbb{C})$ so that $S_{n}$ acts diagonally on tensor space (instead of permuting the tensor factors). Explicitly, this action is given by

$$
\sigma \cdot v_{i_{1}} \otimes \ldots \otimes v_{i_{r}}=v_{\sigma\left(i_{1}\right)} \otimes \ldots \otimes v_{\sigma\left(i_{r}\right)}
$$

for any $\sigma \in S_{n}$ where $v_{1}, \ldots, v_{n}$ is the standard basis of $V$. A natural question to ask is what the centralising partner of this action is

$$
S_{n} \curvearrowright V^{\otimes r} \curvearrowleft ?
$$

This is what we study in this section.
Definition 3.1. Let $r$ be a non-negative integer. A (set) partition is an equivalence relation on the set $\left\{1,2, \ldots, r, 1^{\prime}, 2^{\prime}, \ldots, r^{\prime}\right\}$. The equivalence classes will be referred to as blocks.

We define

$$
A_{r}:=\left\{\text { partitions of }\left\{1,2, \ldots, r, 1^{\prime}, 2^{\prime}, . ., r^{\prime}\right\}\right\}
$$

The partitions in $A_{r}$ are usually drawn as diagrams with $2 r$ dots which are aligned in two rows each containing r dots corresponding to the sets $\{1,2, \ldots, r\}$ and $\left\{1^{\prime}, 2^{\prime}, . ., r^{\prime}\right\}$. We indicate the blocks by connecting the dots that lie in the same block.

Example 3.2. Here are two partitions and diagrams representing them for $r=5$ :
$d_{1}=\left\{\left\{1,2^{\prime}, 3^{\prime}, 5\right\},\left\{1^{\prime}\right\},\{2,3\},\left\{4,4^{\prime}\right\},\left\{5^{\prime}\right\}\right\}, d_{2}=\left\{\{1,5\},\left\{1^{\prime}, 2^{\prime}\right\},\left\{2,3,3^{\prime}, 4^{\prime}\right\},\left\{4,5^{\prime}\right\}\right\}$


When drawing partition diagrams, we will usually omit the labels of the dots. Given two partitions $d_{1}, d_{2} \in A_{r}$, we can define their concatenation $d_{1} \star d_{2}$. This is given by stacking $d_{1}$ on top of $d_{2}$ (i.e. connecting the bottom row of $d_{1}$ with the top row of $d_{2}$ ) and removing all components of that stack that are not connected with the top and the bottom row. We refer to the components removed in the middle of the stack as free blocks.

Example 3.3. Let $d_{1}, d_{2}$ be the two partitions from Example 3.2. Then

is the concatenation of $d_{1}$ and $d_{2}$. Note that we have removed exactly one free block to compute $d_{1} \star d_{2}$ which contains the dots $\left\{1^{\prime}, 5^{\prime}\right\}$ in the bottom row of $d_{1}$ and $\{1,5\}$ in the top row of $d_{2}$.

We now use this concatenation rule for partitions to construct the partition algebra which was introduced in Mar91.

Definition 3.4. For $\delta \in \mathbb{C}$, the partition algebra $P_{r}(\delta)$ is the $\mathbb{C}$-vector space with basis $A_{r}$. On diagrams $d_{1}, d_{2} \in A_{r} \subset P_{r}(\delta)$ the multiplication is defined by

$$
d_{1} \cdot d_{2}:=\delta^{r\left(d_{1}, d_{2}\right)}\left(d_{1} \star d_{2}\right)
$$

with $r\left(d_{1}, d_{2}\right):=\mid\left\{\right.$ free blocks removed in $\left.d_{1} \star d_{2}\right\} \mid$ and we extend this bilinearly to the whole vector space $P_{r}(\delta)$.

It is easy to check that $P_{r}(\delta)$ is an associative $\mathbb{C}$-algebra with unit element

$$
1_{P_{r}(\delta)}=\left\{\left\{1,1^{\prime}\right\},\left\{2,2^{\prime}\right\}, \ldots,\left\{r, r^{\prime}\right\}\right\}=\square \ldots
$$

Example 3.5. For the partitions $d_{1}, d_{2} \in A_{5} \subset P_{5}(\delta)$ from Example 3.2 and Example 3.3, we have

$$
d_{1} \cdot d_{2}=\delta^{r\left(d_{1}, d_{2}\right)} d_{1} \star d_{2} \stackrel{(24)}{=} \delta
$$

Remark 3.6. The reader is probably familiar with the diagrammatic calculus used in the definition of the multiplication of $P_{r}(\delta)$ in another example: Consider the symmetric group $S_{r}$ with its right action on $\{1, \ldots, r\}$. Then a permutation $\sigma \in S_{r}$ can be interpreted as a diagram with $r$ dots on the top and $r$ dots on the bottom and $i$ in the top row connected with $j^{\prime}$ in the bottom row if $i \cdot \sigma=j$. For example, the element $\sigma \in S_{5}$ with $1 \stackrel{\sigma}{\mapsto} 2 \stackrel{\sigma}{\mapsto} 5 \stackrel{\stackrel{\sigma}{\mapsto}}{ } 1$ and $3 \stackrel{\cdot \sigma}{\mapsto} 4 \stackrel{\sigma}{\mapsto} 3$ is represented by the diagram


Multiplication in the symmetric group then just corresponds to stacking diagrams on top of each other. This induces an inclusion $\mathbb{C}\left[S_{r}\right] \hookrightarrow P_{r}(\delta)$ for any $\delta \in \mathbb{C}$.

We will now explain how to equip $V^{\otimes r}$ with the structure of a right $P_{r}(n)$ module (where $n$ is the dimension of $V=\mathbb{C}^{n}$ ). For this we denote the standard basis elements of $V^{\otimes r}$ by

$$
v_{\underline{i}}:=v_{i_{1}} \otimes v_{i_{2}} \otimes \ldots \otimes v_{i_{r}}
$$

where $\underline{i}=\left(i_{1}, i_{2}, \ldots, i_{r}\right) \in\{1,2, \ldots, n\}^{r}$. Then $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ is spanned by the matrices $E_{\underline{j} \underline{,} \underline{ }}$ with $E_{\underline{j} \underline{j} \underline{\underline{d}}} \cdot v_{\underline{k}}=\delta_{\underline{k}, \underline{i}} v_{\underline{j}}$. For any $d \in A_{r}$ and $\underline{i}=\left(i_{1}, \ldots, i_{r}\right), \underline{i^{\prime}}=\left(i_{1^{\prime}}, \ldots, i_{r^{\prime}}\right)$ let

$$
\begin{equation*}
\underline{i} \xrightarrow{d} \underline{i}^{\prime}: \Leftrightarrow\left(t \sim s \text { in } d \Rightarrow i_{t}=i_{s}\right) \text { for any } t, s \in\left\{1, \ldots, r, 1^{\prime}, \ldots, r^{\prime}\right\} . \tag{25}
\end{equation*}
$$

In other words, we have $\underline{i} \xrightarrow{d} \underline{i}^{\prime}$ if labelling the top row of $d$ with $\underline{i}$ and the bottom row of $d$ with $\underline{i}^{\prime}$ induces a well-defined labelling of the blocks of $d$. We call this the $\left(\underline{i}, \underline{i}^{\prime}\right)$-labelling of $d$.

Example 3.7. Let $r=5$ and $n=4$. Consider the partition

from Example 3.2 . We have $(1,2,2,3,1) \xrightarrow{d}(2,1,1,3,4)$ since each block has a unique label with respect to this labelling:


On the other hand, $(1,2,3,3,1) \xrightarrow{d}(2,1,4,3,4)$ since there are two blocks with more than one label with respect to this labelling:


Lemma 3.8. There exists a unique right $P_{r}(n)$-module structure on $V^{\otimes r}$ such that $d \in A_{r}$ acts as $\sum_{\underline{i} \xrightarrow{d} \underline{j}} E_{\underline{j}, \underline{i}} \in \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$.

Proof. Uniqueness is clear, since defining the action on the basis $A_{r}$ already determines the action on the whole algebra. It remains to show that the action from the lemma is well-defined. For this, we need to verify that $\left(v \cdot d_{1}\right) \cdot d_{2}=v \cdot\left(d_{1} \cdot d_{2}\right)$ where $d_{1}, d_{2} \in A_{r}$ and $v \in V^{\otimes r}$. This follows from a calculation in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)^{\mathrm{op}}$ :

$$
\begin{aligned}
& d_{1} \circ \mathrm{op} d_{2}=\left(\sum_{\underline{i} \xrightarrow{d_{1}} \underline{j}} E_{\underline{j}, \underline{i}}\right) \circ \text { op }\left(\sum_{\underline{j} \xrightarrow{d_{2}} \underline{\underline{k}}} E_{\underline{k}, \underline{j}}\right) \\
& =\sum_{\underline{i} \xrightarrow{d_{1}} \underline{j} \xrightarrow{d_{2}} \underline{k}} E_{\underline{k}, \underline{i}} \\
& \stackrel{(*)}{=} n^{r\left(d_{1}, d_{2}\right)} \cdot \sum_{\underline{i} \xrightarrow{d_{1} \star d_{2}} \underline{k}} E_{\underline{k}, \underline{i}} \\
& =n^{r\left(d_{1}, d_{2}\right)} \cdot\left(d_{1} \star d_{2}\right)=d_{1} \cdot d_{2} .
\end{aligned}
$$

For $\stackrel{(*)}{=}$, observe that a sequence $\underline{i} \xrightarrow{d_{1}} \underline{j} \xrightarrow{d_{2}} \underline{k}$ corresponds to a labelling of the blocks of $d_{1} \star d_{2}$ where we also assign a label to each free block removed in $d_{1} \star d_{2}$. For fixed $\underline{i}$ and $\underline{k}$ there are $n^{r\left(d_{1}, d_{2}\right)}$ ways to label the free blocks proving the equality.

Our next goal is to deduce the Schur-Weyl from the bottom row of (2).

Theorem 3.9. The actions $S_{n} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}(n)$ commute and generate each other's centraliser.

Proof. The proof will be done later in this section - see also HR05, Theorem 3.6].
Let us introduce another basis of $P_{r}(\delta)$ which behaves well with respect to the action on tensor space. For this, we introduce a partial order on $A_{r}$ via

$$
\begin{equation*}
d_{1} \leq d_{2}: \Leftrightarrow d_{1} \text { is coarser than } d_{2} \Leftrightarrow\left(t \sim s \text { in } d_{2} \Rightarrow t \sim s \text { in } d_{1}\right) . \tag{26}
\end{equation*}
$$

Definition 3.10. The $\mathbb{C}$-basis $\left\{x_{d} \mid d \in A_{r}\right\}$ of $P_{r}(\delta)$ uniquely determined by the property $\sum_{d^{\prime} \leq d} x_{d^{\prime}}=d$ is called the orbit basis.

Note that this is a well-defined basis since the base-change is given by a unitriangular matrix with entries in $\mathbb{Z}$.

Example 3.11. Consider the partition algebra $P_{1}(\delta)$. We have $A_{1}=\left\{\mathrm{id}, p_{1}\right\}$ where id $=】$ and $p_{1}=\bullet$. Then id $\leq p_{1}$ and hence id $=x_{\mathrm{id}}$ and $p_{1}=x_{\mathrm{id}}+x_{p_{1}}$. In particular, we get $x_{p_{1}}=p_{1}-\mathrm{id}$.

We can also describe the action of the orbit basis on tensor space explicitly. For any $d \in A_{r}$ and $\underline{i}=\left(i_{1}, \ldots, i_{r}\right), \underline{i}^{\prime}=\left(i_{1^{\prime}}, \ldots, i_{r^{\prime}}\right)$ let

$$
\begin{equation*}
\underline{i} \xrightarrow{x_{d}} \underline{i}^{\prime}: \Leftrightarrow\left(t \sim s \text { in } d \Leftrightarrow i_{t}=i_{s}\right) \text { for any } t, s \in\left\{1, \ldots, r, 1^{\prime}, \ldots, r^{\prime}\right\} . \tag{27}
\end{equation*}
$$

In other words, we have $\underline{i} \xrightarrow{x_{d}} \underline{i}^{\prime}$ if the $\left(\underline{i}, \underline{i}^{\prime}\right)$-labelling of $d$ induces a well-defined labelling of the blocks of $d$ and labels for distinct blocks are distinct. In particular, $\underline{i} \xrightarrow{x_{d}} \underline{i}^{\prime}$ implies $\underline{i} \xrightarrow{d} \underline{i}^{\prime}$. Note that for any $\underline{i}, \underline{i}^{\prime} \in\{1, \ldots, n\}^{r}$ there is a unique partition $d\left(\underline{i}, \underline{i}^{\prime}\right) \in A_{r}$ such that $\underline{i} \xrightarrow{x_{d\left(\underline{i}, i^{\prime}\right)}} \underline{i}^{\prime}$. This partition is given by

$$
\begin{equation*}
d\left(\underline{i}, \underline{i}^{\prime}\right):=\left(t \sim s \Leftrightarrow i_{t}=i_{s}\right) . \tag{28}
\end{equation*}
$$

Lemma 3.12. With respect to the action $V^{\otimes r} \curvearrowleft P_{r}(n)$ defined in Lemma 3.8, the element $x_{d}$ acts as $\sum_{\underline{i} \underline{\underline{i}} \underline{\underline{j}}} E_{\underline{j}, \underline{i}} \in \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$.

Proof. We have

$$
\stackrel{i}{\rightarrow} \stackrel{i^{\prime}}{\stackrel{255}{\Rightarrow}}\left(t \sim s \text { in } d \Rightarrow i_{t}=i_{s}\right) \stackrel{(28)}{\Rightarrow}\left(t \sim s \text { in } d \Rightarrow t \sim s \text { in } d\left(\underline{i}, \underline{i}^{\prime}\right)\right) \stackrel{(266)}{\Leftrightarrow} d\left(\underline{i}, \underline{i}^{\prime}\right) \leq d .
$$

Thus, we get in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ :

$$
\sum_{d^{\prime} \leq d} x_{d^{\prime}}=d=\sum_{\substack{\underline{i} \xrightarrow{\underline{j}} \underline{j}}} E_{\underline{j}, \underline{i}}=\sum_{\substack{\underline{i}, j \\ d(i, j \underline{j}) \leq d}} E_{\underline{j}, \underline{i}}=\sum_{d^{\prime} \leq d} \sum_{\underline{i} \xrightarrow{\underline{x_{d^{\prime}}} \underline{j}} \boldsymbol{j}} E_{\underline{j}, \underline{i}} .
$$

It then follows by upwards induction along the partial order that $x_{d}=\sum_{\underline{i} \underline{\underline{i}} \underline{\underline{j}} \underline{\underline{j}}} E_{\underline{j}, \underline{i}}$ in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$.

We will often make use of the following observation when working with the orbit basis. Let $y=\sum_{d \in A_{r}} c_{d} x_{d} \in P_{r}(n)$ and $\underline{i} \xrightarrow{d^{\prime}} \underline{j}$. Since $d^{\prime} \in A_{r}$ is the unique element with $\underline{i} \xrightarrow{x_{d^{\prime}}} \underline{j}$, the coefficient of $E_{\underline{j}, \underline{i}}$ in $y$ (considered as an operator on $V^{\otimes r}$ ) is $c_{d^{\prime}}$.

Corollary 3.13. The kernel of the homomorphism $P_{r}(n) \rightarrow \operatorname{End}\left(V^{\otimes r}\right)^{\text {op }}$ induced by the action $V^{\otimes r} \curvearrowleft P_{r}(n)$ is spanned by $\left\{x_{d} \mid d \in A_{r}\right.$ has more than $n$ blocks $\}$.

Proof. If $d$ has more than $n$ blocks, then there are no $\underline{i}, j$ with $\underline{i} \xrightarrow{x_{d}} j$. This shows that $x_{d}$ acts by 0 on $V^{\otimes r}$. Conversely, if $d$ has at most $n$ blocks, then there exist $\underline{i}, \underline{j}$ such that $\underline{i} \xrightarrow{x_{d}} \underline{j}$ and $x_{d}$ is the unique orbit basis element with non-zero coefficient for $E_{\underline{j}, \underline{i}}$. This shows that the $x_{d}$ with at most $n$ blocks are linearly independent in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$.

We can now show that $S_{n} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}(n)$ commute and generate each other's centraliser.

Proof of Theorem 3.9. For $\sigma \in S_{n}$, we have $\sigma \cdot v_{\underline{i}}=v_{\sigma \cdot \underline{i}}$. Hence, $\sigma \circ E_{\underline{j}, \underline{i}} \circ \sigma^{-1}=$ $E_{\sigma \cdot j, \sigma \cdot \underline{i}}$ in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$. It follows that $\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)$ is spanned by the orbit sums $\sum_{(\underline{i}, \underline{j}) \in \mathcal{O}} E_{\underline{j} \underline{, \underline{l}}}$ where $\mathcal{O}$ is an orbit of the diagonal action $S_{n} \curvearrowright\{1, \ldots, n\}^{r} \times\{1, \ldots, n\}^{r}$. Note that

$$
\left(\underline{i}, \underline{i}^{\prime}\right) \stackrel{S_{n}}{\sim}\left(\underline{j}, \underline{j}^{\prime}\right) \Leftrightarrow\left(i_{t}=i_{s} \Leftrightarrow j_{t}=j_{s}\right) \forall t, s \in\left\{1, \ldots, r, 1^{\prime}, \ldots, r^{\prime}\right\} \Leftrightarrow d\left(\underline{i}, \underline{i}^{\prime}\right)=d\left(\underline{j}, \underline{j}^{\prime}\right)
$$

This shows that $\sum_{(\underline{i}, \underline{j}) \in \mathcal{O}} E_{\underline{j}, \underline{i}}=\sum_{\underline{i} \xrightarrow{x_{d}} j} E_{\underline{j}, \underline{i}}=x_{d}$ for the unique $d \in A_{r}$ with $\underline{i} \xrightarrow{x_{d}} \underline{j}$ for some (and then all) $(\underline{i}, \underline{j}) \in \mathcal{O}$. The $x_{d}$ which do not correspond to an orbit $\mathcal{O}$ as above act by 0 by Corollary 3.13. Since the $x_{d}$ form a basis of $P_{r}(n)$, we have shown that the actions $S_{n} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}(n)$ commute and $P_{r}(n)$ generates $\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)$. Since $\mathbb{C}\left[S_{n}\right]$ is semisimple, the double centraliser theorem implies that $S_{n}$ generates $\operatorname{End}_{P_{r}(n)}\left(V^{\otimes r}\right)$. We have thus shown that the two actions generate each other's centraliser.

We conclude this section with a few extra remarks about the relation between $P_{r}(n)$ and $P_{r}(\delta)$ for arbitrary $\delta \in \mathbb{C}$. To be more precise, we explain an interpolation technique which allows to lift information obtained from acting on tensor space to arbitrary $\delta$. The following lemma, puts this into a formal framework.

Lemma 3.14. Let $B_{1}(\delta), \ldots, B_{l}(\delta), C_{1}(\delta), \ldots, C_{m}(\delta) \in P_{r}(\delta)$ such that

$$
\begin{aligned}
& B_{i}(\delta)=\sum_{d \in A_{r}} p_{i}^{d}(\delta) d \\
& C_{i}(\delta)=\sum_{d \in A_{r}} q_{i}^{d}(\delta) d
\end{aligned}
$$

for some $p_{i}^{d}(x), q_{i}^{d}(x) \in \mathbb{C}[x]$. Assume we are given $p\left(x_{1}, \ldots, x_{l}\right) \in \mathbb{C}\left[x_{1}, \ldots, x_{l}\right]$ and $q\left(y_{1}, \ldots, y_{m}\right) \in \mathbb{C}\left[x_{1}, \ldots, x_{m}\right]$ such that $p\left(B_{1}(n), \ldots, B_{l}(n)\right)=q\left(C_{1}(n), \ldots, C_{m}(n)\right)$ as operators on $V^{\otimes r}$ for infinitely many $n$ (with $n=\operatorname{dim} V$ ). Then $p\left(B_{1}(\delta), \ldots, B_{l}(\delta)\right)=$ $q\left(C_{1}(\delta), \ldots, C_{m}(\delta)\right)$ in $P_{r}(\delta)$ for all $\delta \in \mathbb{C}$.

Proof. Since the multiplication table of the basis $A_{r}$ has entries which are polynomial in $\delta$, there are polynomial $p^{d}(x), q^{d}(x) \in \mathbb{C}[x]$ such that

$$
\begin{aligned}
p\left(B_{1}(\delta), \ldots, B_{l}(\delta)\right) & =\sum_{d \in A_{r}} p^{d}(\delta) d \\
q\left(C_{1}(\delta), \ldots, C_{m}(\delta)\right) & =\sum_{d \in A_{r}} q^{d}(\delta) d
\end{aligned}
$$

By Corollary 3.13 the homomorphism $P_{r}(n) \rightarrow \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)^{\text {op }}$ is injective for $n \gg 0$. Hence, we get $p^{d}(n)=q^{d}(n)$ for infinitely many $n$. This already implies $p^{d}(x)=$ $q^{d}(x)$ which proves the claim.

As an application of the interpolation technique from Lemma 3.14 we compute some products of orbit basis elements. We start with a basic example.

Example 3.15. Consider the partition id $=\left\{\left\{1,1^{\prime}\right\},\left\{2,2^{\prime}\right\}, \ldots,\left\{r, r^{\prime}\right\}\right\}=\lfloor\cdots\rfloor$ in $A_{r}$. By (27), we have $\underline{i} \xrightarrow{x_{\mathrm{id}}} j$ if and only if $\underline{i}=j$ and $i_{k} \neq i_{l}$ for $k \neq l$. By Lemma $\sqrt{3.12}$ this implies that $\bar{x}_{\mathrm{id}}$ acts on $V^{\otimes r}$ as the projection onto the basis vectors $v_{\underline{j}}$ with $i_{k} \neq i_{l}$ for $k \neq l$. In particular, we have $x_{\mathrm{id}}^{2}=x_{\mathrm{id}}$ as operators on $V^{\otimes r}$ for all $n=\operatorname{dim} V \in \mathbb{N}$. Now, the orbit basis is a $\mathbb{Z}$-linear combination of the standard basis $A_{r}$ so we can apply Lemma 3.14 to obtain $x_{\mathrm{id}}^{2}=x_{\mathrm{id}}$ in $P_{r}(\delta)$ for any $\delta \in \mathbb{C}$.

The ideas from Example 3.15 can be easily generalised to compute products of orbit basis elements for so-called propagating partitions.

Definition 3.16. For $d \in A_{r}$, we call a block $B$ of $d$ propagating if $B \cap\{1, \ldots, r\} \neq \emptyset$ and $B \cap\left\{1^{\prime}, . ., r^{\prime}\right\} \neq \emptyset$. We say that $d$ is propagating if all its blocks are propagating. For $d_{1}, d_{2} \in A_{r}$, we say that $d_{1}$ and $d_{2}$ match if $\left(i^{\prime} \sim j^{\prime}\right.$ in $\left.d_{1}\right) \Leftrightarrow\left(i \sim j\right.$ in $\left.d_{2}\right)$ for all $i, j \in\{1, \ldots, r\}$.

Proposition 3.17. Let $d_{1}, d_{2} \in A_{r}$ such that $d_{1}$ or $d_{2}$ is propagating. Then

$$
x_{d_{1}} x_{d_{2}}= \begin{cases}x_{d_{1} \star d_{2}} & \text { if } d_{1} \text { and } d_{2} \text { match }  \tag{29}\\ 0 & \text { otherwise } .\end{cases}
$$

in $P_{r}(\delta)$ for any $\delta \in \mathbb{C}$.
Proof. For $\underline{i} \in\{1, \ldots, n\}^{r}$ we have

$$
v_{\underline{i}} \cdot x_{d_{1}} x_{d_{2}}=\sum_{\underline{\underline{x_{x_{1}}}} \underline{\underline{j}} \underline{\underline{j}}} v_{\underline{j}} \cdot x_{d_{2}}=\sum_{\underline{\underline{i}} \xrightarrow{x_{d_{1}} \underline{j}} \xrightarrow{x_{d_{2}}} \underline{k}} x_{\underline{k}} .
$$

A sequence $\underline{i} \xrightarrow{x_{d_{1}}} \underset{\underline{j}}{ }{ }^{x_{d_{2}}} \underline{k}$ corresponds to a well-defined labelling of the blocks of $d_{1}$ and $d_{2}$ such that distinct blocks of $d_{1}$ (resp. $d_{2}$ ) have distinct labels and the labels of the bottom row of $d_{1}$ and the labels of the top row of $d_{2}$ match. If $d_{1}$ and $d_{2}$ do not match this is not possible and then $x_{d_{1}} x_{d_{2}}=0$ on $V^{\otimes r}$. If $d_{1}$ and $d_{2}$ do match and $\underline{i} \xrightarrow{x_{d_{1}}} \underline{j} \xrightarrow{x_{d_{2}}} \underline{k}$ then $\underline{j}$ is uniquely determined by $\underline{i}$ and $\underline{k}$ since the blocks of $d_{1}$ or $d_{2}$ are all propagating. Hence, such a sequence corresponds to a labelling of the blocks of $d_{1} \star d_{2}$ which assigns to each block a unique label or equivalently $\underline{i} \xrightarrow{x_{d_{1} \star d_{2}}} \underline{k}$. We get

$$
v_{\underline{v_{2}}} \cdot x_{d_{1}} x_{d_{2}}=\sum_{\underline{\underline{i}} \xrightarrow{x_{d_{1} \nless d_{2}}} \underline{\underline{k}}} v_{\underline{k}}=v_{\underline{i}} \cdot x_{d_{1} \nless d_{2}} .
$$

We have thus shown that 29 holds as operators on $V^{\otimes r}$. The claim then follows from Lemma 3.14 using that the orbit basis is $\mathbb{Z}$-linear combination of the standard basis $A_{r}$ of $P_{r}(\delta)$.

Products of orbit basis elements for non-propagating partitions are more complicated than the formula in (29) but they can still be computed using the same techniques, at least for specific examples. To illustrate this, let us start with a low-dimensional example.

Example 3.18. Consider the partition algebra $P_{1}(\delta)$ with the orbit basis $x_{\mathrm{id}}=\mathrm{id}$ and $x_{p_{1}}=p_{1}$-id from Example 3.11. Then

$$
\begin{aligned}
x_{p_{1}}^{2} & =\left(p_{1}-\mathrm{id}\right)^{2}=p_{1}^{2}-2 p_{1}+\mathrm{id}=(\delta-2) p_{1}+\mathrm{id} \\
& =(\delta-2)\left(x_{p_{1}}+x_{\mathrm{id}}\right)+x_{\mathrm{id}}=(\delta-2) x_{p_{1}}+(\delta-1) x_{\mathrm{id}} .
\end{aligned}
$$

We can also verify this equation using Lemma 3.14. In fact, Lemma 3.12, tells us that

$$
\begin{aligned}
v_{i} \cdot x_{p_{1}}^{2} & =\sum_{j \neq i} v_{j} \cdot x_{p_{1}}=\sum_{j \neq i} \sum_{k \neq j} v_{k} \\
& =(n-1) v_{i}+\sum_{k \neq i}(n-2) v_{k}=v_{i} \cdot\left((n-1) x_{\mathrm{id}}+(n-2) x_{p_{1}}\right) .
\end{aligned}
$$

Then applying Lemma 3.14 yields $x_{p_{1}}^{2}=(\delta-1) x_{\mathrm{id}}+(\delta-2) x_{p_{1}}$ for all $\delta \in \mathbb{C}$.
Here is a generalisation of Example 3.18 which will be useful later on.
Lemma 3.19. Consider the partition diagram

$$
\begin{equation*}
d=\bullet \quad \bullet \quad \ldots \quad \text { • } \in A_{r} \tag{30}
\end{equation*}
$$

with $2 r$ blocks. For $d^{\prime} \in A_{r}$ let $\left|d^{\prime}\right|$ be the number of blocks of $d^{\prime}$. Then

$$
\begin{equation*}
x_{d}^{2}=\sum_{d^{\prime}}\left(\prod_{i=1}^{r}\left(\delta-\left|d^{\prime}\right|+1-i\right)\right) x_{d^{\prime}} \tag{31}
\end{equation*}
$$

where the sum runs over all $d^{\prime} \in A_{r}$ with $|B \cap\{1, \ldots, r\}| \leq 1$ and $\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| \leq 1$ for each block $B$ of $d^{\prime}$. For $\delta=2 r-1$, this formula becomes $x_{d}^{2}=(-1)^{r} r!x_{d}$.

Proof. Let $x_{d}^{2}=\sum_{d^{\prime} \in A_{r}} c_{d^{\prime}}(\delta) x_{d^{\prime}}$ for some $c_{d^{\prime}}(\delta) \in \mathbb{C}$. For $n \gg 0$ we can find $\underline{i}, \underline{j} \in\{1, \ldots n\}^{r}$ such that $\underline{i} \xrightarrow{x_{d^{\prime}}} \underline{j}$. Then the coefficient of $E_{\underline{j} \underline{j} \underline{\underline{~}}}$ in $x_{d}^{2} \in \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ is $c_{d^{\prime}}(n)$. By Lemma 3.12 this is the same as the number of possible ways to label the middle row of

with labels $k_{1}, \ldots, k_{r} \in\{1, \ldots, n\}$ such that the $i_{1}, . ., i_{r}, k_{1}, \ldots, k_{r}$ are pairwise distinct and the $k_{1}, \ldots, k_{r}, j_{1}, \ldots, j_{r}$ are pairwise distinct. If $i_{t}=i_{s}$ or $j_{t}=j_{s}$ for some $t \neq s$, it is impossible to find $k_{1}, \ldots, k_{r}$ with this property and then $c_{d^{\prime}}(n)=0$. Otherwise, there are $c_{d^{\prime}}(n)=\frac{(n-m)!}{(n-(r+m)!}$ possible ways to choose $k_{1}, \ldots, k_{r}$ where $m=\left|\left\{i_{1}, \ldots, i_{r}, j_{1}, \ldots, j_{r}\right\}\right|=\left|d^{\prime}\right|$. This proves (31) for $\delta=n \gg 0$. We then get (31) for all $\delta \in \mathbb{C}$ by Lemma 3.14. Note that $\left|d^{\prime}\right| \in\{r, r+1, \ldots, 2 r\}$ for any $d^{\prime} \in A_{r}$ with $|B \cap\{1, \ldots, r\}| \leq 1$ and $\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| \leq 1$ for each block $B$ of $d^{\prime}$. On the other hand, we have $\prod_{i=1}^{r}\left(2 r-1-\left|d^{\prime}\right|+1-i\right)=0$ if $\left|d^{\prime}\right|=r, r+1, \ldots, 2 r-1$ and $\prod_{i=1}^{r}\left(2 r-1-\left|d^{\prime}\right|+1-i\right)=(-1)^{r} r$ ! if $\left|d^{\prime}\right|=2 r$. This shows that (31) becomes $x_{d}^{2}=(-1)^{r} r!x_{d}$ for $\delta=2 r-1$.

### 3.2 The structure of the partition algebra

In this section we summarise a few known facts about the representation theory and the structure of the partition algebra $P_{r}(\delta)$. To be consistent with the Schur-Weyl duality from Theorem 3.9 , we will work with right $P_{r}(\delta)$-modules. Actually, it does not matter whether one prefers left or right modules here since there is an algebra isomorphism $P_{r}(\delta) \cong P_{r}(\delta)^{\text {op }}$ given by flipping diagrams upside down. Note that

$$
\operatorname{dim}_{\mathbb{C}} P_{r}(\delta)=\left|A_{r}\right|=B(2 r)
$$

where $B(m)$ is the number of partitions of the set $\{1, \ldots, m\}$. The number $B(m)$ is often called the $m$-th Bell number.

Remark 3.20. The Bell numbers can be written as $B(m)=\sum_{k=1}^{m} S(m, k)$ where $S(m, k)$ is the number of partitions of the set $\{1, \ldots, m\}$ with $k$ blocks. The numbers $S(m, k)$ are often called the Stirling numbers of the second kind. The $S(m, k)$ can be computed using the recursive formula $S(m, k)=S(m-1, k-1)+m S(m-1, k)$ and there even is an explicit expression: $S(m, k)=\frac{1}{k!} \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^{m}$. This can be used to compute the Bell numbers. Alternatively, the Bell numbers can be computed using the recursion $B(m)=\sum_{k=0}^{m-1}\binom{m-1}{k} B(k)$. These formulas can be found in many text books (see for example [HHM08, (2.96), (2.100) and (2.105)]).

By definition, the $d \in A_{r}$ generate $P_{r}(\delta)$ as an algebra. Clearly, this generating set is far from being minimal and we want to give a more efficient presentation of $P_{r}(\delta)$.

Definition 3.21. For $\delta \in \mathbb{C}$ we define $C_{r}(\delta)$ to be the $\mathbb{C}$-algebra with generators

$$
s_{1}, . ., s_{r-1} \text { and } p_{1}, p_{\frac{3}{2}}, p_{2}, p_{\frac{5}{2}}, \ldots, p_{r-\frac{1}{2}}, p_{r}
$$

and relations (whenever they make sense)
(P1) (i) $s_{i}^{2}=1$
(ii) $s_{i} s_{j}=s_{j} s_{i} \quad$ for $|i-j|>1$
(iii) $s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}$
(i) $p_{i}^{2}=\delta p_{i}$
for $i=1,2, . ., r$
(ii) $p_{i}^{2}=p_{i}$ for $i=\frac{3}{2}, \frac{5}{2}, \ldots, r-\frac{1}{2}$
(iii) $p_{i} p_{i \pm \frac{1}{2}} p_{i}=p_{i}$
(iv) $p_{i} p_{j}=p_{j} p_{i} \quad$ for $|i-j|>\frac{1}{2}$
(P3) (i) $s_{i} p_{i} p_{i+1}=p_{i} p_{i+1} s_{i}=p_{i} p_{i+1}$
(ii) $s_{i} p_{i+\frac{1}{2}}=p_{i+\frac{1}{2}}=p_{i+\frac{1}{2}} s_{i}$
(iii) $s_{i} p_{i} s_{i}=p_{i+1}$
(iv) $s_{i} s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_{i}=p_{i+\frac{3}{2}}$
(v) $s_{i} p_{j}=p_{j} s_{i}$ for $j \neq i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}$.

Theorem 3.22. There is an isomorphism of algebras


Proof. See [HR05, Thm 1.11]
Remark 3.23. Even though the generating set of the presentation above is much smaller than the set $A_{r}$, it is still not minimal. For example, the elements $p_{2}, \ldots, p_{r}$ can be omitted since $p_{i+1}=s_{i} p_{i} s_{i}$ by (P3)-(iii).

Next, we study semisimplicity of $P_{r}(\delta)$. By the double centraliser theorem $\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\text {op }}$ is always semisimple and $P_{r}(n) \cong \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\text {op }}$ for $n \geq 2 r$ by Corollary 3.13. This argument can be extended to $n=2 r-1$ as follows: By Corollary 3.13 the kernel $K$ of the action $V^{\otimes r} \curvearrowleft P_{r}(2 r-1)$ is one-dimensional and spanned by $x_{d}$ where $d$ is the partition with $2 r$ blocks as in (30). Since $P_{r}(2 r-1) / K$ is semisimple we have $\operatorname{Rad}\left(P_{r}(2 r-1)\right) \subset K$ and hence $\operatorname{Rad}\left(P_{r}(2 r-1)\right)=K$ or $\operatorname{Rad}\left(P_{r}(2 r-1)\right)=0$. By Lemma 3.19 we have $x_{d}^{2}=(-1)^{r} r!x_{d}$ in $P_{r}(2 r-1)$. Since $\operatorname{Rad}\left(P_{r}(2 r-1)\right)$ is nilpotent we get $\operatorname{Rad}\left(P_{r}(2 r-1)\right)=0$ and hence $P_{r}(2 r-1)$ is semisimple. We have thus shown the following.

