Geometric approach to KLR algebras and their representation theory

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1 Introduction

1.1 The big picture

Khovanov and Lauda ([KL09]) and Rouquier ([Rou08]) associated to a datum consisting of a quiver and a dimension vector certain infinite-dimensional algebras. They have become known as Khovanov-Lauda-Rouquier (KLR) or quiver Hecke algebras. We will use these names interchangeably. Khovanov and Lauda defined KLR algebras using a calculus of braid-like plane diagrams of interacting strings labelled by the vertices of a quiver. The resulting algebra, as an abelian group, consists of finite linear combinations of such diagrams modulo certain relations, which can also be described diagramatically. Multiplication in this algebra is given by concatenation of diagrams. Rouquier, on the other hand, took a different approach and defined KLR algebras directly by generators and relations.

The motivation for studying KLR algebras is twofold - they categorify quantum groups and yield non-trivial gradings on affine Hecke algebras. Let $\mathfrak{g}$ be a simply-laced Kac-Moody Lie algebra with Dynkin diagram $\mathbf{\Gamma}$. It admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $U_q(\mathfrak{g})$ denote the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}[q, q^{-1}]$. The triangular decomposition of $\mathfrak{g}$ induces a corresponding decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+)$ of the quantized universal enveloping algebra.

We are primarily interested in the categorification of a certain subring of the algebra $U_q(\mathfrak{n}^-)$, defined over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, called Lusztig's integral form of $U_q(\mathfrak{n}^-)$. Let ${}_{\mathcal{A}}\mathbf{f}$ denote Lusztig's integral form of $U_q(\mathfrak{n}^-)$ and let ${}_{\mathcal{A}}\mathbf{f}^*$ denote the graded dual of ${}_{\mathcal{A}}\mathbf{f}$. Before we can explain the connection between Lusztig's integral form of the negative half of the quantum group and KLR algebras we need to introduce some notation. Let $H(\mathbf{\Gamma}, \underline{\mathbf{d}})$ be the KLR algebra associated to the quiver $\mathbf{\Gamma}$ with vertex set $\mathbf{I}$ and dimension vector $\underline{\mathbf{d}}$. Let $K(\underline{\mathbf{d}})$ denote the Grothendieck group of the category of finitely generated graded projective modules over $H(\mathbf{\Gamma}, \underline{\mathbf{d}})$ and let $K^*(\underline{\mathbf{d}})$ denote the Grothendieck group of the category of finite-dimensional graded modules over $H(\mathbf{\Gamma}, \underline{\mathbf{d}})$. We can now state the main categorification results. Khovanov and Lauda ([KL09]) have shown using combinatorial and algebraic methods that there exist graded twisted bialgebra isomorphisms

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \cong \bigoplus_{\underline{\mathbf{d}} \in \mathbb{N}\mathbf{I}} K(\underline{\mathbf{d}}), \quad \gamma^* : {}_{\mathcal{A}}\mathbf{f}^* \cong \bigoplus_{\underline{\mathbf{d}} \in \mathbb{N}\mathbf{I}} K^*(\underline{\mathbf{d}}).$$

Varagnolo and Vasserot ([VV11]) have refined this result by proving that γ^{-1} maps classes of indecomposable projective modules to the canonical basis of ${}_{\mathcal{A}}\mathbf{f}$ and that $(\gamma^*)^{-1}$ maps classes of simple modules to the dual canonical basis of ${}_{\mathcal{A}}\mathbf{f}^*$. Their results required the use of geometric methods. Kato ([Kat13]) and McNamara ([McN13]) have also constructed modules which categorify PBW and dual PBW bases of the quantum group. Furthermore, it has been shown by Kang and Kashiwara ([KK12]) that there exists an isomorphism between each integrable highest weight module over $U_q(\mathfrak{g})$ and the Grothendieck group of the category of finitely generated graded projective modules over a certain cyclotomic quotient of the corresponding KLR algebra.

There also exists a connection between cyclotomic KLR algebras and certain quotients of affine Hecke algebras called cyclotomic Hecke (or sometimes Ariki-Koike) algebras. The latter include group algebras of Coxeter groups and Iwahori-Hecke algebras. Brundan and Kleshchev ([BK09]) have constructed an explicit isomorphism between blocks of (possibly degenerate) cyclotomic Hecke algebras and a sign-modified version of cyclotomic KLR algebras associated to the infinite linear quiver or a cyclic quiver (i.e. in types A_∞ and $\widehat{A}_e$). This isomorphism yields interesting $\mathbb{Z}$-gradings on blocks of symmetric groups and the associated Iwahori-Hecke algebras, thus paving the way to the study of graded representation theory of these algebras. Furthermore, Webster ([Web14]) and Miemietz and Stroppel ([MS15]) have constructed an isomorphism between certain completions of KLR algebras and affine Hecke algebras in types A_∞ and $\widehat{A}_e$.

In this thesis we are primarily interested in the geometric construction of KLR algebras due to Varagnolo and Vasserot. To prove that indecomposable projectives over a KLR algebra categorify the canonical basis of ${}_{\mathcal{A}}\mathbf{f}$, they identified KLR algebras with certain convolution algebras in equivariant Borel-Moore homology. This geometric construction is rather complex but it enables us to apply powerful sheaf-theoretic tools such as the BBD decomposition theorem to study KLR algebras and their representation theory. Convolution algebras provide a uniform approach to the construction of many familiar objects such as group algebras of Weyl groups, affine Hecke algebras, degenerate affine Hecke algebras as well as quotients of universal enveloping algebras and quantized loop algebras. We will now briefly review the classical setting in which convolution algebras occur and explain how this framework can be modified to construct KLR algebras.

Let G be a complex semisimple algebraic group and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathcal{N}$ be the closed subvariety of $\mathfrak{g}$ consisting of all nilpotent elements, i.e., all elements $x \in \mathfrak{g}$ such that $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent endomorphism. The group G acts on $\mathcal{N}$ by conjugation and $\mathbb{C}^\times$ acts on $\mathcal{N}$ by dilations. Let $\mathfrak{B}$ denote the variety of all Borel subalgebras of $\mathfrak{g}$. It is isomorphic to the homogeneous space G/B. The interplay between the varieties $\mathcal{N}$ and $\mathfrak{B}$ is encoded in the following "incidence variety" $\widetilde{\mathfrak{B}} := \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathfrak{B} \mid x \in \mathfrak{b}\}$. If $G = \mathrm{SL}_n(\mathbb{C})$ then we can identify $\mathfrak{B}$ with the variety of complete flags in $\mathbb{C}^n$. Moreover, $\mathfrak{g} = \mathfrak{sl}_n$ acts naturally on $\mathbb{C}^n$ by matrix multiplication. Let $F = (V_k)_{k=0}^n$ be the flag corresponding to a Borel subalgebra $\mathfrak{b}$ under this identification. The condition $x \in \mathfrak{b}$ can then be interpreted as saying that $x(V_k) \subset V_{k-1}$ for each k, i.e., that the flag F is stable under the endomorphism x. We have two canonical projections

$$\begin{tikzcd} & \widetilde{\mathfrak{B}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & \mathfrak{B}. \end{tikzcd}$$

The map μ, called the Springer resolution, is proper and the map π is a G-equivariant vector bundle. The fibred product $Z := \widetilde{\mathfrak{B}} \times_{\mathcal{N}} \widetilde{\mathfrak{B}}$, called the Steinberg variety, is one of the central objects of study in geometric representation theory. Using convolution in Borel-Moore homology or equivariant K-theory of this variety (and related varieties) one can construct many interesting algebras of fundamental importance in representation theory, for example the group algebra of a Weyl group, quotients of the universal enveloping algebra of $\mathfrak{sl}_n$ and the affine Hecke algebra. One can also construct all the irreducible modules over these algebras as quotients of convolution modules in the Borel-Moore homology of fibres of the Springer resolution.

A deeper study of convolution algebras involves intersection cohomology methods. There is an algebra isomorphism $H_*(Z) \cong \mathrm{Ext}^*(\mu_*\mathcal{C}_{\widetilde{\mathfrak{B}}}, \mu_*\mathcal{C}_{\widetilde{\mathfrak{B}}})$ between the convolution algebra $H_*(Z)$ and the geometric extension algebra associated to the direct image of the constant perverse sheaf on $\widetilde{\mathfrak{B}}$. One can therefore apply the Beilinson-Bernstein-Deligne decomposition theorem to deduce many deep representation-theoretic consequences, for example the classification and construction of simple modules or Bernstein-Gelfand-Gelfand-type reciprocities.

We apply this framework to study KLR algebras. The main departure from the classical setting explained above is the introduction of a quiver grading. We replace the variety $\mathfrak{B}$ with a suitably defined quiver flag variety and replace $\mathcal{N}$ with the space of representations of the chosen quiver $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. Varagnolo and Vasserot have shown in [VV11] that the equivariant Borel-Moore homology of the resulting Steinberg variety, equipped with the convolution product, is isomorphic to the quiver Hecke algebras defined diagramatically by Khovanov and Lauda and algebraically by Rouquier. The main idea of the proof is to construct and explicitly calculate a faithful representation of the convolution algebra $H_*^G(Z)$, and show that this faithful representation agrees with the faithful representation of the diagramatic quiver Hecke algebra on a direct sum of polynomial rings.

1.2 Overview of the thesis

The thesis has two main objectives. The first is to give a detailed and self-contained account of the geometric construction of KLR algebras due to Varagnolo and Vasserot. Their paper [VV11] was the main inspiration for us in writing this thesis. We note that [VV11] does not contain much detail and many proofs and calculations are left out. We remedy this by supplying detailed proofs and calculations in this thesis. Our second objective is to use the geometric construction to study certain aspects of the representation theory of KLR algebras.

We now briefly summarize the contents of each chapter.

- Chapter 2: Homology and cohomology.

  We recall the definitions and basic properties of equivariant cohomology and Borel-Moore homology. We also calculate some fundamental examples of equivariant cohomology groups.

- Chapter 3: Quivers.

  The definition of a KLR algebra depends on a quiver and a dimension vector. The purpose of this chapter is introduce various notations pertaining to these input data. We also discuss related objects, for example the space $\mathrm{Rep}_{\underline{\mathbf{d}}}$ of representations of our quiver and an associated reductive linear algebraic group $G_{\underline{\mathbf{d}}}$.

- Chapter 4: Quiver flag varieties.

  To a quiver and a dimension vector we associate a "quiver flag variety" $\mathcal{F}_{\underline{\mathbf{d}}}$, which can be characterized as a certain disjoint union of products of ordinary flag varieties. We also investigate the connections between quiver flag varieties and Lie-theoretic objects such as Weyl groups and root systems.

- Chapter 5: The Steinberg variety.

  We study the interplay between representations of our quiver and the quiver flag variety. We begin by defining what it means for a flag to be stabilized by a representation of the quiver. We then define a vector bundle $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ over the quiver flag variety whose fibre consists of representations stabilizing a given flag. Finally, we define a quiver analogue $\mathcal{Z}_{\underline{\mathbf{d}}}$ of the Steinberg variety. We also prove some basic properties of these varieties. For example, we show that $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is isomorphic to a direct sum of polynomial rings.

- Chapter 6: Convolution.

  We first recall the definition of the convolution product from [CG97, Chapter 2.6-2.7] and adapt it to the equivariant setting. We then apply it to our Steinberg variety $\mathcal{Z}_{\underline{\mathbf{d}}}$. Thereby we obtain the main object of study in this thesis - the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We also show that the algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ naturally acts by convolution on $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$. We call it the "polynomial representation" of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

- Chapter 7: Stratifications.

  Our goal here is to gain a better understanding of the structure of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. In particular, we want to determine a basis of this algebra over the equivariant cohomology ring of a point. To do this, we adapt the theory of affine stratifications of an algebraic variety to take account of the presence of a quiver grading. We define various "quiver Schubert cells" and show that equivariant fundamental classes of their closures form a basis of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. Subsequently we show that the stratification of the variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ which we defined induces a filtration on the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. At the end of the chapter we also describe the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and show that the variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ is $G_{\underline{\mathbf{d}}}$-equivariantly formal. We will need these results later when we study the graded representation theory of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

- Chapter 8: Relationship between $G_{\underline{\mathbf{d}}}$- and $T_{\underline{\mathbf{d}}}$-equivariant (co)homology.

  We choose a maximal torus $T_{\underline{\mathbf{d}}}$ in the reductive algebraic group $G_{\underline{\mathbf{d}}}$. We then recall the standard fact that $G_{\underline{\mathbf{d}}}$-equivariant homology is isomorphic to the invariants of $T_{\underline{\mathbf{d}}}$-equivariant homology under the action of the associated Weyl group.

- Chapter 9: Euler classes and convolution.

  We begin by recalling the "clean intersection formula" which, under appropriate assumptions, allows us to explicitly calculate the convolution product. The formula involves multiplicites which can be identified with Euler classes of certain vector bundles. Our goal in this chapter is to determine these multiplicities for quiver flag varieties and the Steinberg variety. We show that the Euler class in the clean intersection formula can also be identified with a product of the weights of the tangent space to a quiver Schubert variety at a torus fixed point, considered as a module over the Lie algebra of the torus $T_{\underline{\mathbf{d}}}$. The rest of the chapter is devoted to the computation of these Euler classes.

- Chapter 10: Localization to $T_{\underline{\mathbf{d}}}$-fixed points.

  We apply the localization theorem for equivariant cohomology and the results of chapter 9 to compute the convolution product on torus fixed points. We then use this calculation to show that the polynomial representation of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is faithful. The main results of this chapter are due to Varagnolo and Vasserot ([VV11]). However, most of the calculations and detailed proofs are ours.

- Chapter 11: Generators and relations.

  The purpose of this chapter is to translate the geometric results from the previous chapters into algebraic terms. We first define certain elements in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and show that these elements generate $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as an algebra. We then give an explicit description of the faithful polynomial representation of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We use this representation to find a complete set of relations in our convolution algebra. The presentation in terms of generators and relations which we obtain implies that $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic as a graded algebra to the quiver Hecke algebras defined by Rouquier and Khovanov-Lauda. The main results of this chapter are also due to Varagnolo and Vasserot ([VV11]).

- Chapter 12: Representation theory of convolution algebras.

  We now turn our attention to the representation theory of KLR algebras. We also adopt a geometric approach. The purpose of chapter 12 is to give a succinct yet rigorous overview of the main technical tools we require. We carefully define perverse sheaves, intersection cohomology complexes and state the Beilinson-Bernstein-Deligne decomposition theorem. Our next objective is to derive a somewhat stronger version of the decomposition theorem which we will apply to study representations of KLR algebras. To do this we exploit additional information which is contained in the equivariant decomposition theorem of Bernstein-Lunts. We then explain how one can use the decomposition theorem to classify graded simple modules over a convolution algebra. The chapter ends with a discussion of standard modules and their relation to graded simple modules.

- Chapter 13: Representation theory of KLR algebras.

  We begin by applying the results of chapter 12 to KLR algebras, always carefully checking that the relevant assumptions are satisfied. We next turn our attention to the special case of the equioriented A_n quiver and study in detail the interplay between geometry and representation theory. The chapter contains two important results. The first results says that all the graded simple modules obtained from the decomposition theorem are non-zero. Our proof is inspired by a proof of the corresponding result for affine Hecke algebras due to Ginzburg ([CG97, Section 8.8]). The second result states that every standard module over a KLR algebra associated to an equioriented A_n quiver is indecomposable with simple head. This result has been proved by Kato ([Kat12]) using sheaf-theoretic and homological methods. We give a different proof which is more geometric in nature.

2 Homology and cohomology

In this chapter we review the definitions and fundamental properties of equivariant cohomology and Borel-Moore homology.

2.1 The homotopy quotient

We first define the homotopy quotient of a manifold by a group action and discuss examples for tori and general linear groups.

Definition 2.1. Let G be a topological group. A *principal G-bundle* is a fibre bundle $p : E \to B$ together with a continuous G-action $E \times G \to E$ such that G preserves fibres and acts freely and transitively on each fibre. A *universal principal G-bundle*, denoted $EG \to BG$, is a principal G-bundle such that for every paracompact manifold X the map

$$[X, BG] \to \text{G-PBund}(X)/\sim$$

$$[f] \mapsto f^*EG$$

(from the set of homotopy classes of maps from X to BG to the set of isomorphism classes of principal G-bundles) is a set isomorphism. In other words, every principal G-bundle over X is isomorphic to the pullback of the universal bundle along some continuous map $f : X \to BG$, and the correspondence between isomorphism classes of principal G-bundles and homotopy classes of maps $f : X \to BG$ is bijective. The space BG is called the *classifying space* for principal G-bundles. The following theorem is standard.

Theorem 2.2. *Let G be a topological group. Then:*
(i) The space EG exists and is unique up to equivariant homotopy equivalence.
(ii) EG is contractible and the action of G on EG is free.
(iii) Conversely, if E is contractible and G acts freely on E, then $E \to E/G$ is a universal principal bundle.

Proof. See e.g. [Hus91, Chapter 4]. □

Remark 2.3. Let M be a G-manifold. We want to study the cohomology of the orbit space M/G. In general, M/G does not admit the structure of a manifold. However, if the action of G is free, then M/G exists as a manifold. If G doesn't act freely, we want to replace M by a homotopy equivalent space on which G does act freely. Since EG is contractible, M is homotopy equivalent to $EG \times M$. Moreover, since G acts freely on EG, the diagonal action of G on $EG \times M$ is free as well and the quotient by this action exists as a manifold. This motivates the following definition. △

Definition 2.4. Suppose that G acts on EG from the right and on M from the left. Let G act diagonally on the product space $EG \times M$ by the formula $(e, m).g = (eg^{-1}, g.m)$. We call the quotient

$$EG \times^G M := (EG \times M)/G$$

of $EG \times M$ by this diagonal action the *homotopy quotient* of M by G, the homotopy orbit space, or the Borel construction. △

Example 2.5. The most important examples for us will be tori and GL_n.
(i) Let $G = \mathbb{C}^\times$. The group G acts freely on $\mathbb{C}^n\backslash\{0\}$ by scalar multiplication. The quotient $(\mathbb{C}^n\backslash\{0\})/G$ is isomorphic to $\mathbb{CP}^{n-1}$. We obtain a principal bundle $(\mathbb{C}^n\backslash\{0\}) \to \mathbb{CP}^{n-1}$ whose total space is $(2n-2)$-connected, i.e., the homotopy groups $\pi_i(\mathbb{C}^n\backslash\{0\})$ vanish for $i = 1, ..., 2n-2$. We can define the universal bundle by taking direct limits of the total space and the base space:

$$E\mathbb{C}^\times = \varinjlim(\mathbb{C}^n\backslash\{0\}) = \mathbb{C}^\infty\backslash\{0\} \qquad \text{and} \qquad B\mathbb{C}^\times = \varinjlim \mathbb{CP}^{n-1} = \mathbb{CP}^\infty.$$

(ii) Let $G = T^m = (\mathbb{C}^\times)^m$ be a torus. Then

$$ET^m = (E\mathbb{C}^\times)^m = (\mathbb{C}^\infty \backslash \{0\})^m \qquad \text{and} \qquad BT^m = (B\mathbb{C}^\times)^m = (\mathbb{CP}^\infty)^m.$$

(iii) Let $G = \mathrm{GL}_n(\mathbb{C})$ and $m > n$ be an integer. Let $\mathrm{Mat}(m \times n)$ denote the space of all $m \times n$ matrices with complex entries, and $\mathrm{Mat}^{max}(m \times n)$ denote the subset of matrices of maximal rank n. The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Mat}(m \times n)$ from the right preserving $\mathrm{Mat}^{max}(m \times n)$. Let $\mathrm{Gr}(n, m)$ denote the Grassmannian of linear subspaces of $\mathbb{C}^m$ of dimension n. Define a map

$$\mathrm{Mat}^{max}(m \times n) \to \mathrm{Gr}(n, m)$$

$$A \mapsto \mathrm{Im}(A).$$

We can interpret matrices of maximal rank as injective $\mathbb{C}$-linear homomorphisms, i.e., $\mathrm{Mat}^{max}(m \times n) = \mathrm{Hom}_{\mathbb{C}}^{inj}(\mathbb{C}^n, \mathbb{C}^m)$. Precomposing such a homomorphism with a linear automorphism of $\mathbb{C}^n$ does not change its image. On the other hand, if two such homomorphisms have the same image, we can precompose one of them with a linear automorphism of $\mathbb{C}^n$ to obtain the other. Hence $\mathrm{Mat}^{max}(m \times n)/\mathrm{GL}_n = \mathrm{Gr}(n, m)$ and $\mathrm{Mat}^{max}(m \times n) \to \mathrm{Gr}(n, m)$ is a principal GL_n-bundle. Taking the limit $m \to \infty$ we obtain the principal GL_n bundle

$$\mathrm{Mat}^{max}(\infty \times n) \to \mathrm{Gr}(n, \infty).$$

The action of GL_n on $\mathrm{Mat}^{max}(\infty \times n)$ is clearly free, and one can check without much difficulty that $\mathrm{Mat}^{max}(\infty \times n)$ is contractible. Hence

$$E\mathrm{GL}_n = \mathrm{Mat}^{max}(\infty \times n) \qquad \text{and} \qquad B\mathrm{GL}_n = \mathrm{Gr}(n, \infty).$$

Proposition 2.6. *Let H be a closed subgroup of G. Then the quotient EG/H exists and the map $EG \to EG/H$ is a universal bundle for H.*

Proof. See [Bri98, p.4]. □

2.2 Equivariant cohomology

In this section we define equivariant cohomology rings and discuss examples for a point and for homogeneous spaces. For a more thorough treatment of equivariant cohomology the reader is referred to [Bri98].

Definition 2.7. Let M be a topological manifold endowed with a continuous action of a topological group G. Let R be a commutative ring.
(i) We define the *equivariant cohomology ring* $H_G^*(M; R)$ to be

$$H_G^*(M; R) := H^*(EG \times^G M; R),$$

where $H^*(-; R)$ denotes singular cohomology with coefficients in R.
(ii) In particular, if $M = \{pt\}$ is a point, then

$$H_G^*(\{pt\}; R) = H^*(EG/G; R) = H^*(BG; R).$$

We set

$$S_G(R) := H^*(BG; R), \qquad K_G(R) := \mathrm{Frac}(S_G(R)).$$

If we work with a fixed coefficient ring R, we will often abbreviate S_G, K_G.
(iii) Let N be a G-stable subspace of M. Then we define

$$H_G^*(M, N; R) := H^*(EG \times^G M, EG \times^G N; R)$$

to be the *relative equivariant cohomology ring* of the pair (M, N). △

Remark 2.8. Even though we have given a general definition of equivariant cohomology for any topological manifold and topological group, we will most often work with manifolds which are also smooth and groups which are also Lie groups.

Example 2.9. Let $M = \{pt\}$ be a point.
(i) We have

$$H^*_{\mathbb{C}^\times}(\{pt\}; R) = H^*(B\mathbb{C}^\times; R) = H^*(\mathbb{CP}^\infty; R) \cong R[t],$$

where $\deg t = 2$. We also have the following concrete description of this cohomology ring. Let $X^*(\mathbb{C}^\times)$ denote the character group of $\mathbb{C}^\times$. We let $\mathbb{C}^\times$ act on $\mathbb{C}_\lambda$ from the left with weight λ, i.e., $t.v = t^\lambda v$, for $t \in \mathbb{C}^\times, v \in \mathbb{C}_\lambda$. Moreover, $\mathbb{C}^\times$ acts naturally on $\mathbb{C}^\infty \backslash \{0\}$ from the right by scalar multiplication (or matrix multiplication). We let $\mathbb{C}^\times$ act on the product space $(\mathbb{C}^\infty\backslash\{0\}) \times^{\mathbb{C}^\times} \mathbb{C}_\lambda$ diagonally according to the formula $(m, v).t = (m.t^{-1}, t.v) = (m.t^{-1}, t^\lambda v)$. The quotient

$$\mathcal{O}_{\mathbb{CP}^\infty}(\lambda) := (\mathbb{C}^\infty\backslash\{0\}) \times^{\mathbb{C}^\times} \mathbb{C}_\lambda = ((\mathbb{C}^\infty\backslash\{0\}) \times \mathbb{C}_\lambda)/\mathbb{C}^\times$$

is a line bundle over $\mathbb{CP}^\infty$. It is well-known that the following composition

$$\mathbb{Z} \cong X^*(\mathbb{C}^\times) \to \mathrm{Pic}(\mathbb{CP}^\infty) \to H^2(\mathbb{CP}^\infty; \mathbb{Z})$$

$$\lambda \mapsto \mathcal{O}_{\mathbb{CP}^\infty}(\lambda) \mapsto c_1(\mathcal{O}_{\mathbb{CP}^\infty}(\lambda)),$$

where the last map is the first Chern class, is an isomorphism. In particular, the first Chern class $c_1(\mathcal{O}_{\mathbb{CP}^\infty}(1))$ of the canonical line bundle generates $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ as an algebra.
(ii) We have $H^*_{T^m}(\{pt\}; R) = H^*(BT^m; R) = H^*((\mathbb{CP}^\infty)^m; R) \cong R[t_1, ..., t_m]$, where $\deg t_i = 2$. As before, we have isomorphisms

$$\mathbb{Z}^m \cong X^*(T^m) \to \mathrm{Pic}((\mathbb{CP}^\infty)^m) \to H^2((\mathbb{CP}^\infty)^m; \mathbb{Z}).$$

We set

$$\mathcal{O}_{(\mathbb{CP}^\infty)^m}(k; \mu) := \mathcal{O}_{(\mathbb{CP}^\infty)^m}(\lambda),$$

where $\lambda = (0, ..., 0, \mu, 0, ..., 0)$ with μ in the k-th position. Then the first Chern classes of the line bundles $\mathcal{O}_{(\mathbb{CP}^\infty)^m}(1; 1), ..., \mathcal{O}_{(\mathbb{CP}^\infty)^m}(m; 1)$ generate $H^*((\mathbb{CP}^\infty)^m; \mathbb{Z})$ as an algebra.
(iii) $H^*_{\mathrm{GL}_n}(\{pt\}; R) = H^*(B\mathrm{GL}_n; R) = H^*(\mathrm{Gr}(n, \infty); R) \cong R[t_1, ..., t_n]$, where $\deg t_i = 2i$. We can interpret this as the algebra of symmetric polynomials.

Example 2.10. Let H be a closed subgroup of G. Proposition 2.6 allows us to calculate the G-equivariant cohomology ring of the space G/H. We have

$$H^*_G(G/H; R) = H^*(EG \times^G G/H; R) = H^*(EG/H; R) = H^*(BH; R) = H^*_H(\{pt\}; R).$$

Let $G = \mathrm{GL}_n$ and let $H = \mathrm{T}$ be a maximal torus. Then

$$H^*_{\mathrm{GL}_n}(\mathrm{GL}_n/\mathrm{T}; R) = H^*_{\mathrm{T}}(\{pt\}; R) = H^*(B\mathrm{T}; R) = H^*((\mathbb{CP}^\infty)^n; R) = R[t_1, ..., t_n],$$

where $\deg t_i = 2$.

Definition 2.11. Let G be a topological group. We define $\{(\pi_n : E^nG \to B^nG, p_n, q_n) \mid n > 0\}$, where $p_n : E^{n+1}G \to E^nG$, $q_n : B^{n+1}G \to B^nG$ are continuous maps, to be an *approximation* of the universal principal G-bundle $EG \to BG$ if:

- each $\pi_n : E^nG \to B^nG$ is a principal G-bundle and each $p_n : E^{n+1}G \to E^nG$ is a morphism of principal G-bundles,
- each space E^nG is n-connected,
- $\pi_{n+1} \circ q_n = p_n \circ \pi_n$ for each $n > 0$,
- $\{(E^nG, p_n) \mid n > 0\}$, $\{(B^nG, q_n) \mid n > 0\}$ form direct systems,
- $EG = \varinjlim E^nG, \ BG = \varinjlim B^nG.$

We call $E^nG \to B^nG$ an n-th approximation of $EG \to BG$. △

Proposition 2.12. *Let $n > 0$ and let $E^nG \to B^nG$ be an n-th approximation of $EG \to BG$. Then for all $m \leq n$*

$$H^m_G(M; R) = H^m(EG \times^G M; R) = H^m(E^nG \times^G M; R),$$

for any compact topological G-manifold M of dimension at most n.

Proof. See [Bri98, p.4-5] and [Hus91, Chapter 4, Theorem 13.1]. □

2.3 Borel-Moore homology

In this section we define non-equivariant Borel-Moore homology and review its basic properties. For a more detailed but accessible treatment of this topic the reader is referred to [CG97, Chapter 2.6].

2.3.1 Definition of BM homology

Let X be a locally compact topological space which has the homotopy type of a finite CW-complex and admits a closed embedding into a smooth manifold. All homology groups below have coefficients in some fixed ring R.

Definition 2.13. Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. We define the i-th *Borel-Moore homology* group $H_i(X)$ of X to be

$$H_i(X) = H_i^{sing}(\hat{X}, \{\infty\}),$$

where $H_*^{sing}(-,-)$ denotes relative singular homology. △

Remark 2.14. It is also possible to define Borel-Moore homology as the homology of a certain chain complex of locally finite infinite singular chains or the homology of a distribution de Rham complex. Also note that our definition immediately implies that Borel-Moore homology coincides with singular homology for compact spaces.

Proposition 2.15 (Poincaré duality)**.** *Let M be a smooth oriented manifold of real dimension m. Let X be a closed subset of M which has an open neighbourhood $U \subset M$ such that X is a proper deformation retract of U. Then there is a canonical isomorphism*

$$H_i(X) \cong H^{m-i}(M, M \backslash X), \tag{1}$$

where $H^*(-,-)$ denotes relative singular cohomology. In particular, we have

$$H_i(M) \cong H^{m-i}(M). \tag{2}$$

2.3.2 Properties of BM homology

2.3.2.1 Proper pushforward. We claim that Borel-Moore homology is a covariant functor with respect to proper maps. Let $f : X \to Y$ be a proper map. We can extend f to a map $\hat{f} : \hat{X} \to \hat{Y}$ by setting $\hat{f}(\infty) = \infty$. Since f is proper, $\hat{f}$ is continuous. Then, by the functoriality of relative singular homology, we obtain the induced map

$$\hat{f}_*^{sing} : H_*^{sing}(\hat{X}, \{\infty\}) \to H_*^{sing}(\hat{Y}, \{\infty\}).$$

We now apply the one-point compactification definition of Borel-Moore homology to obtain a map

$$f_* : H_*(X) \to H_*(Y).$$

We call f_* the *proper pushforward* along f.

2.3.2.2 Künneth formula. Let M_1, M_2 be arbitrary CW-complexes. Then there is a natural isomorphism

$$\otimes : H_*(M_1) \otimes H_*(M_2) \to H_*(M_1 \times M_2).$$

2.3.2.3 Smooth pullback. Let X be a locally compact space and $p : \tilde{X} \to X$ a locally trivial fibration with smooth oriented fibre F. Assume that p is an oriented fibration, i.e., all transition functions of the fibration preserve the orientation of the fibre. Let $d = \dim F$. There exists a natural pullback morphism

$$p^* : H_*(X) \to H_{*+d}(\tilde{X}).$$

If the fibration is trivial, then the morphism p^* is given by $c \mapsto c \otimes [F]$. In general, if U is an open subset in which p is trivial, then p^* restricts to the map $c \mapsto c \otimes [F]$.

2.3.2.4 Intersection pairing. Let M be a smooth oriented manifold of real dimension m and $Z, \tilde{Z}$ two closed subsets of M. Consider the cup product in singular relative cohomology

$$\cup : H^{m-i}(M, M \backslash Z) \times H^{m-j}(M, M \backslash \tilde{Z}) \to H^{2m-j-i}(M, M \backslash (Z \cap \tilde{Z})).$$

By the Poincaré duality definition of Borel-Moore homology we obtain a bilinear map

$$\cap : H_i(Z) \times H_j(\tilde{Z}) \to H_{i+j-m}(Z \cap \tilde{Z}).$$

We call $\cap$ the *intersection pairing*.

2.4 Equivariant Borel-Moore homology

We now define equivariant Borel-Moore homology using approximations to a homotopy quotient and show that (under some assumptions) equivariant Borel-Moore homology is Poincaré dual to equivariant cohomology. Standard references for equivariant Borel-Moore homology are [Bri00] and [Gra99].

Definition 2.16. Let G be a Lie group. Let $\{E^n G \to B^n G \mid n > 0\}$ be an approximation of the universal bundle $EG \to BG$. Let X be a topological space satisfying the conditions in Section 2.3.1. Further assume that X is also a complex algebraic variety of pure dimension $x/2$. Let $\widetilde{n} = \dim_{\mathbb{R}} E^n G$ and $g = \dim_{\mathbb{R}} G$. The inclusions

$$\iota_n : E^n G \times^G X \to E^{n+1} G \times^G X$$

induce Gysin pullback maps

$$(\iota_n)^* : H_{i+\widetilde{(n+1)}-g}(E^{n+1} G \times^G X) \to H_{i+\widetilde{n}-g}(E^n G \times^G X). \tag{3}$$

Therefore, for each $i \in \mathbb{Z}$,

$$\{(H_{i+\widetilde{n}-g}(E^n G \times^G X), (\iota_n)^*) \mid n > 0\} \tag{4}$$

forms an inverse system. We define the i-th *equivariant Borel-Moore homology* group to be the inverse limit

$$H_i^G(M) := \varprojlim_n H_{i+\widetilde{n}-g}(E^n G \times^G X)$$

of the inverse system (4). This inverse system stabilizes for $\widetilde{n} \geq x - i$ (i.e. the maps (3) become isomorphisms), so for $i \geq x - \widetilde{n}$ we can identify

$$H_i^G(M) = H_{i+\widetilde{n}-g}(E^n G \times^G X).$$

Proposition 2.17 (Equivariant Poincaré duality)**.** *Let M be a smooth oriented manifold of real dimension m. Let X be a closed G-stable subset of M which has an open neighbourhood $U \subset M$ such that X is a proper deformation retract of U. Then there is a canonical isomorphism*

$$H_i^G(X) \cong H_G^{m-i}(M, M\backslash X). \tag{5}$$

In particular, we have

$$H_i^G(M) \cong H_G^{m-i}(M). \tag{6}$$

Proof. Choose n so that $i \geq x - \widetilde{n}$. Then $E^nG \times^G M$ is a manifold of real dimension $m + \widetilde{n} - g$. We can now apply the non-equivariant Poincaré duality isomorphism (1) and the fact that

$$(E^nG \times^G M)/(E^nG \times^G X) = E^n \times^G (M\backslash X)$$

to calculate

$$\begin{aligned} H_i^G(X) &:= H_{i+\widetilde{n}-g}(E^nG \times^G X) \\ &\cong H^{m+\widetilde{n}-g-(i+\widetilde{n}-g)}(E^nG \times^G M, (E^nG \times^G M)/(E^nG \times^G X)) \\ &= H^{m-i}(E^nG \times^G M, E^n \times^G (M\backslash X)) \\ &= H_G^{m-i}(M, M\backslash X). \end{aligned}$$

□

Remark 2.18. We can now compare the different notations we use for the various homology and cohomology groups. We use $H_*(-), H_*^G(-)$ to denote non-equivariant, resp. equivariant, Borel-Moore homology. On the other hand $H^*(-), H_G^*(-)$ denote singular, resp. equivariant, cohomology. We use H_*^{sing} to denote singular homology.

2.5 S_G-action on cohomology and BM homology

We now show that equivariant homology and cohomology are naturally endowed with an action of the cohomology of a point. Let M be smooth oriented manifold of real dimension m endowed with an action of a Lie group G. Consider the map

$$M \to \{pt\}.$$

After taking homotopy quotients it becomes the projection

$$p_M : EG \times^G M \to BG$$

onto the first factor. It is a fibration with fibre M. The map p_M induces a homomorphism of cohomology rings

$$p_M^* : H^*(BG) \to H^*(EG \times^G M)$$

or, equivalently, a homomorphism of G-equivariant cohomology rings

$$p_M^* : H_G^*(\{pt\}) \to H_G^*(M).$$

Hence $H_G^*(M)$ is an algebra over the equivariant cohomology ring of a point $H_G^*(\{pt\})$. More explicitly, we have the following action map

$$H_G^i(\{pt\}) \times H_G^k(M) \to H_G^{k+i}(M) \tag{7}$$

$$(a, b) \mapsto p_M^*(a) \cup b. \tag{8}$$

By applying the Poincaré duality isomorphism (6) to $H_k^G(M)$ and $H_{k+i}^G(M)$ we get an action of $H_G^*(\{pt\})$ on $H_*^G(M)$:

$$H_G^i(\{pt\}) \times H_{m-k}^G(M) \to H_{m-k-i}^G(M). \tag{9}$$

2.6 $H_G^*(M)$-action on $H_*^G(M)$

Equivariant cohomology acts on equivariant Borel-Moore homology. This action arises as follows. We have the cup product on cohomology

$$\cup : H_G^i(M) \times H_G^j(M) \to H_G^{i+j}(M). \tag{10}$$

By applying the Poincaré duality isomorphism (6), we get

$$H^G_{m-j}(M) = H^j_G(M), \qquad H^G_{m-i-j}(M) \cong H^{i+j}_G(M).$$

Setting $k = m - j$, the cup product in (10) gives rise, by means of the identifications above, to the following action map

$$\cdot : H_G^i(M) \times H_k^G(M) \to H_{k-i}^G(M) \tag{11}$$

$$(a, b) \mapsto a \cdot b \quad . \tag{12}$$

3 Quivers

The purpose of this chaper is to introduce notations for quivers, their representations and related objects. We will use these notations throughout the thesis so it's vital that the reader becomes familiar with them.

3.1 Quivers and dimension vectors

By a *quiver* we mean a quadruple $\mathbf{\Gamma} = (\mathbf{I}, \mathbf{H}, \mathbf{s}, \mathbf{t})$, where $\mathbf{I}$ is a set of *vertices*, $\mathbf{H}$ is a set of *arrows*, $\mathbf{s} : \mathbf{H} \to \mathbf{I}$ is a *source function*, i.e., it associates to each arrow h its source $\mathbf{s}(h)$ and $\mathbf{t} : \mathbf{H} \to \mathbf{I}$ is a *target function*, i.e., it associates to each arrow h its target $\mathbf{t}(h)$.

For each $i, j \in \mathbf{I}$, we set

$$\mathbf{H}_{i,j} := \{h \in \mathbf{H} \mid \mathbf{s}(h) = i, \mathbf{t}(h) = j\}, \qquad h_{i,j} = |\mathbf{H}_{i,j}|$$

$$(i, j) = -(h_{i,j} + h_{j,i}) \quad \text{if} \quad i \neq j, \qquad (i, i) = 2.$$

A *dimension vector* for a quiver $\mathbf{\Gamma}$ is a function

$$\begin{aligned} \underline{\mathbf{d}} : \mathbf{I} &\to \mathbb{N} = \{0, 1, 2, ...\} \\ i &\mapsto \mathbf{d}_i. \end{aligned}$$

We can also view it as an $|\mathbf{I}|$-tuple $\underline{\mathbf{d}} = (\mathbf{d}_i)_{i\in\mathbf{I}}$ or an element $\underline{\mathbf{d}} = \sum_{i\in I} \mathbf{d}_i i \in \mathbb{N}\mathbf{I}$ of the semigroup $\mathbb{N}\mathbf{I}$. We call

$$\mathbf{d} = |\underline{\mathbf{d}}| = \sum_{i\in I} \mathbf{d}_i \in \mathbb{N}$$

the *cardinality* of the dimension vector $\underline{\mathbf{d}}$. To a pair $(\mathbf{\Gamma}, \underline{\mathbf{d}})$ we associate a complex $\mathbf{I}$-graded $\mathbf{d}$-dimensional vector space

$$\mathbf{V} = \bigoplus_{i\in\mathbf{I}} \mathbf{V}_i \qquad \text{such that} \qquad \dim \mathbf{V}_i = \mathbf{d}_i.$$

The fact that V has $\mathbf{I}$-graded dimension $\underline{\mathbf{d}}$ can also be written more compactly as

$$\operatorname{grdim} V = \underline{\mathbf{d}}.$$

By the dimension of a vector space we will always mean its dimension as a *complex* vector space. On the other hand, when we talk about the dimension of a variety or a manifold, we will mean its *real* dimension, unless otherwise indicated.

3.2 Representations of quivers

To a triple $(\mathbf{\Gamma}, \underline{\mathbf{d}}, \mathbf{V})$ we can associate a complex vector space

$$\operatorname{Rep}_{\underline{\mathbf{d}}} := \bigoplus_{h\in\mathbf{H}} \operatorname{Hom}_{\mathbb{C}}(\mathbf{V}_{\mathbf{s}(h)}, \mathbf{V}_{\mathbf{t}(h)}).$$

If $\mathbf{H} = \varnothing$, we set $\operatorname{Rep}_{\underline{\mathbf{d}}} = \{0\}$, the trivial vector space. Elements of $\operatorname{Rep}_{\underline{\mathbf{d}}}$, which we will usually denote as

$$\rho = (\rho_h)_{h\in\mathbf{H}} = (\rho_h),$$

are called *representations* of the quiver $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. To the triple $(\mathbf{\Gamma}, \underline{\mathbf{d}}, \mathbf{V})$ we can also associate the following complex algebraic groups

$$G_{\mathbf{d}_i} := \mathrm{GL}(\mathbf{V}_i),$$

$$G_{\underline{\mathbf{d}}} := \prod_{i \in \mathbf{I}} \mathrm{GL}(\mathbf{V}_i) = \prod_{i \in \mathbf{I}} G_{\mathbf{d}_i}.$$

Elements of $G_{\underline{\mathbf{d}}}$ will usually be denoted as

$$g = (g_i)_{i \in \mathbf{I}} = (g_i).$$

The natural action of $G_{\underline{\mathbf{d}}}$ on $\mathbf{V}$ induces a "simultaneous conjugation" action of $G_{\underline{\mathbf{d}}}$ on the vector space $\mathrm{Rep}_{\underline{\mathbf{d}}}$. This action admits the following explicit description. Let $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ and $g \in G_{\underline{\mathbf{d}}}$. Then

$$g.\rho = (g_i).(\rho_h) = (g_{\mathbf{t}(h)} \rho_h g_{\mathbf{s}(h)}^{-1}).$$

We also recall that the quotient $\mathrm{Rep}_{\underline{\mathbf{d}}}/G_{\underline{\mathbf{d}}}$ parametrizes the isomorphism classes of representations of the quiver Γ with dimension vector $\underline{\mathbf{d}}$, i.e., there is a bijective correspondence between elements of $\mathrm{Rep}_{\underline{\mathbf{d}}}/G_{\underline{\mathbf{d}}}$ and such classes.

We also set

$$\mathbb{G}_{\mathbf{d}} := \mathrm{GL}(\mathbf{V}).$$

The notation reflects the fact that $\mathbb{G}_{\mathbf{d}}$ does not depend on the dimension vector $\underline{\mathbf{d}}$, but only its cardinality $\mathbf{d}$. We now present some examples to illustrate the notation we have just introduced.

Example 3.1. Let $\mathbf{\Gamma}$ be the quiver A_1 with one vertex i and no arrows. Let $\underline{\mathbf{d}} = ni$. Since $\mathbf{H} = \varnothing$, $\mathrm{Rep}_{\underline{\mathbf{d}}} = \{0\}$ is the trivial vector space. Moreover, $G_{\underline{\mathbf{d}}} \cong \mathrm{GL}(n, \mathbb{C}) \cong \mathbb{G}_{\mathbf{d}}$ is the full general linear group.

Example 3.2. Let $\mathbf{\Gamma}$ be the equioriented quiver A_n

$$i_1 \to i_2 \to ... \to i_n$$

with dimension vector $\underline{\mathbf{d}} = i_1 + i_2 + ... + i_n$. Then

$$\mathrm{Rep}_{\underline{\mathbf{d}}} = \bigoplus_{h \in \mathbf{H}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong (\mathbb{C})^{n-1}.$$

Moreover, $G_{\mathbf{d}_i} \cong \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^\times$ for each $i \in \mathbf{I}$ and so $G_{\underline{\mathbf{d}}} = (\mathbb{C}^\times)^n$ is a torus. We also have $\mathbb{G}_{\mathbf{d}} \cong \mathrm{GL}(n, \mathbb{C})$.

Example 3.3. Consider the quiver D_4

$$\begin{tikzcd} & & i_3 \\ i_1 \arrow[r] & i_2 \arrow[ur] \arrow[dr] & \\ & & i_4 \end{tikzcd}$$

with dimension vector $\underline{\mathbf{d}} = \underline{\mathbf{d}} = i_1 + 2i_2 + 2i_3 + i_4$. We have

$$\mathrm{Rep}_{\underline{\mathbf{d}}} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_1, \mathbf{V}_2) \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_2, \mathbf{V}_3) \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_2, \mathbf{V}_4) \cong \mathbb{C}^2 \oplus \mathbb{C}^4 \oplus \mathbb{C}^2 \cong \mathbb{C}^8,$$

$G_{\underline{\mathbf{d}}} \cong (\mathbb{C}^\times)^2 \times (\mathrm{GL}(2, \mathbb{C}))^2$ and $\mathbb{G}_{\mathbf{d}} \cong \mathrm{GL}(6, \mathbb{C})$.

3.3 Compositions of a dimension vector

To a quiver and a dimension vector we are going to associate a flag variety and a Steinberg variety. The connected components of this flag variety correspond to certain sequences of vertices of the quiver. These sequences can also be regarded as compositions of the chosen dimension vector. In this section we define the notion of a composition and explain how it's related to quiver-graded flags.

Definition 3.4. Let $d \in \mathbb{N}_{>0}$. Let

$$\overline{y} = (y^1, ..., y^k)$$

be a sequence such that

$$y^1, ..., y^k \in \mathbb{N}_{>0}, \qquad \sum_{l=1}^{k} y^l = d.$$

We call $\overline{y}$ a *composition of the natural number* d and k the *length* of the composition $\overline{y}$. We set

$$\mathrm{Comp}_d := \{\overline{y} \mid \overline{y} \text{ is a composition of } d\}.$$

Remark 3.5. Compositions of d correspond to the different types of partial flags in a d-dimensional vector space. More specifically, they describe the dimensions of the quotients of the successive subspaces in a flag. In particular, the composition $(1, ..., 1)$ of length d corresponds to the complete flag type. This will be explained more thoroughly in Section 4.1. $\triangle$

We now generalize this definition to the quiver-graded setting.

Definition 3.6. Let $\mathbf{\Gamma}$ be a quiver with finitely many vertices and let $\underline{\mathbf{d}}$ be a dimension vector.
(i) Let

$$\overline{y} = (\underline{y}^1, ..., \underline{y}^k)$$

be a sequence such that

$$\underline{y}^1 = (y_i^1)_{i\in\mathbf{I}},\ ...,\ \underline{y}^k = (y_i^k)_{i\in\mathbf{I}} \in \mathbb{N}\mathbf{I}, \qquad \forall\ 1 \le l \le k\ \exists\ i \in \mathbf{I} \text{ with } y_i^l \neq 0 \qquad \text{and} \qquad \sum_{l=1}^{k} \underline{y}^l = \underline{\mathbf{d}}.$$

We call $\overline{y}$ a *composition of the dimension vector* $\underline{\mathbf{d}}$ and k the *length* of the composition $\overline{y}$. We set

$$\mathrm{Comp}_{\underline{\mathbf{d}}} := \{\overline{y} \mid \overline{y} \text{ is a composition of } \underline{\mathbf{d}}\}.$$

(ii) A composition $\overline{y}$ is called *multiplicity-free* if for all $1 \le l \le k$ and $i \in \mathbf{I}$ we have $y_i^l = 0$ or 1.
(iii) A composition $\overline{y}$ is called *strictly multiplicity-free* if for all $1 \le l \le k$, there exists a unique $i \in \mathbf{I}$ such that $y_i^l = 1$ and $y_j^l = 0$ for $j \neq i$. Note that a strictly multiplicity-free composition is just a sequence $(i_1, ..., i_{\mathbf{d}})$ of vertices of $\mathbf{\Gamma}$ such that $\sum_{l=1}^{d} i_l = \underline{\mathbf{d}}$. Strictly multiplicity-free compositions are therefore precisely the compositions of length $\mathbf{d}$. We let

$$Y_{\underline{\mathbf{d}}} := \{\overline{y} \in \mathrm{Comp}_{\underline{\mathbf{d}}} \mid \overline{y} \text{ is strictly multiplicity-free}\}$$

denote the set of all strictly multiplicity-free compositions of $\underline{\mathbf{d}}$. Note that

$$|Y_{\underline{\mathbf{d}}}| = \frac{\mathbf{d}!}{\prod_{i\in\mathbf{I}} \mathbf{d}_i!}.$$

(iv) We call a composition $\overline{y}$ *simple* if for all $1 \le l \le k$, $\underline{y}^l = y_i^l i$ for some vertex $i \in \mathbf{I}$, or, equivalently, if for all $1 \le l \le k$, there exists a unique $i \in \mathbf{I}$ with $y_i^l \neq 0$. $\triangle$

Remark 3.7. (i) A composition is strictly multiplicity-free if and only if it is both simple and multiplicity-free.
(ii) Compositions of $\underline{\mathbf{d}}$ correspond to the different types of quiver-graded partial flags in the $\mathbf{I}$-graded vector space $\mathbf{V}$. They describe the *graded* dimensions of the quotients of the successive graded subspaces in a flag. Strictly multiplicity-free compositions correspond to the different types of complete flags. This will be explained in more detail in Section 4.1.
(iii) Let $i \in \mathbf{I}$. If $\mathbf{d}_i \neq 0$ then the sequence

$$(y_i^1, ..., y_i^k)$$

becomes, after deleting the y_i^l equal to zero, a composition of the natural number $\mathbf{d}_i$. This sequence describes the type of a flag restricted to the graded component $\mathbf{V}_i$.
(iv) The sequence

$$\left(\sum_{i\in\mathbf{I}} y_i^1, \ldots, \sum_{i\in\mathbf{I}} y_i^k\right)$$

is a composition of the natural number $\mathbf{d}$. It describes the type of a quiver-graded flag considered as an ungraded flag.

Example 3.8. Consider the quiver A_2

$$i \to j$$

with dimension vector $\underline{\mathbf{d}} = 2i + 2j$. Then

$$Y_{\underline{\mathbf{d}}} = \{(i,i,j,j),(i,j,j,i),(i,j,i,j),(j,j,i,i),(j,i,j,i),(j,i,i,j)\}$$

are all the six strictly multiplicity-free compositions. The remaining seven (not strictly) multiplicity-free compositions are

$$(i+j,i,j),(i+j,j,i),(i,i+j,j),(j,i+j,i),(i,j,i+j),(j,i,i+j),(i+j,i+j).$$

The eight simple but not strictly multiplicity-free compositions are

$$(2i,j,j),(j,2i,j),(j,j,2i),(2j,i,i),(i,2j,i),(i,i,2j),(2i,2j),(2j,2i).$$

There are five remaining compositions

$$(2i+j,j),(j,2i+j),(2j+i,i),(i,2j+i),(2i+2j)$$

which are neither multiplicity-free nor simple. In total we get twenty-six compositions.

4 Quiver flag varieties

Let us fix once and for all a quiver $\mathbf{\Gamma} = (\mathbf{I}, \mathbf{H}, \mathbf{s}, \mathbf{t})$. All the notations we introduce later will refer to this choice of quiver. Let us assume that $\mathbf{\Gamma}$ is *non-empty*, i.e., $\mathbf{I} \neq \varnothing$, *finite*, i.e., $|\mathbf{I}|, |\mathbf{H}| < \infty$, and *without loops*, i.e., there is no $h \in \mathbf{H}$ such that $\mathbf{s}(h) = \mathbf{t}(h)$. However, multiple edges and cycles (of length at least 2) are allowed.

We also fix once and for all a dimension vector $\underline{\mathbf{d}} = (\mathbf{d}_i)_{i\in\mathbf{I}}$ for the quiver $\mathbf{\Gamma}$ and a complex vector space

$$\mathbf{V} = \bigoplus_{i\in\mathbf{I}} \mathbf{V}_i \qquad \text{such that} \qquad \dim \mathbf{V}_i = \mathbf{d}_i.$$

4.1 Flag varieties

4.1.1 Definitions

Definition 4.1. Let V be a complex vector space of dimension d and let $\overline{y} = (y^1, ..., y^k) \in \mathrm{Comp}_d$ be a composition of d.
(i) An *ordinary* or *ungraded flag F of type $\overline{y}$* in V is a sequence

$$\{0\} = V^0 \subset V^1 \subset ... \subset V^{k-1} \subset V^k = V$$

of (ungraded) linear subspaces of V such that for each $1 \leq l \leq k$, V^l/V^{l-1} is an (ungraded) vector space with

$$\dim V^l/V^{l-1} = y^l.$$

We call an ordinary flag F *complete* or *of complete type* if it is a flag of type $(1, ..., 1)$. Otherwise we call F *partial* or *of partial type.*
(ii) The *ordinary* or *ungraded flag variety of type $\overline{y}$* is the variety of all ordinary flags F of type $\overline{y}$ in $\mathbf{V}$. We let $\mathcal{F}(V)$ denote the ordinary flag variety of all complete flags in V, i.e., the ordinary flag variety of type $(1, ..., 1)$. Since we are primarily interested in ordinary flag varieties of complete type, we do not introduce special notation for ordinary flag varieties of arbitrary type. $\triangle$

Definition 4.2. Let $\overline{y} = (\underline{y}^1, ..., \underline{y}^k) \in \mathrm{Comp}_{\underline{\mathbf{d}}}$ be a composition of $\underline{\mathbf{d}}$.
(i) A *quiver-graded flag F of type $\overline{y}$* in $\mathbf{V}$ is a sequence

$$\{0\} = V^0 \subset V^1 \subset ... \subset V^{k-1} \subset V^k = \mathbf{V}$$

of $\mathbf{I}$-graded linear subspaces of $\mathbf{V}$ such that for each $1 \leq l \leq k$,

$$V^l/V^{l-1}$$

is an $\mathbf{I}$-graded vector space with

$$\mathrm{grdim}\, V^l/V^{l-1} = \underline{y}^l.$$

(ii) A *quiver flag variety of type $\overline{y}$*, denoted $\mathcal{F}_{\overline{y}}$, is the variety of all quiver graded flags F of type $\overline{y}$ in $\mathbf{V}$. $\triangle$

The natural action of $G_{\underline{\mathbf{d}}}$ on $\mathbf{V}$ induces a transitive action on flags which preserves the type of a flag. Hence $G_{\underline{\mathbf{d}}}$ acts transitively on each $\mathcal{F}_{\overline{y}}$. The isotropy group of any flag of type $\overline{y}$ is a parabolic subgroup of $G_{\underline{\mathbf{d}}}$, so $\mathcal{F}_{\overline{y}}$ is a smooth projective variety. This is explained in more detail in Section 4.3.2. We now define one of the central objects of study in this thesis.

Definition 4.3. We define the *quiver flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$* to be the finite disjoint union of the quiver flag varieties of types corresponding to all possible strictly multiplicity-free compositions of $\underline{\mathbf{d}}$, i.e.:

$$\mathcal{F}_{\underline{\mathbf{d}}} = \coprod_{\overline{y}\in Y_{\underline{\mathbf{d}}}} \mathcal{F}_{\overline{y}}.$$

We would now like to study the relationship between ordinary and quiver-graded flags. We can obtain ordinary flags from quiver-graded flags in two ways: by restriction to a graded component or by forgetting the grading.

4.1.2 Restrictions of flags

Consider a quiver-graded flag F

$$\{0\} = V^0 \subset V^1 \subset ... \subset V^{k-1} \subset V^k = \mathbf{V} \tag{13}$$

of type $\overline{y} = (\underline{y}^1, ..., \underline{y}^k)$ in $\mathbf{V}$. Since each subspace V^l is $\mathbf{I}$-graded, we have, for each $1 \leq l \leq k$, decompositions

$$V^l = \bigoplus_{i \in \mathbf{I}} V_i^l.$$

Definition 4.4. Let $i \in \mathbf{I}$. We define $F|_i$ to be the *restriction* of the flag F to the graded component $\mathbf{V}_i \subset \mathbf{V}$, i.e.,

$$F|_i = F \cap \mathbf{V}_i.$$

Explicitly, $F|_i$ is the sequence

$$\{0\} = V_i^0 \subseteq V_i^1 \subseteq ... \subseteq V_i^{k-1} \subseteq V_i^k = \mathbf{V}_i \tag{14}$$

of linear subspaces of $\mathbf{V}_i$. △

Although some of the inclusions in (14) may not be strict, we can always contract the sequence by deleting repeated occurrences of the same subspace to obtain a shorter sequence with strict inclusions. After such a contraction $F|_i$ is an ordinary, i.e., ungraded flag in the ungraded vector space $\mathbf{V}_i$. In fact, since $\dim V_i^l / V_i^{l-1} = y_i^l$, we have:

Fact 4.5. $F|_i$ is an ordinary flag in $\mathbf{V}_i$ of type $\overline{y}_i := (y_i^1, ..., y_i^k)$ (with $y_i^l = 0$ deleted). △

Moreover, $\overline{y}$ is a multiplicity-free composition if and only if each $y_i^l = 0$ or 1. But the latter condition is equivalent to each restriction $F|_i$ being a complete flag. Hence:

Fact 4.6. For each $i \in \mathbf{I}$, $F|_i$ is a complete ordinary flag $\iff$ the composition $\overline{y}$ is multiplicity-free. △

4.1.3 Forgetting the grading

We again consider the quiver-graded flag F from (13) of type $\overline{y} = (\underline{y}^1, ..., \underline{y}^k)$ in $\mathbf{V}$.

Definition 4.7. We define $\hat{F}$ to be the flag F with the I-grading forgotten. We call $\hat{F}$ the *ungraded flag associated to* F. △

For each $1 \leq l \leq k$, we have $\dim V^l / V^{l-1} = \sum_{i \in \mathbf{I}} y_i^l$. Hence:

Fact 4.8. $\hat{F}$ is an ordinary flag in the ungraded (or, more precisely, with the grading forgotten) vector space $\mathbf{V}$ of type

$$\hat{y} := \left(\sum_{i \in \mathbf{I}} y_i^1, ..., \sum_{i \in \mathbf{I}} y_i^k \right).$$

Moreover, it is immediate that:

Fact 4.9. $\hat{F}$ is a complete ordinary flag $\iff$ the composition $\overline{y}$ is strictly multiplicity-free, i.e., $\overline{y} \in Y_{\underline{\mathbf{d}}}$. △

We therefore introduce the following definition.

Definition 4.10. We call a quiver-graded flag F in $\mathbf{V}$ of type $\overline{y}$ *complete* if the ordinary flag $\hat{F}$ is complete, or, equivalently, if $\overline{y} \in Y_{\underline{\mathbf{d}}}$. △

4.1.4 The relationship between ordinary and quiver flag varieties

We have investigated the relationship between ordinary and quiver-graded flags. We would now like to examine how ordinary flag varieties and quiver flag varieties are related. We first observe that restriction of flags gives rise to an isomorphism between a quiver flag variety and a product of ordinary flag varieties.

Corollary 4.11. *If $\overline{y}$ is a multiplicity-free composition, then we have an isomorphism of $G_{\underline{\mathbf{d}}}$-varieties*

$$\mathcal{F}_{\overline{y}} \xrightarrow{\cong} \prod_{i\in\mathbf{I}} \mathcal{F}(\mathbf{V}_i), \qquad F \mapsto (F|_i)_{i\in\mathbf{I}}. \tag{15}$$

Proof. This follows directly from Fact 4.6. □

Let $Z(G_{\underline{\mathbf{d}}})$ denote the centre of $G_{\underline{\mathbf{d}}}$. Note that

$$Z(G_{\underline{\mathbf{d}}}) = \prod_{i\in\mathbf{I}} Z(\mathrm{GL}(\mathbf{V}_i)) = \prod_{i\in\mathbf{I}} \mathbb{C}^\times.$$

Forgetting the grading yields an isomorphism between $\mathcal{F}_{\underline{\mathbf{d}}}$ and the closed subvariety of $\mathcal{F}(\mathbf{V})$ consisting of all ordinary flags fixed under the action of $Z(G_{\underline{\mathbf{d}}})$.

Lemma 4.12. *We have an isomorphism of $G_{\underline{\mathbf{d}}}$-varieties*

$$\mathcal{F}_{\underline{\mathbf{d}}} \xrightarrow{\cong} (\mathcal{F}(\mathbf{V}))^{Z(G_{\underline{\mathbf{d}}})}, \qquad F \mapsto \hat{F}.$$

Proof. Straightforward. □

4.2 Torus fixed points in $\mathcal{F}_{\underline{\mathbf{d}}}$

4.2.1 Choice of basis

Definition 4.13. (i) For each $i \in \mathbf{I}$, let us choose an ordered basis $(e_i^1, ..., e_i^{\mathbf{d}_i})$ of $\mathbf{V}_i$. We also fix an ordering $(i_1, ..., i_{|\mathbf{I}|})$ on the vertices in $\mathbf{I}$. We set $\mathbf{d}_k := \mathbf{d}_{i_k}$. This gives us an ordered basis

$$(e_{i_1}^1, ..., e_{i_1}^{\mathbf{d}_1}, ..., e_{i_{|\mathbf{I}|}}^1, ..., e_{i_{|\mathbf{I}|}}^{\mathbf{d}_{|\mathbf{I}|}}) \tag{16}$$

of $\mathbf{V}$. We will refer to this basis as the *chosen basis* and to the vectors forming this basis as the *chosen basis vectors.*

(ii) For each $1 \leq j \leq \mathbf{d}$, let e_j denote the j-th element of our chosen ordered basis. Using this notation, we can write this basis as

$$(e_1, ..., e_{\mathbf{d}}). \tag{17}$$

Remark 4.14. We use the symbols $i_1, ..., i_{|\mathbf{I}|}$ as constants, or names of particular vertices, rather than as variables. Similarly, the symbols $\mathbf{d}_1, ..., \mathbf{d}_{|\mathbf{I}|}$ are constants denoting specific natural numbers and $e_1, ..., e_{\mathbf{d}}$ are constants denoting specific basis elements. On the other hand, we use the symbol i as a variable ranging over $\mathbf{I}$.

4.2.2 $\mathbb{G}_{\mathbf{d}}$ and $G_{\underline{\mathbf{d}}}$ as matrix groups

We can explicitly identify $\mathbb{G}_{\mathbf{d}} = \mathrm{GL}(\mathbf{V})$ with a matrix group:

$$\mathbb{G}_{\mathbf{d}} = \mathrm{GL}(\mathbf{V}) \xrightarrow{\cong} \mathrm{GL}(\mathbf{d}, \mathbb{C}),$$

sending a linear transformation to the matrix representing it in our chosen ordered basis. This isomorphism restricts to

$$G_{\underline{\mathbf{d}}} = \prod_{i \in \mathbf{I}} \mathrm{GL}(\mathbf{V}_i) \xrightarrow{\cong} \prod_{k=1}^{|\mathbf{I}|} \mathrm{GL}(\mathbf{d}_k, \mathbb{C}),$$

allowing us to explicitly identify $G_{\underline{\mathbf{d}}}$ with a product of matrix groups. Observe that there is a canonical embedding $G_{\underline{\mathbf{d}}} \hookrightarrow \mathbb{G}_{\mathbf{d}}$, so we can regard $G_{\underline{\mathbf{d}}}$ as a subgroup of $\mathbb{G}_{\mathbf{d}}$.

4.2.3 Weyl groups $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$

All the definitions we introduce here relate to the choice of basis and the identification with matrix groups from the previous two sections.

Definition 4.15. We let $T_{\underline{\mathbf{d}}}$ denote the subgroup of $G_{\underline{\mathbf{d}}}$ consisting of the diagonal matrices. It is a maximal torus in $G_{\underline{\mathbf{d}}}$ as well as in $\mathbb{G}_{\mathbf{d}}$. If $t \in T_{\underline{\mathbf{d}}}$, we will write $t = (t_i^j)$ or $t = (t_j)$, in accordance with our two notations (16), (17) for the chosen basis. We let $B_{\underline{\mathbf{d}}}$ denote the subgroup of $G_{\underline{\mathbf{d}}}$ consisting of the upper triangular matrices. It is a Borel subgroup of $G_{\underline{\mathbf{d}}}$. We further let $\mathbb{B}_{\mathbf{d}}$ denote the subgroup of $\mathbb{G}_{\mathbf{d}}$ consisting of the upper triangular matrices. It is a Borel subgroup of $\mathbb{G}_{\mathbf{d}}$. Note that tori and Borel subgroups are unique up to conjugacy. △

Definition 4.16. We let

$$\mathbb{W}_{\mathbf{d}} := N_{\mathbb{G}_{\mathbf{d}}}(T_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}} \quad \text{and} \quad W_{\underline{\mathbf{d}}} := N_{G_{\underline{\mathbf{d}}}}(T_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}}$$

denote the Weyl groups of the pairs $(\mathbb{G}_{\mathbf{d}}, T_{\underline{\mathbf{d}}})$ and $(G_{\underline{\mathbf{d}}}, T_{\underline{\mathbf{d}}})$, respectively. △

Remark 4.17. The choice of notation reflects the fact that $G_{\underline{\mathbf{d}}}, B_{\underline{\mathbf{d}}}, W_{\underline{\mathbf{d}}}$ depend on the dimension vector $\underline{\mathbf{d}}$ while $\mathbb{G}_{\mathbf{d}}, \mathbb{B}_{\mathbf{d}}, \mathbb{W}_{\mathbf{d}}$ depend only on its cardinality $\mathbf{d}$. Even though our chosen maximal torus $T_{\underline{\mathbf{d}}}$ depends only on $\mathbf{d}$, we use the notation $T_{\underline{\mathbf{d}}}$ nonetheless because the choice of $T_{\underline{\mathbf{d}}}$ is unique up to conjugation by elements of $G_{\underline{\mathbf{d}}}$. △

Since $G_{\underline{\mathbf{d}}} \subset \mathbb{G}_{\mathbf{d}}$ and so $N_{G_{\underline{\mathbf{d}}}}(T_{\underline{\mathbf{d}}}) \subset N_{\mathbb{G}_{\mathbf{d}}}(T_{\underline{\mathbf{d}}})$, there is a canonical embedding $W_{\underline{\mathbf{d}}} \subset \mathbb{W}_{\mathbf{d}}$. Recall that $N_{\mathbb{G}_{\mathbf{d}}}(T_{\underline{\mathbf{d}}})$ consists of the monomial matrices in $\mathbb{G}_{\mathbf{d}}$ and $N_{G_{\underline{\mathbf{d}}}}(T_{\underline{\mathbf{d}}})$ consists of the monomial matrices in $G_{\underline{\mathbf{d}}}$. We can choose the permutation matrices in $\mathbb{G}_{\mathbf{d}}$ resp. $G_{\underline{\mathbf{d}}}$ as the coset representatives. Therefore, we can also regard $\mathbb{W}_{\mathbf{d}}$ resp. $W_{\underline{\mathbf{d}}}$ as a subgroup of $\mathbb{G}_{\mathbf{d}}$ resp. $G_{\underline{\mathbf{d}}}$.

It follows that $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$ act naturally on the basis vectors in (16) by permutation. Recall that e_j denotes the j-th element in the ordered basis (16). We have a canonical isomorphism

$$\mathbb{W}_{\mathbf{d}} \xrightarrow{\cong} \mathfrak{S}_{\mathbf{d}}, \qquad w \mapsto \widetilde{w} \tag{18}$$

such that, for each $1 \leq j \leq \mathbf{d}$ and $w \in \mathbb{W}_{\mathbf{d}}$, $w(e_j) = e_{\widetilde{w}(j)}$. This isomorphism restricts to the isomorphism

$$W_{\underline{\mathbf{d}}} \xrightarrow{\cong} \prod_{k=1}^{|\mathbf{I}|} \mathfrak{S}_{\mathbf{d}_k}.$$

In the sequel we will freely identify w with $\widetilde{w}$ under the isomorphism (18) and forget the tilde in the notation. Note that $W_{\underline{\mathbf{d}}}$ consists of those permutations of the chosen basis vectors e_i^j which fix the lower index, i.e., those permutations that, for each $i \in \mathbf{I}$, send the chosen basis vectors in $\mathbf{V}_i$ to chosen basis vectors in $\mathbf{V}_i$.

4.2.4 $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$ as Coxeter systems

Definition 4.18. For $1 \leq l \leq \mathbf{d} - 1$, let $s_l \in \mathbb{W}_{\mathbf{d}}$ be the simple transposition swapping e_l and e_{l+1}. Let $\Pi = \{s_1, ..., s_{\mathbf{d}-1}\}$ denote the set of all simple transpositions in $\mathbb{W}_{\mathbf{d}}$. Moreover, let

$$\Pi_{\underline{\mathbf{d}}} = \Pi \cap W_{\underline{\mathbf{d}}} = \Pi \backslash \{s_{\mathbf{d}_1}, s_{\mathbf{d}_2}, ..., s_{\mathbf{d}_{|\mathbf{I}|-1}}\}$$

be the set of simple transpositions in $W_{\underline{\mathbf{d}}}$. $\triangle$

The groups $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$ are generated by Π and $\Pi_{\underline{\mathbf{d}}}$, respectively. Moreover, $(\mathbb{W}_{\mathbf{d}}, \Pi)$ and $(W_{\underline{\mathbf{d}}}, \Pi_{\underline{\mathbf{d}}})$ are Coxeter systems. Let

$$l : \mathbb{W}_{\mathbf{d}} \to \mathbb{N}_0$$

be the associated length function, which assigns to each w the number of simple transpositions in some reduced decomposition of w (this is independent of the choice of reduced decomposition). Let w_0 denote the unique element of $\mathbb{W}_{\mathbf{d}}$ of maximal length and v_0 the unique element of $W_{\underline{\mathbf{d}}}$ of maximal length. We recall the following lemma.

