Some aspects of the representation theory of limit Lie algebras

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Introduction

In the late 19th century and throughout the 20th century, the theory of finite dimensional semisimple Lie algebras and their representations has been extensively studied, and is by now well-established, see e.g. [6] or [7] for an introduction. If $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over $\mathbb{C}$, some of the classical results in this theory are:

FD1 Schur's Lemma, which says that $\dim \hom_{\mathfrak{g}}(M, M) = \mathbb{C}$ for any simple finite dimensional $\mathfrak{g}$–module M;

FD2 Weyl's complete reducibility theorem, which says that any finite dimensional $\mathfrak{g}$–module decomposes into a direct sum of simple submodules;

FD3 there exists a full classification of the finite dimensional simple $\mathfrak{g}$–modules as highest weight modules;

FD4 if M is a finite dimensional simple $\mathfrak{g}$–module, then so is its algebraic dual M^*.

In the '90s, an extension of this theory to certain limit Lie algebras, not necessarily only over $\mathbb{C}$ found its origins, see e.g. [1], [2], [3], [4], [5]. The most prominent examples of this theory are the infinite dimensional Lie algebras $sl(\infty), o(\infty)$, and $sp(\infty)$, which can be constructed as limits of the classical Lie algebras $sl(n), o(n)$, and $sp(n)$ respectively, and can thus be rightly regarded as their infinite dimensional versions. For instance, given a vector space of countable dimension V over a field k, then $sl(\infty, k)$ can be realized as the Lie subalgebra of $\mathrm{End}(V)$ consisting of matrices with trace 0, and which have only finitely many non- zero entries. One can also establish similar realizations of $o(\infty)$ and $sp(\infty)$. More precisely, there exist natural inclusions of Lie algebras $sl(n, k) \longrightarrow sl(\infty, k)$ such that the diagram

$$\begin{tikzcd} sl(n,k) \arrow[rr] \arrow[dr] & & sl(\infty,k) \\ & sl(n+1,k) \arrow[ur] & \end{tikzcd}$$

commutes, where the morphisms $sl(n, k) \longrightarrow sl(n + 1, k)$ are the canonical ones. In this scenario, we have that $sl(\infty, k)$ is the direct limit of the $sl(n, k)$, thus we have

$$sl(\infty, k) = \varinjlim sl(n, k) = \bigcup_{n \in \mathbb{N}} sl(n, k)$$

and $sl(\infty)$ will in fact be the union of all these Lie subalgebras $sl(n, k)$ with $n \in \mathbb{N}$. This already indicates that the well-known existing theory of finite dimensional semisimple Lie algebras and their representations should generalize to some extent to this infinite dimensional setting. This thesis describes several such instances, but also indicates phenomena which do not transfer nicely to our setting.

A study of representations of certain limit Lie algebras over $\mathbb{C}$, called locally semisimple Lie algebras, and in particular of the three infinite dimensional classical Lie algberas $sl(\infty), o(\infty)$, and $sp(\infty)$ is carried out in the paper [10] by Penkov and Serganova. As the category of all representations of these Lie algebras is vast, one is inclined to restrict it to something more accessible with the tools at hand. The standard approach towards doing this is by imposing certain finiteness conditions on representations, and sometimes considering weight modules. This approach is analogous to the one for finite dimensional semisimple Lie algebras, the general simple representations of which have not been fully classified, as opposed to the finite dimensional ones as FD3 listed above indicates.. The main category introduced in [10] is $\text{Int}_{\mathfrak{g}}$, namely the category of integrable $\mathfrak{g}$−modules. This category is defined with the theme of imposing finiteness conditions in mind, more precisely by considering those representations M of $\mathfrak{g}$ such that for any $g \in \mathfrak{g}$, and $m \in M$, the subspace $\text{span}\{m, g.m, g^2.m, g^3.m, \cdots\}$ of M is finite dimensional. This can be seen as the natural generalization of the finite dimensional representation theory in the semisimple case.

The aim of this thesis is to present an extensive account of the representations of what are called locally semisimple Lie algebras (denote $\mathfrak{g}$), with special focus on the classical examples mentioned above. The framework for this is laid out in [10]. The two standard approaches used here towards accomplishing this are:

i) define and study categories of representations of $\mathfrak{g}$ whose objects satisfy some finiteness condition;

ii) investigate whether there exist analogues for the classical results for finite dimensional semisimple Lie algebra representations in these categories, in particular FD1-FD4 listed above.

Following i) we define explicitly a category $\text{Loc}_{\mathfrak{g}}$ consisting of very naturally arising representations of $\mathfrak{g}$, called *local modules*. This category makes precise a class of modules which goes back to [10]. We will prove the following result which gives a nice characterization of the simple objects of $\text{Int}_{\mathfrak{g}}$:

Theorem 2.8. *All simple objects of $\text{Int}_{\mathfrak{g}}$ are local modules.*

We also give a description of $\text{Loc}_{\mathfrak{g}}$ in terms of $\text{Int}_{\mathfrak{g}}$, namely:

Theorem 2.9. *The countable integrable $\mathfrak{g}$−modules are precisely the local $\mathfrak{g}$−modules.*

In Subsection 1.2.3 we show that there exists a Schur's Lemma for a class of simple objects of $\text{Loc}_{\mathfrak{g}}$. In Subsection 2.1.3 we generalize this statement to all of the simple objects of $\text{Int}_{\mathfrak{g}}$. This way we establish that FD1 listed above generalizes nicely to our infinite dimensional setting.

In Subsection 1.2.4 we show that already $\text{Loc}_{\mathfrak{g}}$ contains objects that are not semisimple, so we see that in particular FD2 fails to generalize to our new setting even in a category as reasonable as $\text{Loc}_{\mathfrak{g}}$. In Example 2.5 we provide another example of a non-splitting short exact sequence in $\text{Int}_{\mathfrak{g}}$, and actually show that the example there is of an integrable $\mathfrak{g}$−module that is of uncountable dimension. This indicates that even though $\text{Int}_{\mathfrak{g}}$ has been defined to resemble the finite dimensional representations of semisimple Lie algebras, there still exist very large objects in this category.

In [10], via tools of homological algebra, it is shown that $\text{Int}_{\mathfrak{g}}$ contains enough injectives. In particular it is shown there that given a module $M \in \text{Int}_{\mathfrak{g}}$, then there exists an integrable submodule of M^*, denoted by $\Gamma_{\mathfrak{g}}(M^*)$ which is an injective object of $\text{Int}_{\mathfrak{g}}$. In Subsection 2.2.1 we show that for $M \neq 0$, the injective object $\Gamma_{\mathfrak{g}}(M^*)$ of $\text{Int}_{\mathfrak{g}}$ will also be non-zero.

Note however that as will be indicated in the following paragraph, $\mathrm{Int}_{\mathfrak{g}}$ is not closed under algebraic dualization.

In Section 2.3 we give an account of weight $\mathfrak{g}$−modules with respect to a splitting Cartan subalgebra $\mathfrak{h}$, denoted by $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$. Particular emphasis will be put on the integrable weight $\mathfrak{g}$−modules, the category consisting of which we denote by $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$. For objects M of $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ one can define a restricted dual $M^{\vee} := \bigoplus_{\lambda\in\mathfrak{h}}(M_{\lambda})^{*}$. In Subsection 2.3.3 we show that if M is an integrable weight module, whose weight spaces are all of finite dimension, then $M^{\vee}$ will also be integrable. We also show that if this M is simple, $M^{\vee}$ is simple as well. Following [10], we then define the category $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{fin}}$ which consists of those integrable weight $\mathfrak{g}$−modules whose weight spaces are finite dimensional. In particular we prove the following remarkable result.

Theorem 2.32. *The category $Int_{\mathfrak{g}}^{fin}$ is semisimple.*

In this way we establish a category of $\mathfrak{g}$−modules where FD2 listed above holds.

To see whether the other two classical results listed above, namely FD3 and FD4 generalize to our infinite dimensional setting, turns out to be a rather challenging task. In Example 2.2 we construct a local module whose dual is not integrable. This shows that FD4 does not translate well even to $\mathrm{Loc}_{\mathfrak{g}}$, while FD3 is in general a difficult question even for infinite dimensional simple representations of finite dimensional semisimple Lie algebras.

Chapter 3 is dedicated towards seeing how FD4 can be generalized in our infinite dimensional setting, namely we study the integrability of duals of integrable modules. In Section 3.1 we state and prove a theorem from [10] which gives a necessary and sufficient condition for M^* to be integrable when $M \in \mathrm{Int}_{\mathfrak{g}}$. This condition is quite tedious to check in practice, but it is useful in showing that the property of having an integrable algebraic dual is closed under tensor products, and algebraic dualization. The approach we take in studying the integrability of the dual is the one laid out in i) above, namely we impose some finiteness condition on the $\mathfrak{g}$−modules M. More precisely, in Section 3.2 we introduce and study the socle functors, and the socle filtration of a $\mathfrak{g}$−module. Using the tools developed there, in Subsection 3.2.3 we show that for a certain class of simple integrable modules Q, their dual Q^* will contain a unique simple submodule.

In Section 3.3 we further investigate properties of the socle functors, and we show in particular that the property of having finite socle length is preserved under many algebraic operations, e.g. arbitrary direct sums, quotients, and extensions. In particular, in Corollary 3.25 we give a computational result on quotients of objects in the socle filtration of a $\mathfrak{g}$−module. The work done in Section 3.3 lays out the groundwork which will be used in Section 3.4 in studying the final category to be mentioned in this thesis, namely $\mathrm{Tens}_{\mathfrak{g}}$, where $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$. In particular we prove the following result.

Theorem 3.30. *$Tens_{\mathfrak{g}}$ is the largest full subcategory of $Int_{\mathfrak{g}}$ closed under algebraic dualization, and such that every object in it has finite socle length.*

In particular we show that this category $\mathrm{Tens}_{\mathfrak{g}}$ contains the natural and conatural representations of $\mathfrak{g}$. Thus, we exhibit a very reasonable category where the classical result listed as FD4 above translates very nicely.

We also prove the following result, which shows that in $\mathrm{Tens}_{\mathfrak{g}}$, FD3 obtains a nice generalization:

Corollary 3.46. *Every simple object of $Tens_{\mathfrak{g}}$ is a $\mathfrak{b}$−highest weight module.*

In Subsection 3.4.4 we give a partial account of the injective objects of $\mathrm{Tens}_{\mathfrak{g}}$. In particular we show that if $M \in \mathrm{Tens}_{\mathfrak{g}}$, and I_M is an injective hull of M in $\mathrm{Int}_{\mathfrak{g}}$, then $I_M \in \mathrm{Tens}_{\mathfrak{g}}$ as well.

One should observe that thus far we have not mentioned the category of finite dimensional representations of $\mathfrak{g}$. This is no accident, as using the theory developed throughout the thesis, and especially the content of Section 3.4, we prove the following result.

Theorem 3.54. *Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$. Let $M \in \mathfrak{g}$−mod be finite dimensional. Then M is a trivial $\mathfrak{g}$−module.*

One can interpret this as that for the classical locally semisimple Lie algebras, their non-trivial representation theory is infinite dimensional.

This thesis was based mostly on the paper [10] by Penkov and Serganova, and Section 3.4 uses developments in [11] as well. The goal has been to study the material in [10] and to present a detailed account of it which should make the material more accessible to the reader. Most of the main results stated and proved here, all come from [10]. Some novelties which appear here, if one can call them such, are the more concise treatment of the countable integrable modules, for which we define their own category $\mathrm{Loc}_{\mathfrak{g}}$; the treatment of certain functors $\mathfrak{g}-\mathrm{mod} \longrightarrow \mathfrak{g}-\mathrm{mod}$ in Section 1.3 which creates a template for studying many such functors to appear throughout the thesis; and the treatment in Section 3.3 where we study the socle filtration of $\mathfrak{g}$−modules. In particular, via examples, we have also tried to put an emphasis in comparing the different categories of $\mathfrak{g}$−modules that appear here.

While the structuring and proofs in some parts of the thesis are original, they mainly came to fruition by trying to understand the proofs in [10].

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Notation and conventions

In this thesis, by $\mathbb{N}$ we will denote the set of natural numbers i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$, and by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. All the vector spaces appearing in this thesis, i.e. also Lie algebras, and representations of Lie algebras, are over the base field $\mathbb{C}$. As a convention, by *countable set* we will mean a set S which injects into $\mathbb{N}$.

In some occasions, we will use results from well-known theories. As indicated in the Introduction, in a few instances we will make comparisons of the results appearing in this thesis with results of the already known theory of finite dimensional Lie algebras. We will adopt the following notation for these two types of notes appearing here:

ref. result X.	=	results from well-known theories
Comparison remark X.	=	notes comparing the finite and infinite dimensional theories

For results denoted by 'ref. result' we will cite a source where one can find proofs of them, but we will not present those proofs here. For notes denoted by 'Comparison remark' we will not provide proofs, but they will rather be presented as logical conclusions/summaries of the discussions preceding them. These results will be numbered independently from the rest of the results in this thesis, so as to not interfere with the continuation of the exposition of the main contents.

Chapter 1

Locally semisimple Lie algebras, and local modules

In this chapter, we will introduce a particular class of infinite Lie algebras, namely locally semisimple Lie algebras, the representations of which are the main object of study in this thesis. We then introduce a category of naturally arising representations of these Lie algebras, namely $\mathrm{Loc}_{\mathfrak{g}}$, and we will see in Chapter 2 that this category can be nicely characterized in terms of a larger category of $\mathfrak{g}$-modules $\mathrm{Int}_{\mathfrak{g}}$. Furthermore, we give an account of a class of simple objects of $\mathrm{Loc}_{\mathfrak{g}}$, and we also compare how some classical results on the finite dimensional representations of finite dimensional Lie algebras translate to our situation. In particular, in Subsection 1.2.3 we see that an analogue of Schur's Lemma holds for class of simple objects in $\mathrm{Loc}_{\mathfrak{g}}$. In the next chapter, more precisely in Subsection 2.1.3, we will see that Schur's Lemma actually holds for all simple objects of $\mathrm{Loc}_{\mathfrak{g}}$. We also see that not all the short exact exact sequences in this category split in Subsection 1.2.4, meaning that $\mathrm{Loc}_{\mathfrak{g}}$ is not semisimple.

In Section 1.3 we prove result which will be useful throughout the thesis in order to show that certain subcategories of $\mathfrak{g}$−mod have enough injectives.

We conclude this chapter with an overview of the universal enveloping algebra in Section 1.4.

The exposition here is based on the early part of [10].

1.1 Locally semisimple Lie algebras

We start off with the most fundamental definition of this thesis, the notion of local Lie algebras. Let $(\mathfrak{g}_i, a_i)_{i\in\mathbb{N}}$ be a direct system in the category Lie of Lie algebras over $\mathbb{C}$, with $\mathfrak{g}_i$ finite dimensional Lie algebras, and $a_i : \mathfrak{g}_i \longrightarrow \mathfrak{g}_{i+1}$ injective Lie algebra homomorphisms. One can think of this direct system as a sequence of inclusions

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n \subset \cdots. \tag{1.1}$$

The limit of such a direct system $\mathfrak{g}$ in $\mathrm{Lie}_{\mathbb{C}}$ is called a *local* Lie algebra, and a sequence like (1.1) is called an *exhaustion* of $\mathfrak{g}$. As $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ and the maps a_i are inclusions, we have $\mathfrak{g} = \cup_i \mathfrak{g}_i$. The general class of local Lie algebras is too broad for the purposes of this thesis, so our focus here will be on the following type of local Lie algebras:

Definition 1.1. A local Lie algebra which admits an exhaustion as in (1.1), with all $\mathfrak{g}_i$

being semisimple, is called a *locally semisimple* Lie algebra. If all the $\mathfrak{g}_i$ are simple, we call $\mathfrak{g}$ a *locally simple* Lie algebra.

Whenever we work with a locally semisimple Lie algebra, we will always assume that it comes with an exhaustion as in (1.1), without necessarily mentioning it.

A standard example for a local Lie algebra is $\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(i)$, where the injections $\mathfrak{gl}(i) \longrightarrow \mathfrak{gl}(i+1)$ are the canonical ones.

Remark 1.1. Note that by taking a finite dimensional semisimple Lie algebra $\mathfrak{g}$, and setting $\mathfrak{g}_i = \mathfrak{g}$ for all i, we see that all finite dimensional semisimple Lie algebras are also locally semisimple. This means that whatever theory will be developed here, one can apply it to finite dimensional semi-simple Lie algebras as well.

We will not be too concerned with the locally semisimple Lie algebras themselves, but rather we will study their representations. However, we do give a couple of results which show that, to an extent, these locally semisimple Lie algebras do behave in a similar fashion to their finite dimensional counterparts.

Proposition 1.1. *Let $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ be a local Lie algebra, and $\mathfrak{a} \subset \mathfrak{g}$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ if and only if there exist ideals $\mathfrak{a}_i \subset \mathfrak{g}_i$ such that $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$ and $\mathfrak{a} = \varinjlim \mathfrak{a}_i$.*

Proof. If $\mathfrak{a} = \varinjlim \mathfrak{a}_i$, then given any $a \in \mathfrak{a}$ and $g \in \mathfrak{g}$, one can find some $i, j \in \mathbb{N}$ such that $a \in \mathfrak{a}_i$ and $g \in \mathfrak{g}_j$. If $k > i, j$, we will have $a \in \mathfrak{a}_k$ and $g \in \mathfrak{g}_k$, and as $\mathfrak{a}_k$ is an ideal in $\mathfrak{g}_k$, we have $[a, g] \in \mathfrak{a}_k \subset \mathfrak{a}$. So $\mathfrak{a} \subset \mathfrak{g}$ will be an ideal.

Conversely, let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Set $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{g}_i$. Then $\mathfrak{a}_i \subset \mathfrak{g}_i$ will be ideals, and $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$, so $\varinjlim \mathfrak{a}_i$ will be an ideal of $\mathfrak{g}$ by the previous part of the proof. But for any $a \in \mathfrak{a}$, we have some k such that $a \in \mathfrak{g}_k \Rightarrow a \in \mathfrak{a} \cap \mathfrak{g}_k = \mathfrak{a}_k$, hence $\mathfrak{a} = \varinjlim \mathfrak{a}_k$. □

Let $\mathfrak{g}$ be a local algebra, and let $\mathrm{rad}(\mathfrak{g}) = \mathfrak{r} \subset \mathfrak{g}$ be the largest solvable ideal of $\mathfrak{g}$. In the spirit of [6], call $\mathfrak{g}$ semisimple if $\mathfrak{r} = 0$, and call $\mathfrak{g}$ simple if its only ideals are 0 and $\mathfrak{g}$ itself. Then from Proposition 1.1 we have the immediate

Corollary 1.2. *A locally semisimple (simple) Lie algebra is semisimple (simple).*

Remark 1.2. While from Corollary 1.2 we see that a locally simple Lie algebra is a simple local Lie algebra, the converse does not hold. In [1] one can find constructions of local Lie algebras that are simple, but do not admit an exhaustion as in (1.1) with all $\mathfrak{g}_i$ simple.

We now define three certain locally simple Lie algebras, which are extensions of the classical finite dimensional Lie algebras.

Definition 1.2. The three classical locally simple Lie algebras are $sl(\infty) := \cup sl(i)$, $o(\infty) := \cup o(i)$, and $sp(\infty) := \cup sp(2i)$, and are respectively called the *infinite special linear, orthogonal*, and *symplectic Lie algebra.*

The theory presented in this thesis will apply to general locally semisimple Lie algebras. In the later parts of Chapter 3, the exposition will be unique to the three classical locally simple Lie algebras. We shall be careful to always distinguish the type of locally semisimple Lie algebras the discussion is about.

Call a locally semisimple Lie algebra *diagonal* if it admits an exhaustion (1.1) with all $\mathfrak{g}_i$ being classical Lie algebras, and for any $i \in \mathbb{N}$ there exist non-negative integers k_i, l_i, s_i such that $V_{i+1}|_{\mathfrak{g}_i} = k_i V_i \oplus l_1 V_i^* \oplus \mathbb{C}^{s_i}$, where V_i is the natural representation of $\mathfrak{g}_i$.
If V_i is the natural representation of $sl(i)$, then one has $V_{i+1}|_{\mathfrak{sl}(i)} = V_i \oplus \mathbb{C}$ so one can see that $sl(\infty)$ is diagonal. Similarly one sees that $o(\infty)$ and $sp(\infty)$ are also diagonal. There exists a full classification of diagonal locally simple Lie algebras, see [5], and while the classical

examples in Definition 1.2 fit this description, they are not the only ones. For instance $sl\left(2^{\infty}\right) = \cup_{i\in\mathbb{N}} sl\left(2^{i}\right)$ with the inclusion maps given by

$$sl\left(2^{i}\right) \longrightarrow sl\left(2^{i+1}\right); \qquad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A, \end{pmatrix}$$

is a locally simple diagonal Lie algebra non-isomorphic to $sl(\infty), o(\infty)$ or $sp(\infty)$.

Call a locally semisimple Lie algebra $\mathfrak{g}$ *finitary* if it admits a faithful representation M of countable dimension, and a map $\pi : \mathfrak{g} \longrightarrow \cup_{m,n}\mathrm{Mat}_{m\times n}(\mathbb{C})$ such that there exists a basis $\{m_1, m_2, \dots\}$ of M, for which $g.m_i = \pi(g)m_i$, for all $g \in \mathfrak{g}$ and $i \in \mathbb{N}$, i.e. $\mathfrak{g}$ acts on a basis of M via finite matrices. It is clear that $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ act on their natural representations V (to be introduced in Example 1.2 below) via finite matrices, so they are finitary. It actually turns out that these are the only finitary locally simple Lie algebras, [4], [3].

1.2 Local modules

The focus in this thesis will not be on studying the structure theory of locally semisimple Lie algebras, but rather the study of their representation theory. We will be particularly interested in the three classical locally semisimple Lie algebras $sl(\infty), sp(\infty), o(\infty)$. We start by defining a naturally arising class of $\mathfrak{g}$-modules.

1.2.1 Definition and examples

Definition 1.3. Call a $\mathfrak{g}$-module M a *locally finite* module (or just a *local module* for convenience) if there exists an exhaustion (1.1) of $\mathfrak{g}$ and finite dimensional $\mathfrak{g}_i$-modules M_i such that $M_i \subset M_{i+1}|_{\mathfrak{g}_i}$ as $\mathfrak{g}_i$-modules and $M = \varinjlim M_i$. We call $\{M_i\}_{i\in\mathbb{N}}$ an *exhaustion* for M. If all M_i can be chosen to be simple $\mathfrak{g}_i$-modules, we call M *locally simple.*

Given a local $\mathfrak{g}$-module M with exhaustion $\{M_i\}_{i\in\mathbb{N}}$, one can write the inclusions $M_i \subset M_{i+1}|_{\mathfrak{g}_i}$ simultaneously, and just think of it as a sequence of inclusions

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots,$$

with M_i finite dimensional $\mathfrak{g}_i$-submodules of M, and then clearly $M = \bigcup_{i\in\mathbb{N}} M_i$.

Remark 1.3. It is clear that as M_i are finite dimensional, and M is a union of countably many finite dimensional $\mathfrak{g}$–modules, then M will be of countable dimension. In Theorem 2.9 we will see that countable dimension is the characterizing property of local modules in the context of what are called integrable modules, defined in Chapter 2.

Denote now by $\mathrm{Loc}_{\mathfrak{g}}$ the largest full subcategory of $\mathfrak{g}$-mod consisting of local modules. Remark 1.3 shows all objects of this category are of countable dimension.

Example 1.1.

a) Let $\mathfrak{g} = sl(\infty)$. We have seen earlier that if V_i is the natural representation of $sl(i)$, then $V_{i+1}|_{\mathfrak{g}_i} = V_i \oplus \mathbb{C}$, thus we have natural inclusions $V_i \longrightarrow V_{i+1}$. Then the local module defined by $V := \varinjlim V_i$ is called the *natural representation* of $sl(\infty)$. As all these V_i's are simple, it is clear that this natural module is a locally simple $\mathfrak{g}$-module.

b) Again if $\mathfrak{g} = sl(\infty)$, we have $V_{i+1}|_{\mathfrak{g}_i} = V_i \oplus \mathbb{C}$. If we dualize this, as everything is finite dimensional, we get $V^*_{i+1}|_{\mathfrak{g}_i} = V^*_i \oplus \mathbb{C}$, hence we also have natural injections $V^*_i \longrightarrow V^*_{i+1}$. Then the local module defined by $V_* := \varinjlim V^*_i$ is called the *conatural representation* of $sl(\infty)$. As V_i's are simple, so are V^*_i, hence V_* will be a locally simple module as well.

c) Given any locally semisimple Lie algebra with exhaustion as in (1.1), one can consider the *adjoint representation* ad$\mathfrak{g}$ given by the same exhaustion as $\mathfrak{g}$.

We note that one can similarly define the natural and conatural representations for $o(\infty)$ and $sp(\infty)$ as well. These representations will play a very important part in Chapter 3 in the study of a particular class of simple $\mathfrak{g}$-modules.

1.2.2 Simple objects in $\mathbf{Loc}_{\mathfrak{g}}$

Now we proceed to investigate the nature of local modules, and the category $\mathrm{Loc}_{\mathfrak{g}}$. In particular, we will give an account of a certain type of simple objects in $\mathrm{Loc}_{\mathfrak{g}}$. Let us begin with the following result, similar in nature to Proposition 1.1.

Proposition 1.3. *Let L be a locally simple $\mathfrak{g}-$module, with exhaustion $\{L_i\}_{i\in\mathbb{N}}$. Then L is a simple $\mathfrak{g}-$module.*

Proof. Let $N \subset L$ be a non-zero $\mathfrak{g}$–submodule, say with exhaustion $\{N_i\}_{i\in\mathbb{N}}$. As $L = \cup_{i\in\mathbb{N}} L_i$ and $N_1 \cap L = N_1 \neq 0$, we have that there exists some natural number i such that $N_1 \cap L_i \neq 0$. Note now that for any $k > i$, since $N_1 \subset N_k$, we get $0 \neq (N_1 \cap L_i) \subset (N_k \cap L_k)$. As L_k is a simple $\mathfrak{g}_k-$module, we get that $L_k \subset N_k$ for all $k > i$. So in particular we get

$$N = \bigcup_{i\in\mathbb{N}} N_i = \bigcup_{k>i} N_k = \bigcup_{k>i} L_k \supset \bigcup_{i\in\mathbb{N}} L_i = L,$$

hence we have $L = N$. Thus the only $\mathfrak{g}-$submodules of L are L itself, and the zero module, hence L is a simple $\mathfrak{g}-$module. □

Remark 1.4. Similar to 1.3, the converse of Proposition 1.3 does not hold in general, i.e. there exist simple objects of $\mathrm{Loc}_{\mathfrak{g}}$ which are not locally simple. See [2],[1].

Remark 1.5. In the category of $\mathbb{Z}-$modules, the $\mathbb{Z}-$module $\mathbb{Z}$ contains no simple submodule. Indeed, the $\mathbb{Z}-$submodules of $\mathbb{Z}$ are all of the form $k\mathbb{Z}$ for $k \in \mathbb{N}_0$. Given any submodule $k\mathbb{Z}$ one sees that, for instance, $(2k)\mathbb{Z}$ is a proper non-zero submodule. In other words, the category $\mathbb{Z}-$mod contains objects which have no simple submodules. The same is true for $\mathrm{Loc}_{\mathfrak{g}}$, and for $\mathfrak{g} = sl(\infty)$, one can construct such examples via the branching rules for $sl(n)$.

As per the previous two remarks, in this Chapter we will be focusing on the locally simple objects of $\mathrm{Loc}_{\mathfrak{g}}$. We now give a result which describes what kind of category $\mathrm{Loc}_{\mathfrak{g}}$ is.

Proposition 1.4. *$\mathrm{Loc}_{\mathfrak{g}}$ is closed under taking submodules, quotients, finite direct sums, tensor products, and extensions. In particular, $\mathrm{Loc}_{\mathfrak{g}}$ is an abelian subcategory of $\mathfrak{g}-$mod.*

Instead of giving a direct argument we will however deduce these properties from a similar statement for larger category of $\mathfrak{g}$-modules to be introduced in Chapter 2 (Proposition 2.3).

Given a local $\mathfrak{g}$-module M, define a relation in the class of its exhaustions by saying $\{M_i\} \sim \{M'_i\}$ if there exists some natural n such that $M_i = M'_i$ for all $i > n$. One can easily see that this $\sim$ is an equivalence relation. If M admits only one exhaustion up to '$\sim$', we will say that M *admits an essentially unique* exhaustion. The following result gives shows that all locally simple objects of $\mathrm{Loc}_{\mathfrak{g}}$ have this property.

Proposition 1.5. *Every locally simple object L of $\mathrm{Loc}_{\mathfrak{g}}$ admits an essentailly unique exhaustion $\{L_i\}_{i\in\mathbb{N}}$ with L_i simple $\mathfrak{g}_i-$modules.*

Proof. Let $L \in \mathrm{Loc}_{\mathfrak{g}}$ be locally simple, with an exhaustion $\{L_i\}_{i\in\mathbb{N}}$ of simple $\mathfrak{g}_i-$modules. Let now $\{L'_i\}_{i\in\mathbb{N}}$ be a different exhaustion of L, with L'_i simple $\mathfrak{g}_i-$modules. We have

$$L = \bigcup_{i\in\mathbb{N}} L_i = \bigcup_{i\in\mathbb{N}} L'_i. \tag{1.2}$$

Set $N_i := L_i \cap L'_i$. We clearly have $N_i \subset N_{i+1}$, so by setting $N = \varinjlim N_i$, we get that $N \subset L$ is a submodule. From (1.2) we see that for some i we have $L_i \cap L'_i \neq 0$, hence we get $N \neq 0$. From the proof of Proposition 1.3 we see that $N = L$, and there exists some $i \in \mathbb{N}$ such that $N_k \supset L_k$ for all $k > i$. Since both L_k and L'_k are simple, we get that $L_k = L'_k$ for all $k > i$, i.e. get $\{L_i\}_{i\in\mathbb{N}} \sim \{L'_i\}_{i\in\mathbb{N}}$, which proves the claim of the proposition. □

From this result, given a locally simple module L, we can always assume that it comes along with its *unique* exhaustion $\{L_i\}_{i\in\mathbb{N}}$ consisting of simple $\mathfrak{g}_i$–modules. For convenience, we will call such an exhaustion *the simple exhaustion* of L.

1.2.3 Schur's Lemma for locally simple modules

The following result shows that morphisms from a locally simple module to a local modules behave locally in a nice way.

Proposition 1.6. *Let $L, M \in Loc_{\mathfrak{g}}$, with L locally simple. Let $\{L_i\}$ be the simple exhaustion of L, and $\{M_i\}$ an exhaustion for M. Then for any non-zero morphism $f : L \to M$ there exists some $n \in \mathbb{N}$ such that $f(L_i) \subset M_i$ for all $i > n$, and $f = \varinjlim f_i$, where $f_i := f|_{L_i}$.*

Proof. As L is simple from Proposition 1.3, f is injective. Looking at f as a morphism of $\mathfrak{g}_i$–modules, we see that $f|_{L_i} : L_i \longrightarrow f(L_i)$ will still be injective. This clearly means that $f(L_i) \cong L_i$ as $\mathfrak{g}_i$–modules. Now for each $i \in \mathbb{N}$ set

$$s(i) := \min\{j \in \mathbb{N} \mid i \leq j \text{ and } f(L_i) \subset M_j\}$$

Note that $i \leq s(i) \leq s(i+1)$. Assume that for all i we have $i < s(i)$. In particular we have $1 < s(1)$, and there exists $i(= s(1))$ for which $s(1) < s(i)$. Let now $1, 2, \ldots, k$ be such that $s(1) = s(k) < s(k+1)$. This means that $f(L_k) \subset M_{s(k)}$, $f(L_{k+1}) \notin M_{s(k)}$, and $f(L_{k+1}) \subset M_{s(k+1)}$. Since $i < s(i)$ for all i, we have $s(k+1), s(k) \geq k+1$ so $M_{s(k)}$ and $M_{s(k+1)}$ are also $\mathfrak{g}_{k+1}$-modules. But this would imply that $T = \big(f(L_{k+1}) \cap M_{s(k)}\big)$ is a $\mathfrak{g}_{k+1}$–submodule of $f(L_{k+1})$ that is non-zero, because $f(L_k) \subset T$, and $T \neq f(L_{k+1})$. Since $f(L_{k+1}) \cong L_{k+1}$ is simple, this is impossible, thus the assumption that $i < s(i)$ for all $i \in \mathbb{N}$ is wrong.

Let now i such that $s(i) = i$. Now if $s(i+1) > i+1$, we would have $f(L_{i+1}) \notin M_{i+1}$, thus we $f(L_i) \subset T = (f(L_{i+1} \cap M_{i+1}) \neq f(L_{i+1})$, i.e. $T \subset f(L_{i+1})$ would be a non-trivial $\mathfrak{g}_{i+1}$-submodule of $f(L_{i+1}) \cong L_{i+1}$, which is impossible as L_{i+1} is a simple $\mathfrak{g}_{i+1}$–module. So we must have $s(i+1) = i+1$. The conclusion then follows by induction on i. □

We now use Proposition 1.6 to prove that Schur's Lemma holds for locally simple $\mathfrak{g}$–modules.

Corollary 1.7. *(Schur's Lemma for locally simple $\mathfrak{g}$-modules) Let $L, N \in Loc_{\mathfrak{g}}$ be simple. Then*

$$\hom_{\mathfrak{g}}(L, N) = \begin{cases} \mathbb{C} & \textit{if } M \cong N, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Note first that given a map $f : L \longrightarrow N$, we have that $\ker f \subset L$ is a submodule, hence $\ker f = 0$ or $\ker f = L$. If L and N are not isomorphic, we cannot have $\ker f = 0$, as then $L \cong f(L) \subset N$ would be a non-trivial submodule of N, which cannot happen due to the simplicity of N. Hence we have $f(L) \neq 0$, which implies $f(L) = N$, i.e. $f = 0$. Therefore $\hom_{\mathfrak{g}}(L, N) = 0$ if $L \ncong N$.

Assume now that $L \cong N$. Here it suffices to check $\hom_{\mathfrak{g}}(L, L)$. Let $\{L_i\}_{i\in\mathbb{N}}$ be the exhaustion of L. Let $f : L \longrightarrow L$ be a non-zero map. As $\ker f \neq L$, we must have $\ker f = 0$. From Proposition 1.6, there exists some $n \in \mathbb{N}$ such that for all $i > n-1$ we have $f(L_i) \subset L_i$, and $f = \varinjlim f_i$. Clearly the maps $f_i := f|_{L_i} : L_i \longrightarrow L_i$ are such that $\ker f_i \neq 0$, in particular f_i are non-zero maps from L_i to itself as a $\mathfrak{g}_i$-module. From Schur's Lemma for the finite dimensional case, we get that there exist some $c_i \in \mathbb{C}$ such that $f_i(m) = c_i m$ for all $m \in L_i$. Let now $m \in L_n$ be some non-zero element. We then have $c_n m = f_n(m)$. Looking at m as an element in L_i for $i > n$ we have $f_i(m) = c_i m$. As $f = \varinjlim f_i$, we get $f(m) = f_n(m) = f_i(m)$. So there exists some $c \in \mathbb{C}$ such that $f_i(m) = cm$ for all $i > n$, and in particular $f(m) = cm = c\mathrm{Id}_M$. Thus we obtain

$$\hom_{\mathfrak{g}}(L, L) = \mathbb{C},$$

which is what we wanted to show. □

Remark 1.6. Here we have shown that Schur's Lemma holds for locally simple $\mathfrak{g}$–modules. From Remark 1.2 we know that there exist simple local modules which are not locally simple. The proof presented above uses Proposition 1.6 which is a statement about locally simple modules only, thus it does not work for general simple objects of $\mathrm{Loc}_{\mathfrak{g}}$. However, in Chapter 2 we will introduce a category denoted by $\mathrm{Int}_{\mathfrak{g}}$, and Corollary 2.4 will show that $\mathrm{Loc}_{\mathfrak{g}}$ is a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$. In Subsection 2.1.3 we will show that Schur's Lemma holds for simple modules in $\mathrm{Int}_{\mathfrak{g}}$, and thus also for all simple objects of $\mathrm{Loc}_{\mathfrak{g}}$.

1.2.4 Splitting of short exact sequences in $\mathrm{Loc}_{\mathfrak{g}}$

We now prove a result which shows a general construction of non-splitting short exact sequences in $\mathrm{Loc}_{\mathfrak{g}}$.

Proposition 1.8. *Let $M \in \mathrm{Loc}_{\mathfrak{g}}$ with exhaustion $\{M_i\}_{i\in\mathbb{N}}$, and assume that there exists some natural number n and simple $\mathfrak{g}_i$-submodules $L_i \subset M$ with $i > n$ such that $\dim \hom_{\mathfrak{g}_i}(L_i, L_{i+1}) > 2$. Then there exists a locally simple module $L = \varinjlim L_i \in \mathrm{Loc}_{\mathfrak{g}}$, and a local module $Z \in \mathrm{Loc}_{\mathfrak{g}}$ which fit into a non-splitting short exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} Z \xrightarrow{\beta} L \longrightarrow 0 \tag{1.3}$$

Proof. Denote by $e_i : M_i \longrightarrow M_{i+1}$ the structure maps for M. Fix a sequence of non-zero $\mathfrak{g}_i$-module maps $f_i : L_i \longrightarrow L_{i+1}$, which will naturally be injective, and let $L = \varinjlim L_i$ with respect to these maps. Let now $t_i : L_i \longrightarrow M_{i+1}$ be a sequence of injective maps. Set $Z_i := L_i \oplus M_i$ as $\mathfrak{g}_i$-modules, and define $a_i : Z_i \longrightarrow Z_{i+1}$ by

$$a_i((x, m)) := (f_i(x), t_i(x) + e_i(m)) \tag{1.4}$$

for $x \in L_i$ and $m \in M_i$, and set $Z = \varinjlim Z_i$. As $a_i((0, m)) = (0, e_i(m))$, we can see that $M \subset Z$ is a submodule, and the map $Z \longrightarrow L$ given by $z = (x, m) \longmapsto x$ is well defined, has kernel M, and is surjective. Thus M, Z and L indeed fit in a short exact sequence as in (1.3). If this sequence splits, we then have $\gamma : L \longrightarrow Z$ such that $\beta \circ \gamma = \mathrm{id}_X$. For any $x \in X$ we have $\gamma(x) \in Z$, i.e. there exists some $x' \in L$ and $m \in M$ such that $\gamma(x) = (x', m)$. Since $\beta \circ \gamma = \mathrm{id}_L$, we must have $x = x'$. Denote this m corresponding to x by $p(x)$, so we obtain a map $p : L \longrightarrow M$. An easy check will show that this p is actually a morphism of $\mathfrak{g}$–modules. From Proposition 1.6 we get that there exist morphisms $p_i : L_i \longrightarrow M_i$ of $\mathfrak{g}_i$- modules such that $p = \varinjlim p_i$ for large enough i. Then for $x \in L_i$ we have $\gamma(x) = (x, p(x)) = (x, p_i(x))$. Now since $x = f_i(x) \in L$, we have $\gamma(x) = \gamma(f_i(x))$, i.e.

$$(x, p_i(x)) = (f_i(x), p_{i+1}(f_i(x))).$$

As the left hand side of this equality lies in Z_i and the right hand side in Z_{i+1}, we obtain

$$(f_i(x), p_{i+1}(f_i(x)) = \gamma(f_i(x)) = a_i(\gamma(x)) = a_i(x, p_i(x)) = (f_i(x), t_i(x) + e_i(p_i(x))),$$

which in particular from the second coordinate gives us

$$t_i(x) = p_{i+1}(f_i(x)) - e_i(p_i(x)),$$

i.e.

$$t_i = p_{i+1} \circ f_i - e_i \circ p_i. \tag{1.5}$$

Assume now that for any choice of $\{t_i\}$'s, our sequence (1.3) splits. Set now $n_i :=$ $= \dim \hom_{\mathfrak{g}_i}(L_i, M_i)$. As f_i and e_i are fixed, (1.5) gives us

$$\dim \hom_{\mathfrak{g}_i}(L_i, M_{i+1}) \leq n_i + n_{i+1}.$$

But on the other hand, if we set $k_i := \dim \hom_{\mathfrak{g}_i}(L_i, L_{i+1})$ we have $\dim \hom_{\mathfrak{g}_i}(L_i, M_{i+1}) \geq k_i n_i$. But then $n_i + n_{i+1} \geq k_i n_{i+1}$ which gives us $n_i \geq (k_i - 1)n_{i+1} > n_{i+1}$, as $k_i > 2$, and since $n_i > 0$ for all i, this is not possible, hence a contradiction. Thus there exists a choice of injections $\{t_i\}$ such that (1.3) is non-split. This completes the proof of the proposition. $\square$

Remark 1.7. If $\mathfrak{g} = sl(\infty)$ for instance, using the branching rules for $sl(n)$ one can construct finite dimensional $sl(n)$–modules (even simple ones) M_n with $M_n \subset M_{n+1}|_{sl(n)}$, such that there exist simple $sl(n)$–submodules $L_n \subset M_n$ with

$$\dim \hom_{sl(n)}(L_n, L_{n+1}) > 2.$$

Then by setting $M = \varinjlim M_n$, we obtain an $sl(\infty)$–module which satisfies the conditions of Proposition 1.8, and thus one can produce non-split short exact sequences in $\mathrm{Loc}_{sl(\infty)}$.

