$6j$-symbols and the $SU(N)$ Case

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alla città di Bonn,
per avermi insegnato tanto
ed essere stata la mia seconda casa.

Contents

Introduction

In his seminal 1940 paper [Wig40], Wigner introduced the $6j$-symbols for the first time, utilizing them as a tool to investigate the irreducible representations of $SO(3)$ and $SU(2)$. Since then, $6j$-symbols have been generalized to other compact Lie groups and applied in different contexts. An example of such applications is the construction of invariants of knots in 3-manifolds, for which we have basically two main approaches: one based on quantum groups and the other on Chern-Simons theory. Two references where we can see these approaches in action are [Tur10] and [GJ15] respectively. The latter reference has been the source which initiated our work: in the context of $SU(N)$ Chern-Simons theory, $6j$-symbols are here shown to be a fundamental algebraic ingredient for the construction of the so called colored HOMFLY polynomials, which are invariants of knots on the 3-sphere S^3. Different branches of mathematics and physics meet around this corner, so we refer to [Gu15] for a more complete discussion about $6j$-symbols and HOMFLY polynomials within topological strings, as well as to the introduction of [Tur10] for a brief description of the link between quantum groups, Chern-Simons theory and quantum topological field theory. Thus, since one of the main goals of [GJ15] is to concretely exhibit HOMFLY polynomials, particular attention is paid to the actual computation of $6j$-symbols. We say that a $6j$-symbol is *computable* or *solvable* when we can provide an algorithm which takes it as input and gives the value of its module as a complex number as output, *computed* or *solved* when we know such value explicitly. At this point, the question arises spontaneously:

are we able to compute arbitrary $6j$-symbols in the particular case of interest, namely for the group $SU(N)$?
Is it possible to do it even for a general compact Lie group?

Our work is then devoted to the attempt of answering these open questions. In order to do so, it is necessary to set the general framework and analyze the different objects coming into play. In what follows, we provide an insight of the structure and the contents of this thesis.

1. $6j$-symbols

If our main goal is the computation of $6j$-symbols, the starting point is their actual definition. Strictly speaking, a $6j$-symbol is a matrix coefficient depending on six objects, but let us be more precise. Consider a compact Lie group G. If μ is a representation of G, we denote its associated G-module by V_μ. Any irreducible G-module is equipped with an inner product and is considered together with a chosen orthonormal basis. Let λ_1, λ_2 be two finite-dimensional irreducible representations of G: the *coupling coefficients* are the coefficients of basis change between the standard orthonormal basis of $V_{\lambda_1} \otimes V_{\lambda_2}$ and the standard orthonormal basis of a fixed decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$ into irreducible G-modules. Consider now a third finite-dimensional irreducible representation λ_3 of G. The following concept lies on the associative property of the tensor product: the *recoupling coefficients* are the coefficients of basis change between the standard orthonormal bases arising from decomposing $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$ into irreducibles after having coupled $V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}$ differently. An example is given by the coefficients of basis change between the standard orthonormal basis of a fixed decomposition of $(V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3}$ (where λ_1 and λ_2 are coupled first) and the standard orthonormal basis of a fixed decomposition of $V_{\lambda_1} \otimes (V_{\lambda_2} \otimes V_{\lambda_3})$ (where λ_2 and λ_3 are coupled first) into irreducibles. Such coefficients depend on $\lambda_1, \lambda_2, \lambda_3$, on an irreducible representation λ in the final decomposition of $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$ and on two intermediate irreducible representations $\lambda_{12}, \lambda_{23}$ in a fixed decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$ and $V_{\lambda_2} \otimes V_{\lambda_3}$ respectively: a total of six representations. A $6j$-symbol is then a specific linear combination of recoupling coefficients, all defined by the same six irreducible representations and differing, in principle, by some other labels on which the sum is carried over. We denote a $6j$-symbol by $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$, where the λ_i are the six finite-dimensional irreducible representations defining it and $r_1, \dots, r_4$ are *multiplicity labels*. One can find several equivalent definitions and viewpoints on this in the literature. Mentioned already, we will follow closely [GJ15], which in turns adopts the style used in [But81] and [But75]. The core intention of Chapter 1 is therefore to present a definition of $6j$-symbols in the same fashion of [But75] and [But81].

The formula that justifies all our efforts in Chapter 1 is given by Equation (1.107):

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \sum_{l_1 l_2 l_3} \sum_{m_1 m_2 m_3} \sum_{n_1 n_2 n_3} (\mu_1)^*_{m_1 n_1} (\mu_2)^*_{m_2 n_2} (\mu_3)^*_{m_3 n_3}$$
$$\times \begin{pmatrix} \lambda_1 & \bar{\mu}_2 & \mu_3 \\ l_1 & n_2 & m_3 \end{pmatrix}_{r_1} \begin{pmatrix} \mu_1 & \lambda_2 & \bar{\mu}_3 \\ m_1 & l_2 & n_3 \end{pmatrix}_{r_2} \begin{pmatrix} \bar{\mu}_1 & \mu_2 & \lambda_3 \\ n_1 & m_2 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4}, \quad (1)$$

where we see a $6j$-symbol expressed as a sum of products of three $2jm$-*symbols* and four $3jm$-*symbols*. The latter are specific linear combinations of coupling coefficients, whereas $2jm$-symbols are particular cases of $3jm$-symbols. Equation (1) is proven in Proposition 1.4.1 by applying the Derome-Sharp Lemma (Lemma 1.3.1) after expressing coupling coefficients in terms of $3jm$-symbols in the sum at the right-hand side of the following equation, i.e. Equation (1.24):

$$\begin{aligned}&\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\rangle = \\ &=\frac{1}{|\lambda|}\sum_{l_1l_2l_3}\sum_{l_{12}l_{23}l}\langle r_{12}\lambda_{12}l_{12}|\lambda_1l_1,\lambda_2l_2\rangle\langle r\lambda l|\lambda_{12}l_{12},\lambda_3l_3\rangle \\ &\qquad\times\langle r_{23}\lambda_{23}l_{23}|\lambda_2l_2,\lambda_3l_3\rangle^*\langle r'\lambda l|\lambda_1l_1,\lambda_{23}l_{23}\rangle^*, \quad (2)\end{aligned}$$

where $|\lambda|$ denotes $\dim V_\lambda$. Taking then some terms to the left-hand side, we get the desired linear combination of recoupling coefficients, i.e. a $6j$-symbol. We did not find an extensive proof of Equation (2) in the literature, so we present one in Section 1.2 for completeness. A consistent part of Chapter 1 is then dedicated to the definition and study of the recoupling coefficients, the $2jm$ and $3jm$-symbols.

Understanding the $2jm$ and $3jm$-symbols has been quite challenging. We wanted to give a definition of $3jm$-symbols written in the modern mathematical language (i.e. making explicit the choice of bases and of morphisms of modules) that could have led to the fundamental property expressed by [DS65, Equation (2.1)], namely Proposition 1.3.1. The difficulty lied in the interpretation of the information given by [DS65], the only paper defining $3jm$-symbols in a way that we found sufficiently justified. After several attempts, we finally realized that the key was a different choice of basis in the cases where the considered irreducible representation is equivalent to its dual or not. Once we understood this aspect, we could provide a suitable definition for the $2jm$-symbols and a smooth procedure to achieve our task. We remark that Proposition A.2.2 was essential to accomplish this goal, since it guarantees the unitarity of certain matrices. We find this our approach to $2jm$ and $3jm$-symbols particularly interesting, since it offers an original, rigorous, detailed and modern way of defining them.

It has been very laborious to comprehend how to define properly another object of great interest: the $2j$-*phase*. First of all, we recall some terminology. A finite-dimensional irreducible representation λ is said to be *self-dual* when it is equivalent to its dual $\bar{\lambda}$, *complex* otherwise. In the case of being self-dual, λ is said to be *real* (resp. *quaternionic*) when there exists a G-invariant symmetric (resp. skew-symmetric) non-degenerate bilinear form on V_λ. A result from Representation Theory tells us that λ is one and only one of the following: real, quaternionic, complex. The Frobenius-Schur indicator of λ, denoted by ι_λ, is defined as 1 when

λ is real, -1 when λ is quaternionic and 0 when λ is complex. We illustrate now a proposition describing the behaviour of $2jm$-symbols (seen as a square matrix) with respect to the associated representation being real, quaternionic or complex:

Proposition 1 (Proposition 1.3.4)**.** *Let λ be a finite-dimensional irreducible representation of a compact Lie group G. Then:*

$$
\begin{aligned}
{}^{t}(\lambda) &= \beta_\lambda(\lambda) && \text{where} && \begin{cases} \beta_\lambda = 1 & \lambda \text{ is either real or complex,} \\ \beta_\lambda = -1 & \lambda \text{ is quaternionic;} \end{cases} \\
(\bar\lambda) &= \gamma_\lambda(\lambda) && \text{where} && \gamma_\lambda = 1 \text{ when } \lambda \text{ is self-dual;} \\
{}^{t}(\bar\lambda) &= \phi_\lambda(\lambda) && \text{where} && \phi_\lambda = \gamma_\lambda\beta_\lambda = \begin{cases} 1 & \lambda \text{ is real,} \\ -1 & \lambda \text{ is quaternionic.} \end{cases}
\end{aligned}
$$

In particular, in the case λ is self-dual we have that $\phi_\lambda = \iota_\lambda$.

The key of the proof is Fact A.2.3, which comes from an observation contained in [Sav99, Section A.2] and consists in analyzing the properties of a unitary matrix that makes another unitary matrix similar to its complex conjugate. Being stated in [DS65], we knew that $2jm$-symbols needed to satisfy the properties outlined in Proposition 1, so this served as a guideline to understand how to define $2jm$-symbols. To conclude, the $2j$-phase associated with a finite-dimensional irreducible representation λ is defined as the complex conjugate of ϕ_λ and denoted by $\{\lambda\}$. In other words, this object is characterized by the property $(\lambda) = \{\lambda\}{}^{t}(\bar\lambda)$. This is how the concept of $2j$-phase is presented in [DS65] and [But75]. Furthermore, in Proposition 1.3.5 we show that it coincides with the notion of $2j$-phase described in [But81, Equation (3.2.1)], despite in the latter only a certain subclass of groups is considered, and it is curious to see that we do that by showing that $2jm$-symbols are particular coupling coefficients:

$$
(\lambda)_{ab} = \sqrt{|\lambda|}\, \langle \mathbf{1} | \lambda b, \bar\lambda a \rangle\,, \quad (\bar\lambda)_{ab} = \sqrt{|\lambda|}\, \langle \bar{\mathbf{1}} | \bar\lambda b, \lambda a \rangle\,, \quad \langle \mathbf{1} | \lambda b, \bar\lambda a \rangle = \{\lambda\}\, \langle \bar{\mathbf{1}} | \bar\lambda a, \lambda b \rangle\,.
$$

To achieve this, in Fact 1.1.1 we offer a precise usage of the unit vector $|\mathbf{1}\rangle$ spanning the trivial submodule of $V_\lambda \otimes V_{\bar\lambda}$.
We remark that the proof of Proposition 1.4.1 plays a very central role, being the playground where to test the correctness of definitions and properties of the aforementioned objects.

The $2jm$ and $3jm$-symbols have by definition and adjunction properties beautiful symmetries (see Propositions 1.3.2, 1.3.3 and 1.3.7) which we want to transfer to the $6j$-symbols via Equation (1): we explain in this way the *symmetry properties* of $6j$-symbols. In particular, we give a full detailed theoretic presentation of the

Biedenharn-Elliot sum rule. The proofs of the symmetry properties of $6j$-symbols are useful to see in action all the objects and statements outlined previously. They make us also understand what are the hypothesis we need to impose on the considered group and how these hypothesis intervene. Particular attention is given to the crucial assumption that our group is *quasi-ambivalent*, i.e. $\{\lambda_1\}\{\lambda_2\}\{\lambda_3\} = 1$ whenever $\lambda_1, \lambda_2, \lambda_3$ are finite-dimensional irreducible representations such that $V_{\bar{\lambda}_i}$ appears as a summand in $V_{\lambda_j} \otimes V_{\lambda_k}$ for $\{i, j, k\} = \{1, 2, 3\}$.

One last aspect is the notion of *simple-phase* group. The $3jm$-symbols can be defined up to a transformation via a unitary matrix. If we can choose such matrices to make all $3jm$-symbols invariant under cyclic permutations of their columns and up to a sign under the exchange of two columns, we then say that the considered group is simple-phase. In this case, the definition of $6j$-symbols becomes more simple: a $6j$-symbol, instead of being a generic linear combination, is just a multiple of a recoupling coefficient. As a consequence, there are symmetry properties of $6j$-symbols that become easier as well, in the sense that some summations carried over certain multiplicity labels disappear. Such summations do not include any change in the representations defining the involved $6j$-symbols, so being simple-phase or not do not affect, either positively or negatively, the specific problem of establishing whether a $6j$-symbol is computable or not. This is why the results of [GJ15] in terms of computation of the needed $6j$-symbols are not compromised, even though the simple-phase version of the symmetry properties is the one used despite the fact that $SU(N)$ is not simple-phase for $N \geq 4$.

2. Computation of $6j$-symbols

At present, no general algorithm exists for the calculation of the $3jm$ or $6j$-symbols, although one can find extensive works and literature on special cases, see e.g. [Sea88] and references therein. In Chapter 2, we like to push these attempts further to a wider class using the so called *bootstrap method*. In other words, we want to see if $6j$-symbols can be computed by exploiting their own symmetry properties. The strategy we propose consists essentially in working by induction on the *power* of representations, a concept based on the existence of a finite-dimensional irreducible faithful representation. We have then to restrict our attention to a quasi-ambivalent group G admitting a representation with these characteristics.

We summarily present the steps of our computational method based on [Sea88].
STEP 0. Fix a finite-dimensional irreducible faithful representation ϵ of G. For computational reasons, we ask ϵ to be of minimal dimension with respect to such properties. If λ is an arbitrary finite-dimensional irreducible representation of G, then V_λ appears as a summand in a fixed decomposition of $V_\epsilon^{\otimes m} \otimes V_{\bar{\epsilon}}^{\otimes n}$ into irre-

ducibles for certain minimal $m, n \in \mathbb{N}$: $p(\lambda) := m + n$ is called the power of λ. If an irreducible representation α has power n, we denote α also by n_p. Notice that $p(\lambda) = 0$ if and only if $\lambda \in \{ \epsilon, \bar{\epsilon} \}$.
STEP 1. We apply the Biedenharn-Elliot sum rule to reduce our study to *primitive* $6j$-symbols only, i.e. to those $6j$-symbols $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$ such that either ϵ or $\bar{\epsilon}$ belongs to $\{ \lambda_i \mid i = 1, \ldots, 6 \}$ but the trivial representation is not included.
STEP 2. We classify a primitive $6j$-symbol $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$ by the power of $\lambda_1, \ldots, \lambda_6$. This results in a partitioning of the primitive $6j$-symbols into five classes: *Type I*, *Type II*, *Type III*, *Type IV* and *simple* $6j$-symbols. Type IV and simple $6j$-symbols are the only ones among these classes to include either ϵ or $\bar{\epsilon}$ with multiplicity at least two, and for this reason we call them *base* $6j$-symbols. Type II and simple $6j$-symbols are also known together as *core* $6j$-symbols.
STEP 3. We demonstrate that it is sufficient to focus on Type II, Type IV and simple $6j$-symbols only. Hence, once base $6j$-symbols are solved as the base case of our induction process, for the induction step it is enough to consider Type II $6j$-symbols.
We provide complete proofs of the computational ingredients used in STEPS 1–3. We do this in a way such that we also provide the computational strategy to actually calculate the $6j$-symbols in practice.

The base case for $SU(N)$ is hereby important and done in detail. We show the computability of all base quantum $6j$-symbols in Subsections 2.3.1–2.3.4 and give the explicit value of the module of many of them. This result is one of our main achievement, being the pillar of our method. It is interesting moreover because base $6j$-symbols are exactly the ones needed to compute the desired HOMFLY polynomials in [GJ15]. It took time to describe all the possible base $6j$-symbols that can occur in the $SU(N)$ case: Young diagrams have been a fundamental tool. A non-trivial part for us was also to prove and utilize correctly Lemma 2.3.4 and Lemma 2.3.5, which concern the uniqueness of a representation at a certain position in a specific $6j$-symbol after having fixed all the other five.
In the induction step we need to reduce any Type II $6j$-symbol to $6j$-symbols involving only representations of lower power. Unfortunately, we are not able to achieve this in full generality, but we solve this in the low power case, by which we mean the following:

Theorem 1 (Proposition 2.3.5). *Let $N \geq 10$. Assume $\mathcal{S}$ is a $6j$-symbol of the following shape* $\mathcal{S} = \begin{Bmatrix} 3_p & 3_p & 2_p \\ 3_p & \epsilon & 2_p \end{Bmatrix}_{0r0s}$, *then $\mathcal{S}$ is computable.*

This is our most significant accomplishment, providing a first successful application of our inductive method on a certain subclass of Type II $6j$-symbols. We give a rough idea of the proof. Due to its shape, we can choose the two free irreducible representations of the Biedenharn-Elliot sum rule as either ϵ or $\bar{\epsilon}$ when we apply such symmetry to $\mathcal{S}$. This choice allows $\mathcal{S}$ to be expressed as a sum of products of known coefficients and four $6j$-symbols, three simple and one correlated via other symmetries to a $6j$-symbol $\mathcal{T}$ of Type III. Proposition 2.2.6, contained in STEP 3, shows that Type III $6j$-symbols can be reduced to core and Type IV $6j$-symbols. Following its proof, we apply again the Biedenharn-Elliot sum rule to $\mathcal{T}$, reducing it to base $6j$-symbols. We now have that $\mathcal{S}$ is entirely expressed in terms of base $6j$-symbols, which are already computed being the base case of the induction.

Although in principle there does not seem to be any theoretical obstruction in completing the induction step of our computational method, several technical difficulties arise when the power of the involved representations becomes too high. In [GJ15], the definition of *descendable* $6j$-symbols is given, showing a non-recursive argument to compute primitive $6j$-symbols in the $SU(N)$ case when we assume a particular restriction on the involved representations. In general, one could fix the value of the power of some representations and via Young diagrams list all possible Type II $6j$-symbols and calculate them one by one, but this does not give any crucial insight nor a general approach. Instead we prefer to have a recursive algorithm. In Remark 2.3.3, we explain in more details the obstacles one may encounter and suggest a possible way on how to proceed, which is trying to understand if, putting together the fourth, fifth and sixth symmetry of $6j$-symbols, we are able to reach a sufficient number of independent equations to solve the desired Type II $6j$-symbol.

We tried and tested various strategies in the attempt of showing the computability of some more complicated Type II $6j$-symbols without being successful. Nevertheless, these failed attempts of ours are actually interesting to be examined, but for length reasons they are not reported here in this thesis.

A. Elements of Representation Theory

Appendix A contains background material for Chapter 1 and 2. As it emerged from the above paragraphs, we are interested in understanding the conditions that are needed on a group G to make our computational strategy well founded from a theoretical point of view. We collect the necessary facts in Section A.3, concluding that being a connected compact simple Lie group not of type D_{even} is sufficient to admit an irreducible finite-dimensional faithful representation. Furthermore, these characteristics imply G to be automatically quasi-ambivalent.

Notations and Conventions

Let us define some notations. Let V, W be vector spaces and $\varphi\colon V \to W$ a linear map.

- The different sum indices under a sum symbol will be listed without commas between them.
- If $z \in \mathbb{C}$, then z^* denotes the complex conjugate of z and $|z|$ its module.
- The dual and bidual of V are denoted by $V^\vee$ and $V^{\vee\vee}$ respectively.
- If $\mathcal{B}$ is a basis of V then $\mathcal{B}^\vee$ denotes the dual basis of $\mathcal{B}$.
- Let V, W be finite-dimensional of dimension n and m. Let $\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{C} = (w_1, \dots, w_m)$ be bases of V and W respectively. We denote:

  $$[v]_{\mathcal{B}} \coloneqq \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix},$$

  where $v \in V$ and $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ is the unique way of writing v as a linear combination of the vectors of $\mathcal{B}$, and:

  $$\mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi) \coloneqq \big([\varphi(v_1)]_{\mathcal{C}}, \dots, [\varphi(v_n)]_{\mathcal{C}}\big)$$

  which goes under the name of the matrix associated with φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. In case $V = W$, we denote $\mathcal{M}_{\mathcal{B}\mathcal{B}}(\varphi)$ simply by $\mathcal{M}_{\mathcal{B}}(\varphi)$. In case $V = W$ and $\varphi = \mathrm{id}_V$ is the identity map, the matrix $\mathcal{M}_{\mathcal{B}\mathcal{C}}(\mathrm{id}_V)$ is called the matrix of change of basis of V from the basis $\mathcal{B}$ to the basis $\mathcal{C}$.
- The transposed of φ is denoted ${}^{\mathrm{t}}\varphi$.
- We denote by $\mathrm{GL}(V)$ the group of invertible linear maps from V to itself.
- If A is a $p \times q$ matrix, we denote the entry of A corresponding to the i^{th} row and j^{th} column as A_{ij}, i.e. $A = (A_{ij})_{i,j}$.

- If M is an $n \times n$ complex matrix, then ${}^{\mathrm{t}}M$ denotes the transposed of M, M^* denotes the matrix obtained by M by complex conjugating every coefficients of M and $M^\dagger$ denotes ${}^{\mathrm{t}}M^*$.

- If A and B are a $p \times q$ and an $r \times s$ matrix respectively, let:

$$(A \oplus B)_{ij} \coloneqq \begin{cases} A_{ij} & \text{if } i \leq p \text{ and } j \leq q, \\ B_{(i-p)(j-q)} & \text{if } i > p \text{ and } j > q, \\ 0 & \text{else,} \end{cases} \qquad (A \otimes B)_{(ik)(jl)} \coloneqq A_{ij} B_{kl}$$

  be the direct sum and the tensor product of A and B respectively, namely:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad A \otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1q}B \\ \vdots & & \vdots \\ A_{p1}B & \dots & A_{pq}B \end{pmatrix}.$$

- If V, W are finite-dimensional and $\mathcal{B} = (v_1, \dots, v_n)$, $\mathcal{C} = (w_1, \dots w_m)$ are bases of V and W respectively, then we can define ordered bases of $V \oplus W$ and $V \otimes W$ as follows:

$$\mathcal{B} \oplus \mathcal{C} \coloneqq ((v_1, 0_W), \dots (v_n, 0_W), (0_V, w_1), \dots, (0_V, w_m)),$$

$$\mathcal{B} \otimes \mathcal{C} \coloneqq (v_1 \otimes w_1, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_2 \otimes w_m, \dots, v_n \otimes w_1, \dots, v_n \otimes w_m).$$

- Throughout the thesis, a **representation** of a group G is a group homomorphism $\lambda \colon G \to \mathrm{GL}(U)$ for some complex vector space U, where U will be said to be a ***G*-module** or the **module associated with $\boldsymbol{\lambda}$** and it will also be denoted by V_λ.

- If λ is a representation of a group G, we define $|\lambda| \coloneqq \dim V_\lambda$.

- Let $\lambda \colon G \to \mathrm{GL}(U)$ be a representation of a group G. The **dual representation** of λ will be denoted by $\bar{\lambda}$. We recall that $\bar{\lambda}$ is defined as follows:

$$\bar{\lambda} \colon G \to \mathrm{GL}(U^\vee); \qquad \bar{\lambda}(g) \coloneqq {}^{\mathrm{t}}\left(\lambda(g)^{-1}\right) \ \forall g \in G.$$

  Therefore, the module associated with $\bar{\lambda}$ is $V_{\bar{\lambda}} = (V_\lambda)^\vee$.

- If $\lambda \colon G \to \mathrm{GL}(V)$ and $\sigma \colon G \to \mathrm{GL}(W)$ are two representations of G, we denote by $\lambda \oplus \sigma$ and $\lambda \otimes \sigma$ those representations of G whose associated modules are $V_\lambda \oplus V_\sigma$ and $V_\lambda \otimes V_\sigma$ respectively: $V_{\lambda \oplus \sigma} = V_\lambda \oplus V_\sigma$ and $V_{\lambda \otimes \sigma} = V_\lambda \otimes V_\sigma$.

- The symbol $\mathbf{1}$ denotes the **trivial representation** of a group G, namely the map $\mathbf{1}\colon G \to \mathbb{C}$ such that $\mathbf{1}(g) = 1$ for any g in G. Then, $V_{\mathbf{1}} = \mathbb{C}$ is called the **trivial module**.

To let the reader become more familiar with the conventions in use, we outline here some basic facts of linear algebra:

Fact 1. *If U, V, W, Z are finite-dimensional vector spaces with bases $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ respectively and $\alpha\colon V \to V$, $\beta\colon W \to W$, $\gamma\colon W \to Z$, $\varphi\colon V \to W$, $\psi\colon U \to Z$ are linear maps, then we have the following:*

$$\mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi) \cdot [v]_{\mathcal{B}} = [\varphi(v)]_{\mathcal{C}} \ \forall v \in V, \qquad \mathcal{M}_{\mathcal{D}\mathcal{C}}(\gamma) \cdot \mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi) = \mathcal{M}_{\mathcal{D}\mathcal{B}}(\gamma \circ \varphi),$$
$$^{\mathrm{t}}\mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi) = \mathcal{M}_{\mathcal{B}^\vee\mathcal{C}^\vee}(^{\mathrm{t}}\varphi), \qquad \varphi \text{ is invertible } \Rightarrow \ \mathcal{M}_{\mathcal{B}\mathcal{C}}(\varphi^{-1}) = \mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi)^{-1},$$
$$\mathcal{M}_{\mathcal{B}\oplus\mathcal{C}}(\alpha \oplus \beta) = \mathcal{M}_{\mathcal{B}}(\alpha) \oplus \mathcal{M}_{\mathcal{C}}(\beta), \quad \mathcal{M}_{(\mathcal{C}\otimes\mathcal{D})(\mathcal{B}\otimes\mathcal{A})}(\varphi \otimes \psi) = \mathcal{M}_{\mathcal{C}\mathcal{B}}(\varphi) \otimes \mathcal{M}_{\mathcal{D}\mathcal{A}}(\psi).$$

Chapter 1

6j-symbols

In this chapter we define the notion of $6j$-symbols and the main objects which are useful to express them. We study different aspects of such objects and prove the so called symmetry properties of $6j$-symbols. All representations we are going to work with will be over the complex numbers and finite-dimensional: if not clearly stated, this will be assumed implicitly.

Throughout the chapter, G will denote a fixed compact Lie group defined over $\mathbb{R}$. We refer to [BtD85] for basic definitions and some more specific details.

1.1 Settings

Let us devote this first section to set our framework by fixing specific concepts and structures and defining some specific notations.

- Throughout this chapter, if $\lambda\colon G \to \mathrm{GL}(V)$ is a finite-dimensional irreducible representation of G, then V will be automatically considered as equipped with a chosen G-invariant inner product (see Definition A.2.1) which will be often denoted by $\langle\cdot|\cdot\rangle_V$ or more simply by $\langle\cdot|\cdot\rangle$ when there is no confusion (such inner products are chosen to be antilinear in the first argument and linear in the second one). This is made possible by Theorem A.2.1. Recall in particular that by Proposition A.2.1 any two G-invariant inner products on an irreducible module differ by a constant factor.
  We do not assign such inner products in full arbitrariness: if $V^\vee$ is the (irreducible) dual module of V we then make the following choice:

  $$\langle\cdot|\cdot\rangle_{V^\vee} \coloneqq \left(\langle\cdot|\cdot\rangle_V\right)^\vee ,$$

  where $\left(\langle\cdot|\cdot\rangle_V\right)^\vee$ is defined by (A.8). This guarantees that if $\mathcal{B}$ is an orthonormal basis of V then $\mathcal{B}^\vee$ is an orthonormal basis of $V^\vee$ (see Remark A.2.2).

Furthermore, we identify V and its bidual $V^{\vee\vee}$ via the linear isometry of modules $\mathcal{J}_{V^\vee} \circ \mathcal{J}_V$ defined at the end of Section A.2.

- If $V_1, \dots, V_m$ are irreducible G-modules with pre-chosen G-invariant inner products $h_1, \dots, h_m$ respectively, then $V_1 \oplus \dots \oplus V_m$ and $V_1 \otimes \dots \otimes V_m$ will be considered together with the G-invariant inner products $h_1 \oplus \dots \oplus h_m$ and $h_1 \otimes \dots \otimes h_m$ defined by (A.11) and (A.12) respectively.

- If $\lambda\colon G \to \mathrm{GL}(V)$ is a representation of G, then the symbols $|\lambda l\rangle$ denote the vectors of an orthonormal basis of V with l running from 1 to $|\lambda|$. We then write $v = \sum_{l=1}^{|\lambda|} \langle \lambda l | v \rangle \, |\lambda l\rangle$ for a generic vector v of V.

- Let λ_1, λ_2 be two finite-dimensional representations of G. Let $\mathcal{B}_i = (|\lambda_i l\rangle : l = 1, \dots, |\lambda_i|)$ be an ordered orthonormal basis of V_{λ_i} for $i = 1, 2$. We then use the following notation for any complex numbers α and β and any l_1, l_2:

$$\begin{aligned} \alpha\,|\lambda_1 l_1\rangle + \beta\,|\lambda_2 l_2\rangle &:= (\alpha\,|\lambda_1 l_1\rangle\,, \beta\,|\lambda_2, l_2\rangle) \in V_{\lambda_1} \oplus V_{\lambda_2}, \\ |\lambda_1 l_1, \lambda_2, l_2\rangle &:= |\lambda_1 l_1\rangle\,|\lambda_2, l_2\rangle := |\lambda_1 l_1\rangle \otimes |\lambda_2, l_2\rangle \in V_{\lambda_1} \otimes V_{\lambda_2}. \end{aligned}$$

  Then, $\mathcal{B}_1 \oplus \mathcal{B}_2$ and $\mathcal{B}_1 \otimes \mathcal{B}_2$ are the ordered orthonormal bases that we will consider on $V_{\lambda_1} \oplus V_{\lambda_2}$ and $V_{\lambda_1} \otimes V_{\lambda_2}$ for the inner products $\langle \cdot | \cdot \rangle_{V_{\lambda_1}} \oplus \langle \cdot | \cdot \rangle_{V_{\lambda_2}}$ and $\langle \cdot | \cdot \rangle_{V_{\lambda_1}} \otimes \langle \cdot | \cdot \rangle_{V_{\lambda_2}}$ respectively.

Before the last observation, we recall the concepts of multiplicity, coupling between representations and triad.

Definition 1.1.1. Let $\lambda\colon G \to \mathrm{GL}(V)$, $\sigma\colon G \to \mathrm{GL}(W)$ be representations of G. If W is irreducible, we call the number $\mathrm{m}_V^W := \dim \mathrm{Hom}_G(W, V)$ the **multiplicity of W in V**. Equivalently, m_V^W is also denoted by $\mathrm{m}_\lambda^\sigma$ and called the **multiplicity of σ in λ**.

We denote by $\mathrm{Irr}(G)$ a complete set of pairwise non-isomorphic G-modules, namely any irreducible G-module is isomorphic to exactly one element of $\mathrm{Irr}(G)$. If V is a finite-dimensional G-module, then Theorem A.2.2 tells us that we can write $V = \bigoplus_j V_j$ for some irreducible submodules V_j of V. Let us explore the meaning of the multiplicity: if W is an irreducible submodule of V then $\mathrm{Hom}_G(W, V) \cong \bigoplus_j \mathrm{Hom}_G(W, V_j)$, so by Schur's Lemma $\dim \mathrm{Hom}_G(W, V)$ is simply the number of V_j that are isomorphic to W. In particular, m_V^W is non-zero if and only if W is isomorphic to some submodule of V, and this happens for only those finitely many W in $\mathrm{Irr}(G)$ which are isomorphic to some V_j.

Definition 1.1.2. Let $\lambda_1, \dots, \lambda_m, \lambda$ be finite-dimensional irreducible representations of G and fix a decomposition of $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_m}$ into irreducibles. If V_λ

occurs as a summand in such decomposition, we say that $\lambda_1, \ldots, \lambda_m$ **couple to** λ or equivalently $V_{\lambda_1}, \ldots, V_{\lambda_m}$ **couple to** V_λ. If this is the case, by an abuse of notation we write $\lambda \in \lambda_1 \otimes \ldots \otimes \lambda_m$ or equivalently $V_\lambda \in V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_m}$. In the case $\lambda_1, \ldots, \lambda_m$ and other finite-dimensional irreducible representations $\mu_1, \ldots, \mu_n$ couple to λ, we write $\lambda \in (\lambda_1 \otimes \ldots \otimes \lambda_m) \cap (\mu_1 \otimes \ldots \otimes \mu_n)$.

Definition 1.1.3. Let $\lambda_1, \lambda_2, \lambda_3$ be finite-dimensional irreducible representations of G. We call $(\lambda_1\lambda_2\lambda_3)$ a **triad** when $\mathbf{1} \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$. If $(\lambda_1\lambda_2\lambda_3)$ is a triad, we denote $(\lambda_1\lambda_2\lambda_3)$ also by $(\lambda_1\lambda_2\lambda_3 r)$, where r is an integer number such that $0 \leq r \leq \mathrm{m}^{\mathbf{1}}_{\lambda_1 \otimes \lambda_2 \otimes \lambda_3} - 1$.

Remark 1.1.1. Let $\lambda_1, \lambda_2, \lambda_3$ be finite-dimensional irreducible representations of G. By Fact A.2.2, $(\lambda_1\lambda_2\lambda_3)$ is a triad when $\bar{\lambda}_k \in \lambda_i \otimes \lambda_j$ for $\{i, j, k\} = \{1, 2, 3\}$.

- Let λ_1, λ_2 be finite-dimensional irreducible representations of G. By Theorem A.2.2, we can decompose $V_{\lambda_1} \otimes V_{\lambda_2}$ into a direct sum of irreducible submodules. Therefore, we have an isomorphism of modules Ψ such that:

$$\Psi \colon V_{\lambda_1} \otimes V_{\lambda_2} \to \bigoplus_{\lambda} \bigoplus_{r} V_\lambda^{(r)}, \tag{1.1}$$

  where the label λ tracks the irreducible representations and r, running from 0 to $\mathrm{m}^{\lambda}_{\lambda_1 \otimes \lambda_2} - 1$, tracks their multiplicity. In particular, we consider each $V_\lambda^{(r)}$ to be equal to V_λ. Furthermore, by Proposition A.2.2 we can consider Ψ to be an isometry (notice that Ψ in general is not unique).
  Now, if we choose an orthonormal basis $\mathcal{B}_{\lambda_i} = (|\lambda_i l_i\rangle : l_i = 1, \ldots, |\lambda_i|)$ of V_{λ_i} for $i = 1, 2$ and an orthonormal basis $\mathcal{B}_{r\lambda} = (|r\lambda l\rangle : l = 1, \ldots, |\lambda|)$ of each module $V_\lambda^{(r)}$, we write:

$$\Psi(|\lambda_1 l_1, \lambda_2 l_2\rangle) = \sum_{r\lambda l} \langle r\lambda l | \Psi(|\lambda_1 l_1, \lambda_2 l_2\rangle)\rangle \, |r\lambda l\rangle \,. \tag{1.2}$$

  Being Ψ an isometry, $\langle r\lambda l | \Psi(|\lambda_1 l_1, \lambda_2 l_2\rangle)\rangle = \langle \Psi^{-1}(|r\lambda l\rangle) \, |\lambda_1 l_1, \lambda_2 l_2\rangle$ for any r, l, l_1, l_2, so we have the following equation as well:

$$|\lambda_1 l_1, \lambda_2 l_2\rangle = \sum_{r\lambda l} \langle \Psi^{-1}(|r\lambda l\rangle) \, |\lambda_1 l_1, \lambda_2 l_2\rangle \, \Psi^{-1}(|r\lambda l\rangle) \,. \tag{1.3}$$

  We will often omit to specify the chosen isometry of modules Ψ and denote the previous two equations simply by writing:

$$|\lambda_1 l_1, \lambda_2 l_2\rangle = \sum_{r\lambda l} \langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2\rangle \, |r\lambda l\rangle \,, \tag{1.4}$$

where the $\langle r\lambda l|\lambda_1 l_1, \lambda_2 l_2\rangle$ are called **coupling coefficients**. Equation (1.4) becomes an equality more than a notation when Ψ is the identity and the modules $V_\lambda^{(r)}$ are all actual subspaces of $V_{\lambda_1} \otimes V_{\lambda_2}$.
In order to maintain the same notation used in the main references, the multiplicity labelling starts from 0. For instance, if an irreducible module occurs with multiplicity 1 in a decomposition into irreducibles then it will be labelled by the multiplicity label 0.

We outline two main facts regarding coupling coefficients.

Fact 1.1.1. *Let λ be an irreducible finite-dimensional representation of G of dimension d. Let $\mathcal{B} = (|\lambda 1\rangle, \ldots, |\lambda d\rangle)$ be an orthonormal basis of V_λ and $\mathcal{B}^\vee = (|\bar\lambda 1\rangle, \ldots, |\bar\lambda d\rangle)$. Consider the vector:*

$$z \coloneqq \frac{1}{\sqrt{|\lambda|}} \big(|\lambda 1, \bar\lambda 1\rangle + \ldots + |\lambda d, \bar\lambda d\rangle\big) \in V_\lambda \otimes V_{\bar\lambda}. \tag{1.5}$$

We have that z is independent of $\mathcal{B}$ and the one-dimensional submodule spanned by z is G-isomorphic to the trivial G-module. It is therefore possible to consider a decomposition of $V_\lambda \otimes V_{\bar\lambda}$ into irreducibles expressed by an isometry of modules:

$$\Psi \colon V_\lambda \otimes V_{\bar\lambda} \to \mathbb{C} \oplus \bigoplus_{r,\mu \neq \mathbf{1}} V_\mu^{(r)} \qquad \textit{such that} \qquad \Psi(z) = (1_{\mathbb{C}}, 0, \ldots, 0). \tag{1.6}$$

Defining $|\mathbf{1}\rangle \coloneqq (k, 0, \ldots, 0) = k\Psi(z) = \Psi(kz)$ for some complex number k of module 1, we have that the following equalities hold for any $l, m = 1, \ldots, d$:

$$\langle \mathbf{1}|\lambda l, \bar\lambda m\rangle = \langle \mathbf{1}|\lambda l, \bar\lambda l\rangle\, \delta_{lm} = \frac{k^*}{\sqrt{d}}\delta_{lm}, \quad \langle \lambda l, \bar\lambda m|\mathbf{1}\rangle = \langle \lambda l, \bar\lambda l|\mathbf{1}\rangle\, \delta_{lm} = \frac{k}{\sqrt{d}}\delta_{lm}, \tag{1.7}$$

$$\langle \mathbf{1}|\lambda l, \bar\lambda m\rangle = \langle \lambda l, \bar\lambda m|\mathbf{1}\rangle^*, \tag{1.8}$$

$$\sum_{l_1 l_2 = 1}^{d} \langle \lambda l_1, \bar\lambda l_2|\mathbf{1}\rangle \langle \mathbf{1}|\lambda l_1, \bar\lambda l_2\rangle = \sum_{l=1}^{d} \langle \lambda l, \bar\lambda l|\mathbf{1}\rangle \langle \mathbf{1}|\lambda l, \bar\lambda l\rangle = \sum_{l=1}^{d} \big|\langle \mathbf{1}|\lambda l, \bar\lambda l\rangle\big|^2 = 1. \tag{1.9}$$

Proof. By Proposition A.1.1, the one-dimensional vector subspace Z of $V_\lambda \otimes V_{\bar\lambda}$ spanned by z is indeed G-isomorphic to the trivial G-module and z is independent of $\mathcal{B}$. Furthermore, z has norm 1, so kz is an orthonormal basis of Z for any complex number k of module 1.
We have $\mathrm{m}^{\mathbf{1}}_{\lambda \otimes \bar\lambda} = 1$ by Fact A.2.2, hence when we write $V_\lambda \otimes V_{\bar\lambda} = Z \oplus Z^\perp$ we know that $Z^\perp \coloneqq \{\, w \in V_\lambda \otimes V_{\bar\lambda} \mid \langle w|z\rangle = 0 \,\}$ is a submodule of V that does not contain

any further submodule which is isomorphic to the trivial G-module. Therefore, a decomposition of $V_\lambda \otimes V_{\bar\lambda}$ into irreducibles can be expressed by an isometry of modules:

$$\Psi\colon V_\lambda \otimes V_{\bar\lambda} = Z \oplus Z^\perp \to \mathbb{C} \oplus \bigoplus_{r,\mu\neq \mathbf{1}} V_\mu^{(r)},$$

where by Schur's Lemma Ψ is forced to send z to:

$$\Psi(z) = (k, 0, \dots, 0) \in \mathbb{C} \oplus \bigoplus_{r,\mu\neq \mathbf{1}} V_\mu^{(r)}$$

for some complex number k of module 1 (Ψ is an isometry and z has norm 1). Fix now $l, m \in \{1, \dots, d\}$. By definition of the notation in use, $\langle \mathbf{1}|\lambda l, \bar\lambda m\rangle = \langle \mathbf{1}|\Psi(|\lambda l, \bar\lambda m\rangle)\rangle$. Since Ψ is an isometry, we get that:

$$\langle \mathbf{1}|\lambda l, \bar\lambda m\rangle = \langle \mathbf{1}|\Psi(|\lambda l, \bar\lambda m\rangle)\rangle = \langle \Psi^{-1}(|\mathbf{1}\rangle)|\lambda l, \bar\lambda m\rangle = \langle kz|\lambda l, \bar\lambda m\rangle = k^* \langle z|\lambda l, \bar\lambda m\rangle\,,$$

which coincides with $\delta_{lm} k^*/\sqrt{d}$ by how z and the induced inner product on tensor products are defined. Similarly:

$$\langle \lambda l, \bar\lambda m|\mathbf{1}\rangle = \langle \Psi(|\lambda l, \bar\lambda m\rangle)|\mathbf{1}\rangle = \langle \lambda l, \bar\lambda m|\Psi^{-1}(|\mathbf{1}\rangle)\rangle = \langle \lambda l, \bar\lambda m|kz\rangle = k \langle \lambda l, \bar\lambda m|z\rangle$$

which equals $\delta_{lm} k/\sqrt{d}$. From these results it follows immediately that $\langle \mathbf{1}|\lambda l, \bar\lambda m\rangle = \langle \lambda l, \bar\lambda m|\mathbf{1}\rangle^*$. Finally, we have:

$$\sum_{l_1 l_2=1}^{d} \langle \lambda l_1, \bar\lambda l_2|\mathbf{1}\rangle \langle \mathbf{1}|\lambda l_1, \bar\lambda l_2\rangle = \sum_{l_1 l_2=1}^{d} \langle \mathbf{1}|\lambda l_1, \bar\lambda l_2\rangle^* \langle \mathbf{1}|\lambda l_1, \bar\lambda l_2\rangle = \sum_{l_1 l_2=1}^{d} \left|\langle \mathbf{1}|\lambda l_1, \bar\lambda l_2\rangle\right|^2 =$$
$$= \sum_{l_1 l_2=1}^{d} \left|\frac{k^*}{\sqrt{d}}\delta_{l_1 l_2}\right|^2 = \frac{1}{d}\sum_{l_1 l_2=1}^{d} \delta_{l_1 l_2} = 1 = \sum_{l=1}^{d} \left|\langle \mathbf{1}|\lambda l, \bar\lambda l\rangle\right|^2,$$

where we have used the newly proven (1.7) and (1.8). □

Throughout the thesis, the notation $|\mathbf{1}\rangle$ will always refer to the situation and choices described in Fact 1.1.1.