Lemma 4.19. *(i) Each right coset $W_{\underline{\mathbf{d}}}w$ contains a unique element u of minimal length. Moreover, u is the unique element in the coset $W_{\underline{\mathbf{d}}}w$ such that, for each $v \in W_{\underline{\mathbf{d}}}$, we have $l(vu) = l(v) + l(u)$.*
(ii) The minimal length right coset representatives are precisely the $(\mathbf{d}_{i_1}, ..., \mathbf{d}_{i_{|\mathbf{I}|}})$-shuffles for the left permutation action of $W_{\underline{\mathbf{d}}}$.

We let $\mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ denote the set of minimal length representatives of the right cosets of $W_{\underline{\mathbf{d}}}$ in $\mathbb{W}_{\mathbf{d}}$. By the lemma, these are precisely the permutations which, applied to our chosen ordered basis in (16), yield another ordered basis which preserves the relative order of the chosen basis vectors in each $\mathbf{V}_i$. In terms of the $\mathbb{W}_{\mathbf{d}}$-action on the set of coordinate flags, which we explain in the next section, the minimal length right coset representatives are precisely those elements u which satisfy the condition that, for each $i \in \mathbf{I}$,

$$F_u|_i = F_e|_i.$$

4.2.5 The action of $\mathbb{W}_{\mathbf{d}}$ on $Y_{\underline{\mathbf{d}}}$

The group $\mathbb{W}_{\mathbf{d}}$, identified with the symmetric group $\mathfrak{S}_{\mathbf{d}}$, acts on $Y_{\underline{\mathbf{d}}}$ in the following way. We can regard a composition $\overline{y} \in Y_{\underline{\mathbf{d}}}$ as a map $\{1, 2, ..., \mathbf{d}\} \to \mathbf{I}$ sending l to $\underline{y}^l$. For each $w \in \mathbb{W}_{\mathbf{d}}$, we define

$$w(\overline{y}) = \overline{y} \circ w^{-1}.$$

We then have

$$w(\overline{y}_e) = \overline{y}_{w^{-1}} \quad \forall w \in \mathbb{W}_{\mathbf{d}},$$

where $\overline{y}_u$ is the composition of the coordinate flag F_u (see next section for the explanation of the notation).

4.2.6 Torus fixed points

Definition 4.20. (i) A flag of the form

$$< e_{j_1} > \subset < e_{j_1}, e_{j_2} > \subset ... \subset < e_{j_1}, e_{j_2}, ..., e_{j_{\mathbf{d}}} >= \mathbf{V} \ ,$$

where each e_{j_k} is a distinct chosen basis element, is called a *coordinate flag.*
(ii) We call the flag

$$< e_1 > \subset < e_1, e_2 > \subset ... \subset < e_1, e_2, ..., e_{\mathbf{d}} >= \mathbf{V}$$

the *standard coordinate flag.* We will denote it with F_e. It is a flag of type

$$(i_1, ..., i_1, i_2, ..., i_2, ..., i_{|\mathbf{I}|}, ..., i_{|\mathbf{I}|}),$$

where each i_j appears consecutively $\dim \mathbf{V}_{i_j}$-many times.
(iii) Moreover, we call $(i_1, ..., i_1, i_2, ..., i_2, ..., i_{|\mathbf{I}|}, ..., i_{|\mathbf{I}|})$ the *standard flag type* or the *standard composition* of the dimension vector $\underline{\mathbf{d}}$. We will denote it with $\overline{y}_e$. $\triangle$

We can also write a coordinate flag as

$$< e_{j_1} > \subset < e_{j_1} > \oplus < e_{j_2} > \subset ... \subset < e_{j_1} > \oplus < e_{j_2} > \oplus ... \oplus < e_{j_{\mathbf{d}}} > \quad . \tag{19}$$

Recall that $\mathbb{G}_{\mathbf{d}}$ acts on the flag variety $\mathcal{F}(\mathbf{V})$ and its subgroup $G_{\underline{\mathbf{d}}}$ acts on the quiver flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$. We now want to consider the restriction of the latter action to $T_{\underline{\mathbf{d}}}$ and determine its fixed points.

Lemma 4.21. *We have*

$$(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{\text{coordinate flags in } \mathcal{F}_{\underline{\mathbf{d}}}\}.$$

Proof. Since each $< e_j >$ is a $T_{\underline{\mathbf{d}}}$-submodule by definition, coordinate flags are fixed under the action of $T_{\underline{\mathbf{d}}}$. On the other hand, suppose that $F = (\varnothing \subset V^1 \subset ... \subset V^d = V)$ is not a coordinate flag. Let

$$< e_{j_1} > \subset < e_{j_1} > \oplus < e_{j_2} > \subset ... \subset < e_{j_1} > \oplus < e_{j_2} > \oplus ... \oplus < e_{j_n} >= V^n$$

be its longest initial segment consisting of subspaces spanned by the chosen basis vectors. Let $V^{n+1} = V^n \oplus < f >$. We can choose the vector $f = \sum_{k=1}^{\mathbf{d}} a_{j_k} e_{j_k}$ so that the coefficients $a_{j_1}, ..., a_{j_n}$ on the chosen basis vectors in V^n are 0. Moreover, since V^{n+1} is not spanned by the chosen basis vectors, there must be two distinct nonzero coefficients $a_{j'}, a_{j''}$ in the sum. We can therefore write $f = a_{j'} e_{j'} + a_{j''} e_{j''} + \text{rest}$. Now take $t = (t_j) \in T_{\underline{\mathbf{d}}}$ with $t_{j'} = 2$ and $t_j = 1$ if $j \neq j'$. Then $t.f = 2a_{j'} e_{j'} + a_{j''} e_{j''} + \text{rest} \notin V^{n+1}$. □

4.2.7 Bijection between $\mathbb{W}_{\mathbf{d}}$ and the torus fixed points

We have explained how the actions of $\mathbb{G}_{\mathbf{d}}$ and $G_{\underline{\mathbf{d}}}$ on the vector space V induce the permutation actions of the Weyl groups $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$ on the chosen basis vectors. The groups $\mathbb{W}_{\mathbf{d}}$ and $W_{\underline{\mathbf{d}}}$ also act by permutation on the set $\{< e_1 >, ..., < e_{\mathbf{d}} >\}$ of lines spanned by the chosen basis vectors. These actions induce, by (19), actions on the set of coordinate flags, which equals the set $(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}$ of torus fixed points. Note that the action of $W_{\underline{\mathbf{d}}}$ preserves the type of each coordinate flag.

Lemma 4.22. *The action of $\mathbb{W}_{\mathbf{d}}$ on $(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}$ is free and transitive. Hence there is a bijection*

$$\mathbb{W}_{\mathbf{d}} \to (\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}, \quad w \mapsto w(F_e).$$

In particular,

$$|(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}| = |\mathbb{W}_{\mathbf{d}}| = \mathbf{d}!.$$

Proof. Obvious. □

Definition 4.23. Recall that F_e denotes the standard coordinate flag. For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$F_w := w(F_e).$$

4.2.8 Bijection between $W_{\underline{\mathbf{d}}} \backslash \mathbb{W}_{\mathbf{d}}$ and $Y_{\underline{\mathbf{d}}}$

Recall that the standard coordinate flag F_e has type

$$\overline{y}_e = (i_1, ..., i_1, i_2, ..., i_2, ..., i_{|I|}, ..., i_{|I|}). \tag{20}$$

Let i_{j_k} denote the k-th element in the standard composition (20). Recall that, under the isomorphism from (18), $\mathbb{W}_{\mathbf{d}}$ can be identified with the symmetric group on $\mathbf{d}$ letters. For each $w \in \mathbb{W}_{\mathbf{d}}$, the coordinate flag F_w has type

$$\overline{y}_w := (i_{j_{w(1)}}, ..., i_{j_{w(\mathbf{d})}}).$$

Lemma 4.24. *We have*

$$w \in W_{\underline{\mathbf{d}}} \iff \overline{y}_w = \overline{y}_e.$$

Moreover, there is a bijection

$$W_{\underline{\mathbf{d}}} \backslash \mathbb{W}_{\mathbf{d}} \to Y_{\underline{\mathbf{d}}}, \quad W_{\underline{\mathbf{d}}} w \mapsto \overline{y}_w.$$

Proof. Obvious. □

To emphasize that $\overline{y}_w$ depends only on the coset $W_{\underline{\mathbf{d}}} w$, we will typically write $\overline{y}_{\overline{w}}$ instead. We will also typically take w to be the minimal coset representative.

Definition 4.25. (i) For each $\overline{y} \in Y_{\underline{\mathbf{d}}}$, let $W_{\overline{y}}$ denote the $W_{\underline{\mathbf{d}}}$-coset

$$W_{\overline{y}} := \{w \in W \mid \overline{y}_w = \overline{y}\}.$$

(ii) For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\mathcal{F}_w := \mathcal{F}_{\overline{y}_w}.$$

To emphasize dependence only on the coset $W_{\underline{\mathbf{d}}} w$, we will typically write $\mathcal{F}_{\overline{w}}$. We will also typically take w to be the minimal coset representative. △

We can describe the torus fixed points of the connected component $\mathcal{F}_{\overline{y}}$ of $\mathcal{F}_{\underline{\mathbf{d}}}$ as

$$(\mathcal{F}_{\overline{y}})^{T_{\underline{\mathbf{d}}}} = \{F_w \mid w \in W_{\overline{y}}\}.$$

Moreover,

$$(\mathcal{F}_w)^{T_{\underline{\mathbf{d}}}} = \{F_u \mid u \in W_{\underline{\mathbf{d}}} w\}.$$

4.3 Connections to Lie theory

4.3.1 Parabolic subgroups

Definition 4.26. For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$B_w := \mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(F_w) < G_{\underline{\mathbf{d}}}, \qquad \mathbb{B}_w := \mathrm{Stab}_{\mathbb{G}_{\mathbf{d}}}(F_w) < \mathbb{G}_{\mathbf{d}}.$$

We let

$$N_w := R_u(B_w)$$

be the unipotent radical of B_w. Let $s \in \Pi$ be a simple transposition such that $wsw^{-1} \in W_{\underline{\mathbf{d}}}$. We set

$$P_{w,ws} := (B_w wsw^{-1} B_w) \cup B_w,$$

and let

$$N_{w,ws} = R_u(P_{w,ws})$$

be the unipotent radical. △

Remark 4.27. (i) We have

$$B_e = B_{\underline{\mathbf{d}}}, \qquad B_w = wB_e w^{-1}, \qquad P_{w,ws} = wP_{e,s} w^{-1},$$

for $w \in W_{\underline{\mathbf{d}}}$ and

$$\mathbb{B}_e = \mathbb{B}_{\mathbf{d}}, \qquad \mathbb{B}_w = w\mathbb{B}_e w^{-1},$$

for $w \in \mathbb{W}_{\mathbf{d}}$.

(ii) Each B_w is a Borel subgroup of $G_{\underline{\mathbf{d}}}$ and each $\mathbb{B}_w$ is a Borel subgroup of $\mathbb{G}_{\mathbf{d}}$. Each $P_{w,ws}$ is a

parabolic subgroup of $G_{\underline{\mathbf{d}}}$ containing B_w.
(iii) The Borel subgroups of $\mathbb{G}_{\mathbf{d}}$ containing $T_{\underline{\mathbf{d}}}$ are classified by $\mathbb{W}_{\mathbf{d}}$, i.e., there is a bijection

$$\begin{array}{ccc} \mathbb{W}_{\mathbf{d}} & \longleftrightarrow & \{ \text{ Borel subgroups of } \mathbb{G}_{\mathbf{d}} \text{ containing } T_{\underline{\mathbf{d}}}\} \\ w & \mapsto & \mathbb{B}_w = w\mathbb{B}_e w^{-1}. \end{array}$$

(iv) Let $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$. Since u is a shuffle, B_e also stabilizes the coordinate flag F_u. Hence $B_e = B_u$. Moreover, if $w = vu \in \mathbb{W}_{\mathbf{d}}$, where $v \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, then $vB_{\underline{\mathbf{d}}}v^{-1}$ is the isotropy group of the coordinate flag F_w, i.e., $B_w = B_v$. Hence the Borel subgroups of $G_{\underline{\mathbf{d}}}$ containing the maximal torus $T_{\underline{\mathbf{d}}}$ are classified by $W_{\underline{\mathbf{d}}}$, i.e., there is a bijection

$$\begin{array}{ccc} W_{\underline{\mathbf{d}}} & \longleftrightarrow & \{ \text{ Borel subgroups of } G_{\underline{\mathbf{d}}} \text{ containing } T_{\underline{\mathbf{d}}}\} \\ v & \mapsto & B_v = vB_e v^{-1}. \end{array}$$

4.3.2 Quotients by a Borel subgroup

The group $\mathbb{G}_{\mathbf{d}}$ acts transitively on $\mathcal{F}(\mathbf{V})$. Moreover, for each $w \in \mathbb{W}_{\mathbf{d}}$, the isotropy group of the flag F_w is $\mathbb{B}_w$. Hence, for each $w \in \mathbb{W}_{\mathbf{d}}$, we have an isomorphism of $\mathbb{G}_{\mathbf{d}}$-varieties

$$\mathbb{G}_{\mathbf{d}}/\mathbb{B}_w \xrightarrow{\cong} \mathcal{F}(\mathbf{V}), \qquad g\mathbb{B}_w/\mathbb{B}_w \mapsto g.F_w.$$

Moreover, for each $v \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, we obtain in an analogous manner an isomorphism of $G_{\underline{\mathbf{d}}}$-varieties

$$G_{\underline{\mathbf{d}}}/B_v \xrightarrow{\cong} \mathcal{F}_{\overline{u}}, \qquad gB_v/B_v \mapsto g.F_{vu}.$$

In particular, for each $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ we have an isomorphism

$$G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \xrightarrow{\cong} \mathcal{F}_{\overline{u}}, \qquad gB_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \mapsto g.F_u.$$

This implies that $\mathcal{F}_{\underline{\mathbf{d}}}$ is isomorphic to the disjoint union of $|W_{\underline{\mathbf{d}}} \backslash \mathbb{W}_{\mathbf{d}}|$-many copies of $G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}$.

4.3.3 Lie algebras

Definition 4.28. For each $w \in \mathbb{W}_{\mathbf{d}}$ and $s \in \Pi$ such that $wsw^{-1} \in W_{\underline{\mathbf{d}}}$ we set

$$\mathfrak{g}_{\underline{\mathbf{d}}} := \mathrm{Lie}(G_{\underline{\mathbf{d}}}), \qquad \mathfrak{g} := \mathrm{Lie}(\mathbb{G}_{\mathbf{d}}),$$

$$\mathfrak{t}_{\underline{\mathbf{d}}} := \mathrm{Lie}(T_{\underline{\mathbf{d}}}), \qquad \mathfrak{b}_w := \mathrm{Lie}(B_w), \qquad \mathfrak{n}_w := \mathrm{Lie}(N_w),$$

$$\mathfrak{p}_{w,ws} := \mathrm{Lie}(P_{w,ws}), \qquad \mathfrak{n}_{w,ws} := \mathrm{Lie}(N_{w,ws}), \qquad \mathfrak{m}_{w,ws} := \mathfrak{n}_w/\mathfrak{n}_{w,ws}.$$

Recall that v_0 denotes the unique element of $W_{\underline{\mathbf{d}}}$ of maximal length. We set

$$\mathfrak{b}_w^- := \mathrm{Lie}(B_{v_0 w}), \qquad \mathfrak{n}_w^- := \mathrm{Lie}(N_{v_0 w}),$$

$$\mathfrak{p}_{w,ws}^- := \mathrm{Lie}(P_{v_0 w, v_0 ws}), \qquad \mathfrak{n}_{w,ws}^- := \mathrm{Lie}(N_{v_0 w, v_0 ws}), \qquad \mathfrak{m}_{w,ws}^- := \mathfrak{n}_w^-/\mathfrak{n}_{w,ws}^-.$$

Note that $\mathfrak{n}_{w,ws} = \mathfrak{n}_w \cap \mathfrak{n}_{ws}$. We generalize the definitions of $\mathfrak{n}_{w,ws}$ and $\mathfrak{m}_{x,xy}$ to arbitrary $x, y \in \mathbb{W}_{\mathbf{d}}$ by setting

$$\mathfrak{n}_{x,xy} := \mathfrak{n}_x \cap \mathfrak{n}_{xy}, \qquad \mathfrak{m}_{x,xy} := \mathfrak{n}_x/\mathfrak{n}_{x,xy}.$$

4.3.4 Root systems

4.3.4.1 The root system Δ. Let $\Delta \subset \mathfrak{t}_{\underline{\mathbf{d}}}^*$ denote the set of roots of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}_{\underline{\mathbf{d}}}$. It is a root system of type $A_{\mathbf{d}-1}$. We write

$$\mathfrak{g} = \mathfrak{t}_{\underline{\mathbf{d}}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space with weight α. Recall that for each $1 \leq j \leq \mathbf{d}$, the line $\mathbb{C}.e_j$ spanned by the chosen basis vector e_j is a $\mathfrak{t}_{\underline{\mathbf{d}}}$-module. Let $\chi_j \in \mathfrak{t}_{\underline{\mathbf{d}}}^*$ denote the weight of this module. Let h_k be the matrix whose (k,k)-th entry is 1 and all the other entries are 0. Then $\{h_1, ..., h_{\mathbf{d}}\}$ is the standard basis of $\mathfrak{t}_{\underline{\mathbf{d}}}$ and we have $\chi_j = h_j^*$. Recall that

$$\Delta = \{\chi_j - \chi_k \mid 1 \leq j \neq k \leq \mathbf{d}\}.$$

We choose

$$\Delta^1 := \{\chi_j - \chi_{j+1} \mid 1 \leq j \leq \mathbf{d} - 1\}$$

as the base of the root system Δ. We refer to its elements as simple roots. We set

$$\beta_j := \chi_j - \chi_{j+1}.$$

Moreover, we let Δ^+ denote the set of the positive roots with respect to this choice of base.

Definition 4.29. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and a $\mathfrak{t}_{\underline{\mathbf{d}}}$-submodule of $\mathfrak{g}$ then we set

$$Д(\mathfrak{h}) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{h}\}.$$

Using this notation we have

$$\Delta^+ = Д(\mathrm{Lie}(\mathbb{B}_{\mathbf{d}})) = \{\chi_j - \chi_k \mid 1 \leq j < k \leq d\}.$$

We also set $\Delta^- := -\Delta^+$.

Let s_{β_k} denote the reflection with respect to the simple root $\beta_k := \chi_k - \chi_{k+1}$ and let $W(\Delta)$ denote the Weyl group of the root system Δ. We have $W(\Delta) = < s_{\beta_k} \mid 1 \leq k \leq d-1 >$. There is a canonical isomorphism

$$\mathbb{W}_{\mathbf{d}} \to W(\Delta), \qquad s_k \mapsto s_{\beta_k}.$$

From now on we will freely identify the two groups. If $j \neq k$, we have

$$w(\chi_j - \chi_k) = \chi_{w(j)} - \chi_{w(k)}.$$

Remark 4.30. We have now seen three incarnations of the group $\mathbb{W}_{\mathbf{d}}$: as the Weyl group of the pair $(\mathbb{G}_{\mathbf{d}}, T_{\underline{\mathbf{d}}})$, the symmetric group $\mathfrak{S}_{\mathbf{d}}$ and the Weyl group of the root system Δ. $\triangle$

Note that, for each $w \in \mathbb{W}_{\mathbf{d}}$,

$$w(\Delta^+) = Д(\mathrm{Lie}(\mathbb{B}_w)) = \{\chi_{w(j)} - \chi_{w(k)} \mid 1 \leq j < k \leq d\}.$$

Remark 4.31. Recall that there is an isomorphism

$$\mathfrak{t}_{\underline{\mathbf{d}}}^* \to H^2((\mathbb{CP}^\infty)^{\mathbf{d}}; k) = H^2_{T_{\underline{\mathbf{d}}}}(\{pt\})$$

sending χ_k to the Chern class of the line bundle $\mathcal{O}_{(\mathbb{CP}^\infty)^{\mathbf{d}}}(\chi_k)$, i.e., the canonical line bundle on the k-th copy of $\mathbb{CP}^\infty$. This isomorphism extends to the isomorphism

$$k[\chi_1, ..., \chi_{\mathbf{d}}] \to H^*_{T_{\underline{\mathbf{d}}}}(\{pt\}).$$

Therefore from now on we will regard elements of $H^*_{T_{\underline{\mathbf{d}}}}(\{pt\})$ as polynomials in the weights $\chi_1, ..., \chi_{\mathbf{d}}$. $\triangle$

4.3.4.2 The root system $\Delta_{\underline{\mathbf{d}}}$. We let $\Delta_{\underline{\mathbf{d}}} \subset \Delta$ denote the set of roots of the Lie algebra $\mathfrak{g}_{\underline{\mathbf{d}}}$ with respect to the Cartan subalgebra $\mathfrak{t}_{\underline{\mathbf{d}}}$. It is a root system of type $A_{d_{i_1}-1} \times ... \times A_{d_{i_{|I|}}-1}$. We have

$$\mathfrak{g}_{\underline{\mathbf{d}}} = \mathfrak{t}_{\underline{\mathbf{d}}} \oplus \bigoplus_{\alpha \in \Delta_{\underline{\mathbf{d}}}} \mathfrak{g}_\alpha.$$

We set

$$\Delta_{\underline{\mathbf{d}}}^+ := \Delta_{\underline{\mathbf{d}}} \cap \Delta^+, \qquad \Delta_{\underline{\mathbf{d}}}^- := -\Delta_{\underline{\mathbf{d}}}^+ = \Delta_{\underline{\mathbf{d}}} \cap \Delta^-, \qquad \Delta_{\underline{\mathbf{d}}}^1 := \Delta_{\underline{\mathbf{d}}} \cap \Delta^1.$$

Observe that $\Delta_{\underline{\mathbf{d}}}^1$ forms a base of the root system $\Delta_{\underline{\mathbf{d}}}$ and $\Delta_{\underline{\mathbf{d}}}^+$ are the positive roots with respect to this base. We have

$$\Delta_{\underline{\mathbf{d}}}^+ = Д(\mathfrak{b}_e) = Д(\mathfrak{n}_e).$$

The group $W_{\underline{\mathbf{d}}}$ is canonically isomorphic to the Weyl group of the root system $\Delta_{\underline{\mathbf{d}}}$. If $v \in W_{\underline{\mathbf{d}}}$, then

$$v(\Delta_{\underline{\mathbf{d}}}^+) = Д(\mathfrak{b}_v) = Д(\mathfrak{n}_v).$$

Moreover, if $w = vu \in \mathbb{W}_{\mathbf{d}}$ with $v \in W_{\underline{\mathbf{d}}}, u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, then we also have

$$v(\Delta_{\underline{\mathbf{d}}}^+) = Д(\mathfrak{b}_w) = Д(\mathfrak{n}_w).$$

5 The Steinberg variety

5.1 The incidence variety

We would like to study the interplay between representations of a quiver and the quiver flag variety associated to a fixed dimension vector. The relation between the two is captured in the "incidence variety", or variety of pairs, which we now proceed to define.

Definition 5.1. (i) Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$ and let

$$F = (\{0\} = V^0 \subset V^1 \subset ... \subset V^{\mathbf{d}-1} \subset V^{\mathbf{d}} = \mathbf{V}) \tag{21}$$

be a flag in $\mathcal{F}_{\overline{y}}$. Let $\rho = (\rho_h)_{h \in \mathbf{H}} \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ be a representation of our quiver $\mathbf{\Gamma}$. We call the flag F *ρ-stable* if

$$\rho_h(V^l_{\mathbf{s}(h)}) \subseteq V^l_{\mathbf{t}(h)} \qquad \text{for all} \quad h \in \mathbf{H} \quad \text{and} \quad l \in \{1, ..., k\}.$$

(ii) We define the *incidence variety $\widetilde{\mathcal{F}}_{\overline{y}}$ of type $\overline{y}$* to be the variety of all pairs (ρ, F) such that F is ρ-stable, i.e.,

$$\widetilde{\mathcal{F}}_{\overline{y}} := \{(\rho, F) \mid F \text{ is } \rho\text{-stable}\} \subset \mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}}.$$

It is a closed subvariety of $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}}$. For $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\widetilde{\mathcal{F}}_w := \widetilde{\mathcal{F}}_{\overline{y}_w}.$$

(iii) We define the *incidence variety $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$* to be the finite disjoint union of the incidence varieties of types corresponding to all possible strictly multiplicity-free compositions:

$$\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} := \coprod_{\overline{y} \in Y_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}}.$$

It is a closed subvariety of $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$. △

Remark 5.2. Suppose that $\overline{y} = (i_{j_1}, ..., i_{j_{\mathbf{d}}})$. For each $1 \le l \le \mathbf{d}$ let us set

$$F|_l := (\{0\} = V^0 \subset V^1 \subset ... \subset V^l).$$

It is a flag of type $\overline{y}|_l := (i_{j_1}, ..., i_{j_l})$. The $\mathbf{I}$-graded subspace V^l of $\mathbf{V}$ has graded dimension $\underline{\mathbf{d}}|_l := \sum_{k=1}^{l} i_{j_k}$. If $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ and $\rho_h(V^l_{\mathbf{s}(h)}) \subseteq V^l_{\mathbf{t}(h)}$ for all $h \in \mathbf{H}$, then it is possible to restrict ρ to the subspace V^l. We set $\rho|_l := \rho|_{V^l}$. We have $\rho|_l \in \mathrm{Rep}_{\underline{\mathbf{d}}|_l}$, i.e., $\rho|_l$ is a subrepresentation of ρ. We can now rephrase the definition of ρ-stability in the following way: a flag F is ρ-stable if and only if each restriction $\rho|_l$ is a subrepresentation of ρ. △

Lemma 5.3. *If $F \in \mathcal{F}_{\overline{y}}$ is ρ-stable then $\rho(V^l) \subseteq V^{l-1}$, for each $1 \le l \le \mathbf{d}$.*

Proof. Since $\mathbf{\Gamma}$ is a quiver without loops, we have $\rho(V^1) = \{0\}$. Arguing by induction, we can suppose that $\rho(V^{l-1}) \subseteq V^{l-2}$. If $\overline{y} = (i_{j_1}, ..., i_{j_{\mathbf{d}}})$, we can write $V^l = V^{l-1} \oplus L$, where $L \subset \mathbf{V}_{i_{j_l}}$. Then $\rho(V^l) = \rho(V^{l-1} \oplus L) = \rho(V^{l-1}) \oplus \rho(L) \subseteq V^{l-2} \oplus \rho(L)$. By ρ-stability, $\rho(L) \subset V^l$. But since $\mathbf{\Gamma}$ has no loops, there is no arrow from i_{j_l} to itself, and so $\rho(L) \cap \mathbf{V}_{i_{j_l}} = \{0\}$. Since $V^l/V^{l-1} \cong L \subset \mathbf{V}_{i_{j_l}}$, we must have $\rho(L) \subset V^{l-1}$. □

Remark 5.4. We can interpret the last lemma as saying that a representation which stabilizes a flag has to be nilpotent. Geometrically, this means that representations stabilizing a given flag must lie inside a nilpotent cone in $\mathrm{Rep}_{\underline{\mathbf{d}}}$.

5.1.1 The action of $G_{\underline{\mathbf{d}}}$

$G_{\underline{\mathbf{d}}}$ acts diagonally on $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}}$ by $g : (\rho, F) \mapsto (g.\rho, g.F)$. Since

$$(g.\rho_h)(g_{\mathbf{s}(h)}.V^l_{\mathbf{s}(h)}) = (g_{\mathbf{t}(h)}\rho_h g^{-1}_{\mathbf{s}(h)})(g_{\mathbf{s}(h)}.V^l_{\mathbf{s}(h)}) = (g_{\mathbf{t}(h)}\rho_h)(V^l_{\mathbf{s}(h)}) = g_{\mathbf{t}(h)}.(\rho_h(V^l_{\mathbf{s}(h)})) \subseteq g_{\mathbf{t}(h)}.V^l_{\mathbf{t}(h)},$$

$g.F$ is $g.\rho$-stable and so the diagonal action of $G_{\underline{\mathbf{d}}}$ descends to an action on $\widetilde{\mathcal{F}}_{\overline{y}}$.

5.1.2 The canonical projections

We now turn our attention to the two canonical projections

$$\begin{tikzcd} & \widetilde{\mathcal{F}}_{\overline{y}} \arrow[dl, "\mu_{\overline{y}}"'] \arrow[dr, "\pi_{\overline{y}}"] & \\ \mathrm{Rep}_{\underline{\mathbf{d}}} & & \mathcal{F}_{\overline{y}}. \end{tikzcd}$$

Proposition 5.5. *(i) The first projection*

$$\mu_{\overline{y}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$$

is a $G_{\underline{\mathbf{d}}}$-equivariant proper map.
(ii) The second projection

$$\pi_{\overline{y}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \mathcal{F}_{\overline{y}}$$

is a $G_{\underline{\mathbf{d}}}$-equivariant vector bundle with fibre

$$\pi^{-1}_{\overline{y}}(F) = \{(\rho, F) \mid F \textit{ is } \rho\textit{-stable}\} \subseteq \mathrm{Rep}_{\underline{\mathbf{d}}} \times \{F\}.$$

In particular, for $g \in G_{\underline{\mathbf{d}}}$, we have

$$\pi^{-1}_{\overline{y}}(g.F) = g.\pi^{-1}_{\overline{y}}(F).$$

Proof. (i) $G_{\underline{\mathbf{d}}}$-equivariance is obvious. The projection $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ is proper because the quiver flag variety $\mathcal{F}_{\overline{y}}$ is compact (one can see this using the Iwasawa decomposition, for example). But $\mu_{\overline{y}}$ is the restriction of this projection to the closed subset $\widetilde{\mathcal{F}}_{\overline{y}}$, so it is proper as well.
(ii) This is clear. □

Proposition 5.5 implies that the dimension of the fibres $\pi^{-1}_{\overline{y}}(F)$ is constant with respect to $F \in \mathcal{F}_{\overline{y}}$ and that $\dim_{\mathbb{C}} \widetilde{\mathcal{F}}_{\overline{y}} = \dim_{\mathbb{C}} \mathcal{F}_{\overline{y}} + \dim_{\mathbb{C}} \pi^{-1}_{\overline{y}}(F)$. While the dimension of $\mathcal{F}_{\overline{y}}$ does not depend on the choice of $\overline{y}$ (it is the same for all $\overline{y}$), the dimension of $\widetilde{\mathcal{F}}_{\overline{y}}$ does vary with $\overline{y}$. Therefore, we introduce the following definition.

Definition 5.6. For $\overline{y} \in Y_{\underline{\mathbf{d}}}$, we let

$$\gamma(\overline{y}) := \dim_{\mathbb{C}}(\widetilde{\mathcal{F}}_{\overline{y}})$$

denote the complex dimension of $\widetilde{\mathcal{F}}_{\overline{y}}$. △

The fact that the dimension of $\widetilde{\mathcal{F}}_{\overline{y}}$ depends on $\overline{y}$ is illustrated in the following example.

Example 5.7. Consider the quiver A_2

$$i \to j$$

with $\underline{\mathbf{d}} = i + j$. Let $V = V_i \oplus V_j = \mathbb{C}e_i \oplus \mathbb{C}e_j$ be the associated vector space. Then $Y_{\underline{\mathbf{d}}} = \{\overline{y}, \overline{y}'\}$ with $\overline{y} = (i, j)$ and $\overline{y}' = (j, i)$. We have $\mathcal{F}_{\overline{y}} \cong \mathcal{F}_{\overline{y}'} \cong \{pt\}$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \mathbb{C}\rho_{ij}$, where $\rho_{ij} : e_i \mapsto e_j$. More precisely, $\mathcal{F}_{\overline{y}} = \{F\}, \mathcal{F}_{\overline{y}'} = \{F'\}$, where

$$F = (\{0\} \subset V_i \subset V),$$

$$F' = (\{0\} \subset V_j \subset V).$$

Since $\rho_{ij}(V_i) = V_j$, F is not ρ_{ij}-stable. On the other hand F' is ρ_{ij}-stable. Let ρ_0 be the zero morphism in $\mathrm{Rep}_{\underline{\mathbf{d}}}$. Then

$$\widetilde{\mathcal{F}}_{\overline{y}} = \{(\rho_0, F)\} \cong \{pt\} \qquad \text{while} \qquad \widetilde{\mathcal{F}}_{\overline{y}'} = \{(\rho, F') \mid \rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}\} = \mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}'} \cong \mathbb{C}.$$

Note that $\widetilde{\mathcal{F}}_{\overline{y}}$ is a variety of (complex) dimension 0 while $\widetilde{\mathcal{F}}_{\overline{y}'}$ is a variety of (complex) dimension 1, i.e., $\gamma(\overline{y}) = 0$ but $\gamma(\overline{y}') = 1$. △

Definition 5.8. We will use the notation

$$\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}, \quad \pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}$$

for the projections from the whole incidence variety $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$. △

It follows immediately from Proposition 5.5 that $\mu_{\underline{\mathbf{d}}}$ is proper and that $\pi_{\underline{\mathbf{d}}}$ is a disjoint union of vector bundles (of various ranks).

Remark 5.9. The fact that the morphism $\mu_{\underline{\mathbf{d}}}$ is proper is very important. It allows us to take pushforwards along $\mu_{\underline{\mathbf{d}}}$ in Borel-Moore homology.

5.1.3 Another interpretation of $\widetilde{\mathcal{F}}_w$

Let $w \in \mathbb{W}_{\mathbf{d}}$. If $\overline{y} = \overline{y}_w$, we set

$$\mu_{\overline{y}} = \mu_w : \widetilde{\mathcal{F}}_w \to \mathrm{Rep}_{\underline{\mathbf{d}}}, \qquad \pi_{\overline{y}} = \pi_w : \widetilde{\mathcal{F}}_w \to \mathcal{F}_w.$$

Definition 5.10. For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\mathfrak{r}_w := \{\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid F_w \text{ is } \rho\text{-stable}\}.$$

Clearly, $\mathfrak{r}_w \cong \pi_w^{-1}(F_w)$ as $\mathfrak{t}_{\underline{\mathbf{d}}}$-modules. For $w, w' \in \mathbb{W}_{\mathbf{d}}$, we also set

$$\mathfrak{r}_{w,w'} = \mathfrak{r}_w \cap \mathfrak{r}_{w'}, \qquad \mathfrak{d}_{w,w'} = \mathfrak{r}_w / \mathfrak{r}_{w,w'}.$$

Let $\rho \in \mathfrak{r}_w$ and $b \in B_w$. Then $b.F_w = F_w$ is also $b.\rho$-stable, so the $G_{\underline{\mathbf{d}}}$-action on $\mathrm{Rep}_{\underline{\mathbf{d}}}$ restricts to an action of B_w on $\mathfrak{r}_w$. Therefore, we can endow the variety $G_{\underline{\mathbf{d}}} \times \mathfrak{r}_w$ with two actions. Firstly, we let $G_{\underline{\mathbf{d}}}$ act from the left by left multiplication on the first factor, i.e., $g.(h, \rho) = (gh, \rho)$. Secondly, we let B_w act diagonally by the formula $b : (h, \rho) \mapsto (hb, b^{-1}.\rho)$. This action is free because the action of B_w on $G_{\underline{\mathbf{d}}}$ by right multiplication is free. We let $G_{\underline{\mathbf{d}}} \times^{B_w} \mathfrak{r}_w$ denote the quotient (orbit space) of $G_{\underline{\mathbf{d}}} \times \mathfrak{r}_w$ by this diagonal action of B_w. The left $G_{\underline{\mathbf{d}}}$-action on $G_{\underline{\mathbf{d}}} \times \mathfrak{r}_w$ descends to a left action on the quotient $G_{\underline{\mathbf{d}}} \times^{B_w} \mathfrak{r}_w$.

Lemma 5.11. *For each $w \in W_{\underline{\mathbf{d}}}$, there is an isomorphism of $G_{\underline{\mathbf{d}}}$-varieties*

$$G_{\underline{\mathbf{d}}} \times^{B_w} \mathfrak{r}_w \xrightarrow{\cong} \widetilde{\mathcal{F}}_w, \qquad (g, \rho) \mapsto (g.F_w, g.\rho).$$

Proof. Everything follows directly from the definitions. □

5.1.4 $T_{\underline{\mathbf{d}}}$-fixed points in $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$

Lemma 5.12. *We have*

$$(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{(\rho_0, F) \mid F \in (\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}\},$$

where ρ_0 denotes the zero endomorphism in $\mathrm{Rep}_{\underline{\mathbf{d}}}$.

Proof. Indeed, every flag F is ρ_0-stable, so $(\rho_0, F) \in \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$. If $t \in T_{\underline{\mathbf{d}}}$, then $t.\rho_0 = \rho_0$, so ρ_0 is fixed by $T_{\underline{\mathbf{d}}}$. Hence if $F \in (\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}$ then the pair (ρ_0, F) is a fixed point under the diagonal action of $T_{\underline{\mathbf{d}}}$. On the other hand, suppose that $\rho_0 \neq \rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$. Then there exists $h \in \mathbf{H}$ such that $\rho_h \neq 0$. We can choose $t \in T_{\underline{\mathbf{d}}}$ such that $t_{\mathbf{s}(h)} := t|_{\mathbf{V}_{\mathbf{s}(h)}} = \mathrm{id}_{\mathbf{V}_{\mathbf{s}(h)}}$ but $t_{\mathbf{t}(h)} := t|_{\mathbf{V}_{\mathbf{t}(h)}} = \lambda \cdot \mathrm{id}_{\mathbf{V}_{\mathbf{t}(h)}}$, where $1 \neq \lambda \in \mathbb{C}^\times$. Then $t_{\mathbf{t}(h)}\rho_h t_{\mathbf{s}(h)}^{-1} = \lambda\rho_h \neq \rho_h$. Hence ρ is not a fixed point of $T_{\underline{\mathbf{d}}}$. □

5.1.5 Canonical line bundles and the cohomology ring of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$

Definition 5.13. Let $l \in \{1, ..., \mathbf{d}\}$. Let $F = (\{0\} = V^0 \subset V^1 \subset ... \subset V^{d-1} \subset V^{\mathbf{d}} = \mathbf{V})$ be a flag in $\mathcal{F}_{\overline{y}}$ and $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ a representation such that F is ρ-stable. We define the l-th *canonical line bundle* over $\mathcal{F}_{\overline{y}}$

$$p_l : \mathcal{O}_{\mathcal{F}_{\overline{y}}}(l) \to \mathcal{F}_{\overline{y}}$$

to be the $G_{\underline{\mathbf{d}}}$-equivariant line bundle over $\mathcal{F}_{\overline{y}}$ with fibre $p_l^{-1}(F) = V^l/V^{l-1}$. We also define the l-th canonical line bundle over $\widetilde{\mathcal{F}}_{\overline{y}}$

$$\widetilde{p}_l : \mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) \to \widetilde{\mathcal{F}}_{\overline{y}}$$

to be the $G_{\underline{\mathbf{d}}}$-equivariant line bundle over $\widetilde{\mathcal{F}}_{\overline{y}}$ with fibre $\widetilde{p}_l^{-1}(\rho, F) = V^l/V^{l-1}$. Note that the fibre does not depend on ρ. It is obvious that

$$\mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) \to \mathcal{O}_{\mathcal{F}_{\overline{y}}}(l)$$

is itself a vector bundle with fibre $\pi_{\overline{y}}^{-1}(F)$ at F. △

By taking homotopy quotients of both the total and base spaces we obtain the line bundles

$$p_l^{G_{\underline{\mathbf{d}}}} : \mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\mathcal{F}_{\overline{y}}}(l) := EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{O}_{\mathcal{F}_{\overline{y}}}(l) \to EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{F}_{\overline{y}},$$

$$\widetilde{p}_l^{G_{\underline{\mathbf{d}}}} : \mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) := EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) \to EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}}.$$

Proposition 5.14. *There exists a homotopy equivalence $EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}} \to (\mathbb{CP}^\infty)^{\mathbf{d}}$. This homotopy equivalence induces a homotopy equivalence from the line bundle $\mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l)$ to the canonical (tautological) line bundle $\mathcal{O}_{(\mathbb{CP}^\infty)^{\mathbf{d}}}(l; 1)$ on $(\mathbb{CP}^\infty)^{\mathbf{d}}$.*

Proof. Step 1 Let us first construct the required homotopy equivalence. We choose

$$EG_{\underline{\mathbf{d}}} = \prod_{k=1}^{|\mathbf{I}|} \mathrm{Mat}^{max}(\infty \times \mathbf{d}_k).$$

Then $EG_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}} = (\mathbb{CP}^\infty)^{\mathbf{d}}$. Let

$$EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}} \xrightarrow{\alpha} EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{F}_{\overline{y}}$$

be the map induced by the vector bundle projection $\pi_{\overline{y}}:\widetilde{\mathcal{F}}_{\overline{y}}\to\mathcal{F}_{\overline{y}}$. Recall that $G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}\to G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}$ is a vector bundle with fibre $R_u(B_{\underline{\mathbf{d}}})$. We have a commutative diagram

$$\begin{tikzcd}
G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}} \arrow[r, "\cong"] \arrow[d] & \prod_{k=1}^{|\mathbf{I}|}\{(L^1_{i_k},...,L^{\mathbf{d}_k}_{i_k}) \mid L^n_{i_k}\in\mathbb{CP}^{\mathbf{d}_k}, L^n_{i_k}\notin\mathbb{P}(L^1_{i_k}\oplus...\oplus L^{n-1}_{i_k})\} \arrow[d] \\
G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \arrow[r, "\cong"'] & \prod_{k=1}^{|\mathbf{I}|}\{(V^1_{i_k},...,V^{\mathbf{d}_k}_{i_k}) \mid V^n_{i_k}\in\mathrm{Gr}(n,\mathbf{d}_k), V^{n-1}_{i_k}\subset V^n_{i_k}\}.
\end{tikzcd}$$

Let us set

$$D^1_{i_k}:=V^1_{i_k},\qquad D^n_{i_k}:=V^n_{i_k}\cap(V^{n-1}_{i_k})^{orth}.$$

$D^n_{i_k}$ is the unique line in $V^n_{i_k}$ orthogonal to the hyperplane $V^{n-1}_{i_k}$. The map

$$\begin{aligned} s_0: G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} &\to G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}} \\ \prod_{k=1}^{|\mathbf{I}|}(V^1_{i_k},...,V^{\mathbf{d}_k}_{i_k}) &\mapsto \prod_{k=1}^{|\mathbf{I}|}(D^1_{i_k},...,D^{\mathbf{d}_k}_{i_k}) \end{aligned}$$

is the zero section of the vector bundle $G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}\to G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}$. Let

$$EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}\xrightarrow{\beta}EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}$$

be the map induced by s_0 on homotopy quotients. We also have the obvious map

$$EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}\xrightarrow{\cong}(EG_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}}=(\mathbb{CP}^\infty)^{\mathbf{d}}.$$

If $e\in EG_{\underline{\mathbf{d}}}$ is a matrix and $\mathcal{L}:=\prod_{k=1}^{|\mathbf{I}|}(L^1_{i_k},...,L^{\mathbf{d}_k}_{i_k})\in G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}$ a sequence of lines, then this map sends the equivalence class of $(e,\mathcal{L})$ to $e.\mathcal{L}:=\prod_{k=1}^{|\mathbf{I}|}(e.L^1_{i_k},...,e.L^{\mathbf{d}_k}_{i_k})\in(\mathbb{CP}^\infty)^{\mathbf{d}}$, where the dots denote matrix multiplication.

It is clear that both α and β are homotopy equivalences. Hence the following composition is also a homotopy equivalence:

$$EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}\widetilde{\mathcal{F}}_{\overline{y}}\xrightarrow{\alpha}EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}\mathcal{F}_{\overline{y}}=EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}\xrightarrow{\beta}EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}\xrightarrow{\cong}(EG_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}}=(\mathbb{CP}^\infty)^{\mathbf{d}}.$$

If $F=\prod_{k=1}^{|\mathbf{I}|}(V^1_{i_k},...,V^{\mathbf{d}_k}_{i_k})$ is a flag, then the composition above sends the equivalence class of the triple (e,ρ,F) to the sequence of lines $\prod_{k=1}^{|\mathbf{I}|}(e.D^1_{i_k},...,e.D^{\mathbf{d}_k}_{i_k})\in(\mathbb{CP}^\infty)^{\mathbf{d}}$.

Step 2 Let us define a canonical line bundle on $G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}$. We take a sequence of lines

$$\mathcal{L}:=\prod_{k=1}^{|\mathbf{I}|}(L^1_{i_k},...,L^{\mathbf{d}_k}_{i_k})\in G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}.$$

The composition $\overline{y}$ induces a total order on these lines, so we may write $\mathcal{L}=(L_1,...,L_{\mathbf{d}})$. We define the l-th canonical line bundle $\mathcal{O}_{G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}}\to G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}$ to be the $G_{\underline{\mathbf{d}}}$-equivariant line bundle over $G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}$ with fibre L_l at $\mathcal{L}$. Observe that $\mathcal{O}_{G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}}\to\mathcal{O}_{\mathcal{F}_{\overline{y}}}$ is itself a vector bundle with fibre $R_u(B_{\underline{\mathbf{d}}})$. Let $\mathcal{O}^{G_{\underline{\mathbf{d}}}}_{G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}}$ denote the homotopy quotient of $\mathcal{O}_{G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}}$.

Step 3 We obtain a diagram of lines bundles

$$\begin{tikzcd}
\mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) \arrow[r] \arrow[d] & \mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\mathcal{F}_{\overline{y}}}(l) \arrow[r] \arrow[d] & \mathcal{O}^{G_{\underline{\mathbf{d}}}}_{G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}}}(l) \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{(\mathbb{CP}^\infty)^{\mathbf{d}}}(l;1) \arrow[d] \\
EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}\widetilde{\mathcal{F}}_{\overline{y}} \arrow[r, "\alpha"] & EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}\mathcal{F}_{\overline{y}} \arrow[r, "\beta"] & EG_{\underline{\mathbf{d}}}\times^{G_{\underline{\mathbf{d}}}}G_{\underline{\mathbf{d}}}/T_{\underline{\mathbf{d}}} \arrow[r, "\cong"] & (\mathbb{CP}^\infty)^{\mathbf{d}}.
\end{tikzcd}$$

Now let $F = (V^1, ..., V_{\mathbf{d}}) = \prod_{k=1}^{|\mathbf{I}|}(V^1_{i_k}, ..., V^{\mathbf{d}_k}_{i_k})$ be a flag. The fibre of (e, ρ, F) is V^l/V^{l-1}. The composition of lower horizontal maps sends (e, ρ, F) to $\mathcal{L} = \prod_{k=1}^{|\mathbf{I}|}(e.D^1_{i_k}, ..., e.D^{\mathbf{d}_k}_{i_k}) \in (\mathbb{CP}^\infty)^{\mathbf{d}}$. The composition $\overline{y}$ induces a total order on these lines, so we may write $\mathcal{L} = (D_1, ..., D_{\mathbf{d}})$. On the other hand, the composition of upper horizontal maps sends the fibre V^l/V^{l-1} to D_l, which is the fibre of $\mathcal{O}_{(\mathbb{CP}^\infty)}(l)$ at D_l. Hence the diagram commutes and preserves fibres. □

Corollary 5.15. *Let $z(l)$ denote the first Chern class of the line bundle $\mathcal{O}_{(\mathbb{CP}^\infty)^{\mathbf{d}}}(l;1)$ and let $x_{\overline{y}}(l)$ denote the first Chern class of the line bundle $\mathcal{O}^{G_{\underline{\mathbf{d}}}}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l)$. The homotopy equivalence from Proposition 5.14 induces a k-algebra isomorphism*

$$H^*((\mathbb{CP}^\infty)^{\mathbf{d}}; k) = k[z(1), ..., z(\mathbf{d})] \xrightarrow{\cong} H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = k[x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})]$$
$$z(l) \mapsto x_{\overline{y}}(l).$$

Proof. This follows directly from Example 2.9 and the fact that vector bundle pullback commutes with taking Chern classes. □

Corollary 5.16. *We have*

$$H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) = \bigoplus_{\overline{y} \in Y_{\underline{\mathbf{d}}}} k[x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})]$$

as a k-algebra.

5.1.6 The action of $\mathbb{W}_{\mathbf{d}}$ on $H^{G_{\underline{\mathbf{d}}}}_*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$

Recall that the group $\mathbb{W}_{\mathbf{d}}$ acts on $Y_{\underline{\mathbf{d}}}$ by $w(\overline{y}) = \overline{y} \circ w^{-1}$. Moreover, $\mathbb{W}_{\mathbf{d}}$ acts naturally on the set $\{1, ..., \mathbf{d}\}$ by permutations. Combining these two actions we obtain an action of $\mathbb{W}_{\mathbf{d}}$ on $H^{G_{\underline{\mathbf{d}}}}_*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$. Each $w \in \mathbb{W}_{\mathbf{d}}$ acts by

$$w : H^{G_{\underline{\mathbf{d}}}}_*(\widetilde{\mathcal{F}}_{\overline{y}}) \to H^{G_{\underline{\mathbf{d}}}}_*(\widetilde{\mathcal{F}}_{w(\overline{y})}), \quad f(x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})) \mapsto f(x_{w(\overline{y})}(w(1)), ..., x_{w(\overline{y})}(w(\mathbf{d})))$$

for a polynomial f.

5.2 The Steinberg variety

We now have all the ingredients to define the Steinberg variety, whose Borel-Moore homology will be the main object of study in this thesis.

Definition 5.17. (i) Let $\overline{y}, \overline{y}' \in Y_{\underline{\mathbf{d}}}$. We define the *Steinberg variety of type* $(\overline{y}, \overline{y}')$ to be the reduced fibre product

$$\mathcal{Z}_{\overline{y},\overline{y}'} := \widetilde{\mathcal{F}}_{\overline{y}} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}'} \subset \widetilde{\mathcal{F}}_{\overline{y}} \times \widetilde{\mathcal{F}}_{\overline{y}'}$$

relative to the maps $\mu_{\overline{y}}$ and $\mu_{\overline{y}'}$.

(ii) We define the *Steinberg variety* to be the finite disjoint union of Steinberg varieties of all types corresponding to all the possible strictly multiplicity-free compositions, i.e.,

$$\mathcal{Z}_{\underline{\mathbf{d}}} := \coprod_{\overline{y},\overline{y}' \in Y_{\underline{\mathbf{d}}}} \mathcal{Z}_{\overline{y},\overline{y}'}.$$

Note that the Steinberg variety of type $(\overline{y}, \overline{y}')$ is the universal object making the following pullback diagram commute

$$\begin{tikzcd}
\mathcal{Z}_{\overline{y},\overline{y}'} \arrow[r] \arrow[d] & \widetilde{\mathcal{F}}_{\overline{y}} \arrow[d, "\mu_{\overline{y}}"] \\
\widetilde{\mathcal{F}}_{\overline{y}'} \arrow[r, "\mu_{\overline{y}'}"'] & \mathrm{Rep}_{\underline{\mathbf{d}}}.
\end{tikzcd}$$

We also have the following explicit description

$$\mathcal{Z}_{\overline{y},\overline{y}'} := \widetilde{\mathcal{F}}_{\overline{y}} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}'} = \{((\rho, F), (\rho', F')) \in \widetilde{\mathcal{F}}_{\overline{y}} \times \widetilde{\mathcal{F}}_{\overline{y}'} \mid \rho = \rho'\}.$$

The variety $\mathcal{Z}_{\overline{y},\overline{y}'}$ is clearly isomorphic to the following *variety of triples*

$$\mathcal{Z}_{\overline{y},\overline{y}'} \cong \{(\rho, F, F') \in \mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}} \times \mathcal{F}_{\overline{y}'} \mid F, F' \text{ are } \rho\text{-stable } \}.$$

$G_{\underline{\mathbf{d}}}$ acts diagonally on $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}} \times \mathcal{F}_{\overline{y}'}$ by the formula $(\rho, F, F') \mapsto (g.\rho, g.F, g.F')$. If F and F' are ρ-stable then $g.F$ and $g.F'$ are $g.\rho$-stable, so the diagonal action of $G_{\underline{\mathbf{d}}}$ descends to an action on $\mathcal{Z}_{\overline{y},\overline{y}'}$.

We have two canonical $G_{\underline{\mathbf{d}}}$-equivariant projections

$$\begin{tikzcd} & \mathcal{Z}_{\overline{y},\overline{y}'} \arrow[dl, "\mu_{\overline{y},\overline{y}'}"'] \arrow[dr, "\pi_{\overline{y},\overline{y}'}"] & \\ \mathrm{Rep}_{\underline{\mathbf{d}}} & & \mathcal{F}_{\overline{y}} \times \mathcal{F}_{\overline{y}'}. \end{tikzcd}$$

Taking the disjoint union over the connected components $\mathcal{Z}_{\overline{y},\overline{y}'}$ we obtain projections

$$\begin{tikzcd} & \mathcal{Z}_{\underline{\mathbf{d}}} \arrow[dl, "\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}}"'] \arrow[dr, "\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}"] & \\ \mathrm{Rep}_{\underline{\mathbf{d}}} & & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}. \end{tikzcd}$$

Note that the first projection $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ is proper while the second projection $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ is a $G_{\underline{\mathbf{d}}}$-equivariant affine *fibration* over $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ (but it is in general not a vector bundle because the dimension of the fibres is not constant, even upon restriction to a connected component, so local triviality does not hold).

5.2.1 $T_{\underline{\mathbf{d}}}$-fixed points in $\mathcal{Z}_{\underline{\mathbf{d}}}$

By the argument of section 5.1.4, ρ_0 is the only $T_{\underline{\mathbf{d}}}$-fixed point in $\mathrm{Rep}_{\underline{\mathbf{d}}}$. Since $T_{\underline{\mathbf{d}}}$ acts diagonally on $\mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$, it follows that

$$(\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{(\rho_0, F, F') \mid F, F' \in (\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}\}.$$

In particular,

$$|(\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}| = |(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}|^2 = (d!)^2.$$

6 Convolution

6.1 Fundamental classes

6.1.1 Non-equivariant fundamental classes

Let X be a connected oriented (but not necessarily compact) manifold of real dimension m. We know that fundamental classes of compact oriented manifolds always exist in singular homology. Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Then the *singular fundamental class*

$$[\hat{X}]^{sing} \in H_m^{sing}(\hat{X})$$

exists and is a generator of the top singular homology group of $\hat{X}$. The inclusion of pairs $j : (\hat{X}, \varnothing) \subset (\hat{X}, \{\infty\})$ gives an induced map

$$j_* : H_*^{sing}(\hat{X}) \to H_*^{sing}(\hat{X}, \{\infty\}).$$

By the "one-point compactification" definition of Borel-Moore homology we obtain a homology class

$$[X] := j_*([\hat{X}]^{sing}) \in H_m^{sing}(\hat{X}, \{\infty\}) = H_m(X),$$

which we call the *Borel-Moore fundamental class* of X. In the sequel we will simply refer to $[X]$ as the *fundamental class* of X. An important property of the fundamental class is that it is Poincaré dual to the unity in the cohomology ring, i.e., we have

$$\begin{aligned} H_m(X) &\cong H^0(X) \\ [X] &\mapsto 1, \end{aligned}$$

under the Poincaré duality isomorphism from (2). If $X = \coprod_{l=1}^p X_l$ is a finite disjoint union of connected components, then we set $[X] := \sum_{l=1}^p [X_l]$. If Y is a closed submanifold of X of real dimension k and $i : Y \to X$ denotes the inclusion, we have a pushforward map $i_* : H_*(Y) \to H_*(X)$. We call

$$i_*([Y]) \in H_k(X)$$

the *fundamental class of the closed submanifold* Y. For simplicity we will use the notation

$$[Y] := i_*([Y]).$$

6.1.2 Equivariant fundamental classes

We now want to generalize these definitions to the equivariant setting. As usual, we apply the non-equivariant concepts to the homotopy quotient of X. Since in singular homology theory the notion of a fundamental class only makes sense for finite-dimensional manifolds, we have to use approximation spaces. Suppose that X is endowed with an action of a Lie group G (or, if X is also a complex algebraic variety, an algebraic action of a complex reductive algebraic group G). Let $g = \dim_{\mathbb{R}} G$ and $m = \dim_{\mathbb{R}} X$. Let $\{E^nG \to B^nG \mid n \geq 0\}$ be an approximation of the universal bundle $EG \to BG$ and let $\widetilde{n} = \dim_{\mathbb{R}} E^nG$. We define the *G-equivariant fundamental class* of X, denoted $[X]^G$, to be

$$[X]^G := \varprojlim_n [E^nG \times^G X] \in \varprojlim_n H_{m+\widetilde{n}-g}(E^nG \times^G X) = H_m^G(X),$$

where the limit is taken with respect to the inverse system

$$H_{m+\widetilde{(n+1)}-g}(E^{n+1}G \times^G X) \to H_{m+\widetilde{n}-g}(E^nG \times^G X).$$

Since the inverse system stabilizes, for $n \geq 1$ we can identify

$$[X]^G = [E^n G \times^G X].$$

Now suppose that Y is a G-stable closed submanifold of X of real dimension k. We have a pushforward map in equivariant homology

$$i_* : H_*^G(Y) \to H_*^G(X)$$

induced by the closed embedding $i : Y \to X$. We call

$$i_*([Y]^G) \in H_k^G(X)$$

the *G-equivariant fundamental class of the closed submanifold Y*. For simplicity we will, as before, use the notation

$$[Y]^G := i_*([Y]^G).$$

6.2 General theory of convolution

6.2.1 Non-equivariant convolution

6.2.1.1 The convolution product. Let M_1, M_2, M_3 be oriented C^∞-manifolds and let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be closed subsets. We define the *set-theoretic composition* of Z_{12} and Z_{23} to be

$$Z_{12} \circ Z_{23} = \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ s.t. } (m_1, m_2) \in Z_{12}, (m_2, m_3) \in Z_{23}\}.$$

Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection on the (i, j)-factor and let

$$\hat{p}_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \tag{22}$$

be the restriction of p_{13} to the subset $p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$. Assume that $\hat{p}_{13}$ is proper. We have

$$p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = (Z_{12} \times M_3) \cap (M_1 \times Z_{23}) = Z_{12} \times_{M_2} Z_{23},$$

so $Z_{12} \circ Z_{23}$ is the image of $\hat{p}_{13}$. Since $\hat{p}_{13}$ is proper, and therefore closed, $Z_{12} \circ Z_{23}$ is a closed subset of $M_1 \times M_3$ and the pushforward $(\hat{p}_{13})_*$ exists. We let

$$\hat{p}_{12} : p_{12}^{-1}(Z_{12}) = Z_{12} \times M_3 \to Z_{12},$$

$$\hat{p}_{23} : p_{23}^{-1}(Z_{23}) = M_1 \times Z_{23} \to Z_{23},$$

denote restrictions of the projections p_{12}, p_{23} to $p_{12}^{-1}(Z_{12})$, $p_{23}^{-1}(Z_{23})$, respectively. Let $m = \dim_{\mathbb{R}} M_2$. We define the *convolution product* in Borel-Moore homology

$$H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-m}(Z_{12} \circ Z_{23}),$$

$$(c_{12}, c_{23}) \mapsto c_{12} * c_{23}$$

by

$$c_{12} * c_{23} = (\hat{p}_{13})_*((c_{12} \otimes [M_3]) \cap ([M_1] \otimes c_{23})) \in H_*(Z_{12} \circ Z_{23}),$$

where $c_{12} \otimes [M_3] = \hat{p}_{12}^* c_{12}$ and $[M_1] \otimes c_{23} = \hat{p}_{23}^* c_{23}$.

Lemma 6.1. *The convolution product is associative.*

Proof. See [CG97, Section 2.7.18]. □

6.2.1.2 The convolution algebra. Let M be a smooth complex manifold, N be a (possibly singular) variety and $\mu : M \to N$ a proper map. We let $Z = M \times_N M$. Now set $M_1 = M_2 = M_3 = M$ and

$$Z_{ij} = M_i \times_N M_j = \{(m, m') \in M_i \times M_j \mid \mu(m) = \mu(m')\},$$

for $1 \leq i < j \leq 3$. It is immediate that $Z_{12} \circ Z_{23} = Z_{13}$. We need to check that the map

$$\hat{p}_{13} : M_1 \times_N M_2 \times_N M_3 \to Z_{13}$$

is proper. Recall that for a continuous map f between locally compact Hausdorff spaces the following conditions are equivalent: (1) f is proper (i.e. the preimage of every compact set is compact), (2) f is universally closed, (3) f is closed and all the fibres of f are compact. The fact that $\mu : M \to N$ is proper, and hence universally closed, immediately implies that $\hat{p}_{13}$ is closed. Now take $(m_1, m_3) \in Z_{13}$. We have $\mu(m_1) = \mu(m_3) = x$ for some $x \in N$ and $\hat{p}_{13}^{-1}(m_1, m_3) = \{(m_1, m_3)\} \times \mu^{-1}(x)$. Since μ is proper, $\mu^{-1}(x)$ is compact, and hence $\hat{p}_{13}^{-1}((m_1, m_3))$ is compact as well. So $\hat{p}_{13}$ is closed and has compact fibres, and, therefore, it's proper. Since Z_{12}, Z_{23} and Z_{13} are all canonically isomorphic to Z, we have a convolution product

$$H_*(Z) \times H_*(Z) \to H_*(Z).$$

Corollary 6.2. *$H_*(Z)$ endowed with the convolution product is a unital associative algebra. The unit is given by the fundamental class of the diagonal $M_\Delta = \{(x, x) \in M \times M\} \subset Z$.*

Proof. Associativity follows from Lemma 6.1. We delay the proof of the fact that M_Δ is the unit until we introduce the clean intersection formula, see Lemma 9.3. □

6.2.1.3 Convolution modules. We can apply the convolution construction to obtain interesting modules over the convolution algebra $H_*(Z)$. Let M, N and Z be as in the previous paragraph. We set $M_1 = M_2 = M$, $M_3 = \{pt\}$, $Z_{12} = M_1 \times_N M_2$ and $Z_{23} = M_2 \times \{pt\}$, $Z_{13} = M_1 \times \{pt\}$. Then $Z \circ Z_{23} = Z_{13}$. One can verify that the map

$$\hat{p}_{13} : Z_{12} \times \{pt\} \to Z_{13}$$

is proper in the same way as in the previous paragraph. Since $\mu : M_2 \to N$ is proper, it is universally closed, so the map $\hat{p}_{13} : M_1 \times_N M_2 \cong Z_{12} \times \{pt\} \to M_1 \cong Z_{13}$ is closed. If $m \in M_1$ then $\hat{p}_{13}^{-1}(m) = \{(m, m') \mid \mu(m) = \mu(m')\} = \{m\} \times \mu^{-1}(\mu(m'))$ is compact because μ is proper. Thus $\hat{p}_{13}$ is closed with compact fibres, so it's proper. Since Z_{12} is canonically isomorphic to Z and Z_{23}, Z_{13} are canonically isomorphic to M, we get a convolution product

$$H_*(Z) \times H_*(M) \to H_*(M).$$

Corollary 6.3. *$H_*(M)$ is a module over $H_*(Z)$ under convolution.*

Now let $x \in N$ and set $M_x = \mu^{-1}(x)$. If we set $M_1 = M_2 = M$, $M_3 = \{pt\}$, $Z_{12} = M_1 \times_N M_2$ and $Z_{23} = (M_2)_x \times \{pt\}$, $Z_{13} = (M_1)_x \times \{pt\}$, then $Z_{12} \circ Z_{23} = Z_{23}$. One can verify that the map

$$\hat{p}_{13} : Z_{12} \times \{pt\} \to Z_{13}$$

is proper as before. Since Z_{12} is canonically isomorphic to Z and Z_{23}, Z_{13} are canonically isomorphic to M_x, we get a convolution product

$$H_*(Z) \times H_*(M_x) \to H_*(M_x).$$

Corollary 6.4. *$H_*(M_x)$ is a module over $H_*(Z)$ under convolution.*

6.2.1.4 The diagonal subalgebra. Let $M_\Delta = \{(x,x) \in M \times M\} \subset Z$ be the diagonal in $Z = M \times_N M$. We set $M_1 = M_2 = M_3 = M$ and

$$Z_{ij} = \{(m,m) \in M_i \times M_j\},$$

for $1 \leq i < j \leq 3$. Then $Z_{12} \circ Z_{23} = Z_{13}$. Since Z_{12}, Z_{23} and Z_{13} are all canonically isomorphic to M_Δ (and M), we obtain a convolution product

$$H_*(M_\Delta) \times H_*(M_\Delta) \to H_*(M_\Delta),$$

which endows $H_*(M_\Delta)$ with the structure of a k-subalgebra of $H_*(Z)$. Moreover, one can see from the definitions that the convolution product in this case reduces to the intersection pairing, which is simply the Poincaré dual of the cohomology cup product. Hence $H_*(M_\Delta)$ is, under Poincaré duality, isomorphic to the cohomology algebra $H^*(M_\Delta)$. Note that the diagonal embedding $M \to Z$ with image M_Δ also induces an isomorphism $H^*(M_\Delta) \xrightarrow{\cong} H^*(M)$.

Corollary 6.5. *$H_*(M_\Delta)$ is a k-subalgebra of $H_*(Z)$ under convolution. Moreover, there is a (grading-reversing) k-algebra isomorphism $H_*(M_\Delta) \cong H^*(M)$.*

Therefore, $H_*(M)$ plays two roles in the convolution framework - it is both a module over $H_*(Z)$ and a subalgebra of $H_*(Z)$.

6.2.2 Equivariant convolution

Now suppose that M_1, M_2, M_3 also have the structure of complex algebraic varieties and are endowed with an algebraic action of an algebraic group G. We equip the products $M_1 \times M_2 \times M_3$ and $M_i \times M_j$ $(1 \leq i < j \leq 3)$ with the diagonal actions. Assume that Z_{12}, Z_{23} are closed G-stable subvarieties.

We now want to define equivariant analogues of set-theoretic composition and the convolution product. The first idea that comes to mind is to replace all the manifolds by their homotopy quotients. But this leads to problems with product spaces because the product of homotopy quotients is not the homotopy quotient of a product. For example, if $M_1 = M_2 = \{pt\}$, then $EG \times^G (M_1 \times M_2) = BG$ but $(EG \times^G M_1) \times (EG \times^G M_2) = BG \times BG$.

The right approach is to consider homotopy quotients of the product spaces $M_1 \times M_2 \times M_3, M_i \times M_j$ $(1 \leq i < j \leq 3)$ and the subvarieties Z_{12}, Z_{23} rather than the factors M_1, M_2, M_3 themselves. We can also give the following, perhaps more elegant, interpretation, in which we take the homotopy quotient only once. Indeed, we take the homotopy quotient $EG \times^G (M_1 \times M_2 \times M_3)$ of the ambient manifold $M_1 \times M_2 \times M_3$, and consider $EG \times^G (M_i \times M_j)$ $(1 \leq i < j \leq 3)$ as images of respective projections. We also consider $EG \times^G Z_{12}, EG \times^G Z_{23}$ as closed subsets of these images.

We define the *G-equivariant set-theoretic composition* of Z_{12} and Z_{23} to be

$$\begin{gathered}(EG \times^G Z_{12}) \circ (EG \times^G Z_{23}) := \\ \Big\{\overline{(a, m_1, m_3)} \in EG \times^G (M_1 \times M_3) \mid \\ \exists\ m_2 \in M_2 \text{ with } \overline{(a, m_1, m_2)} \in EG \times^G Z_{12} \text{ and } \overline{(a, m_2, m_3)} \in EG \times^G Z_{23}\Big\}.\end{gathered}$$

It is immediate that

$$(EG \times^G Z_{12}) \circ (EG \times^G Z_{23}) = EG \times^G (Z_{12} \circ Z_{23}).$$

We now have projections

$$\begin{aligned}\hat{p}_{12}^G &: EG \times^G (Z_{12} \times M_3) \to EG \times^G (M_1 \times M_2), \\ \hat{p}_{23}^G &: EG \times^G (M_1 \times Z_{23}) \to EG \times^G (M_1 \times M_2), \\ \hat{p}_{13}^G &: EG \times^G (Z_{12} \times_{M_2} Z_{23}) \to EG \times^G (M_1 \times M_2),\end{aligned}$$

Let $m = \dim_{\mathbb{R}} M_2$. We define the *G-equivariant convolution product* in G-equivariant Borel-Moore homology

$$H_i^G(Z_{12}) \times H_j^G(Z_{23}) \to H_{i+j-m}^G(Z_{12} \circ Z_{23}),$$
$$(c_{12}, c_{23}) \mapsto c_{12} * c_{23}$$

by

$$c_{12} * c_{23} = (\hat{p}_{13}^G)_*((c_{12} \otimes [M_3]^G) \cap ([M_1]^G \otimes c_{23})) \in H_*^G(Z_{12} \circ Z_{23}).$$

where $c_{12} \otimes [M_3]^G = (\hat{p}_{12}^G)^*(c_{12})$ and $[M_1]^G \otimes c_{23} = (\hat{p}_{23}^G)^*(c_{23})$.

Corollary 6.6. *The convolution product is S_G-linear. Hence, in the set-up of Sections 6.2.1.2, 6.2.1.3 and 6.2.1.4, $H_*^G(Z)$ endowed with the G-equivariant convolution product has the structure of a unital associative S_G-algebra. The unit is given by the G-equivariant fundamental class $[M_\Delta]^G$ of the diagonal $M_\Delta = \{(x, x) \in M \times M\} \subset Z$. Moreover, $H_*^G(M_\Delta)$ forms an S_G-subalgebra of $H_*^G(Z)$ isomorphic to the G-equivariant cohomology algebra $H_G^*(M)$. Furthermore, $H_*^G(M)$ and $H_*^G(M_x)$, for $x \in N$, are $H_*^G(Z)$-modules under G-equivariant convolution.*

Proof. Since pullbacks, pushforwards and the intersection pairing are maps of S_G-modules, the equivariant convolution product must also be S_G-linear. The other assertions follow straightforwardly from the analogous assertions about non-equivariant convolution. $\square$

6.3 Application to the Steinberg variety

6.3.1 The convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$

We work in the set-up of Section 6.2.2. Let k be a field and $M_1 = M_2 = M_3 = \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$. Since $\mathcal{Z}_{\underline{\mathbf{d}}} \subset \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \times \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ we have the closed embeddings

$$\begin{tikzcd} & M_1 \times M_2 \\ \mathcal{Z}_{\underline{\mathbf{d}}} \arrow[ur, "j_{12}"] \arrow[r, "j_{13}"] \arrow[dr, "j_{23}"] & M_1 \times M_3 \\ & M_2 \times M_3. \end{tikzcd}$$

Set-theoretic composition gives

$$j_{12}(\mathcal{Z}_{\underline{\mathbf{d}}}) \circ j_{23}(\mathcal{Z}_{\underline{\mathbf{d}}}) = j_{13}(\mathcal{Z}_{\underline{\mathbf{d}}}).$$

Hence we obtain an equivariant convolution product

$$\star : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k), \tag{23}$$

which equips $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$ with the structure of an associative unital $S_{G_{\underline{\mathbf{d}}}}$-algebra.