We now provide a simple explicit example of a non-split short exact sequence in $\mathrm{Loc}_{\mathfrak{g}}$.

Example 1.2. Let $\mathfrak{g} = sl(\infty)$, and V, V_* respectively be the natural and conatural representations for $sl(\infty)$, and consider the module $M = V \otimes V_*$. Let $\{v_1, v_2, \ldots\}$ be the natural basis of V. Let now $m \in V \otimes V_*$ be any non-zero element. As $V \otimes V_* = \varinjlim V_i \otimes (V_i)^*$, let $n \in \mathbb{N}$ be such that $m \in V_n \otimes (V_n)^*$. Then there exist $a_i \in \mathbb{C}$ and $f_i \in V_n^*$ such that

$$m = \sum_{i=1}^{n} a_i (v_i \otimes f_i).$$

Without loss of generality, assume that $a_n, f_n \neq 0$. Set now $g = E_{n+1,n} \in sl(n+1)$, i.e. the matrix with all entries equal to 0, except for the one in the $n+1$-th row and n-th column, which is equal to 1. We have

$$g.m = g.\sum_{i=1}^{n} a_i(v_i \otimes f_i) = \sum_{i=1}^{n} a_i g.(v_i \otimes f_i) = \sum_{i=1}^{n} a_i(g.v_i \otimes f_i) + \sum_{i=1}^{n} a_i(v_i \otimes g.f_i).$$

Note that for $i < n$ one has $g.v_i = 0$, and $g.v_n = v_{n+1} \in V_{n+1}$, so we get

$$g.m = a_n(v_{n+1} \otimes f_n) + \sum_{i=1}^{n-1} a_i(v_i \otimes g.f_i).$$

Since $a_n \neq 0$, and v_{n+1} is linearly independent of $\{v_i \mid 1 \leq i \leq n-1\}$, we get that $g.m \neq 0$. This way we have shown that for any element of $m \in M$, there exists $g \in sl(\infty)$ such that $g.m \neq 0$, i.e. $M^{sl(\infty)} = \{m \in M \mid g.m = 0 \text{ for all } g \in sl(\infty)\} = 0$. In particular, M does not contain any trivial submodule.

Consider now the natural pairing map $\alpha : M \longrightarrow \mathbb{C}$ given by $\alpha(v \otimes f) = f(v)$, and then extended linearly to all of M. Note that this definition makes sense, as given any elements

$v \in V$ and $f \in V_*$, one can find a large enough $n \in \mathbb{N}$ such that $v \in V_n$ and $f \in (V_n)^*$. This map is linear. For $g \in sl(\infty)$ and $m = v \otimes f$ one has

$$\alpha(g.m) = \alpha(g.v \otimes f) + \alpha(v \otimes g.f) = f(g.v) + (g.f)(v) = f(g.v) - f(g.v) = 0 = g.\alpha(m),$$

hence α is a map of $sl(\infty)$-modules. For a given $v_i \in V_n$ for some $i, n \in \mathbb{N}$ with $i < n$, and $f = v_i^*$ the linear functional in V_n^* corresponding to v_i, one has $v_i^*(v_i) = 1$, hence α is a surjective map. Thus, if we denote $K = \ker \alpha$, we get a short exact sequence:

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\alpha} \mathbb{C} \longrightarrow 0$$

which does not split, as we saw that M contains no trivial submodules.

Proposition 1.8 and Example 1.2 show that $\mathrm{Loc}_{\mathfrak{g}}$ contains non-split short exact sequences, i.e. there exist $\mathfrak{g}$–modules which are not semisimple, hence $\mathrm{Loc}_{\mathfrak{g}}$ is not semisimple. We emphasise this in the following note.

Comparison remark I. *As opposed to the finite dimensional theory of representations of semisimple Lie algebras, for an infinite dimensional locally semisimple Lie algebra $\mathfrak{g}$, not every $\mathfrak{g}$–module will be semisimple. In other words, there does not exist an analogue for Weyl's complete reducibility theorem for locally semisimple Lie algebras.*

Remark 1.8. In Subsection 3.2.2 we will show, in particular, that if M is a simple local module that satisfies the conditions of Proposition 1.8, then any injective integrable object (see Chapter 2) which contains M as a submodule, is going to be such that for any $\mathfrak{g}$–submodule $J \subset I$, I/J is not semisimple. This points to the fact that semisimplicity is a property that is lost in even the most natural categories of representations of locally semisimple Lie algebras.

1.3 A categorical observation on a type of functors $\mathfrak{g}$-mod $\longrightarrow$ $\mathfrak{g}$-mod

In this section, we give a result of a categorical nature concerning a type of functors $F : \mathfrak{g} - \mathrm{mod} \longrightarrow \mathfrak{g} - \mathrm{mod}$. The conclusions of this section will be useful later on to show that some subcategories of $\mathfrak{g}$–mod have enough injectives.

Proposition 1.9. *Let $F : \mathfrak{g}$-mod$\longrightarrow \mathfrak{g}$-mod be a functor and $\eta : F \longrightarrow id_{\mathfrak{g}-mod}$ a natural transformation of functors, such that*

i) $\eta_A : F(A) \longrightarrow A$ is an injection for all $A \in \mathfrak{g}$–mod,

ii) for any injection $i : A \longrightarrow B$ in $\mathfrak{g}$-mod, we have

$$(\eta_B \circ F(i))(F(A)) = i(A) \cap \eta_B(F(B)).$$

Then F is a left exact functor. Moreover, if by $F(\mathfrak{g}$-mod$)$ we denote the full subcategory of $\mathfrak{g}$-mod consisting of objects of the form $F(A)$ for $A \in \mathfrak{g}-mod$, then the functor $F : \mathfrak{g}-mod \longrightarrow F(\mathfrak{g}$-mod$)$ is right adjoint to the inclusion $F(\mathfrak{g}$-mod$) \longrightarrow \mathfrak{g} - mod$.

Proof. Note first of all that η being a natural transformation, we get that for any morphism $f : A \longrightarrow B$ in $\mathfrak{g}$–mod, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \eta_A \downarrow & & \downarrow \eta_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes. This gives us

$$\ker F(f) = F(A) \cap \eta_A^{-1}(\ker f) = \eta_A^{-1}(\ker f). \tag{1.6}$$

Given a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0,$$

consider the induced sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(p)} F(C). \tag{1.7}$$

Since η_A is injective, we obtain from (1.6)

$$\ker F(i) = \eta_A^{-1}(\ker f) = \eta_A^{-1}(0) = 0,$$

hence $F(i)$ will indeed be injective. Again from (1.6) we get

$$\ker F(p) = \eta_B^{-1}(\ker p).$$

As (1.6) is exact, we have $\ker p = i(A)$, and therefore

$$\ker F(p) = \eta_B^{-1}(i(A)) \tag{1.8}$$

Note now that since η_B is injective, from ii), and (1.8) we get

$$F(i)(F(A)) = \eta_B^{-1}(i(A) \cap \eta_B(F(B))) = \eta_B^{-1}(i(A)) \cap F(B) = \eta_B^{-1}(i(A)) = \ker F(p).$$

Hence (1.7) is indeed an exact sequence, and F is a left exact functor.

Let now A be any $\mathfrak{g}$-module, and consider the injection $\eta_A : F(A) \longrightarrow A$. From the commutative diagram

$$\begin{tikzcd} F(F(A)) \arrow[r, "F(\eta_A)"] \arrow[d, "\eta_{F(A)}"'] & F(A) \arrow[d, "\eta_A"] \\ F(A) \arrow[r, "\eta_A"] & A \end{tikzcd}$$

we get

$$\eta_A \circ F(\eta_A) = \eta_A \circ \eta_{F(A)} \Longrightarrow F(\eta_A) = \eta_{F(A)}$$

since η_A is injective. Note now that from ii) we have

$$(\eta_A \circ F(\eta_A))(F(F(A))) = \eta_A(F(A)) \cap \eta_A(F(A)) = \eta_A(F(A)).$$

From i) and these last two identities we get

$$F(A) = F(\eta_A)(F(F(A))) = \eta_{F(A)}(F(F(A))),$$

i.e. $\eta_{F(A)}$ is actually surjective, hence it is bijective.

Let now $s : F(A) \longrightarrow B$ be a morphism in $\mathfrak{g}$-mod. Then from the commutative diagram

$$\begin{tikzcd} F(F(A)) \arrow[r, "F(s)"] \arrow[d, "\eta_{F(A)}"'] & F(B) \arrow[d, "\eta_B"] \\ F(A) \arrow[r, "s"] & B \end{tikzcd},$$

as $\eta_{F(A)}$ is fact bijective, we get that

$$\begin{tikzcd} F(A) \arrow[rr, "s"] \arrow[dr, "u(s)"'] & & B \\ & F(B) \arrow[ur, "\eta_B"'] & \end{tikzcd}$$

is also commutative, where $u(s) = F(s) \circ F(\eta_A)^{-1}$. Consider now the map

$$u : \hom_{\mathfrak{g}-\mathrm{mod}}(F(A), B) \longrightarrow \hom_{F(\mathfrak{g}-\mathrm{mod})}(F(A), F(B))$$

given by $u(s) = F(s) \circ F(\eta_A)^{-1}$. If $s_1, s_2 : A \longrightarrow B$ are such that $u(s_1) = u(s_2)$, by applying η_B we get that $s_1 = \eta_B \circ u(s_1) = \eta_B \circ u(s_2) = s_2$, so u is an injective map. Let now $s' : F(A) \longrightarrow F(B)$ be any morphism in $F(\mathfrak{g} - \mathrm{mod})$. Let $s = \eta_B \circ s'$. We then have that $u(s)$ is such that $s = \eta_B \circ u(s)$. Since η_B is injective, we get that $\eta_B \circ s' = s = \eta_B \circ u(s)$ implies $u(s) = s'$. Hence u is also a surjective map, so we get

$$\hom_{\mathfrak{g}-\mathrm{mod}}(F(A), B) \cong \hom_{F(\mathfrak{g}-\mathrm{mod})}(F(A), F(B)).$$

The naturality of these bijections can be easily checked. This way, we get that $F : \mathfrak{g}-\mathrm{mod} \longrightarrow F(\mathfrak{g} - \mathrm{mod})$ is right adjoint to the natural inclusion $F(\mathfrak{g} - \mathrm{mod}) \longrightarrow \mathfrak{g} - \mathrm{mod}$, which is what we wanted to show.

□

If F is a left-exact functor, we know that it will preserve injectives. If F is also right adjoint, F will also preserve injective objects. Since $\mathfrak{g}$-mod has enough injectives, we get the following

Corollary 1.10. *Let $F : \mathfrak{g}$-mod $\longrightarrow \mathfrak{g}$ – mod be a functor as in Proposition 1.9. Then $F(\mathfrak{g}$ – mod) has enough injectives.*

Proposition 1.9 will be used throughout this thesis to invoke Corollary 1.10 for certain subcategories of $\mathfrak{g}$-mod. The way Proposition 1.9 is presented here is quite tedious. Whenever we will use this section to study certain functors, the injective maps $\eta_A : F(A) \longrightarrow A$ of the natural transformation η will usually be evident, as in $F(A)$ will be naturally identified with a submodule of A, for $A \in \mathfrak{g}$-mod. By this argument, for practical purposes, we write down this proposition in simpler language as follows.

Proposition 1.9'. *Let $F : \mathfrak{g}$-mod $\longrightarrow F(\mathfrak{g}$-mod) be a functor such that:*

i) $F(A) \subset A$,

ii) $F(f) = f|_{F(A)}$ *for any morphism* $f : A \longrightarrow B$,

iii) $F(A) = A \cap F(B)$ *if* $A \subset B$,

then F is a left exact functor. Moreover, if $F(\mathfrak{g}$-mod) denotes the full subcategory of $\mathfrak{g}$-mod consisting of objects of the form $F(A)$ for $A \in \mathfrak{g}$ – mod, then the functor $F : \mathfrak{g}$ – mod $\longrightarrow F(\mathfrak{g}$-mod) is right adjoint to the natural inclusion $F(\mathfrak{g}$-mod) $\longrightarrow \mathfrak{g}$ – mod.

Note that under these conditions, for any $\mathfrak{g}$-module A, the inclusion $F(A) \subset A$ implies

$$F(F(A)) = F(A) \cap F(A) = F(A)$$

In particular we see that F is an essentially surjective functor.

1.4 Universal enveloping algebra

Before we proceed with the next chapter, we give here a short account of a key concept in Lie algebra theory which shall be used throughout, namely the universal enveloping algebra of a given Lie algebra $\mathfrak{g}$. A more concise exposition of what appears in this section can be found in most literature about Lie algebras. See [6] and [8] for instance.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra. A universal enveloping algebra of $\mathfrak{g}$ is a pair $(\mathfrak{U}, i)$, where $\mathfrak{U}$ is an associative algebra with 1 over $\mathbb{C}$, and $i : \mathfrak{g} \longrightarrow \mathfrak{U}$ is a linear map that satisfies

$$i([x,y]) = i(x)i(y) - i(y)i(x) \quad \text{for all} \quad x, y \in \mathfrak{g}, \tag{1.9}$$

and such that given any associative algebra A with 1, and a map $j : \mathfrak{g} \longrightarrow A$ satisfying (1.9) for j, then there exists a unique homomorphism of algebras $\varphi : \mathfrak{U} \longrightarrow A$ such that $\phi \circ j = i$.

Since the universal enveloping algebra satisfies a universal property, it will be unique. We denote it by $U(\mathfrak{g})$.

This definition in this form is of a categorical nature, and not the most convenient to work with. However, one can explicitly construct $U(\mathfrak{g})$ as follows.

Given a vector space V over $\mathbb{C}$ set

$$T^0(V) := \mathbb{C}; \quad T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n-\text{times}} = V^{\otimes n},$$

called the tensor powers of V. Set now

$$T(V) := \bigoplus_{n \in \mathbb{N}_0} T^n(V)$$

and define a product in $T(V)$ by setting

$$\alpha \cdot m = \alpha m \quad \text{for } \alpha \in T^0(V) = \mathbb{C},$$

and for $v_1 \otimes \cdots \otimes v_n \in T^n(V)$, $w_1 \otimes \cdots \otimes w_m \in T^m(V)$ set

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

Then once can check $T(V)$ with this product becomes an associative algebra with 1.

Now if $\mathfrak{g}$ is a Lie algebra, as it is a vector space, one can form its tensor algebra $T(\mathfrak{g})$. Consider now the two-sided ideal $J \subset T(\mathfrak{g})$ generated by

$$\{x \otimes y - y \otimes x - [x,y] \mid x, y \in T^1(\mathfrak{g}) = \mathfrak{g}\},$$

and set

$$U(\mathfrak{g}) = T(\mathfrak{g})/J.$$

We have then that $(U(\mathfrak{g}), i)$, where $i : \mathfrak{g} \xrightarrow{i_{\mathfrak{g}}} T(\mathfrak{g}) \xrightarrow{p} T(\mathfrak{g})/J = U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, where $i_{\mathfrak{g}} : \mathfrak{g} \longrightarrow T(\mathfrak{g})$ is the natural identification of $\mathfrak{g}$ with $T^1(\mathfrak{g})$, and p is the canonical map.

Let now $\mathfrak{g}$ be a Lie algebra of at most countable dimension. Let $\mathcal{B} = \{X_i\}_{i \in I}$ be a basis of $\mathfrak{g}$. As $\mathcal{B}$ is at most countable, one can impose a total order on it. We now state a result which gives a basis of the universal enveloping algebra $U(\mathfrak{g})$.

Theorem PBW. *(Poincare-Birkhoff-Witt). Let $\mathfrak{g}$ be a Lie algebra of at most countable dimension, and let $\mathcal{B} = \{X_i\}_{i \in I}$ be a totally ordered basis of it. Then the set of all monomials*

$$i(X_{i_1})^{j_1} \cdots i(X_{i_k})^{j_k}$$

with $i_1 < \cdots < i_k$ and all $j_k \geq 0$ is a basis of $U(\mathfrak{g})$. In particular, the canonical map $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective.

We now make a couple of observations that shall be useful in the context of the Lie algebras that will be studied in this thesis.

Let $\mathfrak{g}_1 \subset \mathfrak{g}_2$ be Lie algebras of countable dimension, and let $(U(\mathfrak{g}_1), i_1)$ and $(U(\mathfrak{g}_2), i_2)$ denote their respective universal enveloping algebras. Denote by $i : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ the natural inclusion. Since $i_2 \circ i : \mathfrak{g}_1 \longrightarrow U(\mathfrak{g}_2)$ is a map satisfying the conditions of Definition 1.4, we get that there exists a unique morphism $j : U(\mathfrak{g}_1) \longrightarrow U(\mathfrak{g}_2)$ of associative algebras with unit such that the diagram

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{i} & \mathfrak{g}_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ U(\mathfrak{g}_1) & \xrightarrow{j} & U(\mathfrak{g}_2) \end{array}$$

commutes. Let now $\mathcal{B}_1$ and $\mathcal{B}_2$ be totally ordered basis of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, such that $\mathcal{B}_1 \subset \mathcal{B}_2$, and that the total orders on them are compatible with this inclusion. Let now $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively be the basis of $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$ as described in Theorem PBW. Note then that for $X \in \mathcal{B}_1$ and the commutative diagram above we have

$$j(i_1(X)) = i_2(i(X)) \in \mathcal{X}_2.$$

As j is a morphism of associative algebras, we get that for any $m \in \mathcal{X}_1$, we have $j(m) \in \mathcal{X}_2$, thus we get $j(\mathcal{X}_1) \subset \mathcal{X}_2$. In particular this implies that j is an injective map, so we get a naturally induced inclusions

$$U(\mathfrak{g}_1) \subset U(\mathfrak{g}_2); \quad \mathcal{X}_1 \subset \mathcal{X}_2.$$

Let us now look at a similar, but more general situation, which can be applied to locally semisimple Lie algebras. Let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \mathfrak{g}_n \subset \cdots \subset \mathfrak{g}$ be any local Lie algebra (not necessarily semisimple), i.e. such that

$$\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n.$$

Let $\mathcal{B}_n$ be totally ordered basis of $\mathfrak{g}_n$ such that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, and again the total orders on them are compatible with the inclusions. Set now $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. The total orders on the $\mathcal{B}_n$ induce a total order on $\mathcal{B}$ which is compatible with all the inclusions $\mathcal{B}_n \subset \mathcal{B}$. Let now $\mathcal{X}_n$ be the basis of $U(\mathfrak{g}_n)$, and $\mathcal{X}$ the basis of $U(\mathfrak{g})$ as indicated in Theorem PBW. From the previous discussion we get that

$$U(\mathfrak{g}_1) \subset U(\mathfrak{g}_2) \subset \cdots \subset U(\mathfrak{g}_n) \subset \cdots \subset U(\mathfrak{g}),$$

and

$$\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset \mathcal{X}_n \subset \cdots \subset \mathcal{X}$$

For convenience we avoid the notation of these inclusions as map. Let now $m \in \mathcal{X}$, say

$$m = X_{i_1}^{j_1} X_{i_2}^{j_2} \cdots X_{i_k}^{j_k}$$

As $X_{i_s} \in \mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$, for $s = 1, 2, \dots, k$, there exist some $l_s \in \mathbb{N}$ such that $X_{i_s} \in \mathcal{B}_{l_s}$. Set now

$$l = \max\{l_s \mid s = 1, 2, \dots, k\}.$$

We then have $X_{i_s} \in \mathcal{B}_{l_s} \subset \mathcal{B}_l \subset \mathfrak{g}_l$ for $s = 1, 2, \dots, k$, so in particular we get $X_{i_s} \in \mathfrak{g}_l$, which gives us $m \in \mathcal{X}_l$. In other words, we have shown that given $m \in \mathcal{X}$, there exists some natural $l \in \mathbb{N}$ such that $m \in \mathcal{X}_l$. This shows that

$$\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_l; \quad \text{and} \quad U(\mathfrak{g}) = \bigcup_{n \in \mathbb{N}} U(\mathfrak{g}_n).$$

It is clear that this identification of $U(\mathfrak{g})$ can also be used for locally semimiple Lie algebras, and we shall often make use of it in what follows.

Chapter 2

Integrable modules and weight modules

In this chapter, we introduce the main category of representations of locally semisimple Lie algebras that we will study in this thesis, i.e. the category of integrable modules $\mathrm{Int}_{\mathfrak{g}}$. Theorem 2.5 is a very important result, which in essence says that if M is integrable, then for any finite dimensional Lie subalgebra $\mathfrak{g} \subset \mathfrak{f}$, M viewed as an $\mathfrak{f}$–module is semisimple, and all its simple submodules are finite dimensional. This results makes integrable modules easier to work with, and is vaguely in accordance with Weyl's complete reducibility theorem. In Subsection 2.1.2 we show that the simple objects of $\mathrm{Int}_{\mathfrak{g}}$ are local modules in the sense of Chapter 1. We also show there that the category $\mathrm{Loc}_{\mathfrak{g}}$ can be characterized as the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ consisting of modules of countable dimension. In 2.1.3 we generalize the result of 1.2.3, and prove Schur's Lemma for simple objects of $\mathrm{Int}_{\mathfrak{g}}$.

In Section 2.2 we define a natural functor $\Gamma_{\mathfrak{g}} : \mathfrak{g} - \mathrm{mod} \longrightarrow \mathrm{Int}_{\mathfrak{g}}$, and show that it is a functor as discussed in Section 1.3. Thus we get that, in particular, $\mathrm{Int}_{\mathfrak{g}}$ has enough injectives. In Subsection 2.2.1 we show how given an integrable $\mathfrak{g}$–module M, one can construct an injective object $I \in \mathrm{Int}_{\mathfrak{g}}$ for which there exists an injection $M \longrightarrow I$.

In Section 2.3 we will study the category of weight modules, and especially its intersection with $\mathrm{Int}_{\mathfrak{g}}$, which we denote by $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$. In particular, we define a functor $\Gamma_{\mathfrak{g},\mathfrak{h}} : \mathfrak{g} - \mathrm{mod} \longrightarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ for which one can use the discussion in Section 1.3. We also show that there exists a full subcategory of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$, namely $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$, which is semisimple as seen in Theorem 2.32.

In this chapter, $\mathfrak{g}$ will denote a general locally semisimple Lie algebra, unless otherwise specified.

The exposition in this chapter is based on [10].

2.1 Integrable modules

2.1.1 Definitions and properties

Let M be a $\mathfrak{g}$–module. For any subset $A \subset M$ and $g \in \mathfrak{g}$, set

$$g(A) := \mathrm{span}\{g^i.m \mid m \in A\ ,\ i \geq 0\}.$$

If $A = \{m\}$ then we write simply $g(m)$ for $g(A)$. Using linear algebra, one can immediately get the following simple properties.

Proposition 2.1. *The following statements hold:*

i) $A \subset g(A)$ *for any subset* $A \subset M$, *and* $g \in \mathfrak{g}$;

ii) $g(A) \subset g(B)$ *for any subsets* $A \subset B \subset M$, *and* $g \in \mathfrak{g}$;

iii) $g(A) = \sum_{m\in A} g(m)$ *for any subset* $A \subset M$, *and* $g \in \mathfrak{g}$;

iv) $g\left(\sum_{i=1}^{n} c_i m_i\right) \subset \sum_{i=1}^{n} g(m_i)$ *for any* $c_i \in \mathbb{C}$, $m_i \in M$, *and* $g \in \mathfrak{g}$;

v) $g(A) \subset U(\mathfrak{f}).A$ *for any Lie subalgebra* $\mathfrak{f} \subset \mathfrak{g}$, *subset* $A \subset M$, *and* $g \in \mathfrak{f}$.

Definition 2.1. A $\mathfrak{g}$-module M is called *integrable* if for every $g \in \mathfrak{g}$ and $m \in M$ we have $\dim g(m) < \infty$.

Remark 2.1.

a) The condition in Definition 2.1 can be interpreted as follows: for any $m \in M$ and $g \in \mathfrak{g}$ we can find some $n \in \mathbb{N}$ such that there exists a non-trivial linear combination of $\{m, g.m, g^2.m, \cdots, g^n.m\}$ which gives 0.

b) If we replace m by a finite subset $A \subset M$ in Definition 2.1, we still get an equivalent definition. Indeed, part iii) of Proposition 2.1 shows that if $A \subset M$ is a finite subset, say $A = \{m_1, \dots, m_n\}$, then $g(A) = \sum_{i=1}^{n} g(m_i)$. From here one gets

$$\dim g(A) \leq \sum_{i=1}^{n} \dim g(m_i),$$

and if M is integrable, then clearly we get $\dim g(A) < \infty$. The other direction is trivial, by just setting $A = \{m\}$.

c) To check whether a $\mathfrak{g}$–module M is integrable, it suffices to find a basis $\mathcal{B}$ of M such that for all $m \in \mathcal{B}$ one has $\dim g(m) < \infty$ for every $g \in \mathfrak{g}$. Indeed, if such a basis exists, say $\mathcal{B} = \{m_i \mid i \in I\}$, then given any element $m \in M$ there exist $i_1, i_2, \dots, i_k \in I$ and $c_1, c_2, \dots, c_k \in \mathbb{C}$ such that $m = c_{i_1} m_{i_1} + \dots + c_k m_{i_k}$. From property iv) above, we get

$$g(m) \subset \sum_{j=1}^{k} g(m_{i_j}).$$

Clearly then

$$\dim g(m) \leq \sum_{j=1}^{j} \dim g(m_{i_j}) < \infty,$$

hence M will indeed be integrable.

The following result shows that when testing the dimension of $g(m)$, it suffices to check $g(a)$ for any $a \in g(m)$.

Lemma 2.2. *Let* $M \in \mathfrak{g} - mod$, $m \in M$, *and* $g \in \mathfrak{g}$. *Then* $\dim g(m) < \infty \iff \dim g(a) < \infty$ *for any* $a \in g(m)$

Proof. The direction $\Longrightarrow$ is trivial, by just taking $a = m$.

Conversely, let $a \in g(m)$ be such that $\dim g(a) < \infty$. As a is in $g(m)$, we have $a = \sum_{i=0}^{k} x_i(g^i.m)$ with not all x_i zero. We may assume that $x_k \neq 0$. Now from part c) of Remark 2.1, from $\dim g(a) < \infty$ we have some $l \in \mathbb{N}$ and y_i not all zero so that $\sum_{i=0}^{l} y_i(g^i.a) = 0$. Here we can also assume $y_l \neq 0$ without loss of generality. We then have

$$0 = \sum_{i=0}^{l} y_i(g^i.a) = \sum_{i=0}^{l} y_i(g^i.\sum_{j=0}^{k} x_j(g^j.m)) = \sum_{i,j=0}^{l,k} y_i x_j(g^{i+j}.a),$$

with $y_l x_k \neq 0$. Hence we have a non-trivial linear combination of $\{m, g.m, \cdots, g^{k+l}.m\}$ which gives 0, thus $\dim g(m) < \infty$, which is what we wanted to show. □

Denote the full subcategory of $\mathfrak{g}$-mod consisting of integrable modules by $\mathrm{Int}_{\mathfrak{g}}$. As a small general observation, note that if $f : M \longrightarrow N$ is a map of $\mathfrak{g}$-modules, since $f(g^n.m) = g^n.f(m)$ for any $m \in M$ and $n \in \mathbb{N}$, we have that $f(g(m)) = g(f(m))$. In particular, $f|_{g(m)}$ factors through the inclusion $g(f(m)) \subset B$.

Proposition 2.3. *$\mathrm{Int}_{\mathfrak{g}}$ is closed under taking submodules, quotients, tensor products, extensions, and arbitrary direct sums. In particular, $\mathrm{Int}_{\mathfrak{g}}$ is an abelian subcategory of $\mathfrak{g}$−mod.*

Proof. This is evident for submodules, quotients. For arbitrary direct sums, the result follows directly from part iv) of 2.1. Let us now prove this for extensions and tensor products.

For extensions, let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be an exact sequence in $\mathfrak{g}$-mod, with $A, C \in \mathrm{Int}_{\mathfrak{g}}$. Let $g \in \mathfrak{g}$ and $b \in B$. Let $c = p(b)$. As $g(b)$ is finite dimensional, from the small general observation right before the statement of this proposition, we see that $g(c) = g(p(b)) = p(g(b))$ is also finite dimensional. If $p|_{g(b)} : g(b) \longrightarrow g(c)$ is injective, then $g(b)$ is also finite dimensional. If not, then there is some non-zero $x \in g(b)$ such that $p(x) = 0$. But this means $x \in \ker p = A$, and since A is integrable, we have $\dim g(x) < \infty$. From Lemma 2.2 we get that $\dim g(b) < \infty$. Hence B is also integrable.

For tensor products, let $M, N \in \mathrm{Int}_{\mathfrak{g}}$, $m \in M$, $n \in N$, and $g \in \mathfrak{g}$. A simple computation shows that

$$g^k.(m \otimes n) = \sum_{i=0}^{k} \binom{k}{i} g^{k-i}.m \otimes g^i.n,$$

so we see that $g^k.(m \otimes n) \in g(m) \otimes g(n)$, i.e. $g(m \otimes n) \subset g(m) \otimes g(n)$, which gives us $\dim g(m \otimes n) < \infty$, hence $M \otimes N$ is also integrable. □

Let now M be a local module in the sense of Chapter 1, with exhaustion $\{M_i\}_{i\in\mathbb{N}}$. Then for any $m \in M$ and $g \in \mathfrak{g}$, we have some $i, j \in \mathbb{N}$ such that $m \in M_i$ and $g \in \mathfrak{g}_j$. For $k \geq i, j$ we have $m \in M_k$ and $g \in \mathfrak{g}_k$. Now as M_k is a $\mathfrak{g}_k$-module, we have $g^i.m \in M_k$ for all i, hence we have $g(m) \subset M_k$, so $\dim g(m) \leq \dim M_k < \infty$. This way we obtain the following corollary of Proposition 2.3.

Corollary 2.4. *$\mathrm{Loc}_{\mathfrak{g}}$ is a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$.*

In what follows, we will give a result which tells us how integrable $\mathfrak{g}$-modules look like when they are restricted to finite dimensional semisimple Lie subalgebras of $\mathfrak{g}$. Before we do this, we give two identities which will be very useful in computations. One can prove by induction that

$$g^n.(h.m) = \sum_{k=0}^{n} \binom{n}{k} u_k.(g^{n-k}.m); \quad g.(h^n.m) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}.(v_k.m) \tag{2.1}$$

with $m \in M$, $g, h \in \mathfrak{g}$, $n \in \mathbb{N}$, and where $u_0 = h, v_0 = g$, and $u_k = \underbrace{[g,[g,[\cdots[g,h]]]}_{k \text{ brackets}}$, $v_k = \underbrace{[[[\cdots[g,h],h],\cdots h],h]}_{k \text{ brackets}}$ for $k > 0$. Now we are ready to state and prove one of the main results about integrable modules.

Theorem 2.5. *A $\mathfrak{g}$-module M is integrable if and only if for any finite dimensional semisimple Lie subalgebra $\mathfrak{f} \subset \mathfrak{g}$ there exists an index set $I_{\mathfrak{f}}$ and finite dimensional $\mathfrak{f}$-modules M_i for $i \in I_{\mathfrak{f}}$ such that:*

$$M|_{\mathfrak{f}} = \bigoplus_{i \in I_{\mathfrak{f}}} M_i \tag{2.2}$$

Proof. Let M be a $\mathfrak{g}$-module, such that for any finite dimensional semisimple Lie subalgebra $\mathfrak{f} \subset \mathfrak{g}$ one has an index set $I_{\mathfrak{f}}$ and finite dimensional $\mathfrak{f}$-modules M_i for $i \in I_{\mathfrak{f}}$ such that (2.2) holds. Let $m \in M$ and $g \in \mathfrak{g}$. Choose $\mathfrak{f} \subset \mathfrak{g}$ finite dimensional semisimple such that $g \in \mathfrak{f}$ (here one can simply choose one of the $\mathfrak{g}_i$'s which contains g). Then there exists a finite subset $S \subset I_{\mathfrak{f}}$ such that $m \in \bigoplus_{i \in S} M_i = M'$, which is also finite dimensional. As the M_i are finite dimensional $\mathfrak{f}$–modules, for any $g \in \mathfrak{f}$ we get $g(m) \subset U(\mathfrak{f}).m \subset M'$ from part v) of Proposition 2.1, hence $g(m)$ will also be finite dimensional. Thus M is integrable.

Conversely, assume M is integrable. We will show that given a finite dimensional semisimple Lie subalgebra $\mathfrak{f} \subset \mathfrak{g}$, and $m \in M$, then there exists a finite dimensional $\mathfrak{f}$–submodule $R \subset M$ such that $m \in R$. We first prove this for the case when $\mathfrak{f} \cong sl(2,\mathbb{C})$, and then extend the result to general semisimple $\mathfrak{f} \subset \mathfrak{g}$ via the well-known decomposition of $\mathfrak{f}$ into root spaces.

Claim 1. Let $sl(2,\mathbb{C}) \cong \mathfrak{f} \subset \mathfrak{g}$ be a finite dimensional semisimple Lie subalgebra, and $A \subset M$ a finite dimensional subspace. Then $U(\mathfrak{f}).A$ is finite dimensional.

Proof. We prove this for $A = \{m\}$, as the general case then follows from the fact that $U(\mathfrak{f}).A = \sum_{m \in A} U(\mathfrak{f}).m$ immediately. Let $\mathfrak{f} = sl(2,\mathbb{C})$ with the usual basis x, y and h. We know that the Lie bracket here is given by $[x,y] = h, [h,x] = 2x$ and $[h,y] = -2y$. Let us see how the second formula in (2.1) looks like for $g = x$ and $h = h$. We have $v_1 = [x,h] = -2x$, $v_2 = [v_1,h] = 4x$, and in this way we see that $v_k = (-2)^k x$ for all $k = 0, \cdots, n$. Thus we get

$$x.(h^n.m) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}.(v_k.m) = \sum_{k=0}^{n} \binom{n}{k} (-2)^k h^{n-k}.(x.m)$$

from which we see in particular that $x.h(m) \subset h(x.m)$. From this, one obtains $x^n.h(m) \subset h(x^n.m)$, thus we have $x(h(m)) \subset h(x(m))$. Now we look at $h.(x^n.m)$. Again in the second formula in (2.1) for h instead of g, and x instead of h we have $v_0 = h, v_1 = 2x$ and $v_k = 0$ for $k \geq 2$. Thus we obtain

$$h.(x^n.m) = x^n.(h.m + 2nm)$$

In particular, one has that $h.x(m) \subset x((h.m))$. From this one obtains that $h^n.x(m) \subset x((h^n.m))$, so we get $h(x(m)) \subset x(h(m))$. Thus we have shown that

$$x(h(m)) = h(x(m))$$

In a completely similar way, one shows that

$$y(h(m)) = h(y(m)).$$

Note that from part iii) of Proposition 2.1 this still holds true if we replace m by any subset $A \subset M$. Now we investigate how $x.(y^n.m)$ looks like. Again, in the second formula in (2.1) we have $v_1 = [x,y] = h$, $v_2 = [h,y] = -2y$, and $v_k = 0$ for $k \geq 3$. We thus get

$$x.(y^n.m) = y^n.(x.m) + ny^{n-1}(h.m) - n(n-1)y^{n-1}.m$$

and in particular we see that

$$x.(y(m)) \subset y(x.m) + y(h(m)) \subset y(x(h(m))). \tag{2.3}$$

Again, from part iii) of Proposition 2.1 we see that this inclusion still holds if instead of m we take any subset $A \subset M$. Note now that from (2.3) we see that

$$x^2.(y(m)) = x.(x.y(m)) \subset x.y(x(h(m))) \subset y(x(h(x(h(m))))) = y(x(h(m))).$$

This way, one obtains inductively that $x(y(m)) \subset y(x(h(m)))$. In a completely similar way one shows that $y(x(m)) \subset x(y(h(m)))$. This implies

$$x(y(h(m))) \subset y(x(h(h(m)))) = y(x(h(m))),$$

and similarly

$$y(x(h(m))) \subset x(y(h(m)))$$

so we obtain the equality

$$y(x(h(m))) = x(y(h(m))). \tag{2.4}$$

Denote the subspace of M in (2.4) by S. Note that every element of S can be written as a linear combination of elements of the form

$$x^{n_1}.y^{n_2}.h^{n_3}.m$$

with $n_1, n_2, n_3 \geq 0$. It is clear that every element that can be written as a linear combination of elements of this form, also lies in S. Thus we have

$$S = \text{span}\{x^{n_1}.y^{n_2}.h^{n_3}.m \mid n_1, n_2, n_3 \geq 0\} \tag{2.5}$$

Since $\mathcal{B} = \{x, y, h\}$ is a basis of $\mathfrak{f} = sl(2, \mathbb{C}$, we may impose an order on $\mathcal{B}$ by setting $x < y < h$. From Theorem PBW we see that $\{x^{n_1}y^{n_2}h^{n_3} \mid n_1, n_2, n_3 \geq 0\}$ is a basis of $U(\mathfrak{f})$. Thus we have

$$U(\mathfrak{f}) = \text{span}\{x^{n_1}y^{n_2}h^{n_3} \mid n_1, n_2, n_3 \geq 0\},$$

which together with (2.5) gives us

$$S = U(\mathfrak{f}).m$$

so we see that S is indeed an $\mathfrak{f}$–module.

Now we show that S is finite dimensional.Well as M is integrable and $m \in M$, we have that $h(m)$ is finite dimensional as well. Then from part b) of Remark 2.1 we see that $y(h(m))$ will also be finite dimensional, and thus so will be $x(y(h(m))) = S = U(\mathfrak{g}).m$.

The claim for any finite dimensional $A \subset M$ then follows from the remark at the beginning of the proof of this claim. □

Let now $\mathfrak{f}$ be any finite dimensional semisimple Lie subalgebra of $\mathfrak{g}$. Let $\Phi = \{\alpha_1, \cdots, \alpha_n\}$ be the positive roots of $\mathfrak{f}$. For each $i = 1, \cdots, n$ denote by $\mathfrak{s}_i$ the corresponding $sl(2, \mathbb{C})$-triple. We know that

$$\mathfrak{f} = \sum_{i=1}^{n} \mathfrak{s}_i.$$

Let $m \in M$ be any element, and consider the following subspace of M

$$R = \sum_{\pi \in S_n} U(\mathfrak{s}_{\pi(1)}).U(\mathfrak{s}_{\pi(2)}).\cdots.U(\mathfrak{s}_{\pi(n)}).m, \tag{2.6}$$

where S_n denotes the symmetric group on n elements. We claim that R is a finite dimensional $\mathfrak{f}$-module. From Claim 1 for $A = \{m\}$, we see that $U(\mathfrak{s}_i).m$ are all finite dimensional. Again by Claim 1 for $A = U(\mathfrak{s}_i).m$, we see that $U(\mathfrak{s}_j).U(\mathfrak{s}_i).m$ will also be finite dimensional for any $1 \leq j \leq n$. Thus all the summands that appear in (2.6) are also finite dimensional. As this sum is over a finite index set, we get that R is indeed finite dimensional.

Now to show that R is an $\mathfrak{f}$-module, we first prove the following technical result.

Claim 2. Let $k \in \mathbb{N}$, and $x_j \in \bigcup_{i=1}^n \mathfrak{s}_i$ for $j = 1, 2, \cdots, k$. Then:

$$x_1.x_2.\cdots.x_k.m \in R$$

Proof. We prove this by induction on k. If $k = 0, 1$ it is trivially true. Assume now that the claim is true for some $k \in \mathbb{N}$. Let now $x_j \in \bigcup_{i=1}^n$ for $j = 0, \cdots, k$, and consider the element

$$x_0.\cdots.x_{k-1}.x_k.m.$$

From the induction hypothesis, we have that $x_1.x_2.\cdots.x_k.m \in R$. Without loss of generality, let us assume that $x_1.x_2.\cdots.x_k.m \in U(\mathfrak{s}_1).U(\mathfrak{s}_2).\cdots.U(\mathfrak{s}_n).m$. In this case, we may assume that $x_1, \cdots, x_{k_1} \in \mathfrak{s}_1$, $x_{k_1+1}, \cdots, x_{k_2} \in \mathfrak{s}_2$ and so forth for some $k_i \leq k$. If $x_0 \in \mathfrak{s}_1$, then we are done. Assume now that $x_0 \in \mathfrak{s}_l$ for some $l > 1$. We may assume that $l = 2$, as the argument for general l is similar to what will be shown for $l = 2$. We then have

$$x_0.\cdots.x_k.m = x_1.x_0.x_2.\cdots.x_k.m + [x_0, x_1].x_2.\cdots.x_k.m = x_1.x_0.x_2.\cdots.x_k.m + r$$

with $r = [x_0, x_1].x_2.\cdots.x_k.m \in R$ due to the induction hypothesis. Doing now the same reordering process for (x_0, x_2), (x_0, x_3) up to (x_0, x_{k_1}), one obtains an expression of the form

$$x_0.\cdots.x_k.m = x_1.\cdots.x_{k_1}.x_0.x_{k_1+1}.\cdots.x_k.m + r$$

with $r \in R$. Note that the first summand in this expression lies in $U(\mathfrak{s}_1).U(\mathfrak{s}_2).\cdots.U(\mathfrak{s}_n).m \subset R$, and thus we obtain

$$x_0.\cdots.x_k.m \in R.$$

The claim then follows by induction on k. □

What Claim 2 shows in particular, is that for any $x \in \mathfrak{s}_i$, with $i = 1, \cdots, n$, we have $x.R \subset R$. Now as $\mathfrak{f} = \sum_{i=1}^n \mathfrak{s}_i$, for any $g \in \mathfrak{f}$ we get that $g.R \subset R$. Hence altogether we have indeed showed that R is a finite dimensional $\mathfrak{f}$-module.