Fact 1.1.2 (Unitarity of coupling coefficients)**.** *Let λ_1, λ_2 be finite-dimensional irreducible representations of G. Consider a decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$ into irreducibles via an isometry of modules $\Psi\colon V_{\lambda_1} \otimes V_{\lambda_2} \to \bigoplus_{r\lambda} V_\lambda^{(r)}$. Choose orthonormal bases $\mathcal{B}_{r\lambda} = (|r\lambda l\rangle : l = 1, \dots, |\lambda|)$ for each $V_\lambda^{(r)}$ and $\mathcal{B}_i = (|\lambda_i l_i\rangle : l_i = 1, \dots, |\lambda_i|)$ for V_{λ_i}. Then:*

1. $\sum_{l_1 l_2} \langle \lambda_1 l_1, \lambda_2 l_2|r\lambda l\rangle \langle r'\lambda' l'|\lambda_1 l_1, \lambda_2 l_2\rangle = \delta_{rr'}\delta_{\lambda\lambda'}\delta_{ll'}$;

2. $\sum_{r\lambda l} \langle \lambda_1 l_1, \lambda_2 l_2 | r\lambda l \rangle \langle r\lambda l | \lambda_1 l_1', \lambda_2 l_2' \rangle = \delta_{l_1 l_1'} \delta_{l_2 l_2'}$;

3. $\langle \lambda_1 l_1, \lambda_2 l_2 | r\lambda l \rangle = \langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle^*$.

Proof. Consider the following matrices:

$$A := (\langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle)_{r\lambda l, l_1 l_2}, \qquad B := (\langle \lambda_1 l_1, \lambda_2 l_2 | r\lambda l \rangle)_{l_1 l_2, r\lambda l}.$$

Consider the orthonormal bases $\mathcal{B} := \mathcal{B}_1 \otimes \mathcal{B}_2$ of $V_{\lambda_1} \otimes V_{\lambda_2}$ and $\mathcal{C} := \bigotimes_{r\lambda} \mathcal{B}_{r\lambda}$ of $\bigoplus_{r\lambda} V_\lambda^{(r)}$. Then we have that $A = \mathcal{M}_{\mathcal{CB}}(\Psi)$ and $B = \mathcal{M}_{\mathcal{BC}}(\Psi^{-1})$, i.e. A is the matrix associated with Ψ with respect to the bases $\mathcal{B}$ of its domain and $\mathcal{C}$ of its codomain, whereas B is the matrix associated with Ψ^{-1} with respect to the bases $\mathcal{C}$ of its domain and $\mathcal{B}$ of its codomain. By Fact 1, we have that $B = A^{-1}$, which is exactly the content of 1.) and 2.) of the claim. Let us show 1.) more formally in order to familiarize with the ket notation, 2.) is done similarly. We have:

$$|r\lambda l\rangle = \sum_{l_1 l_2} \langle \lambda_1 l_1, \lambda_2 l_2 | r\lambda l \rangle \, |\lambda_1 l_1, \lambda_2 l_2\rangle \, , \tag{1.10}$$

$$|\lambda_1 l_1, \lambda_2 l_2\rangle = \sum_{r\lambda l} \langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle \, |r\lambda l\rangle \, . \tag{1.11}$$

Plugging (1.11) into (1.10) we get:

$$\begin{aligned} |r'\lambda' l'\rangle &= \sum_{l_1 l_2} \langle \lambda_1 l_1, \lambda_2 l_2 | r'\lambda' l' \rangle \, |\lambda_1 l_1, \lambda_2 l_2\rangle \\ &= \sum_{l_1 l_2} \langle \lambda_1 l_1, \lambda_2 l_2 | r'\lambda' l' \rangle \sum_{r\lambda l} \langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle \, |r\lambda l\rangle \\ &= \sum_{r\lambda l} \left(\sum_{l_1 l_2} \langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle \langle \lambda_1 l_1, \lambda_2 l_2 | r'\lambda' l' \rangle \right) |r\lambda l\rangle \, , \end{aligned}$$

which implies the claim by linear independence.
Let us now prove 3). Since $\mathcal{B}, \mathcal{C}$ are orthonormal and Ψ is an isometry, A is unitary by Fact A.2.1. Therefore:

$$\begin{aligned} (\langle \lambda_1 l_1, \lambda_2 l_2 | r\lambda l \rangle)_{l_1 l_2, r\lambda l} &= B = A^{-1} = A^\dagger = {}^{\mathrm{t}}A^* = {}^{\mathrm{t}}(\langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle)_{r\lambda l, l_1 l_2}{}^* = \\ &= (\langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle)_{l_1 l_2, r\lambda l}{}^* = (\langle r\lambda l | \lambda_1 l_1, \lambda_2 l_2 \rangle^*)_{l_1 l_2, r\lambda l}. \end{aligned}$$

□

1.2 Recoupling Coefficients

For this entire section, let us fix three finite-dimensional irreducible representations $\lambda_1, \lambda_2, \lambda_3$ with orthonormal bases $\mathcal{B}_i = (|\lambda_i l_i\rangle : l_i = 1, \ldots, |\lambda_i|)$ of V_{λ_i} for

$i = 1, 2, 3$. We study now three specific ways of coupling V_{λ_1}, V_{λ_2}, V_{λ_3} and the basis change occurring between them. Our goal is to give the definition of the so called recoupling coefficients as outlined in [But81, Equation (3.2.16)] and then study some of their basic properties. We recall once again Proposition A.2.2, which tells us that G-isomorphic G-modules are G-isometric.

- There are isometries of modules:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \cong \left(V_{\lambda_1} \otimes V_{\lambda_2}\right) \otimes V_{\lambda_3} \cong \left(\bigoplus_{r_{12}\lambda_{12}} V_{\lambda_{12}}^{(r_{12})}\right) \otimes V_{\lambda_3} \cong$$
$$\cong \bigoplus_{r_{12}\lambda_{12}} \left(V_{\lambda_{12}}^{(r_{12})} \otimes V_{\lambda_3}\right) \cong \bigoplus_{r_{12}\lambda_{12}} \bigoplus_{r\lambda} V_{\lambda}^{(r)}, \quad (1.12)$$

  where $\bigoplus_{r_{12}\lambda_{12}} V_{\lambda_{12}}^{(r_{12})}$ and $\bigoplus_{r\lambda} V_{\lambda}^{(r)}$ denote fixed decompositions of $V_{\lambda_1} \otimes V_{\lambda_2}$ and $V_{\lambda_{12}}^{(r_{12})} \otimes V_{\lambda_3}$ into irreducible modules respectively (notice that both r and λ depend on λ_{12} and r_{12}). We obtain:

$$|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3\rangle = \sum_{r_{12}\lambda_{12}l_{12}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \, |_{(\lambda_1\lambda_2)} r_{12}\lambda_{12}l_{12}, \lambda_3 l_3\rangle =$$
$$= \sum_{r_{12}\lambda_{12}l_{12}} \sum_{r\lambda l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle \, |_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda l\rangle \,, \quad (1.13)$$

  where $(|r_{12}\lambda_{12}l_{12}\rangle : l_{12} = 1, \ldots, |\lambda_{12}|)$ and $(|_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda l\rangle : l = 1, \ldots, |\lambda|)$ are chosen orthonormal bases of $V_{\lambda_{12}}^{(r_{12})}$ and $V_{\lambda}^{(r)}$ respectively (the little label $_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}$ keeps track of the objects that λ and r depend on).

- There are isometries of modules:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \cong V_{\lambda_1} \otimes \left(V_{\lambda_2} \otimes V_{\lambda_3}\right) \cong V_{\lambda_1} \otimes \left(\bigoplus_{r_{23}\lambda_{23}} V_{\lambda_{23}}^{(r_{23})}\right) \cong$$
$$\cong \bigoplus_{r_{23}\lambda_{23}} \left(V_{\lambda_1} \otimes V_{\lambda_{23}}^{(r_{23})}\right) \cong \bigoplus_{r_{23}\lambda_{23}} \bigoplus_{r'\lambda} V_{\lambda}^{(r')}, \quad (1.14)$$

  where $\bigoplus_{r_{23}\lambda_{23}} V_{\lambda_{23}}^{(r_{23})}$ and $\bigoplus_{r'\lambda} V_{\lambda}^{(r')}$ denote fixed decompositions of $V_{\lambda_2} \otimes V_{\lambda_3}$ and $V_{\lambda_1} \otimes V_{\lambda_{23}}^{(r_{23})}$ into irreducible modules respectively (both r' and λ depend on λ_{23} and r_{23}). We have:

$$|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3\rangle = \sum_{r_{23}\lambda_{23}l_{23}} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \, |\lambda_1 l_1, {}_{(\lambda_2\lambda_3)} r_{23}\lambda_{23}l_{23}\rangle =$$

$$= \sum_{r_{23}\lambda_{23}l_{23}} \sum_{r'\lambda l} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r'\lambda l|\lambda_1 l_1, \lambda_{23}l_{23}\rangle \left|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\right\rangle , \tag{1.15}$$

where $(|r_{23}\lambda_{23}l_{23}\rangle : l_{23} = 1, \ldots, |\lambda_{23}|)$ and $(|{\scriptstyle((\lambda_1\lambda_2)r_{23}\lambda_{23}, \lambda_3)}r'\lambda l\rangle : l = 1, \ldots, |\lambda|)$ are chosen orthonormal bases of $V_{\lambda_{23}}^{(r_{23})}$ and $V_\lambda^{(r')}$ respectively. Regarding the notation in use, the same considerations as above hold.

In the two ways of coupling we analyzed above, we ended up with two bases: one given by the vectors $|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda l\rangle$ and the other one by the vectors $|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\rangle$. Let us analyze the transformation between these two bases. In general we have:

$$\begin{aligned}
&\left|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\right\rangle = \\
&\quad = \sum_{r_{12}\lambda_{12}} \sum_{r\lambda' l'} \left\langle{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda' l'\middle|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\right\rangle \left|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda' l'\right\rangle = \\
&\qquad = \sum_{r_{12}\lambda_{12}r} \left\langle{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda\middle|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda\right\rangle \left|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda l\right\rangle ,
\end{aligned} \tag{1.16}$$

and analogously:

$$\left|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda l\right\rangle = \sum_{r_{23}\lambda_{23}r'} \left\langle{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda\middle|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda\right\rangle \left|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\right\rangle . \tag{1.17}$$

Definition 1.2.1. The coefficients

$$\left\langle{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda\middle|{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda\right\rangle , \qquad \left\langle{\scriptstyle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda\middle|{\scriptstyle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r\lambda\right\rangle \tag{1.18}$$

appearing in (1.16) and (1.17) which define the change of the final bases obtained by decomposing $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$ into irreducible modules by coupling firstly $V_{\lambda_1} \otimes V_{\lambda_2}$ and secondly $V_{\lambda_2} \otimes V_{\lambda_3}$ are called **recoupling coefficients**.

Remark 1.2.1. Notice that the recoupling coefficients do not depend on the basis vectors thanks to Schur's Lemma. Indeed, assume φ and ψ are the isometries of modules used above such that:

$$\bigoplus_{r_{12}\lambda_{12}} \bigoplus_{r\lambda} V_\lambda^{(r)} \xleftarrow{\varphi} V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \xrightarrow{\psi} \bigoplus_{r_{23}\lambda_{23}} \bigoplus_{r'\lambda} V_\lambda^{(r')}.$$

Let $f := \varphi \circ \psi^{-1}$. Rigorously, what we have is:

$$f\left(\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle\right) =$$
$$= \sum_{r_{12}\lambda_{12}r}\sum_{\lambda' l'} \left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda' l'\middle| f\left(\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle\right)\right\rangle \left|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda' l'\right\rangle,$$

which is written as:

$$\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle =$$
$$= \sum_{r_{12}\lambda_{12}r}\sum_{\lambda' l'} \left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda' l'\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle \left|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda' l'\right\rangle.$$

Being an isomorphism of modules, by Schur's Lemma f does not map between different V_λ's or between different l's, so we have:

$$f\left(\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle\right) =$$
$$= \sum_{r_{12}\lambda_{12}r} \left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\right\rangle \left|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda l\right\rangle,$$

which is written as:

$$\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle =$$
$$= \sum_{r_{12}\lambda_{12}r} \left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\right\rangle \left|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda l\right\rangle.$$

Example 1.2.1. Let $G = SU(2)$. Being determined by its dimension, a finite-dimensional irreducible representation of $SU(2)$ of dimension n will be denoted by $\boldsymbol{n}$. Let us focus on the representation $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{3} \cong \mathbf{1}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{5}$: we will analyze two ways of coupling the irreducible representations defining this tensor product. Fix ordered orthonormal weight bases $\mathcal{B} = (e_1, e_2)$, $\mathcal{C} = (v_1, v_2, v_3)$ of $V_{\mathbf{2}}$, $V_{\mathbf{3}}$ respectively, where e_1 and v_1 are maximal vectors. What we will do is to write down orthonormal weight bases of some irreducible submodules of tensor products: this is achieved by starting from a normalized maximal vector (that we will always put as the first vector of a basis) and applying the action of $SU(2)$. To make things more readable, throughout this example we will denote the tensor product of generic vectors by omitting the symbol of tensor product, e.g. $e_i \otimes e_j$ and $e_i \otimes e_j \otimes v_k$ will be denoted by e_ie_j and $e_ie_jv_k$ respectively.
Let us couple $\mathbf{2}\otimes\mathbf{2}$ first. We decompose $V_{\mathbf{2}}\otimes V_{\mathbf{2}}$ into irreducible submodules and write $V_{\mathbf{2}}\otimes V_{\mathbf{2}} = A \oplus B$ with ordered orthonormal weight bases:

$$\mathcal{A} = \left\{\frac{e_1e_2 - e_2e_1}{\sqrt{2}}\right\} \text{ of } A, \qquad \mathcal{B} = \left(e_1e_1, \frac{e_1e_2 + e_2e_1}{\sqrt{2}}, e_2e_2\right) \text{ of } B,$$

where $A \cong V_{\mathbf{1}}$, $B \cong V_{\mathbf{3}}$. We have that $(A \oplus B) \otimes V_{\mathbf{3}} \cong C \oplus (B \otimes V_{\mathbf{3}})$ where $C = A \otimes V_{\mathbf{3}} \cong V_{\mathbf{3}}$ with orthonormal weight basis:

$$\mathcal{C} = \left(\frac{e_1e_2v_i - e_2e_1v_i}{\sqrt{2}} \ : \ i = 1,2,3 \right) \text{ of } C.$$

We decompose $B \otimes V_{\mathbf{3}}$ into irreducible submodules and write $B \otimes V_{\mathbf{3}} = D \oplus E \oplus F$ with ordered orthonormal weight bases:

$$\mathcal{D} = \left\{ \frac{e_1e_1v_3}{\sqrt{3}} - \frac{e_1e_2v_2 + e_2e_1v_2}{\sqrt{6}} + \frac{e_2e_2v_1}{\sqrt{3}} \right\},$$

$$\mathcal{E} = \left(\frac{\sqrt{2}e_1e_1v_2 - e_1e_2v_1 - e_2e_1v_1}{2}, \frac{e_2e_1v_1 - e_2e_2v_1}{\sqrt{2}}, \frac{e_1e_2v_3 + e_2e_1v_3 - \sqrt{2}e_2e_2v_2}{2} \right)$$

of D, E respectively, where $D \cong V_{\mathbf{1}}$, $E \cong V_{\mathbf{3}}$, $F \cong V_{\mathbf{5}}$. In the end, we have obtained a decomposition $(V_{\mathbf{2}} \otimes V_{\mathbf{2}}) \otimes V_{\mathbf{3}} \cong C \oplus D \oplus E \oplus F$ that led to the choice of the aforementioned orthonormal weight bases. These choices are induced naturally by the representation theory of $SU(2)$.

Let us now couple $\mathbf{2} \otimes \mathbf{3}$ first. We have $V_{\mathbf{2}} \otimes V_{\mathbf{3}} = H \oplus K$ with ordered orthonormal weight bases:

$$\mathcal{H} = \left(\sqrt{\frac{2}{3}}e_2v_1 - \frac{e_1v_2}{\sqrt{3}}, \frac{e_2v_2}{\sqrt{3}} - \sqrt{\frac{2}{3}}e_1v_3 \right) \text{ of } H,$$

$$\mathcal{K} = \left(e_1v_1, \frac{e_2v_1}{\sqrt{3}} + \sqrt{\frac{2}{3}}e_1v_2, \sqrt{\frac{2}{3}}e_2v_2 + \frac{e_1v_3}{\sqrt{3}}, e_2v_3 \right) \text{ of } K,$$

where $H \cong V_{\mathbf{2}}$, $K \cong V_{\mathbf{4}}$. We have that $V_{\mathbf{2}} \otimes (H \oplus K) \cong (V_{\mathbf{2}} \otimes H) \oplus (V_{\mathbf{2}} \otimes K)$. We write $V_{\mathbf{2}} \otimes H = L \oplus M$ and $V_{\mathbf{2}} \otimes K = X \oplus Y$ with ordered orthonormal weight bases:

$$\mathcal{L} = \left\{ \frac{e_1e_2v_2}{\sqrt{6}} - \frac{e_1e_1v_3 + e_2e_2v_1}{\sqrt{3}} + \frac{e_2e_1v_2}{\sqrt{6}} \right\},$$

$$\mathcal{M} = \left(\frac{\sqrt{2}e_1e_2v_1 - e_1e_1v_2}{\sqrt{3}}, \frac{e_1e_2v_2 - e_2e_1v_2}{\sqrt{6}} + \frac{e_2e_2v_1 - e_1e_1v_3}{\sqrt{3}}, \frac{e_2e_2v_2 - \sqrt{2}e_2e_1v_3}{\sqrt{3}} \right)$$

of L and M respectively, where $L \cong V_{\mathbf{1}}$, $M \cong X \cong V_{\mathbf{3}}$, $Y \cong V_{\mathbf{5}}$. We have therefore the final decomposition $V_{\mathbf{2}} \otimes (V_{\mathbf{2}} \otimes V_{\mathbf{3}}) \cong L \oplus M \oplus X \oplus Y$ that led to the choice of the orthonormal weight bases above.

Let us see now some examples of recoupling coefficients:

- calling $\mathcal{D} = \{d\}$ and $\mathcal{L} = \{l\}$, we see that d and l are the opposite of each other, so we obtain the recoupling coefficient $\langle d|l \rangle = -1$;

- the following equality holds:

$$\frac{\sqrt{2}e_1e_2v_1 - e_1e_1v_2}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{e_1e_2v_1 - e_2e_1v_1}{\sqrt{2}} - \sqrt{\frac{2}{3}}\frac{\sqrt{2}e_1e_1v_2 - e_1e_2v_1 - e_2e_1v_1}{2},$$

namely, calling $\mathcal{C} = (c_1, c_2, c_3)$, $\mathcal{E} = (f_1, f_2, f_3)$, $\mathcal{M} = (m_1, m_2, m_3)$ we have:

$$m_1 = \frac{1}{\sqrt{3}}c_1 - \sqrt{\frac{2}{3}}f_1,$$

giving us the recoupling coefficients $\langle m_1|c_1\rangle = 1/\sqrt{3}$ and $\langle m_1|f_1\rangle = -\sqrt{2/3}$.

Proposition 1.2.1 (Unitarity of recoupling coefficients)**.** *Recall* (1.12) *and* (1.14) *and the notation we used there.*

1. *Fix irreducible representations* $\lambda_{12}, \lambda'_{12} \in \lambda_1 \otimes \lambda_2$ *with multiplicity labels* r_{12} *and* s_{12} *respectively. Fix an irreducible representation* $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ *with multiplicity labels* r *and* s*. Then:*

$$\sum_{r_{23}\lambda_{23}r'} \left\langle ((\lambda_1\lambda_2)s_{12}\lambda'_{12}, \lambda_3)s\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \left\langle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \middle| ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \right\rangle = \delta_{r_{12}s_{12}}\delta_{\lambda_{12}\lambda'_{12}}\delta_{rs}. \quad (1.19)$$

2. *Fix irreducible representations* $\lambda_{23}, \lambda'_{23} \in \lambda_2 \otimes \lambda_3$ *with multiplicity labels* r_{23} *and* s_{23} *respectively. Fix an irreducible representation* $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ *with multiplicity labels* r' *and* s'*. Then:*

$$\sum_{r_{12}\lambda_{12}r} \left\langle (\lambda_1(\lambda_2\lambda_3)s_{23}\lambda'_{23})s'\lambda \middle| ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \right\rangle \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle = \delta_{r_{23}s_{23}}\delta_{\lambda_{23}\lambda'_{23}}\delta_{r's'}. \quad (1.20)$$

3. *The following relation between recoupling coefficients holds:*

$$\left\langle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \middle| ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \right\rangle = \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle^* . \quad (1.21)$$

Proof. Let us prove 1.) Consider (1.16):

$$\left|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\right\rangle = \sum_{s_{12}\lambda'_{12}s} \left\langle ((\lambda_1\lambda_2)s_{12}\lambda'_{12}, \lambda_3)s\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \left|((\lambda_1\lambda_2)s_{12}\lambda'_{12}, \lambda_3)s\lambda l\right\rangle$$

and let us plug it into (1.17), getting:

$$\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda l\big\rangle = \sum_{r_{23}\lambda_{23}r'} \big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\big\rangle \,\big|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l\big\rangle$$
$$= \sum_{r_{23}\lambda_{23}r'} \big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\big\rangle$$
$$\times \sum_{s_{12}\lambda'_{12}s} \big\langle((\lambda_1\lambda_2)s_{12}\lambda'_{12},\lambda_3)s\lambda\big|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big\rangle \,\big|((\lambda_1\lambda_2)s_{12}\lambda'_{12},\lambda_3)s\lambda l\big\rangle$$
$$= \sum_{s_{12}\lambda'_{12}s}\left(\sum_{r_{23}\lambda_{23}r'} \big\langle((\lambda_1\lambda_2)s_{12}\lambda'_{12},\lambda_3)s\lambda\big|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big\rangle\right.$$
$$\left.\times \big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\big\rangle\right) \big|((\lambda_1\lambda_2)s_{12}\lambda'_{12},\lambda_3)s\lambda l\big\rangle\,.$$

The claim follows by comparing the first and the last term of the equation above. By doing this the other way around, we get 2.), whereas 3.) follows from Fact A.2.1 in the same way we proved 3.) of Fact A.2.2. □

The following result is stated in [But81, Equation (3.2.17)] and [LB92, Equation (28)]. We propose here a slightly different version and give a proof for completeness.

Lemma 1.2.1. *We recall the notation from* (1.12)–(1.15).

1. *Fix irreducible representations $\lambda_{23} \in \lambda_2 \otimes \lambda_3$ and $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels r_{23} and r' respectively. Let $l \in \{1, \dots, |\lambda|\}$. Then:*

$$\sum_{l_{23}} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r'\lambda l|\lambda_1 l_1, \lambda_{23}l_{23}\rangle =$$
$$= \sum_{r_{12}\lambda_{12}l_{12}r} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\big\rangle\,. \tag{1.22}$$

2. *Fix irreducible representations $\lambda_{12} \in \lambda_1 \otimes \lambda_2$ and $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels r_{12} and r respectively. Let $l \in \{1, \dots, |\lambda|\}$. Then:*

$$\sum_{l_{12}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle =$$
$$= \sum_{r' r_{23}\lambda_{23}l_{23}} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r'\lambda l|\lambda_1 l_1, \lambda_{23}l_{23}\rangle$$
$$\times \big\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\big|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\big\rangle\,. \tag{1.23}$$

Proof. The first statement follows by plugging (1.17) into (1.13) and comparing the result with (1.15):

$$\sum_{r_{23}\lambda_{23}l_{23}} \sum_{r'\lambda l} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r'\lambda l|\lambda_1 l_1, \lambda_{23}l_{23}\rangle \, |{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\rangle$$
$$= |\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3\rangle$$
$$= \sum_{r_{12}\lambda_{12}l_{12}} \sum_{r\lambda l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle \, |{\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)}r\lambda l\rangle$$
$$= \sum_{r_{12}\lambda_{12}l_{12}} \sum_{r\lambda l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \sum_{r_{23}\lambda_{23}r'} \langle{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda|{\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)}r\lambda\rangle \, |{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda l\rangle \, .$$

Analogously, the second statement follows by plugging (1.16) into (1.15) and comparing with (1.13). □

The next statement from [GJ15, Equation (3.1)] highlights how to write a recoupling coefficient in terms of coupling coefficients. We give a proof for completeness.

Corollary 1.2.1. *Recall once again (1.12)–(1.15) and the notation introduced there. Fix irreducible representations $\lambda_{12} \in \lambda_1 \otimes \lambda_2$ and $\lambda_{23} \in \lambda_2 \otimes \lambda_3$ with multiplicity labels r_{12} and r_{23} respectively. Fix an irreducible representation $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels r and r'. Then:*

$$|\lambda| \, \langle{\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)}r\lambda|{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})}r'\lambda\rangle =$$
$$= \sum_{l_1 l_2 l_3} \sum_{l_{12}l_{23}l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle^* \langle r'\lambda l|\lambda_1 l_1, \lambda_{23}l_{23}\rangle^* \, . \quad (1.24)$$

Proof. Multiplying both sides of (1.23) by $\langle\lambda_2 l_2, \lambda_3 l_3|s_{23}\mu_{23}m_{23}\rangle \langle\lambda_1 l_1, \mu_{23}m_{23}|s'\lambda l\rangle$ and summing over l_1, l_2, l_3, m_{23}, the left-hand side becomes:

$$\sum_{l_{12}} \sum_{l_1 l_2 l_3 m_{23}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \langle\lambda_2 l_2, \lambda_3 l_3|s_{23}\mu_{23}m_{23}\rangle \langle\lambda_1 l_1, \mu_{23}m_{23}|s'\lambda l\rangle \, ,$$

whereas the right-hand side becomes:

$$\sum_{r' r_{23}\lambda_{23}l_{23}} \left(\sum_{l_2 l_3} \langle\lambda_2 l_2, \lambda_3 l_3|s_{23}\mu_{23}m_{23}\rangle \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \right)$$

$$\times \left(\sum_{m_{23} l_1} \langle \lambda_1 l_1, \mu_{23} m_{23} | s' \lambda l \rangle \langle r' \lambda l | \lambda_1 l_1, \lambda_{23} l_{23} \rangle \right)$$
$$\times \left\langle ((\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) r_{23} \lambda_{23}) r' \lambda \right\rangle =$$
$$= \sum_{r'} \left(\sum_{m_{23} l_1} \langle \lambda_1 l_1, \mu_{23} m_{23} | s' \lambda l \rangle \langle r' \lambda l | \lambda_1 l_1, \mu_{23} m_{23} \rangle \right)$$
$$\times \left\langle ((\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) s_{23} \mu_{23}) r' \lambda \right\rangle =$$
$$= \left\langle ((\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) s_{23} \mu_{23}) s' \lambda \right\rangle ,$$

where we have applied 1.) of Fact 1.1.2 twice. We obtain:

$$\sum_{l_{12} l_1 l_2 l_3 m_{23}} \langle r_{12} \lambda_{12} l_{12} | \lambda_1 l_1, \lambda_2 l_2 \rangle \langle r \lambda l | \lambda_{12} l_{12}, \lambda_3 l_3 \rangle$$
$$\times \langle s_{23} \mu_{23} m_{23} | \lambda_2 l_2, \lambda_3 l_3 \rangle^* \langle s' \lambda l | \lambda_1 l_1, \mu_{23} m_{23} \rangle^* =$$
$$= \left\langle ((\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) s_{23} \mu_{23}) s' \lambda \right\rangle ,$$

where we have applied 3.) of Fact 1.1.2. Summing now over l we get:

$$\sum_{l l_{12} l_1} \sum_{l_2 l_3 m_{23}} \langle r_{12} \lambda_{12} l_{12} | \lambda_1 l_1, \lambda_2 l_2 \rangle \langle r \lambda l | \lambda_{12} l_{12}, \lambda_3 l_3 \rangle$$
$$\times \langle s_{23} \mu_{23} m_{23} | \lambda_2 l_2, \lambda_3 l_3 \rangle^* \langle s' \lambda l | \lambda_1 l_1, \mu_{23} m_{23} \rangle^* =$$
$$= |\lambda| \left\langle ((\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) s_{23} \mu_{23}) s' \lambda \right\rangle .$$

Renaming $s_{23}, \mu_{23}, m_{23}, s'$ to $r_{23}, \lambda_{23}, l_{23}, r'$ respectively, we get the claim. □

Regarding the different ways of coupling $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$, there is of course a third case in which we first couple λ_1 with λ_3:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \cong V_{\lambda_2} \otimes \left(V_{\lambda_1} \otimes V_{\lambda_3} \right) \cong V_{\lambda_2} \otimes \left(\bigoplus_{r_{13} \lambda_{13}} V_{\lambda_{13}}^{(r_{13})} \right) \cong$$
$$\cong \bigoplus_{r_{13} \lambda_{13}} \left(V_{\lambda_2} \otimes V_{\lambda_{13}}^{(r_{13})} \right) \cong \bigoplus_{r_{13} \lambda_{13}} \bigoplus_{s \lambda} V_{\lambda}^{(s)}, \quad (1.25)$$

where $\bigoplus_{r_{13}\lambda_{13}} V_{\lambda_{13}}^{(r_{13})}$ and $\bigoplus_{s\lambda} V_{\lambda}^{(s)}$ denote fixed decompositions of $V_{\lambda_1} \otimes V_{\lambda_3}$ and $V_{\lambda_2} \otimes V_{\lambda_{13}}^{(r_{13})}$ into irreducible modules respectively (both s and λ depend on λ_{13} and r_{13}). We have:

$$|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3\rangle = \sum_{r_{13}\lambda_{13}l_{13}} \langle r_{13}\lambda_{13}l_{13}|\lambda_1 l_1, \lambda_3 l_3\rangle \, |\lambda_2 l_2, {\scriptstyle (\lambda_1\lambda_3)} r_{13}\lambda_{13}l_{13}\rangle =$$

$$= \sum_{r_{13}\lambda_{13}l_{13}} \sum_{s\lambda l} \langle r_{13}\lambda_{13}l_{13}|\lambda_1 l_1, \lambda_3 l_3\rangle \, \langle s\lambda l|\lambda_{13}l_{13}, \lambda_2 l_2\rangle \, |{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle \,, \quad (1.26)$$

where $(|r_{13}\lambda_{13}l_{13}\rangle : l_{13} = 1, \ldots, |\lambda_{13}|)$ and $(|{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle : l = 1, \ldots, |\lambda|)$ are chosen orthonormal bases of $V_{\lambda_{13}}^{(r_{13})}$ and $V_{\lambda}^{(s)}$ respectively. All the considerations regarding the notation hold as above.
In this third way of coupling, we end up with a third basis given by the vectors $|{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle$. Let us analyze the transformation between this third basis and the one we got when we coupled first λ_1, λ_2. We have:

$$|{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle = \sum_{r_{12}\lambda_{12}r} \langle {\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda | {\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda\rangle \, |{\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda l\rangle \,, \quad (1.27)$$

$$|{\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda l\rangle = \sum_{r_{13}\lambda_{13}s} \langle {\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda | {\scriptstyle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)} r\lambda\rangle \, |{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle \,. \quad (1.28)$$

Let us now analyze the transformation between the third basis and the second one, the one obtained coupling first λ_2, λ_3. We have:

$$|{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle = \sum_{r_{23}\lambda_{23}r'} \langle {\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})} r'\lambda | {\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda\rangle \, |{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})} r'\lambda l\rangle \,, \quad (1.29)$$

$$|{\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})} r'\lambda l\rangle = \sum_{r_{13}\lambda_{13}s} \langle {\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda | {\scriptstyle (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})} r'\lambda\rangle \, |{\scriptstyle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})} s\lambda l\rangle \,. \quad (1.30)$$

1.3 jm-symbols and j-phases

In this section we define and study the njm-symbols and nj-phases with $n = 2, 3$, which are presented in [DS65] with other names and notations. They are also defined in different fashions in [But75] and [But81]. The meaning of the terminology is the following: we use the letter j only, e.g. for a $2j$-phase, when we have the dependence exclusively on the representation, whereas we include the letter m as well, e.g. for a $3jm$-symbol, when we have dependence also on the chosen basis. We will make use of the concepts presented in Chapter A.

For this entire section, let us fix two unitary finite-dimensional irreducible representations λ_1, λ_2. Let us set our general framework.

- By Theorem A.2.2, we can decompose $V_{\lambda_1} \otimes V_{\lambda_2}$ into a direct sum of irreducible modules, so let us fix an isomorphism of modules Ψ such that:

$$\Psi\colon V_{\lambda_1} \otimes V_{\lambda_2} \to \bigoplus_{r\lambda} V_\lambda^{(r)}, \tag{1.31}$$

  where λ tracks the irreducible representations and r labels their multiplicity. By Proposition A.2.2, we can consider Ψ to be an isometry of modules. Denote by $\mathcal{I}$ the set of irreducible representations λ we chose for such a specific decomposition. Furthermore, denote by $\mathcal{S}$ the set of representations of $\mathcal{I}$ that are equivalent to their dual and by $\mathcal{C}$ the set of representations of $\mathcal{I}$ which are not. Call $\bar{\mathcal{I}}$ the set obtained by taking the dual to every element of $\mathcal{I}$. In particular, if $\lambda \in \mathcal{I}$ we consider each module $V_\lambda^{(r)}$ to be equal to V_λ.

- Choose orthonormal bases $\mathcal{B}_{\lambda_i} = (|\lambda_i l_i\rangle : l_i = 1, \dots, |\lambda_i|)$ of V_{λ_i} for $i = 1, 2$ and $\mathcal{B}_{r\lambda} = (|r\lambda l\rangle : l = 1, \dots, |\lambda|)$ of $V_\lambda^{(r)}$ for any $\lambda \in \mathcal{I}$, where $\mathcal{B}_{r\lambda}$ coincides with $\mathcal{B}_{s\lambda}$ for any r, s in terms of vectors of V_λ. Let us make further assumptions on the choice of such bases, i.e. if $\lambda \in \mathcal{I}$ we will consider the following prescription:

  – CASE $\lambda \cong \bar{\lambda}$. Fix an isometry of modules $\psi_{r\lambda}\colon V_\lambda^{(r)} \to V_{\bar{\lambda}}^{(r)}$, where $V_{\bar{\lambda}}^{(r)}$ is the dual of $V_\lambda^{(r)}$. We will consider the following basis of $V_{\bar{\lambda}}^{(r)}$:

$$\mathcal{B}_{r\bar{\lambda}} = \psi_{r\lambda}\left(\mathcal{B}_{r\lambda}\right) = \left(\psi_{r\lambda}\left(|r\lambda 1\rangle\right), \dots, \psi_{r\lambda}\left(|r\lambda\, |\lambda|\rangle\right)\right). \tag{1.32}$$

  Such a choice is consistent when the considered isometry from $V_{\bar{\lambda}}^{(r)}$ to $V_\lambda^{(r)}$ is simply $\psi_{r\bar{\lambda}} = \psi_{r\lambda}^{-1}$. It is easy to check the following property:

$$\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g)) = \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g)) \quad \forall g \in G. \tag{1.33}$$

  – CASE $\lambda \ncong \bar{\lambda}$. We will consider the following basis of $V_{\bar{\lambda}}^{(r)}$:

$$\mathcal{B}_{r\bar{\lambda}} = \left(\mathcal{B}_{r\lambda}\right)^\vee. \tag{1.34}$$

  In this case, (A.4) tells us that:

$$\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g)) = \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))^* \quad \forall g \in G. \tag{1.35}$$

1.3.1 Definition of jm-symbols

The goal of this subsection is to formalize the definition of the $2jm$ and $3jm$-symbols. This is done by showing the following proposition, which is introduced in [DS65, Equation (2.1)] and recalled in [But75, Equation (5.8)]:

Proposition 1.3.1. *With the notations and the settings established at the beginning of Section 1.3, there exist some coefficients* $\begin{pmatrix}\lambda_1 & \lambda_2 & \mu\\ l_1 & l_2 & l\end{pmatrix}_r$ *satisfying the following equality:*

$$\begin{aligned}&\left[\mathcal{M}_{\mathcal{B}_{\lambda_1}}(\lambda_1(g))\right]_{i_1j_1}\left[\mathcal{M}_{\mathcal{B}_{\lambda_2}}(\lambda_2(g))\right]_{i_2j_2}=\\&\qquad=\sum_{\mu\in\bar{\mathcal{I}}}\sum_{r,ab}|\mu|\begin{pmatrix}\lambda_1 & \lambda_2 & \mu\\ i_1 & i_2 & a\end{pmatrix}_r^*\left[\mathcal{M}_{\mathcal{B}_{r\mu}}(\mu(g))\right]_{ab}^*\begin{pmatrix}\lambda_1 & \lambda_2 & \mu\\ j_1 & j_2 & b\end{pmatrix}_r.\end{aligned}\tag{1.36}$$

We will derive (1.36) and give an explicit expression for the involved coefficients in the steps described below.

- Set $\mathcal{A}:=\mathcal{B}_{\lambda_1}\otimes\mathcal{B}_{\lambda_2}$ and $\mathcal{B}:=\bigoplus_{r\lambda}\mathcal{B}_{r\lambda}$, which are orthonormal bases of $V_{\lambda_1}\otimes V_{\lambda_2}$ and $\bigoplus_{r\lambda}V_\lambda^{(r)}$ respectively.
- We know that $V_{\lambda_1}\otimes V_{\lambda_2}$ and $\bigoplus_{r\lambda}V_\lambda^{(r)}$ are the modules associated with the representations $\rho:=\lambda_1\otimes\lambda_2$ and $\sigma:=\bigoplus_{r\lambda}\lambda^{(r)}$ respectively, where $\lambda^{(r)}=\lambda$ for any r.
- Fix $g\in G$. Call $A:=\mathcal{M}_{\mathcal{B}\mathcal{A}}(\Psi)$ the matrix associated with Ψ with respect to the bases $\mathcal{A}$ of its domain and $\mathcal{B}$ of its codomain. Notice that A is unitary by Fact A.2.1. By Remark A.1.2, we have that:

$$\mathcal{M}_{\mathcal{A}}(\rho(g))=A^{-1}\cdot\mathcal{M}_{\mathcal{B}}(\sigma(g))\cdot A=A^\dagger\cdot\mathcal{M}_{\mathcal{B}}(\sigma(g))\cdot A.\tag{1.37}$$

  By definition of $\mathcal{A}$, $\mathcal{B}$, ρ, σ, we have that:

$$\mathcal{M}_{\mathcal{A}}(\rho(g))=\mathcal{M}_{\mathcal{B}_{\lambda_1}}(\lambda_1(g))\otimes\mathcal{M}_{\mathcal{B}_{\lambda_2}}(\lambda_2(g)),\tag{1.38}$$

$$\mathcal{M}_{\mathcal{B}}(\sigma(g))=\bigoplus_{r\lambda}\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g)).\tag{1.39}$$

  Putting everything together, we obtain:

$$\mathcal{M}_{\mathcal{B}_{\lambda_1}}(\lambda_1(g))\otimes\mathcal{M}_{\mathcal{B}_{\lambda_2}}(\lambda_2(g))=A^\dagger\cdot\bigoplus_{r\lambda}\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\cdot A,\tag{1.40}$$

  implying:

$$\begin{aligned}&\left[\mathcal{M}_{\mathcal{B}_{\lambda_1}}(\lambda_1(g))\right]_{i_1j_1}\left[\mathcal{M}_{\mathcal{B}_{\lambda_2}}(\lambda_2(g))\right]_{i_2j_2}=\\&\qquad=\sum_{b,c}\left[A^\dagger\right]_{(i_1i_2)b}\cdot\bigoplus_{r\lambda}\left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{bc}\cdot[A]_{c(j_1j_2)}\end{aligned}$$

$$= \sum_{\lambda \in \mathcal{I}} \sum_{r=0}^{\mathrm{m}_{\lambda_1 \otimes \lambda_2}^{\lambda} - 1} \sum_{l,m=1}^{|\lambda|} \left[A^{\dagger}\right]_{(i_1 i_2)(l+r|\lambda|)} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{lm} A_{(m+r|\lambda|)(j_1 j_2)}$$

$$= \sum_{\lambda \in \mathcal{I}} \sum_{r=0}^{\mathrm{m}_{\lambda_1 \otimes \lambda_2}^{\lambda} - 1} \sum_{l,m=1}^{|\lambda|} [A]^{*}_{(l+r|\lambda|)(i_1 i_2)} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{lm} A_{(m+r|\lambda|)(j_1 j_2)}. \quad (1.41)$$

Notice that by definition of A we have:

$$A_{(m+r|\lambda|)(j_1 j_2)} = \langle r\lambda m | \lambda_1 j_1, \lambda_2 j_2 \rangle \, . \quad (1.42)$$

- Let us analyze the case in which an irreducible representation of $\mathcal{I}$ is either equivalent to its dual or not. Fix $\lambda \in \mathcal{I}$ with multiplicity label r.