6.3.2 The convolution module $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k)$

Now set $M_1 = M_2 = \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $M_3 = \{pt\}$. We have a closed embedding

$$\mathcal{Z}_{\underline{\mathbf{d}}} \xrightarrow{\quad j_{12} \quad} M_1 \times M_2$$

and isomorphisms

$$\begin{tikzcd}
 & M_1 \times \{pt\} = M_1 \times M_3 \\
\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \arrow[ur, "j_{13}"] \arrow[dr, "j_{23}"'] & \\
 & M_2 \times \{pt\} = M_2 \times M_3.
\end{tikzcd}$$

Set-theoretic composition gives

$$j_{12}(\mathcal{Z}_{\underline{\mathbf{d}}}) \circ j_{23}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) = j_{13}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}).$$

Hence we obtain an equivariant convolution product

$$\diamond : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k) \to H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k), \tag{24}$$

which equips $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k)$ with the structure of an $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$-module.

Remark 6.7. We remark that $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k)$ also has a natural $S_{G_{\underline{\mathbf{d}}}}$-algebra structure when considered as the equivariant cohomology ring of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ under Poincaré duality. In the next paragraph we show that it can also be regarded as an $S_{G_{\underline{\mathbf{d}}}}$-subalgebra of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$.

6.3.3 The convolution subalgebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}}; k)$

Let $\mathcal{Z}^e_{\underline{\mathbf{d}}}$ denote $(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})_\Delta = \{(m,m) \mid m \in \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\} \subset \mathcal{Z}_{\underline{\mathbf{d}}}$, i.e., the diagonal in $\mathcal{Z}_{\underline{\mathbf{d}}}$. The reason for this notation will become clear in the next chapter, where we discuss cellular fibrations - $\mathcal{Z}^e_{\underline{\mathbf{d}}}$ is the first stratum in the cellular fibration of $\mathcal{Z}_{\underline{\mathbf{d}}}$ and the strata are indexed by the Weyl group $\mathbb{W}_{\mathbf{d}}$. Corollaries 6.5 and 6.6 imply that $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$ is an $S_{G_{\underline{\mathbf{d}}}}$-subalgebra of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and that the diagonal embedding $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{Z}_{\underline{\mathbf{d}}}$ with image $\mathcal{Z}^e_{\underline{\mathbf{d}}}$ induces an isomorphism

$$H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}}) \cong H^*_{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}}) \xrightarrow{\cong} H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}).$$

6.3.4 Algebras and modules associated to connected components

The decompositions into connected components

$$\mathcal{Z}_{\underline{\mathbf{d}}} := \coprod_{\overline{y},\overline{y}' \in Y_{\underline{\mathbf{d}}}} \mathcal{Z}_{\overline{y},\overline{y}'}, \qquad \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} := \coprod_{\overline{y} \in Y_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}}$$

induce $S_{G_{\underline{\mathbf{d}}}}$-module decompositions in homology

$$H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) = \bigoplus_{\overline{y},\overline{y}' \in Y_{\underline{\mathbf{d}}}} H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}'}; k),$$
$$H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k) = \bigoplus_{\overline{y} \in Y_{\underline{\mathbf{d}}}} H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}}; k).$$

We now want to see how the convolution product behaves with respect to these decompositions. Let $M_1 = \widetilde{\mathcal{F}}_{\overline{y}}$, $M_2 = \widetilde{\mathcal{F}}_{\overline{y}'}$, $M_3 = \widetilde{\mathcal{F}}_{\overline{y}''}$. Then

$$\mathcal{Z}_{\overline{y},\overline{y}'} = \widetilde{\mathcal{F}}_{\overline{y}} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}'} \subset M_1 \times M_2, \quad \mathcal{Z}_{\overline{y}',\overline{y}''} = \widetilde{\mathcal{F}}_{\overline{y}'} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}''} \subset M_2 \times M_3,$$

$$\mathcal{Z}_{\overline{y},\overline{y}''} = \widetilde{\mathcal{F}}_{\overline{y}} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}''} \subset M_1 \times M_3.$$

Set-theoretic composition gives

$$\mathcal{Z}_{\overline{y},\overline{y}'} \circ \mathcal{Z}_{\overline{y}',\overline{y}''} = \mathcal{Z}_{\overline{y},\overline{y}''}.$$

Hence we have an equivariant convolution product

$$\star : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}'};k) \times H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y}',\overline{y}''};k) \to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}''};k). \tag{25}$$

Now set $M_1 = \widetilde{\mathcal{F}}_{\overline{y}}$, $M_2 = \widetilde{\mathcal{F}}_{\overline{y}'}$, $M_3 = \{pt\}$. We have isomorphisms

$$\widetilde{\mathcal{F}}_{\overline{y}} \to \widetilde{\mathcal{F}}_{\overline{y}} \times \{pt\} = M_1 \times M_3, \qquad \widetilde{\mathcal{F}}_{\overline{y}'} \to \widetilde{\mathcal{F}}_{\overline{y}'} \times \{pt\} = M_2 \times M_3.$$

Set-theoretic composition yields

$$\mathcal{Z}_{\overline{y},\overline{y}'} \circ (\widetilde{\mathcal{F}}_{\overline{y}'} \times \{pt\}) = \widetilde{\mathcal{F}}_{\overline{y}} \times \{pt\}.$$

We thus obtain an equivariant convolution product

$$\diamond : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}'};k) \times H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}'};k) \to H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}};k). \tag{26}$$

7 Stratifications

The goal of this chapter is to describe a convenient (i.e. Schubert class) basis of $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ and $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We first recall the theory of cellular decompositions and cellular fibrations and later adapt it to the quiver-graded setting. Our main tool will be the "cellular fibration lemma". At the end of the chapter we also discuss how stratifications interact with the convolution product, describe the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and show that $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong H_*^{G_{\underline{\mathbf{d}}}} \otimes_k H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$.

7.1 Cellular decompositions and cellular fibrations

7.1.1 Definitions and examples: cellular decomposition

We begin by defining a cellular decomposition and a cellular fibration. We also show that a flag variety G/B, resp. a product of flag varieties $G/B \times G/B$, satisfy these definitions.

Definition 7.1. Let X be an algebraic variety endowed with an algebraic action of an algebraic group A.
(i) An *A-equivariant partial cellular decomposition* of X is a filtration

$$X = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \varnothing$$

satisfying the following conditions:

- $(C1)$ Each X_r is a closed A-stable subvariety of X.
- $(C2)$ Each $\widetilde{X}_r = X_r - X_{r-1}$ is a (possibly empty) finite disjoint union of A-stable subvarieties of X, each isomorphic to $\mathbb{A}^r$, called *r-cells*.
- $(C3)$ The closure $\overline{U}$ of each r-cell U is the disjoint union of U and some l-cells with $l < r$. We call $\overline{U}$ a *closed cell*.

(ii) An *A-equivariant complete cellular decomposition* of X is a filtration

$$X = X_m \supset X_{m-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$$

satisfying condition $(C1)$ and the following two conditions:

- $(C2')$ Each $\widetilde{X}_r = X_r - X_{r-1}$ is a nonempty, A-stable subvariety of X isomorphic to $\mathbb{A}^k$, for some $k \geq 0$. We call $\widetilde{X}_r$ the *r-stratum*.
- $(C3')$ The closure of $\widetilde{X}_r$ is the disjoint union of $\widetilde{X}_r$ and some l-strata of lower dimension (as varieties) such that $l < r$.

A filtration satisfying only conditions $(C1), (C2)$ resp. $(C1), (C2')$ is called a (partial resp. complete) *weak cellular decomposition*. △

Remark 7.2. (i) One can obtain a complete cellular decomposition from a partial decomposition $X = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \varnothing$ in the following way. First delete all the X_r equal to X_{r-1} from the filtration and renumber. Subsequently choose an ordering on the cells $\{U_1^0, ..., U_{p_0}^0\}$ in $\widetilde{X}_0$, and define a new filtration by setting $Y_0 = U_1^0$, $Y_1 = U_1^0 \cup U_2^0$, ..., $Y_{p_0} = U_1^0 \cup ... \cup U_p^0$. Then choose an ordering on the cells $\{U_1^1, ..., U_{p_1}^1\}$ in $\widetilde{X}_1$, and extend the filtration by setting $Y_{p_0+1} = Y_{p_0} \cup U_1^1$, ..., $Y_{p_0+p_1} = Y_{p_0+p_1-1} \cup U_{p_1}^1$. Continue inductively until all cells have been attached. Note that this procedure is not canonical - it depends on the choice of ordering of the cells.

On the other hand, given a complete cellular decomposition $X = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \varnothing$, we can obtain a partial decomposition by setting Y_r equal to the disjoint union of those strata $\widetilde{X}_r$ which have dimension r. This procedure does not depend on any choices.

(ii) We will need the notion of a complete cellular decomposition to calculate homology. While the notion of a partial cellular decomposition is not needed for this purpose, we have decided to define it because many spaces have a natural partial decomposition yet they lack a canonical complete decomposition. This is the case, for examples, in flag varieties. The transformation of a partial decomposition into a complete one corresponds, in this case, to extending the Bruhat order on a Weyl group to a total one.

(iii) Observe that an A-equivariant cellular decomposition is also an A'-equivariant cellular decomposition for any subgroup A' of A. △

We now discuss the key example of a space with a cellular decomposition, namely a flag variety.

Lemma 7.3. *Let G be a complex reductive connected linear algebraic group with a Borel subgroup B and a maximal torus T contained in B. Let $W = N_G(T)/T$ be the Weyl group of the pair (G, T) and let $\Omega_w = BwB/B$. Moreover, let $\leq$ denote the Bruhat order on W. Then*

$$(G/B)^T = \{wB/B \mid w \in W\}$$

and

$$G/B = \coprod_{w\in W} \Omega_w, \qquad \Omega_w \cong \mathbb{A}^{l(w)}, \qquad \overline{\Omega_w} = \coprod_{u\leq w\in W} \Omega_u.$$

Moreover,

$$\widetilde{X}_r = \coprod_{l(w)=r} \Omega_w, \qquad X_r = \coprod_{l(w)\leq r} \Omega_w$$

gives a B-equivariant partial cellular decomposition of G/B.

Proof. See [Prz14, Theorem 3.21]. □

Definition 7.4. In the setting of the lemma above, we call the $\{\Omega_w \mid w \in W\}$ *Schubert cells* and the $\{\overline{\Omega_w} \mid w \in W\}$ *Schubert (sub)varieties.* △

Example 7.5. Let $G = \mathrm{SL}(2, \mathbb{C})$ and let B be the standard Borel subgroup consisting of invertible upper triangular matrices. Then $G/B \cong \mathbb{CP}^1$. The Weyl group W is isomorphic to $\mathbb{Z}_2 = \{e, s\}$. Our flag variety has two cells: the one-point cell B/B and $BsB/B \cong \mathbb{C}$.

7.1.2 Definitions and examples: cellular fibration

A cellular decomposition is a way of decomposing a variety into affine spaces. We now define a more general notion, that of a cellular fibration, which is a way of decomposing a variety into vector bundles.

Definition 7.6. Let X, Y be algebraic varieties endowed with algebraic actions of an algebraic group A. Let $\pi : X \to Y$ be an A-equivariant morphism of varieties.
(i) An *A-equivariant partial cellular fibration* structure on X over Y is a filtration

$$X = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$$

satisfying the following conditions:

- $(D1)$ Each X_r is a closed A-stable subvariety of X and each restriction $\pi : X_r \to Y$ is an A-equivariant fibre bundle.
- $(D2)$ Each $\widetilde{X}_r = X_r - X_{r-1}$ is a finite disjoint union of subvarieties U_i of X such that each restriction $\pi : U_i \to Y$ is a A-equivariant vector bundle of rank r. We also call the U_i *r-cells.*
- $(D3)$ The closure of each r-cell U_i is the disjoint union of U_i and some l-cells with $l < r$.

(ii) An A-*equivariant complete cellular fibration* structure on X over Y is a filtration

$$X = X_m \supset X_{m-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$$

satisfying condition $(D1)$ and the following two conditions:

- $(D2')$ Each $\widetilde{X}_r = X_r - X_{r-1}$ is a nonempty, A-stable subvariety of X such that the restriction $\pi : \widetilde{X}_r \to Y$ is an A-equivariant vector bundle of rank k, for some $k \geq 0$. We call $\widetilde{X}_r$ the *r-stratum*.
- $(D3')$ The closure of $\widetilde{X}_r$ is the disjoint union of $\widetilde{X}_r$ and some l-strata of lower rank (as vector bundles) such that $l < r$.

A filtration satisfying only conditions $(D1), (D2)$ resp. $(D1), (D2')$ is called a (partial resp. complete) *weak cellular fibration* structure. △

Remark 7.7. (i) A cellular decomposition is a cellular fibration with $Y = \{pt\}$.

(ii) Analogous observations regarding the relationship between partial and complete cellular fibrations apply as in the previous remark about cellular decompositions. △

We will now discuss the key example of a space with a cellular fibration structure, namely a product of two flag varieties. As before, let G be a complex reductive connected linear algebraic group and let B be a Borel subgroup containing a maximal torus T. Let $\pi : G/B \times G/B \to G/B$ be the projection onto the *first* factor. Let $W = N_G(T)/T$ be the Weyl group of the pair (G,T). Then

$$(G/B \times G/B)^T = \{(wB/B, w'B/B) \mid w, w' \in W\}$$

and we have bijections:

$$\begin{array}{ccccc} W & \longleftrightarrow & \{B\text{-orbits on } G/B\} & \longleftrightarrow & \{G\text{-diagonal orbits on } G/B \times G/B\}. \\ w & \longmapsto & \Omega_w & \longmapsto & \boldsymbol{\Omega}_w. \end{array}$$

We already explained the first bijection in Lemma 7.3. Let us now define $\boldsymbol{\Omega}_w$ and explain the second bijection.

Definition 7.8. Let Ω_w be a Schubert cell in G/B.
(i) Let $g \in G$. We call $g.\Omega_w \subset G/B$ the translation of Ω_w by g.

Note that we still have $g.\Omega_w \cong \mathbb{A}^{l(w)}$. Moreover, if $g, g' \in G$ belong to the same coset, i.e., $gB = g'B$, then $g.\Omega_w = g'.\Omega_w$ because Schubert cells are B-stable. Therefore it also makes sense to regard $g.\Omega_w \subset G/B$ as a translation of Ω_w by the element $\overline{g} := gB/B \in G/B$.

(ii) We set

$$\boldsymbol{\Omega}_w := \coprod_{\overline{g} \in G/B} (\overline{g}, g.\Omega_w).$$

The definition directly implies that $\boldsymbol{\Omega}_w$ is stable under the diagonal G-action. We call $\boldsymbol{\Omega}_w$ a *diagonal Schubert cell* and its closure $\overline{\boldsymbol{\Omega}_w}$ a *diagonal Schubert variety.* △

Remark 7.9. We use standard font to denote Schubert cells in G/B and bold font to denote diagonal Schubert cells in $G/B \times G/B$.

Lemma 7.10. *$\boldsymbol{\Omega}_w$ is the G-orbit of the T-fixed point $(B/B, wB/B)$ and it contains precisely the T-fixed points $\{(uB/B, uwB/B) \mid u \in W\}$.*

Proof. This follows directly from the definitions. □

Lemma 7.11. *Let G be a complex reductive connected linear algebraic group and B be a Borel subgroup. Let $\pi : G/B \times G/B \to G/B$ be the projection onto the first factor. Then $G/B \times G/B$ has a natural structure of a G-equivariant partial cellular fibration over G/B.*

More specifically, if G/B is endowed with the partial cellular decomposition from Lemma 7.3, with Schubert cells $\{\Omega_w \mid w \in W\}$, then each restriction

$$\pi_w : \mathbf{\Omega}_w \to G/B$$

of the projection π to a diagonal Schubert cell $\mathbf{\Omega}_w$ is a vector bundle of rank $l(w)$. Moreover,

$$G/B \times G/B = \coprod_{w \in W} \mathbf{\Omega}_w, \qquad \overline{\mathbf{\Omega}_w} = \coprod_{u \leq w \in W} \mathbf{\Omega}_u$$

and

$$\widetilde{X}_r := \coprod_{l(w)=r} \mathbf{\Omega}_w, \qquad X_r := \coprod_{l(w)\leq r} \mathbf{\Omega}_w$$

gives a G-equivariant partial cellular fibration of G/B.

Proof. We first prove that, for each $w \in W$,

$$\pi_w : \mathbf{\Omega}_w \to G/B \tag{27}$$

is a vector bundle of rank $l(w)$. Let $\overline{g} := gB/B \in G/B$ and let $F := \pi^{-1}(\overline{g})$ be a fibre of the projection π. The restriction $\mathbf{\Omega}_w|_F = \mathbf{\Omega}_w \cap F$ of $\mathbf{\Omega}_w$ to F is the translated Schubert cell $g.\Omega_w \cong \mathbb{A}^{l(w)}$. In particular, each fibre is affine. It remains to be shown that (27) satisfies local triviality. Recall that G/B contains a unique cell of highest dimension, the so-called "big cell". This cell is an open subvariety of G/B. There exists an open covering of the base space G/B by translations of the big cell. It is not difficult to check that this covering gives a local trivialization of (27). Hence (27) is indeed a vector bundle of rank $l(w)$.

$G/B \times G/B$ is a disjoint union of the diagonal Schubert cells because they are precisely the G-orbits. The closure of $\mathbf{\Omega}_w$ is the disjoint union of closures of each fibre. The fact that

$$\overline{\mathbf{\Omega}_w} = \coprod_{u \leq w \in W} \mathbf{\Omega}_u$$

now follows from the corresponding claim about (non-diagonal) Schubert cells.

It is now completely straightforward to verify the axioms $(D1)$-$(D3)$. We conclude that $\pi : G/B \times G/B \to G/B$, endowed with the filtration

$$G/B \times G/B = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing,$$

is indeed a partial cellular fibration over G/B. $\square$

Example 7.12. Let $G = \mathrm{SL}(2, \mathbb{C})$ and B be the standard Borel subgroup. Then $G/B \times G/B \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. This "double flag variety" contains two diagonal Schubert cells. One cell is the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. It is isomorphic to $\mathbb{CP}^1$ and contains the torus fixed points $(B/B, B/B)$ and $(sB/B, sB/B)$. The other cell is its complement. It is a complex vector bundle over $\mathbb{CP}^1$ of rank 1 and contains the torus fixed points $(B/B, sB/B)$ and $(sB/B, B/B)$.

7.1.3 Further cellular decompositions

Our goal is to give a $H^*_G(\{pt\})$-basis of the homology group $H^G_*(G/B \times G/B)$. For this purpose we need to define some more cells in $G/B \times G/B$.

Definition 7.13. Let $\{\Omega_w \mid w \in W\}$ be the Schubert cells in G/B and $\{\mathbf{\Omega}_w \mid w \in W\}$ the diagonal Schubert cells in $G/B \times G/B$. Recall that the projection

$$\pi : G/B \times G/B \to G/B$$

onto the first factor restricts to a vector bundle

$$\pi_w : \mathbf{\Omega}_w \to G/B \tag{28}$$

with fibre $\pi_w^{-1}(gB/B) = g\Omega_w$. Now let $\Omega_{w'}$ be a Schubert cell in the base space. Consider the restriction of the bundle (28) to the Schubert cell $\Omega_{w'}$. Its total space is

$$\pi_w^{-1}(\Omega_{w'}) = \pi^{-1}(\Omega_{w'}) \cap \mathbf{\Omega}_w.$$

We denote it by $\mathbf{\Omega}_{w',w}$. It is isomorphic to the affine space $\mathbb{A}^{l(w)+l(w')}$. △

Note that $(w'B/B, w'wB/B)$ is the only T-fixed point in $\mathbf{\Omega}_{w',w}$.

Lemma 7.14. *We have decompositions*

$$G/B \times G/B = \coprod_{w',w\in W} \mathbf{\Omega}_{w,w'}, \qquad \mathbf{\Omega}_w = \coprod_{w'\in W} \mathbf{\Omega}_{w',w}, \qquad \pi^{-1}(\Omega_{w'}) = \coprod_{w\in W} \mathbf{\Omega}_{w',w}.$$

Proof. This is immediate from the definitions. □

Proposition 7.15. *(i) Let $w \in W$ and $n = |W|$. If we set*

$$\widetilde{X}_r = \coprod_{l(w')=r} \mathbf{\Omega}_{w',w}, \qquad X_r = \coprod_{l(w')\leq r} \mathbf{\Omega}_{w',w}$$

then

$$\mathbf{\Omega}_w = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$$

is a T-equivariant partial cellular decomposition of $\mathbf{\Omega}_w$.
(ii) Let $w' \in W$. If we set

$$\widetilde{X}_r = \coprod_{l(w)=r} \mathbf{\Omega}_{w',w}, \qquad X_r = \coprod_{l(w)\leq r} \mathbf{\Omega}_{w',w}$$

then

$$\pi^{-1}(\Omega_{w'}) = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$$

is a T-equivariant partial cellular decomposition of $\pi^{-1}(\Omega_{w'})$.
(iii) The closure of $\mathbf{\Omega}_{w',w}$ is

$$\overline{\mathbf{\Omega}_{w',w}} = \coprod_{u\leq w, u'\leq w'} \mathbf{\Omega}_{u',u}.$$

Proof. This is also clear from the definitions. □

7.1.4 Thom isomorphism

An important property of cellular fibrations is that their homology can be recovered from the homology of the affine cells. Moreover, the filtration associated to a complete cellular fibration induces a filtration on homology. To show this we will need the Thom isomorphism.

Proposition 7.16 (Thom isomorphism)**.** *Suppose that $\pi : E \to X$ is a smooth real vector bundle of rank k and that $i : X \to E$ is the inclusion of the zero section. Then the pullback homomorphisms*

$$\pi^* : H_*(X) \to H_{*+k}(E), \qquad i^* : H_*(E) \to H_{*-k}(X)$$

are mutually inverse isomorphisms. Similarly, if $\pi : E \to X$ is in addition a G-equivariant bundle, then

$$\pi^* : H_*^G(X) \to H_{*+k}^G(E), \qquad i^* : H_*^G(E) \to H_{*-k}^G(X)$$

are mutually inverse isomorphisms.

Proof. See [CG97, Proposition 2.6.23]. □

7.1.5 The cellular fibration lemma

In this section we will prove a technical result of fundamental importance. It will allow us to connect the geometry of flag varieties and products of flag varieties with their homology. We will work in the following set-up.

Let A be an algebraic group, and let $\pi : X \to Y$ be an A-equivariant map of algebraic varieties endowed with an algebraic A-action, and suppose that the filtration $X = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \varnothing$ defines a weak A-equivariant *complete cellular fibration* structure on X over Y. We have the following diagram of nested fibre bundles.

$$\begin{tikzcd}
X = X_n \arrow[d, "\pi"'] & \supset X_{n-1} \arrow[dl, "\pi"] & \supset \;... \supset & X_0 \arrow[dlll, "\pi"] \\
Y & & &
\end{tikzcd}$$

Let $E_r = X_r \backslash X_{r-1}$ and let $\overline{E}_r$ denote the closure of E_r. Note that $\overline{E}_r$ is a closed subset of X_r but needn't equal X_r. Let us further introduce the following notation.

$$\begin{tikzcd}
X_{r-1} \arrow[r, hook, "i"] & X_r & E_r \arrow[l, hook', "j"'] \arrow[r, hook, "\alpha_r"] \arrow[d, "\pi_r"'] & \overline{E}_r \arrow[r, hook, "\beta_r"] \arrow[dl, "\overline{\pi}_r"] & X \\
& & Y & &
\end{tikzcd} \tag{29}$$

Lemma 7.17 (Cellular fibration lemma). *(i) For each $r = 1, ..., n$ there is a canonical short exact sequence*

$$0 \to H_*^A(X_{r-1}) \xrightarrow{i_*} H_*^A(X_r) \xrightarrow{j^*} H_*^A(E_r) \to 0. \tag{30}$$

(ii) Moreover, if $H_*^A(Y)$ is a free S_A-module with basis $y_1, ..., y_m$ then each short exact sequence (30) *is (non-naturally) split and $H_*^A(X_r)$ is a free S_A-module with basis $\{\beta_{r*}\overline{\pi}_r^*(y_l) \mid 1 \leq l \leq m, 1 \leq r \leq n\}$.*

Proof. Recall that we have the following inclusions

$$X_{r-1} \xhookrightarrow{i} X_r \xhookleftarrow{j} E_r,$$

where i is a closed embedding. Suppose that $m = \dim X_r$ and that the bundle $\pi_r : E_r \to Y$ has rank k. Then we get a long exact sequence in equivariant Borel-Moore homology

$$0 \to H_m^A(X_{r-1}) \xrightarrow{i_*} H_m^A(X_r) \xrightarrow{j^*} H_m^A(E_r) \xrightarrow{\partial} H_{m-1}^A(X_{r-1}) \xrightarrow{i_*} ...$$

We need to show that all the boundary maps ∂ vanish, or, equivalently, that the maps j^* are surjective. Consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & H_m^A(X_{r-1}) \arrow[r, "i_*"] & H_m^A(X_r) \arrow[rr, "j^*"] & & H_m^A(E_r) \arrow[r, "\partial"] & H_{m-1}^A(X_{r-1}) \arrow[r, "i_*"] & ... \\
& & & H_{m-k}^A(Y) \arrow[ul, "\pi_r^*"] \arrow[ur, "\pi_r^*"', "\cong"] & & &
\end{tikzcd}$$

Since the pullback maps commute and $\pi_r^* : H_{m-k}^A(Y) \to H_m^A(E_r)$ is an isomorphism, it follows that j^* is surjective, as desired. The same argument works in each degree, so all boundary maps vanish.

For the second part, we argue by induction on r. Since $X_0 = E_0$, Thom isomorphism implies $H_*^A(X_0) \cong H_*^A(Y)$. Now assume the result holds for X_{r-1}. Let β_k^p denote the following closed embedding

$$\beta_k^p : \overline{E}_k \hookrightarrow X_p.$$

Note that β_k^p is obtained from β_k by restricting its range to X_p. We have $i \circ \beta_k^{r-1} = \beta_k^r$. By induction $H_*^A(X_{r-1})$ is a free S_A-module with basis $\{(\beta_k^{r-1})_* \overline{\pi}_k^*(y_l) \mid 1 \leq l \leq m, 1 \leq k \leq r-1\}$. By the Thom isomorphism $H_*^A(E_r) \cong H_*^A(Y)$, and so $H_*^A(E_r)$ is a free S_A-module with basis $\{\pi_r^*(y_1), ..., \pi_r^*(y_m)\}$. Since both the left and the right ends of the exact sequence (30) are free S_A-modules, it follows that the sequence splits. If s is a section of j^*, then $\{(\beta_k^r)_* \overline{\pi}_k^*(y_l) \mid 1 \leq l \leq m, 1 \leq k \leq r-1\} \cup \{s\pi_r^*(y_1), ..., s\pi_r^*(y_m)\}$ is a basis of $H_*^A(X_r)$. We thus need to find an appropriate section so that $s\pi_r^*(y_l) = \overline{\pi}_r^*(y_l)$ holds for each l.

Now observe that the cellular fibration structure on X induces a cellular fibration structure on the closed subvariety X_r, which also induces a cellular fibration structure on the closed subvariety $\overline{E}_r$, by axiom (C3') in the definition of a cellular fibration. We can choose a filtration of X_r so that the filtration of $\overline{E}_r$ forms an initial segment of the filtration of X_r. Repeating the argument of the first part of the proof we can conclude that the map $(\beta_r^r)^* : H_*^A(\overline{E}_r) \to H_*^A(X_r)$ is injective, so we can regard $H_*^A(\overline{E}_r)$ as a subgroup of $H_*^A(X_r)$. Moreover, we can choose a section s so that its image is contained in $H_*^A(\overline{E}_r)$. We can therefore regard s as a section of α_r^*. Now let

$$\overline{E}_r = F_r \supset F_{r-1} \supseteq F_{r-2} \supseteq ... \supseteq F_0$$

be a filtration of $\overline{E}_r$. Let i' denote the closed embedding $i' : F_{r-1} \hookrightarrow \overline{E}_r$. Repeating the argument of the first part of the proof for the cellular fibration of $\overline{E}_r$, we conclude that there is a split short exact sequence

$$0 \to H_*^A(F_{r-1}) \xrightarrow{i'_*} H_*^A(\overline{E}_r) \xrightarrow{\alpha_r^*} H_*^A(E_r) \to 0. \tag{31}$$

Note that the commutativity of diagram (29) implies that for each l, $\pi_r^*(y_l) = \alpha_r^* \overline{\pi}_r^*(y_l)$. We can now choose a section s of α_r^* such that for each l, $\overline{\pi}^*(y_l) = s\alpha_r^* \overline{\pi}_r^*(y_l) = s\pi_r^*(y_l)$. But this is the section we were looking for. □

We have the following immediate corollaries of the cellular fibration lemma.

Corollary 7.18. *Suppose that the total filtration of X has length n, i.e., X contains n cells and that $H_*^A(Y)$ is a free S_A-module of rank m. Then $H_*^A(X)$ is a free S_A-module of rank $n \cdot m$. Moreover, the filtration of X induces a filtration on homology*

$$H_*^A(X) \supset H_*^A(X_{n-1}) \supset ... \supset H_*^A(X_0).$$

Corollary 7.19. *We have*

$$H_*(G/B) \cong \bigoplus_{w \in W} k \left[\Omega_w\right], \qquad H_*(G/B \times G/B) \cong \bigoplus_{w,w' \in W} k \left[\mathbf{\Omega}_{w,w'}\right],$$

$$H_*^T(G/B) \cong \bigoplus_{w \in W} S_T \left[\Omega_w\right]^T, \qquad H_*^T(G/B \times G/B) \cong \bigoplus_{w,w' \in W} S_T \left[\mathbf{\Omega}_{w,w'}\right]^T,$$

7.2 Stratification of $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$

7.2.1 The cells $\Omega_w^{\overline{u}}, \mathbf{\Omega}_w^{\overline{u}',\overline{u}}$ and $\mathbf{\Omega}_{w',w}^{\overline{u}',\overline{u}}$

We first apply the results of the previous section to $G = G_{\underline{\mathbf{d}}}, B = B_{\underline{\mathbf{d}}}, T = T_{\underline{\mathbf{d}}}, W = W_{\underline{\mathbf{d}}}$. We get bijections

$$W_{\underline{\mathbf{d}}} \longleftrightarrow \{B_{\underline{\mathbf{d}}}\text{-orbits on } G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}\} \longleftrightarrow \{G_{\underline{\mathbf{d}}}\text{-diagonal orbits on } G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \times G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}\}.$$

This applies to every connected component of the quiver flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$. Recall that $\mathcal{F}_{\underline{\mathbf{d}}} = \coprod_{\overline{y}\in Y_{\underline{\mathbf{d}}}} \mathcal{F}_{\overline{y}}$ and that there are bijections

$$Y_{\underline{\mathbf{d}}} \longleftrightarrow W_{\underline{\mathbf{d}}}\backslash \mathbb{W}_{\mathbf{d}} \longleftrightarrow \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}}).$$

Furthermore, we have fixed a $T_{\underline{\mathbf{d}}}$-fixed flag, which we called the standard coordinate flag, F_e of type $\overline{y}_e$. We have set, for each $w \in \mathbb{W}_{\mathbf{d}}$, $F_w = w.F_e$ and, for each $w \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, $\mathcal{F}_{\overline{w}} = \mathcal{F}_{w(\overline{y}_e)}$.

Definition 7.20. For each $u, u' \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ and $w, w' \in W_{\underline{\mathbf{d}}}$ we set

$$\Omega_w^{\overline{u}} := B_{\underline{\mathbf{d}}}.F_{wu}, \qquad \boldsymbol{\Omega}_w^{\overline{u}',\overline{u}} := G_{\underline{\mathbf{d}}}.(F_{u'}, F_{wu}),$$

Let

$$\pi^{\overline{u}',\overline{u}} : \mathcal{F}_{\overline{u}'} \times \mathcal{F}_{\overline{u}} \to \mathcal{F}_{\overline{u}'}$$

denote the projection onto the first factor. We set

$$\boldsymbol{\Omega}_{w',w}^{\overline{u}',\overline{u}} := (\pi^{\overline{u}',\overline{u}})^{-1}(\Omega_{w'}^{\overline{u}'}) \cap \boldsymbol{\Omega}_w^{\overline{u}',\overline{u}}.$$

The notation has been chosen in such a way that the upper indices indicate the connected component in which the cell is contained, and lower indices indicate relative position within that connected component. Bold font is used for diagonal cells. $\triangle$

Then for each $u, u' \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ we have bijections

$$\begin{array}{ccccc} W_{\underline{\mathbf{d}}}u & \longleftrightarrow & \{B_{\underline{\mathbf{d}}}\text{-orbits on } \mathcal{F}_{\overline{u}}\} & \longleftrightarrow & \{G_{\underline{\mathbf{d}}}\text{-diagonal orbits on } \mathcal{F}_{\overline{u}'} \times \mathcal{F}_{\overline{u}}\}. \\ wu & \longmapsto & \Omega_w^{\overline{u}} & \longmapsto & \boldsymbol{\Omega}_w^{\overline{u}',\overline{u}}. \end{array}$$

We have

$$(\Omega_w^{\overline{u}})^{T_{\underline{\mathbf{d}}}} = \{F_{wu}\}, \qquad (\boldsymbol{\Omega}_w^{\overline{u}',\overline{u}})^{T_{\underline{\mathbf{d}}}} = \{(F_{vu'}, F_{vwu}) \mid v \in W_{\underline{\mathbf{d}}}\}, \qquad (\boldsymbol{\Omega}_{w',w}^{\overline{u}',\overline{u}})^{T_{\underline{\mathbf{d}}}} = \{(F_{w'u'}, F_{w'wu})\}.$$

7.2.2 The cells $\mho_w, \boldsymbol{\mho}_w, O_w$ and $\mathbf{O}_w$

For each connected component $\mathcal{F}_{\overline{u}'} \times \mathcal{F}_{\overline{u}}$ we have a projection

$$\pi^{\overline{u}',\overline{u}} : \mathcal{F}_{\overline{u}'} \times \mathcal{F}_{\overline{u}} \to \mathcal{F}_{\overline{u}'}$$

onto the first factor, which, for each $w \in W_{\underline{\mathbf{d}}}$, restricts to a vector bundle

$$\pi_w^{\overline{u}',\overline{u}} : \boldsymbol{\Omega}_w^{\overline{u}',\overline{u}} \to \mathcal{F}_{\overline{u}'}. \tag{32}$$

Summing over all the connected components we get the projection

$$\pi : \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}. \tag{33}$$

We want to combine the cells $\boldsymbol{\Omega}_w^{\overline{u}',\overline{u}}$ to obtain "quiver Schubert cells" in $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ satisfying the following properties:

- they are disjoint unions of the cells $\boldsymbol{\Omega}_w^{\overline{u}',\overline{u}}$,
- they are stable under the diagonal $G_{\underline{\mathbf{d}}}$-action,
- the projection (33), restricted to a quiver Schubert cell, constitutes a vector bundle over $\mathcal{F}_{\underline{\mathbf{d}}}$,
- it is possible to define an ordering on the quiver Schubert cells which encodes closure relations.

Observe that for each connected component $\mathcal{F}_{\overline{u}'}$ in the base space $\mathcal{F}_{\underline{\mathbf{d}}}$, the fibre $\pi^{-1}(\mathcal{F}_{\overline{u}'}) = \coprod_{u\in \mathrm{Min}(\mathbb{W}_{\mathbf{d}},W_{\underline{\mathbf{d}}})} \mathcal{F}_{\overline{u}'} \times \mathcal{F}_{\overline{u}}$ consists of $|W_{\underline{\mathbf{d}}}\backslash \mathbb{W}_{\mathbf{d}}|$-many connected components. Therefore, to determine a quiver diagonal Schubert cell, we need to make two kinds of choices. In each fibre, we need to choose a connected component, and in that connected component, we need to choose a diagonal Schubert cell.

There is a natural way to make these choices. Recall that $\mathcal{F}_{\underline{\mathbf{d}}}$ is a closed subvariety of the flag variety $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$. We can apply the analysis from the previous section (with $G = \mathbb{G}_{\mathbf{d}}, B = \mathbb{B}_{\mathbf{d}}, T = T_{\underline{\mathbf{d}}}, W = \mathbb{W}_{\mathbf{d}}$) to obtain bijections

$$\mathbb{W}_{\mathbf{d}} \longleftrightarrow \{\mathbb{B}_{\mathbf{d}}\text{-orbits on } \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}\} \longleftrightarrow \{\mathbb{G}_{\mathbf{d}}\text{-diagonal orbits on } \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}\}.$$

Let

$$\varpi : \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \to \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$$

denote the projection onto the first factor.

Definition 7.21. For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\mho_w := \mathbb{B}_{\mathbf{d}}.F_w, \qquad \boldsymbol{\mho}_w := \mathbb{G}_{\mathbf{d}}.(F_e, F_w), \qquad \boldsymbol{\mho}_{w,w'} = \varpi^{-1}(\mho_w) \cap \boldsymbol{\mho}_{w'}.$$

$\mho_w$ is a Schubert cell in $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$ and $\boldsymbol{\mho}_w$ is a diagonal Schubert cell in $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$. We now obtain the desired quiver Schubert cells by restricting the Schubert cells in $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$ resp. $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$ to $\mathcal{F}_{\underline{\mathbf{d}}}$ resp. $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$:

$$O_w := \mho_w \cap \mathcal{F}_{\underline{\mathbf{d}}}, \quad \mathbf{O}_w := \boldsymbol{\mho}_w \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}), \quad \mathbf{O}_{w,w'} := \boldsymbol{\mho}_{w,w'} \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}),$$

We also set

$$O_{\leq w} := \coprod_{w'\leq w} O_{w'}, \quad \mathbf{O}_{\leq w} := \coprod_{w'\leq w} \mathbf{O}_{w'}.$$

We have

$$(O_w)^{T_{\underline{\mathbf{d}}}} = \{F_w\}, \qquad (\mathbf{O}_w)^{T_{\underline{\mathbf{d}}}} = \{(F_{w'}, F_{w'w}) \mid w' \in \mathbb{W}_{\mathbf{d}}\}, \qquad (\mathbf{O}_{w,w'})^{T_{\underline{\mathbf{d}}}} = \{(F_w, F_{ww'})\}.$$

Remark 7.22. If we set $X_r = \coprod_{l(w)\leq r} \mho_w$ and $n = l(w_0)$, where w_0 is the unique longest element of $\mathbb{W}_{\mathbf{d}}$, then

$$\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \varnothing$$

is a partial cellular fibration structure on $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$ over $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$. Now set $X_r = \coprod_{l(w)\leq r} \mathbf{O}_w$. Then

$$G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \times G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \varnothing$$

is only a partial *weak* cellular fibration structure on $G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}} \times G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}$ over $G_{\underline{\mathbf{d}}}/B_{\underline{\mathbf{d}}}$ (i.e. the closure of a cell may be a disjoint union of proper subsets of cells rather than entire cells). This is due to the fact that in general

$$\overline{\mathbf{O}_w} \subsetneq \mathbf{O}_{\leq w}.$$

This fact can be explained as follows. There is a difference between restricting a cell $\boldsymbol{\mho}_w$ to $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ and subsequently taking (Zariski) closure on the one hand, and taking (Zariski) closure of $\boldsymbol{\mho}_w$ first and then restricting $\overline{\boldsymbol{\mho}_w}$ to $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$. We have

$$\overline{\mathbf{O}_w} = \overline{\boldsymbol{\mho}_w \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})} \ \subseteq\ \overline{\boldsymbol{\mho}_w} \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}) = \mathbf{O}_{\leq w}$$

and this inclusion is in most cases proper. We believe that this point was overlooked in Section 2.5 of [VV11]. For example, let $\mathbf{\Gamma}$ be the quiver with two vertices and no arrows and let $\underline{\mathbf{d}} = (1, 1)$. Then $\mathcal{F}(\mathbf{V}) = \mathbb{CP}^1$ and $\mathcal{F}_{\underline{\mathbf{d}}}$ consists of two flags F and F'. We have $\mathbb{W}_{\mathbf{d}} \cong \mathfrak{S}_2 = \{e, s\}, W_{\underline{\mathbf{d}}} = \{e\}$.

The double flag variety $\mathcal{F}(\mathbf{V}) \times \mathcal{F}(\mathbf{V}) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ decomposes into two cells $\mho_e$ and $\mho_s$. The cell $\mho_e$ is the diagonal and $\mho_s$ is its complement. We have

$$\overline{\mho_e} = \mho_e, \quad \overline{\mho_s} = \mathbb{CP}^1 \times \mathbb{CP}^1.$$

The double quiver flag variety $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ consists of four points, i.e.,

$$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}} = \{(F,F), (F',F'), (F,F'), (F',F)\}.$$

We have

$$\overline{\mathbf{O}_e} = \mathbf{O}_e = \{(F,F), (F',F')\},$$
$$\overline{\mathbf{O}_s} = \mathbf{O}_s = \{(F,F'), (F',F)\} \subsetneq \{(F,F), (F',F'), (F,F'), (F',F)\} = \overline{\mho_s} \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}).$$

7.2.3 The cells $\mathbf{O}_w^{\overline{u}}$

We would like to see how the cells $\mathbf{O}_w$, inherited from the cellular fibration of $\mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}} \times \mathbb{G}_{\mathbf{d}}/\mathbb{B}_{\mathbf{d}}$, behave with respect to the decomposition of $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ into connected components. In particular, we would like to decompose each cell O_w into cells of the form $\Omega_x^{\overline{u}}$ and each cell $\mathbf{O}_w$ into cells of the form $\mathbf{\Omega}_x^{\overline{u},\overline{u}'}$. The former is easy - if $w = xu$, with $x \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, then we have

$$O_w = \Omega_x^{\overline{u}}.$$

The latter is more complicated.

Definition 7.23. For each $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ and $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\mathbf{O}_w^{\overline{u}} := G_{\underline{\mathbf{d}}}.(F_u, F_{uw}).$$

$\triangle$

It is immediate that

$$\mathbf{O}_w^{\overline{u}} = \mathbf{O}_w \cap (\mathcal{F}_{\overline{u}} \times \mathcal{F}_{\overline{uw}}) = \mathbf{O}_w \cap \pi^{-1}(\mathcal{F}_u)$$

and

$$\mathbf{O}_w = \coprod_{u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})} \mathbf{O}_w^{\overline{u}}.$$

Suppose that $uw = xu'$, where $x \in W_{\underline{\mathbf{d}}}$ and $u' \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$. Then

$$\mathbf{O}_w^{\overline{u}} := G_{\underline{\mathbf{d}}}.(F_u, F_{uw}) = G_{\underline{\mathbf{d}}}.(F_u, F_{xu'}) = \mathbf{\Omega}_x^{\overline{u},\overline{u}'}.$$

Definition 7.24. The transition between the notations $\mathbf{O}_w^{\overline{u}}$ and $\mathbf{\Omega}_x^{\overline{u},\overline{u}'}$ can be expressed by the following functions:

$$\begin{aligned} ж_1 : \mathbb{W}_{\mathbf{d}} \times \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}}) &\to W_{\underline{\mathbf{d}}} \\ (w,u) &\mapsto x, \\ ж_2 : \mathbb{W}_{\mathbf{d}} \times \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}}) &\to \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}}) \\ (w,u) &\mapsto u', \end{aligned}$$

where $uw = xu'$. $\triangle$

Then

$$\mathbf{O}_w = \coprod_{u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})} \mathbf{\Omega}_{ж_1(w,u)}^{\overline{u},\overline{ж_2(w,u)}}.$$

Unfortunately it is difficult to describe the functions $ж_1, ж_2$ explicitly.

Example 7.25. Consider the quiver
$$i \to j$$
with dimension vector $\underline{\mathbf{d}} = i + 2j$. Then $\mathbb{W}_{\mathbf{d}} \cong \mathfrak{S}_3 = < s, t >$ and $W_{\underline{\mathbf{d}}} \cong \mathfrak{S}_2 = < t >$, where $t = (1)(23)$ and $s = (12)(3)$. The following diagram illustrates the Bruhat ordering on $\mathbb{W}_{\mathbf{d}}$, the cosets with respect to $W_{\underline{\mathbf{d}}}$ (the cosets are elements designated with the same colour) and the minimal length coset representatitves (the elements designated with bold font).

$$\begin{tikzcd}
& sts \arrow[dl] \arrow[dr] & \\
\mathbf{st} \arrow[d] \arrow[drr] & & ts \arrow[d] \arrow[dll] \\
\mathbf{s} \arrow[dr] & & t \arrow[dl] \\
& \mathbf{e} & .
\end{tikzcd}$$

We have three compositions ijj, jii and jij in $Y_{\underline{\mathbf{d}}}$. The coset $\{st, sts\}$ corresponds to the composition jji, the coset $\{s, ts\}$ corresponds to jij and the coset $\{e, t\}$ to ijj. Let $u, u' \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$. We have $\mathcal{F}_{\overline{u}} \times \mathcal{F}_{\overline{u'}} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. We set

$$(\mathcal{F}_{\overline{u}} \times \mathcal{F}_{\overline{u'}})_{\Delta} = \{(F, u'u^{-1}F) \mid F \in \mathcal{F}_{\overline{u}}\} = G_{\underline{\mathbf{d}}}.(F_u, F_{u'}),$$
$$(\mathcal{F}_{\overline{u}} \times \mathcal{F}_{\overline{u'}})_{\nabla} = \{(F, F') \mid F \in \mathcal{F}_{\overline{u}}, F \in \mathcal{F}_{\overline{u'}}, F' \neq u'u^{-1}F\} = G_{\underline{\mathbf{d}}}.(F_u, F_{u't}).$$

We have

$$\begin{array}{lll}
\mathbf{O}_e^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{e}})_{\Delta} = \mathbf{\Omega}_e^{\overline{e},\overline{e}} & & \mathbf{O}_t^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{e}})_{\nabla} = \mathbf{\Omega}_t^{\overline{e},\overline{e}} \\
\mathbf{O}_e^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{s}})_{\Delta} = \mathbf{\Omega}_e^{\overline{s},\overline{s}} & & \mathbf{O}_t^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{st}})_{\Delta} = \mathbf{\Omega}_e^{\overline{s},\overline{st}} \\
\mathbf{O}_e^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{st}})_{\Delta} = \mathbf{\Omega}_e^{\overline{st},\overline{st}} & & \mathbf{O}_t^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{s}})_{\Delta} = \mathbf{\Omega}_e^{\overline{st},\overline{s}} \\
 & & \\
\mathbf{O}_s^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{s}})_{\Delta} = \mathbf{\Omega}_e^{\overline{e},\overline{s}} & & \mathbf{O}_{ts}^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{s}})_{\nabla} = \mathbf{\Omega}_t^{\overline{e},\overline{s}} \\
\mathbf{O}_s^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{e}})_{\Delta} = \mathbf{\Omega}_e^{\overline{s},\overline{e}} & & \mathbf{O}_{ts}^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{st}})_{\nabla} = \mathbf{\Omega}_t^{\overline{s},\overline{st}} \\
\mathbf{O}_s^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{st}})_{\nabla} = \mathbf{\Omega}_t^{\overline{st},\overline{st}} & & \mathbf{O}_{ts}^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{e}})_{\Delta} = \mathbf{\Omega}_e^{\overline{st},\overline{e}} \\
 & & \\
\mathbf{O}_{st}^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{st}})_{\Delta} = \mathbf{\Omega}_e^{\overline{e},\overline{st}} & & \mathbf{O}_{sts}^{\overline{e}} = (\mathcal{F}_{\overline{e}} \times \mathcal{F}_{\overline{st}})_{\nabla} = \mathbf{\Omega}_t^{\overline{e},\overline{st}} \\
\mathbf{O}_{st}^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{e}})_{\nabla} = \mathbf{\Omega}_t^{\overline{s},\overline{e}} & & \mathbf{O}_{sts}^{\overline{s}} = (\mathcal{F}_{\overline{s}} \times \mathcal{F}_{\overline{s}})_{\nabla} = \mathbf{\Omega}_t^{\overline{s},\overline{s}} \\
\mathbf{O}_{st}^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{s}})_{\nabla} = \mathbf{\Omega}_t^{\overline{st},\overline{s}} & & \mathbf{O}_{sts}^{\overline{st}} = (\mathcal{F}_{\overline{st}} \times \mathcal{F}_{\overline{e}})_{\nabla} = \mathbf{\Omega}_t^{\overline{st},\overline{e}}.
\end{array}$$

The closures of the quiver Schubert cells $\mathbf{O}_w$ are given by

$$\begin{array}{ll}
\overline{\mathbf{O}_e} = \mathbf{O}_e & = \mathbf{O}_e^{\overline{e}} \sqcup \mathbf{O}_e^{\overline{s}} \sqcup \mathbf{O}_e^{\overline{st}} \\
\overline{\mathbf{O}_t} = \mathbf{O}_t \sqcup \mathbf{O}_e^{\overline{e}} & = \mathbf{O}_t^{\overline{e}} \sqcup \mathbf{O}_t^{\overline{s}} \sqcup \mathbf{O}_t^{\overline{st}} \sqcup \mathbf{O}_e^{\overline{e}} \\
\overline{\mathbf{O}_s} = \mathbf{O}_s \sqcup \mathbf{O}_e^{\overline{st}} & = \mathbf{O}_s^{\overline{e}} \sqcup \mathbf{O}_s^{\overline{s}} \sqcup \mathbf{O}_s^{\overline{st}} \sqcup \mathbf{O}_e^{\overline{st}} \\
\overline{\mathbf{O}_{ts}} = \mathbf{O}_{ts} \sqcup \mathbf{O}_s^{\overline{e}} \sqcup \mathbf{O}_t^{\overline{s}} & = \mathbf{O}_{ts}^{\overline{e}} \sqcup \mathbf{O}_{ts}^{\overline{s}} \sqcup \mathbf{O}_{ts}^{\overline{st}} \sqcup \mathbf{O}_s^{\overline{e}} \sqcup \mathbf{O}_t^{\overline{s}} \\
\overline{\mathbf{O}_{st}} = \mathbf{O}_{st} \sqcup \mathbf{O}_s^{\overline{s}} \sqcup \mathbf{O}_t^{\overline{st}} & = \mathbf{O}_{st}^{\overline{e}} \sqcup \mathbf{O}_{st}^{\overline{s}} \sqcup \mathbf{O}_{st}^{\overline{st}} \sqcup \mathbf{O}_s^{\overline{s}} \sqcup \mathbf{O}_t^{\overline{st}} \\
\overline{\mathbf{O}_{sts}} = \mathbf{O}_{sts} \sqcup \mathbf{O}_{st}^{\overline{e}} \sqcup \mathbf{O}_e^{\overline{s}} \sqcup \mathbf{O}_{ts}^{\overline{st}} & = \mathbf{O}_{sts}^{\overline{e}} \sqcup \mathbf{O}_{sts}^{\overline{s}} \sqcup \mathbf{O}_{sts}^{\overline{st}} \sqcup \mathbf{O}_{st}^{\overline{e}} \sqcup \mathbf{O}_e^{\overline{s}} \sqcup \mathbf{O}_{ts}^{\overline{st}}.
\end{array}$$

7.3 Stratification of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}_{\underline{\mathbf{d}}}$

We have considered in detail the stratifications of $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$. Recall that we have projections

$$\pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}, \qquad \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}} : \mathcal{Z}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$$

forgetting the stabilizing representations. We are going to use these projections to obtain stratifications on $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}_{\underline{\mathbf{d}}}$.

Definition 7.26. Let us set

$$\widetilde{O}_w := \pi_{\underline{\mathbf{d}}}^{-1}(O_w), \qquad \widetilde{O}_{\leq w} := \coprod_{w' \leq w} \widetilde{O}_{w'},$$

$$\widetilde{\mathbf{O}}_w := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_w), \qquad \widetilde{\mathbf{O}}_w^{\overline{u}} := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_w^{\overline{u}}), \qquad \widetilde{\mathbf{O}}_{w,w'} := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_{w,w'}),$$

$$\mathcal{Z}_{\underline{\mathbf{d}}}^w := \overline{\widetilde{\mathbf{O}}_w}, \qquad \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w} := \bigcup_{w' \leq w} \mathcal{Z}_{\underline{\mathbf{d}}}^{w'} = \coprod_{w' \leq w} \widetilde{\mathbf{O}}_w,$$

$$\mathcal{Z}_{\overline{y},\overline{y}'}^w := \mathcal{Z}_{\underline{\mathbf{d}}}^w \cap \mathcal{Z}_{\overline{y},\overline{y}'}, \qquad \mathcal{Z}_{\overline{y},\overline{y}'}^{\leq w} := \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w} \cap \mathcal{Z}_{\overline{y},\overline{y}'}.$$

Note that $\bigcup_{w' \leq w} \mathcal{Z}_{\underline{\mathbf{d}}}^{w'}$ is in general *not* a disjoint union.

Remark 7.27. It is important to note that there is a difference between pulling back a cell along $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ and subsequently taking (Zariski) closure on the one hand, and taking (Zariski) closure of a cell and then pulling back a closed cell along $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ on the other hand. In other words, taking closure does not commute with pulling back along the projection $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$. We have

$$\mathcal{Z}_{\underline{\mathbf{d}}}^w = \overline{\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_w)} \subseteq \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}\left(\overline{\mathbf{O}_w}\right)$$

and this inclusion is in most cases proper. This holds because a pair of flags in a cell $\mathbf{O}_{w'}$ with $w' < w$ may have a higher-dimensional fibre of stabilizing representations than a pair of flags in the cell $\mathbf{O}_w$. This entire higher-dimensional fibre would be included in $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}\left(\overline{\mathbf{O}_w}\right)$, but only a lower-dimensional subspace would be included in $\overline{\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_w)}$. △

Suppose that $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ stabilizes flags F and F'. Let $g \in G_{\underline{\mathbf{d}}}$. Then $g.\rho$ stabilizes flags $g.F$ and $g.F'$. In particular, for each $g \in G_{\underline{\mathbf{d}}}, u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ and $w \in \mathbb{W}_{\mathbf{d}}$, we have an isomorphism of vector spaces

$$\mathfrak{r}_{u,uw} \xrightarrow{\cong} \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}((g.F_u, g.F_{uw})), \quad \rho \mapsto g.\rho$$

(see Definition 5.10 for the definition of $\mathfrak{r}_{u,uw}$). This implies that the restricted projection

$$\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}} : \widetilde{\mathbf{O}}_w^{\overline{u}} \to \mathbf{O}_w^{\overline{u}}$$

has fibers of constant dimension and is a vector bundle over $\mathbf{O}_w^{\overline{u}}$ (which is itself a vector bundle over $\mathcal{F}_u$). In particular, $\widetilde{\mathbf{O}}_w^{\overline{u}}$ is smooth.

7.3.1 $H_*^A(\{pt\})$-basis

We are now going to use the stratifications we have defined to obtain bases of homology groups of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}_{\underline{\mathbf{d}}}$. The following result is a corollary of the cellular fibration lemma (Lemma 7.17). Recall that we set $S_{T_{\underline{\mathbf{d}}}} = H_*^{T_{\underline{\mathbf{d}}}}(\{pt\})$.

Corollary 7.28. *We have*

$$H_*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} k\left[\,\overline{\widetilde{O}_w}\,\right], \qquad H_*(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w,w' \in \mathbb{W}_{\mathbf{d}}} k\left[\,\overline{\widetilde{\mathbf{O}}_{w,w'}}\,\right],$$

$$H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w\in\mathbb{W}_{\mathbf{d}}} S_{T_{\underline{\mathbf{d}}}}\left[\,\overline{\widetilde{O}_w}\,\right]^{T_{\underline{\mathbf{d}}}}, \qquad H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w,w'\in\mathbb{W}_{\mathbf{d}}} S_{T_{\underline{\mathbf{d}}}}\left[\,\overline{\widetilde{\mathbf{O}}_{w,w'}}\,\right]^{T_{\underline{\mathbf{d}}}},$$

In particular, $H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is an $S_{T_{\underline{\mathbf{d}}}}$-module of rank $|\mathbb{W}_{\mathbf{d}}|$ and $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is an $S_{T_{\underline{\mathbf{d}}}}$-module of rank $|\mathbb{W}_{\mathbf{d}}|^2$.

Proof. Let us extend the Bruhat order on $\mathbb{W}_{\mathbf{d}}$ to a total order $w_1 < w_2 < ... < w_{\mathbf{d}!}$. Then

$$\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \supset \widetilde{O}_{\leq w_{\mathbf{d}!-1}} \supset ... \supset \widetilde{O}_{\leq w_2} \supset \widetilde{O}_{\leq w_1}$$

is a $T_{\underline{\mathbf{d}}}$-equivariant complete weak cellular decomposition of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ in the sense of Definition 7.1 and

$$\mathcal{Z}_{\underline{\mathbf{d}}} \supset \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w_{\mathbf{d}!-1}} \supset ... \supset \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w_2} \supset \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w_1}$$

is a $T_{\underline{\mathbf{d}}}$-equivariant complete weak cellular fibration of $\mathcal{Z}_{\underline{\mathbf{d}}}$ in the sense of Definition 7.6. The statement now follows from Lemma 7.17. □

7.3.2 $H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$-basis

To prove the next proposition we will need some tools (reduction to the torus and the clean intersection formula) which we discuss in the later sections.

Proposition 7.29. *(i) Let $q : \mathcal{Z}_{\underline{\mathbf{d}}} \to \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ be the projection onto the first factor, i.e., $q(F,\rho,F',\rho) = (F,\rho)$. Then for each $w \in \mathbb{W}_{\mathbf{d}}$, $A \in \{\{e\}, T_{\underline{\mathbf{d}}}, G_{\underline{\mathbf{d}}}\}$ the diagram*

$$\begin{tikzcd} H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^e) \times H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w}) \arrow[r, "\star"] & H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w}) \\ H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \times H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \arrow[r, "\star"] \arrow[u, "q^*\times q^*"] & H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \arrow[u, "q^*"'] \end{tikzcd}$$

commutes.
(ii) We have

$$H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w}) = \bigoplus_{u\in\mathbb{W}_{\mathbf{d}}, u\leq w} H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \star \left[\mathcal{Z}_{\underline{\mathbf{d}}}^u\right]^A,$$

i.e. $H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w})$ is a free left $H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$-module with basis $\left\{\left[\mathcal{Z}_{\underline{\mathbf{d}}}^u\right]^A \mid u \leq w, u \in \mathbb{W}_{\mathbf{d}}\right\}$.

Proof. (i) Let $A \in \{\{e\}, T_{\underline{\mathbf{d}}}\}$. The pullbacks q^* and convolution maps $\star$ are maps of S_A-modules. Therefore, it suffices to check commutativity on an S_A-basis of $H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$, for example the basis $\left\{\left[\,\overline{\widetilde{O}_x}\,\right]^A \mid x \in \mathbb{W}_{\mathbf{d}}\right\}$. We have

$$q^*\left(\left[\,\overline{\widetilde{O}_x}\,\right]^A \star \left[\,\overline{\widetilde{O}_y}\,\right]^A\right) = q^*\left(c.\left[\,\overline{\widetilde{O}_x} \cap \overline{\widetilde{O}_y}\,\right]^A\right) = q^*(c).\left[((\overline{\widetilde{O}_x} \cap \overline{\widetilde{O}_y}) \times \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cap \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w}\right]^A,$$

$$\left(q^*\left[\,\overline{\widetilde{O}_x}\,\right]^A \star q^*\left[\,\overline{\widetilde{O}_y}\,\right]^A\right) = \left[\,\overline{\widetilde{\mathbf{O}}_{x,e}}\,\right]^A \star \left[\,\overline{\widetilde{\mathbf{O}}_{y,\leq w}}\,\right]^A = c'.\left[((\overline{\widetilde{O}_x} \cap \overline{\widetilde{O}_y}) \times \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cap \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w}\right]^A,$$

where $c \in H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}), c' \in H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w})$ are A-equivariant Euler classes of vector bundles $\mathcal{T}, \mathcal{T}'$ as in Lemma 9.3. It is straightforward to check that $q^*(c) = c'$. To prove commutativity for $A = G_{\underline{\mathbf{d}}}$, take $W_{\underline{\mathbf{d}}}$-invariants in the $T_{\underline{\mathbf{d}}}$-equivariant diagram.
(ii) This follows from the cellular fibration lemma, where we now consider $H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$, $H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w})$ as $H_*^A(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$-modules. □

7.3.3 Convolution preserves the stratification

We will later need the following result to determine a set of multiplicative generators of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. It shows that the stratification of $\mathcal{Z}_{\underline{\mathbf{d}}}$ gives rise to a certain kind of filtration on $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Lemma 7.30. *(i) For each $x \in \mathbb{W}_{\mathbf{d}}$, $A \in \{\{e\}, T_{\underline{\mathbf{d}}}, G_{\underline{\mathbf{d}}}\}$ the closed embedding $\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq x} \subset \mathcal{Z}_{\underline{\mathbf{d}}}$ induces an injective left graded S_A-module homomorphism $H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq x}) \to H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}})$.*
(ii) For $x, y \in \mathbb{W}_{\mathbf{d}}$ such that $l(xy) = l(x) + l(y)$, we have $H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq x}) \star H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq y}) \subseteq H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq xy})$.

Proof. (i) This follows directly from the cellular fibration lemma.
(ii) For $x' \leq x, y' \leq y$, we have

$$\mathbb{G}_{\mathbf{d}} \cdot (F_e, F_{x'}) \circ \mathbb{G}_{\mathbf{d}} \cdot (F_e, F_{y'}) = \mathbb{G}_{\mathbf{d}} \cdot (F_e, F_{x'}) \circ \mathbb{G}_{\mathbf{d}} \cdot (F_{x'}, F_{x'y'}) = \mathbb{G}_{\mathbf{d}}(F_e, F_{x'y'}).$$

Hence

$$\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq x} \circ \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq y} = \left(\coprod_{x' \leq x} \widetilde{\mathbf{O}}_{x'} \right) \circ \left(\coprod_{y' \leq y} \widetilde{\mathbf{O}}_{y'} \right) = \coprod_{x' \leq x, y' \leq y} \widetilde{\mathbf{O}}_{x'y'}.$$

But $x'y' \leq xy$ if $l(xy) = l(x) + l(y)$. Hence

$$\coprod_{x' \leq x, y' \leq y} \widetilde{\mathbf{O}}_{x'y'} \subseteq \coprod_{z \leq xy} \widetilde{\mathbf{O}}_z$$

and so

$$H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq x}) \star H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq y}) \subseteq H_*^A(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq xy}).$$

□

7.3.4 The centre

We now want to determine the centre of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We will identify it with the $G_{\underline{\mathbf{d}}}$-equivariant cohomology of a point, i.e., with the algebra of symmetric polynomials.

Definition 7.31. Let $1_{\overline{y}, \overline{y}'} := [\mathcal{Z}^e_{\overline{y}, \overline{y}'}], 1_{\overline{y}} := [\widetilde{\mathcal{F}}_{\overline{y}}]$. Further, let $\mathfrak{Z}$ denote $H_*^{G_{\underline{\mathbf{d}}}}(\{pt\}) \left[\mathcal{Z}^e_{\underline{\mathbf{d}}}\right]^{G_{\underline{\mathbf{d}}}} \subset H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. Here $H_*^{G_{\underline{\mathbf{d}}}}(\{pt\})$ acts on $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as explained in Section 2.5. △

We will use the following lemma.

Lemma 7.32. *Let $\overline{y}, \overline{y}', \overline{y}'' \in Y_{\underline{\mathbf{d}}}$. The maps*

$$\begin{aligned} H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y}, \overline{y}'}) &\to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y}, \overline{y}''}) \\ z_{\overline{y}, \overline{y}'} &\mapsto z_{\overline{y}, \overline{y}'} \star 1_{\overline{y}', \overline{y}''} =: \widetilde{z}_{\overline{y}, \overline{y}''}, \end{aligned}$$

$$\begin{aligned} H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y}, \overline{y}'}) &\to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y}'', \overline{y}'}) \\ z_{\overline{y}, \overline{y}'} &\mapsto 1_{\overline{y}'', \overline{y}} \star z_{\overline{y}, \overline{y}'} =: \widetilde{z}_{\overline{y}'', \overline{y}'} \end{aligned}$$

are injective.

Proof. Standard. Calculate the convolution product on $T_{\underline{\mathbf{d}}}$-equivariant Schubert classes by applying the clean intersection formula (Lemma 9.3) and take $W_{\underline{\mathbf{d}}}$-invariants. See also [KL09, Theorem 2.9]. □

We also need the following definition.

Definition 7.33. Let NH_m denote the NilHecke ring, i.e., the unital ring of endomorphisms of $k[y(1),...,y(m)]$ generated by multiplication with $y(1),...,y(m)$ and Demazure operators

$$\partial_l(f) = \frac{f - s_l f}{y(l) - y(l+1)},$$

for $1 \le l \le m-1$, where s_l is the transposition switching $y(l)$ and $y(l+1)$. The endomorphisms which act by multiplication with $y(1),...,y(m)$ generate a subring which is canonically isomorphic to $k[y(1),...,y(m)]$. Moreover, it is well-known that the ring of endomorphisms which act by mutiplication by a symmetric polynomial equals the centre of NH_m.

We are now ready to determine the centre of the convolution algebra $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We follow the proof of [KL09].

Proposition 7.34. *$\mathfrak{Z}$ is the centre of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.*

Proof. We have $\mathfrak{Z} = H_*^{G_{\mathbf{d}}}(\{pt\}) \left[\mathcal{Z}^e_{\underline{\mathbf{d}}}\right]^{G_{\mathbf{d}}}$. But $\left[\mathcal{Z}^e_{\underline{\mathbf{d}}}\right]^{G_{\mathbf{d}}}$ is the unity in the convolution algebra $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and convolution is $H_*^{G_{\mathbf{d}}}(\{pt\})$-linear, so $\mathfrak{Z}$ is contained in the centre of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

We now prove the reverse inclusion, which is slightly more difficult. It follows from Example 11.28 that for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we have an isomorphism of k-algebras

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\overline{y},\overline{y}}) \cong \bigotimes_{l=1}^{|\mathbf{I}|} NH_{\mathbf{d}_l}, \tag{34}$$

where the LHS is a convolution subalgebra of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. The image of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}^e_{\overline{y},\overline{y}})$ under this isomorphism is $\bigotimes_{l=1}^{|\mathbf{I}|} k[y(1),...,y(\mathbf{d}_l)] \cong k[x_{\overline{y}}(1),...,x_{\overline{y}}(\mathbf{d})]$, i.e., the subring consisting of endomorphisms acting by multiplication with a polynomial. It is well-known that the centre of a NilHecke ring consists of the endomorphisms which act by multiplication with a symmetric polynomial. The centre of a tensor product of NilHecke rings is isomorphic to the tensor product of centres of NilHecke rings. Hence the centre of $\bigotimes_{l=1}^{|\mathbf{I}|} NH_{\mathbf{d}_l}$ is $k[x_{\overline{y}}(1),...,x_{\overline{y}}(\mathbf{d})]^{W_{\mathbf{d}}}$.

Summing over connected components, we get a k-algebra isomorphism

$$\bigoplus_{\overline{y}\in Y_{\underline{\mathbf{d}}}} H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\overline{y},\overline{y}}) \cong \bigoplus_{\overline{y}\in Y_{\underline{\mathbf{d}}}} \bigotimes_{l=1}^{|\mathbf{I}|} NH_{\mathbf{d}_l}. \tag{35}$$

If we set $x_{\underline{\mathbf{d}}}(l) := \sum_{\overline{y}\in Y_{\underline{\mathbf{d}}}} x_{\overline{y}}(l)$, then the image of $\mathfrak{Z}$ is $k[x_{\underline{\mathbf{d}}}(1),...,x_{\underline{\mathbf{d}}}(\mathbf{d})]^{W_{\underline{\mathbf{d}}}}$.