In particular, as all finite dimensional $\mathfrak{f}$-modules are semisimple, we have that every element of M lies in some semisimple finite dimensional $\mathfrak{f}$-submodule of M. So if we set $\mathcal{C} := \{\text{finite dimensional simple } \mathfrak{f} - \text{submodules of } M\}$, we get that

$$\sum_{S \in \mathcal{C}} S = M. \tag{2.7}$$

Note now that if $S \neq 0$ is any simple $\mathfrak{f}$-submodule of M, for any $m \in S$ non-zero, there exists a finite dimensional $\mathfrak{f}$-submodule $R \subset M$ such that $m \in R$. Since $R \cap S \neq 0$, one has $R \cap S = S$, hence S is finite dimensional. Thus we actually have $\mathcal{C} = \{\text{simple } \mathfrak{f}\text{−submodules of } M\}$, and then (2.7) shows that M is a semisimple $\mathfrak{f}$-module, i.e. there exists an index set $I_\mathfrak{f}$, and $M_i \in \mathcal{C}$ for $i \in I_\mathfrak{f}$ such that

$$M|_\mathfrak{f} = \bigoplus_{i \in I_\mathfrak{f}} M_i$$

which concludes the proof of the theorem. □

Remark 2.2. The $\mathfrak{f}$−module R defined in (2.6) is actually equal to $U(\mathfrak{f}).m$. This is evident because given a permutation $\pi \in S_n$, and $q_i \in U(\mathfrak{s}_{\pi(i)})$ for $i = 1, 2, \ldots, n$, we see that $q_i \in U(\mathfrak{s}_{\pi(i)}) \subset U(\mathfrak{f})$, and thus $q_1 q_2 \cdots q_n \in U(\mathfrak{f})$, so we get

$$q_1 \cdot q_2 \cdots q_n.m \in U(\mathfrak{f}).m$$

This gives us $R \subset U(\mathfrak{f}).m$. Conversely, since R was shown to be a $\mathfrak{f}$−module and $m \in R$, we have

$$U(\mathfrak{f}).m \subset R,$$

thus we indeed get $R = U(\mathfrak{f}).m$.

Remark 2.3. Note that the proof of Theorem 2.5 actually shows that $\mathfrak{f}$−modules M_i in (2.2) are simple modules. However, we will often use a decomposition as in (2.2) without assuming that the M_i are simple.

We note that the proof of Theorem 2.5 shows that every integrable module will be semisimple as a $\mathfrak{f}$−module for any finite dimensional semismple Lie subalgebra $\mathfrak{f} \subset \mathfrak{g}$. We emphasize this in the following note.

Comparison remark II. *In a weak analogy to the finite dimensional theory of representations of semisimple Lie algebras, we see that $Int_{\mathfrak{g}}$ is a category of representations of locally semisimple Lie algebras such that every object of it is semisimple as a representation of any of its finite dimensional semisimple Lie subalgebras.*

From Theorem 2.5 we get the following direct consequence.

Corollary 2.6. *Let $M \in Int_{\mathfrak{g}}$, $n \in \mathbb{N}$, and $S \subset M$ a finite dimensional subspace of M. Then*

$$\dim U(\mathfrak{g}_n).S < \infty.$$

Moreover, if we denote $U(\mathfrak{g}_n).S = N$, there exists a $\mathfrak{g}_n$−submodule $R \subset M$ such that $M|_{\mathfrak{g}_n} = N \bigoplus R$.

Proof. From Theorem 2.5, let I be an index set, and M_i finite dimensional $\mathfrak{g}_n$−modules such that

$$M|_{\mathfrak{g}_n} = \bigoplus_{i \in I} M_i \tag{2.8}$$

As $S \subset M$ is finite dimensional, there exists a finite subset of indices $I_0 = \{i_1, i_2, \dots, i_k\}$ such that $S \subset \bigoplus_{i \in I_0} M_i = T$. It is clear that T is a finite dimensional $\mathfrak{g}_n$−module. Note now that

$$U(\mathfrak{g}_n).S \subset U(\mathfrak{g}_n).T = T \tag{2.9}$$

so we get $\dim U(\mathfrak{g}_n).S \leq \dim T < \infty$, which proves the first part of the corollary.

From our definition of T, and (2.8), we get

$$M|_{\mathfrak{g}_n} = T \oplus \bigoplus_{i \in I \setminus I_0} M_i = T \oplus R',$$

where $R' = \bigoplus_{i \in I \setminus I_0} M_i$. From Weyl's complete reducibility theorem, since $N \subset T$ are finite dimensional $\mathfrak{g}_n$−modules, we have that there exists some $\mathfrak{g}_n$−submodule $U \subset T$ such that $T = U \oplus N$. We then get

$$M|_{\mathfrak{g}_n} = T \oplus R' = N \oplus U \oplus R' = N \oplus R,$$

where $R = U \oplus R'$, which completes the proof of this corollary. □

2.1.2 Integrable modules of countable dimension

In this section we study the countable dimensional integrable modules, and also the simple objects of $\mathrm{Int}_{\mathfrak{g}}$. We start with the following observation.

Proposition 2.7. *Every integrable $\mathfrak{g}$-module contains a local submodule.*

Proof. Let $i \in \mathbb{N}$ and $0 \neq m \in M$ non-zero. From Corollary 2.6 for $S = \mathbb{C}m$ we see that is a finite dimensional $\mathfrak{g}_i$-module. Since $\mathfrak{g}_\mathrm{i} \subset \mathfrak{g}_{\mathrm{i+1}}$ we have $N_i \subset N_{i+1}$. By setting $N = \cup N_i$, we get a local module which is a submodule of M such that $m \in N$. □

Remark 2.4. Note that the proof of Proposition 2.7 actually shows that given an integrable $\mathfrak{g}$–module M, for any $m \in M$ there exists some local submodule $N \subset M$ such that $m \in N$.

From Proposition 2.7 we get the following direct consequence.

Theorem 2.8. *All simple objects of $\mathrm{Int}_{\mathfrak{g}}$ are local modules, i.e. we have*

$$\{\textit{simple objects of } \mathrm{Int}_{\mathfrak{g}}\} = \{\textit{simple objects of } \mathrm{Loc}_{\mathfrak{g}}\}.$$

Proof. If $L \in \mathrm{Loc}_{\mathfrak{g}}$ is a simple local module, from Corollary 2.4 we see that $L \in \mathrm{Int}_{\mathfrak{g}}$, hence we get one of the inclusions.

Conversely, let $L \in \mathrm{Int}_{\mathfrak{g}}$ be simple. From Proposition 2.7 we get that there exists a non-zero $\mathfrak{g}$–submodule $N \subset L$ with $N \in \mathrm{Loc}_{\mathfrak{g}}$. As L is simple, we get that $N = L$, hence $L \in \mathrm{Loc}_{\mathfrak{g}}$ □

We have already seen in Corollary 2.4 that $\mathrm{Loc}_{\mathfrak{g}}$ is a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$, and Remark 1.3 showed that all objects of $\mathrm{Loc}_{\mathfrak{g}}$ are of countable dimension. The following result shows that these two properties are actually the characterizing properties of $\mathrm{Loc}_{\mathfrak{g}}$ as a subcategory of $\mathrm{Int}_{\mathfrak{g}}$.

Theorem 2.9. *The countable integrable $\mathfrak{g}$-modules are precisely the local $\mathfrak{g}$-modules.*

Proof. If $M \in \mathrm{Loc}_{\mathfrak{g}}$, we saw in the preceding discussion that M will be an integrable $\mathfrak{g}$–module of countable dimension, thus we get

$$\mathrm{Loc}_{\mathfrak{g}} \subset \{\text{integrable } \mathfrak{g} - \text{modules of countable dimension}\}.$$

Let now $M \in \mathrm{Int}_{\mathfrak{g}}$ be of countable dimension. From Theorem 2.5, there exists an index set I and non-zero finite dimensional $\mathfrak{g}_1$–modules T_i such that

$$M|_{\mathfrak{g}_1} = \bigoplus_{i \in I} T_i. \tag{2.10}$$

As M is of countable dimension, we see that I has to be a countable set, so without loss of generality we may assume that $I = \mathbb{N}$. Thus we have

$$M|_{\mathfrak{g}_1} = T_1 \oplus T_2 \oplus \cdots \oplus T_n \oplus \cdots$$

Set now

$$S_n := \bigoplus_{i=1}^{n} T_i$$

as a vector subspace of M, and let

$$M_n = U(\mathfrak{g}_n).S_n.$$

Since the S_n's are clearly finite dimensional, from Corollary 2.6 we see that all the M_n's will be finite dimensional $\mathfrak{g}_n$–modules. Note now that since $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ and $S_n \subset S_{n+1}$ we have

$$M_n = U(\mathfrak{g}_n).S_n \subset U(\mathfrak{g}_{n+1}).S_{n+1} = M_{n+1}.$$

Let us now set $N = \bigcup_{n \in \mathbb{N}} M_n$. We clearly see that N will be a local $\mathfrak{g}$–submodule of M. Note also that since $S_n \subset U(\mathfrak{g}_n).S_n = M_n$, we have

$$M = M|_{\mathfrak{g}_1} = \bigcup_{n \in \mathbb{N}} S_n \subset \bigcup_{n \in \mathbb{N}} U(\mathfrak{g}_n).S_n = N,$$

hence we get $M = N = \varinjlim M_i$, thus M is also a local module. This proves the inclusion

$$\{\text{integrable } \mathfrak{g} - \text{modules of countable dimension}\} \subset \mathrm{Loc}_{\mathfrak{g}},$$

and thus completes the proof of the theorem. □

Remark 2.5. In Theorem 2.9 we have seen that

$$\mathrm{Loc}_{\mathfrak{g}} = \{\text{integrable } \mathfrak{g} - \text{modules of countable dimension}\}\mathfrak{g}.$$

Since all submodules, quotients, extensions, finite direct sums, and tensor products of local modules are integrable modules of countable dimension from Proposition 2.3 and Theorem 2.9, we get that they are all also local modules. This way we obtain Proposition 1.4.

Remark 2.6. We note here that for $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ in Theorem 3.54 we will see that $\mathfrak{g}$–mod contains no non-trivial finite dimensional modules. Hence, for $\mathfrak{g}$ a classical locally semisimple Lie algebra, the representation theory for $\mathfrak{g}$ is always infinite dimensional.

2.1.3 Schur's Lemma in $\mathrm{Int}_{\mathfrak{g}}$

We now prove a Schur's Lemma for the category $\mathrm{Int}_{\mathfrak{g}}$. We already saw in Subsection 1.2.3 that Schur's Lemma holds for locally simple $\mathfrak{g}$–modules. However as noted in Remark 1.5, not all simple local modules are locally simple, thus the proof presented there does not cover all the simple objects of $\mathrm{Loc}_{\mathfrak{g}}$. In particular, Theorem 2.8 shows that the result here will generalize the result of Subsection 1.2.3 to all of simple objects of $\mathrm{Loc}_{\mathfrak{g}}$.

Theorem 2.10. *(Schur's Lemma in $Int_{\mathfrak{g}}$) Let $L, N \in Int_{\mathfrak{g}}$ be simple. Then*

$$\hom_{\mathfrak{g}}(L, N) = \begin{cases} \mathbb{C} & \text{if } M \cong N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume first that $L \ncong N$. Let $f : L \longrightarrow N$ be a $\mathfrak{g}$–module morphism. Since $\ker f \subset L$ and L is simple, we have that $\ker f = 0$ or $\ker f = L$. Assume that $\ker f = 0$. We then have that f is injective, and $f(L) \subset N$ is a non-zero submodule of N. Since $L \ncong N$, we cannot have $f(L) = N$. But this would mean that N, as a simple module, contains a submodule which is neither 0 nor N, and this is impossible. Hence we must have $\ker f = L$. This gives us $f = 0$, hence $\hom_{\mathfrak{g}}(L, N) = 0$ in this case.

Assume now that $L \cong N$. Without loss of generality, we may only consider $\hom_{\mathfrak{g}}(L, L)$. From Theorem 2.8 we know that L as a simple integrable $\mathfrak{g}$–module is going to be a local module. Pick $x \in L$ non-zero, and set $L_i := U(\mathfrak{g}_i).x$ for $i \in \mathbb{N}$ which are clearly $\mathfrak{g}_i$–modules. Since L is integrable, from Corollary 2.6 we see that these L_is will all be finite dimensional. From the discussion in Section 1.4 we see that $L_i \subset L_{i+1}$, and actually $L = \bigcup_{i \in \mathbb{N}} L_i = U(\mathfrak{g}).x$ (one can also see that this is true also from the fact that L is simple).

Let now $f : L \longrightarrow L$ be any map. Similar to the first paragraph of the proof, we see that either $f = 0$, or f is an automorphism of the $\mathfrak{g}$–module L. Assume that $f \neq 0$. We then have $f(x) \neq 0$, and since $f(x) \in L = U(\mathfrak{g}).x = \bigcup_{i \in \mathbb{N}} L_i$, there exists some $n \in \mathbb{N}$ such that $f(x) \in L_n = U(\mathfrak{g}_n).x$. Let now $q \in U(\mathfrak{g}_n)$ be such that $f(x) = qx$. Clearly we have $q \neq 0$. Note now that if $y \in L_n$, we have that $L = rx$ for some $r \in U(\mathfrak{g}_n)$, and since f is a morphism of $\mathfrak{g}$–modules, it is also a morphism of $\mathfrak{g}_n$–modules, so we get

$$f(y) = f(rx) = rf(x) = rqx \in L_n$$

This means that $f|_{L_n} \subset L_n$. As f is an automorphism, and L_n is finite dimensional we get that $f(L_n) \subset L_n$, and $\dim f(L_n) = \dim L_n$, thus $f|_{L_n} : L_n \longrightarrow L_n$ is an automorphism of finite dimensional $\mathfrak{g}_n$–modules. In particular, it is an automorphism of finite dimensional vector spaces. As f is an invertible linear map, it will have some non-zero eigenvalue, say $\lambda \in \mathbb{C}$, and some non-zero λ–eigenvector $z \in L_n$, i.e. we have

$$f(z) = \lambda z$$

As in the second paragraph of this proof, we have that $L = U(\mathfrak{g})z$. Note now that for any $m \in L$, there exists some $s \in U(\mathfrak{g})$ such that $m = sz$. We then have

$$f(m) = f(sz) = sf(z) = s \cdot \lambda z = \lambda sz = \lambda m.$$

This shows that there exists some $\lambda \in \mathbb{C}$ such that $f(m) = \lambda m$ for all $m \in L$. Hence we really do have that

$$\hom_{\mathfrak{g}}(L, L) = \mathbb{C}$$

which is what we wanted to show, hence the proof of the theorem is complete. $\square$

This way we have generalized a classical result from the finite dimensional theory of Lie algebras to the theory of locally semisimple Lie algebras. We emphasize this in the following note.

Comparison remark III. *Analogous to the finite dimensional theory of representations of semisimple Lie algebras, there exists a Schur's Lemma in a reasonable category of locally semisimple Lie algebras.*

2.2 The functor $\Gamma_{\mathfrak{g}} : \mathfrak{g}-\text{mod} \longrightarrow \text{Int}_{\mathfrak{g}}$ and injectives in $\text{Int}_{\mathfrak{g}}$

Let $M \in \mathfrak{g}-\text{mod}$ be any module. Define

$$\Gamma_{\mathfrak{g}}(M) := \{m \in M \mid \dim g(m) < \infty \text{ for all } g \in \mathfrak{g}\}. \tag{2.11}$$

Lemma 2.11. *$\Gamma_{\mathfrak{g}}(M)$ is an integrable submodule of M.*

Proof. Let $m \in \Gamma_{\mathfrak{g}}(M)$ and $h \in \mathfrak{g}$. We want to show that $h.m \in \Gamma_{\mathfrak{g}}(M)$. For this, let $g \in \mathfrak{g}$ be any element, and let i be natural number such that $g, h \in \mathfrak{g}_i$. As both $\mathfrak{g}_i$ and $g(m)$ are finite dimensional, we have that $\mathfrak{g}_i.g(m)$ is also finite dimensional. From (2.1) one can see that $g^n.(h.m) \in \mathfrak{g}_i.g(m)$, since $u_k \in \mathfrak{g}_i$. This means that $g(h.m) \subset \mathfrak{g}_i.g(m)$, so we get $\dim g(h.m) < \infty$. Thus we have $h.m \in \Gamma_{\mathfrak{g}}(M)$, which proves that $\Gamma_{\mathfrak{g}}(M)$ is indeed a submodule of M. It is evidently integrable, and in fact it is the largest integrable submodule of M. $\square$

Remark 2.7. We note here specifically the last sentence of the proof of Lemma 2.11. If $M \in \mathfrak{g}$–mod, and $N \subset M$ is integrable, then $N \subset \Gamma_{\mathfrak{g}}(M)$. Note that in particular this implies that

$$\Gamma_{\mathfrak{g}}(M/\Gamma_{\mathfrak{g}}(M)) = 0$$

for any $M \in \mathfrak{g}$–mod.

Let now M, N be $\mathfrak{g}$-modules, and $f : M \longrightarrow N$ a morphism. From Proposition 2.3 we get that $f(\Gamma_{\mathfrak{g}}(M))$ is also integrable (as a quotient of $\Gamma_{\mathfrak{g}}(M)$), hence it lies in $\Gamma_{\mathfrak{g}}(N)$ from Remark 2.7. Thus f restricted to $\Gamma_{\mathfrak{g}}(M)$ factors through $\Gamma_{\mathfrak{g}}(N)$. This allows us to make the following definition.

Definition 2.2. The functor $\Gamma_{\mathfrak{g}} : \mathfrak{g}-\text{mod} \longrightarrow \mathfrak{g}-\text{mod}$ as defined on objects via (2.11), and by $\Gamma_{\mathfrak{g}}(f) = f|_{\Gamma_{\mathfrak{g}}(M)}$ on morphisms $f : M \longrightarrow N$, is called the *functor of $\mathfrak{g}$-integrable vectors.*

We now apply the theme set in Section 1.3 to this functor $\Gamma_{\mathfrak{g}}$.

Lemma 2.12. *The following statements hold:*

i) $\Gamma_{\mathfrak{g}}(A) \subset A$ *for any* $A \in \mathfrak{g}$*–mod;*

ii) $\Gamma_{\mathfrak{g}}(f) = f|_{\Gamma_{\mathfrak{g}}(M)}$ *for any morphism* $f : A \longrightarrow B$ *in* $\mathfrak{g}$*–mod;*

iii) $\Gamma_{\mathfrak{g}}(A) = A \cap \Gamma_{\mathfrak{g}}(B)$ *if* $A \subset B$ *in* $\mathfrak{g}$*-mod.*

Proof. i) and ii) are clear from the definition of $\Gamma_{\mathfrak{g}}$ as a functor. For iii), we have immediately that $\Gamma_{\mathfrak{g}}(A) \subset A$, and as $\Gamma_{\mathfrak{g}}(A) \subset B$ is an integrable module, from Remark 2.7 we have $\Gamma_{\mathfrak{g}}(A) \subset \Gamma_{\mathfrak{g}}(B)$. Thus we have

$$\Gamma_{\mathfrak{g}}(A) \subset A \cap \Gamma_{\mathfrak{g}}(B).$$

For the other inclusion, note that $A \cap \Gamma_{\mathfrak{g}}(B)$ is an integrable module, as a submodule of $\Gamma_{\mathfrak{g}}(B)$, from Proposition 2.3. Then by Remark 2.7 we get that $A \cap \Gamma_{\mathfrak{g}}(B) \subset \Gamma_{\mathfrak{g}}(A)$ Thus altogether we obtain

$$\Gamma_{\mathfrak{g}}(A) = A \cap \Gamma_{\mathfrak{g}}(B)$$

which completes the proof of the lemma. □

Lemma 2.12 says that $\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$ satisfies the conditions of Proposition 1.9'. Then, in particular, $\Gamma_{\mathfrak{g}}$ will be essentially surjective, hence $\Gamma_{\mathfrak{g}}(\mathfrak{g}\text{-mod}) = \mathrm{Int}_{\mathfrak{g}}$. This gives us the following result.

Proposition 2.13. *$\Gamma_{\mathfrak{g}}$ is a left-exact functor, and it is right-adjoint to the inclusion $Int_{\mathfrak{g}} \longrightarrow \mathfrak{g} - mod$.*

Applying now Corollary 1.10 in our situation with the functor $\Gamma_{\mathfrak{g}}$, we get the following result.

Corollary 2.14. *$Int_{\mathfrak{g}}$ has enough injectives.*

While these arguments show the existence of enough injectives in $\mathrm{Int}_{\mathfrak{g}}$, they are not of a constructive nature. However, there exists a nice way of producing injective objects in $\mathrm{Int}_{\mathfrak{g}}$ by looking at the duals of integrable modules. Before we do this, we digress into recalling some facts from the homology and cohomology theories of finite dimensional representations of semisimple Lie algebras. One can find these results in many treatments of Lie algebras, for instance [12] and [7].

ref. result 2A. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $\mathbb{C}$ the trivial $\mathfrak{g}$-module. Then $H_i(\mathfrak{g}, \mathbb{C}) = \mathfrak{g}^{\mathrm{ab}}$ where $\mathfrak{g}^{\mathrm{ab}} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.*

ref. result 2B. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, and $M \neq \mathbb{C}$ a simple $\mathfrak{g}$-module. Then $H_i(\mathfrak{g}, M) = H^i(\mathfrak{g}, M) = 0$.*

Note that if $\mathfrak{g}$ is any finite dimensional semisimple Lie algebra, and M a finite dimensional $\mathfrak{g}$-module, by Weyl's complete reducibility theorem we can write

$$M = \bigoplus_{i=1}^{n} M_i \oplus \mathbb{C}^s$$

for some $n, s \in \mathbb{N}_0$, where the M_i are non-trivial simple $\mathfrak{g}$-modules. We then have:

$$H_1(\mathfrak{g}, M) = \bigoplus_{i=1}^{n} H_1(\mathfrak{g}, M_i) \oplus H_1(\mathfrak{g}, \mathbb{C})^s = 0$$

as $\mathfrak{g}^{\mathrm{ab}} = 0$ for $\mathfrak{g}$ semisimple. So in particular we have $H_1(\mathfrak{g}, M) = 0$ for all finite dimensional $\mathfrak{g}$-modules. This result, also known as Whitehead's Lemma, is a well known result. See [12] for instance.

ref. result 2C. *(Poincare duality) Let $\mathfrak{g}$ be any Lie algebra, and M a $\mathfrak{g}$-module. Then there exists an isomorphism of vector spaces*

$$H_n(\mathfrak{g}, M)^* \cong H^n(\mathfrak{g}, M^*)^*.$$

Let's now get back to our category of interest, namely $\mathrm{Int}_{\mathfrak{g}}$. Let $M \in \mathrm{Int}_{\mathfrak{g}}$. The homology modules of M are computed via the chain complex

$$0 \longrightarrow M \longrightarrow \mathfrak{g} \otimes M \longrightarrow \bigwedge^2 \mathfrak{g} \otimes M \longrightarrow \cdots.$$

Since M is integrable, for any natural i, by Theorem 2.5 we get a decomposition

$$M|_{\mathfrak{g}_i} = \bigoplus_{j \in I_i} M_{ij}$$

for some index set I_j, such that all M_{ij} are finite dimensional. Using the discussion following ref. result 2B, we get

$$H_1(\mathfrak{g}_i, M|_{\mathfrak{g}_i}) = \bigoplus_{j \in I_i} H_1(\mathfrak{g}_i, M_{ij}) = 0.$$

As this is true for any natural i, by taking direct limits we see that $H_1(\mathfrak{g}, M) = 0$, and by Poincare duality we get that

$$H^1(\mathfrak{g}, M^*) = H_1(\mathfrak{g}, M)^* = 0. \tag{2.12}$$

Using this, we are ready to prove the following result.

Theorem 2.15. *Let $X, M \in \mathrm{Int}_{\mathfrak{g}}$. Then the following holds:*

$$\mathrm{Ext}_{\mathfrak{g}}(X, M^*) = 0.$$

Proof. From Proposition 2.3, we have $X \otimes M \in \mathrm{Int}_{\mathfrak{g}}$ as well, and from (2.12) we get $H^1(\mathfrak{g}, (X \otimes M)^*) = 0$. Then we have

$$\mathrm{Ext}_{\mathfrak{g}}(X, M^*) \cong H^1(\mathfrak{g}, \hom_{\mathbb{C}}(X, M^*)) = H^1(\mathfrak{g}, (X \otimes M)^*) = 0,$$

which is what we wanted to show. □

In other words, this result says that by dualizing integrable modules, we obtain objects in $\mathfrak{g}$-mod which are $\mathrm{Int}_{\mathfrak{g}}$-injective. However M^* may not necessarily be an object of $\mathrm{Int}_{\mathfrak{g}}$ as we shall see in Example 2.2. The following consequence of Theorem 2.15, shows that we can actually obtain $\mathrm{Int}_{\mathfrak{g}}$-injective objects which lie in $\mathrm{Int}_{\mathfrak{g}}$.

Corollary 2.16. *For any $M \in \mathrm{Int}_{\mathfrak{g}}$, $\Gamma_{\mathfrak{g}}(M^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$.*

Proof. Let $X \in \mathrm{Int}_{\mathfrak{g}}$. Since $\Gamma_{\mathfrak{g}}(M^*) \subset M^*$, we have the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{g}}(M^*) \longrightarrow M^* \longrightarrow M^*/\Gamma_{\mathfrak{g}}(M^*) \longrightarrow 0 \tag{2.13}$$

in $\mathfrak{g}$-mod. Let now $X \in \mathrm{Int}_{\mathfrak{g}}$ be any module. By applying $\hom_{\mathfrak{g}}(X, -)$ to (2.13), we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow \hom_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \longrightarrow \hom_{\mathfrak{g}}(X, M^*) \longrightarrow \hom_{\mathfrak{g}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \longrightarrow \mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0. \end{aligned}$$

Note that from Remark 2.7 we have that $\Gamma_{\mathfrak{g}}(M^*/\Gamma_{\mathfrak{g}}(M^*) = 0$. Since $X \in \mathrm{Int}_{\mathfrak{g}}$, from Proposition 2.13 we get

$$\hom_{\mathfrak{g}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) = \hom_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*/\Gamma_{\mathfrak{g}}(M^*))) = \hom_{\mathfrak{g}}(X, 0) = 0.$$

This way we obtain that

$$0 \longrightarrow \mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \longrightarrow 0$$

is an exact sequence, thuswe get $\mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) = 0$. As this is true for all $X \in \mathrm{Int}_{\mathfrak{g}}$, it means that $\Gamma_{\mathfrak{g}}(M^*)$ is indeed an injective object of $\mathrm{Int}_{\mathfrak{g}}$. □

2.2.1 Constructing injective objects for integrable module

In this subsection we show how given an integrable module M, one can find an injective object $I \in \mathrm{Int}_{\mathfrak{g}}$ such that there exists an injective map $M \longrightarrow I$. To do this, we first begin with the following result.

Proposition 2.17. *Let $M \in Int_{\mathfrak{g}}$ and $m \in M$ any fixed non-zero element. If $f_m = m^* : M \longrightarrow \mathbb{C}$ is given by $f(m) = 1$, and $f_m(M/\mathbb{C}m) = 0$, then there exists a local submodule of M^* which contains f_m. In particular $\Gamma_{\mathfrak{g}}(M^*) \neq 0$.*

Proof. From Remark 2.4, there exists a local module $N = \lim N_i \subset M$ such that $m \in N$. As M is integrable and $N_i \subset M$ are $\mathfrak{g}_i$-submodules, and from Corollary 2.6 we get that there exist $\mathfrak{g}_i$-modules $R_i \subset M$ such that

$$M|_{\mathfrak{g}_i} = N_i \oplus R_i.$$

As N_i are finite dimensional $\mathfrak{g}_i$-modules, there exist $\mathfrak{g}_i$-modules $S_i \subset N_{i+1}$ such that

$$N_{i+1}|_{\mathfrak{g}_i} = N_i \oplus S_i.$$

Dualizing these equalities, we get

$$M^*|_{\mathfrak{g}_i} = N_i^* \oplus R_i^*; \quad \text{and} \quad N_{i+1}^*|_{\mathfrak{g}_i} = N_i^* \oplus S_i^*.$$

Note that these give us natural injective maps of $\mathfrak{g}_i$-modules $s_i : N_i^* \longrightarrow N_{i+1}^*$ and $t_i : N_i^* \longrightarrow M^*$ given by

$$s_i(f)|_{N_i} = t_i(f)|_{N_i} = f; \quad \text{and} \quad s_i(f)(S_i) = t_i(f)(R_i) = 0.$$

for $f \in N_i^*$. Since $R_i = R_{i+1} \oplus S_i$, we get $(t_{i+1} \circ s_i)(f)|_{N_i} = f$ and $(t_{i+1} \circ s_i)(f)(R_i) = 0$. Hence the diagram

$$\begin{tikzcd} N_i^* \arrow[rr, "t_i"] \arrow[dr, "s_i"'] & & M^* \\ & N_{i+1}^* \arrow[ur, "t_{i+1}"'] & \end{tikzcd}$$

is commutative. This way we can set $N_* := \varinjlim N_i^*$ and we get that this N_* is a well defined submodule of M^*. Clearly N_* is a local $\mathfrak{g}$−module. Since $m \in N$, there exists some $i \in \mathbb{N}$ such that $m \in N_i$. We then have $f_m \in N_i^*$, hence $f_m \in N_*$. Thus, we have proved the existence of a local module submodule of M^* as indicated in the statement of the proposition.

Since N_* is a local module, it is also integrable, hence we have $N_* \subset \Gamma_{\mathfrak{g}}(M^*)$, so in particular $\Gamma_{\mathfrak{g}}(M^*) \neq 0$, which completes the proof of the proposition. □

This result shows that the injective objects produced in Corollary 2.16 are always non-zero for $M \neq 0$.

Using Corollary 2.16 and Proposition 2.17 we will now show how given an integrable module M, one can produce an injective object $I \in \mathrm{Int}_{\mathfrak{g}}$ such that there exists an injective morphism of $\mathfrak{g}$−modules $M \longrightarrow I$.

Let $M \in \mathrm{Int}_{\mathfrak{g}}$. One then always has the natural injective map

$$i' : M \longrightarrow M^{**} \quad \text{given by} \quad i'(m)(f) = f(m) \quad \text{for all } f \in M^*. \tag{2.14}$$

As M is integrable, from Proposition 2.3 its image will also be integrable as a quotient of M, thus i' factors through the inclusion $\Gamma_{\mathfrak{g}}(M^{**}) \subset M^{**}$. Denote now $i = \Gamma_{\mathfrak{g}}(i')$ i.e. $i : M \longrightarrow \Gamma_{\mathfrak{g}}(M^{**})$ is given as in (2.14). Let now $\Gamma_{\mathfrak{g}}(M^*) \subset M^*$ be the natural inclusion, and denote by

$$\pi' : M^{**} \longrightarrow \Gamma_{\mathfrak{g}}(M^*)^*$$

its dual map, which is given by $\pi'(F)(f) = F(f)$ for all $F \in M^{**}$ and $f \in \Gamma_{\mathfrak{g}}(M^*) \subset M^*$. Denote now again by $\Gamma_{\mathfrak{g}}(\pi') = \pi : \Gamma_{\mathfrak{g}}(M^{**}) \longrightarrow \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*)$ the corresponding map of π' in $\mathrm{Int}_{\mathfrak{g}}$, which again will be given by the same expression as π'. Composing now i with π we get the following map

$$\phi = \pi \circ i : M \longrightarrow \Gamma_{\mathfrak{g}}(M^{**}) \longrightarrow \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*),$$

given by

$$\phi(m)(f) = f(m) \quad \text{for } m \in M \text{ and } f \in \Gamma_{\mathfrak{g}}(M^*). \tag{$\star$}$$

Note now that $\ker \phi = \{m \in M \mid f(m) = 0 \text{ for all } f \in \Gamma_{\mathfrak{g}}(M^*)\}$. From Proposition 2.17 we saw that given a non-zero $m \in M$ and $f_m = m^* \in M^*$ its corresponding linear map, we have that $f_m \in \Gamma_{\mathfrak{g}}(M^*)$. Since $f_m(m) = 1$, we see that $m \notin \ker \phi$. Hence we get $\ker \phi = 0$. This way we have proved the following corollary.

Corollary 2.18. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Then the map $\phi : M \longrightarrow \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*)$ given by ($\star$) is injective.*

Corollary 2.18 shows that $\mathrm{Int}_{\mathfrak{g}}$ has enough injectives in a constructive fashion, which is something that Corollary 2.16 did not do.

2.2.2 Further examples

In this section we give two examples. The first is of a non-integrable $\mathfrak{g}$–module, while the second one will be an integrable module whose dual is not integrable.

Example 2.1. Let $\mathfrak{g} = sl(\infty)$, and let $M = U(\mathfrak{g})$ be its universal enveloping algebra viewed as a $\mathfrak{g}$–module. For clarity of exposition, we denote the elements of $M = U(\mathfrak{g})$ as tensors. Here the action is given by

$$x.(x_1 \otimes x_2 \otimes \cdots \otimes \ldots x_n) = x \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

We claim that M is not integrable. For this, let $x \in sl(\infty)$ any non-zero element. From Section 1.4, there exists a totally ordered basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n, \ldots\}$ of $sl(\infty)$, say with order $x_i < x_j$ if $i < j$ for $i, j \in \mathbb{N}$. Without loss of generality we may assume that $x = x_1$. From Theorem PBW the set consisting of monomials

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \tag{2.15}$$

with $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ and $n \in \mathbb{N}_0$ forms a basis $\mathcal{X}$ of M. As $u \in M$, u will be a linear combination of elements of the form (2.15). Let now $u \in \mathcal{X}$, say

$$u = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$$

for some indices i_j, $j = 1, 2, \ldots, n$. In particular, we have $x = x_1 \leq x_{i_j}$ for all $j = 1, 2, \ldots, n$. Consider now the subspace $x(u)$. Note that

$$x^k.u = x_1^k.u = x_1 \otimes x_1 \otimes \cdots \otimes x_1 \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$$

and as $x_1 \leq x_1 \leq \cdots \leq x_1 \leq x_{i_1} \leq \cdots \leq x_{i_n}$, we have that $x^k.u \in \mathcal{X}$ is actually a basis vector for M. This gives us $x(u) \subset \mathcal{B}$, and as $\mathcal{X}$ is a basis, we have that $x(u)$ is a linearly independent set. As it is an infinite, we get that $\dim x(u) = \infty$. Thus we have that $\mathcal{X} \cap \Gamma_{\mathfrak{g}}(M) = 0$, so M cannot be integrable. Note that Remark 2.7 $N = M/\Gamma_{\mathfrak{g}}(M)$ is a non-trivial module such that $\Gamma_{\mathfrak{g}}(N) = 0$.

We will now investigate an example of a local module, the dual of which will be shown to not be integrable via a result to be proved later on in Chapter 3, namely Theorem 3.4.

Example 2.2. Let $\mathfrak{g} = sl(\infty)$. Denote, as earlier, by V_n the natural representation of $sl(n)$, and let $\{v_1, \dots, v_n\}$ be its natural basis. Consider now the n-th symmetric power of V_n given as a vector space by

$$S^n(V_n) := V^{\otimes n}/\sim,$$

where the relation $\sim$ is generated linearly by the expressions

$$v_1 \otimes v_2 \otimes \cdots \otimes \ v_n \sim v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

for all permutations $\sigma \in S_n$, where S_n denotes the symmetric group on n elements. One can show that a basis of $S^n(V_n)$ consists of elements of the form

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$$

with $i_1 \leq i_2 \leq \cdots \leq i_n$. By descending the action of $sl(n)$ on $V^{\otimes n}$, one makes $S^n(V_n)$ into an $sl(n)$–module. This action is described by

$$x.(v_{i_1} \otimes \cdots \otimes v_{i_n}) = (x.v_{i_1}) \otimes \cdots \otimes v_{i_n} + v_{i_1} \otimes (x.v_{i_2}) \cdots \otimes v_{i_n} + \cdots + v_{i_1} \otimes \cdots \otimes (x.v_{i_n}).$$

Consider the map $s_n : S^n(V_n) \longrightarrow S^{n+1}(V_{n+1})$ given by $s_n(w) = w \otimes v_{n+1}$. One can see that this map is linear, and since for $x \in sl(n)$ we have $x.v_{n+1} = 0$, we get that

$$x.s_n(w) = (x.w) \otimes v_{n+1} + w \otimes (x.v_{n+1}) = (x.w) \otimes v_{n+1} = s_n(x.w),$$

i.e. s_n is a map of $sl(n)$–modules. This way we can define a local $\mathfrak{g}$–module by setting $S(V) = \varinjlim S^i(V_i)$. Since $S(V) \in \mathrm{Loc}_{\mathfrak{g}}$, we have that $S(V)$ is integrable.

We now want to look at $S(V)$ as an $sl(2)$–module. For this, let us investigate $S^n(V_n)$ as $sl(2)$–modules. Denote the standard basis of $sl(2)$ by h, x, and y. Recall first the action of $sl(2)$ on V_n

$$x.v_1 = y.v_2 = 0; \quad y.v_1 = v_2; \quad x.v_2 = v_1;$$

$$h.v_1 = v_1; \quad h.v_2 = -v_2; \quad sl(2).v_i = 0 \text{ for } i > 2$$

Consider now an element of the form

$$s = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \in S^n(V_n).$$

Note that if $i_1 > 2$, then $i_n \geq \cdots \geq i_2 \geq i_1 > 2$, so we have $h.s = 0$, thus we get an $sl(2)$–module we have

$$\mathrm{span}\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mid i_1 > 2\} \subset S^n(V_n)_0 := S^n_0 \tag{2.16}$$

where $S^n(V_n)_0$ denotes the 0-weight space of $S^n(V_n)$. Assume now that $i_1 \leq 2$. Let $1 \leq l, k \leq n$ be such that $i_1 = i_2, \cdots = i_l = 1$ and $i_{l+1} = i_{l+2} = \cdots = i_{l+k} = 2 \neq i_{k+1}$, so we have

$$s = \underbrace{v_1 \otimes \cdots \otimes v_1}_{l \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{k \text{ times}} \otimes w$$

where w is such that $sl(2).w = 0$. Note now that we have

$$x.s = \underbrace{v_1 \otimes \cdots \otimes v_1}_{l \text{ times}} \otimes \underbrace{(x.v_2) \otimes \cdots \otimes v_2}_{k \text{ times}} \otimes w + \cdots + \underbrace{v_1 \otimes \cdots \otimes v_1}_{l \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes (x.v_2)}_{k \text{ times}} \otimes w =$$

$$= \underbrace{v_1 \otimes \cdots \otimes v_1 \otimes v_1}_{l+1 \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{k-1 \text{ times}} \otimes w + \cdots + \underbrace{v_1 \otimes \cdots \otimes v_1}_{l \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{k-1 \text{ times}} \otimes v_1 \otimes w =$$

$$= k\,\underbrace{v_1 \otimes \cdots \otimes v_1 \otimes v_1}_{l+1 \text{ times}} \otimes \cdots \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{k-1 \text{ times}} \otimes w.$$

Iterating this argument we see that

$$x^k.s = \underbrace{v_1 \otimes \cdots \otimes v_1}_{l+k \text{ times}} \otimes w.$$

It is clear then that $x^{k+1}.s = 0$. One can similarly show that

$$y^l.s = \underbrace{v_2 \otimes \cdots \otimes v_2}_{k+l \text{ times}} \otimes w,$$

and $y^{l+1}.s = 0$. Set now

$$S_t^n = \operatorname{span}\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mid i_t \le 2 < i_{t+1}\}.$$

From the previous calculations, we can see that for any $s \in S_t^n$ we have $sl(2,\mathbb{C}).s = S_t^n$, so S_t^n is a simple $sl(2,\mathbb{C})$–submodule of $S^n(V)$. If we set $s_t = \underbrace{v_1 \otimes \cdots \otimes v_1}_{t \text{ times}} \otimes w.$, since $x.s_t = 0$, we see that s_t is the highest weight vector of S_t^n. Note that

$$h.s_t = ts_t$$

hence the highest weight of S_t^n is t, thus $\dim S_t^n = t+1$. It is clear that these s_t's along with $\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mid i_1 > 2\}$ generate $S^n(V_n)$, thus we actually have an equality in (2.16)

$$\operatorname{span}\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mid i_1 > 2\} = S_0$$

which is trivial as an $sl(2)$–module, hence the decomposition into simples of $S^n(V_n)$ as an $sl(2)$–module is

$$S^n(V_n)|_{sl(2)} = S^n(V_n)_0 \oplus \bigoplus_{t=1}^{n} S_t$$

In particular, the largest dimension of a simple submodule of $S^n(V_n)$ is $n+1 = \dim S_n^n$.