Proposition 3.24. The partition algebra $P_{r}(n)$ is semisimple for any integer $n$ with $n \geq 2 r-1$.

Seeing this argument, one might think that $P_{r}(\delta)$ is always semisimple or at least for positive integers $\delta=n$. However, this is not the case.

Proposition 3.25. The algebra $P_{r}(1)$ is not semisimple for any $r>1$.
Proof. There is an algebra homomorphism $P_{r}(1) \xrightarrow{e v_{1}} \mathbb{C}, d \mapsto 1$. This is well-defined since $d \cdot d^{\prime} \in A_{r} \subset P_{r}(1)$ for any $d, d^{\prime} \in A_{r}$. The homomorphism $e v_{1}$ defines a (right) $P_{r}(1)$-module structure on $\mathbb{C}$ which we denote by $L^{(r)}(\emptyset)$. The result then follows if we can show the following claim.
Claim: The homomorphism of right $P_{r}(1)$-modules $P_{r}(1) \xrightarrow{e v_{1}} L^{(r)}(\emptyset)$ does not split. Proof of claim: If the homomorphism does split, we can find a non-zero element $y=\sum c_{d} d \in P_{r}(1)$ such that $y \cdot d^{\prime}=y$ for all $d^{\prime} \in A_{r}$. For any $d \in A_{r}$ we have $1^{\prime} \sim 2^{\prime}$ in $d \star p_{\frac{3}{2}}$. Since $y=y \cdot p_{\frac{3}{2}}$, we get that $c_{d}=0$ if $1^{\prime} \nsim 2^{\prime}$ in $d$. On the other hand, $1^{\prime} \nsim 2^{\prime}$ in $d \star p_{1}$ for any $d \in A_{r}$. Since $y=y \cdot p_{1}$ we get $c_{d}=0$ if $1^{\prime} \sim 2^{\prime}$ in $d$. This implies $y=0$ which is a contradiction.

There also is a general semisimplicity result for the partition algebra.

Theorem 3.26. For any $\delta \in \mathbb{C}$ the partition algebra $P_{r}(\delta)$ is semisimple if and only if $\delta \notin\{0,1, \ldots, 2 r-2\}$

Proof. It is not hard to prove that $P_{r}(\delta)$ is semisimple for all but finitely many $\delta \in \mathbb{C}$ (see for example [CO11, proof of Thm. 3.15]). The key idea is to use that a finite-dimensional algebra $A$ is semisimple if and only if its trace form is non-degenerate. One then uses that the trace form of $P_{r}(\delta)$ is represented by a matrix whose entries are polynomial in $\delta$. This is degenerate for only finitely many $\delta$ where the determinant of this matrix vanishes. Finding the explicit values where the algebra is not semisimple is a bit harder. This was done in [MS94].

Our next goal is to parametrise the irreducible representations of $P_{r}(\delta)$. Let $A$ be a finite-dimensional algebra and let $e \in A$ be an idempotent. If $A e A=A$ there is an equivalence of categories $e A e-\bmod \cong A$ - $\bmod$ by Corollary 2.12. In particular, if $A e A=A$ we have a bijection

$$
\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple } \\
A \text {-modules }
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple } \\
e A e \text {-modules }
\end{array}\right\}
$$

For $A e A \neq A$, there are more simple $A$-modules than simple $e A e$-modules. In fact, the next result states that we are missing exactly the simple $A$-modules which are the restriction of a simple $A /(e)$-module along the projection $A \rightarrow A /(e)$ (where $(e)=A e A$ is the two-sided ideal generated by $e$ ). We formulate this result for right modules which is more natural in the partition algebra setting.

Lemma 3.27. Let $A$ be a finite-dimensional $\mathbb{C}$-algebra and $e \in A$ an idempotent. Then there is a bijection

$$
\begin{aligned}
\left.\begin{array}{c}
\left.\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
A \text {-modules }
\end{array}\right\}
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
A /(e) \text {-modules }
\end{array}\right\} \sqcup\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
e A e \text {-modules }
\end{array}\right\} \\
\operatorname{Res}_{A}^{A /(e)}(S) \longleftrightarrow S \in \bmod -A /(e) \\
\operatorname{Head}\left(S \otimes_{e A e} e A\right) \longleftrightarrow S \in \bmod -e A e
\end{aligned}
$$

Proof. The simple right $A /(e)$-modules are in a 1:1 correspondence with the simple right $A$-modules that are annihilated by $e$. One can show that the simple right $A$-modules $S$ with $S e \neq 0$ are in a 1:1 correspondence with the simple right $e A e-$ modules using the functors $M \mapsto M e$ and $N \mapsto N \otimes_{e A e} e A$ from (11). Details can be found in Gre06, Thm 6.2g].

Consider the idempotent $e=p_{r-\frac{1}{2}}$ of $P_{r}(\delta)$. There is an algebra isomorphism

$$
\begin{equation*}
e P_{r}(\delta) e \xrightarrow{\sim} P_{r-1}(\delta), \quad d \mapsto d \backslash\left\{r, r^{\prime}\right\} \tag{32}
\end{equation*}
$$

where $d \backslash\left\{r, r^{\prime}\right\}$ is the partition in $A_{r-1}$ obtained from $d$ by forgetting the dots with label $r$ and $r^{\prime}$. Hence, we can describe the irreducible representations of $P_{r}(\delta)$ inductively using Lemma 3.27 if we can describe the irreducible representations of the algebra $P_{r}(\delta) /(e)$.

Lemma 3.28. For the idempotent $e=p_{r-\frac{1}{2}}$, we have $(e)=\operatorname{Span}_{\mathbb{C}} A_{r} \backslash S_{r}$.
Proof. Let $d \in A_{r} \backslash S_{r}$. If $d$ has a block $B$ with $|B \cap\{1, \ldots, r\}|>1$ we can find $\sigma \in S_{r}$ such that $r \sim r-1$ in $\sigma d$. Then $d=\sigma^{-1} e \sigma d \in(e)$. Similarly, if $\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|>1$
we can find $\sigma \in S_{r}$ such that $r^{\prime} \sim(r-1)^{\prime}$ in $d \sigma$ and then $d=d \sigma e \sigma^{-1} \in(e)$. If all blocks of $d$ satisfy $|B \cap\{1, \ldots, r\}| \leq 1$ and $\mid B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\} \leq 1$ there must be a block $B$ of $d$ with $|B|=1$. Then $d=p_{i} d^{\prime}$ if $B=\{i\}$ and $d=d^{\prime} p_{i}$ if $B=\left\{i^{\prime}\right\}$ where $d^{\prime} \in A_{r}$ is the partition obtained from $d$ by connection $i$ with $i^{\prime}$. We have $p_{r}=p_{r} e p_{r}$ and $p_{i}=\sigma p_{r} \sigma$ for some $\sigma \in S_{r}$. This shows that $p_{i} \in(e)$ and hence $d \in(e)$. This proves $\operatorname{Span}_{\mathbb{C}} A_{r} \backslash S_{r} \subset(e)$. For $d \in A_{r}$ let $\mathrm{pn}(d)$ be the number of propagating blocks of $d$. Then $\operatorname{pn}\left(d \star d^{\prime}\right) \leq \min \left\{\operatorname{pn}(d), \operatorname{pn}\left(d^{\prime}\right)\right\}$ since any propagating block of $d \star d^{\prime}$ is obtained by merging at least one propagating block of $d$ with at least one propagating block of $d^{\prime}$. Moreover, $\operatorname{pn}(d)=r \Leftrightarrow d \in S_{r}$ and hence $\operatorname{Span}_{\mathbb{C}} A_{r} \backslash S_{r}=\operatorname{Span}_{\mathbb{C}}\left\{d \in A_{r} \mid \operatorname{pn}(d)<r\right\}$ is an ideal containing $e$. This shows that $(e)=\operatorname{Span}_{\mathbb{C}} A_{r} \backslash S_{r}$.

Here is the classification theorem of the simple $P_{r}(\delta)$-modules.
Theorem 3.29. For $\delta \neq 0$ there is a bijection

$$
\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
P_{r}(\delta) \text {-modules }
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { Young diagrams } \lambda \\
\text { with } 0 \leq|\lambda| \leq r
\end{array}\right\} .
$$

For $\delta=0$ and $r>0$, there is a bijection

$$
\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
P_{r}(0) \text {-modules }
\end{array}\right\} \stackrel{\stackrel{1: 1}{\longleftrightarrow}}{\longleftrightarrow}\left\{\begin{array}{c}
\text { Young diagrams } \lambda \\
\text { with } 0<|\lambda| \leq r
\end{array}\right\} .
$$

Proof. See also [Mar96, Corollary 5.1] or [CO11, Theorem 3.4]. The claim is obvious for $r=0$ since $P_{0}(\delta)=\mathbb{C}$. We have $P_{1}(\delta)=\operatorname{Span}_{\mathbb{C}}\left\{1, p_{1}\right\} \cong \mathbb{C}[X] /\left(X^{2}-\delta X\right)$. For $\delta \neq 0$ this has two irreducible representations which we can be identified with $\{\emptyset, \square\}$. For $\delta=0$ we have $P_{1}(0) \cong \mathbb{C}[X] /\left(X^{2}\right)$. This has one irreducible representation which we identify with $\{\square\}$. Hence, we have shown the claim for $r=1$. The composition $\mathbb{C}\left[S_{r}\right] \rightarrow P_{r}(\delta) \rightarrow P_{r}(\delta) /(e)$ is an isomorphism by 3.28 and the irreducible representations of $S_{r}$ are indexed by Young diagrams $\lambda$ with $|\lambda|=r$. Moreover $e P_{r}(\delta) e \cong P_{r-1}(\delta)$ by (32). Lemma 3.27 then tells us that

$$
\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
P_{r}(\delta) \text {-modules }
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
P_{r-1}(\delta) \text {-modules }
\end{array}\right\} \sqcup\left\{\begin{array}{c}
\text { Young diagrams } \lambda \\
\text { with }|\lambda|=r
\end{array}\right\} .
$$

The claim now follows by induction.
Remark 3.30. Lemma 3.27 also gives a recipe to recursively construct the simple $P_{r}(\delta)$-modules. For $r=1$ the irreducible $P_{1}(\delta)$-modules are

$$
\begin{aligned}
L^{(1)}(\emptyset) & :=P_{1}(\delta) /\left(p_{1}-\delta\right) \\
L^{(1)}(\square) & :=P_{1}(\delta) /\left(p_{1}\right)
\end{aligned}
$$

(which are equal for $\delta=0$ ). Now assume we have constructed the simple $P_{r-1}(\delta)$ modules $L^{(r-1)}(\lambda)$ for $0 \leq|\lambda| \leq r-1$. Then the simple $P_{r}(\delta)$-modules are defined as:

$$
L^{(r)}(\lambda):= \begin{cases}\operatorname{Res}_{P_{r}(\delta)}^{\mathbb{C}\left[S_{r}\right]} S(\lambda) & \text { if }|\lambda|=r \\ \operatorname{Head}\left(L^{(r-1)}(\lambda) \otimes_{e P_{r}(\delta) e} e P_{r}(\delta)\right) & \text { if }|\lambda|<r\end{cases}
$$

Here $e=p_{r-\frac{1}{2}}$, the module $S(\lambda)$ is the Specht module of $S_{r}$ corresponding to $\lambda$ and the restriction is along the algebra homomorphism $P_{r}(\delta) \rightarrow P_{r}(\delta) /(e) \cong \mathbb{C}\left[S_{r}\right]$.

The double centraliser theorem tells us that decomposing $V^{\otimes r}$ as a $\left(S_{n}, P_{r}(n)\right)$ bimodule gives a bijection between simple $\mathbb{C}\left[S_{n}\right]$-modules and simple $P_{r}(n)$-modules appearing in $V^{\otimes r}$. We can now ask what this bijection does with respect to the indexing set from Theorem 3.29, Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{l}\right)$ be a Young diagram with $0 \leq|\lambda| \leq r$. We define

$$
\lambda_{[n]}:=\left(n-|\lambda|, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}\right)
$$

which defines a Young diagram if and only if $n-|\lambda| \geq \lambda_{1}$.
Example 3.31. We decompose $V$ as a $\left(\mathbb{C}\left[S_{n}\right], P_{1}(n)\right.$ )-bimodule for $n \geq 2$. Note that $v_{1}+\ldots+v_{n}$ spans a copy of the trivial representation which corresponds to the Young diagram $(n)$. On the other hand, $\left(v_{1}+\ldots+v_{n}\right) \cdot p_{1}=n\left(v_{1}+\ldots+v_{n}\right)$ so $v_{1}+\ldots+v_{n}$ spans a copy of the $P_{1}(n)$-representation $L^{(1)}((0))=L^{(1)}(\emptyset)$. The space $W:=\left\{\sum_{i=1}^{n} a_{i} v_{i} \mid \sum_{i=1}^{n} a_{i}=0\right\} \subset V$ is also an irreducible $S_{n}$-representation which corresponds to the Young diagram $(n-1,1)$ (see Example 4.16). Note that $x \cdot p_{1}=0$ for all $x \in W$. Hence, $W$ is a direct sum of $n-1$ copies of the $P_{1}(n)$-representation $L^{(1)}((1))=L^{(1)}(\square)$. This shows that

$$
V \cong\left(S((n)) \otimes L^{(1)}((0))\right) \oplus\left(S((n-1,1)) \otimes L^{(1)}((1))\right)
$$

as $\left(\mathbb{C}\left[S_{n}\right], P_{1}(n)\right)$-bimodules. Note that $(n)=(0)_{[n]}$ and $(n-1,1)=(1)_{[n]}$, so $V \cong \bigoplus_{0 \leq|\lambda| \leq 1} S\left(\lambda_{[n]}\right) \otimes L^{(r)}(\lambda)$.

Here is the bimodule decomposition of $V^{\otimes r}$ for general $r \in \mathbb{N}_{0}$.
Proposition 3.32. There is an isomorphism of $\left(\mathbb{C}\left[S_{n}\right], P_{r}(n)\right)$-bimodules

$$
V^{\otimes r} \cong \bigoplus_{\lambda} S\left(\lambda_{[n]}\right) \otimes L^{(r)}(\lambda)
$$

where the sum runs over all Young diagrams $\lambda$ with $0 \leq|\lambda| \leq r$ and $n-|\lambda| \geq \lambda_{1}$.
Proof. This is a bit technical. Nonetheless, we outline the main ideas of the proof since this is omitted in the literature. By the double centraliser theorem, we have a decomposition of the $\left(\mathbb{C}\left[S_{n}\right], P_{r}(n)\right)$-bimodule

$$
V^{\otimes r} \cong \bigoplus_{0 \leq|\lambda| \leq r} A^{(r)}(\lambda) \otimes L^{(r)}(\lambda)
$$

where the $A^{(r)}(\lambda)$ are distinct simple $\mathbb{C}\left[S_{n}\right]$-modules or 0 . We need to show that

$$
A^{(r)}(\lambda)= \begin{cases}S\left(\lambda_{[n]}\right) & \text { if } n-|\lambda| \geq \lambda_{1}  \tag{33}\\ 0 & \text { otherwise. }\end{cases}
$$

For $r=0,1$ the claim can be checked by hand as in Example 3.31. We proceed by induction. It is not hard to check that

$$
L(\lambda)^{(r)} p_{r-\frac{1}{2}} \cong \begin{cases}L^{(r-1)}(\lambda) & \text { if }|\lambda|<r \\ 0 & \text { if }|\lambda|=r\end{cases}
$$

as right $p_{r-\frac{1}{2}} P_{r}(n) p_{r-\frac{1}{2}} \cong P_{r-1}(n)$-modules. In particular

$$
\bigoplus_{0 \leq|\lambda|<r} A^{(r-1)}(\lambda) \otimes L^{(r-1)}(\lambda) \cong V^{\otimes r-1} \cong V^{\otimes r} \cdot p_{r-\frac{1}{2}} \cong \bigoplus_{0 \leq|\lambda|<r} A^{(r)}(\lambda) \otimes L^{(r-1)}(\lambda)
$$

as $\left(\mathbb{C}\left[S_{n}\right], P_{r-1}(n)\right.$ )-bimodules. This proves 33$)$ for $0 \leq|\lambda|<r$ using the induction hypothesis. Let

$$
W:=\operatorname{Span}_{\mathbb{C}}\left\{v_{\underline{i}} \mid i_{j} \neq i_{k} \text { for } j \neq k\right\} \supset\left\{v \in V^{\otimes r} \left\lvert\, v \cdot\left(p_{r-\frac{1}{2}}\right)=0\right.\right\} .
$$

Then $A^{(r)}(\lambda) \otimes L^{(r)}(\lambda) \subset W$ for $|\lambda|=r$ since $L^{(r)}(\lambda) \cdot\left(p_{r-\frac{1}{2}}\right)=0$. A combinatorial argument as in Del07, Prop. 6.4] shows that

$$
W \cong \bigoplus_{\lambda, \mu} S(\mu) \otimes S(\lambda)
$$

as $\left(\mathbb{C}\left[S_{n}\right], \mathbb{C}\left[S_{r}\right]\right.$ )-bimodules where the sum runs over all $\lambda, \mu$ with $|\lambda|=r,|\mu|=n$ such that $\lambda \subset \mu$ and $\mu / \lambda$ is a horizontal strip. The $S(\mu)$ appearing together with $S(\lambda)=\operatorname{Res}_{S_{r}}^{P_{r}(n)} L^{(r)}(\lambda)$ are then of the form $S(\mu)=S\left(\lambda_{[n]}^{\prime}\right)$ where $\left|\lambda^{\prime}\right|<r$ or $\lambda^{\prime}=\lambda$. Here the latter case occurs if and only if $n-|\lambda| \geq \lambda_{1}$. Since we already know that $S\left(\lambda_{[n]}^{\prime}\right)=A^{(r)}\left(\lambda^{\prime}\right)$ if $\left|\lambda^{\prime}\right|<r$, we get $A^{(r)}(\lambda)=S\left(\lambda_{[n]}\right)$ if $n-|\lambda| \geq \lambda_{1}$ and $A^{(r)}(\lambda)=0$ otherwise. This proves the claim.

### 3.3 A Schur-Weyl duality for $S_{n}^{\text {aff }}$

Recall the affine symmetric group $S_{n}^{\text {aff }}:=\mathbb{Z}^{n} \rtimes S_{n}$ from Section 1.3. In Lemma 1.20 we have seen that for any $x \in \mathbb{C}^{\times}$the inclusion $S_{n} \hookrightarrow \mathrm{GL}_{n}(\mathbb{C})$ sending $\sigma \in S_{n}$ to the corresponding permutation matrix $P_{\sigma}$ extends to a group homomorphism

$$
\begin{aligned}
& S_{n}^{\text {aff }} \rightarrow \mathrm{GL}_{n}(\mathbb{C}) \\
& S_{n} \ni \sigma \mapsto P_{\sigma} \\
& \mathbb{Z}^{n} \ni\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(\begin{array}{cccc}
x^{a_{1}} & 0 & \cdots & 0 \\
0 & x^{a_{2}} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & x^{a_{n}}
\end{array}\right)
\end{aligned}
$$

This induces a diagonal action of $S_{n}^{\text {aff }}$ on tensor space and we can ask as before what the centralising partner of this action is

$$
\begin{equation*}
S_{n}^{\mathrm{aff}} \curvearrowright V^{\otimes r} \curvearrowleft ? \tag{34}
\end{equation*}
$$

This is what we study in this section.
Let $\Phi_{x}: \mathbb{C}\left[S_{n}^{\text {aff }}\right] \rightarrow \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ be the homomorphism induced by the $S_{n}^{\text {aff }}$-action on $V^{\otimes r}$. The first thing we have to worry about is how the answer to (34) depends on $x$. For this, we look at the action of $\mathbb{Z}^{n} \subset S_{n}^{\text {aff }}$ on $V^{\otimes r}$. The abelian group $\mathbb{Z}^{n}$ is generated by the standard basis vectors $\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{n}$. These act on $V^{\otimes r}$ via

$$
\begin{equation*}
\epsilon_{k} \cdot\left(v_{i_{1}} \otimes \ldots \otimes v_{i_{r}}\right)=x^{\#\left\{| | i_{i}=k\right\}} \cdot v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} . \tag{35}
\end{equation*}
$$

For $x \in \mathbb{C}^{\times}$, we denote by $\operatorname{ord}(x)$ the multiplicative order of $x$.
Lemma 3.33. Let $x^{\prime} \in \mathbb{C}^{\times}$. Then we have $\operatorname{Im}\left(\Phi_{x^{\prime}}\right) \subset \operatorname{Im}\left(\Phi_{x}\right)$ for any $x \in \mathbb{C}^{\times}$with $\operatorname{ord}(x)>r$. In particular, $\operatorname{Im}\left(\Phi_{x}\right)=\operatorname{Im}\left(\Phi_{x^{\prime}}\right)$ whenever $\operatorname{ord}\left(x^{\prime}\right)>r$ and ord $(x)>r$.

Proof. Let $k \in\{1, \ldots, n\}, m \in\{0, \ldots, r\}$ and

$$
W_{m}=\operatorname{Span}_{\mathbb{C}}\left\{v_{i_{1}} \otimes \ldots \otimes v_{i_{r}} \in V^{\otimes r} \mid \#\left\{l \mid i_{l}=k\right\}=m\right\} .
$$

Then $V^{\otimes r}=\bigoplus_{m=0}^{r} W_{m}$ is the eigendecomposition of $V^{\otimes r}$ with respect to the endomorphism $\Phi_{x}\left(\epsilon_{k}\right)$ where $W_{m}$ is the eigenspace corresponding to the eigenvalue $x^{m}$. We then have

$$
\Phi_{x}\left(\epsilon_{k}\right)=\sum_{m=0}^{r} x^{m} \operatorname{pr}_{W_{m}} \text { and } \operatorname{pr}_{W_{m}}=\prod_{\substack{0 \leq l \leq \leq \\ l \neq m}} \frac{\Phi_{x}\left(\epsilon_{k}\right)-x^{l}}{x^{m}-x^{l}}
$$

This shows that the subalgebra of $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ generated by $\Phi_{x}\left(\epsilon_{k}\right)$ is the same the subalgebra generated by the projections $\mathrm{pr}_{W_{0}}, \ldots, \mathrm{pr}_{W_{m}}$. Note that

$$
\Phi_{x^{\prime}}\left(\epsilon_{k}\right)=\sum_{m=0}^{r}\left(x^{\prime}\right)^{m} \operatorname{pr}_{W_{m}}
$$

so $\Phi_{x^{\prime}}\left(\epsilon_{k}\right)$ is contained in the subalgebra of $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ generated by $\Phi_{x}\left(\epsilon_{k}\right)$. Moreover, $\Phi_{x}(\sigma)=\Phi_{x^{\prime}}(\sigma)$ for all $\sigma \in S_{n}$. Since $S_{n}^{\text {aff }}$ is generated by $S_{n}$ and the $\epsilon_{k}$ we get $\operatorname{Im}\left(\Phi_{x^{\prime}}\right) \subset \operatorname{Im}\left(\Phi_{x}\right)$.

Let $\mathcal{M}_{n} \subset \mathrm{GL}_{n}(\mathbb{C})$ be the set of invertible monomial matrices acting diagonally on $V^{\otimes r}$. Let $\Phi: \mathcal{M}_{n} \rightarrow \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ be the induced algebra homomorphism. We can use $\mathcal{M}_{n}$ to reinterpret the $S_{n}^{\text {aff }}$-action in a way that is independent of $x$.

Corollary 3.34. Let $x \in \mathbb{C}^{\times}$with ord $(x)>r$. Then $\operatorname{Im}\left(\Phi_{x}\right)=\operatorname{Im}(\Phi)$.
Proof. We have $S_{n}^{\text {aff }} \hookrightarrow \mathcal{M}_{n}$, so $\operatorname{Im}\left(\Phi_{x}\right) \subset \operatorname{Im}(\Phi)$. On the other hand, $\mathcal{M}_{n}$ is generated by $S_{n}$ and the diagonal matrices with exactly one diagonal entry $y \neq 1$. The homomorphism $\Phi$ maps such a diagonal matrix into $\operatorname{Im}\left(\Phi_{y}\right)$ and $\operatorname{Im}\left(\Phi_{y}\right) \subset \operatorname{Im}\left(\Phi_{x}\right)$ for all $y \in \mathbb{C}^{\times}$by Lemma 3.33. This shows that $\operatorname{Im}(\Phi) \subset \operatorname{Im}\left(\Phi_{x}\right)$ and the claim follows.

Let us now return to the question what the centralising partner of the $S_{n}^{\text {aff }}$-action (for $\operatorname{ord}(x)>r)$ or equivalently the $\mathcal{M}_{n}$-action is. By (35), we have in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ :

$$
\begin{equation*}
\epsilon_{k} E_{\underline{j}, \underline{i}} \epsilon_{k}^{-1}=x^{\#\left\{l \mid j_{l}=k\right\}-\#\left\{l \mid i_{l}=k\right\}} E_{\underline{j} \underline{,} \underline{ }} \tag{36}
\end{equation*}
$$

If $\underline{i} \xrightarrow{x_{d}} \underline{j}$ for a partition $d \in A_{r}$, we have

$$
\begin{equation*}
\#\left\{l \mid j_{l}=k\right\}-\#\left\{l \mid i_{l}=k\right\}=\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|-|B \cap\{1, \ldots, r\}| \tag{37}
\end{equation*}
$$

where $B$ is the block of $d$ with label $k$ in the $(\underline{i}, \underline{j})$-labelling of d (and $B=\emptyset$ if the label $k$ does not occur). Hence, $E_{\underline{j}, \underline{i}} \in \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ commutes with the $\mathbb{Z}^{n}$-action on $V^{\otimes r}$ if $|B \cap\{1, \ldots, r\}|=\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|$ for all blocks $B$ of $d$.

Definition 3.35. A block $B$ of a partition $d \in A_{r}$ is called balanced if contains the same number of dots from the top row of $d$ as from the bottom row of $d$, i.e.

$$
|B \cap\{1, \ldots, r\}|=\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|
$$

We call $d \in A_{r}$ balanced if all blocks of $d$ are balanced and we denote the set of balanced partitions by $A_{r}^{\text {bal }}$.

Example 3.36. The following two partitions are balanced:


On the other hand, the following two partitions are not balanced and we have marked the non-balanced blocks in red:



Here are a few basic properties of balanced partitions.
Lemma 3.37. For any two balanced partitions $d_{1}, d_{2} \in A_{r}^{\text {bal }}$ the following hold:

1) The concatenation $d_{1} \star d_{2}$ is balanced.
2) In the concatenation $d_{1} \star d_{2}$ no blocks have to be removed, i.e. $r\left(d_{1}, d_{2}\right)=0$.
3) If $d \in A_{r}$ and $d \leq d_{1}$ then $d$ is balanced.

Proof. Since $d_{1}$ and $d_{2}$ are balanced, their blocks are propagating. Hence, any block $B$ of $d_{1}$ is connected with the top row of $d_{1}$ and thus it cannot be removed in $d_{1} \star d_{2}$. Similarly, any block $B$ of $d_{2}$ is connected to the bottom row of $d_{2}$ and thus it cannot be removed from $d_{1} \star d_{2}$ either. Hence, we have $r\left(d_{1}, d_{2}\right)=0$ which proves 2$)$. Let $B$ be a block of $d_{1} \star d_{2}$. Then $B$ is obtained by fusing a collection of blocks $B_{1}, \ldots, B_{k}$ of $d_{1}$ with a collection of blocks $\widetilde{B}_{1}, \ldots, \widetilde{B}_{k^{\prime}}$ of $d_{2}$. We then have

$$
\begin{aligned}
|B \cap\{1, \ldots, r\}| & =\left|\left(B_{1} \cup \ldots \cup B_{k}\right) \cap\{1, \ldots, r\}\right| \\
& =\left|\left(B_{1} \cup \ldots \cup B_{k}\right) \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| \\
& =\left|\left(\widetilde{B}_{1} \cup \ldots \cup \widetilde{B}_{k^{\prime}}\right) \cap\{1, \ldots, r\}\right| \\
& =\left|\left(\widetilde{B}_{1} \cup \ldots \cup \widetilde{B}_{k^{\prime}}\right) \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|=\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| .
\end{aligned}
$$

This proves 1). For 3), observe that any block of $d$ is a union of blocks of $d_{1}$ which are all balanced. Hence the blocks of $d$ are also balanced.

Definition 3.38. The balanced partition algebra $P_{r}^{\text {bal }}$ is the $\mathbb{C}$-vector space with basis $A_{r}^{\text {bal }}$ and multiplication $d_{1} \cdot d_{2}:=d_{1} \star d_{2}$ for $d_{1}, d_{2} \in A_{r}^{\text {bal }}$, extended bilinearly to the vector space $P_{r}^{\mathrm{bal}}$.

By Lemma 3.37, balanced partitions are closed under concatenation, so the multiplication of $P_{r}^{\text {bal }}$ is well-defined. $P_{r}^{\text {bal }}$ is also associative and unital. In fact, we can consider $P_{r}^{\text {bal }}$ as a subalgebra of $P_{r}(\delta)$ in the obvious way for any $\delta \in \mathbb{C}$. This inclusion is compatible with the multiplication since free blocks never occur when multiplying two balanced partitions.

Remark 3.39. We have seen in Remark 3.20 that the dimension of the partition algebra can be computed recursively. A similar technique works for the balanced partition algebra. For this, let

$$
B^{\mathrm{bal}}(r):=\left|A_{r}^{\mathrm{bal}}\right|=\operatorname{dim}_{\mathbb{C}} P_{r}^{\mathrm{bal}}
$$

be the r-th balanced Bell number. There is a map $A_{r}^{\text {bal }} \rightarrow \bigsqcup_{0 \leq k<r} A_{k}^{\text {bal }}$ which sends $d \in A_{r}^{\text {bal }}$ to the partition obtained by removing the block of $d$ that contains the dot $\stackrel{r}{\bullet}$ in the top right corner. For example, for $r=5$ we have


We claim that the fibre of an element $d^{\prime} \in A_{k}^{\text {bal }}$ under the map above has size $\binom{r-1}{k}\binom{r}{k}$. In fact, any $d \in A_{r}^{\text {bal }}$ in the fibre of $d^{\prime}$ differs from $d^{\prime}$ by a single block which contains the dot ${ }^{r}$ as well as $r-k-1$ other dots from the top of $d$ and $r-k$ other dots from the bottom of $d$. There are $\binom{r-1}{r-k-1}=\binom{r-1}{k}$ possible ways to choose the position of the $r-k-1$ dots on the top and $\binom{r}{r-k}=\binom{r}{k}$ ways to choose the position of the $r-k$ dots on the bottom. This proves the claim about the size of the fibre. In particular, we get the recursion

$$
B^{\mathrm{bal}}(r)=\sum_{k=0}^{r-1}\binom{r-1}{k}\binom{r}{k} B^{\mathrm{bal}}(k) .
$$

Using this recursion, one can check that the values of $B^{\text {bal }}(r)=\operatorname{dim}_{\mathbb{C}} P_{r}^{\text {bal }}$ for $r=0,1,2,3,4,5,6, \ldots$ are $1,1,3,16,131,1496,22482, \ldots$.

For $\delta=n \in \mathbb{C}$, we have defined an action $V^{\otimes r} \curvearrowleft P_{r}(n)$ and via the inclusion $P_{r}^{\text {bal }} \hookrightarrow P_{r}(n)$, this induces an action $V^{\otimes r} \curvearrowleft P_{r}^{\text {bal }}$. We now show that this is the centralising partner of the $S_{n}^{\text {aff }}$-action. For this we make use of the orbit basis of $P_{r}(\delta)$ from Definition 3.10 which was denoted by $\left\{x_{d} \mid d \in A_{r}\right\}$.
Lemma 3.40. For any balanced partition $d \in A_{r}^{\text {bal }}$ we have $x_{d} \in P_{r}^{\text {bal }}$. Moreover, $\left\{x_{d} \mid d \in A_{r}^{\text {bal }}\right\}$ is a basis of $P_{r}^{\text {bal }}$.
Proof. We have $d=\sum_{d^{\prime} \leq d} x_{d^{\prime}}$ in $P_{r}(\delta)$. By Lemma 3.37 the $d^{\prime}$ occurring in this sum are balanced if $d$ is balanced. It then follows by upwards induction along the partial order that $x_{d} \in P_{r}^{\text {bal }}$. The $\left\{x_{d} \mid d \in A_{r}^{\text {bal }}\right\}$ form a basis of $P_{r}^{\text {bal }}$ since the base change to the standard basis $A_{r}^{\text {bal }}$ is unitriangular.

We now come to the main theorem of this section.
Theorem 3.41. The actions $S_{n}^{\text {aff }} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}^{\text {bal }}$ commute. If $\operatorname{ord}(x)>r$ the two actions generate each other's centraliser.

## Proof.

Step 1: The actions commute.
For $d \in A_{r}^{\text {bal }}$ and $\underline{i} \xrightarrow{x_{d}} \underline{j}$, we have that seen in 36 and (37) that $E_{\underline{j}, \underline{i}}$ commutes with the $\mathbb{Z}^{n}$-action. Moreover, $x_{d}=\sum_{\underline{i} \xrightarrow{x_{d}} \underline{j}} E_{\underline{j}, \underline{\underline{L}}}$ in $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)^{\text {op }}$ and thus $x_{d}$ commutes with the $\mathbb{Z}^{n}$-action. Since $\bar{x}_{d}$ also commutes with the $S_{n}$-action we get that $x_{d}$ commutes with the $S_{n}^{\text {aff }}$-action. By Lemma 3.40 the $x_{d}$ span $P_{r}^{\text {bal }}$, so Step 1 follows.
Step 2: $P_{r}^{\text {bal }}$ generates $\operatorname{End}_{S_{n}^{\text {aff }}}\left(V^{\otimes r}\right)^{\text {op }}$ if $\operatorname{ord}(x)>r$.
Let $y \in \operatorname{End}_{S_{n}^{\text {aff }}}\left(V^{\otimes r}\right)^{\text {op }}$. By Theorem 3.9 the endomorphism $y$ is induced by an element $\sum_{d \in A_{r}} c_{d} x_{d} \in P_{r}(n)$ for some $c_{d} \in \mathbb{C}$. By Corollary 3.13 the element $x_{d}$ acts by multiplying with 0 if $d$ has more than $n$ blocks, so in this case we may assume $c_{d}=0$. For $d \in A_{r}$ with at most $n$ blocks and $c_{d} \neq 0$, we can find $\underline{i}, \underline{j} \in\{1, \ldots, n\}^{r}$ such that $\underline{i} \xrightarrow{x_{d}} \underline{j}$. Using (36) and (37) we see that the coefficient of $E_{j, \underline{i} \underline{i}}$ in $\epsilon_{k} y \epsilon_{k}^{-1}=y$ is

$$
c_{d} x^{\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|-|B \cap\{1, \ldots, r\}|}=c_{d} \neq 0
$$

where $B$ is the block with label $k$ in the $(\underline{i}, \underline{j})$-labelling of $d$. Since

$$
\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|-|B \cap\{1, \ldots, r\}| \in\{-r, \ldots,-1,0,1, \ldots, r\}
$$

and $\operatorname{ord}(x)>r$, the block $B$ has to be balanced. Since $k$ and hence also the block of $B$ was arbitrary, we deduce that $d$ is a balanced partition. This shows that $\sum_{d \in A_{r}} c_{d} x_{d} \in P_{r}^{\text {bal }}$ proving Step 2.