Suppose that z lies in the centre of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We can write

$$z = \sum_{\overline{y},\overline{y}'\in Y_{\underline{\mathbf{d}}}} z_{\overline{y},\overline{y}'}, \qquad z_{\overline{y},\overline{y}'} \in H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\overline{y},\overline{y}'}).$$

Let us pick $\overline{\mathbf{y}} \neq \overline{\mathbf{y}}' \in Y_{\underline{\mathbf{d}}}$. We have

$$z \star 1_{\overline{\mathbf{y}},\overline{\mathbf{y}}'} = \sum_{\overline{y}\in Y_{\underline{\mathbf{d}}}} \widetilde{z}_{\overline{y},\overline{\mathbf{y}}'} = 1_{\overline{\mathbf{y}},\overline{\mathbf{y}}'} \star z = \sum_{\overline{y}'\in Y_{\underline{\mathbf{d}}}} \widetilde{z}_{\overline{\mathbf{y}},\overline{y}'},$$

where $\widetilde{z}_{\overline{y},\overline{y}'} \in H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\overline{y},\overline{y}'})$. The only common summand is $\widetilde{z}_{\overline{\mathbf{y}},\overline{\mathbf{y}}'}$, so it's the only possibly non-zero summand. However, convolution with $1_{\overline{\mathbf{y}},\overline{\mathbf{y}}'}$ is injective. Hence, the fact that $\widetilde{z}_{\overline{y},\overline{\mathbf{y}}'} = 0$, for $\overline{y} \neq \overline{\mathbf{y}}$, implies that $z_{\overline{y},\overline{\mathbf{y}}} = 0$ as well. Varying $\overline{\mathbf{y}}$ we see that only the summands $z_{\overline{y},\overline{y}}$ can be non-zero. Hence we can write

$$z = \sum_{\overline{y} \in Y_{\underline{\mathbf{d}}}} z_{\overline{y},\overline{y}}, \qquad z_{\overline{y},\overline{y}} \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}}).$$

Since z is central in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$, $z_{\overline{y},\overline{y}}$ is central in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\overline{y},\overline{y}})$. Under the isomorphism (34), we can identify $z_{\overline{y},\overline{y}}$ with a symmetric polynomial in $k[x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})]^{W_{\underline{\mathbf{d}}}}$. Subtracting an appropriate central element $f \in k[x_{\underline{\mathbf{d}}}(1), ..., x_{\underline{\mathbf{d}}}(\mathbf{d})]^{W_{\underline{\mathbf{d}}}}$ we can assume that $z_{\overline{y},\overline{y}} = 0$. For all $\overline{y}' \in Y_{\underline{\mathbf{d}}}$, we have

$$0 = z_{\overline{y},\overline{y}} \star 1_{\overline{y},\overline{y}'} = z \star 1_{\overline{y},\overline{y}'} = 1_{\overline{y},\overline{y}'} \star z = 1_{\overline{y},\overline{y}'} \star z_{\overline{y}',\overline{y}'}.$$

But convolution with $1_{\overline{y},\overline{y}'}$ is injective, so $z_{\overline{y}',\overline{y}'} = 0$. It follows that $\mathfrak{Z} \cong k[x_{\underline{\mathbf{d}}}(1), ..., x_{\underline{\mathbf{d}}}(\mathbf{d})]^{W_{\underline{\mathbf{d}}}}$ equals the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. □

7.4 Equivariant formality

In this section we will show that the variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ is $G_{\underline{\mathbf{d}}}$-equivariantly formal. This result will prove of great importance in the study of graded finite-dimensional representation theory of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We will later show that the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ acts trivially on any graded simple module. This will allow us to identify graded finite-dimensional representations of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ with those of the finite-dimensional non-equivariant algebra $H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Definition 7.35. A G-space X is called *G-equivariantly formal* if $H_*^G(X) \cong H_*^G(\{pt\}) \otimes_k H_*(X)$.

Proposition 7.36. *The variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ is $T_{\underline{\mathbf{d}}}$-equivariantly formal.*

Proof. Recall that by the cellular fibration lemma, $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a free $H_*^{T_{\underline{\mathbf{d}}}}(\{pt\})$-module with basis $\left\{ \left[\overline{\widetilde{\mathbf{O}}_{w,w'}} \right]^{T_{\underline{\mathbf{d}}}} \mid w, w' \in \mathbb{W}_{\mathbf{d}} \right\}$. This basis restricts to a k-basis $\left\{ \left[\overline{\widetilde{\mathbf{O}}_{w,w'}} \right] \mid w, w' \in \mathbb{W}_{\mathbf{d}} \right\}$ of $H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$. Hence, by the Leray-Hirsch theorem, we obtain the following explicit isomorphism of $H_*^{T_{\underline{\mathbf{d}}}}(\{pt\})$-modules

$$\begin{aligned} H_*^{T_{\underline{\mathbf{d}}}}(\{pt\}) \otimes_k H_*(\mathcal{Z}_{\underline{\mathbf{d}}}) &\to H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \\ \alpha \otimes \left[\overline{\widetilde{\mathbf{O}}_{w,w'}} \right] &\mapsto \alpha \left[\overline{\widetilde{\mathbf{O}}_{w,w'}} \right]^{T_{\underline{\mathbf{d}}}}. \end{aligned}$$

□

Proposition 7.37. *$\mathcal{Z}_{\underline{\mathbf{d}}}$ is $G_{\underline{\mathbf{d}}}$-equivariantly formal.*

Proof. Recall that we can relate the homology of the total space of a fibre bundle to the homology of the fibres and the base using the Leray-Serre spectral sequence. Consider the fibration

$$\eta : EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{Z}_{\underline{\mathbf{d}}} \to BG_{\underline{\mathbf{d}}}$$

with fibre $\mathcal{Z}_{\underline{\mathbf{d}}}$. The homology of the fibres forms a local coefficient system on $BG_{\underline{\mathbf{d}}}$, which we denote by $\{H_*(\mathcal{Z}_{\underline{\mathbf{d}}})\}$. There exists a spectral sequence $(E^r_{p,q})$ with

$$E^2_{p,q} = H_p(BG_{\underline{\mathbf{d}}}; \{H_q(\mathcal{Z}_{\underline{\mathbf{d}}})\})$$

converging to

$$\bigoplus_{p+q=n} E^\infty_{p,q} \cong H_n(EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{Z}_{\underline{\mathbf{d}}}),$$

where the isomorphism is an isomorphism of $H_*(BG_{\underline{\mathbf{d}}})$-modules. Since $G_{\underline{\mathbf{d}}}$ is connected and $EG_{\underline{\mathbf{d}}}$ is contractible, the long exact sequence of homotopy groups

$$1 = \pi_1(EG_{\underline{\mathbf{d}}}) \to \pi_1(BG_{\underline{\mathbf{d}}}) \to \pi_0(G_{\underline{\mathbf{d}}}) = 1$$

associated to the fibration $EG_{\underline{\mathbf{d}}} \to BG_{\underline{\mathbf{d}}}$ with fibre $G_{\underline{\mathbf{d}}}$ implies that $BG_{\underline{\mathbf{d}}}$ is simply-connected. But isomorphism classes of representations of $\pi_1(BG_{\underline{\mathbf{d}}})$ are in a one-to-one correspondence with isomorphism classes of local systems on $BG_{\underline{\mathbf{d}}}$. Hence the only local system on $BG_{\underline{\mathbf{d}}}$ is the constant system $H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$. The universal coefficient theorem now implies that

$$E^2_{p,q} = H_p(BG_{\underline{\mathbf{d}}}; H_q(\mathcal{Z}_{\underline{\mathbf{d}}})) \cong H_p(BG_{\underline{\mathbf{d}}}) \otimes_k H_q(\mathcal{Z}_{\underline{\mathbf{d}}}).$$

Recall that $(E^2_{p,q}, \partial^2)$ forms a bigraded chain complex with the differential

$$\partial^2 : E^2_{p,q} \to E^2_{p-2,q+1}.$$

In our situation this translates to

$$\partial^2 : H_p(BG_{\underline{\mathbf{d}}}) \otimes_k H_q(\mathcal{Z}_{\underline{\mathbf{d}}}) \to H_{p-2}(BG_{\underline{\mathbf{d}}}) \otimes_k H_{q+1}(\mathcal{Z}_{\underline{\mathbf{d}}}).$$

Recall that the odd-dimensional homology of $\mathcal{Z}_{\underline{\mathbf{d}}}$ vanishes . Hence the differential ∂^2 vanishes and the spectral sequence degenerates at E^2. In particular,

$$H_n(EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{Z}_{\underline{\mathbf{d}}}) \cong \bigoplus_{p+q=n} E^\infty_{p,q} = \bigoplus_{p+q=n} E^2_{p,q} = \bigoplus_{p+q=n} H_p(BG_{\underline{\mathbf{d}}}) \otimes_k H_q(\mathcal{Z}_{\underline{\mathbf{d}}}).$$

Thus

$$H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong H^{G_{\underline{\mathbf{d}}}}_*(\{pt\}) \otimes_k H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$$

as modules over $H^{G_{\underline{\mathbf{d}}}}_*(\{pt\})$. □

Recall that we have the fibration

$$\mathcal{Z}_{\underline{\mathbf{d}}} \hookrightarrow EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \mathcal{Z}_{\underline{\mathbf{d}}} \xrightarrow{\eta} BG_{\underline{\mathbf{d}}}.$$

The projection η induces an injective homomorphism $\eta^* : H^{G_{\underline{\mathbf{d}}}}_*(\{pt\}) \to H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ with image $S_{G_{\underline{\mathbf{d}}}}[\mathcal{Z}_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}}$. Note that $[\mathcal{Z}_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}}$ is the unity in $H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ regarded as the algebra endowed with the intersection pairing $\cap$ (i.e., the Poincaré dual of the cohomology algebra endowed with the cup product), and $[\mathcal{Z}^e_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}}$ is the unity in $H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ regarded as a convolution algebra. Let $S^-_{G_{\underline{\mathbf{d}}}}$ and $\mathfrak{Z}^-$ denote the ideals of $S_{G_{\underline{\mathbf{d}}}}$, resp. $\mathfrak{Z}$ generated by elements in strictly negative degrees. Then we have the following equalities:

$$\eta^*(H^{G_{\underline{\mathbf{d}}}}_-(\{pt\}) = S^-_{G_{\underline{\mathbf{d}}}}[\mathcal{Z}_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}} \cap H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}}) = S^-_{G_{\underline{\mathbf{d}}}}[\mathcal{Z}^e_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}} \star H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}}) = \mathfrak{Z}^- \star H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}}). \quad (36)$$

Corollary 7.38. *The quotient of the equivariant convolution algebra $H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ by the right ideal generated by $\mathfrak{Z}^-$ is isomorphic to the non-equivariant convolution algebra $H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$, i.e.,*

$$H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})/(\mathfrak{Z}^- \star H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})) \cong H_*(\mathcal{Z}_{\underline{\mathbf{d}}}). \quad (37)$$

as k-algebras.

Proof. By (36) we can identify the centre of $H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ with $H^{G_{\underline{\mathbf{d}}}}_*(\{pt\})$. The corollary now follows directly from Proposition 7.37. □

8 Relationship between $G_{\underline{\mathbf{d}}}$- and $T_{\underline{\mathbf{d}}}$-equivariant (co)homology

8.1 Schubert and Borel models of cohomology of a flag variety

Recall from Corollary 5.16 that

$$H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong \bigoplus_{\overline{y}\in Y_{\underline{\mathbf{d}}}} k[x_{\overline{y}}(1),...,x_{\overline{y}}(\mathbf{d})] \tag{38}$$

as a k-algebra. It is also well-known that there is a k-algebra isomorphism

$$H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})/(S^+_{G_{\underline{\mathbf{d}}}} H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})) \cong H^*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}). \tag{39}$$

One can prove it using a spectral sequence argument in an analogous way to Corollary 7.38. The isomorphism (39) is often referred to as the *Borel isomorphism*. We will say that the two isomorphisms (38) and (39) form the *Borel model* of the cohomology of a flag variety. We have also shown in Corollary 7.28 that there is a k-vector space isomorphism

$$H^*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w\in\mathbb{W}_{\mathbf{d}}} k\left[\,\overline{\widetilde{O}_w}\,\right] \tag{40}$$

and an $S_{T_{\underline{\mathbf{d}}}}$-module isomorphism

$$H^*_{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w\in\mathbb{W}_{\mathbf{d}}} S_{T_{\underline{\mathbf{d}}}}\left[\,\overline{\widetilde{O}_w}\,\right]^{T_{\underline{\mathbf{d}}}}. \tag{41}$$

We will say that these two isomorphisms form the *Schubert model* of the cohomology of our flag variety. The connection between these two models is not so easy to describe - see [BGG73] for details. The inclusions of a typical fibre in fibrations

$$\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \hookrightarrow ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to BT_{\underline{\mathbf{d}}}, \qquad \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \hookrightarrow EG_{\underline{\mathbf{d}}} \times^{G_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to BG_{\underline{\mathbf{d}}}$$

induce homomorphisms

$$H^*_{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \to H^*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}), \qquad H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \to H^*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}).$$

We refer to these maps as the *forgetful maps* because they "forget" equivariance. They admit a more concrete description. The first map sends each equivariant Schubert class $\left[\,\overline{\widetilde{O}_w}\,\right]^{T_{\underline{\mathbf{d}}}}$ to the correspoding non-equivariant Schubert class $\left[\,\overline{\widetilde{O}_w}\,\right]$. The second map has kernel $S^+_{G_{\underline{\mathbf{d}}}} H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ and induces the Borel isomorphism (39). In particular, both the forgetful maps are surjective. Unfortunately, it is difficult to give an explicit description of the first forgetful map in terms of the Borel model, or an explicit description of the second forgetful map in terms of the Schubert model.

The Borel model is therefore best suited to analyzing the $G_{\underline{\mathbf{d}}}$-equivariant case. It has two advantages. Firstly, it provides a concrete algebraic description of the $G_{\underline{\mathbf{d}}}$-equivariant and nonequivariant cohomology rings. Secondly, the action of the Weyl group $\mathbb{W}_{\mathbf{d}}$ on the $G_{\underline{\mathbf{d}}}$-equivariant cohomology ring is very explicit in the Borel model - $\mathbb{W}_{\mathbf{d}}$ acts naturally by permuting indeterminates $x_{\overline{y}}(l)$ and compositions $\overline{y}$. The main disadvantage of the Borel model is that it is useless for computing the convolution product. The Schubert model is much better suited for this purpose. It yields a basis consisting of fundamental classes, to which we can apply localization and the clean intersection formula (Lemma 9.3). However, the Schubert model is only available in the $T_{\underline{\mathbf{d}}}$-equivariant case because Schubert varieties are not $G_{\underline{\mathbf{d}}}$-stable. Therefore, we will perform most calculations using the Schubert model and $T_{\underline{\mathbf{d}}}$-equivariant homology. However, to use these calculations we need a way to relate $T_{\underline{\mathbf{d}}}$-equivariant homology to $G_{\underline{\mathbf{d}}}$-equivariant homology.

8.2 Reduction to the torus

In this section we will explain how $G_{\underline{\mathbf{d}}}$-equivariant homology can instead be derived from $T_{\underline{\mathbf{d}}}$-equivariant homology.

Proposition 8.1. *Let G be a connected reductive linear algebraic group with a maximal torus T. Let $W = N_G(T)/T$ we the Weyl group of (G,T). Let X be a G-variety. Then W acts on $H_*^G(X)$ and*

$$H_*^G(X) \cong H_*^T(X)^W, \qquad H_G^*(X) \cong H_T^*(X)^W.$$

Proof. See [Bri98, Proposition 1]. □

We now apply this proposition to $X = \{pt\}$, $G = G_{\underline{\mathbf{d}}}$, $T = T_{\underline{\mathbf{d}}}$ and $W = W_{\underline{\mathbf{d}}}$. Recall that

$$H^*_{T_{\underline{\mathbf{d}}}}(\{pt\}) =: S_{T_{\underline{\mathbf{d}}}} \cong k[\mathfrak{t}_{\underline{\mathbf{d}}}] = k[\chi_1, ..., \chi_{\mathbf{d}}].$$

$\mathbb{W}_{\mathbf{d}}$ acts on $k[\mathfrak{t}_{\underline{\mathbf{d}}}]$ by permuting the weights. More precisely, each $w \in \mathbb{W}_{\mathbf{d}}$ acts by

$$w : f = f(\chi_1, ..., \chi_{\mathbf{d}}) \mapsto w(f) = f(\chi_{w(1)}, ..., \chi_{w(\mathbf{d})}). \tag{42}$$

This action restricts to an action of $W_{\underline{\mathbf{d}}}$. The cohomology ring $H^*_{G_{\underline{\mathbf{d}}}}(\{pt\})$ consists of those polynomials in the weights $\chi_1, ..., \chi_{\mathbf{d}}$ which are invariant under $W_{\underline{\mathbf{d}}}$.

9 Euler classes and convolution

9.1 General theory

In this section we state an equivariant version of the so-called "clean intersection" formula and apply it to calculate the convolution product of equivariant fundamental classes of closed subvarieties. At the end we will also explain the connection between topological Euler classes associated to vector bundles and abstract Euler classes associated to certain representations of abelian Lie algebras.

9.1.1 Topological Euler classes

Theorem 9.1. *(i) Let $\pi : E \to B$ be an oriented real vector bundle of rank n. Then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$ and $H^n(E, E_0; \mathbb{Z}) = \mathbb{Z}u$, where u is a unique cohomology class whose restriction*

$$u|(F, F_0) \in H^n(F, F_0; \mathbb{Z})$$

is equal to the preferred generator u_F for every fibre F.
(ii) $y \mapsto y \cup u$ is an isomorphism $H^k(E; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$ for every $k \in \mathbb{Z}$.

Proof. See [MS74, Theorem 9.1]. □

Since $\pi : E \to B$ is a retraction, it is a homotopy equivalence. Hence we obtain the following isomorphism, called the Thom isomorphism:

$$\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$$

$$\phi(x) = (\pi^* x) \cup u.$$

The inclusion $j : (E, \varnothing) \subset (E, E_0)$ induces a restriction homomorphism

$$j^* : H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z}).$$

By applying this homomorphism to the class u we obtain $j^*(u) \in H^n(E; \mathbb{Z})$.

Definition 9.2. (i) We define the *Euler class* of the vector bundle $\pi : E \to B$ to be the cohomology class

$$e(E) = (\pi^*)^{-1} j^*(u).$$

(ii) If E and G are in addition endowed with algebraic actions of a complex linear algebraic group G, and π is a G-equivariant vector bundle, then we define the *G-equivariant Euler class* $e^G(E)$ of the vector bundle $\pi : E \to B$ to be the Euler class of the vector bundle

$$\pi^G : EG \times^G E \to EG \times^G B.$$

We will often make use of the following properties of the Euler class.

(Whitney sum formula) If E, E' are two oriented real vector bundles over B, then $e(E \oplus E') = e(E) \cup e(E')$.
(Top Chern class) If E is a complex vector bundle, then $e(E) = c_{top}(E)$, i.e., the Euler class of E equals the top Chern class of E.

9.1.2 Clean intersection formula

Lemma 9.3. *Let X be a smooth oriented manifold and let Y_1, Y_2 be two closed oriented submanifolds. Assume that $Y := Y_1 \cap Y_2$ is smooth. Let $\mathcal{T}$ be the quotient vector bundle*

$$\mathcal{T} := TX|_Y/(TY_1|_Y + TY_2|_Y)$$

on Y. Assume, moreover, that the intersection of Y_1 and Y_2 is "clean" in the sense that

$$T_yY_1 \cap T_yY_2 = T_yY, \qquad \forall y \in Y.$$

Then

$$[Y_1] \cap [Y_2] = e(\mathcal{T}).[Y],$$

where $\cap : H_*(Y_1) \times H_*(Y_2) \to H_*(Y)$ is the intersection pairing (in the ambient space X), $e(\mathcal{T})$ is the Euler class of the vector bundle $\mathcal{T}$ and the dot on the right hand side stands for the action of $H^*(Y)$ on $H_*(Y)$.

Proof. See [CG97, Proposition 2.6.47], □

Corollary 9.4. *Under the same assumptions as in Lemma 9.3 we have*

$$[Y_1]^G \cap [Y_2]^G = e^G(\mathcal{T}).[Y]^G.$$

Proof. We apply the lemma to the approximation space $E^nG \times^G X$ instead of X, for $n >> 0$. The approximations $E^nG \times^G Y_1$ and $E^nG \times^G Y_2$ of the homotopy quotients of Y_1 and Y_2 are closed oriented submanifolds of $E^nG \times^G X$ with smooth intersection $E^nG \times^G Y$. We also have the approximation $E^nG \times^G \mathcal{T} \to E^nG \times^G Y$ to the homotopy quotient of the vector bundle $\mathcal{T}$. Note that $e(E^nG \times^G \mathcal{T}) \in H^*(E^nG \times^G Y) \subset H^*_G(Y)$ is the G-equivariant Euler class $e^G(\mathcal{T})$ of $\mathcal{T}$. By the lemma,

$$[Y_1]^G \cap [Y_2]^G = [E^nG \times^G Y_1] \cap [E^nG \times^G Y_2] = e(E^nG \times^G \mathcal{T}).[E^nG \times^G Y] = e^G(\mathcal{T}).[Y]^G.$$

□

9.1.3 Application to the equivariant convolution product

Now recall our general convolution set-up. We have three connected oriented smooth manifolds M_1, M_2, M_3 and two closed submanifolds $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$. Let $Y_{12} \subset Z_{12}$ and $Y_{23} \subset Z_{23}$ be closed oriented submanifolds. We consider $[Y_{12}]^G, [Y_{23}]^G$ as classes in $H^G_*(Z_{12})$ resp. $H^G_*(Z_{23})$ and want to compute their convolution product $[Y_{12}]^G \star [Y_{23}]^G \in H^G_*(Z_{12} \circ Z_{23})$. We have

$$\hat{p}^*_{12}([Y_{12}]^G) = [Y_{12} \times M_3]^G, \qquad \hat{p}^*_{23}([Y_{23}]^G) = [M_1 \times Y_{23}]^G.$$

Now $Y_{12} \times M_3, M_1 \times Y_{23}$ are closed oriented submanifolds of $M_1 \times M_2 \times M_3$. Suppose that all the assumptions of lemma 9.3 hold. Then

$$[Y_{12} \times M_3]^G \cap [M_1 \times Y_{23}]^G = e^G(\mathcal{T}).[Y_{12} \times_{M_2} Y_{23}]^G,$$

where

$$\mathcal{T} = \frac{T(M_1 \times M_2 \times M_3)|_{Y_{12} \times_{M_2} Y_{23}}}{(T(Y_{12} \times M_3)|_{Y_{12} \times_{M_2} Y_{23}} + T(M_1 \times Y_{23})|_{Y_{12} \times_{M_2} Y_{23}})}.$$

But $\hat{p}_{13}(Y_{12} \times_{M_2} Y_{23}) = Y_{12} \circ Y_{23}$. Hence

$$(\hat{p}_{13})_*([Y_{12} \times_{M_2} Y_{23}]^G) = [\hat{p}_{13}(Y_{12} \times_{M_2} Y_{23})]^G = [Y_{12} \circ Y_{23}]^G.$$

Therefore we get

$$[Y_{12}]^G \star [Y_{23}]^G = \left((\hat{p}_{13})_*(e^G(\mathcal{T}))\right).[Y_{12} \circ Y_{23}]^G. \tag{43}$$

It follows that the calculation of the equivariant convolution product reduces to the calculation of equivariant Euler classes. We are eventually going to apply the framework we have developed here to the special case in which G is the torus $T_{\underline{\mathbf{d}}}$, Y_{12} and Y_{23} are closed subvarieties each consisting of a single torus fixed point and $\mathcal{T}$ is a vector bundle over a point. The Euler class of the bundle $ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathcal{T}$ will then define a cohomology class in $S_{T_{\underline{\mathbf{d}}}} := H^*_{T_{\underline{\mathbf{d}}}}(\{pt\}; k)$, the $T_{\underline{\mathbf{d}}}$-equivariant cohomology ring of a point. Recall that there is an isomorphism $S_{T_{\underline{\mathbf{d}}}} \cong k[\mathfrak{t}_{\underline{\mathbf{d}}}]$. We will now define some special elements in $k[\mathfrak{t}_{\underline{\mathbf{d}}}]$ arising from certain representations of the Lie algebra $\mathfrak{t}_{\underline{\mathbf{d}}}$ induced by the action of $T_{\underline{\mathbf{d}}}$ on X. We call these elements "abstract Euler classes". We will later show that in the aforementioned special case these abstract Euler classes coincide with the topological Euler classes and provide us with a tool to compute the multiplicities in the clean intersection formula.

9.1.4 Abstract Euler classes

Let G be a complex reductive linear algebraic group with maximal torus T. Let $\mathfrak{t}$ denote the Lie algebra of T. Suppose that M is a finite-dimensional $\mathfrak{t}$-module. Then $M = \bigoplus_{\lambda \in \mathfrak{t}^*} M_\lambda$, where M_λ is the weight space associated to λ. Let $\Lambda = \{\lambda \in \mathfrak{t}^* \mid M_\lambda \neq \{0\}\}$.

Definition 9.5. We define the *Euler class* of the T-module M to be

$$\mathrm{eu}(M) := \prod_{\lambda \in \Lambda} \lambda^{\dim M_\lambda} \in \mathrm{Sym}_k(\mathfrak{t}^*) \cong k[\mathfrak{t}],$$

where $\mathrm{Sym}_k(\mathfrak{t}^*)$ denotes the symmetric algebra of $\mathfrak{t}^*$ and $k[\mathfrak{t}]$ the algebra of polynomials functions on $\mathfrak{t}$. Note that $\mathrm{eu}(M)$ is a homogeneous polynomial of degree $\dim(M)$ on $\mathfrak{t}$. △

Recall there is a canonical isomorphism

$$k[\mathfrak{t}] \to S_T := H_T^*(\{pt\}; k)$$

which doubles degrees. We can therefore consider $\mathrm{eu}(M)$ as a homogeneous polynomial of degree $2 \cdot \dim(M)$ in S_T.

Now assume that X is a quasi-projective variety equipped with an algebraic action of T. Let $x \in X^T$ be a smooth point of X. The tangent space T_xX at x, i.e., the fibre of the tangent bundle TX at x, naturally carries the structure of a T-module. Indeed, every $t \in T$ defines an automorphism

$$t : X \to X$$

of X which induces a linear automorphism

$$T_x(t) : T_xX \to T_{t(x)}X = T_xX$$

of the tangent space T_xX since x is a smooth fixed point. Hence T_xX is a representation of T. By differentiating the map

$$T \to \mathrm{GL}(T_xX)$$

we obtain a representation

$$\mathfrak{t} \to \mathfrak{gl}(T_xX)$$

of the Lie algebra $\mathfrak{t}$ on T_xX.

Definition 9.6. We define the *abstract Euler class* $\mathrm{eu}(X, x)$ associated to the pair (X, x) to be

$$\mathrm{eu}(X, x) := \mathrm{eu}(T_xX) \in k[\mathfrak{t}] \cong S_T.$$

Remark 9.7. We have defined the homotopy quotient $ET \times^T X$ as the quotient of the product $ET \times X$ by the diagonal action of T, where T acts by $(e,x).t = (e.t^{-1}, t.x)$. We could also have defined the homotopy quotient as the quotient of the product $ET \times X$ by the diagonal action $(e,x).t = (e.t, t^{-1}.x)$. In that case the second factor is endowed with the dual of the T-action on X. This is the convention adopted in [VV11]. If we worked with this convention, we would have to make some adjustments to the framework introduced above. In particular, we would be interested in the representation of $\mathfrak{t}$ on the cotangent space T_x^*X. This representation is the dual of the $\mathfrak{t}$-module T_xX. If $T_xX = \bigoplus_\lambda V_\lambda$, then $T_x^*X = \bigoplus_\lambda V_{-\lambda}$. Hence $\mathrm{eu}(T_x^*X) = (-1)^{\dim T_xX}\mathrm{eu}(T_xX)$.

9.2 Applications

We are primarily interested in abstract Euler classes associated to the varieties $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}^x_{\underline{\mathbf{d}}}$, for $x \in \mathbb{W}_{\mathbf{d}}$, and $T_{\underline{\mathbf{d}}}$-fixed points.

Definition 9.8. For each $w, w', x \in \mathbb{W}_{\mathbf{d}}$ we set

$$\Lambda_w := \mathrm{eu}(\mathcal{F}_{\underline{\mathbf{d}}}, F_w), \qquad \widetilde{\Lambda}_w := \mathrm{eu}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, (\rho_0, F_w)),$$
$$\Lambda^x_{w,w'} := \mathrm{eu}(\overline{\mathbf{O}_x}, (F_w, F_{w'})), \qquad \widetilde{\Lambda}^x_{w,w'} := \mathrm{eu}(\mathcal{Z}^x_{\underline{\mathbf{d}}}, (\rho_0, F_w, F_{w'}))$$

whenever the definition makes sense. If $(F_w, F_{w'}) \notin \overline{\mathbf{O}_x}$ then we set $\Lambda^x_{w,w'} = \widetilde{\Lambda}^x_{w,w'} = 0$. If $(F_w, F_{w'}) \in \overline{\mathbf{O}_x}$ but is not a smooth point we set $\Lambda^x_{w,w'} = \widetilde{\Lambda}^x_{w,w'} = 1$.

Since $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}^x_{\underline{\mathbf{d}}}$ are vector bundles, tangent spaces are direct sums of the tangent space to the base space and the tangent space to the fibre.

Definition 9.9. For each $w \in \mathbb{W}_{\mathbf{d}}$, we set

$$\mathfrak{r}_w := \{x \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid F_w \text{ is } x\text{-stable}\}.$$

Clearly, $\mathfrak{r}_w \cong \pi_{\underline{\mathbf{d}}}^{-1}(F_w)$ as $\mathfrak{t}_{\underline{\mathbf{d}}}$-modules, where $\pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}$ is the obvious projection. For $w, w' \in \mathbb{W}_{\mathbf{d}}$, we also set

$$\mathfrak{r}_{w,w'} = \mathfrak{r}_w \cap \mathfrak{r}_{w'}, \qquad \mathfrak{d}_{w,w'} = \mathfrak{r}_w / \mathfrak{r}_{w,w'}.$$

Note that $\mathfrak{r}_{w,w'}$ is symmetric in w and w'. △

Lemma 9.10. *We have*

$$\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{r}_w) \cdot \Lambda_w, \qquad \widetilde{\Lambda}^x_{w,w'} = \mathrm{eu}(\mathfrak{r}_{w,w'}) \cdot \Lambda^x_{w,w'}.$$

Proof. Recall that the projection $\pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}$ is a vector bundle with fibre $\pi_{\underline{\mathbf{d}}}^{-1}(F_w) \cong \mathfrak{r}_w$ at F_w. Hence, by local triviality, we have an isomorphism of $\mathfrak{t}_{\underline{\mathbf{d}}}$-modules

$$T_{(\rho_0, F_w)}\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \xrightarrow{\cong} T_{\rho_0}\pi_{\underline{\mathbf{d}}}^{-1}(F_w) \oplus T_{F_w}\mathcal{F}_{\underline{\mathbf{d}}}.$$

Therefore

$$\mathrm{eu}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, (\rho_0, F_w)) = \mathrm{eu}(\pi_{\underline{\mathbf{d}}}^{-1}(F_w), \rho_0) \cdot \mathrm{eu}(\mathcal{F}_{\underline{\mathbf{d}}}, F_w),$$

i.e., $\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{r}_w) \cdot \Lambda_w$. The projection $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}} : \mathcal{Z}^x_{\underline{\mathbf{d}}} \to \overline{\mathbf{O}_x}$ is also a vector bundle with fibre $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}((F_w, F_{w'})) \cong r_{w,w'}$ at $(F_w, F_{w'})$, provided that $(F_w, F_{w'}) \in \overline{\mathbf{O}_x}$. Hence, by local triviality, we have an isomorphism of $\mathfrak{t}_{\underline{\mathbf{d}}}$-modules

$$T_{(\rho_0, F_w, F_{w'})}\mathcal{Z}^x_{\underline{\mathbf{d}}} \xrightarrow{\cong} T_{\rho_0}\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}((F_w, F_{w'})) \oplus T_{(F_w, F_{w'})}(\overline{\mathbf{O}_x}).$$

Therefore

$$\mathrm{eu}(\mathcal{Z}^x_{\underline{\mathbf{d}}}, (\rho_0, F_w, F_{w'})) = \mathrm{eu}(\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}((F_w, F_{w'})), \rho_0) \cdot \mathrm{eu}(\overline{\mathbf{O}_x}, (F_w, F_{w'})),$$

i.e., $\widetilde{\Lambda}^x_{w,w'} = \mathrm{eu}(\mathfrak{r}_{w,w'}) \cdot \Lambda^x_{w,w'}$. □

9.2.1 A lemma about Coxeter systems

Lemma 9.11. *Let $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ and $s \in \Pi$. Then:*

- *if $us \in W_{\underline{\mathbf{d}}}u$, then $us = \widetilde{s}u$, for some $\widetilde{s} \in \Pi_{\underline{\mathbf{d}}}$,*
- *if $us \notin W_{\underline{\mathbf{d}}}u$, then $us \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$.*

Proof. First suppose that $us \in W_{\underline{\mathbf{d}}}u$. Then $l(us) > l(u)$. But s is a simple reflection, so $l(us) = l(u) \pm 1$. It follows that $l(us) = l(u) + 1$. Moreover, since $us \in W_{\underline{\mathbf{d}}}u$, we have $us = vu$, for some $v \in W_{\underline{\mathbf{d}}}$. Then $l(u)+1 = l(us) = l(vu) = l(v)+l(u)$. This implies that $l(v) = 1$, which is equivalent to saying that $v \in \Pi_{\underline{\mathbf{d}}}$.

Now suppose that $us \notin W_{\underline{\mathbf{d}}}u$. Then $us \in W_{\underline{\mathbf{d}}}\widetilde{u}$, for some $u \neq \widetilde{u} \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$. Hence

$$us = v\widetilde{u}, \tag{44}$$

for some $v \in W_{\underline{\mathbf{d}}}$. Assume that $v \neq e \in W_{\underline{\mathbf{d}}}$. We have $l(v\widetilde{u}) = l(us) = l(u) \pm 1$. Moreover, (44) implies that $v^{-1}us = \widetilde{u}$. Hence

$$l(\widetilde{u}) = l(v) + l(u) \pm 1 > l(u) \pm 1 = l(us) = l(v\widetilde{u}).$$

But this is a contradiction because $\widetilde{u}$ is the element of minimal length in the coset $W_{\underline{\mathbf{d}}}\widetilde{u}$. Hence $v = e$. □

9.2.2 Some isomorphisms of varieties

Our goal now is to compute the tangent spaces to varieties $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\overline{\mathbf{O}_s}$ at $T_{\underline{\mathbf{d}}}$-fixed points. Of course, it suffices to consider the connected component in which a given fixed point is contained. We will use the following isomorphisms in our calculations.

Lemma 9.12. *Let $w \in \mathbb{W}_{\mathbf{d}}, s \in \Pi$. Suppose that $w = vu$ with $v \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$.*
(i) We have an isomorphism of $G_{\underline{\mathbf{d}}}$-varieties

$$G_{\underline{\mathbf{d}}}/B_w \to \mathcal{F}_{\overline{u}}, \qquad g \mapsto g.F_w. \tag{45}$$

(ii) Suppose that $ws \in W_{\underline{\mathbf{d}}}w$. Then there are isomorphisms of $G_{\underline{\mathbf{d}}}$-varieties

$$G_{\underline{\mathbf{d}}} \times^{B_w} (P_{w,ws}/B_w) \to \overline{\mathbf{O}^{\overline{u}}_s}, \qquad (g,p) \mapsto (g.F_w, gp.F_w), \tag{46}$$

$$G_{\underline{\mathbf{d}}} \times^{B_w} (P_{w,ws}/B_{ws}) \to \overline{\mathbf{O}^{\overline{u}}_s}, \qquad (g,p) \mapsto (g.F_w, gp.F_{ws}). \tag{47}$$

where $G_{\underline{\mathbf{d}}}$ acts on the LHS through its natural action on the first factor and acts on the RHS diagonally.

Proof. (i) The isomorphism follows from the fact that the group $G_{\underline{\mathbf{d}}}$ acts on $\mathcal{F}_{\overline{u}}$ transitively and the isotropy group of the flag F_w is B_w.
(ii) By Lemma 9.11, $ws = v\widetilde{s}u$, for some $\widetilde{s} \in \Pi_{\underline{\mathbf{d}}}$. Hence

$$P_{w,ws} = P_{v,v\widetilde{s}}, \qquad B_w = B_v, \qquad B_{ws} = B_{v\widetilde{s}}.$$

The isomorphisms (46), (47) now reduce to

$$G_{\underline{\mathbf{d}}} \times^{B_v} (P_{v,v\widetilde{s}}/B_v) \to \overline{\mathbf{O}^{\overline{u}}_s}, \qquad (g,p) \mapsto (g.F_{vu}, gp.F_{vu}), \tag{48}$$

$$G_{\underline{\mathbf{d}}} \times^{B_v} (P_{v,v\widetilde{s}}/B_{v\widetilde{s}}) \to \overline{\mathbf{O}^{\overline{u}}_s}, \qquad (g,p) \mapsto (g.F_{vu}, gp.F_{v\widetilde{s}u}). \tag{49}$$

respectively. We first show that (48) holds if $v = e$. We have

$$\overline{\mathbf{O}^{\overline{u}}_{s}} = \overline{\mathbf{\Omega}^{\overline{u},\overline{u}}_{\widetilde{s}}}.$$

It follows from Lemma 7.11 that $\overline{\mathbf{\Omega}^{\overline{u},\overline{u}}_{\widetilde{s}}}$ is a fibre bundle over $G_{\underline{\mathbf{d}}}/B_e$ with fibre

$$\overline{\Omega^{\overline{u}}_{\widetilde{s}}} = \Omega^{\overline{u}}_{\widetilde{s}} \cup \Omega^{\overline{u}}_{e} = (B_e\widetilde{s}B_e/B_e) \cup (B_e/B_e) = P_{e,\widetilde{s}}/B_e$$

at F_u. This yields the desired isomorphism

$$G_{\underline{\mathbf{d}}} \times^{B_e} (P_{e,\widetilde{s}}/B_e) \to \overline{\mathbf{O}^{\overline{u}}_{e}}, \qquad (g,p) \mapsto (g.F_u, gp.F_u). \tag{50}$$

We obtain the isomorphism (48) by conjugating the LHS of (50) by v and acting on the RHS diagonally by v. To be more precise, let $\mathrm{aut}(v) : G_{\underline{\mathbf{d}}} \to G_{\underline{\mathbf{d}}}$ denote conjugation by v. Then we have the following commutative diagram

$$\begin{tikzcd}
G_{\underline{\mathbf{d}}} \times^{B_e} P_{e,\widetilde{s}}/B_e \arrow[r] \arrow[d, "\mathrm{aut}(v)\times\mathrm{aut}(v)"'] & \overline{\mathbf{O}^{\overline{u}}_{s}} \arrow[d, "v.\times v."] & (g,p) \arrow[r, mapsto] \arrow[d, mapsto] & (g.F_u, gp.F_u) \arrow[d, mapsto] \\
G_{\underline{\mathbf{d}}} \times^{B_v} P_{v,v\widetilde{s}}/B_v \arrow[r] & \overline{\mathbf{O}^{\overline{u}}_{s}} & (vgv^{-1}, vpv^{-1}) \arrow[r, mapsto] & (vg.F_u, vgp.F_u),
\end{tikzcd}$$

where $(vg.F_u, vgp.F_u) = (vgv^{-1}.F_{vu}, vgv^{-1}vpv^{-1}.F_{vu})$. We have already established that the upper horizontal arrow is an isomorphism and the vertical arrows are clearly isomorphisms as well. Commutativity implies that the lower horizontal arrow is also an isomorphism, as desired. Equivariance follows directly from the formula describing this isomorphism. Isomorphism (47) follows in an analogous fashion from the following commutative diagram

$$\begin{tikzcd}
G_{\underline{\mathbf{d}}} \times^{B_e} P_{e,\widetilde{s}}/B_e \arrow[r] \arrow[d, "\mathrm{aut}(v)\times\mathrm{aut}(v\widetilde{s})"'] & \overline{\mathbf{O}^{\overline{u}}_{s}} \arrow[d, "v.\times vg\widetilde{s}g^{-1}."] & (g,p) \arrow[r, mapsto] \arrow[d, mapsto] & (g.F_u, gp.F_u) \arrow[d, mapsto] \\
G_{\underline{\mathbf{d}}} \times^{B_v} P_{v,v\widetilde{s}}/B_{v\widetilde{s}} \arrow[r] & \overline{\mathbf{O}^{\overline{u}}_{s}} & (vgv^{-1}, v\widetilde{s}p\widetilde{s}v^{-1}) \arrow[r, mapsto] & (vg.F_u, vg\widetilde{s}p.F_u).
\end{tikzcd}$$

Note that conjugating by $\widetilde{s}$ corresponds to a shift in the fibre, while conjugating by v corresponds to a diagonal shift.

□

Lemma 9.13. *For $x, y \in \mathbb{W}_{\mathbf{d}}$ we have isomorphisms of $G_{\underline{\mathbf{d}}}$-varieties*

$$G_{\underline{\mathbf{d}}}/(B_{xy} \cap B_x) \xrightarrow{\cong} \mathbf{O}^{\overline{x}}_{y} = G_{\underline{\mathbf{d}}}.(F_x, F_{xy}), \tag{51}$$

$$B_e/(B_e \cap B_x) \xrightarrow{\cong} O_x = B_e.F_x. \tag{52}$$

Proof. $G_{\underline{\mathbf{d}}}$ acts transitively on $\mathbf{O}^{\overline{x}}_{y}$ with stabilizer $B_{xy} \cap B_x$. Similarly, B_e acts transitively on O_x with stabilizer $B_e \cap B_x$. □

9.2.3 Abstract Euler classes associated to Steinberg and flag varieties

We can now compute the tangent spaces at torus fixed point to the various Schubert varieties considered in the previous section.

Lemma 9.14. *Let $w \in \mathbb{W}_{\mathbf{d}}$.*
(i) We have

$$\Lambda_w = \mathrm{eu}(\mathfrak{n}_w^-).$$

(ii) Let $x, y \in \mathbb{W}_{\mathbf{d}}$. Then

$$\Lambda^y_{x,xy} = \mathrm{eu}(\mathfrak{n}_x^- \oplus \mathfrak{m}_{x,xy}). \tag{53}$$

(iii) Let $s \in \Pi$. If $ws \in W_{\underline{\mathbf{d}}} w$, then

$$\Lambda^s_{w,w} = \mathrm{eu}(\mathfrak{m}_{ws,w} \oplus \mathfrak{n}_w^-), \tag{54}$$

$$\Lambda^s_{w,ws} = \mathrm{eu}(\mathfrak{m}_{w,ws} \oplus \mathfrak{n}_w^-). \tag{55}$$

If $ws \notin W_{\underline{\mathbf{d}}} w$ then

$$\Lambda^s_{w,ws} = \mathrm{eu}(\mathfrak{n}_w^-). \tag{56}$$

Proof. (i) By (45)

$$\begin{aligned} T_{F_w}\mathcal{F}_{\underline{\mathbf{d}}} = T_{F_w}\mathcal{F}_{\overline{u}} &= T_{(eB_w/B_w)}(G_{\underline{\mathbf{d}}}/B_w) \\ &= \mathfrak{g}_{\underline{\mathbf{d}}}/\mathfrak{b}_w = \mathfrak{n}_w^-. \end{aligned}$$

Hence

$$\Lambda_w = \mathrm{eu}(\mathcal{F}_{\underline{\mathbf{d}}}, F_w) = \mathrm{eu}(T_{F_w}\mathcal{F}_{\underline{\mathbf{d}}}) = \mathrm{eu}(\mathfrak{n}_w^-).$$

(ii) Using the isomorphism (51), we get

$$T_{(F_x,F_{xy})}\overline{\mathbf{O}_y} = T_{(F_x,F_{xy})}\mathbf{O}_y^{\overline{x}} = T_e(G_{\underline{\mathbf{d}}}/(B_{xy} \cap B_x)) = \mathfrak{g}_{\underline{\mathbf{d}}}/(\mathfrak{b}_{xy} \cap \mathfrak{b}_x) = \mathfrak{m}_{x,xy} \oplus \mathfrak{n}_x^-.$$

Hence

$$\Lambda^y_{x,xy} = \mathrm{eu}(\overline{\mathbf{O}_y}, (F_x, F_{xy})) = \mathrm{eu}(T_{(F_x,F_{xy})}\overline{\mathbf{O}_y}) = \mathrm{eu}(\mathfrak{m}_{x,xy} \oplus \mathfrak{n}_x^-).$$

(iii) Suppose that $ws \in W_{\underline{\mathbf{d}}} w$. The equality (55) is a special case of (53). We could also have computed it using the isomorphism (47). Indeed, by (47), we have

$$T_{(F_w,F_{ws})}\overline{\mathbf{O}_s} = \mathfrak{p}_{w,ws}/\mathfrak{b}_{ws} \oplus \mathfrak{g}_{\underline{\mathbf{d}}}/\mathfrak{b}_w = \mathfrak{m}_{w,ws} \oplus \mathfrak{n}_w^-.$$

Hence

$$\Lambda^s_{w,ws} = \mathrm{eu}(\overline{\mathbf{O}_s}, (F_w, F_{ws})) = \mathrm{eu}(T_{(F_{ws},F_w)}\overline{\mathbf{O}_s}) = \mathrm{eu}(\mathfrak{m}_{w,ws} \oplus \mathfrak{n}_w^-).$$

To prove (54) we use the isomorphism (46). Indeed, we have

$$T_{(F_w,F_w)}\overline{\mathbf{O}_s} = \mathfrak{p}_{w,ws}/\mathfrak{b}_w \oplus \mathfrak{g}_{\underline{\mathbf{d}}}/\mathfrak{b}_w = \mathfrak{m}_{ws,w} \oplus \mathfrak{n}_w^-.$$

Hence

$$\Lambda^s_{w,w} = \mathrm{eu}(\overline{\mathbf{O}_s}, (F_w, F_w)) = \mathrm{eu}(T_{(F_w,F_w)}\overline{\mathbf{O}_s}) = \mathrm{eu}(\mathfrak{m}_{ws,w} \oplus \mathfrak{n}_w^-).$$

Now suppose that $ws \notin W_{\underline{\mathbf{d}}} w$. Then, by Lemma 9.11, $B_w = B_{ws}$ and so $\mathfrak{n}_w = \mathfrak{n}_{ws}$. Therefore, $\mathfrak{m}_{w,ws} = \mathfrak{n}_w/(\mathfrak{n}_w \cap \mathfrak{n}_{ws}) \cong \{0\}$. Then, by (53), we get

$$\Lambda^s_{w,ws} = \mathrm{eu}(\mathfrak{n}_w^- \oplus \mathfrak{m}_{w,ws}) = \mathrm{eu}(\mathfrak{n}_w^-).$$

$\square$

9.2.4 Euler classes of $\mathfrak{t}_{\underline{\mathbf{d}}}$-modules

We now compute the Euler classes of the tangent spaces from the previous section. If X is a set of weights, then we set

$$\prod(X) := \prod_{x \in X} x.$$

Lemma 9.15. *Let $w \in \mathbb{W}_{\mathbf{d}}$ and $s = s_l \in \Pi$.*

- *If $ws \in W_{\underline{\mathbf{d}}} w$, then*

$$\mathrm{eu}(\mathfrak{n}_{ws}) = -\mathrm{eu}(\mathfrak{n}_w), \qquad \mathrm{eu}(\mathfrak{m}_{w,ws}) = -\mathrm{eu}(\mathfrak{m}_{ws,w}) = \chi_{w(l)} - \chi_{w(l+1)}.$$

- *If $ws \notin W_{\underline{\mathbf{d}}} w$, then*

$$\mathfrak{n}_{ws} = \mathfrak{n}_w, \qquad \mathrm{eu}(\mathfrak{m}_{w,ws}) = \mathrm{eu}(\mathfrak{m}_{ws,w}) = 0.$$

Proof. Suppose that $w = vu$, where $v \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$.
(i) Let $ws \in W_{\underline{\mathbf{d}}} w$. Then $ws = vus \in W_{\underline{\mathbf{d}}} w = W_{\underline{\mathbf{d}}} u$. Hence $us \in W_{\underline{\mathbf{d}}} u$. Then, by Lemma 9.11, we have $us = \widetilde{s} u$, for some $\widetilde{s} \in \Pi_{\underline{\mathbf{d}}}$. Thus $ws = v\widetilde{s}u$. Since $v\widetilde{s}, v \in W_{\underline{\mathbf{d}}}$ and $u \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$, we have

$$Д(\mathfrak{n}_{ws}) = Д(\mathfrak{n}_{v\widetilde{s}}) = v\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}}), \qquad Д(\mathfrak{n}_w) = Д(\mathfrak{n}_v) = v(\Delta^+_{\underline{\mathbf{d}}}). \tag{57}$$

Hence

$$Д(\mathfrak{n}_{w,ws}) = Д(\mathfrak{n}_{v\widetilde{s},v}) = v\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}}) \cap v(\Delta^+_{\underline{\mathbf{d}}}). \tag{58}$$

Since $\widetilde{s} \in \Pi_{\underline{\mathbf{d}}}$, it holds that $\widetilde{s} = s_\beta$ for some simple root $\beta \in \Delta^1_{\underline{\mathbf{d}}}$. Then $\widetilde{s}(\beta) = -\beta$ and the set $\Delta^+_{\underline{\mathbf{d}}} \backslash \{\beta\}$ is stable under $\widetilde{s}$. Hence

$$\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}}) = (\Delta^+_{\underline{\mathbf{d}}} \backslash \{\beta\}) \cup \{-\beta\}, \tag{59}$$

$$v\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}}) = v((\Delta^+_{\underline{\mathbf{d}}} \backslash \{\beta\}) \cup \{-\beta\}) = (v(\Delta^+_{\underline{\mathbf{d}}}) \backslash \{v(\beta)\}) \cup \{-v(\beta)\}. \tag{60}$$

(57) and (60) imply that

$$\mathrm{eu}(\mathfrak{n}_{ws}) = \prod \left(v\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}})\right) = \prod \left((v(\Delta^+_{\underline{\mathbf{d}}}) \backslash \{v(\beta)\}) \cup \{-v(\beta)\}\right) = -\prod \left(v(\Delta^+_{\underline{\mathbf{d}}})\right) = -\mathrm{eu}(\mathfrak{n}_w),$$

which proves the first statement. Moreover, (58) and (60) imply that

$$Д(\mathfrak{n}_{w,ws}) = v\widetilde{s}(\Delta^+_{\underline{\mathbf{d}}}) \cap v(\Delta^+_{\underline{\mathbf{d}}}) = v(\Delta^+_{\underline{\mathbf{d}}}) \backslash \{v(\beta)\},$$

$$Д(\mathfrak{m}_{w,ws}) = Д(\mathfrak{n}_w) \backslash Д(\mathfrak{n}_{w,ws}) = \{v(\beta)\}, \qquad Д(\mathfrak{m}_{ws,w}) = Д(\mathfrak{n}_{ws}) \backslash Д(\mathfrak{n}_{w,ws}) = \{-v(\beta)\}. \tag{61}$$

We have to compute the root $v(\beta)$. Recall that $us = \widetilde{s}u$ and $s = s_l = s_{\beta_l}$. Hence $\widetilde{s}u(\beta_l) = us(\beta_l) = u(-\beta_l) = -u(\beta_l)$, which means that $\widetilde{s}$ is the reflection with respect to the root $u(\beta_l)$. Hence $\beta = u(\beta_l)$. It follows that $v(\beta) = vu(\beta_l) = w(\beta_l)$. The equalities (61) now imply

$$\mathrm{eu}(\mathfrak{m}_{w,ws}) = v(\beta) = w(\beta_l) = \chi_{w(l)} - \chi_{w(l+1)} = -\mathrm{eu}(\mathfrak{m}_{ws,w}).$$

(ii) Let $ws = vus \notin W_{\underline{\mathbf{d}}} w$. Then, by Lemma 9.11, $us \in \mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$. Hence

$$\mathfrak{n}_{ws} = \mathfrak{n}_v = \mathfrak{n}_w \tag{62}$$

and so $\mathrm{eu}(\mathfrak{n}_{ws}) = \mathrm{eu}(\mathfrak{n}_w)$. Moreover, (62) implies that $\mathfrak{n}_{ws} = \mathfrak{n}_w = \mathfrak{n}_{ws} \cap \mathfrak{n}_w = \mathfrak{n}_{w,ws}$. Hence $\mathfrak{m}_{w,ws} = \mathfrak{m}_{ws,w} = \{0\}$ and

$$\mathrm{eu}(\mathfrak{m}_{w,ws}) = \mathrm{eu}(\mathfrak{m}_{ws,w}) = 0.$$

□

Let $s = s_l \in \Pi$. We fix a $w \in \mathbb{W}_{\mathbf{d}}$ and write

$$\overline{y}_w = (i_1^w, i_2^w, ..., i_{\mathbf{d}}^w), \qquad D_{w(m)} = < e_{w(m)} >, \qquad V_w^k = D_{w(1)} \oplus D_{w(2)} \oplus ... \oplus D_{w(k)}.$$

We have, for each $1 \leq k \leq \mathbf{d}$,

$$F_w = (V_w^0 \subset V_w^1 \subset ... \subset V_w^{\mathbf{d}} = \mathbf{V}),$$

$$D_{w(k)} \subset \mathbf{V}_{i_k^w},$$

$$V_w^k = V_{ws}^k \quad \text{if} \quad k \neq l, \qquad V_{ws}^l = V_w^{l-1} \oplus D_{w(l+1)}.$$

Moreover, by Lemma 5.3, $\rho(V_w^k) \subseteq V_w^{k-1}$ for every $\rho \in \mathfrak{r}_w$ and $\rho(V_{ws}^k) \subseteq V_{ws}^{k-1}$ for every $\rho \in \mathfrak{r}_{ws}$. In particular, if $\rho \in \mathfrak{r}_{w,ws}$ then $\rho(V_{ws}^l) \subseteq V_{ws}^{l-1} = V_w^{l-1}$ and so

$$\rho(D_{w(l+1)}) \subseteq V_w^{l-1}. \tag{63}$$

On the other hand, if $\rho \in \mathfrak{r}_w$ and (63) holds, then

$$\rho(V_{ws}^l) = \rho(V_w^{l-1} \oplus D_{w(l+1)}) \subset V_w^{l-1} = V_{ws}^{l-1},$$

so $\rho \in \mathfrak{r}_{w,ws}$. Hence

$$\mathfrak{r}_w = \{\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid \forall k \ \rho(V_w^k) \subset V_w^{k-1}\},$$

$$\mathfrak{r}_{w,ws} = \{\rho \in \mathfrak{r}_w \mid \rho(D_{w(l+1)}) \subseteq V_w^{l-1}\}.$$

We will use the following lemma to describe the action of the fundamental classes $\left[\mathcal{Z}_{s(\overline{y}),\overline{y}}^s\right]^{G_{\underline{\mathbf{d}}}}$ on $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$.

Lemma 9.16. *We have*

$$\mathrm{eu}(\mathfrak{d}_{w,ws}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{i_{l+1}^w, i_l^w}}.$$

Proof. Consider the following map of $T_{\underline{\mathbf{d}}}$-modules

$$\mathfrak{r}_w \to \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, V_w^l), \qquad \rho \mapsto (\rho_h|_{D_{w(l+1)}}). \tag{64}$$

The image of $\mathfrak{r}_{w,ws}$ under this map is contained in $\bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, V_w^{l-1})$. Moreover,

$$\bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, V_w^l) \ / \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, V_w^{l-1}) \cong$$

$$\cong \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, V_w^l / V_w^{l-1})$$

$$\cong \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} \mathrm{Hom}(D_{w(l+1)}, D_{w(l)})$$

$$\cong \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} D_{w(l+1)}^* \otimes D_{w(l)}$$

as $T_{\underline{\mathbf{d}}}$-modules. Since the map (64) is surjective and its kernel is contained in $\mathfrak{r}_{w,ws}$, it follows that the induced map of quotient modules

$$\mathfrak{d}_{w,ws} = \mathfrak{r}_w / \mathfrak{r}_{w,ws} \to \bigoplus_{h \in \mathbf{H}_{i_{l+1}^w, i_l^w}} D_{w(l+1)}^* \otimes D_{w(l)}. \tag{65}$$

is an isomorphism. Hence

$$\mathrm{eu}(\mathfrak{d}_{w,ws}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{i_{l+1}^w, i_l^w}}.$$

□

Lemma 9.17. *(i) Let $ws \in W_{\underline{\mathbf{d}}}w$. Then*

$$\widetilde{\Lambda}^s_{w,w} = \mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_w, \qquad \widetilde{\Lambda}^s_{w,ws} = \mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_{ws},$$

$$\widetilde{\Lambda}_w = -\widetilde{\Lambda}_{ws}.$$

(ii) Let $ws \notin W_{\underline{\mathbf{d}}}w$. Then

$$(\widetilde{\Lambda}^s_{w,ws})^{-1}\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{d}_{w,ws}) = \left(\chi_{w(l)} - \chi_{w(l+1)}\right)^{h_{i^w_{l+1},i^w_l}}.$$

Proof. (i) Since $ws \in W_{\underline{\mathbf{d}}}w$, $\overline{y}_w = s(\overline{y}_w)$ and so $i^w_{l+1} = i^w_l$. By (65) $\mathfrak{d}_{w,ws}$ is trivial since there is no arrow joining i^w_{l+1} to i^w_l (because our quiver has no loops). Hence

$$\mathfrak{r}_w = \mathfrak{r}_{w,ws} = \mathfrak{r}_{ws}. \tag{66}$$

Equation (66) together with Lemmas 9.14 and 9.15 imply

$$\widetilde{\Lambda}^s_{w,w} = \mathrm{eu}(\mathfrak{r}_w)\Lambda^s_{w,w} = \mathrm{eu}(\mathfrak{r}_w)\mathrm{eu}(\mathfrak{m}_{ws,w})\mathrm{eu}(\mathfrak{n}^-_w) = \mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_w,$$

$$\widetilde{\Lambda}^s_{w,ws} = \mathrm{eu}(\mathfrak{r}_{w,ws})\Lambda^s_{w,ws} = \mathrm{eu}(\mathfrak{r}_{ws})\mathrm{eu}(\mathfrak{m}_{w,ws})\mathrm{eu}(\mathfrak{n}^-_w) = \mathrm{eu}(\mathfrak{r}_{ws})\mathrm{eu}(\mathfrak{m}_{ws,w})\mathrm{eu}(\mathfrak{n}^-_{ws}) = \mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_{ws},$$

$$\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{r}_w)\Lambda_w = -\mathrm{eu}(\mathfrak{r}_{ws})\Lambda_{ws} = -\widetilde{\Lambda}_{ws}.$$

(ii) By Lemma 9.14, $\Lambda^s_{w,ws} = \Lambda_w$. Hence

$$\begin{aligned}
(\widetilde{\Lambda}^s_{w,ws})^{-1}\widetilde{\Lambda}_w &= (\mathrm{eu}(\mathfrak{r}_{w,ws}))^{-1}(\Lambda^s_{w,ws})^{-1}\mathrm{eu}(\mathfrak{r}_w)\Lambda_w \\
&= (\mathrm{eu}(\mathfrak{r}_{w,ws}))^{-1}\mathrm{eu}(\mathfrak{r}_w) \\
&= \mathrm{eu}(\mathfrak{r}_w/\mathfrak{r}_{w,ws}) \\
&= \mathrm{eu}(\mathfrak{d}_{w,ws}) \\
&= \left(\chi_{w(l)} - \chi_{w(l+1)}\right)^{h_{i^w_{l+1},i^w_l}}.
\end{aligned}$$

The last equality follows from Lemma 9.16. □

Lemma 9.18. *For all $w, x, y \in \mathbb{W}_{\mathbf{d}}$ such that $l(xy) = l(x) + l(y)$ we have*

$$\mathrm{eu}(\overline{\mathbf{O}_{xy}}, (F_w, F_{wxy}))\mathrm{eu}(\mathcal{F}_{\underline{\mathbf{d}}}, (F_{wx})) = \mathrm{eu}(\overline{\mathbf{O}_x}, (F_w, F_{wx}))\mathrm{eu}(\overline{\mathbf{O}_y}, (F_{wx}, F_{wxy})),$$

$$\mathrm{eu}(\mathfrak{r}_{w,wxy} \oplus \mathfrak{r}_{wx}) = \mathrm{eu}(\mathfrak{r}_{w,wx} \oplus \mathfrak{r}_{wx,wxy}).$$

Proof. See [VV11, Lemma 3.8]. □

10 Localization to $T_{\underline{\mathbf{d}}}$-fixed points

Our goal now is to compute the convolution product. Moreover, we want to show that $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is a faithful module over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$, identify $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ with a subalgebra of $\mathrm{End}_{S_{G_{\underline{\mathbf{d}}}}}(H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}))$ and use this fact to find a set of multiplicative generators of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

10.1 The localization theorem and the localization formula

Theorem 10.1 (Localization theorem)**.** *Let X be a complex algebraic variety which is also a smooth oriented manifold endowed with an algebraic action of a torus $T \cong (\mathbb{C}^*)^m$. Let X^T be the set of fixed points under the action of T, and let $i : X^T \to X$ denote the inclusion. Let $S_T := H_T^*(\{pt\}; k)$ and let K_T be the field of fractions of S_T. Then the induced homomorphisms i_*, i^* of S_T-modules*

$$\begin{array}{ccccc} H_*^T(X^T;k) & \xrightarrow{i_*} & H_*^T(X;k) & \xrightarrow{i^*} & H_*^T(X^T;k) \\ \cong\big\downarrow PD & & \cong\big\downarrow PD & & \cong\big\downarrow PD \\ H_T^*(X^T;k) & \xrightarrow{i_*} & H_T^*(X;k) & \xrightarrow{i^*} & H_T^*(X^T;k) \end{array}$$

become isomorphisms after inverting finitely many characters of T. In particular, all horizontal maps in the diagram below

$$\begin{array}{ccccc} H_*^T(X^T;k)\otimes_{S_T} K_T & \xrightarrow{i_*} & H_*^T(X;k)\otimes_{S_T} K_T & \xrightarrow{i^*} & H_*^T(X^T;k)\otimes_{S_T} K_T \\ \cong\big\downarrow PD & & \cong\big\downarrow PD & & \cong\big\downarrow PD \\ H_T^*(X^T;k)\otimes_{S_T} K_T & \xrightarrow{i_*} & H_T^*(X;k)\otimes_{S_T} K_T & \xrightarrow{i^*} & H_T^*(X^T;k)\otimes_{S_T} K_T \end{array}$$

are isomorphisms of K_T-vector spaces.

Proof. See [Bri98, Theorem 3]. □

Theorem 10.2 (Localization formula)**.** *Suppose that the same assumptions as in the theorem above hold. Suppose that X^T is finite and let $X^T = \{x_1, ..., x_m\}$. For each $1 \leq l \leq m$ let $i_l : \{x_l\} \to X$ denote the inclusion. Let $\alpha \in H_*^T(X;k)\otimes_{S_T} K_T \cong H_T^*(X;k)\otimes_{S_T} K_T$. Then*

$$\alpha = \sum_{l=1}^{m} \frac{(i_l)_* i_l^*(\alpha)}{\mathrm{eu}(X, x_l)}.$$

More generally, if X is not necessarily smooth, we also have the following formula

$$[X]^T = \sum_{l=1}^{m} f_l[\{x_l\}]^T,$$

where $f_l \in K_T$ and $f_l = (\mathrm{eu}(X, x_l))^{-1}$ whenever x_l is a smooth point of X.

Proof. See [EG96, Theorem 3]. □

Let us denote

$$\mathcal{H}_*^T(X;k) := H_*^T(X;k)\otimes_{S_T} K_T, \qquad \mathcal{H}_T^*(X;k) := H_T^*(X;k)\otimes_{S_T} K_T$$

If c is a homology class in $H_*^T(X;k)$, then we will also abbreviate

$$c := c \otimes 1 \in \mathcal{H}^T(X;k).$$

10.2 Applications of localization

10.2.1 Change of basis

Now let $X = \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ or $\mathcal{Z}_{\underline{\mathbf{d}}}$ and $T = T_{\underline{\mathbf{d}}}$. The isomorphisms from Corollary 7.28 become, after localization (i.e., applying the functor $K_{T_{\underline{\mathbf{d}}}} \otimes_{S_{T_{\underline{\mathbf{d}}}}} \_$),

$$\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) = \bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \left[\, \overline{\widetilde{O}_w} \,\right]^{T_{\underline{\mathbf{d}}}}, \qquad \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) = \bigoplus_{w,w' \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \left[\, \overline{\widetilde{\mathbf{O}}_{w,w'}} \,\right]^{T_{\underline{\mathbf{d}}}}.$$

Recall that

$$(\mathcal{F}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{F_w \mid w \in \mathbb{W}_{\mathbf{d}}\}, \quad (\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{(\rho_0, F_w) \mid w \in \mathbb{W}_{\mathbf{d}}\}, \quad (\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} = \{(\rho_0, F_w, F_{w'}) \mid w, w' \in \mathbb{W}_{\mathbf{d}}\}.$$

For each $w, w' \in \mathbb{W}_{\mathbf{d}}$ let

$$\psi_w := [\{(\rho_0, F_w)\}]^{T_{\underline{\mathbf{d}}}} \in H_*^{T_{\underline{\mathbf{d}}}}((\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}), \qquad \psi_{w,w'} := [\{(\rho_0, F_w, F_{w'})\}]^{T_{\underline{\mathbf{d}}}} \in H_*^{T_{\underline{\mathbf{d}}}}((\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}})$$

denote the $T_{\underline{\mathbf{d}}}$-equivariant fundamental classes of the singleton sets $\{(\rho_0, F_w)\}$ and $\{(\rho_0, F_w, F_{w'})\}$. These fundamental classes generate the cohomology rings of $(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}$ and $(\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}$. After localization we get

$$\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}((\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}) = \bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \psi_w, \qquad \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}((\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}) = \bigoplus_{w,w' \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \psi_{w,w'}.$$

Given the inclusions $i : (\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} \to \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$, $i : (\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}} \to \mathcal{Z}_{\underline{\mathbf{d}}}$, we will also use the notation

$$\psi_w = i_* \psi_w, \qquad \psi_{w,w'} = i_* \psi_{w,w'}$$

for the fundamental classes of the point subvarieties $\{(\rho_0, F_w)\}$ and $\{(\rho_0, F_w, F_{w'})\}$ of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ resp. $\mathcal{Z}_{\underline{\mathbf{d}}}$.

We have the following immediate corollary of the localization theorem.

Corollary 10.3. *The inclusions $i : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{T_{\underline{\mathbf{d}}}} \to \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$, $i : \mathcal{Z}_{\underline{\mathbf{d}}}^{T_{\underline{\mathbf{d}}}} \to \mathcal{Z}_{\underline{\mathbf{d}}}$ induce isomorphisms of $K_{T_{\underline{\mathbf{d}}}}$-vector spaces*

$$\bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \psi_w = \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}((\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}) \xrightarrow{i_*} \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) = \bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \left[\, \overline{\widetilde{O}_w} \,\right]^{T_{\underline{\mathbf{d}}}},$$

$$\bigoplus_{w,w' \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \psi_{w,w'} = \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}((\mathcal{Z}_{\underline{\mathbf{d}}})^{T_{\underline{\mathbf{d}}}}) \xrightarrow{i_*} \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) = \bigoplus_{w,w' \in \mathbb{W}_{\mathbf{d}}} K_{T_{\underline{\mathbf{d}}}} \left[\, \overline{\widetilde{O}_{w,w'}} \,\right]^{T_{\underline{\mathbf{d}}}}.$$

We now have two bases of the localized cohomology rings of $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}_{\underline{\mathbf{d}}}$: a basis consisting of fundamental classes of the torus fixed points and a basis consisting of the fundamental classes of the Schubert subvarieties. We would like to know more about how these two bases are related.

Corollary 10.4. *For each $w \in \mathbb{W}_{\mathbf{d}}$, let $i_w : \{(\rho_0, F_w)\} \to \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ be the inclusion. We have*

$$[\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}]^{T_{\underline{\mathbf{d}}}} = \sum_{w \in \mathbb{W}_{\mathbf{d}}} \widetilde{\Lambda}_w^{-1} \psi_w.$$

Proof. By the localization formula,

$$[\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}]^{T_{\underline{\mathbf{d}}}} = \sum_{w \in \mathbb{W}_{\mathbf{d}}} \widetilde{\Lambda}_w^{-1} (i_w)_* i_w^* [\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}]^{T_{\underline{\mathbf{d}}}}.$$

For each $w \in \mathbb{W}_{\mathbf{d}}$, the projection formula yields

$$\widetilde{\Lambda}_w^{-1} (i_w)_* i_w^* [\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}]^{T_{\underline{\mathbf{d}}}} = \widetilde{\Lambda}_w^{-1} ([\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}]^{T_{\underline{\mathbf{d}}}} \cap (i_w)_* \psi_w) = \widetilde{\Lambda}_w^{-1} \psi_w.$$

□

Corollary 10.5. *Let $s \in \Pi$. Then*

$$[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\underline{\mathbf{d}}}} = \sum_{w \in \mathbb{W}_{\mathbf{d}}} (\widetilde{\Lambda}_{w,ws}^s)^{-1} \psi_{w,ws} + \sum_{w \ | \ ws \in W_{\underline{\mathbf{d}}} w} (\widetilde{\Lambda}_{w,w}^s)^{-1} \psi_{w,w}. \tag{67}$$

Proof. The variety $\mathcal{Z}_{\underline{\mathbf{d}}}^s$ is smooth because it is a disjoint union of fibre bundles over flag varieties with fibre trivial or isomorphic to $\mathbb{CP}^1$. Hence we can apply the localization formula 10.2 with $X = \mathcal{Z}_{\underline{\mathbf{d}}}^s$. The $T_{\underline{\mathbf{d}}}$-fixed points in $\mathcal{Z}_{\underline{\mathbf{d}}}^s$ are

$$(\mathcal{Z}_{\underline{\mathbf{d}}}^s)^{T_{\underline{\mathbf{d}}}} = \{(\rho_0, F_w, F_{ws}) \mid w \in \mathbb{W}_{\mathbf{d}}\} \cup \{(\rho_0, F_w, F_w) \mid w \in \mathbb{W}_{\mathbf{d}} \text{ s.t. } ws \in W_{\underline{\mathbf{d}}} w\}.$$

Let $i_{w,x} : \{(\rho_0, F_w, F_x)\} \hookrightarrow \mathcal{Z}_{\underline{\mathbf{d}}}^s$, where $x = w$ or ws, denote the inclusion. Then, by the projection formula,

$$(i_{w,x})_* i_{w,x}^* [\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\underline{\mathbf{d}}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\underline{\mathbf{d}}}} \cap (i_{w,x})_* \psi_{w,x} = \psi_{w,x}.$$

Instead of using the projection formula, one could also simply observe that the class $[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\underline{\mathbf{d}}}}$ is the unity in $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^s)$, and since $i_{w,x}^*$ is a homomorphism of cohomology rings, it must map it to $\psi_{w,x}$, the unity in $H_*^{T_{\underline{\mathbf{d}}}}\left((\mathcal{Z}_{\underline{\mathbf{d}}}^s)^{T_{\underline{\mathbf{d}}}}\right)$. The formula (67) for $[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\underline{\mathbf{d}}}}$ in $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^s)$ now follows directly from the localization formula. But the cellular fibration lemma implies that there is a canonical inclusion $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^s) \hookrightarrow \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$, so the formula (67) also holds in $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. □

In general, Schubert varieties are not smooth, so we cannot apply the localization formula directly to find an expression for their fundamental classes in the new basis. Instead, we would need to embed them into a smooth variety, for example $\mathcal{Z}_{\underline{\mathbf{d}}}$, and apply the localization formula to this smooth variety. However, if we do that, the trick with the projection formula no longer works, and in some cases it is rather hard to find explicit coefficients in the new basis.