If we look now at $S(V)$ as an $sl(2)$–module, since $S^n(V_n) \subset S(V)$, we have that for any natural number n, there exists a simple submodule $S_n^n \subset S(V)$ such that $\dim S_n^n = n+1$. In particular, $S(V)$ consists infinitely many non-isomorphic simple $sl(2)$–submodules.

To make this more precise, there exist infinitely many non-isomorphic $sl(2,\mathbb{C})$–modules N such that $\hom_{sl(2,\mathbb{C})}(N, S(V)) \neq 0$. Theorem 3.4 which will be proved early on in Chapter 3 shows that in this situation, $S(V)^*$ will not be integrable.

We have this way exhibited the first example of an integrable $\mathfrak{g}$–module for which its dual is not in $\mathrm{Int}_{\mathfrak{g}}$. We emphasize this in the following note.

Comparison remark IV. *As opposed to the finite dimensional theory of representations of semisimple Lie algebras, even a category as reasonable as* $\mathrm{Int}_{\mathfrak{g}}$ *of representations of locally semisimple Lie algebras is not closed under algebraic dualization.*

This shows that algebraic dualization is an operation which can lead to wild behavior even of local modules, as $S(V)$ illustrates this. In the next section we will introduce a restricted dual which behave in a much nicer way under certain conditions. Chapter 3 will mostly be devoted towards studying the integrability of the duals of objects in $\mathrm{Int}_{\mathfrak{g}}$.

2.3 Weight modules

Here we will introduce the category of weight $\mathfrak{g}$-modules, and in particular study its intersection with $\mathrm{Int}_{\mathfrak{g}}$. In particular we will define a restricted dual for integrable modules, which turns out to have some nice properties. Before we proceed with the main contents of this section, let us first make note of a technical observation.

Let I be an index set, and $M_i \in \mathfrak{g}$–mod for all $i \in I$. Set now

$$M = \bigoplus_{i \in I} M_i.$$

For each $i \in I$ let $p_i : M \longrightarrow M_i$ be the natural projections. Note first that given $m \in M$, there exists a finite subset $I_0 = \{i_1, \dots, i_n\} \subset I$ and $m_j \in M_{i_j}$ such that

$$m = m_1 + \dots + m_n.$$

It is clear that $p_{i_j}(m) = m_j$ for $j = 1, \dots, n$ and $i_j \in I_0$. We also see that $p_i(m) = 0$ for $i \in I \backslash I_0$. This means that

$$m = \bigoplus_{i \in I} p_i(m).$$

Define now maps $\beta_i : M^* \longrightarrow M_i^*$ by $\beta_i(f) = f|_{M_i}$. These are clearly linear maps. We want to show that they are actually $\mathfrak{g}$–module morphisms. For this purpose, let $x \in \mathfrak{g}$, and $f \in M^*$. Let $m \in M_i$. Note now that

$$\beta_i(x.f)(m) = (x.f)|_{M_i}(m) = (x.f)(m) = -f(x.m) = -f|_{M_i}(x.m) =$$

$$= -\beta_i(f)(x.m) = (x.\beta_i(f))(m)$$

hence we really get $\beta_i(x.f) = x.\beta_i(f)$, thus these β_i's are actually morphisms of $\mathfrak{g}$–modules.

Consider now the map

$$s : M^* \longrightarrow \prod_{i \in I} M_i^* \tag{2.17}$$

given by $s = \prod_{i \in I} \beta_i = (\beta_i)_{i \in I}$. One can see that this will be a morphisms of $\mathfrak{g}$–modules. We claim that this is actually an isomorphism.

First let $f \in M^*$ be such that $s(f) = 0$. Let now $m \in M$, and let $I_0 \subset I$ consist of those indices i such that $p_i(m) \neq 0$. Naturally I_0 will be a finite subset of I. Note now that $s(f) = 0$ means that $\beta_i(f) = 0$ for all $i \in I_0$. In particular for all $i \in I_0$ we have that

$$0 = \beta_i(f)(p_i(m)) = f|_{M_i}(p_i(m)) = f(p_i(m)),$$

thus we obtain $f(m) = f\left(\sum_{i \in I_0} p_i(m)\right) = 0$. Since $m \in M$ was arbitrary, we get that $f = 0$, thus $\ker s = 0$, so s is an injective map.

Let now $(f_i)_{i \in I} \in \prod_{i \in I} M_i^*$. Define $f \in M^*$ by

$$f(m) = \sum_{i \in I} f_i(p_i(m)).$$

Since we have that $p_i(m) \neq 0$ for only finitely many indices $i \in I$, we can see that f is well-defined. One can then easily compute to see that f is a linear map, and note that for a given $j \in I$ we have for $m \in M_j$

$$\beta_j(f)(m) = f|_{M_j}(m) = \sum_{i \in I} f_i(p_i(m)) = f_j(p_j(m)) = f_j(m),$$

because $p_j(m) = m$ for $m \in M_j$. Thus we get that $s(f) = (f_i)_{i \in I}$, i.e. s is a surjective map.

Hence we really have that s in (2.17) is an isomorphism of $\mathfrak{g}$–modules.

Remark 2.8. We remark that this discussion still holds for any Lie algebra $\mathfrak{g}$.

2.3.1 Definitions, properties, and examples

Let $\mathfrak{g}$ be a semisimple Lie algebra. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *splitting Cartan subalgebra* if $\mathfrak{h}$ is abelian, and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \tag{2.18}$$

where

$$\mathfrak{g}_\alpha := \{ g \in \mathfrak{g} \ \mid \ [h, g] = \alpha(h) g \ \text{ for all } \ h \in \mathfrak{h} \},$$

If $\alpha \in \mathfrak{h}^*$ is such that $\mathfrak{g}_\alpha \neq 0$, we call α a *root* of $\mathfrak{g}$. In other words, we say that $\mathfrak{g}$ admits a splitting Cartan subalgebra $\mathfrak{h}$ as above if it decompososes as a direct sum of $\mathfrak{h}$ and its root spaces.

Example 2.3. Let $\mathfrak{g} = sl(\infty)$. One can naturally identify $sl(\infty)$ as a subspace of $\mathrm{Mat}_\infty(\mathbb{C}) = \{(m_{ij})_{i,j \in \mathbb{N}} \mid m_{ij} \in \mathbb{C}\}$. Let now $\mathfrak{h} \subset sl(\infty)$ be the subspace consisting of finitary diagonal matrices, i.e. the diagonal matrices in $\mathrm{Mat}_\infty(\mathbb{C})$ which have only finitely many non-zero entries. One can see that $[\mathfrak{h}, \mathfrak{h}] = 0$, hence $\mathfrak{h}$ is an abelian subalgebra of $sl(\infty)$. Clearly a natural basis of $\mathfrak{h}$ is $\{h_i = E_{ii} - E_{i+1,i+1} \mid i \geq 1\}$. For each natural i, define a map $\epsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$ by

$$\epsilon_i \left(\sum_{j=1}^{n} x_j E_{jj} \right) = x_i.$$

Set now $\alpha_{ij} := \epsilon_i - \epsilon_j : \mathfrak{h} \longrightarrow \mathbb{C}$ for $i \neq j$. Assume that $i < j$. Then if $i+1 < j$ one has

$$\alpha_{ij}(h_k) = \begin{cases} 1 & \text{if } k \in \{i, j-1\}, \\ -1 & \text{if } k \in \{i-1, j\}, \\ 0 & \text{otherwise.} \end{cases}$$

and if $i + 1 = j$

$$\alpha_{i,i+1}(h_k) = \alpha_{i,i+1}(h_k) = \begin{cases} -1 & \text{if } k = i-1, \\ 2 & \text{if } k = i, \\ 1 & \text{if } k = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

We now want to compute $[h_k, E_{ij}]$. First note that

$$[E_{kk}, E_{ij}] = E_{kk}E_{ij} - E_{ij}E_{kk} = \delta_{ki}E_{kj} - \delta_{kj}E_{ik},$$

so we have

$$[E_{kk}, E_{ij}] = \begin{cases} E_{ij} & \text{for } k = i, \\ -E_{ij} & \text{for } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Using this, we obtain first for $i + 1 < j$:

$$[h_k, E_{ij}] = [E_{kk}, E_{ij}] - [E_{k+1,k+1}, E_{ij}] = \begin{cases} E_{ij} & \text{for } k \in \{i, j-1\}, \\ -E_{ij} & \text{for } k \in \{i-1, j\}, \\ 0 & \text{otherwise.} \end{cases} \quad = \quad \alpha_{ij}(h_k) E_{ij}$$

and get for $i + 1 = j$

$$[h_k, E_{i,i+1}] = \begin{cases} -E_{i,i+1} & \text{if } k = i-1, \\ 2E_{i,i+1} & \text{if } k = i, \\ E_{i,i+1} & \text{if } k = i+1, \\ 0 & \text{otherwise.} \end{cases} \quad = \quad \alpha_{i,i+1}(h_k) E_{i,i+1}$$

Thus for any $i < j$ we have that $E_{ij} \in \mathfrak{g}_{\alpha_{ij}}$. If $j < i$, one can similarly show that $E_{ij} \in \mathfrak{g}_{\alpha_{ij}}$. This actually shows that $\mathbb{C}E_{ij} = \mathfrak{g}_{\alpha_{ij}}$ and therefore

$$sl(\infty) = \mathfrak{h} \oplus \bigoplus_{1 \leq i \neq j} \mathfrak{g}_{\alpha_{ij}}.$$

Hence $sl(\infty)$ admits a splitting Cartan subalgebra. Note that this root space decomposition of $sl(\infty)$ is very similar to that of the finite dimensional special linear algebras. This is no coincidence, as in fact one can obtain the root decomposition of $sl(\infty)$ with regards to this $\mathfrak{h}$ also from the analogous root decomposition in the finite dimensional case. Indeed, we know that

$$sl(n) = \mathfrak{h}_n \oplus \mathfrak{sl}(\mathfrak{n})_{\alpha_{ij}^{(n)}}$$

where $\mathfrak{h}_n$ is the subalgebra of $sl(n)$ consisting of diagonal matrices. Comparing this with the root decomposition of $sl(n+1)$, with the standard Cartan subalgebra $\mathfrak{h}_{n+1}$, if $i, j < n+1$, from

$$[h_k, E_{ij}] = \alpha_{ij}^{(n)}(h_k)E_{ij} \quad \text{in } sl(n) \quad \text{and}$$

$$[h_k, E_{ij}] = \alpha_{ij}^{(n+1)}(h_k)E_{ij} \quad \text{in } sl(n+1)$$

we get that

$$\alpha_{ij}^{(n+1)}|_{sl(n)} = \alpha_{ij}^{(n)}.$$

Note that we also have $sl(n)_{\alpha_{ij}^{(n)}} = \mathbb{C}E_{ij} = sl(n+1)_{\alpha_{ij}^{(n+1)}}$. This means that for the natural inclusions $i_n : sl(n) \longrightarrow sl(n+1)$, we have that $i_n(\mathfrak{h}_n) \subset \mathfrak{h}_{n+1}$, and $i_n(sl(n)_{\alpha_{ij}^{(n)}}) \subset sl(n+1)_{\alpha_{ij}^{(n+1)}}$. Now by setting $\mathfrak{h} = \varinjlim \mathfrak{h}_n$, $\alpha_{ij} = \varinjlim \alpha_{ij}^{(n)}$, and $\mathfrak{g}_{\alpha_{ij}} = \varinjlim \mathfrak{g}_{\alpha_{ij}^{(n)}}$, since $sl(\infty) = \varinjlim sl(n)$ we get that

$$sl(\infty) = \mathfrak{h} \oplus \bigoplus_{1 \leq i \neq j} \mathfrak{g}_{\alpha_{ij}}.$$

which gives another way of decomposing $sl(\infty)$ into a direct sum of its Cartan subalgebra, and its root spaces.

Remark 2.9. In [11] it is shown that if a locally semisimple Lie algebra $\mathfrak{g}$ admits a splitting Cartan subalgebra, then there exists index sets I_1, I_2, I_3, and I_4 and finite dimensional simple Lie algebras $\mathfrak{r}_i$ for $i \in I_4$, such that

$$\mathfrak{g} = \bigoplus_{i \in I_1} sl(\infty) \oplus \bigoplus_{i \in I_2} o(\infty) \oplus \bigoplus_{i \in I_3} sp(\infty) \uplus \bigoplus_{i \in I_4} \mathfrak{r}_i.$$

However this result will not be particularly relevant in this thesis.

Let us note that for $g_1 \in \mathfrak{g}_\alpha$ and $g_2 \in \mathfrak{g}_\beta$ for all $h \in \mathfrak{h}$ we have that $[h[g_1, g_2]] = [[h, g_1], g_2] + [g_1, [h, g_2]] = \alpha(h)[g_1, g_2] + \beta(h)[g_1, g_2] = (\alpha + \beta)(h)[g_1, g_2]$, which gives us $[g_1, g_2] \in \mathfrak{g}_{\alpha+\beta}$.

Let now M be any $\mathfrak{g}$-module. For any $\lambda \in \mathfrak{h}^*$ define

$$M_\lambda := \{m \in M \mid h.m = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}.$$

If $\lambda \in \mathfrak{h}^*$ is such that $M_\lambda \neq 0$, we call λ a *weight* for M, and call M_λ the λ-*weight space* of M. If λ is not a weight for M, clearly $M_\lambda = 0$. We first note some computational properties of weight spaces.

Lemma 2.19. *The following statements hold:*

i) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *for any roots* α, β *of* $\mathfrak{g}$*;*

ii) $\mathfrak{g}_\alpha.M_\lambda \subset M_{\alpha+\lambda}$ *for any root* α *of* $\mathfrak{g}$*, and weight* λ *of* M*;*

iii) $M_\lambda \cap \left(\sum_{\mu\neq\lambda} M_\mu\right) = 0$*, for any weight* λ *of* M*, where* μ *runs over all the weights of* M *different from* λ*.*

Proof. Part i) follows directly from the earlier computation of $[h,[g_1,g_2]]$. For part ii), let $g \in \mathfrak{g}_\alpha$, and $m \in M_\lambda$. For any $h \in \mathfrak{h}$ we have

$$h.(g.m) = g.(h.m) + [h,g].m = \lambda(h)(g.m) + \alpha(h)(g.m) = (\alpha+\lambda)(h)(g.m)$$

hence $g.m \in M_{\alpha+\lambda}$.

iii) Let us first note that for $\lambda \neq \mu$, one has $M_\lambda \cap M_\mu = 0$. Indeed, let $m \in M_\lambda$ non-zero, and let $h \in \mathfrak{h}$ be such that $\lambda(h) \neq \mu(h)$. We then have $h.m = \lambda(h)m \neq \mu(h)m$, i.e. $m \notin M_\mu$, thus $M_\lambda \cap M_\mu = 0$. Assume now that $M_\lambda \cap \left(\sum_{\mu\neq\lambda} M_\mu\right) \neq 0$, i.e. there is some $m \in M_\lambda$ so that $m \in \sum_{\mu\neq\lambda} M_\mu$. One then has pariwise distinct $\mu_1, \cdots, \mu_n \neq \lambda$, and $m_i \in M_{\mu_i}$ all non-zero, such that

$$m = m_1 + \cdots + m_n. \tag{2.19}$$

Denote $\lambda = \mu_0$ for convenience. Let $m_0 \in M_\lambda$ be such that it can be written as in (2.19) with n minimal. From the previous discussion, we have $n > 1$. Let $k \in \mathbb{N}$ be such that $\mu_i' = \mu_i|_{\mathfrak{h}_k} \neq \mu_j|_{\mathfrak{h}_k} = \mu_j'$ for all $i,j \in \{0,1,\cdots,n\}$ with $i \neq j$, where $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}_k$. This is possible because of our assumption that the μ_i are pairwise distinct. Assume that $\mu_1' \neq 0$. If $x\mu_1' \neq \mu_i'$ for all $x \in \mathbb{C}$, then $\ker \mu_1' \neq \ker \mu_i'$, so one can find some $h_1 \in \mathfrak{h}_k$ such that $\mu_1'(h_1) = 1$ and $\mu_i'(h_1) = 0$. Let $1 \leq s \leq n$ be such that for $i = 1, \cdots, s$ one has $\mu_i' = x_i\mu_1'$ for some $x_i \in \mathbb{C}$, and for $i = s+1, \cdots, n$ one has $\mu_i' \neq x\mu_1'$ for all $x \in \mathbb{C}$. If $s = 1$, we have

$$\lambda(h_1)m_0 = h_1.m_0 = h_1.m_1 = m_1$$

which cannot be possible, as $\lambda \neq \mu_1$ implies $M_\lambda \cap M_{\mu_1} = 0$ from the beginning of the proof of iii). Assume now that $s \geq 2$. Then we have

$$\lambda(h_1)m_0 = h_1.m_0 = h_1.m_1 + \cdots + h_1.m_k = m_1 + x_1m_2 + \cdots x_km_k$$

i.e.

$$h_1.m_0 - m_0 = (x_2 - 1)m_2 + \cdots + (x_s - 1)m_s \tag{2.20}$$

But $h_1.m_0 \in M_\lambda$ from ii), so $h_1.m_0 - m_0 \in M_\lambda$ as well. Since $\mu_1' \neq \mu_2'$, we have that $x_2 - 1 \neq 0$, hence in (2.20) we have exhibited an element of M_λ which admits an expression as in (2.19), with $s - 1 < n$ summands, and this contradicts our assumption on the minimality of n. Thus we indeed have

$$M_\lambda \left(\cap \sum_{\mu\neq\lambda} M_\mu\right) = 0,$$

which is what we wanted to show. □

Note that from Lemma 2.19, if $A, B \subset \mathfrak{h}^*$ are such that $A \cap B = \varnothing$, we get

$$\left(\sum_{\lambda\in A} M_\lambda\right) \cap \left(\sum_{\mu\in B} M_\mu\right) = 0 \tag{2.21}$$

2.3.2 The category $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ and the functor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}} : \mathfrak{g}-\mathbf{mod} \longrightarrow \mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$

Let now $\mathfrak{g}$ be a locally semisimple Lie algebra that admits a splitting Cartan subalgebra $\mathfrak{h}$. Let M be a $\mathfrak{g}$−module, and consider the subspace of M

$$\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) := \mathrm{span}\{M_\lambda \mid \lambda \in \mathfrak{h}^*\}.$$

As $\mathfrak{g}$ admits a decomposition as in (2.18), from (2.21) we can see that

$$\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \tag{2.22}$$

and part ii) of Lemma 2.19 shows that $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$ will be a $\mathfrak{g}$-submodule of M. In fact this decomposition is a decomposition into $\mathfrak{h}$-submodules. Call M an $\mathfrak{h}$-*weight module* if $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) = M$, and denote by $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ the largest full subcategory of $\mathfrak{g}$ consisting of $\mathfrak{h}$-weight modules. We then can define a functor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}} : \mathfrak{g}\text{-mod} \longrightarrow \mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ given on objects by (2.22), and $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(f) := f|_{\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)}$ for morphisms $f : A \longrightarrow B$.

In this thesis, we are not especially concerned with the category $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ as a whole, but rather in its intersection with $\mathrm{Int}_{\mathfrak{g}}$. However, we shall note a few examples and properties of it which are useful in our context.

Example 2.4. Let $\mathfrak{g} = sl(\infty)$. In Example 2.3 we saw that $\mathfrak{h} = \{\text{diagonals in } sl(\infty)\}$ is a splitting Cartan subalgebra, and have the root space decomposition

$$sl(\infty) = \mathfrak{h} \oplus \bigoplus_{1 \le i \ne j} \mathfrak{g}_{\alpha_{ij}}$$

Let now V be the natural representaion of $sl(\infty)$. Let $\{v_1, \dots, v_n, \dots\}$ be the standard basis of V. Then we have

$$h_k.v_1 = \begin{cases} v_1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and for $i > 1$ we have:

$$h_k.v_i = \begin{cases} -v_i & \text{if } k = i-1, \\ v_i & \text{if } k = i. \end{cases}$$

So we have that every v_i is a weight vector, with weight $\lambda_i : \mathfrak{h} \longrightarrow \mathbb{C}$ given by:

$$\lambda_1(h_1) = 1; \quad \lambda_1(h_k) = 0 \quad \text{for } k > 1$$

and for $i > 1$

$$\lambda_i(h_k) = \begin{cases} -1 & \text{if } k = i-1, \\ 1 & \text{if } k = i. \end{cases}$$

This means that $V_{\lambda_i} = \mathbb{C}v_i$, and thus

$$V = \mathrm{span}\{v_1, v_2, \dots\} = \mathrm{span}\{V_{\lambda_i} \mid i \in \mathbb{N}\} = \bigoplus_{i=1}^{\infty} V_{\lambda_i}$$

hence V is a weight module. One could actually obtain this weight space decomposition of V from the well known weight space decompositions of the natural representations of $sl(n)$'s, in a similar way to what we did with the root space decomposition of $sl(\infty)$. One can also show similarly that the conatural representation V_* is also a weight module, but we will soon see this in Example 2.6 after having investigated the funtor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ more.

We now proceed with noting a few more properties of the category $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$.

Proposition 2.20. $\mathfrak{g}_{\mathfrak{h}}^{wt}$ *is closed under taking submodules, quotients, finite direct sums, and tensors. In particular,* $\mathfrak{g}_{\mathfrak{h}}^{wt}$ *is an abelian subcategory of* $\mathfrak{g}$*–mod.*

Proof. For direct sums this follows immediately from $(M \oplus N)_\lambda = M_\lambda \oplus N_\lambda$.

We now prove the proposition for quotients. Let $M \in \mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$, so we have

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$

and let $N \subset M$ be a $\mathfrak{g}$-submodule of M. Consider the quotient M/N. Note that for every $\lambda \in \mathfrak{h}^*$ we have that $(M_\lambda + N)/N \subset (M/N)_\lambda$. Really, given $s \in (M_\lambda + N)/N$, say $s = m + N$ with $m \in M_\lambda$, one has for any $h \in \mathfrak{h}$ that $h.s = h.(m+N) = h.m + N = \lambda(h)m + N = \lambda(h)(m+N) = \lambda(h)s$, i.e. $s \in (M/N)_\lambda$. This implies

$$M/N \supset \bigoplus_{\lambda \in \mathfrak{h}^*} (M/N)_\lambda \supset \mathrm{span}\{(M_\lambda + N)/N \mid \lambda \in \mathfrak{h}^*\} =$$

$$= \mathrm{span}\{M_\lambda + N \mid \lambda \in \mathfrak{h}^*\}/N = M/N,$$

i.e. $M/N = \bigoplus_{\lambda \in \mathfrak{h}^*} (M/N)_\lambda$, so M/N will also be an $\mathfrak{h}$-weight module.

In fact from this proof one can also see that $(M_\lambda + N)/N = (M/N)_\lambda$ for all $\lambda \in \mathfrak{h}^*$, so in particular we have that for any pairwise distinct $\lambda_1, \cdots, \lambda_k \in \mathfrak{h}^*$ we have

$$\mathrm{span}\{(M_{\lambda_i} + N)/N \mid i = 1, \cdots, k\} = \bigoplus_{i=1}^{k} (M_{\lambda_i} + N)/N.$$

We now show that any submodule N will also be an $\mathfrak{h}$-weight module. For this, let $m \in N \subset M$. As M is an $\mathfrak{h}$-weight module, there exist pairwise distinct $\lambda_1, \cdots, \lambda_k \in \mathfrak{h}^*$, and $m_i \in M_{\lambda_i}$ for $i = 1, \cdots, k$ such that $n = m_1 + \cdots + m_k$. Consider now the image of this element m under the canonical map $p : M \longrightarrow M/N$. We have

$$0 = p(m) = p(m_1) + \cdots + p(m_k) \in \bigoplus_{i=1}^{k} (M_{\lambda_i} + N)/N$$

with $p(m_i) \in (M_{\lambda_i} + N)/N$. Clearly then one must have $p(m_i) = 0$, i.e. $m_i + N = p(m_i) = N$, hence $m_i \in N$. Thus we have that $m_i \in N \cap M_{\lambda_i} = N_{\lambda_i}$, so $n \in \bigoplus_{\lambda \in \mathfrak{h}^*} N_\lambda$. This shows that

$$N = \bigoplus_{\lambda \in \mathfrak{h}^*} N_\lambda$$

i.e. N is also an $\mathfrak{h}$-weight module.

As for tensor products, given M and N weight modules, one can easily compute to show that $M_\lambda \otimes N_\beta \subset (M \otimes N)_{\lambda+\beta}$, hence we get

$$M \otimes N = \left(\bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \right) \otimes \left(\bigoplus_{\lambda \in \mathfrak{h}^*} N_\lambda \right) = \bigoplus_{\lambda, \mu \in \mathfrak{h}^*} (M_\lambda \otimes N_\mu) \subset \bigoplus_{\lambda \in \mathfrak{h}^*} (M \otimes N)_\lambda$$

hence $M \otimes N$ will also be an $\mathfrak{h}$-weight module. □

If M is a $\mathfrak{g}$-module, since $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) \subset M$, we have $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)_\lambda \subset M_\lambda$ for any $\lambda \in \mathfrak{h}^*$. Moreover, given $m \in M_\lambda$, from Proposition 2.20 we have $m \in \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$ with $h.m = \lambda(h)m$ for all $h \in \mathfrak{h}$, hence $m \in \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)_\lambda$. This shows that $M_\lambda = \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)_\lambda$, thus we get $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)) = \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$. Now if $N \subset M$, one has $N_\lambda = N \cap M_\lambda$, hence $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(N) \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$. We note a consequence of this in the following remark.

Remark 2.10. Let N be an $\mathfrak{h}$-weight submodule of M We then have $N = \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(N) \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$, i.e. $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M)$ is the largest $\mathfrak{h}$-weight submodule of M.

Now using Remark 2.10 we can prove the following result.

Lemma 2.21. *The following statements hold:*

i) $\Gamma_{\mathfrak{h}}^{wt}(A) \subset A$ *for any* $A \in \mathfrak{g}$*-mod;*

ii) $\Gamma_{\mathfrak{h}}^{wt}(f) = f|_{\Gamma_{\mathfrak{h}}^{wt}(M)}$ *for any morphism* $f : A \longrightarrow B$ *in* $\mathfrak{g}$*-mod;*

iii) $\Gamma_{\mathfrak{h}}^{wt}(A) = A \cap \Gamma_{\mathfrak{h}}^{wt}(B)$ *if* $A \subset B$ *in* $\mathfrak{g}$*-mod.*

Proof. i) and ii) are direct consequences of the definition of $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$. For iii), as $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(A) \subset B$ is a weight module, from Remark 2.10 we see that $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(A) \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B)$. This gives us $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(A) \subset A \cap \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B)$. Conversely, note that from Proposition 2.20 we have that $A \cap \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B)$ will be a weight module, as a submodule of $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B)$. Again by Remark 2.20 we have that

$$A \cap \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B) \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(A).$$

This way, altogether we obtain

$$\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(A) = A \cap \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(B),$$

which is what we wanted to show. □

Lemma 2.21 says that the functor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ satisfies the conditions of Proposition 1.9'. Then $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ will also be essentially surjective, and $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(\mathfrak{g}\text{-mod}) = \mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$. In particular, Proposition 1.9' implies the following result.

Proposition 2.22. $\Gamma_{\mathfrak{h}}^{wt}$ *is a left-exact functor, and it is right adjoint to the inclusion* $\subset$: $\mathfrak{g}_{\mathfrak{h}}^{wt} \longrightarrow \mathfrak{g}$*-mod*

Applying Corollary 1.10 to our situation for the functor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ we get:

Corollary 2.23. *The category* $\mathfrak{g}_{\mathfrak{h}}^{wt}$ *has enough injectives.*

This concludes our investigation of $\mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$ – mod as a category in itself, as our main focus will be on integrable modules which are also weight modules.

2.3.3 Integrable weight modules

We start off by the following result, which is not of high importance in the context of what follows, but it shows a similarity of what we are doing here with a different well-known theory.

Proposition 2.24. *Let* $M \in \mathfrak{g}_{\mathfrak{h}}^{wt}$ *– mod,* $m \in M$ *and* $g \in \mathfrak{g}_\alpha$ *with* $\alpha \neq 0$*. Then*

$$\dim g(m) < \infty \Longleftrightarrow g^n.m = 0 \text{ for some } n \in \mathbb{N}.$$

Proof. If $g^n.m = 0$ for some natural n, then clearly $\dim g(m) < \infty$.

Conversely, let $\dim g(m) < \infty$. Assume that $m \in M_\lambda$. As $g \in \mathfrak{g}_\alpha$, we have from part ii) of Lemma 2.19 that $g^i.m \in M_{\lambda+i\alpha}$. Since $\dim g(m) < \infty$, from part a) of Remark 2.1 we have that there exists some natural $n \in \mathbb{N}_0$ and $x_i \in \mathbb{C}$ for $i = 0, 1, \cdots, n$ not all zero such that

$\sum_{i=1}^n x_i g^i.m = 0$. Without loss of generality, we may assume that $x_n \neq 0$. Then by setting $y_i = -\frac{x_i}{x_n}$ we get

$$g^n.m = \sum_{i=0}^{n-1} y_i g^i.m.$$

The left hand side of this equation lies in $M_{\lambda+n\alpha}$, while the right hand side lies in $\bigoplus_{i=0}^{n-1} M_{\lambda+i\alpha}$. From Lemma 2.19 part iii), we then must have $g^n.m = 0$, which proves the claim. $\square$

This shows that if a weight module M is integrable, then each $g \in \mathfrak{g}_\alpha$ for $\alpha \neq 0$ acts locally nilpotently on M. This echoes the definition of integrable weight modules in the theory of quantum groups, see for instance [9], and can be thought of as a justification for the terminology 'integrable module' in the context of locally semisimple Lie algebras.

We define $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ to be the largest full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ consisting of $\mathfrak{h}$-weight modules. Naturally one can describe $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ also as the largest full subcategory of $\mathfrak{g}^{\mathrm{wt}}_{\mathfrak{h}}$ consisting of integrable modules. In fact let M be any $\mathfrak{g}$-module, and consider the modules $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M))$ and $\Gamma_{\mathfrak{g}}(\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M))$. As $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M))$ is an $\mathfrak{h}$-weight submodule of M, we have that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M)) \subset \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M)$, and as $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M))$ is also integrable, as a submodule of $\Gamma_{\mathfrak{g}}(M)$, we have that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M)) \subset \Gamma_{\mathfrak{g}}(\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M))$. Arguing symmetrically, we get that

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M)) = \Gamma_{\mathfrak{g}}(\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M)).$$

We can then define a functor $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} : \mathfrak{g} - \mathrm{mod} \longrightarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ by

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M) := \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M))$$

on objects, and $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(f) = f|_{\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M)}$ on morphisms. Note now that if $N \subset M \in \mathfrak{g}$−mod, since $N \subset M$, from Lemma 2.12 we have

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(N) = \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(N)) = \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(N \cap \Gamma_{\mathfrak{g}}(M)),$$

and since $N \cap \Gamma_{\mathfrak{g}}(M) \subset \Gamma_{\mathfrak{g}}(M)$, from Lemma 2.21 we get

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(N) = N \cap \Gamma_{\mathfrak{g}}(M) \cap \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(M)) = N \cap \Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M),$$

because $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M) \subset \Gamma_{\mathfrak{g}}(M)$. This way we have proved the following result.

Lemma 2.25. *The following statements hold:*

i) $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(A) \subset A$ *for any* $A \in \mathfrak{g}$*-mod,*

ii) $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(f) = f|_{\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(M)}$ *for any morphism* $f : A \longrightarrow B$ *in* $\mathfrak{g}$*-mod,*

iii) $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(A) = A \cap \Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(B)$ *if* $A \subset B$ *in* $\mathfrak{g}$*-mod.*

Lemma 2.25 says that the functor $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ satisfies the conditions of 1.9'. Then $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ will also be essentially surjective, and $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(\mathfrak{g} - \mathrm{mod}) = \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$. In particular, Proposition 1.9' implies the following result.

Proposition 2.26. $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}$ *is a left-exact functor, and it is right adjoint to the inclusion* $\subset$: $Int^{wt}_{\mathfrak{g},\mathfrak{h}} \longrightarrow \mathfrak{g} - mod$.

Applying Corollary 1.10 to our situation for the funtor $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ we get the following result.

Corollary 2.27. *The category* $Int^{wt}_{\mathfrak{g},\mathfrak{h}}$ *has enough injectives.*

Remark 2.11. Note that given $M \in \mathrm{Int}_{\mathfrak{g}}$, Theorem 2.15 shows that M^* is an $\mathrm{Int}_{\mathfrak{g}}$−injective object. Since $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*) \in \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} \subset \mathrm{Int}_{\mathfrak{g}}$ and $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ is right adjoint to the inclusion $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} \subset$ $\mathfrak{g}$−mod, we have that $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)$ is going to be an injective object of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.

We have seen that $\mathrm{Int}_{\mathfrak{g}}$ is not a semisimple category, as we have exhibited non-splitting exact sequences in it. The following example shows that the subcategory $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ of $\mathrm{Int}_{\mathfrak{g}}$ is not semisimple.

Example 2.5. Let $\mathfrak{g} = sl(\infty)$, and denote by $M = \mathrm{Mat}_\infty(\mathbb{C})$ the vector space consisting of all infinite complex matrices $m = (m_{ij})_{i,j\in\mathbb{N}}$. Note that one embeds $\mathfrak{gl}(n) \longrightarrow M$ via the identification

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}.$$

Via this, one then gets natural embeddings of $sl(\infty) \subset \mathfrak{gl}(\infty)$ as subspaces of M. M is made into an $sl(\infty)$-module naturally, by letting $sl(\infty)$ act on it via commutators, i.e. for $g \in sl(\infty)$ and $m \in M$, using the above identification $sl(\infty) \subset M$, we set

$$g.m \coloneqq [g, m] = gm - mg \quad \text{with the usual product of matrices in } M$$

Denote by D the subspace of M consisting of diagonal matrices. Set now $N = \mathfrak{g} + D$. We want to show that $N \subset M$ is a submodule, and actually

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M) = N.$$

Let us again take the standard root space decomposition of $sl(\infty)$. In that case, we have $\mathfrak{h} = \{\text{finitary matrices in } D\}$. Note first that given any $d = (d_{ii})_{i\in\mathbb{N}} \in D$, for any $h \in \mathfrak{h}$ as both h and d are diagonal matrices, they commute, so we have

$$h.d = [h, d] = 0.$$

In other words, we have that 0 is a weight of M, and $D \subset M_0$. Let now $m \in M$, such that $m \in M_0$. For $n \in \mathbb{N}$ let $m_1^{(n)}, m_2^{(n)}, m_3^{(n)}, m_4^{(n)}$ be matrices such that $m_1^{(n)}$ is of type $n \times n$ and

$$m = \begin{pmatrix} m_1^{(n)} & m_2^{(n)} \\ m_3^{(n)} & m_4^{(n)} \end{pmatrix}. \tag{2.23}$$

Then for $i \le n-1$ and $h_i = E_{ii} - E_{i+1,i+1}$ as in Example 2.3 we have

$$0 = h_i.m = [h_i, m] = \begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} m_1^{(n)} & m_2^{(n)} \\ m_3^{(n)} & m_4^{(n)} \end{pmatrix} - \begin{pmatrix} m_1^{(n)} & m_2^{(n)} \\ m_3^{(n)} & m_4^{(n)} \end{pmatrix}\begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \left[h_i, m_1^{(n)}\right] & h_i m_2^{(n)} \\ -m_3^{(n)} h_i & 0 \end{pmatrix}.$$

In particular we have $\left[h_i, m_1^{(n)}\right] = 0$ for all $i \le n-1$. Let $m_1^{(n)} = r + d$, where d is a diagonal matrix in $\mathfrak{gl}(n)$, and $r \in sl(n)$. We then have that for each $i \le n-1$

$$0 = \left[h_i, m_1^{(n)}\right] = [h_i, r] + [h_i, d] = [h_i, r].$$

As the Cartan subalgebra $\mathfrak{h}_n$ is spanned by h_i for $i \le n-1$, and $r \in sl(n)$, we have that r is a 0-weight vector in $sl(n)$. In particular, this means that r is a diagonal matrix, which means that $m_1^{(n)} = r + d$ is also a diagonal matrix.

Thus we have shown that given an element $m \in M_0$, writing it as in (2.23) for any natural n, we get that $m_1^{(n)}$ is a diagonal matrix. This implies that m will itself be a diagonal matrix, hence $m \in D$. This proves

$$M_0 = D.$$

Let us now compute the other weight spaces of M. Let $\lambda \neq 0$ and assume that $m \in M$ is non-zero and such that $m \in M_\lambda$. Let n be a natural number such that when m is written as in (2.23), one has $m_1^{(n)} \neq 0$ and $\lambda(h_i) \neq 0$ for some $i \leq n-1$. Write $m_i = m_i^{(n)}$ for this n. Computing $\lambda(h)m = h.m$ for $h \in \mathfrak{h} \cap sl(n) = \mathfrak{h}_n$ as we did above, we get

$$\lambda(h)\begin{pmatrix} m_1^{(n)} & m_2^{(n)} \\ m_3^{(n)} & m_4^{(n)} \end{pmatrix} = \begin{pmatrix} \left[h, m_1^{(n)}\right] & hm_2^{(n)} \\ -m_3^{(n)}h & 0 \end{pmatrix}.$$

In particular we can see that $\lambda(h)m_1 = [h, m_1]$ for all $h \in \mathfrak{h}_{n+1}$. We have $m_1 \in \mathfrak{gl}(n) \subset M$. As $\lambda|_{\mathfrak{h}_n} \neq 0$, we get that m_1 is a weight vector for $\mathfrak{gl}(n)$ as an $sl(n)$-module, of non-zero weight. The weight space decomposition of $\mathfrak{gl}(n)$ as an $sl(n)$-module is

$$\mathfrak{gl}(n) = \bigoplus_{\alpha \in \mathfrak{h_n}^*} gl(n)_\alpha = \mathfrak{d} \oplus \bigoplus_{\alpha \in \Phi} sl(n)_\alpha$$

where $\mathfrak{d} \subset \mathfrak{gl}(n)$ is the subspace of diagonal matrices, and Φ is the root system for $sl(n)$. In particular, the weight spaces of $\mathfrak{gl}(n)$ are $\mathbb{C}E_{ij}$ for $1 \leq i \neq j \leq n$. This means $m_1 = xE_{pq}$ for some non-zero $x \in \mathbb{C}$, and $1 \leq p \neq q \leq n$, or in other words the only non-zero entry m_{ij} in m for $1 \leq i, j \leq n$ is $m_{pq} = x$

Let now now $k > n$, and write m as in (2.23) for k. Naturally this k satisfies the conditions of $m_1^{(k)} \neq 0$, and $\alpha(h_i) \neq 0$ for some $i < k$. This way, get $m_1^{(k)} = yE_{uv}$ for some non-zero $y \in \mathbb{C}$ and $1 \leq u \neq v \leq k$, and $m_{uv} = y$ is the only non-zero entry in m with $1 \leq u, v \leq k$. This implies that $u = p, v = q$ and $y = x$. This way we have shown that there exists a natural n and $p \neq q \leq n$ such that for any $k \geq n$, the only non-zero entry m_{ij} of m with $1 \leq i, j \leq k$ is $E_{pq} = x$. This means that m has no non-zero entries m_{ij} with $i, j > n$. Hence we have:

$$M_\lambda = \mathbb{C}E_{pq}$$

for some natural $p \neq q$, and from this we get

$$\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) = D \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} M_\alpha = sl(\infty) + D = N.$$

We claim now that N is an integrable module. Let $g \in sl(\infty)$ and $m \in N$. If $m \in sl(\infty)$, then $g(m) \subset sl(\infty)$, so $\dim g(m) < \infty$, as $sl(\infty)$ is an integrable module. Let now $m = d \in D$, and let n be natural such that $g \in sl(n)$. Write d as in (2.23) for n. We then have:

$$g.d = \left[\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\right] = \begin{pmatrix} [g, d_1] & 0 \\ 0 & 0 \end{pmatrix}$$

i.e. we have $g.d \in sl(n) \subset N$. Iterating this argument, we get $g(d) \subset sl(n) \subset M$, so in particular we get $\dim g(d) < \infty$. Now if $m \in N$ is any element, there is some $s \in sl(n)$ and $d \in D$ such that $m = s + d$. We then have $g(s+d) \subset g(s) + g(d)$, hence $g(s+d)$ is also finite dimensional. This proves that N is indeed integrable.

Remark 2.12. Note that this module N is of uncountable dimension, and it is the first example of such an object of $\mathrm{Int}_{\mathfrak{g}}$ in this thesis. This in particular shows that (see also Theorem 2.9)

$$\mathrm{Loc}_{\mathfrak{g}} \neq \mathrm{Int}_{\mathfrak{g}}.$$

Let us consider the action of $sl(\infty)$ on N a bit more. We saw that $sl(\infty).D \subset sl(\infty)$, so we get $sl(\infty).N \subset sl(\infty)$. Consider now the short exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow N \longrightarrow N/\mathfrak{g} \longrightarrow 0.$$

One can see that $T = N/\mathfrak{g} = D/\mathfrak{h}$, and from the previous discussion we have that T is a trivial $sl(\infty)$-module. An easy check will show that M does not contain any trivial submodules, hence neither does N.