  - CASE $\lambda \cong \bar{\lambda}$.
    Recall the isometry of modules $\psi_{r\lambda} \colon V_{\lambda}^{(r)} \to V_{\bar{\lambda}}^{(r)}$ we fixed at the beginning of the current section and the choice of the following basis of $V_{\bar{\lambda}}^{(r)}$: $\mathcal{B}_{r\bar{\lambda}} = \psi_{r\lambda}(\mathcal{B}_{r\lambda})$. Consider the matrix associated with $\psi_{r\lambda}$ with respect to the bases $\mathcal{B}_{r\lambda}$ of its domain and the dual basis $\mathcal{B}_{r\lambda}^{\vee}$ of its codomain:

    $$T := \mathcal{M}_{\mathcal{B}_{r\lambda}^{\vee} \mathcal{B}_{r\lambda}}(\psi_{r\lambda}). \quad (1.43)$$

    Notice that T is unitary by Fact A.2.1. By definition of T and (A.4), we have:

    $$\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g)) = T^{-1} \cdot \mathcal{M}_{\mathcal{B}_{r\lambda}^{\vee}}(\bar{\lambda}(g)) \cdot T = T^{-1} \cdot \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))^{*} \cdot T. \quad (1.44)$$

    If F is any other matrix such that $\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h)) = F^{-1} \cdot \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h))^{*} \cdot F$ for any h in G, we have that TF^{-1} commutes with $\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h))$ for any h in G, making TF^{-1} a multiple of the identity matrix by Schur's Lemma, implying $F = kT$ for some $k \in \mathbb{C}$. Hence, since we want to define a unitary matrix (λ) satisfying:

    $$\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h)) = \left({}^{\mathrm{t}}(\lambda)\right)^{-1} \cdot \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h)) \cdot {}^{\mathrm{t}}(\lambda) = (\lambda)^{*} \cdot \mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(h)) \cdot {}^{\mathrm{t}}(\lambda) \quad (1.45)$$

    for any h in G, we have only one option, that is to choose a unitary scalar $d_{\lambda} \in \mathbb{C}$ and define:

    $$(\lambda) := d_{\lambda} {}^{\mathrm{t}}T. \quad (1.46)$$

    By definition of (λ) and the unitarity of d_{λ}, we get:

    $$\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))_{lm} = \sum_{a,b=1}^{|\lambda|} (\lambda)^{*}_{la} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]^{*}_{ab} (\lambda)_{mb}. \quad (1.47)$$

We define $(\bar{\lambda})$ analogously, getting:

$$\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))_{lm} = \sum_{a,b=1}^{|\lambda|} (\bar{\lambda})^{*}_{la} \left[\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))\right]^{*}_{ab} (\bar{\lambda})_{mb}, \tag{1.48}$$

but since (1.33) implies $\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g)) = \mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))$, we can set:

$$(\bar{\lambda}) = (\lambda) \tag{1.49}$$

and write:

$$\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))_{lm} = \sum_{a,b=1}^{|\lambda|} (\lambda)^{*}_{la} \left[\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))\right]^{*}_{ab} (\lambda)_{mb}. \tag{1.50}$$

– CASE $\lambda \ncong \bar{\lambda}$.
Recall the choice of the following basis of $V_{\bar{\lambda}}^{(r)}$: $\mathcal{B}_{r\bar{\lambda}} = (\mathcal{B}_{r\lambda})^{\vee}$. We consider an isometry of modules $\psi_{r\lambda} \colon V_{\lambda}^{(r)} \to V_{\lambda}^{(r)}$. By Schur's Lemma, $\psi_{\lambda} = c_{\lambda} \operatorname{id}_{V_{\lambda}}$ for some complex number c_{λ} of module 1. Therefore, $\mathcal{M}_{\mathcal{B}_{r\lambda}}(\psi_{r\lambda}) = c_{\lambda} I_{|\lambda|}$ where $I_{|\lambda|}$ is the $|\lambda| \times |\lambda|$ identity matrix. We define the following unitary matrix:

$$(\lambda) \coloneqq {}^{\mathrm{t}}\!\left(c_{\lambda} I_{|\lambda|}\right) = c_{\lambda} I_{|\lambda|}. \tag{1.51}$$

Then:

$$\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))_{lm} = \sum_{l,m=1}^{|\lambda|} (\lambda)^{*}_{la} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{ab} (\lambda)_{mb}, \tag{1.52}$$

or equivalently, by (1.35):

$$\left[\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))\right]^{*}_{lm} = \sum_{l,m=1}^{|\lambda|} (\lambda)^{*}_{la} \left[\mathcal{M}_{\mathcal{B}_{r\bar{\lambda}}}(\bar{\lambda}(g))\right]^{*}_{ab} (\lambda)_{mb}. \tag{1.53}$$

- Let us continue (1.41):

$$\begin{aligned}
&\left[\mathcal{M}_{\mathcal{B}_1}(\lambda_1(g))\right]_{i_1 j_1} \left[\mathcal{M}_{\mathcal{B}_2}(\lambda_2(g))\right]_{i_2 j_2} = \\
&= \sum_{\lambda \in \mathcal{I}} \sum_{r,lm} [A]^{*}_{(l+r|\lambda|)(i_1 i_2)} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{lm} A_{(m+r|\lambda|)(j_1 j_2)} \\
&= \sum_{\lambda \in \mathcal{S}} \sum_{r,lm} [A]^{*}_{(l+r|\lambda|)(i_1 i_2)} \left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{lm} A_{(m+r|\lambda|)(j_1 j_2)}
\end{aligned}$$

$$+\sum_{\lambda\in\mathcal{C}}\sum_{r,lm}[A]^*_{(l+r|\lambda|)(i_1i_2)}\left[\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))\right]_{lm}A_{(m+r|\lambda|)(j_1j_2)},$$

apply (1.33) and (1.35):

$$=\sum_{\lambda\in\mathcal{S}}\sum_{r,lm}[A]^*_{(l+r|\lambda|)(i_1i_2)}\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]_{lm}A_{(m+r|\lambda|)(j_1j_2)}$$
$$+\sum_{\lambda\in\mathcal{C}}\sum_{r,lm}[A]^*_{(l+r|\lambda|)(i_1i_2)}\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]^*_{lm}A_{(m+r|\lambda|)(j_1j_2)},$$

apply (1.50) and (1.53):

$$=\sum_{\lambda\in\mathcal{S}}\sum_{r,lm}\sum_{a,b}[A]^*_{(l+r|\lambda|)(i_1i_2)}(\lambda)^*_{la}\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]^*_{ab}(\lambda)_{mb}A_{(m+r|\lambda|)(j_1j_2)}$$
$$+\sum_{\lambda\in\mathcal{C}}\sum_{r,lm}\sum_{a,b}[A]^*_{(l+r|\lambda|)(i_1i_2)}(\lambda)^*_{la}\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]^*_{lm}(\lambda)_{mb}A_{(m+r|\lambda|)(j_1j_2)}$$
$$=\sum_{\lambda\in\mathcal{I}}\sum_{r,lm}\sum_{a,b}[A]^*_{(l+r|\lambda|)(i_1i_2)}(\lambda)^*_{la}\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]^*_{ab}(\lambda)_{mb}A_{(m+r|\lambda|)(j_1j_2)}$$
$$=\sum_{\lambda\in\mathcal{I}}\sum_{r,ab}|\lambda|\begin{pmatrix}\lambda_1&\lambda_2&\bar\lambda\\ i_1&i_2&a\end{pmatrix}^*_r\left[\mathcal{M}_{\mathcal{B}_{r\bar\lambda}}(\bar\lambda(g))\right]^*_{ab}\begin{pmatrix}\lambda_1&\lambda_2&\bar\lambda\\ j_1&j_2&b\end{pmatrix}_r$$
$$=\sum_{\mu\in\bar{\mathcal{I}}}\sum_{r,ab}|\mu|\begin{pmatrix}\lambda_1&\lambda_2&\mu\\ i_1&i_2&a\end{pmatrix}^*_r\left[\mathcal{M}_{\mathcal{B}_{r\mu}}(\mu(g))\right]^*_{ab}\begin{pmatrix}\lambda_1&\lambda_2&\mu\\ j_1&j_2&b\end{pmatrix}_r,$$

where, recalling (1.42):

$$\begin{pmatrix}\lambda_1&\lambda_2&\bar\lambda\\ j_1&j_2&b\end{pmatrix}_r:=\frac{1}{\sqrt{|\lambda|}}\sum_{m=1}^{|\lambda|}(\lambda)_{mb}\left\langle r\lambda m|\lambda_1j_1,\lambda_2j_2\right\rangle, \tag{1.54}$$

which is exactly the definition used in [But75, Equation (5.2)]. This shows now Proposition 1.3.1.

Definition 1.3.1. The coefficients $(\lambda)_{ij}$ described by (1.46) and (1.51) are called **$2jm$-symbols**, whereas the coefficients $\begin{pmatrix}\lambda_1&\lambda_2&\bar\lambda\\ j_1&j_2&b\end{pmatrix}_r$ defined by (1.54) are called **$3jm$-symbols**.

Remark **1.3.1.** 1. The term $2jm$-symbol we presented in Definition 1.3.1 is the one used in [But81]. The same object is called $1j$*-symbol* in [DS65] and $1jm$*-symbol* in [But75].
The term $3jm$-symbol we proposed in Definition 1.3.1 is the one used in

[But75] and [But81]. The same object is called $3j$-*symbol* in [DS65]. The reason to call $\begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda} \\ j_1 & j_2 & b \end{pmatrix}_r$ a $3jm$-symbol comes from its dependence on three representations $\lambda_1, \lambda_2, \bar{\lambda}$ (expressed by the "$3j$") and on three basis indices j_1, j_2, b (expressed by the use of the letter "m").

2. In (1.54), we denoted the $3jm$-symbols through a notation that goes under the name of *Wigner's notation* (see [But75]). This is also the choice made in [But81]. In [DS65], the following symbol is used instead:

$$(\lambda_1 \lambda_2 \mu)_{r, i_1 i_2 a} = \begin{pmatrix} \lambda_1 & \lambda_2 & \mu \\ i_1 & i_2 & a \end{pmatrix}_r . \tag{1.55}$$

Remark 1.3.2. Mentioned in [DS65, Equation (2.4)] and [But75, Equation (5.1)], in principle we could have defined a $3jm$-symbol even more generally in the following way:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda} \\ j_1 & j_2 & b \end{pmatrix}_r = \sum_s H(\lambda_1 \lambda_2 \lambda)_{rs} \frac{1}{\sqrt{|\lambda|}} \sum_{m=1}^{|\lambda|} (\lambda)_{mb} \langle s \lambda m | \lambda_1 j_1, \lambda_2 j_2 \rangle \tag{1.56}$$

for some unitary matrix $H(\lambda_1 \lambda_2 \lambda)$ depending on $\lambda_1, \lambda_2, \lambda$ only. Indeed, the terms $H(\lambda_1 \lambda_2 \lambda)_{rs}$ get cancelled in (1.36) by their unitarity.

Remark 1.3.3. Notice that $2jm$-symbols are particular cases of $3jm$-symbols. Indeed, with the notations above, we have:

$$\begin{pmatrix} \lambda & \mathbf{1} & \bar{\lambda} \\ j & 0 & b \end{pmatrix}_0 = \frac{1}{\sqrt{|\lambda|}} (\lambda)_{jb}. \tag{1.57}$$

This is the reason behind the name "$2jm$-symbols": as it is done in [But81], we could define the symbol $\begin{pmatrix} \lambda & \mu \\ j & b \end{pmatrix}$ for finite-dimensional irreducible representations λ, μ to be equal to 0 when $\mu \ncong \bar{\lambda}$ and to $|\lambda|^{-1/2} (\lambda)_{jb}$ when $\mu \cong \bar{\lambda}$. The "$2j$" expresses the dependence on two representations λ and μ, whereas the letter "m" expresses the dependence on the two basis indices j, b.
To conclude, if we consider the matrix H as well in the definition of $3jm$-symbols, more generally we get:

$$\begin{pmatrix} \lambda & \mathbf{1} & \bar{\lambda} \\ j & 0 & b \end{pmatrix}_0 = \frac{1}{\sqrt{|\lambda|}} H(\lambda, \mathbf{1}, \bar{\lambda}) (\lambda)_{jb}. \tag{1.58}$$

Remark 1.3.4. From the definition of $3jm$-symbols and the unitarity of the $2jm$-symbols, it follows that:

$$\langle r\lambda m|\lambda_1 j_1, \lambda_2 j_2\rangle = \sqrt{|\lambda|}\sum_{b=1}^{|\lambda|}(\lambda)^*_{mb}\begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\lambda} \\ j_1 & j_2 & b\end{pmatrix}_r. \tag{1.59}$$

More generally, we find:

$$\langle r\lambda m|\lambda_1 j_1, \lambda_2 j_2\rangle = \sqrt{|\lambda|}\sum_s K(\lambda_1\lambda_2\lambda)_{rs}\sum_{b=1}^{|\lambda|}(\lambda)^*_{mb}\begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\lambda} \\ j_1 & j_2 & b\end{pmatrix}_s, \tag{1.60}$$

where the matrix $K(\lambda_1\lambda_2\lambda)$ is the inverse of $H(\lambda_1\lambda_2\lambda)$, introduced by (1.56).

1.3.2 Properties of jm-symbols and Definition of j-phases

We want now to analyze the various properties of the jm-symbols.

Proposition 1.3.2 (Unitarity of $2jm$-symbols)**.** *Let $\lambda \in \mathcal{I}$. Then (λ) is a unitary matrix, namely:*

$$\sum_{a=1}^{|\lambda|}(\lambda)^*_{al}(\lambda)_{am} = \sum_{a=1}^{|\lambda|}(\lambda)^*_{la}(\lambda)_{ma} = \delta_{lm}. \tag{1.61}$$

Proof. The unitarity of (λ) follows directly from the way it is defined by (1.46) in the case $\lambda \cong \bar{\lambda}$ and by (1.51) in the case $\lambda \ncong \bar{\lambda}$. □

Proposition 1.3.3 (Unitarity of $3jm$-symbols)**.** *Consider the framework and the notations established so far. Let $\lambda, \mu \in \mathcal{I}$. Then the following equalities hold:*

$$\sum_{l_1 l_2}\begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\lambda} \\ l_1 & l_2 & l\end{pmatrix}_r \begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\mu} \\ l_1 & l_2 & m\end{pmatrix}_s^* = \frac{1}{|\lambda|}\delta_{rs}\delta_{lm}\delta_{\lambda\mu}, \tag{1.62}$$

$$\sum_{r\lambda l}\begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\lambda} \\ l_1 & l_2 & l\end{pmatrix}_r \begin{pmatrix}\lambda_1 & \lambda_2 & \bar{\lambda} \\ m_1 & m_2 & l\end{pmatrix}_r^* = \frac{1}{|\lambda|}\delta_{l_1 m_1}\delta_{l_2 m_2}. \tag{1.63}$$

Proof. The left-hand side of (1.62) indeed coincides with:

$$\frac{1}{\sqrt{|\lambda|}}\frac{1}{\sqrt{|\mu|}}\sum_{a=1}^{|\lambda|}\sum_{b=1}^{|\mu|}(\lambda)_{al}(\mu)^*_{bm}\sum_{l_1 l_2}\langle r\lambda a|\lambda_1 l_1, \lambda_2 l_2\rangle\langle\lambda_1 l_1, \lambda_2 l_2|s\mu b\rangle$$

$$=\frac{1}{\sqrt{|\lambda|}}\frac{1}{\sqrt{|\mu|}}\sum_{a=1}^{|\lambda|}\sum_{b=1}^{|\mu|}(\lambda)_{al}(\mu)^*_{bm}\delta_{rs}\delta_{\lambda\mu}\delta_{ab} = \frac{1}{|\lambda|}\sum_{a=1}^{|\lambda|}(\lambda)_{al}(\lambda)^*_{am}\delta_{rs}\delta_{\lambda\mu} = \frac{1}{|\lambda|}\delta_{lm}\delta_{rs}\delta_{\lambda\mu},$$

where we have used the definition of $3jm$-symbols via (1.54), 3.) and 1.) of Fact 1.1.2 and the unitarity of $2jm$-symbols via (1.61).
The left-hand side of (1.63) indeed coincides with:

$$\frac{1}{|\lambda|}\sum_{r\lambda}\sum_{a,b=1}^{|\lambda|}\left(\sum_{l=1}^{|\lambda|}(\lambda)_{al}(\lambda)^*_{bl}\right)\langle\lambda_1 m_1,\lambda_2 m_2|r\lambda b\rangle\langle r\lambda a|\lambda_1 l_1,\lambda_2 l_2\rangle$$
$$=\frac{1}{|\lambda|}\sum_{r\lambda a}\langle\lambda_1 m_1,\lambda_2 m_2|r\lambda a\rangle\langle r\lambda a|\lambda_1 l_1,\lambda_2 l_2\rangle=\frac{1}{|\lambda|}\delta_{l_1 m_1}\delta_{l_2 m_2},$$

where we have used the definition of $3jm$-symbols via (1.54), the unitarity of $2jm$-symbols via (1.61), 3.) and 2.) of Fact 1.1.2. □

Recall the meaning of the real, quaternionic, self-dual, complex representation terminology illustrated by Definition A.2.3.

Proposition 1.3.4. *Let $\lambda\in\mathcal{I}$. Then:*

$$^{\mathrm{t}}(\lambda)=\beta_\lambda(\lambda)\qquad\textit{where}\qquad\begin{cases}\beta_\lambda=1 & \lambda\textit{ is either real or complex,}\\ \beta_\lambda=-1 & \lambda\textit{ is quaternionic;}\end{cases}\tag{1.64}$$

$$(\bar\lambda)=\gamma_\lambda(\lambda)\qquad\textit{where}\qquad\begin{cases}\gamma_\lambda=1 & \lambda\textit{ is self-dual,}\\ \gamma_\lambda=c_{\bar\lambda}/c_\lambda & \lambda\textit{ is complex;}\end{cases}\tag{1.65}$$

$$^{\mathrm{t}}(\bar\lambda)=\phi_\lambda(\lambda)\qquad\textit{where}\qquad\phi_\lambda=\gamma_\lambda\beta_\lambda=\begin{cases}1 & \lambda\textit{ is real,}\\ -1 & \lambda\textit{ is quaternionic,}\\ c_{\bar\lambda}/c_\lambda & \lambda\textit{ is complex.}\end{cases}\tag{1.66}$$

Furthermore, we have the following relations:

$$\beta_{\bar\lambda}=\beta^*_\lambda=\beta^{-1}_\lambda=\beta_\lambda,\qquad\gamma_{\bar\lambda}=\gamma^*_\lambda=\gamma^{-1}_\lambda,\qquad\phi_{\bar\lambda}=\phi^*_\lambda=\phi^{-1}_\lambda.\tag{1.67}$$

In particular, in the case λ is self-dual we have that $\phi_\lambda=\phi_{\bar\lambda}$ coincides with the Frobenius-Schur indicator of λ.

Proof. CASE $\lambda\cong\bar\lambda$. Recall that (λ) was defined by (1.46) as $d_\lambda{}^{\mathrm{t}}T$ where d_λ is a chosen unitary scalar and $T=\mathcal{M}_{\mathcal{B}^\vee_{r\lambda}\mathcal{B}_{r\lambda}}(\psi_{r\lambda})$ was the matrix associated with an isometry of modules $\psi_{r\lambda}\colon V^{(r)}_\lambda\to V^{(r)}_{\bar\lambda}$. The matrix T had the property to make $\mathcal{M}_{\mathcal{B}_{r\lambda}}(\lambda(g))$ similar to its complex conjugate for any g in G, as expressed by (1.44). Therefore, by Fact A.2.3 we get that ${}^{\mathrm{t}}T=\iota_\lambda T$ where ι_λ is 1 when λ is real and -1 when λ is quaternionic. By how it is defined, the same property holds for (λ), i.e. ${}^{\mathrm{t}}(\lambda)=\iota_\lambda(\lambda)$. Furthermore, $(\bar\lambda)=(\lambda)$ as set by (1.49).
CASE $\lambda\ncong\bar\lambda$. By (1.51), we have that (λ) is a multiple of the unit matrix by a factor c_λ of module 1, then (λ) is symmetric and $(\bar\lambda)=(c_{\bar\lambda}/c_\lambda)(\lambda)$. □

Throughout the thesis, the symbols β_λ, γ_λ, ϕ_λ will always denote the quantities that they define in Proposition 1.3.4.

Definition 1.3.2. Let λ be a finite-dimensional irreducible representation of G. Consider the $2jm$-symbols $(\lambda)_{ab}$ associated with λ. The coefficient:

$$\{\lambda\} \coloneqq \phi_\lambda^* \tag{1.68}$$

such that the equality $(\lambda)_{ab} = \{\lambda\}(\bar{\lambda})_{ba}$ holds for any a, b is called the **$2j$-phase associated with $\boldsymbol{\lambda}$**.

Remark **1.3.5.** Let λ be a finite-dimensional irreducible representation of G.

1. By Proposition 1.3.4 and Definition 1.3.2, we have:

$$\{\bar{\lambda}\} = \{\lambda\}^* = \{\lambda\}^{-1}, \tag{1.69}$$

$$(\lambda) = \{\lambda\}^{\mathrm{t}}(\bar{\lambda}), \tag{1.70}$$

$$\{\lambda\} = \begin{cases} 1 & \lambda \text{ is real,} \\ -1 & \lambda \text{ is quaternionic,} \\ c_\lambda / c_{\bar{\lambda}} & \lambda \text{ is complex.} \end{cases} \tag{1.71}$$

2. The notion and notation $\{\lambda\}$ of the $2j$-phase we presented in Definition 1.3.2 is the one used in [But81, Equation (3.2.3)]. In other references we may find different symbols and terminologies: in [But75, Equation (4.6)] the author refers to ϕ_λ as the *$1j$-symbol*, whereas in [DS65] the author does not assign a name but when an irreducible finite-dimensional representation j is considered then he denotes our ϕ_j by λ_j.

In the following proposition we want to recover the behaviour of the $2j$-phase outlined in [But81, Equation (3.2.1)] with respect to the braket notation:

Proposition 1.3.5. *Let $\lambda \in \mathcal{I}$ and call $n = |\lambda|$. Consider the basis $\mathcal{B}_{r\lambda} = (|\lambda 1\rangle, \ldots, |\lambda n\rangle)$ of $V_\lambda^{(r)}$ and $\mathcal{B}_{r\bar{\lambda}} = (|\bar{\lambda} 1\rangle, \ldots, |\bar{\lambda} n\rangle)$ of $V_{\bar{\lambda}}^{(r)}$ accordingly to the prescription given at the beginning of Section 1.3. For simplicity, we will drop the multiplicity label r in what follows. For any a, b we then have that:*

$$(\lambda)_{ab} = \sqrt{|\lambda|}\,\langle \mathbf{1} | \lambda b, \bar{\lambda} a\rangle\,, \qquad (\bar{\lambda})_{ab} = \sqrt{|\lambda|}\,\langle \bar{\mathbf{1}} | \bar{\lambda} b, \lambda a\rangle\,, \tag{1.72}$$

with the appropriate choice of a unit vector $|\mathbf{1}\rangle$ (respectively, $|\bar{\mathbf{1}}\rangle$) spanning the submodule of $V_\lambda \otimes V_{\bar{\lambda}}$ (respectively, $V_{\bar{\lambda}} \otimes V_\lambda$) isomorphic to the trivial module. Furthermore, for any a, b we have:

$$\langle \mathbf{1} | \lambda b, \bar{\lambda} a\rangle = \{\lambda\}\,\langle \bar{\mathbf{1}} | \bar{\lambda} a, \lambda b\rangle\,. \tag{1.73}$$

Proof. To make the notation easier in this proof, we write $\mathcal{B}_{r\lambda} = (e_1, \dots, e_n)$ (so basically $|\lambda l\rangle = e_l$ where we omit the multiplicity label) and $\mathcal{B}_{r\lambda}^{\vee} = (\theta_1, \dots, \theta_n)$.

CASE $\lambda \cong \bar{\lambda}$. Recall the fixed isometry of modules $\psi \coloneqq \psi_{r\lambda} \colon V_{\lambda}^{(r)} \to V_{\bar{\lambda}}^{(r)}$. Consider the matrix $T = \mathcal{M}_{\mathcal{B}_{r\lambda}^{\vee}\mathcal{B}_{r\lambda}}(\psi_{r\lambda})$ such that $(\lambda) = d_{\lambda}{}^{\mathrm{t}}T$, consider the basis $\mathcal{B}_{r\bar{\lambda}} = (\psi(e_1), \dots, \psi(e_n))$ of $V_{\bar{\lambda}}^{(r)}$ (namely $|\bar{\lambda}l\rangle = \psi(e_l)$ where we are omitting the label r) and consider its dual basis $\mathcal{D} \coloneqq \mathcal{B}_{r\bar{\lambda}}^{\vee} = (f_1, \dots, f_n)$ (through the identification of $V_{\lambda}^{(r)}$ and its bidual, $\mathcal{D}$ is the basis of $V_{\lambda}^{(r)}$ such that $\mathcal{D}^{\vee} = \mathcal{B}_{r\bar{\lambda}}$). Then:

$$(\lambda)_{ab} = d_{\lambda} \left({}^{\mathrm{t}}T\right)_{ab} = d_{\lambda} T_{ba} = d_{\lambda} \langle \theta_b | \psi(e_a) \rangle \tag{1.74}$$

by definition of T and the orthonormality of $\mathcal{B}_{r\lambda}^{\vee}$. By Fact 1.1.1, we can make the following choice:

$$|\mathbf{1}\rangle = \frac{d_{\lambda}^{*}}{\sqrt{|\lambda|}} \sum_{l=1}^{|\lambda|} e_l \otimes \theta_l \tag{1.75}$$

and therefore derive (1.72):

$$\sqrt{|\lambda|}\, \langle \mathbf{1} | \lambda b, \bar{\lambda} a \rangle = \sqrt{|\lambda|} \left\langle \frac{d_{\lambda}^{*}}{\sqrt{|\lambda|}} \sum_{l=1}^{|\lambda|} e_l \otimes \theta_l \middle| e_b \otimes \psi(e_a) \right\rangle =$$

$$= d_{\lambda} \sum_{l=1}^{|\lambda|} \langle e_l | e_b \rangle \langle \theta_l | \psi(e_a) \rangle = d_{\lambda} \sum_{l=1}^{|\lambda|} \delta_{lb} \langle \theta_l | \psi(e_a) \rangle = d_{\lambda} \langle \theta_b | \psi(e_a) \rangle = (\lambda)_{ab},$$

where we have used (1.74). Similarly, we have the analogous equality for $\bar{\lambda}$.

CASE $\lambda \ncong \bar{\lambda}$. In this case, we have $(\lambda)_{ab} = c_{\lambda}\delta_{ab}$, $(\bar{\lambda})_{ab} = c_{\bar{\lambda}}\delta_{ab}$ and $\mathcal{B}_{r\bar{\lambda}} = \mathcal{B}_{r\lambda}^{\vee}$. By Fact 1.1.1, we can make the following choice:

$$|\mathbf{1}\rangle = \frac{c_{\lambda}^{*}}{\sqrt{|\lambda|}} \sum_{l=1}^{|\lambda|} e_l \otimes \theta_l \tag{1.76}$$

and therefore derive (1.72):

$$\sqrt{|\lambda|}\, \langle \mathbf{1} | \lambda b, \bar{\lambda} a \rangle = \sqrt{|\lambda|} \left\langle \frac{c_{\lambda}^{*}}{\sqrt{|\lambda|}} \sum_{l=1}^{|\lambda|} e_l \otimes \theta_l \middle| e_b \otimes \theta_a \right\rangle = c_{\lambda} \sum_{l=1}^{|\lambda|} \langle e_l | e_b \rangle \langle \theta_l | \theta_a \rangle =$$

$$= c_{\lambda} \sum_{l=1}^{|\lambda|} \delta_{lb} \langle \theta_l | \theta_a \rangle = c_{\lambda} \langle \theta_b | \theta_a \rangle = c_{\lambda}\delta_{ab} = (\lambda)_{ab}.$$

Choosing:

$$|\bar{\mathbf{1}}\rangle = \frac{c_{\bar{\lambda}}^*}{\sqrt{|\lambda|}} \sum_{l=1}^{|\lambda|} \theta_l \otimes e_l, \tag{1.77}$$

we prove similarly the analogous equality for $\bar{\lambda}$, i.e. $\sqrt{|\lambda|}\,\langle \bar{\mathbf{1}}|\bar{\lambda}b, \lambda a\rangle = (\bar{\lambda})_{ab}$.

In the end, we can conclude (1.73):

$$\langle \mathbf{1}|\lambda b, \bar{\lambda}a\rangle = \frac{1}{\sqrt{|\lambda|}}(\lambda)_{ab} = \frac{1}{\sqrt{|\lambda|}}\phi_\lambda^*(\bar{\lambda})_{ba} = \phi_\lambda^*\,\langle \bar{\mathbf{1}}|\bar{\lambda}a, \lambda b\rangle\,,$$

where we have used (1.72) for λ, Proposition 1.3.4 and (1.72) for $\bar{\lambda}$. □

Recall now the notion of triad given by Definition 1.1.3 and let us present the following concept:

Definition 1.3.3. A group is said to be **quasi-ambivalent** if, whenever three irreducible representations ρ_1, ρ_2, ρ_3 form a triad, there is a choice of $2j$-phases such that the product of the three $2j$-phases is the following:

$$\{\rho_1\}\{\rho_2\}\{\rho_3\} = 1. \tag{1.78}$$

Remark 1.3.6. 1. The concept of quasi-ambivalent group we chose to give in Definition 1.3.3 is the one from [But81, Equation (3.2.20)]. Anyway, (1.78) appears already in [But75, Equation (8.10)]: in this paper the author gives also ideas and choices for the $2j$-phases to show that any simple compact Lie group is indeed quasi-ambivalent. This claim is supported in more details in [BK74, Section 4]. In [But81] and [But75] the author refers to a specific finite-group of order 24 which is not quasi-ambivalent.

2. Another definition of quasi-ambivalent group is given in [Sha75], but it is not clear yet if it is correlated in some way with Definition 1.3.3. In this paper, the authors prove that all simple compact Lie groups different from E_6 satisfy their definition of quasi-ambivalence, which therefore could imply Definition 1.3.3 but cannot be equivalent to it.

3. In the case three finite-dimensional irreducible representations ρ_1, ρ_2, ρ_3 form a triad, the quantity $\{\rho_1\}\{\rho_2\}\{\rho_3\}$ is denoted by $\mu(\rho_1\rho_2\rho_3)^*$ in [DS65].

Next result shows what happens when we take the dual of the representations in a $3jm$-symbol:

Lemma 1.3.1 (Derome-Sharp Lemma)**.** *Let* $\lambda_3 \in \bar{\mathcal{I}}$. *Then:*

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}_r^* = \sum_{s,m_1m_2m_3} A(\lambda_1\lambda_2\lambda_3)^*_{rs}(\lambda_1)^*_{l_1m_1}(\lambda_2)^*_{l_2m_2}(\lambda_3)^*_{l_3m_3}\begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_s, \tag{1.79}$$

where:

$$A(\lambda_1\lambda_2\lambda_3)_{rs} := \sum_{\substack{a_1a_2a_3 \\ b_1b_2b_3}} (\lambda_1)^*_{a_1b_1}(\lambda_2)^*_{a_2b_2}(\lambda_3)^*_{a_3b_3}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_r \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ b_1 & b_2 & b_3 \end{pmatrix}_s. \tag{1.80}$$

If G is quasi-ambivalent, then we can make the following choice:

$$A(\lambda_1\lambda_2\lambda_3)_{rs} = \delta_{rs}. \tag{1.81}$$

Proof. The proof of (1.79) can be found in [DS65].
The fact that $A(\lambda_1\lambda_2\lambda_3)_{rs}$ can be chosen to be the Kronecker delta δ_{rs} for quasi-ambivalent (compact Lie) groups is a theorem stated and proved in [But75]. The link between $A(\lambda_1\lambda_2\lambda_3)$ and the notion of quasi-ambivalence consists in the following relation given in [DS65, Equation (4.7)]:

$$A(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)_{rs} = \{\lambda_1\}\{\lambda_2\}\{\lambda_3\}A(\lambda_1\lambda_2\lambda_3)_{sr}, \tag{1.82}$$

where we recall that $\{\lambda_1\}\{\lambda_2\}\{\lambda_3\} = \phi^*_{\lambda_1}\phi^*_{\lambda_2}\phi^*_{\lambda_3}$. In [But75, Equation (8.9)] we find written:

$$A(\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)_{rs} = \phi_{\lambda_1}\phi_{\lambda_2}\phi_{\lambda_3}A(\lambda_1\lambda_2\lambda_3)_{sr},$$

but this is only because it was already assumed previously that $\phi_\lambda = \pm 1$ for any finite-dimensional irreducible representation of G. □

Remark 1.3.7. If λ is a finite-dimensional irreducible representation of G which is not self-dual, in principle we can choose $c_\lambda = c_{\bar{\lambda}} (= 1)$ and therefore have $\phi_\lambda = 1$. However, in the case G is quasi-ambivalent it is useful to have $\{\lambda_1\}\{\lambda_2\}\{\lambda_3\}$ equal to 1 whenever $(\lambda_1\lambda_2\lambda_3)$ is a triad, as Derome-Sharp Lemma clearly shows. Therefore, in general and throughout the thesis we will always choose c_λ and $c_{\bar{\lambda}}$ in order to have $\{\lambda\} = \pm 1$.
For the reasons above, for an arbitrary finite-dimensional irreducible representation λ of G, we will have that:

$$\{\lambda\} = \pm 1, \tag{1.83}$$

which implies by (1.69) that:

$$\{\bar{\lambda}\} = \{\lambda\}. \tag{1.84}$$

Let us look now into the symmetry properties of $3jm$-symbols. Before to start with the main result, recall that since G is a compact Lie group, up to sign there is exactly one left-invariant volume form $d\mu$ such that $\int_G d\mu = 1$, determining a well-defined normalized integral of functions: the *invariant (Haar-) integral*. We will write the invariant integral using the notation $\int_{g\in G} f(g)\, d\mu$.

Proposition 1.3.6. *If $\lambda_3 \in \bar{\mathcal{I}}$, then for any a_i, b_j we have that:*

$$\int_{g\in G} \left[\mathcal{M}_{\mathcal{B}_{\lambda_1}}(\lambda_1(g))\right]_{a_1b_1} \left[\mathcal{M}_{\mathcal{B}_{\lambda_2}}(\lambda_2(g))\right]_{a_2b_2} \left[\mathcal{M}_{\mathcal{B}_{\lambda_3}}(\lambda_3(g))\right]_{a_3b_3} d\mu = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_r^* \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \end{pmatrix}_r . \tag{1.85}$$

Proof. See [DS65, Equation (2.5)] and [But75, Equation (5.12)]. □

Denote the permutation group of three objects by S_3. A direct consequence of Proposition 1.3.6 is given by the following corollary:

Corollary 1.3.1. *Let $\lambda_3 \in \bar{\mathcal{I}}$ and $a_i \in \{1, \ldots, |\lambda_i|\}$ for $i = 1, 2, 3$. Then:*

$$\sum_r \left| \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_r \right|^2 = \sum_r \left| \begin{pmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ a_2 & a_1 & a_3 \end{pmatrix}_r \right|^2 = \sum_r \left| \begin{pmatrix} \lambda_2 & \lambda_3 & \lambda_1 \\ a_2 & a_3 & a_1 \end{pmatrix}_r \right|^2 . \tag{1.86}$$

It follows that there exists a unitary matrix $M(\pi, \lambda_1\lambda_2\lambda_3)$ (depending on the representations $\lambda_1, \lambda_2, \lambda_3$ and on the permutation π only) such that:

$$\begin{pmatrix} \lambda_{\pi(1)} & \lambda_{\pi(2)} & \lambda_{\pi(3)} \\ a_{\pi(1)} & a_{\pi(2)} & a_{\pi(3)} \end{pmatrix}_r = \sum_s M(\pi, \lambda_1\lambda_2\lambda_3)_{rs} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_s , \tag{1.87}$$

where π is an element of S_3. By the unitarity of $3jm$-symbols, we get:

$$M(\pi, \lambda_1\lambda_2\lambda_3)_{rs} = \begin{pmatrix} \lambda_{\pi(1)} & \lambda_{\pi(2)} & \lambda_{\pi(3)} \\ a_{\pi(1)} & a_{\pi(2)} & a_{\pi(3)} \end{pmatrix}_r \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_s^* . \tag{1.88}$$

In particular, we have that:

$$M(\pi, \lambda_{\sigma(1)}\lambda_{\sigma(2)}\lambda_{\sigma(3)}) \cdot M(\sigma, \lambda_1\lambda_2\lambda_3) = M(\sigma \circ \pi, \lambda_1\lambda_2\lambda_3) \tag{1.89}$$

for any π, σ in S_3.

Proof. See [DS65, Equations (2.6), (2.7) and (2.8)], [Der66], [But75, Equations (6.1) and (6.2)]. □

Definition 1.3.4. Let $\lambda_3 \in \bar{\mathcal{I}}$. The unitary matrices $M(\pi, \lambda_1\lambda_2\lambda_3)$ with $\pi \in S_3$ described in Corollary 1.3.1 by the equation:

$$\begin{pmatrix} \lambda_{\pi(1)} & \lambda_{\pi(2)} & \lambda_{\pi(3)} \\ a_{\pi(1)} & a_{\pi(2)} & a_{\pi(3)} \end{pmatrix}_r = \sum_s M(\pi, \lambda_1\lambda_2\lambda_3)_{rs} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1 & a_2 & a_3 \end{pmatrix}_s \tag{1.90}$$

are called **permutation matrices**.

Proposition 1.3.7. *Let $\lambda_3 \in \bar{\mathcal{I}}$. Consider 3jm-symbols as defined via (1.56), where a unitary matrix $H(\lambda_1\lambda_2\lambda_3)$ is introduced.*

1. *If $\lambda_1 \ncong \lambda_2 \ncong \lambda_3 \ncong \lambda_1$ then we can make some choices on $H(\lambda_1\lambda_2\lambda_3)$ to obtain either*
$$M(\pi, \lambda_1\lambda_2\lambda_3)_{rs} = \delta_{rs} \quad \forall \pi \in S_3 \tag{1.91}$$
or
$$M(i, \lambda_1\lambda_2\lambda_3)_{rs} = \{\lambda_1\lambda_2\lambda_3 r\}\delta_{rs} \quad \text{and} \quad M(c, \lambda_1\lambda_2\lambda_3)_{rs} = \delta_{rs} \tag{1.92}$$
for any interchange i and cycle c of S_3, where $\{\lambda_1\lambda_2\lambda_3 r\} = \pm 1$.

2. *If $\lambda_1 \cong \lambda_2 \ncong \lambda_3$, then we can make some choices on $H(\lambda_1\lambda_2\lambda_3)$ to obtain:*
$$M(i, \lambda_1\lambda_2\lambda_3) = \{\lambda_1\lambda_2\lambda_3 r\}\delta_{rs} \quad \text{and} \quad M(c, \lambda_1\lambda_2\lambda_3) = \delta_{rs} \tag{1.93}$$
for any interchange i and cycle c of S_3, where $\{\lambda_1\lambda_2\lambda_3 r\} = \pm 1$. More precisely, $\{\lambda_1\lambda_2\lambda_3 r\} = 1$ when $\bar{\lambda}_3$ occurs in the symmetric tensor product of $\lambda_1 \otimes \lambda_2$ and $\{\lambda_1\lambda_2\lambda_3 r\} = -1$ when $\bar{\lambda}_3$ occurs in the anti-symmetric tensor product of $\lambda_1 \otimes \lambda_2$.

3. *If $\lambda_1 \cong \lambda_2 \cong \lambda_3$, then we can make some choices on $H(\lambda_1\lambda_2\lambda_3)$ to get $M(\pi, \lambda_1\lambda_2\lambda_3)$ to be a real (orthogonal) matrix for any π in S_3 and to obtain*
$$M((12), \lambda_1\lambda_2\lambda_3)_{rs} = \{\lambda_1\lambda_2\lambda_3 r\}\delta_{rs}, \tag{1.94}$$
where $\{\lambda_1\lambda_2\lambda_3 r\} = \pm 1$.

4. *In all the cases above, it is possible to have $M(\pi, \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3) = M(\pi, \lambda_1\lambda_2\lambda_3)$.*

Proof. See [Der66] and [But75, Section 6]. In the latter, the author gives explicit values for $M(\pi, \lambda_1\lambda_2\lambda_3)$ for any π in S_3 in the case $\lambda_1 \cong \lambda_2 \cong \lambda_3$. □

If $(\lambda_1\lambda_2\lambda_3)$ is a triad, then the notation and the choices outlined in Proposition 1.3.7 regarding the matrices $M(\pi, \lambda_1\lambda_2\lambda_3)$ will always be assumed throughout the entire thesis. In particular, $\{\lambda_1\lambda_2\lambda_3 r\}$ will denote $M((12), \lambda_1\lambda_2\lambda_3)_{rr}$ in the case $\lambda_1 \cong \lambda_2 \cong \lambda_3$ and denote $M(i, \lambda_1\lambda_2\lambda_3)_{rr}$ for any involution i of S_3 in all the other cases.