Step 3: $S_{n}^{\text {aff }}$ generates $\operatorname{End}_{P_{r}^{\text {bal }}}\left(V^{\otimes r}\right)$ if $\operatorname{ord}(x)>r$.
If $r<\operatorname{ord}(x)<\infty$, then $\Phi_{x}: \mathbb{C}\left[S_{n}^{\text {aff }}\right] \rightarrow \operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ factors through

where $m=\operatorname{ord}(x)$. Since $(\mathbb{Z} / m)^{n} \rtimes S_{n}$ is a finite group, we see that $\operatorname{Im}\left(\Phi_{x}\right)$ is semisimple. The result then follows from the double centraliser theorem. If $\operatorname{ord}(x)=\infty$, we pick $x^{\prime} \in \mathbb{C}^{\times}$with $r<\operatorname{ord}\left(x^{\prime}\right)<\infty$. By Lemma 3.33, we have $\operatorname{Im}\left(\Phi_{x}\right)=\operatorname{Im}\left(\Phi_{x^{\prime}}\right)$ and the result follows from the previous case.

We can also rephrase this as the following duality for the monomial matrices $\mathcal{M}_{n}$.

Corollary 3.42. The commuting actions $\mathcal{M}_{n} \curvearrowright V^{\otimes r} \curvearrowleft S_{r}^{\text {aff }}$ generate each other's centraliser.

Proof. Let $x \in \mathbb{C}^{\times}$with $\operatorname{ord}(x)>r$. Then the actions of $\mathcal{M}_{n}$ and $S_{n}^{\text {aff }}$ on $V^{\otimes r}$ generate the same subalgebra of $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$ by Corollary 3.34. The result now follows from Theorem [3.41,

Let us briefly outline a Schur-Weyl duality for the $S_{n}^{\text {aff }}$-action where $x \in \mathbb{C}^{\times}$is root of unity of order $m$ (possibly $\leq r$ ).

Definition 3.43. A block $B$ of a partition $d \in A_{r}$ is called $m$-balanced if

$$
|B \cap\{1, \ldots, r\}| \equiv\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| \quad \bmod m .
$$

We call $d \in A_{r} m$-balanced if all blocks of $d$ are $m$-balanced and we denote the set of $m$-balanced partitions by $A_{r}^{m \text {-bal }}$.

Clearly, balanced partitions are also $m$-balanced for any $m \in \mathbb{N}$. However, free blocks can occur when concatenating $m$-balanced partitions. For example, we have to remove $r(d, d)=1$ free block to compute $d \star d$ where

$$
d=\underset{i_{1}}{\stackrel{2}{\sim}} \cdots \xrightarrow{2} \underset{\mathrm{~m}^{\prime}}{\mathrm{m}} \in A_{m}^{m-\text { bal }} .
$$

Apart from this, the analogous statements of Lemma 3.37 hold for $m$-balanced partitions.

Lemma 3.44. For any two $m$-balanced partitions $d_{1}, d_{2} \in A_{r}^{m-b a l}$ the following hold:

1) The concatenation $d_{1} \star d_{2}$ is $m$-balanced.
2) If $d \in A_{r}$ and $d \leq d_{1}$ then $d$ is $m$-balanced.

Proof. The proof is word by word the same as the proof of Lemma 3.37 when replacing equalities by equalities $\bmod m$.

Hence, we can define the $m$-balanced partition algebra

$$
P_{r}^{m-\mathrm{bal}}(\delta):=\operatorname{Span}_{\mathbb{C}} A_{r}^{m-\mathrm{bal}} \subset P_{r}(\delta)
$$

for any $\delta \in \mathbb{C}$. We then have the following Schur-Weyl duality.
Proposition 3.45. Let $x \in \mathbb{C}^{\times}$be a root of unity of order $m$. Then the actions $S_{n}^{\text {aff }} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}^{m-\mathrm{bal}}(n)$ commute and generate each other's centraliser.

Proof. The proof is essentially the same as the proof of Theorem 3.41 when replacing balanced by $m$-balanced.

Remark 3.46. 1. If $m>r$ any $m$-balanced partition $d$ is balanced since we have $|B \cap\{1, \ldots, r\}|,\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right| \in\{0, \ldots, r\}$ for each block $B$ of $d$. In this case Proposition 3.45 recovers the Schur-Weyl duality from Theorem 3.41
2. If $m=1$ (i.e. $x=1$ ), any partition is $m$-balanced and the $S_{n}$ and $S_{n}^{\text {aff }}$-actions generate the same subalgebra of $\operatorname{End}_{\mathbb{C}}\left(V^{\otimes r}\right)$. Hence, we recover the duality $S_{n} \curvearrowright V^{\otimes r} \curvearrowleft P_{r}(n)=P_{r}^{1-\mathrm{bal}}(n)$ in this case.

### 3.4 A presentation of the balanced partition algebra

This section is devoted to deriving an efficient presentation of the balanced partition algebra similar to the one given in Theorem 3.22. The first step will be to derive a presentation for a sublagebra of $P_{r}^{\text {bal }}$.

Definition 3.47. A partition $d \in A_{r}$ is called horizontal if $k \sim k^{\prime}$ in $d$ for $k=1, \ldots, r$.
Let

$$
\begin{aligned}
& A_{r}^{\text {hor }}:=\left\{d \in A_{r} \mid d \text { is horizontal }\right\} \\
& P_{r}^{\text {hor }}:=\operatorname{Span}_{\mathbb{C}}\left\{d \mid d \in A_{r}^{\text {hor }}\right\} .
\end{aligned}
$$

Note that horizontal partitions are always balanced and a concatenation of horizontal partitions is again horizontal. Hence, $P_{r}^{\text {hor }} \subset P_{r}^{\text {bal }}$ inherits an algebra structure from $P_{r}^{\text {bal }}$.

Definition 3.48. We call $P_{r}^{\text {hor }}$ the horizontal partition algebra.
We want to derive a presentation of the horizontal partition algebra.
Definition 3.49. $C_{r}^{\text {hor }}$ is the $\mathbb{C}$-algebra with generators

$$
q_{i, j}=q_{j, i} \text { for } 1 \leq i<j \leq r
$$

and relations

$$
\begin{array}{ll}
\text { (HOR1) } q_{i, j}^{2}=q_{i, j} & \text { for } 1 \leq i<j \leq r ; \\
\text { (HOR2) } q_{i, j} q_{k, l}=q_{k, l} q_{i, j} & \text { for } 1 \leq i<j \leq r, 1 \leq k<l \leq r ; \\
\text { (HOR3) } q_{i, j} q_{j, k}=q_{i, k} q_{j, k}=q_{i, j} q_{i, k} & \text { for } 1 \leq i<j<k \leq r .
\end{array}
$$

Proposition 3.50. There is an isomorphism of algebras

$$
\begin{aligned}
& \Phi_{r}^{\mathrm{hor}}: C_{r}^{\mathrm{hor}} \xrightarrow{\sim} P_{r}^{\mathrm{hor}} \\
& q_{i, j} \mapsto
\end{aligned}
$$

Proof.

- $\Phi_{r}^{\text {hor }}$ is well-defined:

It is straightforward to check that the $\Phi_{r}^{\text {hor }}$ is compatible with (HOR1)-(HOR3).

- $\Phi_{r}^{\mathrm{hor}}$ is surjective:

The standard basis of $P_{r}^{\text {hor }}$ is indexed by partitions of the set $\{1, \ldots, r\}$. Let $Q$ be a partition of this set and $d_{Q} \in A_{r}^{\text {hor }}$ the corresponding basis element. Let

$$
q_{Q}:=\prod_{\substack{i<j \text { s.t. } \\ i \sim j \text { in } Q}} q_{i, j} .
$$

Then $\Phi_{r}^{\text {hor }}\left(q_{Q}\right)=d_{Q}$.

- $\Phi_{r}^{\mathrm{hor}}$ is injective:

The map $\Phi_{r}^{\text {hor }}$ sets up a bijection between the basis $A_{r}^{\text {hor }}$ of $P_{r}^{\text {hor }}$ and the $q_{Q} \in C_{r}^{\text {hor }}$ as above. It suffices to show that the $q_{Q}$ span $C_{r}^{\text {hor }}$ as a vector space since then the $q_{Q}$ must also be a basis of $C_{r}^{\text {hor }}$ and $\Phi_{r}^{\text {hor }}$ must be an isomorphism. Monomials of the form $q=q_{i_{1}, j_{1}} \cdot \ldots \cdot q_{i_{m}, j_{m}}$ span $C_{r}^{\text {hor }}$ and $\Phi_{r}^{\mathrm{hor}}(q)=d_{Q}$ for some $Q$. Hence, it suffices to prove the following claim.
Claim: We have $q=q_{Q}$.
By (HOR1) and (HOR3), we may assume that the $q_{i_{k}, j_{k}}$ are pairwise distinct. Then all the factors of the product $q$ appear also in the product $q_{Q}$ since $\Phi_{r}^{\mathrm{hor}}(q)=d_{Q}$ implies $i_{k} \sim j_{k}$ in $Q$. We show that multiplying $q$ with the extra factors from $q_{Q}$ gives the same element in $C_{r}^{\text {hor }}$. If $\left\{i_{l}, j_{l}\right\} \cap\left\{i_{k}, j_{k}\right\} \neq 0$ the $q_{a, b}$ with $a<b$ and $a, b \in\left\{i_{l}, j_{l}, i_{k}, j_{k}\right\}$ appear in $q_{Q}$. These can inserted in the product $q$ using (HOR1)-(HOR3). In fact, for $i<j<k$, we have

$$
\begin{aligned}
q_{i, j} q_{j, k} & \stackrel{(\mathrm{HOR} 3)}{=} q_{i, k} q_{j, k} \stackrel{(\mathrm{HOR} 3)}{=} q_{i, j} q_{i, k} \stackrel{(\mathrm{HOR} 1)}{=} q_{i, j}^{2} q_{i, k}^{2} \stackrel{(\mathrm{HOR} 2)}{=}\left(q_{i, j} q_{i, k}\right)^{2} \\
& \stackrel{(\mathrm{HOR} 3)}{=} q_{i, j} q_{i, k} q_{i, j} q_{j, k} \stackrel{(\mathrm{HOR} 2)}{=} q_{i, j}^{2} q_{i, k} q_{j, k} \stackrel{(\mathrm{HOR} 1)}{=} q_{i, j} q_{i, k} q_{j, k}
\end{aligned}
$$

Repeating this argument, wee see that multiplying $q$ with all the $q_{i, j}$ where $i \sim j$ in $Q$ gives the same element in $C_{r}^{\text {hor }}$. This shows that $q=q_{Q}$.

We can now return to our original goal, which was to give a presentation of $P_{r}^{\text {bal }}$ by generators and relations.
Definition 3.51. $C_{r}^{\text {bal }}$ is the $\mathbb{C}$-algebra with generators

$$
s_{1}, \ldots, s_{r-1} \text { and } p_{\frac{3}{2}}, p_{\frac{5}{2}}, \ldots ., p_{r-\frac{1}{2}}
$$

and relations
(i) $s_{i}^{2}=1$
(ii) $s_{i} s_{j}=s_{j} s_{i}$
(iii) $s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}$
(BAL2)
(i) $\left(p_{i+\frac{1}{2}}\right)^{2}=p_{i+\frac{1}{2}}$
(ii) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}}=p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}$
(BAL3)
(i) $s_{i} p_{i+\frac{1}{2}}=p_{i+\frac{1}{2}}=p_{i+\frac{1}{2}} s_{i}$
for $i=1, \ldots, r-1$;
for $|i-j|>1$;
for $i=1, \ldots, r-2$;
for $i=1, \ldots, r-1$;
for $i, j=1, \ldots, r-1$;
for $i=1, \ldots, r-1$;

$$
\begin{array}{ll}
\text { (ii) } s_{i} s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_{i}=p_{i+\frac{3}{2}} & \text { for } i=1, \ldots, r-2 \\
\text { (iii) } s_{i} p_{j+\frac{1}{2}}=p_{j+\frac{1}{2}} s_{i} & \text { for } j \neq i-1, i, i+1
\end{array}
$$

There is an algebra homomorphism

$$
\left.\begin{array}{rl}
\Phi_{r}^{\mathrm{bal}}: C_{r}^{\mathrm{bal}} \longrightarrow P_{r}^{\mathrm{bal}} \\
s_{i} \mapsto & \ldots
\end{array}\right]
$$

One can check by hand that this is compatible with the defining relations of $C_{r}^{\mathrm{bal}}$. However, it is probably easier to observe that the generators of $C_{r}^{\text {bal }}$ are a subset of the generators of the algebra $C_{r}(\delta)$ (from Definition 3.21) and the relations of $C_{r}^{\mathrm{bal}}$ are a subset of the relations from $C_{r}(\delta)$. Hence, there is a natural algebra homomorphism $C_{r}^{\text {bal }} \rightarrow C_{r}(\delta)$ and the map $\Phi_{r}^{\text {bal }}$ is the composition $C_{r}^{\mathrm{bal}} \rightarrow C_{r}(\delta) \xrightarrow{\sim}$ $P_{r}(\delta)$ whose image is contained in $P_{r}^{\mathrm{bal}} \subset P_{r}(\delta)$. There also is an inclusion

$$
\mathbb{C}\left[S_{r}\right] \hookrightarrow C_{r}^{\mathrm{bal}}, \quad s_{i} \mapsto s_{i}
$$

This is injective since the composition $\mathbb{C}\left[S_{r}\right] \rightarrow C_{r}^{\text {bal }} \xrightarrow{\Phi_{r}^{\text {bal }}} P_{r}^{\text {bal }}$ is the diagrammatic inclusion from Remark 3.6. Let $B_{r}$ be the subalgebra of $C_{r}^{\text {bal }}$ generated by the $\sigma p_{i+\frac{1}{2}} \sigma^{-1}$ where $\sigma \in S_{r}$ and $i=1, \ldots, r-1$. We then have $\sigma B_{r} \sigma^{-1}=B_{r}$ for any $\sigma \in S_{r}$. Moreover, $C_{r}^{\mathrm{bal}}$ is generated by $B_{r}$ and $S_{r}$. In particular,

$$
\begin{equation*}
C_{r}^{\mathrm{bal}}=B_{r} \cdot \mathbb{C}\left[S_{r}\right] \quad \text { and } \quad P_{r}^{\mathrm{bal}}=P_{r}^{\mathrm{hor}} \cdot \mathbb{C}\left[S_{r}\right] \tag{38}
\end{equation*}
$$

where the second equality follows from the fact that any balanced partition has a decomposition $q \cdot \sigma$ with a unique (upper) horizontal part $q \in A_{r}^{\text {hor }}$ and a (not necessarily unique) permutation part $\sigma \in S_{r}$. For example,


We will use $(38)$ to prove that $\Phi_{r}^{\text {bal }}$ is an isomorphism. For this we analyse the restriction of $\Phi_{r}^{\mathrm{bal}}$ to $B_{r}$. Note that $\Phi_{r}^{\mathrm{bal}}(\sigma) P_{r}^{\mathrm{hor}} \Phi_{r}^{\mathrm{bal}}\left(\sigma^{-1}\right)=\sigma P_{r}^{\text {hor }} \sigma^{-1}=P_{r}^{\text {hor }}$ for all $\sigma \in S_{r}$. Since $\Phi_{r}^{\text {bal }}\left(p_{i+\frac{1}{2}}\right) \in P_{r}^{\text {hor }}$ we get $\operatorname{Im}\left(\left.\Phi_{r}^{\text {bal }}\right|_{B_{r}}\right) \subset P_{r}^{\text {hor }}$. In fact, we even have $\operatorname{Im}\left(\left.\Phi_{r}^{\mathrm{bal}}\right|_{B_{r}}\right)=P_{r}^{\text {hor }}$ since

$$
\begin{equation*}
x_{i, j}=x_{j, i}:=s_{j-1} s_{j-2} \ldots s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \ldots s_{j-2} s_{j-1} \in B_{r} \tag{39}
\end{equation*}
$$

is a preimage of $q_{i, j} \in C_{r}^{\mathrm{hor}} \cong P_{r}^{\mathrm{hor}}$ under $\Phi_{r}^{\mathrm{bal}}$ for $1 \leq i<j \leq r$.

Lemma 3.52. For $1 \leq i<j \leq r$ and $\sigma \in S_{r}$, we have $\sigma x_{i, j} \sigma^{-1}=x_{\sigma(i), \sigma(j)}$.
Proof. It suffices to prove the claim for $\sigma=s_{k}$ a simple reflection.
Case 1: $k<i-1$ or $k>j$.
In this case $s_{k}$ commutes with $p_{i+\frac{1}{2}}, s_{i+1}, \ldots, s_{j-1}$ and hence also with $x_{i j}$.
We then get $s_{k} x_{i, j} s_{k}=x_{i, j}=x_{s_{k}(i), s_{k}(j)}$.
Case 2: $i<k<j-1$.
In this case, we have

$$
\begin{aligned}
s_{k} x_{i, j} s_{k} & =s_{j-1} \ldots s_{k+2} s_{k} s_{k+1} s_{k} \ldots s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \ldots s_{k} s_{k+1} s_{k} s_{k+2} \ldots s_{j-1} \\
& =s_{j-1} \ldots s_{k+1} s_{k} s_{k+1} s_{k-1} \ldots s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \ldots s_{k-1} s_{k+1} s_{k} s_{k+1} \ldots s_{j-1} \\
& =s_{j-1} s_{j-2} \ldots s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \ldots s_{j-2} s_{j-1} \\
& =x_{i, j}=x_{s_{k}(i), s_{k}(j)}
\end{aligned}
$$

Case $3: k=i \neq j-1$ or $k=i-1$.
This case follows from

$$
\begin{aligned}
s_{i} x_{i, j} s_{i} & =s_{j-1} \ldots s_{i+2} s_{i} s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_{i} s_{i+2} \ldots s_{j-1} \\
& =s_{j-1} \ldots s_{i+2} s_{i} s_{i+1} s_{i} p_{i+\frac{1}{2}} s_{i} s_{i+1} s_{i} s_{i+2} \ldots s_{j-1} \\
& =s_{j-1} \ldots s_{i+2} s_{i+1} s_{i} s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_{i} s_{i+1} s_{i+2} \ldots s_{j-1} \\
& =s_{j-1} \ldots s_{i+2} s_{i+1} p_{i+\frac{3}{2}} s_{i+1} s_{i+2} \ldots s_{j-1} \\
& =s_{j-1} \ldots s_{i+2} p_{i+\frac{3}{2}} s_{i+2} \ldots s_{j-1}=x_{i+1, j}
\end{aligned}
$$

Case 4: $k=j$ or $k=j-1 \neq i$.
This case follows from $s_{j} x_{i, j} s_{j}=s_{j} s_{j-1} \ldots s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \ldots s_{j-1} s_{j}=x_{i, j+1}$.
Case 5: $k=i=j-1$.

$$
\text { Here, } s_{k} x_{i, i+1} s_{k}=s_{i} p_{i+\frac{1}{2}} s_{i}=p_{i+\frac{1}{2}}=x_{i+1, i}=x_{s_{k}(i), s_{k}(i+1)}
$$

The claim now follows for $\sigma=s_{k}$.
Corollary 3.53. The restriction of $\Phi_{r}^{\text {bal }}$ to $B_{r}$ induces an isomorphism

$$
\left.\Phi_{r}^{\mathrm{bal}}\right|_{B_{r}}: B_{r} \xrightarrow{\sim} P_{r}^{\mathrm{hor}}
$$

Proof. We have already seen that $\operatorname{Im}\left(\left.\Phi_{r}^{\mathrm{bal}}\right|_{B_{r}}\right)=P_{r}^{\text {hor }}$ using the elements from 39). It remains to show that $\left.\Phi_{r}^{\mathrm{bal}}\right|_{B_{r}}$ is injective.
Claim: $\Psi: P_{r}^{\text {hor }} \cong C_{r}^{\text {hor }} \rightarrow B_{r}, q_{i, j} \mapsto x_{i, j}$ is a well-defined algebra homomorphism. Assuming this claim, we get

$$
\begin{aligned}
\Psi\left(\Phi_{r}^{\mathrm{bal}}\left(\sigma p_{i+\frac{1}{2}} \sigma^{-1}\right)\right) & =\Psi\left(\sigma q_{i, i+1} \sigma^{-1}\right) \\
& =\Psi\left(q_{\sigma(i), \sigma(i+1)}\right) \\
& =x_{\sigma(i), \sigma(i+1)} \\
& \stackrel{3.52}{=} \sigma x_{i, i+1} \sigma^{-1}=\sigma p_{i+\frac{1}{2}} \sigma^{-1}
\end{aligned}
$$

for any $\sigma \in S_{r}$ and $i=1, \ldots, r-1$. This shows that $\left.\Psi \circ \Phi_{r}^{\mathrm{bal}}\right|_{B_{r}}=\mathrm{id}_{B_{r}}$ and hence $\Phi_{r}^{\mathrm{bal}}{ }_{B_{r}}$ is injective. It remains to show the claim that $\Psi$ is well defined, i.e. that the $x_{i j}$ satisfy (HOR1)-(HOR3). We have

$$
x_{1,2}^{2}=\left(p_{\frac{3}{2}}\right)^{2}=p_{\frac{3}{2}}=x_{1,2}
$$

Let $\sigma \in S_{r}$ such that $\sigma(1)=i$ and $\sigma(2)=j$. Then, by Lemma 3.52 we have

$$
x_{i, j}^{2}=\left(\sigma x_{1,2} \sigma^{-1}\right)^{2}=\sigma x_{1,2} \sigma^{-1}=x_{i, j}
$$

so the $x_{i j}$ satisfy (HOR1). Similarly, (HOR2) follows from

$$
\begin{aligned}
& x_{1,2} x_{3,4}=p_{\frac{3}{2}} p_{\frac{7}{2}}=p_{\frac{7}{2}} p_{\frac{3}{2}}=x_{3,4} x_{1,2} \\
& x_{1,2} x_{2,3}=p_{\frac{3}{2}} p_{\frac{5}{2}}=p_{\frac{5}{2}} p_{\frac{3}{2}}=x_{2,3} x_{1,2}
\end{aligned}
$$

and (HOR3) follows from

$$
x_{1,2} x_{2,3}=p_{\frac{3}{2}} p_{\frac{5}{2}}=p_{\frac{5}{2}} p_{\frac{3}{2}}=s_{2} p_{\frac{5}{2}} p_{\frac{3}{2}}=s_{2} p_{\frac{3}{2}} p_{\frac{5}{2}}=s_{2} p_{\frac{3}{2}} s_{2} p_{\frac{5}{2}}=x_{1,3} x_{2,3}
$$

by conjugating with an appropriate $\sigma \in S_{r}$. We have thus shown that $\Psi$ is welldefined. This finishes the proof.

Now, we have all the ingredients to prove the presentation theorem.
Theorem 3.54. The map $\Phi_{r}^{\text {bal }}: C_{r}^{\text {bal }} \rightarrow P_{r}^{\text {bal }}$ is an isomorphism of algebras.
Proof. We have already seen that $\Phi_{r}^{\text {bal }}$ is well-defined so it remains to show that $\Phi_{r}^{\mathrm{bal}}$ is an isomorphism. Note that $\Phi_{r}^{\mathrm{bal}}$ is the identitiy on $\mathbb{C}\left[S_{r}\right]$ and an isomorphism on $\left.\Phi_{r}^{\text {bal }}\right|_{B_{r}}: B_{r} \xrightarrow{\sim} P_{r}^{\text {hor }}$ by Corollary 3.53 . We have $P_{r}^{\text {bal }}=P_{r}^{\text {hor }} \cdot \mathbb{C}\left[S_{r}\right]$ as well as $C_{r}^{\mathrm{bal}}=B_{r} \cdot \mathbb{C}\left[S_{r}\right]$ by 38 , so $\Phi_{r}^{\mathrm{bal}}$ is surjective. Hence, $\Phi_{r}^{\mathrm{bal}}$ is an isomorphism if and only if we can lift the basis $A_{r}^{\mathrm{bal}}$ of $P_{r}^{\mathrm{bal}}$ to a spanning set (and then also a basis) of $C_{r}^{\mathrm{bal}}$. We prove that such a lift exists:
By (38) the algebra $C_{r}^{\text {bal }}$ is spanned by the elements $q \cdot \sigma$ where $q \in\left(\Phi_{r}^{\text {bal }}\right)^{-1}\left(A_{r}^{\text {hor }}\right)$ and $\sigma \in S_{r}$. These elements satisfy $\Phi_{r}^{\mathrm{bal}}(q \cdot \sigma) \in A_{r}^{\mathrm{bal}}$ and any $d \in A_{r}^{\mathrm{bal}}$ has a lift of this form. Hence, it suffices to show that each $d \in A_{r}^{\text {bal }}$ has at most one lift of this form, i.e.

$$
\Phi_{r}^{\mathrm{bal}}(q \sigma)=\Phi_{r}^{\mathrm{bal}}\left(q^{\prime} \sigma^{\prime}\right) \Rightarrow q \sigma=q^{\prime} \sigma^{\prime}
$$

for any $q, q^{\prime} \in\left(\Phi_{r}^{\mathrm{bal}}\right)^{-1}\left(A_{r}^{\text {hor }}\right)$ and $\sigma, \sigma^{\prime} \in S_{r}$. After multiplying with $\sigma^{\prime-1}$ on the right in both equations, we may assume $\sigma^{\prime}=1$. Note that $\Phi_{r}^{\mathrm{bal}}(q) \sigma=\Phi_{r}^{\mathrm{bal}}(q \sigma)=\Phi_{r}^{\mathrm{bal}}\left(q^{\prime}\right)$ implies $\Phi_{r}^{\mathrm{bal}}(q)=\Phi_{r}^{\mathrm{bal}}\left(q^{\prime}\right)$ since the (upper) horizontal part of a balanced partition is unique. This shows that $q=q^{\prime}$ by Corollary 3.53. Let $Q$ be the partition of $\{1, \ldots, r\}$ corresponding to the element $\Phi_{r}^{\text {bal }}(q) \in A_{r}^{\text {hor }}$. We can then find $\tau \in S_{r}$ such that $\tau \cdot Q=\left\{\left\{1, \ldots, i_{1}\right\},\left\{i_{1}+1, \ldots, i_{1}+i_{2}\right\}, \ldots\right\}$. Then $\Phi_{r}^{\text {hor }}\left(\tau q \tau^{-1}\right)$ is the horizontal partition corresponding to $\tau \cdot Q$ which can be written as

$$
\tau q \tau^{-1}=\left(p_{\frac{3}{2}} \cdot \ldots \cdot p_{i_{1}-\frac{1}{2}}\right) \cdot\left(p_{i_{1}+\frac{3}{2}} \cdot \ldots \cdot p_{i_{1}+i_{2}-\frac{1}{2}}\right) \cdot \ldots
$$

Then $\Phi_{r}^{\text {bal }}\left(\tau q \tau^{-1}\right) \tau \sigma \tau^{-1}=\Phi_{r}^{\text {bal }}\left(\tau q \tau^{-1}\right)$ implies $\tau \sigma \tau^{-1} \in S_{i_{1}} \times S_{i_{2}} \times \ldots$ and we get $\left(\tau q \tau^{-1}\right) \cdot\left(\tau \sigma \tau^{-1}\right)=\tau q \tau^{-1}$ by (BAL3)-(i). This shows that $q \sigma=q=q^{\prime} \sigma^{\prime}$ and the proof is complete.

### 3.5 The representation theory of the balanced partition algebra

By Corollary 3.13 and Lemma 3.40 the kernel of the action $V^{\otimes r} \curvearrowleft P_{r}^{\text {bal }}$ is spanned by the $x_{d}$ where $d \in A_{r}^{\text {bal }}$ has more than $n$ blocks. Since any balanced partition has at most $r$ blocks, $P_{r}^{\text {bal }}$ acts faithfully on $V^{\otimes r}$ for $n \geq r$. In this case $\operatorname{End}_{S_{n}^{\text {aff }}}\left(V^{\otimes r}\right)^{\mathrm{op}} \cong$ $P_{r}^{\text {bal }}$ is semisimple by the double centraliser theorem. We have thus shown the following.

Proposition 3.55. The balanced partition algebra $P_{r}^{\text {bal }}$ is semisimple for any $r \in \mathbb{N}$.

Our next goal is to parametrise the irreducible $P_{r}^{\text {bal }}$-representations. To any $d \in A_{r}^{\text {bal }}$ we can assign a partition $\lambda(d):=\left(1^{l_{1}}, 2^{l_{2}}, \ldots,\right)$ (meaning a partition of an integer and not a set) where $l_{i}$ is the number of blocks $B$ of $d$ with $|B|=2 i$. Note that

$$
|\lambda(d)|=\sum l_{i} \cdot i=\sum_{B \text { block of } d}|B| / 2=r .
$$

Since balanced blocks are always propagating, Proposition 3.17 tells us that

$$
x_{d} x_{d^{\prime}}= \begin{cases}x_{d \star d^{\prime}} & \text { if } d \text { and } d^{\prime} \text { match }  \tag{40}\\ 0 & \text { otherwise }\end{cases}
$$

for any $d, d^{\prime} \in A_{r}^{\text {bal }}$. Note that $d$ and $d^{\prime}$ can only match if $\lambda(d)=\lambda\left(d^{\prime}\right)$. Hence, $P_{r}^{\text {bal }}$ decomposes into a product of algebras

$$
\begin{equation*}
P_{r}^{\mathrm{bal}}=\prod_{\lambda \vdash r} B_{\lambda} \tag{41}
\end{equation*}
$$

where $B_{\lambda}=\operatorname{Span}_{\mathbb{C}}\left\{x_{d} \mid d \in A_{r}^{\text {bal }}\right.$ with $\left.\lambda(d)=\lambda\right\}$. Moreover, for any $\lambda \vdash r$ there is a distinguished element $e_{\lambda} \in A_{r}^{\text {hor }}$ corresponding to the set partition

$$
Q(\lambda):=\left\{\left\{1, \ldots, \lambda_{1}\right\},\left\{\lambda_{1}+1, \ldots, \lambda_{1}+\lambda_{2}\right\}, \ldots\right\}
$$

of $\{1, \ldots, r\}$. The element $e_{\lambda}$ is an idempotent in $P_{r}^{\text {bal }}$ and $x_{e_{\lambda}}$ is an idempotent in $B_{\lambda}$ by 40).

Lemma 3.56. We have $B_{\lambda}=B_{\lambda} x_{e_{\lambda}} B_{\lambda}$.
Proof. Let $d \in A_{r}^{\text {bal }}$ with $\lambda(d)=\lambda$. The top row of $d$ corresponds to a partition of the set $\{1, \ldots, r\}$ and hence to an idempotent $q \in A_{r}^{\text {hor }}$ such that $q$ and $d$ match with $q d=d$. In particular, $x_{d}=x_{q} x_{d}$ by 40 . We can then find $\sigma \in S_{r}$ with $\sigma e_{\lambda} \sigma^{-1}=q$. Note that

$$
q=\sigma e_{\lambda} \cdot e_{\lambda} \sigma^{-1}=\sigma e_{\lambda} \cdot e_{\lambda} \cdot e_{\lambda} \sigma^{-1}
$$

Moreover, $\sigma e_{\lambda}$ and $e_{\lambda} \sigma^{-1}$ (resp. $\sigma e_{\lambda}$ and $e_{\lambda}$ ) match. Hence,

$$
x_{d}=x_{q} x_{d}=x_{\sigma e_{\lambda}} x_{e_{\lambda} \sigma^{-1}} x_{d}=x_{\sigma e_{\lambda}} x_{e_{\lambda}} x_{e_{\lambda} \sigma^{-1}} x_{d} \in P_{r}^{\text {bal }} x_{e_{\lambda}} P_{r}^{\text {bal }} \stackrel{41]}{=} B_{\lambda} x_{e_{\lambda}} B_{\lambda}
$$

by (40). This shows that $B_{\lambda}=B_{\lambda} x_{e_{\lambda}} B_{\lambda}$.
As a direct consequence, Corollary 2.12 (or the analogous statement for right modules to be precise) tells us that there is an equivalence of categories

$$
\begin{equation*}
\bmod -B_{\lambda} \cong \bmod -x_{e_{\lambda}} B_{\lambda} x_{e_{\lambda}} . \tag{42}
\end{equation*}
$$

The following proposition determines the right hand side of this equivalence.
Proposition 3.57. There is an algebra isomorphism

$$
x_{e_{\lambda}} B_{\lambda} x_{e_{\lambda}} \cong \mathbb{C}\left[S_{l_{1}} \times S_{l_{2}} \times \ldots \times S_{l_{m}}\right]
$$

where $\lambda=\left(1^{l_{1}}, 2^{l_{2}}, \ldots, m^{l_{m}}\right)$.
Proof. Let $d \in A_{r}^{\text {bal }}$ with $\lambda(d)=\lambda$. By 40 we have that $x_{e_{\lambda}} x_{d}=x_{d}$ if the top row of $d$ is given by the set partition $Q(\lambda)$ and $x_{e_{\lambda}} x_{d}=0$ otherwise. Similarly, $x_{d} x_{e_{\lambda}}=x_{d}$ if the bottom row of $d$ is given by the set partition $Q(\lambda)$ and $x_{d} x_{e_{\lambda}}=0$ otherwise. Hence $x_{e_{\lambda}} B_{\lambda} x_{e_{\lambda}}$ is spanned by all those $x_{d}$ where the top and the bottom
row of $d \in A_{r}^{\text {bal }}$ are given by $Q(\lambda)$. Since the blocks of $d$ are propagating, we see that $d$ connects each block of $Q(\lambda)$ in the top row of $d$ with a unique block of $Q(\lambda)$ in the bottom row of $d$. In other words, $d$ permutes the blocks of $Q(\lambda)$. The fact that $d$ is balanced is equivalent to the fact that $d$ only permutes blocks of $Q(\lambda)$ which are of the same size. Moreover, the composition of two such diagrams $d$ and $d^{\prime}$ is just given by composing the corresponding permutations of the blocks of $Q(\lambda)$. This shows that $x_{e_{\lambda}} B_{\lambda} x_{e_{\lambda}} \cong \mathbb{C}\left[S_{l_{1}} \times S_{l_{2}} \times \ldots \times S_{l_{m}}\right]$ since there are $l_{i}$ blocks of size $i$ in $Q(\lambda)$.

This already completely determines the representation theory of $P_{r}^{\text {bal }}$. In fact, we get equivalences of categories

$$
\begin{align*}
& \bmod -P_{r}^{\text {bal }} \stackrel{\stackrel{41]}{=}}{\rightleftharpoons} \bigoplus_{\lambda \vdash r} \bmod -B_{\lambda} \stackrel{\sqrt[422]{=}}{=} \bigoplus_{\lambda \vdash r} \bmod -x_{e_{\lambda}} B_{\lambda} x_{e_{\lambda}} \\
& \stackrel{\boxed{3.57}}{\cong} \bigoplus_{\lambda=\left(1^{1}, 2^{l_{2}}, \ldots\right) \vdash r} \bmod -\mathbb{C}\left[S_{l_{1}} \times S_{l_{2}} \times \ldots\right] . \tag{43}
\end{align*}
$$

Remark 3.58. This gives another proof of the fact that $P_{r}^{\text {bal }}$ is semisimple from Proposition 3.55

To give a nice parametrisation of the irreducible $P_{r}^{\text {bal }}$-representations, we need the following definition.