Remark 2.13. Note that this example is of a non-splitting short exact sequence in $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$, hence also in $\mathrm{Int}_{\mathfrak{g}}$. Example 2.4 showed that $V \in \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$, and Example 2.6 will showh that $V_* \in \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ as well. This means that the non-splitting short exact sequence in Example 1.2 is also an example in this category $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.

We will soon introduce a subcategory of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ which turns out to be semisimple, and which will be introduced soon. Before we do it, let us collect some facts about duals of weight modules, and integrable weight modules.

Let $M \in \mathfrak{g}^{\mathrm{wt}}_{\mathfrak{h}}$, i.e.

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

From the discussion at the beginning of Section 2.3, viewing M and M_λ in this decomposition of M as $\mathfrak{h}$−modules, we get that

$$M^* = \prod_{\alpha \in \mathfrak{h}^*} (M_\alpha)^*. \tag{2.24}$$

This expression for the dual of a $\mathfrak{g}$-module which is a direct sum of a type of its subspaces will also be used later. In this scenario, (2.24) coupled with the following lemma, gives us a nice way of writing M^* as a direct product of it weight spaces.

Lemma 2.28. *Let $M \in \mathfrak{g}^{wt}_{\mathfrak{h}}$-mod. Then $(M^*)_\lambda = (M_{-\lambda})^*$ for any weight λ of M.*

Proof. Let $f \in (M_{-\lambda})^*$, i.e. $f \in M^*$ such that $f(m) = 0$ for all $m \in M_\mu$ with $\mu \neq -\lambda$. One naturally exhibits f as an element of M^*. Let $h \in \mathfrak{h}$ be any element. For $\mu \neq -\lambda$, and $m \in M_\mu$, one has $h.m \in M_\mu$, hence $(h.f)(m) = -f(h.m) = 0 = \lambda(h)f(m)$. For $m \in M_{-\lambda}$ we get

$$(h.f)(m) = -f(h.m) = \lambda(h)f(m).$$

As M_μ for $\mu \in \mathfrak{h}^*$ span M, we get that $(h.f)(m) = \lambda(h)m$ for all $m \in M$, i.e. $h.f = \lambda(h)f$, so $f \in (M^*)_\lambda$, which gives us $(M_{-\lambda})^* \subset (M^*)_\lambda$.

For the other inclusion, let $f \in (M^*)_\lambda$. As $f \in M^*$, from (2.24) one has $f|_{M_\mu} = f_\mu \in (M_\mu)^*$ for $\mu \in \mathfrak{h}^*$ are such that $f = (f_\mu)_{\mu \in \mathfrak{h}^*}$. Let $h \in \mathfrak{h}$ be any element, and let $\mu \neq -\lambda$. Now for $m \in M_\mu$ we have

$$\mu(h)f(m) = f(\mu(h)m) = f(h.m) = -h.f(m) = -(h.f)(m) = -\lambda(h)f(m).$$

As this is true for all $h \in \mathfrak{h}$, we must have $f(m) = 0$, i.e. $f_\mu = f|_{M_\mu} = 0$. Hence the determining component of f is $f_{-\lambda}$, and this is naturally unique. This way we get $(M^*)_\lambda \subset (M_{-\lambda})^*$, thus $(M^*)_\lambda \subset (M_{-\lambda})^*$, which is what we wanted to show. □

As an immediate consequence of Lemma 2.28, we get the following corollary.

Corollary 2.29. *Let $M \in \mathfrak{g}^{wt}_{\mathfrak{h}}$-mod. Then*

$$\Gamma^{wt}_{\mathfrak{h}}(M^*) = \bigoplus_{\lambda \in \mathfrak{h}^*} (M_\lambda)^*.$$

Example 2.6. Let $\mathfrak{g} = sl(\infty)$ and let V_* be the conatural representation of $sl(\infty)$. We know that $V_* = \varinjlim(V_n)^*$, where V_n is the natural representation of $sl(n)$. As $\{v_1, \dots, v_n\}$ is a basis for V_n, using Example 2.4 we get

$$V_n = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n},$$

and this gives us $(V_n)^* = V_{\lambda_1}^* \oplus \cdots \oplus V_{\lambda_n}^*$ From Lemma 2.28, we get that

$$(V_n)^* = (V^*)_{-\lambda_1} \oplus \cdots \oplus (V^*)_{-\lambda_n} \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(V^*)$$

As V_* is the direct limit of these V_n^*, we get that $V^* \subset \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(V^*)$. Now since $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(V^*)$ is a weight module by definition, from Proposition 2.20, we have that V_*, as a submodule of it, will also be a weight module. Note that one has for all natural i

$$(V^*)_{-\lambda_i} = (V_{\lambda_i})^* \subset (V_i)^* \subset V_*,$$

hence we get $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(V^*) \subset V_*$. This gives us

$$V_* = \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(V^*).$$

Let us now simplify the notation slightly. Given $M \in \mathfrak{g}_{\mathfrak{h}}^{\mathrm{wt}}$–mod, we denote

$$M^\vee := \bigoplus_{\alpha \in \mathfrak{h}^*} (M_\alpha)^* = \Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M^*)$$

and call it the *restricted dual* of M. Clearly $M^\vee$ is an $\mathfrak{h}$-weight module, and $M_\lambda^\vee = (M_{-\lambda})^*$. In general, if M is integrable, we cannot expect that $M^\vee$ will also be integrable. However, there exists a condition which allows us to deduce the integrability of $M^\vee$ in that situation.

Proposition 2.30. *Let $M \in Int_{\mathfrak{g},\mathfrak{h}}$ be such that $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$. Then $M^\vee$ is integrable.*

Proof. Let $f \in (M^*)_{-\lambda} \subset M^\vee$, and $g \in \mathfrak{g}_n$. From Lemma 2.28 we see that $(M^*)_{-\lambda} = (M_\lambda)^*$. Since $M_\lambda \subset M$ is a finite dimensional subspace, from Corollary 2.6 we have that $N = U(\mathfrak{g}_n).M_\lambda \subset M$ is a finite dimensional $\mathfrak{g}_n$–submodule, and there exists a $\mathfrak{g}_n$–submodule $R \subset M$ such that

$$M|_{\mathfrak{g}_n} = N \oplus R.$$

Since $f \in (M_\lambda)^*$, there exists some $m \in M_\lambda$ such that $f(m) = 1$ and $f(M/\mathbb{C}m) = 0$. Since $m \in M_\lambda \subset N$, we get that $m \in N$, hence $f \in N^*$. This means that for any $r \in R$ we have $f(r) = 0$.

Note now that for any $r \in R$, since $g^i.r \in U(\mathfrak{g}_n).r \subset U(\mathfrak{g}_n).R = R$, we have

$$(g^i.f)(r) = -f(g^i.r) = 0.$$

This means that $g^i.f \in N^*$ for all $i \in \mathbb{N}$. This clearly implies

$$g(f) \subset N^*,$$

and since N^* is finite dimensional, we get that $\dim g(f) < \infty$.

Let now $\Lambda = \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$. Let $B_\lambda \subset (M^*)_\lambda = (M_{-\lambda})^*$ be a basis for $(M_{-\lambda})^*$. Naturally B_λ are finite and $B_\lambda \cap B_\mu = \varnothing$ if $\lambda \neq \mu$. Set now $\mathcal{B} = \bigcup_{\lambda\in\Lambda} B_\lambda$, and one can see that this will be a basis of $M^\vee$. Let now $g \in \mathfrak{g}$. In the first part of the proof we have shown that $\dim g(f) < \infty$ for all $f \in \mathcal{B}$, and from Remark 2.1 c), we get that $M^\vee$ is indeed integrable. $\square$

Next we prove another result regarding the modules $M^\vee$ for which M has finite dimensional weight spaces. For this, we will take note of the fact that

$$(M^\vee)^\vee = \bigoplus_{\alpha\in\mathfrak{h}^*}(M^\vee_\alpha)^* = \bigoplus_{\alpha\in\mathfrak{h}^*}(M^*_{-\alpha})^* = \bigoplus_{\alpha\in\mathfrak{h}^*} M_\alpha = M.$$

Note also that given two modules $N \subset M \in \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ which have finite dimensional weight spaces, there exists a natural surjective map $M^\vee \longrightarrow N^\vee$ given by

$$(M_\lambda)^* \ni f \longrightarrow f|_{N_{-\lambda}}$$

and then extended to $M^\vee$ linearly.

Proposition 2.31. *Let $L \in Int^{wt}_{\mathfrak{g},\mathfrak{h}}$ be such that $\dim L_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$. If L is simple, then so is $L^\vee$.*

Proof. Let $N \subset L^\vee$ be a simpe submodule of $L^\vee$. From the discussion preceding this proposition, we obtain a surjective map

$$s : L = (L^\vee)^\vee \longrightarrow N^\vee$$

But this means that L is a quotient of N, i.e. there exists some submodule $S \subset N$ such that $M = N/S$. As N is simple, we have $S = 0$ or $S = N$, hence we get $N = 0$ or $N = L^\vee$, which proves that $L^\vee$ is indeed a simple module. □

These two propositions show once again how by imposing finiteness conditions on representations, we tend to get nicer objects. These results serve as motivation the following definition.

Definition 2.3. Denote by $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ the largest full subcategory of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ consisting of $\mathfrak{h}$-weight modules M, such that for any $\lambda \in \mathfrak{h}^*$ one has $\dim M_\lambda < \infty$.

We now prove the following result for $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

Theorem 2.32. *The category $Int^{fin}_{\mathfrak{g},\mathfrak{h}}$ is semisimple.*

Proof. Let $L \in \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ be a simple module. Note that

$$L^\vee = \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(L^*) = \Gamma_{\mathfrak{g}}(\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(L^*)) = \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(\Gamma_{\mathfrak{g}}(L^*))$$

will be an injective object of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ from Remark 2.11, hence also of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$. From Proposition 2.30 and Proposition 2.31, we know that $L^\vee$ will also be a simple object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$, hence we get that $L = (L^\vee)^\vee$ will be an injective object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$. Since every simple object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ turns out to be an injective object, we have that $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ is indeed a semisimple category. □

From Example 2.4 and Example 2.6 we saw that the natural and conatural representations V, and V_* for $\mathfrak{g} = sl(\infty)$ are elements of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$. However in Example 1.2 we have seen that $V \otimes V_*$ is not a semisimple object, thus $V \otimes V_* \notin \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$. One can also see that this is true as follows. Let λ be a weight of V. We saw that $-\lambda$ will be a weight of V_*. Note that if $v \in V_\lambda$ and $f \in (V_*)_{-\lambda}$, for any $h \in \mathfrak{h}$ we get

$$h.(v \otimes f) = (h.v) \otimes f + v \otimes (h.f) = \lambda(h)v \otimes f - \lambda(h)v \otimes f = 0.$$

This shows that $V_\lambda \otimes (V_*)_{-\lambda} \subset (V \otimes V_*)_0$, and as V has infinitely many weights λ, we get that

$$\dim(V \otimes V_*)_0 = \infty,$$

thus really $V \otimes V_* \notin \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.

Remark 2.14. This discussion shows $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ is not closed under taking tensor products, and is the first category of $\mathfrak{g}$–modules that we have mentioned so far which does not have this property.

We now emphasize the importance of Theorem 2.32 in the following note.

Comparison remark V. *Analogous to the finite dimensional theory of representations of Lie algebras, every module in $Int^{wt}_{\mathfrak{g},\mathfrak{h}}$ is semisimple. This establishes an analogue of Weyl's complete reducibility theorem for representations of locally semisimple Lie algebras.*

Chapter 3

Integrability of the dual, the socle functors, and the category $\mathrm{Tens}_{\mathfrak{g}}$

In this chapter the main aim is to investigate the integrability of duals of integrable $\mathfrak{g}$–modules. In Section 3.1 we give a necessary and sufficient condition for the integrability of M^* when $M \in \mathrm{Int}_{\mathfrak{g}}$. While useful, see for instance Example 2.2, this criterion is too difficult to implement in practice, and does not give too much information on what kind of subcategory of $\mathrm{Int}_{\mathfrak{g}}$ may be closed under algebraic dualization. It is however useful in deducing that the property of having an integrable dual is closed under some algebraic operations.

In Section 3.2 we introduce the socle functors, and the socle filtration, and describe some of their elementary properties. In Subsection 3.2.2 we give a generalization of Proposition 1.8, and for the case when M is simple we use this result to show how and injective object $I \in \mathrm{Int}_{\mathfrak{g}}$ containing M may look like. In particular we show that if a simple module $L \in \mathrm{Int}_{\mathfrak{g}}$ is such that $\Gamma_{\mathfrak{g}}(L^*)$ has finite socle length, then its algebraic dual L^* will contain a unique simple submodule, which is integrable. The construction of this simple submodule, which may be regarded as a restricted dual of M, is vaguely similar to the construction in 2.2.1.

In Section 3.3 we further investigate the socle functors, and in particular show that the property of having finite socle length is preserved under many algebraic operations. We also show how quotients of objects in the socle filtration of a $\mathfrak{g}$–module look like, and in particular Corollary **??** gives a computational identity on the socle length of a $\mathfrak{g}$–module.

In Section 3.4 we introduce the final category of $\mathfrak{g}$–modules to appear in this thesis, namely $\mathrm{Tens}_{\mathfrak{g}}$. This category is closed under algebraic dualization, and it turns that it contains many $\mathfrak{g}$–modules that we have already encountered, including the natural representation V. We give a characterization of the simple objects of this category. In particular, $\mathrm{Tens}_{\mathfrak{g}}$ has the property that all of its simple objects are highest weight modules for some Borel subalgebra. We conclude this chapter with Theorem 3.54, which shows that if $\mathfrak{g} \cong sl(\infty), o(\infty), sp(\infty)$, then the only finite dimensional $\mathfrak{g}$–modules are trivial.

Unless otherwise specified, in Subsection 3.2.1 and Section 3.3, $\mathfrak{g}$ can be any Lie algebra. In Section 3.4 $\mathfrak{g}$ will be a classical locally semisimple Lie algebra. In all the other parts, $\mathfrak{g}$ is a general locally semisimple Lie algebra.

Sections 3.1, 3.2 are based on [10], while Section 3.4 also uses developments in [11].

3.1 A criterion for integrability of the dual

We begin this chapter by stating and proving a result which gives a necessary and sufficient condition for the dual of an integrable module to be integrable itself. In Example 2.2, we constructed an integrable $\mathfrak{g}$–module M, whose dual M^* contains infinitely many non-isomorphic simple $sl(2,\mathbb{C})$–submodules. Theorem 3.4 shows why this fact leads to the non-integrability of M^*. Before we state this result, we make a few observation on isomorphisms of some $\mathfrak{g}$-modules and also prove a technical linear algebraic result.

Lemma 3.1. *Let A, B be $\mathfrak{g}$-modules. One always has a natural inclusion morphism of $\mathfrak{g}$-modules*

$$A^* \otimes B^* \xrightarrow{\phi} (A \otimes B)^* \quad \text{given by} \quad \phi(f \otimes g)(a \otimes b) = f(a)g(b) \tag{3.1}$$

This map is an isomorphism, unless both A and B are infinite dimensional.

Proof. We show first that this is a map of $\mathfrak{g}$-modules. Indeed, for $x \in \mathfrak{g}, f \in A^*, g \in B^*$ one has

$$(x.\phi(f \otimes g))(a \otimes b) = -\phi(f \otimes g)(x.(a \otimes b)) = -\phi(f \otimes g)(x.a \otimes b + a \otimes x.b) =$$
$$= -f(x.a)g(b) - f(a)g(x.b) = (x.f)(a)g(b) + f(a)(x.g)(b) = \phi(x.(f \otimes g))(a \otimes b),$$

i.e. we have $x.\phi(f \otimes g) = \phi(x.(f \otimes g))$, so ϕ indeed is a map of $\mathfrak{g}$-modules. Note now that if $f, g \neq 0$, one can find $a \in A$ and $b \in B$ with $f(a), g(b) \neq 0$, and we would have $\phi(f \otimes g) \neq 0$. This shows that ϕ is always an injective map.

Now if both A and B are of finite dimension, by cardinality reasons we have that ϕ will be an isomorphism. Assume now that one of A or B is of finite dimension, say B. We show now that ϕ is surjective. Pick a basis $f_1, \cdots, f_n$ of B^*. Let $\alpha \in (A \otimes B)^*$, and $a \in A$. Consider $\alpha_a \in B^*$ given by $\alpha_a(b) = \alpha(a \otimes b)$. An easy check will show that this map is linear. As $\alpha_a \in B^*$, there exist $x_i(a) \in \mathbb{C}$ such that $\alpha_a = x_1(a)f_1 + \cdots + x_n(a)f_n$. As these coefficients $x_i(a)$ are unique, one obtains maps $x_i : A \longrightarrow \mathbb{C}$. Since $\alpha_{a_1+a_2} = \alpha_{a_1} + \alpha_{a_2}$, and $\alpha_{\lambda a} = \lambda \alpha_a$, these maps are linear, i.e. $x_i \in A^*$. On the other hand, for any $a \in A$ and $b \in B$ we have

$$\alpha(a \otimes b) = \sum_i x_i(a) f_i(b) = \sum_i \phi(x_i \otimes f_i)(a \otimes b) = \phi(\sum_i x_i \otimes f_i)(a \otimes b),$$

hence $\alpha = \phi(\sum_i x_i \otimes f_i)$, so ϕ is indeed a surjective map.

Now if both A and B are of infinite dimension, then the cardinality of $(A \otimes B)^*$ will be strictly larger than that of $A^* \otimes B^*$, so the map ϕ cannot be an isomorphism. □

Next we prove another result which shows how one can express a certain direct sum of simple $\mathfrak{g}$-modules in a slightly more convenient way.

Lemma 3.2. *Let $L \in Int_{\mathfrak{g}}$ be a simple module, and I some index set. Set*

$$N := \bigoplus_{i \in I} L_i; \quad \textit{and} \quad P := \hom_{\mathfrak{g}}(L, N)$$

where $L_i \cong L$ for all $i \in I$, and P is trivial as a $\mathfrak{g}$-module. Then the map $\phi : P \otimes L \longrightarrow N$ given by $\phi(f \otimes m) = f_i(m)$, for $f \in P, m \in L$, where $f_i : L \longrightarrow L_i$ are fixed isomorphisms for all $i \in I$, is an isomorphism of $\mathfrak{g}$-modules.

Proof. As P is a trivial module, in order to understand how P looks like, it suffices to find a basis of it as a vector space, i.e. want to find a basis of $\hom_{\mathfrak{g}}(L, N)$. We will show that $\{f_i \mid i \in I\}$ is a basis for P. Let us first treat a simpler case.

Case of finite I. Let I be finite, say $I = \{1, \ldots, k\}$. In this case we have

$$N = L_1 \oplus L_2 \oplus \cdots \oplus L_k$$

Now if $f_i : L \longrightarrow N$ are the fixed maps $f_i : L \cong L_i$, one can see $\{f_1, \cdots, f_k\} \subset P$ is a linearly independent set. Let now $f : L \longrightarrow N$ be any non-zero map. For each $m \in L$ we have some $s_i(m) \in L_i$ such that

$$f(m) = s_1(m) + \cdots + s_k(m). \tag{3.2}$$

Note that these $s_i(m)$ are well defined and unique, so we have maps $s_i : L \longrightarrow L_i$ such that (3.2) holds for every $m \in L$. One can see that these s_i will actually be morphisms of $\mathfrak{g}$-modules. Indeed, linearity is obvious, and for $x \in \mathfrak{g}$ we have

$$x.s_1(m) + \cdots + x.s_k(m) = x.f(m) = f(x.m) = s_1(x.m) + \cdots + s_k(x.m),$$

and as $L_i \subset N$ are submodules, we get $x.s_i(m) = s_i(x.m)$ for all $i = 1, 2, \ldots, k$. Since $L = L_i$ are simple integrable modules, from Schur's Lemma for simple integrable $\mathfrak{g}-$ modules, see Theorem 2.10, we get $s_i = 0$ or $s_i : L \cong L_i$ and in particular there exist $\alpha_1, \cdots, \alpha_k \in \mathbb{C}$ (not all zero) such that $s_i(m) = \alpha_i f_i(m)$. One then has $f = \sum s_i = \sum \alpha_i f_i$. Thus every element of $\hom_{\mathfrak{g}}(L, N)$ is a linear combination of $\{f_1, \cdots, f_k\}$, which means that this set is actually a basis of P, hence we get $P \cong \mathbb{C}^n$.

Case for general I. Let I be any index set. One can see that $\{f_i \mid i \in I\}$ with $f_i : L \cong L_i$ is a linearly independent subset of P, as every finite subset of it will be linearly independent by the first case. Then given any $f \in P$, one can again define $s_i : L \longrightarrow L_i$ as above, and show that they are maps of $\mathfrak{g}$-modules. Pick now any non-zero $m \in L$. As

$$f(m) = \sum_{i \in I} s_i(m),$$

we have that there exists a finite subset $J \subset I$ such that $s_i(m) = 0$ for all $i \in I \backslash J$. Since $\ker s_i \neq 0$ for $i \notin J$, and L is a simple $\mathfrak{g}$-module, we have $\ker s_i = M$, hence $s_i = 0$. This gives us

$$f = \sum_{j \in J} s_j.$$

As in the previous case, we have $\alpha_j \in \mathbb{C}$ for $j \in J$, not all zero, such that $s_j = \alpha_j f_j(m)$, hence we get that f really lies in $\operatorname{span}\{f_i \mid i \in I\}$, i.e. $P = \operatorname{span}\{f_i \mid i \in I\} = \mathbb{C}^I$.

Consider now the map

$$P \otimes L \xrightarrow{\phi} N$$

given by $\phi(f_i \otimes m) = f_i(m)$, and then extended linearly to $P \otimes L$. A simple check will show that this is a morphism of $\mathfrak{g}$-modules. Note that if $0 = \phi(f_i \otimes m) = f_i(m)$, we get $m = 0$, hence $f_i \otimes m = 0$. So $\ker \phi = 0$, i.e. ϕ is injective. As for surjectivity, note that $\phi(f_i \otimes L) = L_i$, hence we indeed see that ϕ is surjective. Hence the map ϕ is an isomorphism of $\mathfrak{g}$-modules., which is what we wanted to show. □

The following is a technical result from linear algebra, which will be useful in the proof of Theorem 3.4 below.

Lemma 3.3. *Let V be a vector space, and $S \subset V^*$ a finite subset consisting of non-zero elements. Then there exists some $v \in V$ such that $f(v) \neq 0$ for all $f \in S$.*

Proof. We prove this by induction on the cardinality of S, denoted $\#S$. For $\#S = 1$ the result is clear. Assume now that we know the lemma holds if $\#S = n$. Let now $\#S = n + 1$. Let $S = S' \cup \{g\}$, where $\#S' = n$. From the induction hypothesis, we have some $v \in V$ such that $f(v) \neq 0$ or all $f \in S'$. If $g(v) \neq 0$ we are done. If $g(v) = 0$, let $w \in V$ such that $g(w) \neq 0$. For $f \in S'$ set $A_f := \{a \in \mathbb{C} \mid f(v + aw) = 0\}$. One can see that $\#A_f \leq 1$. Pick then some $a \in \mathbb{C}$ such that $a \notin A_f$ for any $f \in S'$ and set $v' = v + aw$. We then have $f(v') \neq 0$ for all $f \in S'$, and $g(v') \neq 0$, thus the lemma is true for S with $\#S = n + 1$ as well. The statement then follows by induction. □

We are now ready to state and prove the main result of this section, which gives a characterization of those integrable modules M which have integrable duals M^*.

Theorem 3.4. *Let $M \in Int_{\mathfrak{g}}$. Then M^* is integrable if and only if for all $i \in \mathbb{N}$ there exists only finitely many non-isomorphic finite dimensional simple $\mathfrak{g}_i$-modules N such that* $\hom_{\mathfrak{g}_i}(N, M) \neq 0$

Proof. Let $i \in \mathbb{N}$, and fix a Cartan subalgebra and Borel subalgebra $\mathfrak{h}_i \subset \mathfrak{b}_i \subset \mathfrak{g}_i$. Adopt the following notation

$$\begin{array}{rcc} \Lambda_i & = & \text{the set of dominant integral weights of } \mathfrak{g}_i, \\ V_i^\lambda & = & \text{the } \lambda - \text{highest weight simple } \mathfrak{g}_i - \text{module}, \\ \Lambda_i(M) & = & \text{the subset of } \Lambda_i \text{ consisting of those } \lambda \text{ such that } \hom_{\mathfrak{g}_i}(V_i^\lambda, M) \neq 0. \end{array}$$

From Theorem 2.5 we know that we can write

$$M|_{\mathfrak{g}_i} = \bigoplus_{j \in Ij} M_{ij}$$

for some index set I_j. We may assume these M_{ij} to be simple, and by our definition of $\Lambda_i(M)$ we get a map $s_\lambda : I_j \longrightarrow \Lambda_i(M)$ such that $s(j) = \lambda \iff M_{ij} \cong V_i^\lambda$. One then has

$$M|_{\mathfrak{g}_i} = \bigoplus_{j \in Ij} M_{ij} = \bigoplus_{\lambda \in \Lambda_i(M)} \bigoplus_{s(j)=\lambda} M_{ij}.$$

Note that if we set $Q^\lambda := \hom_{\mathfrak{g}}(V_i^\lambda, M) = \hom_{\mathfrak{g}}(V_i^\lambda, \oplus_{s(j)=\lambda} M_{ij})$, from Lemma 3.2 we get:

$$\bigoplus_{s(j)=\lambda} M_{ij} = \bigoplus_{s(j)=\lambda} V_i^\lambda = Q^\lambda \otimes V_i^\lambda$$

This gives us:

$$M|_{\mathfrak{g}_i} = \bigoplus_{\lambda \in \Lambda_i(M)} Q^\lambda \otimes V_i^\lambda$$

As the V_i^λ are finite dimensional, from the discussion at the beginning of Section 2.3 and Lemma 3.1, we obtain

$$M^* = \prod_{\lambda \in \Lambda_i(M)} (V_i^\lambda \otimes Q^\lambda)^* = \prod_{\lambda \in \Lambda_i(M)} (V_i^\lambda)^* \otimes (Q^\lambda)^*.$$

Now we can start to prove the theorem. Assume that $\Lambda_i(M)$ is finite. Then given any $g \in \mathfrak{g}$ one has a finite degree polynomial $p_\lambda \in \mathbb{C}[z]$ such that for any $g \in \mathfrak{g}_i$ we have $p_\lambda(g).(V_i^\lambda)^* = 0$. Set then $p(z) = \prod_{\lambda \in \Lambda_i(M)} p_\lambda(z)$, and one gets that $p(g).M^* = 0$, hence M^* will also be integrable.

Conversely, let $\Lambda_i(M)$ is infinite. Let now $v_\lambda \in V_i^\lambda$ be a vector of weight λ, and set $w_\lambda = v_\lambda \otimes q$ for some non-zero $q \in Q^\lambda$. Note that for any $h \in \mathfrak{h}_i$ we have

$$h.w_\lambda = h.(v_\lambda \otimes q) = h.v_\lambda \otimes q + v_\lambda \otimes (h.q) = \lambda(h) v_\lambda \otimes q = \lambda(h) w_\lambda, \tag{3.3}$$

i.e. $w_\lambda \in V_i^\lambda \otimes Q^\lambda$ is a λ-weight vector. Consider now the dual map $w_\lambda^* = f_\lambda \in (V_i^\lambda \otimes Q^\lambda)^*$ corresponding to w_λ. As V_i^λ is a finite dimensional $\mathfrak{g}_i$−module, we know that it is a weight module, with respect to a Cartan splitting subalgebra $\mathfrak{h}_i \subset \mathfrak{g}_i$. Since Q^λ is a trivial $\mathfrak{g}_i$-module, we have that $h.q = 0$ for all $h \in \mathfrak{h}_i$ and $q \in Q^\lambda$, thus Q^λ is also a weight $\mathfrak{g}_i$−module, and in fact it has only one weight space, namely the 0-weight space. From Remark 1.1 we see that Proposition 2.20 holds for $\mathfrak{g}_i$ as well, hence we have that $V_i^\lambda \otimes Q^\lambda$ is also a $\mathfrak{g}_i$-weight module, and from (3.3)its weight space decomposition is

$$V_i^\lambda \otimes Q^\lambda = \bigoplus_\mu (V_i^\lambda)_\mu \otimes Q^\lambda,$$

where μ runs over the weights of V_i^λ.

We now want to show that f_λ as defined in the previous paragraph, is a $-\lambda$-weight vector of $(V_i^\lambda \otimes Q^\lambda)^*$. To this end, note that for any $h \in \mathfrak{h}_i$ and $m \in (V_i^\lambda)_\lambda \otimes Q^\lambda$ we have

$$(h.f_\lambda)(m) = -f_\lambda(h.m) = -f_\lambda(\lambda(h)m) = -\lambda(h)f_\lambda(m),$$

and for $m \in (V_i^\lambda)_\mu \otimes Q^\lambda$ with $\mu \neq \lambda$ we have that in particular $m \notin \mathbb{C}w_\lambda$, hence $f(m) = 0$. So we indeed have that

$$(h.f_\lambda)(m) = -\mu(h)f_\lambda(m) = 0 = -\lambda(h)f_\lambda(m)$$

holds for all $m \in V_i^\lambda \otimes Q^\lambda$, hence $h.f_\lambda = -\lambda(h)f_\lambda$. Since $h \in \mathfrak{h}_i$ was arbitrary, we have that f_λ is a non-zero $-\lambda$ weight vector of $(V_i^\lambda \otimes Q^\lambda)^*$.

Now using these weight vectors of $(V_i^\lambda \otimes Q^\lambda)^*$, we aim to investigate the integrability of M^*. For this purpose, set $f = (f_\lambda)_{\lambda \in \Lambda_i(M)}$. Note first that for any $h \in \mathfrak{h}_i$ we have

$$h^i.v = (h^i.f_\lambda)_{\lambda \in \Lambda_i(M)} = ((-\lambda(h))^i f_\lambda)_{\lambda \in \Lambda_i(M)}.$$

Assume now that M^* is integrable. Then as $f \in M^*$, from Corollary 2.6, there exists a finite dimensional $\mathfrak{g}_i$-submodule $R \subset M^*$ such that $f \in R$. Let $n - 1 = \dim R$, and pick now pairwise distinct $\lambda_1, \dots, \lambda_n \in \Lambda_i(M)$. Consider the set

$$S := \{\alpha_{ij} = \lambda_i - \lambda_j \mid 1 \leq i < j \leq n\}.$$

From Lemma 3.3 we can find a non-zero $h_n \in \mathfrak{h}_i$ such that $\alpha_{ij}(h_n) \neq 0$ for all $i < j$, i.e. such that $\lambda_i(h_n) \neq \lambda_j(h_n)$. Now as $h_n \in \mathfrak{h}_i$ and $f \in R$, we have that $h_n(f) \subset R$, hence $\dim h_n(f) \leq \dim R = n$. In particular, the set $\{f, h_n.f, h_n^2.f, \dots, h_n^{n-1}.f\} \subset R$, as it contains more vectors than the dimension of R, is linearly dependent. Hence there exist $\alpha_0, \dots, \alpha_n \in \mathbb{C}$ not all zero such that

$$0 = \sum_{i=0}^{n-1} \alpha_i h_n^i.f = \left(\sum_{i=0}^{k} \alpha_i(-\lambda(h_n))^i f_\lambda\right)_{\lambda \in \Lambda_i(M)}$$

i.e. we have for all $\lambda \in \Lambda_i(M)$

$$\sum_{i=0}^{n-1} \lambda(h_n)^i \gamma_i = 0 \tag{3.4}$$

where $\gamma_i = (-1)^i \alpha_i$, so not all γ_i are zero. In particular, for $j = 1, \dots, n$ one can simultaneously write all the respective equalities as in (3.4) for λ_j, which looks like

$$\operatorname{Van}\gamma = \begin{pmatrix} 1 & \lambda_1(h_n) & \cdots & \lambda_1(h_n)^{n-1} \\ 1 & \lambda_2(h_n) & \cdots & \lambda_2(h_n)^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_n(h_n) & \cdots & \lambda_n(h_n)^{n-1} \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = 0 \tag{3.5}$$

But here Van is the Vandermonde matrix, for which one computes the determinant via

$$\det \operatorname{Van} = \prod_{i>j} (\lambda_i(h_n) - \lambda j(h_n)).$$

Thus we see that $\det \operatorname{Van} \neq 0$, which means that (3.5) has only one solution, the trivial one, which contradicts the fact that not all γ_i are zero. Hence our assumption that M^* is integrable leads to a contradiction, thus $\Lambda_i(M)$ being infinite indeed implies that M^* cannot be integrable. This concludes the proof of the theorem. $\square$

Using this theorem, one gets the following consequence, which shows how in some cases $\mathrm{Int}_{\mathfrak{g}}$ can be closed under taking duals.

Corollary 3.5. *Let $M, N \in Int_{\mathfrak{g}}$. If $M^*, N^* \in Int_{\mathfrak{g}}$, then $M \otimes N, M^{**} \in Int_{\mathfrak{g}}$ as well.*

Proof. Let $s \in \mathbb{N}$. As $M, N \in \mathrm{Int}_{\mathfrak{g}}$ are such that $M^*, N^* \in \mathrm{Int}_{\mathfrak{g}}$, from Theorem 2.5 and Theorem 3.4 we get that there exist finite dimensional simple $\mathfrak{g}_s$-modules $M_1, \cdots, M_k, N_1, \cdots, N_l$ and index sets I, J such that

$$M|_{\mathfrak{g}_s} = \bigoplus_{i \in I} M_i \quad \text{and} \quad N|_{\mathfrak{g}_s} = \bigoplus_{j \in J} N_j,$$

where for each $i \in I$ and $j \in J$ we have $M_i \in \{M_1, \cdots, M_k\}$ and $N_j \in \{N_1, \cdots, N_l\}$. Note now that

$$(M \otimes N)|_{\mathfrak{g}_s} = M|_{\mathfrak{g}_s} \otimes N|_{\mathfrak{g}_s} = \bigoplus_{i \in I, j \in J} (M_i \otimes N_j),$$

and from the discussion at the beginning of Section 2.3, we have

$$M^*|_{\mathfrak{g}_s} = (M|_{\mathfrak{g}_i})^* = \prod_{i \in I} M_i^*.$$

By decomposing $M_i \otimes N_j$'s into direct sums of simple $\mathfrak{g}_{\mathfrak{s}}$-modules, we see that all the possible finite dimensional simple $\mathfrak{g}_s$-modules K for which $\hom_{\mathfrak{g}_s}(K, M \otimes N) \neq 0$ are such that $\hom_{\mathfrak{g}_s}(K, M_i \otimes N_j) \neq 0$ for some $1 \leq i \leq k$ and $1 \leq j \leq l$. Clearly then there exist only finitely many such possible K's, and by virtue of the previous theorem, we must have $M \otimes N \in \mathrm{Int}_{\mathfrak{g}_{\mathfrak{i}}}$.

As for M^*, we see that all the M_i^* are finite dimensional simple $\mathfrak{g}_{\mathfrak{s}}$-modules. Let K be a finite dimensional simple $\mathfrak{g}_{\mathfrak{s}}$-module for which $\hom_{\mathfrak{g}_{\mathfrak{s}}}(K, M^*) \neq 0$, and let $f \in \hom_{\mathfrak{g}_{\mathfrak{i}}}(K, M^*)$ non-zero. Since K simple and finite dimensional, f will be injective and $f(K)$ finite dimensional. Denote by $p_i : M^* \longrightarrow M_i^*$ the natural projections, which will in particular be morphisms of $\mathfrak{g}_s$–modules. Then there exists some $i \in I$ such that $p_i(f(K)) \neq 0$. This means that $p_i \circ f : K \longrightarrow f(K) \longrightarrow M_i^*$ is a non-zero map of $\mathfrak{g}_s$–modules, thus it will be an isomorphism, so we get that $K \cong M_i^*$. As there are finitely many non-isomorphic $\mathfrak{g}_s$–modules M_i^*, from Theorem 3.4, we get that $(M^*)^*$ will also be integrable. ◻

Remark 3.1. Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Adopt the notation

$$M^{*0} = M; \quad M^{*n} = \left(M^{*(n-1)}\right)^*,$$

i.e. M^{*n} is M dualized n times. Note that Corollary 3.5 says that if M satisfies the conditions of Theorem 3.4, then so does M^*. Iterating this argument, we see that M^{*n} satisfies the conditions of Theorem 3.4 for all $n \in \mathbb{N}_0$, so in particular we have

$$M^{*n} \in \mathrm{Int}_{\mathfrak{g}}$$

for all $n \in \mathbb{N}_0$.

3.2 The socle functors; socle length

Section 3.1 gives a nice result towards the problem of understanding when the dual of an integrable module is itself integrable. However, the necessary and sufficient condition of Theorem 3.4 can be a tedious one to check in practice. In this section, we first introduce a class of functors and also a certain filtration for objects of $\mathfrak{g}$–mod. By imposing some finiteness conditions on these filtrations, we will be able to give some conclusions on when the dual of an integrable module is also integrable.

3.2.1 Definitions and injective objects in $\mathrm{soc}^i(\mathfrak{g}-\mathrm{mod})$

Let $\mathfrak{g}$ be any Lie algebra.

Given $M \in \mathfrak{g}$–mod, set $C(M) := \{S \text{ simple } \mathfrak{g}-\text{submodule of } M\}$.

Definition 3.1. The functor $\mathrm{soc} : \mathfrak{g}-\mathrm{mod} \longrightarrow \mathfrak{g}-\mathrm{mod}$ given by

$$\mathrm{soc}(M) := \sum_{S \in C(M)} S = \text{largest semisimple } \mathfrak{g}-\text{submodule of } M \tag{3.6}$$

on objects, and $\mathrm{soc}(f) = f|_{\mathrm{soc}(M)}$ on morphisms $f : M \longrightarrow N$, is called the *socle functor.*

One sees immediately that if $S \subset M$ is a semisimple submodule, then $S \subset \mathrm{soc}(M)$. We can then also define the *higher socle functors* $\mathrm{soc}^i : \mathfrak{g}-\mathrm{mod} \longrightarrow \mathfrak{g}-\mathrm{mod}$ inductively by setting $\mathrm{soc}^0 = \mathrm{soc}$ and

$$\mathrm{soc}^{i+1}(M) := p_i^{-1}(\mathrm{soc}(M/\mathrm{soc}^i(M))),$$

on objects, where $p_i : M \longrightarrow M/\mathrm{soc}^i(M)$ is the canonical map. On morphisms $f : M \longrightarrow N$, we define the maps $\mathrm{soc}^i(f)$ to be just the adequate restrictions of f. Since $p_i(\mathrm{soc}^i(M)) = 0$, we have $\mathrm{soc}^i(M) \subset \mathrm{soc}^{i+1}(M)$, thus we obtain what we call the *socle filtration* of M

$$0 \subset \mathrm{soc}(M) \subset \mathrm{soc}^1(M) \subset \cdots \subset \mathrm{soc}^i(M) \subset \cdots.$$

We say that the socle filtration of M is *exhaustive* if

$$\varinjlim \mathrm{soc}^i(M) = \bigcup_i \mathrm{soc}^i(M) = M.$$

Say that M has *finite socle length* if there exists some $k \in \mathbb{N}_0$ such that $\mathrm{soc}^k(M) = M$. In that case, denote by $ll(M)$ the smallest natural number k such that $\mathrm{soc}^k(M) = M$, and call it the *socle length* of M. Note that Remark 1.5 says that for $\mathfrak{g}$ a locally semisimple Lie algebra, there exist non-zero $\mathfrak{g}$–modules, even local ones, such that their socle filtration is just 0. Let us now note a property of the socle filtration of a $\mathfrak{g}$-module.

Proposition 3.6. *Let $i \in \mathbb{N}_0$. Then for any $M \in \mathfrak{g}$–mod, the following holds*

$$soc^{i+1}(M)/soc^i(M) = soc(M/soc^i(M)),$$

Proof. By definition, we have

$$\mathrm{soc}^{i+1}(M) = p_i^{-1}(\mathrm{soc}(M/\mathrm{soc}^i(M))),$$

where $p_i : M \longrightarrow M/\mathrm{soc}^i(M)$ is the canonical map. The restriction of this p_i to $\mathrm{soc}^{i+1}(M)$ gives us a surjective map

$$p_i' = p_i|_{\mathrm{soc}^{i+1}(M)} : \mathrm{soc}^{i+1}(M) \longrightarrow \mathrm{soc}(M/\mathrm{soc}^i(M)).$$

As p_i' is the restriction of p_i, we get

$$\ker p_i' = \ker p_i \cap \mathrm{soc}^{i+1}(M) = \mathrm{soc}^i(M) \cap \mathrm{soc}^{i+1}(M) = \mathrm{soc}^i(M),$$

hence we get really get that p_i' induces the equality

$$\mathrm{soc}^{i+1}(M)/\mathrm{soc}^i(M) = \mathrm{soc}(M/\mathrm{soc}^i(M)),$$

which is what we wanted to show. □

Remark 3.2. Proposition 3.6 says that the appropriate quotient of any two neighboring elements in the socle filtration of a $\mathfrak{g}$−module M is a semisimple module. In Proposition 3.23 we will give a generalization of the result in Proposition 3.6.