Accordingly to [Sea88], we give the following definition:

Definition 1.3.5. Let $(\lambda_1\lambda_2\lambda_3)$ be a triad and $\pi \in S_3$. If $M(\pi, \lambda_1\lambda_2\lambda_3)$ is diagonal, then its diagonal entries are called **3*j*-phases**.

Remark **1.3.8.** Notice that $2j$-phases are a particular type of $3j$-phases. Indeed, by (1.57) and Definition 1.3.2 we find:

$$\begin{pmatrix} \lambda & \mathbf{1} & \bar{\lambda} \\ a & 0 & b \end{pmatrix}_0 = \frac{(\lambda)_{ab}}{\sqrt{|\lambda|}} = \{\lambda\} \frac{(\bar{\lambda})_{ba}}{\sqrt{|\lambda|}} = \{\lambda\} \begin{pmatrix} \bar{\lambda} & \mathbf{1} & \lambda \\ b & 0 & a \end{pmatrix}_0 . \tag{1.95}$$

By (1.54) and (1.73), we find also that:

$$\begin{pmatrix} \lambda & \bar{\lambda} & \mathbf{1} \\ a & b & 0 \end{pmatrix}_0 = \langle \mathbf{1} | \lambda a, \bar{\lambda} b \rangle = \{\lambda\} \langle \mathbf{1} | \bar{\lambda} b, \lambda a \rangle = \{\lambda\} \begin{pmatrix} \bar{\lambda} & \lambda & \mathbf{1} \\ b & a & 0 \end{pmatrix}_0 . \tag{1.96}$$

From the results above, together with the choices made in Remark 1.3.7, we have:

$$M(i, \lambda\bar{\lambda}\mathbf{1}) = \{\lambda\} \qquad \text{and} \qquad M(c, \lambda\bar{\lambda}\mathbf{1}) = 1 \tag{1.97}$$

for any interchange i and any cycle c of S_3. In particular, we can write:

$$\{\lambda\} = \{\lambda\bar{\lambda}\mathbf{1}0\}. \tag{1.98}$$

The same thing holds for any permutation of $\lambda, \bar{\lambda}, \mathbf{1}$.

Remark **1.3.9.** Let $(\lambda_1\lambda_2\lambda r)$ be a triad. By Proposition 1.3.7, we have that:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda \\ l_1 & l_2 & l \end{pmatrix}_r = \{\lambda_1\lambda_2\lambda_3 r\} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda \\ l_1 & l_2 & l \end{pmatrix}_r$$

for arbitrary l_1, l_2, l. Then, by (1.59) we get:

$$\langle r\bar{\lambda}l | \lambda_2 l_1, \lambda_1 l_1 \rangle = \{\lambda_1\lambda_2\lambda_3 r\} \langle r\bar{\lambda}l | \lambda_1 l_1, \lambda_2 l_2 \rangle . \tag{1.99}$$

Definition 1.3.6. Let λ be a finite-dimensional irreducible representation of G. We say that λ is **simple-phase** either when $(\lambda\lambda\lambda)$ is not a triad or when $(\lambda\lambda\lambda)$ is a triad and it is still possible to make some choices on the matrix $H(\lambda\lambda\lambda)$ introduced in (1.56) in order for the permutation matrices to behave as follows:

$$M(i, \lambda\lambda\lambda)_{rs} = \{\lambda\lambda\lambda r\}\delta_{rs} \quad \text{and} \quad M(c, \lambda\lambda\lambda)_{rs} = \delta_{rs} \tag{1.100}$$

for any interchange i and cycle c of S_3. The group G is called **simple-phase** when all its finite-dimensional irreducible representations are simple-phase. A triad $(\alpha\beta\nu)$ of G is called **simple-phase** whenever either α, β, ν are all non-equivalent or exactly two of them are equivalent or $\alpha \cong \beta \cong \nu$ with α being simple-phase.

Throughout the thesis, whenever λ is a simple-phase representation such that $(\lambda\lambda\lambda)$ is a valid triad, the choices expressed by (1.100) will always be assumed.

Remark 1.3.10. 1. The terms *simple-phase representation* and *simple-phase group* are defined in [BK74], [But75, Section 7], [But81]. The term *simple-phase triad* is something we have introduced directly in this thesis to make certain statements easier to present.

2. Let λ be a finite-dimensional irreducible representation of G such that $(\lambda\lambda\lambda r)$ is a valid triad. Assume λ to be simple-phase. Then $\{\lambda\lambda\lambda r\} = \pm 1$ since $M(\pi, \lambda\lambda\lambda)$ is a diagonal, unitary, real matrix for any $\pi \in S_3$.

3. Let λ be a finite-dimensional irreducible representation of G. Let $\lambda \otimes \{21\}$ denote the mixed symmetry product of λ in $\lambda \otimes \lambda \otimes \lambda$. Define the following:

$$\kappa_\lambda := \int_{g\in G} \left[\chi_\lambda(g)^3 - \chi_\lambda(g^3)\right] d\mu, \tag{1.101}$$

where χ_λ is the character associated with λ (see Definition A.2.5). Then, by [DS65, Equation (11)] and [BK74] we have that:

- λ is simple-phase $\Leftrightarrow \kappa_\lambda = 0 \Leftrightarrow \mathbf{1} \in \lambda \otimes \{21\}$;
- λ is not simple-phase $\Leftrightarrow \kappa_\lambda \neq 0 \Leftrightarrow \mathbf{1} \notin \lambda \otimes \{21\}$.

4. All point groups are simple-phase (see [But81]).

5. The groups $SU(2)$ and $SU(3)$ are simple-phase (see [BK74] and [Der67] respectively). All simple compact Lie groups apart from $SU(2)$ and $SU(3)$ are not simple-phase. In particular, $SU(N)$ for $N \geq 4$, $SO(N)$ for $N \geq 5$, $Sp(N)$ for $N \geq 4$ are not simple-phase (see [BK74, Section 5] and [But75]).

6. The permutation groups S_n for $n \geq 6$ are not simple-phase (see [BK74]).

1.4 Definition of $6j$-symbols

Next goal is to define the $6j$-symbols as specific linear combinations of recoupling coefficients. Such linear combinations are chosen based on the desire to express a $6j$-symbol via $2jm$ and $3jm$-symbols (recall Definition 1.3.1), since these present more clear symmetry properties summarized by Propositions 1.3.2, 1.3.3 and 1.3.7. Whenever an orthonormal basis of an irreducible module will be chosen, it will be implicitly assumed to follow the prescription given at the beginning of Section 1.3. All $2j$-phases will be equal to ± 1 accordingly to Remark 1.3.7.

Furthermore, we assume G to be quasi-ambivalent (see Definition 1.3.3).

Throughout this section, we will consider three fixed finite-dimensional irreducible representations $\lambda_1, \lambda_2, \lambda_3$ of G. Let $\mathcal{B}_i = (|\lambda_i l_i\rangle : l_i = 1, \ldots, |\lambda_i|)$ be an orthonormal basis of V_{λ_i} for $i = 1, 2, 3$. Recall again (1.12), (1.14), (1.25) expressing a chosen decomposition of $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$ into irreducible modules obtained via coupling $\lambda_1, \lambda_2, \lambda_3$ differently. We will use the same notations introduced in these equations.

The first result we want to illustrate here is highlighted in [But75, Equations (9.6) and (9.16)]. We give a proof for completeness and for a better understanding of the objects and properties outlined in the previous subsections.

Proposition 1.4.1. *Fix an irreducible representation $\lambda_{12} \in \lambda_1 \otimes \lambda_2$ with multiplicity label t_{12} and an orthonormal basis $(|t_{12}\lambda_{12}l_{12} \colon l_{12} = 1, \ldots, |\lambda_{12}|\rangle)$ of $V_{\lambda_{12}}^{(t_{12})}$. Fix an irreducible representation $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels t and s'. Then:*

$$\begin{aligned}
&\frac{\{\lambda_2\}\{\lambda_3\lambda_{12}\bar{\lambda}t\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}} \sum_{s_{12}s_{23}} M((23), \lambda_1\bar{\lambda}_{12}\lambda_2)_{t_{12}s_{12}} M((132), \bar{\lambda}_3\lambda_{23}\bar{\lambda}_2)_{t_{23}s_{23}} \\
&\qquad \times \sum_{r_{12}rr_{23}r'} H(\lambda_1\lambda_2\lambda_{12})_{s_{12}r_{12}} H(\lambda_{12}\lambda_3\lambda)_{tr} H(\lambda_2\lambda_3\lambda_{23})^*_{s_{23}r_{23}} H(\lambda_1\lambda_{23}\lambda)^*_{s'r'} \\
&\qquad\qquad \times \langle((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\rangle \\
&= \sum_{\substack{l_1l_2l_3\\ l_{12}l_{23}b}} \sum_{b_3b_{12}b_2} (\bar{\lambda}_3)^*_{b_3l_3} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_2)^*_{l_2b_2} \begin{pmatrix} \lambda_1 & \bar{\lambda}_{12} & \lambda_2 \\ l_1 & b_{12} & l_2 \end{pmatrix}_{t_{12}} \begin{pmatrix} \lambda_3 & \lambda_{12} & \bar{\lambda} \\ l_3 & l_{12} & b \end{pmatrix}_{t} \\
&\qquad \times \begin{pmatrix} \bar{\lambda}_3 & \lambda_{23} & \bar{\lambda}_2 \\ b_3 & l_{23} & b_2 \end{pmatrix}_{t_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & b \end{pmatrix}^*_{s'}.
\end{aligned} \tag{1.102}$$

Proof. Consider the following:

$$\begin{aligned}
&\sum_{\substack{l_1l_2l_3\\ l_{12}l_{23}l}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1l_1, \lambda_2l_2\rangle \langle r\lambda l|\lambda_{12}l_{12}, \lambda_3l_3\rangle \\
&\qquad \times \langle r_{23}\lambda_{23}l_{23}|\lambda_2l_2, \lambda_3l_3\rangle^* \langle r'\lambda l|\lambda_1l_1, \lambda_{23}l_{23}\rangle^* \\
&= |\lambda| \sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|} \sum_{s_{12}ss_{23}s'} K(\lambda_1\lambda_2\lambda_{12})_{r_{12}s_{12}} K(\lambda_{12}\lambda_3\lambda)_{rs} \\
&\qquad \times K(\lambda_2\lambda_3\lambda_{23})^*_{r_{23}s_{23}} K(\lambda_1\lambda_{23}\lambda)^*_{r's'} \\
&\quad \times \sum_{\substack{l_1l_2l_3\\ l_{12}l_{23}l}} \sum_{b_{12}bb_{23}c} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_{23})_{l_{23}b_{23}} (\lambda)^*_{lb} (\lambda)_{lc} \begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda}_{12} \\ l_1 & l_2 & b_{12} \end{pmatrix}_{s_{12}} \begin{pmatrix} \lambda_{12} & \lambda_3 & \bar{\lambda} \\ l_{12} & l_3 & b \end{pmatrix}_{s}
\end{aligned}$$

$$\times \begin{pmatrix} \lambda_2 & \lambda_3 & \bar{\lambda}_{23} \\ l_2 & l_3 & b_{23} \end{pmatrix}^*_{s_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & c \end{pmatrix}^*_{s'}$$

$$= |\lambda| \sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|} \sum_{s_{12} s s_{23} s'} K(\lambda_1\lambda_2\lambda_{12})_{r_{12}s_{12}} K(\lambda_{12}\lambda_3\lambda)_{rs}$$

$$\times K(\lambda_2\lambda_3\lambda_{23})^*_{r_{23}s_{23}} K(\lambda_1\lambda_{23}\lambda)^*_{r's'}$$

$$\times \sum_{\substack{l_1 l_2 l_3 \\ l_{12} l_{23} l}} \sum_{b_{12} b b_{23} c} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_{23})_{l_{23}b_{23}} (\lambda)^*_{lb} (\lambda)_{lc} \begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda}_{12} \\ l_1 & l_2 & b_{12} \end{pmatrix}_{s_{12}} \begin{pmatrix} \lambda_{12} & \lambda_3 & \bar{\lambda} \\ l_{12} & l_3 & b \end{pmatrix}_s$$

$$\times \sum_{b_3 c_{23} b_2} (\lambda_2)^*_{l_2 b_2} (\lambda_3)^*_{l_3 b_3} (\bar{\lambda}_{23})^*_{b_{23} c_{23}} \begin{pmatrix} \bar{\lambda}_2 & \bar{\lambda}_3 & \lambda_{23} \\ b_2 & b_3 & c_{23} \end{pmatrix}_{s_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & c \end{pmatrix}^*_{s'}$$

$$= |\lambda| \sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|} \{\lambda_3\}^* \{\bar{\lambda}_{23}\}^*$$

$$\times \sum_{s_{12} s s_{23} s'} K(\lambda_1\lambda_2\lambda_{12})_{r_{12}s_{12}} K(\lambda_{12}\lambda_3\lambda)_{rs} K(\lambda_2\lambda_3\lambda_{23})^*_{r_{23}s_{23}} K(\lambda_1\lambda_{23}\lambda)^*_{r's'}$$

$$\times \sum_{\substack{l_1 l_2 l_3 \\ l_{12} l_{23} l}} \sum_{\substack{b_{12} b b_{23} c \\ b_3 c_{23} b_2}} (\bar{\lambda}_3)^*_{b_3 l_3} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_2)^*_{l_2 b_2} (\lambda)^*_{lb} (\lambda)_{lc} (\lambda_{23})^*_{c_{23}b_{23}} (\lambda_{23})_{l_{23}b_{23}}$$

$$\times \begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda}_{12} \\ l_1 & l_2 & b_{12} \end{pmatrix}_{s_{12}} \begin{pmatrix} \lambda_{12} & \lambda_3 & \bar{\lambda} \\ l_{12} & l_3 & b \end{pmatrix}_s \begin{pmatrix} \bar{\lambda}_2 & \bar{\lambda}_3 & \lambda_{23} \\ b_2 & b_3 & c_{23} \end{pmatrix}_{s_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & c \end{pmatrix}^*_{s'}$$

$$= |\lambda| \sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|} \{\lambda_3\} \{\lambda_{23}\}$$

$$\times \sum_{s_{12} s s_{23} s'} K(\lambda_1\lambda_2\lambda_{12})_{r_{12}s_{12}} K(\lambda_{12}\lambda_3\lambda)_{rs} K(\lambda_2\lambda_3\lambda_{23})^*_{r_{23}s_{23}} K(\lambda_1\lambda_{23}\lambda)^*_{r's'}$$

$$\times \sum_{\substack{l_1 l_2 l_3 \\ l_{12} l_{23}}} \sum_{\substack{b_{12} b \\ b_3 b_2}} (\bar{\lambda}_3)^*_{b_3 l_3} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_2)^*_{l_2 b_2} \begin{pmatrix} \lambda_1 & \lambda_2 & \bar{\lambda}_{12} \\ l_1 & l_2 & b_{12} \end{pmatrix}_{s_{12}} \begin{pmatrix} \lambda_{12} & \lambda_3 & \bar{\lambda} \\ l_{12} & l_3 & b \end{pmatrix}_s$$

$$\times \begin{pmatrix} \bar{\lambda}_2 & \bar{\lambda}_3 & \lambda_{23} \\ b_2 & b_3 & l_{23} \end{pmatrix}_{s_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & b \end{pmatrix}^*_{s'}$$

$$= |\lambda| \sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|} \{\lambda_2\}$$

$$\times \sum_{s_{12} s_{23} s'} K(\lambda_1\lambda_2\lambda_{12})_{r_{12}s_{12}} K(\lambda_{12}\lambda_3\lambda)_{rt} K(\lambda_2\lambda_3\lambda_{23})^*_{r_{23}s_{23}} K(\lambda_1\lambda_{23}\lambda)^*_{r's'}$$

$$\times \sum_{t_{12} t t_{23}} \{\lambda_3 \lambda_{12} \bar{\lambda} t\} M((23), \lambda_1 \bar{\lambda}_{12} \lambda_2)_{s_{12}t_{12}} M((132), \bar{\lambda}_3 \lambda_{23} \bar{\lambda}_2)_{s_{23}t_{23}}$$

$$\times \sum_{\substack{l_1 l_2 l_3 \\ l_{12} l_{23}}} \sum_{\substack{b_{12} b \\ b_3 b_2}} (\bar{\lambda}_3)^*_{b_3 l_3} (\lambda_{12})^*_{l_{12}b_{12}} (\lambda_2)^*_{l_2 b_2} \begin{pmatrix} \lambda_1 & \bar{\lambda}_{12} & \lambda_2 \\ l_1 & b_{12} & l_2 \end{pmatrix}_{t_{12}} \begin{pmatrix} \lambda_3 & \lambda_{12} & \bar{\lambda} \\ l_{12} & l_3 & b \end{pmatrix}_t$$

$$\times \begin{pmatrix} \bar{\lambda}_3 & \lambda_{23} & \bar{\lambda}_2 \\ b_3 & l_{23} & b_2 \end{pmatrix}_{t_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & b \end{pmatrix}^*_{s'} . \quad (1.103)$$

In the above, we have done the following: we have applied (1.60) to every coupling coefficient and Derome-Sharp Lemma to the third $3jm$-symbol in the sum; we have inserted some $2j$-phases accordingly to Definition 1.3.2; we have utilized the unitarity of $2jm$-symbols via (1.61); we have inserted the permutation matrices accordingly to (1.90), noticing that $M((12), \lambda_3\lambda_{12}\bar{\lambda})_{ts} = \{\lambda_3\lambda_{12}\bar{\lambda}t\}\delta_{ts}$ by Proposition 1.3.7; we have used the fact that $2j$-phases are chosen to be ± 1 (see Remark 1.3.7) and that $\{\lambda_3\}\{\lambda_{23}\} = \{\lambda_3\}\{\lambda_{23}\}\{\lambda_2\}^2 = \{\lambda_2\}$ by quasi-ambivalence. By (1.24), the left-hand side of (1.103) coincides with:

$$|\lambda| \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle .$$

We therefore get the conclusion by definition of H (being the inverse matrix of K), by the orthogonality of the permutation matrices (see Proposition 1.3.7) and by the relation $\{\lambda_2\}^{-1} = \{\lambda_2\}$. □

Remark 1.4.1. Dealing with simple-phase triads only (see Definition 1.3.6), under the hypothesis of Proposition 1.4.1 we find:

$$\frac{\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}r\}\{\lambda_2\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}} \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle$$
$$= \sum_{\substack{l_1l_2l_3 \\ l_{12}l_{23}b}} \sum_{b_3b_{12}b_2} (\bar{\lambda}_3)^*_{b_3l_3}(\lambda_{12})^*_{l_{12}b_{12}}(\lambda_2)^*_{l_2b_2} \begin{pmatrix} \lambda_1 & \bar{\lambda}_{12} & \lambda_2 \\ l_1 & b_{12} & l_2 \end{pmatrix}_{r_{12}} \begin{pmatrix} \lambda_3 & \lambda_{12} & \bar{\lambda} \\ l_3 & l_{12} & b \end{pmatrix}_r$$
$$\times \begin{pmatrix} \bar{\lambda}_3 & \lambda_{23} & \bar{\lambda}_2 \\ b_3 & l_{23} & b_2 \end{pmatrix}_{r_{23}} \begin{pmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ l_1 & l_{23} & b \end{pmatrix}^*_{r'} . \quad (1.104)$$

Definition 1.4.1. Fix irreducible representations $\lambda_{12} \in \lambda_1 \otimes \lambda_2$ and $\lambda_{23} \in \lambda_2 \otimes \lambda_3$ with multiplicity labels t_{12} and t_{23} respectively. Fix an irreducible representation $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels t and s'. The quantity:

$$\frac{\{\lambda_2\}\{\lambda_3\lambda_{12}\bar{\lambda}t\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}} \sum_{s_{12}s_{23}} M((23), \lambda_1\bar{\lambda}_{12}\lambda_2)_{t_{12}s_{12}} M((132), \bar{\lambda}_3\lambda_{23}\bar{\lambda}_2)_{t_{23}s_{23}}$$
$$\times \sum_{r_{12}rr_{23}r'} H(\lambda_1\lambda_2\lambda_{12})_{s_{12}r_{12}} H(\lambda_{12}\lambda_3\lambda)_{tr} H(\lambda_2\lambda_3\lambda_{23})^*_{s_{23}r_{23}} H(\lambda_1\lambda_{23}\lambda)^*_{s'r'}$$
$$\times \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \quad (1.105)$$

is denoted by the following symbol, which is called a **6*j*-symbol**:

$$\begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{t_{12}t_{23}ts'} . \quad (1.106)$$

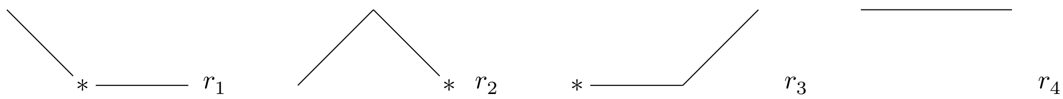
Corollary 1.4.1. *With the assumptions of Proposition 1.4.1, using a more general notation we have:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{l_1l_2l_3} \sum_{m_1m_2m_3} \sum_{n_1n_2n_3} (\mu_1)^*_{m_1n_1} (\mu_2)^*_{m_2n_2} (\mu_3)^*_{m_3n_3}$$
$$\times \begin{pmatrix} \lambda_1 & \bar{\mu}_2 & \mu_3 \\ l_1 & n_2 & m_3 \end{pmatrix}_{r_1} \begin{pmatrix} \mu_1 & \lambda_2 & \bar{\mu}_3 \\ m_1 & l_2 & n_3 \end{pmatrix}_{r_2} \begin{pmatrix} \bar{\mu}_1 & \mu_2 & \lambda_3 \\ n_1 & m_2 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4} . \quad (1.107)$$

The right-hand side of (1.107) consists of a sum of products of three $2jm$-symbols and four $3jm$-symbols. Each of these four $3jm$-symbols corresponds to one and only one of the four triads involving the six representations defining a $6j$-symbol. More explicitly, if we consider a $6j$-symbol:

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} ,$$

then the four involved triads are:



$(\lambda_1\bar{\mu}_2\mu_3r_1)$ $(\mu_1\lambda_2\bar{\mu}_3r_2)$ $(\bar{\mu}_1\mu_2\lambda_3r_3)$ $(\lambda_1\lambda_2\lambda_3r_4)$

where $*$ stands for conjugate representation.

If some of the six irreducible representations in a $6j$-symbol is trivial, then we know automatically the value of such symbol. Here an example:

Fact 1.4.1. *The following relation holds:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mathbf{1} \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\lambda}_2 & \lambda_1 & \mathbf{1} \end{Bmatrix}_{00r_3r_4} = \frac{\{\lambda_3\}\{\lambda_1\lambda_2\lambda_3r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}}\delta_{r_3r_4}. \quad (1.108)$$

Proof. The $6j$-symbol $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mathbf{1} \end{Bmatrix}_{r_1r_2r_3r_4}$ involves four triads, let us analyze two of them. Being $(\lambda_1\bar{\mu}_2\mathbf{1}r_1)$ a triad forces μ_2 to be equivalent to λ_1 and r_1 to be 0 by Fact A.2.2. Similarly, being $(\mu_1\lambda_2\mathbf{1}r_2)$ a triad forces μ_1 to be equivalent to $\bar{\lambda}_2$ and r_2 to be 0 for the same reason. Then we have that $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\lambda}_2 & \lambda_1 & \mathbf{1} \end{Bmatrix}_{00r_3r_4}$ is equal to:

$$\sum_{l_1l_2l_3} \sum_{m_1m_2n_1n_2} (\bar{\lambda}_2)^*_{m_1n_1} (\lambda_1)^*_{m_2n_2} \begin{pmatrix} \lambda_1 & \bar{\lambda}_1 & \mathbf{1} \\ l_1 & n_2 & 0 \end{pmatrix}_0 \begin{pmatrix} \bar{\lambda}_2 & \lambda_2 & \mathbf{1} \\ m_1 & l_2 & 0 \end{pmatrix}_0$$

$$
\times \begin{pmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ n_1 & m_2 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4}
$$

$$
= \sum_{l_1 l_2 l_3} \sum_{m_1 m_2 n_1 n_2} (\bar{\lambda}_2)^*_{m_1 n_1} (\lambda_1)^*_{m_2 n_2} \begin{pmatrix} \bar{\lambda}_1 & \mathbf{1} & \lambda_1 \\ n_2 & 0 & l_1 \end{pmatrix}_0 \begin{pmatrix} \lambda_2 & \mathbf{1} & \bar{\lambda}_2 \\ l_2 & 0 & m_1 \end{pmatrix}_0
$$

$$
\times \{\lambda_1 \lambda_2 \lambda_3 r_3\} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ m_2 & n_1 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4}
$$

$$
= \frac{\{\bar{\lambda}_1\}^* \{\bar{\lambda}_2\}^*}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}} \sum_{l_1 l_2 l_3} \sum_{m_1 m_2 n_1 n_2} (\lambda_1)_{l_1 n_2} (\lambda_1)^*_{m_2 n_2} (\lambda_2)_{l_2 m_1} (\lambda_2)^*_{n_1 m_1}
$$

$$
\times \{\lambda_1 \lambda_2 \lambda_3 r_3\} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ m_2 & n_1 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4}
$$

$$
= \frac{\{\lambda_1\}\{\lambda_2\}\{\lambda_1 \lambda_2 \lambda_3 r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}} \sum_{l_3} \sum_{l_1 l_2} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{r_4}
$$

$$
= \frac{\{\lambda_1\}\{\lambda_2\}\{\lambda_1 \lambda_2 \lambda_3 r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}} \sum_{l_3} \frac{\delta_{r_3 r_4}}{|\lambda_3|} = \frac{\{\lambda_1\}\{\lambda_2\}\{\lambda_1 \lambda_2 \lambda_3 r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}} \delta_{r_3 r_4}.
$$

In the above we have done the following: we have applied (1.24) and Proposition 1.3.7, we have then used (1.57), (1.69), the unitarity of $2jm$-symbols via (1.61) and of $3jm$-symbols via (1.62). Finally, by quasi-ambivalence we get $\{\lambda_1\}\{\lambda_2\} = \{\lambda_3\}$, since $2j$-phases are all assumed to be ± 1, as explained by Remark 1.3.7. □

Definition 1.4.2. A $6j$-symbol is called **trivial** whenever one of the six irreducible representations defining it is the trivial one.

1.5 Symmetries of $6j$-symbols

$6j$-symbols satisfy interesting properties. In this section, we present the ones that go under the name of *symmetries* of $6j$-symbols, which can be found in [DS65], [But75], [Sea88] in full generality and in [But81] in the case of simple-phase groups.

1.5.1 First symmetry: permutation of columns

Let us see what happens to a $6j$-symbol when we play with its columns.

Proposition 1.5.1. *$6j$-symbols behave as follows under a cyclic permutation c of the columns:*

$$
\begin{Bmatrix} \lambda_{c(1)} & \lambda_{c(2)} & \lambda_{c(3)} \\ \mu_{c(1)} & \mu_{c(2)} & \mu_{c(3)} \end{Bmatrix}_{r_{c(1)} r_{c(2)} r_{c(3)} r_4} = \sum_{s_1 s_2 s_3 s_4} M(c, \lambda_1 \bar{\mu}_2 \mu_3)_{r_1 s_1} M(c, \mu_1 \lambda_2 \bar{\mu}_3)_{r_2 s_2}
$$

$$\times M(c,\bar{\mu}_1\mu_2\lambda_3)_{r_3s_3}M(c,\lambda_1\lambda_2\lambda_3)_{r_4s_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{s_1s_2s_3s_4}. \quad (1.109)$$

Proof. The $6j$-symbol $\begin{Bmatrix}\lambda_2 & \lambda_3 & \lambda_1\\ \mu_2 & \mu_3 & \mu_1\end{Bmatrix}_{r_2r_3r_1r_4}$ is equal to:

$$=\sum_{l_1l_2l_3}\sum_{m_1m_2m_3}\sum_{n_1n_2n_3}(\mu_2)^*_{m_2n_2}(\mu_3)^*_{m_3n_3}(\mu_1)^*_{m_1n_1}$$
$$\times\begin{pmatrix}\lambda_2 & \bar{\mu}_3 & \mu_1\\ l_2 & n_3 & m_1\end{pmatrix}_{r_2}\begin{pmatrix}\mu_2 & \lambda_3 & \bar{\mu}_1\\ m_2 & l_3 & n_1\end{pmatrix}_{r_3}\begin{pmatrix}\bar{\mu}_2 & \mu_3 & \lambda_1\\ n_2 & m_3 & l_1\end{pmatrix}_{r_1}\begin{pmatrix}\lambda_2 & \lambda_3 & \lambda_1\\ l_2 & l_3 & l_1\end{pmatrix}^*_{r_4}$$
$$=\sum_{s_1s_2s_3}M((123),\mu_1\lambda_2\bar{\mu}_3)_{r_2s_2}M((123),\bar{\mu}_1\mu_2\lambda_3)_{r_3s_3}M((123),\lambda_1\bar{\mu}_2\mu_3)_{r_1s_1}$$
$$\times M((123)\lambda_1\lambda_2\lambda_3)_{r_4s_4}\sum_{l_1l_2l_3}\sum_{m_1m_2m_3}\sum_{n_1n_2n_3}(\mu_1)^*_{m_1n_1}(\mu_2)^*_{m_2n_2}(\mu_3)^*_{m_3n_3}$$
$$\times\begin{pmatrix}\mu_1 & \lambda_2 & \bar{\mu}_3\\ m_1 & l_2 & n_3\end{pmatrix}_{s_2}\begin{pmatrix}\bar{\mu}_1 & \mu_2 & \lambda_3\\ n_1 & m_2 & l_3\end{pmatrix}_{s_3}\begin{pmatrix}\lambda_1 & \bar{\mu}_2 & \mu_3\\ l_1 & n_2 & m_3\end{pmatrix}_{s_1}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ l_1 & l_2 & l_3\end{pmatrix}^*_{s_4}$$
$$=\sum_{s_1s_2s_3s_4}M((123),\lambda_1\bar{\mu}_2\mu_3)_{r_1s_1}M((123),\mu_1\lambda_2\bar{\mu}_3)_{r_2s_2}M((123),\bar{\mu}_1\mu_2\lambda_3)_{r_3s_3}$$
$$\times M((123)\lambda_1\lambda_2\lambda_3)_{r_4s_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{s_1s_2s_3s_4}.$$

In the above, we have used (1.107), definition and properties of the permutation matrices (see Proposition 1.3.7). The other cases are proved similarly. □

Proposition 1.5.2. *$6j$-symbols behave as follows under an interchange i of the columns:*

$$\begin{Bmatrix}\lambda_{i(1)} & \lambda_{i(2)} & \lambda_{i(3)}\\ \mu_{i(1)} & \mu_{i(2)} & \mu_{i(3)}\end{Bmatrix}_{r_{i(1)}r_{i(2)}r_{i(3)}r_4}=\{\mu_1\}\{\mu_2\}\{\mu_3\}\sum_{s_1s_2s_3s_4}M(i,\lambda_1\bar{\mu}_2\mu_3)_{r_1s_1}$$
$$\times M(i,\mu_1\lambda_2\bar{\mu}_3)_{r_2s_2}M(i,\bar{\mu}_1\mu_2\lambda_3)_{r_3s_3}M(i,\lambda_1\lambda_2\lambda_3)_{r_4s_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3\end{Bmatrix}_{s_1s_2s_3s_4}. \quad (1.110)$$

Proof. The $6j$-symbol $\begin{Bmatrix}\lambda_2 & \lambda_1 & \lambda_3\\ \mu_2 & \mu_1 & \mu_3\end{Bmatrix}_{r_2r_1r_3r_4}$ is equal to:

$$\sum_{l_1l_2l_3}\sum_{m_1m_2m_3}\sum_{n_1n_2n_3}(\mu_2)^*_{m_1n_1}(\mu_1)^*_{m_2n_2}(\mu_3)^*_{m_3n_3}$$
$$\times\begin{pmatrix}\lambda_2 & \bar{\mu}_1 & \mu_3\\ l_1 & n_2 & m_3\end{pmatrix}_{r_2}\begin{pmatrix}\mu_2 & \lambda_1 & \bar{\mu}_3\\ m_1 & l_2 & n_3\end{pmatrix}_{r_1}\begin{pmatrix}\bar{\mu}_2 & \mu_1 & \lambda_3\\ n_1 & m_2 & l_3\end{pmatrix}_{r_3}\begin{pmatrix}\lambda_2 & \lambda_1 & \lambda_3\\ l_1 & l_2 & l_3\end{pmatrix}^*_{r_4}$$

$$
\begin{aligned}
=&\{\mu_1\}^*\{\mu_2\}^*\{\mu_3\}^* \sum_{s_2 s_1 s_3 s_4} M((12),\lambda_2\bar{\mu}_1\mu_3)_{r_2 s_2} M((12),\mu_2\lambda_1\bar{\mu}_3)_{r_1 s_1} \\
&\times M((12),\bar{\mu}_2\mu_1\lambda_3)_{r_3 s_3} M((12),\lambda_2\lambda_1\lambda_3)_{r_4 s_4} \sum_{l_1 l_2 l_3} \sum_{\substack{m_1 m_2 m_3 \\ n_1 n_2 n_3}} (\bar{\mu}_2)^*_{n_1 m_1} (\bar{\mu}_1)^*_{n_2 m_2} (\bar{\mu}_3)^*_{n_3 m_3} \\
&\times \begin{pmatrix} \lambda_1 & \mu_2 & \bar{\mu}_3 \\ l_1 & n_2 & m_3 \end{pmatrix}_{s_1} \begin{pmatrix} \bar{\mu}_1 & \lambda_2 & \mu_3 \\ m_1 & l_2 & n_3 \end{pmatrix}_{s_2} \begin{pmatrix} \mu_1 & \bar{\mu}_2 & \lambda_3 \\ n_1 & m_2 & l_3 \end{pmatrix}_{s_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^*_{s_4} \\
=&\{\mu_1\}\{\mu_2\}\{\mu_3\} \sum_{s_2 s_1 s_3 s_4} M((12),\lambda_2\bar{\mu}_1\mu_3)_{r_2 s_2} M((12),\mu_2\lambda_1\bar{\mu}_3)_{r_1 s_1} \\
&\qquad\qquad \times M((12),\bar{\mu}_2\mu_1\lambda_3)_{r_3 s_3} M((12),\lambda_2\lambda_1\lambda_3)_{r_4 s_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}_{s_1 s_2 s_3 s_4} .
\end{aligned}
$$

In the computations above, we have used (1.107), definition and properties of the permutation matrices (see Proposition 1.3.7), $2j$-phases being ± 1 (see Remark 1.3.7). The other cases are proved similarly. □

Remark 1.5.1. In the case we interchange the first and the second column, by Proposition 1.3.7 we can write (1.110) as follows:

$$
\begin{aligned}
\begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ \mu_2 & \mu_1 & \mu_3 \end{Bmatrix}_{r_2 r_1 r_3 r_4} =&\{\mu_2\}\{\mu_1\}\{\mu_3\}\{\mu_2\lambda_1\bar{\mu}_3 r_1\}\{\lambda_2\bar{\mu}_1\mu_3 r_2\} \\
&\times \{\bar{\mu}_2\mu_1\lambda_3 r_3\}\{\lambda_2\lambda_1\lambda_3 r_4\} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} . \qquad (1.111)
\end{aligned}
$$

In the case we work with simple-phase triads only (see Definition 1.3.6), the results above simplify into the following:

Corollary 1.5.1. *Assume to deal with simple-phase triads only. Then the following relations hold:*

$$
\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \begin{Bmatrix} \lambda_2 & \lambda_3 & \lambda_1 \\ \mu_2 & \mu_3 & \mu_1 \end{Bmatrix}_{r_2 r_3 r_1 r_4} = \begin{Bmatrix} \lambda_3 & \lambda_1 & \lambda_2 \\ \mu_3 & \mu_1 & \mu_2 \end{Bmatrix}_{r_3 r_1 r_2 r_4} ; \qquad (1.112)
$$

$$
\begin{aligned}
\begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ \mu_2 & \mu_1 & \mu_3 \end{Bmatrix}_{r_2 r_1 r_3 r_4} =& \{\mu_2\}\{\mu_1\}\{\mu_3\}\{\mu_2\lambda_1\bar{\mu}_3 r_1\}\{\lambda_2\bar{\mu}_1\mu_3 r_2\} \\
&\times \{\bar{\mu}_2\mu_1\lambda_3 r_3\}\{\lambda_2\lambda_1\lambda_3 r_4\} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} . \qquad (1.113)
\end{aligned}
$$

Proof. The claim follows simply by using the simple-phase hypothesis when applying Proposition 1.3.7 to Proposition 1.5.1 and 1.5.2. □

1.5.2 Second symmetry: exchange of rows in two neighbouring columns

The next symmetry regards the exchange of rows in two neighbouring columns, where neighbouring columns means first and second, second and third, first and third columns.

Proposition 1.5.3. *The following relations hold:*

$$\begin{aligned}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4} &= \begin{Bmatrix}\bar\lambda_1 & \mu_2 & \bar\mu_3\\ \bar\mu_1 & \lambda_2 & \bar\lambda_3\end{Bmatrix}_{r_4r_3r_2r_1} = \begin{Bmatrix}\bar\mu_1 & \bar\lambda_2 & \mu_3\\ \bar\lambda_1 & \bar\mu_2 & \lambda_3\end{Bmatrix}_{r_3r_4r_1r_2}\\ &= \begin{Bmatrix}\mu_1 & \bar\mu_2 & \bar\lambda_3\\ \lambda_1 & \bar\lambda_2 & \bar\mu_3\end{Bmatrix}_{r_2r_1r_4r_3}.\end{aligned} \tag{1.114}$$

Proof. The $6j$-symbol $\begin{Bmatrix}\bar\lambda_1 & \mu_2 & \bar\mu_3\\ \bar\mu_1 & \lambda_2 & \bar\lambda_3\end{Bmatrix}_{r_4r_3r_2r_1}$ is equal to:

$$\begin{aligned}&\sum_{l_1l_2l_3}\sum_{m_1m_2m_3}\sum_{n_1n_2n_3}(\bar\mu_1)^*_{m_1n_1}(\lambda_2)^*_{m_2n_2}(\bar\lambda_3)^*_{m_3n_3}\\ &\qquad\times\begin{pmatrix}\bar\lambda_1 & \bar\lambda_2 & \bar\lambda_3\\ l_1 & n_2 & m_3\end{pmatrix}_{r_4}\begin{pmatrix}\bar\mu_1 & \mu_2 & \lambda_3\\ m_1 & l_2 & n_3\end{pmatrix}_{r_3}\begin{pmatrix}\mu_1 & \lambda_2 & \bar\mu_3\\ n_1 & m_2 & l_3\end{pmatrix}_{r_2}\begin{pmatrix}\bar\lambda_1 & \mu_2 & \bar\mu_3\\ l_1 & l_2 & l_3\end{pmatrix}^*_{r_1}\\ =&\{\bar\lambda_1\}^*\{\bar\lambda_3\}^*\{\bar\mu_1\}^*\{\bar\mu_3\}^*\sum_{\substack{l_1l_2l_3\\ m_1m_2m_3}}\sum_{\substack{n_1n_2n_3\\ a_1a_2a_3}}(\mu_1)^*_{n_1m_1}(\mu_2)^*_{l_2a_2}(\mu_3)^*_{a_3l_3}\begin{pmatrix}\lambda_1 & \bar\mu_2 & \mu_3\\ a_1 & a_2 & a_3\end{pmatrix}_{r_1}\\ &\times\begin{pmatrix}\mu_1 & \lambda_2 & \bar\mu_3\\ n_1 & m_2 & l_3\end{pmatrix}_{r_2}\begin{pmatrix}\bar\mu_1 & \mu_2 & \lambda_3\\ m_1 & l_2 & n_3\end{pmatrix}_{r_3}(\lambda_1)^*_{a_1l_1}(\lambda_2)^*_{m_2n_2}(\lambda_3)^*_{n_3m_3}\begin{pmatrix}\bar\lambda_1 & \bar\lambda_2 & \bar\lambda_3\\ l_1 & n_2 & m_3\end{pmatrix}_{r_4}\\ =&\{\bar\lambda_1\}^*\{\bar\lambda_3\}^*\{\bar\mu_1\}^*\{\bar\mu_3\}^*\sum_{\substack{l_2l_3\\ m_1m_2}}\sum_{\substack{n_1n_3\\ a_1a_2a_3}}(\mu_1)^*_{n_1m_1}(\mu_2)^*_{l_2a_2}(\mu_3)^*_{a_3l_3}\begin{pmatrix}\lambda_1 & \bar\mu_2 & \mu_3\\ a_1 & a_2 & a_3\end{pmatrix}_{r_1}\\ &\qquad\qquad\times\begin{pmatrix}\mu_1 & \lambda_2 & \bar\mu_3\\ n_1 & m_2 & l_3\end{pmatrix}_{r_2}\begin{pmatrix}\bar\mu_1 & \mu_2 & \lambda_3\\ m_1 & l_2 & n_3\end{pmatrix}_{r_3}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ a_1 & m_2 & n_3\end{pmatrix}_{r_4}\\ =&\{\lambda_1\}\{\lambda_3\}\{\mu_1\}\{\mu_3\}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4}.\end{aligned}$$

Here, we have applied (1.107) at the beginning and at the end, we have used definition and properties of $2j$-phase (see Definition 1.3.2 and (1.69)), we have applied Derome-Sharp Lemma twice. Quasi-ambivalence implies $\{\lambda_1\}\{\lambda_3\}\{\mu_1\}\{\mu_3\} = \{\lambda_1\}\{\lambda_3\}\{\lambda_2\}^2\{\mu_1\}\{\mu_3\} = 1$. The other relations are proved similarly. □

Definition 1.5.1. These first two types of symmetries, i.e. permutation of columns and exchange of rows in two neighbouring columns, are known as **tetrahedral symmetries**.