Definition 3.59. A multipartition is a tuple of partitions $\underline{\lambda}=\left(\lambda^{(1)}, \lambda^{(2)}, \ldots\right)$ such that $\left|\lambda^{(i)}\right|=0$ for $i \gg 0$.

Corollary 3.60. There is a bijection

$$
\left\{\begin{array}{c}
\text { iso. classes } \\
\text { of simple right } \\
P_{r}^{\text {bal_-modules }}
\end{array}\right\} \stackrel{1: 1}{\longleftrightarrow}\left\{\begin{array}{c}
\text { multipartitions } \\
\underset{\lambda}{\lambda}\left(\lambda^{(1)}, \lambda^{(2)}, \ldots\right) \\
\text { with } \sum\left|\lambda^{(i)}\right| \cdot i=r
\end{array}\right\} .
$$

Proof. The irreducible representations of $\mathbb{C}\left[S_{l_{1}} \times S_{l_{2}} \times \ldots \times S_{l_{m}}\right]$ are parametrised by multipartitions $\underline{\lambda}=\left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}\right)$ with $\left|\lambda^{(i)}\right|=l_{i}$. Moreover,

$$
\sum\left|\lambda^{(i)}\right| \cdot i=\sum l_{i} \cdot i=\left|\left(1^{l_{1}}, 2^{l_{2}}, \ldots, m^{l_{m}}\right)\right| .
$$

so $\sum\left|\lambda^{(i)}\right| \cdot i=r$ if and only if $\left(1^{l_{1}}, 2^{l_{2}}, \ldots, m^{l_{m}}\right)$ is a partition of $r$. The claim now follows from (43).

### 3.6 Dualities for other diagram algebras

We have already seen a few dualities with diagram algebras like the symmetric group algebra or the (balanced) partition algebra. More generally, we refer to any subalgebra of the partition algebra as a diagram algebra if it is spanned by a set of diagrams $S \subset A_{r}$. In this section we briefly discuss a few more important examples of diagram algebras and their associated Schur-Weyl dualities. Explaining all of these in detail would be beyond the scope of this thesis so proofs will be omitted.

The Brauer algebra: Having seen classical Schur-Weyl duality for $\mathfrak{s l}_{n}$ and $\mathrm{GL}_{n}(\mathbb{C})$, the first question one might ask (putting the affine and quantum generalisations we already explained aside) is whether there are similar dualities for other simple Lie algebras or algebraic groups. In type $B, C, D$, this can be answered using the Brauer algebra which was introduced in Bra37.

Definition 3.61. For any $\delta \in \mathbb{C}$, the Brauer algebra $B_{r}(\delta)$ is the subalgebra of $P_{r}(\delta)$ spanned by all diagrams $d \in A_{r}$ with $|B|=2$ for each block $B$ of $d$.

Clearly this algebra contains the symmetric group algebra $\mathbb{C}\left[S_{r}\right]$ (when considered as a subalgebra of $P_{r}(\delta)$ as in Remark 3.6). It is not hard to see that $B_{r}(\delta)$ is generated as an algebra by $\mathbb{C}\left[S_{r}\right]$ and the elements


In fact, there is the following presentation of the Brauer algebra.
Proposition 3.62. The Brauer algebra $B_{r}(\delta)$ is the algebra with generators

$$
s_{1}, \ldots, s_{r-1}, e_{1}, \ldots, e_{r-1}
$$

and relations (whenever they make sense)
(BR1)
(i) $s_{i}^{2}=1$
(ii) $s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}$
(iii) $s_{i} s_{j}=s_{j} s_{i} \quad$ if $|\overline{i-j}|>1$
(BR2) (i) $e_{i}^{2}=\delta e_{i}$
(ii) $e_{i} e_{i \pm 1} e_{i}=e_{i}$
(BR3) (i) $s_{i} e_{i}=e_{i}=e_{i} s_{i}$
(ii) $s_{i} e_{j}=e_{j} s_{i} \quad$ if $|\overline{i-j}|>1$
(iii) $s_{i} e_{i \pm 1} e_{i}=s_{i \pm 1} e_{i}$
(iv) $e_{i} e_{i \pm 1} s_{i}=e_{i} s_{i \pm 1}$

Proof. See [GW09, Section 9 and 10].
Let $V$ be an $n$-dimensional $\mathbb{C}$-vector space with a non-degenerate bilinear form $\langle\cdot, \cdot\rangle: V \otimes V \rightarrow \mathbb{C}$. Let

$$
\mathfrak{g}:=\{x \in \mathfrak{g l}(V) \mid\langle x v, w\rangle+\langle v, x w\rangle=0 \quad \forall v, w \in V\}
$$

Assume further that $\langle x, y\rangle=\epsilon_{\mathfrak{g}}\langle y, x\rangle$ where $\epsilon_{\mathfrak{g}} \in\{ \pm 1\}$. If $\epsilon_{\mathfrak{g}}=1$, then $\mathfrak{g}=\mathfrak{s o}_{n}$ and if $\epsilon_{\mathfrak{g}}=-1$ then $n$ is even and $\mathfrak{g}=\mathfrak{s p}_{n}$. Pick a basis $v_{1}, \ldots, v_{n}$ of $V$ and let $v^{1}, \ldots, v^{n}$ be the basis defined by $\left\langle v_{i}, v^{j}\right\rangle=\delta_{i, j}$. Let $S_{r}$ act on $V^{\otimes r}$ by permuting the tensor factors and $e_{i}$ as $\mathrm{id}^{\otimes i-1} \otimes e \otimes \mathrm{id}^{r-i-1}$ where

$$
\begin{equation*}
v \otimes w \cdot e_{i}:=\epsilon_{\mathfrak{g}}\langle v, w\rangle \cdot \sum_{i=1}^{n} v_{i} \otimes v^{i} \tag{44}
\end{equation*}
$$

One can check that this extends uniquely to an action $V^{\otimes r} \curvearrowleft B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$.
Remark 3.63. If $\mathfrak{g}=\mathfrak{s o}_{n}$ and $\left\langle v_{i}, v_{j}\right\rangle=\delta_{i, j}$, the $B_{r}(n)$-action on $V^{\otimes r}$ is just the restriction of the $P_{r}(n)$-action on $V^{\otimes r}$ along the inclusion $B_{r}(n) \hookrightarrow P_{r}(n)$. Note that $\mathbb{C}\left[S_{r}\right] \subset B_{r}(n) \subset P_{r}(n)$ all act on $V^{\otimes r}$. On the other hand $S_{n} \subset \mathrm{O}_{n}(\mathbb{C}) \subset \mathrm{GL}_{n}(\mathbb{C})$ and we get

$$
\mathbb{C}\left[S_{r}\right] \cong \operatorname{End}_{\mathrm{GL}_{n}(\mathbb{C})}\left(V^{\otimes r}\right)^{\mathrm{op}} \subset \operatorname{End}_{\mathrm{O}_{n}(\mathbb{C})}\left(V^{\otimes r}\right)^{\mathrm{op}} \subset \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\mathrm{op}} \cong P_{r}(n)
$$

for $n \gg 0$. This is already a hint that the Brauer algebra might be a good candidate for the centralising partner of the $\mathrm{O}_{n}(\mathbb{C})$-action.

Let $x \in \mathfrak{g}$ and $x v_{i}=\sum_{j=1}^{n} x_{j i} v_{j}$. Then $x_{j i}=\left\langle x v_{i}, v^{j}\right\rangle$ and

$$
x v^{i}=\sum_{j=1}^{n}\left\langle v_{j}, x v^{i}\right\rangle \cdot v^{j}=-\sum_{j=1}^{n}\left\langle x v_{j}, v^{i}\right\rangle \cdot v^{j}=\sum_{j=1}^{n}-x_{i j} v^{j}
$$

Hence,

$$
\begin{align*}
x \cdot(v \otimes w \cdot e) & =\epsilon_{\mathfrak{g}}\langle v, w\rangle x \cdot \sum_{i=1}^{n} v_{i} \otimes v^{i} \\
& =\epsilon_{\mathfrak{g}}\langle v, w\rangle \cdot \sum_{i, j=1}^{n} x_{j i} v_{j} \otimes v^{i}-\epsilon_{\mathfrak{g}}\langle v, w\rangle \cdot \sum_{i, j=1}^{n} x_{i j} v_{i} \otimes v^{j}  \tag{45}\\
& =0 .
\end{align*}
$$

and

$$
\begin{align*}
(x \cdot v \otimes w) \cdot e & =(x v \otimes w+v \otimes x w) \cdot e \\
& =\epsilon_{\mathfrak{g}}(\langle x v, w\rangle+\langle v, x w\rangle) \cdot \sum_{i=1}^{n} v_{i} \otimes v^{i}  \tag{46}\\
& =0 .
\end{align*}
$$

It follows that the $B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$-action commutes with the $\mathfrak{g}$-action on $V^{\otimes r}$. Actually, one can prove the following Schur-Weyl duality

Theorem 3.64. For $\epsilon_{\mathfrak{g}} \in\{ \pm 1\}$, the commuting actions $\mathfrak{g} \curvearrowright V^{\otimes r} \curvearrowleft B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$ generate each other's centraliser.

Proof. See [GW09, Chapter 10].
Similar to the partition algebra, the Brauer algebra $B_{r}(\delta)$ is almost always semisimple.

Proposition 3.65. The Brauer algebra $B_{r}(\delta)$ is semisimple for $\delta \notin \mathbb{Z}$.
Proof. See Wen88, Cor. 3.3].
Remark 3.66. It can be shown that $B_{r}(\delta)$ is not semisimple for some integers $\delta \in \mathbb{Z}$. For more precise semisimplicity criteria we refer to [Rui05] and AST17.

There is a nice explanation for the failure of semisimplicity of $B_{r}(n)$ using another kind of Schur-Weyl duality. For this wee need to dive into the world of Lie super algebras.

Super Schur-Weyl duality for $\mathfrak{g l}(V)$ :
Definition 3.67. A vector superspace $V=V_{\overline{0}} \oplus V_{\overline{1}}$ is a $\mathbb{Z} / 2$-graded vector space. For $v \in V$ homogeneous, let $|v| \in \mathbb{Z} / 2$ denote the degree of $v$. A Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{1}$ is a vector superspace with a Lie superbracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. This means that $\left[\mathfrak{g}_{i}, \mathfrak{g}_{j}\right] \subset \mathfrak{g}_{i+j}$ and that $[\cdot, \cdot]$ satisfies super skew-symmetry

$$
[x, y]=-(-1)^{|x||y|}[y, x]
$$

and the super Jacobi identity

$$
(-1)^{|x||z|}[x,[y, z]]+(-1)^{|y||x|}[y,[z, x]]+(-1)^{|z||y|}[z,[x, y]]=0
$$

with $x, y, z \in \mathfrak{g}$ homogeneous. For Lie superalgebras $\mathfrak{g}, \mathfrak{g}^{\prime}$ a homomorphism of Lie superalgebras is a linear map $f: \mathfrak{g} \rightarrow \mathfrak{g}^{\prime}$ satisfying $f([x, y])=[f(x), f(y)]$.

The most prominent example of a Lie superalgebra is the Lie superalgebra $\mathfrak{g l}(V)$ where $V=V_{\overline{0}} \oplus V_{\overline{1}}$ is a vector superspace. The grading on $\mathfrak{g l}(V)$ is given by

$$
\begin{aligned}
& \mathfrak{g l}(V)_{\overline{\overline{0}}}:=\mathfrak{g l l}\left(V_{\overline{0}}\right) \oplus \mathfrak{g l}\left(V_{\overline{1}}\right) \\
& \mathfrak{g l r}(V)_{\overline{1}}:=\operatorname{Hom}_{\mathbb{C}}\left(V_{0}, V_{1}\right) \oplus \operatorname{Hom}_{\mathbb{C}}\left(V_{1}, V_{0}\right)
\end{aligned}
$$

and we equip $\mathfrak{g l}(V)$ with the super commutator bracket

$$
[x, y]:=x y-(-1)^{|x||y|} y x
$$

for $x, y \in \mathfrak{g l}(V)$ homogeneous extended bilinearly to the whole space $\mathfrak{g l}(V)$.
Definition 3.68. For a Lie superalgebra $\mathfrak{g}$ and a vector superspace $V$, a $\mathfrak{g}$-module structure on $V$ is a Lie superalgebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{g l}(V)$. We then also call $V$ a (super) representation of $\mathfrak{g}$.

Given two $\mathfrak{g}$-modules $V, W$ we can define a $\mathfrak{g}$-module structure on $V \oplus W$ via $x \cdot(v+w)=x v+x w$ and on $V \otimes W$ via $x \cdot v \otimes w=x v \otimes w+(-1)^{|x||v|} v \otimes x w$. Hence we have an action of $\mathfrak{g l}(V)$ on $V^{\otimes r}$ and we can resume to our business of constructing commuting actions and Schur-Weyl dualities. In fact, defining the super swap operator $s \in \operatorname{End}(V \otimes V)$ by

$$
\begin{equation*}
v \otimes w \cdot s:=(-1)^{|v||w|} w \otimes v \tag{47}
\end{equation*}
$$

induces a right action of $S_{r}$ on $V^{\otimes r}$. With the same techniques as in the classical case (but keeping track of signs) one can prove the following Schur-Weyl duality.

Theorem 3.69. For any vector superspace $V$ the actions $\mathfrak{g l}(V) \curvearrowright V^{\otimes r} \curvearrowleft S_{r}$ commute and generate each other's centraliser. Moreover, $V^{\otimes r}$ is completely reducible as a $\mathfrak{g l}(V)$-module.

Proof. See [CW12, Thm. 3.10].
This might not look surprising, given the dualities we have seen so far. However, the category of completely reducible representations of a Lie superalgebra is usually not closed under taking tensor products and $V^{\otimes r}$ may not be completely reducible for some other Lie superalgebras (like for the duality that follows next).

Super Schur-Weyl duality for $\mathfrak{o s p}(V)$ : Let us now investigate the super analogue of the classical dualities in types $B, C, D$. For this, let $\langle\cdot, \cdot\rangle$ be a non-degenerate bilinear form on the vector superspace $V$ such that $\langle\cdot, \cdot\rangle$ is symmetric on $V_{0} \times V_{0}$, skew symmetric on $V_{1} \times V_{1}$ and 0 on mixed products. We define the orthosymplectic Lie superalgebra to be

$$
\mathfrak{o s p}(V):=\left\{x \in \mathfrak{g l}(V) \mid\langle x v, w\rangle+(-1)^{|x| v \mid}\langle v, x w\rangle=0 \quad \forall v, w \in V \text { homogeneous }\right\} .
$$

It is easy to check that this inherits the structure of a Lie superalgebra from $\mathfrak{g l}(V)$ and we have an action $\mathfrak{o s p}(V) \curvearrowright V^{\otimes r}$. Let $\left\{v_{i} \mid i \in I\right\}$ be a homogeneous basis of $V$ and denote by $\left\{v^{i} \mid i \in I\right\}$ the basis with $\left\langle v_{i}, v^{j}\right\rangle=\delta_{i, j}$. The action of the symmetric group algebra $\mathbb{C}\left[S_{r}\right]$ on $V^{\otimes r}$ from (47) can be extended to an action of the Brauer algebra $B_{r}\left(\operatorname{dim} V_{0}-\operatorname{dim} V_{1}\right)$ by letting $e_{i}$ acts as $\operatorname{id}^{\otimes i-1} \otimes e \otimes \operatorname{id}^{r-i-1}$ where

$$
v \otimes w \cdot e:=\langle v, w\rangle \cdot \sum_{i \in I}(-1)^{\left|v_{i}\right|} v_{i} \otimes v^{i} .
$$

A direct computation (c.f. [ES16, Section 3]) shows that the $\mathfrak{o s p}(V)$-action and the $B_{r}\left(\operatorname{dim} V_{0}-\operatorname{dim} V_{1}\right)$-action commute. This means that there is a canonical
homomorphism $B_{r}\left(\operatorname{dim} V_{0}-\operatorname{dim} V_{1}\right) \rightarrow \operatorname{End}_{\text {osp }(V)}\left(V^{\otimes r}\right)^{\mathrm{op}}$. Note that $V^{\otimes r}$ is in general not completely reducible as a $\mathfrak{o s p}(V)$-module. In particular, the bimodule $V^{\otimes r}$ will not decompose into a sum of outer tensor products of simples as in the double centraliser theorem. Nonetheless, one can prove the following.

Theorem 3.70. Let $m, n \in \mathbb{N}_{0}$ such that $\operatorname{dim} V_{0} \in\{2 m, 2 m+1\}$ and $\operatorname{dim} V_{1}=2 n$. If $\left(\operatorname{dim} V_{0}, \operatorname{dim} V_{1}\right) \neq(2 m, 0)$ and $r \leq n+m$, the canonical algebra homomorphism $B_{r}\left(\operatorname{dim} V_{0}-\operatorname{dim} V_{1}\right) \rightarrow \operatorname{End}_{\text {osp }(V)}\left(V^{\otimes r}\right)^{\text {op }}$ is an isomorphism.

Proof. This was proved by Ehrig and Stroppel [ES16].
Remark 3.71. There is a nice explanation for the failure of semisimplicity of $B_{r}(n)$ for some $n \in \mathbb{Z}$ (c.f. Remark 3.66) using the duality from Theorem 3.70: By choosing $V_{0}$ and $V_{1}$ with $\operatorname{dim} V_{0}-\operatorname{dim} V_{1}=n$ and $\operatorname{dim} V_{0}+\operatorname{dim} V_{1}$ large enough, the Brauer algebra $B_{r}(n)$ can be realised as the endomorphism algebra of the $\mathfrak{o s p}(V)$-module $V^{\otimes r}$. $B_{r}(n)$ not being semisimple then corresponds to $V^{\otimes r}$ not being completely reducible as an $\mathfrak{o s p}(V)$-module. It would be interesting to have a similar explanation for the failure of semisimplicity of the partition algebra $P_{r}(\delta)$ for some $\delta \in \mathbb{N}_{0}$ but no such argument is known to the author.

The walled Brauer algebra: We want to talk about one more type of Schur-Weyl duality fitting into the classical setting. The motivation for this comes from the observation that not every irreducible representation of the (algebraic) group $\mathrm{GL}_{n}(\mathbb{C})$ appears in some tensor power of $V=\mathbb{C}^{n}$. On the other hand, it is known that any finite-dimensional $\mathrm{GL}_{n}(\mathbb{C})$-representation appears in $V^{\otimes r} \otimes V^{* \otimes s}$ for some nonnegative integers $r, s$. We explain how to construct a Schur-Weyl duality for these mixed tensor powers.

Definition 3.72. The walled Brauer algebra $B_{r, s}(\delta)$ is the subalgebra of the Brauer algebra $B_{r+s}(\delta)$ spanned by all diagrams $d$ with the property that any block $B$ of $d$ contains elements from both $\left\{1, \ldots, r, 1^{\prime}, \ldots, r^{\prime}\right\}$ and $\left\{r+1, \ldots, r+s,(r+1)^{\prime}, \ldots,(r+s)^{\prime}\right\}$ if and only if $B$ is not propagating.

This is called the walled Brauer algebra since it has a standard basis which can be interpreted as Brauer diagrams with a wall separating the dots in $\left\{1, \ldots, r, 1^{\prime}, \ldots, r^{\prime}\right\}$ from the dots in $\left\{r+1, \ldots, r+s,(r+1)^{\prime}, \ldots,(r+s)^{\prime}\right\}$ such that strands cross the wall if and only if they are not propagating. Here is an example of a walled Brauer diagram in $B_{3,2}(\delta)$


Clearly $\mathbb{C}\left[S_{r} \times S_{s}\right]$ is a subalgebra of $B_{r, s}(\delta)$ and there is a distinguished diagram


In fact, it is not hard to see that $B_{r, s}(\delta)$ is generated by $\mathbb{C}\left[S_{r}, \times S_{s}\right]$ and the element $e_{r}$. Let $v_{1}, \ldots, v_{n}$ be a basis of $V$ and let $v^{1}, \ldots, v^{n}$ be the dual basis in $V^{*}$. We can act with $S_{r} \times S_{s}$ on $V^{\otimes r} \otimes V^{* \otimes s}$ by letting $S_{r}$ permute the tensor factors of $V^{\otimes r}$ and letting $S_{s}$ permute the tensor factors of $V^{* \otimes s}$. This extends uniquely to an action $V^{\otimes r} \otimes V^{* \otimes s} \curvearrowleft B_{r, s}(n)$ by letting $e_{r}$ act as $\mathrm{id}_{V}^{\otimes r-1} \otimes e \otimes \mathrm{id}_{V^{*}}^{\otimes s-1}$ where $e \in \operatorname{End}_{\mathbb{C}}\left(V \otimes V^{*}\right)$ acts by

$$
v \otimes f \cdot e=f(v) \cdot \sum_{i=1}^{n} v_{i} \otimes v^{i}
$$

Remark 3.73. Identifying $v_{i}$ with $v^{i}$ gives (non-canonical) isomorphisms of vector spaces $V \cong V^{*}$ and $V^{\otimes r} \otimes V^{* \otimes s} \cong V^{\otimes r+s}$. The induced action of $B_{r, s}(n)$ on $V^{\otimes r+s}$ is then just the restriction of the $P_{r+s}(n)$ along the inclusion $B_{r, s}(n) \subset P_{r+s}(n)$.

One can show by direct computation that the $\mathrm{GL}_{n}(\mathbb{C})$-action and the $B_{r, s}(n)$ action on $V^{\otimes r} \otimes V^{* \otimes s}$ commute. We even have the following mixed Schur-Weyl duality

Theorem 3.74. The commuting actions $\mathrm{GL}_{n}(\mathbb{C}) \curvearrowright V^{\otimes r} \otimes V^{* \otimes s} \curvearrowleft B_{r, s}(n)$ generate each other's centraliser.

Proof. The endofunctors $V \otimes(-)$ and $V^{*} \otimes(-)$ on $\operatorname{Rep}\left(\mathrm{GL}_{n}(\mathbb{C})\right)$ form a biadjoint pair. This induces an isomorphism $\operatorname{End}_{\mathrm{GL}_{n}(\mathbb{C})}\left(V^{\otimes r+s}\right) \cong \operatorname{End}_{\mathrm{GL}_{n}(\mathbb{C})}\left(V^{\otimes r} \otimes V^{* \otimes s}\right)$. One can check that on diagrams this is the same as flipping the subdiagram on the dots $\left\{r+1, \ldots, r+s,(r+1)^{\prime}, \ldots,(r+s)^{\prime}\right\}$ upside down. Hence, this isomorphism identifies permutation diagrams with walled Brauer diagrams and the result follows from classical Schur-Weyl duality. For more details see [Nik07].

Remark 3.75. There also is a super version of mixed Schur-Weyl duality (see BS12, Thm 7.8]).

## 4 Towers of algebras and Jucys-Murphy elements

Most of the diagram algebras from the previous sections come in the form of a tower. For example, there is the tower $\mathbb{C}\left[S_{0}\right] \subset \mathbb{C}\left[S_{1}\right] \subset \mathbb{C}\left[S_{2}\right] \subset \mathbb{C}\left[S_{3}\right] \subset \ldots$ of symmetric group algebras induced by the inclusions $S_{i} \hookrightarrow S_{i+1}$ with $s_{k} \mapsto s_{k}$. We will explain how the structure of towers like this can be analysed using the so-called Jucys-Murphy elements. Moreover, these elements can be used to construct so-called higher Schur-Weyl dualities which also involve infinite-dimensional representations of Lie algebras like Verma modules. Our main sources in Section 4.1 will be OV96 and CSST10] and in Section 4.2 (resp. Section 4.3) we will follow AS98, BK08 (resp. [ES18]). Motivated by the results from Section 4.1-Section 4.3, we will study the Jucys-Murphy elements of the partition algebra in Section 4.4. These elements were introduced by HR05 but our construction will be slightly different using Schur-Weyl duality. We then use an interpolation argument to give new proofs of some formulas for the Jucys-Murphy elements (in particular Lemma 4.54 and Proposition 4.55) which were also verified in Eny13] using other techniques. We will also show that the relations between the Jucys-Murphy elements and the standard generators of the partition algebra are not local (see Proposition 4.56) explaining why deriving a higher Schur-Weyl duality for the partition algebra might be a more difficult task.

### 4.1 The Okounkov-Vershik approach

The representation theory of a semisimple Lie algebra $\mathfrak{g}$ is usually developed by analysing the action of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ on the representations of $\mathfrak{g}$. In fact, $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and any finite-dimensional representation decomposes into simultaneous eigenspaces for the action of $\mathfrak{h}$. These eigenspaces are called weight spaces and any finite-dimensional irreducible $\mathfrak{g}$-representation is uniquely determined by its weight space decomposition. One can try and imitate this approach to develop the representation theory of a finite-dimensional semisimple algebra $A$. The strategy is to look at the action of a maximal commutative subalgebra of $A$ on the irreducible representation of $A$. In OV96, Okounkov and Vershik used this approach (which was pioneered by Nazarov) to develop the representation theory of $S_{n}$ only assuming a few basic facts about the representation theory of finite groups. In this thesis, we are less interested in the Okounkov-Vershik approach as a new way to develop the representation theory of the symmetric group but rather as a general framework to study finite-dimensional semisimple algebras (whose irreducible representations might already be known by other methods). Let us explain these ideas in a bit more detail.

Consider an arbitrary tower of finite-dimensional semisimple algebras

$$
\begin{equation*}
\mathbb{C} \cong A_{0} \subset A_{1} \subset A_{2} \subset A_{3} \subset \ldots \tag{48}
\end{equation*}
$$

Let $\hat{A}_{i}$ be an indexing set of the simple $A_{i}$-modules $V^{\lambda}\left(\lambda \in \hat{A}_{i}\right)$
Definition 4.1. The branching graph of a tower as in (48) of finite-dimensional semisimple algebras is the multigraph with vertices

$$
\hat{A}_{0} \sqcup \hat{A}_{1} \sqcup \hat{A}_{2} \sqcup \hat{A}_{3} \sqcup \ldots
$$

such that $\left[\operatorname{Res}_{A_{i-1}}^{A_{i}} V^{\lambda}: V^{\mu}\right]$ is the number of edges between $\mu \in \hat{A}_{i-1}$ and $\lambda \in \hat{A}_{i}$. We say that the branching is multiplicity-free if no multiple edges occur.

Remark 4.2. Note that

$$
\begin{aligned}
{\left[\operatorname{Res}_{A_{n-1}}^{A_{n}} V^{\lambda}: V^{\mu}\right] } & =\operatorname{dim} \operatorname{Hom}_{A_{n-1}}\left(V^{\mu}, \operatorname{Res}_{A_{n-1}}^{A_{n}} V^{\lambda}\right) \\
& =\operatorname{dim} \operatorname{Hom}_{A_{n}}\left(\operatorname{Ind}_{A_{n-1}}^{A_{n}} V^{\mu}, V^{\lambda}\right) \\
& =\left[\operatorname{Ind}_{A_{n-1}}^{A_{n}} V^{\mu}: V^{\lambda}\right]
\end{aligned}
$$

by Frobenius reciprocity. Hence, the branching graph encodes both the decomposition of restriction and induction applied to irreducible representations.

In what follows, we make the following assumption:
(A): The tower of semisimple algebras $A_{0} \subset A_{1} \subset \ldots$ has multiplicity-free branching.

In other words, we assume that for any $\lambda \in \hat{A}_{i}$ there is a multiplicity-free decomposition

$$
V^{\lambda}=\bigoplus_{\mu \nearrow \lambda} V^{\mu}
$$

where $\mu \nearrow \lambda$ means that $\mu \in \hat{A}_{i-1}$ and there is an edge between $\mu$ and $\lambda$ in the branching graph. Note that $A_{0} \cong \mathbb{C}$ has a unique irreducible representation $V^{\lambda_{0}}$
which is one-dimensional. Applying the branching rule recursively, we get that any irreducible representation $V^{\lambda}$ of $A_{n}$ has a canonical decomposition

$$
V^{\lambda}=\bigoplus_{T} V_{T}
$$

where the sum runs through all paths $T=\lambda_{0} \nearrow \lambda_{1} \nearrow \ldots \nearrow \lambda_{n}$ in the branching graph starting at $\lambda_{0} \in \hat{A}_{0}$ and terminating at $\lambda_{n}=\lambda . V_{T}$ is the one-dimensional subspace of $V^{\lambda}$ uniquely determined by the property that $V_{T}$ lies in the $V^{\lambda_{i} \text {-isotypical }}$ component of $V^{\lambda}$ for all $i=0,1, \ldots, n$.

Definition 4.3. By choosing non-zero vectors $v_{T} \in V_{T}$ for each path $T$ as above, we get a basis of $V^{\lambda}$ that is unique up to scaling. This is called the Gelfand-Tsetlin basis.

Note that

$$
A_{n} \cong \bigoplus_{\lambda \in \hat{A}_{n}} \operatorname{End}_{\mathbb{C}}\left(V^{\lambda}\right) \subset \operatorname{End}_{\mathbb{C}}\left(\bigoplus_{\lambda \in \hat{A}_{n}} V^{\lambda}\right)
$$

and we can consider the Gelfand-Tsetlin basis of $\bigoplus_{\lambda \in \hat{A}_{n}} V^{\lambda}$. This basis is indexed by the set of all $n$-step paths in the branching graph

$$
\operatorname{Path}(n):=\left\{\lambda_{0} \nearrow \lambda_{1} \nearrow \ldots \nearrow \lambda_{n} \mid \lambda_{i} \in \hat{A}_{i}\right\} .
$$

Let $Z(n)$ be the centre of $A_{n}$.
Definition 4.4. The Gelfand-Tsetlin algebra $G Z(n)$ is the subalgebra of $A_{n}$ generated by $Z(1), \ldots, Z(n)$.

Lemma 4.5. The algebra $G Z(n) \subset A_{n} \subset \operatorname{End}_{\mathbb{C}}\left(\bigoplus_{\lambda \in \hat{A}_{n}} V^{\lambda}\right)$ is the algebra of $\mathbb{C}$-linear operators on $\bigoplus_{\lambda \in \hat{A}_{n}} V^{\lambda}$ that are diagonal in the Gelfand-Tsetlin basis. Moreover, $G Z(n) \subset A_{n}$ is a maximal commutative subalgebra.

Proof. See OV96, Prop. 1.1] or [CSST10, Thm. 2.2.2].
Assume now that we are given generators $X_{1}, \ldots, X_{m}$ of $G Z(n)$. In particular, we have $X_{i} \cdot v_{T}=a_{i} v_{T}$ for some $a_{i} \in \mathbb{C}$ and we call

$$
\alpha\left(v_{T}\right):=\left(a_{1}, \ldots, a_{m}\right)
$$

the weight of $v_{T}$. Let $\operatorname{Spec}(n)=\left\{\alpha\left(v_{T}\right) \mid T \in \operatorname{Path}(n)\right\}$ be the set of weights. Since the $X_{i}$ generate $G Z(n)$, we see that $v_{T}$ is uniquely determined (up to scaling) by its weight. In fact, if $\alpha\left(v_{T}\right)=\alpha\left(v_{T^{\prime}}\right)$ and $X \in G Z(n)$, then $X \cdot v_{T}=c v_{T}$ and $X \cdot v_{T^{\prime}}=c v_{T^{\prime}}$ for the same $c \in \mathbb{C}$. In particular, $\operatorname{pr}_{V_{T}}\left(v_{T}\right)=v_{T}$ implies that $\operatorname{pr}_{V_{T}}\left(v_{T^{\prime}}\right)=v_{T^{\prime}}$ and hence $v_{T^{\prime}} \in V_{T}$. We see that there is a bijection

$$
\begin{aligned}
\operatorname{Path}(n) & \stackrel{1: 1}{\longleftrightarrow} \operatorname{Spec}(n) \\
T & \longmapsto \alpha\left(v_{T}\right) .
\end{aligned}
$$

If we equip the set $\operatorname{Path}(n)$ with an equivalence relation where $T \sim T^{\prime}$ if $T$ and $T^{\prime}$ terminate at the same vertex, we get an induced equivalence relation $\sim$ on $\operatorname{Spec}(n)$ such that the equivalence classes parametrise the irreducible $A_{n}$-modules. This can be used to study the representation theory of $A_{n}$ by looking at the weight structure of irreducible representations. The art is of course to find a particularly nice generating set $X_{1}, \ldots, X_{m}$ of $G Z(n)$ for which there is a good combinatorial description of
$\operatorname{Spec}(n)$. Here, 'particularly nice' is not a term we can make mathematically precise and its meaning will depend on the application. Hence, from this point onwards, we have to work with specific examples (though there are still some similarities in most applications).

As a first example, let us look at the tower of symmetric group algebras

$$
\begin{equation*}
\mathbb{C} \cong \mathbb{C}\left[S_{0}\right] \subset \mathbb{C}\left[S_{1}\right] \subset \mathbb{C}\left[S_{2}\right] \subset \mathbb{C}\left[S_{3}\right] \subset \ldots \tag{49}
\end{equation*}
$$

induced by the inclusions $S_{i} \hookrightarrow S_{i+1}, s_{k} \mapsto s_{k}$. One can show that this tower satisfies assumption (A).

Proposition 4.6. The branching graph of the tower (49) is multiplicity-free, i.e. $\left[\operatorname{Res}_{S_{n-1}}^{S_{n}} V^{\lambda}: V^{\mu}\right] \in\{0,1\}$ for any $n \geq 1$.

Proof. This can be done by elementary methods using the fact that $A \subset B$ has multiplicity-free branching if and only if $Z(B, A):=\{b \in B \mid a b=b a \quad \forall a \in A\}$ is commutative. For more details see OV96 or [CSST10, Thm. 2.1.20, Cor. 3.2.2].

For $0 \leq k \leq n$ let

$$
\begin{equation*}
Z_{k}:=\sum_{1 \leq i<j \leq k}(i j) \in \mathbb{C}\left[S_{k}\right] \tag{50}
\end{equation*}
$$

with $Z_{0}=Z_{1}=0$.
Definition 4.7. For $1 \leq i \leq n$ the elements

$$
X_{i}:=Z_{i}-Z_{i-1}=(1 i)+(2 i)+\ldots+(i-1 i) .
$$

are called the Jucys-Murphy elements of $\mathbb{C}\left[S_{n}\right]$.
Note that $X_{1}=0$. Clearly, $Z_{i}$ lies in the centre $Z(i)$ of $\mathbb{C}\left[S_{i}\right]$ and hence we have $X_{i} \in G Z(n)$ for all $1 \leq i \leq n$. One can even show the following.

Proposition 4.8. The Jucys-Murphy elements $X_{1}, \ldots, X_{n}$ generate the GelfandTsetlin algebra $G Z(n)$.

Proof. See OV96 or [CSST10, Cor. 3.2.7].
It follows from the general set-up that any Gelfand-Tsetlin basis vector is uniquely determined (up to scaling) by its weight for the action of the Jucys-Murphy elements. To see what weights can occur, one needs to understand how the Jucys-Murphy elements interact with the standard generators $s_{1}, \ldots, s_{n-1}$ of $\mathbb{C}\left[S_{n}\right]$.