10.2.2 Reduction to the torus revisited

Corollary 10.6. *The image of the inclusion $H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \hookrightarrow \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is*

$$\bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} S_{T_{\underline{\mathbf{d}}}} \left[\overline{\widetilde{O}_w} \right]^{T_{\underline{\mathbf{d}}}} = \bigoplus_{w \in \mathbb{W}_{\mathbf{d}}} S_{T_{\underline{\mathbf{d}}}} \widetilde{\Lambda}_w^{-1} \psi_w. \tag{68}$$

Proof. By the localization formula,

$$\left[\overline{\widetilde{O}_w} \right]^{T_{\underline{\mathbf{d}}}} = \sum_{u \leq w} \widetilde{\Lambda}_u^{-1} (i_u)_* i_u^* \left[\overline{\widetilde{O}_u} \right]^{T_{\underline{\mathbf{d}}}}.$$

For each $u \leq w$, the projection formula yields

$$\widetilde{\Lambda}_u^{-1} (i_u)_* i_u^* \left[\overline{\widetilde{O}_u} \right]^{T_{\underline{\mathbf{d}}}} = \widetilde{\Lambda}_u^{-1} \left(\left[\overline{\widetilde{O}_u} \right]^{T_{\underline{\mathbf{d}}}} \cap (i_u)_* \psi_u \right) = \widetilde{\Lambda}_u^{-1} f \psi_u,$$

for some $f \in S_{T_{\underline{\mathbf{d}}}}$. When (ρ_0, F_u) is a smooth point of $\overline{\widetilde{O}_w}$, then $f = \widetilde{\Lambda}_u / \mathrm{eu}(\overline{\widetilde{O}_w}, (\rho_0, F_u))$, by the localization formula or the clean intersection formula. We have therefore shown that $\left[\overline{\widetilde{O}_w} \right]^{T_{\underline{\mathbf{d}}}}$ can be expressed in the $\{\psi_w \mid w \in \mathbb{W}_{\mathbf{d}}\}$ basis with coefficients in $S_{T_{\underline{\mathbf{d}}}} \widetilde{\Lambda}_w^{-1}$. □

$W_{\underline{\mathbf{d}}}$ acts on the fundamental classes $\{\psi_w \mid w \in W_{\overline{y}}\}$ by permuting them, i.e., each $w \in W_{\underline{\mathbf{d}}}$ acts by

$$w : \psi_u \mapsto \psi_{wu}, \tag{69}$$

where $u \in W_{\overline{y}}$. Combining (42), (69) and (68) we obtain the $W_{\underline{\mathbf{d}}}$-action on $H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Each $w \in W_{\underline{\mathbf{d}}}$ acts by

$$w : f\widetilde{\Lambda}_u^{-1}\psi_u \mapsto w(f)\widetilde{\Lambda}_{wu}^{-1}\psi_{wu}, \tag{70}$$

for all $f \in S_{T_{\underline{\mathbf{d}}}}$, $u \in W_{\overline{y}}$. Hence

$$\left(H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})\right)^{W_{\underline{\mathbf{d}}}} = \left\{ \sum_{w \in W_{\overline{y}}} f(\chi_{w(1)}, ..., \chi_{w(\mathbf{d})})\widetilde{\Lambda}_w^{-1}\psi_w \;\middle|\; f \in k[\mathfrak{t}_{\underline{\mathbf{d}}}] \right\}.$$

We have an isomorphism

$$H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}}) \to \left(H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})\right)^{W_{\underline{\mathbf{d}}}} \tag{71}$$

$$f(x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})) \mapsto \sum_{w \in W_{\overline{y}}} f(\chi_{w(1)}, ..., \chi_{w(d)})\widetilde{\Lambda}_w^{-1}\psi_w = \sum_{w \in W_{\overline{y}}} w(f)\widetilde{\Lambda}_w^{-1}\psi_w. \tag{72}$$

Observe that this isomorphism is not canonical. For example, we could also have chosen

$$f \mapsto \sum_{w \in W_{\overline{y}}} w(f)\widetilde{\Lambda}_{uw}^{-1}\psi_{uw}$$

for any $u \in W_{\underline{\mathbf{d}}}$. In the sequel we will always use the isomorphism (72).

10.2.3 Calculation of the convolution product

Let $\star, \diamond$ also denote the convolution products

$$\star : H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \to H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k),$$

$$\diamond : H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k) \to H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k).$$

Our goal now is to compute convolution products of the basis elements $\psi_{w,w'}$ and ψ_w. The following important theorem will allow us to prove that the representation of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ on $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is faithful.

Theorem 10.7. *For $w, w', w'', w''' \in \mathbb{W}_{\mathbf{d}}$ we have*

$$\psi_{w''',w''} \star \psi_{w',w} = \delta_{w'',w'}\widetilde{\Lambda}_{w'}\psi_{w''',w}, \qquad \psi_{w'',w'} \diamond \psi_w = \delta_{w',w}\widetilde{\Lambda}_w\psi_{w''},$$

where $\delta_{w',w}$ is the Kronecker delta.

Proof. We use the notation and results of section 9.1.3. Let $M_1 = M_2 = M_3 = \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and set

$$Z_{12} = M_1 \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} M_2 \cong \mathcal{Z}_{\underline{\mathbf{d}}}, \qquad Z_{23} = M_2 \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} M_3 \cong \mathcal{Z}_{\underline{\mathbf{d}}}.$$

Then

$$Z_{12} \circ Z_{23} \cong \mathcal{Z}_{\underline{\mathbf{d}}}.$$

Moreover, set

$$Y_{12} = \{((\rho_0, F_{w'''}), (\rho_0, F_{w''}))\} \subset Z_{12}, \qquad Y_{23} = \{((\rho_0, F_{w'}), (\rho_0, F_w))\} \subset Z_{23}.$$

Clearly

$$Y_{12} \circ Y_{23} = \{((\rho_0, F_{w'''}), (\rho_0, F_w))\} \subset Z_{12} \circ Z_{23} \cong \mathcal{Z}_{\underline{\mathbf{d}}}$$

if $w' = w''$ or is empty otherwise. Hence

$$[Y_{12} \circ Y_{23}]^{T_{\underline{\mathbf{d}}}} = \delta_{w'',w,}\psi_{w''',w}.$$

From now on suppose that $w' = w''$. The equivariant clean intersection formula (43) implies that

$$\psi_{w''',w'} \star \psi_{w',w} = [Y_{12}]^{T_{\underline{\mathbf{d}}}} \star [Y_{23}]^{T_{\underline{\mathbf{d}}}} = e^{T_{\underline{\mathbf{d}}}}(\mathcal{T}).[Y_{12} \circ Y_{23}]^{T_{\underline{\mathbf{d}}}} = e^{T_{\underline{\mathbf{d}}}}(\mathcal{T}).\psi_{w''',w}.$$

We now need to calculate the vector bundle $\mathcal{T}$. Observe that

$$Y_{12} \times_{M_2} Y_{23} = (Y_{12} \times M_3) \cap (M_1 \times Y_{23}) = \{((\rho_0, F_{w'''}), (\rho_0, F_{w'}), (\rho_0, F_w))\}.$$

Let us abbreviate

$$y := ((\rho_0, F_{w'''}), (\rho_0, F_{w'}), (\rho_0, F_w)),$$

$$y_1 = (\rho_0, F_{w'''}), \qquad y_2 = (\rho_0, F_{w'}), \qquad y_3 = (\rho_0, F_w).$$

We have

$$T(M_1 \times M_2 \times M_3)|_{\{y\}} = T_{y_1}M_1 \oplus T_{y_2}M_2 \oplus T_{y_3}M_3,$$

$$T(Y_{12} \times M_3)|_{\{y\}} = T_{y_3}M_3, \qquad T(M_1 \times Y_{23})|_{\{y\}} = T_{y_1}M_1.$$

Hence

$$\mathcal{T} = \frac{T_{y_1}M_1 \oplus T_{y_2}M_2 \oplus T_{y_3}M_3}{T_{y_1}M_1 + T_{y_3}M_3} = T_{y_2}M_2 := T_{(\rho_0, F_{w'})}\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}.$$

$\mathcal{T}$ is thus a bundle over a point, i.e., just a vector space. The vector space $T_{y_2}M_2$ is naturally endowed with the structure of a $T_{\underline{\mathbf{d}}}$-module. It can be decomposed into one-dimensional representations of $T_{\underline{\mathbf{d}}}$

$$\mathcal{T} = T_{y_2}M_2 = \bigoplus_{\lambda \in Д(\mathcal{T})} \mathbb{C}_\lambda.$$

We now want to calculate its equivariant Euler class. We pass to homotopy quotients and get the following vector bundle

$$ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathcal{T} \to BT_{\underline{\mathbf{d}}}.$$

It can be decomposed as a direct sum of line bundles:

$$ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathcal{T} = ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \left(\bigoplus_{\lambda \in Д(\mathcal{T})} \mathbb{C}_\lambda \right) = \bigoplus_{\lambda \in Д(\mathcal{T})} \left(ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathbb{C}_\lambda \right).$$

We have

$$\begin{aligned} e^{T_{\underline{\mathbf{d}}}}(\mathcal{T}) = e(ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathcal{T}) = c_{top}(ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathcal{T}) \\ = \prod_{\lambda \in Д(\mathcal{T})} c_1(ET_{\underline{\mathbf{d}}} \times^{T_{\underline{\mathbf{d}}}} \mathbb{C}_\lambda) = \prod_{\lambda \in Д(\mathcal{T})} \lambda = \mathrm{eu}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, (\rho_0, F_{w'})) = \widetilde{\Lambda}_{w'}. \end{aligned}$$

The proof of the second formula is analogous. □

Remark 10.8. We have finally shown that in our special case topological Euler classes coincide with the abstract Euler classes.

10.2.4 Implications for $G_{\underline{\mathbf{d}}}$-equivariant convolution

The calculations above have implications for our original algebras $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}), H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ in view of the following lemma.

Lemma 10.9. *The forgetful maps*

$$H_*^{G_{\underline{\mathbf{d}}}}\left(\mathcal{Z}_{\underline{\mathbf{d}}}\right) \to H_*^{T_{\underline{\mathbf{d}}}}\left(\mathcal{Z}_{\underline{\mathbf{d}}}\right), \qquad H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\right) \to H_*^{T_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\right).$$

commute with the convolution product $\star$ resp. $\diamond$.

Proof. The pullback and pushforward maps in the definition of the convolution product are, in our case, $W_{\underline{\mathbf{d}}}$-equivariant, so the convolution product commutes with taking invariants. $\square$

Theorem 10.10. *The left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$-module $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is faithful.*

Proof. First consider the $K_{T_{\underline{\mathbf{d}}}}$-linear map

$$\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k) \to \mathrm{End}_{K_{T_{\underline{\mathbf{d}}}}}\left(\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)\right). \tag{73}$$

Take a nonzero element $0 \neq \sum_{w,w' \in \mathbb{W}_{\mathbf{d}}} a_{w',w}\psi_{w',w} \in \mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k)$, where $a_{w',w} \in K_{T_{\underline{\mathbf{d}}}}$. Then there exists at least one nonzero coefficient $a_{u',u} \neq 0$. By Theorem 10.7 We have

$$\sum_{w,w' \in \mathbb{W}_{\mathbf{d}}} a_{w',w}\psi_{w',w} \diamond \psi_u = \sum_{w' \in \mathbb{W}_{\mathbf{d}}} a_{w',u}\widetilde{\Lambda}_u \psi_{w'} \neq 0.$$

Therefore, the map (73) has trivial kernel, i.e., the left $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k)$-module $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)$ is faithful. Since the $S_{T_{\underline{\mathbf{d}}}}$-modules $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k)$ and $H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)$ are free, we also obtain a faithful representation

$$H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k) \hookrightarrow \mathrm{End}_{S_{T_{\underline{\mathbf{d}}}}}\left(H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)\right).$$

Taking $W_{\underline{\mathbf{d}}}$-invariants yields a faithful representation

$$H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k) = (H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k))^{W_{\underline{\mathbf{d}}}} \hookrightarrow \left(\mathrm{End}_{S_{T_{\underline{\mathbf{d}}}}}\left(H_*^{T_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)\right)\right)^{W_{\underline{\mathbf{d}}}} = \mathrm{End}_{S_{G_{\underline{\mathbf{d}}}}}\left(H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}};k)\right).$$

$\square$

11 Generators and relations

The purpose of this chapter is to reinterpret the geometric results from the previous chapters in algebraic terms. It contains two very important results about the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. The first result is a description of its faithful polynomial representation $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$. The second result, deduced from the first, is a presentation of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ in terms of generators and relations. In particular, we give a complete list of relations for this algebra.

11.1 Generators of the convolution algebra

We begin by defining some elements in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We will later show that these elements are multiplicative generators of the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Definition 11.1. (i) Recall that for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we set $1_{\overline{y},\overline{y}} = [Z^e_{\overline{y},\overline{y}}]^{G_{\underline{\mathbf{d}}}}$.
(ii) Recall that $\Pi = \{s_1, ..., s_{\mathbf{d}-1}\}$ denotes the set of simple reflections in W. Now fix a simple reflection $s = s_l$. Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. We set

$$\sigma_{\overline{y}}(l) := [\mathcal{Z}^s_{s(\overline{y}),\overline{y}}]^{G_{\underline{\mathbf{d}}}} \in H_*^{G_{\underline{\mathbf{d}}}}\left(\mathcal{Z}^{\leq s}_{s(\overline{y}),\overline{y}}\right).$$

(iii) Let $k \in \{1, 2, ..., \mathbf{d}\}$ and $\overline{y} \in Y_{\underline{\mathbf{d}}}$. Recall the k-th canonical line bundle $\mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(k)$ over $\widetilde{\mathcal{F}}_{\overline{y}}$. We have defined

$$e^{G_{\underline{\mathbf{d}}}}\left(\mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(k)\right) =: x_{\overline{y}}(k) \in H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}}).$$

Now consider the maps $\mathcal{Z}^e_{\overline{y},\overline{y}} \overset{i}{\hookrightarrow} \widetilde{\mathcal{F}}_{\overline{y}} \times \widetilde{\mathcal{F}}_{\overline{y}} \xrightarrow{p_1} \widetilde{\mathcal{F}}_{\overline{y}}$. Let $\pi = p_1 \circ i$. It is an isomorphism of varieties. We define

$$\varkappa_{\overline{y}}(k) := \pi^*(x_{\overline{y}}(k)) \in H_*^{G_{\underline{\mathbf{d}}}}\left(\mathcal{Z}^e_{\overline{y},\overline{y}}\right).$$

Since $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^{\leq s}_{s(\overline{y}),\overline{y}}), H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\overline{y},\overline{y}}) \subset H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ by the cellular fibration lemma, we can regard $\sigma_{\overline{y}}(l), \varkappa_{\overline{y}}(k)$ as homology classes in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

We consider a *provisional generating set* consisting of the following elements:

- $1_{\overline{y},\overline{y}}$
- $\sigma_{\overline{y}}(1), ..., \sigma_{\overline{y}}(\mathbf{d}-1)$
- $\varkappa_{\overline{y}}(1), ..., \varkappa_{\overline{y}}(\mathbf{d})$

where $\overline{y}$ ranges over $Y_{\underline{\mathbf{d}}}$.

Remark 11.2. The generating set defined above is the one most convenient to work with, but it is not a smallest generating set. Let $s = s_l \in \Pi$. Define

$$\sigma(l) := [\mathcal{Z}^s_{\underline{\mathbf{d}}}]^{G_{\underline{\mathbf{d}}}} \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^{\leq s}_{\underline{\mathbf{d}}}).$$

Then for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we have $\sigma_{\overline{y}}(l) = 1_{s(\overline{y}),s(\overline{y})} \star \sigma(l) \star 1_{\overline{y},\overline{y}}$. Now let $1 \leq k \leq \mathbf{d}$. Define

$$\varkappa(k) = \sum_{\overline{y} \in Y_{\underline{\mathbf{d}}}} \varkappa_{\overline{y}}(k).$$

Then for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we have $\varkappa_{\overline{y}}(k) = 1_{\overline{y},\overline{y}} \star \varkappa(k) \star 1_{\overline{y},\overline{y}}$. Therefore, the following set

- $1_{\overline{y},\overline{y}}$
- $\sigma(1), ..., \sigma(\mathbf{d}-1)$

- $\varkappa(1), ..., \varkappa(\mathbf{d})$,

where $\overline{y}$ ranges over $Y_{\underline{\mathbf{d}}}$, generates the same subalgebra of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as the generating set from Definition 11.1.

11.2 Completeness of the generating set

We are now going to prove the following theorem.

Theorem 11.3. *The elements* $\{1_{\overline{y},\overline{y}}, \sigma(1), ..., \sigma(\mathbf{d}-1), \varkappa(1), ..., \varkappa(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ *generate* $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$ *as a* k*-algebra.*

The proof relies on the following idea. Observe that the elements $\{1_{\overline{y},\overline{y}}, \varkappa(1), ..., \varkappa(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ generate the subalgebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^e) \cong H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$. Moreover, recall from Proposition 7.29 that $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a free left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^e)$-module with basis $\left\{\left[\mathcal{Z}_{\underline{\mathbf{d}}}^w\right]^{G_{\underline{\mathbf{d}}}} \mid w \in \mathbb{W}_{\mathbf{d}}\right\}$. Therefore, it suffices to express these basis elements in terms of our generators. We will use the following lemma.

Lemma 11.4. *Let* $s = s_l \in \Pi$ *and* $w \in \mathbb{W}_{\mathbf{d}}$*. If* $l(sw) = l(w) + 1$ *then* $[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{G_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^w]^{G_{\underline{\mathbf{d}}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}]^{G_{\underline{\mathbf{d}}}}$ *in the quotient vector space* $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$.

Proof. By Lemma 7.30, there is a unique element $c \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^e)$ such that

$$[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{G_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^w]^{G_{\underline{\mathbf{d}}}} = c \star [\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}]^{G_{\underline{\mathbf{d}}}}$$

in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$. We need to show that $c = 1$.

For each $x \in \mathbb{W}_{\mathbf{d}}$ we abbreviate $[\mathcal{Z}_{\underline{\mathbf{d}}}^x]^{T_{\mathbf{d}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^x]^{T_{\mathbf{d}}} \otimes 1 \in \mathcal{H}_*^{T_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We have

$$[\mathcal{Z}_{\underline{\mathbf{d}}}^x]^{T_{\mathbf{d}}} = \sum_{y,z \in \mathbb{W}_{\mathbf{d}}} f_{y,z}^x \psi_{y,z},$$

for some uniquely determined $f_{y,z}^x \in K_{T_{\underline{\mathbf{d}}}}$. Since $\mathcal{Z}_{\underline{\mathbf{d}}}^x$ is the closure of the cell $\widetilde{\mathbf{O}}_x$, each point contained in the cell $\widetilde{\mathbf{O}}_x$ is a smooth point of $\mathcal{Z}_{\underline{\mathbf{d}}}^x$. In particular, for each $y \in \mathbb{W}_{\mathbf{d}}$, the $T_{\underline{\mathbf{d}}}$-fixed point (ρ_0, F_y, F_{yx}) is a smooth point of $\mathcal{Z}_{\underline{\mathbf{d}}}^x$. Hence, by the localization formula,

$$f_{y,yx}^x = (\widetilde{\Lambda}_{y,yx}^x)^{-1} = (\mathrm{eu}(\mathcal{Z}_{\underline{\mathbf{d}}}^x, (\rho_0, F_y, F_{yx})))^{-1}.$$

Substituting sw for x, we deduce that if we expand the class $[\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}]^{T_{\mathbf{d}}}$ in the $K_{T_{\underline{\mathbf{d}}}}$-basis $\{\psi_{y,z} \mid y, z \in \mathbb{W}_{\mathbf{d}}\}$, then, for each $y \in \mathbb{W}_{\mathbf{d}}$, the coefficient on $\psi_{y,ysw}$ is

$$f_{y,ysw}^{sw} = (\widetilde{\Lambda}_{y,ysw}^{sw})^{-1} = (\mathrm{eu}(\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}, (\rho_0, F_y, F_{ysw})))^{-1}.$$

We can also expand $[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\mathbf{d}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^w]^{T_{\mathbf{d}}}$ in the $K_{T_{\underline{\mathbf{d}}}}$-basis $\{\psi_{y,z} \mid y, z \in \mathbb{W}_{\mathbf{d}}\}$:

$$[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\mathbf{d}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^w]^{T_{\mathbf{d}}} = \sum_{y,z \in \mathbb{W}_{\mathbf{d}}} g_{y,z} \psi_{y,z},$$

for some uniquely determined $g_{y,z} \in K_{T_{\underline{\mathbf{d}}}}$. We want to compare, for each $y \in \mathbb{W}_{\mathbf{d}}$, $f_{y,ysw}^{sw}$ with $g_{y,ysw}$ and show that they are equal. Recall that

$$[\mathcal{Z}_{\underline{\mathbf{d}}}^s]^{T_{\mathbf{d}}} = \sum_{y \in \mathbb{W}_{\mathbf{d}}} (\widetilde{\Lambda}_{y,ys}^s)^{-1} \psi_{y,ys} + \sum_{y \ \mid \ ys \in W_{\underline{\mathbf{d}}} y} (\widetilde{\Lambda}_{y,y}^s)^{-1} \psi_{y,y}.$$

Moreover,
$$[\mathcal{Z}_{\underline{\mathbf{d}}}^{w}]^{T_{\underline{\mathbf{d}}}} = \sum_{y,z\in\mathbb{W}_{\mathbf{d}}} f_{y,z}^{w}\psi_{y,z},$$
where $f_{y,yw}^{w} = (\mathrm{eu}(\mathcal{Z}_{\underline{\mathbf{d}}}^{w},(\rho_0,F_y,F_{yw})))^{-1}$, by our previous remarks about smoothness. Note that since $l(sw) > l(w)$, the fixed points (ρ_0,F_y,F_{ysw}) are not contained in $\mathcal{Z}_{\underline{\mathbf{d}}}^{w}$, so $f_{y,ysw}^{w} = 0$. Hence, by Theorem 10.7, for each $y \in \mathbb{W}_{\mathbf{d}}$ we have
$$g_{y,ysw} = (\widetilde{\Lambda}_{y,ys}^{s}\widetilde{\Lambda}_{ys,ysw}^{w})^{-1}\widetilde{\Lambda}_{ys} = (\mathrm{eu}(\mathcal{Z}_{\underline{\mathbf{d}}}^{s},(\rho_0,F_y,F_{ys})))^{-1}(\mathrm{eu}(\mathcal{Z}_{\underline{\mathbf{d}}}^{w},(\rho_0,F_{ys},F_{ysw})))^{-1}\widetilde{\Lambda}_{ys}.$$
The equality of $f_{y,ysw}^{sw} = g_{y,ysw}$ now follows from Lemma 9.18. Therefore
$$[\mathcal{Z}_{\underline{\mathbf{d}}}^{s}]^{T_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^{w}]^{T_{\underline{\mathbf{d}}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}]^{T_{\underline{\mathbf{d}}}} \tag{74}$$
in $\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/\mathcal{H}_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$. But the $S_{T_{\underline{\mathbf{d}}}}$-module $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$ is free, so (74) also holds in $H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/H_*^{T_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$. Since the forgetful maps commute with convolution, we conclude that
$$[\mathcal{Z}_{\underline{\mathbf{d}}}^{s}]^{G_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^{w}]^{G_{\underline{\mathbf{d}}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^{sw}]^{G_{\underline{\mathbf{d}}}} \tag{75}$$
holds in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sw})/H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sw})$. □

We can now return to the proof of Theorem 11.3.

Proof of Theorem 11.3. The elements $\{1_{\overline{y},\overline{y}}, \varkappa(1), ..., \varkappa(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ generate the subalgebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{e};k)$ as a k-algebra. Moreover, $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a free left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{e})$-module with basis $\left\{\left[\mathcal{Z}_{\underline{\mathbf{d}}}^{w}\right]^{G_{\underline{\mathbf{d}}}} \mid w \in \mathbb{W}_{\mathbf{d}}\right\}$. So it suffices to show that, for each $w \in \mathbb{W}_{\mathbf{d}}$,
$$\left[\mathcal{Z}_{\underline{\mathbf{d}}}^{w}\right]^{G_{\underline{\mathbf{d}}}} = f^{\beta} \star \left[\mathcal{Z}_{\underline{\mathbf{d}}}^{e}\right]^{G_{\underline{\mathbf{d}}}} + \sum_{\alpha} f_1^{\alpha} \star \left[\mathcal{Z}_{\underline{\mathbf{d}}}^{s_1^{\alpha}}\right]^{G_{\underline{\mathbf{d}}}} \star f_2^{\alpha} \star ... \star f_{n^{\alpha}}^{\alpha} \star \left[\mathcal{Z}_{\underline{\mathbf{d}}}^{s_{n^{\alpha}}^{\alpha}}\right]^{G_{\underline{\mathbf{d}}}} \star f_{n^{\alpha}+1}^{\alpha}, \tag{76}$$
where each $s_k^{\alpha} \in \Pi$, $n^{\alpha} \geq 1$, $f_k^{\alpha}, f^{\beta} \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{e})$ and α ranges over a finite index set. We show this by induction on the length of w. The claim obviously holds for the simple transpositions $s \in \Pi$. Suppose that we have shown that (76) holds for all $u \in \mathbb{W}_{\mathbf{d}}$ with $l(u) \leq m$. Let $l(w) = m+1$. Then $w = sv$ for some $v \in \mathbb{W}_{\mathbf{d}}$ with $l(v) = m$ and $s \in \Pi$. Hence, by Lemma 11.4,
$$[\mathcal{Z}_{\underline{\mathbf{d}}}^{s}]^{G_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^{v}]^{G_{\underline{\mathbf{d}}}} = [\mathcal{Z}_{\underline{\mathbf{d}}}^{sv}]^{G_{\underline{\mathbf{d}}}}$$
in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq sv})/H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sv})$, i.e.,
$$[\mathcal{Z}_{\underline{\mathbf{d}}}^{s}]^{G_{\underline{\mathbf{d}}}} \star [\mathcal{Z}_{\underline{\mathbf{d}}}^{v}]^{G_{\underline{\mathbf{d}}}} - r = [\mathcal{Z}_{\underline{\mathbf{d}}}^{sv}]^{G_{\underline{\mathbf{d}}}}, \qquad r \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sv}). \tag{77}$$
Since $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{< sv})$ is a free left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{e})$-module with basis $\left\{\left[\mathcal{Z}_{\underline{\mathbf{d}}}^{u}\right]^{G_{\underline{\mathbf{d}}}} \mid u \in \mathbb{W}_{\mathbf{d}}, u < sv\right\}$ and since $l(v) = m$, it follows by induction that both r and $[\mathcal{Z}_{\underline{\mathbf{d}}}^{v}]^{G_{\underline{\mathbf{d}}}}$ can be expanded as in the RHS of (76). Therefore, by (77), the same holds for $[\mathcal{Z}_{\underline{\mathbf{d}}}^{sv}]^{G_{\underline{\mathbf{d}}}}$. □

Using Lemma 11.4 we can also construct another basis of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^{e})$-module.

Definition 11.5. For each $w \in \mathbb{W}_{\mathbf{d}}$ choose a reduced decomposition $w = s_{l_1}s_{l_2}...s_{l_r}$, where $r \geq 0$ and $l_1, ..., l_r \in \{1, ..., \mathbf{d}-1\}$. Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. If $r = 0$ set $\sigma_{\overline{y}}(w) = 1_{\overline{y},\overline{y}}$. Otherwise set
$$\sigma_{\overline{y}}(w) = \sigma_{s_{l_1}(\overline{y})}(l_1) \star \sigma_{s_{l_2}s_{l_1}(\overline{y})}(l_2) \star ... \star \sigma_{w^{-1}(\overline{y})}(l_r).$$
Moreover, set
$$\sigma(w) = \sum_{\overline{y}\in Y_{\underline{\mathbf{d}}}} \sigma_{\overline{y}}(w) = \sigma(l_1) \star ... \star \sigma(l_r).$$
Note that $\sigma(w)$ in general *does* depend on the choice of reduced decomposition of w.

Corollary 11.6. *$H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a free left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}^e)$-module with basis $\{\sigma(w) \mid w \in \mathbb{W}_{\mathbf{d}}\}$.*

Proof. This follows directly from Lemma 11.4. □

11.3 Faithful polynomial representation of the convolution algebra

Recall that we have a k-vector space isomorphism $H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\right) \cong \bigoplus_{\overline{y} \in Y_{\underline{\mathbf{d}}}} k[x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})]$ and that $H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\right)$ is a faithful $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$-module. We are now going to calculate the action of the generators of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ on $H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}\right)$. The following theorem was first proved by Varagnolo and Vasserot in [VV11, Proposition 2.23].

Theorem 11.7. *Fix $\overline{y}, \overline{y}' \in Y_{\underline{\mathbf{d}}}$ and a polynomial $f \in H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right)$.*
(i) We have $1_{\overline{y}',\overline{y}'} \diamond f = 0$ unless $\overline{y}' = \overline{y}$ and $1_{\overline{y},\overline{y}} \diamond f = f$.
(ii) We have $\varkappa_{\overline{y}'}(k) \diamond f = 0$ unless $\overline{y}' = \overline{y}$ and $\varkappa_{\overline{y}}(k) \diamond f = x_{\overline{y}}(k) f$.
(iii) We have $\sigma_{\overline{y}'}(l) \diamond f = 0$ unless $\overline{y}' = \overline{y}$. Suppose that $\overline{y} = \overline{y}_w$ and set $s = s_l$.
If $s(\overline{y}) = \overline{y}$ then the action of $\sigma_{\overline{y}}(l)$ is given by the following Demazure operator

$$\sigma_{\overline{y}}(l) \diamond f = \frac{f - s(f)}{x_{\overline{y}}(l+1) - x_{\overline{y}}(l)}.$$

If $s(\overline{y}) \neq \overline{y}$ then

$$\sigma_{\overline{y}}(l) \diamond f = (x_{s(\overline{y})}(l) - x_{s(\overline{y})}(l+1))^{h(i_l^w, i_{l+1}^w)} s(f).$$

Proof. We have $\mathcal{Z}_{\overline{y}',\overline{y}'} \circ \widetilde{\mathcal{F}}_{\overline{y}} = \varnothing$ unless $\overline{y} = \overline{y}'$. This explains why $1_{\overline{y}',\overline{y}'} \diamond f = 0$ and $\varkappa_{\overline{y}'}(k) \diamond f = 0$ unless $\overline{y} = \overline{y}'$. In the latter case we have the convolution product

$$\diamond : H_*^{G_{\underline{\mathbf{d}}}}\left(\mathcal{Z}_{\overline{y},\overline{y}}\right) \times H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right) \to H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right).$$

(i) By the definition of the convolution product and by the projection formula we get

$$1_{\overline{y}} \diamond f = (p_1)_* \left([\mathcal{Z}_{\overline{y},\overline{y}}^e]^{G_{\underline{\mathbf{d}}}} \cap p_1^*(f)\right) = (p_1)_* \left([\mathcal{Z}_{\overline{y},\overline{y}}^e]^{G_{\underline{\mathbf{d}}}}\right) \cap f = [\widetilde{\mathcal{F}}_{\overline{y}}]^{G_{\underline{\mathbf{d}}}} \cap f = f.$$

(ii) We have

$$\varkappa_{\overline{y}}(k) \diamond f = (p_1)_*(i_*(\varkappa_{\overline{y}}(k)) \cap p_1^*(f)) = ((p_1)_*(i_*(\varkappa_{\overline{y}}(k)))) \cap f,$$

where the second equality is implied by the projection formula. We can apply the projection formula again to get

$$\begin{aligned}(p_1)_*(i_*(\varkappa_{\overline{y}}(k))) = \pi_*(\varkappa_{\overline{y}}(k)) &= \pi_*(\varkappa_{\overline{y}}(k) \cap [\mathcal{Z}_{\overline{y},\overline{y}}^e]^{G_{\underline{\mathbf{d}}}}) \\ &= \pi_*(\pi^*(x_{\overline{y}}(k)) \cap [\mathcal{Z}_{\overline{y},\overline{y}}^e]^{G_{\underline{\mathbf{d}}}}) \\ &= x_{\overline{y}}(k) \cap \pi_*([\mathcal{Z}_{\overline{y},\overline{y}}^e]^{G_{\underline{\mathbf{d}}}}) = x_{\overline{y}}(k) \cap [\widetilde{\mathcal{F}}_{\overline{y}}]^{G_{\underline{\mathbf{d}}}} = x_{\overline{y}}(k).\end{aligned}$$

Hence

$$\varkappa_{\overline{y}}(k) \diamond f = x_{\overline{y}}(k) \cap f = x_{\overline{y}}(k) f,$$

where the second equality reinterprets the intersection product $\cap$ as multiplication of polynomials.

(iii) The convolution product

$$\diamond : H_*^{G_{\underline{\mathbf{d}}}}\left(\mathcal{Z}_{s(\overline{y}),\overline{y}}\right) \times H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right) \to H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{s(\overline{y})}\right)$$

gives rise to an $S_{T_{\underline{\mathbf{d}}}}$-linear operator

$$H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right) \to H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{s(\overline{y})}\right), \qquad f \mapsto \sigma_{\overline{y}} \diamond f. \tag{78}$$

Recall that we have a k-vector space isomorphism

$$H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right) = k[x_{\overline{y}}(1), ..., x_{\overline{y}}(d)].$$

By Section 10.2.2, we have injective homomorphisms

$$H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{\overline{y}}\right) \to \bigoplus_{w \in W_{\overline{y}}} K_{T_{\underline{\mathbf{d}}}}\psi_w, \qquad f(x_{\overline{y}}(1), ..., x_{\overline{y}}(\mathbf{d})) \mapsto f_{\overline{y}} := \sum_{w \in W_{\overline{y}}} w(f)\widetilde{\Lambda}_w^{-1}\psi_w,$$

$$H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{s(\overline{y})}\right) \to \bigoplus_{w \in W_{\overline{y}'}} K_{T_{\underline{\mathbf{d}}}}\psi_w, \qquad f(x_{s(\overline{y})}(1), ..., x_{s(\overline{y})}(\mathbf{d})) \mapsto f_{s(\overline{y})} := \sum_{w \in W_{s(\overline{y})}} w(f)\widetilde{\Lambda}_w^{-1}\psi_w. \tag{79}$$

Under these injections, the operator (78) is given by

$$f_{\overline{y}} \mapsto \sum_{w' \in W_{s(\overline{y})}} g_{w'}\psi_{w'}, \qquad g_{w'} = \sum_{w \in W_{\overline{y}}} w(f)\widetilde{\Lambda}^s_{w',w}. \tag{80}$$

Observe that the RHS of (80) is the image $g_{s(\overline{y})}$ of some $g(x_{s(\overline{y})}(1), ..., x_{s(\overline{y})}(\mathbf{d})) \in H_*^{G_{\underline{\mathbf{d}}}}\left(\widetilde{\mathcal{F}}_{s(\overline{y})}\right)$ under (79), i.e.,

$$\sum_{w' \in W_{s(\overline{y})}} g_{w'}\psi_{w'} = \sum_{w' \in W_{s(\overline{y})}} w'(g)\widetilde{\Lambda}_{w'}^{-1}\psi_{w'}. \tag{81}$$

We now compute the polynomial g. It follows from (81) that for each $w' \in W_{\overline{y}}$,

$$g_{w'} = w'(g)\widetilde{\Lambda}_{w'}^{-1}. \tag{82}$$

Suppose that $s(\overline{y}) = \overline{y}$. Then $W_{\overline{y}} = W_{s(\overline{y})}$ and for each $w' \in W_{\overline{y}}$ we have $w's \in W_{\underline{\mathbf{d}}}w'$. Hence, by Corollary 10.5, the second sum in (80) reduces to

$$g_{w'} = w'(f)(\widetilde{\Lambda}^s_{w',w'})^{-1} + w's(f)(\widetilde{\Lambda}^s_{w',w's})^{-1}.$$

By Lemma 9.17 and (82) we have

$$\begin{aligned} g_{w'} &= w'(f)(\mathrm{eu}(\mathfrak{m}_{w's,w'}))^{-1}\widetilde{\Lambda}_{w'}^{-1} + w's(f)(\mathrm{eu}(\mathfrak{m}_{w's,w'}))^{-1}\widetilde{\Lambda}_{w's}^{-1} \\ &= \frac{w'(f)\widetilde{\Lambda}_{w'}^{-1} + w's(f)\widetilde{\Lambda}_{w's}^{-1}}{\chi_{w'(l+1)} - \chi_{w'(l)}} \\ &= \frac{w'(f)\widetilde{\Lambda}_{w'}^{-1} - w's(f)\widetilde{\Lambda}_{w'}^{-1}}{\chi_{w'(l+1)} - \chi_{w'(l)}} \\ &= w'(g)\widetilde{\Lambda}_{w'}^{-1}. \end{aligned}$$

Hence

$$g = \frac{f - s(f)}{\chi_{l+1} - \chi_l}.$$

Now suppose $s(\overline{y}) \neq \overline{y}$. Then for each $w' \in W_{\overline{y}}$ we have $w's \notin W_{\underline{\mathbf{d}}}w'$. Hence, by Corollary 10.5, the second sum in (80) reduces to

$$g_{w'} = w's(f)(\widetilde{\Lambda}^s_{w',w's})^{-1} = w's(f)((\widetilde{\Lambda}^s_{w',w's})^{-1}\widetilde{\Lambda}_{w'})\widetilde{\Lambda}_{w'}^{-1}.$$

By Lemma 9.17 and (82) we have

$$\begin{aligned} g_{w'} &= w's(f)\mathrm{eu}(\mathfrak{d}_{w',w's})\widetilde{\Lambda}_{w'}^{-1} \\ &= w's(f)(\chi_{w'(l)} - \chi_{w'(l+1)})^{h_{i^{w'}_{l+1},i^{w'}_l}}\widetilde{\Lambda}_{w'}^{-1} \\ &= w'(g)\widetilde{\Lambda}_{w'}^{-1}. \end{aligned}$$

Hence

$$g = s(f)(\chi_l - \chi_{l+1})^{h_{i^{w'}_{l+1},i^{w'}_l}}.$$

Since $i^{w'}_l = i^{w's}_{l+1}$, $i^{w'}_{l+1} = i^{w's}_l$ and $\overline{y}_{w's} = \overline{y}_w$ we get

$$g = s(f)(\chi_l - \chi_{l+1})^{h_{i^w_l,i^w_{l+1}}}.$$

□

11.4 The grading

We are now going to define two interesting gradings on the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. To do this we need the following definition.

Definition 11.8. Let $\overline{y} = \overline{y}_w \in Y_{\underline{\mathbf{d}}}$. We write $\overline{y} = (i^w_1, ..., i^w_{\mathbf{d}})$. Set

$$h_{\overline{y}}(l) = \begin{cases} h_{i^w_l,i^w_{l+1}} & \text{if } s_l(\overline{y}) \neq \overline{y} \quad \text{(i.e., if } i^w_l \neq i^w_{l+1}), \\ -1 & \text{if } s_l(\overline{y}) = \overline{y} \quad \text{(i.e., if } i^w_l = i^w_{l+1}). \end{cases}$$

$$a_{\overline{y}}(l) = h_{\overline{y}}(l) + h_{s_l(\overline{y})}(l) = \begin{cases} -(i^w_l, i^w_{l+1}) & \text{if } s_l(\overline{y}) \neq \overline{y}, \\ -2 & \text{if } s_l(\overline{y}) = \overline{y}. \end{cases}$$

If $s_l(\overline{y}) \neq \overline{y}$ then $h_{i^w_l,i^w_{l+1}}$ is the number of arrows from i^w_l to i^w_{l+1} and $a_{\overline{y}}(l)$ is the number of edges between i^w_l and i^w_{l+1} in the undirected graph obtained from the quiver Γ by forgetting orientations of the edges. △

First observe that the groups $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}), H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ are endowed with a natural homological grading. However, given this grading, $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ is not a *graded* module over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. This motivates us to define different gradings which make $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ a *graded* module over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Definition 11.9 (Grading 1)**.** We endow $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ with the cohomological grading, i.e.,

$$\deg_1 x_{\overline{y}}(k) = 2, \qquad y \in Y_{\underline{\mathbf{d}}},\ k \in \{1, ..., \mathbf{d}\}.$$

We now endow $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ with the grading uniquely determined by setting

$$\deg_1 1_{\overline{y},\overline{y}} = 0, \qquad \deg_1 \varkappa_{\overline{y}}(k) = 2, \qquad \deg_1 \sigma_{\overline{y}}(l) = 2h_{\overline{y}}(l).$$

Proposition 11.10. *If we endow $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ with the gradings from Definition 11.9, the vector space $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ is a graded module over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.*

Proof. The proposition follows directly from the explicit description of the faithful polynomial representation in Theorem 11.7. □

It is also possible to define another grading on $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ which is more symmetric in the sense that it does not depend on the orientation of the quiver.

Definition 11.11 (Grading 2)**.** Recall that $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) = \bigoplus_{\overline{y}\in Y_{\underline{\mathbf{d}}}} H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Let $m_{\overline{y}} := \dim \widetilde{\mathcal{F}}_{\overline{y}}$ be the dimension of $\widetilde{\mathcal{F}}_{\overline{y}}$ as a variety. We endow $H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ with the cohomological grading shifted by $m_{\overline{y}}$, i.e., if $f \in H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ then we set

$$\deg_2 f = \deg_1 f - m_{\overline{y}}.$$

We now endow $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ with the grading uniquely determined by setting

$$\deg_2 1_{\overline{y},\overline{y}} = 0, \qquad \deg_2 \varkappa_{\overline{y}}(k) = 2, \qquad \deg_2 \sigma_{\overline{y}}(l) = a_{\overline{y}}(l).$$

Proposition 11.12. *If we endow $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ with the gradings from Definition 11.11, the vector space $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ is a graded module over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.*

To prove the proposition we will use the following lemma.

Lemma 11.13. *Let $\overline{y} = (i_1, ..., i_{\mathbf{d}}) \in Y_{\underline{\mathbf{d}}}$ and set $\overline{y}(l) := i_l$. We have*

$$m_{\overline{y}} := \dim \widetilde{\mathcal{F}}_{\overline{y}} = \dim \mathcal{F}_{\overline{y}} + \sum_{l'\leq l} h_{\overline{y}(l),\overline{y}(l')}.$$

Proof. See [Lus91, Lemma 1.6(c)]. □

Proof of Proposition 11.12. By Lemma 11.13, for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we have

$$m_{s(\overline{y})} - m_{\overline{y}} = h_{\overline{y}(l),\overline{y}(l+1)} - h_{\overline{y}(l+1),\overline{y}(l)} = h_{\overline{y}}(l) - h_{s_l(\overline{y})}(l).$$

Let $f \in H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ and suppose that $s_l(\overline{y}) \neq \overline{y}$. Then

$$\begin{aligned}\deg_2(\sigma_{\overline{y}}(l) \diamond f) - \deg_2 f = &\ \deg_2 \left((x_{s_l(\overline{y})}(l) - x_{s_l(\overline{y})})^{h_{\overline{y}}(l)} s_l(f)\right) - \deg_2 f \\ &= 2h_{\overline{y}}(l) + \deg_1 f - m_{s(\overline{y})} - \deg_1 f + m_{\overline{y}} \\ &= 2h_{\overline{y}}(l) - h_{\overline{y}}(l) + h_{s_l(\overline{y})}(l) = h_{\overline{y}}(l) + h_{s_l(\overline{y})}(l) = \deg_2 \sigma_{\overline{y}}(l).\end{aligned}$$

The fact that the other generators - $1_{\overline{y},\overline{y}}$ and $\varkappa_{\overline{y}}(l)$ - act in a homogeneous way with respect to the grading is obvious. □

From now on we will always consider $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as a graded algebra endowed with the "symmetric" grading from Definition 11.11. We finally remark that there is another reason for using this grading. In the next chapter we will show that the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic to a certain naturally graded geometric extension algebra. This isomorphism is in fact an isomorphism of graded algebras, if we endow $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ with the grading from Definition 11.11.

11.5 Relations

We are now ready to give a complete list of relations for the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. We will use the following lemma.

Lemma 11.14. *Let $\partial_{\overline{y},l}$ denote the Demazure operator*

$$\partial_{\overline{y},l} : f \mapsto \frac{f - s_l(f)}{x_{\overline{y}}(l+1) - x_{\overline{y}}(l)},$$

where $f \in H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Then

$$\partial_{\overline{y},l}^2 = 0, \qquad \partial_{\overline{y},l}(fg) = \partial_{\overline{y},l}(f)g + s(g)\partial_{\overline{y},l}(g).$$

We call the first relation the quadratic relation and the second relation the twisted derivation relation for Demazure operators. Moreover, Demazure operators satisfy the following braid relations

$$\partial_{\overline{y},l}\partial_{\overline{y},l'} = \partial_{\overline{y},l'}\partial_{\overline{y},l}, \quad \textit{if} \quad |l-l'|>1, \qquad \partial_{\overline{y},l+1}\partial_{\overline{y},l}\partial_{\overline{y},l+1} = \partial_{\overline{y},l}\partial_{\overline{y},l+1}\partial_{\overline{y},l}.$$

Proof. Easy calculation left to the reader. □

The relations in our convolution algebra will involve the following polynomials which depend on the quiver $\mathbf{\Gamma}$ and dimension vector $\underline{\mathbf{d}}$.

Definition 11.15. Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$ be a composition of the dimension vector $\underline{\mathbf{d}}$ and $l \in \{1, ..., \mathbf{d}-1\}$. We define the following polynomials

$$Q_{\overline{y},l}(u,v) = \begin{cases} (-1)^{h_{\overline{y}}(l)}(u-v)^{a_{\overline{y}}(l)} & \text{if } s_l(\overline{y}) \neq \overline{y} \\ 0 & \text{else.} \end{cases}$$

Definition 11.15 says that the polynomial $Q_{\overline{y},l}(u,v)$ is non-zero only if the l-th and $l+1$-th vertices in the composition $\overline{y}$ are distinct. In that case $Q_{\overline{y},l}(u,v)$ equals $(u-v)$ to the power of $a_{\overline{y}}(l)$, where $a_{\overline{y}}(l)$ equals the number of edges between the vertices $\overline{y}(l)$ and $\overline{y}(l+1)$, multiplied by -1 to the power of $h_{\overline{y}}(l)$, where $h_{\overline{y}}(l)$ equals the number of arrows from $\overline{y}(l)$ to $\overline{y}(l+1)$. Note that the polynomials $Q_{\overline{y},l}(u,v)$ depend up to sign only on the underlying undirected graph of $\mathbf{\Gamma}$, and not on the orientation of $\mathbf{\Gamma}$.

We now state one of the main theorems about the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Theorem 11.16. *The following relations hold in the algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}};k)$ (we omit the $\star$ signs for ease of reading):*

(1) Idempotents:

- $1_{\overline{y},\overline{y}}1_{\overline{y}',\overline{y}'} = \delta_{\overline{y},\overline{y}'}1_{\overline{y},\overline{y}}$
- $\varkappa_{\overline{y}}(l) = 1_{\overline{y},\overline{y}}\varkappa_{\overline{y}}(l)1_{\overline{y},\overline{y}}$
- $1_{s_l(\overline{y}),s_l(\overline{y})}\sigma_{\overline{y}}(k)1_{\overline{y},\overline{y}} = \sigma_{\overline{y}}(k)$.

(2) Polynomial subalgebra:

- $\varkappa_{\overline{y}}(k)\varkappa_{\overline{y}'}(k') = \varkappa_{\overline{y}'}(k')\varkappa_{\overline{y}}(k)$.

(3) The straightening rule:

- $\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(k) - \varkappa_{s_l(\overline{y})}(s_l(k))\sigma_{\overline{y}}(l) = \begin{cases} -1_{\overline{y},\overline{y}} & k=l, s_l(\overline{y})=\overline{y}, \\ 1_{\overline{y},\overline{y}} & k=l+1, s_l(\overline{y})=\overline{y}, \\ 0 & \textit{else.} \end{cases}$

(4) The quadratic relation:

- $\sigma_{s_l(\overline{y})}(l)\sigma_{\overline{y}}(l) = Q_{\overline{y},l}(\varkappa_{\overline{y}}(l), \varkappa_{\overline{y}}(l+1))$.

(5) "Braid relations": let us write $\overline{y} = (i_1^{\overline{y}}, ..., i_{\mathbf{d}}^{\overline{y}})$, then

- $\sigma_{s_l(\overline{y})}(l')\sigma_{\overline{y}}(l) = \sigma_{s_{l'}(\overline{y})}(l)\sigma_{\overline{y}}(l') \quad \textit{if} \quad |l-l'|>1.$
- $\sigma_{s_ls_{l+1}(\overline{y})}(l+1)\sigma_{s_{l+1}(\overline{y})}(l)\sigma_{\overline{y}}(l+1) - \sigma_{s_{l+1}s_l(\overline{y})}(l)\sigma_{s_l(\overline{y})}(l+1)\sigma_{\overline{y}}(l) =$ $\frac{Q_{\overline{y},l}(\varkappa_{\overline{y}}(l+2),\varkappa_{\overline{y}}(l+1)) - Q_{\overline{y},l}(\varkappa_{\overline{y}}(l),\varkappa_{\overline{y}}(l+1))}{\varkappa_{\overline{y}}(l+2)-\varkappa_{\overline{y}}(l)} \quad \textit{if} \quad i_l^{\overline{y}} = i_{l+2}^{\overline{y}} \neq i_{l+1}^{\overline{y}}$ *and* 0 *otherwise.*

Remark 11.17. We can interpret the quadratic relation as saying that $(\sigma(l))^2 = 0$ up to a polynomial. Similarly, the second "braid relation" says that $\sigma(l+1)\sigma(l)\sigma(l+1) = \sigma(l)\sigma(l+1)\sigma(l)$ modulo a polynomial. In this sense, these relations mirror the usual relations $s_l^2 = e$ and $s_{l+1}s_ls_{l+1} = s_ls_{l+1}s_l$ in the symmetric group $\mathbb{W}_{\mathbf{d}}$. Moreover, as one can easily see, all the relations in the above theorem respect the grading on $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Proof. (1) These relations follow directly from the definition of the convolution product.
(2) We have proven that $H_*^{G_{\mathbf{d}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$ is a subalgebra isomorphic to $H^*_{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$. Hence the relation.
(3) Let $f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Suppose that $k = l$ and $s_l(\overline{y}) = \overline{y}$. Then

$$\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(k) - \varkappa_{s_l(\overline{y})}(s_l(k))\sigma_{\overline{y}}(l) = \sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(l) - \varkappa_{\overline{y}}(l+1)\sigma_{\overline{y}}(l).$$

By Lemma 11.14 (the twisted derivation rule for Demazure operators applied to $\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(l) \diamond f$) we have

$$\begin{aligned}(\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(l) - \varkappa_{\overline{y}}(l+1)\sigma_{\overline{y}}(l)) \diamond f &= [(\sigma_{\overline{y}}(l) \diamond x_{\overline{y}}(l))f + x_{\overline{y}}(l+1)(\sigma_{\overline{y}}(l) \diamond f)] - x_{\overline{y}}(l+1)(\sigma_{\overline{y}}(l) \diamond f)\\ &= (\sigma_{\overline{y}}(l) \diamond x_{\overline{y}}(l))f = -f = -1_{\overline{y},\overline{y}} \diamond f.\end{aligned}$$

Now suppose $k = l+1$ and $s_l(\overline{y}) = \overline{y}$. Then

$$\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(k) - \varkappa_{s_l(\overline{y})}(s_l(k))\sigma_{\overline{y}}(l) = \sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(l+1) - \varkappa_{\overline{y}}(l)\sigma_{\overline{y}}(l).$$

By Lemma 11.14 we have

$$\begin{aligned}(\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(l+1) - \varkappa_{\overline{y}}(l)\sigma_{\overline{y}}(l)) \diamond f &= [(\sigma_{\overline{y}}(l) \diamond x_{\overline{y}}(l+1))f + x_{\overline{y}}(l)(\sigma_{\overline{y}}(l) \diamond f)] - x_{\overline{y}}(l)(\sigma_{\overline{y}}(l) \diamond f)\\ &= (\sigma_{\overline{y}}(l) \diamond x_{\overline{y}}(l+1))f = f = 1_{\overline{y},\overline{y}} \diamond f.\end{aligned}$$

Suppose $k \notin \{l, l+1\}$ and $s_l(\overline{y}) = \overline{y}$. Then

$$(\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(k) - \varkappa_{\overline{y}}(k)\sigma_{\overline{y}}(l)) \diamond f = \frac{x_{\overline{y}}(k)f - s(x_{\overline{y}}(k)f)}{x_{\overline{y}}(l+1) - x_{\overline{y}}(l)} - x_{\overline{y}}(k)\frac{f - s(f)}{x_{\overline{y}}(l+1) - x_{\overline{y}}(l)} = 0.$$

Now suppose that $s_l(\overline{y}) = \overline{y}' \neq \overline{y}$. Then

$$(\sigma_{\overline{y}}(l)\varkappa_{\overline{y}}(k) - \varkappa_{\overline{y}'}(k)\sigma_{\overline{y}}(l)) \diamond f =$$

$$= [(x_{\overline{y}'}(l) - x_{\overline{y}'}(l+1))^{h_{\overline{y}}(l)} s(x_{\overline{y}}(k)f)] - [x_{\overline{y}'}(s_l(k))(x_{\overline{y}'}(l) - x_{\overline{y}'}(l+1))^{h_{\overline{y}}(l+1)} s(f)] = 0.$$

(4) Suppose $s(\overline{y}) = \overline{y}$. Then

$$\sigma_{s(\overline{y})}(l)\sigma_{\overline{y}}(l) = \sigma_{\overline{y}}(l)\sigma_{\overline{y}}(l) = 0$$

by Lemma 11.14 (the quadratic rule for Demazure operators). If $s(\overline{y}) \neq \overline{y}$ then

$$\begin{aligned}\sigma_{s(\overline{y})}(l)\sigma_{\overline{y}}(l) &= (x_{\overline{y}}(l) - x_{\overline{y}}(l+1))^{h_{s(\overline{y})}(l)} s[(x_{s(\overline{y})}(l) - x_{s(\overline{y})}(l+1))^{h_{\overline{y}}(l)} s(f)]\\ &= (x_{\overline{y}}(l) - x_{\overline{y}}(l+1))^{h_{s(\overline{y})}(l)} (x_{\overline{y}}(l+1) - x_{\overline{y}}(l))^{h_{\overline{y}}(l)} f\\ &= (-1)^{h_{\overline{y}}(l)} (x_{\overline{y}}(l) - x_{\overline{y}}(l+1))^{h_{s(\overline{y})}(l) + h_{\overline{y}}(l)} f\\ &= (-1)^{h_{\overline{y}}(l)} (x_{\overline{y}}(l) - x_{\overline{y}}(l+1))^{a_{\overline{y}}(l)} f\\ &= (-1)^{h_{\overline{y}}(l)} (\varkappa_{\overline{y}}(l) - \varkappa_{\overline{y}}(l+1))^{a_{\overline{y}}(l)} \diamond f.\end{aligned}$$

(5)(i) Let us write $s_l = s, s_{l'} = s'$. First suppose that $\overline{y}' := s(\overline{y}) \neq y, \overline{y}'' := s'(\overline{y}) \neq \overline{y}, \overline{y}''' := s's(\overline{y})$. Then

$$\begin{aligned}\sigma_{\overline{y}'}(l')\sigma_{\overline{y}}(l) \diamond f &= (x_{\overline{y}'''}(l') - x_{\overline{y}'''}(l'+1))^{h_{\overline{y}'}(l')} s'[(x_{\overline{y}'}(l) - x_{\overline{y}'}(l+1))^{h_{\overline{y}}(l)} s(f)]\\ &= (x_{\overline{y}'''}(l') - x_{\overline{y}'''}(l'+1))^{h_{\overline{y}'}(l')} (x_{\overline{y}'''}(l) - x_{\overline{y}'''}(l+1))^{h_{\overline{y}}(l)} s's(f)\\ &= (x_{\overline{y}'''}(l) - x_{\overline{y}'''}(l+1))^{h_{\overline{y}'}(l')} s[(x_{\overline{y}''}(l') - x_{\overline{y}''}(l'+1))^{h_{\overline{y}}(l)} s'(f)]\\ &= \sigma_{\overline{y}''}(l)\sigma_{\overline{y}}(l') \diamond f.\end{aligned}$$

The other cases are similar - one repeatedly has to use the commutation relation $ss' = s's$ and the fact that $s(l') = l', s(l'+1) = l'+1, s'(l) = l, s'(l'+1) = l+1$. We leave the calculation to the reader.

(ii) Suppose that $i^{\overline{y}}_l = i^{\overline{y}}_{l+2} \neq i^{\overline{y}}_{l+1}$.

$$
\begin{aligned}
X &:= \sigma_{s_l s_{l+1}(\overline{y})}(l+1)\sigma_{s_{l+1}(\overline{y})}(l)\sigma_{\overline{y}}(l+1) \diamond f \\
&= \sigma_{s_l s_{l+1}(\overline{y})}(l+1)\sigma_{s_{l+1}(\overline{y})}(l) \diamond \Big[[x_{s_{l+1}(\overline{y})}(l+1) - x_{s_{l+1}(\overline{y})}(l+2)]^{h_{\overline{y}}(l+1)} s_{l+1}(f)\Big] \\
&= \sigma_{s_l s_{l+1}(\overline{y})}(l+1) \diamond \Bigg[\frac{[x_{s_{l+1}(\overline{y})}(l+1) - x_{s_{l+1}(\overline{y})}(l+2)]^{h_{\overline{y}}(l+1)} - [x_{s_{l+1}(\overline{y})}(l) - x_{s_{l+1}(\overline{y})}(l+2)]^{h_{\overline{y}}(l+1)}}{x_{s_{l+1}(\overline{y})}(l+1) - x_{s_{l+1}(\overline{y})}(l)} s_{l+1}(f) \\
&+ \frac{[x_{s_{l+1}(\overline{y})}(l) - x_{s_{l+1}(\overline{y})}(l+2)]^{h_{\overline{y}}(l+1)}}{x_{s_{l+1}(\overline{y})}(l+1) - x_{s_{l+1}(\overline{y})}(l)} (s_{l+1}(f) - s_l s_{l+1}(f))\Bigg] \\
&= \Big[[x_{\overline{y}}(l+1) - x_{\overline{y}}(l+2)]^{h_{s_{l+1}(\overline{y})}(l+1)}\Big] \Bigg[\frac{[x_{\overline{y}}(l+2) - x_{\overline{y}}(l+1)]^{h_{\overline{y}}(l+1)} - [x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{h_{\overline{y}}(l+1)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} f \\
&+ \frac{[x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{h_{\overline{y}}(l+1)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} (f - s_{l+1}s_l s_{l+1}(f))\Bigg] \\
&= \frac{(-1)^{h_{s_{l+1}(\overline{y})}(l+1)}[x_{\overline{y}}(l+2) - x_{\overline{y}}(l+1)]^{a_{\overline{y}}(l+1)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} f \\
&- \frac{[x_y(l+1) - x_{\overline{y}}(l+2)]^{h_{s_{l+1}(\overline{y})}(l+1)}[x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{h_{\overline{y}}(l+1)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} s_{l+1}s_l s_{l+1}(f).
\end{aligned}
$$

$$
\begin{aligned}
Y &:= \sigma_{s_{l+1}s_l(\overline{y})}(l)\sigma_{s_l(\overline{y})}(l+1)\sigma_{\overline{y}}(l) \diamond f \\
&= \sigma_{s_{l+1}s_l(\overline{y})}(l)\sigma_{s_l(\overline{y})}(l+1) \diamond \Big[[x_{s_l(\overline{y})}(l) - x_{s_l(\overline{y})}(l+1)]^{h_{\overline{y}}(l)} s_l(f)\Big] \\
&= \sigma_{s_{l+1}s_l(\overline{y})}(l) \diamond \Bigg[\frac{[x_{s_l(\overline{y})}(l) - x_{s_l(\overline{y})}(l+1)]^{h_{\overline{y}}(l)} - [x_{s_l(\overline{y})}(l) - x_{s_l(\overline{y})}(l+2)]^{h_{\overline{y}}(l)}}{x_{s_l(\overline{y})}(l+2) - x_{s_l(\overline{y})}(l+1)} s_l(f) \\
&+ \frac{[x_{s_l(\overline{y})}(l) - x_{s_l(\overline{y})}(l+2)]^{h_{\overline{y}}(l)}}{x_{s_l(\overline{y})}(l+2) - x_{s_l(\overline{y})}(l+1)} (s_l(f) - s_{l+1}s_l(f))\Bigg] \\
&= \Big[[x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{h_{s_l(\overline{y})}(l)}\Big] \Bigg[\frac{[x_{\overline{y}}(l+1) - x_{\overline{y}}(l)]^{h_{\overline{y}}(l)} - [x_{\overline{y}}(l+1) - x_{\overline{y}}(l+2)]^{h_{\overline{y}}(l)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} f \\
&+ \frac{[x_{\overline{y}}(l+1) - x_{\overline{y}}(l+2)]^{h_{\overline{y}}(l)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} (f - s_l s_{l+1} s_l(f))\Bigg] \\
&= \frac{(-1)^{h_{\overline{y}}(l)}[x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{a_{\overline{y}}(l)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} f - \frac{[x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{h_{s_l(\overline{y})}(l)}[x_{\overline{y}}(l+1) - x_{\overline{y}}(l+2)]^{h_{\overline{y}}(l)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} s_l s_{l+1} s_l(f)
\end{aligned}
$$

We have $s_{l+1}s_l s_{l+1}(f) = s_l s_{l+1} s_l(f)$. Moreover, since $i^{\overline{y}}_l = i^{\overline{y}}_{l+2}$, we have

$$
h_{\overline{y}}(l) = h_{s_{l+1}(\overline{y})}(l+1), \qquad h_{s_l(\overline{y})}(l) = h_{\overline{y}}(l+1), \qquad a_{\overline{y}}(l) = a_{\overline{y}}(l+1).
$$

Hence

$$
X - Y = (-1)^{h_{\overline{y}}(l)} \frac{[x_{\overline{y}}(l+2) - x_{\overline{y}}(l+1)]^{a_{\overline{y}}(l)} - [x_{\overline{y}}(l) - x_{\overline{y}}(l+1)]^{a_{\overline{y}}(l)}}{x_{\overline{y}}(l+2) - x_{\overline{y}}(l)} f.
$$

We leave the other cases to the reader (use the braid relations for Demazure operators). □

11.6 The main theorem

We now prove that the list of relations we have given is complete.

Theorem 11.18. *The algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is generated by $\{1_{\overline{y},\overline{y}}, \sigma_{\overline{y}}(1), ..., \sigma_{\overline{y}}(\mathbf{d}-1), \varkappa_{\overline{y}}(1), ..., \varkappa_{\overline{y}}(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ subject to the relations in Theorem 11.16 (i.e., the relations in Theorem 11.16 generate all the relations).*

Proof. Let A be the k-algebra generated by $\{1_{\overline{y},\overline{y}}, \sigma_{\overline{y}}(1), ..., \sigma_{\overline{y}}(\mathbf{d}-1), \varkappa_{\overline{y}}(1), ..., \varkappa_{\overline{y}}(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ subject to the relations in Theorem 11.16. Then there is an obvious surjective k-algebra homomorphism

$$A \to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}). \tag{83}$$

We know that $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-module of rank $\mathbf{d}!$ with basis $\{\sigma(w) \mid w \in \mathbb{W}_{\mathbf{d}}\}$ (for some choices of reduced decompositions). It is clear that the map (83) is $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-linear. Therefore, it suffices to show that A also has rank $\mathbf{d}!$ as a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-module. In fact, it is easy to show that $\{\sigma(w) \mid w \in \mathbb{W}_{\mathbf{d}}\}$ is a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-basis of A. By the straightening rule, we know that we can express any element $a \in A$ in the form

$$a = f^\beta + \sum_\alpha f^\alpha \sigma(l_1^\alpha)...\sigma(l_{r^\alpha}^\alpha),$$

where each $l_k^\alpha \in \{1, ..., \mathbf{d}-1\}$, $r^\alpha \geq 1$, $f^\alpha, f^\beta \in < 1_{\overline{y},\overline{y}}, \varkappa_{\overline{y}}(1), ..., \varkappa_{\overline{y}}(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}} > \cong H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$ and α ranges over a finite index set. Let $w \in \mathbb{W}_{\mathbf{d}}$ and let $w = s_{l_1}...s_{l_r}$ be its chosen reduced decomposition. Now the quadratic and the braid relations imply that if $w = s_{t_1}...s_{t_r}$ is another reduced decomposition of w then

$$\sigma(t_1)...\sigma(t_r) = \sum_{u \leq w} f_u \sigma(u)$$

for some polynomials $f_u \in < 1_{\overline{y},\overline{y}}, \varkappa_{\overline{y}}(1), ..., \varkappa_{\overline{y}}(\mathbf{d}) \mid \overline{y} \in Y_{\underline{\mathbf{d}}} > \cong H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$. Hence $\{\sigma(w) \mid w \in \mathbb{W}_{\mathbf{d}}\}$ generate A as a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-module. They're also $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-linearly independent because they are independent in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. □

Corollary 11.19. *The algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ endowed with the grading from Definition 11.11 is isomorphic as a graded k-algebra to the algebra defined by Khovanov and Lauda in [KL09] and the algebra defined by Rouquier in [Rou12].*

Proof. All three algebras have the same presentation in terms of generators and relations. □

In light of this corollary, we will from now on refer to the convolution algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ as the KLR algebra or the quiver Hecke algebra associated to the quiver $\mathbf{\Gamma}$ and dimension vector $\underline{\mathbf{d}}$.

11.7 Some corollaries

We now present some easy corollaries which follow from the presentation of our convolution algebra in terms of generators and relations.

Corollary 11.20 (k-basis theorem)**.** *Choose a reduced decomposition for each element $w \in \mathbb{W}_{\mathbf{d}}$. The sets*

$$\{\varkappa(1)^{\alpha_1}\varkappa(2)^{\alpha_2}...\varkappa(\mathbf{d})^{\alpha_{\mathbf{d}}}\sigma(w)1_{\overline{y},\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, w \in \mathbb{W}_{\mathbf{d}}, \alpha_m \in \mathbb{N}_{\geq 0}\},$$
$$\{\sigma(w)\varkappa(1)^{\alpha_1}\varkappa(2)^{\alpha_2}...\varkappa(\mathbf{d})^{\alpha_{\mathbf{d}}}1_{\overline{y},\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, w \in \mathbb{W}_{\mathbf{d}}, \alpha_m \in \mathbb{N}_{\geq 0}\}$$

form k-bases of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Proof. It is obvious that

$$\{\varkappa(1)^{\alpha_1}\varkappa(2)^{\alpha_2}...\varkappa(\mathbf{d})^{\alpha_\mathbf{d}}1_{\overline{y},\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, \alpha_m \in \mathbb{N}_{\geq 0}\},$$

forms a k-basis of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$. But $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a left $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}^e_{\underline{\mathbf{d}}})$-module with basis $\{\sigma(w) \mid w \in \mathbb{W}_\mathbf{d}\}$. Hence $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is a k-module with basis

$$\{\varkappa(1)^{\alpha_1}\varkappa(2)^{\alpha_2}...\varkappa(\mathbf{d})^{\alpha_\mathbf{d}}1_{\overline{y},\overline{y}}\sigma(w) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, w \in \mathbb{W}_\mathbf{d}, \alpha_m \in \mathbb{N}_{\geq 0}\}.$$

Using the idempotents relations and the straightening rule one obtains the two bases in the statement of the corollary. $\square$

Definition 11.21. Let $\mathbf{\Gamma}^0$ denote the underlying unoriented graph of the quiver $\mathbf{\Gamma}$. We can view it as a triple $(\mathbf{I}, \mathbf{H}, \mathbf{st})$, where $\mathbf{st}$ is a function from $\mathbf{H}$ to the set of two-element (and one-element, if we allow loops) subsets of $\mathbf{I}$ such that $\mathbf{st}(h) = \{\mathbf{s}(h), \mathbf{t}(h)\}$. We say that the source and target functions $\mathbf{s}, \mathbf{t}$ are the *orientation* of the graph $\mathbf{\Gamma}^0$. Let us denote a choice of orientation of $\mathbf{\Gamma}^0$ with the symbol $\mathcal{O}$. So far we have assumed that we are working with a fixed quiver $\mathbf{\Gamma}$ and suppressed it from the notation. Occasionally we will want to compare algebras and varieties arising from different choices of a quiver or different choices of an orientation of a fixed underlying unoriented graph. We therefore introduce the following notation. Let $\mathcal{Z}(\mathbf{\Gamma}, \underline{\mathbf{d}})$, $\widetilde{\mathcal{F}}(\mathbf{\Gamma}, \underline{\mathbf{d}})$ and $\mathcal{F}(\mathbf{\Gamma}, \underline{\mathbf{d}})$ denote the quiver Steinberg variety, the incidence variety and the quiver flag variety, resp., arising from the quiver $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. Also let $\mathcal{Z}(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}})$, $\widetilde{\mathcal{F}}(\mathbf{\Gamma}, \mathcal{O}, \underline{\mathbf{d}})$ and $\mathcal{F}(\mathbf{\Gamma}, \mathcal{O}, \underline{\mathbf{d}})$ denote the quiver Steinberg variety, the incidence variety and the quiver flag variety, resp., arising from the unoriented graph $\mathbf{\Gamma}$ with orientation $\mathcal{O}$ and dimension vector $\underline{\mathbf{d}}$. We set

$$H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k) := H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}(\mathbf{\Gamma}, \underline{\mathbf{d}}); k), \quad H(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}}; k) := H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}}); k).$$

Note that we have not defined anything new here - we have merely made the dependency on the quiver and the orientation explicit. In the sequel we will continue to suppress the quiver and the choice of orientation from the notation whenever we can assume that these choice are fixed, i.e., essentially when we are not directly comparing results for different choices of quivers or orientations.