We now want to study the socle functors. Before we do this, we give a technical result from commutative algebra, which will be useful in proving Proposition 3.8.

Lemma 3.7. *Let $A \subset B$, $P \subset A$, and $Q \subset B$ be $\mathfrak{g}$−modules with $Q \cap A = P$. Then there exists a natural injection $\alpha : A/P \longrightarrow B/Q$. Let $f : A \longrightarrow A/P$, and $g : B \longrightarrow B/Q$ be the canonical maps. If $R \subset A/P$ and $S \subset B/Q$ are such that $S \cap \alpha(A/P) = \alpha(R)$, then $f^{-1}(R) = A \cap g^{-1}(S)$.*

Proof. Consider the map $\alpha' : A \longrightarrow B/Q$ given by $\alpha(a) = a + Q$. Note that if $\alpha'(a) = 0$, we have $a + Q = Q$, i.e. $a \in Q$. This means $a \in A \cap Q = P$. Thus $\ker \alpha' = P$, so we get an induced injection $\alpha : A/P \longrightarrow B/Q$.

Let now $R \subset A/P$ and $S \subset B/Q$ be such that

$$\alpha(R) = S \cap \alpha(A/P)$$

Note first that we have

$$\alpha(R) = (f^{-1}(R) + Q)/Q.$$

Indeed, if $r \in R$, we have some $x \in f^{-1}(R)$ such that $r = f(x) = x + P$. Then $\alpha(r) = x + Q$, so we see that $\alpha(R) \subset (f^{-1}(R) + Q)/Q$. Conversely let $u \in (f^{-1}(R) + Q)/Q$. This means that there exists some $x \in f^{-1}(R)$ and $q \in Q$ such that $u = x + q + Q$. But note that since $q \in Q$, we have $u = x + Q = \alpha(x + P) = \alpha(f(x)) \in \alpha(R)$. So we indeed get that $\alpha(R) = (f^{-1}(R) + Q)/Q$. Applying this argument for $R = A/P$, we see that we also have $\alpha(A/P) = (A + Q)/Q$. Note now that since $S = g^{-1}(S)/Q$, we have

$$S \cap \alpha(A/P) = g^{-1}(S)/Q \cap (A + Q)/Q = (g^{-1}(S) \cap (A + Q))/Q,$$

thus $g^{-1}(S \cap \alpha(A/P)) = g^{-1}(S) \cap (A + Q)$. Now since $\alpha(R) = g^{-1}(\alpha(R))/Q$, from $\alpha(R) = S \cap \alpha(A/P)$ we obtain

$$f^{-1}(R) + Q = g^{-1}(\alpha(R)) = g^{-1}(S \cap \alpha(A/P)) = g^{-1}(S) \cap (A + Q).$$

Let now $U \subset A$ any subspace. If $x \in (U + Q) \cap A$, we have that there exists some $u \in U$ and $q \in Q$ such that $x = u + q$. Since $x \in A$, we get that $q = x - u \in A$, i.e. $q \in Q \cap A = P$. Thus we get that $x \in U + P$. This gives us $(U + Q) \cap A \subset U + P$. If $P \subset U$, it is clear that $(U + Q) \cap A \subset U + P = U \subset (U + Q) \cap A$, thus $(U + Q) \cap A = U$ in that case. Since $P \subset f^{-1}(R) \subset A$, intersecting both sides of(3.2.1) with A, and using the discussion here for $U = f^{-1}(R)$, and $U = A$ since $P \subset U$ in both cases, we get

$$f^{-1}(R) = g^{-1}(S) \cap (A + Q) \cap A = g^{-1}(S) \cap A,$$

which is what we wanted to show. ∎

We are now ready to prove the following result

Proposition 3.8. *For any $i \in \mathbb{N}$, the following statements hold:*

i) $soc^i(M) \subset M$ *for any* $M \in \mathfrak{g}$−*mod;*

ii) $soc^i(f) = f|_{soc^i(M)}$ *for any morphism* $f : A \longrightarrow B$ *in* $\mathfrak{g}$−*mod;*

iii) $soc^i(M) = M \cap soc^i(N)$ *if* $M \subset N$ *in* $\mathfrak{g}$−*mod.*

Proof. i) and ii) are evident by the definition of soc^i. We now prove iii) by induction on i. For $i = 0$, we have that $\mathrm{soc}(M)$ is a semisimple submodule of M, hence also of N, so we have $\mathrm{soc}(M) \subset M \cap \mathrm{soc}(N)$. Conversely, we have that $M \cap \mathrm{soc}(N)$ is a submodule of M, and is actually semisimple as a submodule of $\mathrm{soc}(N)$, and as such, we will have $M \cap \mathrm{soc}(N) \subset \mathrm{soc}(M)$, i.e. $\mathrm{soc}(M) = M \cap \mathrm{soc}(N)$.

Assume now that this result holds for $i \in \mathbb{N}$, i.e. that $\mathrm{soc}^i(N) \cap M = \mathrm{soc}^i(M)$. We want to show that our claim is also true for $i + 1$. In the context of Lemma 3.7, set

$$A = M; \quad B = N; \quad P = \mathrm{soc}^i(M) \subset M = A; \quad Q = \mathrm{soc}^i(N) \subset N = B;$$

$$p_i : M \longrightarrow M/P; \quad q_i : N \longrightarrow N/Q \qquad \text{the canonical maps}$$

From the induction hypothesis, we have $Q \cap A = \mathrm{soc}^i(N) \cap M = \mathrm{soc}^i(M) = P$. Set now $R = \mathrm{soc}(A/P)$ and $S = \mathrm{soc}(B/Q)$. Note that as our canonical map $\alpha : A/P \longrightarrow B/Q$ is injective, we have that $\alpha(A/P) \subset B/Q$. Now using the statement of iii) for $i = 0$ which we proved above, we see that

$$S \cap \alpha(A/P) = \mathrm{soc}(B/Q) \cap \alpha(A/P) = \mathrm{soc}(\alpha(A/P)).$$

Consider $\alpha : A/P \longrightarrow \alpha(A/P)$, which is an isomorphism of $\mathfrak{g}$–modules. As $\mathrm{soc}(A/P)$ is a semisimple submodule of A/P, we have $\alpha(\mathrm{soc}(A/P)) \subset \mathrm{soc}(\alpha(A/P))$. If $L \subset \mathrm{soc}(\alpha(A/P))$ is simple, as α is an isomorphism, we have that $\alpha^{-1}(L) \subset A/P$ is also simple, hence $\alpha^{-1}(L) \subset \mathrm{soc}(A/P)$. This means that $\alpha^{-1}(\mathrm{soc}(\alpha(A/P))) \subset \mathrm{soc}(A/P)$. Thus we indeed get

$$\alpha(\mathrm{soc}(A/P)) = \mathrm{soc}(\alpha(A/P)),$$

which gives us

$$S \cap \alpha(A/P) = \alpha(\mathrm{soc}(A/P)) = \alpha(R).$$

Since all the conditions of the Lemma 3.7 are satisfied, we get

$$\mathrm{soc}^{i+1}(M) = p_i^{-1}(R) = A \cap q_i^{-1}(S) = M \cap \mathrm{soc}^{i+1}(N).$$

The proposition then follows by induction on i, which is what we wanted to show. $\square$

Proposition 3.47 says that the functors soc^i satsify the conditions of Proposition 1.9'. Then soc^i will also be essentially surjective. In particular, Proposition 1.9' implies the following result.

Corollary 3.9. *$soc^i : \mathfrak{g} - mod \longrightarrow \mathfrak{g} - mod$ are left-exact functors, and right adjoint to the inclusions $\subset$: $soc^i(\mathfrak{g} - mod) \longrightarrow \mathfrak{g} - mod$, $i \in \mathbb{N}$.*

Applying Corollary 1.10 to our situation for the functors soc^i we get

Corollary 3.10. *$soc^i(\mathfrak{g} - mod)$ has enough injectives, $i \in \mathbb{N}$.*

Remark 3.3. Let $\mathcal{S} \subset \mathfrak{g} - \mathrm{mod}$ denote the full subcategory of $\mathfrak{g}$–mod consisting of semisimple submodules. Note that if M is semisimple, then $\mathrm{soc}(M) = M$, so we have $\mathrm{soc}(\mathfrak{g} - \mathrm{mod}) = \mathcal{S}$. Then from Corollary 3.10 we get that $\mathcal{S}$ has enough injectives.

Note now that if one has $M \subset N \in \mathfrak{g}$-mod, and $\mathrm{soc}^k(N) = N$, then $\mathrm{soc}^k(M) = M \cap \mathrm{soc}^k(N) = M \cap N = M$. From this observation one gets the following consequence of Proposition 3.47

Corollary 3.11. *Let $M \in \mathfrak{g}$-mod have finite socle length. Then every submodule $K \subset M$ also has finite socle length, and $ll(K) \leq ll(M)$ for any submodule $K \subset M$.*

3.2.2 A generalization of Proposition 1.8

From now on in this section, let $\mathfrak{g}$ be a locally semisimple Lie algebra.

We will now give a stronger result from the setting of Proposition 1.8, and use it to try and understand how injective hulls of a particular type of simple integrable modules might look like. There, we saw that given a local module $M = \varinjlim M_i$, with M_i's containing simple $\mathfrak{g}_i$-submodules L_i such that $\dim \hom_{\mathfrak{g}_i}(L_i, L_{i+1}) > 2$, there will then exist a local module $Z = \varinjlim Z_i$ which fits into a non-split short exact sequence

$$0 \longrightarrow M \longrightarrow Z \longrightarrow L \longrightarrow 0,$$

where $Z_i = M_i \oplus L_i$, and $L = \varinjlim L_i$ for some already fixed morphisms $f_i : L_i \longrightarrow L_{i+1}$ of $\mathfrak{g}_i$−modules. Denote $Z^{(-1)} := 0$, $Z^{(0)} := M$, and $Z^{(1)} := Z$. Iterating this process for the newly obtained extensions we obtain non-split short exact sequences

$$0 \longrightarrow Z^{(k)} \longrightarrow Z^{(k+1)} \longrightarrow L \longrightarrow 0. \tag{3.7}$$

Note that can use the same L on (3.7) for all $k \in \mathbb{N}_0 \cup \{-1\}$. We will soon show that under the conditions of Proposition 1.8, one can get an even stronger conclusion for the short exact sequences in (3.7).

For this purpose, let us digress shortly into an observation from commutative algebra. Let

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 0 \tag{$\star$}$$

be a short exact sequence in $\mathfrak{g}$−mod, and let $D \subset A$ be a $\mathfrak{g}$−submodule. Consider now the map $\alpha' : A \longrightarrow B/D$ given by $\alpha'(a) = a + D$. Clearly we have $\ker \alpha' = D$, so this induces an injective map $\alpha : A/D \longrightarrow B/D$. Consider now the map $\beta : B/D \longrightarrow C$ given by $\beta(b + D) = p(b)$. Note that if $b_1 + D = b_2 + D$, there exists some $d \in D$ such that $b_2 = b_1 + d$. This gives us

$$\beta(b_2 + D) = p(b_2) = p(b_1 + d) = p(b_1) = \beta(b_1 + D)$$

because $p(d) = 0$ since $d \in A$. This means that β is well defined. Note that $(\beta \circ \alpha)(a + D) = \beta(a + D) = p(a) = 0$, hence we have $\mathrm{im}\alpha \subset \ker \beta$. Note also that from $0 = \beta(b + D) = p(b)$, we get that $b \in \ker p$, hence $b \in A$, thus we have $b + D \in A/D$. This shows what $\ker \beta \subset \mathrm{im}\alpha$. Hence we have obtain the following sequence

$$0 \longrightarrow A/D \longrightarrow B/D \longrightarrow B/A \longrightarrow 0 \tag{$\star\star$}$$

which is also short exact. Assume now that ($\star$) splits, i.e. that $B = A \oplus C$. Note then that

$$B/D = (A \oplus C)/D = A/D \oplus C$$

since $D \cap C \subset A \cap C = 0$, thus we have that ($\star\star$) also splits. This way we have proved the following result.

Lemma 3.12. *If ($\star$) splits, then ($\star\star$) also splits.*

Applying this discussion to (3.7) for the inclusion $Z^{(k-1)} \subset Z^{(k)}$ one gets a new short exact sequence:

$$0 \longrightarrow L = Z^{(k)}/Z^{(k-1)} \longrightarrow Z^{(k+1)}/Z^{(k-1)} \longrightarrow L = Z^{(k+1)}/Z^{(k)} \longrightarrow 0. \tag{3.8}$$

We are now ready to state and prove a generalization of Proposition 1.8.

Proposition 3.13. *Let $M \in Int_{\mathfrak{g}}$ be a local module with exhaustion $\{M_i\}_{i\in\mathbb{N}}$, and assume that there exists some natural number n and simple $\mathfrak{g}_i$−modules $L_i \subset M_i$ with $i > n$ such that* $\dim \hom_{\mathfrak{g}_i}(L_i, L_{i+1}) > 2$. *Then there exists a locally simple module $L = \varinjlim L_i \in Int_{\mathfrak{g}}$, and local modules $Z^{(k)} \in Int_{\mathfrak{g}}$ which fit into a short exact sequence as in (3.7). Furthermore, one can choose $Z^{(k)}$ such that (3.7) and (3.8) are non-split.*

Proof. Set $Z^{(0)} = M$. As we did in Proposition 1.8, one can get $Z^{(1)} = Z$ such that

$$0 \longrightarrow Z^{(0)} \longrightarrow Z^{(1)} \longrightarrow L \longrightarrow 0$$

is a non-split short exact sequence, and since $Z^{(1)}/Z^{(-1)} \cong Z^{(1)}$, we get that (3.8) for $k = 0$ is the same as (3.7), hence it will also be non-split. We note now how ($\star$) and ($\star\star$) in the context of our situation look like

$$0 \longrightarrow Z^{(k-1)} \longrightarrow Z^{(k)} \longrightarrow L \longrightarrow 0, \tag{3.7$'$}$$

$$0 \longrightarrow L \longrightarrow Z^{(k)}/Z^{(k-2)} \xrightarrow{p} L \longrightarrow 0 \tag{3.8$'$}$$

for $k \in \mathbb{N}$. Assume now that $k \in \mathbb{N}$ is such that (3.7$'$) and (3.8$'$) do not split for $l = 1, \ldots, k-1$. From the discussion at the beginning of this subsection, we know that these extensions were defined via

$$Z_i^{(l)} \coloneqq L_i \oplus Z_i^{(l-1)} = L_i \oplus (L_i \oplus Z_i^{(l-2)})$$

and the structure maps $a_i^{(l)} : Z_i^{(l)} \longrightarrow Z_i^{(l+1)}$ given by

$$a_i^{(l)}(x, x', z) = (f_i(x), r_i^{(l-1)}(q) + f_i(x'), t_i^{(l-2)}(x') + a_i^{(l-2)}(z)),$$

where $x, x' \in L_i$, $z \in Z_i^{(l-2)}$,$f_i : L_i \longrightarrow L_{i+1}$ are the structure maps for L, while $t_i^{(l-2)} : L_i \longrightarrow Z_{i+1}^{(l-2)}$ and $r_i^{(l-1)} : L_i \longrightarrow L_{i+1}$ are the chosen injective morphisms (in the spirit of Proposition 1.8) that make (3.7$'$) and (3.8$'$) non-split.

Let us now set, as previously

$$Z_i^{(k)} \coloneqq L_i \oplus Z_i^{(k-1)} = L_i \oplus (L_i \oplus Z_i^{(k-2)}),$$

and define the structure maps

$$a_i^{(k)}(x, x', z) = (f_i(x), r_i^{(k-1)}(x) + f_i(x'), t_i^{(k-2)}(x') + a_i^{(k-2)}(z)),$$

where $x, x' \in L_i$, $z \in Z_i^{(k-2)}$, and again $t_i^{(k-2)} : L_i \longrightarrow Z_{i+1}^{(k-2)}$ and $r_i^{(k-1)} : L_i \longrightarrow L_{i+1}$ being non-zero injections. An easy check will show that $a_i^{(k)}$ are injective maps, so one can set $Z^{(k)} = \varinjlim Z_i^{(k)}$. We then get two short exact sequences as in (3.7$'$) and (3.8$'$). We want to find $\{t_i^{(k-2)}\}_{i>n}$ and $\{r_i^{(k-1)}\}_{i>n}$ such that these two sequences do not split. From Lemma 3.12, we see that it suffices to find families of maps $\{t_i^{(k-2)}\}_{i>n}$ and $\{r_i^{(k-1)}\}_{i>n}$ so that (3.8$'$) does not split.

Let us assume for a moment that (3.8$'$) does split. We then have a map $\alpha : L \longrightarrow Z^{(k)}/Z^{(k-2)}$ such that

$$id_L = \sigma : L \xrightarrow{\alpha} Z^{(k)}/Z^{(k-2)} \xrightarrow{p} L.$$

From Proposition 1.6, as L is locally simple, we may assume that $\alpha(L_i) \subset Z_i^{(k)}/Z_i^{(k-2)}$. We now want to describe this map α in more explicit terms. Given $x \in L$, say $x \in L_i$ we have $\alpha(x) = z + Z_i^{(k-2)}$ for some $z \in Z_i^{(k)}$, say $z = (a, b, c)$ for some $a, b \in L_i$ and $c \in Z_i^{(k-2)}$. We then have $\sigma(x) = p(z + Z_i^{(k-2)}) = (a, b, c) + Z_i^{(k-1)} = (a, 0, 0) + Z_i^{(k-1)}$ as $b, c \in Z^{(k-1)}$. As $\sigma = \mathrm{id}_L$, we get that $(x, 0, 0) + Z_i^{(k-1)} = id_L(x) = \sigma(x) = (a, 0, 0) + Z_i^{(k-1)}$, so we obtain $x = a$.

Thus far we have that for every $x \in L$ there exists some $b \in L$, and $c \in Z^{(k-2)}$ such that

$$\alpha(x) = (x, b, c) + Z^{(k-2)}$$

Assume now that $(x, b, c) + Z^{(k-2)} = (x, b', c') + Z^{(k-2)}$ for some $b, b' \in L$ and $c, c' \in Z^{(k-2)}$. Again passing to the local $\mathfrak{g}_i$-module for which $x \in L_i$, we get that $(0, b - b', c - c') + Z_i^{(k-2)}$, which gives us $b - b' \in Z_i^{(k-2)}$. As $b - b' \in L_i$, and $L_i \cap Z_i^{(k-2)} = 0$, we get that $b = b'$. Hence for any $x \in L$, there exists a unique $b \in L$ such that $\alpha(x) = (x, b, 0) + Z^{(k-2)}$, where have removed c from the notation, as $c \in Z^{(k-2)}$. Thus we have a map $\beta : L \longrightarrow L$ such that for every $x \in L$ we have

$$\alpha(x) = (x, \beta(x), 0) + Z^{(k-2)}.$$

An easy check shows that β is indeed a map of $\mathfrak{g}$-modules. From Proposition 1.6, for large enough i we have $\beta(L_i) \subset L_i$, hence we get maps $\beta_i : L_i \longrightarrow L_i$ such that $\beta = \varinjlim \beta_i$. Let now $x \in L_i$. Since $(x, 0, 0) + Z^{(k-2)} = (f_i(q), 0, 0) + Z^{(k-2)}$, we get

$$(x, \beta_i(x), 0) + Z^{(k-2)} = \alpha(x) = \alpha(f_i(x)) = (f_i(x), \beta_{i+1}(f_i(x)), 0) + Z^{(k-2)}.$$

Hence, there exists some $z \in Z^{(k-2)}$ (may as well assume $z \in Z_i^{(k-2)}$ for large enough i) such that $(x, \beta_i(x), 0) = (f_i(x), \beta_{i+1}(f_i(x)), z)$ in $Z^{(k)}$. From the structure maps of $Z^{(k)}$, we get

$$(f_i(x), \beta_{i+1}(f_i(x)), z) = a_i^{(k)}(x, \beta_i(x), 0) = (f_i(q), r_i^{(k-1)}(x) + f_i(\beta_i(x)), \cdots)$$

which in particular from the second coordinates gives us $r_i^{(k-1)}(x) + f_i(\beta_i(x)) = \beta_{i+1}(f_i(x))$, i.e.

$$r_i = \beta_{i+1} \circ f_i - f_i \circ \beta_i. \tag{3.9}$$

From Schur's Lemma from the finite dimensional theory of representations of Lie algebras, we have $\dim \hom_{\mathfrak{g}_i}(L_i, L_i) = 1$. If by $S_i \subset \hom_{\mathfrak{g}_i}(L_i, L_{i+1})$ we denote the subspace of maps $r_i^{(k-1)} : L_i \longrightarrow L_{i+1}$ which make (3.8′) split (hence also (3.7′)), knowing that the f_i are fixed, from (3.9) we get that $\dim S_i \leq 2 < \dim \hom_{\mathfrak{g}_i}(L_i, L_{i+1})$. Hence not all choices of $\{r_i^{(k-1)}\}_{i>n}$ make these (3.7′) split. Thus there exists a choice of $\{t_i^{(k-2)}\}_{i>n}$ and $\{r_i^{(k-1)}\}_{i>n}$ which make (3.7′) and (3.8′) for k non-split.

The statement of the theorem then follows by induction on k. □

This, in itself, is a tedious proposition to prove. However, one can see it as a stronger version of Proposition 1.6. The importance of this technical result lies in that it tells us something about injective hulls of certain simple $\mathfrak{g}$-modules in $\mathrm{Int}_{\mathfrak{g}}$.

Note first that Proposition 3.13 gives us a sequence of non-split inclusions

$$0 \subset M \subset Z^{(1)} \subset Z^{(2)} \subset \cdots \subset Z^{(n)} \subset \cdots. \tag{3.10}$$

The following result discusses the socle lengths of these $\mathfrak{g}$-modules $Z^{(n)}$ in (3.10).

Proposition 3.14. *Let $Q \in Int_{\mathfrak{g}}$ be a simple module that satisfies the conditions of Proposition 3.13. Then the following statements hold:*

i)

$$0 \subset Q \subset Z \subset \cdots \subset Z^{(n)}$$

is a sequence consisting of all the submodules of $Z^{(n)}$ for any $n \in \mathbb{N}_0$;

ii) $soc^i(Z^{(n)}) = Z^{(i)}$ *for all i, n;*

iii) $ll(Z^{(n)}) = n$.

Proof. i) We prove this by induction on n. For $n = 0$, we have $Z^{(0)} = Q$ simple so the claim is trivially true.

Let $n = 1$. We then have $Z^{(1)} = Z$ fits in a non-split short exact sequence

$$0 \longrightarrow Q \longrightarrow Z \longrightarrow L \longrightarrow 0.$$

Let $M \subset Z$ be any submodule, and consider $N = M + Q \subset Z$. We then have $N/Q \subset Z/Q \cong L$, so we must have $N/Q = 0$ or $N/Q = L$, i.e. $N = Q$ or $N = Z$. If $N = Q$ we have that $M \subset Q$, hence we get $M \in \{0, Q\}$. Let now $N = Z$. Since $M + Q = Z$, and the short exact sequence above does not split, we cannot have $M \cap Q = 0$, so we get $Q \cap M \neq 0$, and since Q is simple, we get $Q \subset M$. But then $Z = N = M + Q = M$. So really the only submodules of Z are $0, Q$, and Z.

Assume now that the statement i) is true for $n \in \mathbb{N}_0$, so that

$$0 \subset Q \subset Z \subset \cdots \subset Z^{(n)}$$

is a sequence consisting of all the submodules of $Z^{(n)}$. We now want to look at the submodules of $Z^{(n+1)}$. Let $M \subset Z^{(n+1)}$ be a submodule. Let $N = M + Z^{(n)} \subset Z^{(n+1)}$. We then have $N/Z^{(n)} \subset L$, hence we get $N/Z^{(n)} = 0$ or $N/Z^{(n)} = L$, i.e. $N = Z^{(n)}$ or $N = Z^{(n+1)}$. If $N = Z^{(n)}$, we have $M + Z^{(n)} = N = Z^{(n)}$, i.e. get $M \subset Z^{(n)}$, hence $M \in \{0, Q, \dots, Z^{(n)}\}$ by the induction hypothesis. If $N = Z^{(n+1)}$, we get that $M + Z^{(n)} = N = Z^{(n+1)}$. Now since the inclusion $Z^{(n)} \longrightarrow Z^{(n+1)}$ is non-split, we have $M \cap Z^{(n)} \neq 0$. We want to show that $M \cap Z^{(n)} \in \{M, Z^{(n)}\}$. Assume not. Then $M \cap Z^{(n)}$ is a submodule of $Z^{(n)}$ different from $Z^{(n)}$, hence we have $M \cap Z^{(n)} \subset Z^{(n-1)}$ from the induction hypothesis. Set now $R = M + Z^{(n-1)}$. We then get $R + Z^{(n)} = M + Z^{(n)} = Z^{(n+1)}$ and since $R \cap Z^{(n)}$ is a proper submodule of $Z^{(n)}$ (as equality would imply $M \subset Z^{(n)}$, which we have assumed to not be true) which contains $Z^{(n-1)}$, we have $R \cap Z^{(n)} = Z^{(n-1)}$. But this would imply

$$Z^{(n+1)}/Z^{(n-1)} = R/Z^{(n-1)} \oplus Z^{(n)}/Z^{(n-1)} = R/Z^{(n-1)} \oplus L$$

with both summands being non-zero, as we have $M \notin Z^{(n-1)}$. This however is impossible, as by assumption the inclusion $L \longrightarrow Z^{(n+1)}/Z^{(n-1)}$ does not split. Hence we indeed have $M \cap Z^{(n)} \in \{M, Z^{(n)}\}$. If $M \cap Z^{(n)} = M$, we get that $M \subset Z^{(n)}$, hence $M \in \{0, Q, \dots, Z^{(n)}\}$. If $M \cap Z^{(n)} = Z^{(n)}$, we get that $Z^{(n)} \subset M$. Then as we did for the case $n = 1$, one will see that this implies $M \in \{Z^{(n)}, Z^{(n+1)}\}$. So really we get that

$$0 \subset Q \subset Z \subset \cdots \subset Z^{(n)}$$

is a sequence consisting of all the submodules of $Z^{(n+1)}$. The claim then follows by induction on n.

ii) For $n = 0$ this is trivially true. For $n = 1$, we know that the inclusion $Q \subset Z$ does not split, hence Z is not semisimple. Since $Q \subset \operatorname{soc}(Z) \neq Z$, from i) we get that $\operatorname{soc}(Z) = Q$. Note now that

$$\operatorname{soc}^1(Z) = p^{-1}(\operatorname{soc}(Z/Q)) = p^{-1}(\operatorname{soc}(L)) = p^{-1}(L) = Z,$$

where $p : Z \longrightarrow X$ is the canonical map. Hence the claim of ii) is also true for $n = 1$.

Fix now i, and assume that ii) holds for n. From $Z^{(n)} \subset Z^{(n+1)}$ and Proposition 3.8 we have

$$Z^{(i)} = \operatorname{soc}^i(Z^{(n)}) = Z^{(n)} \cap \operatorname{soc}^i(Z^{(n+1)})$$

Since $\operatorname{soc}^i(Z^{(n+1)})$ is a submodule of of $Z^{(n+1)}$, from i) we can see that this implies $\operatorname{soc}^i(Z^{(n+1)}) = Z^{(i)}$. The claim then follows by induction, and by noting that this works for any natural i.

iii) Since $\operatorname{soc}^i(Z^{(n)}) = Z^{(i)}$, we can see that the sequence in i) is actually the socle filtration for $Z^{(n)}$, so in particular we see that $ll(Z^{(n)}) = n$.

□

3.2.3 Socle length of $\Gamma_{\mathfrak{g}}(M^*)$

We now use the discussion of the previous subsection to show how given a simple $Q \in \mathrm{Int}_{\mathfrak{g}}$, the socle length of $\Gamma_{\mathfrak{g}}(M^*)$ can impact its behavior. This theme will be explored even more in Section 3.4. In particular, the following result discusses integrable injective modules I for which there exists an inclusion $Q \longrightarrow I$ with Q as in the statement of Proposition 3.13.

Corollary 3.15. *Let $Q \in Int_{\mathfrak{g}}$ be as in the statement of Proposition 3.13, and $I \in Int_{\mathfrak{g}}$ an injective module such that there exists an injection $q : Q \longrightarrow I$. Then for any $n \in \mathbb{N}$ we have natural injections $f_n : Z^{(n)} \longrightarrow I$ such that $f_n|_Q = q$ for all natural n. In particular I does not have finite socle length.*

Proof. We prove this by induction on n. For $n = 0$ this is trivially true. For $n = 1$, we have the inclusions $Q \subset Z$ and $Q \subset I$. Since I is an injective object, there exists a morphism $f_1 : Z \longrightarrow I$ such that $f_1|_Q = q$. We want to show that this f_1 is injective. Since $f_1|_Q = q$, we have that $f_1 \neq 0$, hence $\ker f_1 \neq Z$. As $\ker f_1 \subset Z$ is a submodule, and from Proposition 3.14 we get that $\ker f_1 \subset Q$ so $\ker f_1 = \ker(f_1|_Q) = \ker q = 0$. Hence f_1 is indeed an injective map.

Assume now that the claim is true for $n \in \mathbb{N}$. Again, from the inclusion $Z^{(n)} \subset Z^{(n+1)}$ and the map $f_n : Z^{(n)} \longrightarrow I$ we get a map $f_{n+1} : Z^{(n+1)} \longrightarrow I$ such that $f_{n+1}|_{Z^{(n)}} = f_n$. As f_n is injective, we clearly have that $\ker f_{n+1} \neq Z^{(n+1)}$. Again from Proposition 3.14, we get that $\ker f_{n+1} \subset Z^{(n+1)}$, i.e. $\ker f_{n+1} = \ker(f_{n+1}|_{Z^{(n+1)}}) = \ker f_n = 0$, hence f_{n+1} is injective. Note that we also have $f_{n+1}|_Q = (f_{n+1}|_{Z^{(n)}})|_Q = f_n|_Q = q$. Thus the first part of the proposition follows by induction.

This way we have shown that in the setting of the statement of this corollary, we obtain a sequence of inclusions

$$0 \subset Q \subset Z \subset \cdots \subset Z^{(n)} \subset \cdots \subset I.$$

It is clear then that if I were of finite socle length, from Corollary 3.11 we would get $ll(I) \geq ll(Z^{(n)})$ for all natural n. Since $ll(Z^{(n)})$ from part iii) of Proposition 3.14, we get that $ll(I) \geq ll(Z^{(n)}) = n$ for all natural n, which is impossible. Hence I is indeed not a $\mathfrak{g}$−module of finite socle length. ☐

Remark 3.4. In Remark 1.5 we noted that one can construct $\mathfrak{g}$−modules M that satisfy the conditions of Proposition 3.13. In fact one can also construct such modules $M = Q$ that are simple. If Q is such a module, from Corollary 2.18, we know that $I = \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(Q^*)^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ for which there exists an injective morphism $Q \longrightarrow I$. From Corollary 3.15 we see that I will not be of finite socle length. This shows how algebraic dualization of even simple integrable modules can lead to wild behavior of $\mathfrak{g}$−modules.

Corollary 3.15 shows that given a simple module Q which satisfies the conditions of Proposition 3.13, then it can not be embedded into any injective module of finite socle length. The following result, the proof of which one can find in [10], gives a result on a contrapositive assumption to that of Proposition 3.13.

Lemma 3.16. *Let $Q \in Int_{\mathfrak{g}}$ be a simple module which admits an embedding into an injective object $I \in Int_{\mathfrak{g}}$ with $ll(I) < \infty$. Then there exists a simple exhaustion $\{Q_i\}_{i \in \mathbb{N}}$ of Q and $n \in \mathbb{N}$ such that for any $j > i > n$ we have*

$$\dim \hom_{\mathfrak{g}_i}(Q_i, Q_j) = 1.$$

Remark 3.5. Lemma 3.16 says in particular that if $Q \in \mathrm{Int}_{\mathfrak{g}}$ is simple, such that $\Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(Q^*)^*)$ has finite socle length, then Q is a locally simple module. In Remark 1.4 we noted that not all simple local modules are locally simple, so this result gives a conditional converse of Proposition 1.3.

We now want to look at the duals of simple modules Q which satisfy the conclusion of Lemma 3.16.

Lemma 3.17. *Let $Q \in Int_{\mathfrak{g}}$ be a simple module that admits a simple exhaustion $\{Q_i\}_{i\in\mathbb{N}}$ such that for some natural n, we have $\dim \hom_{\mathfrak{g}_i}(Q_i, Q_j) = 1$ for all $j > i > n_0$. Then Q^* has a unique simple submodule Q_*, and $Q_* \in Int_{\mathfrak{g}}$.*

Proof. For convenience, let all the indices i that appear in what follows be such that $i > n$. Let now $f, g : Q_i \longrightarrow Q$ be non-zero maps of $\mathfrak{g}_i$−modules. Since Q_i is finite dimensional and $Q = \varinjlim Q_i$ is a simple exhaustion of Q, there exists some $j > i$ such that $f(Q_i), g(Q_i) \subset Q_j$, i.e. f and g factor through the inclusion $q_j : Q_j \subset Q$. This means that there exist maps of $\mathfrak{g}_i$−modules $f', g' : Q_i \longrightarrow Q_j$ such that $f = q_j \circ f'$ and $g = q_j \circ g'$. But since $\dim \hom_{\mathfrak{g}_i}(Q_i, Q_j) = 1$, we have that there exist $x, y \in \mathbb{C}$ not both zero such that $xf' + yg' = 0$. But then we have

$$xf + yg = q_j \circ (xf' + yg') = 0.$$

This shows that we have

$$\dim \hom_{\mathfrak{g}_i}(Q_i, Q) = 1. \tag{3.11}$$

From (3.11) and Theorem 2.5, we get that

$$Q|_{\mathfrak{g}_i} = Q_i \oplus \bigoplus_{t\in T} M_t$$

for some index set T, and M_t simple finite dimensional $\mathfrak{g}_i$−modules such that $M_t \ncong Q_i$ for all $t \in T$. Dualizing this equality, we get

$$Q^*|_{\mathfrak{g}_i} = Q_i^* \oplus \prod_{t\in T} M_t^*. \tag{3.12}$$

By an argument similar to that in the last part of the proof of Corollary 3.5, we get that

$$\dim \hom_{\mathfrak{g}_i}(Q_i^*, Q^*) = 1. \tag{3.11'}$$

Completely analogously we get that $\dim \hom_{\mathfrak{g}_i}(Q_i^*, Q_j^*) = 1$ for all $j > i > n_0$, hence we can define a submodule $Q_* = \varinjlim Q_i^* \subset Q^*$ uniquely. Clearly this Q_* is a locally simple module, hence it is simple and $Q_* \in \mathrm{Int}_{\mathfrak{g}}$. We do not show here that Q_* is the unique simple submodule of Q^*, and we refer to [10] for the proof of this. □

With all of these results, we are now able to state and prove the main result of this section.

Theorem 3.18. *Let $Q \in Int_{\mathfrak{g}}$ be simple such that $ll(\Gamma_{\mathfrak{g}}(Q^*)) < \infty$. Then there exists a simple exhaustion $\{Q_i\}_{i\in\mathbb{N}}$ of Q and a natural number n such that for all $i > n$ we have $\dim \hom_{\mathfrak{g}_i}(Q_i, Q_j) = 1$ for all $j > i > n_0$.*

Proof. Note that as $\Gamma_{\mathfrak{g}}(Q^*)$ has finite socle length, we get in particular that $\mathrm{soc}(\Gamma_{\mathfrak{g}}(Q^*)) \neq 0$, so $\Gamma_{\mathfrak{g}}(Q^*)$ contains some simple submodule L. As this L is a submodule of $\Gamma_{\mathfrak{g}}(Q^*)$, it is in fact simple integrable submodule of Q^*. As $\Gamma_{\mathfrak{g}}(Q^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ from Corollary 2.16, has finite socle length, and contains S, we get that this simple integrable module S satisfies the conditions of Lemma 3.16. Hence it also satisfies the conditions of Lemma 3.17, thus we have that S^* has a unique simple submodule S_*, and $S_* \in \mathrm{Int}_{\mathfrak{g}}$. Note that the morphism obtained by composing the natural injection $Q \longrightarrow Q^{**}$ as in Subsection 2.2.1 with the dual of the map $S \longrightarrow Q^*$, i.e. the morphism

$$\alpha : Q \longrightarrow Q^{**} \longrightarrow S^*$$

is given by

$$\alpha(q)(f) = f(q).$$

As $S \neq 0$, there exists some non-zero $f \in S$. Let now $q \in Q$ be such that $f(q) \neq 0$. We then have $\alpha(q)(f) = f(q) \neq 0$, i.e. $\alpha(q) \neq 0$. Since Q is simple, and $\alpha \neq 0$, we have that this α is an injective map. In particular, $\alpha(Q) \subset S^*$ is a simple integrable module. From Lemma 3.17 we have $Q \cong \alpha(Q) = S_*$. By the same lemma, we know that S_* admits an exhaustion as we desire to find for Q, and since they are isomorphic modules, the proof of this theorem is completed. □

One can see that the condition on Q in Theorem 3.18 implies the condition on Q in Lemma 3.17, so we get the following consequence.

Corollary 3.19. *Let $Q \in Int_{\mathfrak{g}}$ be a simple module such that $\Gamma_{\mathfrak{g}}(Q^*)$ has finite socle length. Then Q^* has a unique simple submodule Q_*, and $Q_* \in Int_{\mathfrak{g}}$.*

Remark 3.6. Let Q be as in Corollary 3.19. We saw in the proof of Lemma 3.17 that if $\{Q_i\}_{i\in\mathbb{N}}$ is the simple exhaustion of Q, with $\dim \hom_{\mathfrak{g}_i}(Q_i, Q_j) = 1$ for all $j > i > n$, for some $n \in \mathbb{N}$, then $\{Q_i^*\}_{i\in\mathbb{N}}$ is an exhaustion of Q_* with $\dim \hom_{\mathfrak{g}_i}(Q_i^*, Q_j*) = 1$ for all $j > i > n$. Note that if $\Gamma_{\mathfrak{g}}((Q_*)^*)$ has finite socle length, from the construction in the proof of Lemma 3.17 we see that we will have $(Q_*)_* \cong Q$. From Corollary 3.19, we get that the only simple submodule of $(Q_*)^*$ is Q.

3.3 Further properties of the socle functors

In this section, let $\mathfrak{g}$ be any Lie algebra.

In this section we give a survey of some properties of the socle functors. In particular we will want to see how finite socle length behaves under algebraic operations, specifically under taking arbitrary direct sums, quotients, and extensions.

Let us start first with the following simple observation.

Lemma 3.20. *Let M be a $\mathfrak{g}$−module, and $N \subset M$ a $\mathfrak{g}$−submodule such that $N \in soc^i(\mathfrak{g}-mod)$. Then $N \subset soc^i(M)$.*

Proof. Since $N \in \mathrm{soc}^i(\mathfrak{g} - \mathrm{mod})$ we have that there exists some $\mathfrak{g}$−module A such that $N \cong \mathrm{soc}^i(A)$. Without loss of generality, we may assume that $N \subset A$. Note then that from Proposition 3.8 we have

$$\mathrm{soc}^i(N) = N \cap \mathrm{soc}^i(A) = N \cap N = N.$$

From this and Proposition 3.8 again we get

$$N = \mathrm{soc}^i(N) = N \cap \mathrm{soc}^i(M),$$

which clearly gives us that $N \subset \mathrm{soc}^i(M)$, which is what we wanted to show. □

In other words, Lemma 3.20 says that $\mathrm{soc}^i(M)$ is the largest submodule of M which lies in $\mathrm{soc}^i(\mathfrak{g} - \mathrm{mod})$.

Remark 3.7. Note that a trivial consequence of Lemma 3.20 is that given an isomorphism $\alpha : A \longrightarrow B$, the restriction of α to $\mathrm{soc}^k(A)$ induces an isomorphism $\mathrm{soc}^k(A) \longrightarrow \mathrm{soc}^k(B)$. Really, as $\alpha(\mathrm{soc}^k(A))$ is just a copy of $\mathrm{soc}^k(A)$, we have that $\alpha(\mathrm{soc}^k(A)) \in \mathrm{soc}^k(\mathfrak{g} - \mathrm{mod})$, hence by the Lemma we have $\alpha(\mathrm{soc}^k(A)) \subset \mathrm{soc}^k(B)$. Arguing similarly for α^{-1} we get that $\alpha^{-1}(\mathrm{soc}^k(B)) \subset \mathrm{soc}^k(A)$. Thus really $\alpha(\mathrm{soc}^k(A)) = \mathrm{soc}^k(B)$.