Lemma 4.9. We have $s_{i} X_{j}=X_{j} s_{i}$ if $j \neq i, i+1$ and $s_{i} X_{i} s_{i}+s_{i}=X_{i+1}$.
Proof. $X_{j}$ commutes with $\mathbb{C}\left[S_{j-1}\right]$ since $X_{j}=Z_{j}-Z_{j-1}$ and $Z_{k}$ is a central element of $\mathbb{C}\left[S_{k}\right]$. This proves that $s_{i}$ and $X_{j}$ commute for $i<j-1$. The generator $s_{i}$ commutes with $\mathbb{C}\left[S_{i-1}\right]$ and hence with $X_{j}$ for $j<i$. Finally, we have

$$
s_{i} X_{i} s_{i}+s_{i}=\sum_{j=1}^{i-1}(j i+1)+s_{i}=\sum_{j=1}^{i}(j i+1)=X_{i+1} .
$$

Here is an example that illustrates how the relations above may be used to study the weight structure of the Jucys-Murphy elements.

Proposition 4.10. Let $V$ be a finite-dimensional representation of $S_{n}$. Then the eigenvalues of $X_{i}$ on $V$ are all integral for $i=1, \ldots, n$.

Proof. The claim is obvious for $i=1$ since $X_{1}=0$. We proceed by induction. Assume we have shown the claim for $X_{i}$. The commuting operators $X_{i}$ and $X_{i+1}$ diagonalise simultaneously in the Gelfand-Testlin basis of $V$. Let $v \in V$ be such a simultaneous eigenvector with $X_{i} v=a v$ and $X_{i+1} v=b v$ for $a, b \in \mathbb{C}$. Then

$$
\begin{aligned}
X_{i}\left((a-b) s_{i} v+v\right) & =(a-b) X_{i} s_{i} v+X_{i} v \stackrel{4.9}{=}(a-b)\left(s_{i} X_{i+1}-1\right) v+X_{i} v \\
& =(a-b) \cdot b s_{i} v-(a-b) v+a v=b\left((a-b) s_{i} v+v\right) .
\end{aligned}
$$

If $(a-b) s_{i} v+v \neq 0$ this shows that $b$ is an eigenvalue of $X_{i}$. In this case $b$ is integral by the induction assumption. If $(a-b) s_{i} v+v=0$, then $s_{i} v=c v$ for some $c \in \mathbb{C}$ and since $s_{i}^{2}=1$, we get $s_{i} v= \pm v$. Then $(a-b) v= \pm v$ and hence $b=a \pm 1$. Since $a$ is integral by induction assumption, we get that $b$ is integral.

The relations from Lemma 4.9 can also be studied in the following more formal set-up.

Definition 4.11. The degenerate affine Hecke algebra $\mathcal{H}_{n}^{\mathrm{deg}}$ is the $\mathbb{C}$-algebra with generators $s_{1}, . ., s_{n-1}, x_{1}, \ldots, x_{n}$ and relations
(HDEG1)
(i) $s_{i}^{2}=1$
for $i=1, \ldots, n-1$
(ii) $s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}$
for $i=1, \ldots, n-2$
(iii) $s_{i} s_{j}=s_{j} s_{i}$
(HDEG2)
(i) $s_{i} x_{i} s_{i}+s_{i}=x_{i+1}$
(ii) $s_{i} x_{j}=x_{j} s_{i}$
for $i=1, \ldots, n-1$
(iii) $x_{i} x_{j}=x_{j} x_{i}$
if $j \neq i, i+1$
for $i, j=1, \ldots, n$.
if $|i-j|>1$

Remark 4.12. There also is a diagrammatic interpretation of the degenerate affine Hecke algebra (which is also explained in Kho14, Section 2]). We still interpret the elements of the symmetric group as permutation diagrams. However, in contrast to the partition algebra, we omit the dots at the end of strands which usually indicate the elements of a block. Instead, the generator $x_{i}$ corresponds to the identity diagram with a dot on the $i$-th strand, i.e.

$$
x_{i}=|\cdots| \begin{aligned}
& \mathrm{i} \\
& \mathrm{i} \\
& \mathrm{i} \\
&
\end{aligned}|\cdots| .
$$

Similarly, a monomial $x_{i}^{k}$ corresponds to the identity diagram with $k$ dots on the $i$-th strand. The defining relations of the degenerate affine Hecke algebra can then be read as the constraint that dots may be moved freely along strands but an error term has to be introduced when passing through a crossing. In fact, (HDEG2)-(i) can be rewritten as $s_{i} x_{i}=x_{i+1} s_{i}-1$ which can be interpreted (locally) as


By Lemma 4.9 there is a surjective algebra homomorphism

$$
\begin{align*}
\mathcal{H}_{n}^{\operatorname{deg}} & \longrightarrow \mathbb{C}\left[S_{n}\right] \\
s_{i} & \longmapsto s_{i}  \tag{51}\\
x_{i} & \longmapsto X_{i} .
\end{align*}
$$

Note that there is an inclusion $\mathbb{C}\left[S_{n}\right] \hookrightarrow \mathcal{H}_{n}^{\mathrm{deg}}$ with $s_{i} \mapsto s_{i}$. This is injective since the composition $\mathbb{C}\left[S_{n}\right] \rightarrow \mathcal{H}_{n}^{\operatorname{deg}} \stackrel{[51]}{\Longrightarrow} \mathbb{C}\left[S_{n}\right]$ is the identity.

Lemma 4.13. The kernel of the canonical homomorphism $\mathcal{H}_{n}^{\mathrm{deg}} \rightarrow \mathbb{C}\left[S_{n}\right]$ from 51) is generated by $x_{1}$. In particular, we have $\mathcal{H}_{n}^{\operatorname{deg}} /\left(x_{1}\right) \cong \mathbb{C}\left[S_{n}\right]$.
Proof. Clearly, the ideal $I=\left(x_{i}-X_{i} \mid 1 \leq i \leq n\right) \subset \mathcal{H}_{n}^{\text {deg }}$ is contained in the kernel of the homomorphism from (51). Moreover, $\mathcal{H}_{n}^{\mathrm{deg}}$ is generated by $\mathbb{C}\left[S_{n}\right]$ and the $x_{i}$ which are identified with the $X_{i} \in \mathbb{C}\left[S_{n}\right]$ after modding out $I$. Hence, $\mathcal{H}_{n}^{\mathrm{deg}} / I$ is generated by $\mathbb{C}\left[S_{n}\right]$ and the canonical map $\mathcal{H}_{n}^{\mathrm{deg}} / I \rightarrow \mathbb{C}\left[S_{n}\right]$ is an isomorphism. Note that $x_{1}-X_{1}=x_{1}$ and

$$
s_{i}\left(x_{i}-X_{i}\right) s_{i}=s_{i} x_{i} s_{i}-s_{i} X_{i} s_{i}=\left(x_{i+1}-s_{i}\right)-\left(X_{i+1}-s_{i}\right)=x_{i+1}-X_{i+1} .
$$

This shows that $I=\left(x_{1}\right)$ and the claim follows.
Using the algebra homomorphism from (51), any $\mathbb{C}\left[S_{n}\right]$-module can be considered as a $\mathcal{H}_{n}^{\text {deg }}$-module by letting $x_{i} \in \mathcal{H}_{n}^{\text {deg }}$ act by multiplying with the Jucys-Murphy element $X_{i}$. Using the relations of the degenerate affine Hecke algebra, one can give a complete combinatorial description of the set $\operatorname{Spec}(n)$. Moreover, this can be used to determine the branching graph of the tower in (49). To state this result, we need to define the Young graph.

Definition 4.14. The Young graph is the graph with vertices indexed by partitions

$$
\{\lambda \vdash 0\} \sqcup\{\lambda \vdash 1\} \sqcup\{\lambda \vdash 2\} \sqcup\{\lambda \vdash 3\} \sqcup \ldots
$$

and an edge between $\mu \vdash i-1$ and $\lambda \vdash i$ if and only if $\mu \subset \lambda$.
Here are the first 5 layers of the Young graph


To any box $(x, y)$ in a Young diagram $\lambda \vdash n$, we can associate content $c((x, y))=y-x$ (where by convention the $x$-coordinate increases from top to bottom and the $y$
coordinate increases from left to right). For example, the contents of the boxes in the diagram ( $4,4,3,2,1,1$ ) are given by

| 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| -1 | 0 | 1 | 2 |
| -2 | -1 | 0 |  |
| -3 | -2 |  |  |
| -4 |  |  |  |
| -5 |  |  |  |

To any path $T=\lambda_{0} \nearrow \lambda_{1} \nearrow \ldots \nearrow \lambda_{n}$ in the Young graph we can associate a content vector $c(T) \in \mathbb{C}^{n}$ where $c(T)_{i}$ is the content of the box of $\lambda_{i} / \lambda_{i-1}$. Let $\operatorname{Cont}(n)$ be the set off all contents $c(T)$ of $n$-step paths $T$ in the Young graph. The $S_{n}$-orbits define an equivalence relation $\approx$ on $\operatorname{Cont}(n)$, i.e.

$$
\left(a_{1}, \ldots, a_{n}\right) \approx\left(b_{1}, \ldots, b_{n}\right): \Leftrightarrow \exists \sigma \in S_{n} \text { such that } a_{i}=b_{\sigma(i)} \text { for all } i .
$$

Using the Jucys-Murphy elements, the following is shown in OV96 (see also CSST10, Thm. 3.3.7]).

Theorem 4.15. The Young graph is the branching graph of the symmetric group. We have $\operatorname{Cont}(n)=\operatorname{Spec}(n)$ and the content vector $c(T)$ of a path $T$ is the same as the weight $\alpha(T)$. Moreover, the equivalence relations $\sim$ and $\approx$ coincide.

This is now an explicit tool that can be used to decompose a representation of $S_{n}$ into irreducibles once it has been decomposed into simultaneous eigenspaces for the action of the Jucys-Murphy elements.

Example 4.16. Consider the permutation representation $V=\mathbb{C}^{n}$ of $S_{n}$ with standard basis $v_{1}, \ldots, v_{n}$. For $i=1, \ldots, n-1$, the vectors

$$
w_{i}=v_{1}+v_{2}+\ldots+v_{i}-i v_{i+1}
$$

are common eigenvectors of the Jucys-Murphy elements. In fact, one can check that

$$
X_{k} \cdot w_{i}= \begin{cases}(k-1) w_{i} & \text { if } k \leq i \\ -w_{i} & \text { if } k=i+1 \\ (k-2) w_{i} & \text { if } k>i+1\end{cases}
$$

Hence, $V$ contains an irreducible representation corresponding to a Young diagram whose boxes have content $\{-1,0,1, \ldots, n-2\}$. This is the Young diagram

| 0 | 1 | 2 | $n-3 \mid n-2$ |
| :---: | :---: | :---: | :---: |
| -1 |  |  |  |

The vector

$$
w_{n}=v_{1}+v_{2}+\ldots+v_{n}
$$

is also a common eigenvector with $X_{k} \cdot w_{n}=(k-1) w_{n}$. This correspond to the Young diagram

| 0 | 1 | 2 | $\cdots$ | $n-2$ |
| :--- | :--- | :--- | :--- | :--- |
|  | $n-1$ |  |  |  |

The vectors $w_{1}, \ldots, w_{n}$ form a basis of $V$ and their respective weights are given by

$$
(0,-1,1,2 \ldots, n-1),(0,1,-1,2, \ldots, n-1), \ldots,(0,1, \ldots, n-1,-1),(0,1, \ldots, n)
$$

Since these weights are pairwise distinct, every irreducible in $V$ appears with multiplicity 1. This shows that there is a decomposition

$$
V=\operatorname{Span}_{\mathbb{C}}\left\{w_{1}, \ldots, w_{n-1}\right\} \oplus \operatorname{Span}_{\mathbb{C}}\left\{w_{n}\right\} \cong S(\square \square \ldots \square) \oplus S(\square \square \ldots \square)
$$

### 4.2 Higher Schur-Weyl duality

As mentioned in the previous section, the degenerate affine Hecke algebra $\mathcal{H}_{r}^{\text {deg }}$ plays an important role in the representation theory of the symmetric group via the ring homomorphism $\mathcal{H}_{r}^{\mathrm{deg}} \rightarrow \mathbb{C}\left[S_{r}\right], x_{1} \mapsto 0$. In this section, we consider a generalisation of classical Schur-Weyl duality, often called higher Schur-Weyl duality, that relates the degenerate affine Hecke algebra with $\mathfrak{g l} l_{n}$. Apart from the algebra $\mathcal{H}_{r}^{\text {deg }}$ being interesting by itself, this generalisation can also be motivated by a desire to look at infinite-dimensional representations of $\mathfrak{g l}{ }_{n}$ like Verma modules. In fact, following AS98, we will construct commuting actions

$$
\mathfrak{g l}_{n} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\mathrm{deg}}
$$

where $M$ is an arbitrary (not necessarily finite-dimensional) $\mathfrak{g l}_{n}$-module and $V=\mathbb{C}^{n}$ is the defining $\mathfrak{g l}_{n}$-representation. For the $\mathfrak{g l}_{n}$-action on $M \otimes V^{\otimes r}$ we just take the usual action on tensor products. The more difficult part is the construction of an interesting $\mathcal{H}_{r}^{\mathrm{deg}}$-action on $M \otimes V^{\otimes r}$ that commutes with the $\mathfrak{g l}_{n}$-action.

For this, recall the construction of the Casimir element: Assume we are given a non-degenerate symmetric invariant bilinear form $(\cdot, \cdot)$ of $\mathfrak{g l}_{n}$ where invariant means that $([x, y], z)=(x,[y, z])$ for all $x, y, z \in \mathfrak{g l}_{n}$. Pick a basis $\left\{X_{\gamma} \mid \gamma \in B\right\}$ of $\mathfrak{g l}_{n}$ and let $\left\{X^{\gamma} \mid \gamma \in B\right\}$ be the dual basis with respect to $(\cdot, \cdot)$. Then the Casimir element for $(\cdot, \cdot)$ is defined as

$$
C:=\sum_{\gamma \in B} X_{\gamma} X^{\gamma} \in U\left(\mathfrak{g l}_{n}\right)
$$

It follows from the invariance of $(\cdot, \cdot)$ that $C$ is a central element of $U\left(\mathfrak{g l}_{n}\right)$. Moreover, $C$ is independent of the choice of a basis which follows from the next lemma (using the multiplication map $\left.\mathfrak{g l} \otimes \mathfrak{g l}_{n} \rightarrow U\left(\mathfrak{g l}_{n}\right)\right)$.

Lemma 4.17. Let $V$ be a finite-dimensional vector space with a basis $v_{1}, \ldots, v_{m}$. Let $v^{1}, \ldots, v^{m} \in V^{*}$ be the dual basis. Then $\sum_{i=1}^{m} v_{i} \otimes v^{i} \in V \otimes V^{*}$ does not depend on the choice of the basis $v_{1}, \ldots, v_{m}$.

Proof. The canonical isomorphism $V \otimes V^{*} \cong \operatorname{End}_{\mathbb{C}}(V)$ sends $\sum_{i=1}^{m} v_{i} \otimes v^{i}$ to the element $\mathrm{id}_{V} \in \operatorname{End}_{\mathbb{C}}(V)$. This is independent of the chosen basis.

As a central element of $U\left(\mathfrak{g l}_{n}\right)$, the Casimir operator defines a $\mathfrak{g l}_{n}$-endomorphism of any $\mathfrak{g l}_{n^{\prime}}$-module $M$. We would like to have a similar element that induces a $\mathfrak{g l}_{n^{-}}$ endomorphisms on tensor products $M \otimes N$.

Definition 4.18. The pseudo Casimir element in $U\left(\mathfrak{g l}_{n}\right) \otimes U\left(\mathfrak{g l}_{n}\right)$ is defined as

$$
\Omega:=\sum_{\gamma \in B} X_{\gamma} \otimes X^{\gamma} .
$$

By Lemma 4.17 the pseudo Casimir element $\Omega$ is independent of the choice of the basis. In particular, we have $\Omega=\sum_{\gamma \in B} X^{\gamma} \otimes X_{\gamma}$. Using this, we can relate the Casimir element with the pseudo Casimir element via

$$
\begin{align*}
\Delta(C)-C \otimes 1-1 \otimes C= & \sum_{\gamma \in B}\left(X_{\gamma} \otimes 1+1 \otimes X_{\gamma}\right)\left(X^{\gamma} \otimes 1+1 \otimes X^{\gamma}\right) \\
& -\sum_{\gamma \in B} X_{\gamma} X^{\gamma} \otimes 1-\sum_{\gamma \in B} 1 \otimes X_{\gamma} X^{\gamma}  \tag{52}\\
= & \sum_{\gamma \in B}\left(X_{\gamma} \otimes X^{\gamma}+X^{\gamma} \otimes X_{\gamma}\right) \\
= & 2 \Omega .
\end{align*}
$$

Lemma 4.19. The pseudo Casimir element $\Omega$ commutes with $\Delta(x)$ for any $x \in \mathfrak{g l}_{n}$. Proof. The Casimir element $C \in U\left(\mathfrak{g l}_{n}\right)$ is central. Hence, $C \otimes 1$ and $1 \otimes C$ are central elements of $U\left(\mathfrak{g l}_{n}\right) \otimes U\left(\mathfrak{g l}_{n}\right)$ and therefore they commute with $\Delta(x)$. Moreover,

$$
\Delta(x) \Delta(C)=\Delta(x C)=\Delta(C x)=\Delta(C) \Delta(x) .
$$

This shows that $\Delta(x)$ commutes with $\Omega \stackrel{[52]}{=} \frac{1}{2}(\Delta(C)-C \otimes 1-1 \otimes C)$.
This also shows that $\Omega$ commutes with the $\mathfrak{g l}_{n}$ action on an arbitrary tensor product $M \otimes N$ of $\mathfrak{g l}_{n}$-modules. This is already a hint that $\Omega$ might play an important role in the construction of the $\mathcal{H}_{r}^{\operatorname{deg}}$-action on $M \otimes V^{\otimes r}$. We now take a specific choice for the invariant bilinear form $(\cdot, \cdot)$, namely

$$
(x, y):=\operatorname{Tr}(x y) .
$$

The Lie algebra $\mathfrak{g l}_{n}$ has the standard basis $\left\{E_{i j} \mid 1 \leq i, j \leq n\right\}$ and $E_{j i}$ is the dual basis vector of $E_{i j}$ with respect to $(\cdot, \cdot)$. Hence, we can express the pseudo Casimir element as

$$
\begin{equation*}
\Omega=\sum_{1 \leq i, j \leq n} E_{i j} \otimes E_{j i} . \tag{53}
\end{equation*}
$$

Considered as an operator on $V \otimes V$, this is very familiar.
Lemma 4.20. The pseudo Casimir element $\Omega$ acts as the swap operator on $V \otimes V$.
Proof. It suffices to check this on the standard basis of $V \otimes V$. There we have $\Omega \cdot v_{i} \otimes v_{j}=v_{j} \otimes v_{i}$ by (53) which is the swap.

We can use the pseudo Casimir element to describe the action of the JucysMurphy elements of $\mathbb{C}\left[S_{r}\right]$ on $M \otimes V^{\otimes r}$ (where $S_{r}$ acts on $M \otimes V^{\otimes r}$ by permuting the tensor factors of $V^{\otimes r}$ ). For $0 \leq j<i \leq r$, we define

$$
\Omega_{j i}=1^{\otimes j} \otimes X_{\gamma} \otimes 1^{\otimes i-j-1} \otimes X^{\gamma} \otimes 1^{\otimes r-i} \in U\left(\mathfrak{g l}_{n}\right)^{\otimes r+1} .
$$

By Lemma 4.20 the element $\Omega_{j i}$ acts as $(j i) \in S_{r}$ on $M \otimes V^{\otimes r}$ for $1 \leq j<i \leq r$ where $S_{r}$ acts on $M \otimes V^{\otimes r}$ by permuting the tensor factors of $V^{\otimes r}$. Hence,

$$
\begin{equation*}
X_{i}=\sum_{1 \leq j<i} \Omega_{j i} \tag{54}
\end{equation*}
$$

as operators on $M \otimes V^{\otimes r}$ where $X_{i}=(1 i)+\ldots+(i-1 i)$ is the $i$-th Jucys-Murphy element of $\mathbb{C}\left[S_{r}\right]$. To construct an $\mathcal{H}_{r}^{\text {deg }}$-action that also interacts with the $M$ component of $M \otimes V^{\otimes r}$, we will add the element $\Omega_{0 i}$ the sum in (54).

Theorem 4.21. There is a unique right action of $\mathcal{H}_{r}^{\text {deg }}$ on $M \otimes V^{\otimes r}$ such that

1) the $\mathcal{H}_{r}^{\text {deg }}$-action extends the $\mathbb{C}\left[S_{r}\right]$-action on $M \otimes V^{\otimes r}$
2) $x_{i} \in \mathcal{H}_{r}^{\mathrm{deg}}$ acts as $\sum_{0 \leq j<i} \Omega_{j i}$.

Moreover, the actions $\mathfrak{g l}_{n} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\text {deg }}$ commute.
Proof. Uniqueness is clear since $\mathcal{H}_{r}^{\text {deg }}$ is generated by $\mathbb{C}\left[S_{r}\right]$ and the $x_{i}$. We need to verify that the actions of the generators of $\mathcal{H}_{r}^{\mathrm{deg}}$ are compatible with the defining relations of $\mathcal{H}_{r}^{\mathrm{deg}}$. For the symmetric group relations, this is obvious. Note that $x_{i}$ acts as $\Omega_{0 i}+X_{i}$ on $M \otimes V^{\otimes r}$. We then get in $\operatorname{End}_{\mathbb{C}}\left(M \otimes V^{\otimes r}\right)$ that

$$
s_{i} x_{i} s_{i}+s_{i}=s_{i} \Omega_{0 i} s_{i}+s_{i} X_{i} s_{i}+s_{i}=\Omega_{0 s_{i}(i)}+X_{i+1}=x_{i+1}
$$

by Lemma 4.9 and

$$
s_{j} x_{i} s_{j}=s_{j} \Omega_{0 i} s_{j}+s_{j} X_{i} s_{j}=\Omega_{0 i}+X_{i}=x_{i}
$$

for $j \neq i, i+1$. It remains to show that the actions of the $x_{i}$ commute. This follows if we can show that $\Omega_{j i}$ commutes $\sum_{0 \leq k<l} \Omega_{k l}$ for any $0 \leq j<i<l \leq r$. If $i, j \neq k$ then $\Omega_{j i}$ and $\Omega_{k l}$ commute. We claim that $\Omega_{j i}$ also commutes with $\Omega_{j l}+\Omega_{i l}$. Without loss of generality we have $r=2, j=0, i=1$ and $l=2$. Then

$$
\Omega_{02}+\Omega_{12}=\sum_{\gamma \in B} X_{\gamma} \otimes 1 \otimes X^{\gamma}+\sum_{\gamma \in B} 1 \otimes X_{\gamma} \otimes X^{\gamma}=\sum_{\gamma \in B} \Delta\left(X_{\gamma}\right) \otimes X^{\gamma}
$$

This commutes with $\Omega_{01}=\Omega \otimes 1$ by Lemma 4.19 . This proves that the $\mathcal{H}_{r}^{\mathrm{deg}}$-action is well-defined.
For the statement about the commuting actions, observe that the $\mathbb{C}\left[S_{r}\right]$-action on $M \otimes V^{\otimes r}$ clearly commutes with the $\mathfrak{g l}_{n}$-action. The $\Omega_{j i}$ commute with the $\mathfrak{g l}_{n}$-action by Lemma 4.19. This finishes the proof.

Example 4.22. 1. For $M=\mathbb{C}$ the trivial $\mathfrak{g l}_{n}$-module we can interpret $M \otimes V^{\otimes r}$ as the tensor space $V^{\otimes r}$. Then the element $\Omega_{0 i}$ acts by 0 and $x_{i}$ acts by multiplying with the $i$-th Jucys-Murphy element $X_{i} \in \mathbb{C}\left[S_{r}\right]$ (see (54)). We conclude that the $\mathcal{H}_{r}^{\mathrm{deg}}$-action on $V^{\otimes r}$ is induced by the $S_{r}$-action along the algebra homomorphism $\mathcal{H}_{r}^{\text {deg }} \rightarrow \mathbb{C}\left[S_{r}\right]$ from 51 and we recover the classical Schur-Weyl duality

$$
\mathfrak{g l}_{n} \curvearrowright V^{\otimes r} \curvearrowleft \mathbb{C}\left[S_{r}\right] \leftarrow \mathcal{H}_{r}^{\mathrm{deg}}
$$

2. For $M=V$ we have $M \otimes V^{\otimes r}=V^{\otimes r+1}$. By classical Schur-Weyl duality, the canonical $\mathbb{C}\left[S_{r+1}\right]$-action on $V^{\otimes r+1}$ is a centralising partner of the $\mathfrak{g l}_{n}$-action. By 54 the $x_{i} \in \mathcal{H}_{r}^{\text {deg }}$ act by multiplying with $X_{i+1} \in \mathbb{C}\left[S_{r+1}\right]$. Note that $x_{1}$ acts as $X_{2}=s_{1}$. On the other hand, the $s_{i} \in \mathcal{H}_{r}^{\text {deg }}$ act as $s_{i+1} \in \mathbb{C}\left[S_{r+1}\right]$. Hence, the $\mathcal{H}_{r}^{\text {deg }}$-action is induced along the surjective algebra homomorphism $\mathcal{H}_{r}^{\text {deg }} \rightarrow \mathbb{C}\left[S_{r+1}\right]$ with $s_{i} \mapsto s_{i+1}$ and $x_{i} \mapsto X_{i+1}$. Note that this homomorphism is well-defined by Lemma 4.9.

Remark 4.23. Given a $\mathfrak{g l}_{n}$-module $M$, it is natural to ask whether there is a (double) centraliser property for the commuting actions $\mathfrak{g l}_{n} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathcal{H}_{r}^{\mathrm{deg}}$. It is of course hopeless to establish such a property for arbitrary $M$. However, it turns out that $\mathcal{H}_{r}^{\text {deg }} \rightarrow \operatorname{End}_{\mathfrak{g r}_{n}}\left(M \otimes V^{\otimes r}\right)^{\text {op }}$ is surjective for certain parabolic Verma modules $M$. Moreover, one can show that the kernel of this map is generated by a polynomial of the form $p\left(x_{1}\right)=\prod_{i=1}^{l}\left(x_{1}-u_{i}\right)$ for some $l \in \mathbb{N}$ and $u_{i} \in \mathbb{C}$. Hence, we
obtain an isomorphism $\mathcal{H}_{r}^{\mathrm{deg}} / p\left(x_{1}\right) \cong \operatorname{End}_{\mathfrak{g l}_{n}}\left(M \otimes V^{\otimes r}\right)^{\text {op }}$ establishing a link between the representation theory of the degenerate affine Hecke algebra and parabolic Verma modules. The algebras $\mathcal{H}_{r}^{\mathrm{deg}} / p\left(x_{1}\right)$ are also often referred to as degenerate cyclotomic Hecke algebras and one calls $l$ the level of $\mathcal{H}_{r}^{\mathrm{deg}} / p\left(x_{1}\right)$ (see BK08, Introduction and Section 3]). As we have already seen in Example 4.22, classical Schur-Weyl duality can be interpreted as a duality for the degenerate cyclotomic Hecke algebra $\mathcal{H}_{r}^{\mathrm{deg}} /\left(x_{1}\right) \cong \mathbb{C}\left[S_{r}\right]$ of level 1 . For other parabolic Verma modules $M$, one will encounter degenerate cyclotomic Hecke algebras of higher levels. This explains why we refer to the construction in this section as higher Schur-Weyl duality. More details on this and proofs can be found in BK08].

### 4.3 The affine VW-algebra

There also is a version of higher Schur-Weyl duality in types B,C,D. To describe this, we first need to explain what the Brauer analogue of the degenerate affine Hecke algebra is. Our definition of the degenerate affine Hecke algebra was motivated by certain relations between the standard generators and the Jucys-Murphy elements of $\mathbb{C}\left[S_{r}\right]$. We proceed in a similar way for the Brauer algebra. Recall the standard generators $s_{1}, \ldots, s_{r-1}, e_{1}, \ldots, e_{r-1}$ of $B_{r}(\delta)$ subject to the relations from Proposition 3.62. The Jucys-Murphy elements for the tower of algebras

$$
\mathbb{C} \cong B_{0}(\delta) \subset B_{1}(\delta) \subset B_{2}(\delta) \subset B_{3}(\delta) \subset \ldots
$$

were defined by Nazarov in [Naz96]. Here the inclusion $B_{i}(\delta) \subset B_{i+1}(\delta)$ is given by adding the strand $\left\{i+1,(i+1)^{\prime}\right\}$ to the diagrams in $B_{i}(\delta)$. This tower is semisimple if $\delta \notin \mathbb{Z}$, but $B_{r}(\delta)$ might not be semisimple for $\delta \in \mathbb{Z}$ (see Proposition 3.65 and Remark 3.66). Still, the definition of the Jucys-Murphy elements makes sense for any $\delta \in \mathbb{C}$. For any $1 \leq i<j \leq r$ let
and

$$
\widehat{Z}_{k}:=\sum_{1 \leq i<j \leq k}(i j)-\sum_{1 \leq i<j \leq k}(\overline{i j}) \in B_{k}(\delta)
$$

for any $k \geq 0$ (with $\left.\widehat{Z}_{0}=\widehat{Z}_{1}=0\right)$.
Definition 4.24. For $1 \leq i \leq r$ the elements

$$
\widehat{X}_{i}:=\frac{\delta-1}{2}+\widehat{Z}_{i}-\widehat{Z}_{i-1}=\frac{\delta-1}{2}+\sum_{1 \leq j<i}(j i)-\sum_{1 \leq j<i} \overline{(j i)}
$$

are called the Jucys-Murphy elements of $B_{r}(\delta)$.
Note that $\widehat{X}_{1}=\frac{\delta-1}{2}$. The following observation is useful for understanding the structure of the Jucys-Murphy elements.

Lemma 4.25. The element $\widehat{Z}_{k}$ lies in the centre of $B_{k}(\delta)$.
Proof. For any $\sigma \in S_{k}$, we have

$$
\begin{equation*}
\sigma(i j) \sigma^{-1}=(\sigma(i) \sigma(j)) \quad \text { and } \quad \sigma \overline{(i j)} \sigma^{-1}=\overline{(\sigma(i) \sigma(j))} \tag{55}
\end{equation*}
$$

This shows that $\widehat{Z}_{k}$ commutes with the elements of $S_{k}$. On the other hand, $\overline{(a b)}$ commutes with $(i j)$ and $\overline{(i j)}$ if $\{a, b\} \cap\{i, j\}=\emptyset$. We claim that $\overline{(a b)}$ also commutes with $(a i)-\overline{(b i)}$ for any distinct $a, b, i \in\{1, \ldots, k\}$. In fact, we show that

$$
\overline{(a b)} \cdot((a i)-\overline{(b i)})=0=((a i)-\overline{(b i)}) \cdot \overline{(a b)} .
$$

By (55) it suffices to prove this for $a=1, b=2$ and $i=3$. This case follows from

and the equation $(13) \cdot \overline{(12)}=\overline{(23)} \cdot \overline{(12)}$ which is obtained by flipping the diagrams above upside down. This proves that $\overline{(a b)}$ commutes with $(a i)-\overline{(b i)}$. By a similar argument $\overline{(a b)}$ commutes with $(b i)-\overline{(a i)}$ and hence also with $\widehat{Z}_{k}$. Since $S_{r}$ generates $B_{r}(\delta)$ together with the $\overline{(a b)}$ we see that $\widehat{Z}_{k}$ lies in the centre of $B_{r}(\delta)$.

Here are some relations between the $\widehat{X}_{i}, s_{i}$ and $e_{i}$.
Lemma 4.26. The Jucys Murphy elements $\widehat{X}_{1}, \ldots, \widehat{X}_{r}$ of $B_{r}(\delta)$ commute pairwise. Moreover, $\widehat{X}_{j}$ commutes with $s_{i}$ and $e_{i}$ for $j \neq i, i+1$ and

$$
\begin{equation*}
s_{i} \widehat{X}_{i} s_{i}+s_{i}-e_{i}=\widehat{X}_{i+1} \quad e_{i}\left(\widehat{X}_{i}+\widehat{X}_{i+1}\right)=0=\left(\widehat{X}_{i}+\widehat{X}_{i+1}\right) e_{i} . \tag{56}
\end{equation*}
$$

Proof. The $\widehat{Z}_{i}$ are central in $B_{i}(\delta)$. Hence, the $\widehat{X}_{i}=\widehat{Z}_{i}-\widehat{Z}_{i-1}$ commute pairwise and $\widehat{X}_{j}$ commutes with $s_{i}$ and $e_{i}$ for $i<j-1$. Moreover $s_{i}$ and $e_{i}$ commute with $B_{i-1}(\delta)$. This shows that $s_{i}$ and $e_{i}$ commute with $\widehat{X}_{j}$ for $j<i$. The relations from (56) can be checked by direct computation (see [Naz96, Prop. 2.3])

This motivates the following definition.
Definition 4.27. Let $r \in \mathbb{N}$ and fix a system of complex parameters $\Xi=\left(\omega_{k}\right)_{k \geq 0}$. The affine $V W$-algebra $\mathbb{W}_{r}(\Xi)$ is the $\mathbb{C}$-algebra with generators

$$
s_{1}, \ldots, s_{r-1}, \quad e_{1}, \ldots, e_{r-1}, \quad y_{1}, \ldots, y_{r}
$$

subject to the Brauer algebra relations from Proposition 3.62 and the relations

$$
e_{1} y_{1}^{k} e_{1}=\omega_{k} e_{1}
$$

$$
\begin{equation*}
\text { for } k \geq 0 \tag{VW1}
\end{equation*}
$$

(i) $s_{i} y_{j}=y_{j} s_{i}$
for $j \neq i, i+1$
(ii) $e_{i} y_{j}=y_{j} e_{i}$
for $j \neq i, i+1$
(iii) $y_{i} y_{j}=y_{j} y_{i}$
for $i, j=1, \ldots, n$.
(VW3)
(i) $s_{i} y_{i} s_{i}+s_{i}-e_{i}=y_{i+1}$
for $i=1, \ldots, n-1$
(ii) $e_{i}\left(y_{i}+y_{i+1}\right)=0$
for $i=1, \ldots, n-1$
(iii) $\left(y_{i}+y_{i+1}\right) e_{i}=0$
for $i=1, \ldots, n-1$
Remark 4.28. We do not impose any conditions on the $\omega_{k}$ in our definition of the affine VW-algebra. However, it can be shown that the relations of $\mathbb{W}_{r}(\Xi)$ imply that

$$
\omega_{k} e_{1}=e_{1} y_{1}^{k} e_{1}=\ldots=\left(-\omega_{k}-\omega_{k-1}+\sum_{j=1}^{k}(-1)^{j-1} \omega_{j-1} \omega_{k-j}\right) e_{1}
$$

for any odd positive integer $k$ (see [Naz96, Lemma 2.5 and (4.6)]). In particular, we have $e_{1}=0$ in $W_{r}(\Xi)$ unless the parameters $\omega_{k}$ satisfy the admissibility condition $-2 \omega_{k}=\omega_{k-1}+\sum_{j=1}^{k}(-1)^{j} \omega_{j-1} \omega_{k-j}$ for any odd positive integer $k$.
Remark 4.29. As for the degenerate affine Hecke algebra, there is a diagrammatic interpretation of the affine VW-algebra. We interpret $s_{i}$ and $e_{i}$ as the usual permutation and Brauer diagrams (omitting the dots at the end of strands that usually indicate the elements of the blocks). The element $y_{i}$ corresponds to the identity diagram with a dot on the $i$-th strand. Then the relations (VW3) can be rewritten as

$$
s_{i} y_{i}=y_{i+1} s_{i}+e_{i}-1, \quad e_{i} y_{i}=-e_{i} y_{i+1}, \quad y_{i} e_{i}=-y_{i+1} e_{i}
$$

which have the diagrammatic interpretation


In other words, we can move dots freely along strands but passing through crossings introduces an error term and sliding trough cups or caps introduces a sign. Moreover, (VW1) tells us how to remove bubbles, at least if the bubble is at the left end of the underlying Brauer diagram:


For arbitrary bubbles one has to be a bit more careful. However, one can show that the relations of $\mathbb{W}_{r}(\Xi)$ imply that $e_{i} y_{i}^{k} e_{i}=\omega_{k}^{(i)} e_{i}$ for some $\omega_{k}^{(i)} \in \mathbb{W}_{r}(\Xi)$ (see [Naz96, Prop. 4.2]).