Corollary 11.22 (Change of orientation)**.** *Let $\mathcal{O} = (\mathbf{s}, \mathbf{t})$ and $\mathcal{O}' = (\mathbf{s}', \mathbf{t}')$ be two choices of orientation of the unoriented graph $\mathbf{\Gamma}^0$. For $i, j \in \mathbf{I}$ let*

$$h_{i,j} = |\{h \in \mathbf{H} \mid \mathbf{s}(h) = i, \mathbf{t}(h) = j\}|, \quad h'_{i,j} = |\{h \in \mathbf{H} \mid \mathbf{s}'(h) = i, \mathbf{t}'(h) = j\}|,$$

$$\beta(i,j) = \begin{cases} (-1)^{h_{i,j}+h'_{i,j}} & \text{if } i \neq j, h_{i,j} \geq h'_{i,j} \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$H(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}}; k) \xrightarrow{\cong} H(\mathbf{\Gamma}^0, \mathcal{O}', \underline{\mathbf{d}}; k), \quad 1_{\overline{y},\overline{y}} \mapsto 1'_{\overline{y},\overline{y}}, \quad \varkappa_{\overline{y}}(l) \mapsto \varkappa'_{\overline{y}}(l), \quad \sigma_{\overline{y}}(l) \mapsto \beta(i^{\overline{y}}_l, i^{\overline{y}}_{l+1})\sigma'_{\overline{y}}(l)$$

is a k-algebra isomorphism (where $\{1_{\overline{y},\overline{y}}, \varkappa_{\overline{y}}(l), \sigma_{\overline{y}}(m) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, 1 \leq l \leq \mathbf{d}, 1 \leq m \leq \mathbf{d}-1\}$ are the standard generators of $H(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}}; k)$ and $\{1'_{\overline{y},\overline{y}}, \varkappa'_{\overline{y}}(l), \sigma'_{\overline{y}}(m) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, 1 \leq l \leq \mathbf{d}, 1 \leq m \leq \mathbf{d}-1\}$ are the standard generators of $H(\mathbf{\Gamma}^0, \mathcal{O}', \underline{\mathbf{d}}; k)$).

Proof. This is obviously a vector space isomorphism. To prove that this is also an algebra homomorphism, we directly check the relations. It is clear that the idempotents and polynomial relations as well as the straightening rule are preserved. Let us check the quadratic relation. If $s_l(\overline{y}) = \overline{y}$ then $\sigma_{s_l(\overline{y})}(l)\sigma_{\overline{y}}(l) = 0 = \sigma'_{s_l(\overline{y})}(l)\sigma'_{\overline{y}}(l)$. So suppose that $s_l(\overline{y}) \neq \overline{y}$. Let $i := i^{\overline{y}}_l, j := i^{\overline{y}}_{l+1}$. If $h_{i,j} \geq h'_{i,j}$ then

$$\beta(j,i)\sigma'_{s_l(\overline{y})}(l)\beta(i,j)\sigma'_{\overline{y}}(l) = (-1)^{h'_{i,j}+h_{i,j}+h'_{i,j}}(\varkappa'_{\overline{y}}(l) - \varkappa'_{\overline{y}}(l+1))^{a_{\overline{y}}(l)} = (-1)^{h_{i,j}}(\varkappa'_{\overline{y}}(l) - \varkappa'_{\overline{y}}(l+1))^{a_{\overline{y}}(l)}.$$

If $h_{i,j} < h'_{i,j}$ then

$$\beta(j,i)\sigma'_{s_l(\overline{y})}(l)\beta(i,j)\sigma'_{\overline{y}}(l) = (-1)^{h'_{i,j}+h_{j,i}+h'_{j,i}}(\varkappa'_{\overline{y}}(l)-\varkappa'_{\overline{y}}(l+1))^{a_{\overline{y}}(l)} = (-1)^{h_{i,j}}(\varkappa'_{\overline{y}}(l)-\varkappa'_{\overline{y}}(l+1))^{a_{\overline{y}}(l)}$$

because $h_{i,j} + h_{j,i} = h'_{i,j} + h'_{j,i}$ and so $h_{i,j} = h'_{i,j} + h'_{j,i} + h_{j,i} \bmod 2$. We leave checking the braid relations to the reader. □

Remark 11.23. The corollary implies that up to isomorphism the graded algebra $H(\mathbf{\Gamma}^0, \mathcal{O}, \underline{\mathbf{d}}; k)$ depends only on the underlying undirected graph $\mathbf{\Gamma}^0$. Note, however, that these isomorphisms do not commute with the faithful polynomial representation (as can easily be seen from the description of this representation).

Corollary 11.24. *Given $\overline{y} \in Y_{\underline{\mathbf{d}}}$, let $\overline{y}^* \in Y_{\underline{\mathbf{d}}}$ be such that $i_l^{\overline{y}^*} = i^{\overline{y}}_{\mathbf{d}+1-l}$. There is an involutive algebra automorphism*

$$H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k) \xrightarrow{\cong} H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k), \quad 1_{\overline{y},\overline{y}} \mapsto 1_{\overline{y},\overline{y}}, \quad \varkappa_{\overline{y}}(l) \mapsto \varkappa_{\overline{y}^*}(\mathbf{d}+1-l), \quad \sigma_{\overline{y}}(l) \mapsto -\sigma_{\overline{y}^*}(\mathbf{d}-l).$$

Proof. A straightforward calculation left to the reader. □

Remark 11.25. There is also an involutive graded vector space automorphism

$$H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k) \xrightarrow{\cong} H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k), \quad x_{\overline{y}}(l) \mapsto x_{\overline{y}^*}(\mathbf{d}+1-l).$$

These automorphisms commute with the action of $H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k)$ on $H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k)$.

Corollary 11.26. *There is an isomorphism*

$$H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k) \xrightarrow{\cong} H(\mathbf{\Gamma}, \underline{\mathbf{d}}; k)^{opp}, \quad 1_{\overline{y},\overline{y}} \mapsto 1_{\overline{y},\overline{y}}, \quad \varkappa_{\overline{y}}(l) \mapsto \varkappa_{\overline{y}}(l), \quad \sigma_{\overline{y}}(l) \mapsto \sigma_{s_l(\overline{y})}(l).$$

Proof. A straightforward calculation left to the reader. □

11.8 Examples

We can obtain some familiar algebras as quiver Hecke algebras, for example, matrix rings with polynomial entries and NilHecke rings.

Example 11.27 (Matrix ring with polynomial entries)**.** Set $\mathbf{I} = \{i_1, ..., i_n\}$, $\mathbf{H} = \varnothing$ and $\underline{\mathbf{d}} = i_1 + ... + i_n$. Then $\mathbb{W}_{\mathbf{d}} \cong \mathfrak{S}_n$, $W_{\underline{\mathbf{d}}} = \{e\}$, $|Y_{\underline{\mathbf{d}}}| = n!$, $G_{\underline{\mathbf{d}}} = T_{\underline{\mathbf{d}}} \cong (\mathbb{C}^\times)^n$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \{0\}$. Moreover, $\mathcal{F}_{\overline{y}} \cong \{pt\}$ for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$, $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} = \mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{Z}_{\underline{\mathbf{d}}} = \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$. We have $S_{G_{\underline{\mathbf{d}}}} = S_{T_{\underline{\mathbf{d}}}} = k[x_{\underline{\mathbf{d}}}(1), ..., x_{\underline{\mathbf{d}}}(n)]$ and $H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ is a free $S_{G_{\underline{\mathbf{d}}}}$-module of rank $n!$ with basis $\{1_{\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$. Let $\phi_{\overline{y},\overline{y}'}$ be the $S_{G_{\underline{\mathbf{d}}}}$-linear endomorphism of $H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}})$ sending $1_{\overline{y}}$ to $1_{\overline{y}'}$ and all the other basis elements $1_{\overline{y}''}$ to 0. Then $\mathrm{End}_{S_{G_{\underline{\mathbf{d}}}}}\left(H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}})\right)$ is a free $S_{G_{\underline{\mathbf{d}}}}$-module of rank $(n!)^2$ with basis $\{\phi_{\overline{y},\overline{y}'} \mid \overline{y}, \overline{y}' \in Y_{\underline{\mathbf{d}}}\}$. Since for each $s_l \in \Pi$ and $\overline{y} \in Y_{\underline{\mathbf{d}}}$, we have $s_l(\overline{y}) \neq \overline{y}$, the elements $\sigma_{\overline{y}}(l)$ never act as Demazure operators. Moreover, since $h_{\overline{y}}(l) = 0$ for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ and l, we have $\sigma_{\overline{y}}(l) \diamond f = s_l(f)$, for $f \in H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\overline{y}})$. Hence, if $\overline{y}' = w(\overline{y})$, then $\sigma_{\overline{y}}(w) = \phi_{\overline{y},\overline{y}'}$ as $S_{G_{\underline{\mathbf{d}}}}$-linear operators on $H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}})$. It follows that

$$\{\phi_{\overline{y},\overline{y}'} \mid \overline{y}, \overline{y}' \in Y_{\underline{\mathbf{d}}}\} = \{\sigma_{\overline{y}}(w) \mid \overline{y} \in Y_{\underline{\mathbf{d}}}, w \in \mathbb{W}_{\mathbf{d}}\}$$

and thus

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong \mathrm{End}_{S_{G_{\underline{\mathbf{d}}}}}\left(H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}})\right) \cong \mathrm{Mat}(n! \times n!, k[x(1), ..., x(n)]).$$

In particular, if $n = 1$ then $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong k[x]$. Now let us consider in detail the case $n = 2$. We have $Y_{\underline{\mathbf{d}}} = \{\overline{y}, \overline{y}'\}$, where $\overline{y} = (i_1, i_2)$ and $\overline{y}' = (i_2, i_1)$. We consider $H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}) \cong k[x_{\overline{y}}(1), x_{\overline{y}}(2)] \oplus k[x_{\overline{y}'}(1), x_{\overline{y}'}(2)]$ as a free module over $k[x_{\overline{y}}(1) + x_{\overline{y}}(1), x_{\overline{y}}(2) + x_{\overline{y}'}(2)]$ with ordered basis $1_{\overline{y}}, 1_{\overline{y}'}$.

Setting $x(1) = x_{\overline{y}}(1) + x_{\overline{y}'}(1), x(2) = x_{\overline{y}}(2) + x_{\overline{y}'}(2)$ we can thus interpret an element of $H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ as a two-row vector with entries in $k[x(1), x(2)]$. We want to explicitly describe the isomorphism

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong \mathrm{Mat}(2 \times 2, k[x(1), x(2)]).$$

It is given by the following map

$$\begin{aligned} 1_{\overline{y},\overline{y}} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & 1_{\overline{y}',\overline{y}'} &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \varkappa_{\overline{y}}(l) &\mapsto \begin{pmatrix} x(l) & 0 \\ 0 & 0 \end{pmatrix}, & \varkappa_{\overline{y}'}(l) &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & x(l) \end{pmatrix}, \\ \sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \sigma_{\overline{y}'}(1) &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

for $l = 1, 2$.

Example 11.28 (NilHecke ring)**.** Set $\mathbf{I} = \{i\}$, $\mathbf{H} = \varnothing$ and $\underline{\mathbf{d}} = ni$. Then $\mathbb{W}_{\mathbf{d}} = W_{\underline{\mathbf{d}}} \cong \mathfrak{S}_n$, $|Y_{\underline{\mathbf{d}}}| = 1$, $Y_{\underline{\mathbf{d}}} = \{\overline{y}\}$, where $\overline{y} = (i, i, ..., i)$, $G_{\underline{\mathbf{d}}} = \mathbb{G}_{\mathbf{d}} \cong \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \{0\}$. Moreover, $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} = \mathcal{F}_{\underline{\mathbf{d}}} = \mathcal{F}_{\overline{y}}$, $H_*^{G_{\mathbf{d}}}(\mathcal{F}_{\overline{y}}) = k[x_{\overline{y}}(1), ..., x_{\overline{y}}(n)]$ and $\mathcal{Z}_{\underline{\mathbf{d}}} = \mathcal{F}_{\overline{y}} \times \mathcal{F}_{\overline{y}}$. Since for each $s_l \in \Pi$, we have $s_l(\overline{y}) = \overline{y}$, the elements $\sigma_{\overline{y}}(l)$ always act as Demazure operators. Hence $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is the ring of endomorphisms of $k[x_{\overline{y}}(1), ..., x_{\overline{y}}(n)]$ generated by endomorphisms $\varkappa_{\overline{y}}(l)$ which act by multiplication with $x_{\overline{y}}(l)$ and Demazure operators $\sigma_{\overline{y}}(l)$. Therefore

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong NH_n,$$

i.e., the convolution algebra $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic to the NilHecke ring (see Definition 7.33). Note that the quadratic relation and the "braid relations" for elements $\sigma_{\overline{y}}(l)$ simplify to the braid relations for Demazure operators.

It is well known that the NilHecke ring NH_n is also isomorphic to the matrix algebra $\mathrm{Mat}\left(n! \times n!, k[x(1), ..., x(n)]^{\mathfrak{S}_n}\right)$ (see for example [KL09], p.11). Let us set $n = 2$ and construct an isomorphism

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong \mathrm{Mat}\left(2 \times 2, k[x(1)x(2), x(1) + x(2)]\right). \tag{84}$$

We consider the polynomial ring $k[x(1), x(2)]$ as a $k[x(1)x(2), x(1)+x(2)]$-module of rank 2 with ordered basis $1, x(1)$. The algebra $\mathrm{Mat}\left(2 \times 2, k[x(1)x(2), x(1) + x(2)]\right)$ acts naturally on $k[x(1), x(2)]$ endowed with this basis by matrix multiplication. Consider the map

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \to \mathrm{Mat}\left(2 \times 2, k[x(1)x(2), x(1) + x(2)]\right) \tag{85}$$

defined by

$$\begin{aligned} 1_{\overline{y},\overline{y}} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \varkappa_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & -x(1)x(2) \\ 1 & x(1) + x(2) \end{pmatrix}, \\ \sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, & \varkappa_{\overline{y}}(2) &\mapsto \begin{pmatrix} x(1) + x(2) & x(1)x(2) \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

This map intertwines the actions of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and $\mathrm{Mat}\left(2 \times 2, k[x(1)x(2), x(1) + x(2)]\right)$ on $k[x(1), x(2)]$. Indeed, we have, for example, $\varkappa_{\overline{y}}(1) \diamond 1 = x(1)$, $\varkappa_{\overline{y}}(1) \diamond x(1) = x(1)^2$ and

$$\begin{pmatrix} 0 & -x(1)x(2) \\ 1 & x(1) + x(2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x(1),$$

$$\begin{pmatrix} 0 & -x(1)x(2) \\ 1 & x(1) + x(2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -x(1)x(2) \\ x(1) + x(2) \end{pmatrix} = -x(1)x(2) + (x(1) + x(2))x(1) = x(1)^2.$$

It follows that the map (85) is injective (since the elements $1_{\overline{y},y}, \varkappa_{\overline{y}}(1), \varkappa_{\overline{y}}(2), \sigma_{\overline{y}}(1)$ and their images act in the same way on $k[x(1), x(2)]$). To show surjectivity, we find pre-images of a basis of $\mathrm{Mat}\,(2 \times 2, k[x(1)x(2), x(1) + x(2)])$ over its centre (which is isomorphic to $k[x(1)x(2), x(1)+x(2)]$):

$$\begin{aligned} \sigma_{\overline{y}}(1)\varkappa_{\overline{y}}(2) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \sigma_{\overline{y}}(1)\varkappa_{\overline{y}}(1)\sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ -\sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & -\varkappa_{\overline{y}}(1)\sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Let us also look at some examples with arrows.

Example 11.29. Set $\mathbf{I} = \{i, j\}, \mathbf{H} = \{i \xrightarrow{n} j\}$ (n arrows from i to j) and $\underline{\mathbf{d}} = i+j$. Then $\mathbb{W}_{\mathbf{d}} \cong \mathfrak{S}_2$, $W_{\underline{\mathbf{d}}} = \{e\}$, $|Y_{\underline{\mathbf{d}}}| = 2$, $Y_{\underline{\mathbf{d}}} = \{\overline{y} = (i,j), \overline{y}' = (j,i)\}$, $G_{\underline{\mathbf{d}}} = T_{\mathbf{d}} \cong (\mathbb{C}^\times)^2$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \mathbb{C}^n$. Moreover, $\mathcal{F}_{\overline{y}} \cong \mathcal{F}_{\overline{y}'} \cong \{pt\}$, $\widetilde{\mathcal{F}}_{\overline{y}} \cong \mathcal{F}_{\overline{y}}$, $\widetilde{\mathcal{F}}_{\overline{y}'} \cong \mathbb{C}^n$, and $\mathcal{Z}_{\overline{y},\overline{y}} \cong \mathcal{Z}_{\overline{y},\overline{y}'} \cong \mathcal{Z}_{\overline{y}',\overline{y}} \cong \{pt\}$, $\mathcal{Z}_{\overline{y}',\overline{y}'} \cong \mathbb{C}^n$. We have $S_{G_{\underline{\mathbf{d}}}} = S_{T_{\mathbf{d}}} = k[x_{\underline{\mathbf{d}}}(1), x_{\underline{\mathbf{d}}}(2)]$ and $H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is a free $S_{G_{\underline{\mathbf{d}}}}$-module of rank 2 with basis $\{1_{\overline{y}}, 1_{\overline{y}'}\}$. The convolution algebra $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is generated by the idempotents, $\varkappa_{\overline{y}}(1), \varkappa_{\overline{y}}(2), \varkappa_{\overline{y}'}(1), \varkappa_{\overline{y}'}(2)$ and $\sigma_{\overline{y}}(1), \sigma_{\overline{y}'}(1)$. We have

$$\sigma_{\overline{y}}(1) \diamond f = (x_{\overline{y}'}(1) - x_{\overline{y}'}(2))^n s_l(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}),$$

$$\sigma_{\overline{y}'}(1) \diamond f = s_l(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}'}).$$

Note that, unlike in the previous two examples, $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic to a proper subring of $\mathrm{End}_{S_{G_{\underline{\mathbf{d}}}}}\left(H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})\right) \cong \mathrm{Mat}(2 \times 2, k[x_{\underline{\mathbf{d}}}(1), x_{\underline{\mathbf{d}}}(2)])$. The inclusion

$$H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \hookrightarrow \mathrm{Mat}(2 \times 2, k[x_{\underline{\mathbf{d}}}(1), x_{\underline{\mathbf{d}}}(2)])$$

is given by

$$\begin{aligned} 1_{\overline{y},\overline{y}} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & 1_{\overline{y}',\overline{y}'} &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \varkappa_{\overline{y}}(l) &\mapsto \begin{pmatrix} x(l) & 0 \\ 0 & 0 \end{pmatrix}, & \varkappa_{\overline{y}'}(l) &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & x(l) \end{pmatrix}, \\ \sigma_{\overline{y}}(1) &\mapsto \begin{pmatrix} 0 & 0 \\ (x(1) - x(2))^n & 0 \end{pmatrix}, & \sigma_{\overline{y}'}(1) &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

for $l = 1, 2$.

Example 11.30. Set $\mathbf{I} = \{i, j\}$, $\mathbf{H} = \{i \xrightarrow{n} j\}$ and $\underline{\mathbf{d}} = 2i + j$. Then $\mathbb{W}_{\mathbf{d}} \cong \mathfrak{S}_3$, $W_{\underline{\mathbf{d}}} \cong \mathfrak{S}_2$, $|Y_{\underline{\mathbf{d}}}| = 3$, $Y_{\underline{\mathbf{d}}} = \{\overline{y} = (i,i,j), \overline{y}' = (i,j,i), \overline{y}'' = (j,i,i)\}$, $G_{\underline{\mathbf{d}}} \cong \mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^\times$, $T_{\mathbf{d}} \cong (\mathbb{C}^\times)^3$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \mathbb{C}^{2n}$. Moreover, $\mathcal{F}_{\overline{y}} \cong \mathcal{F}_{\overline{y}'} \cong \mathcal{F}_{\overline{y}''} \cong \mathbb{CP}^1 \cong \widetilde{\mathcal{F}}_{\overline{y}}$, $\widetilde{\mathcal{F}}_{\overline{y}'}$ is a complex vector bundle over $\mathcal{F}_{\overline{y}'}$ of rank n and $\widetilde{\mathcal{F}}_{\overline{y}''}$ is a complex vector bundle over $\mathcal{F}_{\overline{y}''}$ of rank $2n$. Furthermore, $\mathcal{Z}_{\overline{y},\overline{y}} \cong \mathcal{Z}_{\overline{y},\overline{y}'} \cong \mathcal{Z}_{\overline{y}',\overline{y}} \cong \mathcal{Z}_{\overline{y}'',\overline{y}} \cong \mathcal{Z}_{\overline{y},\overline{y}''} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathcal{Z}_{\overline{y}',\overline{y}'} \cong \mathcal{Z}_{\overline{y}',\overline{y}''} \cong \mathcal{Z}_{\overline{y}'',\overline{y}'}$ is a complex vector bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ of rank n and $\mathcal{Z}_{\overline{y}'',\overline{y}''}$ is a complex vector bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ of rank $2n$. We have $S_{T_{\underline{\mathbf{d}}}} = k[x_{\underline{\mathbf{d}}}(1), x_{\underline{\mathbf{d}}}(2), x_{\underline{\mathbf{d}}}(3)], S_{G_{\underline{\mathbf{d}}}} = k[x_{\underline{\mathbf{d}}}(1)x_{\underline{\mathbf{d}}}(2), x_{\underline{\mathbf{d}}}(1) + x_{\underline{\mathbf{d}}}(2), x_{\underline{\mathbf{d}}}(3)]$ and $H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is a free $S_{G_{\underline{\mathbf{d}}}}$-module of rank 6. We now have several different kinds of σ operators:

$$\begin{aligned}
\sigma_{\overline{y}}(1) \diamond f &= \frac{f - s_1(f)}{x_{\overline{y}}(2) - x_{\overline{y}}(1)}, \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}),\\
\sigma_{\overline{y}''}(2) \diamond f &= \frac{f - s_2(f)}{x_{\overline{y}''}(3) - x_{\overline{y}''}(2)}, \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}''}),\\
\sigma_{\overline{y}}(2) \diamond f &= (x_{\overline{y}'}(2) - x_{\overline{y}'}(3))^n s_2(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}),\\
\sigma_{\overline{y}'}(1) \diamond f &= (x_{\overline{y}''}(1) - x_{\overline{y}''}(2))^n s_1(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}'}),\\
\sigma_{\overline{y}'}(2) \diamond f &= s_2(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}'}),\\
\sigma_{\overline{y}''}(1) \diamond f &= s_1(f), \quad f \in H_*^{G_{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}''}).
\end{aligned}$$

11.9 Quivers with loops

We are now going to generalize the results of this section to quivers with loops. Suppose that $\mathbf{\Gamma}$ is now a quiver which may have loops. Let $\mathbf{H} = \mathbf{H}^{\circlearrowleft} \sqcup \mathbf{H}^{\rightarrow}$, where $\mathbf{H}^{\circlearrowleft} = \{h \in \mathbf{H} \mid \mathbf{s}(h) = \mathbf{t}(h)\}$ is the set of loops and $\mathbf{H}^{\rightarrow} = \{h \in \mathbf{H} \mid \mathbf{s}(h) \neq \mathbf{t}(h)\}$ is the set of all the other arrows.

Let $\rho = (\rho_h) \in \mathrm{Rep}_{\underline{\mathbf{d}}}$. Define $\rho^{\rightarrow} = (\rho_h^{\rightarrow})$, $\rho^{\circlearrowleft} = (\rho_h^{\circlearrowleft})$ by setting

$$\rho_h^{\rightarrow} = \begin{cases} \rho_h & \text{if } h \in \mathbf{H}^{\rightarrow}, \\ 0 & \text{if } h \in \mathbf{H}^{\circlearrowleft}, \end{cases} \qquad \rho_h^{\circlearrowleft} = \begin{cases} 0 & \text{if } h \in \mathbf{H}^{\rightarrow}, \\ \rho_h & \text{if } h \in \mathbf{H}^{\circlearrowleft}. \end{cases}$$

We of course have $\rho = \rho^{\rightarrow} + \rho^{\circlearrowleft}$. We further define

$$\mathrm{Rep}_{\underline{\mathbf{d}}}^{\rightarrow} = \{\rho^{\rightarrow} \mid \rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}\}, \quad \mathrm{Rep}_{\underline{\mathbf{d}}}^{\circlearrowleft} = \{\rho^{\circlearrowleft} \mid \rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}\}.$$

Let $F = (V^l)$ be a quiver flag. We call F ρ-stable if $\rho(V^l) \subseteq V^{l-1}$ for each l. This is equivalent to requiring that $\rho^{\rightarrow}(V^l) \subseteq V^l$ and $\rho^{\circlearrowleft}(V^l) \subseteq V^{l-1}$ for each l and hence consistent with our prior definition of stability. For each $w \in \mathbb{W}_{\mathbf{d}}$ we also define

$$\mathfrak{r}_w^{\rightarrow} = \mathfrak{r}_w \cap \mathrm{Rep}_{\underline{\mathbf{d}}}^{\rightarrow}, \quad \mathfrak{r}_w^{\circlearrowleft} = \mathfrak{r}_w \cap \mathrm{Rep}_{\underline{\mathbf{d}}}^{\circlearrowleft}.$$

We of course have

$$\mathrm{Rep}_{\underline{\mathbf{d}}} = \mathrm{Rep}_{\underline{\mathbf{d}}}^{\rightarrow} \oplus \mathrm{Rep}_{\underline{\mathbf{d}}}^{\circlearrowleft}, \quad \mathfrak{r}_w = \mathfrak{r}_w^{\rightarrow} \oplus \mathfrak{r}_w^{\circlearrowleft}.$$

Observe that Lemma 9.16 still holds for quivers with loops. Let us recall it here.

Lemma 11.31. *We have*

$$\mathrm{eu}(\mathfrak{d}_{w,ws}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{i^w_{l+1}, i^w_l}}.$$

We now modify the statement and proof of Lemma 9.17.

Lemma 11.32. *(i) Let $ws \in W_{\underline{\mathbf{d}}}w$. Then*

$$\widetilde{\Lambda}^s_{w,w} = (\mathrm{eu}(\mathfrak{d}_{w,ws}))^{-1}\mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_w = -(\chi_{w(l)} - \chi_{w(l+1)})^{1-h_{i^w_l, i^w_l}}\widetilde{\Lambda}_w,$$

$$\widetilde{\Lambda}^s_{ws,w} = (\mathrm{eu}(\mathfrak{d}_{ws,w}))^{-1}\mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_{ws} = (-1)^{1+h_{i^w_l, i^w_l}}(\chi_{w(l)} - \chi_{w(l+1)})^{1-h_{i^w_l, i^w_l}}\widetilde{\Lambda}_{ws},$$

$$\widetilde{\Lambda}_w = (-1)^{1+h_{i^w_l, i^w_l}}\widetilde{\Lambda}_{ws},$$

$$\widetilde{\Lambda}^s_{w,ws} = (\chi_{w(l)} - \chi_{w(l+1)})^{1-h_{i^w_l, i^w_l}}\widetilde{\Lambda}_w.$$

(ii) Let $ws \notin W_{\underline{\mathbf{d}}}w$. Then

$$(\widetilde{\Lambda}^s_{w,ws})^{-1}\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{d}_{w,ws}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{i^w_{l+1}, i^w_l}}.$$

Proof. (i) Since $ws \in W_{\underline{\mathbf{d}}}w$, $\overline{y}_w = s(\overline{y}_w)$ and so $i^w_{l+1} = i^w_l$.We have

$$\mathfrak{r}_w = \mathfrak{r}_{w,ws} \oplus \mathfrak{d}_{w,ws}, \quad \mathfrak{r}_{ws} = \mathfrak{r}_{w,ws} \oplus \mathfrak{d}_{ws,w},$$

$$\mathrm{eu}(\mathfrak{d}_{w,ws}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{i^w_l,i^w_l}} = (-1)^{h_{i^w_l,i^w_l}} \mathrm{eu}(\mathfrak{d}_{ws,w}),$$

$$\mathrm{eu}(\mathfrak{r}_w) = (-1)^{h_{i^w_l,i^w_l}} \mathrm{eu}(\mathfrak{r}_{ws}).$$

Lemmata 9.14 and 9.15 imply

$$\begin{aligned}\widetilde{\Lambda}^s_{w,w} = \mathrm{eu}(\mathfrak{r}_{w,ws})\Lambda^s_{w,w} &= (\mathrm{eu}(\mathfrak{d}_{w,ws}))^{-1}\mathrm{eu}(\mathfrak{r}_w)\mathrm{eu}(\mathfrak{m}_{ws,w})\mathrm{eu}(\mathfrak{n}^-_w)\\ &= (\mathrm{eu}(\mathfrak{d}_{w,ws}))^{-1}\mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_w\\ &= -(\chi_{w(l)} - \chi_{w(l+1)})^{1-h_{i^w_l,i^w_l}}\widetilde{\Lambda}_w,\end{aligned}$$

$$\begin{aligned}\widetilde{\Lambda}^s_{w,ws} = \mathrm{eu}(\mathfrak{r}_{w,ws})\Lambda^s_{w,ws} &= \mathrm{eu}(\mathfrak{d}_{ws,w})^{-1}\mathrm{eu}(\mathfrak{r}_{ws})\mathrm{eu}(\mathfrak{m}_{w,ws})\mathrm{eu}(\mathfrak{n}^-_w)\\ &= \mathrm{eu}(\mathfrak{d}_{ws,w})^{-1}\mathrm{eu}(\mathfrak{r}_{ws})\mathrm{eu}(\mathfrak{m}_{ws,w})\mathrm{eu}(\mathfrak{n}^-_{ws})\\ &= \mathrm{eu}(\mathfrak{d}_{ws,w})^{-1}\mathrm{eu}(\mathfrak{m}_{ws,w})\widetilde{\Lambda}_{ws}\\ &= (-1)^{1+h_{i^w_l,i^w_l}}(\chi_{w(l)} - \chi_{w(l+1)})^{1-h_{i^w_l,i^w_l}}\widetilde{\Lambda}_{ws},\end{aligned}$$

$$\begin{aligned}\widetilde{\Lambda}_w = \mathrm{eu}(\mathfrak{r}_w)\Lambda_w &= -\mathrm{eu}(\mathfrak{r}_{ws})(\mathrm{eu}(\mathfrak{r}_w)/\mathrm{eu}(\mathfrak{r}_{ws}))\Lambda_{ws}\\ &= -(\mathrm{eu}(\mathfrak{r}_w)/\mathrm{eu}(\mathfrak{r}_{ws}))\widetilde{\Lambda}_{ws}\\ &= (-1)^{1+h_{i^w_l,i^w_l}}.\end{aligned}$$

(ii) The calculation is the same as for quivers without loops in this case. $\square$

The action of $H^{G_{\mathbf{d}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$ on $H^{G_{\mathbf{d}}}_*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}})$ is the same as for quivers without loops except the following modification.

Proposition 11.33. *Let $\overline{y} = \overline{y}_w \in Y_{\underline{\mathbf{d}}}$, $s_l \in \Pi$, $f \in H^{G_{\mathbf{d}}}_*(\widetilde{\mathcal{F}}_{\overline{y}})$. Suppose that $s_l(\overline{y}) = \overline{y}$. Then*

$$\sigma_{\overline{y}}(l) \diamond f = (\chi_{w(l)} - \chi_{w(l+1)})^{(h_{i^w_l,i^w_l})-1}(s(f) - f).$$

Proof. The proof is the same as in Theorem 11.7(iii) but one uses Lemma 11.32 rather than Lemma 9.17. $\square$

We invite the reader to work out how the relations change.

Example 11.34 (The skew group ring)**.** Let $\mathbf{\Gamma}$ be the Jordan quiver, i.e., $\mathbf{I} = \{i\}$, $\mathbf{H} = \{i \to i\}$. Let $\underline{\mathbf{d}} = ni$. Everything is as in Example 11.28 (the NilHecke ring) except that the elements $\sigma_{\overline{y}}(l)$ don't act as Demazure operators. In fact,

$$\sigma_{\overline{y}}(l) \diamond f = s_l(f) - f, \quad (\sigma_{\overline{y}}(l) + 1_{\overline{y},\overline{y}}) \diamond f = s_l(f), \qquad f \in H^{G_{\mathbf{d}}}_*(\widetilde{\mathcal{F}}_{\overline{y}}) = H^{G_{\mathbf{d}}}_*(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}).$$

The skew group ring of $\mathfrak{S}_n$, denoted $k[x_1, ..., x_n] \rtimes \mathfrak{S}_n$, is defined to be the abelian group $k[x_1, ..., x_n] \times \mathfrak{S}_n$ endowed with the product $(p, w) \cdot (r, u) = (pw(r), wu)$. We have the following algebra isomorphism

$$H^{G_{\mathbf{d}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}}) \to k[x_1, ..., x_n] \rtimes \mathfrak{S}_n, \quad \varkappa_{\overline{y}}(l) \mapsto (x_l, e), \quad \sigma_{\overline{y}}(l) + 1_{\overline{y},\overline{y}} \mapsto (1, s_l).$$

12 Representation theory of convolution algebras

12.1 Perverse sheaves and the decomposition theorem

12.1.1 Derived categories

We will only consider *complex* algebraic varieties, even though some of the definitions and results we state hold more generally. Let Y be a complex algebraic variety. We first fix notation pertaining to the various categories we will consider. Let $Sh(Y)$ denote the abelian category of sheaves of *complex* vector spaces on Y and let $D^b(Y)$ denote the bounded derived category of $Sh(Y)$. We denote the constant sheaf on Y by $\mathbb{C}_Y$. A sheaf $\mathcal{G} \in Sh(Y)$ is called *constructible* if there exists an algebraic stratification of $Y = \sqcup_\lambda S_\lambda$ such that the restriction of $\mathcal{G}$ to each stratum S_λ is a locally constant sheaf of finite dimensional complex vector spaces, i.e., a local system. If $Q \in D^b(Y)$ is a complex of sheaves then let $\mathcal{H}^i(Q)$ denote the i-th cohomology sheaf of this complex. We call a complex $Q \in D^b(Y)$ *constructible* if all the cohomology sheaves $\mathcal{H}^i(Q)$ are constructible. Finally, let $D^b_c(Y)$ be the full subcategory of $D^b(Y)$ whose objects are constructible complexes.

We follow a standard convention and do not distinguish in notation between functors $F : Sh(X) \to Sh(Y)$ and the corresponding derived functors $RF : D^b(X) \to D^b(Y)$ or $LF : D^b(X) \to D^b(Y)$. Let $f : X \to Y$ be a (proper) morphism of algebraic varieties. We let $f_*, f_!, f^*, f^!$ denote the *derived* direct image, proper direct image, inverse image and proper (or exceptional) inverse image functors. Moreover, let $\otimes$ and $\mathcal{H}om$ denote the *derived* internal tensor product and internal Hom functors. When referring to these functors, we will typically omit the word "derived" and just call them them "direct image", "proper direct image", etc.

We take for granted and do not explicitly recall basic definitions and facts concerning sheaves, derived categories and triangulated categories. We do however recall that main concepts and results needed for a precise statement of the decomposition theorem.

12.1.2 Local systems

Definition 12.1. By a *local system* on a complex algebraic variety Y we mean a locally free sheaf of finite dimensional complex vector spaces on Y. We denote by $\mathrm{Loc}(Y)$ the category of local systems on Y.

Proposition 12.2. *Suppose that Y is path-connected, locally path-connected and locally simply connected. Let us choose a base-point $y \in Y$. Then there is an equivalence of categories*

$$\begin{array}{rcl} \mathrm{Loc}(Y) & \longleftrightarrow & \{\text{finite-dimensional representations of } \pi_1(Y, y)\} \\ L & \longmapsto & L_y \end{array}$$

sending a local system to its stalk at y. The fundamental group $\pi_1(Y, y)$ acts naturally on L_y by monodromy.

Proof. See [Rie03, Proposition 1.5] □

Definition 12.3. The fundamental group $\pi_1(Y, y)$ acts on a local system L on Y by automorphisms, i.e., we have a group homomorphism $\phi : \pi_1(Y, y) \to \mathrm{Aut}(L)$. We say that the local system L has *finite monodromy* if $\phi(\pi_1(Y, y))$ is a finite group.

12.1.3 Perverse sheaves

Recall that the *support* of a sheaf is the closure of the set of points where the sheaf has non-trivial stalks.

Definition 12.4. Let Y be a complex algebraic variety and let $Q \in D_c^b(Y)$ be a constructible complex. It satisfies the *support condition* if

- (SUP) $\dim \operatorname{supp}\left(\mathcal{H}^{-i}(Q)\right) \leq i \quad$ for all $\quad i \in \mathbb{Z}$.

The complex Q satisfies the *cosupport condition* if its Verdier dual $Q^\vee$ satisfies the support condition, i.e., if

- (COSUP) $\dim \operatorname{supp}\left(\mathcal{H}^{-i}(Q^\vee)\right) \leq i \quad$ for all $\quad i \in \mathbb{Z}$.

A *perverse sheaf* on Y is a constructible complex $Q \in D_c^b(Y)$ which satisfies the support and cosupport conditions. Let $Perv(Y)$ denote the full subcategory of $D_c^b(Y)$ whose objects are perverse sheaves. Moreover, let ${}^{\mathfrak{p}}D_c^{\leq 0}(Y)$ (resp. ${}^{\mathfrak{p}}D_c^{\geq 0}(Y)$) denote the full subcategory of $D_c^b(Y)$ whose objects are constructible complexes which satisfy the support (resp. cosupport) condition. △

We obviously have ${}^{\mathfrak{p}}D_c^{\leq 0}(Y) \cap {}^{\mathfrak{p}}D_c^{\geq 0}(Y) = Perv(Y)$. The category of perverse sheaves can also be characterised in the following way.

Proposition 12.5. *The pair* $({}^{\mathfrak{p}}D_c^{\leq 0}(Y), {}^{\mathfrak{p}}D_c^{\geq 0}(Y))$ *is a t-structure (truncation structure) on* $D_c^b(Y)$ *and* $Perv(Y)$ *is the heart* ${}^{\mathfrak{p}}D_c^{\leq 0}(Y) \cap {}^{\mathfrak{p}}D_c^{\geq 0}(Y)$ *of this t-structure.*

Proof. See [HTT08, Theorem 8.1.27]. □

Corollary 12.6. *The category* $Perv(Y)$ *is abelian.*

Proof. It is well known that the heart of a t-structure on a derived category forms an abelian category (see [HTT08, Theorem 8.1.9]). □

Definition 12.7. The t-structure $({}^{\mathfrak{p}}D_c^{\leq 0}(Y), {}^{\mathfrak{p}}D_c^{\geq 0}(Y))$ is called the *middle perversity t-structure* on $D_c^b(Y)$. We set ${}^{\mathfrak{p}}D_c^{\leq i}(Y) := {}^{\mathfrak{p}}D_c^{\leq 0}(Y)[-i]$ and ${}^{\mathfrak{p}}D_c^{\geq i}(Y) := {}^{\mathfrak{p}}D_c^{\geq 0}(Y)[-i]$. Let

$$ {}^{\mathfrak{p}}\tau_{\leq i} : D_c^b(Y) \to {}^{\mathfrak{p}}D_c^{\leq i}(Y), \quad {}^{\mathfrak{p}}\tau_{\geq i} : D_c^b(Y) \to {}^{\mathfrak{p}}D_c^{\geq i}(Y) $$

be the *truncation functors* associated to our t-structure. The functor ${}^{\mathfrak{p}}\tau_{\leq i}$ is right adjoint to the inclusion ${}^{\mathfrak{p}}D_c^{\leq i}(Y) \to D_c^b(Y)$ and ${}^{\mathfrak{p}}\tau_{\geq i}$ is left adjoint to the inclusion ${}^{\mathfrak{p}}D_c^{\geq i}(Y) \to D_c^b(Y)$. We also define a functor

$$ {}^{\mathfrak{p}}\mathcal{H}^i : D_c^b(Y) \to Perv(Y), \quad Q \mapsto {}^{\mathfrak{p}}\tau_{\leq 0} \circ {}^{\mathfrak{p}}\tau_{\geq 0}(Q[i]) $$

called the i-th *perverse cohomology functor*.

Definition 12.8. Let $j : U \to Y$ be a locally closed embedding and let $i : \overline{U}\backslash U =: Z \to Y$ be the inclusion of the boundary Z of U. Let $Q \in Perv(U)$ be a perverse sheaf on U. Considering Q as an object in $D_c^b(U)$, we have a natural map $j_!Q \to j_*Q$. It induces a map in perverse cohomology $a : {}^{\mathfrak{p}}\mathcal{H}^0(j_!Q) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_*Q)$. The *intermediate extension* of Q is the perverse sheaf

$$ j_{!*}Q := \operatorname{Im}(a) \in Perv(\overline{U}) \subseteq Perv(Y). $$

The intermediate extension $j_{!*}Q$ can also be characterized as the unique extension of Q to $Perv(\overline{U})$ with neither subobjects nor subquotients supported on Z, or as the unique extension $\overline{Q}$ of Q to $Perv(\overline{U})$ such that $i^*\overline{Q} \in {}^{\mathfrak{p}}D_c^{\leq -1}(Z)$ and $i^!\overline{Q} \in {}^{\mathfrak{p}}D_c^{\geq 1}(Z)$.

Definition 12.9. Let X, Y be algebraic varieties. Suppose that $F : D_c^b(Y) \to D_c^b(X)$ is a functor of triangulated categories. We define a functor ${}^{\mathfrak{p}}F : Perv(Y) \to Perv(X)$ to be the composite of the functors

$$ Perv(Y) \hookrightarrow D_c^b(Y) \xrightarrow{F} D_c^b(X) \xrightarrow{{}^{\mathfrak{p}}\mathcal{H}^0} Perv(X). $$

More generally, for $k \in \mathbb{Z}$ we define a functor ${}^{\mathfrak{p}}\mathcal{H}^k(F) : Perv(Y) \to Perv(X)$ to be the composite of the functors

$$ Perv(Y) \hookrightarrow D_c^b(Y) \xrightarrow{F} D_c^b(X) \xrightarrow{{}^{\mathfrak{p}}\mathcal{H}^k} Perv(X). $$

We have ${}^{\mathfrak{p}}F = {}^{\mathfrak{p}}\mathcal{H}^0(F)$. If $X \to Y$ is a (proper) morphism of algebraic varieties, we will be particularly interested in the functors ${}^{\mathfrak{p}}f_*, {}^{\mathfrak{p}}f_!, {}^{\mathfrak{p}}f^*, {}^{\mathfrak{p}}f^!$ and ${}^{\mathfrak{p}}\otimes, {}^{\mathfrak{p}}\mathcal{H}om$, which we call the *perverse direct image, perverse proper direct image*, etc.

12.1.4 Intersection cohomology complexes

Definition 12.10 (Axiomatic definition)**.** Let Y be a complex algebraic variety of dimension n. Let L be a local system on a smooth Zariski dense open subvariety U of Y. The *intersection cohomology complex* $IC(Y,L)$ is defined to be an object in $D_c^b(Y)$ satisfying the following conditions:

- $\mathcal{H}^i(IC(Y,L)) = 0 \quad \text{if} \quad i < -n,$
- $\mathcal{H}^{-n}(IC(Y,L))|_U = L,$
- $\dim\ \mathrm{supp}\left(\mathcal{H}^i(IC(Y,L))\right) < -i \quad \text{if} \quad i > -n,$
- $\dim\ \mathrm{supp}\left(\mathcal{H}^i(IC(Y,L)^\vee)\right) < -i \quad \text{if} \quad i > -n.$

Proposition 12.11. *(i) For any local system L on U there exists a unique, up to isomorphism, object in $D_c^b(Y)$ satisfying the conditions in Definition 12.10.*
(ii) The complex $IC(Y,L)$ does not depend, up to canonical isomorphism, on the choice of U. That is, if U and U' are smooth Zariski dense open subvarieties of Y, L is a local system on U, L' is a local system on U' and $L|_{U\cap U'} \cong L'|_{U\cap U'}$ then the associated intersection cohomology complexes are canonically isomorphic.
(iii) If Y is smooth and connected, and $L = \mathbb{C}_U$ is the constant sheaf on a Zariski dense open subset of Y, then $IC(Y,\mathbb{C}_U)$ is isomorphic to $\mathbb{C}_Y[n]$, the shift of the constant sheaf on Y by the dimension of Y.
(iv) $IC(Y,L)$ is an object in $Perv(Y)$.

Proof. One can prove (i) and (ii) using the explicit constructions of $IC(Y,L)$ given below. For (iii) one can check directly that $\mathbb{C}_Y[n]$ satisfies the conditions of Definition 12.10. Part (iv) is obvious - the conditions in Definition 12.10 are strictly stronger then the support and cosupport conditions in Definition 12.4. □

Definition 12.12 (Explicit definition 1)**.** Let Y be a complex algebraic variety of dimension n and let $j: U \hookrightarrow Y$ be the inclusion of a smooth Zariski dense open subvariety U of Y. Let $L \in \mathrm{Loc}(U)$ be a local system on U. We can regard it as an object in $D_c^b(U)$, i.e., as a constructible complex concentrated in one degree. We define $IC(Y,L)$ to be the intermediate extension of the complex $L[n]$:

$$IC(Y,L) := j_{!*}(L[n]).$$

Definition 12.13 (Explicit definition 2)**.** Choose a Whitney stratification $Y = \bigsqcup_{\lambda\in\Lambda} Y_\lambda$ of Y such that U is the unique open stratum in this stratification. Set $Y_k = \bigsqcup_{\dim Y_\lambda \leq k} Y_\lambda$ for each $k \in \mathbb{Z}$. We have a filtration

$$\varnothing \subset Y_0 \subset ... \subset Y_{n-1} \subset Y_n = Y$$

of Y by closed subvarieties. Set $U_k = Y \backslash Y_{k-1} = \bigsqcup_{\dim Y_\alpha \geq k} Y_\alpha$. We have the following sequence

$$U = U_n \xrightarrow{j_n} U_{n-1} \xrightarrow{j_{n-1}} ... \xrightarrow{j_2} U_1 \xrightarrow{j_1} U_0 = Y$$

of inclusions of open subsets in Y. The category $D_c^b(Y)$ admits, beside the middle perversity t-structure, also the *standard t-structure* $(D_c^{\leq 0}(Y), D_c^{\geq 0}(Y))$, where

$$D_c^{\leq 0}(Y) := \{Q \in D_c^b(Y) \mid \mathcal{H}^k(Q) = 0 \text{ for all } k > 0\},$$
$$D_c^{\geq 0}(Y) := \{Q \in D_c^b(Y) \mid \mathcal{H}^k(Q) = 0 \text{ for all } k < 0\}.$$

Let $\tau_{\leq k}, \tau_{\geq k}$ be the truncation functors associated to the standard t-structure on $D_c^b(Y)$. Finally, we define

$$IC(Y,L) := (\tau_{\leq -1} \circ j_{1*}) \circ ... \circ (\tau_{\leq -n} \circ j_{n*})\,(L[n]).$$

Proposition 12.14. *Definitions 12.12 and 12.13 are equivalent, i.e., there exists an isomorphism in $D_c^b(Y)$:*

$$j_{!*}(L[n]) \cong (\tau_{\leq -1} \circ j_{1*}) \circ ... \circ (\tau_{\leq -n} \circ j_{n*})\,(L[n]).$$

Proof. See [HTT08, Proposition 8.2.11]. □

Definition 12.15. We are particularly interested in the intersection cohomology complex $IC(X, \mathbb{C}_U)$, where X is an irreducible complex algebraic variety of dimension n and $\mathbb{C}_U$ is the constant sheaf on a smooth Zariski dense open subvariety of X. Since this complex is independent of the choice of U, we will also denote it by $IC(X, \mathbb{C}_X)$. We call $IC(X, \mathbb{C}_X)$ the *constant perverse sheaf* on X. Note that if X is smooth then $IC(X, \mathbb{C}_X) = \mathbb{C}_X[n]$ by Proposition 12.11.

Lemma 12.16. *Suppose that Y is an irreducible complex algebraic variety of dimension n and let L be an irreducible local system on a Zariski dense open subset U. Then $IC(Y, L)$ is a simple object in $Perv(Y)$.*

Proof. Let $j : U \hookrightarrow Y$ be the inclusion. The locally constant perverse sheaf $L[n]$ on U is a simple object in $Perv(U)$ by [HTT08, Lemma 8.2.24]. But the intermediate extension $j_{!*}(L[n])$ of a simple object is again a simple object in $Perv(Y)$, by [HTT08, Corollary 8.2.10]. □

Proposition 12.17. *Let $i : Y \hookrightarrow X$ be an inclusion of a closed subvariety Y. Then the functor $i_* = i_!$ is t-exact with respect to the middle perversity t-structure and induces an exact functor*

$$^{\mathfrak{p}}i_* = {}^{\mathfrak{p}}i_! : Perv(Y) \to Perv(X).$$

Let $Perv_Y(X)$ denote the full subcategory of $Perv(X)$ whose objects are perverse sheaves on X whose support is contained in Y. Then the functor $^{\mathfrak{p}}i_*$ induces an equivalence of categories between $Perv(Y)$ and $Perv_Y(X)$. The quasi-inverse of $^{\mathfrak{p}}i_*$ is $^{\mathfrak{p}}i^*$.

Proof. See [HTT08, Corollary 8.1.44]. □

In light of Proposition 12.17, we can naturally regard a perverse sheaf on a closed subvariety Y of X as a perverse sheaf on X. To simplify notation, if $Q \in Perv(Y)$ we will also denote $^{\mathfrak{p}}i_*Q \in Perv(X)$ by Q. Informally, we can think of $^{\mathfrak{p}}i_*Q$ as an extension by zero of the perverse sheaf Q to X.

Theorem 12.18. *Let X be a complex algebraic variety. The simple objects in $Perv(X)$ are precisely the intersection cohomology complexes $IC(Y, L)$, where Y is an irreducible closed subvariety of X and L is an irreducible local system on a smooth Zariski dense open subvariety U of Y.*

Proof. This is [BBD82, Theorem 4.3.1]. The theorem is stated in the context of l-adic sheaves, but it is explained in Section 6 of [BBD82] how to deduce corresponding results for the complex case. □

The following proposition summarizes the main properties of the category $Perv(X)$.

Proposition 12.19. *Let X be a complex algebraic variety. The category $Perv(X)$ is noetherian and artinian. In particular, every object is of finite length, i.e., it admits a composition series.*

Proof. See [BBD82, Theorem 4.3.1]. □

Definition 12.20. Let $Q \in Perv(X)$. Let

$$Q = Q_1 \supset Q_2 \supset ... \supset Q_n = 0$$

be a composition series of Q. We call the simple subquotients Q_i/Q_{i+1} the *constituents* of the perverse sheaf Q. This definition is obviously independent of the choice of composition series.

By Theorem 12.18 every constituent of Q is of the form $IC(Y, L)$, for some irreducible closed subvariety Y of X and a simple local system on a smooth Zariski dense open subvariety U of Y.

12.1.5 Semi-simple complexes of geometric origin

Definition 12.21. Let X be a complex algebraic variety. Let $Q \in Perv(X)$ be a simple perverse sheaf on X. We say that Q is *of geometric origin* if it belong to the smallest set which

- contains the constant sheaf on a point

and is stable under the following operations:

- for every morphism f of varieties, take the constituents of ${}^{\mathfrak{p}}\mathcal{H}^k(f^*)(-)$, ${}^{\mathfrak{p}}\mathcal{H}^k(f^!)(-)$, ${}^{\mathfrak{p}}\mathcal{H}^k(f_*)(-)$, ${}^{\mathfrak{p}}\mathcal{H}(f_!)(-)$,
- take the constituents of ${}^{\mathfrak{p}}\mathcal{H}^k(\otimes)(-,-), {}^{\mathfrak{p}}\mathcal{H}^k(\mathcal{H}om)(-,-)$.

Now suppose that $Q \in Perv(X)$ is an arbitrary perverse sheaf. We say that Q is *semi-simple of geometric origin* if it is a finite direct sum of simple perverse sheaves of geometric origin. More generally, if $K \in D^b_c(X)$ is a constructible complex on X, we say that K is semi-simple of geometric origin if there is an isomorphism $K \cong \bigoplus_{i\in\mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i]$ in $D^b_c(X)$ and each perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^i(K)$ is semi-simple of geometric origin.

Lemma 12.22. *Let Y be an n-dimensional irreducible subvariety of X and suppose that L is an irreducible local system on a smooth Zariski dense open subvariety U of Y with finite monodromy. Then $IC(Y, L)$ is a simple perverse sheaf of geometric origin.*

Proof. Since L has finite monodromy, there exists a finite etale morphism $\pi : \widetilde{U} \to U$ trivializing L, i.e., $\pi^* L = (\mathbb{C}_{\widetilde{U}})^{\oplus n}$. Since π^* and π_* form an adjoint pair we have $\mathrm{Hom}_{\mathrm{Loc}(U)}\left(L, \pi_*(\mathbb{C}_{\widetilde{U}})^{\oplus n}\right) \cong \mathrm{Hom}_{\mathrm{Loc}(\widetilde{U})}\left(\pi^*L, (\mathbb{C}_{\widetilde{U}})^{\oplus n}\right) = \mathrm{Hom}_{\mathrm{Loc}(\widetilde{U})}\left((\mathbb{C}_{\widetilde{U}})^{\oplus n}, (\mathbb{C}_{\widetilde{U}})^{\oplus n}\right) \neq \{0\}$. Because L is simple, it is a subobject of $\pi_*(\mathbb{C}_{\widetilde{U}})^{\oplus n}$ (by an analogue of Schur's lemma). But we have $\pi_*(\mathbb{C}_{\widetilde{U}})^{\oplus n} = \left(\pi_*\mathbb{C}_{\widetilde{U}}\right)^{\oplus n}$ so, again by the simplicity of L, it follows that L is in fact a subobject of $\pi_*\mathbb{C}_{\widetilde{U}}$.

Let $p : \widetilde{U} \to \{pt\}$ be the projection. By Theorem 12.18 we know that $\mathbb{C}_{\widetilde{U}}[n]$ is a simple perverse sheaf on U. Moreover, it is of geometric origin because ${}^{\mathfrak{p}}\mathcal{H}^n(p^*)(\mathbb{C}_{\{pt\}}) = {}^{\mathfrak{p}}\mathcal{H}^0(p^*)(\mathbb{C}_{\{pt\}}[n]) = p^*\mathbb{C}_{\{pt\}}[n] = \mathbb{C}_{\widetilde{U}}[n]$ (since the map p is semi-small). Since $\mathbb{C}_{\widetilde{U}}$ is a locally free sheaf and π is finite and flat, $\pi_*\mathbb{C}_{\widetilde{U}}$ is also a locally free sheaf. Hence $\pi_*\mathbb{C}_{\widetilde{U}}[n]$ is a perverse sheaf on U. Therefore $\pi_*\mathbb{C}_{\widetilde{U}}[n] = {}^{\mathfrak{p}}\mathcal{H}^0(\pi_*)(\mathbb{C}_{\widetilde{U}}[n])$. But we have shown that $L[n]$ is a subobject of $\pi_*\mathbb{C}_{\widetilde{U}}[n]$. Since $L[n]$ is simple, it is in fact a constituent of $\pi_*\mathbb{C}_{\widetilde{U}}[n] = {}^{\mathfrak{p}}\mathcal{H}^0(\pi_*)(\mathbb{C}_{\widetilde{U}}[n])$, and so it is simple of geometric origin.

Let $j : U \hookrightarrow Y$ be the inclusion. The intersection cohomology complex $IC(Y, L) := j_{!*}(L[n])$ is defined as the image of the natural map ${}^{\mathfrak{p}}\mathcal{H}^0(j_!)(L[n]) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_*)(L[n])$. In particular, $IC(Y, L)$ is a subobject of ${}^{\mathfrak{p}}\mathcal{H}^0(j_*)(L[n])$. By Theorem 12.18 we know that $IC(Y, L)$ is simple, so it is a constituent of ${}^{\mathfrak{p}}\mathcal{H}^0(j_*)(L[n])$. But since $L[n]$ is simple of geometric origin, the constituents of ${}^{\mathfrak{p}}\mathcal{H}^0(j_*)(L[n])$ are also simple of geometric origin (by definition). Hence $IC(Y, L)$ is simple of geometric origin. □

Corollary 12.23. *Suppose that X is an irreducible complex algebraic variety. Then $IC(X, \mathbb{C}_X)$ is a simple perverse sheaf of geometric origin.*

Proof. The constant sheaf has trivial monodromy, so the corollary follows immediately from Lemma 12.22. □

12.1.6 The decomposition theorem

Now we can state the original decomposition theorem ([BBD82, Theorem 6.2.5]).

Theorem 12.24. *Let $f : Z \to X$ be a proper morphism of complex algebraic varieties. Suppose that $K \in D^b_c(Z)$ is semisimple of geometric origin. Then $f_*K \in D^b_c(X)$ is also semisimple of geometric origin.*

Proof. There are three known approaches to proving this theorem. The original proof of Beilinson, Bernstein, Deligne and Gabber ([BBD82]) uses etale cohomology of l-adic sheaves and arithemtic properties of varieties defined over finite fields. There is also a proof of Saito ([Sai90]) which uses mixed Hodge modules and a proof by Cataldo and Migliorini ([CM05]) based on classical Hodge theory. We refer the reader to these publications for details. $\square$

Corollary 12.25. *Let $f : Z \to X$ be a proper morphism of complex algebraic varieties. Suppose that Z is irreducible. Then there is an isomorphism*

$$f_*\left(IC(Z,\mathbb{C}_Z)\right) \cong \bigoplus_{i\in\mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i\left(f_*\left(IC(Z,\mathbb{C}_Z)\right)\right)[-i] \tag{86}$$

in $D^b_c(X)$. Moreover, each perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^i\left(f_*\left(IC(Z,\mathbb{C}_Z)\right)\right)$ is semi-simple of geometric origin, i.e., for each $i \in \mathbb{Z}$ there exists an isomorphism

$${}^{\mathfrak{p}}\mathcal{H}^i\left(f_*\left(IC(Z,\mathbb{C}_Z)\right)\right) \cong \bigoplus_{(Y,L)} V_{(Y,L)}(i) \otimes IC(Y,L) \tag{87}$$

in $Perv(X)$, where

- *(A1) Y ranges over irreducible closed subvarieties Y of X and L ranges over irreducible local systems on a smooth Zariski dense open subvariety U of Y,*
- *(A2) each $IC(Y,L)$ is of geometric origin,*
- *(A3) each $V_{(Y,L)}(i)$ is a finite-dimensional complex vector space which is nonzero for only finitely many pairs (Y,L).*

The vector space $V_{(Y,L)}(i)$ encodes the multiplicity with which $IC(Y,L)$ occurs in the decomposition (87)*, i.e., we have $V_{(Y,L)}(i) \otimes IC(Y,L) \cong IC(Y,L)^{\oplus \dim V_{(Y,L)}(i)}$. Now set $V_{(Y,L)} = \bigoplus_{i\in\mathbb{Z}} V_{(Y,L)}(i)[-i]$. It is a $\mathbb{Z}$-graded vector space. We can combine the two decompositions* (86) *and* (87) *to obtain an isomorphism*

$$f_*\left(IC(Z,\mathbb{C}_Z)\right) \cong \bigoplus_{(Y,L)} V_{(Y,L)} \otimes IC(Y,L) \tag{88}$$

in $D^b_c(X)$, where Y and L satisfy (A1) and (A2) and each $V_{(Y,L)}$ is a finite-dimension $\mathbb{Z}$-graded complex vector space which is nonzero for only finitely many pairs (Y,L). The vector space $V_{(Y,L)}$ encodes the graded multiplicity with which $IC(Y,L)$ occurs in the decomposition (88)*, i.e., $V_{(Y,L)} \otimes IC(Y,L) \cong \bigoplus_{i\in\mathbb{Z}}(IC(Y,L)[-i])^{\oplus \dim V_{(Y,L)}(i)}$.*

Proof. This is immediate from Theorem 12.24 and the description of simple perverse sheaves in Theorem 12.18. $\square$

Lemma 12.26. *Let $f : Z \to X$ be a surjective proper morphism of complex algebraic varieties which is also a locally trivial topological fibration. Let Z be smooth of dimension n. Then each cohomology sheaf $\mathcal{H}^i\left(f_*(IC(Z,\mathbb{C}_Z))\right)$ is locally trivial and its stalk at $x \in X$ is canonically isomorphic to $H^{i-n}(f^{-1}(x))$.*

Proof. Since Z is smooth of dimension n we have $IC(Z,\mathbb{C}_Z) = \mathbb{C}_Z[n]$, by Proposition 12.11. Recall that there is a canonical isomorphism $(f_*\mathbb{C}_Z)_x \cong \Gamma(f^{-1}(x), \mathbb{C}_Z|_{f^{-1}(x)})$. Hence

$$\begin{aligned}
\mathcal{H}^i_x\left(f_*(IC(Z,\mathbb{C}_Z))\right) = \mathcal{H}^i_x\left(f_*\mathbb{C}_Z[n]\right) &= \mathcal{H}^{i-n}_x\left(f_*\mathbb{C}_Z\right) \\
&= (R^{i-n}f_*\mathbb{C}_Z)_x \\
&\cong R^{i-n}\Gamma(f^{-1}(x), \mathbb{C}_Z|_{f^{-1}(x)}) \\
&= H^{i-n}(f^{-1}(x), \mathbb{C}_{f^{-1}(x)}) = H^{i-n}(f^{-1}(x)).
\end{aligned}$$

Since f is a locally trivial topological fibration, in a sufficiently small connected open neighbourhood U of x we have

$$\mathcal{H}^i\left(f_*(IC(Z,\mathbb{C}_Z))\right)|_U \cong (\mathbb{C}_U)^{\dim H^{i-n}(f^{-1}(x))}$$

so $\mathcal{H}^i\left(f_*(IC(Z,\mathbb{C}_Z))\right)$ is locally trivial. □

Corollary 12.27. *Let $f : Z \to X$ be a surjective proper morphism of complex algebraic varieties. Let Z be smooth. Suppose that there exists an algebraic stratification $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ of X such that $f : f^{-1}(X_\lambda) \to X_\lambda$ is a locally trivial topological fibration. Then $f_*(IC(Z,\mathbb{C}_Z))$ is constructible with respect to the stratification $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$. Hence there is an isomorphism*

$$f_*\left(IC(Z,\mathbb{C}_Z)\right) \cong \bigoplus_{(\overline{X_\lambda},L)} V_{(\overline{X_\lambda},L)} \otimes IC(\overline{X_\lambda},L) \tag{89}$$

in $D^b_c(X)$, where λ ranges over Λ and L ranges over irreducible local systems on X_λ.

Proof. The constructibility of $f_*(IC(Z,\mathbb{C}_Z))$ with respect to the stratification $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ follows by applying Lemma 12.26 to the restriction of each cohomology sheaf $\mathcal{H}^i\left(f_*(IC(Z,\mathbb{C}_Z))\right)$ to each stratum X_λ. If Z is connected, by Corollary 12.25 we have a decomposition

$$f_*\left(IC(Z,\mathbb{C}_Z)\right) \cong \bigoplus_{(Y,L)} V_{(Y,L)} \otimes IC(Y,L). \tag{90}$$

Since the complex on the LHS of (90) is constructible with respect to the stratification $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$, each intersection cohomology complex on the RHS of (90) must also be constructible with respect to this stratification. But only intersection cohomology complexes of the form $IC(\overline{X_\lambda},L)$ are constructible with respect to the aforementioned stratification.

Now suppose that Z is not connected and $Z = \bigsqcup Z_\phi$ is a decomposition of Z into connected components. Then $f_*(IC(Z,\mathbb{C}_Z)) = f_*(\bigoplus IC(Z_\phi,\mathbb{C}_{Z_\phi})) = \bigoplus f_*(IC(Z_\phi,\mathbb{C}_{Z_\phi}))$ so we can apply the preceding argument to each $f_*(IC(Z_\phi,\mathbb{C}_{Z_\phi}))$. □

The importance of the corollary lies in the fact that it puts a restriction on the intersection cohomology complexes which can occur in the decomposition (88). They are precisely the intersection cohomology complexes associated to closures of the strata X_λ.

12.1.7 Equivariant sheaves

Our goal now is to derive an even stronger statement about the decomposition (88) which also imposes some restrictions on the local systems L. We will show that under appropriate hypotheses the only intersection cohomology complexes which can occur in the decomposition (88) are those associated to the constant sheaf on each stratum X_λ. To do this we will need to exploit equivariant techniques. In the next few sections we briefly discuss the equivariant derived category of Bernstein and Lunts, state the equivariant decomposition theorem, and finally deduce the version of the decomposition theorem which we will apply to study the finite-dimensional representation theory of quiver Hecke algebras.

Let G be a reductive complex linear algebraic group. All varieties we consider in this section are complex algebraic varieties endowed with a left algebraic action of G.

Definition 12.28. We call a complex algebraic variety endowed with an algebraic action of G a G-variety. We call a G-variety X *free* if G acts freely on X and the quotient map $X \to G\backslash X$ is a locally trivial fibration with fibre G.

Definition 12.29. Let X be a G-variety. We have natural maps

$$d_{i,n} : G^n \times X \to G^{n-1} \times X, \quad s_0 : X \to G \times X$$

defined by

$$\begin{aligned} d_{0,n}(g_1, ..., g_n, x) &= (g_2, ..., g_n, g_1^{-1}x), \\ d_{i,n}(g_1, ..., g_n, x) &= (g_1, ..., g_i g_{i+1}, ..., g_n, x), \quad 1 \leq i \leq n-1, \\ d_{n,n}(g_1, ..., g_n, x) &= (g_1, ..., g_{n-1}, x), \\ s_0(x) &= (e, x). \end{aligned}$$

A *G-equivariant sheaf* on X is a pair (F, θ), where $F \in Sh(X)$ is a sheaf on X and θ is an isomorphism

$$\theta : d_{1,1}^* F \cong d_{0,1}^* F$$

in $Sh(G \times X)$ satisfying the following cocycle condition

$$d_{0,2}^* \theta \circ d_{2,2}^* \theta = d_{1,2}^* \theta, \quad s_0^* \theta = id_F,$$

where $\circ$ stands for composition of morphisms. A morphism of G-equivariant sheaves is a morphism of sheaves which commutes with θ. Let $Sh_G(X)$ denote the category of G-equivariant sheaves on X. We have a forgetful functor

$$For : Sh_G(X) \to Sh(X), \quad (F, \theta) \mapsto F.$$

We call a G-equivariant sheaf (F, θ) on X a *G-equivariant local system* if F is a local system on X. Let $\mathrm{Loc}_G(X)$ denote the full subcategory of $Sh_G(X)$ whose objects are G-equivariant local systems.

Suppose that G acts freely on X. Consider the quotient map $q : X \to G\backslash X$. Let $F \in Sh(G\backslash X)$. Then $q^*(F)$ is naturally a G-equivariant sheaf. This defines a functor

$$q^* : Sh(G\backslash X) \to Sh_G(X).$$

Now let $H \in Sh_G(X)$. Then $q_* H \in Sh(G\backslash X)$ has a natural action of G. Let $q_*^G H = (q_* H)^G$ denote the subsheaf of G-invariants of $q_* H$. This defines a functor

$$q_*^G : Sh_G(X) \to Sh(G\backslash X).$$

Lemma 12.30. *Let X be a free G-variety. Then the functor $q^* : Sh(G\backslash X) \to Sh_G(X)$ is an equivalence of categories with quasi-inverse $q_*^G : Sh_G(X) \to Sh(G\backslash X)$*

Proof. See [BL94, Lemma 0.3]. □

Remark 12.31. If G acts freely on X, one can identify $Sh_G(X)$ with $Sh(G\backslash X)$. It is then possible to define the equivariant derived category as the derived category of the abelian category $Sh_G(X)$, i.e., $D_G(X) := D(Sh_G(X)) = D(Sh(G\backslash X))$. In general, this approach is too naive and does not yield the right equivariant derived category.

Now let H be a closed subgroup of G and let $X = G/H$ be a homogeneous space. The fibration $H \hookrightarrow G \twoheadrightarrow G/H$ gives rise to a long exact sequence of homotopy groups

$$... \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1.$$

Suppose that G is connected. Then we have a surjective map $\pi_1(G/H) \twoheadrightarrow \pi_0(H) = H/H^o$. Note that the identity component H^o is a normal subgroup of H so H/H^o is actually a group.