We now want to see under what algebraic operations will finite socle length be preserved. Before we state and prove a result on the socle filtration of arbitrary direct sum, namely Lemma 3.21, let us make the following observation. Let I be an index set, and $B_i \subset U_i \subset A_i$ be $\mathfrak{g}$−modules for all $i \in I$, and set

$$A = \bigoplus_{i \in I} A_i; \;\; B = \bigoplus_{i \in I} B_i.$$

Let now $r_i : A \longrightarrow A_i$ and $p_i : A_i \longrightarrow A_i/B_i$ be the natural projections. Consider now the map

$$p := \bigoplus_{i \in I} (p_i \circ r_i) : A \longrightarrow \bigoplus_{i \in I} (A_i/B_i).$$

Since the p_i and r_i are surjective for all $i \in I$, it follows that p is also a surjective morphism. Let now $a \in A$ be such that $p(a) = 0$. This means that $p_i(r_i(a)) = 0$ for all $i \in I$, so we get $r_i(a) \in B_i$ for all $i \in I$. Since we have $a = \bigoplus_{i \in I} r_i(a)$, we get that $a \in B$. It is also evident that $p(B) = 0$. Hence we get $\ker p = B$, thus the map p induces a natural isomorphism

$$A/B \equiv \bigoplus_{i \in I} (A_i/B_i) \tag{3.13}$$

Let now $u \in p^{-1} \left(\bigoplus_{i \in I} (U_i/B_i) \right)$. This means that

$$\bigoplus_{i \in I} p_i(r_i(u)) = p(u) \in \bigoplus_{i \in I} (U_i/B_i)$$

so in particular we get that $r_i(u) \in U_i$ for every $i \in I$. Since $U_i = p_i^{-1}(U_i/B_i)$, this gives us

$$u = \bigoplus_{i \in I} r_i(u) \in \bigoplus_{i \in I} U_i = \bigoplus_{i \in I} p_i^{-1}(U_i/B_i).$$

This way we have shown that with A_i and B_i as in our setting, for any submodules $S_i \subset A_i/B_i$ for $i \in I$, we will have

$$p^{-1} \left(\bigoplus_{i \in I} S_i \right) \subset \bigoplus_{i \in I} p_i^{-1}(S_i). \tag{3.14}$$

Using this observation, we can now prove the following result.

Lemma 3.21. *Let $A_i \in \mathfrak{g} - mod$ for $i \in I$, where I is some index set, and set*

$$A := \bigoplus_{i \in I} A_i.$$

Then

$$soc^k(A) = \bigoplus_{i \in I} soc^k(A_i)$$

for all $k \in \mathbb{N}_0$.

Proof. First of all note that for any $i \in I$ we have $\mathrm{soc}^k(A_i) \subset \mathrm{soc}^k(A)$, as soc^k are left exact functors from Corollary 3.9. This way we get

$$\bigoplus_{i \in I} \mathrm{soc}^k(A_i) = \sum_{i \in I} \mathrm{soc}^k(A_i) \subset \mathrm{soc}^k(A),$$

thus one of the inclusions in the statement of the lemma is evident.

The other inclusion we prove by induction on k. Let $k = 0$, and let $S \subset A$ be a simple submodule. Denote by $u : S \longrightarrow A$ the natural inclusion, and by $p_i : A \longrightarrow A_i$ the natural projections. Set now

$$q_i := p_i \circ u : S \longrightarrow A_i; \;\; \text{and} \;\; q_i(S) = S_i.$$

Note that as S is simple, we have $S_i = 0$ or $q_i : S \longrightarrow S_i$ is an isomorphism. Let now $s \in S$ non-zero, and as $s \in A$, let $I_0 = \{i_1, \dots, i_n\}$ be such that there exist non-zero $a_{i_j} \in A_{i_j}$ with $s = a_{i_1} + \dots + a_{i_n}$. Clearly this means that $a_{i_j} = q_{i_j}(s)$. Note now that if $i \in I \backslash I_0$ we have $q_i(s) = 0$, hence $q_i : S \longrightarrow S_i$ cannot be an isomorphism, thus it must be the zero map, so we get $q_i = 0$ and $S_i = 0$ for $i \in I \backslash I_0$. This gives us that for any $s' \in S$, we have $q_i(s') = 0$ for all $i \in I \backslash I_0$. Since $s' = \bigoplus_{i \in I} q_i(s')$, this means that $s' = \bigoplus_{i \in I_0} q_i(s')$, thus we get

$$S \subset \bigoplus_{i \in I_0} S_i.$$

As $S_i \subset A_i$ are simple, we have that $S_i \subset \operatorname{soc}(A_i)$, and we get

$$S \subset \bigoplus_{i \in I_0} \operatorname{soc}(A_i) \subset \bigoplus_{i \in I} \operatorname{soc}(A_i).$$

Clearly then we have

$$\operatorname{soc}(A) = \sum_{S \subset A \text{ simple}} S \subset \bigoplus_{i \in I} \operatorname{soc}(A_i),$$

so indeed we get

$$\operatorname{soc}(A) = \bigoplus_{i \in I} \operatorname{soc}(A_i).$$

Assume now that the claim of the lemma is true for $k \in \mathbb{N}$. Note now that from (3.13) we have

$$A/\operatorname{soc}^k(A) = \left(\bigoplus_{i \in I} A_i \right) / \left(\bigoplus_{i \in I} \operatorname{soc}^k(A_i) \right) = \bigoplus_{i \in I} \left(A_i / \operatorname{soc}^k(A_i) \right)$$

and from first part of the proof of this lemma for $k = 0$ we get

$$\operatorname{soc}\left(A/\operatorname{soc}^k(A) \right) = \operatorname{soc}\left(\bigoplus_{i \in I} \left(A_i/\operatorname{soc}^k(A_i) \right) \right) = \bigoplus_{i \in I} \operatorname{soc}\left(A_i/\operatorname{soc}^k(A_i) \right).$$

Now if $p : A \longrightarrow A/\operatorname{soc}^k(A)$ and $p_i : A_i \longrightarrow A_i/\operatorname{soc}^k(A_i)$ are the canonical quotient maps, from (3.14), with $B_i = \operatorname{soc}^k(A_i)$, $S_i = \operatorname{soc}\left(A_i/\operatorname{soc}^k(A_i) \right)$ we see that

$$\begin{aligned}\operatorname{soc}^{k+1}(A) = p^{-1}(\operatorname{soc}(A/\operatorname{soc}^k(A)) &= p^{-1} \left(\bigoplus_{i \in I} \operatorname{soc}\left(A_i/\operatorname{soc}^k(A_i) \right) \right) \\ &\subset \bigoplus_{i \in I} p_i^{-1} \left(A_i/\operatorname{soc}^k(A_i) \right) = \bigoplus_{i \in I} \operatorname{soc}^{k+1}(A_i),\end{aligned}$$

which proves the inclusion in the other direction. From this and the first part of the proof of this lemma, we indeed get

$$\operatorname{soc}^{k+1}(A) = \bigoplus_{i \in I} \operatorname{soc}^{k+1}(A_i).$$

The statement of the lemma then follows by induction. □

Keeping with the theme of observing under which algebraic operations is finite socle length preserved, we note the following direct consequence of Lemma 3.21.

Corollary 3.22. *Let $A_1, A_2, \dots, A_n \in \mathfrak{g}$–mod be modules of finite socle length. Then $A = A_1 \oplus \dots \oplus A_n$ also has finite socle length.*

Proof. As all A_i for $1 \le i \le n$ have finite socle length, let $k \in \mathbb{N}$ be such that for all such i we have $\mathrm{soc}^k(A_i) = A_i$. Then from Lemma 3.21 we have

$$\mathrm{soc}^k(A) = \bigoplus_{i=1}^{n} \mathrm{soc}^k(A_i) = \bigoplus_{i=1}^{n} A_i = A,$$

hence A does have finite socle length. In fact this proof shows that if

$$k = \max\{ll(A_i) \mid i = 1, \dots, n\},$$

then $ll(A) = k$. □

Remark 3.8. Note that one can prove Corollary 3.22 via only the inclusion $\bigoplus_{i=1}^{n} \mathrm{soc}^k(A_i) \subset \mathrm{soc}^k(A)$ shown in the first part of the proof of Lemma 3.21. Indeed, if k is as in the proof of Corollary 3.22, we get

$$\bigoplus_{i=1}^{n} \mathrm{soc}^k(A_i) = \bigoplus_{i=1}^{n} A_i = A \subset \mathrm{soc}^k(A),$$

which clearly gives us $A = \mathrm{soc}^k(A)$.

Next we will give a generalization of Proposition 3.6. Before we do this, let us make an observation in the spirit of the one preceding Lemma 3.21. Let $C \subset B \subset A$ be $\mathfrak{g}$–modules, and let $q : A \longrightarrow A/B$, $p : A/C \longrightarrow (A/C)/(B/C)$ be the canonical maps. Denote by $\alpha : (A/C)/(B/C) \longrightarrow A/B$ the canonical isomorphism. Let now $S \subset (A/C)/(B/C)$ be a $\mathfrak{g}$–submodule. Set $P = p^{-1}(S)$ so that we have $P/(B/C) = S$. As $P \subset A/C$ is a submodule, there exists some submodule $R \subset A$ such that $P = R/C$. Note now that

$$\alpha(S) = \alpha(P/(B/C)) = \alpha((R/C)/(B/C)) = R/B,$$

so in particular we get $q^{-1}(\alpha(S)) = R$. So in our setting, we obtain

$$p^{-1}(S) = P = R/C = q^{-1}(\alpha(S))/C. \tag{3.15}$$

We now proceed with two computational results of the socle functors. These results will be useful in showing that finite socle length is preserved under further algebraic operations.

Proposition 3.23. *Let $k \in \mathbb{N}$. Then for any $A \in \mathfrak{g}$–mod we have*

$$soc^k(A)/soc(A) = soc^{k-1}(A/soc(A)).$$

Proof. We prove this by induction on k. Proposition 3.6 shows that the statement is true for $k = 1$. Assume now that the statement of the proposition holds for $k \in \mathbb{N}$. Consider now the case for $k+1$. Note now that by definition and the induction hypothesis we have

$$\begin{aligned} \mathrm{soc}^k(A/\mathrm{soc}(A)) &= p^{-1}\left[\mathrm{soc}\left[(A/\mathrm{soc}(A))/\mathrm{soc}^{k-1}(A/\mathrm{soc}(A))\right]\right] \\ &= p^{-1}\left[\mathrm{soc}\left[(A/\mathrm{soc}(A))/(\mathrm{soc}^k(A)/\mathrm{soc}(A))\right]\right], \end{aligned} \tag{3.16}$$

where p is the canonical map. Now by letting $A = A$, $B = \mathrm{soc}^k(A)$, $C = \mathrm{soc}(A)$, and $S = \mathrm{soc}((A/\mathrm{soc}(A))/(\mathrm{soc}^k(A)/\mathrm{soc}(A)))$ in the setting of the discussion preceding this proposition, from (3.15) we get

$$\mathrm{soc}^k(A/\mathrm{soc}(A)) = p^{-1}(S) = q^{-1}(\alpha(S))/C.$$

Note that as α is an isomorphism, from Remark 3.7 we get $\alpha(S) = \mathrm{soc}(A/B)$, thus

$$\mathrm{soc}^k(A/\mathrm{soc}(A)) = q^{-1}\left[\mathrm{soc}(A/\mathrm{soc}^k(A))\right]/\mathrm{soc}(A) = \mathrm{soc}^{k+1}(A)/\mathrm{soc}(A).$$

The proposition then follows by induction. □

Using this result, we can now write and prove an even more general statement on how one can identify quotients in the socle filtration of a $\mathfrak{g}$−module.

Corollary 3.24. *Let $k, l \in \mathbb{N}$ be such that $k > l$. Then for any $A \in \mathfrak{g}$−mod, we have*

$$soc^k(A)/soc^l(A) = soc^{k-l-1}(A/soc^l(A)).$$

Proof. We prove this by induction on l. Proposition 3.23 shows that this is true for $l = 0$. Assume now that this is true for $l-1 \in \mathbb{N}$. Consider now the case for l. Denote by $\alpha : A/\mathrm{soc}^l(A) \longrightarrow (A/\mathrm{soc}(A))/(\mathrm{soc}^l(A)/\mathrm{soc}(A)$ the canonical isomorphism. Let now $k > l$, so we have also in particular $k-1 > l-1$.Then by the induction hypothesis and Proposition 3.23 we have

$$\mathrm{soc}^k(A)/\mathrm{soc}^l(A) \overset{\alpha}{\cong} (\mathrm{soc}^k(A)/\mathrm{soc}(A))/(\mathrm{soc}^l(A)/\mathrm{soc}(A)) =$$

$$\overset{3.23}{=} \mathrm{soc}^{k-1}((A/\mathrm{soc}(A))/\mathrm{soc}^{l-1}(A/\mathrm{soc}(A)) =$$

$$\overset{\text{i.h.}}{=} \mathrm{soc}^{(k-1)-(l-1)-1}((A/\mathrm{soc}(A))/\mathrm{soc}^{l-1}(A/\mathrm{soc}(A))) =$$

$$\overset{3.23}{=} \mathrm{soc}^{k-l-1}((A/\mathrm{soc}(A))/(\mathrm{soc}^l(A)/\mathrm{soc}(A))) \overset{\alpha^{-1}}{\cong} \mathrm{soc}^{k-l-1}(A/\mathrm{soc}^l(A)),$$

where 'i.h.' indicates the induction hypothesis, and where the last isomorphism holds because of Remark 3.7. It is clear then that

$$\mathrm{soc}^k(A)/\mathrm{soc}^l(A) = \mathrm{soc}^{k-l-1}(A/\mathrm{soc}^l(A))$$

which is what we wanted to show. The corollary then follows by induction. □

We can now use Corollary 3.24 to give a result which can be practically useful when one wants to check the socle length of a $\mathfrak{g}$−module.

Corollary 3.25. *Let $A \in \mathfrak{g}$−mod, and $k \in \mathbb{N}$. Then A has finite socle length if and only if $A/soc^k(A)$ has finite socle length. In that case, if $k \leq ll(A)$, the following holds*

$$ll(A) = ll(A/soc^k(A)) + k + 1.$$

Proof. From Corollary 3.24, for any natural $n > k$ we have

$$\mathrm{soc}^n(A)/\mathrm{soc}^k(A) = \mathrm{soc}^{n-k-1}(A/\mathrm{soc}^k(A))$$

If A has finite socle length, there exists some natural t, which we may assume to be $t > k$ without loss of generality, such that $\mathrm{soc}^t(A) = A$. We then get

$$\mathrm{soc}^{t-k-1}(A/\mathrm{soc}^k(A)) = \mathrm{soc}^t(A)/\mathrm{soc}^k(A) = A/\mathrm{soc}^k(A),$$

so we get that $A/\mathrm{soc}^k(A)$ also has finite socle length. If $A/\mathrm{soc}^k(A)$ has finite socle length, there exists some natural $n_0 > k$ such that $\mathrm{soc}^{n_0}(A/\mathrm{soc}^k(A)) = A/\mathrm{soc}^k(A)$. Then for $t = n_0 + k + 1$ we get

$$\mathrm{soc}^t(A)/\mathrm{soc}^k(A) = \mathrm{soc}^{t-k-1}(A/\mathrm{soc}^k(A)) = \mathrm{soc}^{n_0}(A/\mathrm{soc}^k(A)) = A/\mathrm{soc}^k(A)$$

so we get $\mathrm{soc}^t(A) = A$, hence A also does have finite socle length.

Assume now that A has finite socle length, say $ll(A) = t$. This means that $\mathrm{soc}^{t-1}(A) \neq \mathrm{soc}^t(A) = A$. Note now that

$$\mathrm{soc}^{t-k-2}(A/\mathrm{soc}^k(A)) = \mathrm{soc}^{t-1}(A)/\mathrm{soc}^k(A) \neq A/\mathrm{soc}^k(A)$$

and

$$\mathrm{soc}^{t-k-1}(A/\mathrm{soc}^k(A)) = \mathrm{soc}^t(A)/\mathrm{soc}^k(A) = A/\mathrm{soc}^k(A)$$

so we get that $ll(A/\mathrm{soc}^k(A)) = t-k-1$, i.e.

$$ll(A) = ll(A/\mathrm{soc}^k(A)) + k + 1,$$

which is what we wanted to show. □

Remark 3.9. Let A be a $\mathfrak{g}$–module of finite socle length, say $ll(A) = k$. Note now that since $\mathrm{soc}(A)$ is semisimple, we naturally have $ll(\mathrm{soc}(A)) = 0$. Assume that $l < k$ is such that $ll(\mathrm{soc}^l(A)) = l$. From Proposition 3.6 we have that $\mathrm{soc}^{l+1}(A)/\mathrm{soc}^l(A)$ is semisimple, hence has socle length 0. Applying Corollary 3.25 to $\mathrm{soc}^{l+1}(A)$ we get

$$ll(\mathrm{soc}^{l+1}(A)) = ll(\mathrm{soc}^{l+1}(A)/\mathrm{soc}^l(A)) + l + 1 = l + 1.$$

Hence, by induction, we have that for any $t \le k$ the following holds

$$ll(\mathrm{soc}^t(A)) = t.$$

So far we have seen how finite socle length behaves with respect to direct sums. In what follows, we want to show that:

i) finite socle length is preserved under taking extensions;

ii) finite socle length is preserved under taking quotients.

The proofs for these presented here follow a slightly unconventional route. More precisely, we first prove a weaker version of i), from which ii) follows, which then in turn implies the full version of i).

Lemma 3.26. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in $\mathfrak{g}$–mod, with C semisimple, and $ll(A) = k$. Then $ll(B) = k$ or $ll(B) = k+1$. In particular, B has finite socle length.

Proof. As $ll(A) = k$, we have $A = \mathrm{soc}^k(A) \subset \mathrm{soc}^k(B)$. If $\mathrm{soc}^k(B) = B$, we get $ll(B) \le k$. Since $A \subset B$ implies $ll(A) \le ll(B)$ we get $ll(B) = k$ and we are done. Assume now that $\mathrm{soc}^k(B) \ne B$. If $A = \mathrm{soc}^k(B)$ we have that

$$B/\mathrm{soc}^k(B) = B/A \cong C$$

has finite socle length from Corollary 3.25, and we get

$$ll(B) = ll(B/\mathrm{soc}^k(B)) + k + 1 = ll(C) + k + 1 = k + 1$$

so we are also done in this case.

Assume now that $A \ne \mathrm{soc}^k(B) \ne B$. Since $A \subset \mathrm{soc}^k(B)$, we have that

$$\mathrm{soc}^k(B)/A \subset B/A = C$$

is a non-trivial submodule of C. As C is semisimple, there exists some $V \subset C$, also semisimple, such that

$$\mathrm{soc}^k(B)/A \oplus V = C = B/A.$$

Note then that

$$B/\mathrm{soc}^k(B) \cong (B/A)/(\mathrm{soc}^k(B)/A) \cong V,$$

which gives us

$$ll(B/\mathrm{soc}^k(B)) = ll(V) = 0$$

because $0 \leq ll(V) \leq ll(C) = 0$. Thus from Corollary 3.25 we get

$$ll(B) = ll(B/\mathrm{soc}^k(B)) + k + 1 = k + 1,$$

which is what we wanted to show. □

Now using this weaker version of i) above, we first prove the following slightly weaker version of ii).

Lemma 3.27. *Let $B \in \mathfrak{g}$−mod with $ll(B) = k$, and $A \subset B$ a $\mathfrak{g}$−submodule with $ll(A) < k$. Then $ll(B/A) < \infty$.*

Proof. We prove this by induction on k. Let first $k = 1$. Note that in this case, from Corollary 3.25 we have $ll(B/\mathrm{soc}(B)) = ll(B) - 0 - 1 = 1 - 1 = 0$, hence $B/\mathrm{soc}(B)$ is semisimple. Since $ll(A) < 1$ we have that A is also semisimple, hence $A \subset \mathrm{soc}(B)$. Then

$$0 \longrightarrow \mathrm{soc}(B)/A \longrightarrow B/A \longrightarrow B/\mathrm{soc}(B) \longrightarrow 0$$

is a short exact sequence, where $\mathrm{soc}(B)$ and $B/\mathrm{soc}(B)$ are both semisimple. Then from Lemma 3.26 we have that in particular B/A has finite socle length.

Assume now that the statement of the lemma is true for $k \in \mathbb{N}$. Consider now $B \in \mathfrak{g}$−mod with $ll(B) = k + 1$, and let $A \subset B$ with $ll(A) \leq k$. We then have $A = \mathrm{soc}^k(A) \subset \mathrm{soc}^k(B) \neq B$. Since $\mathrm{soc}^k(B)$ has socle length k by Remark 3.9, from the induction hypothesis we get that $\mathrm{soc}^k(B)/A$ has finite socle length, and again by Proposition 3.6 we get that $B/\mathrm{soc}^k(B) = \mathrm{soc}^{k+1}(B)/\mathrm{soc}^k(B)$ is semisimple. Then since B/A fits in a short exact sequence

$$0 \longrightarrow \mathrm{soc}^k(B)/A \longrightarrow B/A \longrightarrow B/\mathrm{soc}^k(B) \longrightarrow 0$$

we get that B/A is also of finite socle length from Lemma 3.26. □

Using this result, we are now ready to prove ii) in full.

Theorem 3.28. *Let $B \in \mathfrak{g}$−mod have finite socle length, and $A \subset B$ a $\mathfrak{g}$−submodule. Then B/A also has finite socle length.*

Proof. Since $A \subset B$, we have that A also has finite socle length. Let $ll(A) = k$. Pick now $B' \in \mathfrak{g}$−mod with $ll(B') > k$. One can find such a module for example from the construction in Subsection 3.2.2. We then have $A \subset B \oplus B'$. From Corollary 3.11 we see that

$$ll(B \oplus B') \geq ll(B') > k.$$

Now since $A \subset B \oplus B'$ then $ll(A) < ll(B \oplus B')$ again by Corollary 3.11. From Lemma 3.27 we get that $ll((B \oplus B')/A) < \infty$. Clearly we have

$$(B \oplus B')/A = B/A \oplus B',$$

and since $B/A \subset (B \oplus B')/A$, we get that $ll(B/A) < \infty$ as well from Corollary 3.11, which is what we wanted to show. □

Now using all of these results, we can prove i) in full as well.

Theorem 3.29. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in $\mathfrak{g}$−mod, with A and C of finite socle lengths. Then B also has finite socle length.

Proof. Let $ll(A) = k$. If $\mathrm{soc}^k(B) = B$ then we are done. Assume now that $\mathrm{soc}^k(B) \neq B$. If $A = \mathrm{soc}^k(B)$, we have
$$B/\mathrm{soc}^k(B) = B/A \cong C.$$
From Corollary 3.25 we get $ll(B) = ll(B/\mathrm{soc}^k(B)) + k + 1 = ll(C) + k + 1 = k + 1 < \infty$, and we are done.

Assume now that $A \neq \mathrm{soc}^k(B) \neq B$. We then have
$$B/\mathrm{soc}^k(B) \cong (B/A)/(\mathrm{soc}^k(B)/A) \cong C/D,$$
where D is the submodule of C corresponding to $\mathrm{soc}^k(B)/A$. Now since C has finite socle length, from Theorem 3.28 we get that C/D also has finite socle length, hence we get $ll(B/\mathrm{soc}^k(B)) < \infty$, and from Corollary 3.25 we get $ll(B) < \infty$ also, which is what we wanted to show. □

3.4 $\mathrm{Tens}_{\mathfrak{g}}$

In this section we introduce a subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which turns out to be very useful in studying the integrable modules whose duals are also integrable. The exposition in this section follows [10]. In this section $\mathfrak{g}$ will denote one of the classical locally semisimple Lie algebras $sl(\infty), sp(\infty), o(\infty)$. However the discussion in 3.4.1 works for general locally semisimple Lie algebras $\mathfrak{g}$.

3.4.1 Definition and properties

Let $\mathfrak{g}$ be a locally semisimple Lie algebra.

Let us first recall the adopted notation as mentioned in Remark 3.1. We set
$$M^{*0} := M; \quad M^{*n} = \left(M^{*(n-1)}\right)^*,$$
i.e. M^{*n} is the $\mathfrak{g}$–module when M is dualized n times.

Definition 3.2. Denote by $\mathrm{Tens}_{\mathfrak{g}}$ the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ consisting of those modules $M \in \mathrm{Int}_{\mathfrak{g}}$ that satisfy the two following properties

T1. $M^* \in \mathrm{Int}_{\mathfrak{g}}$.

T2. $ll(M^{*n}) < \infty$ for all $n \in \mathbb{N}$.

It is clear that if $M \in \mathrm{Tens}_{\mathfrak{g}}$ then $M = M^{*0}$ has finite socle length. Note also that $M \in \mathrm{Tens}_{\mathfrak{g}}$ gives us $M^* \in \mathrm{Int}_{\mathfrak{g}}$, and from Corollary 3.5 we see that $M^{**} \in \mathrm{Int}_{\mathfrak{g}}$, so M^* also satisfies **T1**. Since $(M^*)^{*n} = M^{*(n+1)}$ we get that M^* also satisfies **T2**, hence we get $M^* \in \mathrm{Tens}_{\mathfrak{g}}$. This shows that $\mathrm{Tens}_{\mathfrak{g}}$ is a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ that is closed under algebraic dualization, and such that every object in it has finite socle length. In fact there exists a characterization of $\mathrm{Tens}_{\mathfrak{g}}$ in these terms, as the following results shows.

Theorem 3.30. *$\mathrm{Tens}_{\mathfrak{g}}$ is the largest full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ closed under algebraic dualization, and such that every object in it has finite socle length.*

Proof. The previous discussion shows that $\mathrm{Tens}_{\mathfrak{g}}$ indeed satisfies these two conditions. Let now $\mathcal{C} \subset \mathrm{Int}_{\mathfrak{g}}$ be a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ that is closed under algebraic dualization, and that every object in $\mathcal{C}$ has finite socle length. Let $M \in \mathcal{C}$. Since $\mathcal{C}$ is closed under algebraic dualization, we have that $M^* \in \mathcal{C}$, so in particular $M^* \in \mathrm{Int}_{\mathfrak{g}}$, hence M satisfies **T1**. We also see that $M^{*n} \in \mathcal{C}$ for all $n \in \mathbb{N}$, so in particular we have that $ll(M^{*n}) < \infty$ for all $n \in \mathbb{N}$. This shows that M satisfies **T2** as well, thus we have $M \in \mathrm{Tens}_{\mathfrak{g}}$. This implies that $\mathcal{C}$ is actually a full subcategory of $\mathrm{Tens}_{\mathfrak{g}}$, and this proves the theorem. □

Before we proceed with stating and proving a few properties of the category $\mathrm{Tens}_{\mathfrak{g}}$, let us now make an observation about the dualization of short exact sequences. TO this end, let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in $\mathfrak{g}$−mod. For convenience, we use C and B/A interchangeably. Let now $p : B^* \longrightarrow A^*$ and $s : (B/A)^* \longrightarrow B^*$ be the induced dual maps of $A \longrightarrow B$ and $B \longrightarrow B/A$ respectively, given as usual by

$$p(f)(a) = f(a); \text{ and } s(F)(b) = F(b + A),$$

and consider the dualized sequence

$$0 \longrightarrow C^* \xrightarrow{s} B^* \xrightarrow{p} A^* \longrightarrow 0. \tag{3.17}$$

Note that given $F \in (B/A)^*$, for any $a \in A$ we have

$$(p \circ s)(F)(a) = p(s(F))(a) = s(F)(a) = F(a + A) = F(A) = 0,$$

i.e. $p(s(F)) = 0$ for all $F \in C^*$, thus we have $s(F) \in \ker p$ for all $F \in C^*$ which gives us $\mathrm{im}(s) \subset \ker(p)$. Let now $f \in \ker(p)$, i.e. $f : B \longrightarrow \mathbb{C}$ linear such that $f(a) = 0$ for all $a \in A$. Let us now define $F : B/A \longrightarrow \mathbb{C}$ by $F(b + A) = f(b)$. Note that for $b_1, b_2 \in B$ such that $b_1 + A = b_2 + A$, there exists some $a \in A$ with $b_2 = b_1 + a$, thus we get

$$F(b_1 + A) = f(b_1) = f(b_1 + a) = f(b_2) = F(b_2 + A)$$

because $f(a) = 0$. Hence this F is well defined. Note now that

$$s(F)(b) = F(b + A) = f(b)$$

for all $b \in B$, so we have $s(F) = f$, thus $\mathrm{im}(s) = \ker(p)$. Hence (3.17) is also a short exact sequence.

Dualizing (3.17) repeatedly, we get short exact sequences:

$$\begin{gathered} 0 \longrightarrow A^{*2n} \longrightarrow B^{*2n} \longrightarrow C^{*2n} \longrightarrow 0; \\ 0 \longrightarrow C^{*(2n-1)} \longrightarrow B^{*(2n-1)} \longrightarrow A^{*(2n-1)} \longrightarrow 0. \end{gathered} \tag{3.18}$$

Using this observation, we can now prove the following result.

Proposition 3.31. *$\mathrm{Tens}_{\mathfrak{g}}$ is closed under taking submodules, quotients, extensions, and finite direct sums. In particular, $\mathrm{Tens}_{\mathfrak{g}}$ is an abelian subcategory of $\mathrm{Int}_{\mathfrak{g}}$.*

Proof. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{3.19}$$

be a short exact sequence in $\mathfrak{g}$−mod. Assume first that $B \in \mathrm{Tens}_{\mathfrak{g}}$. Note that from (3.17) we get that $A^* \cong B^*/C^*$. From Proposition 2.3, since $B^* \in \mathrm{Int}_{\mathfrak{g}}$, we get that $A^* \in \mathrm{Int}_{\mathfrak{g}}$ as well, hence A satisfies **T1**. Note that for $n \in \mathbb{N}$, from (3.18) we have

$$A^{*2n} \subset B^{*2n}; \quad A^{*(2n-1)} \cong B^{*(2n-1)}/C^{*(2n-1)},$$

or in other words, A^{*n} is either a submodule or a quotient of B^{*n}. From Corollary 3.11 and Theorem 3.28 we get that $ll(A^{*n}) < \infty$ for all natural n, hence A satisfies **T2** as well, thus $A \in \mathrm{Tens}_{\mathfrak{g}}$. This proves that $\mathrm{Tens}_{\mathfrak{g}}$ is closed under taking submodules.

Note now that from (3.17) we have $C^* \subset B^*$, which in combination with Proposition 2.3 implies that $C^* \in \mathrm{Int}_{\mathfrak{g}}$, so C satisfies **T1**. Similar to the first paragraph of this proof, we have

$$C^{*(2n-1)} \subset B^{*(2n-1)}; \quad C^{*2n} \cong B^{*2n}/A^{*2n},$$

or in other words C^{*n} is either a submodule or a quotient of B^{*n}. Again from Corollary 3.11 and Theorem 3.28 we get that $ll(C^{*n}) < \infty$ for all natural n, hence C satisfies **T2** as well, thus $C \in \mathrm{Tens}_{\mathfrak{g}}$. This proves that $\mathrm{Tens}_{\mathfrak{g}}$ is closed under taking quotients.

Assume now that in (3.19) $A, C \in \mathrm{Tens}_{\mathfrak{g}}$. Since $A^*, C^* \in \mathrm{Int}_{\mathfrak{g}}$, from Proposition 2.3 we get that $B^* \in \mathrm{Int}_{\mathfrak{g}}$ as well, hence B satisfies **T1**. From (3.18) one can see that B^{*n} is always an extension of two $\mathfrak{g}$–modules of finite socle length, and from Theorem 3.29 we get that B^{*n} will also have finite socle length, hence B will satisfy **T2** as well, thus $B \in \mathrm{Tens}_{\mathfrak{g}}$. This proves that $\mathrm{Tens}_{\mathfrak{g}}$ is closed under taking extensions.

As for finite direct sums, let $A_1, \dots, A_k \in \mathrm{Tens}_{\mathfrak{g}}$, and set

$$A = \bigoplus_{i=1}^{k} A_i$$

Note first that as this is a finite direct sum, we have for all natural n

$$A^{*n} = \bigoplus_{i=1}^{k} A_i^{*n}$$

In particular, since $A_i^* \in \mathrm{Int}_{\mathfrak{g}}$, from Proposition 2.3 we have $A^* \in \mathrm{Int}_{\mathfrak{g}}$, hence A satisfies **T1**. Now from $ll(A_i^{*n}) < \infty$ for $i = 1, 2, \dots, k$, and Corollary 3.22, we get $ll(A^{*n}) < \infty$, hence A satisfies **T2** as well, thus $A \in \mathrm{Tens}_{\mathfrak{g}}$. This proves that $\mathrm{Tens}_{\mathfrak{g}}$ is closed under finite direct sums, and thus completes the proof of this proposition. □

3.4.2 Tensor modules

From here on out, our locally semisimple Lie algebras $\mathfrak{g}$ will be the classical ones, i.e. $\mathfrak{g} \cong sl(\infty), o(\infty), sp(\infty)$.

In this subsection we introduce a type of integrable $\mathfrak{g}$–modules, called tensor modules, which turn out to be very important objects of $\mathrm{Tens}_{\mathfrak{g}}$. These modules have been studied in [11], and the exposition in this subsection is meant mostly as an overview of the results in that paper, and in [10].

For $p, q \in \mathbb{N}$ set

$$T^{p,q} := V^{\otimes p} \otimes V_*^{\otimes q},$$

and call such a module a *tensor module*. Since V and V_* are local modules, from Proposition 2.3 we see that $T^{p,q} \in \mathrm{Int}_{\mathfrak{g}}$ for all $p, q \in \mathbb{N}$. It is known, see [11], that these modules will be of finite socle length. In particular, for $\mathfrak{g} = sl(\infty)$ we have

$$ll(T^{p,q}) = \min\{p, q\} + 1.$$

The simple modules which arise as subquotients of tensor modules will play a key role in what follows, so we give the following definition.

Definition 3.3. A simple module M for which there exist natural numbers p, q and modules $A \subset B \subset T^{p,q}$ with $M \cong B/A$ are called *simple tensor modules.*

In [11] it is shown that there exists a chain of nested Borel subalgebras $\mathfrak{b}_i \subset \mathfrak{g}_i$, such that for $\mathfrak{b} = \varinjlim \mathfrak{b}_i = \bigcup_{i \in \mathbb{N}} \mathfrak{b}_i$, every simple tensor module is a $\mathfrak{b}$-highest weight module.

Denote by Θ the set of all $\mathfrak{b}$-highest weights of simple tensor modules. For $\lambda \in \Theta$, denote by V_λ the simple tensor module with λ as its highest weight, and if one considers λ as a weight of $\mathfrak{g}_i$, denote by V^i_λ the corresponding highest weight $\mathfrak{g}_i$-module. In [11] it is shown that there exists a finite set $S = \{\gamma_1, \dots, \gamma_n\}$ of linearly independent weights of the natural representation V such that

$$\Theta \subset \operatorname{span}_{\mathbb{Z}}\{\gamma_1, \dots, \gamma_n\}.$$

Since S is linearly independent, for any $\lambda \in \Theta$ there exist unique integers $a_1, \dots, a_n$ such that $\lambda = a_1\gamma_1 + \dots + a_n\gamma_n$. Define now $|\lambda| := \sum_{i=1}^n |a_i|$, and for a given $k \in \mathbb{N}$ set $\Theta_k := \{\lambda \in \Theta \mid |\lambda| \le k\}$. It is clear then that these Θ_k are all finite, and

$$\Theta = \bigcup_{k \in \ltimes} \Theta_k.$$

Note that Example 1.2 shows that $\mathbb{C}$ is a simple quotient of $T^{1,1} = V \otimes V_*$, which means that $\mathbb{C}$ is a simple tensor module. Assume that λ is the $\mathfrak{b}$−highest weight of $\mathbb{C}$. Then for $x \in \mathbb{C}$ non-zero, and any $b \in \mathfrak{b}$ we have $0 = b.x = \lambda(b)x$, hence we have $\lambda = 0$. This means that $V_0 = \mathbb{C}$. In [11] it is shown that if $\mu \in \Theta$ is such that V_μ is a simple subquotient of $T^{p,q}$, we have $|\mu| \le p + q$, and if it is a submodule of $T^{p,q}$ we have $|\mu| = p + q$. It is also known that given a finite dimensional simple $\mathfrak{g}_i$-module N such that $\hom_{\mathfrak{g}_i}(N, V_\lambda) \neq 0$, we have that $N \cong V^i_\mu$ for some $\mu \in \Theta$ with $|\mu| \le |\lambda|$. Using this last remark, we can prove that duals of tensor modules are integrable.

To this end, we begin with the following result.

Lemma 3.32. *Let $A \subset B \in Int_{\mathfrak{g}}$ and $i > 0$. Let N be a simple $\mathfrak{g}_i$-module such that*

$$\hom_{\mathfrak{g}_i}(N, A) = 0 \neq \hom_{\mathfrak{g}_i}(N, B).$$

Then

$$\hom_{\mathfrak{g}_i}(N, B/A) \neq 0.$$

Proof. As $\hom_{\mathfrak{g}_i}(N, B) \neq 0$, let $\alpha : N \longrightarrow B$ be a non-zero map of $\mathfrak{g}_i$-modules, so that $N \cong \alpha(N) \subset B$. If $\alpha(N) \cap A \neq 0$, since $\alpha(N)$ is simple, we would have $\alpha(N) \subset A$, hence $\alpha \in \hom_{\mathfrak{g}_i}(N, A)$ which cannot be true, so we must have $\alpha(N) \cap A = 0$. This means that the map

$$\beta = \alpha \circ p : N \longrightarrow B \xrightarrow{p} B/A,$$

where p is the canonical map, is such that $\ker \beta = \alpha(N) \cap \ker p = \alpha(N) \cap A = 0$, hence $\beta \neq 0$, which gives us $\hom_{\mathfrak{g}_i}(N, B/A) \neq 0$, which is what we wanted to show. ☐

Lemma 3.33. *Let $A \in Int_{\mathfrak{g}}$ be semisimple, $k \in \mathbb{N}$, and N a simple $\mathfrak{g}_k$-module such that $\hom_{\mathfrak{g}_k}(N, A) \neq 0$. Then there exists a simple $\mathfrak{g}$−submodule $Q \subset A$ such that $\hom_{\mathfrak{g}_k}(N, Q) \neq 0$.*

Proof. As A is semisimple, let I be an index set and Q_i simple $\mathfrak{g}$−modules for $i \in I$ such that

$$A = \bigoplus_{i \in I} Q_i.$$

For each $i \in I$, let $p_i : A \longrightarrow Q_i$ be the natural projection. Naturally this p_i is a map of $\mathfrak{g}$−modules, so in particular it is also a map of $\mathfrak{g}_k$-modules. Let now $\alpha : N \longrightarrow A$ be a non-zero map of $\mathfrak{g}_k$-modules, and let $n \in N$ be non-zero. As N is simple, α will be injective, so $\alpha(n) \in A$ is non-zero. This means that there exist indices $\{i_1, \dots, i_s\} \subset I$ and $a_j \in Q_{i_j}$ non-zero for $j = 1, \dots, s$ such that $\alpha(n) = a_1 + \dots + a_s$. In particular we have

$$p_{i_1}(\alpha(n)) = a_1 \neq 0.$$

By setting $Q = Q_{i_1}$ and $p = p_{i_1}$, we get that $\beta = p \circ \alpha : N \longrightarrow Q$ is a non-zero map of $\mathfrak{g}_k$-modules, so we indeed get that

$$\hom_{\mathfrak{g}_k}(N, Q) \neq 0$$

which is what we wanted to show. □

Now we prove a more general version of Lemma 3.33.

Proposition 3.34. *Let $M \in Int_{\mathfrak{g}}$ have finite socle length, $i \in \mathbb{N}$, and N a simple $\mathfrak{g}_i$-module such that $\hom_{\mathfrak{g}_i}(N, M) \neq 0$. Then there exists a simple subquotient Q of M such that $\hom_{\mathfrak{g}_i}(N, Q) \neq 0$.*

Proof. Let $\alpha : N \longrightarrow M$ be a non-zero map of $\mathfrak{g}_i$-modules, so we have $\alpha(N) \subset M$. If $ll(M) = s$, we have $\alpha(N) \subset M = \operatorname{soc}^s(M)$. Let us now set

$$k = \min\{n \in \mathbb{N} \mid \alpha(N) \subset \operatorname{soc}^n(M)\}.$$

Clearly $k \leq s$. If $k = 0$, the claim follows directly from Lemma 3.33, by taking Q to be the appropriate simple submodule. Assume now that $k > 0$. We then have $\alpha(N) \not\subset \operatorname{soc}^{k-1}(M)$, and since $\alpha(N)$ is simple, we get that

$$\hom_{\mathfrak{g}_i}(N, \operatorname{soc}^{k-1}(M)) = 0 \neq \hom_{\mathfrak{g}_i}(N, \operatorname{soc}^k(M)).$$

From Lemma 3.32 we then get that

$$\hom_{\mathfrak{g}_i}(N, \operatorname{soc}^k(M)/\operatorname{soc}^{k-1}(M)) \neq 0.$$

From Proposition 3.6 we have that $\operatorname{soc}^k(M)/\operatorname{soc}^{k-1}(M) = \operatorname{soc}(M/\operatorname{soc}^{k-1}(M))$ is semisimple, so from Lemma 3.33 we get that there exists a simple submodule $Q \subset \operatorname{soc}(M/\operatorname{soc}^{k-1}(M))$ with $\hom_{\mathfrak{g}_i}(N, Q) \neq 0$. Note that Q being a simple submodule in $M/\operatorname{soc}^{k-1}(M)$, we get that there exists some $U \subset M$ such that $Q = U/\operatorname{soc}^{k-1}(M)$, so in particular we get that Q is a simple subquotient of M, and this concludes the proof of our proposition. □

We are now ready to give a result on the integrability of duals of tensor modules.