With the use of the tetrahedral symmetries, we can easily compute all possible trivial $6j$-symbols in the case of simple-phase triads:

Corollary 1.5.2. *Assume to deal with simple-phase triads only. Then the following relations hold:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mathbf{1} \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\lambda}_2 & \lambda_1 & \mathbf{1} \end{Bmatrix}_{00r_3r_4} = \frac{\{\lambda_3\}\{\lambda_1\lambda_2\lambda_3 r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_2|}}\delta_{r_3r_4}; \tag{1.115}$$

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mathbf{1} & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & \mathbf{1} & \bar{\lambda}_1 \end{Bmatrix}_{0r_20r_4} = \frac{\{\lambda_2\}\{\lambda_1\lambda_2\lambda_3 r_2\}}{\sqrt{|\lambda_1|}\sqrt{|\lambda_3|}}\delta_{r_2r_4}; \tag{1.116}$$

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mathbf{1} & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mathbf{1} & \bar{\lambda}_3 & \lambda_2 \end{Bmatrix}_{r_100r_4} = \frac{\{\lambda_1\}\{\lambda_1\lambda_2\lambda_3 r_1\}}{\sqrt{|\lambda_2|}\sqrt{|\lambda_3|}}\delta_{r_1r_4}; \tag{1.117}$$

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \mathbf{1} \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \bar{\lambda}_1 & \mathbf{1} \\ \mu_1 & \mu_1 & \mu_3 \end{Bmatrix}_{r_1r_200} = \frac{\{\mu_3\}\{\bar{\mu}_1\bar{\lambda}_1\mu_3 r_2\}}{\sqrt{|\lambda_1|}\sqrt{|\mu_1|}}\delta_{r_1r_2}; \tag{1.118}$$

$$\begin{Bmatrix} \lambda_1 & \mathbf{1} & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \lambda_1 & \mathbf{1} & \bar{\lambda}_1 \\ \mu_1 & \mu_2 & \mu_1 \end{Bmatrix}_{r_10r_30} = \frac{\{\mu_2\}\{\bar{\mu}_1\mu_2\bar{\lambda}_1 r_3\}}{\sqrt{|\lambda_1|}\sqrt{|\mu_1|}}\delta_{r_1r_3}; \tag{1.119}$$

$$\begin{Bmatrix} \mathbf{1} & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \mathbf{1} & \lambda_2 & \bar{\lambda}_2 \\ \mu_1 & \mu_2 & \mu_2 \end{Bmatrix}_{0r_2r_30} = \frac{\{\mu_1\}\{\mu_1\lambda_2\bar{\mu}_3 r_2\}}{\sqrt{|\lambda_2|}\sqrt{|\mu_2|}}\delta_{r_2r_3}. \tag{1.120}$$

Proof. The statement follows simply by applying the suitable tetrahedron symmetries to the result of Fact 1.4.1. □

1.5.3 Third symmetry: complex conjugation

In this subsection we want to understand what happens when we take the complex conjugate of a $6j$-symbol.

Proposition 1.5.4. *The following relation holds:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r_2r_3r_4} = \begin{Bmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}_{r_1r_2r_3r_4}. \tag{1.121}$$

Proof. By (1.107), the $6j$-symbol $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r_2r_3r_4}$ is equal to:

$$\sum_{l_1l_2l_3}\sum_{m_1m_2m_3}\sum_{n_1n_2n_3}(\mu_1)_{m_1n_1}(\mu_2)_{m_2n_2}(\mu_3)_{m_3n_3}$$

$$\times \begin{pmatrix} \lambda_1 & \bar{\mu}_2 & \mu_3 \\ l_1 & n_2 & m_3 \end{pmatrix}^*_{r_1} \begin{pmatrix} \mu_1 & \lambda_2 & \bar{\mu}_3 \\ m_1 & l_2 & n_3 \end{pmatrix}^*_{r_2} \begin{pmatrix} \bar{\mu}_1 & \mu_2 & \lambda_3 \\ n_1 & m_2 & l_3 \end{pmatrix}^*_{r_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ l_1 & l_2 & l_3 \end{pmatrix}^{**}_{r_4}$$

$$= \sum_{l_1 l_2 l_3} \sum_{m_1 m_2 m_3} \sum_{n_1 n_2 n_3} \sum_{\substack{a_1 a_2 a_3 \\ b_1 b_2 b_3}} \sum_{\substack{c_1 c_2 c_3 \\ i_1 i_2 i_3}} (\mu_1)_{m_1 n_1} (\mu_2)_{m_2 n_2} (\mu_3)_{m_3 n_3}$$

$$\times (\lambda_1)^*_{l_1 a_1} (\bar{\mu}_2)^*_{n_2 c_2} (\mu_3)^*_{m_3 b_3} \begin{pmatrix} \bar{\lambda}_1 & \mu_2 & \bar{\mu}_3 \\ a_1 & c_2 & b_3 \end{pmatrix}_{r_1} (\mu_1)^*_{m_1 b_1} (\lambda_2)^*_{l_2 a_2} (\bar{\mu}_3)^*_{n_3 c_3} \begin{pmatrix} \bar{\mu}_1 & \bar{\lambda}_2 & \mu_3 \\ b_1 & a_2 & c_3 \end{pmatrix}_{r_2}$$

$$\times (\bar{\mu}_1)^*_{n_1 c_1} (\mu_2)^*_{m_2 b_2} (\lambda_3)^*_{l_3 a_3} \begin{pmatrix} \mu_1 & \bar{\mu}_2 & \bar{\lambda}_3 \\ c_1 & b_2 & a_3 \end{pmatrix}_{r_3} (\lambda_1)_{l_1 i_1} (\lambda_2)_{l_2 i_2} (\lambda_3)_{l_3 i_3} \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^*_{r_4}$$

$$= \sum_{a_1 a_2 a_3} \sum_{b_1 b_2 b_3} \sum_{c_1 c_2 c_3} (\bar{\mu}_1)^*_{b_1 c_1} (\bar{\mu}_2)^*_{b_2 c_2} (\bar{\mu}_3)^*_{b_3 c_3} \begin{pmatrix} \bar{\lambda}_1 & \mu_2 & \bar{\mu}_3 \\ a_1 & c_2 & b_3 \end{pmatrix}_{r_1} \begin{pmatrix} \bar{\mu}_1 & \bar{\lambda}_2 & \mu_3 \\ b_1 & a_2 & c_3 \end{pmatrix}_{r_2}$$
$$\times \begin{pmatrix} \mu_1 & \bar{\mu}_2 & \bar{\lambda}_3 \\ c_1 & b_2 & a_3 \end{pmatrix}_{r_3} \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ a_1 & a_2 & a_3 \end{pmatrix}^*_{r_4}$$

which coincides with $\begin{Bmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$ again by (1.107). In the above, we have applied Derome-Sharp Lemma on each $3jm$-symbol in the sum and we have used the unitarity of $2jm$-symbols (see (1.61)). □

1.5.4 Fourth symmetry: unitarity

The previous symmetries regard properties of a single $6j$-symbol. Here we want to study particular expressions that involve more $6j$-symbols at once.

Proposition 1.5.5 (Unitarity of $\mathbf{6j}$-symbols)**.** *The following relation holds:*

$$\sum_{\mu_2 r_1 r_3} |\mu_2| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} \begin{Bmatrix} \lambda_1 & \lambda_2' & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1 r_2' r_3 r_4'} = \frac{\delta_{\lambda_2 \lambda_2'} \delta_{r_2 r_2'} \delta_{r_4 r_4'}}{|\lambda_2|}. \quad (1.122)$$

Proof. For better comprehension, we will assume to work with simple-phase triads only. The proof in the general case goes similarly.
Let us fix two irreducible representations $\lambda_{23}, \lambda_{23}' \in \lambda_2 \otimes \lambda_3$ with multiplicity labels r_{23}, s_{23} respectively. Fix an irreducible representation $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels r', s'. Recall the following definition of $6j$-symbol:

$$\begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12} r_{23} r r'} = \frac{\{\lambda_{23}\}\{\lambda_3\}\{\lambda_1 \lambda_2 \bar{\lambda}_{12} r_{12}\}\{\lambda_{12} \lambda_3 \bar{\lambda} r\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}}$$
$$\times \left\langle ((\lambda_1 \lambda_2) r_{12} \lambda_{12}, \lambda_3) r \lambda \middle| (\lambda_1 (\lambda_2 \lambda_3) r_{23} \lambda_{23}) r' \lambda \right\rangle .$$

Then we have:

$$\sum_{r_{12}\lambda_{12}r} |\lambda_{12}| \begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12}r_{23}rr'} \begin{Bmatrix} \lambda_1 & \lambda'_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}^*_{r_{12}s_{23}rs'}$$

$$= \sum_{r_{12}\lambda_{12}r_{23}} |\lambda_{12}| \frac{\{\lambda_{23}\}\{\lambda_3\}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}r\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}} \big\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \big| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \big\rangle$$

$$\times \frac{\{\lambda'_{23}\}\{\lambda_3\}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}r\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda'_{23}|}} \big\langle (\lambda_1(\lambda_2\lambda_3)s_{23}\lambda'_{23})s'\lambda \big| ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \big\rangle$$

$$= \frac{\{\lambda_{23}\}\{\lambda'_{23}\}}{\sqrt{|\lambda_{23}|}\sqrt{|\lambda'_{23}|}} \sum_{r_{12}\lambda_2 r_{23}} \big\langle (\lambda_1(\lambda_2\lambda_3)s_{23}\lambda'_{23})s'\lambda \big| ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \big\rangle$$

$$\times \big\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \big| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \big\rangle$$

$$= \frac{\{\lambda_{23}\}\{\lambda'_{23}\}}{\sqrt{|\lambda_{23}|}\sqrt{|\lambda'_{23}|}} \delta_{r_{23}s_{23}}\delta_{\lambda_{23}\lambda'_{23}}\delta_{r's'} = \frac{1}{|\lambda_{23}|}\delta_{r_{23}s_{23}}\delta_{\lambda_{23}\lambda'_{23}}\delta_{r's'},$$

where we have used that $2j$ and $3j$-phases are ± 1 and we have applied 3.) and 2.) of Proposition 1.2.1. □

Corollary 1.5.3. *The following relations hold:*

$$\sum_{\lambda_1 r_1 r_4} |\lambda_1| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu'_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r'_2r'_3r_4} = \frac{\delta_{\mu_1\mu'_1}\delta_{r_2r'_2}\delta_{r_3r'_3}}{|\mu_1|}; \tag{1.123}$$

$$\sum_{\lambda_2 r_2 r_4} |\lambda_2| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu'_2 & \mu_3 \end{Bmatrix}^*_{r'_1r_2r'_3r_4} = \frac{\delta_{\mu_2\mu'_2}\delta_{r_1r'_1}\delta_{r_3r'_3}}{|\mu_2|}; \tag{1.124}$$

$$\sum_{\lambda_3 r_3 r_4} |\lambda_3| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu'_3 \end{Bmatrix}^*_{r'_1r'_2r_3r_4} = \frac{\delta_{\mu_3\mu'_3}\delta_{r_1r'_1}\delta_{r_2r'_2}}{|\mu_3|}; \tag{1.125}$$

$$\sum_{\mu_1 r_2 r_3} |\mu_1| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda'_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r'_1r_2r_3r'_4} = \frac{\delta_{\lambda_1\lambda'_1}\delta_{r_1r'_1}\delta_{r_4r'_4}}{|\lambda_1|}; \tag{1.126}$$

$$\sum_{\mu_2 r_1 r_3} |\mu_2| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda_1 & \lambda'_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r'_2r_3r'_4} = \frac{\delta_{\lambda_2\lambda'_2}\delta_{r_2r'_2}\delta_{r_4r'_4}}{|\lambda_2|}; \tag{1.127}$$

$$\sum_{\mu_3 r_1 r_2} |\mu_3| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda'_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r_2r'_3r'_4} = \frac{\delta_{\lambda_3\lambda'_3}\delta_{r_3r'_3}\delta_{r_4r'_4}}{|\lambda_3|}. \tag{1.128}$$

Proof. The proof is given simply by applying the tetrahedron symmetries to the result of Proposition 1.5.5, taking into account the orthogonality of the permutation matrices (see Proposition 1.3.7). □

1.5.5 Fifth symmetry: generalized Racah-backcoupling rule

There are three ways of coupling $\lambda_1, \lambda_2, \lambda_3$ when we tensor them together. As we discussed in Section 1.2, such process provides us in general with three different orthonormal bases of $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$. Of course, one way to go from the first basis to the second basis is going from the first to the third one and then from the third to the second one. This will give us a simple relation between recoupling coefficients that is presented in [But81, Equation (3.3.24)] and proved in the following fact:

Fact 1.5.1. *The following relation holds:*

$$\sum_{\lambda_{13} r_{13} s} \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda \right\rangle \left\langle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle$$
$$= \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle . \quad (1.129)$$

Proof. Let us plug (1.27) into (1.30):

$$\left| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l \right\rangle =$$
$$= \sum_{r_{13}\lambda_{13}s} \left\langle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \left| (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda l \right\rangle$$
$$= \sum_{\substack{r r_{12} \\ \lambda_{12}}} \left(\sum_{r_{13}\lambda_{13}s} \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda \right\rangle \left\langle (\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \right)$$
$$\times \left| ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda l \right\rangle .$$

Compare what we have obtained with (1.16), which was:

$$\left| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda l \right\rangle = \sum_{r_{12}\lambda_{12}r} \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle \left| ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda l \right\rangle .$$

By linear independence, we get the claim. □

Before to prove the main result, consider the following lemma:

Lemma 1.5.1. *The following relations hold:*

$$\left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle$$
$$= \{\lambda_1\lambda_2\lambda_{12}r_{12}\} \left\langle ((\lambda_2\lambda_1)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle , \quad (1.130)$$
$$\left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle$$
$$= \{\lambda_2\lambda_3\lambda_{23}r_{23}\}^* \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_3\lambda_2)r_{23}\lambda_{23})r'\lambda \right\rangle . \quad (1.131)$$

Proof. Using (1.24) and (1.99), we get:

$$
\begin{aligned}
&|\lambda|\left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\right\rangle = \\
&= \sum_{\substack{l_1l_2l_3\\ l_{12}l_{23}l}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1,\lambda_2 l_2\rangle \langle r\lambda l|\lambda_{12}l_{12},\lambda_3 l_3\rangle \\
&\qquad\qquad\qquad\qquad \times \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2,\lambda_3 l_3\rangle^* \langle r'\lambda l|\lambda_1 l_1,\lambda_{23}l_{23}\rangle^* \\
&= \sum_{\substack{l_1l_2l_3\\ l_{12}l_{23}l}} \{\lambda_1\lambda_2\lambda_{12}r_{12}\} \langle r_{12}\lambda_{12}l_{12}|\lambda_2 l_2,\lambda_1 l_1\rangle \langle r\lambda l|\lambda_{12}l_{12},\lambda_3 l_3\rangle \\
&\qquad\qquad\qquad\qquad \times \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2,\lambda_3 l_3\rangle^* \langle r'\lambda l|\lambda_1 l_1,\lambda_{23}l_{23}\rangle^* \\
&= |\lambda|\,\{\lambda_1\lambda_2\lambda_{12}r_{12}\}\left\langle((\lambda_2\lambda_1)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\right\rangle .
\end{aligned}
$$

The second expression is proved analogously. □

We are now ready to prove the fifth symmetry of $6j$-symbols:

Corollary 1.5.4 (Generalized Racah-backcoupling rule)**.** *The following relation holds:*

$$
\begin{aligned}
\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4} &= \{\lambda_3\}\{\lambda_1\bar{\mu}_2\mu_3r_1\}\{\mu_1\lambda_2\bar{\mu}_3r_2\} \\
&\times \sum_{\nu rs}|\nu|\,\{\bar{\lambda}_1\mu_1\nu r\}\begin{Bmatrix}\mu_3 & \nu & \lambda_3\\ \mu_1 & \mu_2 & \lambda_1\end{Bmatrix}_{r_1rr_3s}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \bar{\mu}_3 & \nu & \bar{\mu}_1\end{Bmatrix}_{rr_2sr_4} . \qquad (1.132)
\end{aligned}
$$

Proof. For simplicity, we will assume to work with simple-phase triads only, i.e. permutation matrices are diagonal and their values on the diagonal, namely the $3j$-phases, are ± 1. The proof in the general case goes similarly.
By Lemma 1.5.1, the left-hand side of (1.129) is:

$$
\begin{aligned}
&\sum_{\lambda_{13}r_{13}s}\left\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda\right\rangle\left\langle(\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda\middle|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda\right\rangle \\
&= \sum_{\lambda_{13}r_{13}s}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_2\lambda_{13}\bar{\lambda}s\}\{\lambda_2\lambda_3\bar{\lambda}_{23}r_{23}\} \\
&\qquad\times\left\langle((\lambda_2\lambda_1)r_{12}\lambda_{12},\lambda_3)r\lambda\middle|(\lambda_2(\lambda_1\lambda_3)r_{13}\lambda_{13})s\lambda\right\rangle\left\langle((\lambda_1\lambda_3)r_{13}\lambda_{13},\lambda_2)s\lambda\middle|(\lambda_1(\lambda_3\lambda_2)r_{23}\lambda_{23})r'\lambda\right\rangle \\
&= \frac{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}\{\lambda_2\lambda_3\bar{\lambda}_{23}r_{23}\}}{\{\lambda_3\}\{\lambda_{23}\}\{\lambda_2\}\{\lambda_{12}\lambda_3\bar{\lambda}r\}} \\
&\qquad\times\sum_{\lambda_{13}r_{13}s}\frac{|\lambda_{13}|}{\{\lambda_{13}\}\{\lambda_1\lambda_3\bar{\lambda}_{13}r_{13}\}}\begin{Bmatrix}\lambda_2 & \lambda_{13} & \bar{\lambda}\\ \bar{\lambda}_3 & \lambda_{12} & \lambda_1\end{Bmatrix}_{r_{12}r_{13}rs}\begin{Bmatrix}\lambda_1 & \lambda_{23} & \bar{\lambda}\\ \bar{\lambda}_2 & \lambda_{13} & \lambda_3\end{Bmatrix}_{r_{13}r_{23}sr'} .
\end{aligned}
$$

Then (1.129) is equivalent to the following:

$$\sum_{\lambda_{13} r_{13} s} \frac{|\lambda_{13}| \, \{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_2\lambda_3\bar{\lambda}_{23}r_{23}\}}{\{\lambda_{13}\}\{\lambda_2\}\{\lambda_1\lambda_3\bar{\lambda}_{13}r_{13}\}} \begin{Bmatrix} \lambda_2 & \lambda_{13} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_1 \end{Bmatrix}_{r_{12}r_{13}rs} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_2 & \lambda_{13} & \lambda_3 \end{Bmatrix}_{r_{13}r_{23}sr'}$$
$$= \frac{\{\lambda_{23}\}\{\lambda_3\}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}r\}}{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}} \left\langle ((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda \middle| (\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})r'\lambda \right\rangle ,$$

where the right-hand side coincides with $\begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12}r_{23}rr'}$ by definition. Recalling once again that $2j$ and $3j$-phases are ± 1, we finally get:

$$\begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12}r_{23}rr'} = \{\lambda_2\}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_2\lambda_3\bar{\lambda}_{23}r_{23}\}$$
$$\times \sum_{\lambda_{13} r_{13} s} |\lambda_{13}| \, \{\lambda_{13}\}\{\lambda_1\lambda_3\bar{\lambda}_{13}r_{13}\} \begin{Bmatrix} \lambda_2 & \lambda_{13} & \bar{\lambda} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_1 \end{Bmatrix}_{r_{12}r_{13}rs} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda} \\ \bar{\lambda}_2 & \lambda_{13} & \lambda_3 \end{Bmatrix}_{r_{13}r_{23}sr'} .$$

Rewriting the equation above in a more general notation, we get the claim, provided that by quasi-ambivalence we have $\{\lambda_2\}\{\lambda_{13}\} = \{\lambda_2\}\{\lambda_{13}\}\{\lambda\}^2 = \{\lambda\}$. □

Exploiting the previous symmetries, we can rewrite the Racah-backcoupling rule in the following way:

Corollary 1.5.5. *The following relation holds:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{\nu rs} \# \, |\nu| \begin{Bmatrix} \mu_1 & \nu & \lambda_1 \\ \mu_2 & \mu_3 & \lambda_2 \end{Bmatrix}_{r_1rr_3s} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_2 & \bar{\mu}_1 & \nu \end{Bmatrix}_{r_2srr_4} , \tag{1.133}$$

where # denotes the product of specific coefficients which are known.

1.5.6 Sixth symmetry: Biedenharn-Elliot sum rule

In this subsection we give a complete proof of the last symmetry of $6j$-symbols based on the strategy sketched in [But81]. For this purpose, let us fix a fourth irreducible representation λ_4 with an orthonormal basis $(|\lambda_4 l_4\rangle : l_4 = 1, \dots, |\lambda_4|)$ of V_{λ_4}. We will consider more ways of coupling the modules V_{λ_i} for $i = 1, \dots, 4$ in the tensor product $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4}$. Regarding the notation about basis vectors, the same considerations made in Section 1.2 hold.

Consider firstly:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4} \cong \big((V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3}\big) \otimes V_{\lambda_4}$$

$$\cong \left(\left(\left(\bigoplus_{r_{12}\lambda_{12}} V_{\lambda_{12}}^{(r_{12})}\right)\otimes V_{\lambda_3}\right)\otimes V_{\lambda_4} \cong \left(\bigoplus_{r_{12}\lambda_{12}} \left(V_{\lambda_{12}}^{(r_{12})}\otimes V_{\lambda_3}\right)\right)\otimes V_{\lambda_4}\right.$$
$$\cong \left(\bigoplus_{r_{12}\lambda_{12}}\bigoplus_{r_{123}\lambda_{123}} V_{\lambda_{123}}^{(r_{123})}\right)\otimes V_{\lambda_4} \cong \bigoplus_{r_{12}\lambda_{12}}\bigoplus_{r_{123}\lambda_{123}} \left(V_{\lambda_{123}}^{(r_{123})}\otimes V_{\lambda_4}\right)$$
$$\cong \bigoplus_{r_{12}\lambda_{12}}\bigoplus_{r_{123}\lambda_{123}}\bigoplus_{r\lambda} V_{\lambda}^{(r)},$$

where:

$$|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3, \lambda_4 l_4\rangle = \sum_{r_{12}\lambda_{12}l_{12}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \,|_{(\lambda_1\lambda_2)}r_{12}\lambda_{12}l_{12}, \lambda_3 l_3, \lambda_4 l_4\rangle$$
$$= \sum_{r_{12}\lambda_{12}l_{12}}\sum_{r_{123}\lambda_{123}l_{123}} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r_{123}\lambda_{123}l_{123}|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \,|_{((\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3)}r_{123}\lambda_{123}l_{123}, \lambda_4 l_4\rangle$$
$$= \sum_{r_{12}\lambda_{12}l_{12}}\sum_{r_{123}\lambda_{123}l_{123}}\sum_{r\lambda l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r_{123}\lambda_{123}l_{123}|\lambda_{12}l_{12}, \lambda_3 l_3\rangle$$
$$\times \langle r\lambda l|\lambda_{123}l_{123}, \lambda_4 l_4\rangle \,|_{(((\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3)r_{123}\lambda_{123},\,\lambda_4)}r\lambda l\rangle\,.$$

Consider secondly:

$$V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}\otimes V_{\lambda_4} \cong \left(V_{\lambda_1}\otimes\left(V_{\lambda_2}\otimes V_{\lambda_3}\right)\right)\otimes V_{\lambda_4} \cong \bigoplus_{r_{23}\lambda_{23}}\bigoplus_{s_{123}\lambda_{123}}\bigoplus_{r\lambda} V_{\lambda}^{(r)},$$

where we have that $|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3, \lambda_4 l_4\rangle$ equals:

$$\sum_{r_{23}\lambda_{23}l_{23}}\sum_{s_{123}\lambda_{123}l_{123}}\sum_{r\lambda l} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle s_{123}\lambda_{123}l_{123}|\lambda_1 l_1, \lambda_{23}l_{23}\rangle$$
$$\times \langle r\lambda l|\lambda_{123}l_{123}, \lambda_4 l_4\rangle \,|_{((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\,\lambda_4)}r\lambda l\rangle\,.$$

Consider in third place:

$$V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}\otimes V_{\lambda_4} \cong \left(V_{\lambda_2}\otimes V_{\lambda_3}\right)\otimes\left(V_{\lambda_1}\otimes V_{\lambda_4}\right) \cong \bigoplus_{r_{23}\lambda_{23}}\bigoplus_{r_{14}\lambda_{14}}\bigoplus_{t\lambda} V_{\lambda}^{(t)},$$

where we have that $|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3, \lambda_4 l_4\rangle$ equals:

$$\sum_{r_{23}\lambda_{23}l_{23}}\sum_{r_{14}\lambda_{14}l_{14}}\sum_{t\lambda l} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r_{14}\lambda_{14}l_{14}|\lambda_1 l_1, \lambda_4 l_4\rangle$$
$$\times \langle t\lambda l|\lambda_{23}l_{23}, \lambda_{14}l_{14}\rangle \,|_{((\lambda_2\lambda_3)r_{23}\lambda_{23},\,(\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda l\rangle\,.$$

Consider in fourth place:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4} \cong V_{\lambda_3} \otimes \big((V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_4} \big) \cong \bigoplus_{r_{12}\lambda_{12}} \bigoplus_{r_{124}\lambda_{124}} \bigoplus_{s\lambda} V_\lambda^{(s)},$$

where we have that $|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3, \lambda_4 l_4\rangle$ equals:

$$\sum_{r_{12}\lambda_{12}l_{12}} \sum_{r_{124}\lambda_{124}l_{124}} \sum_{s\lambda l} \langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1, \lambda_2 l_2\rangle \langle r_{124}\lambda_{124}l_{124}|\lambda_{12}l_{12}, \lambda_4 l_4\rangle \\ \times \langle s\lambda l|\lambda_3 l_3, \lambda_{124}l_{124}\rangle \left|(\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l\right\rangle .$$

Consider in fifth place:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4} \cong V_{\lambda_3} \otimes \big(V_{\lambda_2} \otimes (V_{\lambda_1} \otimes V_{\lambda_4}) \big) \cong \bigoplus_{r_{14}\lambda_{14}} \bigoplus_{s_{124}\lambda_{124}} \bigoplus_{s\lambda} V_\lambda^{(s)},$$

where we have that $|\lambda_1 l_1, \lambda_2 l_2, \lambda_3 l_3, \lambda_4 l_4\rangle$ equals:

$$\sum_{r_{14}\lambda_{14}l_{14}} \sum_{s_{124}\lambda_{124}l_{124}} \sum_{s\lambda l} \langle r_{14}\lambda_{14}l_{14}|\lambda_1 l_1, \lambda_4 l_4\rangle \langle s_{124}\lambda_{124}l_{124}|\lambda_2 l_2, \lambda_{14}l_{14}\rangle \\ \times \langle s\lambda l|\lambda_3 l_3, \lambda_{124}l_{124}\rangle \left|(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l\right\rangle .$$

Let us now analyze the transformation between the first and second final bases:

$$\left|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle = \\ = \sum_{r_{12}\lambda_{12}r_{123}} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle \\ \times \left|(((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle , \quad (1.134)$$

$$\left|(((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle = \\ = \sum_{r_{23}\lambda_{23}s_{123}} \left\langle ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda \middle| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle \\ \times \left|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle . \quad (1.135)$$

Let us analyze the transformation between the first and third final bases:

$$\left|((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l\right\rangle = \\ = \sum_{\substack{r_{12}\lambda_{12} \\ r_{123}\lambda_{123}r}} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle \\ \times \left|(((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle , \quad (1.136)$$

$$\left|(((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l\right\rangle =$$

$$= \sum_{\substack{r_{23}\lambda_{23} \\ r_{14}\lambda_{14}t}} \left\langle ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \middle| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle$$

$$\times \left| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l \right\rangle . \qquad (1.137)$$

Let us analyze the transformation between the second and third final bases:

$$\left| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l \right\rangle =$$

$$= \sum_{s_{123}\lambda_{123}r} \left\langle ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle$$

$$\times \left| ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle , \qquad (1.138)$$

$$\left| ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle =$$

$$= \sum_{r_{14}\lambda_{14}t} \left\langle ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \middle| ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle$$

$$\times \left| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l \right\rangle . \qquad (1.139)$$

Let us analyze the transformation between the first and fourth final bases:

$$\left| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l \right\rangle =$$

$$= \sum_{\lambda_{123}r_{123}r} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \right\rangle$$

$$\times \left| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle , \qquad (1.140)$$

$$\left| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle =$$

$$= \sum_{\lambda_{124}r_{124}s} \left\langle (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \middle| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle$$

$$\times \left| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l \right\rangle . \qquad (1.141)$$

Let us analyze the transformation between the fourth and fifth final bases:

$$\left| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l \right\rangle =$$

$$= \sum_{\lambda_{12}r_{12}r_{124}} \left\langle (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \middle| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \right\rangle$$

$$\times \left| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l \right\rangle , \qquad (1.142)$$

$$\left| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l \right\rangle =$$

$$= \sum_{\lambda_{14}r_{14}s_{124}} \left\langle (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \middle| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \right\rangle$$

$$\times \left| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l \right\rangle . \qquad (1.143)$$

Let us analyze the transformation between the third and fifth final bases:

$$\begin{aligned}
&\big|(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l\big\rangle = \\
&= \sum_{\lambda_{23}r_{23}t} \big\langle((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big|(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda\big\rangle \\
&\qquad\qquad \times \big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l\big\rangle\,, \qquad (1.144)\\
&\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l\big\rangle = \\
&= \sum_{\lambda_{124}s_{124}s} \big\langle(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big\rangle \\
&\qquad\qquad \times \big|(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l\big\rangle\,. \qquad (1.145)
\end{aligned}$$

Remark 1.5.2. In the expressions above, we sum up on some indices but not on others accordingly to Schur's Lemma, as already explained by Remark 1.2.1.

The statements of the following lemmas, facts and corollaries refer to fixed irreducible representations $\lambda_{12} \in \lambda_1 \otimes \lambda_2$, $\lambda_{23} \in \lambda_2 \otimes \lambda_3$, $\lambda_{14} \in \lambda_1 \otimes \lambda_4$ with multiplicity labels r_{12}, r_{23}, r_{14} respectively, $\lambda_{123} \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3$ with multiplicity labels r_{123} and s_{123}, $\lambda \in \lambda_1 \otimes \lambda_2 \otimes \lambda_3 \otimes \lambda_4$ with multiplicity labels r and t.

Lemma 1.5.2. *The following relation holds:*

$$\begin{aligned}
&\sum_{s_{123}} \big\langle(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle \\
&\quad\times \big\langle((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big\rangle \\
&\qquad = \big\langle(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big\rangle\,. \qquad (1.146)
\end{aligned}$$

Proof. We will write $\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l\big\rangle$ in two ways: we first use (1.136) and then (1.138), in which we insert (1.134). What we get is the following:

$$\begin{aligned}
&\sum_{\substack{r_{12}\lambda_{12}\\ r_{123}\lambda_{123}r}} \big\langle(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big\rangle \\
&\qquad\qquad \times \big|(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda l\big\rangle \\
&= \big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l\big\rangle = \\
&= \sum_{s_{123}\lambda_{123}r} \big\langle((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big\rangle \\
&\qquad\qquad \times \big|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda l\big\rangle \\
&= \sum_{\substack{r_{12}\lambda_{12}\\ r_{123}\lambda_{123}r}} \Bigg(\sum_{s_{123}} \big\langle(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle
\end{aligned}$$

$$\times \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)r\lambda \right\rangle \Bigg)$$
$$\times \left| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle .$$

We get the claim by comparing the first and the last term of the equation above. □

Lemma 1.5.3. *The following equality holds:*

$$\begin{aligned} &\left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle \\ &= \sum_{r_{124}\lambda_{124}ss_{124}} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \right\rangle \\ &\qquad \times \left\langle (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \middle| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \right\rangle \\ &\qquad \times \left\langle (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle . \end{aligned} \tag{1.147}$$

Proof. Notice that $\left| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda l \right\rangle$ is equal to both:

$$\sum_{r_{12}\lambda_{12}} \sum_{r_{123}\lambda_{123}r} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle$$
$$\times \left| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle$$

by (1.136) and:

$$\begin{aligned} &\sum_{\lambda_{124}s_{124}s} \left\langle (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle \\ &\qquad\qquad \times \left| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda l \right\rangle \\ &= \sum_{\lambda_{124}s_{124}s} \left\langle (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle \\ &\times \sum_{r_{12}r_{124}r} \left\langle (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \middle| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \right\rangle \\ &\qquad\qquad \times \left| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda l \right\rangle \\ &= \sum_{\lambda_{124}s_{124}s} \left\langle (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \middle| ((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda \right\rangle \\ &\times \sum_{\lambda_{12}r_{12}r_{124}} \left\langle (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \middle| (\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})s\lambda \right\rangle \\ &\times \sum_{\lambda_{123}r_{123}r} \left\langle (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda \middle| (\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})s\lambda \right\rangle \\ &\qquad\qquad \times \left| (((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)r\lambda l \right\rangle \end{aligned}$$

by (1.145), (1.142), (1.140). The claim follows by linear independence. □

Fact 1.5.2. *The following equality holds:*

$$\big\langle((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big|(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle = \big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123}\big\rangle\,. \tag{1.148}$$

Proof. Notice that $|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123}l_{123},\lambda_4 l_4\rangle$ is equal to both:

$$\begin{aligned}&\sum_{r\lambda l}\langle r\lambda l|\lambda_{123}l_{123},\lambda_4 l_4\rangle\,|(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda l\rangle\\ &=\sum_{\substack{r\lambda l\\ r_{23}\lambda_{23}s_{123}}}\Bigg(\langle r\lambda l|\lambda_{123}l_{123},\lambda_4 l_4\rangle\\ &\qquad\times\big\langle((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big|(((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle\Bigg)\\ &\qquad\qquad\times|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda l\rangle\end{aligned}$$

by (1.135) and:

$$\begin{aligned}&\sum_{r_{23}\lambda_{23}s_{123}}\big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123}\big\rangle\\ &\qquad\qquad\times|(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}l_{123},\lambda_4 l_4\rangle\\ &=\sum_{\substack{r\lambda l\\ r_{23}\lambda_{23}s_{123}}}\Bigg(\langle r\lambda l|\lambda_{123}l_{123},\lambda_4 l_4\rangle\\ &\qquad\times\big\langle(\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}\big|((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123}\big\rangle\Bigg)\\ &\qquad\qquad\times|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda l\rangle\end{aligned}$$

by (1.17). The claim follows by linear independence. □

Fact 1.5.3. *The following equality holds:*

$$\big\langle((\lambda_2\lambda_3)r_{23}\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle = \big\langle(\lambda_{23}(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\big|((\lambda_1\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda\big\rangle\,. \tag{1.149}$$

Proof. Knowing that $|\lambda_1 l_1,\lambda_2 l_2,\lambda_2 l_3,\lambda_4 l_4\rangle$ equals both:

$$\begin{aligned}&\sum_{r_{23}\lambda_{23}l_{23}}\sum_{s_{123}\lambda_{123}l_{123}}\sum_{r\lambda l}\langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2,\lambda_3 l_3\rangle\,\langle s_{123}\lambda_{123}l_{123}|\lambda_1 l_1,\lambda_{23}l_{23}\rangle\\ &\qquad\times\langle r\lambda l|\lambda_{123}l_{123},\lambda_4 l_4\rangle\,|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123},\lambda_4)r\lambda l\rangle\end{aligned}$$

and

$$= \sum_{r_{23}\lambda_{23}l_{23}} \sum_{r_{14}\lambda_{14}l_{14}} \sum_{t\lambda l} \langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2, \lambda_3 l_3\rangle \langle r_{14}\lambda_{14}l_{14}|\lambda_1 l_1, \lambda_4 l_4\rangle$$
$$\times \langle t\lambda l|\lambda_{23}l_{23}, \lambda_{14}l_{14}\rangle \,|_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda l\rangle ,$$

by 1.) of Fact 1.1.2 we have that $|_{((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23})s_{123}\lambda_{123}, \lambda_4)}r\lambda l\rangle$ equals:

$$\sum_{r_{14}\lambda_{14}t} |\lambda|^{-1} \sum_{l_2 l_3 l_4} \sum_{l_{23}l_{123}l_{14}} \langle s_{123}\lambda_{123}l_{123}|\lambda_1 l_1, \lambda_{23}l_{23}\rangle^* \langle r\lambda l|\lambda_{123}l_{123}, \lambda_4 l_4\rangle^*$$
$$\times \langle r_{14}\lambda_{14}l_{14}|\lambda_1 l_1, \lambda_4 l_4\rangle \langle t\lambda l|\lambda_{23}l_{23}, \lambda_{14}l_{14}\rangle \,|_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda l\rangle$$
$$= \sum_{r_{14}\lambda_{14}t} \left\langle {}_{(\lambda_{23}(\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda \middle| {}_{((\lambda_1\lambda_{23})s_{123}\lambda_{123}, \lambda_4)}r\lambda \right\rangle |_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda l\rangle ,$$

where we have applied (1.24). The claim follows by comparing with (1.139). □

Fact 1.5.4. *The following equalities hold:*

$$\left\langle {}_{(((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)}r\lambda \middle| {}_{(\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})}s\lambda \right\rangle$$
$$= \left\langle {}_{((\lambda_{12}, \lambda_3)r_{123}\lambda_{123}, \lambda_4)}r\lambda \middle| {}_{(\lambda_3(\lambda_{12}, \lambda_4)r_{124}\lambda_{124})}s\lambda \right\rangle ; \quad (1.150)$$
$$\left\langle {}_{(\lambda_3((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)r_{124}\lambda_{124})}s\lambda \middle| {}_{(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})}s\lambda \right\rangle$$
$$= \left\langle {}_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)}r_{124}\lambda_{124} \middle| {}_{(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})}s_{124}\lambda_{124} \right\rangle ; \quad (1.151)$$
$$\left\langle {}_{(\lambda_3(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124})}s\lambda \middle| {}_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, (\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda \right\rangle$$
$$= \left\langle {}_{(\lambda_3(\lambda_2\lambda_{14})s_{124}\lambda_{124})}s\lambda \middle| {}_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, \lambda_{14})}t\lambda \right\rangle . \quad (1.152)$$

Proof. The first overlap is independent of the $(\lambda_1\lambda_2\bar{\lambda}_{12}r_{12})$ coupling, the second of $(\lambda_3\lambda_{124}\bar{\lambda}s)$, the third of $(\lambda_1\lambda_4\bar{\lambda}_{14}r_{14})$. □

Putting together the two lemmas and three facts above, we finally get:

Corollary 1.5.6. *The following relation holds:*

$$\sum_{s_{123}} \left\langle {}_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3)}r_{123}\lambda_{123} \middle| {}_{((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23}))}s_{123}\lambda_{123} \right\rangle$$
$$\times \left\langle {}_{((\lambda_1\lambda_{23})s_{123}\lambda_{123}, \lambda_4)}r\lambda \middle| {}_{(\lambda_{23}(\lambda_1\lambda_4)r_{14}\lambda_{14})}t\lambda \right\rangle$$
$$= \sum_{r_{124}\lambda_{124}ss_{124}} \left\langle {}_{((\lambda_{12}\lambda_3)r_{123}\lambda_{123}, \lambda_4)}r\lambda \middle| {}_{(\lambda_3(\lambda_{12}\lambda_4)r_{124}\lambda_{124})}s\lambda \right\rangle$$
$$\times \left\langle {}_{((\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_4)}r_{124}\lambda_{124} \middle| {}_{(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})}s_{124}\lambda_{124} \right\rangle$$
$$\times \left\langle {}_{(\lambda_3(\lambda_2\lambda_{14})s_{124}\lambda_{124})}s\lambda \middle| {}_{((\lambda_2\lambda_3)r_{23}\lambda_{23}, \lambda_{14})}t\lambda \right\rangle .$$

The corollary above states a very general relation between recoupling coefficients, where no particular assumption is made. In what follows, we will convert our results in terms of $6j$-symbols, where all the conditions regarding phases, jm-symbols, quasi-ambivalence stated at the beginning of Section 1.4 are assumed.