Lemma 12.32. *Let H be a closed subgroup of a connected reductive complex linear algebraic group G and let $X = G/H$ be a homogeneous space. Let us choose a base-point $x \in G/H$. Suppose that $L \in \mathrm{Loc}(G/H)$ is a local system on G/H. The local system L is G-equivariant if and only if the monodromy representation of $\pi_1(G/H, x)$ on the stalk L_x is the pullback of a finite-dimensional representation of H/H^o by means of the map $\pi_1(G/H, x) \twoheadrightarrow H/H^0$. In particular, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{simple } G\textit{-equivariant} \\ \textit{local systems on } G/H \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{simple representations of} \\ \textit{the component group } H/H^o \end{array} \right\}$$

Proof. See [Jan04, Proposition 12.10]. □

12.1.8 The equivariant derived category

Definition 12.33. (i) Suppose that X is a G-variety and P a free G-variety. We call a G-equivariant morphism $p : P \to X$ of varieties a *resolution* of X.
(ii) Let $n \geq 0$. We say that a morphism $f : X \to Y$ of varieties is n-acyclic if:

- (a) for any sheaf $B \in Sh(Y)$ the adjunction morphism $B \to R^0 f_* f^*(B)$ is an isomorphism and $R^i f_* f^*(B) = 0$ for $1 \leq i \leq n$.
- (b) for any base change $\widetilde{Y} \to Y$ the induced map $f : \widetilde{X} = X \times_Y \widetilde{Y} \to \widetilde{Y}$ satisfies property (a).

Definition 12.34. (i) Let $p : P \to X$ be a resolution of a G-variety X. Consider the following diagram of G-varieties

$$X \overset{p}{\leftarrow} P \overset{q}{\rightarrow} \overline{P} = G\backslash P.$$

We define the category $D^b_G(X, P)$ in the following way:

- an object F of $D^b_G(X, P)$ is a triple $(F_X, \overline{F}, \beta)$ where $F_X \in D^b(X), \overline{F} \in D^b(\overline{P})$ and $\beta : p^* F_X \cong q^* \overline{F}$ is an isomorphism in $D^b(P)$,
- a morphism $\alpha : F \to H$ in $D^b_G(X, P)$ is a pair $(\alpha_X, \overline{\alpha})$, where $\alpha_X : F_X \to H_X$ and $\overline{\alpha} : \overline{F} \to \overline{H}$ satisfy $\beta \circ p^*(\alpha_X) = q^*(\overline{\alpha}) \circ \beta$.

Let

$$For_P : D^b_G(X) \to D^b(X), \quad F \to F_X$$

denote the forgetful functor.

(ii) Let $I = [a, b] \subset \mathbb{Z}$ and let $p : P \to X$ be a resolution. Let $D^I(X)$ be the full subcategory $D^{\geq a}(X) \cap D^{\leq b}(X)$ of $D^b(X)$, i.e., the full subcategory of $D^b(X)$ consisting of objects Q satisfying $\mathcal{H}^i(Q) = 0$ for $i > b$ and $i < a$. We define $D^I_G(X, P)$ to be the full subcategory of $D^b_G(X, P)$ whose objects F satisfy $For_P(F) = F_X \in D^I(X)$.

(iii) Let $I = [a, b] \subset \mathbb{Z}$ and let $p : P \to X$ be some n-acyclic resolution of X with $n \geq |I|$. We define the category $D^I_G(X)$ to be $D^I_G(X, P)$. One can show that this definition is independent of the choice of the resolution P, up to a canonical equivalence of categories. If $J \subset I$, we have an obvious fully faithful functor $D^J_G(X) \to D^I_G(X)$ defined uniquely up to a canonical isomorphism. We finally define the *equivariant derived category* $D^b_G(X)$ to be the limit

$$D^b_G(X) = \lim_I D^I_G(X).$$

Let $For : D^b_G(X) \to D^b(X)$ be the associated forgetful functor.

Remark 12.35. We could also describe the category $D^b_G(X)$ in the following way. We regard the space EG as a free G-ind-variety. The projection $EG \times X \to X$ is an ∞-acyclic resolution of X. Therefore we have $D^b_G(X) = D^b_G(X, EG \times X)$. Recall that we also have the projection $EG \times^G X \to G\backslash BG$ with fibre X. An object of $D^b_G(X)$ is then essentially an object in $D^b(EG \times^G X)$

whose restrictions to all fibres are isomorphic. For example, the category $D^b_G(\{pt\})$ is equivalent to the full subcategory of $D^b(BG)$ consisting of complexes with locally constant cohomology sheaves. If G is a connected Lie group, then this subcategory consists of complexes with constant cohomology sheaves (see [BL94, Proposition 2.7.2] for a proof).

If $f : X \to Y$ is a G-equivariant map one can define functors $f^*, f^!, f_*, f_!, \otimes, \mathcal{H}om, ^\vee$ between/in the categories $D^b_G(X)$ and $D^b_G(Y)$ (see [BL94, Section 3]).

Definition 12.36. We define the *constructible equivariant derived category* $D^b_{G,c}(X)$ in the same way as in Definition 12.34, replacing $D^b(X), D^b(\overline{P})$ and $D^I(X)$ everywhere by $D^b_c(X), D^b_c(\overline{P}), D^I_c(X)$, respectively.

Definition 12.37. Let $p : EG \times X \to X$ and $q : EG \times X \to EG \times^G X$ be the canonicaal projections. There is a natural functor

$$\iota : D^b_c(Sh_G(X)) \to D^b_{G,c}(X), \quad (F, \theta) \mapsto (F, q^G_* p^* F, \beta),$$

where $\beta : p^*F \to q^* q^G_* p^* F = p^*F$ is the identity.

12.1.9 Equivariant perverse sheaves

We are now ready to define the category of equivariant perverse sheaves on X.

Definition 12.38. We define $Perv_G(X)$ to be the full subcategory of $D^b_{G,c}(X)$ consisting of those $F \in D^b_{G,c}(X)$ which satisfy $For(F) \in Perv(X)$.

The category $Perv_G(X)$ has the same basic properties as the non-equivariant category $Perv(X)$. In particular, it is the heart of a perverse t-structure, it is abelian and every object in it has finite length.

Definition 12.39. Let Y be a closed G-stable irreducible subvariety of X and let $L \in \mathrm{Loc}_G(U)$ be a G-equivariant local system on a G-stable smooth Zariski dense open subvariety U of Y. Let $j : U \hookrightarrow X$ be the inclusion. We define the *equivariant intersection cohomology complex* $IC_G(Y, L) \in Perv_G(X)$ to be the intermediate extension $j_{!*}(\iota(L)[n])$ (where $j_{!*}$ is a functor $j_{!*} : Perv_G(U) \to Perv_G(X)$ between equivariant categories).

We have an analogous result to the non-equivariant case (see [BL94, Section 5.2]).

Proposition 12.40. *The simple objects in $Perv_G(X)$ are precisely the equivariant cohomology complexes $IC_G(Y, L)$, where Y is a G-stable irreducible subvariety of X and L is an irreducible G-equivariant local system on a G-stable smooth Zariski dense open subvariety U of Y.*

We also have an equivariant version of the decomposition theorem. We call an object $Q \in D^b_{G,c}(X)$ semi-simple if it is isomorphic to a direct sum of shifts of simple G-equivariant perverse sheaves on X, i.e., $Q = \bigoplus IC_G(Y, L)[n_{Y,L}]$, or equivalently, if $For(Q)$ is semi-simple. We say that Q is semi-simple of geometric origin if $For(Q) \in D^b_c(X)$ is semi-simple of geometric origin.

Theorem 12.41. *Let $f : Z \to X$ be a proper G-equivariant morphism of complex algebraic varieties. Let $F \in D^b_{G,c}(Z)$ be semi-simple of geometric origin. Then $f_*F \in D^b_{G,c}(X)$ is also semi-simple of geometric origin.*

Proof. See [BL94, Theorem 5.3]. □

Corollary 12.42. *Suppose that G is connected. Let $f : Z \to X$ be a surjective G-equivariant proper morphism of complex algebraic varieties. Let Z be smooth. Let $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ be an algebraic stratification of X by finitely many G-orbits. Then there is an isomorphism*

$$f_*\left(IC(Z, \mathbb{C}_Z)\right) \cong \bigoplus_{(\overline{X_\lambda}, L)} V_{(\overline{X_\lambda}, L)} \otimes IC(\overline{X_\lambda}, L) \tag{91}$$

in $D^b_c(X)$, where λ ranges over Λ and L ranges over G-equivariant irreducible local systems on $\overline{X_\lambda}$.

Proof. Let $\lambda \in \Lambda$ and $x_\lambda \in X_\lambda$. Then $f : f^{-1}(X_\lambda) \to X_\lambda = G/\operatorname{Stab}(x_\lambda)$ is a locally trivial topological fibration with fibre $\operatorname{Stab}(x_\lambda)$. By Corollary 12.27 we have an isomorphism

$$f_*\left(IC(Z, \mathbb{C}_Z)\right) \cong \bigoplus_{(\overline{X_\lambda}, L)} V_{(\overline{X_\lambda}, L)} \otimes IC(\overline{X_\lambda}, L) \tag{92}$$

in $D^b_c(X)$, where λ ranges over Λ and L ranges over irreducible local systems on X_λ. On the other hand, we can apply Theorem 12.41 to the pushforward $f_*\left(IC_G(Z, \mathbb{C}_Z)\right)$ of the G-equivariant constant perverse sheaf $IC_G(Z, \mathbb{C}_Z)$ to deduce that $f_*\left(IC_G(Z, \mathbb{C}_Z)\right)$ is semi-simple in $D^b_{G,c}(X)$, i.e., there is an isomorphism

$$f_*\left(IC_G(Z, \mathbb{C}_Z)\right) \cong \bigoplus IC_G(Y, L)[i], \tag{93}$$

for some irreducible closed subvarieties Y and G-equivariant irreducible local systems L on smooth Zariski dense open subvarieties U of Y. We now apply the forgetful functor to both sides of (93) to get an isomorphism

$$f_*\left(IC(Z, \mathbb{C}_Z)\right) \cong \bigoplus IC(Y, L)[i], \tag{94}$$

in $D^b_c(X)$, where all the local systems L are G-equivariant. But the intersection cohomology complexes occurring in the decomposition (92) and their graded multiplicities are uniquely determined, so the two decompositions (92) and (94) must agree. In particular, every local system L in (92) must be G-equivariant. □

We can finally deduce the version of the decomposition theorem which we will use.

Corollary 12.43. *Suppose that G is connected and that for every $x \in X$ the isotropy group* $\operatorname{Stab}(x)$ *is connected. Let $f : Z \to X$ be a surjective G-equivariant proper morphism of complex algebraic varieties. Let Z be smooth. Let $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ be an algebraic stratification of X by finitely many G-orbits. Then there is an isomorphism*

$$f_*\left(IC(Z, \mathbb{C}_Z)\right) \cong \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes IC(\overline{X_\lambda}, \mathbb{C}_{X_\lambda}) \tag{95}$$

in $D^b_c(X)$, where V_λ describes the graded multiplicity with which $IC(\overline{X_\lambda}, \mathbb{C}_{X_\lambda})$ occurs in the decomposition (95)*, i.e. $V_\lambda = \bigoplus_{i\in\mathbb{Z}} V_\lambda(i)[-i]$ and*

$$V_\lambda \otimes IC(\overline{X_\lambda}, \mathbb{C}_{X_\lambda}) \cong \bigoplus_{i\in\mathbb{Z}} \left(IC(\overline{X_\lambda}, \mathbb{C}_{X_\lambda})[-i]\right)^{\oplus \dim V_\lambda(i)}.$$

Proof. Since every stabilizer $\operatorname{Stab}(x)$ is connected, Lemma 12.32 implies that there is only one G-equivariant local system on each orbit X_λ, the constant sheaf $\mathbb{C}_{X_\lambda}$. The corollary now follows directly from Corollary 12.42. □

12.2 Geometric extension algebras

We will now work in the following framework. Let G be a connected complex reductive linear algebraic group, M and N be complex G-varieties, let M be smooth (but not necessarily connected), and let $\mu : M \to N$ be a proper G-equivariant morphism. Moreover, assume that N has finitely many G-orbits. Later we will make the additional assumption that for each $x \in N$ the stabilizer group $\operatorname{Stab}(x)$ is connected. Set $Z = M \times_N M$. Recall that the G-equivariant Borel-Moore homology $H^G_*(Z; \mathbb{C})$ of Z with complex coefficients, endowed with the convolution product, has the structure of an associative algebra. We are now going to identify this algebra with a certain geometric extension algebra, which we define below.

Let Let $A_1, A_2, A_3 \in D^b(N)$. The composition of morphisms in $D^b(N)$ yields a bilinear product

$$\mathrm{Hom}(A_1, A_2[p]) \times \mathrm{Hom}(A_2[p], A_2[p+q]) \to \mathrm{Hom}(A_1, A_3[p+q]).$$

This composition can also be expressed as a bilinear product of Ext-groups, called the *Yoneda product*:

$$\mathrm{Ext}^p(A_1, A_2) \otimes \mathrm{Ext}^q(A_2, A_3) \to \mathrm{Ext}^{p+q}(A_1, A_3).$$

Definition 12.44. Let $A \in D^b(N)$. We call $\mathrm{Ext}^*(A, A)$, endowed with the Yoneda product, the *Yoneda algebra* or *geometric extension algebra* associated to A. △

Analogously, we have a G-equivariant Yoneda product

$$\mathrm{Ext}^p_G(A_1, A_2) \otimes \mathrm{Ext}^q_G(A_2, A_3) \to \mathrm{Ext}^{p+q}_G(A_1, A_3),$$

where $A_1, A_2, A_3 \in D^b_G(N)$.

Definition 12.45. Let $A \in D^b_G(N)$. We call $\mathrm{Ext}^*_G(A, A)$, endowed with the Yoneda product, the *G-equivariant Yoneda algebra* or *geometric extension algebra* associated to A. △

We are particularly interested in the Yoneda algebra $\mathrm{Ext}^*_G(\mu_* IC(M, \mathbb{C}_M), \mu_* IC(M, \mathbb{C}_M))$ associated to the direct image of the constant perverse sheaf $IC(M, \mathbb{C}_M)$ on M.

12.2.1 Geometric extension algebras and convolution algebras

Theorem 12.46. *There is an $H^*_G(\{pt\})$-algebra isomorphism*

$$H^G_*(Z; \mathbb{C}) \cong \mathrm{Ext}^*_G(\mu_* IC(M, \mathbb{C}_M), \mu_* IC(M, \mathbb{C}_M)), \tag{96}$$

where the LHS is endowed with the convolution product and the RHS is endowed with the Yoneda product. Similarly, there is a $\mathbb{C}$-algebra isomorphism

$$H_*(Z; \mathbb{C}) \cong \mathrm{Ext}^*(\mu_* IC(M, \mathbb{C}_M), \mu_* IC(M, \mathbb{C}_M)). \tag{97}$$

Proof. The proof is not difficult but rather long and technical. See [CG97, Proposition 8.6.35]. □

Remark 12.47. We will *never* consider the convolution algebras $H_*(Z; \mathbb{C}), H^G_*(Z; \mathbb{C})$ endowed with the homological grading. Instead we use the isomorphisms from Theorem 12.46 to import the gradings from the corresponding geometric extension algebras. From now on we will always consider $H_*(Z; \mathbb{C}), H^G_*(Z; \mathbb{C})$ as graded algebras endowed with this "geometric extension algebra" grading.

Let Λ be a set parametrizing the G-orbits in N, i.e., $N = \bigsqcup_{\lambda \in \Lambda} \mathbb{O}_\lambda$. From now on assume that for each $x \in N$ the stabilizer group $\mathrm{Stab}(x)$ is *connected.* In light of Corollary 12.43 we have an isomorphism

$$\mu_*(IC(M, \mathbb{C}_M)) \cong \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes IC(\overline{\mathbb{O}_\lambda}, \mathbb{C}_{\mathbb{O}_\lambda}) \tag{98}$$

in $D^b_c(N)$. To simplify notation let us set, for each $\lambda \in \Lambda$, $IC_\lambda = IC(\overline{\mathbb{O}_\lambda}, \mathbb{C}_{\mathbb{O}_\lambda})$.

We can use the isomorphism (98) together with Theorem 12.46 to obtain the following isomorphism of algebras. We will later use it to find all simple modules over $H^G_*(Z)$.

Lemma 12.48. *We have the following isomorphism of $\mathbb{C}$-algebras*

$$H_*(Z; \mathbb{C}) \cong \left(\bigoplus_{\lambda \in \Lambda} \mathrm{End}_{\mathbb{C}} V_\lambda \right) \bigoplus \left(\bigoplus_{k>0,\ \phi, \psi \in \Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\phi, V_\psi) \otimes \mathrm{Ext}^k(IC_\phi, IC_\psi) \right), \tag{99}$$

where the LHS is endowed with the convolution product and the RHS is endowed with the Yoneda product (note that the direct sums in the formula are direct sums of vector spaces, not algebras).

Proof.

$$\begin{aligned}
H_*(Z;\mathbb{C}) &\cong \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^k(\mu_* IC(M,\mathbb{C}_M), \mu_* IC(M,\mathbb{C}_M)) \\
&\cong \bigoplus_{k\in\mathbb{Z},\ \phi,\psi\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\phi, V_\psi)\otimes \mathrm{Ext}^k(IC_\phi, IC_\psi) \\
&= \bigoplus_{k\geq 0,\ \phi,\psi\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\phi, V_\psi)\otimes \mathrm{Ext}^k(IC_\phi, IC_\psi) \\
&= \left(\bigoplus_{\lambda\in\Lambda} \mathrm{End}_{\mathbb{C}} V_\lambda\right) \bigoplus \left(\bigoplus_{k>0,\ \phi,\psi\in\Phi} \mathrm{Hom}_{\mathbb{C}}(V_\phi, V_\psi)\otimes \mathrm{Ext}^k(IC_\phi, IC_\psi)\right).
\end{aligned}$$

In the first equality we used Theorem 12.46, in the second equality we used the isomorphism (98), in the third equality we used the fact that $\mathrm{Ext}^k(IC_\phi, IC_\psi) = 0$ for $k < 0$ and in the fourth equality we used the fact that $\dim \mathrm{Hom}(IC_\phi, IC_\psi) = \delta_{\phi,\psi}$. The last two facts follow from the fact that the intersection cohomology complexes are simple objects in the category $Perv(N)$. □

12.2.2 Classification of simple modules over a convolution algebra

We first classify simple modules over the non-equivariant convolution algebra $H_*(Z)$.

Theorem 12.49. *The non-zero members of the set $\{V_\lambda \mid \lambda\in\Lambda\}$ form a complete and irredundant set of representatives of isomorphism classes of simple modules over $H_*(Z)$.*

Proof. By Lemma 12.48 we have a $\mathbb{C}$-algebra isomorphism

$$H_*(Z;\mathbb{C}) \cong \left(\bigoplus_{\lambda\in\Lambda} \mathrm{End}_{\mathbb{C}} V_\lambda\right) \bigoplus \left(\bigoplus_{k>0,\ \phi,\psi\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\phi, V_\psi)\otimes \mathrm{Ext}^k(IC_\phi, IC_\psi)\right). \tag{100}$$

Observe that the second direct summand on the RHS of (100) is an ideal of the geometric extension algebra. Let us denote this ideal with $\mathcal{I}$. Using the isomorphism (100) we can also regard it as an ideal of $H_*(Z;\mathbb{C})$. Since the geometric extension algebra is finite-dimensional over $\mathbb{C}$ (because $H_*(Z;\mathbb{C})$ is), there exists an $m \geq 0$ such that $\mathrm{Ext}^n(IC_\phi, IC_\psi) = \{0\}$ for all $n \geq m,\ \phi,\psi\in\Lambda$. Now let us take any m elements $a_1, ..., a_m$ of the ideal $\mathcal{I}$ such that each $a_l \in \mathrm{Hom}_{\mathbb{C}}(V_{\phi_l}, V_{\phi_{l+1}}) \otimes \mathrm{Ext}^{k_l}(IC_{\phi_l}, IC_{\phi_{l+1}})$ for some $k_1, ..., k_m > 0,\ \phi_1, ..., \phi_{m+1} \in \Lambda$. Then

$$a_1 a_2 ... a_m \in \mathrm{Hom}_{\mathbb{C}}(V_{\phi_1}, V_{\phi_{m+1}}) \otimes \mathrm{Ext}^{\sum_{l=1}^m k_l}(IC_{\phi_1}, IC_{\phi_{m+1}}) = \{0\}$$

because $\sum_{l=1}^m k_l \geq m$. Therefore $\mathcal{I}^m = \{0\}$, i.e., the ideal $\mathcal{I}$ is nilpotent. Therefore $\mathcal{I}$ is contained in the Jacobson radical of $H_*(Z;\mathbb{C})$. Since the quotient of $H_*(Z;\mathbb{C})$ by this ideal is isomorphic to a direct sum of matrix algebras (by (100)) and hence semisimple, our ideal equals the Jacobson radical of $H_*(Z;\mathbb{C})$.

Since the Jacobson radical annihilates every simple module, the action of $H_*(Z;\mathbb{C})$ on every simple module factors over the maximal semi-simple quotient $H_*(Z)/\mathcal{I} \cong \bigoplus_{\lambda\in\Lambda} \mathrm{End}_{\mathbb{C}} V_\lambda$. But we know that all simple modules over $\bigoplus_{\phi\in\Lambda} \mathrm{End}_{\mathbb{C}} V_\phi$ are of the form V_ψ. For each $\psi\in\Lambda$ we now obtain an irreducible representation of $H_*(Z)$ on V_ψ by composing the obvious projections:

$$H_*(Z) \twoheadrightarrow \bigoplus_{\lambda\in\Lambda} \mathrm{End}_{\mathbb{C}} V_\lambda \twoheadrightarrow \mathrm{End}_{\mathbb{C}} V_\psi.$$

□

We are most interested in *graded* simple modules over the *equivariant* convolution algebra $H_*^G(Z)$. We are now going to show that under appropriate hypotheses $\{V_\lambda \mid \lambda \in \Lambda\}$ also form a complete and irredundant set of representatives of isomorphism classes of graded simple modules over $H_*^G(Z)$, up to grading shifts. We will use the following general lemma about graded algebras.

Lemma 12.50. *Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded algebra such that there exists a $j \in \mathbb{Z}$ with $A_k = \{0\}$ for all $k < j$. Let $B = \bigoplus_{i\in\mathbb{Z}} B_i \subset A$ be its centre and assume that $B_k = \{0\}$ for $k < 0$. Then $B_+ = \bigoplus_{i>0} B_i$ annihilates any left graded simple A-module M. Moreover, M is a graded simple module over A/B_+A.*

Proof. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a left graded simple A-module. We claim that the grading of M must be bounded from below, i.e., there exists a $d \in \mathbb{Z}$ such that $M_k = \{0\}$ for $k < d$. Let us pick an integer e such that $M_{\geq e} = \bigoplus_{i\geq e} M_i \neq \{0\}$. Then $A.M_{\geq e}$ is a graded submodule of M. Since M is simple we have $A.M_{\geq e} = M$. But we also have $A.M_{\geq e} \subseteq M_{\geq(e+j)}$ because the grading of A is bounded from below. This proves the claim.

Since B_+ lies in the centre of A, we have $AB_+.M = B_+A.M \subseteq B_+.M$, so $B_+.M$ is a graded submodule of M. Since M is simple, we deduce that $B_+.M = M$ or $\{0\}$. Suppose that $B_+.M = M$. Since the grading of M is bounded from below, there exists a uniquely determined minimal integer d such that $M_d \neq \{0\}$. But B_+ is positively graded, so $(B_+.M)_d = \{0\}$. This contradicts the equality $B_+.M = M$. Hence $B_+.M = \{0\}$.

It follows that the ideal $B_+A = B_+A$ annihilates M. Hence the action of A on M factors over the quotient algebra A/B_+A. □

Corollary 12.51. *Suppose that the variety Z is G-equivariantly formal, i.e., $H_*^G(Z) \cong H_*^G(\{pt\})\otimes_{\mathbb{C}} H_*(Z)$, and that the centre of $H_*^G(Z)$ is isomorphic as a graded algebra to $H_G^*(\{pt\})$. Then the non-zero members of the set $\{V_\lambda \mid \lambda \in \Lambda\}$ form a complete and irredundant set of representatives of isomorphism classes of* graded *simple modules over $H_*^G(Z)$, up to grading shifts. In particular, there are, up to isomorphism and grading shifts, finitely many graded simple modules over $H_*^G(Z)$ and every such module is finite-dimensional.*

Proof. Since the centre of $H_*^G(Z)$ is isomorphic as a graded algebra to $H_G^*(\{pt\})$ it is concentrated in non-negative degrees. Hence Lemma 12.50 implies that the action of $H_*^G(Z)$ on any graded simple module factors over the quotient algebra $H_*^G(Z)/H_G^+(\{pt\})H_*^G(Z)$. But, by equivariant formality, this quotient algebra is isomorphic to the non-equivariant convolution algebra $H_*(Z)$. By Theorem 12.49 we know that the non-zero members of the set $\{V_\lambda \mid \lambda \in \Lambda\}$ form a complete and irredundant set of representatives of isomorphism classes of simple modules over $H_*(Z)$. Moreover, it is clear by the definition of these modules that they are graded. This proves the first part of the corollary.

The fact that the set $\{V_\lambda \mid \lambda \in \Lambda\}$ is finite follows from the fact that, by assumption, there are only finitely many G-orbits in N. The fact that each V_λ is finite-dimensional follows from the fact that V_λ is a simple module over the finite-dimensional algebra $H_*(Z)$. □

Definition 12.52. From now on we will *change the notation* and denote, for each $\lambda \in \Lambda$, the module V_λ as L_λ. We do this in order to emphasise that L_λ is a graded module over $H_*^G(Z)$ rather than merely a graded vector space. By using the new notation we also comply with the quite widespread convention to use the letter L to denote simple modules over a geometric extension algebra. Please note that this new notation has nothing to do with our rather similar notation for local systems in Section 12.1.2. There will fortunately be no scope for confusion since we will not explicitly discuss local systems again.

12.2.3 Standard modules over a convolution algebra

We have classified the graded simple modules over $H_*^G(Z)$. This raises the following question: can we, in principle, also construct these modules? In fact, this is possible. The graded simple modules L_λ can be constructed as quotients of the so-called *standard modules* K_λ. The standard modules admit a beautiful geometric interpretation as convolution modules in the homology of the fibres of the map $\mu : M \to N$. In this section we will define standard modules and state their main properties.

Recall that $N = \bigsqcup_{\lambda \in \Lambda} \mathbb{O}_\lambda$ is the decomposition of N into finitely many G-orbits.

Definition 12.53. Let $\lambda \in \Lambda$ and $x \in \mathbb{O}_\lambda$. We define the *standard module* K_λ to be the vector space

$$K_\lambda := H_*(\mu^{-1}(x); \mathbb{C})$$

endowed with the convolution action of $H_*(Z)$ (see paragraph 6.2.1.3). We consider K_λ as a module over $H_*^G(Z)$ by composing the projection $H_*^G(Z) \twoheadrightarrow H_*(Z)$ with the convolution action:

$$H_*^G(Z) \twoheadrightarrow H_*(Z) \to \mathrm{End}_{\mathbb{C}}(K_\lambda).$$

In other words, we let the ideal $H_G^+(\{pt\})H_*^G(Z)$ act trivially on K_λ. Note that K_λ is a *finite-dimensional* module.

Proposition 12.54. *The definition of K_λ does not, up to isomorphism, depend on the choice of $x \in \mathbb{O}_\lambda$.*

Proof. See [CG97, Theorem 3.5.7(b)]. □

Proposition 12.55. *Suppose that M is connected and let m be the dimension of m as a variety. Let $x \in \mathbb{O}_\lambda$ and let $i_x : \{x\} \hookrightarrow N$ be the inclusion. Then there is a vector space isomorphism*

$$K_\lambda := H_*(\mu^{-1}(x)) \cong H^{*-m}(i_x^! \mu_* IC(M, \mathbb{C}_M)).$$

Moreover, it is possible to define an action of the geometric extension algebra $\mathrm{Ext}^*\left(\mu_* IC(M, \mathbb{C}_M), \mu_* IC(M, \mathbb{C}_M)\right)$ *on $H^{*-m}(i_x^! \mu_* IC(M, \mathbb{C}_M))$ such that the isomorphism* (97) *intertwines this action with the convolution action of $H_*(Z)$ on $H_*(\mu^{-1}(x))$.*

Proof. See [CG97, Proposition 8.6.16]. □

To establish the relationship between standard and simple modules we need the notion of a transverse slice.

Definition 12.56. Let X be an algebraic variety which admits an embedding into some smooth algebraic variety. Let us fix an algebraic stratification of X, i.e., a finite partition $X = \bigsqcup_{j \in J} X_j$ into smooth locally closed subvarieties (called strata) such that the closure of each stratum is a disjoint union of strata. More precisely, for each $j \in J$ there exists a $J' \subseteq J$ such that $\overline{X_j} = \bigsqcup_{i \in J'} X_i$.

Let $j \in J$ and $y \in X_j$. A locally closed (in the Hausdorff topology) complex analytic subset $S \subset X$ containing y is called a *transverse slice* to X_j at y if there exists an open neighbourhood $U \subset X$ of y (in the Hausdorff topology) and an analytic isomorphism $f : (X_j \cap U) \times S \xrightarrow{\cong} U$ such that

- f restricts to the tautological maps

$$f : \{y\} \times S \xrightarrow{\cong} S, \quad (X_j \cap U) \times \{y\} \xrightarrow{\cong} X_j \cap U.$$

- for each $i \in J$ we have

$$f\left((X_j \cap U) \times (S \cap X_i)\right) \subseteq X_i \cap U.$$

The following general proposition guarantees that for each G-orbit $\mathbb{O}_\lambda \subset N$ and any point $x_\lambda \in \mathbb{O}_\lambda$ there exists a transverse slice to $\mathbb{O}_\lambda$ at x_λ.

Proposition 12.57. *Let $\widetilde{X}$ be a smooth algebraic G-variety and $X \subset \widetilde{X}$ a G-stable algebraic subvariety consisting of finitely many G-orbits. Then for each G-orbit $\mathbb{O}$ and any $y \in \mathbb{O}$ there exists a transverse slice to $\mathbb{O}$ at y.*

Proof. See [CG97, Proposition 3.2.24]. □

Fix $\lambda \in \Lambda$ and choose an element $x_\lambda \in \mathbb{O}_\lambda$. Let S_λ be a transverse slice to $\mathbb{O}_\lambda$ at x_λ and let $\widehat{S}_\lambda := \mu^{-1}(S_\lambda)$. We have an inclusion $i_\lambda : \mu^{-1}(x_\lambda) \hookrightarrow \widehat{S}_\lambda$. This inclusion induces a map on homology

$$(i_\lambda)_* : H_*(\mu^{-1}(\rho_\lambda)) \to H_*(\widehat{S}_\lambda). \tag{101}$$

Let $M = \bigsqcup M_\alpha$ be the decomposition of M into connected components and let $\mu_\alpha : M_\alpha \to N$ denote the restriction of μ to M_α. Let $m(\alpha)$ denote the dimension of $\widehat{S}_\lambda \cap M_\alpha$ as a variety. For each α we have a bilinear pairing

$$< \, , \, >^{\widehat{S}_\lambda} : \quad H_{m(\alpha)+*}(\mu_\alpha^{-1}(x_\lambda)) \times H_{m(\alpha)-*}(\mu_\alpha^{-1}(x_\lambda)) \xrightarrow{\cap} \mathbb{C}$$

given by intersection in the ambient space $\widehat{S}_\lambda \cap M_\alpha$ (see paragraph 2.3.2.4 for the definition of the intersection pairing). The following crucial result says that every simple module L_λ is a quotient of the corresponding standard module K_λ.

Proposition 12.58. *The image of the map $(i_\lambda)_*$ equals L_λ and the kernel of $(i_\lambda)_*$ equals the radical of the bilinear form $< \, , \, >^{\widehat{S}_\lambda}$ on $H_*(\mu^{-1}(x_\lambda))$. Hence we have a natural isomorphism*

$$H_*(\mu^{-1}(x_\lambda))/\mathrm{rad} < \, , \, >^{\widehat{S}_\lambda} \quad \xrightarrow{\cong} \quad L_\lambda.$$

Moreover, this is an isomorphism of modules over $H_*^G(Z)$.

Proof. See [CG97, Proposition 8.5.10]. □

13 Representation theory of KLR algebras

In this chapter we are going to apply the general results from the previous chapter to study graded simple modules and standard modules over KLR algebras associated to Dynkin quivers.

13.1 Graded simple modules over KLR algebras

We work in the framework of Section 12.2. Let us fix a *Dynkin* quiver (i.e. a quiver whose underlying undirected graph is of type ADE) and a dimension vector. We set

$$M = \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \quad N = \mathrm{Rep}_{\underline{\mathbf{d}}}, \quad Z = \mathcal{Z}_{\underline{\mathbf{d}}}, \quad \mu = \mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}, \quad G = G_{\underline{\mathbf{d}}}.$$

To apply the results of Section 12.2 we need to check that the following assumptions hold:

(1) the map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ is proper and $G_{\underline{\mathbf{d}}}$-equivariant,

(2) the variety $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ is smooth,

(3) there are finitely many $G_{\underline{\mathbf{d}}}$-orbits in $\mathrm{Rep}_{\underline{\mathbf{d}}}$,

(4) for each $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ the stabilizer group $\mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(\rho)$ is connected,

(5) the variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ is $G_{\underline{\mathbf{d}}}$-equivariantly formal,

(6) the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic as a graded algebra to $H^*_{G_{\underline{\mathbf{d}}}}(\{pt\})$.

We know that the map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ is proper and $G_{\underline{\mathbf{d}}}$-equivariant by Proposition 5.5. The variety $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ is smooth because $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ is a disjoint union of vector bundles over complete flag varieties. The variety $\mathcal{Z}_{\underline{\mathbf{d}}}$ is $G_{\underline{\mathbf{d}}}$-equivariantly formal by Proposition 7.37. We also know that the centre of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is isomorphic to $H^*_{G_{\underline{\mathbf{d}}}}(\{pt\})$ as a graded algebra by Proposition 7.34 and the description of the grading on $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ in Definition 11.11.

Therefore, it only remains to check (3) and (4). We first check that there are finitely many $G_{\underline{\mathbf{d}}}$-orbits in $\mathrm{Rep}_{\underline{\mathbf{d}}}$.

Let $\mathfrak{g}$ be the complex semisimple Lie algebra corresponding to the quiver $\mathbf{\Gamma}$. Let us choose a Cartan subalgebra and a Borel subalgebra of $\mathfrak{g}$ and let Δ be the corresponding root system, Δ^0 the corresponding set of simple roots and Δ^+ the corresponding set of positive roots. We can identify the vertices of the quiver $\mathbf{\Gamma}$ with the simple roots, i.e., identify $\mathbf{I}$ with Δ^0. We can then view the dimension vector $\underline{\mathbf{d}}$ as an element in the root semilattice $\mathbb{N}\Delta^0$.

Definition 13.1. (i) Let us choose an order on the set of positive roots Δ^+. A *partition* of a dimension vector $\underline{\mathbf{d}}$ is a non-decreasing (with respect to the chosen order) sequence $(\alpha_1, ..., \alpha_k)$ of positive roots such that $\sum_{i=1}^k \alpha_k = \underline{\mathbf{d}}$. Alternatively, without choosing an ordering on Δ^+, we can define a partition of $\underline{\mathbf{d}}$ to be a function $p : \Delta^+ \to \mathbb{N}$ such that $\sum_{\alpha \in \Delta^+} p(\alpha) \cdot \alpha = \underline{\mathbf{d}}$. Informally, we think of a partition as a way to write the dimension vector as a sum of positive roots.
(ii) *Kostant's partition function*, denoted kpf, is a function $\mathrm{kpf} : \mathbb{N}\Delta^0 \to \mathbb{N}$ which assigns to a dimension vector $\underline{\mathbf{d}}$ the number of partitions of $\underline{\mathbf{d}}$.

Definition 13.2. A quiver is called *of finite type* if it has only finitely many isomorphism classes of indecomposable representations.

Theorem 13.3 (Gabriel's theorem)**.** *A connected quiver $\mathbf{\Gamma}$ is of finite type if and only if it is a Dynkin quiver. Moreover, the assignment* dim *sending an indecomposable representation to its dimension vector establishes a one-to-one correspondence between the set of isomorphism classes of indecomposable representations of $\mathbf{\Gamma}$ and the set of positive roots Δ^+.*

Proof. See for example [Kra07, Theorem 5.1.1]. □

Corollary 13.4. *There are finitely many $G_{\underline{\mathbf{d}}}$-orbits in* $\mathrm{Rep}_{\underline{\mathbf{d}}}$.

Proof. The space $\mathrm{Rep}_{\underline{\mathbf{d}}}$ is the space of representations of the Dynkin quiver $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. The $G_{\underline{\mathbf{d}}}$-orbits in $\mathrm{Rep}_{\underline{\mathbf{d}}}$ are in one-to-one correspondence with isomorphism classes of representations of $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. The latter are, in turn, in one-to-one correspondence with partitions of the dimension vector $\underline{\mathbf{d}}$ into positive roots, by Gabriel's theorem. But the number of such partitions is obviously finite. $\square$

We now check that for each $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ the stabilizer group $\mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(\rho)$ is connected.

Proposition 13.5. *For each $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$, the isotropy group $\mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(\rho)$ is connected.*

Proof. The group $\mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(\rho)$ is isomorphic to the group $\mathrm{Aut}_{\mathbf{\Gamma}}(\rho)$ of automorphisms of the representation ρ of the quiver $\mathbf{\Gamma}$. But the latter is an open dense subset of the affine space $\mathrm{End}_{\mathbf{\Gamma}}(\rho)$ of endomorphisms of the representation ρ, defined by the non-vanishing of the determinant. Hence it is connected. $\square$

We have hereby verified that all the relevant assumptions from Section 12.2 hold.

Definition 13.6. Let $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ denote the set of partitions of the dimension vector $\underline{\mathbf{d}}$ into positive roots. By Gabriel's theorem, we can identify $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ with the set of isomorphism classes of representations of $\mathbf{\Gamma}$ with dimension vector $\underline{\mathbf{d}}$. If $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$, let $\mathbb{O}_\lambda$ denote the corresponding $G_{\underline{\mathbf{d}}}$-orbit in $\mathrm{Rep}_{\underline{\mathbf{d}}}$. More precisely, for each $\alpha \in \Delta^+$ let ρ_α be an indecomposable representation of $\mathbf{\Gamma}$ with dimension vector α. Set $\rho_\lambda = \bigoplus_{\alpha \in \Delta^+} \lambda(\alpha)\rho_\alpha$. Then $\mathbb{O}_\lambda$ is the set of all representations of $\mathbf{\Gamma}$ which are isomorphic to ρ_λ.

We can now deduce the classification of graded simple modules over KLR algebras associated to Dynkin quivers.

Theorem 13.7. *Let $\mathbf{\Gamma}$ be a Dynkin quiver and $\underline{\mathbf{d}}$ a dimension vector.*
(i) There is an isomorphism

$$(\mu_{\underline{\mathbf{d}}})_* \left(IC(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}}) \right) \cong \bigoplus_{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})} L_\lambda \otimes IC(\overline{\mathbb{O}_\lambda}, \mathbb{C}_{\mathbb{O}_\lambda}) \tag{102}$$

in $D^b_c(\mathrm{Rep}_{\underline{\mathbf{d}}})$.
(ii) The non-zero members of the set $\{L_\lambda \mid \lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})\}$ form a complete and irredundant set of representatives of isomorphism classes of graded *simple modules over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$, up to grading shifts. In particular, there are, up to isomorphism and grading shifts, finitely many graded simple modules over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ and every such module is finite-dimensional.*

Proof. The theorem follows directly from Corollary 12.42 and Corollary 12.51 because we have verified that all the assumptions of these corollaries are satisfied in the KLR setting when $\mathbf{\Gamma}$ is a Dynkin quiver. $\square$

13.2 The equioriented A_n quiver

We are now going to study in more detail graded simple modules and standard modules over KLR algebras associated to the equioriented A_n quiver. In particular, our goal is to prove that each graded simple module L_λ is non-zero and that each standard module K_λ is indecomposable and has simple head L_λ. We emphasize that these facts do not follow from the general theory of convolution algebras in [CG97, Chapter 8]. Our proof that each module L_λ is non-zero is similar in flavour to the corresponding proof for affine Hecke algebras ([CG97, Proposition 8.1.14, Theorem 8.8.1, Proposition 8.8.2]). The fact that each standard module K_λ is indecomposable with simple head L_λ follows from results proved by Kato ([Kat12, Theorem 1.8(2)]). He proved these results using sheaf-theoretic and homological methods. We give a different proof which relies on a study of the geometry of fibers of the map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$.

13.2.1 The order on orbits

We begin with some more general remarks about the algebraic stratification of $\mathrm{Rep}_{\underline{\mathbf{d}}}$. Recall that

$$\mathrm{Rep}_{\underline{\mathbf{d}}} = \bigsqcup_{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})} \mathbb{O}_\lambda.$$

Definition 13.8. The set $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ is naturally endowed with a partial order. Suppose that $\lambda, \lambda' \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. We set

$$\lambda' \leq \lambda :\iff \mathbb{O}_{\lambda'} \subseteq \overline{\mathbb{O}_\lambda}. \tag{103}$$

We call this partial order the *closure ordering*.

We will find the following easy lemmata useful.

Lemma 13.9. *There is a unique maximal and minimal stratum in* $\mathrm{Rep}_{\underline{\mathbf{d}}}$ *with respect to the closure ordering.*

Proof. The variety $\mathrm{Rep}_{\underline{\mathbf{d}}}$ is isomorphic to an affine space, so it's irreducible. If there existed two or more maximal strata then their closures would constitute distinct irreducible components, contradicting irreducibility. The unique minimal stratum is the one-point stratum containing the zero representation. □

Lemma 13.10. *The map* $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ *maps each conneted component* $\widetilde{\mathcal{F}}_{\overline{y}}$ *onto the closure of some orbit* $\mathbb{O}_\lambda$*, i.e.,* $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = \overline{\mathbb{O}_\lambda}$ *for some* $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$.

Proof. Recall that we also have a vector bundle projection $\pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}$. Let $F \in \mathcal{F}_{\overline{y}}$ be a flag. The fibre $\pi_{\underline{\mathbf{d}}}^{-1}(F) = \{F\} \times \mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F)) \cong \mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F))$ consists of representations stabilizing F. Regarded as a vector subspace of $\mathrm{Rep}_{\underline{\mathbf{d}}}$, it inherits a stratification

$$\mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F)) = \coprod_{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})} \mathbb{O}_\lambda^F,$$

where $\mathbb{O}_\lambda^F := \mathbb{O}_\lambda \cap \mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F))$. Since $\mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F))$ is a vector space and is therefore irreducible, it must contain a unique nonempty maximal stratum $\mathbb{O}_\lambda^F$. Let $\rho \in \mathbb{O}_\lambda^F$. If $g \in G_{\underline{\mathbf{d}}}$ then $g.F \in \mathcal{F}_{\overline{y}}$ and $g.F$ is $g.\rho$-stable. Therefore, $g.\rho \in \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. This implies that $\mathbb{O}_\lambda \subset \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Now suppose that $\lambda' > \lambda$ or that λ' is unrelated to λ. Suppose further that $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) \cap \mathbb{O}_{\lambda'} \neq \varnothing$. Then there exists a flag $F' \in \widetilde{\mathcal{F}}_{\overline{y}}$ and representation $\rho' \in \mathbb{O}_{\lambda'}$ stabilizing F'. Since $G_{\underline{\mathbf{d}}}$ acts transitively on $\mathcal{F}_{\overline{y}}$, there exists a $g' \in G_{\underline{\mathbf{d}}}$ with $g'.F' = F$. But then F is $g'.\rho'$-stable. Since $g'.\rho' \in \mathbb{O}_{\lambda'}$, this contradicts the fact that $\mathbb{O}_\lambda^F$ is the unique maximal stratum in $\mu_{\underline{\mathbf{d}}}(\pi_{\underline{\mathbf{d}}}^{-1}(F))$. Therefore, $\mathbb{O}_\lambda$ is the unique maximal stratum in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Now, since the map $\mu_{\underline{\mathbf{d}}}$ is proper and hence closed, and $\widetilde{\mathcal{F}}_{\overline{y}}$ is a connected component, we can conclude that $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ is also closed. Hence $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = \overline{\mathbb{O}_\lambda}$. □

13.2.2 The equioriented A_n quiver

A deeper insight into the representation theory of KLR algebras requires a good understanding of the partial order (103), i.e., an understanding of the closure relations between orbits. It is difficult to describe these relations explicitly for arbitrary Dynkin quivers and orientations. Zwara has proven the following result.

Theorem 13.11. *Let $\lambda, \lambda' \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ and $\rho \in \mathbb{O}_\lambda, \rho' \in \mathbb{O}_{\lambda'}$. Then $\lambda \leq \lambda'$ if and only if there exists a short exact sequence*

$$0 \to x \to x \oplus \rho' \to \rho \to 0$$

for some representation x of the quiver $\mathbf{\Gamma}$.

Proof. See [Zwa00]. □

There are no known bounds on the size of x, so this theorem is not very useful to us in practice. It is, however, possible to describe closures of orbits explicitly for the A_n quiver with some special choices of orientation. In particular, the equioriented A_n quiver

$$\mathbf{\Gamma}_{A_n} = \quad i_1 \to i_2 \to \ldots \to i_n$$

admits an elegant description of orbit closures. Let $\underline{\mathbf{d}} = (\mathbf{d}_1, \ldots, \mathbf{d}_n)$ be a dimension vector for this quiver. Let us also assign to each vertex i_l a complex vector space $\mathbf{V}_l$ of dimension $\mathbf{d}_l$.

Definition 13.12. (i) A *rank matrix* for the equioriented A_n quiver $\mathbf{\Gamma}_{A_n}$ with dimension vector $\underline{\mathbf{d}}$ is an $n \times n$ matrix $\mathbf{r} = (r_{kl})$ such that $r_{ll} = \mathbf{d}_l$, $r_{kl} = 0$ if $k > l$ and r_{kl} is a non-negative integer if $k < l$.
(ii) We say that a rank matrix $\mathbf{r} = (r_{kl})$ *weakly dominates* another rank matrix $\mathbf{r}' = (r'_{kl})$, denoted $\mathbf{r} \geq \mathbf{r}'$ if $r_{kl} \geq r'_{kl}$ for each $1 \leq k, l \leq n$. A rank matrix $\mathbf{r}$ *strictly dominates* $\mathbf{r}'$, denoted $\mathbf{r} > \mathbf{r}'$, if $\mathbf{r} \geq \mathbf{r}'$ and there exist $1 \leq k < l \leq n$ such that $r_{kl} > r'_{kl}$.
(iii) If $\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}}$ is a representation of the quiver $\mathbf{\Gamma}_{A_n}$ with dimension vector $\underline{\mathbf{d}}$, we can write it as a sequence $\rho = (\rho_1, \rho_2, \ldots, \rho_{n-1})$, where each ρ_l is a linear map $\rho_l : \mathbf{V}_l \to \mathbf{V}_{l+1}$. If for each $1 \leq k < l \leq n$ we set

$$r_{kl} = \mathrm{rk}(\rho_{l-1} \circ \rho_{l-2} \circ \ldots \circ \rho_k),$$

$r_{ll} = \mathbf{d}_l$ and $r_{kl} = 0$ otherwise then the resulting matrix (r_{kl}) is a rank matrix. We will denote it with $\mathbf{Rk}(\rho)$ and refer to it as the *rank matrix for the representation* ρ. The rank matrix depends only on the isomorphism class of ρ. Therefore, if $\rho \in \mathbb{O}_\lambda$ we will also write $\mathbf{Rk}(\lambda)$ for $\mathbf{Rk}(\rho)$.
(iv) We define

$$Y(\mathbf{r}) := \{\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid \mathbf{Rk}(\rho) = \mathbf{r}\}.$$

Note that for some choices of $\mathbf{r}$ these sets may be empty.

Proposition 13.13. *Let $\mathbf{\Gamma} = \mathbf{\Gamma}_{A_n}$ be the equioriented A_n quiver.*
(i) The $G_{\underline{\mathbf{d}}}$-orbits in $\mathrm{Rep}_{\underline{\mathbf{d}}}$ are precisely the sets $Y(\mathbf{r})$ for a rank matrix $\mathbf{r} = (r_{kl})$ such that

$$r_{k,l} - r_{k,l+1} - r_{k-1,l} + r_{k-1,l+1} \geq 0 \qquad \text{for all} \quad 1 \leq k < l \leq n.$$

(ii) The Zariski closure of $Y(\mathbf{r})$ is

$$\overline{Y(\mathbf{r})} = \{\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid \mathbf{Rk}(\rho) \leq \mathbf{r}\}.$$

Proof. See e.g. [LR08, Prop. 13.5.3.1]. □

We now have two classifications of orbits for the equioriented A_n quiver - by rank and by partitions of the dimension vector. We want to relate them to each other. If α and β are positive roots, we call α a *subroot* of β if $\beta = \alpha + \gamma$ and γ is a sum of positive roots. Recall that positive roots in type A_n correspond to segments in the corresponding Dynkin diagram. A subroot corresponds to a subsegment.

Proposition 13.14. *Let α_k be the simple root corresponding to the vertex i_l in the quiver $\mathbf{\Gamma}_{A_n}$. We have*

$$Y(\mathbf{r}) = \mathbb{O}_\lambda$$

if and only if, whenever $k < l$, r_{kl} equals the number of positive roots (including each instance of a root which occurs several times) in the partition λ for which the positive root $\alpha_k + \alpha_{k+1} + \ldots + \alpha_l$ is a subroot.

Proof. Let $m_{i,j}$ be the number of times the positive root $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + ... + \alpha_j$ occurs in the partition λ. If $\rho \in \mathbb{O}_\lambda$ then we have an isomorphism of quiver representations $\rho \cong \bigoplus_{1\le i\le j\le n}(\rho_{\alpha_{i,j}})^{\oplus m_{i,j}}$. Therefore,

$$\mathrm{rk}(\rho_l \circ \rho_{l-1} \circ ... \circ \rho_k) = \sum_{1\le i\le j\le n} m_{i,j}\,\mathrm{rk}((\rho_{\alpha_{i,j}})_l \circ (\rho_{\alpha_{i,j}})_{l-1} \circ ... \circ (\rho_{\alpha_{i,j}})_k) = \sum_{1\le i\le j\le n} m_{i,j} 1_{i\le k} 1_{l\le j},$$

where $1_{i\le k}$ is the indicator function taking value 1 if $i \le k$ and 0 otherwise. But $\alpha_{k,l}$ is a subroot of $\alpha_{i,j}$ if and only if $1 \le k < l \le j$ so the last expression is precisely the number of positive roots in the partition λ of which $\alpha_{k,l}$ is a subroot. Since $\rho \in Y(\mathbf{r})$ if and only if $\mathrm{rk}(\rho_l \circ \rho_{l-1} \circ ... \circ \rho_k) = r_{kl}$ for each $k < l$, this proves the proposition. $\square$

From now on we let $\mathbf{\Gamma}$ be the equioriented A_n quiver.

Example 13.15. Let us consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = i+2j+k$. The Lie algebra corresponding to this quiver is $\mathfrak{sl}_4$. Let α, β, γ be the simple roots corresponding to the vertices i, j, k, respectively. There are five partitions of the dimension vector $\underline{\mathbf{d}}$: $\lambda_0 = (\alpha, \beta, \beta, \gamma), \lambda_1 = (\alpha+\beta, \beta, \gamma), \lambda_2 = (\alpha, \beta, \beta+\gamma), \lambda_3 = (\alpha+\beta, \beta+\gamma), \lambda_4 = (\alpha + \beta + \gamma, \beta)$. The corresponding rank matrices are

$$\mathbf{Rk}(\lambda_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{Rk}(\lambda_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{Rk}(\lambda_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{Rk}(\lambda_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{Rk}(\lambda_4) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The closure ordering on the partitions therefore is

$$\begin{tikzcd} & \lambda_4 \arrow[d] & \\ & \lambda_3 \arrow[dl] \arrow[dr] & \\ \lambda_1 \arrow[dr] & & \lambda_2 \arrow[dl] \\ & \lambda_0. & \end{tikzcd}$$

It is also not hard to give an explicit description of the $G_{\underline{\mathbf{d}}}$-orbits on $\mathrm{Rep}_{\underline{\mathbf{d}}}$. We have $\mathrm{Rep}_{\underline{\mathbf{d}}} \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) \cong \mathbb{C}^4$. We can identify it with the set of all pairs of complex matrices of the form

$$\left(\begin{bmatrix} * \\ * \end{bmatrix}, \begin{bmatrix} * & * \end{bmatrix} \right).$$

We have

$$\mathbb{O}_{\lambda_0} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right) \right\}, \quad \mathbb{O}_{\lambda_1} = \left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right) \,\middle|\, a \text{ or } b \neq 0 \right\},$$

$$\mathbb{O}_{\lambda_2} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} \right) \,\middle|\, c \text{ or } d \neq 0 \right\},$$

$$\mathbb{O}_{\lambda_3} = \left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c & d \end{bmatrix} \right) \,\middle|\, a \text{ or } b \neq 0,\ c \text{ or } d \neq 0,\ ac + bd = 0 \right\},$$

$$\mathbb{O}_{\lambda_4} = \left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c & d \end{bmatrix} \right) \,\middle|\, a \text{ or } b \neq 0,\ c \text{ or } d \neq 0,\ ac + bd \neq 0 \right\}.$$

13.2.3 Weights

Recall that $\{1_{\overline{y},\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$ forms a complete set of primitive orthogonal idempotents of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Definition 13.16. Let M be a graded $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$-module and let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. Define the *$\overline{y}$-weight space* $M_{\overline{y}}$ of M to be $1_{\overline{y},\overline{y}}M$. Let $\mathfrak{C}$ be the $\mathbb{C}$-subalgebra of $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ generated by the idempotents $\{1_{\overline{y},\overline{y}} \mid \overline{y} \in Y_{\underline{\mathbf{d}}}\}$. The vector space $M_{\overline{y}}$ is a $\mathfrak{C}$-submodule of M. We have an equality of $\mathfrak{C}$-modules:

$$M = \bigoplus_{\overline{y} \in Y_{\underline{\mathbf{d}}}} M_{\overline{y}}.$$

We call $\overline{y}$ a weight of M if $M_{\overline{y}} \neq \{0\}$. We call the set of weights of M

$$\mathrm{supp} M := \{\overline{y} \in Y_{\underline{\mathbf{d}}} \mid 1_{\overline{y},\overline{y}}M \neq \{0\}\}$$

the *support* of M.

It follows from the relations in $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ that

$$\sigma(r).M_{\overline{y}} \subseteq M_{s_r.\overline{y}}, \quad \varkappa(r).M_{\overline{y}} \subseteq M_{\overline{y}}.$$

Moreover, $1_{\overline{y},\overline{y}}$ acts on $M_{\overline{y}}$ by the identity endomorphism and if $\overline{y}' \neq \overline{y}$ then $1_{\overline{y}',\overline{y}'}$ acts by zero on $M_{\overline{y}}$.

13.2.4 The order on weights

Definition 13.17. (i) Recall that $i_1, ..., i_n$ are the vertices of our quiver. We set $i_1 \succ i_2 \succ ... \succ i_n$ and extend this ordering on $\mathbf{I}$ to a total lexicographic order on $Y_{\underline{\mathbf{d}}}$. If $\overline{y}, \overline{y}' \in Y_{\underline{\mathbf{d}}}$ and $\overline{y}'$ is greater than $\overline{y}$ in this ordering we write $\overline{y}' \succ \overline{y}$.
(ii) We say that $\overline{y}'$ is *accessible* from $\overline{y}$, denoted $\overline{y} < \overline{y}'$, if there exist simple transpositions $s_{k_1}, ..., s_{k_m} \in \Pi$ such that $\overline{y}' = s_{k_1} \circ ... \circ s_{k_m}(\overline{y})$ and $s_{k_l} \circ s_{k_{l+1}} \circ .... \circ s_{k_m}(\overline{y}) \prec s_{k_{l+1}} \circ .... \circ s_{k_m}(\overline{y})$ in the lexicographic ordering, for each $1 \leq l \leq m$. This defines a partial order on $Y_{\underline{\mathbf{d}}}$, which we call the *accessibility ordering.*
(iii) We say that $\overline{y}'$ is *directly accessible* from $\overline{y}$, denoted $\overline{y} \lhd \overline{y}'$ if there exists a simple transposition $s \in \Pi$ such that $\overline{y}' = s(\overline{y})$ and $s(\overline{y}) \prec \overline{y}$, i.e., if $\overline{y}$ is an immediate predecessor of $\overline{y}'$ in the accessibility ordering.

Example 13.18. Let us again consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = i + 2j + k$. The accessibility ordering on $Y_{\underline{\mathbf{d}}}$ looks as follows

$$\begin{tikzcd}
 & & kjji \arrow[dl] \arrow[dr] & & \\
 & jkji \arrow[dl] \arrow[dr] & & kjij \arrow[dl] \arrow[dr] & \\
jjki \arrow[d] & & jkij \arrow[d] & & kijj \arrow[d] \\
jjik \arrow[dr] & & jikj \arrow[dl] \arrow[dr] & & ikjj \arrow[dl] \\
 & jijk \arrow[dr] & & ijkj \arrow[dl] & \\
 & & ijjk. & &
\end{tikzcd}$$

13.2.5 The functions Φ and Ψ

Definition 13.19. We set

$$\begin{aligned}
&\Phi : Y_{\underline{\mathbf{d}}} \to \mathcal{P}(\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})), \quad \overline{y} \mapsto \{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}}) \mid \text{there exist } \rho \in \mathbb{O}_\lambda, F \in \mathcal{F}_{\overline{y}} \text{ s.t. } F \text{ is } \rho\text{-stable}\},\\
&\Psi : \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}}) \to \mathcal{P}(Y_{\underline{\mathbf{d}}}), \quad \lambda \mapsto \{\overline{y} \in Y_{\underline{\mathbf{d}}} \mid \text{there exist } \rho \in \mathbb{O}_\lambda, F \in \mathcal{F}_{\overline{y}} \text{ s.t. } F \text{ is } \rho\text{-stable}\}.
\end{aligned}$$

We have

$$\Phi(\overline{y}) = \{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}}) \mid \mathbb{O}_\lambda \subseteq \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})\}, \quad \Psi(\lambda) = \{\overline{y} \in Y_{\underline{\mathbf{d}}} \mid \mathbb{O}_\lambda \subseteq \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})\}.$$

Example 13.20. Let us again consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = i + 2j + k$. One easily sees that

$$\begin{aligned}
\Phi(ijjk) =&\{\lambda_0\}, & \Phi(jijk) =&\{\lambda_0, \lambda_1\}, & \Phi(ijkj) =&\{\lambda_0, \lambda_2\},\\
\Phi(jjik) =&\{\lambda_0, \lambda_1\}, & \Phi(jikj) =&\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}, & \Phi(ikjj) =&\{\lambda_0, \lambda_2\},\\
\Phi(jjki) =&\{\lambda_0, \lambda_1\}, & \Phi(jkij) =&\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}, & \Phi(kijj) =&\{\lambda_0, \lambda_2\},\\
\Phi(jkji) =&\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}, & \Phi(kjij) =&\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}, & \Phi(kjii) =&\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\},
\end{aligned}$$

$$\begin{aligned}
\Psi(\lambda_0) =&Y_{\underline{\mathbf{d}}}, & \Psi(\lambda_1) =&Y_{\underline{\mathbf{d}}} \backslash \{ijjk, ikjj, kijj\}, \quad \Psi(\lambda_2) =Y_{\underline{\mathbf{d}}} \backslash \{ijjk, jijk, jjki\},\\
\Psi(\lambda_3) =&\{jikj, jkij, jjkji, kjij, kjji\}, & \Psi(\lambda_4) =&\{jkij, kjij, kjji\}.
\end{aligned}$$

Lemma 13.21. *Suppose that $\overline{y} \lhd \overline{y}' = s_l(\overline{y})$. If $F \in \mathcal{F}_{\overline{y}}$ is ρ-stable then $s_l.F$ is also ρ-stable. Hence $s_l.(\pi_{\underline{\mathbf{d}}} \circ \mu_{\overline{y}}^{-1}(\rho)) \subseteq \pi_{\underline{\mathbf{d}}} \circ \mu_{\overline{y}'}^{-1}(\rho)$ and $\mu_{\underline{\mathbf{d}}} \circ \pi_{\underline{\mathbf{d}}}^{-1}(F) \subseteq \mu_{\underline{\mathbf{d}}} \circ \pi_{\underline{\mathbf{d}}}^{-1}(s_l.F)$.*

Proof. We can write $F = (V^k)$, where $V^k = D_1 \oplus ... \oplus D_k$ and each D_n is a one-dimensional subspace in some graded component $\mathbf{V}_i$ of $\mathbf{V}$. We want to show that $s_l.F$ is still ρ-stable. Let us write $s_l.F = (W^k)$. It is clear that $V^k = W^k$ unless $k = l$. Therefore, we need to check that

$\rho(W^l) \subseteq W^{l-1}$ and $\rho(W^{l+1}) \subseteq W^l$. We have $W^l = D_1 \oplus ... \oplus D_{l-1} \oplus D_{l+1} = V^{l-1} \oplus D_{l+1}$. Moreover,

$$\rho(W^l) = \rho(V^{l-1} \oplus D_{l+1}) \subseteq \rho(V^{l-1}) + \rho(D_{l+1}) \subseteq V^{l-2} + \rho(D_{l+1}). \tag{104}$$

We know that $\rho(D_{l+1}) \subseteq V^l = V^{l-1} \oplus D_l$. We also have $D_l \subseteq V_{\overline{y}(l)}, D_{l+1} \subseteq V_{\overline{y}(l+1)}$. Now since $\overline{y} \lhd \overline{y}' = s_l(\overline{y})$, we get $\overline{y}(l) > \overline{y}(l+1)$. This implies that in our quiver there is no arrow from $\overline{y}(l+1)$ to $\overline{y}(l)$. Hence $\rho(D_{l+1}) \cap D_l = \{0\}$ and so $\rho(D_{l+1}) \subseteq V^{l-1}$. This, together with (104), implies that $\rho(W^l) \subseteq V^{l-1} = W^{l-1}$. Furthermore,

$$\rho(W^{l+1}) = \rho(V^l \oplus D_{l+1}) \subseteq \rho(V^l) + \rho(D_{l+1}) \subseteq V^{l-1} = W^{l-1}.$$

Therefore, $s_l.F$ is ρ-stable. □

Lemma 13.22. *Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. There exists a $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ such that*

$$\Phi(\overline{y}) = \{\lambda' \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}}) \mid \lambda' \leq \lambda\}.$$

Proof. This follows directly from Lemma 13.10 . □

Lemma 13.23. *The functions Φ, Ψ have the following "monotonicity" properties*

(a) if $\overline{y} < \overline{y}'$ then $\Phi(\overline{y}) \subseteq \Phi(\overline{y}')$,

(b) if $\lambda < \lambda'$ then $\Psi(\lambda') \subseteq \Psi(\lambda)$.

Proof. Suppose that $\overline{y} \lhd \overline{y}'$. Let $\lambda \in \Phi(\overline{y})$. There exist $\rho \in \mathbb{O}_\lambda, F \in \mathcal{F}_{\overline{y}}$ such that F is ρ-stable. Since $\overline{y} \lhd \overline{y}'$, there exists a simple transposition $s_l \in \Pi$ such that $\overline{y}' = s_l(\overline{y})$. By Lemma 13.21, $s_l.F$ is ρ-stable. Since $s_l.F \in \mathcal{F}_{\overline{y}'}$ we have $\lambda \in \Phi(\overline{y}')$.

Now suppose that $\lambda < \lambda'$. Let $\overline{y} \in \Psi(\lambda')$. We have $\mathbb{O}_{\lambda'} \subseteq \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Since $\lambda < \lambda'$, Lemma 13.10 implies that $\mathbb{O}_\lambda \subseteq \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Hence $\overline{y} \in \Psi(\lambda)$. □

Corollary 13.24. *(i) Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. Then the set $\Psi(\lambda)$ is* closed from above *in the following sense. If $\overline{y} \in \Psi(\lambda)$ and $\overline{y}' > \overline{y}$ then $\overline{y}' \in \Psi(\lambda)$.*
(ii) Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. Then the set $\Phi(\overline{y})$ is closed from below *in the following sense. If $\lambda \in \Phi(\lambda)$ and $\lambda' < \lambda$ then $\lambda' \in \Phi(\overline{y})$.*

Definition 13.25. Let us set

$$\widehat{\Psi}(\lambda) = \Psi(\lambda) \backslash \left(\bigcup_{\lambda' > \lambda} \Psi(\lambda') \right).$$

By Lemma 13.23, the definition makes sense.

Lemma 13.26. *Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. Then*

$$\widehat{\Psi}(\lambda) = \{\overline{y} \in Y_{\underline{\mathbf{d}}} \mid \lambda \textit{ is the unique maximal element in } \Phi(\overline{y})\}.$$

Proof. This is clear. □

Lemma 13.27. *We have a disjoint union decomposition*

$$Y_{\underline{\mathbf{d}}} = \bigsqcup_{\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})} \widehat{\Psi}(\lambda). \tag{105}$$

Proof. Let us first prove disjointness. It is immediately clear from the definition that if $\lambda < \lambda'$ then $\widehat{\Psi}(\lambda)$ and $\widehat{\Psi}(\lambda')$ are disjoint. So suppose λ and λ' are unrelated. Let $\overline{y} \in \widehat{\Psi}(\lambda) \cap \widehat{\Psi}(\lambda')$. Then $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ contains both $\mathbb{O}_\lambda$ and $\mathbb{O}_{\lambda'}$. Moreover, these two orbits are maximal in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$, which contradicts Lemma 13.10 (the Lemma implies there is a unique maximal orbit in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$).

Now let us prove the equality in (105). For each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ there exists a unique maximal $\lambda \in \Phi(\overline{y})$, by Lemma 13.10 . Hence $\overline{y} \in \Psi(\lambda)$ and $\overline{y} \notin \Psi(\lambda')$ for any $\lambda' > \lambda$. Therefore, $\overline{y} \in \widehat{\Psi}(\lambda)$. □

Corollary 13.28. *We have a disjoint union decomposition*

$$\Psi(\lambda) = \bigsqcup_{\lambda' \geq \lambda} \widehat{\Psi}(\lambda).$$

Proof. This follows directly from the definitions and the disjointness in the decomposition (105). ∎

Example 13.29. Let us again consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = i + 2j + k$. We designate $\widehat{\Psi}(\lambda_0)$ with black colour, $\widehat{\Psi}(\lambda_1)$ with red, $\widehat{\Psi}(\lambda_2)$ with green, $\widehat{\Psi}(\lambda_3)$ with pink and $\widehat{\Psi}(\lambda_4)$ with blue:

$$\begin{tikzcd}
 & & {\color{blue}kjji} \arrow[dl] \arrow[dr] & & \\
 & {\color{blue}jkji} \arrow[dl] \arrow[dr] & & {\color{blue}kjij} \arrow[dl] \arrow[dr] & \\
{\color{red}jjki} \arrow[d] & & {\color{magenta}jkij} \arrow[d] & & {\color{green}kijj} \arrow[d] \\
{\color{red}jjik} \arrow[dr] & & {\color{magenta}jikj} \arrow[dl] \arrow[dr] & & {\color{green}ikjj} \arrow[dl] \\
 & {\color{red}jijk} \arrow[dr] & & {\color{green}ijkj} \arrow[dl] & \\
 & & ijjk. & &
\end{tikzcd}$$

Definition 13.30. If $A, B \subseteq Y_{\underline{\mathbf{d}}}$, we write $A < B$ if for all $a \in A, b \in B$ we have $a \not\geq b$ and there exist $a \in A, b \in B$ such that $a < b$.

Lemma 13.31. *Let $\lambda' > \lambda$. Then $\widehat{\Psi}(\lambda') > \widehat{\Psi}(\lambda)$.*

Proof. Let $\overline{y} \in \widehat{\Psi}(\lambda)$ and $\overline{y}' \in \widehat{\Psi}(\lambda')$. Since $\widehat{\Psi}(\lambda'), \widehat{\Psi}(\lambda)$ are disjoint, by Lemma 13.27, we can't have $\overline{y} = \overline{y}'$. Suppose that $\overline{y} > \overline{y}'$. By Lemma 13.23, $\Phi(\overline{y}') \subseteq \Phi(\overline{y})$. But λ is the unique maximal element in $\Phi(\overline{y})$, which contradicts the fact that $\lambda' \in \Phi(\overline{y}') \subseteq \Phi(\overline{y})$. Hence $\overline{y} \not\geq \overline{y}'$.

To prove that there exist $\overline{y} \in \widehat{\Psi}(\lambda)$ and $\overline{y}' \in \widehat{\Psi}(\lambda')$ such that $\overline{y} < \overline{y}'$ we need to develop some more theory. This fact will follow immediately from Lemma 13.47. ∎

Definition 13.32. Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. We define

$$\mu_{\widehat{\Psi}(\lambda)} : \bigsqcup_{\overline{y} \in \widehat{\Psi}(\lambda)} \widetilde{\mathcal{F}}_{\overline{y}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$$

to be the restriction of the map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ to $\bigsqcup_{\overline{y} \in \widehat{\Psi}(\lambda)} \widetilde{\mathcal{F}}_{\overline{y}}$.

13.2.6 Characteristic compositions

Definition 13.33. Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. We call $\overline{y} \in Y_{\underline{\mathbf{d}}}$ a λ-characteristic composition, and $\widetilde{\mathcal{F}}_{\overline{y}}$ a λ-characteristic component, if $\overline{y}$ satisfies the following two properties:

(P1) $\overline{y}$ is a minimal (but not necessarily the least) element in $\Psi(\lambda)$,

(P2) λ is a maximal element in $\Phi(\overline{y})$.

Note that if λ is a maximal element in $\Phi(\overline{y})$ then it is automatically the unique maximal (i.e. the greatest) element. Moreover, (P2) is equivalent to requiring that $\overline{y} \in \widehat{\Psi}(\lambda)$. Therefore, $\overline{y}$ is a λ-characteristic composition if and only if $\overline{y}$ is a minimal element in $\widehat{\Psi}(\lambda)$.

Example 13.34. Let us return to the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = i+2j+k$. The λ_0-characteristic composition is $ijjk$, the λ_1-characteristic composition is $jijk$, the λ_2-characteristic composition is $ijkj$, the λ_3-characteristic composition is $jikj$ and the λ_4-characteristic compositions are $jkji$ and $kjij$.