For $\omega_{0}=\delta$ there is an algebra homomorphism $B_{r}(\delta) \rightarrow \mathbb{W}_{r}(\Xi)$ with $s_{i} \mapsto s_{i}$ and $e_{i} \mapsto e_{i}$. Moreover, if we have $\omega_{k}=\delta\left(\frac{\delta-1}{2}\right)^{k}$ for all $k \geq 0$ then there is a surjective algebra homomorphism

$$
\begin{align*}
W_{r}(\Xi) & \longrightarrow B_{r}(\delta) \\
s_{i} & \longmapsto s_{i}  \tag{57}\\
e_{i} & \longmapsto e_{i} \\
y_{i} & \longmapsto \widehat{X}_{i} .
\end{align*}
$$

It follows from $e_{1} \widehat{X}_{1}^{k} e_{1}=e_{1}\left(\frac{\delta-1}{2}\right)^{k} e_{1}=\delta\left(\frac{\delta-1}{2}\right)^{k} e_{1}$ and Lemma 4.26 that this homomorphism is well-defined. Let $\mathfrak{g}$ be one of the type B,C,D Lie algebras and let $V$ be the defining representation of $\mathfrak{g}$. We would like to construct commuting actions

$$
\mathfrak{g} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathbb{W}_{r}(\Xi)
$$

for a $\mathfrak{g}$-module $M$ (which will not quite be arbitrary) and an appropriate choice of parameters $\Xi$. As in type A, this will make use of the pseudo Casimir element. Let
$(\cdot, \cdot)$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. Let $\left\{X_{\gamma} \mid \gamma \in B\right\}$ be a basis of $\mathfrak{g}$ and $\left\{X^{\gamma} \mid \gamma \in B\right\}$ the dual basis. Then $C:=\sum_{\gamma \in B} X_{\gamma} X^{\gamma}$ is a central element of $U(\mathfrak{g})$ called the Casimir element.

Definition 4.30. The pseudo Casimir element in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined as

$$
\Omega:=\sum_{\gamma \in B} X_{\gamma} \otimes X^{\gamma} \in U(\mathfrak{g})
$$

By the same arguments as in type $A$, one checks that

- $\Omega=\frac{1}{2}(\Delta(C)-1 \otimes C-C \otimes 1)$;
- $\Omega$ commutes with $\Delta(x)$ for all $x \in \mathfrak{g}$;
- $\Omega$ does not depend on the choice of the basis $\left\{X_{\gamma} \mid \gamma \in B\right\}$.

Let us now take a specific choice for the invariant bilinear form $(\cdot, \cdot)$, namely

$$
(x, y)=\frac{1}{2} \operatorname{Tr}(x y)
$$

We determine the action of $\Omega$ on $V \otimes V$. Let $\mathfrak{g}$ be defined with respect to the nondegenerate bilinear form $\langle\cdot, \cdot\rangle$ on $V \otimes V$. Recall that $\langle x, y\rangle=\epsilon_{\mathfrak{g}}\langle y, x\rangle$ for all $x, y \in \mathfrak{g}$ where $\epsilon_{\mathfrak{g}}=1$ for $\mathfrak{g}=\mathfrak{s o}_{n}$ and $\epsilon_{\mathfrak{g}}=-1$ for $\mathfrak{g}=\mathfrak{s p}_{n}$. Let $\left\{v_{i} \mid i \in I\right\}$ be a basis of $V$ and $\left\{v^{i} \mid i \in I\right\}$ the basis defined by $\left\langle v_{i}, v^{j}\right\rangle=\delta_{i, j}$.

Lemma 4.31. The pseudo Casimir element acts as $s-e$ on $V \otimes V$ where $s$ is the swap operator and $v \otimes w \cdot e=\langle v, w\rangle \epsilon_{\mathfrak{g}} \sum_{i \in I} v_{i} \otimes v^{i}$.
Proof. See also [ES18, Remark 2.5]. To avoid having to deal with a lot of signs, we assume $\mathfrak{g}=\mathfrak{s o}_{2 m}$ or $\mathfrak{g}=\mathfrak{s o}_{2 m+1}$. The $\mathfrak{s p}_{2 m}$ case works similarly. We work with $I:=\{-m, \ldots,-1,1, \ldots, m\}$ if $\mathfrak{g}=\mathfrak{s o}_{2 m}$ and $I:=\{-m, \ldots,-1,0,1, \ldots, m\}$ if $\mathfrak{g}=\mathfrak{s o}_{2 m+1}$. Assume that $\mathfrak{g}$ is defined with respect to the the non-degenerate bilinear form on $V$ given by $\left\langle v_{i}, v_{j}\right\rangle=\delta_{i,-j}$. This implies $v^{i}=v_{-i}$. Let

$$
F_{i, j}:=E_{i, j}-E_{-j,-i} \in \mathfrak{g}
$$

for all $i, j \in I$. One can check that

$$
2 \Omega=\sum_{i, j \in I} F_{i, j} \otimes F_{j, i}
$$

We have $F_{k, l} \otimes F_{l, k} \cdot v_{i} \otimes v_{j}=0$ unless $i \in\{l,-k\}$ and $j \in\{k,-l\}$. Hence, for $i \neq-j$, we get

$$
2 \Omega \cdot v_{i} \otimes v_{j}=\left(F_{j, i} \otimes F_{i, j}+F_{-i,-j} \otimes F_{-j,-i}\right) \cdot v_{i} \otimes v_{j}=2 v_{j} \otimes v_{i}=2 v_{i} \otimes v_{j} \cdot(s-e)
$$

Noting that $F_{-k, k}=0$, we get

$$
\begin{aligned}
2 \Omega \cdot v_{i} \otimes v_{-i} & =\left(\sum_{k \in I \backslash\{-i\}} F_{k, i} \otimes F_{i, k}+\sum_{k \in I \backslash\{i\}} F_{-i, k} \otimes F_{k,-i}\right) \cdot v_{i} \otimes v_{-i} \\
& =-\sum_{k \in I \backslash\{-i\}} v_{k} \otimes v_{-k}-\sum_{k \in I \backslash\{i\}} v_{-k} \otimes v_{k} \\
& =2 v_{-i} \otimes v_{i}-2 \sum_{k \in I} v_{k} \otimes v_{-k} \\
& =2 v_{i} \otimes v_{-i} \cdot(s-e)
\end{aligned}
$$

This proves the claim.

Recall the action of $B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$ on $V^{\otimes r}$ from Theorem 3.64 which induces an action $M \otimes V^{\otimes r} \curvearrowleft B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$ fixing the $M$ component. Lemma 4.31 tells us that

$$
\widehat{X}_{i}=\frac{\epsilon_{\mathfrak{g}} n-1}{2}+\sum_{1 \leq j<i} \Omega_{j i}
$$

as operators on $M \otimes V^{\otimes r}$. As in type A, we would like to add the element $\Omega_{0 i}$ to this sum so that we also interact with $M$. There is one more technicality though that did not appear in type A: We have to explain how to choose the parameter set $\Xi$ appearing in relation (VW1).
Lemma 4.32. If $M$ is a highest weight module of $\mathfrak{g}$ and $y \in \operatorname{End}_{\mathfrak{g}}(M \otimes V \otimes V)$, then there are complex numbers $\left(\alpha_{k}(M)\right)_{k \geq 0}$ with $e_{1} y^{k} e_{1}=\alpha_{k}(M) e_{1}$ as operators on $M \otimes V \otimes V$.

Proof. The vector $\sum_{k \in I} v_{i} \otimes v^{i} \in V \otimes V$ spans a copy of the trivial representation and induces an inclusion $L(0) \hookrightarrow V \otimes V$. Consider the composition

$$
M \otimes L(0) \hookrightarrow M \otimes V \otimes V \xrightarrow{y^{k}} M \otimes V \otimes V \xrightarrow{e_{l}} M \otimes L(0) .
$$

By general highest weight theory, this endomorphism (as any other endomorphism of $M \cong M \otimes L(0))$ is of the form $\alpha_{k}(M) \cdot \operatorname{id}_{M}$ for some $\alpha_{k}(M) \in \mathbb{C}$. Precomposing with $e_{1}$ yields $e_{1} y^{k} e_{1}=\alpha_{k}(M) e_{1}$ on $M \otimes V \otimes V$.

Note that $e_{1} y^{0} e_{1}=e_{1}^{2}=\epsilon_{\mathfrak{g}} n e_{1}$ and hence $\alpha_{0}(M)=\epsilon_{\mathfrak{g}} n$ in the setting of the lemma above. It is now clear how to construct commuting actions for a higher Schur-Weyl duality of $\mathfrak{g}$.
Theorem 4.33. Let $M$ be a highest weight module of $\mathfrak{g}$ and $\Xi=\left(\alpha_{k}(M)\right)_{k \geq 0}$ as in Lemma 4.32 for $y=\frac{\epsilon_{\mathfrak{g}} n-1}{2}+\Omega_{01}$. Then there is a unique right action of $\mathbb{W}_{r}(\Xi)$ on $M \otimes V^{\otimes r}$ such that

1) the $\mathbb{W}_{r}(\Xi)$-action extends the $B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$-action on $M \otimes V^{\otimes r}$
2) $y_{i} \in \mathbb{W}_{r}(\Xi)$ acts as $\frac{\epsilon_{\mathrm{G}} n-1}{2}+\sum_{0 \leq j<i} \Omega_{j i}$.

Moreover, the actions $\mathfrak{g} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathbb{W}_{r}(\Xi)$ commute.
Proof. It is clear that 1) and 2) uniquely determine the action of $\mathbb{W}_{r}(\Xi)$. Hence, we need to prove this action is well defined, i.e. that it is compatible with the defining relations of $\mathbb{W}_{r}(\Xi)$. (VW1) holds by Lemma 4.32. Observe that $y_{i}$ acts as $\Omega_{0 i}+\widehat{X}_{i}$ on $M \otimes V^{\otimes r}$. Relations (VW2)-(i), (VW2)-(ii) and (VW3)-(i) then follow from Lemma 4.26 using that $s_{i} \Omega_{0 j}=\Omega_{0 s_{i}(j)} s_{j}$ and $e_{i} \Omega_{0 j}=\Omega_{0 j} e_{i}$ for $j \neq i, i+1$. (VW2)-(iii), (VW3)-(ii) and (VW3)-(iii) need a few extra calculations as in ES18, Appendix] (see also DRV13, Thm. 2.2]).
The $B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$-action commutes with the $\mathfrak{g}$-action by the classical type B,C,D SchurWeyl duality (Theorem 3.64). Moreover, $y_{i}$ commutes with the $\mathfrak{g}$-action since $\Omega$ commutes with $\Delta(x)$ for all $x \in \mathfrak{g}$. Hence, the actions $\mathfrak{g} \curvearrowright M \otimes V^{\otimes r} \curvearrowleft \mathbb{W}_{r}(\Xi)$ commute.

Example 4.34. 1. For $M=L(0)$ the trivial module, we can identify $M \otimes V^{\otimes r}$ with the tensor space $V^{\otimes r}$. Then $\Omega_{0 i}$ acts by 0 and $y_{i} \in \mathbb{W}_{r}(\Xi)$ acts as the $i$-th Jucys-Murphy element $\widehat{X}_{i} \in B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$ by Lemma 4.31 . Hence, the $\mathbb{W}_{r}(\Xi)$ action is induced along the algebra homomorphism $\mathbb{W}_{r}(\Xi) \rightarrow B_{r}\left(\epsilon_{\mathfrak{g}} n\right)$ from (57) and we recover the classical duality

$$
\mathfrak{g} \curvearrowright V^{\otimes r} \curvearrowleft B_{r}\left(\epsilon_{\mathfrak{g}} n\right) \longleftarrow \mathbb{W}_{r}(\Xi) .
$$

2. For $M=V$, we have $M \otimes V^{\otimes r}=V^{\otimes r+1}$. The algebra $B_{r+1}\left(\epsilon_{\mathfrak{g}} n\right)$ is a centralising partner for the $\mathfrak{g}$-action on $V^{\otimes r+1}$. The element $y_{i} \in \mathbb{W}_{r}(\Xi)$ acts as $\widehat{X}_{i+1} \in B_{r+1}\left(\epsilon_{\mathfrak{g}} n\right)$ by Lemma 4.31 and $s_{i}, e_{i} \in W_{r}(\Xi)$ act as $s_{i+1}, e_{i+1} \in$ $B_{r+1}\left(\epsilon_{\mathfrak{g}} n\right)$. In particular, the $\mathbb{W}_{r}(\Xi)$-action is induced along the algebra homomorphism $\mathbb{W}_{r}(\Xi) \rightarrow B_{r+1}\left(\epsilon_{\mathfrak{g}} n\right)$ with $y_{i} \mapsto \widehat{X}_{i+1}, s_{i} \mapsto s_{i+1}$ and $e_{i} \mapsto e_{i+1}$. This homomorphism is well-defined by Lemma 4.26.
Remark 4.35. We already noted for type A that $\mathcal{H}_{r}^{\text {deg }}$ surjects onto the algebra $\operatorname{End}_{\mathfrak{g l}_{n}}\left(M \otimes V^{\otimes r}\right)^{\mathrm{op}}$ for certain parabolic Verma modules $M$ (Remark 4.23). A similar statement holds for the affine VW-algebra and the Lie algebra $\mathfrak{g}=\mathfrak{s o}_{2 m}$. In fact, one can show that for certain parabolic Verma modules $M$ the homomorphism $W_{r}(\Xi) \rightarrow \operatorname{End}_{\mathfrak{g}}\left(M \otimes V^{\otimes r}\right)^{\mathrm{op}}$ is surjective with kernel $p\left(y_{1}\right)=\prod_{i=1}^{l}\left(y_{1}-u_{i}\right)$ for some $l \in \mathbb{N}$ and $u_{i} \in \mathbb{C}$. This establishes a link between parabolic Verma modules and the so-called cyclotomic affine VW-algebras $\mathbb{W}_{r}(\Xi) / p\left(y_{1}\right)$. For more details, see ES18].

### 4.4 Jucys-Murphy elements in the partition algebra

Having seen Jucys-Murphy elements and higher Schur-Weyl duality for $\mathbb{C}\left[S_{r}\right]$ and $B_{r}(\delta)$, it is natural to attempt a similar construction for the partition algebra. This is surprisingly more difficult and to the knowledge of the author an analogue of the degenerate affine Hecke algebra or the affine VW-algebra has not been defined in the partition algebra setting, yet. In this section we introduce the Jucys-Murphy elements of $P_{r}(\delta)$ and try to explain why these elements are more difficult than in the previous examples.

Consider the tower of algebras

$$
\begin{equation*}
\mathbb{C}=P_{0}(\delta) \subset P_{1}(\delta) \subset P_{2}(\delta) \subset P_{3}(\delta) \subset \ldots \tag{58}
\end{equation*}
$$

where $P_{i}(\delta) \subset P_{i+1}(\delta)$ is the usual diagrammatic inclusion which adds the block $\left\{\left(i+1,(i+1)^{\prime}\right\}\right.$ to the diagrams in $A_{i} \subset P_{i}(\delta)$. Recall that these algebras are semisimple for $\delta \notin \mathbb{N}_{0}$ and also for $\delta \in \mathbb{N}_{0}$ if $\delta \geq 2 r-1$. To apply the techniques from the Okounkov-Vershik approach, we would like to have that the branching of $P_{i}(\delta) \subset P_{i+1}(\delta)$ is multiplicity free whenever $P_{i}(\delta)$ and $P_{i+1}(\delta)$ are semisimple. Unfortunately, this is not the case.

Example 4.36. We show that $P_{1}(\delta) \subset P_{2}(\delta)$ does not have multiplicity-free branching for any $\delta \in \mathbb{C}$ such that $P_{1}(\delta)$ and $P_{2}(\delta)$ are semisimple. We use the fact that an inclusion of semisimple algebras $A \subset B$ has multiplicity-free branching if and only if the centraliser $Z(B, A)$ is commutative (c.f. [CSST10, Thm. 2.1.10]). For $P_{1}(\delta)=\operatorname{Span}_{\mathbb{C}}\left\{1, p_{1}\right\} \subset P_{2}(\delta)$ we have $p_{1} p_{2}=p_{2} p_{1}$ and

$$
\begin{aligned}
p_{1}\left(\delta p_{\frac{3}{2}}-p_{1} p_{\frac{3}{2}}-p_{\frac{3}{2}} p_{1}\right) & =\delta p_{1} p_{\frac{3}{2}}-p_{1}^{2} p_{\frac{3}{2}}-p_{1} p_{\frac{3}{2}} p_{1} \\
& =-p_{1} p_{\frac{3}{2}} p_{1} \\
& =\delta p_{\frac{3}{2}} p_{1}-p_{1} p_{\frac{3}{2}} p_{1}-p_{\frac{3}{2}} p_{1}^{2} \\
& =\left(\delta p_{\frac{3}{2}}-p_{1} p_{\frac{3}{2}}-p_{\frac{3}{2}} p_{1}\right) p_{1}
\end{aligned}
$$

Hence, $p_{2}$ and $\delta p_{\frac{3}{2}}-p_{1} p_{\frac{3}{2}}-p_{\frac{3}{2}} p_{1}$ lie in $Z\left(P_{2}(\delta), P_{1}(\delta)\right)$. On the other hand,


This shows that $Z\left(P_{2}(\delta), P_{1}(\delta)\right)$ is not commutative and the branching $P_{1}(\delta) \subset P_{2}(\delta)$ is not multiplicity-free.

To get multiplicity-free branching, we need a finer filtration than the tower in (58). For this we analyse the structure of $V^{\otimes r}$ as an $S_{n}$ representation. Note that $V \cong \mathbb{C}\left[S_{n} / S_{n-1}\right]$ is a permutation module with basis $\left\{\bar{g} \mid g \in S_{n} / S_{n-1}\right\}$.

Lemma 4.37. For any $S_{n}$ representation $M$ the map

$$
\begin{aligned}
\operatorname{Ind}_{S_{n-1}}^{S_{n}} \operatorname{Res}_{S_{n-1}}^{S_{n}} M=\mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} M & \longrightarrow M \otimes_{\mathbb{C}} V \\
g \otimes m & \longmapsto g m \otimes \bar{g}
\end{aligned}
$$

is an isomorphism of $S_{n}$ representations.
Proof. It is straightforward to check that this is a well-defined $S_{n}$-linear map with inverse $m \otimes \bar{g} \mapsto g \otimes g^{-1} m$.

Hence, we have

$$
\begin{equation*}
V^{\otimes r} \cong \operatorname{Ind}_{S_{n-1}}^{S_{n}} \operatorname{Res}_{S_{n-1}}^{S_{n}} V^{\otimes r-1} \cong \ldots \cong\left(\operatorname{Ind}_{S_{n-1}}^{S_{n}} \operatorname{Res}_{S_{n-1}}^{S_{n}}\right)^{r} \mathbf{1}_{n} \tag{59}
\end{equation*}
$$

where $\mathbf{1}_{n}$ is the trivial representation of $S_{n}$. We can therefore refine the tower of partition algebras by inserting the centralising partners of $S_{n-1} \curvearrowright \operatorname{Res}_{S_{n-1}}^{S_{n}} V^{\otimes r}$ for $r \in \mathbb{N}_{0}$ into our tower. Note that $\operatorname{Res}_{S_{n-1}}^{S_{n}} V^{\otimes r}$ can be identified with the $S_{n-1^{-}}$ representation $V^{\otimes r} \otimes \mathbb{C} v_{n} \subset V^{\otimes r+1}$.
Definition 4.38. We define the half tensor space by $V^{\otimes r+\frac{1}{2}}:=V^{\otimes r} \otimes \mathbb{C} v_{n} \subset V^{\otimes r+1}$.
We also define the set

$$
A_{r+\frac{1}{2}}:=\left\{d \in A_{r+1} \mid r+1 \text { and }(r+1)^{\prime} \text { lie in the same block of } d\right\}
$$

Then any $d \in A_{r+\frac{1}{2}}$ stabilises $V^{\otimes r+\frac{1}{2}}$ and hence we can consider $d$ as an element of $\operatorname{End}_{S_{n-1}}\left(V^{\otimes r+\frac{1}{2}}\right)$ which induces commuting actions $S_{n-1} \curvearrowright V^{\otimes r+\frac{1}{2}} \curvearrowleft \operatorname{Span}_{\mathbb{C}} A_{r+\frac{1}{2}}$.
Definition 4.39. The algebra $P_{r+\frac{1}{2}}(\delta):=\operatorname{Span}_{\mathbb{C}} A_{r+\frac{1}{2}} \subset P_{r+1}(\delta)$ is called the half partition algebra.
Remark 4.40. It is easy to check that $P_{r+\frac{1}{2}}(\delta)$ is a subalgebra of $P_{r+1}(\delta)$ which is generated by $P_{r}(\delta)$ and $p_{r+\frac{1}{2}}$.

Recall that $A_{r+1}$ is partially ordered via $d^{\prime} \leq d$ if $d^{\prime}$ is coarser than $d$. Note that, $d \in A_{r+\frac{1}{2}}$ and $d^{\prime} \leq d$ imply that $d^{\prime} \in A_{r+\frac{1}{2}}$. Recall also that $\left\{x_{d} \mid d \in A_{r+1}\right\}$ is the unique basis of $P_{r+1}(\delta)$ such that $d=\sum_{d^{\prime} \leq d} x_{d^{\prime}}$. It follows by induction along the partial order that $d \in A_{r+\frac{1}{2}}$ implies $x_{d} \in P_{r+\frac{1}{2}}^{-}(\delta)$. Hence, $\left\{x_{d} \left\lvert\, d \in A_{r+\frac{1}{2}}\right.\right\}$ is a basis of $P_{r+\frac{1}{2}}(\delta)$ which we call the orbit basis of $P_{r+\frac{1}{2}}(\delta)$.
Example 4.41. Consider the algebra $P_{\frac{3}{2}}(\delta)$ with basis $A_{\frac{3}{2}}=\left\{\operatorname{id}, p_{1}, p_{\frac{3}{2}}, p_{\frac{3}{2}} p_{1}, p_{1} p_{\frac{3}{2}}\right\}$ and partial order $p_{\frac{3}{2}} \leq p_{1} p_{\frac{3}{2}}$, id, $p_{\frac{3}{2}} p_{1} \leq p_{1}$. Then, using that $x_{d}=\stackrel{2}{d}-\sum_{d^{\prime}<d}^{2} x_{d^{\prime}}^{2}$, the orbit basis can be computed as follows:

$$
\begin{aligned}
x_{p_{\frac{3}{2}}} & =p_{\frac{3}{2}} \\
x_{p_{1} p_{\frac{3}{2}}} & =p_{1} p_{\frac{3}{2}}-x_{p_{\frac{3}{2}}}=p_{1} p_{\frac{3}{2}}-p_{\frac{3}{2}} \\
x_{\mathrm{id}} & =\mathrm{id}-x_{p_{\frac{3}{2}}}=\mathrm{id}-p_{\frac{3}{2}} \\
x_{p_{\frac{3}{2}} p_{1}} & =p_{\frac{3}{2}} p_{1}-x_{p_{\frac{3}{2}}}=p_{\frac{3}{2}} p_{1}-p_{\frac{3}{2}} \\
x_{p_{1}} & =p_{1}-x_{p_{1} p_{\frac{3}{2}}}-x_{\mathrm{id}}-x_{p_{\frac{3}{2}} p_{1}}-x_{p_{\frac{3}{2}}}=p_{1}-p_{1} p_{\frac{3}{2}}-\mathrm{id}-p_{\frac{3}{2}} p_{1}-4 p_{\frac{3}{2}} .
\end{aligned}
$$

Theorem 4.42. The commuting actions $S_{n-1} \curvearrowright V^{\otimes r+\frac{1}{2}} \curvearrowleft P_{r+\frac{1}{2}}(n)$ generate each other's centraliser. Moreover, the set $\left\{x_{d} \left\lvert\, d \in A_{r+\frac{1}{2}}\right.\right.$ has more than $n$ blocks $\}$ spans the kernel of the induced algebra homomorphism $P_{r+\frac{1}{2}}(n) \rightarrow \operatorname{End}_{S_{n-1}}\left(V^{\otimes r+\frac{1}{2}}\right)^{\mathrm{op}}$. In particular, we have $P_{r+\frac{1}{2}}(n) \cong \operatorname{End}_{S_{n-1}}\left(V^{\otimes r+\frac{1}{2}}\right)^{\text {op }}$ if $n \geq 2\left(r+\frac{1}{2}\right)$.
Proof. This is analogous to the proof of Theorem 3.9 (see also [HR05, Thm. 3.6]).

We can now study the refined tower of algebras

$$
\mathbb{C} \cong P_{0}(\delta) \subset P_{\frac{1}{2}}(\delta) \subset P_{1}(\delta) \subset P_{\frac{3}{2}}(\delta) \subset P_{2}(\delta) \subset P_{\frac{5}{2}}(\delta) \subset \ldots
$$

For $\delta=n$ and $r \in \mathbb{N}_{0}$, these algebras fit into the Schur-Weyl dualities


Each of the dualities in (60) gives a bimodule decomposition into outer tensor products of simples

$$
\begin{aligned}
V^{\otimes r} & \cong \sum_{i} A_{i} \otimes B_{i} \\
V^{\otimes r+\frac{1}{2}} & \cong \sum_{j} A_{j}^{\prime} \otimes B_{j}^{\prime} \\
V^{\otimes r+1} & \cong \sum_{k} A_{k}^{\prime \prime} \otimes B_{k}^{\prime \prime}
\end{aligned}
$$

which satisfy the following property.
Proposition 4.43 (Seesaw resciprocity). We have

1) $\left[\operatorname{Res}_{P_{r}(n)}^{P_{r+\frac{1}{2}}(n)} B_{j}^{\prime}: B_{i}\right]=\left[\operatorname{Res}_{S_{n-1}}^{S_{n}} A_{i}: A_{j}^{\prime}\right]$
2) $\left[\operatorname{Res}_{P_{r+\frac{1}{2}}(n)}^{P_{r+1}(n)} B_{k}^{\prime \prime}: B_{j}^{\prime}\right]=\left[\operatorname{Res}_{S_{n-1}}^{S_{n}} A_{k}^{\prime \prime}: A_{j}^{\prime}\right]$.

Proof. See also CSST10, Sections 7.5.3 and 7.5.4]. We prove 1) (the proof of 2) is analogous). We can decompose $V^{\otimes r}$ as a $\left(\mathbb{C}\left[S_{n-1}\right], P_{r}(n)\right)$-bimodule in two ways. Restricting from the $\left(\mathbb{C}\left[S_{n-1}\right], P_{r+\frac{1}{2}}(n)\right)$-bimodule $V^{\otimes r+\frac{1}{2}}$, we get

$$
\begin{equation*}
V^{\otimes r} \cong \sum_{j, i}\left[\operatorname{Res}_{P_{r}(n)}^{P_{r+\frac{1}{2}}(n)} B_{j}^{\prime}: B_{i}\right] \cdot A_{j}^{\prime} \otimes B_{i} \tag{61}
\end{equation*}
$$

Restricting instead from the $\left(\mathbb{C}\left[S_{n}\right], P_{r}(n)\right)$-bimodule $V^{\otimes r}$, we get

$$
\begin{equation*}
V^{\otimes r}=\sum_{j, i}\left[\operatorname{Res}_{S_{n-1}}^{S_{n}} A_{i}: A_{j}^{\prime}\right] \cdot A_{j}^{\prime} \otimes B_{i} \tag{62}
\end{equation*}
$$

Comparing the coefficients in (61) and (62) implies 1).

For $2 r \in\{0,1, \ldots, n\}$ the $P_{r}(n)$ action on tensor space is faithful by Theorem 3.9 and Theorem 4.42 and in this case $P_{r}(n)$ is semisimple. Hence, the representation theory of the (finite) tower of semisimple algebras

$$
\begin{equation*}
P_{0}(n) \subset P_{\frac{1}{2}}(n) \subset P_{1}(n) \subset \ldots \subset P_{\frac{n-1}{2}} \subset P_{\frac{n}{2}}(n) \tag{63}
\end{equation*}
$$

is completely determined by Schur-Weyl duality.
Proposition 4.44. Let $2 r \in\{0,1, \ldots, n\}$. Then the irreducible $P_{r}(n)$-representations $V_{r}^{\lambda}$ can be indexed by the partitions $\lambda \vdash n$ (resp. $\lambda \vdash n-1$ if $r \in \mathbb{Z}+\frac{1}{2}$ ) with the property that the Specht module $S(\lambda)$ appears in $V^{\otimes r}$. Moreover, the branching of the truncated tower in 63 is multiplicity-free with

$$
\operatorname{Res}_{P_{r}(n)}^{P_{r+\frac{1}{2}}(n)} V_{r+\frac{1}{2}}^{\lambda}= \begin{cases}\bigoplus_{\substack{\lambda \subset \mu \\ \mu / \lambda=\square}} V_{r}^{\mu} & \text { if } r \in \mathbb{Z} \\ \bigoplus_{\substack{\mu \subset \lambda \\ \lambda / \mu=\square}} V_{r}^{\mu} & \text { if } r \in \mathbb{Z}+\frac{1}{2}\end{cases}
$$

Proof. The claim about the indexing set of the irreducible representations of $P_{r}(n)$ follows from the double centraliser theorem using that $P_{r}(n) \cong \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\mathrm{op}}$ (resp. $P_{r}(n) \cong \operatorname{End}_{S_{n-1}}\left(V^{\otimes r}\right)^{\text {op }}$ if $\left.r \in \mathbb{Z}+\frac{1}{2}\right)$ for $n \geq 2 r$. The branching rule now follows from Proposition 4.43 and the branching rule of $S_{n-1} \subset S_{n}$.

Note that for $r \in \mathbb{Z}$ the irreducible $S_{n}$ representations appearing in $V^{\otimes r}$ (resp. the irreducible $S_{n-1}$-representations for $r \in \mathbb{Z}+\frac{1}{2}$ ) can be determined recursively using the branching rule of $S_{n-1} \subset S_{n}$. This can be used to determine the branching graph of the tower from (63).

Example 4.45. The following graph is the branching graph of the tower from 63 for $n=5$.


Remark 4.46. We have already seen that the irreducible representations $L^{(r)}(\lambda)$ of $P_{r}(n)$ can be indexed by Young diagrams $\lambda$ with $0 \leq|\lambda| \leq r$. This can be connected
with the set of irreducibles $V_{r}^{\mu}$ from Proposition 4.44 which are indexed by the partitions $\mu \vdash n$ such that $S(\mu)$ appears in $V^{\otimes r}$. In fact, Proposition 3.32 tells us that the irreducible representation $L^{(r)}(\lambda)$ for the first indexing set corresponds to $V_{r}^{\lambda_{[n]}}$ for the second indexing set. The branching graph of $P_{r}(n)$ with respect to the first indexing set is then just obtained from the branching graph in the second indexing set by deleting the first row of each partition.

We now use Schur-Weyl duality to construct Jucys-Murphy elements for the algebras $P_{r}(n)$ for $n \geq 2 r$. Recall the central elements $Z_{n} \in \mathbb{C}\left[S_{n}\right]$ from (50).

Definition 4.47. Let $2 r \in \mathbb{N}_{0}$ and $n \geq 2 r$. Then $\widetilde{Z}_{r}(n)=\widetilde{Z}_{r}$ is defined to be the unique element of $\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\mathrm{op}} \cong P_{r}(n)$ that acts on $V^{\otimes r}$ by multiplying with $Z_{n} \in \mathbb{C}\left[S_{n}\right]$ if $r \in \mathbb{N}_{0}$ and by multiplying with $Z_{n-1} \in \mathbb{C}\left[S_{n-1}\right]$ if $r \in \mathbb{N}_{0}+\frac{1}{2}$.

Note that $Z_{n}\left(\operatorname{resp} Z_{n-1}\right)$ commutes with both the $S_{n}$ (resp. $S_{n-1}$ )-action and the $P_{r}(n)$-action on $V^{\otimes r}$. This shows that $\widetilde{Z}_{r}(n)$ is a well-defined central element of $P_{r}(n)$ for $n \geq 2 r$.

Definition 4.48. Let $2 r \in \mathbb{N}_{0}$ and $n \geq 2 r$. Then for $i \in\left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r\right\}$ the elements

$$
\widetilde{X}_{i}=\widetilde{X}_{i}(n):=\widetilde{Z}_{i}(n)-\widetilde{Z}_{i-\frac{1}{2}}(n) \in P_{r}(n)
$$

are called the Jucys-Murphy elements of $P_{r}(n)$.
Note that $Z_{n}$ (resp. $Z_{n-1}$ ) acts as the scalar $\binom{n}{2}$ (resp. $\binom{n-1}{2}$ ) on the trivial representation $V^{\otimes 0}$ (resp. $V^{\otimes \frac{1}{2}}$ ). In particular, $\widetilde{X}_{\frac{1}{2}}(n)=\binom{n-1}{2}-\binom{n}{2}=-(n-1)$ is also just a scalar.

Remark 4.49. Our normalisation of the Jucys-Murphy differs from the normalisations in HR05] and Eny13 by a constant factor but agrees with the normalisation in CSST10, Rem. 8.3.19]. In fact, we will see that our normalistation is adapted to the indexing set of the irreducible $P_{r}(n)$-modules from Proposition 4.44 . The normalisation in HR05], on the other hand, is adapted to the indexing set of simple $P_{r}(n)$ modules from Theorem 3.29.