Corollary 1.5.7. *The following relation holds:*

$$\begin{aligned}
&\sum_h \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1c_2c_3h} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}^*_{d_1d_2d_3h} \\
&= \{\alpha_1\}\{\nu_1\}\{\beta_2\bar{\beta}_1\alpha_3c_3\} \sum_{c_1'c_2'd_1'd_2'} M((13),\beta_3\bar{\beta}_2\alpha_1)_{c_1c_1'} M((23),\beta_1\bar{\beta}_3\alpha_2)_{c_2c_2'} \\
&\qquad\qquad \times M((123),\nu_2\bar{\nu}_3\bar{\alpha}_1)_{d_1d_1'} M((132),\nu_3\bar{\nu}_1\bar{\alpha}_2)_{d_2d_2'} \\
&\times \sum_{\xi e_1e_2e_3} |\xi| \{\bar{\beta}_1\nu_1\xi e_1\}\{\bar{\beta}_2\nu_2\xi e_2\}\{\bar{\beta}_3\nu_3\xi e_3\} \begin{Bmatrix} \nu_2 & \bar{\beta}_2 & \xi \\ \beta_3 & \nu_3 & \bar{\alpha}_1 \end{Bmatrix}_{d_1'c_1'e_3e_2} \\
&\qquad \times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2'c_2'e_1e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3c_3e_2e_1}.
\end{aligned} \tag{1.153}$$

Assuming to deal with simple-phase triads only, (1.153) *becomes:*

$$\begin{aligned}
&\sum_h \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1c_2c_3h} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}^*_{d_1d_2d_3h} \\
&= \{\alpha_1\}\{\nu_1\}\{\beta_3\bar{\beta}_2\alpha_1c_1\}\{\beta_1\bar{\beta}_3\alpha_2c_2\}\{\beta_2\bar{\beta}_1\alpha_3c_3\} \\
&\times \sum_{\xi e_1e_2e_3} |\xi| \{\bar{\beta}_1\nu_1\xi e_1\}\{\bar{\beta}_2\nu_2\xi e_2\}\{\bar{\beta}_3\nu_3\xi e_3\} \begin{Bmatrix} \nu_2 & \bar{\beta}_2 & \xi \\ \beta_3 & \nu_3 & \bar{\alpha}_1 \end{Bmatrix}_{d_1c_1e_3e_2} \\
&\qquad \times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2c_2e_1e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3c_3e_2e_1}.
\end{aligned} \tag{1.154}$$

Proof. For simplicity, we will prove (1.154) only. The general case is done similarly. By Lemma 1.5.1 and the definition of $6j$-symbols, we get:

$$\begin{aligned}
&\sum_{r_{124}\lambda_{124}ss_{124}} \langle((\lambda_{12}\lambda_3)r_{123}\lambda_{123},\lambda_4)r\lambda|(\lambda_3(\lambda_{12}\lambda_4)r_{124}\lambda_{124})s\lambda\rangle \\
&\times \langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_4)r_{124}\lambda_{124}|(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124}\rangle \langle(\lambda_3(\lambda_2\lambda_{14})s_{124}\lambda_{124})s\lambda|((\lambda_2\lambda_3)r_{23}\lambda_{23},\lambda_{14})t\lambda\rangle \\
&= \sum_{r_{124}\lambda_{124}ss_{124}} \{\lambda_{12}\lambda_3\bar{\lambda}_{123}r_{123}\} \langle((\lambda_3\lambda_{12})r_{123}\lambda_{123},\lambda_4)r\lambda|(\lambda_3(\lambda_{12}\lambda_4)r_{124}\lambda_{124})s\lambda\rangle \\
&\qquad \times \{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\} \langle((\lambda_2\lambda_1)r_{12}\lambda_{12},\lambda_4)r_{124}\lambda_{124}|(\lambda_2(\lambda_1\lambda_4)r_{14}\lambda_{14})s_{124}\lambda_{124}\rangle \\
&\qquad \times \{\lambda_2\lambda_3\bar{\lambda}_{23}r_{23}\} \langle((\lambda_3\lambda_2)r_{23}\lambda_{23},\lambda_{14})t\lambda|(\lambda_3(\lambda_2\lambda_{14})s_{124}\lambda_{124})s\lambda\rangle^*
\end{aligned}$$

$$= \frac{\sqrt{|\lambda_{123}|}\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{14}|}\sqrt{|\lambda_{23}|}}{\{\lambda_{123}\lambda_4\bar{\lambda}r\}\{\lambda_{23}\lambda_{14}\bar{\lambda}t\}} \sum_{r_{124}\lambda_{124}ss_{124}} |\lambda_{124}| \{\lambda_{12}\lambda_4\bar{\lambda}_{124}r_{124}\}$$
$$\times \begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_{12} \end{Bmatrix}_{r_{123}r_{124}rs} \begin{Bmatrix} \lambda_2 & \lambda_{14} & \bar{\lambda}_{124} \\ \bar{\lambda}_4 & \lambda_{12} & \lambda_1 \end{Bmatrix}_{r_{12}r_{14}r_{124}s_{124}} \begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_{14} & \lambda_{23} & \lambda_2 \end{Bmatrix}^*_{r_{23}s_{124}ts}.$$

By Corollary 1.5.6 and Lemma 1.5.1, the latter is equal to:

$$\sum_{s_{123}} \langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r_{123}\lambda_{123}|((\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23}))s_{123}\lambda_{123}\rangle$$
$$\times \{\lambda_1\lambda_{23}\bar{\lambda}_{123}r_{123}\} \langle((\lambda_{23}\lambda_1)s_{123}\lambda_{123},\lambda_4)r\lambda|(\lambda_{23}(\lambda_1\lambda_4)r_{14}\lambda_{14})t\lambda\rangle$$
$$= \sum_{s_{123}} \frac{\sqrt{|\lambda_{12}|}\sqrt{|\lambda_{23}|}}{\{\lambda_{23}\}\{\lambda_3\}\{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}_{123}r_{123}\}} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda}_{123} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12}r_{23}r_{123}s_{123}}$$
$$\times \frac{\sqrt{|\lambda_{123}|}\sqrt{|\lambda_{14}|}}{\{\lambda_{14}\}\{\lambda_4\}\{\lambda_{123}\lambda_4\bar{\lambda}r\}} \begin{Bmatrix} \lambda_{23} & \lambda_{14} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_1 \end{Bmatrix}_{s_{123}r_{14}rt}.$$

Therefore we have:

$$\sum_{s_{123}} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \bar{\lambda}_{123} \\ \bar{\lambda}_3 & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r_{12}r_{23}r_{123}s_{123}} \begin{Bmatrix} \lambda_{23} & \lambda_{14} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_1 \end{Bmatrix}_{s_{123}r_{14}rt} = \{\lambda_{14}\}\{\lambda_4\}\{\lambda_{23}\}\{\lambda_3\}$$
$$\times \{\lambda_1\lambda_2\bar{\lambda}_{12}r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda}_{123}r_{123}\}\{\lambda_{23}\lambda_{14}\bar{\lambda}t\} \sum_{r_{124}\lambda_{124}ss_{124}} |\lambda_{124}| \{\lambda_{12}\lambda_4\bar{\lambda}_{124}r_{124}\}$$
$$\times \begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_{12} \end{Bmatrix}_{r_{123}r_{124}rs} \begin{Bmatrix} \lambda_2 & \lambda_{14} & \bar{\lambda}_{124} \\ \bar{\lambda}_4 & \lambda_{12} & \lambda_1 \end{Bmatrix}_{r_{12}r_{14}r_{124}s_{124}} \begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_{14} & \lambda_{23} & \lambda_2 \end{Bmatrix}^*_{r_{23}s_{124}ts}. \tag{1.155}$$

We will now use former symmetries to manipulate what we have obtained. Exchanging rows in the last two columns, permuting the columns cyclically and applying the third symmetry, we have:

$$\begin{Bmatrix} \lambda_{23} & \lambda_{14} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_1 \end{Bmatrix}_{s_{123}r_{14}rt} = \{\lambda_1\}\{\lambda_4\}\{\lambda_{23}\}\{\lambda\} \begin{Bmatrix} \bar{\lambda}_1 & \lambda_{23} & \bar{\lambda}_{123} \\ \bar{\lambda} & \bar{\lambda}_4 & \bar{\lambda}_{14} \end{Bmatrix}^*_{r_{14}trs_{123}};$$

exchanging the rows in the first two columns, we get:

$$\begin{Bmatrix} \lambda_2 & \lambda_{14} & \bar{\lambda}_{124} \\ \bar{\lambda}_4 & \lambda_{12} & \lambda_1 \end{Bmatrix}_{r_{12}r_{14}r_{124}s_{124}} = \{\lambda_{14}\}\{\lambda_{124}\}\{\lambda_{12}\}\{\lambda_1\} \begin{Bmatrix} \bar{\lambda}_4 & \bar{\lambda}_{12} & \lambda_{124} \\ \lambda_2 & \bar{\lambda}_{14} & \bar{\lambda}_1 \end{Bmatrix}_{r_{14}r_{12}s_{124}r_{124}};$$

applying the third symmetry, exchanging the second and the third column and exchanging the rows in the first two columns, we get:

$$\begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_{14} & \lambda_{23} & \lambda_2 \end{Bmatrix}^*_{r_{23}s_{124}ts} = \{\lambda_{14}\}\{\lambda_{124}\}\{\lambda\}\{\bar{\lambda}_3\lambda_{23}\bar{\lambda}_2r_{23}\}\{\bar{\lambda}_{14}\lambda\bar{\lambda}_{23}t\}$$

$$\times \{\lambda_{14}\lambda_2\bar{\lambda}_{124}s_{124}\}\{\bar{\lambda}_3\lambda\bar{\lambda}_{124}s\} \begin{Bmatrix} \bar{\lambda}_{14} & \bar{\lambda}_2 & \lambda_{124} \\ \bar{\lambda}_3 & \bar{\lambda} & \bar{\lambda}_{23} \end{Bmatrix}_{tr_{23}s_{124}s} ;$$

permuting cyclically the columns, we have:

$$\begin{Bmatrix} \lambda_3 & \lambda_{124} & \bar{\lambda} \\ \bar{\lambda}_4 & \lambda_{123} & \lambda_{12} \end{Bmatrix}_{r_{123}r_{124}rs} = \begin{Bmatrix} \bar{\lambda} & \lambda_3 & \lambda_{124} \\ \lambda_{12} & \bar{\lambda}_4 & \lambda_{123} \end{Bmatrix}_{rr_{123}r_{124}s} .$$

Rewriting (1.155) using the last observations gives the conclusion, where we recall that nj-phases are ± 1 and that by quasi-ambivalence we have:

$$\begin{aligned} &\{\lambda_3\}\{\lambda_{14}\}\{\lambda_{12}\} = \{\lambda_3\}\{\lambda_{14}\}\{\lambda_{12}\} \cdot \{\lambda_1\}^2\{\lambda_2\}^2\{\lambda_{23}\}^2\{\lambda\}^2 \\ =&\{\lambda_1\}\{\lambda\}(\{\lambda_{23}\}\{\lambda_{14}\}\{\lambda\})(\{\lambda_2\}\{\lambda_3\}\{\lambda_{23}\})(\{\lambda_1\}\{\lambda_2\}\{\lambda_{12}\}) = \{\lambda_1\}\{\lambda\}. \end{aligned}$$

□

Corollary 1.5.8 (Biedenharn-Elliot sum rule)**.** *The following symmetry property holds:*

$$\begin{aligned} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1c_2c_3k} &= |\alpha_3|\,\{\alpha_1\}\{\nu_1\}\{\beta_2\bar{\beta}_1\alpha_3c_3\} \sum_{c_1'c_2'd_1'd_2'd_1d_2} M((13),\beta_3\bar{\beta}_2\alpha_1)_{c_1c_1'} \\ \times M((23),\beta_1\bar{\beta}_3\alpha_2)_{c_2c_2'} &\sum_{\nu_3} |\nu_3|\, M((123),\nu_2\bar{\nu}_3\bar{\alpha}_1)_{d_1d_1'} M((132),\nu_3\bar{\nu}_1\bar{\alpha}_2)_{d_2d_2'} \\ &\times \sum_{\xi e_1e_2e_3} |\xi|\,\{\bar{\beta}_1\nu_1\xi e_1\}\{\bar{\beta}_2\nu_2\xi e_2\}\{\bar{\beta}_3\nu_3\xi e_3\} \begin{Bmatrix} \nu_2 & \bar{\beta}_2 & \xi \\ \beta_3 & \nu_3 & \bar{\alpha}_1 \end{Bmatrix}_{d_1'c_1'e_3e_2} \\ &\times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2'c_2'e_1e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3c_3e_2e_1} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{d_1d_2d_3k} . \end{aligned} \quad (1.156)$$

Assuming to deal with simple-phase triads only, (1.156) *becomes:*

$$\begin{aligned} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1c_2c_3k} &= |\alpha_3|\,\{\alpha_1\}\{\nu_1\}\{\beta_3\bar{\beta}_2\alpha_1c_1\}\{\beta_1\bar{\beta}_3\alpha_2c_2\}\{\beta_2\bar{\beta}_1\alpha_3c_3\} \\ \times &\sum_{\substack{\xi\nu_3 \\ e_1e_2e_3d_1d_2}} |\xi|\,|\nu_3|\,\{\bar{\beta}_1\nu_1\xi e_1\}\{\bar{\beta}_2\nu_2\xi e_2\}\{\bar{\beta}_3\nu_3\xi e_3\} \begin{Bmatrix} \nu_2 & \bar{\beta}_2 & \xi \\ \beta_3 & \nu_3 & \bar{\alpha}_1 \end{Bmatrix}_{d_1c_1e_3e_2} \\ &\times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2c_2e_1e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3c_3e_2e_1} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{d_1d_2d_3k} . \end{aligned} \quad (1.157)$$

In the equations above, ν_1, ν_2 are irreducible representations chosen such that the right-hand side does not vanish identically, namely the triads involving ν_1 or ν_2 must be valid triads.

Proof. Multiply both sides of (1.153) by $|\nu_3| \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3' \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{d_1 d_2 d_3' k}$ and sum over ν_3, d_1, d_2: the claim follows then by applying the fourth symmetry to the left-hand side of the obtained expression. □

Definition 1.5.2. Equation (1.156) is known as the **Biedenharn-Elliot sum rule** as well as the **pentagon relation**.

With the appropriate choices, (1.156) can be seen as a recursive relation. We will exploit this explicitly in the next chapter, together with the following:

Corollary 1.5.9. *The following version of the Biedenharn-Elliot sum rule holds:*

$$\begin{aligned}\begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 c_2 c_3 k} &= |\alpha_3| \{\beta_1\}\{\beta_2 \bar{\beta}_1 \alpha_3 c_3\} \sum_{\substack{c_1' c_2' d_2' d_1 d_2 \\ s_1 s_4}} \sum_{\substack{\nu_3 \xi \\ e_1 e_2 e_3}} |\nu_3|\,|\xi| \\ &\times \{\bar{\beta}_1 \nu_1 \xi e_1\}\{\bar{\beta}_2 \nu_2 \xi e_2\}\{\alpha_1 \nu_3 \bar{\nu}_2 d_1\} M((13), \xi \bar{\beta}_2 \nu_2)_{e_2 s_4} M((23), \beta_1 \bar{\beta}_3 \alpha_2)_{c_2 c_2'} \\ &\quad \times M((123), \xi \nu_3 \bar{\beta}_3)_{e_3 s_1} M((132), \nu_3 \bar{\nu}_1 \bar{\alpha}_2)_{d_2 d_2'} \begin{Bmatrix} \alpha_1 & \nu_3 & \bar{\nu}_2 \\ \xi & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 s_1 s_4 d_1} \\ &\times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2' c_2' e_1 e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3 c_3 e_2 e_1} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{d_1 d_2 d_3 k} . \end{aligned} \tag{1.158}$$

In case we deal with simple-phase triads only, we get:

$$\begin{aligned}\begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 c_2 c_3 k} &= |\alpha_3| \{\beta_1\}\{\beta_1 \bar{\beta}_3 \alpha_2 c_2\}\{\beta_2 \bar{\beta}_1 \alpha_3 c_3\} \\ &\quad \times \sum_{\substack{\xi \nu_3 \\ e_1 e_2 e_3 d_1 d_2}} |\nu_3|\,|\xi| \{\bar{\beta}_1 \nu_1 \xi e_1\}\{\alpha_1 \nu_3 \bar{\nu}_2 d_1\} \begin{Bmatrix} \alpha_1 & \nu_3 & \bar{\nu}_2 \\ \xi & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 e_3 e_2 d_1} \\ &\times \begin{Bmatrix} \nu_3 & \bar{\beta}_3 & \xi \\ \beta_1 & \nu_1 & \bar{\alpha}_2 \end{Bmatrix}_{d_2 c_2 e_1 e_3} \begin{Bmatrix} \nu_1 & \bar{\beta}_1 & \xi \\ \beta_2 & \nu_2 & \bar{\alpha}_3 \end{Bmatrix}_{d_3 c_3 e_2 e_1} \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{d_1 d_2 d_3 k} . \end{aligned} \tag{1.159}$$

Proof. Exchanging first and third column and exchanging rows in the first and second column, the $6j$-symbol $\begin{Bmatrix} \nu_2 & \bar{\beta}_2 & \xi \\ \beta_3 & \nu_3 & \bar{\alpha}_1 \end{Bmatrix}_{d_1' c_1' e_3 e_2}$ in (1.156) is equal to:

$$\begin{aligned} \{\bar{\alpha}_1\}\{\nu_3\}\{\beta_3\} \sum_{s_1 s_2 s_3 s_4} & M((13), \xi \nu_3 \bar{\beta}_3)_{e_3 s_1} M((13), \alpha_1 \bar{\beta}_2 \beta_3)_{c_1' s_2} \\ & \times M((13), \bar{\alpha}_1 \bar{\nu}_3 \nu_2)_{d_1' s_3} M((13), \xi \bar{\beta}_2 \nu_2)_{e_2 s_4} \begin{Bmatrix} \alpha_1 & \nu_3 & \bar{\nu}_2 \\ \xi & \beta_2 & \beta_3 \end{Bmatrix}_{s_2 s_1 s_4 s_3} . \end{aligned}$$

Substituting this into (1.156), we get the claim via the following equalities given by Proposition 1.3.7, Corollary 1.3.1, Remark 1.3.7 and quasi-ambivalence:

$$\sum_{c_1'} M((13), \beta_3\bar{\beta}_2\alpha_1)_{c_1c_1'} M((13), \alpha_1\bar{\beta}_2\beta_3)_{c_1's_2} = \delta_{c_1s_2};$$

$$\sum_{d_1'} M((123), \nu_2\bar{\nu}_3\bar{\alpha}_1)_{d_1d_1'} M((13), \bar{\alpha}_1\bar{\nu}_3\nu_2)_{d_1's_3} = \{\alpha_1\nu_3\bar{\nu}_2d_1\}\delta_{d_1s_3};$$

$$\sum_{e_3}\{\bar{\beta}_3\nu_3\xi e_3\}M((13), \xi\nu_3\bar{\beta}_3)_{e_3s_1} = \sum_{e_3} M((123), \xi\nu_3\bar{\beta}_3)_{e_3s_1};$$

$$\{\bar{\alpha}_1\}\{\alpha_1\} = \{\alpha_1\}\{\alpha_1\} = \{\alpha_1\}^2 = 1; \quad \{\nu_1\}(\{\nu_3\}\{\beta_3\}) = \{\nu_1\}\{\xi\} = \{\bar{\beta}_1\} = \{\beta_1\}.$$

□

1.6 Quantum 6*j*-symbols

It is possible to generalize the concepts of coupling and recoupling coefficients to one-parameter deformations of universal enveloping algebras, i.e. to *quantum groups*. Usually, the deformation parameter is denoted by the letter q and we have that in the limit as $q \to 1$ the classical algebra is retrieved. For a detailed introduction to this topic, we refer to [Kas95] and [LB92]. In the context of quantum groups, we can then define the so called *quantum* $6j$*-symbols*. With a slightly different definition of $3jm$-symbols proposed in [LB92], one can check that all the same symmetry properties analyzed in Section 1.5 apply to quantum $6j$-symbols as well, apart from the following modification of the fifth symmetry:

Proposition 1.6.1 (Generalized Racah-backcoupling rule)**.** *The following relation holds:*

$$q^{(C_{\lambda_1}+C_{\lambda_3}+C_{\mu_1}+C_{\mu_3})/2}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4} = \{\mu_3\}\{\lambda_1\bar{\mu}_2\mu_3r_1\}\{\mu_1\lambda_2\bar{\mu}_3r_2\}$$
$$\times\sum_{\nu rs} q^{(C_\nu+C_{\lambda_2}+C_{\mu_2})/2}\,|\nu|\,\{\nu\}\{\bar{\lambda}_1\mu_1\nu r\}\begin{Bmatrix}\mu_3 & \nu & \lambda_3\\ \mu_1 & \mu_2 & \lambda_1\end{Bmatrix}_{r_1rr_3s}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \bar{\mu}_3 & \nu & \bar{\mu}_1\end{Bmatrix}_{rr_2sr_4}, \tag{1.160}$$

where C_α is the quadratic Casimir operator acting on the module V_α associated with the representation α. Assuming quasi-ambivalence, we get $\{\mu_3\}\{\nu\} = \{\lambda_3\}$.

Proof. See [LB92]. □

As in Subsection 1.5.5, applying different symmetries to the result above leads to:

Corollary 1.6.1. *Denoting by # the product of specific known coefficients, we get:*

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \sum_{\nu r s} \# \, |\nu| \begin{Bmatrix} \mu_1 & \nu & \lambda_1 \\ \mu_2 & \mu_3 & \lambda_2 \end{Bmatrix}_{r_1 r r_3 s} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_2 & \bar{\mu}_1 & \nu \end{Bmatrix}_{r_2 s r r_4} . \tag{1.161}$$

A last remark on the notation: within this quantum frame, if x is an operator or a number, we define the following:

$$[x] := \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}. \tag{1.162}$$

Chapter 2

Computation of $6j$-symbols

The present chapter wants to deal with the following question: are $6j$-symbols computable? Is there a way to exploit the symmetry properties of $6j$-symbols in order to be able to know explicitly the absolute value of an arbitrary $6j$-symbol? Stated like this, the question addresses the problem very generically. We will focus on certain types of groups admitting particular irreducible representations that will allow us to categorize $6j$-symbols in different classes. We will then analyze separately these classes more in details in the case of the group $SU(N)$. Another goal is to give detailed and rigorous proofs, since they will also serve as a computational strategy in case the reader is interested in computing a specific $6j$-symbol.

Throughout the chapter, G will denote a simple connected compact Lie group not of type D_{even}. These conditions imply G to be quasi-ambivalent (see Definition 1.3.3 and Remark 1.3.6), enabling us to utilize all the results of Chapter 1, and they guarantee the existence of a finite-dimensional faithful representation which is irreducible (see Section A.3 for details). We therefore fix a lowest-dimensional irreducible faithful representation ϵ of G. Although it is not a common reference, when we use the expression **primitive irreducible representation** in this chapter we refer either to ϵ or to $\bar{\epsilon}$.

Before to continue, we define some other notation. If V is a vector space and λ is a representation of G, we write the following for any $k, l \in \mathbb{N}$:

$$V(k,l) := V^{\otimes k} \otimes \left(V^{\vee}\right)^{\otimes l}, \qquad V^{\vee}(k,l) := \left(V^{\vee}\right)^{\otimes k} \otimes V^{\otimes l}; \tag{2.1}$$

$$\lambda(k,l) := \lambda^{\otimes k} \otimes \left(\bar{\lambda}\right)^{\otimes l}, \qquad \bar{\lambda}(k,l) := \left(\bar{\lambda}\right)^{\otimes k} \otimes \lambda^{\otimes l}. \tag{2.2}$$

2.1 Power of Representations

We present now the notion of power of an irreducible representation and study some basic properties of it. This concept is illustrated in [But81, Chapter 3].

Definition 2.1.1. Let λ be an irreducible representation of G. By Theorem A.3.1, we have that $\lambda \in \epsilon(m,n)$ for some $m,n \in \mathbb{N}$. Choosing m,n minimal with respect to this property, we define the **power** of λ as $p(\lambda) \coloneqq m+n$.

Fact 2.1.1. *Let λ, μ, ν be finite-dimensional irreducible representations of G and let $\varepsilon \in \{\epsilon, \bar{\epsilon}\}$. Then:*

1. $p(\lambda) = p(\bar{\lambda})$;
2. $(\lambda\mu\bar{\nu})$ *is a triad* $\Rightarrow |p(\lambda) - p(\mu)| \leq p(\nu) \leq p(\lambda) + p(\mu)$;
3. $(\varepsilon\lambda\bar{\mu})$ *is a triad* $\Rightarrow p(\lambda) - 1 \leq p(\mu) \leq p(\lambda) + 1$.

Proof.

1. Assume $\lambda \in \epsilon(m,n)$ with m,n minimal, so that $p(\lambda) = m+n$. Then $\bar{\lambda} \in \bar{\epsilon}(m,n) \cong \epsilon(n,m)$, hence $p(\bar{\lambda}) \leq n+m$ (by the minimality of $p(\bar{\lambda})$). Consider $\bar{\lambda} \in \epsilon(h,l)$ with h,l minimal, so that $p(\bar{\lambda}) = h+l$. Then $\lambda \in \bar{\epsilon}(h,l) \cong \epsilon(l,h)$, so $p(\lambda) \leq l+h$ (by the minimality of $p(\lambda)$). We conclude that $p(\bar{\lambda}) \leq n+m = p(\lambda) \leq l+h = p(\bar{\lambda})$, getting $p(\bar{\lambda}) = p(\lambda)$.

2. Assume $\lambda \in \epsilon(a,b)$, $\mu \in \epsilon(c,d)$ with $p(\lambda) = a+b$, $p(\mu) = c+d$. Suppose $(\lambda\mu\bar{\nu})$ is a triad. We have that $\nu \in \lambda \otimes \mu$, which implies $\nu \in \epsilon(a,b) \otimes \epsilon(c,d) \cong \epsilon(a+c, b+d)$, implying $p(\nu) \leq a+c+b+d = p(\lambda)+p(\mu)$. By Fact A.2.2, we have that $\bar{\mu} \in \lambda \otimes \bar{\nu}$, therefore by the result we have just proven we have $p(\mu) = p(\bar{\mu}) \leq p(\lambda) + p(\bar{\nu}) = p(\lambda) + p(\nu)$, which implies $p(\mu) - p(\lambda) \leq p(\nu)$. Always by Fact A.2.2, we get $\bar{\lambda} \in \mu \otimes \bar{\nu}$, hence $p(\lambda) = p(\bar{\lambda}) \leq p(\mu) + p(\bar{\nu}) = p(\mu) + p(\nu)$, which implies $p(\lambda) - p(\mu) \leq p(\nu)$. Therefore we can conclude $|p(\lambda) - p(\mu)| \leq p(\nu)$.

3. The claim follows directly from 2.). ☐

We now define the concept of ordered triad and ordering between ordered triads, as illustrated in [GJ15]:

Definition 2.1.2. A triad $(\lambda_1\lambda_2\lambda_3)$ is said to be **ordered** or **in standard order** when the following condition is satisfied: $p(\lambda_1) \geq p(\lambda_2) \geq p(\lambda_3)$.
If $(\lambda_1\lambda_2\lambda_3)$ and $(\mu_1\mu_2\mu_3)$ are ordered triads, we say that $(\lambda_1\lambda_2\lambda_3)$ is greater than $(\lambda_1\lambda_2\lambda_3)$ and write $(\lambda_1\lambda_2\lambda_3) > (\mu_1\mu_2\mu_3)$ when:

$$\begin{cases} p(\lambda_3) > p(\mu_3) & \text{if } p(\lambda_3) \neq p(\mu_3); \\ p(\lambda_2) > p(\mu_2) & \text{if } p(\lambda_3) = p(\mu_3); \\ p(\lambda_1) > p(\mu_1) & \text{if } p(\lambda_3) = p(\mu_3),\ p(\lambda_2) = p(\mu_2). \end{cases}$$

We illustrate a useful notation for the power of a representation. We will also use the symbol n_p with $n \in \mathbb{N}$ to denote a generic irreducible representation of power n. More precisely, if α is an irreducible representation such that $p(\alpha) = n$, we can write $\alpha = n_p$. In particular, if β is another irreducible representation such that $p(\alpha) = p(\beta) = n$, we write $\alpha = n_p$ and $\beta = n_p$ without necessarily implying that α and β are equal. For instance, if the symbol n_p is used more times in the same $6j$-symbol or among different $6j$-symbols, the representations denoted by it do not necessarily coincide.

Remark 2.1.1. With the settings established above, the $2j$-phase of a finite-dimensional irreducible representation λ of G can be easily determined from the $2j$-phase of ϵ. Indeed, exploiting quasi-ambivalence, it is not difficult to see that the following relation holds:

$$\{\lambda\} = \{\epsilon\}^{p(\lambda)}. \tag{2.3}$$

2.2 Primitive 6*j*-symbols

In this section we focus on the class of the so called primitive $6j$-symbols. We will prove that any $6j$-symbol can be expressed in terms of a further subclass of primitive $6j$-symbols, i.e. the core $6j$-symbols. The definitions we adopt in this section are the ones used in [Sea88], apart from the terms *simple* and *base* $6j$-symbol which are introduced by us in this thesis.

Definition 2.2.1. A triad is said to be **primitive** when it includes either ϵ or $\bar{\epsilon}$.

Definition 2.2.2. A $6j$-symbol is said to be **primitive** whenever it is not trivial and contains either ϵ or $\bar{\epsilon}$. A $6j$-symbol is said to be **non-primitive** when it is neither primitive nor trivial.

Next result, stated in [GJ15] together with the main idea of the proof, shows that the study of $6j$-symbols can be reduced to that of primitive $6j$-symbols.

Proposition 2.2.1. *Every non-primitive $6j$-symbol can be computed in terms of primitive $6j$-symbols.*

Proof. Pick a non-primitive $6j$-symbol $\mathcal{S} = \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 c_2 c_3 k}$, namely none of the six irreducible representations of $\mathcal{S}$ are $\mathbf{1}$, ϵ or $\bar{\epsilon}$. Let us assume α_3 to be the smallest representation of $\mathcal{S}$, i.e. the representation of lowest power (we can do this by eventually applying the tetrahedral symmetries of $6j$-symbols seen in Definition 1.5.1).
Let us utilize now the Biedenharn-Elliot sum rule in the version given by (1.158).

In such equation, two irreducible representations ν_1 and ν_2 can be chosen. What we want to do now is to specify ν_1 and ν_2 in the different cases that may occur. Consider $\alpha_3 \in \epsilon(m,n)$ with $m,n \in \mathbb{N}$ minimal with respect to this property, so that $p(\alpha_3) = m+n$. Since $\mathcal{S}$ is non-primitive, we have that $p(\alpha_3) \geq 2$.

CASE $n = 0$. We have $\alpha_3 \in \epsilon^{\otimes m}$, hence $m \geq 2$. Consider the following expression:

$$\epsilon^{\otimes m} \cong (\epsilon^{\otimes(m-1)}) \otimes \epsilon \cong \left(\bigoplus_\eta \eta\right) \otimes \epsilon \cong \bigoplus_\eta (\eta \otimes \epsilon),$$

where $\epsilon^{\otimes(m-1)} \cong \bigoplus_\eta \eta$ is a decomposition of $\epsilon^{\otimes(m-1)}$ into irreducible representations. Therefore, there exists an irreducible representation η' among the representations η in the decomposition above such that $\alpha_3 \in \eta' \otimes \epsilon$. We then set $\nu_1 := \epsilon$, $\nu_2 := \bar{\eta}'$. In this way the triad $(\bar{\nu}_1 \nu_2 \alpha_3)$ is valid and $p(\nu_2) < p(\alpha_3)$.

CASE $m = 0$. We proceed as in the previous case: we have that $\alpha_3 \in \bar{\epsilon}^{\otimes n}$ with $n \geq 2$ and that $\alpha_3 \in \eta' \otimes \bar{\epsilon}$ for some $\eta' \in \bar{\epsilon}^{\otimes(n-1)}$. We set $\nu_1 := \bar{\epsilon}$, $\nu_2 := \bar{\eta}'$ and get that the triad $(\bar{\nu}_1 \nu_2 \alpha_3)$ is valid and $p(\nu_2) < p(\alpha_3)$.

CASE $m, n > 0$. This case is analogous to the previous ones:
– in the subcase in which $\alpha_3 \in \epsilon \otimes \eta'$ for some $\eta' \in \epsilon(m-1, n)$ we set $\nu_1 := \epsilon$, $\nu_2 := \bar{\eta}'$ getting that the triad $(\bar{\nu}_1 \nu_2 \alpha_3)$ is valid and $p(\nu_2) < p(\alpha_3)$;
– in the subcase in which $\alpha_3 \in \eta' \otimes \bar{\epsilon}$ for some $\eta' \in \epsilon(m, n-1)$ we set $\nu_1 := \bar{\epsilon}$, $\nu_2 := \bar{\eta}'$ getting that the triad $(\bar{\nu}_1 \nu_2 \alpha_3)$ is valid and $p(\nu_2) < p(\alpha_3)$.

In all the above cases, we end up with an expression of the type:

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 c_2 c_3 k} = \sum_{\xi \nu_3} \sum_{s_1 s_4 d_1} C(c_2 c_3 k) \cdot \begin{Bmatrix} \alpha_1 & \nu_3 & \bar{\nu}_2 \\ \xi & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 s_1 s_4 d_1}, \tag{2.4}$$

where $C(c_2 c_3 k)$ is a sum of products of terms consisting of $2j$-phases, $3j$-phases, coefficients of permutation matrices and primitive $6j$-symbols depending on the multiplicity labels c_2, c_3, k. Call $\mathcal{T} := \begin{Bmatrix} \alpha_1 & \nu_3 & \bar{\nu}_2 \\ \xi & \beta_2 & \beta_3 \end{Bmatrix}_{c_1 s_1 s_4 d_1}$. If $\mathcal{T}$ is primitive, we are done. Otherwise, notice that $\bar{\nu}_2$ is part of $\mathcal{T}$, implying that the smallest irreducible representation of $\mathcal{T}$ has power strictly less than $p(\alpha_3)$ (since $p(\bar{\nu}_2) = p(\nu_2) < p(\alpha_3)$), enabling us to repeat the whole process of this proof knowing that it will end in a finite number of steps. □

Next task is to characterize primitive $6j$-symbols depending on the number of primitive triads they have. This characterization is found in [Sea88] and here we give complete proofs together with some properties of the different subclasses of $6j$-symbols that will arise. Of course, primitive $6j$-symbols have at least two primitive triads.

Proposition 2.2.2. *A 6j-symbol with exactly two primitive triads can be related by the various symmetries to only one of the following:*

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon & \beta' & \alpha' \end{Bmatrix}_{r_1 r_2 r_3 r_4} \qquad s.t. \qquad p(\lambda) > p(\alpha), \tag{2.5}$$

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \varepsilon & \lambda' \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \tag{2.6}$$

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \alpha' & \lambda' & \varepsilon \end{Bmatrix}_{r_1 r_2 r_3 r_4} \qquad s.t. \qquad p(\alpha) > p(\beta), \tag{2.7}$$

where the triad at the top row is in standard order and the greatest among all four.

Proof. Fix a primitive $6j$-symbol $\mathcal{S}$ with exactly two primitive triads. Via an exchange of rows in two neighbouring columns, we can move the primitive irreducible representation to the bottom row of $\mathcal{S}$. Call T and V the two non-primitive triads of $\mathcal{S}$. Either T or V is the triad at the top row, so let us say that T is such. Non-primitive triads are greater than the primitive ones, so we only need to focus on T and V. Call μ the irreducible representation directly above the primitive representation in $\mathcal{S}$, then T and V have μ in common. If $T \geq V$ then we permute the columns of $\mathcal{S}$ to have the representations of T in standard order and we are done. If $T \leq V$ then we exchange rows in the neighbouring columns of $\mathcal{S}$ which do not contain μ and ε so that V (up to take the dual for certain representations) is now at the top row; we then permute the columns of $\mathcal{S}$ to have the representations of V in standard order and we are done. We have therefore obtained the following three types of primitive $6j$-symbols:

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon & \beta' & \alpha' \end{Bmatrix}_{r_1 r_2 r_3 r_4}; \tag{a}$$

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \varepsilon & \lambda' \end{Bmatrix}_{r_1 r_2 r_3 r_4}; \tag{b}$$

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \alpha' & \lambda' & \varepsilon \end{Bmatrix}_{r_1 r_2 r_3 r_4}. \tag{c}$$

When $p(\lambda) = p(\alpha)$ in (a), we can exchange the first two columns to relate this $6j$-symbol to (b). Analogously, (b) and (c) are related when $p(\alpha) = p(\beta)$. Adding the restriction $p(\lambda) > p(\alpha)$ in (a) and $p(\alpha) > p(\beta)$ in (c) distinguishes the various forms. □

Definition 2.2.3. Primitive $6j$-symbols like in (2.5) are said to be of **Type I**, like in (2.6) of **Type II**, like in (2.7) of **Type III**.

Lemma 2.2.1. *Let* $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon & \beta' & \alpha' \end{Bmatrix}_{r_1r_2r_3r_4}$ *be a* $6j$*-symbol of Type I. We then have that* $p(\lambda) \geq p(\alpha'), p(\beta')$.

Proof. Being $\mathcal{S}$ of Type I requires $p(\lambda) > p(\alpha)$, so since $(\varepsilon\alpha\bar{\alpha}')$ is a triad we have $p(\alpha') \leq p(\alpha) + 1$, implying $p(\alpha') \leq p(\alpha) + 1 < p(\lambda) + 1$ hence $p(\alpha') \leq p(\lambda)$.
Analogously, being $\mathcal{S}$ of Type I requires $p(\lambda) > p(\alpha) \geq p(\beta)$ therefore $p(\lambda) > p(\beta)$, so since $(\bar{\varepsilon}\beta'\beta)$ is a triad we have $p(\beta') \leq p(\beta) + 1$, implying $p(\beta') \leq p(\beta) + 1 < p(\lambda) + 1$ hence $p(\beta') \leq p(\lambda)$. □

Lemma 2.2.2. *Let* $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \varepsilon & \lambda' \end{Bmatrix}_{r_1r_2r_3r_4}$ *be a* $6j$*-symbol of Type II. Then, the following implication holds:*

$$p(\lambda') > p(\lambda) \;\Rightarrow\; p(\beta') < p(\beta). \tag{2.8}$$

By contraposition, we have equivalently that:

$$p(\beta') \geq p(\beta) \;\Rightarrow\; p(\lambda') \leq p(\lambda). \tag{2.9}$$

Proof. Call $A = (\lambda\alpha\beta)$, $B = (\beta'\alpha\bar{\lambda}')$, where A is already in standard order. Being $\mathcal{S}$ of Type II implies that $A \geq B$.
Assume $p(\lambda') > p(\lambda)$. We then have that $p(\lambda') > p(\lambda) \geq p(\alpha) \geq p(\beta)$ and therefore $p(\lambda') \geq p(\beta')$ since $p(\beta') \leq p(\beta) + 1$.
Let us show that $p(\beta') \leq p(\alpha)$. Indeed, if $p(\beta') > p(\alpha)$ then the following things happen:
– we have that $p(\lambda') \geq p(\beta') > p(\alpha)$, enabling us to order B as $B = (\lambda'\beta'\alpha)$;
– the power of the third representation of A is greater or equal than the power of the third representation of B since $A \geq B$, i.e. we get $p(\beta) \geq p(\alpha)$;
– we have that $p(\alpha) = p(\beta)$ since $p(\alpha) \geq p(\beta)$ (A is in standard order);
– the power of the second representation of A is greater or equal than the power of the second representation of B since $A \geq B$, i.e. we get $p(\beta) \geq p(\beta')$, against the assumption $p(\beta') > p(\alpha)$.
Since we proved that $p(\alpha) \geq p(\beta')$, we can order B as $B = (\lambda'\alpha\beta')$. Having $p(\beta') \geq p(\beta)$ would imply $A < B$, against the condition $A \geq B$. Therefore, we conclude $p(\beta') < p(\beta)$. □

Lemma 2.2.3. *Let* $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \alpha' & \lambda' & \varepsilon \end{Bmatrix}_{r_1r_2r_3r_4}$ *be a* $6j$*-symbol of Type III. We then have that* $p(\beta) \leq p(\lambda'), p(\alpha')$.

Proof. Since $\mathcal{S}$ is of Type III, we have that $p(\alpha) - 1 \geq p(\beta)$ and since $(\alpha'\alpha\bar{\varepsilon})$ is a triad, we have that $p(\alpha') \geq p(\alpha) - 1$. Putting together these two facts, we obtain

that $p(\alpha') \geq p(\alpha) - 1 \geq p(\beta)$.
Since $\mathcal{S}$ is of Type III, we have that $p(\lambda) \geq p(\alpha)$ and since $(\lambda\bar{\lambda}'\varepsilon)$ is a triad, we have that $p(\lambda') \geq p(\lambda) - 1$. Putting together these two facts, we get that $p(\lambda') \geq p(\lambda) - 1 \geq p(\alpha) - 1 \geq p(\beta)$. □

Proposition 2.2.3. *A 6j-symbol with exactly three primitive triads can be related by the various symmetries to only one of the following:*

$$\begin{Bmatrix} \lambda & \alpha & 2_p \\ \varepsilon_1 & \varepsilon_2 & \alpha' \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.10) \qquad \begin{Bmatrix} 2_p & 2_p & 2_p \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.11)$$

where the triad at the top row is in standard order and the greatest among all four.

Proof. Fix a primitive $6j$-symbol $\mathcal{S}$ with exactly three primitive triads. The only chance is that among the six irreducible representations defining $\mathcal{S}$ only two or three of them are primitive. Via an exchange of rows in two neighbouring columns, we can move all primitive irreducible representations to the bottom row of $\mathcal{S}$. In this way, the only non-primitive triad T of $\mathcal{S}$ is at the top row. Automatically, T is the greatest triad. We then permute the columns in order to put T in standard order. We have therefore obtained the following types of primitive $6j$-symbols:

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon_1 & \varepsilon_2 & \alpha' \end{Bmatrix}_{r_1r_2r_3r_4}, \quad \text{(d)} \qquad \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \varepsilon_1 & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad \text{(e)}$$

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon_1 & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad \text{(f)} \qquad \begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{Bmatrix}_{r_1r_2r_3r_4}. \quad \text{(g)}$$

These classes are not totally distinguished, hence we require $p(\lambda) > p(\beta)$ in (e) and $p(\lambda) > p(\alpha) > p(\beta)$ in (f).
Furthermore, $p(\lambda) = 2$ in (e) and since the triad at the top row is ordered we have $p(\alpha) = p(\beta) = 2$ as well, violating the requirement $p(\lambda) > p(\beta)$, so this form vanishes.
In (f) we also have $p(\alpha) = p(\beta) = 2$ which does not fit the condition $p(\lambda) > p(\alpha) > p(\beta)$, so this form vanishes as well.
Finally, we have $p(\beta) = 2$ in (d) and $p(\lambda) = p(\alpha) = p(\beta) = 2$ in (g). □

Proposition 2.2.4. *A 6j-symbol with no non-primitive triads can be related by the various symmetries to only one of the following:*

$$\begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.12) \qquad \begin{Bmatrix} 2_p & \varepsilon_1 & \varepsilon_2 \\ 2_p & \varepsilon_3 & \varepsilon_4 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.13)$$

$$\begin{Bmatrix} 3_p & 2_p & \varepsilon_1 \\ \varepsilon_2 & 2_p & \varepsilon_3 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.14) \qquad \begin{Bmatrix} 2_p & 2_p & \varepsilon_1 \\ 2_p & \varepsilon_2 & \varepsilon_3 \end{Bmatrix}_{r_1r_2r_3r_4}, \quad (2.15)$$

$$\begin{Bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \quad (2.16) \qquad \begin{Bmatrix} 2_p & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \quad (2.17)$$

$$\begin{Bmatrix} 2_p & 2_p & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \quad (2.18) \qquad \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_3 & \lambda' \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \quad (2.19)$$

where the triad at the top row is in standard order and the greatest among all four (6j-symbols like in (2.16)–(2.19) only occur for the groups where a triad of the type $(\varepsilon_1\varepsilon_2\varepsilon_3)$ *exists).*

Proof. Let $\mathcal{S}$ be a primitive $6j$-symbol with no non-primitive triads. Then $\mathcal{S}$ must have at least two primitive irreducible representations. We classify $\mathcal{S}$ based on the number of primitive irreducible representations that $\mathcal{S}$ contains.