Corollary 3.35. *Let $p, q \in \mathbb{N}$. Then $(T^{p,q})^*$ is integrable.*

Proof. Fix $i > 0$. Let N be a simple finite dimensional $\mathfrak{g}_i$–module such that

$$\hom_{\mathfrak{g}_i}(N, T^{p,q}) \neq 0.$$

Then from Proposition 3.34, we get that there exists a simple subquotient Q of $T^{p,q}$ such that

$$\hom_{\mathfrak{g}_i}(N, Q) \neq 0.$$

It is clear that this Q is a simple tensor module, so there exists some $\lambda \in \Theta$ such that $Q \cong V_\lambda$. From the discussion preceding Lemma 3.32, we see that $|\lambda| \leq p+q$, and that $\hom_{\mathfrak{g}_i}(N, V_\lambda) \neq 0$ implies that there exists some $\mu \in \Theta_{p+q}$ such that $N \cong V^i_\mu$. Since Θ_{p+q} is finite, this means that $\hom_{\mathfrak{g}_i}(N, T^{p,q}) \neq 0$ for only finitely many non-isomorphic simple finite dimensional $\mathfrak{g}_i$-modules N, and from Theorem 3.4 we get that $(T^{p,q})^* \in \mathrm{Int}_{\mathfrak{g}}$, which is what we wanted to show. □

We also note here that in [11] it is shown that given $\lambda \in \Theta$ is non-zero, there exists some $n_0 \in \mathbb{N}$ such that for any $i > n_0$, one has

$$\dim \hom_{\mathfrak{g}_i}(V^i_\lambda, V_\lambda) = 1,$$

and if $|\mu| < |\lambda|$ is such that $\hom_{\mathfrak{g}_i}(V^i_\mu, V_\lambda) \neq 0$, then

$$\dim \hom_{\mathfrak{g}_i}(V^i_\mu, V_\lambda) = \infty. \tag{3.20}$$

Remark 3.10. Earlier in this subsection we saw that $V_0 = \mathbb{C}$ is a simple tensor module. From (3.20) one can see that if $\lambda \in \Theta$ is non-zero, we will have $\dim V_\lambda = \infty$. In other words, the only simple tensor module of finite dimension in $\mathrm{Tens}_{\mathfrak{g}}$ is the trivial $\mathfrak{g}$–module.

3.4.3 A sufficient condition for $M \in \mathbf{Tens}_{\mathfrak{g}}$

We have previously just seen that tensor modules have finite socle length, and integrable duals, so that in particular they satisfy **T1**. In this subsection our goal is to give a sufficient condition, consisting of two parts, which guarantees that a $\mathfrak{g}$–module $M \in \mathrm{Int}_{\mathfrak{g}}$ is in $\mathrm{Tens}_{\mathfrak{g}}$. In particular we will see that $T^{p,q} \in \mathrm{Tens}_{\mathfrak{g}}$. To this end, we note the following two properties that a module $M \in \mathrm{Int}_{\mathfrak{g}}$ may satisfy.

P1. M has finite socle length.

P2. there exists a $k \in \mathbb{N}$ such that $Q \in \{V_\mu \mid \mu \in \Theta_k\}$ for any simple subquotient Q of M.

Note that Corollary 3.11, Corollary 3.22, Theorem 3.28, and Theorem 3.29 show that the property **P1** is closed under taking submodules, quotients, extensions, and finite direct sums. We now want to see that the same holds true for **P2**.

Proposition 3.36. *Property **P2** is closed under taking submodules, quotients, extensions, and finite direct sums.*

Proof. Let
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0 \tag{3.21}$$
be a short exact sequence in $\mathrm{Int}_{\mathfrak{g}}$. Assume first that B satisfies **P2** with $k \in \mathbb{N}$. Let now Q be a simple subquotient of A, i.e. there exists some $V \subset U \subset A$ such that $U/V \cong Q$ is simple. But clearly we have that $V \subset U \subset B$, hence Q is also a simple subquotient of B, so $Q = V_\mu$ for some $\mu \in \Theta_k$, i.e. A also satisfies **P2**. Thus property **P2** is closed under taking submodules.

Let now Q be a simple subquotient of C, i.e there exist $V \subset U \subset C$ such that $U/V \cong Q$ is simple. Let now $P = p^{-1}(V)$ and $R = p^{-1}(U)$, so that we have $P/A = V$, and $R/A = U$. We then have $Q \cong U/V = (R/A)/(P/A) \cong R/P$, i.e. Q is isomorphic to a simple subquotient of B, thus we have $Q \cong V_\mu$ for some $\mu \in \Theta_k$, so C also satisfies **P2**. Thus property **P2** is closed under taking quotients.

Assume now that A and C satisfy **P2** with $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ respectively. Let $k = \max\{k_1, k_2\}$. Let now Q be a simple subquotient of B, and let $D \subset S \subset B$ be submodules such that $Q \cong S/D$. We now distinguish two cases of the intersection of A with D.

1°$A \cap D = 0$. Set now $U = S \cap A$. If $U = 0$, we have that

$$\ker p|_D = \ker p \cap D = A \cap D = 0; \text{ and } \ker p|_S = \ker p \cap S = A \cap S = U = 0$$

hence we get that $p : D \xrightarrow{\cong} p(D)$ and $p : S \xrightarrow{\cong} p(S)$. We then have that

$$Q \cong S/D \cong p(S)/p(D)$$

is a subquotient of C, hence $Q \cong V_\mu$ for some $\mu \in \Theta_{k_2} \subset \Theta_k$. Now if $U \neq 0$, we have $U, D \subset S$ and $U \cap D = S \cap A \cap D = 0$, so we get $U + D = U \oplus D \subset S$. Since $U \oplus D \neq D$ and S/D simple, we get that $(U \oplus D)/D \subset S/D$ implies $U \oplus D = S$. This gives us

$$Q \cong S/D = (U \oplus D)/D \cong U$$

is simple. Since $U \subset A$ we get that $U \cong V_\mu$ for some $\mu \in \Theta_{k_1} \subset \Theta_k$.

$2^\circ A \cap D \neq 0$. Set now again $U = S \cap A$, and $V = D \cap A$. Assume that $V \neq U$. Consider now the map

$$s : U \xrightarrow{i_U} S \xrightarrow{p'} S/D.$$

where i_U and p' are the canonical injection and surjection respectively. We then have

$$\ker s = i_U(U) \cap \ker p' = U \cap D = S \cap A \cap D = S \cap V = V$$

hence we get an injection $U/V \longrightarrow S/D$ with $U/V \neq 0$. As S/D is simple, we must have $Q \cong S/D \cong U/V$ which is a subquotient of A, hence we get $Q \cong V_\mu$ for some $\mu \in \Theta_{k_1} \subset \Theta_k$. Assume now that $U = V$. Note that

$$\ker p|_S = S \cap \ker p = S \cap A = V,$$

thus we get that p induces an isomorphism $i : S/V \longrightarrow p(S)$, such that $i(D/V) = p(D)$. We then have

$$Q \cong S/D = (S/V)/(D/V) \cong p(S)/p(D)$$

which is a subquotient of C, hence we have $Q \cong V_\mu$ for some $\mu \in \Theta_{k_2} \subset \Theta_k$.

This way we have shown that given any simple subquotient Q of B, we always have $Q \cong V_\mu$ for some $\mu \in \Theta_k$, and this proves that B also satisfies **P2**. Thus, property **P2** is closed under taking extensions as well.

As for finite direct sums, note that if $A, B \in \mathrm{Int}_\mathfrak{g}$ satisfy **P2**, since

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

is a short exact sequence, and $A \oplus B$ is an extension of modules that satisfy **P2**, from the previous paragraph, we have that $A \oplus B$ also satisfies **P2**. It follows by induction that property **P2** is preserved under taking finite direct sums, and thus we complete the proof of this proposition. $\square$

For convenience, say that $M \in \mathrm{Int}_\mathfrak{g}$ *satisfies property* ***P*** if it satisfies both properties **P1** and **P2**. Note now that Proposition 3.36 in combination with Corollary 3.11, Corollary 3.22, Theorem 3.28, and Theorem 3.29 gives us the following consequence, which we note here, again for convenience purposes.

Corollary 3.37. *Property* ***P*** *is closed under taking submodules, quotients, extensions, and finite direct sums.*

Note now that given $M \in \mathrm{Int}_\mathfrak{g}$ that satisfies **P2** with $k \in \mathbb{N}$, from Proposition 3.34 we see that given $i > 0$ and N a finite dimensional simple $\mathfrak{g}_i$-module such that $\hom_{\mathfrak{g}_i}(N, M) \neq 0$, there exists a simple subqoutient Q of M such that $\hom_{\mathfrak{g}_i}(N, Q) \neq 0$. Since $Q \cong V_\mu$ for some $\mu \in \Theta_k$, we have $\hom_{\mathfrak{g}_i}(N, V_\mu) \neq 0$, and from the discussion in Subsection 3.4.2 we get that $N \cong V^i_\mu$. As Θ_k is finite, from Theorem 3.4 we get that $M^* \in \mathrm{Int}_\mathfrak{g}$. This way we have proven the following result.

Corollary 3.38. *If $M \in \mathrm{Int}_\mathfrak{g}$ satisfies property* ***P2****, it satisfies property* ***T1*** *as well.*

We have seen in Subsection 3.4.2 that the tensor modules $T^{p,q}$ satisfy property **P**. In what follows we want to show that this property is closed under more algebraic operations than listed in Corollary 3.37. To this end, we start by citing following result from [10].

Lemma 3.39. *Let $p, q \in \mathbb{N}$, I an index set, and $T_i = T^{p,q}$ for all $i \in$. Then the $\mathfrak{g}$−modules*

$$T := \prod_{i \in I} T_i; \quad (T^{p,q})^*$$

both satisfy property ***P****.*

Let us now digress shortly into the following observation. Let I be an index set, and

$$0 \longrightarrow B_i \xrightarrow{s_i} A_i \xrightarrow{p_i} C_i \longrightarrow 0$$

be short exact sequences for all $i \in I$. Set now

$$A = \prod_{i\in I} A_i; \quad B = \prod_{i\in I} B_i; \quad C = \prod_{i\in I} C_i,$$

and

$$s = \prod_{i\in I} s_i; \text{ and } p = \prod_{i\in I} p_i.$$

Note that $p \circ s = \prod_{i\in I}(p_i \circ s_i) = 0$, so that $\operatorname{im}(s) \subset \ker(p)$. Now if $b = (b_i)_{i\in I} \in \ker(p)$, we have $0 = p(b) = \prod_{i\in I} p_i(b_i)$, thus $p_i(b_i) = 0$ for all $i \in I$, hence we get $b_i \in \operatorname{im}(s_i)$. If $a_i \in A_i$ are such that $s_i(a_i) = b_i$, by setting $a = (a_i)_{i\in I}$ we get $s(a) = b$, thus $\operatorname{im}(s) = \ker(p)$, and we get a short exact sequence

$$0 \longrightarrow A \xrightarrow{s} B \xrightarrow{p} C \longrightarrow 0.$$

Note also that by setting $A' = \bigoplus_{i\in I} A_i$, the natural inclusions $\alpha_i : A_i \longrightarrow A$ induce an injection $\alpha : A' \longrightarrow A$.

We now apply this observation to the following situation. Let $\lambda \in \Theta$. Then there exists natural numbers p, q such that V_λ is a subquotient of $T^{p,q}$. Let $B \subset A \subset T^{p,q}$ be such that $V_\lambda \cong A/B$. Let now I be an index set, T as in Lemma 3.39, and

$$S = \prod_{i\in I} A; \quad R = \prod_{i\in I} B; \quad U = \prod_{i\in I} V_\lambda,$$

i.e. as products of the same modules over the index set I. We can then fit these $\mathfrak{g}$−modules into a short exact sequence

$$0 \longrightarrow R \longrightarrow S \longrightarrow U \longrightarrow 0. \tag{3.22}$$

Since $S \subset T$, from Lemma 3.39, and Corollary 3.37 for submodules, we get that S also satisfies property **P**. Then again from the previous observation applied to (3.22), and from Corollary 3.37 for subquotients (i.e. submodules and quotients) we get that U also satisfies property **P**. This way we have proved one part of the following result.

Corollary 3.40. *Let $\lambda \in \Theta$, I an index set, and $U_i = V_\lambda$ for all $i \in I$. Then*

$$U = \prod_{i\in I} U_i \quad \textit{and} \quad (V_\lambda)^*$$

*both satisfy property **P**.*

Proof. For U we saw that this holds in the discussion preceding this corollary.

Let now $V_\lambda = A/B$ as above. We then have a short exact sequence

$$0 \longrightarrow V_\lambda \longrightarrow T^{p,q}/B \longrightarrow X \longrightarrow 0,$$

where X is the appropriate quotient. Dualizing this, we get another short exact sequence

$$0 \longrightarrow X^* \longrightarrow (T^{p,q}/B)^* \longrightarrow (V_\lambda)^* \longrightarrow 0.$$

From the discussion in Subsection 3.4.1, particularly (3.17), by dualizing the surjection $T^{p,q} \longrightarrow T^{p,q}/B$ we get an injection $(T^{p,q}/B)^* \longrightarrow (T^{p,q})^*$. From Lemma 3.39 we know that $(T^{p,q})^*$ satisfies property **P**, hence $(T^{p,q}/B)^*$ will also satisfy property **P**. Now as V_λ^* is a quotient of $(T^{p,q}/B)^*$, from Corollary 3.37 we get that V_λ^* also satisfies property **P**, which is what we wanted to show. □

We are now ready to give a result which shows than an integrable module M satisfying property **P** will be in $\mathrm{Tens}_{\mathfrak{g}}$.

Theorem 3.41. *Let $M \in Int_{\mathfrak{g}}$ satisfy property* ***P****. Let I be an index set, and $M_i = M$ for all $i \in I$. Then*
i) $S = \prod_{i\in I} M_i$ satisfies property ***P****,*
ii) M^* satisfies property ***P****,*
iii) $M \in Tens_{\mathfrak{g}}$.

Proof. Assume that M satisfies **P2** for $k \in \mathbb{N}$.

i) We prove this by induction on $ll(M) = n$. Assume first that $ll(M) = 0$, i.e. that M is semisimple. As M satisfies **P2**, we have that if $Q \subset M$ is simple, there exists some $\mu \in \Theta_k$ such that $Q \cong V_\mu$. We then have and index set J and simple submodules $Q_j \subset M$ for all $j \in J$ such that

$$M = \mathrm{soc}(M) = \bigoplus_{j\in J} Q_j = \bigoplus_{\mu\in\Theta_k} \bigoplus_{j\in J_\mu} V_\mu, \tag{3.23}$$

where $J_\mu = \{j \in J \mid Q_j \cong V_\mu\}$. Note now that since Θ_k is finite, we have

$$S = \prod_{i\in I} M = \prod_{i\in I} \bigoplus_{\mu\in\Theta_k} \bigoplus_{j\in J_\mu} V_\mu = \bigoplus_{\mu\in\Theta_k} \prod_{i\in I} \bigoplus_{j\in J_\mu} V_\mu.$$

One can see from the discussion following Lemma 3.39 that we have a natural inclusion $\bigoplus_{j\in J_\mu} V_\mu \subset \prod_{j\in J_\mu} V_\mu$, which gives us

$$\prod_{i\in I} \bigoplus_{j\in J_\mu} V_\mu \subset \prod_{(i,j)\in I\times J_\mu} V_\mu.$$

From Corollary 3.40 we see that $\prod_{(i,j)\in I\times J_\mu} V_\mu$ satisfies **P**, hence from Corollary 3.37 for submodules we see that $\prod_{i\in I} \bigoplus_{j\in J_\mu} V_\mu$ also satisfies **P**. Then, again from Corollary 3.37 for finite direct sums we get that S, as a finite direct sum of modules who satisfy **P**, will also satisfy this property, which proves the claim for $ll(M) = 0$.

Assume now that i) is true for M with $ll(M) = n$. Let now $M \in \mathrm{Int}_{\mathfrak{g}}$ satisfy **P**, and $ll(M) = n + 1$. From Remark 3.9 we know that $N = \mathrm{soc}^n(M)$ has socle length n, so by induction hypothesis i) holds for N. Since $M/N = \mathrm{soc}(M/N)$ is semisimple from Proposition 3.6, we have that i) also holds for M/N by the previous paragraph. From the discussion following Lemma 3.39 we obtain a short exact sequence

$$0 \longrightarrow \prod_{i\in I} N \longrightarrow \prod_{i\in I} M \longrightarrow \prod_{i\in I} (M/N) \longrightarrow 0.$$

Since i) holds for N and M/N, we have that both $\prod_{i\in I} N$ and $\prod_{i\in I}(M/N)$ satisfy **P** by the induction hypothesis, so by Corollary 3.37 for extensions, we get that $\prod_{i\in I} M$ also satisfies **P**. The claim then follows by induction.

ii) We prove this by induction on $ll(M) = n$ as well. Let $ll(M) = 0$, so that M is semisimple. From (3.23), the discussion at the beginning of Section 2.3, specifically (2.17), we get

$$M^* = \bigoplus_{\mu\in\Theta_k} \prod_{i\in I} (V_\mu)^*,$$

because Θ_k is finite. From Corollary 3.40 we see that $(V_\mu)^*$ satisfies **P** for any $\mu \in \Theta_k$, so from part i) of this theorem we get that $\prod_{i\in I}(V_\mu)^*$ satisfies **P** for any $\mu \in \Theta_k$. Using Corollary 3.37 for finite direct sums we see that M^* also does satisfy **P**.

Assume now that the claim is true for M with $ll(M) = n$. Let now $M \in \mathrm{Int}_{\mathfrak{g}}$ be with $ll(M) = n + 1$. By setting N as in i), we get a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Dualizing this we obtain

$$0 \longrightarrow (M/N)^* \longrightarrow M^* \longrightarrow N^* \longrightarrow 0.$$

As M/N is semisimple, and $ll(N) = n$, by the induction hypothesis we have that $(M/N)^*$ and N^* satisfy **P**. Then by Corollary 3.37 for extensions we get that M^* also satisfies **P**. The claim then follows by induction.

iii) Let $M \in \mathrm{Int}_{\mathfrak{g}}$ satisfy **P**. Since M satisfies **P2**, from Corollary 3.38 we have that M satisfies **T1**. Note that part ii) of this theorem shows the implication

$$M \text{ satisfies } \mathbf{P} \implies M^* \text{ satisfies } \mathbf{P},$$

and this clearly implies that M^{*n} satisfies **P** for all natural n. In particular, all M^{*n} satisfy **P1**, so that $ll(M^{*n}) < \infty$. In other words, M satisfies **T2** as well. Thus we get $M \in \mathrm{Tens}_{\mathfrak{g}}$, which is what we wanted to show. □

Now from Lemma 3.39 and Theorem 3.41 we get the following immediate consequence.

Corollary 3.42. *For any natural $p, q \in Int_{\mathfrak{g}}$ we have $T^{p,q} \in Tens_{\mathfrak{g}}$.*

Example 3.1. From Corollary 3.42 we see that $T^{1,0} = V$, and $T^{0,1} = V_*$ are both objects of $\mathrm{Tens}_{\mathfrak{g}}$. This means that $V^*, (V_*)^* \in \mathrm{Int}_{\mathfrak{g}}$. In particular we see that $\Gamma_{\mathfrak{g}}(V^*) = V^*$ has finite socle length. As V is simple, from Corollary 3.19 we see that $V_* \subset V^*$ will be the unique simple submodule of V^*. Thus we get

$$\mathrm{soc}(V^*) = V_*$$

Note that Corollary 3.42 provides us with the first examples of $\mathrm{Tens}_{\mathfrak{g}}$. From Corollary 3.37 we see that every simple tensor module V_λ is an object of $\mathrm{Tens}_{\mathfrak{g}}$ as well. In particular, we have that the trivial representations $\mathbb{C} = V_0$ are also in $\mathrm{Tens}_{\mathfrak{g}}$, something that could be noticed even by directly checking properties **T1** and **T2**. As every simple tensor module is a simple module by definition, we get the following result.

Corollary 3.43. *Every simple tensor module is a simple module in $Tens_{\mathfrak{g}}$.*

The importance of Corollary 3.42 also stands in that it shows that $\mathrm{Tens}_{\mathfrak{g}}$ is an interesting category closed under algebraic dualization, i.e. it does not consist of only trivial $\mathfrak{g}$–modules. As Example 3.1, this category contains even $\mathfrak{g}$–modules as basic as the natural and conatural representations, so in a sense one can see that $\mathrm{Tens}_{\mathfrak{g}}$ is a very reasonable category of $\mathfrak{g}$–modules. We emphasize this in the following note.

Comparison remark VI. *Analogous to the finite dimensional theory of representations of semisimple Lie algebras, for $\mathfrak{g} \cong sl(\infty), o(\infty), sp(\infty)$, there exists a reasonable category $Tens_{\mathfrak{g}}$ of $\mathfrak{g}$–representations which is closed under algebraic dualization.*

3.4.4 Simple objects and injective objects in $\mathrm{Tens}_{\mathfrak{g}}$

In Subsection 3.2.3 we gave an account of integrable modules Q for which $\Gamma(Q^*)$ has finite socle length. We now cite a result from [10] which characterizes all such simple modules.

Theorem 3.44. *Let $Q \in Int_{\mathfrak{g}}$ be a simple module, so that $\Gamma(Q^*)$ has finite socle length. Then Q is a simple tensor module.*

Note now that given a simple module $Q \in \mathrm{Tens}_{\mathfrak{g}}$, Q satisfies properties **T1** and **T2**. From **T1** we get that $Q^* = \Gamma_{\mathfrak{g}}(Q^*)$, and from **T2** we get that $ll(\Gamma_{\mathfrak{g}}(Q^*)) = ll(Q^*) < \infty$. In other words, Q satisfies the conditions of Theorem 3.44, so it must be a simple tensor module. From this, and Corollary 3.43 we get the following result.

Corollary 3.45. *The following holds*

$$\{\textit{simple objects of } Tens_{\mathfrak{g}}\} = \{\textit{simple tensor modules}\}.$$

Remark 3.11. In other words, this corollary says that a simple module $Q \in \mathrm{Int}_{\mathfrak{g}}$ is in $\mathrm{Tens}_{\mathfrak{g}}$ if and only if $ll(\Gamma(Q^*)) < \infty$. Note that from Remark 3.5 we see that if $Q \in \mathrm{Tens}_{\mathfrak{g}}$ is simple, then Q is locally simple

In Subsection 3.4.2 we saw that all simple tensor modules are $\mathfrak{b}-$highest weight modules for some Borel subalgebra $\mathfrak{b} = \varinjlim \mathfrak{b}_i$. This way we get the following direct consequence of Corollary 3.45.

Corollary 3.46. *Every simple object of $Tens_{\mathfrak{g}}$ is a $\mathfrak{b}-$highest weight module.$\mathfrak{g}-$module.*

Remark 3.12. Let $\mathfrak{g} = sl(\infty)$, and let $Q \in \mathrm{Int}_{\mathfrak{g}}$ be a simple module satisfying the conditions of Proposition 3.23. In Remark 1.7 we noted that such modules can be constructed. In Corollary 2.18 we have seen that there exists an injective map

$$Q \longrightarrow \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(Q^*)^*),$$

with $\Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(Q^*)^*) = I$ and injective object of $\mathrm{Int}_{\mathfrak{g}}$. From Corollary 3.15, we know that I will be a $\mathfrak{g}-$module of infinite socle length. If we had $Q \in \mathrm{Tens}_{\mathfrak{g}}$, then we would have $Q^*, Q^{**} \in \mathrm{Tens}_{\mathfrak{g}}$ as well, so we would get $I = Q^{**}$. However, $Q \in \mathrm{Tens}_{\mathfrak{g}}$ would imply that I would have finite socle length, which is clearly not possible. Hence we get $Q \notin \mathrm{Tens}_{\mathfrak{g}}$. This way we have shown that

$$\{\text{simple objects of } \mathrm{Tens}_{\mathfrak{g}}\} \neq \{\text{simple objects of } \mathrm{Int}_{\mathfrak{g}}\}.$$

We now emphasize the result in Corollary 3.46 in the following note.

Comparison remark VII. *Analogous to the finite dimensional theory of representations of semisimple Lie algebras, for $\mathfrak{g} \cong sl(\infty), o(\infty), sp(\infty)$, there exists a reasonable category $Tens_{\mathfrak{g}}$ of $\mathfrak{g}-$representations, every simple object of which is a highest weight module.*

In what follows in this subsection we give an account of the injective objects in $\mathrm{Tens}_{\mathfrak{g}}$. Note that if $M \in \mathrm{Tens}_{\mathfrak{g}}$ is a non-zero module, **T2** tells us that $ll(M) < \infty$. If we had $\mathrm{soc}(M) = 0$, we would have $\mathrm{soc}^1(M) = p^{-1}(\mathrm{soc}(M/\mathrm{soc}(M))) = p^{-1}(\mathrm{soc}(M)) = p^{-1}(0) = 0$ where $p : M \longrightarrow M/\mathrm{soc}(M)$ is the canonical map. Inductively then one can show that this implies $\mathrm{soc}^n(M) = 0$ for all natural $n \in \mathbb{N}$, which cannot be true. This proves the following.

Proposition 3.47. *$soc(M) \neq 0$ for all non-zero $M \in Tens_{\mathfrak{g}}$.*

Remark 3.13. In particular, every module in $\mathrm{Tens}_{\mathfrak{g}}$ contains some simple submodule, which is a statement that is not true even in $\mathrm{Loc}_{\mathfrak{g}}$. Since $T^{1,1} = V \otimes V_* \in \mathrm{Tens}_{\mathfrak{g}}$, and in Example 1.2 we saw that $T^{1,1}$ is not a semisimple $\mathfrak{g}-$module, we have that $\mathrm{Tens}_{\mathfrak{g}}$ is not a semisimple category. However, Proposition 3.47 shows that at least one does not have wild objects which contain no simple submodules in $\mathrm{Tens}_{\mathfrak{g}}$.

Now using Proposition 3.47, we can prove the following result.

Corollary 3.48. *Let $I \in Tens_{\mathfrak{g}}$ be an injective object, and let $Q \subset I$ be simple. Then I is an injective hull for Q if and only if $Q = soc(I)$.*

Proof. Assume that I is an injective hull for Q. We then have naturally $Q \subset \mathrm{soc}(I)$. Assume that $Q \neq \mathrm{soc}(I)$. Then there exists some simple sumbodule $Q' \subset I$ such that $Q \oplus Q' \subset \mathrm{soc}(I)$. But then $Q \cap Q' = 0$, and $Q' \neq 0$, which would imply that I is not an essential extension of Q, hence I is not an injective hull for Q. This is a contradiction, so we indeed get $\mathrm{soc}(I) = Q$.

Conversely, let $Q = \mathrm{soc}(I)$. Let now $M \subset I$ be a submodule such that $M \cap Q = 0$. Assume that $M \neq 0$. Since $M \in \mathrm{Tens}_{\mathfrak{g}}$ as a submodule of I, from Proposition 3.47 there exists some simple submodule $R \subset M$. This implies $R \subset \mathrm{soc}(M) \subset \mathrm{soc}(I) = Q$, i.e. we would have $R \subset Q \cap M$, which cannot be possible. Hence we get $M = 0$. This shows that I is an essential extension of Q, thus it is an injective hull for Q. $\square$

We now show that there exists a natural way of constructing injective hulls for the simple objects of $\mathrm{Tens}_{\mathfrak{g}}$. Let $Q \in \mathrm{Tens}_{\mathfrak{g}}$ be a simple module. Since Q satisfies **T1** and **T2**, we get that $ll(\Gamma(Q^*)) = ll(Q^*) < \infty$. From Remark 3.6 we see that $Q \cong (Q_*)_*$ is the unique simple submodule of $(Q_*)^*$. We note this specifically in the following result.

Lemma 3.49. *Let $Q \in \mathrm{Tens}_{\mathfrak{g}}$ be a simple module. Then*

$$\mathrm{soc}(Q_*)^* \cong Q.$$

Note that for $Q \in \mathrm{Tens}_{\mathfrak{g}}$ simple, from $Q_* \subset Q^*$ we get that $Q_* \in \mathrm{Tens}_{\mathfrak{g}}$ as well. Using this, we can now prove the following result.

Proposition 3.50. *Let $Q \in \mathrm{Tens}_{\mathfrak{g}}$ be a simple module. Then $(Q_*)^*$ is an injective hull of Q in $\mathrm{Tens}_{\mathfrak{g}}$.*

Proof. Since $Q_* \in \mathrm{Int}_{\mathfrak{g}}$, from Corollary 2.16 we get that $(Q_*)^*$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$. As $Q_* \in \mathrm{Tens}_{\mathfrak{g}}$ we get that $(Q_*)^*$ is also an injective object of $\mathrm{Tens}_{\mathfrak{g}}$. From Lemma 3.49 we see that $\mathrm{soc}((Q_*)^*) \cong Q$. Then Lemma 3.48 shows that $(Q_*)^*$ is indeed an injective hull for Q in $\mathrm{Tens}_{\mathfrak{g}}$. $\square$

We now give another characterization of injective modules of $\mathrm{Tens}_{\mathfrak{g}}$ which have simple socles.

Lemma 3.51. *Let $I \in \mathrm{Tens}_{\mathfrak{g}}$ be an injective object. Then I is indecomposable if and only if $\mathrm{soc}(I)$ is simple.*

Proof. Note that if I is decomposable, we have that $I = A \oplus B$ for some non-zero submodules $A, B \subset I$. Since $A, B \in \mathrm{Tens}_{\mathfrak{g}}$ as submodules, from Proposition 3.47 we get that there exist simple submodules $Q_1 \subset A$ and $Q_2 \subset B$. It is clear then that $Q_1 \oplus Q_2 = Q_1 + Q_2 \subset \mathrm{soc}(I)$, hence $\mathrm{soc}(I)$ will not be simple. Thus if an injective object $I \in \mathrm{Tens}_{\mathfrak{g}}$ has a simple socle, it cannot be decomposable, hence it will indecomposable.

Conversely, assume that I is indecomposable. Let now $Q \subset I$ be simple. From Proposition 3.50 let $I_Q = (Q_*)^*$ be an injective hull of Q. Denote by $j : Q \longrightarrow I_Q$ the natural inclusion. Since I is injective, the inclusion $i : Q \longrightarrow I$ induces a map $s : I_Q \longrightarrow I$ such that $s \circ j = i$. Let now $R = \ker s$. If $R \neq 0$, from Proposition 3.47 we have that $0 \neq \mathrm{soc}(j^{-1}((R)) \subset \mathrm{soc}(I_Q) = Q$, thus we get $Q = \mathrm{soc}(j^{-1}(R)) \subset j^{-1}(R)$. But this would imply that for any $q \in Q$ we would have $i(q) = (s \circ j)(q) = s(j(q)) = 0$, which is impossible. Hence we really must have that $R = 0$, i.e. that s is an injective map. Now since $s : I_Q \longrightarrow I$ is an injective map of injective modules, we know that the short exact sequence

$$0 \longrightarrow I_Q \longrightarrow I \longrightarrow I/I_Q \longrightarrow 0$$

splits, i.e. $I \cong I_Q \oplus I/I_Q$. But since I is indecomposable, we must have $I/I_Q = 0$, i.e. $I = I_Q$. It is clear from Lemma 3.49 then that $\mathrm{soc}(I) = Q$ is simple. $\square$

Note that Proposition 3.47, Corollary 3.48, and Lemma 3.51 show that an injective object $I \in \mathrm{Tens}_{\mathfrak{g}}$ is indecomposable if and only if it is an injective hull of its unique simple submodule. Using this, we are now ready to give the following result which characterizes indecomposable injective objects of $\mathrm{Tens}_{\mathfrak{g}}$.

Corollary 3.52. *i) Any indecomposable injective object I in $Tens_{\mathfrak{g}}$ is isomorphic to Q^* for some simple module $Q \in Tens_{\mathfrak{g}}$.*

ii) Let $M \in Tens_{\mathfrak{g}}$ and $I_M \in Int_{\mathfrak{g}}$ an injective hull for M. Then $I_M \in Tens_{\mathfrak{g}}$.

Proof. i) Let I be an indecomposable injective object in $\mathrm{Tens}_{\mathfrak{g}}$. In the proof of Lemma 3.51 we saw that if $Q' = \mathrm{soc}(I)$, then $I \cong (Q'_*)^*$. If we set $M = Q'_*$, we see that $I \cong M^*$, and this proves i).

ii) Let $M \in \mathrm{Tens}_{\mathfrak{g}}$, and let I_M be an injective hull of M in $\mathrm{Int}_{\mathfrak{g}}$. From Corollary **??** we know that there exists an injective map

$$\phi : M \longrightarrow \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*).$$

Since $M \in \mathrm{Tens}_{\mathfrak{g}}$, we have that $M^*, M^{**} \in \mathrm{Int}_{\mathfrak{g}}$, thus this injection becomes

$$\phi : M \longrightarrow M^{**}$$

with $M^{**} = \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*)$ an injective object of $\mathrm{Tens}_{\mathfrak{g}}$. Now since I_M is an injective hull of M, there exists a natural map $s : I_M \longrightarrow M^{**}$ such that $s|_M = \phi$. Note now that

$$0 = \ker \phi = \ker(s|_M) = M \cap \ker s,$$

and since I_M is an essential extension of M, we have that $\ker s = 0$. Now since s realizes I_M as a submodule of $M^{**} \in \mathrm{Tens}_{\mathfrak{g}}$, we get that $I_M \in \mathrm{Tens}_{\mathfrak{g}}$ as well, which is what we wanted to show. □

3.4.5 Finite dimensional representations of $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$

Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$.

In this subsection we want to use the theory we have developed so far about the representations of the classical locally semisimple Lie algebras $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ to classify their finite dimensional modules. In particular we will use Theorem 3.44.

Let us begin with the following result.

Lemma 3.53. *Let $M \in \mathfrak{g}$−mod be finite dimensional. Then $M \in Loc_{\mathfrak{g}}$, and M is semisimple.*

Proof. Note that for any $g \in \mathfrak{g}$ and $m \in M$ we have $g(f) \subset M$, hence

$$\dim g(f) \leq \dim M < \infty$$

so $M \in \mathrm{Int}_{\mathfrak{g}}$. Since M is an integrable $\mathfrak{g}$-module of countable dimension, from Theorem 2.9 we see that M will actually be a local module, hence $M \in \mathrm{Loc}_{\mathfrak{g}}$, and this proves the first part of the lemma.

As M is a local module, there exists an exhaustion $\{M_i\}_{i\in\mathbb{N}}$. Let us now set $d = \dim M$ and $d_i = \dim M_i$ for all $i \in \mathbb{N}$. It is clear then that

$$d_1 \leq d_2 \leq \cdots \leq d_i \leq \cdots \leq d.$$

As this is a bounded sequence of natural numbers, it stabilizes, i.e. there exists some $n_0 \in \mathbb{N}$ such that $d_i = d_j$ for all $i, j \geq n_0$. Since $M_{n_0} \subset M_i$ for $i > n_0$ and $\dim M_{n_0} = d_{n_0} = d_i = \dim M_i$, we get that $M_{n_0} = M_i$ for all $i \geq n_0$. Since $\{M_i\}_{i\in\mathbb{N}}$ is an exhaustion for M, we get that

$$M_{n_0} = \bigcup_{i\in\mathbb{N}} = M.$$

It is clear then that $M|_{\mathfrak{g}_i} = M_i$ for every $i > n_0$.

Given $i > n_0$, from Theorem 2.5 let $L_{i1}, L_{i2}, \dots, L_{in_i}$ be simple $\mathfrak{g}_i$-modules such that

$$M|_{\mathfrak{g}_i} = M_i = L_{i1} \oplus L_{i2} \oplus \cdots \oplus L_{in_i},$$

so that M as a $\mathfrak{g}_i$–module can be written as a direct sum of n_i simple $\mathfrak{g}_i$–submodules. We know that this decomposition is not necessarily unique, however the n_i will be a constant throughout the different such decompositions. Consider now M as a $\mathfrak{g}_{i+1}$–module, i.e. we have

$$M|_{\mathfrak{g}_{i+1}} = M_i = L_{(i+1)1} \oplus L_{(i+2)2} \oplus \cdots \oplus L_{(i+1)n_{i+1}}.$$

Note that for any $1 \le k \le n_{i+1}$, we have that there exists some simple $\mathfrak{g}_i$– submodule $L'_{ik} \subset L_{(i+1)k}|_{\mathfrak{g}_i}$. It is clear then that

$$\sum_{k=1}^{n_{i+1}} L'_{ik} = \bigoplus_{k=1}^{n_{i+1}} L'_{ik} \subset M|_{\mathfrak{g}_i}.$$

This way we see that $M|_{\mathfrak{g}_i}$ contains a direct sum of at least n_{i+1} simple $\mathfrak{g}_i$–modules, so we get that $n_i \ge n_{i+1}$. This way we obtain an infinite sequence of natural numbers

$$n_i \ge n_{i+1} \ge \cdots \ge n_k \ge \cdots > 0$$

It is clear then that this sequence stabilizes, i.e. there exists some $t \in \mathbb{N}$ such that $n_t = n_j$ for all $j \ge t$. Set now $n_t = n$. From the previous construction, we see that for $j \ge t$, we have that $L_{(j+1)k}|_{\mathfrak{g}_j}$ for $k = 1, 2, \dots, n$ contains a unique simple $\mathfrak{g}_j$–submodule L_{jk}, hence we get $L_{(j+1)k}|_{\mathfrak{g}_j} = L_{jk}$ is simple as a $\mathfrak{g}_j$–module. This gives us that for all $j \ge t$ and $k = 1, 2, \dots, n$ we have $L_{jk} = L'_{tk}$. Let us now set $L_k := \bigcup_{j \ge t} L_{jk} = L'_{tk}$, i.e. L_k are locally simple $\mathfrak{g}$–modules, hence they are also simple $\mathfrak{g}$–modules. It is clear then that

$$M = L_1 \oplus L_2 \oplus \cdots \oplus L_n. \tag{3.24}$$

As these L_k were shown to be simple, $k = 1, 2, \dots, n$, (3.24) clearly shows that M is indeed semisimple, which concludes the proof of this lemma. □

Now using this result, we are able to classify all finite dimensional $\mathfrak{g}$–modules as follows.

Theorem 3.54. *Let $M \in \mathfrak{g}$–mod be finite dimensional. Then M is a trivial $\mathfrak{g}$–module.*

Proof. From Lemma 3.53, we have that M will be semisimple, i.e. there exist simple submodules $L_1, L_2, \dots, L_n \subset M$ such that (3.24) holds. It is clear that if we prove that L_i are trivial $\mathfrak{g}$–modules, then M will also be a trivial $\mathfrak{g}$–module, so Lemma 3.53 reduces the proof of this theorem to only the case where M is simple.

Consider the algebraic dual M^* as a $\mathfrak{g}$–module. Since $\dim M < \infty$, we have $\dim M^* = \dim M < \infty$. From 3.53 we know that $M^* \in \mathrm{Int}_{\mathfrak{g}}$, and $M^* = \Gamma(M^*)$ is semisimple, so in particular it has finite socle length. This shows that M satisfies the conditions of Theorem 3.44, so it must be a simple tensor module, so $M \cong V_\lambda$ for some $\lambda \in \Theta$. From Remark 3.10 we have seen that if $\lambda \ne 0$, V_λ will have infinite dimension, which cannot be the case for our M, hence we have $\lambda = 0$. We saw in Subsection 3.4.2 that $M = V_0 = \mathbb{C}$, i.e. our M will be the trivial simple $\mathfrak{g}$–module. The theorem than follows from the first paragraph of the proof. □

Bibliography

[1] Y. Bahturin, H. Strade, Some examples of locally finite simple Lie algebras, Archiv der Mathematik 1995, Volume 65, Issue 1, pp 23-26

[2] Y. Bahturin, H. Strade, Locally finite-dimensional simple Lie algebras, Russian Acad. Sci. Sb. Math, Volume 81 (1995)

[3] A. Baranov, Finitary simple Lie algebras, J. Algebra 219(1999), 299- 329

[4] A. Baranov, Complex finitary simple Lie algebras, Archiv der Math. 71(1998), 1-6

[5] A. Baranov, A. Zhilinski, Diagonal direct limits of simple Lie algebras, Comm. Algebra 27 (1998), 2749-2766

[6] James E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag New York Inc., 1972

[7] Anthony W. Knapp, Lie Groups, Lie Algebras, and Cohomology, Princeton University Press 1988

[8] Anthony W. Knapp, Lie Groups Beyond an Introduction, Second Edition, Birkäuser, 2002

[9] Georg Lusztig, Introduction to Quantum Groups, Birkäuser, 1994

[10] Ivan Penkov, Vera Serganova, Categories of integrable $sl(\infty)-, o(\infty)-, sp(\infty)-$modules, in Representation Theory and Mathematical Physics, Contemporary Mathematics 557, AMS 2011, 335-357.

[11] I. Penkov, K. Styrkas, Tensor representations of infinite-dimensional root-reductive Lie algebras, in Developments and Trends in Infinite-Dimensional Lie Theory, Progress in Mathematics 288, Birkhäuser, 2011, pp. 127-150

[12] Charles A. Weibel, An introduction to homological algebra, Cambridge University Press, 1994