Lemma 4.50. For $2 r \in \mathbb{N}_{0}$ and $n \geq 2 r$. Let $T=\lambda_{0} \nearrow \lambda_{\frac{1}{2}} \nearrow \lambda_{1} \nearrow \ldots \nearrow \lambda_{r}$ be a path in the branching graph of $P_{r}(n)$ as described in Proposition 4.44. Let $v_{T}$ be a corresponding Gelfand-Tsetlin basis vector. Then

$$
\widetilde{X}_{i} \cdot v_{T}= \begin{cases}c\left(\lambda_{i} / \lambda_{i-\frac{1}{2}}\right) & \text { if } i \in \mathbb{Z}  \tag{64}\\ -c\left(\lambda_{i-\frac{1}{2}} / \lambda_{i}\right) & \text { if } i \in \mathbb{Z}+\frac{1}{2}\end{cases}
$$

were $c(\lambda / \mu)$ denotes the content of the box $\lambda / \mu$ (if $\lambda$ and $\mu$ differ by a single box).
Proof. Let $\lambda \vdash k$ and let $S(\lambda)$ be the corresponding irreducible $S_{k}$-representation. Then the element $Z_{k}=X_{1}+X_{2}+\ldots+X_{k}$ acts as $\sum_{b \in \lambda} c(b)$ on $S(\lambda)$ by Theorem4.15. Now $v_{T}$ lies in the $\lambda_{i}$-isotypical component of $\operatorname{Res}_{P_{i}(n)}^{P_{r}(n)} V^{\otimes r}$ and in the $\lambda_{i-\frac{1}{2}}$-isotypical component of $\operatorname{Res}_{P_{i-\frac{1}{2}}(n)}^{P_{r}(n)} V^{\otimes r}$. Then the element $\widetilde{Z}_{i}$ acts as $\sum_{b \in \lambda_{i}} c(b)$ on $v_{T}$ and $\widetilde{Z}_{i-\frac{1}{2}}$ acts as $\sum_{b \in \lambda_{i-\frac{1}{2}}} c(b)$ on $v_{T}$. It follows that $\widetilde{X}_{i}=\widetilde{Z}_{i}-\widetilde{Z}_{i-\frac{1}{2}}$ acts by the formulas in (64).

We have determined the branching graph, the Jucys-Murphy elements and the weight structure of $P_{r}(n)$ for integers $n \geq 2 r$. Now, we want to interpolate this (or at least the Jucys-Murphy elements) to arbitrary $\delta \in \mathbb{C}$. The main ingredient is the following lemma.

Lemma 4.51. For any $2 r \in \mathbb{N}_{0}$ there are polynomials $p_{r}^{d}(x) \in \mathbb{C}[x]$ such that $\widetilde{Z}_{r}(n)=\sum_{d \in A_{r}} p_{r}^{d}(n) d \in P_{r}(n)$ for all $n \geq 2 r$.
Proof. See also CO11, Section 4.1]. We give the argument for $r \in \mathbb{Z}$ with the $r \in \mathbb{Z}+\frac{1}{2}$ case being similar. Since the base change between the standard basis and the orbit basis is unitriangular and integral, it suffices to show that there are $q_{r}^{d}(x) \in \mathbb{C}[x]$ such that $\widetilde{Z}_{r}(n)=\sum_{d \in A_{r}} q_{r}^{d}(n) x_{d}$ for all $n \geq 2 r$. For this, write

$$
\widetilde{Z}_{r}(n)=\sum_{d \in A_{r}} c_{r}^{d}(n) x_{d} .
$$

with $c_{r}^{d}(n) \in \mathbb{C}$. For $d \in A_{r}$ let $\underline{i}, \underline{j} \in\{1, \ldots, 2 r\}^{r} \subset\{1, \ldots, n\}^{r}$ such that $\underline{i} \xrightarrow{d} \underline{j}$. Then $c_{r}^{d}(n)$ is the coefficient of $v_{j}$ in $v_{\underline{i}} \cdot \widetilde{Z}_{r}(n)=Z_{n} \cdot v_{\underline{i}}$. This is the same as the number of all pairs $1 \leq a<b \leq n$ such that $(a b) \cdot \underline{i}=\underline{j}$. If $\underline{i}=\underline{j}$ this number is $c_{r}^{d}(n)=\binom{n-|d|}{2}$ where $|d|=\left|\left\{i_{1}, \ldots, i_{r}, j_{1}, \ldots, j_{r}\right\}\right|$ is the number of blocks of d. If $\underline{i} \neq \underline{j}$ then we can find $k \in\{1, \ldots, r\}$ such that $i_{k} \neq j_{k}$. Then ( $\left.a b\right) \cdot \underline{i}=\underline{j}$ implies $\{a, b\}=\left\{i_{k}, j_{k}\right\}$. We see that $c_{r}^{d}(n) \in\{0,1\}$ and this value is independent of $n$ since $\underline{i}$ and $\underline{j}$ are independent of $n$. In particular, $c_{r}^{d}(n)$ is polynomial in $n$ for all $d \in A_{r}$ as claimed. This finishes the proof.

We can use this to define Jucys-Murphy elements for arbitrary $\delta \in \mathbb{C}$ using the interpolated central elements

$$
\widetilde{Z}_{r}=\widetilde{Z}_{r}(\delta):=\sum_{d \in A_{k}} p_{r}^{d}(\delta) d \in P_{r}(\delta)
$$

with the $p_{r}^{d}(x)$ from Lemma 4.51 .
Definition 4.52. Let $2 r \in \mathbb{N}_{0}$ and $\delta \in \mathbb{C}$. Then for $i \in\left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, r\right\}$ the elements

$$
\widetilde{X}_{i}=\widetilde{X}_{i}(\delta):=\tilde{Z}_{i}(\delta)-\tilde{Z}_{i-1}(\delta) \in P_{r}(\delta)
$$

are called the Jucys-Murphy elements of $P_{r}(\delta)$.
Note that by definition, the Jucys-Murphy elements are of the form

$$
\widetilde{X}_{i}(\delta)=\sum_{d \in A_{k}} q_{i}^{d}(\delta) d
$$

for some $q_{i}^{d}(x) \in \mathbb{C}[x]$ and that this recovers the Jucys-Murphy elements from Definition 4.48 for $\delta=n \in \mathbb{N}$ with $n \geq 2 r$. Since $\widetilde{X}_{\frac{1}{2}}(n)=-(n-1)$ for infinitely many $n$, we get $q_{\frac{1}{2}}^{d}(x)=0$ for all $d \neq$ id and $q_{\frac{1}{2}}^{\text {id }}(x)=-(x-1)$. This shows that $\widetilde{X}_{\frac{1}{2}}(\delta)=\delta-1$ for all $\delta \in \mathbb{C}$. In fact, such deformation arguments (which we formalised in Lemma 3.14) are generally useful to do computations with the JucysMurphy elements $\widetilde{X}_{i}$. We use this to prove some basic properties of the $\widetilde{X}_{i}$ for all $\delta \in \mathbb{C}$. Most of these were also verified in Eny13 by other methods.
Lemma 4.53. The element $\widetilde{Z}_{k}(\delta) \in P_{k}(\delta)$ is central for any $\delta \in \mathbb{C}$. In particular, the Jucys-Murphy elements $\widetilde{X}_{\frac{1}{2}}, \widetilde{X}_{1}, \widetilde{X}_{\frac{3}{2}}, \ldots, \widetilde{X}_{r} \in P_{r}(\delta)$ commute pairwise.

Proof. We already know that the elements $\widetilde{Z}_{k}(n) \in P_{k}(n)$ are central for $n \geq 2 k$. Moreover, by construction, the element $\widetilde{Z}_{k}(\delta)$ is polynomial in the standard basis. The equation $d \widetilde{Z}_{k}(\delta)=\widetilde{Z}_{k}(\delta) d$ then follows for any $\delta \in \mathbb{C}$ and $d \in A_{k}$ by the standard deformation argument from Lemma 3.14. Hence $\widetilde{Z}_{k}(\delta) \in P_{k}(\delta)$ is central. It is then clear that the Jucys-Murphy elements commute since $\widetilde{X}_{i}=\tilde{Z}_{i}-\tilde{Z}_{i-1}$.

Note that this implies that $\widetilde{X_{\frac{1}{2}}}+\widetilde{X}_{1}+\ldots+\widetilde{X}_{r}=\widetilde{Z}_{r}-\widetilde{Z}_{0}$ is central in $P_{r}(\delta)$ (keeping in mind that $\widetilde{Z}_{0} \in P_{0}(\delta)=\mathbb{C}$ ). One might also wonder whether there is a degenerate affine version of the partition algebra. This would require some relations involving the Jucys-Murphy elements and the standard generators of the partition algebra. The following relations hold.

Lemma 4.54. For any $i \in\left\{1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, r\right\}$ and $\delta \in \mathbb{C}$ we have

$$
\begin{equation*}
\left(\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}\right) p_{i}=0=p_{i}\left(\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}\right) . \tag{65}
\end{equation*}
$$

in $P_{r}(\delta)$. Moreover,

$$
\begin{aligned}
& \widetilde{X}_{j} p_{i}=p_{i} \widetilde{X}_{j} \quad \text { if } j \neq i-\frac{1}{2}, i \\
& \widetilde{X}_{j} s_{i}=s_{i} \widetilde{X}_{j} \quad \text { if } j \neq i-\frac{1}{2}, i, i+\frac{1}{2}, i+1 \text { and } i \in \mathbb{Z}
\end{aligned}
$$

Proof. Since the $\widetilde{Z}_{k} \in P_{k}(\delta)$ are central and $\widetilde{X}_{j}=\widetilde{Z}_{j}-\widetilde{Z}_{j-\frac{1}{2}}$, we see that $\widetilde{X}_{j}$ commutes with $p_{i}$ for $j>i$ and $s_{i}$ for $j>i+1$. Moreover, $p_{i}$ and $s_{i}$ commute with $P_{i-1}(\delta)$ and hence with $\widetilde{X}_{j}$ for $j \leq i-1$. It remains to prove 65). By our standard interpolation argument (Lemma 3.14) it suffices to prove these equations as operators on $V^{\otimes i}$ for $n \gg 0$. Note that $X_{i-\frac{1}{2}}+\widetilde{X}_{i}=\widetilde{Z}_{i}-\widetilde{Z}_{i-1}$. For $i \in \mathbb{Z}$, let $\underline{j}=\left(j_{1}, \ldots, j_{i}\right) \in\{1, \ldots, n\}^{i}$ and $\underline{j}^{\prime}=\left(j_{1}, \ldots, j_{i-1}\right)$. Then

$$
\begin{aligned}
v_{j} \cdot\left(\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}\right) p_{i} & =v_{\underline{j}} \cdot\left(\widetilde{Z}_{i}-\widetilde{Z}_{i-1}\right) p_{i} \\
& \stackrel{4.47}{=} \sum_{1 \leq a<b \leq n}\left(v_{(a b) \cdot \underline{j}}-v_{(a b) \cdot j^{\prime}} \otimes v_{j_{i}}\right) \cdot p_{i} \\
& =\sum_{a \neq j_{i}}\left(v_{\left(a j_{i}\right) \cdot \dot{j}^{\prime}} \otimes v_{a}-v_{\left(a j_{i}\right) \cdot \dot{j}^{\prime}} \otimes v_{j_{i}}\right) \cdot p_{i} \\
& =\sum_{\substack{a \neq j_{i} \\
1 \leq k \leq n}} v_{\left(a j_{i}\right) \cdot \underline{j^{\prime}}} \otimes v_{k}-v_{\left(a j_{i}\right) \cdot \dot{j}^{\prime}} \otimes v_{k}=0 .
\end{aligned}
$$

By Lemma 3.14 this implies $\left(\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}\right) p_{i}=0$ in $P_{i}(\delta)$. Moreover, $p_{i}$ commutes with $\widetilde{Z}_{i-1}$ and $Z_{i}$ and hence also with $\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}$. This proves $p_{i}\left(\widetilde{X}_{i-\frac{1}{2}}+\widetilde{X}_{i}\right)=0$ and (65) follows for $i \in \mathbb{Z}$.
To prove 65 for $i+\frac{1}{2}$ (with $i \in \mathbb{Z}$ ) it suffices to verify the equation as operators on $V^{\otimes i+\frac{1}{2}}=V^{\otimes i} \otimes \mathbb{C} v_{n}$ :

$$
\begin{aligned}
& v_{\underline{j}} \otimes v_{n} \cdot p_{i+\frac{1}{2}}\left(\widetilde{X}_{i}+\widetilde{X}_{i+\frac{1}{2}}\right) \\
= & v_{\underline{j}} \otimes v_{n} \cdot p_{i+\frac{1}{2}}\left(\widetilde{Z}_{i+\frac{1}{2}}-\widetilde{Z}_{i-\frac{1}{2}}\right) \\
= & \delta_{j_{i}, n} \cdot v_{\underline{j}^{\prime}} \otimes v_{n} \otimes v_{n} \cdot\left(\widetilde{Z}_{i+\frac{1}{2}}-\widetilde{Z}_{i-\frac{1}{2}}\right) \\
= & \delta_{j_{i}, n} \cdot \sum_{1 \leq a<b \leq n-1}\left(v_{(a b) \cdot \dot{j}^{\prime}} \otimes v_{(a b) \cdot n} \otimes v_{n}-v_{(a b) \cdot j^{\prime}} \otimes v_{n} \otimes v_{n}\right)=0 .
\end{aligned}
$$

This proves $p_{i+\frac{1}{2}}\left(\widetilde{X}_{i}+\widetilde{X}_{i+\frac{1}{2}}\right)=0$ on $V^{\otimes i+\frac{1}{2}}$. Moreover, $p_{i+\frac{1}{2}}$ commutes with $\widetilde{Z}_{i+\frac{1}{2}}$ and $\widetilde{Z}_{i-\frac{1}{2}}$ and hence also with $\widetilde{X}_{i+\frac{1}{2}}+\widetilde{X}_{i}$. This implies $\left(\widetilde{X}_{i}^{2}+\widetilde{X}_{i-\frac{1}{2}}\right) p_{i+\frac{1}{2}}=0$ and the proof is complete.

Lemma 4.54 gives us nearly all the relations one would like to have for a good definition of a degenerate affine version of the partition algebra. The only relation that is missing is a relation that tells us how the $s_{i}$ commute/interact with $\widetilde{X}_{j}$ for $j=i-\frac{1}{2}, i, i+\frac{1}{2}, i+1$. In fact, this is the crucial point where the analogy to $\mathbb{C}\left[S_{r}\right]$ and $B_{r}(\delta)$ breaks down. The main problem is that the quasi-commutator $s_{i} \widetilde{X}_{j}-\widetilde{X}_{j+1} s_{i}$ cannot be expressed locally for $j=i-\frac{1}{2}, i$. Let us make a bit more precise what we mean by local here. The Jucys-Murphy elements of the symmetric group satisfy the relation $s_{i} X_{i}-X_{i+1} s_{i}=-1$ and in the Brauer algebra case we have $s_{i} \widehat{X}_{i}-\widehat{X}_{i+1} s_{i}=e_{i}-1$. These are expressions which only involve neighbouring indices. Moreover, we can consider the shift homomorphisms

$$
\begin{aligned}
\mathcal{S}_{k}: B_{r}(\delta) & \longrightarrow B_{r+k}(\delta) \\
s_{i} & \longmapsto s_{i+k} \\
e_{i} & \longmapsto e_{i+k} .
\end{aligned}
$$

Then

$$
\begin{aligned}
s_{i+k} X_{i+k}-X_{i+k+1} s_{i+k} & =-1=\mathcal{S}_{k}(-1)=\mathcal{S}_{k}\left(s_{i} X_{i}-X_{i+1} s_{i}\right) \\
\text { and } s_{i+k} \widehat{X}_{i+k}-\widehat{X}_{i+k+1} s_{i+k} & =e_{i+k}-1=\mathcal{S}_{k}\left(e_{i}-1\right)=\mathcal{S}_{k}\left(s_{i} \widehat{X}_{i}-\widehat{X}_{i+1} s_{i}\right)
\end{aligned}
$$

in $\mathbb{C}\left[S_{r}\right]$ and $B_{r}(\delta)$ for all $i$ and $k$. In pictures this means that the error term for the expression

$$
x->
$$

is a picture without dots which does not depend on where the crossing takes place within the whole diagram (explaining why we call this local). The shift operator above extends to the partition algebra via

$$
\begin{aligned}
\mathcal{S}_{k}: P_{r}(\delta) & \longrightarrow P_{r+k}(\delta) \\
s_{i} & \longmapsto s_{i+k} \\
p_{i} & \longmapsto p_{i+k} .
\end{aligned}
$$

We will show that the expression $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}$ for the Jucys-Murphy elements of the partition algebra is not local. Let us first derive a general formula for the action of the $\widetilde{X}_{i}$ on tensor space.
Proposition 4.55. Let $r \in \mathbb{N}, \underline{i}=\left(i_{1}, \ldots, i_{r}\right) \in\{1, \ldots, n\}^{r}$ and $\underline{i}^{\prime}=\left(i_{1}, \ldots, i_{r-1}\right)$. Then for $n \geq 2 r$ and $r \in \mathbb{N}$ the Jucys-Murphy elements $\widetilde{X}_{r}$ and $\widetilde{X}_{r-\frac{1}{2}}$ act on $V^{\otimes r}$ as

$$
\begin{aligned}
v_{\underline{i}} \cdot \widetilde{X}_{r} & =\sum_{j \neq i_{r}} v_{\left(j i_{r}\right) \cdot \underline{i}} \\
v_{\underline{i}} \cdot \widetilde{X}_{r-\frac{1}{2}} & =-\sum_{j \neq i_{r}} v_{\left(j i_{r}\right) \cdot \dot{i}^{\prime}} \otimes v_{i_{r}} .
\end{aligned}
$$

Proof. By Definition 4.47, we have

$$
\begin{aligned}
v_{\underline{i}} \cdot \widetilde{Z}_{r} & =\sum_{1 \leq a<b \leq n} v_{(a b) \cdot \underline{i}} \\
v_{\underline{i}} \cdot \widetilde{Z}_{r-1} & =\sum_{1 \leq a<b \leq n} v_{(a b) \cdot \underline{i}^{\prime}} \otimes v_{i_{r}}
\end{aligned}
$$

We also need the action of $\widetilde{Z}_{r-\frac{1}{2}}$ on $V^{\otimes r}$. We claim that

$$
v_{\underline{i}} \cdot \widetilde{Z}_{r-\frac{1}{2}}=\sum_{\substack{1 \leq a<b \leq n \\ a, b \neq i_{r}}} v_{(a b) \cdot \underline{i}^{\prime}} \otimes v_{i_{r}} .
$$

In fact, the claim is true for $i_{r}=n$ by the definition of the element $\widetilde{Z}_{r-\frac{1}{2}}$. For general $i_{r} \in\{1, \ldots, n\}$, this then follows from the $S_{n}$-equivariance of $\widetilde{Z}_{r-\frac{1}{2}} \in P_{r}(n)$. The proposition now follows since $\widetilde{X}_{r}=\widetilde{Z}_{r}-\widetilde{Z}_{r-\frac{1}{2}}$ and $\widetilde{X}_{r-\frac{1}{2}}=\widetilde{Z}_{r-\frac{1}{2}}-\widetilde{Z}_{r-1}$.
Proposition 4.56. The expression $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}$ is not local. In fact, we have

$$
\begin{equation*}
s_{i+k} \widetilde{X}_{i+k}-\widetilde{X}_{i+k+1} s_{i+k} \neq \mathcal{S}_{k}\left(s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}\right) \tag{66}
\end{equation*}
$$

in $P_{r+k+1}(\delta)$ for all $k>0,1 \leq i \leq r-1$ and all but finitely many $\delta \in \mathbb{C}$.
Proof. If the expression $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}$ is local for infinitely many $\delta$ then it is local for all $\delta$ by our standard interpolation argument. Hence it suffices to show (66) for some $\delta \in \mathbb{C}$. We will show this by acting on tensor space for $\delta=n \geq 2 r$. Consider the vector $v:=v_{1}^{\otimes i+k} \otimes v_{2}^{\otimes r+1-i} \cdot s_{i+k} \in V^{\otimes r+k+1}$. Using the formulas from Proposition 4.55, we get

$$
\begin{aligned}
v \cdot s_{i+k} \widetilde{X}_{i+k} & =v_{1}^{\otimes i+k} \otimes v_{2}^{\otimes r+1-i} \cdot \widetilde{X}_{i+k}=\sum_{j \neq 1} v_{j}^{\otimes i+k} \otimes v_{2}^{\otimes r+1-i} \\
v \cdot \widetilde{X}_{i+k+1} s_{i+k} & =v_{1}^{\otimes i+k-1} \otimes v_{2} \otimes v_{1} \otimes v_{2}^{\otimes r-i} \cdot \widetilde{X}_{i+k+1} s_{i+k} \\
& =\sum_{j \neq 1} v_{(1 j) \cdot 1}^{\otimes i+k-1} \otimes v_{(1 j) \cdot 2} \otimes v_{(1 j) \cdot 1} \otimes v_{2}^{\otimes r-i} \cdot s_{i+k} \\
& =\sum_{j \neq 1} v_{(1 j) \cdot 1}^{\otimes i+k} \otimes v_{(1 j) \cdot 2} \otimes v_{2}^{\otimes r-i} \\
& =\sum_{j \neq 1,2} v_{j}^{\otimes i+k} \otimes v_{2}^{\otimes r+1-i}+v_{2}^{\otimes i+k} \otimes v_{1} \otimes v_{2}^{\otimes r-i} .
\end{aligned}
$$

by Proposition 4.55 and hence

$$
\begin{equation*}
v \cdot\left(s_{i+k} \widetilde{X}_{i+k}-\widetilde{X}_{i+k+1} s_{i+k}\right)=v_{2}^{\otimes i+k+1}-v_{2}^{\otimes i+k} \otimes v_{1} \otimes v_{2}^{\otimes r-i} . \tag{67}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
v \cdot \mathcal{S}_{k}\left(s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}\right)=v_{1}^{\otimes k} \otimes w \tag{68}
\end{equation*}
$$

for some $w \in V^{\otimes r+1-k}$. We see that $\left.(67) \neq 68\right)$ for $k>0$. This proves the claim.
Remark 4.57. Similar calculations to the ones above show that $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i-\frac{1}{2}} s_{i}$ and $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+\frac{1}{2}} s_{i}$ are not local.

The only way out for a local expression of the error term for $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}$ seems to be to allow other Jucys-Murphy elements (or pictorially dots) in the error term. These should at least have index $<i$ so that one can give a basis of the corresponding degenerate affine algebra in terms of dotted diagrams. However, no such nice relation is known to the author.

Remark 4.58. Recursive formulas for the Jucys-Murphy elements which also give an expression for $s_{i} \widetilde{X}_{i}-\widetilde{X}_{i+1} s_{i}$ can be found in Eny13, (3.1), (3.3)]. However, the error term in Eny13 is not local (not even when allowing other Jucys-Murphy
elements) since they involve elements $\sigma_{k}$ which are not local in the standard generators. These recursive formulas (and most other formulas in Eny13) can also be verified using our standard interpolation technique after one has checked them on tensor space and after one has shown that the matrix representing the action of $\sigma_{k}$ on tensor space (with respect to the standard basis) has entries which depend polynomially on $n$.

We conclude this section by describing the action of the Jucys-Murphy elements along the isomorphism

$$
\begin{aligned}
\mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \ldots \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbf{1}_{n} & \stackrel{\sim}{\longrightarrow}\left[S_{n} / S_{n-1}\right]^{\otimes r} \cong V^{\otimes r} \\
g_{r} \otimes g_{r-1} \otimes \ldots \otimes g_{1} \otimes 1 & \longmapsto \overline{g_{r} g_{r-1} \cdots g_{1}} \otimes \overline{g_{r} g_{r-1} \cdots g_{2}} \otimes \ldots \otimes \overline{g_{r}}
\end{aligned}
$$

from 59 . For this we act on elements of the form

$$
v=g_{r} \otimes \ldots \otimes g_{1} \otimes g_{0} \in \mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \cdots \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbf{1}_{n} \cong V^{\otimes r}
$$

with $g_{i} \in S_{n}$ for $i>0$ and $g_{0}=1 \in \mathbf{1}_{n}$.
Proposition 4.59. We have

$$
\begin{aligned}
v \cdot \widetilde{X}_{i} & =\ldots \otimes g_{i+1} \otimes g_{i} X_{n} \otimes g_{i-1} \otimes \ldots \\
v \cdot \widetilde{X}_{i-\frac{1}{2}} & =\ldots \otimes g_{i} \otimes\left(-X_{n}\right) g_{i-1} \otimes g_{i-2} \otimes \ldots
\end{aligned}
$$

for $i=1, \ldots, r$.
Proof. Recall that $\widetilde{Z}_{i}$ acts on $V^{\otimes i}$ by multiplying with $Z_{n}$ on the left and $\widetilde{Z}_{i-\frac{1}{2}}$ acts on $V^{\otimes i-\frac{1}{2}}$ by multiplying with $Z_{n-1}$ on the left. Note that for any $\mathbb{C}\left[S_{n}\right]$-module $M$ and $f \in \operatorname{End}_{S_{n}}(M)$ the isomorphism from Lemma 4.37 identifies id ${ }_{V} \otimes f \in \operatorname{End}_{S_{n}}(V \otimes M)$ with $\operatorname{id}_{\mathbb{C}\left[S_{n}\right]} \otimes f \in \operatorname{End}_{S_{n}}\left(\mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} M\right)$. In particular, we have

$$
\begin{aligned}
v \cdot \widetilde{X}_{i} & =v \cdot\left(\widetilde{Z}_{i}-\widetilde{Z}_{i-\frac{1}{2}}\right) \\
& =\ldots \otimes g_{i+1} \otimes Z_{n} g_{i} \otimes g_{i-1} \otimes \ldots-\ldots \otimes g_{i} \otimes Z_{n-1} g_{i-1} \otimes g_{i-2} \otimes \ldots \\
& =\ldots \otimes g_{i+1} \otimes g_{i} Z_{n} \otimes g_{i-1} \otimes \ldots-\ldots \otimes g_{i+1} \otimes g_{i} Z_{n-1} \otimes g_{i-1} \otimes \ldots \\
& =\ldots \otimes g_{i+1} \otimes g_{i} X_{n} \otimes g_{i-1} \otimes \ldots
\end{aligned}
$$

and

$$
\begin{aligned}
v \cdot \widetilde{X}_{i-\frac{1}{2}} & =v \cdot\left(\widetilde{Z}_{i-\frac{1}{2}}-\widetilde{Z}_{i-1}\right) \\
& =\ldots \otimes g_{i} \otimes Z_{n-1} g_{i-1} \otimes g_{i-2} \otimes \ldots-\ldots \otimes g_{i} \otimes Z_{n} g_{i-1} \otimes g_{i-2} \otimes \ldots \\
& =\ldots \otimes g_{i} \otimes\left(-X_{n}\right) g_{i-1} \otimes g_{i-2} \otimes \ldots
\end{aligned}
$$

The standard generators of $P_{r}(n)$ can also be described explicitly as operators on $\left(\operatorname{Ind}_{S_{n-1}}^{S_{n}} \operatorname{Res}_{S_{n-1}}^{S_{n}}\right)^{r} \mathbf{1}_{n}$.

Proposition 4.60. We have

1) $v \cdot p_{i-\frac{1}{2}}=\ldots \otimes g_{i} \otimes \operatorname{pr}_{S_{n-1}}\left(g_{i-1}\right) \otimes g_{i-2} \otimes \ldots \quad$ for $i=2, \ldots, r$
2) $v \cdot p_{i}=\sum_{g \in S_{n} / S_{n-1}} \ldots \otimes g_{i+1} \otimes g \otimes g^{-1} g_{i} g_{i-1} \otimes g_{i-2} \otimes \ldots \quad$ for $i=1, \ldots, r$
3) $v \cdot s_{i}=\ldots \otimes g_{i+2} \otimes g_{i+1} g_{i} \otimes g_{i}^{-1} \otimes g_{i} g_{i-1} \otimes g_{i-2} \otimes \ldots \quad$ for $i=1, \ldots, r-1$
where $\operatorname{pr}_{S_{n-1}}: \mathbb{C}\left[S_{n}\right] \rightarrow \mathbb{C}\left[S_{n-1}\right], \quad g \mapsto\left\{\begin{array}{ll}g & \text { if } g \in S_{n-1} \\ 0 & \text { otherwise }\end{array}\right.$.
Proof. All of this can be done by direct calculation. We give the argument for 1). Recall that, $p_{i-\frac{1}{2}}$ acts on $V^{\otimes r}$ by projecting onto the basis vectors $v_{\underline{j}}$ with $\underline{j} \in\{1, \ldots, n\}^{r}$ such that $j_{i}=j_{i-1}$. Note that $v$ corresponds to the basis vector $\bar{v}_{j} \in V^{\otimes r}$ with $j_{i}=g_{r} g_{r-1} \cdots g_{i}(n)$. In particular, we have

$$
j_{i}=j_{i-1} \Leftrightarrow g_{r} g_{r-1} \cdots g_{i}(n)=g_{r} g_{r-1} \cdots g_{i-1}(n) \Leftrightarrow g_{i-1}(n)=n \Leftrightarrow g_{i-1} \in S_{n-1}
$$

which implies 1 ).
Proposition 4.59 and Proposition 4.60 are an alternative to the formulas from Proposition 4.55 if one wants to do calculations with Jucys-Murphy elements. For example, we have

$$
\begin{aligned}
v \cdot p_{i+\frac{1}{2}}\left(\widetilde{X}_{i}+\widetilde{X}_{i+\frac{1}{2}}\right) & =\left(\ldots \otimes \operatorname{pr}_{S_{n-1}}\left(g_{i}\right) \otimes \ldots\right) \cdot\left(\widetilde{X}_{i}+\widetilde{X}_{i+\frac{1}{2}}\right) \\
& =\ldots \otimes \operatorname{pr}_{S_{n-1}}\left(g_{i}\right) X_{n} \otimes \ldots+\ldots \otimes\left(-X_{n}\right) \operatorname{pr}_{S_{n-1}}\left(g_{i}\right) \otimes \ldots \\
& \stackrel{(*)}{=} \ldots \otimes X_{n} \operatorname{pr}_{S_{n-1}}\left(g_{i}\right) \otimes \ldots+\ldots \otimes\left(-X_{n}\right) \operatorname{pr}_{S_{n-1}}\left(g_{i}\right) \otimes \ldots \\
& =0
\end{aligned}
$$

where $(*)$ uses that $X_{n}$ commutes with $\mathbb{C}\left[S_{n-1}\right]$. This gives another way to verify the equations from Lemma 4.54.

Remark 4.61. The formulas from Proposition 4.60 might also be useful if one is interested in quantising the duality between the symmetric group $S_{n}$ and the partition algebra $P_{r}(n)$. The natural representation of the Hecke algebra is the $n$-dimensional parabolic Hecke module

$$
V_{q}:=\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1} q^{-1}} \mathbb{C}(q)=\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} x_{n}
$$

where $x_{n}=\sum_{x \in S_{n}} q^{-l(x)} H_{x}$. However, defining a 'diagonal' action of the Hecke algebra on $V_{q}^{\otimes r}$ is not straightforward, since we do not have an obvious candidate for a comultiplication on $\mathcal{H}_{n}$ which specialises to the comultiplication of $\mathbb{C}\left[S_{n}\right]$ for $q \rightarrow 1$. On the other hand, restriction and induction along $\mathcal{H}_{n-1} \subset \mathcal{H}_{n}$ makes perfect sense for the Hecke algebra and we can define the 'tensor space' for the Hecke algebra by

$$
T_{q}(r):=\underbrace{\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \cdots \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n}}_{r} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} x_{n}
$$

We could now try to define a quantum partition algebra by finding analogues of the elements $p_{i}, p_{i-\frac{1}{2}}, s_{i} \in P_{r}(n)$ as described in Proposition 4.60. In fact, the map

$$
\operatorname{pr}_{\mathcal{H}_{n-1}}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n-1}, \quad H_{x} \mapsto \begin{cases}H_{x} & \text { if } x \in S_{n-1} \\ 0 & \text { otherwise }\end{cases}
$$

is an $\left(\mathcal{H}_{n-1}, \mathcal{H}_{n-1}\right)$-bimodule homomorphism (which follows from Proposition 1.11 ) and hence induces an endomorphism

$$
P_{i-\frac{1}{2}}:=\operatorname{id}_{\mathcal{H}_{n}}^{r-i+1} \otimes \operatorname{pr}_{\mathcal{H}_{n-1}} \otimes \operatorname{id}_{\mathcal{H}_{n}}^{i-1} \in \operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right)
$$

Let $\mathcal{D}$ be the set of shortest left coset representatives of $S_{n-1}$ in $S_{n}$. Consider the element $\sum_{x \in \mathcal{D}} H_{x} \otimes H_{x^{-1}} \in \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n}$. One can check that the map

$$
\Phi: \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} \rightarrow \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n}, \quad a \otimes b \mapsto \sum_{x \in \mathcal{D}} H_{x} \otimes H_{x^{-1}} a b
$$

is an $\left(\mathcal{H}_{n}, \mathcal{H}_{n}\right)$-bimodule homomorphism (see [LS13, Lemma 5.3]). In particular, this induces an endomorphism

$$
P_{i}:=\operatorname{id}_{\mathcal{H}_{n}}^{r-i} \otimes \Phi \otimes \operatorname{id}_{\mathcal{H}_{n}}^{i-1} \in \operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right) .
$$

Unfortunately, defining an analogue of the element $s_{i} \in P_{r}(n)$ is not that simple since obvious candidates like the assignment $H_{a} \otimes H_{x} \otimes H_{b} \mapsto H_{a} H_{x} \otimes H_{x^{-1}} \otimes H_{x} H_{b}$ on $\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n}$ are not well-defined. For instance, if $s \in S_{n-1}$ is a simple reflection and $s x>x$, then

$$
H_{s} \otimes H_{x} \otimes 1 \mapsto H_{s} H_{x} \otimes H_{x^{-1}} \otimes H_{x}
$$

but $1 \otimes H_{s} H_{x} \otimes 1=1 \otimes H_{s x} \otimes 1 \mapsto H_{s x} \otimes H_{(s x)^{-1}} \otimes H_{s x}=H_{s} H_{x} \otimes H_{x^{-1}} \otimes H_{s}^{2} H_{x}$.
Nonetheless, one can construct a basis of $\operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right)$. Note that the endofunctor $V \otimes_{\mathbb{C}}(-) \cong \mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]}(-)$ on $\operatorname{Rep}\left(S_{n}\right)$ is self-adjoint since $V \cong V^{*}$ as $S_{n^{-}}$ representations. Hence,

$$
\begin{aligned}
\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right) & \cong \operatorname{Hom}_{S_{n}}\left(\mathbf{1}_{n}, V^{\otimes 2 r}\right) \\
& \cong \operatorname{Hom}_{S_{n}}\left(e_{n} \mathbb{C}\left[S_{n}\right], \mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \cdots \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbb{C}\left[S_{n}\right] e_{n}\right) \\
& \cong e_{n} \mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \underbrace{\left.\mathbb{C}\left[S_{n}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]}\right] \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbb{C}\left[S_{n}\right]}_{2 r-1} \otimes_{\mathbb{C}\left[S_{n-1}\right]} \mathbb{C}\left[S_{n}\right] e_{n}
\end{aligned}
$$

by Lemma 2.7 where $e_{n}=\sum_{\sigma \in S_{n}} \sigma$. This space has basis $\left\{e_{n} \otimes g_{1} \otimes \ldots \otimes g_{2 r-1} \otimes e_{n}\right\}$ where the ( $\left.g_{1}, \ldots ., g_{2 r-1}\right)$ are representatives of

$$
\begin{equation*}
\{*\} \times_{S_{n-1}} \underbrace{S_{n} \times_{S_{n-1}} \ldots . \times_{S_{n-1}} S_{n}}_{2 r-1} \times_{S_{n-1}}\{*\} . \tag{69}
\end{equation*}
$$

A similar argument can be used to describe $\operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right)$. In fact, one can check that $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n}}$ and $\operatorname{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n}}$ form a biadjoint pair of functors (see LS13, Prop. 5.4]). Hence, the endofunctor $\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}}(-)=\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n}} \operatorname{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n}}(-)$ on $\mathcal{H}_{n}$-mod is selfadjoint. In particular,

$$
\begin{aligned}
\operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right) & \cong \operatorname{Hom}_{\mathcal{H}_{n}}\left(\mathcal{H}_{n} x_{n}, T_{q}(2 r)\right) \\
& \cong x_{n} \mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1}} \underbrace{\mathcal{H}_{n} \otimes_{\mathcal{H}_{n-1} \ldots \otimes_{\mathcal{H}_{n-1}}} \mathcal{H}_{n}}_{2 r-1} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n} x_{n} .
\end{aligned}
$$

This space is spanned by the elements $\left\{e_{n} \otimes H_{x_{1}} \otimes \ldots \otimes H_{x_{2 r-1}} \otimes e_{n}\right\}$ where the $\left(x_{1}, \ldots ., x_{2 r-1}\right)$ are representatives of (69). In fact, a straightforward specialisation argument (as in the proof of Lemma 2.23) shows that these elements are linearly independent and therefore form a basis. This shows that $\operatorname{dim}_{\mathbb{C}(q)} \operatorname{End}_{\mathcal{H}_{n}}\left(T_{q}(r)\right)=$ $\operatorname{dim}_{\mathbb{C}} \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)$.