- Assume $\mathcal{S}$ has exactly two primitive irreducible representations. Then the only configuration for which we have no non-primitive triads is when the two primitive irreducible representations are in the same column, column that must be put at the third place if we want the triad at the top row to be in standard order. Therefore, we obtain (2.12).

- Assume $\mathcal{S}$ has exactly three primitive irreducible representations.
  In the case where we have two primitive irreducible representations in the same column, this column must be put as the third one (same reason as above) and $\mathcal{S}$ can be related to:

$$\begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \varepsilon_2 & \lambda' & \varepsilon_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}, \quad (2.20)$$

  for which we have $\alpha = 2_p$, $\lambda' = 2_p$ and so $\lambda = 2_p, 3_p$. In the case $\lambda = 2_p$ we further permute the first two columns to get (2.15), whereas in the case $\lambda = 3_p$ we have (2.14).
  In the case we do not have two primitive irreducible representations in the same column, we put the two of them that are in the same row at the bottom row. The primitive irreducible representation that we then find at the top row is sent to the third column to get the triad at the top row in standard order. In this way we have (2.19).

- Assume $\mathcal{S}$ has exactly four primitive irreducible representations. Then there are two cases with the triad at the top row as the greatest one and in standard order: either we find the four primitive irreducible representations in the last two columns or we find three of them at the bottom row and the other one in the third position of the top row. The other irreducible representations are forced to be of power 2. In this way we get (2.13) and (2.18).

- Assume $\mathcal{S}$ has exactly five primitive irreducible representations. Requiring the top row to be the greatest and in standard order, we get only one option, that is (2.17).

- If $\mathcal{S}$ has exactly six primitive irreducible representations, we have (2.16).

$\square$

Definition 2.2.4. Primitive $6j$-symbols like in (2.12) are said to be of **Type IV**. Primitive $6j$-symbols like in (2.10), (2.11), (2.13)–(2.19) are said to be **simple**. A $6j$-symbol which is either simple or of Type II is called a **core** $6j$-symbol. A $6j$-symbol which is either simple or of Type IV is called a **base** $6j$-symbol.

Lemma 2.2.4. *Let* $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4}$ *be a Type IV $6j$-symbol. We then have that $p(\lambda'), p(\alpha') \leq p(\lambda)$.*

Proof. Call $A = (\lambda\alpha\varepsilon_1)$, $B = (\lambda'\bar{\alpha}'\varepsilon_1)$. Since $\mathcal{S}$ is of Type IV, we have that $p(\lambda) \geq p(\alpha) > 1$, hence A is already in standard order and $A \geq B$.
If $p(\lambda') > p(\lambda)$ then the only way we can have $A > B$ is to have $p(\alpha') < p(\alpha)$. This implies $p(\lambda') > p(\lambda) \geq p(\alpha) > p(\alpha')$, hence $p(\lambda') > p(\alpha') + 1$, against $p(\alpha') - 1 \leq p(\lambda') \leq p(\alpha') + 1$. We therefore conclude $p(\lambda') \leq p(\lambda)$.
Let us show now the implication: $p(\alpha') > p(\lambda) \Rightarrow A < B$. Assume therefore that $p(\alpha') > p(\lambda)$. We then get:

$$p(\alpha') > p(\lambda) \geq p(\alpha) \quad \text{and} \quad p(\alpha) - 1 \leq p(\alpha') \leq p(\alpha) + 1,$$

implying $p(\alpha') = p(\alpha) + 1$ and $p(\alpha) = p(\lambda)$. Being B a triad, we have $p(\alpha') - 1 \leq p(\lambda') \leq p(\alpha') + 1$, hence the following considerations cover all possible cases:

$$\begin{array}{llll} \text{if} & p(\lambda') = p(\alpha') + 1 & \text{then} & p(\lambda') = p(\alpha') + 1 > p(\lambda) + 1 \text{ (against Fact 2.1.1)}, \\ \text{if} & p(\lambda') = p(\alpha') & \text{then} & A < B, \\ \text{if} & p(\lambda') = p(\alpha') - 1 & \text{then} & p(\lambda') = p(\alpha') - 1 = p(\alpha) = p(\lambda) \ \text{ then } \ A < B, \end{array}$$

making us conclude $A < B$. Now, since the lemma is assuming $A \geq B$, the just proven implication $p(\alpha') > p(\lambda) \Rightarrow A < B$ entails $p(\alpha') \leq p(\lambda)$. $\square$

Let us show now that in principle it is sufficient to reduce our attention to core $6j$-symbols only. The following results are again stated in [Sea88] and here we give complete proofs.

Proposition 2.2.5. *$6j$-symbols of Type I can be computed in terms of Type II and Type III $6j$-symbols.*

Proof. Let $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon & \beta' & \alpha' \end{Bmatrix}_{r_1r_2r_3r_4}$ be a Type I $6j$-symbol: $p(\lambda) > p(\alpha) \geq p(\beta)$. Lemma 2.2.1 tells us that $p(\lambda) \geq p(\alpha'), p(\beta')$. We will work in two cases.
CASE $p(\lambda) > p(\alpha')$. The Racah-backcoupling rule (see (1.160)) reads:

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon & \beta' & \alpha' \end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{\nu rs} \# \begin{Bmatrix} \alpha' & \nu & \beta \\ \varepsilon & \beta' & \lambda \end{Bmatrix}_{r_1rr_3s} \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\alpha}' & \nu & \bar{\varepsilon} \end{Bmatrix}_{rr_2sr_4},$$

where # denotes the product of specific coefficients which are known. Regarding this last expression, we have that:

$$\begin{aligned} &p(\nu) \geq p(\lambda) - 1 \geq p(\alpha), p(\beta); \\ &p(\nu) \geq p(\lambda) - 1 \geq p(\alpha'); \\ &\nu \neq \epsilon, \bar{\epsilon} \ \text{ since } \ p(\nu) \geq p(\alpha) > 1. \end{aligned}$$

We prove that the $6j$-symbols in the sum above are related either to Type II or to Type III $6j$-symbols.

- The first term in the sum above is related to $\mathcal{U} = \begin{Bmatrix} \nu & \beta & \alpha' \\ \beta' & \lambda & \varepsilon \end{Bmatrix}_{rr_3r_1s}$. Call $A = (\lambda\alpha'\bar{\beta}')$, $B = (\nu\beta\alpha')$.
  Assume $A \geq B$. Via the exchange of rows in the second and third column first and a permutation of columns afterwards, we relate $\mathcal{U}$ either to $\begin{Bmatrix} \bar{\lambda} & \bar{\alpha}' & \beta' \\ \bar{\beta} & \bar{\varepsilon} & \nu \end{Bmatrix}_{rsr_3r_1}$ if $p(\alpha') \geq p(\beta')$ or to $\begin{Bmatrix} \bar{\lambda} & \beta' & \bar{\alpha}' \\ \beta & \bar{\nu} & \varepsilon \end{Bmatrix}_{rr_3sr_1}$ if $p(\beta') > p(\alpha')$, which are respectively a Type II and Type III $6j$-symbol.
  Assume now $A < B$. Being $p(\nu) \geq p(\beta), p(\alpha')$, $\mathcal{U}$ can be related either to the Type II $6j$-symbol $\begin{Bmatrix} \nu & \alpha' & \beta \\ \bar{\beta}' & \bar{\varepsilon} & \bar{\lambda} \end{Bmatrix}_{rr_1r_3s}$ if $p(\alpha') \geq p(\beta)$ or to the Type III $6j$-symbol $\begin{Bmatrix} \nu & \beta & \alpha' \\ \beta' & \lambda & \varepsilon \end{Bmatrix}_{rr_3r_1s}$ if $p(\alpha') < p(\beta)$.

- Let us focus on the second term $\mathcal{T} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\alpha}' & \nu & \bar{\varepsilon} \end{Bmatrix}_{rr_2sr_4}$. Call $A = (\lambda\alpha\beta)$, $B = (\nu\alpha'\beta)$.
  If $A \geq B$, then $\mathcal{T}$ is already of Type III if $p(\alpha) > p(\beta)$, otherwise is related to the Type II $6j$-symbol $\begin{Bmatrix} \lambda & \beta & \alpha \\ \alpha' & \varepsilon & \bar{\nu} \end{Bmatrix}_{rsr_2r_4}$ if $p(\beta) \geq p(\alpha)$.
  Assume now $A < B$. We then move the triad B at the top row and being $p(\nu) \geq p(\alpha'), p(\beta)$, $\mathcal{T}$ can only be related either to the Type II $6j$-

symbol $\begin{Bmatrix} \bar{\nu} & \bar{\beta} & \bar{\alpha}' \\ \bar{\alpha} & \varepsilon & \lambda \end{Bmatrix}_{rr_2sr_4}$ if $p(\beta) \geq p(\alpha')$ or to the Type III $6j$-symbol $\begin{Bmatrix} \bar{\nu} & \bar{\alpha}' & \bar{\beta} \\ \alpha & \bar{\lambda} & \bar{\varepsilon} \end{Bmatrix}_{rr_2r_4s}$ if $p(\alpha') > p(\beta)$.

CASE $p(\alpha') = p(\lambda)$. Let us permute the columns of $\mathcal{S}$ via the cycle (132) and then apply the Racah-backcoupling rule:

$$\begin{Bmatrix} \alpha & \beta & \lambda \\ \beta' & \alpha' & \varepsilon \end{Bmatrix}_{r_2r_3r_1r_4} = \sum_{\nu rs} \# \begin{Bmatrix} \varepsilon & \nu & \lambda \\ \beta' & \alpha' & \alpha \end{Bmatrix}_{r_2rr_1s} \begin{Bmatrix} \alpha & \beta & \lambda \\ \bar{\varepsilon} & \nu & \bar{\beta}' \end{Bmatrix}_{rr_3sr_4}.$$

Again, $\#$ denotes the product of some known coefficients and $p(\nu) \geq 2, p(\alpha)$. In what follows we prove that the $6j$-symbols in the sum above are related either to Type II or to Type III $6j$-symbols.

- The first term $\begin{Bmatrix} \varepsilon & \nu & \lambda \\ \beta' & \alpha' & \alpha \end{Bmatrix}_{r_2rr_1s}$ is related to $\mathcal{U} = \begin{Bmatrix} \bar{\lambda} & \bar{\alpha}' & \beta' \\ \alpha & \nu & \bar{\varepsilon} \end{Bmatrix}_{sr_2rr_1}$. Call $A = (\lambda\alpha'\bar{\beta}')$, $B = (\beta'\nu\bar{\alpha})$.
  If $A \geq B$, being $p(\lambda) = p(\alpha') \geq p(\beta')$ we have that $\mathcal{U}$ is either of Type III if $p(\alpha') > p(\beta')$ or it can be related to the Type II $6j$-symbol $\begin{Bmatrix} \bar{\lambda} & \beta' & \bar{\alpha}' \\ \bar{\alpha} & \varepsilon & \bar{\nu} \end{Bmatrix}_{srr_2r_1}$ if $p(\beta') = p(\alpha')$.
  Assume now $A < B$. Then we put B at the top row. The only problem may occur in the case $p(\beta') > p(\nu), p(\alpha)$, because in this way $\mathcal{U}$ would be related to a Type I $6j$-symbol. Anyway, this case does not occur. Indeed:

$$\begin{aligned} p(\lambda) \geq p(\beta) + 1 \geq p(\beta') > p(\nu) \geq p(\lambda) - 1 \quad &\Rightarrow \quad p(\beta') = p(\lambda), \\ p(\lambda) = p(\beta') > p(\nu) \geq p(\lambda) - 1 \quad &\Rightarrow \quad p(\nu) = p(\lambda) - 1; \end{aligned}$$

$$\begin{aligned} p(\beta') > p(\alpha) \geq p(\beta) \quad &\Rightarrow \quad p(\beta) = p(\beta') - 1 = p(\lambda) - 1, \\ p(\lambda) > p(\alpha) \geq p(\beta) = p(\lambda) - 1 \quad &\Rightarrow \quad p(\alpha) = p(\lambda) - 1, \\ A < B \quad &\Rightarrow \quad p(\alpha') = p(\lambda) - 2, \end{aligned}$$

  but this last condition is excluded by the assumption $p(\alpha') = p(\lambda)$. Therefore, we can move β to the second or to the third column preserving the triad B in standard order, making $\mathcal{U}$ related either to the Type II $6j$-symbol $\begin{Bmatrix} \bar{\nu} & \bar{\beta}' & \alpha \\ \alpha' & \varepsilon & \bar{\lambda} \end{Bmatrix}_{sr_1r_2r}$ if $p(\beta') \geq p(\alpha)$ or to the Type III $6j$-symbol $\begin{Bmatrix} \bar{\nu} & \alpha & \bar{\beta}' \\ \bar{\alpha}' & \lambda & \bar{\varepsilon} \end{Bmatrix}_{sr_2r_1r}$ if $p(\beta') < p(\alpha)$.

- The second term $\begin{Bmatrix} \alpha & \beta & \lambda \\ \bar{\varepsilon} & \nu & \bar{\beta}' \end{Bmatrix}_{rr_3sr_4}$ is related to $\mathcal{T} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\beta}' & \bar{\varepsilon} & \nu \end{Bmatrix}_{srr_3r_4}$. Call $A = (\lambda\alpha\beta)$, $B = (\nu\bar{\alpha}\beta')$.
If $A \geq B$, then $\mathcal{T}$ is of Type II.
Assume now $A < B$. We show that $p(\beta') \leq p(\nu)$:

$$p(\beta') > p(\nu) \quad \Rightarrow \quad p(\beta') = p(\lambda), p(\nu) = p(\beta) = p(\alpha) = p(\lambda) - 1 \quad \Rightarrow \quad A = B$$

against the assumption $A < B$. Being $p(\nu) \geq p(\alpha), p(\beta')$, $\mathcal{T}$ is related either to the Type II $6j$-symbol $\begin{Bmatrix} \nu & \bar{\alpha} & \beta' \\ \bar{\beta} & \bar{\varepsilon} & \lambda \end{Bmatrix}_{sr_4r_3r}$ if $p(\alpha) \geq p(\beta')$ or to the Type III $6j$-symbol $\begin{Bmatrix} \nu & \beta' & \bar{\alpha} \\ \beta & \bar{\lambda} & \varepsilon \end{Bmatrix}_{sr_3r_4r}$ if $p(\beta') > p(\alpha) (= p(\beta))$.

□

Proposition 2.2.6. *$6j$-symbols of Type III can be computed in terms of core and Type IV $6j$-symbols.*

Proof. Let $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \alpha' & \lambda' & \varepsilon_1 \end{Bmatrix}_{c_1c_2c_3k}$ be of Type III: $p(\lambda) \geq p(\alpha) > p(\beta) > 1$. Let us apply the pentagon relation to $\mathcal{S}$ in the version given by (1.158). In order to do this, we specify the choice of the two free irreducible representations ν_1, ν_2 appearing in such equation. Consider $\beta \in \epsilon(m,n)$ with m, n minimum with such property. Being $\mathcal{S}$ of Type III requires $p(\beta) > 1$, i.e. $m + n \geq 2$. Hence we have $\beta \in \varepsilon_2 \otimes \beta'$ for some irreducible representation β' with $p(\beta') = p(\beta) - 1$. We therefore choose ν_2 to be $\bar{\varepsilon}_2$ (we recall ε_2 to be either ϵ or $\bar{\epsilon}$) and ν_1 to be β'. Let us write the pentagon relation with these choices:

$$\begin{Bmatrix} \lambda & \alpha & \beta \\ \alpha' & \lambda' & \varepsilon_1 \end{Bmatrix}_{c_1c_2c_3k} = \sum_{\substack{\xi\nu_3e_1e_2e_3 \\ d_1d_2c_1'c_2'd_2' \\ s_1s_4}} \# \begin{Bmatrix} \lambda & \nu_3 & \varepsilon_2 \\ \xi & \lambda' & \varepsilon_1 \end{Bmatrix}_{c_1s_1s_4d_1} \begin{Bmatrix} \nu_3 & \bar{\varepsilon}_1 & \xi \\ \alpha' & \beta' & \bar{\alpha} \end{Bmatrix}_{d_2'c_2'e_1e_3}$$
$$\times \begin{Bmatrix} \beta' & \bar{\alpha}' & \xi \\ \lambda' & \bar{\varepsilon}_2 & \bar{\beta} \end{Bmatrix}_{d_3c_3e_2e_1} \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \bar{\varepsilon}_2 & \nu_3 \end{Bmatrix}_{d_1d_2d_3k}, \qquad (*)$$

where $\#$ denotes the product of specific coefficients which are known. In the following, we are going to prove that each of the four $6j$-symbols in an arbitrary addend of the sum above is related either to a core or to a Type IV $6j$-symbol.
Notice that $p(\lambda) \geq 3$, $p(\xi) \geq p(\lambda') - 1 \geq p(\beta) - 1 \geq 1$ and $p(\nu_3) \geq p(\lambda) - 1 \geq p(\beta) > 1$. In particular, if $p(\xi) = 1$ is admitted for some ξ in the sum, we can conclude:

$$\lambda = 3_p, \quad \nu_3 = 2_p, \quad \lambda' = 2_p, \quad \alpha' = 2_p, \quad \alpha = 3_p, \quad \beta = 2_p, \quad p(\beta') = 1.$$

- In the case $p(\xi) = 1$, we have that $\mathcal{R} = \begin{Bmatrix} \lambda & \nu_3 & \varepsilon_2 \\ \xi & \lambda' & \varepsilon_1 \end{Bmatrix}_{c_1 s_1 s_4 d_1}$ is a core $6j$-symbol of the type $\begin{Bmatrix} 3_p & 2_p & 1_p \\ 1_p & 2_p & 1_p \end{Bmatrix}_{c_1 s_1 s_4 d_1}$. In the case $p(\xi) > 1$, we eventually apply some tetrahedron symmetries to put the greatest triad at the top row and in standard order, making $\mathcal{R}$ related to a Type IV $6j$-symbol.

- Focus now on the second $6j$-symbol appearing in an arbitrary addend of the sum in (*): $\mathcal{T} = \begin{Bmatrix} \nu_3 & \bar{\varepsilon}_1 & \xi \\ \alpha' & \beta' & \bar{\alpha} \end{Bmatrix}_{d_2' c_2' e_1 e_3}$. Assume $p(\beta') = 1$. If $p(\xi) = 1$ then $\mathcal{T}$ is related to the core $6j$-symbol $\begin{Bmatrix} \alpha & \bar{\nu}_3 & \beta' \\ \bar{\xi} & \bar{\alpha}' & \bar{\varepsilon}_1 \end{Bmatrix}_{c_2' e_3 e_1 d_2'} = \begin{Bmatrix} 3_p & 2_p & 1_p \\ 1_p & 2_p & 1_p \end{Bmatrix}_{c_2' e_3 e_1 d_2'}$, whereas if $p(\xi) > 1$ then $\mathcal{T}$ is related to a Type IV $6j$-symbol.
Assume now $p(\beta') > 1$. We have:

$$\begin{aligned} \text{Lemma 2.2.3} \;\; \Rightarrow \quad & p(\lambda'), p(\alpha') \geq p(\beta) = p(\beta') + 1 > p(\beta'), \\ p(\lambda') - 1 \leq p(\xi) \leq p(\lambda') + 1 \quad \Rightarrow \quad & p(\xi) \geq p(\beta'), \\ \mathcal{S} \text{ is of Type III} \;\; \Rightarrow \;\; p(\alpha) > p(\beta) \quad \Rightarrow \quad & p(\alpha) > p(\beta'), \\ p(\lambda) - 1 \leq p(\nu_3) \leq p(\lambda) + 1, \;\; p(\lambda) > p(\beta) > p(\beta') \quad \Rightarrow \quad & p(\nu_3) > p(\beta'). \end{aligned}$$

Consider the only two non-primitive triads of $\mathcal{T}$: $A = (\xi \bar{\alpha}' \beta')$ and $B = (\alpha \bar{\nu}_3 \beta')$. If we want to put them in standard order, the above proves that β' can be put at the last place.

  - CASE $p(\xi) = p(\beta')$. We have $B > A$ by the relations above, therefore $\mathcal{T}$ is related to the Type III $6j$-symbol $\begin{Bmatrix} \alpha & \bar{\nu}_3 & \beta' \\ \bar{\xi} & \bar{\alpha}' & \bar{\varepsilon}_1 \end{Bmatrix}_{c_2' e_3 e_1 d_2'}$. Anyway, the greatest triad of the latter is strictly less than the greatest triad of the initial $6j$-symbol $\mathcal{S}$, i.e. $B < (\lambda \alpha \beta)$ since $p(\beta') < p(\beta)$, so we can apply again the same whole procedure.
  - CASE $p(\xi) > p(\beta')$. If $B \geq A$, then $\mathcal{T}$ is related to the Type III $6j$-symbol $\begin{Bmatrix} \alpha & \bar{\nu}_3 & \beta' \\ \bar{\xi} & \bar{\alpha}' & \bar{\varepsilon}_1 \end{Bmatrix}_{c_2' e_3 e_1 d_2'}$. If $B < A$, then $\mathcal{T}$ is related to the Type III $6j$-symbol $\begin{Bmatrix} \bar{\xi} & \alpha' & \bar{\beta}' \\ \alpha & \nu_3 & \varepsilon_1 \end{Bmatrix}_{e_3 c_2' d_2' e_1}$. Since $A, B < (\lambda \alpha \beta)$, in both situations the greatest triad is strictly less than the one in $\mathcal{S}$, making possible to consistently apply the whole procedure again.

- Call $A = (\lambda' \bar{\alpha}' \beta)$, $B = (\bar{\alpha}' \xi \beta')$. Observe that $A > B$: indeed by Lemma 2.2.3 β may be considered as the smallest irreducible representation of A and β'

can be considered the smallest irreducible representation of B because of:

$$\begin{aligned} p(\alpha') \geq p(\beta) = p(\beta') + 1 > p(\beta'), \\ p(\lambda') \geq p(\beta) = p(\beta') + 1 > p(\beta'), \\ p(\xi) \geq p(\lambda') - 1 \geq p(\beta'). \end{aligned}$$

The third $6j$-symbol in an addend of the sum in (*) is therefore related to $\mathcal{U} = \begin{Bmatrix} \bar{\lambda}' & \bar{\alpha}' & \beta \\ \bar{\beta}' & \bar{\varepsilon}_2 & \bar{\xi} \end{Bmatrix}_{e_2 e_1 d_3 c_3}$.

If $p(\beta') = 1$, namely $\beta' = \varepsilon_3$, then $\mathcal{U} = \begin{Bmatrix} \bar{\lambda}' & \bar{\alpha}' & 2_p \\ \bar{\varepsilon}_3 & \bar{\varepsilon}_2 & \bar{\xi} \end{Bmatrix}_{e_2 e_1 d_3 c_3}$ which is a core $6j$-symbol. From now on let us assume $p(\beta') > 1$.

If $p(\lambda') \geq p(\alpha')$ then $\mathcal{U}$ is of Type II.

Assume now $p(\alpha') > p(\lambda')$. $\mathcal{U}$ is then related to $\mathcal{U}' = \begin{Bmatrix} \bar{\alpha}' & \bar{\lambda}' & \beta \\ \varepsilon_2 & \beta' & \xi \end{Bmatrix}_{e_1 e_2 d_3 c_3}$ which is of Type I. We then proceed similarly to the proof of the previous statement. Let us permute the columns of $\mathcal{U}'$ via the cycle (132) and then apply the Racah-backcoupling rule:

$$\begin{Bmatrix} \bar{\lambda}' & \beta & \bar{\alpha}' \\ \beta' & \xi & \varepsilon_2 \end{Bmatrix}_{e_2 d_3 e_1 c_3} = \sum_{\nu r s} \# \begin{Bmatrix} \varepsilon_2 & \nu & \bar{\alpha}' \\ \beta' & \xi & \bar{\lambda}' \end{Bmatrix}_{e_2 r e_1 s} \begin{Bmatrix} \bar{\lambda}' & \beta & \bar{\alpha}' \\ \bar{\varepsilon}_2 & \nu & \bar{\beta}' \end{Bmatrix}_{r d_3 s c_3}.$$

Again, $\#$ denotes the product of specific known coefficients and $p(\nu) \geq 2, p(\lambda')$. In what follows we prove that the $6j$-symbols in the sum above are related either to Type II $6j$-symbols or to Type III $6j$-symbols with greatest triad strictly less than the greatest triad $(\lambda\alpha\beta)$ of the initial $6j$-symbol $\mathcal{S}$.

- The first term $\begin{Bmatrix} \varepsilon_2 & \nu & \bar{\alpha}' \\ \beta' & \xi & \bar{\lambda}' \end{Bmatrix}_{e_2 r e_1 s}$ is related to $\mathcal{A} = \begin{Bmatrix} \alpha' & \bar{\xi} & \beta' \\ \lambda' & \nu & \bar{\varepsilon}_2 \end{Bmatrix}_{s e_2 r e_1}$ and to $\mathcal{B} = \begin{Bmatrix} \bar{\nu} & \lambda' & \bar{\beta}' \\ \bar{\xi} & \bar{\alpha}' & \bar{\varepsilon}_2 \end{Bmatrix}_{s e_2 e_1 r}$. Call $A = (\alpha'\bar{\xi}\beta')$, $B = (\nu\bar{\lambda}'\beta')$ and notice that these two triads are already in standard order exactly as we have just defined them.
  If $A \geq B$ we consider $\mathcal{A}$. If $p(\xi) > p(\beta')$ then $\mathcal{A}$ is a Type III $6j$-symbol whose greatest triad is strictly less than the greatest triad $(\lambda\alpha\beta)$ of $\mathcal{S}$ from which we started. Otherwise, if $p(\xi) = p(\beta')$ then $\mathcal{A}$ is related to the Type II $6j$-symbol $\begin{Bmatrix} \alpha' & \beta' & \bar{\xi} \\ \bar{\lambda}' & \varepsilon_2 & \bar{\nu} \end{Bmatrix}_{s r e_2 e_1}$.
  If $A < B$ we consider $\mathcal{B}$, which is of Type III since $p(\lambda') \geq p(\lambda) - 1 \geq p(\beta) = p(\beta') + 1 > p(\beta')$. But there are no problems, since B is strictly less than the greatest triad $(\lambda\alpha\beta)$ of $\mathcal{S}$.

– Focus now on the second term $\begin{Bmatrix} \bar{\lambda}' & \beta & \bar{\alpha}' \\ \bar{\varepsilon}_2 & \nu & \bar{\beta}' \end{Bmatrix}_{rd_3sc_3}$ which is related to $\mathcal{C} = \begin{Bmatrix} \bar{\alpha}' & \bar{\lambda}' & \beta \\ \bar{\beta}' & \bar{\varepsilon}_2 & \nu \end{Bmatrix}_{srd_3c_3}$. Call $A = (\alpha'\lambda'\bar{\beta})$, $B = (\nu\lambda'\beta')$ and notice that these two triads are already in standard order exactly as we have just defined them. Therefore, $\mathcal{C}$ is of Type II. In particular, $p(\beta) > p(\beta) - 1 = p(\beta')$ implies $A > B$ so there are no other cases to consider.

- In the end, consider the $6j$-symbol $\mathcal{V} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \bar{\varepsilon}_2 & \nu_3 \end{Bmatrix}_{d_1d_2d_3k}$ from (*). When we order the triad $(\alpha\bar{\nu}_3\beta')$, we see that β' is strictly the lowest representation, since the inequality $p(\alpha), p(\nu_3) > p(\beta')$ is implied by the conditions:

$$p(\lambda) \geq p(\alpha) > p(\beta), \quad p(\lambda) - 1 \leq p(\nu_3) \leq p(\lambda) + 1, \quad p(\beta') = p(\beta) - 1.$$

Hence, $(\lambda\alpha\beta) > (\alpha\bar{\nu}_3\beta')$, so $\mathcal{V}$ is a core $6j$-symbol. If $p(\beta) > 2$, then $\mathcal{V}$ is in particular of Type II.

□

Proposition 2.2.7. *$6j$-symbols of Type IV can be computed in terms of trivial and core $6j$-symbols.*

Proof. Let $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4}$ be of Type IV. Call $A = (\lambda\alpha\varepsilon_1)$, $B = (\lambda'\bar{\alpha}'\varepsilon_1)$. The triad A is in standard order and $A \geq B$. The Racah-backcoupling rule reads:

$$\begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{\nu rs} \# \begin{Bmatrix} \varepsilon_2 & \nu & \varepsilon_1 \\ \alpha' & \lambda' & \lambda \end{Bmatrix}_{r_1rr_3s} \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \bar{\varepsilon}_2 & \nu & \bar{\alpha}' \end{Bmatrix}_{rr_2sr_4},$$

where # denotes some specific known coefficients and $0 \leq p(\nu) \leq 2$. Let us show that the $6j$-symbols in the sum above are either trivial or related to core $6j$-symbols.

- The first $6j$-symbol $\begin{Bmatrix} \varepsilon_2 & \nu & \varepsilon_1 \\ \alpha' & \lambda' & \lambda \end{Bmatrix}_{r_1rr_3s}$ we see in the sum above is related to $\mathcal{T} = \begin{Bmatrix} \lambda & \bar{\alpha}' & \bar{\nu} \\ \varepsilon_1 & \bar{\varepsilon}_2 & \bar{\lambda}' \end{Bmatrix}_{r_1r_3sr}$. If $p(\nu) = 0$, $\mathcal{T}$ is trivial and we are done. If $p(\nu) = 1$, Lemma 2.2.4 assures the triad at the top row to be in standard order and the greatest one, making $\mathcal{T}$ a core $6j$-symbol like in (2.19). If $p(\nu) = 2$, the triad at the top row is strictly the greatest one and it is in standard order by Lemma 2.2.4, making $\mathcal{T}$ a core $6j$-symbol like in (2.10).

- The second $6j$-symbol we see in the sum above is $\mathcal{U} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \bar{\varepsilon}_2 & \nu & \bar{\alpha}' \end{Bmatrix}_{rr_2sr_4}$. If $p(\nu) = 0$ then $\mathcal{U}$ is trivial and we are done. If $p(\nu) = 1$ then $\mathcal{U}$ is a core $6j$-symbol like in (2.19). If $p(\nu) = 2$ then $\mathcal{U}$ is related to $\begin{Bmatrix} \bar{\lambda} & \alpha' & \nu \\ \bar{\varepsilon}_2 & \varepsilon_1 & \bar{\alpha} \end{Bmatrix}_{r_4r_2sr}$. The latter is a core $6j$-symbol like in (2.10), since the triad at the top row is in standard order by Lemma 2.2.4 and it is strictly the greatest one.

□

We formally state our result in the following theorem:

Theorem 2.2.1. *If G is a connected simple compact Lie group not of type D_{even}, then any $6j$-symbol can be computed in terms of core $6j$-symbols. In this case, the computability of all core $6j$-symbols implies the computability of any $6j$-symbol.*

2.3 The $SU(N)$ Case

The goal of this section is to investigate our initial question concerning the $SU(N)$ case: we will try to understand if giving six finite-dimensional irreducible representations of $SU(N)$ defining a $6j$-symbol is enough to know its absolute value. We will see that this is indeed possible for base $6j$-symbols.

Denote the fundamental representation of $SU(N)$ by ϵ. Notice that ϵ is an irreducible faithful representation of minimal dimension, therefore primitive. We will use ε, ε_i to denote either ϵ or its dual $\bar{\epsilon}$. Furthermore, we denote the adjoint representation of $SU(N)$ by ad and the quantum parameter by q.
If N is high enough, the irreducible representations at quantum level behave like their classical counterparts (see [GJ15]). Therefore, when working with quantum $6j$-symbols the hypothesis of having N high enough will always be implicit. In particular, working with $SU(N)$ allows us to use an important tool: Young Tableaux. We define the notation $Y(\lambda)$ to denote the Young diagram of an irreducible representation λ of $SU(N)$. The reader may find further details in Section A.4, especially regarding the notation we are going to utilize.
Despite having stated that we will work with N high enough, throughout the results of this section we will distinguish different cases for N: the reader may either look at the cases with highest N only if interested in quantum $6j$-symbols or consider all the cases if interested in classic $6j$-symbols (for which we need to be more careful since some situations may not occur when N is not high enough). In other words, we present the results with lower N to take into account the limit to the classic case.

The value of the quantum dimension of the following representations will be used:

$$|\epsilon| = [N], \qquad |\yng(2)| = \frac{[N][N+1]}{2}, \qquad |\mathrm{ad}| = [N-1][N+1], \qquad |\yng(1,1)| = \frac{[N-1][N]}{2},$$
$$|\yng(3)| = \frac{[N][N+1][N+2]}{[2][3]}, \quad |\yng(2,1)| = \frac{[N-1][N][N+1]}{[3]}, \quad |\yng(1,1,1)| = \frac{[N-2][N-1][N]}{[2][3]}.$$

Remark 2.3.1. The multiplicity label for a primitive triad is automatically 0. Indeed, it is easy to see via Young Tableaux theory that if λ is a generic finite-dimensional irreducible representation then all summands of a decomposition into irreducibles of $\lambda \otimes \epsilon$ have multiplicity one.

Lemma 2.3.1. *Fix an ordered primitive triad $A = (\lambda\alpha\varepsilon)$ of $SU(N)$. If N is even then $p(\lambda) > p(\alpha)$. If N is odd and $Y(\lambda)$ has a column of hight $(N-1)/2$ then a decomposition of $\lambda \otimes \epsilon$ into irreducibles has exactly one summand of power $p(\lambda)$.*

Proof. It follows by Young Tableaux theory. □

Lemma 2.3.2. *Let N be even. Let $A = (\lambda\alpha\varepsilon)$ be an ordered triad, then $p(\lambda) = p(\alpha) + 1$. It follows that $6j$-symbols like in (2.15) do not show up for $SU(N)$.*

Proof. It follows by Young Tableaux theory. □

Corollary 2.3.1. *Let N be even. Let $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \varepsilon & \lambda' \end{Bmatrix}_{0r_2 0r_4}$ be a Type II $6j$-symbol. Then we have that either $p(\lambda) > p(\lambda')$ or $p(\beta) > p(\beta')$.*

Proof. If $p(\lambda) < p(\lambda')$ then by Lemma 2.2.2 we conclude that $p(\beta') < p(\beta)$. The case $p(\lambda) = p(\lambda')$ does not occur by Lemma 2.3.2, so the remaining possibility is therefore $p(\lambda) > p(\lambda')$. □

Corollary 2.3.2. *Let N be even. Let $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$ be a Type IV $6j$-symbol. Then $p(\alpha) = p(\lambda) - 1$, $p(\lambda') = p(\alpha)$ and $p(\alpha') \in \{p(\lambda), p(\alpha) - 1\}$.*

Proof. By Lemma 2.2.4 we have $p(\lambda') \leq p(\lambda)$, which implies $p(\lambda') < p(\lambda)$ by Lemma 2.3.2. Since the the triad at the top row of $\mathcal{S}$ is in standard order, we have $p(\alpha) < p(\lambda)$ again by Lemma 2.3.2. Always by Lemma 2.3.2, the options left for $p(\alpha')$ are then either $p(\alpha) + 1 = p(\lambda)$ or $p(\alpha) - 1$. □

Fact 2.3.1. *$6j$-symbols like in (2.16)–(2.19) do not exist for $SU(N)$ with $N > 3$.*

Proof. It follows by Lemma A.4.2. □

Fact 2.3.2. *$6j$-symbols like in (2.15) do not exist for $SU(N)$ with $N > 5$.*

Proof. It follows by Lemma A.4.3. □

2.3.1 Subcase (2.13)

Let us focus on $6j$-symbols like in (2.13), hence call:

$$\mathcal{S} = \begin{Bmatrix} \lambda & \varepsilon_1 & \varepsilon_2 \\ \mu & \varepsilon_3 & \varepsilon_4 \end{Bmatrix}_{0000} \tag{2.21}$$

where $p(\lambda) = p(\mu) = 2$.

Proposition 2.3.1. *$\mathcal{S}$ can assume the following values only:*

$$\begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \bar{\epsilon} & \epsilon \\ \mathrm{ad} & \bar{\epsilon} & \bar{\epsilon} \end{Bmatrix}_{0000}, \tag{2.22}$$

$$\begin{Bmatrix} \mathrm{ad} & \bar{\epsilon} & \epsilon \\ \yng(2) & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \overline{\yng(2)} & \bar{\epsilon} & \bar{\epsilon} \end{Bmatrix}_{0000}, \tag{2.23}$$

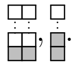
$$\begin{Bmatrix} \overline{\yng(2)} & \epsilon & \epsilon \\ \mathrm{ad} & \bar{\epsilon} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(2) & \bar{\epsilon} & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \bar{\epsilon} \end{Bmatrix}_{0000} \tag{2.24}$$

and if $N \geq 4$ we have also:

$$\begin{Bmatrix} \mathrm{ad} & \bar{\epsilon} & \epsilon \\ \yng(1,1) & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \overline{\yng(1,1)} & \bar{\epsilon} & \bar{\epsilon} \end{Bmatrix}_{0000}, \tag{2.25}$$

$$\begin{Bmatrix} \overline{\yng(1,1)} & \epsilon & \epsilon \\ \mathrm{ad} & \bar{\epsilon} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(1,1) & \bar{\epsilon} & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \bar{\epsilon} \end{Bmatrix}_{0000}. \tag{2.26}$$

Proof. We analyze all possible combinations for the values of ε_1, ε_2, ε_3, ε_4.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \epsilon \\ \mu & \epsilon & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\epsilon\epsilon)$ and $(\lambda\bar{\epsilon}\epsilon)$ must be valid triads, implying $\bar{\lambda} \in (\epsilon \otimes \epsilon) \cap (\bar{\epsilon} \otimes \epsilon)$, but the latter is empty, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\epsilon} & \bar{\epsilon} \\ \mu & \bar{\epsilon} & \bar{\epsilon} \end{Bmatrix}_{0000}$ is allowable, then the same is true for its conjugate, which would fit the previous discarded case, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \bar{\epsilon} \\ \mu & \epsilon & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\epsilon\bar{\epsilon})$ and $(\bar{\mu}\epsilon\bar{\epsilon})$ must be valid triads, implying $\bar{\lambda} \in \epsilon \otimes \bar{\epsilon}$ and $\mu \in \epsilon \otimes \bar{\epsilon}$, implying $\lambda = \mu = \mathrm{ad}$.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\epsilon} & \epsilon \\ \mu & \epsilon & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\bar{\epsilon}\epsilon)$ and $(\bar{\mu}\epsilon\epsilon)$ must be valid triads, implying $\bar{\lambda} \in \epsilon \otimes \bar{\epsilon}$ and $\mu \in \epsilon \otimes \epsilon$, implying $\lambda = \mathrm{ad}$ and $\mu = \yng(2), \yng(1,1)$.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \epsilon \\ \mu & \epsilon & \bar{\epsilon} \end{Bmatrix}_{0000}$ we see that $(\lambda\epsilon\epsilon)$ and $(\lambda\bar{\epsilon}\bar{\epsilon})$ must be valid triads, implying $\bar{\lambda} \in (\epsilon \otimes \epsilon) \cap (\bar{\epsilon} \otimes \bar{\epsilon})$, but the latter is empty, therefore $\mathcal{S}$ cannot have this shape.