Proposition 13.35. *For each λ, the set $\widehat{\Psi}(\lambda)$ is non-empty. In particular, a λ-characteristic composition exists.*

Proof. By Lemma 13.10, it suffices to find a sequence $\overline{y} \in Y_{\underline{\mathbf{d}}}$ such that $\mathbb{O}_\lambda$ is a maximal orbit in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Let α_k be the simple root corresponding to the vertex i_k in the quiver $\mathbf{\Gamma}_{A_n}$. We identify the isomorphism class of quiver representations λ with the corresponding partition of $\underline{\mathbf{d}}$, i.e., we regard it as a function $\lambda : \Delta^+ \to \mathbb{N}$.

We first define a total order on Δ^+ by induction. Let Δ_k^+ be the subset of Δ^+ consisting of those positive roots which do not contain $\alpha_1, ..., \alpha_k$ as a subroot. We have $\Delta_{n-1}^+ = \{\alpha_n\}$, so there is a unique choice of ordering on Δ_{n-1}^+. Assuming that we have defined the order on Δ_{n-l}^+, we extend it to Δ_{n-l-1}^+ by setting $\alpha_{n-l-1} < \alpha_{n-l-1} + \alpha_{n-l} < \alpha_{n-l-1} + \alpha_{n-l} + \alpha_{n-l+1} < ... < \alpha_{n-l-1} + \alpha_{n-l} + \alpha_{n-l+1} + ... + \alpha_n < \alpha_{n-l}$, where α_{n-l} is (by induction) the least element in Δ_{n-l}^+.

Let us denote concatenation of sequences with $\circ$. If $\alpha_{k,l} := \alpha_k + \alpha_{k+1} + ... + \alpha_l$ is a positive root, we associate to it the sequence of vertices $\phi(\alpha_{k,l}) := (i_l, i_{l-1}, ..., i_{k+1}, i_k)$. Let $\beta_1 < \beta_2 < ... < \beta_{n(n+1)/2}$ be the enumeration of the elements of Δ^+ in the order we have just defined. We finally define the sequence $\overline{y}$ in the following way

$$\overline{y} := \phi(\beta_1)^{\circ\lambda(\beta_1)} \circ \phi(\beta_2)^{\circ\lambda(\beta_2)} \circ ... \circ \phi(\beta_{n(n+1)/2})^{\circ\lambda(\beta_{n(n+1)/2})}.$$

We encourage the reader to look at the example following the proof to build an intuition for the various constructions and definitions we introduce here. We now have to check that $\mathbb{O}_\lambda$ is in fact a maximal orbit in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ (by Lemma 13.10 such an orbit is automatically unique). We first show that $\mathbb{O}_\lambda \subset \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. Let us choose a representation $\rho \in \mathbb{O}_\lambda$ and decompose it into indecomposable representations $\rho = \bigoplus_{j=1}^{n(n+1)/2} \bigoplus_{m=1}^{m=\lambda(\beta_j)} \rho_{(j,m)}$. Here the $\rho_{(j,1)}, ..., \rho_{(j,\lambda(\beta_j))}$ are $\lambda(\beta_j)$ indecomposable subrepresentations of ρ with dimension vector β_j (obviously the choice of subrepresentations is not unique, but the number of isomorphism classes of indecomposable representations corresponding to each positive root occurring in this decomposition is uniquely determined). Now fix j and m. Suppose that $\beta_j = \alpha_{k,l} = \alpha_k + \alpha_{k+1} + ... + \alpha_l$. For $k \le p \le l$ let $W^{(j,m,p)} = \rho_{(j,m)}[p]$ be the one-dimensional vector subspace of $\mathbf{V}$ that $\rho_{(j,m)}$ assigns to the vertex i_p. We define sequences of one-dimensional $\mathbf{I}$-graded subspaces of $\mathbf{V}$ as follows

$$E^{(j,m)} = (W^{(j,m,l)}, W^{(j,m,l-1)}, ..., W^{(j,m,k)}), \quad E^j = E^{(j,1)} \circ E^{(j,2)} \circ ... \circ E^{(j,\lambda(\beta_j))},$$

$$E = E^1 \circ E^2 \circ ... \circ E^{n(n+1)/2} =: (U^1, ..., U^{\mathbf{d}}),$$

where U^r is the r-th member of the sequence E. Let $F = (V_t)_{t=1}^{\mathbf{d}}$ be the flag defined by $V_t = \bigoplus_{r=1}^{t} U^r$. One can easily see from the definition of the flag F that $F \in \mathcal{F}_{\overline{y}}$ and that F is indeed ρ-stable. Therefore $\rho \in \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. By $G_{\underline{\mathbf{d}}}$-equivariance of $\mu_{\underline{\mathbf{d}}}$, it follows that $\mathbb{O}_\lambda \subset \mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$.

We now show that $\mathbb{O}_\lambda$ is a maximal orbit in $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$. For the sake of contradiction suppose that $\mathbb{O}_\lambda$ were not maximal. Then there would exist $\lambda' > \lambda$, $\rho' \in \mathbb{O}_{\lambda'}$ and $F \in \mathcal{F}_{\overline{y}}$ such that F is ρ'-stable. Let $r_{k,l} = (\mathbf{Rk}(\rho))_{k,l}$ and $r'_{k,l} = (\mathbf{Rk}(\rho'))_{k,l}$. The fact that $\lambda' > \lambda$ implies that for all $1 \leq k < l \leq n$ we have $r'_{k,l} \geq r_{k,l}$ and there exist $1 \leq k < l \leq n$ such that $r'_{k,l} > r_{k,l}$.

Observe that the maximal possible value of $r'_{k,l}$ equals the maximal number of mutually disjoint subsequences of $\overline{y}$ of the form $(i_l, i_{l-1}, ..., i_k)$. But we constructed the sequence $\overline{y}$ in such a way that the maximal number of mutually disjoint subsequences of $\overline{y}$ of the form $(i_l, i_{l-1}, ..., i_k)$ equals $r_{k,l}$.

Indeed, we can easily prove this by induction on the inductive definition of $\overline{y}$. Let $\overline{y}_b$ be the sequence obtained from $\overline{y}$ by deleting all the subsequences $\phi(\beta)$ containing the vertex i_1 and let $\overline{y}_a$ be the sequence obtained from $\overline{y}$ by deleting all the subsequences $\phi(\beta)$ which do not contain the vertex i_1. We have $\overline{y} = \overline{y}_a \circ \overline{y}_b$. Similarly, we can decompose ρ as $\rho = \rho_a \oplus \rho_b$, where ρ_a is a direct sum of indecomposable representations ρ_β such that α_1 is a subroot of $\beta = \dim \rho_\beta$ and ρ_a is a direct sum of indecomposable representations ρ_β such that α_1 is not a subroot of $\beta = \dim \rho_\beta$. By induction, we can assume that $(\mathbf{Rk}(\rho_b))_{k,l}$ equals the maximal number of mutually disjoint subsequences of $\overline{y}_b$ of the form $(i_l, i_{l-1}, ..., i_k)$. Now observe that

$$\overline{y}_a = (i_1)^{\circ\lambda(\alpha_1)} \circ (i_2, i_1)^{\circ\lambda(\alpha_1+\alpha_2)} \circ (i_n, i_{n-1}, ..., i_1)^{\circ\lambda(\alpha_1+...+\alpha_n)}$$

and $\rho_a \cong \bigoplus_{j=1}^{n} (\rho_{\alpha_1+...+\alpha_j})^{\oplus\lambda(\alpha_1+...+\alpha_j)}$, where $\dim \rho_{\alpha_1+...+\alpha_j} = \alpha_1 + ... + \alpha_j$. Hence $(\mathbf{Rk}(\rho_a))_{k,l}$ also equals the maximal number of mutually disjoint subsequences of $\overline{y}_a$ of the form $(i_l, i_{l-1}, ..., i_k)$. We conclude that $r_{k,l}$ equals the maximal number of mutually disjoint subsequences of $\overline{y}$ of the form $(i_l, i_{l-1}, ..., i_k)$ by observing that $r_{k,l} = (\mathbf{Rk}(\rho_a))_{k,l} + (\mathbf{Rk}(\rho_b))_{k,l}$ and the maximal number of mutually disjoint subsequences of $\overline{y}$ of the form $(i_l, i_{l-1}, ..., i_k)$ equals the sum of maximal numbers of such mutually disjoint subsequences in $\overline{y}_a$ and $\overline{y}_b$. $\square$

Example 13.36. Consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = 2i + 3j + 2k$. The Lie algebra corresponding to this quiver is $\mathfrak{sl}_4$. Let α, β, γ be the simple roots corresponding to the vertices i, j, k, respectively. The positive roots, given in the order we defined in the proof above, are: $\alpha < \alpha + \beta < \alpha + \beta + \gamma < \beta < \beta + \gamma < \gamma$. Let us consider for example a representation $\rho_{\alpha+\beta} \oplus \rho_{\alpha+\beta+\gamma} \oplus \rho_\beta \oplus \rho_\gamma$ of our quiver associated to the partition $\lambda = (\alpha + \beta, \alpha + \beta + \gamma, \beta, \gamma)$. The corresponding λ-characteristic composition is (j, i, k, j, i, j, k).

Corollary 13.37. *For each $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ there exists a connected component $\widetilde{\mathcal{F}}_{\overline{y}}$ such that $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = \overline{\mathbb{O}_\lambda}$.*

Proof. By the preceding proposition, some λ-characteristic composition $\overline{y}$ exists. By the definition of a λ-characteristic composition, $\mathbb{O}_\lambda$ is the unique maximal orbit in $\mu_{\underline{\mathbf{d}}}(\mathbb{O}_\lambda)$. But by Lemma 13.10 $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}})$ equals the closure of some orbit. This proves the corollary. $\square$

We can now prove that every graded simple module L_λ is non-zero.

Theorem 13.38. *Let $\mathbf{\Gamma}$ be the equioriented A_n quiver. Then for every $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$, the graded simple module L_λ is non-zero.*

Proof. Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. By Corollary 13.37, there exists a connected component $\widetilde{\mathcal{F}}_{\overline{y}}$ such that $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = \overline{\mathbb{O}_\lambda}$. Let $\widetilde{\mathcal{F}}_{\overline{y}}^{\perp} = \bigsqcup_{\overline{y} \neq \overline{y}' \in Y_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}'}$. We have

$$IC(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}}) = IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}}) \oplus IC(\widetilde{\mathcal{F}}_{\overline{y}}^{\perp}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}^{\perp}}).$$

Therefore

$$(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}}) \cong (\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}}) \oplus (\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}^{\perp}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}^{\perp}}).$$

Let us investigate the direct summand $(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}})$. We can apply the decomposition theorem (Corollary 12.43) to the restricted map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \overline{\mathbb{O}_\lambda}$ to obtain

$$(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}}) \cong \Big(\widetilde{L}_\lambda \otimes IC(\mathbb{O}_\lambda, \mathbb{C}_{\mathbb{O}_\lambda})\Big) \oplus \left(\bigoplus_{\lambda' < \lambda} \widetilde{L}_{\lambda'} \otimes IC(\mathbb{O}_{\lambda'}, \mathbb{C}_{\mathbb{O}_{\lambda'}}) \right), \tag{106}$$

where the $\widetilde{L}_{\lambda'}$ are finite-dimensional graded vector spaces encoding the graded multiplicity with which each intersection cohomology complex $IC(\mathbb{O}_{\lambda'}, \mathbb{C}_{\mathbb{O}_{\lambda'}})$ occurs in the decomposition (106). Now let $x \in \mathbb{O}_\lambda$ and let $i_x : \{x\} \hookrightarrow \overline{\mathbb{O}_\lambda}$ denote the inclusion. Since, for each $\lambda' < \lambda$, the complex $IC(\mathbb{O}_{\lambda'}, \mathbb{C}_{\mathbb{O}_{\lambda'}})$ is supported on the boundary of $\overline{\mathbb{O}_\lambda}$, we have $i_x^* IC(\mathbb{O}_{\lambda'}, \mathbb{C}_{\mathbb{O}_{\lambda'}}) = 0$. Hence if we apply the functor $H^* i_x^*$ to the decomposition (106) we get

$$H^{m+*}(\mu_{\underline{\mathbf{d}}}^{-1}(x)) \cong H^*(i_x^*(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}})) \cong \widetilde{L}_\lambda \otimes H^*(i_x^* IC(\mathbb{O}_\lambda, \mathbb{C}_{\mathbb{O}_\lambda})), \tag{107}$$

where $m = \dim \widetilde{\mathcal{F}}_{\overline{y}}$ (for the proof of the first isomorphism above see [CG97, Lemma 8.5.4]). But $\mu_{\underline{\mathbf{d}}}(\widetilde{\mathcal{F}}_{\overline{y}}) = \overline{\mathbb{O}_\lambda}$ so the fibre $\mu_{\underline{\mathbf{d}}}^{-1}(x)$ is non-trivial and so the cohomology group $H^{m+*}(\mu_{\underline{\mathbf{d}}}^{-1}(x))$ is non-trivial too. We conclude from the isomorphism (107) that $\widetilde{L}_\lambda$ must be non-zero as well. Since $(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\overline{y}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\overline{y}}})$ is a direct summand of $(\mu_{\underline{\mathbf{d}}})_* IC(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \mathbb{C}_{\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}})$, we have $\widetilde{L}_\lambda \subseteq L_\lambda$, so L_λ is nonzero. □

Corollary 13.39. *There are exactly* $\mathrm{kpf}(\underline{\mathbf{d}})$ *isomorphism classes of graded simple modules over the KLR algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ associated to the equioriented A_n quiver.*

Proof. We have shown that the non-zero members of $\{L_\lambda \mid \lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})\}$ form a complete and irredundant set of representatives of isomorphism classes of graded simple modules over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. But by Theorem 13.38 each module L_λ is non-zero. Hence there are exactly $|\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})|$-many isomorphism classes of graded simple modules over $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. But $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ is the set of partitions of $\underline{\mathbf{d}}$ into positive roots, and therefore its cardinality equals $\mathrm{kpf}(\underline{\mathbf{d}})$. □

Corollary 13.40. *Theorem 13.38 also holds for A_n quivers with an arbitrary orientation.*

Proof. By Corollary 11.22, the algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is already determined, up to isomorphism, by the underlying graph of $\mathbf{\Gamma}$ and dimension vector $\underline{\mathbf{d}}$. In particular, the number of isomorphism classes of graded simple modules is independent of the choice of orientation. □

13.2.7 Composition series of a standard module

Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. From now on let us fix $\rho_\lambda \in \mathbb{O}_\lambda$ and set $K_\lambda = H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda))$. By Prop. 12.58, there is a $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$-module isomorphism $K_\lambda / (\mathrm{rad} < \ , \ >^{\widehat{S}_\lambda}) \cong L_\lambda$. Let us choose a vector space complement of $\mathrm{rad} < \ , \ >^{\widehat{S}_\lambda}$ in K_λ in such a way that this complement is $Y_{\underline{\mathbf{d}}}$-graded and denote it with $\widehat{L}_\lambda$. In other words, $\widehat{L}_\lambda$ is a complement of $\mathrm{rad} < \ , \ >^{\widehat{S}_\lambda}$ as a $\mathfrak{C}$-module.

Lemma 13.41. *The surjection $K_\lambda \twoheadrightarrow L_\lambda$ restricts to an isomorphism of $\mathfrak{C}$-modules $\widehat{L}_\lambda \cong L_\lambda$.*

Proof. This is obvious. □

Recall that for each $\overline{y} \in Y_{\underline{\mathbf{d}}}$, the map $\mu_{\overline{y}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ is the restriction of $\mu_{\underline{\mathbf{d}}}$ to the connected component $\widetilde{\mathcal{F}}_{\overline{y}}$.

Lemma 13.42. *Let $\overline{y} \in Y_{\underline{\mathbf{d}}}$. Then $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda))$ is the $\overline{y}$-weight space of $H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda))$, i.e.,*

$$(H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda))_{\overline{y}} = H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)).$$

Proof. This is also obvious. □

Lemma 13.43. *Let $\overline{y} \in \widehat{\Psi}(\lambda)$. Then $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) \subseteq \widehat{L}_\lambda$. In particular, $\widehat{\Psi}(\lambda) \subseteq$ $\mathrm{supp} L_\lambda$.*

Proof. Let S_λ be a transverse slice to $\mathbb{O}_\lambda$ at ρ_λ and let $\widehat{S}_\lambda := \mu_{\underline{\mathbf{d}}}^{-1}(S_\lambda)$. The slice S_λ inherits an algebraic stratification from $\mathrm{Rep}_{\underline{\mathbf{d}}}$. More specifically, let us set $S_\lambda^{\lambda'} := \mathbb{O}_{\lambda'} \cap S_\lambda$. Then

$$S_\lambda = \bigsqcup_{\lambda' \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})} S_\lambda^{\lambda'}.$$

By the definition of a transverse slice, we have $S_\lambda^\lambda = \{\rho_\lambda\}$ and $S_\lambda^{\lambda'} = \varnothing$ if $\lambda' < \lambda$.

Now recall that rad $< \, , \, >^{\widehat{S}_\lambda}$ is the kernel of the map $H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda)) \to H_*(\mu_{\underline{\mathbf{d}}}^{-1}(S_\lambda)) = H_*(\widehat{S}_\lambda)$ induced by the inclusion $\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda) \hookrightarrow \mu_{\underline{\mathbf{d}}}^{-1}(S_\lambda)$. Let $\overline{y} \in \widehat{\Psi}(\lambda)$. This means that flags in $\mathcal{F}_{\overline{y}}$ are only stabilized by (some) representations in $\mathbb{O}_\lambda$ and orbits lying in the closure of $\mathbb{O}_\lambda$. In particular, if $\lambda' > \lambda$ or λ' and λ are unrelated then $\mu_{\underline{\mathbf{d}}}^{-1}(\mathbb{O}_{\lambda'}) \cap \widetilde{\mathcal{F}}_{\overline{y}} = \varnothing$. This implies that

$$\mu_{\overline{y}}^{-1}(S_\lambda) = \mu_{\overline{y}}^{-1}(S_\lambda \cap \mathbb{O}_\lambda) = \mu_{\overline{y}}^{-1}(\rho_\lambda).$$

Therefore, the inclusion $\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda) \hookrightarrow \mu_{\underline{\mathbf{d}}}^{-1}(S_\lambda)$ restricts to the identity map $\mu_{\overline{y}}^{-1}(\rho_\lambda) \hookrightarrow \mu_{\overline{y}}^{-1}(S_\lambda)$. Hence the induced map on homology $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) \to H_*(\mu_{\overline{y}}^{-1}(S_\lambda))$ must also be the identity. Hence (rad $< \, , \, >^{\widehat{S}_\lambda}) \cap$ $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) = \{0\}$. Since $\widehat{L}_\lambda$ is a complement of (rad $< \, , \, >^{\widehat{S}_\lambda}$) as a $\mathfrak{C}$-module, we must have $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) \subseteq \widehat{L}_\lambda$. □

Proposition 13.44. *Suppose that $\lambda' < \lambda$ or that λ and λ' are unrelated. Then $[K_\lambda : L_{\lambda'}] = 0$. Moreover, $[K_\lambda : L_\lambda] = 1$.*

Proof. We first prove that $[K_\lambda : L_\lambda] = 1$. By Proposition 12.58, $[K_\lambda : L_\lambda] \geq 1$. Let $\overline{y} \in \widehat{\Psi}(\lambda)$. By Proposition 13.2.6 such a $\overline{y}$ exists. Lemma 13.43 implies that $H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) \subseteq \widehat{L}_\lambda$. Since $\widehat{L}_\lambda$ and L_λ are isomorphic as $\mathfrak{C}$-modules, $\dim(L_\lambda)_{\overline{y}} = \dim(\widehat{L}_\lambda)_{\overline{y}} = \dim H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda)) = \dim(K_\lambda)_{\overline{y}}$. Therefore, $[K_\lambda : L_\lambda] = 1$.

Now suppose that $\lambda' < \lambda$ or λ and λ' are unrelated. Let $\overline{y}' \in \widehat{\Psi}(\lambda')$. Then $\mu_{\underline{\mathbf{d}}}^{-1}(\rho_\lambda) \cap \widetilde{\mathcal{F}}_{\overline{y}'} = \varnothing$. Therefore $\overline{y}' \notin \mathrm{supp} K_\lambda$. But Lemma 13.43 implies that $\overline{y}' \in \mathrm{supp} L_{\lambda'}$. Hence $[K_\lambda : L_{\lambda'}] = 0$. □

We immediately obtain the folllowing corollary. It is a geometric analogue of [KR09, Corollary 7.5].

Corollary 13.45. *If λ is the unique maximal partition in $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$ then $K_\lambda \cong L_\lambda$.*

Let $c(\lambda, \lambda') = [K_\lambda : L_{\lambda'}]$. Each $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$-module $L_{\lambda'}$ occurring in the composition series of K_λ can be realized as a subquotient of K_λ. We can lift these subquotients to $\mathfrak{C}$-submodules of K_λ. These lifts are of course not necessarily unique. For each $\lambda' > \lambda$ let $\widehat{L}_{\lambda'}^1, ..., \widehat{L}_{\lambda'}^{c(\lambda,\lambda')}$ be the lifts of the $c(\lambda, \lambda')$-many copies of the simple module L_λ occurring in the composition series of K_λ. We have an equality of $\mathfrak{C}$-modules

$$K_\lambda = \bigoplus_{\lambda' \geq \lambda, 1 \leq l \leq c(\lambda,\lambda')} \widehat{L}_{\lambda'}^l.$$

13.2.8 Preparations for the inductive step

We collect some lemmas for a proposition which will play a crucial role in the inductive step of our main proof. Let $\widetilde{\sigma}_{\overline{y}}(l) = [\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}}] \in H_*(\mathcal{Z}_{\underline{\mathbf{d}}})$.

Lemma 13.46. *Suppose that $\overline{y}'$ is directly accessible from $\overline{y} \neq \overline{y}'$ and $\overline{y}' = s_l(\overline{y})$. Let $[Y] \in H_*(\mu_{\overline{y}}^{-1}(\rho_\lambda))$. Then $\widetilde{\sigma}_{\overline{y}}(l) \diamond [Y] = [\{(s_l.F, \rho_\lambda) \mid F \in \pi(Y)\}]$.*

Proof. Let us recall the convolution setup. The ambient manifold is $\widetilde{\mathcal{F}}_{\overline{y}'} \times \widetilde{\mathcal{F}}_{\overline{y}}$. By the clean intersection formula, we have

$$[\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}}] \cap [\widetilde{\mathcal{F}}_{\overline{y}'} \times Y] = e(\mathcal{T}).[\{((s_l.F, \rho_\lambda), (F, \rho_\lambda)) \mid F \in \pi(Y)\}]. \tag{108}$$

Let us abbreviate $\widetilde{Y} := \{((s_l.F, \rho_\lambda), (F, \rho_\lambda)) \mid F \in \pi(Y)\}$. Using this notation, $\mathcal{T}$ is the following vector bundle

$$\mathcal{T} = \frac{T(\widetilde{\mathcal{F}}_{\overline{y}'} \times \widetilde{\mathcal{F}}_{\overline{y}})|_{\widetilde{Y}}}{T(\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}})|_{\widetilde{Y}} + T(\widetilde{\mathcal{F}}_{\overline{y}'} \times Y)|_{\widetilde{Y}}}$$

over $\widetilde{Y}$. Let $y = ((s_l.\mathbf{F}, \rho_\lambda), (\mathbf{F}, \rho_\lambda)) \in \widetilde{Y}$ (note that this y has nothing to do with $\overline{y}$ and $\overline{y}'$ despite the similarity of notation). We have

$$\mathcal{T}|_{\{y\}} = \frac{T(\widetilde{\mathcal{F}}_{\overline{y}'})|_{\{(s_l.\mathbf{F},\rho_\lambda)\}} \oplus T(\widetilde{\mathcal{F}}_{\overline{y}})|_{\{(\mathbf{F},\rho_\lambda)\}}}{T(\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}})|_{\{y\}} + \Big(T(\widetilde{\mathcal{F}}_{\overline{y}'})|_{\{(s_l.\mathbf{F},\rho_\lambda)\}} \oplus T(Y)|_{\{(\mathbf{F},\rho_\lambda)\}}\Big)}. \tag{109}$$

Moreover,

$$\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}} = \{(s_l.F, F, \rho) \mid F \in \mathcal{F}_{\overline{y}}, F \text{ is } \rho\text{-stable}\}. \tag{110}$$

Let $\mathfrak{e} = \{\rho \mid \mathbf{F} \text{ is } \rho\text{-stable}\}$ and $\mathfrak{e}' = \{\rho \mid s_l.\mathbf{F} \text{ is } \rho\text{-stable}\}$. We have $\mathfrak{e} \subset \mathfrak{e}'$ by Lemma 13.21. The map

$$\widetilde{\mathcal{F}}_{\overline{y}} \to \widetilde{\mathcal{F}}_{\overline{y}'}, \quad F \mapsto s_l.F$$

induces a pushforward isomorphism on tangent bundles $T(\widetilde{\mathcal{F}}_{\overline{y}}) \to T(\widetilde{\mathcal{F}}_{\overline{y}'})$. Let us denote this isomorphism with ξ. It follows from (110) that

$$T(\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}})|_{\{y\}} = \{(\xi(v), v) \mid v \in T(\mathcal{F}_{\overline{y}})|_{\{\mathbf{F}\}}\} \oplus T(\mathfrak{e})|_{\{\rho_\lambda\}}. \tag{111}$$

Moreover,

$$T(\widetilde{\mathcal{F}}_{\overline{y}'})|_{\{(s_l.\mathbf{F},\rho_\lambda)\}} = T(\mathcal{F}_{\overline{y}'})|_{\{s_l.\mathbf{F}\}} \oplus T(\mathfrak{e}')|_{\{\rho_\lambda\}}. \tag{112}$$

Therefore

$$\begin{aligned} T(\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}})|_{\{y\}} \oplus T(\widetilde{\mathcal{F}}_{\overline{y}'})|_{\{(s_l.\mathbf{F},\rho_\lambda)\}} &= T(\mathcal{F}_{\overline{y}'})|_{\{s_l.\mathbf{F}\}} \oplus T(\mathcal{F}_{\overline{y}})|_{\{\mathbf{F}\}} \oplus T(\mathfrak{e}')|_{\{\rho_\lambda\}} \oplus T(\mathfrak{e})|_{\{\rho_\lambda\}} \\ &= T(\widetilde{\mathcal{F}}_{\overline{y}'})|_{\{(s_l.\mathbf{F},\rho_\lambda)\}} \oplus T(\widetilde{\mathcal{F}}_{\overline{y}})|_{\{(\mathbf{F},\rho_\lambda)\}}. \end{aligned}$$

It now follows directly from (109) that $\mathcal{T}|_{\{y\}} = \{0\}$. Therefore $\mathcal{T}$ is a trivial zero-dimensional vector bundle over $\widetilde{Y}$. In other words, $\mathcal{T}$ is isomorphic to $\widetilde{Y}$. Hence the Euler class of $\mathcal{T}$ is $1 \in H^*(\widetilde{Y})$. Therefore (108) reduces to $[\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}}] \cap [\widetilde{\mathcal{F}}_{\overline{y}'} \times Y] = [\widetilde{Y}]$. Let p_1 be the projection $p_1 : \widetilde{Y} \to \widetilde{\mathcal{F}}_{\overline{y}}$. Then

$$[\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}}] \diamond [Y] = (p_1)_* \Big([\mathcal{Z}^{s_l}_{\overline{y}',\overline{y}}] \cap [\widetilde{\mathcal{F}}_{\overline{y}'} \times Y]\Big) = (p_1)_*[\widetilde{Y}] = [p_1(\widetilde{Y})] = [\{(s_l.F, \rho_\lambda) \mid F \in \pi(Y)\}].$$

□

Lemma 13.47. *Let $\lambda' > \lambda$. Suppose that $\overline{y}' \in Y_{\underline{\mathbf{d}}}$ is a λ'-characteristic composition. Then for every flag $F \in \pi \circ \mu_{\overline{y}'}^{-1}(\rho_\lambda)$ there exists a simple transposition $s \in \Pi$ such that $\overline{y} := s(\overline{y}') \lhd \overline{y}'$ and $s.F$ is ρ_λ-stable, i.e., $s.F \in \pi \circ \mu_{\overline{y}}^{-1}(\rho_\lambda)$.*

Proof. Let $F \in \pi \circ \mu_{\overline{y}'}^{-1}(\rho_\lambda)$. Let us choose a $\rho_{\lambda'}$ such that $\rho_{\lambda'} \in \mathbb{O}_{\lambda'}$ and F is $\rho_{\lambda'}$. Let $r_{k,l}(\lambda) := \mathrm{rk}((\rho_\lambda)_{l-1} \circ ... \circ (\rho_\lambda)_k)$. By Proposition 13.13, this is independent of the choice of $\rho_\lambda \in \mathbb{O}_\lambda$. Since $\lambda' > \lambda$ there exist $l > k$ such that $r_{k,l}(\lambda') > r_{k,l}(\lambda)$. Hence there exists a one-dimensional subspace $D \subseteq \mathbf{V}_{i_k}$ such that $(\rho_\lambda)_{l-1} \circ ... \circ (\rho_\lambda)_k(D) = \{0\}$ but $(\rho_{\lambda'})_{l-1} \circ ... \circ (\rho_{\lambda'})_k(D) \neq \{0\}$. Letting $a = l - k$, we have, in particular, $(\rho_\lambda)^a(D) = \{0\}$ but $(\rho_{\lambda'})^a(D) \neq \{0\}$. Let W be the smallest subspace in the flag F such that $D \subset W$. Writing $F = (V^n)_{n=0}^{\mathbf{d}}$ suppose that $W = V^t = V^{t-1} \oplus D$. Since the flag F is ρ_λ-stable, we have

$$(\rho_\lambda)^a(W) = (\rho_\lambda)^a(V^{t-1}) + (\rho_\lambda)^a(D) = (\rho_\lambda)^a(V^{t-1}) \subseteq V^{t-a-1}.$$

Let b be the smallest natural number such that $(\rho_\lambda)^b(W) \subseteq V^{t-b-1}$. We have

$$V^{t-1} = V^{t-2} \oplus \rho_\lambda(D), \quad ..., \quad V^{t-b+1} = V^{t-b} \oplus (\rho_\lambda)^{b-1}(D).$$

Moreover, since $\rho_{\lambda'}(D) \neq \{0\}$ and $\overline{y}'$ is a λ'-characteristic composition, we have $\overline{y}'(t) = i_k, \overline{y}'(t-1) = i_{k+1}, ..., \overline{y}'(t-a) = i_{k+a} = i_l$.

We choose $s = s_{t-b}$. Since $b \leq a$ we have $(s(\overline{y}'))(t-b) = \overline{y}'(t-b+1) = i_{k+b-1} > i_{k+b} = \overline{y}'(t-b)$, so $s(\overline{y}') \succ \overline{y}'$ in the lexicographic ordering. Hence $s(\overline{y}') \lhd \overline{y}'$ in the accessibility ordering. We now show that $s.F$ is ρ_λ-stable. We write $s.F = (U^n)$. It is clear that $U^n = V^n$ unless $n = t - b$. We need to check that $\rho_\lambda(U^{t-b+1}) \subset U^{t-b}$ and $\rho_\lambda(U^{t-b}) \subset U^{t-b-1}$. We have

$$\begin{aligned}\rho_\lambda(U^{t-b+1}) = \rho_\lambda(V^{t-b+1}) &= \rho_\lambda(V^{t-b} \oplus (\rho_\lambda)^{b-1}(D)) \\ &\subset \rho_\lambda(V^{t-b}) + (\rho_\lambda)^b(D) \subset V^{t-b-1} = U^{t-b-1}\end{aligned}$$

because $(\rho_\lambda)^b(W) \subseteq V^{t-b-1}$ by our choice of b. Since $U^{t-b} \subset U^{t-b+1}$ we also have $\rho_\lambda(U^{t-b}) \subset U^{t-b-1}$. □

Proposition 13.48. *Let $\lambda' > \lambda$ and let $\overline{y}'$ be a λ'-characteristic composition. Then $\bigoplus_{\lambda \leq \lambda'' < \lambda'} H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda'')}(\rho_\lambda)\right) \subset K_\lambda$ generates, under the action of $H_*^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$, the homology group $H_*(\mu_{\overline{y}'}^{-1}(\rho_\lambda)) \subset K_\lambda$.*

Proof. Let $\Pi_{\overline{y}'}$ denote the set of all simple transpositions s in Π such that $s(\overline{y}') \lhd \overline{y}'$ and $\mu^{-1}_{s(\overline{y}')}(\rho_\lambda) \neq \{0\}$. If $s \in \Pi_{\overline{y}'}$ then representations in $\mathbb{O}_{\lambda'}$ do not stabilize any flags in $\mathcal{F}_{s(\overline{y}')}$ because $\overline{y}'$ is a λ'-characteristic composition. Hence the greatest element λ'' in $\Phi(s(\overline{y}'))$ must satisfy $\lambda'' < \lambda'$ (it is not possible that λ'' and λ' are unrelated - in that case $\phi(\overline{y}')$ would contain both λ' and λ'', and so would not contain a greatest element, contradicting Lemma 13.22). It follows that if $s \in \Pi_{\overline{y}'}$ then $s(\overline{y}') \in \widehat{\Psi}(\lambda'')$ for some $\lambda \leq \lambda'' < \lambda$.

For each $s \in \Pi_{\overline{y}'}$ let us define a map

$$\zeta_s : \widetilde{\mathcal{F}}_{s(\overline{y}')} \to \widetilde{\mathcal{F}}_{\overline{y}'}, \quad (F, \rho) \mapsto (s.F, \rho).$$

This map is clearly an isomorphism of varieties. Let us also define $X_s = \mu^{-1}_{s(\overline{y}')}(\rho_\lambda)$. By Lemma 13.21, $\zeta_s(X_s) \subseteq \mu^{-1}_{\overline{y}'}(\rho_\lambda)$. Moreover, by Lemma 13.47,

$$\bigcup_{s \in \Pi_{\overline{y}'}} \zeta_s(X_s) = \mu^{-1}_{\overline{y}'}(\rho_\lambda).$$

Observe that X_s is a closed subvariety of $\widetilde{\mathcal{F}}_{s(\overline{y}')}$. Hence $\zeta_s(X_s)$ is a closed subvariety of $\mu^{-1}_{\overline{y}'}(\rho_\lambda)$. Let $\widetilde{\Pi_{\overline{y}'}}$ be a subset of $\Pi_{\overline{y}'}$ such that

$$\bigcup_{s \in \widetilde{\Pi_{\overline{y}'}}} \zeta_s(X_s) = \mu^{-1}_{\overline{y}'}(\rho_\lambda) \quad \text{and} \quad \bigcup_{s \in A} \zeta_s(X_s) \neq \mu^{-1}_{\overline{y}'}(\rho_\lambda)$$

for any proper subset A of $\widetilde{\Pi_{\overline{y}'}}$. It follows that $\bigcup_{s\in\widetilde{\Pi_{\overline{y}'}}} \zeta_s(X_s) = \mu_{\overline{y}'}^{-1}(\rho_\lambda)$ is a decomposition of $\mu_{\overline{y}'}^{-1}(\rho_\lambda)$ into (possibly unions of) irreducible components. We have a corresponding decomposition in homology

$$\sum_{s\in\widetilde{\Pi_{\overline{y}'}}} H_*(\zeta_s(X_s)) = H_*(\mu_{\overline{y}'}^{-1}(\rho_\lambda)). \tag{113}$$

Note that the sum is not necessarily direct. Decomposition (113) implies that in order to generate $H_*(\mu_{\overline{y}'}^{-1}(\rho_\lambda))$ it suffices to generate each homology group $H_*(\zeta_s(X_s))$. Let $s = s_l \in \widetilde{\Pi_{\overline{y}'}}$. If Y is a closed subvariety of X_s then $\widetilde{\sigma}_{s_l(\overline{y}')}(l) \diamond [Y] = [\zeta_s(Y)]$, by Lemma 13.46 . But $H_*(X_s)$ can be given a basis consisting of fundamental classes of closed subvarieties, so the map

$$\widetilde{\sigma}_{s_l(\overline{y}')}(l) : H_*(X_s) \to H_*(\zeta_s(X_s))$$

takes a basis of $H_*(X_s)$ to a basis of $H_*(\zeta_s(X_s))$, i.e., it is a linear isomorphism. In particular, $H_*(\zeta_s(X_s))$ can be generated from $H_*(X_s)$ under the action of the element $\sigma_{s_l(\overline{y}')}(l) = 1 \otimes \widetilde{\sigma}_{s_l(\overline{y}')}(l) \in H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. Finally, it follows from the remarks at the very beginning of the proof that

$$H_*(X_s) \subseteq \bigoplus_{\lambda\leq\lambda''<\lambda'} H_*\left(\mu_{\widehat{\Psi}(\lambda'')}^{-1}(\rho_\lambda)\right).$$

This completes the proof.

□

Example 13.49. Let us consider the quiver

$$i \to j \to k$$

with dimension vector $\underline{\mathbf{d}} = 2i+2j+2k$. Let us fix a complex graded vector space $\mathbf{V} = \mathbf{V}_i \oplus \mathbf{V}_j \oplus \mathbf{V}_k$ with $\dim \mathbf{V}_i = 2, \dim \mathbf{V}_j = 2$ and $\dim \mathbf{V}_k = 2$. The Lie algebra corresponding to the quiver is $\mathfrak{sl}_4$. Let α, β, γ be simple roots corresponding to the vertices i, j, k, respectively. We define the following partitions $\lambda_1 = (\alpha+\beta, \alpha+\beta, \gamma, \gamma)$, $\lambda_2 = (\alpha, \alpha+\beta, \beta, \gamma, \gamma)$ of $\underline{\mathbf{d}}$. Let us choose a representation $\rho \in \mathbb{O}_{\lambda_2}$. We consider the λ_1-characteristic composition $jijikk$. Note that this composition has two immediate predecessors in the accessibility ordering:

$$\begin{tikzcd}
& jijikk \arrow[dl] \arrow[dr] & \\
ijjikk & & jiijkk.
\end{tikzcd}$$

We now want to determine the fibre $\mu_{jijikk}^{-1}(\rho)$.

Note that to describe a flag $F \in \mathcal{F}_{jijikk}$ uniquely it suffices to give one-dimensional subspaces D_j, D_i, D_k of $\mathbf{V}_j$, $\mathbf{V}_i$ and $\mathbf{V}_k$, respectively. The corresponding flag F is $(D_j, D_j \oplus D_i, \mathbf{V}_j \oplus D_i, \mathbf{V}_j \oplus \mathbf{V}_i, \mathbf{V}_j \oplus \mathbf{V}_i \oplus D_k, \mathbf{V})$. To simplify notation we will therefore identify every flag with the corresponding triple (D_j, D_i, D_k). We will use similar notation for flags in other connected components we consider.

Let us write $\rho = (\rho_a, \rho_b)$, where $\rho_a : \mathbf{V}_i \to \mathbf{V}_j$ and $\rho_b : \mathbf{V}_j \to \mathbf{V}_k$. We have

$$\begin{aligned}
&\pi_{\underline{\mathbf{d}}}\left(\mu_{jijikk}^{-1}(\rho)\right) = \\
&= \{(D_j, \ker\rho_a, D_k) \mid D_j \in \mathbb{P}(\mathbf{V}_j), D_k \in \mathbb{P}(\mathbf{V}_k)\} \cup \{(\operatorname{Im}\rho_a, D_i, D_k) \mid D_i \in \mathbb{P}(\mathbf{V}_j), D_k \in \mathbb{P}(\mathbf{V}_k)\} \\
&\cong (\mathbb{CP}^1 \vee \mathbb{CP}^1) \times \mathbb{CP}^1.
\end{aligned}$$

We now determine the fibres $\mu^{-1}_{ijjikk}(\rho)$ and $\mu^{-1}_{jiijkk}(\rho)$. We have

$$\begin{aligned}\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{ijjikk}(\rho)\right) &= \{(\ker\rho_a, D_j, D_k) \mid D_j \in \mathbb{P}(\mathbf{V}_j), D_k \in \mathbb{P}(\mathbf{V}_k)\}\\ &\cong \mathbb{CP}^1 \times \mathbb{CP}^1.\end{aligned}$$

$$\begin{aligned}\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jiijkk}(\rho)\right) &= \{(\operatorname{Im}\rho_a, D_i, D_k) \mid D_i \in \mathbb{P}(\mathbf{V}_j), D_k \in \mathbb{P}(\mathbf{V}_k)\}\\ &\cong \mathbb{CP}^1 \times \mathbb{CP}^1.\end{aligned}$$

Now let $s_1 = (12)(3)(4)(5)(6)$ and $s_3 = (1)(2)(34)(5)(6)$ be simple transpositions in $\mathfrak{S}_6$. Then

$$s_1.\left(\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{ijjikk}(\rho)\right)\right) \bigcup s_3.\left(\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jiijkk}(\rho)\right)\right) = \pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jijikk}(\rho)\right)$$

is a decomposition of $\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jijikk}(\rho)\right)$ into irreducible components. We have a corresponding decomposition in homology

$$H_*\left(s_1.\left(\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{ijjikk}(\rho)\right)\right)\right) + H_*\left(s_3.\left(\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jiijkk}(\rho)\right)\right)\right) = H_*\left(\pi_{\underline{\mathbf{d}}}\left(\mu^{-1}_{jijikk}(\rho)\right)\right).$$

13.2.9 The main argument

We will argue by induction on the poset $\{\lambda' \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}}) \mid \lambda' \geq \lambda\}$. This poset has a least and greatest element. The least element is λ, the greatest element is the same as the greatest element in $\Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. We have an equality of $\mathfrak{E}$-modules

$$K_\lambda = H_*(\mu^{-1}_{\underline{\mathbf{d}}}(\rho_\lambda)) = \bigoplus_{\lambda' \geq \lambda} H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right).$$

The idea of the proof is to show inductively that each subspace $H_*(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda))$ can be generated (under the action of $H^{G_{\underline{\mathbf{d}}}}_*(\mathcal{Z}_{\underline{\mathbf{d}}})$) from $\widehat{L}_\lambda$.

Theorem 13.50. *Let $\lambda \in \Lambda(\mathbf{\Gamma}, \underline{\mathbf{d}})$. The standard module K_λ is generated by the subspace $\widehat{L}_\lambda$. Therefore, K_λ is indecomposable and has simple head L_λ.*

Proof. By Lemma 13.43, $H_*(\mu^{-1}_{\widehat{\Psi}(\lambda)}(\rho_\lambda)) \subseteq \widehat{L}_\lambda$. Therefore, $H_*(\mu^{-1}_{\widehat{\Psi}(\lambda)}(\rho_\lambda))$ can (trivially) be generated from $\widehat{L}_\lambda$. Moreover, by Proposition 13.2.6, we know that $\widehat{\Psi}(\lambda)$ is non-empty. Therefore, $\mu^{-1}_{\widehat{\Psi}(\lambda)}(\rho_\lambda) \neq \varnothing$ and $H_*(\mu^{-1}_{\widehat{\Psi}(\lambda)}(\rho_\lambda)) \neq \{0\}$.

Now let $\lambda' > \lambda$ and inductively assume that for all $\lambda \leq \lambda'' < \lambda'$ and $1 \leq l \leq c(\lambda, \lambda'')$ the subspaces $\widehat{L}^l_{\lambda''}$ and $H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda'')}(\rho_\lambda)\right)$ have already been generated. We want to generate $H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right)$ and $\widehat{L}^l_{\lambda'}$.

We first show that we can generate all the $\widehat{L}^l_{\lambda'}$. Let $\overline{y}$ be a λ'-characteristic composition. By Proposition 13.48, we can generate $H_*\left(\mu^{-1}_{\overline{y}}(\rho_\lambda)\right)$. Since $\overline{y} \in \widehat{\Psi}(\lambda') \subseteq \operatorname{supp} L_{\lambda'}$, we know that each $\widehat{L}^l_{\lambda'}$ intersects $H_*\left(\mu^{-1}_{\overline{y}}(\rho_\lambda)\right)$ non-trivially. But $\widehat{L}^l_{\lambda'}$ is a lift of a simple module, so it is generated by any non-trivial subspace. Hence we can generate all the subspaces $\widehat{L}^l_{\lambda'}$.

We now show that we can generate $H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right)$. The vector space $H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right)$ can only non-trivially intersect some $\widehat{L}^l_{\lambda''}$ if $\lambda \leq \lambda'' \leq \lambda'$ because $\operatorname{supp} L^l_{\widehat{L}_{\lambda''}} = \operatorname{supp} L_{\lambda''} \subseteq \operatorname{supp} K_{\lambda''} =$

$\Psi(\lambda'')$ and $\widehat{\Psi}(\lambda') \subseteq \Psi(\lambda'')$ only if $\lambda' \geq \lambda''$. For each $\lambda \leq \lambda'' \leq \lambda'$ and $1 \leq l \leq c(\lambda, \lambda'')$ let $H_*^{(\lambda'',l)}\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right) = H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right) \cap \widehat{L}^l_{\lambda''}$. We have an equality of vector spaces

$$H_*\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right) = \bigoplus_{\lambda \leq \lambda'' \leq \lambda', 1 \leq l \leq c(\lambda, \lambda'')} H_*^{(\lambda'',l)}\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right).$$

By induction, we can assume to have already generated $H_*^{(\lambda'',l)}\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right)$ for $\lambda \leq \lambda'' < \lambda'$, $1 \leq l \leq c(\lambda, \lambda'')$. We can also generate the subspaces $H_*^{(\lambda',l)}\left(\mu^{-1}_{\widehat{\Psi}(\lambda')}(\rho_\lambda)\right)$, for $1 \leq l \leq c(\lambda, \lambda')$, because we have already generated the subspaces $\widehat{L}^l_{\lambda'}$. This completes the inductive step.

It follows that $\widehat{L}_\lambda$ generates all the lifts $\widehat{L}^l_{\lambda'}$ of the simple modules in the composition series of K_λ, so it generates all of K_λ. Therefore, K_λ is indecomposable and has simple head L_λ. $\square$

Let us finish with a simple but non-trivial example which brings together the various threads of our argument.

Example 13.51. Let us consider the quiver

$$i \to j$$

with dimension vector $\underline{\mathbf{d}} = 2i + 2j$. Let us fix a complex graded vector space $\mathbf{V} = \mathbf{V}_i \oplus \mathbf{V}_j$ with $\dim \mathbf{V}_i = 2, \dim \mathbf{V}_j = 2$. The Lie algebra corresponding to the quiver is $\mathfrak{sl}_3$. Let α, β be the simple roots corresponding to the vertices i, j, respectively. There are three partitions of $\underline{\mathbf{d}}$: $\lambda_0 = (\alpha, \alpha, \beta, \beta)$, $\lambda_1 = (\alpha, \alpha+\beta, \beta)$, $\lambda_2 = (\alpha+\beta, \alpha+\beta)$. The closure ordering on these partitions is $\lambda_2 > \lambda_1 > \lambda_0$. We have $\mathrm{Rep}_{\underline{\mathbf{d}}} \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \cong \mathbb{C}^4$. The orbit $\mathbb{O}_{\lambda_m}$ consists of representations of rank m, for $0 \leq m \leq 2$. The following diagram illustrates the ordering on the weights:

$$\begin{tikzcd}
& jjii \arrow[d] & \\
& jiji \arrow[dl] \arrow[dr] & \\
ijji \arrow[dr] & & jiij \arrow[dl] \\
& ijij \arrow[d] & \\
& iijj &
\end{tikzcd}$$

where $\widehat{\Psi}(\lambda_0)$ is designated with red colour, $\widehat{\Psi}(\lambda_1)$ with green colour and $\widehat{\Psi}(\lambda_2)$ with blue colour. For each $\overline{y} \in Y_{\underline{\mathbf{d}}}$ we have $\mathcal{F}_{\overline{y}} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. Moreover, $\widetilde{\mathcal{F}}_{iijj} \cong \mathcal{F}_{iijj}$ and $\widetilde{\mathcal{F}}_{ijij}$ is a vector bundle over $\mathcal{F}_{ijij}$ of rank one, $\widetilde{\mathcal{F}}_{ijji}$ and $\widetilde{\mathcal{F}}_{jiij}$ are vector bundles over $\mathcal{F}_{ijji}$ resp. $\mathcal{F}_{jiij}$ of rank two, $\widetilde{\mathcal{F}}_{jiji}$ is a vector bundle over $\mathcal{F}_{jiji}$ of rank three and $\widetilde{\mathcal{F}}_{jjii}$ is a vector bundle over $\mathcal{F}_{jjii}$ of rank four.

Suppose that $F = (D^1, D^1 \oplus D^2, D^1 \oplus D^2 \oplus D^3, D^1 \oplus D^2 \oplus D^3 \oplus D^4) \in \mathcal{F}_{\underline{\mathbf{d}}}$ is a flag, where each D^k is a one-dimensional subspace of $\mathbf{V}_i$ or $\mathbf{V}_j$. Let m_i be the lowest integer such that $D^{m_i} \subset \mathbf{V}_i$ and let m_j be the lowest integer such that $D^{m_j} \subset \mathbf{V}_j$. Since the vector spaces $\mathbf{V}_i, \mathbf{V}_j$ are two-dimensional, the flag F is already determined uniquely by D^{m_i} and D^{m_j}. To simplify the notation in what follows we will therefore identify the flag F with the tuple (D^{m_i}, D^{m_j}) if $m_i < m_j$ or the tuple (D^{m_j}, D^{m_i}) if $m_i > m_j$.

Let ρ_0 be the unique representation of rank zero and let us choose $\rho_1 \in \mathbb{O}_{\lambda_1}$ and $\rho_2 \in \mathbb{O}_{\lambda_2}$. We describe the fibers of the map $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ in the table below. It should be read in the following manner. The entry in row $\overline{y} = ijij$ and column ρ_1, for example, is the fibre $\pi_{\underline{\mathbf{d}}}(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_1))$ restricted to the connected component $\mathcal{F}_{ijij}$.

$\overline{y}$	ρ_0	ρ_1	ρ_2
iijj	$\mathcal{F}_{iijj}$	$\varnothing$	$\varnothing$
ijij	$\mathcal{F}_{ijij}$	$\{(\ker\rho_1, \mathrm{Im}\,\rho_1)\} \cong \{pt\}$	$\varnothing$
ijji	$\mathcal{F}_{ijji}$	$\{(\ker\rho_1, D) \mid D \in \mathbb{P}(\mathbf{V}_j)\} \cong \mathbb{CP}^1$	$\varnothing$
jiij	$\mathcal{F}_{jiij}$	$\{(\mathrm{Im}\,\rho_1, D) \mid D \in \mathbb{P}(\mathbf{V}_i)\} \cong \mathbb{CP}^1$	$\varnothing$
jiji	$\mathcal{F}_{jiji}$	$\{(\mathrm{Im}\,\rho_1, D) \mid D \in \mathbb{P}(\mathbf{V}_i)\} \vee \{(D, \ker\rho_1) \mid D \in \mathbb{P}(\mathbf{V}_j)\} \cong \mathbb{CP}^1 \vee \mathbb{CP}^1$	$\{(D, \rho_2^{-1}(D)) \mid D \in \mathbb{P}(\mathbf{V}_j)\} \cong \mathbb{CP}^1$
jjii	$\mathcal{F}_{jjii}$	$\mathcal{F}_{jjii}$	$\mathcal{F}_{jjii}$

Note that $\pi_{\underline{\mathbf{d}}}(\mu_{jiji}^{-1}(\rho_1)) \cong \mathbb{CP}^1 \vee \mathbb{CP}^1$ is not smooth and is the wedge sum of $s_1.\big(\pi_{\underline{\mathbf{d}}}(\mu_{ijji}^{-1}(\rho_1))\big)$ and $s_3.\big(\pi_{\underline{\mathbf{d}}}(\mu_{jiij}^{-1}(\rho_1))\big)$, where $s_1 = (12)(3)(4), s_3 = (1)(2)(34)$ are simple transpositions in $\mathfrak{S}_4$.

We are now going to work out the composition series of the standard modules, compute their dimensions, compute the dimensions of the corresponding simple modules as well as the dimensions of the different weight spaces.

There are three simple graded modules over the KLR algebra $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$. Since λ_2 is the top partition, the simple module $L(\lambda_2)$ is isomorphic (up to a shift in the grading) to the standard module $K(\lambda_2) = H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_2)) \cong H_*(\mathbb{CP}^1) \oplus H_*(\mathbb{CP}^1 \times \mathbb{CP}^1)$. In particular, it follows that $L(\lambda_2)$ is six-dimensional. The $jjii$-weight space $L(\lambda_2)_{jjii}$ is four-dimensional and the $jiji$-weight space $L(\lambda_2)_{jiji}$ is two-dimensional.

Let us now consider the standard module $K(\lambda_1)$. We have an isomorphism $K(\lambda_1) = H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_1)) \cong H_*(\{pt\}) \oplus H_*(\mathbb{CP}^1) \oplus H_*(\mathbb{CP}^1) \oplus H_*(\mathbb{CP}^1 \vee \mathbb{CP}^1) \oplus H_*(\mathbb{CP}^1 \times \mathbb{CP}^1)$. It follows that $K(\lambda_1)$ is twelve-dimensional. Since $[K(\lambda_1) : L(\lambda_1)] = 1$ it follows from dimension considerations ($\dim L(\lambda_1) + x \cdot 6 = \dim L(\lambda_1) + x \cdot \dim L(\lambda_2) = \dim K(\lambda_1) = 12$ for some $x \geq 0$) that $[K(\lambda_1) : L(\lambda_2)] = 1$ and that $L(\lambda_1)$ is a six-dimensional simple module. The $ijij$-weight space $L(\lambda_1)_{ijij}$ is one-dimensional, the weight spaces $L(\lambda_1)_{ijji}$ and $L(\lambda_1)_{jiij}$ are two-dimensional and the $jiji$-weight space $L(\lambda_1)_{jiji}$ is one-dimensional.

Finally, let us consider the standard module $K(\lambda_0)$. We have an isomorphism $K(\lambda_0) = H_*(\mu_{\underline{\mathbf{d}}}^{-1}(\rho_0)) \cong H_*(\mathcal{F}_{\underline{\mathbf{d}}}) \cong \big(H_*(\mathbb{CP}^1 \times \mathbb{CP}^1)\big)^{\oplus 6}$. Therefore, the standard module $K(\lambda_0)$ is 24-dimensional. We can now analyze the dimensions of the weight spaces of the simple modules $L(\lambda_1)$ and $L(\lambda_2)$ to conclude that $[K(\lambda_0) : L(\lambda_1)] = 2$ and $[K(\lambda_0) : L(\lambda_2)] = 1$. This implies that $L(\lambda_0)$ is six-dimensional. The $iijj$-weight space $L(\lambda_0)_{iijj}$ is four-dimensional and the $ijij$-weight space $L(\lambda_0)_{ijij}$ is two-dimensional.

14 Notation

For the reader's convenience we collect some of the frequently used notations here.

1. Homology and cohomology

- EG - the universal principal G-bundle (unique up to homotopy)
- $BG = EG/G$ - the classifying space for principal G-bundles
- $EG \times^G M := (EG \times M)/G$ - the homotopy quotient of a manifold M by a group G
- $\mathrm{Gr}(n, m)$ - the Grassmannian of linear n-dimensional subspaces of $\mathbb{C}^m$
- $H^*(M)$ - singular cohomology ring of M
- $H^*_G(M) := H^*(EG \times^G M)$ - the equivariant cohomology ring of M
- $S_G := H^*_G(\{pt\}) = H^*(BG)$
- $K_G := \mathrm{Frac}(S_G)$
- $E^nG \to B^nG$ - an n-th approximation to the universal principal G-bundle $EG \to BG$
- $H^{sing}_*(M)$ - singular homology of M
- $H_*(M)$ - Borel-Moore homology of M
- $H^G_*(M)$ - G-equivariant Borel-Moore homology of M

2. Quivers

- $\mathbf{\Gamma} = (\mathbf{I}, \mathbf{H}, \mathbf{s}, \mathbf{t})$ - a quiver (vertices, arrows, source function, target function)
- $\underline{\mathbf{d}} = (\mathbf{d}_i)_{i \in \mathbf{I}}$ - dimension vector for $\mathbf{\Gamma}$
- $\mathbf{V} = \bigoplus_{i \in \mathbf{I}} \mathbf{V}_i$ - fixed vector space with $\mathrm{grdim}\, \mathbf{V} = \underline{\mathbf{d}}$
- $\mathrm{Rep}_{\underline{\mathbf{d}}} := \bigoplus_{h \in \mathbf{H}} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_{\mathbf{s}(h)}, \mathbf{V}_{\mathbf{t}(h)})$
- $G_{\mathbf{d}_i} := \mathrm{GL}(\mathbf{V}_i)$
- $G_{\underline{\mathbf{d}}} := \prod_{i \in \mathbf{I}} \mathrm{GL}(\mathbf{V}_i) = \prod_{i \in \mathbf{I}} G_{\mathbf{d}_i}$
- $\mathbb{G}_{\mathbf{d}} := \mathrm{GL}(\mathbf{V})$
- $\overline{y} = (y^1, ..., y^k)$ - a composition of $\mathbf{d}$
- $\overline{y} = (\underline{y}^1, ..., \underline{y}^k)$ - a composition of $\underline{\mathbf{d}}$
- $\mathrm{Comp}_{\mathbf{d}}$ - the set of compositions of the natural number $\mathbf{d}$
- $\mathrm{Comp}_{\underline{\mathbf{d}}}$ - the set of compositions of the vector $\underline{\mathbf{d}}$
- $Y_{\underline{\mathbf{d}}}$ - the set of strictly multiplicity-free compositions of $\underline{\mathbf{d}}$

3. Flag varieties

- F - generic notation for flags
- $\mathcal{F}_{\overline{y}}$ - the quiver flag variety of type $\overline{y}$
- $\mathcal{F}(V)$ - the ordinary flag variety of all complete flags in V
- $\mathcal{F}_{\underline{\mathbf{d}}} = \coprod_{\overline{y} \in Y_{\underline{\mathbf{d}}}} \mathcal{F}_{\overline{y}}$ - the quiver flag variety
- $F|_i = F \cap \mathbf{V}_i$ - restriction of a flag F to the graded component $\mathbf{V}_i$ of $\mathbf{V}$
- $\hat{F}$ - the ungraded flag associated to a quiver-graded flag F

- $Z(G_{\underline{\mathbf{d}}})$ - the centre of $G_{\underline{\mathbf{d}}}$

4. Weyl groups and torus fixed points
   - $(i_1, ..., i_{|\mathbf{I}|})$ - a fixed chosen ordering of the vertices in $\mathbf{I}$
   - $(e_{i_1}^1, ..., e_{i_1}^{\mathbf{d}_1}, ..., e_{i_{|\mathbf{I}|}}^1, ..., e_{i_{|\mathbf{I}|}}^{\mathbf{d}_{|\mathbf{I}|}})$ - a fixed chosen basis of $\mathbf{V}$
   - $T_{\underline{\mathbf{d}}}$ - the subgroup of diagonal matrices in $G_{\underline{\mathbf{d}}}$ wrt. the chosen basis
   - $B_{\underline{\mathbf{d}}}$ - the subgroup of upper triangular matrices in $G_{\underline{\mathbf{d}}}$ wrt. the chosen basis
   - $\mathbb{B}_{\mathbf{d}}$ - the subgroup of upper triangular matrices in $\mathbb{G}_{\mathbf{d}}$ wrt. the chosen basis
   - $\mathbb{W}_{\mathbf{d}} := N_{\mathbb{G}_{\mathbf{d}}}(T_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}}$
   - $W_{\underline{\mathbf{d}}} := N_{G_{\underline{\mathbf{d}}}}(T_{\underline{\mathbf{d}}})/T_{\underline{\mathbf{d}}}$
   - $\mathfrak{S}_n$ - the symmetric group on n letters
   - $\Pi = \{s_1, ..., s_{\mathbf{d}}\}$ - the set of simple transpositions in $\mathbb{W}_{\mathbf{d}}$
   - $\Pi_{\underline{\mathbf{d}}} = \Pi \cap W_{\underline{\mathbf{d}}}$ - the set of simple transpositions in $W_{\underline{\mathbf{d}}}$
   - $l : \mathbb{W}_{\mathbf{d}} \to \mathbb{N}_{\geq 0}$ - the length function
   - F_e - the standard coordinate flag
   - $\mathrm{Min}(\mathbb{W}_{\mathbf{d}}, W_{\underline{\mathbf{d}}})$ - the set of minimal length right coset representatives
   - $F_w := w(F_e)$
   - $\overline{y}_e$ - the type of the standard coordinate flag
   - $\overline{y}_w$ - the type of the coordinate flag F_w
   - $\mathcal{F}_{\overline{w}} := \mathcal{F}_{\overline{y}_w}$
5. Algebraic groups and Lie algebras
   - $B_w := \mathrm{Stab}_{G_{\underline{\mathbf{d}}}}(F_w)$
   - $\mathbb{B}_w := \mathrm{Stab}_{\mathbb{G}_{\mathbf{d}}}(F_w)$
   - $N_w := R_u(B_w)$ - the unipotent radical of B_w
   - $P_{w,ws} := (B_w wsw^{-1} B_w) \cup B_w$
   - $N_{w,ws} := R_u(P_{w,ws})$ - the unipotent radical of $P_{w,ws}$
   - $\mathfrak{g}_{\underline{\mathbf{d}}} := \mathrm{Lie}(G_{\underline{\mathbf{d}}})$
   - $\mathfrak{g} := \mathrm{Lie}(\mathbb{G}_{\mathbf{d}})$
   - $\mathfrak{t}_{\underline{\mathbf{d}}} := \mathrm{Lie}(T_{\underline{\mathbf{d}}})$
   - $\mathfrak{b}_w := \mathrm{Lie}(B_w)$
   - $\mathfrak{n}_w := \mathrm{Lie}(N_w)$
   - $\mathfrak{p}_{w,ws} := \mathrm{Lie}(P_{w,ws})$
   - $\mathfrak{n}_{w,ws} := \mathrm{Lie}(N_{w,ws})$
   - $\mathfrak{m}_{w,ws} := \mathfrak{n}_w/\mathfrak{n}_{w,ws}$
6. Root systems
   - Δ - the set of roots of $\mathfrak{g}$ wrt. $\mathfrak{t}_{\underline{\mathbf{d}}}$

- $\mathfrak{g}_\alpha$ - the root space of weight α
- χ_j - the weight of the $\mathfrak{t}_{\underline{\mathbf{d}}}$-module $\mathbb{C}.e_j$
- $\Delta^1 := \{\chi_j - \chi_{j+1} \mid 1 \leq j \leq d-1\}$ - the base of Δ
- $\beta_j := \chi_j - \chi_{j+1}$ - simple root
- Δ^+ - the set of positive roots in Δ wrt. base Δ^1
- Δ^- - the set of negative roots in Δ wrt. base Δ^1
- $Д(\mathfrak{h}) := \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{h}\}$
- $\Delta_{\underline{\mathbf{d}}}$ - the set of roots of $\mathfrak{g}_{\underline{\mathbf{d}}}$ wrt. $\mathfrak{t}_{\underline{\mathbf{d}}}$
- $\Delta^+_{\underline{\mathbf{d}}} := \Delta_{\underline{\mathbf{d}}} \cap \Delta^+$
- $\Delta^-_{\underline{\mathbf{d}}} := -\Delta^+_{\underline{\mathbf{d}}} = \Delta_{\underline{\mathbf{d}}} \cap \Delta^-$
- $\Delta^1_{\underline{\mathbf{d}}} := \Delta_{\underline{\mathbf{d}}} \cap \Delta^1$

7. The Steinberg variety

- $\widetilde{\mathcal{F}}_{\overline{y}} := \{(\rho, F) \mid F \text{ is } \rho\text{-stable}\} \subset \mathrm{Rep}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\overline{y}}$ - the incidence variety of type $\overline{y}$
- $\widetilde{\mathcal{F}}_w := \widetilde{\mathcal{F}}_{\overline{y}_w}$
- $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} := \coprod_{\overline{y} \in Y_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}}$
- $\mu_{\overline{y}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ - first projection
- $\pi_{\overline{y}} : \widetilde{\mathcal{F}}_{\overline{y}} \to \mathcal{F}_{\overline{y}}$ - second projection
- $\gamma(\overline{y}) := \dim_{\mathbb{C}}(\widetilde{\mathcal{F}}_{\overline{y}})$
- $\mu_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ - first projection
- $\pi_{\underline{\mathbf{d}}} : \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}$ - second projection
- $\mathfrak{r}_w := \{\rho \in \mathrm{Rep}_{\underline{\mathbf{d}}} \mid F_w \text{ is } \rho\text{-stable}\}$
- $\mathfrak{r}_{w,w'} := \mathfrak{r}_w \cap \mathfrak{r}_{w'}$
- $\mathfrak{d}_{w,w'} = \mathfrak{r}_w / \mathfrak{r}_{w,w'}$
- $\widetilde{p}_l : \mathcal{O}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l) \to \widetilde{\mathcal{F}}_{\overline{y}}$ - the l-th canonical line bundle over $\widetilde{\mathcal{F}}_{\overline{y}}$
- $x_{\overline{y}}(l) := c_1(\mathcal{O}^{G_{\mathbf{d}}}_{\widetilde{\mathcal{F}}_{\overline{y}}}(l))$
- $\mathcal{Z}_{\overline{y},\overline{y}'} := \widetilde{\mathcal{F}}_{\overline{y}} \times_{\mathrm{Rep}_{\underline{\mathbf{d}}}} \widetilde{\mathcal{F}}_{\overline{y}'} \subset \widetilde{\mathcal{F}}_{\overline{y}} \times \widetilde{\mathcal{F}}_{\overline{y}'}$ - the Steinberg variety of type $(\overline{y}, \overline{y}')$
- $\mathcal{Z}_{\underline{\mathbf{d}}} := \coprod_{\overline{y},\overline{y}' \in Y_{\underline{\mathbf{d}}}} \mathcal{Z}_{\overline{y},\overline{y}'}$
- $\mu_{\overline{y},\overline{y}'} : \mathcal{Z}_{\overline{y},\overline{y}'} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ - first projection
- $\pi_{\overline{y},\overline{y}'} : \mathcal{Z}_{\overline{y},\overline{y}'} \to \widetilde{\mathcal{F}}_{\overline{y}} \times \widetilde{\mathcal{F}}_{\overline{y}'}$ - first projection
- $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}} : \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}} \to \mathrm{Rep}_{\underline{\mathbf{d}}}$ - first projection
- $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}} : \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}} \times \widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ - first projection

8. Convolution

- $[X]$ - (Borel-Moore) fundamental class of X

- $[X]^G$ - G-equivariant fundamental class of X
- $\star : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \to H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k)$ - convolution product
- $\diamond : H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}; k) \times H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k) \to H_*^{G_{\underline{\mathbf{d}}}}(\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}}; k)$ - convolution product

9. Generic notation for Schubert cells
- $\Omega_w := BwB/B$ - Schubert cell
- $\boldsymbol{\Omega}_w := G.(B/B, wB/B)$ - diagonal Schubert cell
- $\boldsymbol{\Omega}_{w',w} := \pi^{-1}(\Omega_{w'}) \cap \boldsymbol{\Omega}_w$

10. Cells in the quiver flag and Steinberg varieties
- $\Omega_w^{\overline{u}} := B_{\underline{\mathbf{d}}}.F_{wu}$
- $\boldsymbol{\Omega}_w^{\overline{u},\overline{u}'} := G_{\underline{\mathbf{d}}}.(F_{u'}, F_{wu})$
- $\boldsymbol{\Omega}_{w',w}^{\overline{u}',\overline{u}} := (\pi^{\overline{u}',\overline{u}})^{-1}(\Omega_{w'}^{\overline{u}'}) \cap \boldsymbol{\Omega}_w^{\overline{u}',\overline{u}}$
- $\mho_w := \mathbb{B}_{\mathbf{d}}.F_w$
- $\boldsymbol{\mho}_w := \mathbb{G}_{\mathbf{d}}.(F_e, F_w)$
- $\boldsymbol{\mho}_{w,w'} := \varpi^{-1}(\mho_w) \cap \boldsymbol{\mho}_{\boldsymbol{w'}}$
- $O_w := \mho_w \cap \mathcal{F}_{\underline{\mathbf{d}}}$
- $\mathbf{O}_w := \boldsymbol{\mho}_w \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})$
- $\mathbf{O}_{w,w'} := \boldsymbol{\mho}_{w,w'} \cap (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})$
- $\widetilde{O}_w := \pi_{\underline{\mathbf{d}}}^{-1}(O_w)$
- $\widetilde{\mathbf{O}}_w := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_w)$
- $\widetilde{\mathbf{O}}_{w,w'} := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{-1}(\mathbf{O}_{w,w'})$
- $O_{\leq w} := \coprod_{w' \leq w} O_{w'}$
- $\mathbf{O}_{\leq w} := \coprod_{w' \leq w} \mathbf{O}_{w'}$
- $\mathbf{O}_w^{\overline{u}} := G_{\underline{\mathbf{d}}}.(F_u, F_{uw})$
- $\mathcal{Z}_{\underline{\mathbf{d}}}^w := \overline{\widetilde{\mathbf{O}}_w}$
- $\mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w} := \bigcup_{w' \leq w} \mathcal{Z}_{\underline{\mathbf{d}}}^{w'} = \coprod_{w' \leq w} \widetilde{\mathbf{O}}_w$
- $\mathcal{Z}_{\overline{y},\overline{y}'}^w := \mathcal{Z}_{\underline{\mathbf{d}}}^w \cap \mathcal{Z}_{\overline{y},\overline{y}'}$
- $\mathcal{Z}_{\overline{y},\overline{y}'}^{\leq w} := \mathcal{Z}_{\underline{\mathbf{d}}}^{\leq w} \cap \mathcal{Z}_{\overline{y},\overline{y}'}$

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