## 5 Diagram categories and their monoidal structure

This last section will serve as an outlook on applications of the monoidal category formalism to Schur-Weyl duality. This gives a more conceptual framework for studying Schur-Weyl dualities for tensor spaces of varying size at the same time. Section 5.1 introduces monoidal categories and the partition category following [EGNO16] and CO11. In Section 5.2 we introduce a new monoidal category which we call the balanced partition category. This category generalises the balanced partition algebra from Definition 3.38. We also give a presentation of the balanced partition category by generators and relations.

### 5.1 Monoidal categories

When studying Schur-Weyl dualities in the previous sections, we only considered the morphism spaces $\operatorname{Hom}_{A}\left(V^{\otimes r}, V^{\otimes s}\right)$ for $r=s$ where usually $A$ is an algebra/group/Lie algebra with a left action $A \curvearrowright V^{\otimes r}$ for all $r \in \mathbb{N}_{0}$. In fact, for classical Schur-Weyl duality it might seem that all the interesting information is already contained in these endomorphism spaces. For example, $\operatorname{Hom}_{\mathrm{GL}_{n}(\mathbb{C})}\left(V^{\otimes r}, V^{\otimes s}\right)=0$ for $r \neq s$ since $c \cdot I_{n}$ acts by multiplying with $c^{r}$ on $V^{\otimes r}$ and by multiplying with $c^{s}$ on $V^{\otimes s}$ for any $c \in \mathbb{C}^{\times}$so that there are no non-zero homomorphisms that commute with all $c \cdot I_{n}$ if $r \neq s$. However, thinking about it this way does not take into account the additional structure that comes with the isomorphism of $\mathrm{GL}_{n}(\mathbb{C})$-representations

$$
V^{\otimes r} \otimes_{\mathbb{C}} V^{\otimes s} \xrightarrow{\sim} V^{\otimes r+s}
$$

and the associated homomorphism of algebras

$$
\operatorname{End}_{\operatorname{GL}_{n}(\mathbb{C})}\left(V^{\otimes r}\right) \otimes_{\mathbb{C}} \operatorname{End}_{\operatorname{GL}_{n}(\mathbb{C})}\left(V^{\otimes s}\right) \longrightarrow \operatorname{End}_{\mathrm{GL}_{n}(\mathbb{C})}\left(V^{\otimes r+s}\right) .
$$

On top of that, the space $\operatorname{Hom}_{A}\left(V^{\otimes r}, V^{\otimes s}\right)$ will actually be non-zero for most other Schur-Weyl dualities and some $r \neq s$. Forcing ourself to only look at the $r=s$ case is sometimes even a bit unnatural. Take for example the relation $e_{1}^{2}=\delta e_{1}$ from the Brauer algebra. In diagrams this is the bubble removal axiom

which should really have nothing to do with the cup and the cap that remain on the top and on the bottom. Moreover, the action $v \otimes w \cdot e_{1}=\epsilon_{\mathfrak{g}}\langle v, w\rangle \sum_{i=1}^{n} v_{i} \otimes v^{i}$ from (44) is actually the composition of two actions, namely first computing the pairing $\epsilon_{\mathfrak{g}}\langle v, w\rangle$ and then multiplying with $\sum_{i=1}^{n} v_{i} \otimes v^{i}$. The better picture for the bubble removal axiom would be

$$
\oint=\delta \cdot \emptyset .
$$

As homomorphisms between tensor spaces this can be interpreted as

$$
\begin{aligned}
\Omega & =\left(c \mapsto c \cdot \sum_{i=1}^{n} v_{i} \otimes v^{i}\right) \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, V \otimes V) \\
\mho & =\epsilon_{\mathfrak{g}}(-,-\rangle \in \operatorname{Hom}_{\mathbb{C}}(V \otimes V, \mathbb{C}) \\
\emptyset & =\operatorname{id}_{\mathbb{C}} \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) .
\end{aligned}
$$

It follows from the calculations in (45) and (46) that these are actually $\mathfrak{g}$-module homomorphisms (where $\mathfrak{g}=\mathfrak{s o}_{n}$ or $\mathfrak{g}=\mathfrak{s p}_{n}$ ). We can also interpret this as a categorical approach to Schur-Weyl duality as follows: Where before we were just studying one object $V^{\otimes r}$ and its endomorphisms, we are now looking at the collection of objects $V^{\otimes r}$ with $r \in \mathbb{N}_{0}$ and morphisms between any two objects of this kind, i.e. the full subcategory of $\mathfrak{g}$-mod whose objects are the $V^{\otimes r}$. Let us put this into a more axiomatic framework to take full advantage these ideas.

Recall that a category $\mathcal{C}$ is called $\mathbb{C}$-linear if all its Hom-spaces have the structure of a $\mathbb{C}$-vector space and composition of morphisms is bilinear. From now on, all categories are assumed to be $\mathbb{C}$-linear. Moreover, all functors are assumed to be $\mathbb{C}$-linear as well (i.e. $\mathbb{C}$-linear on Hom-spaces).

Definition 5.1. A monoidal category $\mathcal{T}=(\mathcal{T}, \otimes, a, \mathbf{1}, l, r)$ is a category $\mathcal{T}$ together with:

- a bifunctor $\otimes: \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$
- a natural isomorphism $a:(-\otimes-) \otimes-\xrightarrow{\sim}-\otimes(-\otimes-)$
- an object $\mathbf{1} \in \mathcal{T}$
- two natural isomorphisms $l: \mathbf{1} \otimes-\xrightarrow{\sim}-$ and $r:-\otimes \mathbf{1} \xrightarrow{\sim}-$.
satisfying the pentagon axiom

and the triangle axiom

for all $W, X, Y, Z \in \mathcal{T}$. The functor $\otimes$ is often called the tensor bifunctor, $\mathbf{1}=(\mathbf{1}, l, r)$ is called the unit object and $a$ is called the associator. The monoidal category $\mathcal{T}$ is called strict if $a, l$ and $r$ are the identity transformations.
Given two monoidal categories ( $\mathcal{T}, \otimes, \mathbf{1}, a, l, r)$ and $\left(\mathcal{T}^{\prime}, \otimes^{\prime}, \mathbf{1}^{\prime}, a^{\prime}, l^{\prime}, r^{\prime}\right)$, a (monoidal) functor between these categories a pair $F=(F, J)$ consisting of a functor $F: \mathcal{T} \rightarrow \mathcal{T}^{\prime}$ and a natural isomorphism

$$
J: F(-) \otimes^{\prime} F(-) \xrightarrow{\sim} F(-\otimes-)
$$

with $F(\mathbf{1}) \cong \mathbf{1}^{\prime}$ such that the following diagram commutes

$$
\begin{aligned}
& \begin{array}{r}
\left(F(X) \otimes^{\prime} F(Y)\right) \otimes^{\prime} F(Z) \xrightarrow{a_{F(X), F(Y), F(Z)}^{\prime}} F(X) \otimes^{\prime}\left(F(Y) \otimes^{\prime} F(Z)\right) \\
\downarrow J_{X, Y} \otimes^{\prime} \mathrm{id}_{F(Z)}
\end{array} \\
& F(X \otimes Y) \otimes^{\prime} F(Z) \quad F(X) \otimes^{\prime} F(Y \otimes Z) \\
& \downarrow_{J_{X \otimes Y, Z}} \quad \downarrow^{J_{X, Y \otimes Z}} \\
& F((X \otimes Y) \otimes Z) \xrightarrow{F\left(a_{X, Y, Z}\right)} F(X \otimes(Y \otimes Z)) \text {. }
\end{aligned}
$$

We call $(F, J)$ an equivalence of monoidal categories if $F$ is an equivalence of the underlying categories.

Remark 5.2. One can define and study many interesting properties of monoidal categories such as abelian, $\operatorname{End}_{\mathcal{C}}(\mathbf{1})=\mathbb{C}$, symmetric (which means that $\otimes$ is symmetric) or rigid (which means that objects have duals). A monoidal category satisfying all of these properties (or some of these depending on the author) is called a tensor category. Not all our monoidal categories will satisfy these properties (except for symmetric and $\operatorname{End}_{\mathcal{C}}(\mathbf{1})=\mathbb{C}$ ) and we will not talk about these in further detail to not overcomplicate things. For an introduction to tensor categories and monoidal categories in general, we refer to EGNO16.

The usual tensor product constructions for rings, modules and representations give rise to many examples of monoidal categories.

Example 5.3. Let $G$ be a group and let $\operatorname{Rep}(G)$ be the category of finite-dimensional representations of $G$. This is a monoidal category with the tensor bifunctor $\otimes=\otimes_{\mathbb{C}}$, i.e. the tensor product of representations with the diagonal $G$-action. The unit object is the trivial representation.

Let us look at another example of a monoidal category that is adapted to the Schur-Weyl duality setting. Let $\mathcal{C}_{n}$ be the full subcategory of $\operatorname{Rep}\left(S_{n}\right)$ whose objects are the representations of $S_{n}$ which are isomorphic to $V^{\otimes r}$ for some $r \geq 0$. The tensor bifunctor of $\operatorname{Rep}\left(S_{n}\right)$ restricts to a tensor bifunctor on $\mathcal{C}_{n}$ and the trivial representation $\mathbb{C}=V^{\otimes 0}$ is a unit object of $\mathcal{C}_{n}$. It follows that $\mathcal{C}_{n}$ inherits the structure of a monoidal category. We can think of $\mathcal{C}_{n}$ as a generalisation of all the algebras $\operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)$ at the same time. Let us now define the analogue of the partition algebra in this categorical setting.

Definition 5.4. For any $\delta \in \mathbb{C}$, the partition category $\mathcal{P}(\delta)$ is the category with objects $[r]$ indexed by integers $r \geq 0$ and morphism spaces

$$
\operatorname{Hom}_{\mathcal{P}(\delta)}([r],[s]):=\operatorname{Span}_{\mathbb{C}} A_{s, r}
$$

where $A_{s, r}$ is the set of all partitions of the set $\{1, \ldots, s\} \cup\left\{1^{\prime}, \ldots, r^{\prime}\right\}$. We interpret these as diagrams with $r$ dots on the bottom and $s$ dots on the top. For $d_{1} \in A_{s, r}$ and $d_{2} \in A_{r, t}$ we define the composition

$$
d_{1} \circ d_{2}:=\delta^{r\left(d_{1}, d_{2}\right)} d_{1} \star d_{2}
$$

where $d_{1} \star d_{2}$ is the diagram obtained from stacking $d_{1}$ on top of $d_{2}$ and removing all free blocks in the middle where $r\left(d_{1}, d_{2}\right)$ is the number of these free blocks removed. This is a strict monoidal category with unit object [0] and tensor bifunctor

$$
[r] \otimes[s]:=[r+s]
$$

where

$$
\otimes: \operatorname{Hom}_{\mathcal{P}(\delta)}([r],[s]) \times \operatorname{Hom}_{\mathcal{P}(\delta)}\left(\left[r^{\prime}\right],\left[s^{\prime}\right]\right) \rightarrow \operatorname{Hom}_{\mathcal{P}(\delta)}\left(\left[r+r^{\prime}\right],\left[s+s^{\prime}\right]\right)
$$

is given by putting diagrams next to each other without overlapping.
Example 5.5. Here is an example of a composition and a tensor product of morphisms in $\mathcal{P}(\delta)$ :



In the world of algebras, constructing a right action $V^{\otimes r} \curvearrowleft P_{r}(n)$ corresponds to the construction of an algebra homomorphism $P_{r}(n) \rightarrow \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\mathrm{op}}$. In the monoidal category setting this corresponds to a contravariant monoidal functor

$$
F: \mathcal{P}(n) \longrightarrow \mathcal{C}_{n}
$$

which is given by

$$
F([r]):=V^{\otimes r}
$$

on objects. To explain what this functor does on morphisms, consider $d \in A_{s, r}$. Recall that for $r=s$ this acts on $V^{\otimes s}$ as

$$
\begin{equation*}
\sum_{\underline{i} \xrightarrow{\underline{i}} \underline{j}} E_{\underline{j}, \underline{i}} \in \operatorname{Hom}_{S_{n}}\left(V^{\otimes s}, V^{\otimes r}\right)=\operatorname{Hom}_{\mathcal{C}_{n}^{\text {op }}}\left(V^{\otimes r}, V^{\otimes s}\right) \tag{70}
\end{equation*}
$$

with $\underline{i} \in\{1, \ldots, n\}^{s}$ and $\underline{j} \in\{1, \ldots, n\}^{r}$. Here the notation $\underline{i} \xrightarrow{d} \underline{j}$ means that labelling the top row of $d$ with $\underline{i}$ and the bottom row $d$ with $j$ induces a well-defined labelling of the blocks of $d$ (see (25). This notation extends without problem to the $r \neq s$ case and we can define $F(d) \in \operatorname{Hom}_{S_{n}}\left(V^{\otimes r}, V^{\otimes s}\right)^{\text {op }}$ by (70). The same argument as in the $r=s$ case shows that this is $S_{n}$-equivariant and compatible with composition. It is also clear that this is monoidal functor.
Schur-Weyl duality states that the algebra homomorphism $P_{r}(n) \rightarrow \operatorname{End}_{S_{n}}\left(V^{\otimes r}\right)^{\text {op }}$ is surjective. In the monoidal categorical set-up this is the following statement.

Theorem 5.6. The contravariant monoidal functor $F: \mathcal{P}(n) \rightarrow \mathcal{C}_{n}$ is full, i.e.

$$
F: \operatorname{Hom}_{\mathcal{P}(n)}([r],[s]) \rightarrow \operatorname{Hom}_{S_{n}}\left(V^{\otimes s}, V^{\otimes r}\right)
$$

is surjective for all $r, s \geq 0$.
Proof. In the $r=s$ case this is Theorem 3.9. For $r \neq s$ the proof works exactly the same.

Remark 5.7. Any finite-dimensional $S_{n}$-representation appears as a direct summand of a direct sum of some $V^{\otimes r}$ (this follows for example from the branching rule in Theorem 4.15). In other words, closing up the category $\mathcal{C}_{n}$ under taking direct sums and direct summands recovers the whole category $\operatorname{Rep}\left(S_{n}\right)$. More formally, one can define the additive closure $\operatorname{Add}(\mathcal{C})$ of a category which consists of formal direct sums $X_{1} \oplus \ldots \oplus X_{m}$ with $X_{i} \in \mathcal{C}$ and $\operatorname{Hom}_{\operatorname{Add}(\mathcal{C})}\left(\bigoplus_{j=1}^{k} X_{j}, \bigoplus_{i=1}^{l} Y_{i}\right)$ consists of all $l \times k$ matrices where the $(i, j)$-th entry is an element of $\operatorname{Hom}_{\mathcal{C}}\left(X_{j}, Y_{i}\right)$. This makes the notion of closing up a category under taking direct sums precise. Similarly, one can define the Karoubian envelope $\operatorname{Kar}(\mathcal{C})$ of an additive category $\mathcal{C}$. The objects in this category are pairs $(X, e)$ where $e \in \operatorname{End}_{\mathcal{C}}(X)$ is an idempotent morphism and $\operatorname{Hom}_{\operatorname{Kar}(\mathcal{A})}((X, e),(Y, f)):=f \operatorname{Hom}_{\mathcal{A}}(X, Y) e$. This makes the notion of closing up a category under taking direct summands precise (see Kar08, Section I-6] for more details). Moreover, if $\mathcal{C}$ is a monoidal category there is a canonical monoidal structure on $\operatorname{Add}(\mathcal{C})$ and $\operatorname{Kar}(\operatorname{Add}(\mathcal{C}))$. It is not hard to show that $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{C}_{n}\right)\right) \cong \operatorname{Rep}\left(S_{n}\right)$ as monoidal categories and there is a monoidal functor $F^{\prime}: \operatorname{Kar}(\operatorname{Add}(\mathcal{P}(n))) \rightarrow \operatorname{Rep}\left(S_{n}\right)$. The functor $F^{\prime}$ is not quite an equivalence of
categories, but it is surjective on objects and morphisms. Hence, one can think of $\operatorname{Kar}(\operatorname{Add}(\mathcal{P}(t)))$ as the category that interpolates $\operatorname{Rep}\left(S_{t}\right)$ for $t \in \mathbb{C}$ not necessarily an integer. The category $\operatorname{Kar}(\operatorname{Add}(\mathcal{P}(t)))$ was introduced in Del07] is nowadays called a Deligne category. This Deligne category can be used to study properties of $\operatorname{Rep}\left(S_{t}\right)$ which are stable under varying $t$. For more details, we refer to Del07 and CO11.

One can also interpret other Schur-Weyl dualities and diagram algebras in the monoidal category setting. For example, we can define the Brauer category $\mathcal{B}(\delta)$ to be the subcategory of $\mathcal{P}(\delta)$ with the same objects but morphism spaces spanned by Brauer diagrams, that is

$$
\operatorname{Hom}_{\mathcal{B}(\delta)}([r],[s])=\operatorname{Span}_{\mathbb{C}}\left\{d \in A_{s, r}| | B \mid=2 \text { for each block } B \text { of } d\right\}
$$

Note that $\operatorname{Hom}_{\mathcal{B}(\delta)}([r],[s])=0$ unless $r+s$ is even. It is straightforward to check that $\mathcal{B}(\delta)$ inherits the structure of a monoidal category from $\mathcal{P}(\delta)$.

### 5.2 The balanced partition category

We now apply the techniques from the previous section to the balanced partition algebra.

Definition 5.8. The balanced partition category $\mathcal{P}^{\text {bal }}$ is the strict monoidal category with objects $[r]$ for $r \geq 0$ and morphism spaces

$$
\operatorname{Hom}([r],[s])=\operatorname{Span}_{\mathbb{C}} A_{s, r}^{\mathrm{bal}}
$$

where $A_{s, r}^{\mathrm{bal}}$ is the set of all $d \in A_{s, r}$ such that

$$
\mid\left\{B \cap \{ 1 , \ldots , s \} \left|=\left|B \cap\left\{1^{\prime}, \ldots, r^{\prime}\right\}\right|\right.\right.
$$

for each block $B$ of $d$. Composition is again defined by stacking diagrams on top of each other and the the tensor bifunctor $\otimes$ puts diagrams next to each other.

Note that $A_{s, r}=\emptyset$ for $r \neq s$ and hence the underlying category of $\mathcal{P}^{\text {bal }}$ decomposes as

$$
\begin{equation*}
\mathcal{P}^{\mathrm{bal}}=\bigsqcup_{r \geq 0}\left([r], P_{r}^{\mathrm{bal}}\right) \tag{71}
\end{equation*}
$$

where $\left([r], P_{r}^{\text {bal }}\right)$ is the category with one object $[r]$ and $\operatorname{End}([r])=P_{r}^{\text {bal }}$. However, this is not a decomposition of $\mathcal{P}^{\text {bal }}$ as a monoidal category since $\left([r], P_{r}^{\text {bal }}\right)$ is not closed under the tensor bifunctor.

We can also write down diagram categories by generators and relations which is more efficient than for the underlying algebras. In fact, it allows us to state relations locally without repeating generators and relations over again. Let us explain this for $\mathcal{P}^{\text {bal }}$. Recall from Theorem 3.54 that the endomorphism algebra $\operatorname{End}_{\mathcal{P} \text { bal }}([r])=P_{r}^{\text {bal }}$ has generators $s_{1}^{(r)}, \ldots, s_{r-1}^{(r)}$ and $p_{\frac{3}{2}}^{(r)}, p_{\frac{5}{2}}^{(r)}, \ldots, p_{r-\frac{1}{2}}^{(r)}$. Using our monoidal structure, we can rewrite these as

$$
\begin{equation*}
s_{i}^{(r)}=\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes s_{1}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1} \quad \text { and } \quad p_{i+\frac{1}{2}}^{(r)}=\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes p_{\frac{3}{2}}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1} \tag{72}
\end{equation*}
$$

We can also express the relations of $P_{r}^{\text {bal }}$ more efficiently using the monoidal structure. For example, it suffices to know the relation $s_{i}^{(r)} p_{i+\frac{1}{2}}^{(r)}=p_{i+\frac{1}{2}}^{(r)}$ in the $r=2$ case
since then

$$
\begin{align*}
s_{i}^{(r)} p_{i+\frac{1}{2}}^{(r)} & =\left(\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes s_{1}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1}\right) \circ\left(\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes p_{\frac{3}{2}}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1}\right) \\
& =\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes s_{1}^{(2)} \circ p_{\frac{3}{2}}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1} \\
& =\mathrm{id}_{\mathbf{1}}^{\otimes i-1} \otimes p_{\frac{3}{2}}^{(2)} \otimes \mathrm{id}_{\mathbf{1}}^{\otimes r-i-1}  \tag{73}\\
& =p_{i+\frac{1}{2}}^{(r)}
\end{align*}
$$

Let us now explain how to define a strict monoidal category by generators and relations. First, we briefly outline the construction of a free strict monoidal category $\mathcal{F}$ (for a rigorous construction we refer to [Kas12, Section XII.1]):
(i) Given a collection of generating objects $a_{i}$ for $i \in I$ the objects of $\mathcal{F}$ are formal tensor products $a_{i_{1}} \otimes \ldots \otimes a_{i_{r}}$ (including the empty tensor product $\mathbf{1}$ ). We define the tensor product of two such objects in the obvious way, i.e. by juxtaposition of the two formal tensor products.
(ii) Given a collection of generating morphisms $S(X, Y)$ for all objects $X, Y \in$ $\mathcal{F}$, we construct from this a collection $\bar{S}(X, Y)$ for all $X, Y \in \mathcal{F}$. This is obtained by closing up the $S(X, Y)$ under formal compositions and formal tensor products (also including the empty composition $\mathrm{id}_{X}$ ). To be more precise, we recursively introduce formal elements $f \circ g$ (when $f$ and $g$ are composable) and $f \otimes g$ subject to the formal condition

$$
\left(f \circ f^{\prime}\right) \otimes\left(g \circ g^{\prime}\right)=(f \otimes g) \circ\left(f^{\prime} \otimes g^{\prime}\right)
$$

We define $\operatorname{Hom}_{\mathcal{F}}(X, Y):=\operatorname{Span}_{\mathbb{C}} \bar{S}(X, Y)$.
$\mathcal{F}$ is a strict monoidal category with the obvious composition and tensor bifunctor. By construction the category $\mathcal{F}$ has the following universal property:

Let $\mathcal{T}$ be a strict monoidal category with a collection of objects $F\left(a_{i}\right) \in \mathcal{T}$ for all $i \in I$. Then there is a unique assignment on the level of objects $F: \mathcal{F} \longrightarrow \mathcal{T}$ with $a_{i} \stackrel{F}{\mapsto} F\left(a_{i}\right)$ and $F(X \otimes Y)=F(X) \otimes F(Y)$ for all $X, Y \in \mathcal{F}$. Assume further that we are given a morphism $F(f) \in \operatorname{Hom}_{\mathcal{T}}(F(X), F(Y))$ for any $X, Y \in \mathcal{F}$ and $f \in S(X, Y)$. Then $F: \mathcal{F} \rightarrow \mathcal{T}$ extends in a unique way to a monoidal functor with $f \stackrel{F}{\mapsto} F(f)$.

To introduce relations on the free category $\mathcal{F}$, we explain what the analogue of an ideal in our categorical setting is.

Definition 5.9. Let $\mathcal{T}$ be a monoidal category. A tensor ideal $\mathcal{N}$ of $\mathcal{T}$ is a collection of subspaces $\mathcal{N}(X, Y) \subset \operatorname{Hom}_{\mathcal{T}}(X, Y)$ for all $X, Y \in \mathcal{T}$ with the property that $\mathcal{N}$ is closed under left and right composition as well as left and right tensoring. More precisely, we require for any $W, X, Y, Z \in \mathcal{T}$ and $g \in \mathcal{N}(X, Y)$ that

$$
\begin{aligned}
f \circ g \in \mathcal{N}(X, Z) & \forall f \in \operatorname{Hom}_{\mathcal{T}}(Y, Z) \\
g \circ f \in \mathcal{N}(W, Y) & \forall f \in \operatorname{Hom}_{\mathcal{T}}(W, X) \\
f \otimes g \in \mathcal{N}(W \otimes X, Z \otimes Y) & \forall f \in \operatorname{Hom}_{\mathcal{T}}(W, Z) \\
g \otimes f \in \mathcal{N}(X \otimes W, Y \otimes Z) & \forall f \in \operatorname{Hom}_{\mathcal{T}}(W, Z) .
\end{aligned}
$$

Example 5.10. Let $F: \mathcal{T} \rightarrow \mathcal{T}^{\prime}$ be a functor of monoidal categories and let $\operatorname{ker}(F)(X, Y):=\left\{f \in \operatorname{Hom}_{\mathcal{T}}(X, Y) \mid F(f)=0\right\}$. Then $\operatorname{ker}(F)$ is a tensor ideal in $\mathcal{T}$.

Given a tensor ideal $\mathcal{N}$ in a monoidal category $\mathcal{T}$ we can define the quotient category $\mathcal{T} / \mathcal{N}$ which is the category with the same objects as $\mathcal{T}$ but

$$
\operatorname{Hom}_{\mathcal{T} / \mathcal{N}}(X, Y):=\operatorname{Hom}_{\mathcal{T}}(X, Y) / \mathcal{N}(X, Y)
$$

It follows from the definition of tensor ideals that this inherits the structure of a monoidal category from $\mathcal{T}$. Furthermore, given tensor ideals $\mathcal{N}$ and $\mathcal{N}^{\prime}$ of $\mathcal{T}$ we can define their intersection $\mathcal{N} \cap \mathcal{N}^{\prime}$ by

$$
\left(\mathcal{N} \cap \mathcal{N}^{\prime}\right)(X, Y):=\mathcal{N}(X, Y) \cap \mathcal{N}^{\prime}(X, Y)
$$

It is clear that this is again a tensor ideal. Hence it makes sense to talk about the tensor ideal generated by a collection of morphisms by intersecting all tensor ideals containing these morphisms. We now know how to make sense of a strict monoidal category with generators and relations by constructing a free strict monoidal category generated by a collection of objects and morphisms and then modding out a tensor ideal generated by some relations. Let us explain this for the balanced partition category
Definition 5.11. We define $\mathcal{F}^{\text {bal }}$ to be the free strict monoidal category generated by a single object $*$ and two morphisms

$$
\begin{aligned}
& s=\square \in \operatorname{Hom}_{\mathcal{F}}(* \otimes *, * \otimes *), \\
& p=\square \in \operatorname{Hom}_{\mathcal{F}}(* \otimes *, * \otimes *) .
\end{aligned}
$$

Moreover, we define $\mathcal{N}^{\text {bal }}$ to be the tensor ideal in $\mathcal{F}^{\text {bal }}$ generated by the relations
(CBAL1) (i) $s \circ s=\mathrm{id}_{* \otimes *}$
(ii) $\left(s \otimes \operatorname{id}_{\mathbf{1}}\right) \circ\left(\mathrm{id}_{\mathbf{1}} \otimes s\right) \circ\left(s \otimes \mathrm{id}_{\mathbf{1}}\right)=\left(\mathrm{id}_{\mathbf{1}} \otimes s\right) \circ\left(s \otimes \operatorname{id}_{\mathbf{1}}\right) \circ\left(\mathrm{id}_{\mathbf{1}} \otimes s\right)$
(CBAL2) (i) $p \circ p=p$
(ii) $\left(p \otimes \mathrm{id}_{\mathbf{1}}\right) \circ\left(\mathrm{id}_{\mathbf{1}} \otimes p\right)=\left(\mathrm{id}_{\mathbf{1}} \otimes p\right) \circ\left(p \otimes \mathrm{id}_{\mathbf{1}}\right)$
(CBAL3) (i) $s \circ p=p=p \circ s$
(ii) $\left(\mathrm{id}_{\mathbf{1}} \otimes s\right) \circ\left(p \otimes \mathrm{id}_{\mathbf{1}}\right) \circ\left(\mathrm{id}_{\mathbf{1}} \otimes s\right)=\left(s \otimes \mathrm{id}_{\mathbf{1}}\right) \circ\left(\mathrm{id}_{\mathbf{1}} \otimes p\right) \circ\left(s \otimes \mathrm{id}_{\mathbf{1}}\right)$

Remark 5.12. The relations (CBAL1)-(CBAL3) have the following diagrammatic interpretation:

(CBAL3) :



The categorical analogue of the presentation Theorem 3.54 is then the following.
Theorem 5.13. There is an equivalence of monoidal categories

$$
\begin{aligned}
\mathcal{F}^{\text {bal }} / \mathcal{N}^{\text {bal }} & \xrightarrow{\sim} \mathcal{P}^{\text {bal }} \\
* & \longmapsto[1] \\
s & \longmapsto s_{1}^{(2)}=\left\{\in \operatorname{Hom}_{\mathcal{P} \text { bal }}([2],[2])\right. \\
p & \longmapsto p_{\frac{3}{2}}^{(2)}=\square \in \operatorname{Hom}_{\mathcal{P} \text { bal }}([2],[2]) .
\end{aligned}
$$

Proof. By the universal property of the free strict monoidal category $\mathcal{F}^{\text {bal }}$ there is a monoidal functor $G: \mathcal{F}^{\text {bal }} \rightarrow \mathcal{P}^{\text {bal }}$ with $G(*)=[1], G(s)=s_{1}^{(2)}$ and $G(p)=p_{\frac{3}{2}}^{(2)}$. Clearly, $G$ is bijective on objects. $G$ is also full since the standard generators of $\operatorname{End}_{\mathcal{P} \text { bal }}([r])=P_{r}^{\text {bal }}$ lie in the image of $G$ by 72$)$ and $\operatorname{Hom}_{\mathcal{P} \text { bal }}([r],[s])=0$ for $r \neq s$. Hence, the induced functor $\mathcal{F}^{\text {bal }} / \operatorname{ker}(G) \cong \mathcal{P}^{\text {bal }}$ is an equivalence of monoidal categories. The theorem follows if we can show that $\mathcal{N}^{\text {bal }}=\operatorname{ker}(G)$. Note that $\mathcal{N}^{\text {bal }}(X, Y)=0=\operatorname{ker}(G)(X, Y)$ whenever $X \neq Y$ since $\operatorname{Hom}_{\mathcal{F} \text { bal }}(X, Y)=0$ in this case. The relations (CBAL1)-(CBAL3) clearly hold for $G(s)$ and $G(p)$ and hence $\mathcal{N}^{\text {bal }}(X, X) \subset \operatorname{ker}(G)(X, X)$ for all $X \in \mathcal{F}^{\text {bal }}$. By Theorem 3.54 we have that $\operatorname{ker}(G)(X, X)$ is generated by the relations (BAL1)-(BAL3). Since these relations can be built from (CBAL1)-(CBAL3) using the monoidal structure (as in $\sqrt[73]{7}$ ), we get that $\operatorname{ker}(G)(X, X) \subset \mathcal{N}^{\text {bal }}(X, X)$ for all $X \in \mathcal{F}^{\text {bal }}$. This proves the claim.

Remark 5.14. One can give similar persentations of other diagram categories like the Brauer category $\mathcal{B}(\delta)$ or the partition category $\mathcal{P}(\delta)$. Defining a strict monoidal category by generators and relations can also be very useful when it is not obvious how to write down a diagrammatic basis with a multiplication table. For example, this can be used to define a diagrammatic version of the affine VW-algebras $W_{r}(\Xi)$ (see [RS19]) which also gives a more natural framework for the admissibility conditions from Remark 4.28 (see [RS19, Lemma 3.4]).

Remark 5.15. We can also apply the formalism of additive closures and Karoubian envelopes from Remark 5.7 to $\mathcal{P}^{\text {bal }}$. However, $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\text {bal }}\right)\right)$ should not be thought of as the category that interpolates $\operatorname{Rep}\left(S_{t}^{\text {aff }}\right)\left(\operatorname{or} \operatorname{Rep}\left(\mathcal{M}_{t}\right)\right)$ for $t \in \mathbb{C}$ since not all irreducible representations of $S_{n}^{\text {aff }}$ (resp. $\mathcal{M}_{n}$ ) appear as a direct summand of a direct sum of some $V^{\otimes r}$. Let us still say a few more words about the category $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\text {bal }}\right)\right)$. Forgetting about the monoidal structure of $\mathcal{P}^{\text {bal }}$, we have

$$
\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\mathrm{bal}}\right)\right) \stackrel{\sqrt[71 \mathrm{l}]{ }}{=} \operatorname{Kar}\left(\operatorname{Add}\left(\bigsqcup_{r \geq 0}\left(\star, P_{r}^{\mathrm{bal}}\right)\right)\right) \cong \bigoplus_{r \geq 0} \operatorname{Kar}\left(\operatorname{Add}\left(\left(\star, P_{r}^{\mathrm{bal}}\right)\right)\right)
$$

where $\left(\star, P_{r}^{\text {bal }}\right.$ ) is the category with a single object $\star$ and $\operatorname{End}(\star)=P_{r}^{\text {bal }}$. One can check that for an algebra $A$, the category $\operatorname{Add}((\star, A))$ is equivalent to the category of
free right $A$-modules of finite rank and $\operatorname{Kar}(\operatorname{Add}((\star, A)))$ is equivalent to the category of finitely-generated projective right $A$-modules (the latter follows for example from Kar08, Thm. I-6.12]). In particular, we have that

$$
\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\mathrm{bal}}\right)\right) \cong \bigoplus_{r \geq 0} \bmod -P_{r}^{\mathrm{bal}}
$$

using that the balanced partition algebra is semisimple by Proposition 3.55. Note, however, that this is not a decomposition of $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\text {bal }}\right)\right)$ as a monoidal category. Nonetheless, this shows that the category $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\text {bal }}\right)\right)$ is abelian and semisimple. Moreover, Corollary 3.60 implies that the simple objects of $\operatorname{Kar}\left(\operatorname{Add}\left(\mathcal{P}^{\text {bal }}\right)\right)$ are indexed by the set of all multipartitions (of arbitrary size).

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