If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \epsilon \\ \mu & \bar{\epsilon} & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\epsilon\epsilon)$ and $(\bar{\mu}\bar{\epsilon}\epsilon)$ must be valid triads, implying $\bar{\lambda} \in \epsilon \otimes \epsilon$ and $\mu \in \bar{\epsilon} \otimes \epsilon$, implying $\mu = \mathrm{ad}$ and $\lambda =$ .
If $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\epsilon} & \bar{\epsilon} \\ \mu & \epsilon & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\bar{\epsilon}\bar{\epsilon})$ and $(\lambda\bar{\epsilon}\epsilon)$ must be valid triads, implying $\bar{\lambda} \in (\bar{\epsilon} \otimes \bar{\epsilon}) \cap (\bar{\epsilon} \otimes \epsilon)$, but the latter is empty, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \epsilon \\ \mu & \bar{\epsilon} & \bar{\epsilon} \end{Bmatrix}_{0000}$ is allowable, then the same is true for its conjugate, which would fit the previous discarded case, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\epsilon} & \epsilon \\ \mu & \bar{\epsilon} & \epsilon \end{Bmatrix}_{0000}$ we see that $(\lambda\bar{\epsilon}\epsilon)$ and $(\lambda\epsilon\epsilon)$ must be valid triads, implying $\bar{\lambda} \in (\bar{\epsilon} \otimes \epsilon) \cap (\epsilon \otimes \epsilon)$, but the latter is empty, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \bar{\epsilon} \\ \mu & \epsilon & \bar{\epsilon} \end{Bmatrix}_{0000}$ is allowable, then the same is true for its conjugate, which would fit the previous discarded case, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\epsilon} & \epsilon \\ \mu & \epsilon & \bar{\epsilon} \end{Bmatrix}_{0000}$ we see that $(\lambda\bar{\epsilon}\epsilon)$ and $(\lambda\bar{\epsilon}\bar{\epsilon})$ must be valid triads, implying $\bar{\lambda} \in (\bar{\epsilon} \otimes \epsilon) \cap (\bar{\epsilon} \otimes \bar{\epsilon})$, but the latter is empty, therefore $\mathcal{S}$ cannot have this shape.
If $\mathcal{S} = \begin{Bmatrix} \lambda & \epsilon & \bar{\epsilon} \\ \mu & \bar{\epsilon} & \epsilon \end{Bmatrix}_{0000}$ is allowable, then the same is true for its conjugate, which would fit the previous discarded case, therefore $\mathcal{S}$ cannot have this shape.
The four cases left, where three of ε_1, ε_2, ε_2, ε_4 are $\bar{\epsilon}$ and one is ϵ, are obtained by conjugating $\mathcal{S}$ in the above cases where three of them were ϵ and one was $\bar{\epsilon}$. □

Corollary 2.3.3. *The module of $\mathcal{S}$ is computable.*

Proof. By Proposition 2.3.1, we see that either λ or μ is the adjoint representation, which can be substituted only by the trivial one. Therefore, utilizing the unitarity symmetry we can compute the module of $\mathcal{S}$. All cases are similar, so we give just one example with $\lambda = \mu = \mathrm{ad}$, for which we use (1.123) and (1.120):

$$\frac{1}{|\mathrm{ad}|} = \sum_\lambda |\lambda| \left| \begin{Bmatrix} \lambda & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = |\mathbf{1}| \left| \begin{Bmatrix} \mathbf{1} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 + |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2$$
$$= |\mathbf{1}| \left| \frac{\{\mathrm{ad}\}\{\mathrm{ad}\,\epsilon\bar{\epsilon}0\}}{\sqrt{|\epsilon|}\sqrt{|\epsilon|}} \right|^2 + |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{|\mathbf{1}|}{|\epsilon|^2} + |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2,$$

implying

$$\left| \begin{Bmatrix} \mathrm{ad} & \epsilon & \bar{\epsilon} \\ \mathrm{ad} & \epsilon & \epsilon \end{Bmatrix}_{0000} \right| = \frac{\sqrt{|\epsilon|^2 - |\mathrm{ad}|\,|\mathbf{1}|}}{|\mathrm{ad}|\,|\epsilon|} = \frac{\sqrt{[N]^2 - [N-1][N+1]}}{[N-1][N+1][N]}$$

$$= \frac{\sqrt{(q^{N/2}-q^{-N/2})^2-(q^{(N-1)/2}-q^{-(N-1)/2})(q^{(N+1)/2}-q^{-(N+1)/2})}}{[N-1][N+1][N]\sqrt{q^{1/2}-q^{-1/2}}}$$

$$= \frac{\sqrt{(q^{1/2}-q^{-1/2})^2}}{[N-1][N+1][N]\sqrt{q^{1/2}-q^{-1/2}}} = \frac{1}{[N-1][N][N+1]}.$$

$\square$

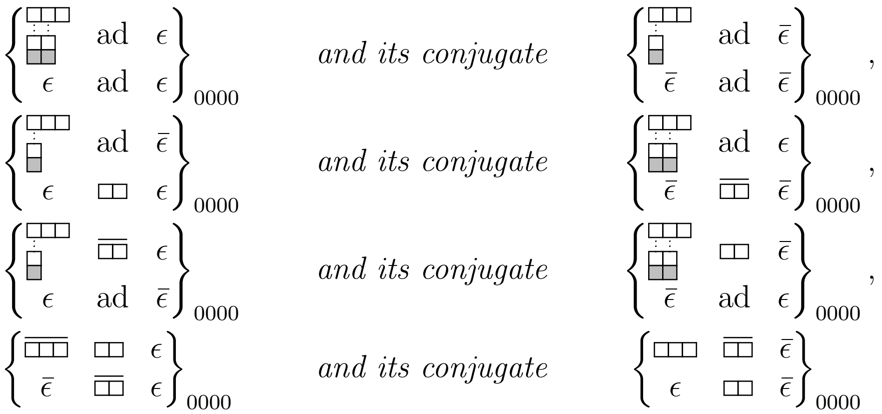
2.3.2 Subcase (2.14)

Let us focus on $6j$-symbols like in (2.14), hence call:

$$\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \varepsilon_2 & \mu & \varepsilon_3 \end{Bmatrix}_{0000} \tag{2.27}$$

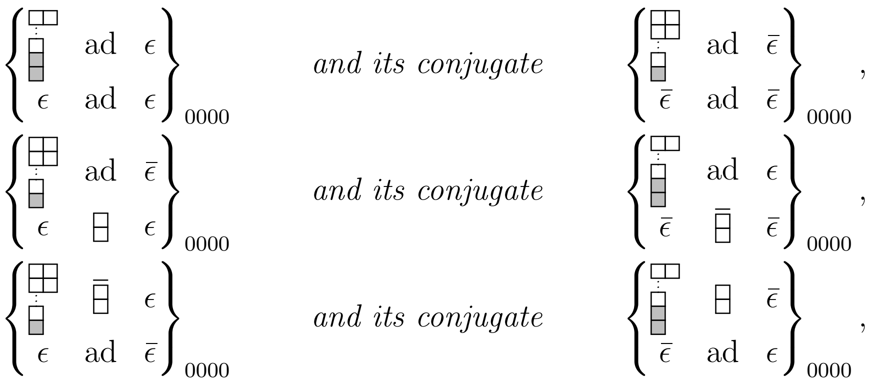
where $p(\lambda) = 3$ and $p(\alpha) = p(\mu) = 2$.

Proposition 2.3.2. *$\mathcal{S}$ can assume the following values only:*



(2.28) (2.29) (2.30) (2.31)

and if $N \geq 4$ we have also:



(2.32) (2.33) (2.34)

$$\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(2,1) & \overline{\yng(2)} & \bar\epsilon \\ \epsilon & \yng(1,1) & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.35}$$

$$\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(2,1) & \overline{\yng(2)} & \bar\epsilon \\ \epsilon & \yng(2) & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.36}$$

$$\begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(2,1) & \overline{\yng(1,1)} & \bar\epsilon \\ \epsilon & \yng(2) & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.37}$$

$$\begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(2,1) & \overline{\yng(1,1)} & \bar\epsilon \\ \epsilon & \yng(1,1) & \bar\epsilon \end{Bmatrix}_{0000} \tag{2.38}$$

and if $N \geq 6$ we have also:

$$\begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \yng(1,1,1) & \overline{\yng(1,1)} & \bar\epsilon \\ \epsilon & \yng(1,1) & \bar\epsilon \end{Bmatrix}_{0000}. \tag{2.39}$$

Proof. We analyze all possible combinations for the values of ε_1, ε_2, ε_3.

CASE 1: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mu & \epsilon \end{Bmatrix}_{0000}$. We see that $(\epsilon\alpha\bar\epsilon)$ and $(\bar\epsilon\mu\epsilon)$ must be valid triads, implying $\bar\alpha, \bar\mu \in \epsilon \otimes \bar\epsilon$, hence $\alpha = \mathrm{ad} = \mu$ and so $\mathcal{S} = \begin{Bmatrix} \lambda & \mathrm{ad} & \epsilon \\ \epsilon & \mathrm{ad} & \epsilon \end{Bmatrix}_{0000}$. We observe that $(\lambda\,\mathrm{ad}\,\epsilon)$ is a triad, implying $\bar\lambda \in \mathrm{ad} \otimes \epsilon$, hence λ is either $\yng(3,1,\vdots,2)$ or (if $N \geq 4$) $\yng(2,1,\vdots,1)$.

CASE 2: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \bar\epsilon \\ \epsilon & \mu & \epsilon \end{Bmatrix}_{0000}$. As above, we see that $\alpha = \mathrm{ad}$ and so $\mathcal{S} = \begin{Bmatrix} \lambda & \mathrm{ad} & \bar\epsilon \\ \epsilon & \mu & \epsilon \end{Bmatrix}_{0000}$. We have that $(\lambda\,\mathrm{ad}\,\bar\epsilon)$ is a triad, implying $\lambda \in \mathrm{ad} \otimes \epsilon$, hence λ is either $\yng(3,1,\vdots,1)$ or (if $N \geq 4$) $\yng(2,2,\vdots,1)$. In the first case we have $\mu = \yng(2)$ whereas in the second $\mu = \yng(1,1)$, since $(\lambda\bar\mu\epsilon)$ being a triad implies $\bar\lambda \in \bar\mu \otimes \epsilon$ (recall $p(\mu) = 2$).

CASE 3: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mu & \bar\epsilon \end{Bmatrix}_{0000}$. As in CASE 1, we see that $\mu = \mathrm{ad}$ and so $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathrm{ad} & \bar\epsilon \end{Bmatrix}_{0000}$. We have that $(\lambda\,\mathrm{ad}\,\bar\epsilon)$ is a triad, so λ is either $\yng(3,1,\vdots,1)$ or (if $N \geq 4$) $\yng(2,2,\vdots,1)$ as in CASE 2. In the first case we have $\alpha = \overline{\yng(2)}$ whereas in the second $\alpha = \overline{\yng(1,1)}$, since $(\lambda\alpha\epsilon)$ being a triad implies $\bar\lambda \in \alpha \otimes \epsilon$ (recall $p(\alpha) = 2$).

CASE 4: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \bar\epsilon & \mu & \epsilon \end{Bmatrix}_{0000}$. We see that $(\bar\epsilon\alpha\bar\epsilon)$ is a triad, implying $\alpha \in \epsilon \otimes \epsilon$ so α is either $\yng(2)$ or (if $N \geq 4$) $\yng(1,1)$ (the power of $\yng(1,1)$ is 2 for $N \geq 4$ and 1 for $N = 3$ since in this case we have $\yng(1,1) = \bar\epsilon$).

SUBCASE $\alpha = \yng(2)$. We have $\mathcal{S} = \begin{Bmatrix} \lambda & \yng(2) & \epsilon \\ \bar\epsilon & \mu & \epsilon \end{Bmatrix}_{0000}$. Here we see that $(\lambda\yng(2)\epsilon)$ is

a triad, so $\bar{\lambda} \in \ydiagram{2} \otimes \epsilon$. Hence λ is either $\overline{\ydiagram{3}}$ or (if $N \geq 4$) $\overline{\ydiagram{2,1}}$, therefore either $\mathcal{S} = \begin{Bmatrix} \overline{\ydiagram{3}} & \ydiagram{2} & \epsilon \\ \bar{\epsilon} & \mu & \epsilon \end{Bmatrix}_{0000}$ or $\mathcal{S} = \begin{Bmatrix} \overline{\ydiagram{2,1}} & \ydiagram{2} & \epsilon \\ \bar{\epsilon} & \mu & \epsilon \end{Bmatrix}_{0000}$. In the first case $\mu = \overline{\ydiagram{2}}$, whereas in the second μ can be either $\overline{\ydiagram{2}}$ or $\overline{\ydiagram{1,1}}$, since $(\lambda\bar{\mu}\epsilon)$ being a triad implies $\bar{\lambda} \in \bar{\mu} \otimes \epsilon$.
SUBCASE $\alpha = \ydiagram{1,1}$ (recall that here we are assuming $N \geq 4$). We have that $\mathcal{S} = \begin{Bmatrix} \lambda & \ydiagram{1,1} & \epsilon \\ \bar{\epsilon} & \mu & \epsilon \end{Bmatrix}_{0000}$. Here we see that $(\lambda\ydiagram{1,1}\epsilon)$ is a triad, so $\bar{\lambda} \in \ydiagram{1,1} \otimes \epsilon$. Hence λ is either $\overline{\ydiagram{2,1}}$ or (if $N \geq 6$) $\overline{\ydiagram{1,1,1}}$, therefore either $\mathcal{S} = \begin{Bmatrix} \overline{\ydiagram{2,1}} & \ydiagram{1,1} & \epsilon \\ \bar{\epsilon} & \mu & \epsilon \end{Bmatrix}_{0000}$ or $\mathcal{S} = \begin{Bmatrix} \overline{\ydiagram{1,1,1}} & \ydiagram{1,1} & \epsilon \\ \bar{\epsilon} & \mu & \epsilon \end{Bmatrix}_{0000}$.
In the first case μ can be either $\overline{\ydiagram{2}}$ or $\overline{\ydiagram{1,1}}$ whereas in the second $\mu = \overline{\ydiagram{1,1}}$, since $(\lambda\bar{\mu}\epsilon)$ being a triad implies $\bar{\lambda} \in \bar{\mu} \otimes \epsilon$. $\square$

Corollary 2.3.4. *The module of $\mathcal{S}$ is computable.*

Proof. Consider all different values for $\mathcal{S}$ computed in Proposition 2.3.2.
CASE 1. If a representation of $\mathcal{S}$ is the adjoint, then we see it appearing in a triad of the type $(\mathrm{ad}\,\epsilon\bar{\epsilon})$, where it could be substituted only by $\mathbf{1}$, but when we consider the other triad containing ad we realize that $\mathbf{1}$ cannot be part of this other triad. In other words, if the other five representations are fixed then the only irreducible representation that fits the position in which ad is located is only ad itself. Therefore, the module of $\mathcal{S}$ can be easily computed with the unitarity symmetry. Let us give an example using (1.124):

$$\frac{1}{|\ydiagram{2}|} = \sum_{\lambda} |\lambda| \begin{Bmatrix} \mathrm{ad}_{\ydiagram{3}} & \lambda & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \epsilon \end{Bmatrix}_{0000} \begin{Bmatrix} \mathrm{ad}_{\ydiagram{3}} & \lambda & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \epsilon \end{Bmatrix}^{*}_{0000} = |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad}_{\ydiagram{3}} & \mathrm{ad} & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \epsilon \end{Bmatrix}_{0000} \right|^2,$$

implying

$$\left| \begin{Bmatrix} \mathrm{ad}_{\ydiagram{3}} & \mathrm{ad} & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{[2]}{[N-1][N][N+1]^2}.$$

CASE 2: $\mathcal{S} = \begin{Bmatrix} \ydiagram{3} & \overline{\ydiagram{2}} & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \bar{\epsilon} \end{Bmatrix}_{0000}$. In this case we see that the representation at position $(2,1)$ is unique if we fix the other five, in the sense that if $\begin{Bmatrix} \ydiagram{3} & \overline{\ydiagram{2}} & \bar{\epsilon} \\ \epsilon & \mu & \bar{\epsilon} \end{Bmatrix}_{0000}$ is a well defined $6j$-symbol, then $\mu = \ydiagram{2}$ because this value of μ is the only one that fits the triad $(\ydiagram{3}\bar{\mu}\bar{\epsilon})$ (by (A.16)). Therefore, we can use the unitarity symmetry via (1.127):

$$\frac{1}{|\overline{\ydiagram{2}}|} = \sum_{\mu} |\mu| \begin{Bmatrix} \ydiagram{3} & \overline{\ydiagram{2}} & \bar{\epsilon} \\ \epsilon & \mu & \bar{\epsilon} \end{Bmatrix}_{0000} \begin{Bmatrix} \ydiagram{3} & \overline{\ydiagram{2}} & \bar{\epsilon} \\ \epsilon & \mu & \bar{\epsilon} \end{Bmatrix}^{*}_{0000} = |\ydiagram{2}| \left| \begin{Bmatrix} \ydiagram{3} & \overline{\ydiagram{2}} & \bar{\epsilon} \\ \epsilon & \ydiagram{2} & \bar{\epsilon} \end{Bmatrix}_{0000} \right|^2,$$

implying

$$\left|\begin{Bmatrix} \yng(3) & \overline{\yng(2)} & \bar\epsilon \\ \epsilon & \yng(2) & \bar\epsilon \end{Bmatrix}_{0000}\right| = \frac{1}{|\yng(2)|} = \frac{[2]}{[N][N+1]}.$$

CASE 3: $\mathcal{S} = \begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}$. If $\begin{Bmatrix} \lambda & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}$ is a valid $6j$-symbol then $\bar\lambda \in (\yng(2) \otimes \epsilon) \cap (\yng(1,1) \otimes \epsilon)$, where $(\yng(2) \otimes \epsilon) \cap (\yng(1,1) \otimes \epsilon) = \{\yng(2,1)\}$ by what we see in (A.16), making us conclude $\lambda = \overline{\yng(2,1)}$. Therefore, similarly as above we use the unitarity symmetry in the form of (1.123):

$$\frac{1}{|\bar\epsilon|} = \sum_\lambda |\lambda| \begin{Bmatrix} \lambda & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}^*_{0000} = |\overline{\yng(2,1)}| \left|\begin{Bmatrix} \yng(3) & \overline{\yng(2)} & \bar\epsilon \\ \epsilon & \yng(2) & \bar\epsilon \end{Bmatrix}_{0000}\right|^2,$$

implying

$$\left|\begin{Bmatrix} \yng(3) & \overline{\yng(2)} & \bar\epsilon \\ \epsilon & \yng(2) & \bar\epsilon \end{Bmatrix}_{0000}\right|^2 = \frac{1}{|\epsilon|\,|\yng(2,1)|} = \frac{[3]}{[N-1][N]^2[N+1]}.$$

CASE 4: $\mathcal{S} = \begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}$. This case follows from the previous one, because here $\mathcal{S}$ is simply obtained by $\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}$ via exchanging the rows in columns 2 and 3 and then conjugating.

CASE 5: $\mathcal{S} = \begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}$. Similarly to the above cases, the irreducible representation at position $(1,1)$ can be either $\overline{\yng(2,1)}$ or $\overline{\yng(3)}$. The latter gives us a $6j$-symbol which has been already computed in CASE 2. We apply the unitarity symmetry via (1.123), finding that $|\bar\epsilon|^{-1}$ is equal to:

$$\sum_\lambda |\lambda| \left|\begin{Bmatrix} \lambda & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right|^2 = |\overline{\yng(2,1)}| \left|\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right|^2 + |\overline{\yng(3)}| \left|\begin{Bmatrix} \overline{\yng(3)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right|^2,$$

implying

$$\begin{aligned} \left|\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right|^2 &= \frac{1}{|\yng(2,1)|}\left(\frac{1}{|\epsilon|} - |\yng(3)| \left|\begin{Bmatrix} \overline{\yng(3)} & \yng(2) & \epsilon \\ \bar\epsilon & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right|^2\right) \\ &= \frac{[3][N+1]-[2][N+2]}{|\yng(2,1)|\,[N][3][N+1]} = \frac{1}{[N]^2[N+1]^2}, \end{aligned}$$

where a straightforward computation shows that $[3][N+1]-[2][N+2]=[N-1]$. We then conclude:

$$\left|\begin{Bmatrix} \overline{\yng(2,1)} & \yng(2) & \epsilon \\ \bar{\epsilon} & \overline{\yng(2)} & \epsilon \end{Bmatrix}_{0000}\right| = \frac{1}{[N][N+1]}.$$

CASE 6: $\mathcal{S} = \begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}$ (recall that here we are assuming $N \geq 6$). The irreducible representation at position $(1,2)$ is unique when the other five representations are considered as fixed, since it has to lie in $\left(\yng(1,1,1) \otimes \bar{\epsilon}\right) \cap (\epsilon \otimes \epsilon) = \{\yng(1,1)\}$. Hence, by using the unitarity symmetry via (1.124) we get:

$$\frac{1}{\left|\overline{\yng(1,1)}\right|} = \sum_{\lambda} |\lambda| \left|\begin{Bmatrix} \overline{\yng(1,1,1)} & \lambda & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2 = \left|\yng(1,1)\right| \left|\begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2,$$

implying

$$\left|\begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right| = \frac{1}{\left|\overline{\yng(1,1)}\right|} = \frac{[2]}{[N-1][N]}.$$

CASE 7: $\mathcal{S} = \begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}$. If $N = 4, 5$ then the result is straightforward. Assume $N \geq 6$. The irreducible representation at position $(1,1)$ can be either $\overline{\yng(2,1)}$ or $\overline{\yng(1,1,1)}$, where the latter concerns the already solved CASE 6. Hence:

$$\frac{1}{|\bar{\epsilon}|} = \sum_{\lambda} |\lambda| \left|\begin{Bmatrix} \lambda & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2 = \left|\overline{\yng(2,1)}\right| \left|\begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2 + \left|\overline{\yng(1,1,1)}\right| \left|\begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2,$$

implying

$$\begin{aligned} \left|\begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2 &= \frac{1}{\left|\overline{\yng(2,1)}\right|} \left(\frac{1}{|\epsilon|} - \left|\overline{\yng(1,1,1)}\right| \left|\begin{Bmatrix} \overline{\yng(1,1,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000}\right|^2 \right) \\ &= \frac{1}{\left|\overline{\yng(2,1)}\right|} \cdot \frac{[3][N-1]-[2][N-2]}{[3][N-1][N]} = \frac{1}{[N-1]^2[N]^2}, \end{aligned}$$

where $[3][N-1]-[2][N-2] = [N+1]$. We then have:

$$\begin{Bmatrix} \overline{\yng(2,1)} & \yng(1,1) & \epsilon \\ \bar{\epsilon} & \overline{\yng(1,1)} & \epsilon \end{Bmatrix}_{0000} = \frac{1}{[N-1][N]}.$$

$\square$

2.3.3 Subcase (2.11)

Let us focus on $6j$-symbols like in (2.11), hence call:

$$\mathcal{S} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{Bmatrix}_{000r_4} \tag{2.40}$$

where $p(\lambda_1) = p(\lambda_2) = p(\lambda_3) = 2$.

Proposition 2.3.3. *$\mathcal{S}$ can assume the following values only:*

$$\begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \overline{\square\!\square} & \square\!\square \\ \epsilon & \bar\epsilon & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.41}$$

$$\begin{Bmatrix} \mathrm{ad} & \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} & \begin{smallmatrix}\square\\ \square\end{smallmatrix} \\ \epsilon & \bar\epsilon & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.42}$$

$$\begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \overline{\square\!\square} & \begin{smallmatrix}\square\\ \square\end{smallmatrix} \\ \epsilon & \bar\epsilon & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.43}$$

$$\begin{Bmatrix} \mathrm{ad} & \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} & \square\!\square \\ \epsilon & \bar\epsilon & \bar\epsilon \end{Bmatrix}_{0000}, \tag{2.44}$$

$$\begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{000r_4} \quad \textit{and its conjugate} \quad \begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \bar\epsilon & \bar\epsilon & \bar\epsilon \end{Bmatrix}_{000r_4}, \tag{2.45}$$

where $r_4 = 0, 1$, together with all the symbols obtained by permuting the columns cyclically.

Proof. CASE $\mathcal{S} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{000r_4}$. Here the first triad is $(\lambda_1 \bar\epsilon \epsilon)$ and since $\epsilon \otimes \bar\epsilon \cong \mathbf{1} \oplus \mathrm{ad}$ we deduce that $\lambda_1 = \mathrm{ad}$. The second triad is $(\lambda_2 \bar\epsilon \bar\epsilon)$, implying $\lambda_2 \in \epsilon \otimes \epsilon$ and therefore $\lambda_2 = \square\!\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}$. The third triad is $(\lambda_3 \epsilon\epsilon)$, implying $\bar\lambda_3 \in \epsilon \otimes \epsilon$ and therefore $\lambda_3 = \overline{\square\!\square}, \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$. The fourth triad $(\mathrm{ad}\, \lambda_2 \lambda_3)$ carries no multiplicity, since ad has multiplicity one as a summand in a fixed decomposition of $\square\!\square \otimes \overline{\square\!\square}$, $\begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$, $\square\!\square \otimes \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$, $\begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \overline{\square\!\square}$ into irreducibles.
CASE $\mathcal{S} = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{000r_4}$. Here we see that apart from the fourth triad $(\lambda_1\lambda_2\lambda_3)$, all other triads are the same: $(\lambda_i \epsilon \bar\epsilon)$. We can therefore conclude that $\lambda_1 = \lambda_2 = \lambda_3 = \mathrm{ad}$. Notice now that ad has multiplicity 2 as a summand in a fixed decomposition of $\mathrm{ad} \otimes \mathrm{ad}$ into irreducibles, hence we have that the fourth triad $(\mathrm{ad}\,\mathrm{ad}\,\mathrm{ad})$ carries multiplicity 2. □

Corollary 2.3.5. *The module of $\mathcal{S}$ is computable.*

Proof. Consider all different values for $\mathcal{S}$ computed in Proposition 2.3.3.

CASE 1: $\mathcal{S} = \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}$. By the unitarity symmetry, $|\bar\epsilon|^{-1}$ is equal to:

$$\sum_\lambda |\lambda| \left| \begin{Bmatrix} \lambda & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 + |\mathbf{1}| \left| \begin{Bmatrix} \mathbf{1} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 .$$

Utilizing (1.120) together with nj-phases being ± 1, we get:

$$\left| \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\mathrm{ad}|} \left(\frac{1}{|\epsilon|} - \left| \begin{Bmatrix} \mathbf{1} & \square\!\square & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 \right) = \frac{1}{|\mathrm{ad}|\,|\epsilon|} \left(1 - \frac{1}{|\square\!\square|} \right) .$$

CASE 2: $\mathcal{S} = \begin{Bmatrix} \mathrm{ad} & \begin{smallmatrix}\square\\\square\end{smallmatrix} & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}$. Similarly as above, we get:

$$\left| \begin{Bmatrix} \mathrm{ad} & \begin{smallmatrix}\square\\\square\end{smallmatrix} & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\mathrm{ad}|\,|\epsilon|} \left(1 - \frac{1}{|\begin{smallmatrix}\square\\\square\end{smallmatrix}|} \right) .$$

CASE 3: $\mathcal{S} = \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}$. In this case, the irreducible representation at position $(1,1)$ is unique, so we get:

$$\frac{1}{|\bar\epsilon|} = \sum_\lambda |\lambda| \left| \begin{Bmatrix} \lambda & \square\!\square & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = |\mathrm{ad}| \left| \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 ,$$

implying

$$\left| \begin{Bmatrix} \lambda & \square\!\square & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\epsilon|\,|\mathrm{ad}|} = \frac{1}{[N-1][N][N+1]} .$$

CASE 4: $\mathcal{S} = \begin{Bmatrix} \mathrm{ad} & \begin{smallmatrix}\square\\\square\end{smallmatrix} & \overline{\square\!\square} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}$. Similarly as above, we get:

$$\left| \begin{Bmatrix} \mathrm{ad} & \square\!\square & \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ \bar\epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\epsilon|\,|\mathrm{ad}|} = \frac{1}{[N-1][N][N+1]} .$$

CASE 5: $\mathcal{S} = \begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{000r_4}$. By the fourth symmetry, $|\bar\epsilon|^{-1}$ equals:

$$\sum_{\lambda r_4} |\lambda| \left| \begin{Bmatrix} \lambda & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{000r_4} \right|^2 = \left| \begin{Bmatrix} \mathbf{1} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000} \right|^2 + |\mathrm{ad}| \sum_{i=0,1} \left| \begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{000i} \right|^2 ,$$

implying via (1.120) that:

$$\left|\begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}\right|^2 + \left|\begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0001}\right|^2 = \frac{1}{|\mathrm{ad}|\,|\epsilon|}\left(1 - \frac{1}{|\mathrm{ad}|}\right).$$

Let us choose a multiplicity separation scheme in which we set:

$$\begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0001} = 0$$

and therefore have:

$$\left|\begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}\right|^2 = \frac{1}{|\mathrm{ad}|\,|\epsilon|}\left(1 - \frac{1}{|\mathrm{ad}|}\right).$$

We now explain why we can do this. Since $\mathrm{m}_{\mathrm{ad}\otimes\mathrm{ad}}^{\mathrm{ad}} = 2$, we do not have a unique solution for the modules of the involved $6j$-symbols due to a $SU(2)$ uncertainty. Let $M \in SU(2)$ and call $A := \begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0000}$, $B := \begin{Bmatrix} \mathrm{ad} & \mathrm{ad} & \mathrm{ad} \\ \epsilon & \epsilon & \epsilon \end{Bmatrix}_{0001}$, $\begin{pmatrix} A' \\ B' \end{pmatrix} := M \cdot \begin{pmatrix} A \\ B \end{pmatrix}$. What happens, of course, is that $|A|^2 + |B|^2 = |A'|^2 + |B'|^2$, so by using matrices of $SU(2)$ we can move from $\begin{pmatrix} A \\ B \end{pmatrix}$ to other vectors keeping $|A|^2 + |B|^2$ invariant. We decide to move to $\begin{pmatrix} A' \\ B' \end{pmatrix}$ with $B' = 0$. Geometrically, we are moving along a hyperbole from a generic point to the point of intersection with an axis. □

2.3.4 Subcase (2.12) - Type IV $6j$-symbols

In the $SU(N)$ case we are able to compute Type IV $6j$-symbols directly, without recurring to core $6j$-symbols. We make explicit every passage of the strategy proposed in [GJ15]. Call:

$$\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \alpha' & \lambda' & \varepsilon_2 \end{Bmatrix}_{0000}. \tag{2.46}$$

Denoting by # the product of some specific known coefficients, the Racah-backcoupling reads:

$$q^{(C_\lambda + C_{\varepsilon_1} + C_{\alpha'} + C_{\varepsilon_2})/2}\mathcal{S} = \sum_{\nu,r} q^{(C_\nu + C_\alpha + C_{\lambda'})/2}\,|\nu|\,\#\begin{Bmatrix} \varepsilon_2 & \nu & \varepsilon_1 \\ \alpha' & \lambda' & \lambda \end{Bmatrix}_{0r00}\begin{Bmatrix} \lambda & \alpha & \varepsilon_1 \\ \bar{\varepsilon}_2 & \nu & \bar{\alpha}' \end{Bmatrix}_{r000}. \tag{2.47}$$

Let us analyze the case $\alpha' = \lambda$ first.

Lemma 2.3.3. *Let $\varepsilon_1 = \bar{\epsilon}$ and $\varepsilon_2 = \epsilon$. If $\alpha' = \lambda$ then the module of $\mathcal{S}$ is computable.*

Proof. By our assumptions, ν in (2.47) can be either ad or $\mathbf{1}$. Therefore, denoting by # the product of some specific known coefficients, (2.47) becomes:

$$q^{C_\lambda + C_\epsilon} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} = q^{(C_{\mathrm{ad}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \bar{\epsilon} & \mathrm{ad} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathrm{ad} & \bar{\lambda} \end{Bmatrix}_{0000} + q^{(C_{\mathbf{1}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \bar{\epsilon} & \mathbf{1} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathbf{1} & \bar{\lambda} \end{Bmatrix}_{0000},$$

implying:

$$q^{C_\lambda + C_\epsilon} q^{-(C_{\mathrm{ad}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} = \begin{Bmatrix} \bar{\epsilon} & \mathrm{ad} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathrm{ad} & \bar{\lambda} \end{Bmatrix}_{0000} + q^{(C_{\mathbf{1}} + C_\alpha + C_{\lambda'})/2} q^{-(C_{\mathrm{ad}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \bar{\epsilon} & \mathbf{1} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathbf{1} & \bar{\lambda} \end{Bmatrix}_{0000}. \quad (2.48)$$

Replacing q by q^{-1} in the last equation we obtain:

$$q^{-(C_\lambda + C_\epsilon)} q^{(C_{\mathrm{ad}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} = \begin{Bmatrix} \bar{\epsilon} & \mathrm{ad} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathrm{ad} & \bar{\lambda} \end{Bmatrix}_{0000} + q^{-(C_{\mathbf{1}} + C_\alpha + C_{\lambda'})/2} q^{(C_{\mathrm{ad}} + C_\alpha + C_{\lambda'})/2} \# \begin{Bmatrix} \bar{\epsilon} & \mathbf{1} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathbf{1} & \bar{\lambda} \end{Bmatrix}_{0000}. \quad (2.49)$$

Subtracting (2.49) to (2.48) we have:

$$\# \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} = \# \begin{Bmatrix} \bar{\epsilon} & \mathbf{1} & \epsilon \\ \lambda & \lambda' & \lambda \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \epsilon & \mathbf{1} & \bar{\lambda} \end{Bmatrix}_{0000}.$$

□

Corollary 2.3.6. *If $\alpha' = \lambda$ then the module of $\mathcal{S}$ is computable.*

Proof. In the proof of Lemma 2.3.3 we did not use any fact regarding the ordering of the triads, so we can always start from $\mathcal{S}$ and eventually exchange the first and the second column to make ε_1 and ε_2 be the conjugate of each other, since this symmetry requires all the second row to be conjugated. We then then eventually conjugate the whole symbol to have ϵ at position $(1,3)$. □

Let us see now the case in which α' differs from λ.

Lemma 2.3.4. *Let $\varepsilon_1 = \epsilon$ and $\varepsilon_2 = \bar{\epsilon}$. Consider the five irreducible representations of $\mathcal{S}$ at position different from position $(2,1)$ as fixed. If $\alpha' \neq \lambda$ then α' is uniquely determined*

Proof. We are assuming that $\mathcal{S}$ is a well-defined $6j$-symbol, that $\alpha' \neq \lambda$ and that λ, λ', α are fixed. Consider the following triads:

$$\begin{aligned} (\lambda\bar{\lambda}'\bar{\epsilon}) &\Rightarrow \lambda \in \lambda' \otimes \epsilon, & (\bar{\alpha}'\lambda'\epsilon) &\Rightarrow \alpha' \in \lambda' \otimes \epsilon, \\ (\lambda\alpha\epsilon) &\Rightarrow \bar{\alpha} \in \lambda \otimes \epsilon, & (\alpha'\alpha\epsilon) &\Rightarrow \bar{\alpha} \in \alpha' \otimes \epsilon \end{aligned}$$

and consider the following decomposition:

$$\begin{aligned} (\lambda' \otimes \epsilon) \otimes \epsilon \cong &(\lambda \oplus \alpha' \oplus \nu_3 \oplus \ldots \oplus \nu_d) \otimes \epsilon \\ \cong &(\lambda \otimes \epsilon) \oplus (\alpha' \otimes \epsilon) \oplus (\nu_3 \otimes \epsilon) \oplus \ldots \oplus (\nu_d \otimes \epsilon) \\ \cong &(\nu_{11} \oplus \ldots \oplus \nu_{1m_1}) \oplus \ldots \oplus (\nu_{d1} \oplus \ldots \oplus \nu_{dm_d}) \\ \cong &(\bar{\alpha} \oplus \ldots \oplus \nu_{1m_1}) \oplus (\bar{\alpha} \oplus \ldots \oplus \nu_{2m_2}) \oplus \ldots \oplus (\nu_{d1} \oplus \ldots \oplus \nu_{dm_d}), \end{aligned}$$

where $\lambda = \nu_1$, $\alpha' = \nu_2$, $\nu_i \otimes \epsilon = \nu_{i1} \oplus \ldots \oplus \nu_{im_i}$. Each ν_i has multiplicity 1. Notice that $\bar{\alpha}$ belongs to a fixed decomposition of $\lambda \otimes \epsilon$ and of $\alpha' \otimes \epsilon$, but it cannot belong to other decompositions $\nu_j \otimes \epsilon$. We can see this with Young diagrams: $Y(\nu_i)$ is obtained by adding one box A to $Y(\lambda')$ and $Y(\nu_{ij})$ is obtained adding one box B to $Y(\nu_i)$; if B is added to the right or below A, then ν_{ij} appears only once as a summand in the final decomposition, otherwise it appears only twice since we can build $Y(\nu_{ij})$ as $Y(\lambda')$ to which we add first the box B getting a diagram $Y(\nu_a)$ and then adding the box A getting a diagram $Y(\nu_{ab})$ which equals $Y(\nu_{ij})$.
Since $\bar{\alpha}$ is fixed we have that $\bar{\alpha}$ is known, then $\bar{\alpha} \in \nu_i \otimes \epsilon$ for only two choices of i. One choice corresponds to $\nu_i = \lambda$ and the other determines α' uniquely. In other words, $\bar{\alpha}$ selects λ and α', which are then the only two irreducible representations that can fit the position $(2, 1)$ of $\mathcal{S}$. □

Corollary 2.3.7. *Let $\varepsilon_1 = \epsilon$ and $\varepsilon_2 = \bar{\epsilon}$. Then the module of $\mathcal{S}$ is computable.*

Proof. By the unitarity of $6j$-symbols and Lemma 2.3.4, we get:

$$\frac{1}{|\lambda|} = |\lambda| \left| \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} \right|^2 + |\alpha'| \left| \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \alpha' & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} \right|^2,$$

where $\left| \begin{Bmatrix} \lambda & \alpha & \epsilon \\ \lambda & \lambda' & \bar{\epsilon} \end{Bmatrix}_{0000} \right|^2$ is known by Corollary 2.3.6. □

2.3.5 Subcase (2.10)

Let us focus on $6j$-symbols like in (2.11), hence call:

$$\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \varepsilon_1 & \varepsilon_2 & \mu \end{Bmatrix}_{000r} \tag{2.50}$$

where $p(\lambda) \geq p(\alpha) \geq p(\beta) = 2$.

Proposition 2.3.4. *Up to conjugation, $\mathcal{S}$ can assume the following values only:*

$$\begin{Bmatrix} \lambda & \alpha & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \quad \textit{if} \;\; \alpha \neq \bar{\lambda}, \qquad (2.51) \qquad\qquad \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r}, \qquad (2.52)$$

$$\begin{Bmatrix} \lambda & \alpha & \yng(2) \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{0000}, \qquad (2.53) \qquad\qquad \begin{Bmatrix} \lambda & \alpha & \yng(1,1) \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{0000}. \qquad (2.54)$$

Proof. We classify $\mathcal{S}$ accordingly to the different values that ε_1 and ε_2 can assume, namely $(\varepsilon_1, \varepsilon_2) = (\epsilon, \epsilon)$ and $(\varepsilon_1, \varepsilon_2) = (\bar{\epsilon}, \epsilon)$ up to conjugation.
CASE $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r}$: the triad $(\bar{\epsilon}\epsilon\beta)$ implies $\beta = \mathrm{ad}$. We then have either $\alpha \neq \bar{\lambda}$ or $\alpha = \bar{\lambda}$. If $\alpha \neq \bar{\lambda}$ then Corollary 4 of [Szc80] implies $\mathrm{m}^{\mathrm{ad}}_{\lambda\otimes\alpha} = 1$, so $r = 0$.
CASE $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{000r}$: the triad $(\epsilon\epsilon\beta)$ implies $\beta = \yng(2), \yng(1,1)$. Since $Y(\bar{\beta})$ is either a row or a column, $\mathrm{m}^{\bar{\beta}}_{\lambda\otimes\alpha} = 1$ by Corollary 1 and 2 of [Szc80], so $r = 0$. □

Lemma 2.3.5. *Consider the case $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{0000}$. Consider all representations apart from μ as been fixed. Then only two things can happen: either we have a unique choice for μ (i.e. μ is uniquely determined) or we have two choices for μ.*

Proof. Consider the following triads:

$$(\lambda\bar{\epsilon}\mu) \;\Rightarrow\; \mu \in \bar{\lambda} \otimes \epsilon, \qquad (\bar{\epsilon}\alpha\bar{\mu}) \Rightarrow\; \alpha \in \mu \otimes \epsilon.$$

Consider the following decomposition:

$$\begin{aligned}(\bar{\lambda} \otimes \epsilon) \otimes \epsilon \cong & (\nu_1 \oplus \ldots \oplus \nu_d) \otimes \epsilon \cong (\nu_1 \otimes \epsilon) \oplus \ldots \oplus (\nu_d \otimes \epsilon) \\ \cong & (\nu_{11} \oplus \ldots \oplus \nu_{1m_1}) \oplus \ldots \oplus (\nu_{d1} \oplus \ldots \oplus \nu_{dm_d}) \\ \cong & (\alpha \oplus \ldots \oplus \nu_{1m_1}) \oplus \ldots \oplus (\nu_{d1} \oplus \ldots \oplus \nu_{dm_d}),\end{aligned}$$

where $\alpha = \nu_{11}$, $\nu_i \otimes \epsilon = \nu_{i1} \oplus \ldots \oplus \nu_{im_i}$. Each ν_i has multiplicity 1. We have that λ is given, so we know all the ν_i's and ν_{ij}'s. The representation α is given as well. The question is: what is the multiplicity of α in the final decomposition? In other words, how many ν_{ij}'s are equivalent to the given α? The answer is either one or two. We can see this with Young diagrams: generally speaking, $Y(\nu_i)$ is obtained by adding one box A to $Y(\bar{\lambda})$ and $Y(\nu_{ij})$ is obtained adding one box B to $Y(\nu_i)$; if B is added to the right of A or below A then ν_{ij} appears only once as a summand in the final decomposition, otherwise it appears only twice since we can build $Y(\nu_{ij})$ as $Y(\bar{\lambda})$ to which we add first the box B getting a diagram $Y(\nu_a)$ and then adding the box A getting a diagram $Y(\nu_{ab})$ which equals $Y(\nu_{ij})$.

Hence, if α appears only once as a summand in the final decomposition then $\alpha = \nu_{ij}$ for a unique i, identifying a unique ν_i and making μ be this ν_i. If instead α appears twice in the final decomposition then $\alpha = \nu_{ij}$ for only two choices of i, say $\alpha = \nu_{ab} = \nu_{cd}$ with $a \neq c$, so μ can be either ν_a or ν_c. □

Corollary 2.3.8. *The module of $\mathcal{S}$ is computable.*

Proof. Consider all different values for $\mathcal{S}$ computed in Proposition 2.3.4.
CASE 1: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000}$, $\alpha \neq \bar{\lambda}$. By unitarity, we get:

$$\left| \begin{Bmatrix} \lambda & \alpha & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\mathrm{ad}|\,|\mu|},$$

since the only irreducible representation that could replace ad would be $\mathbf{1}$, but this cannot happen since $\alpha \neq \bar{\lambda}$.
CASE 2: $\mathcal{S} = \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r}$. By unitarity, we get:

$$\frac{1}{|\mu|} = |\mathrm{ad}| \sum_r \left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r} \right|^2 + |\mathbf{1}| \left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathbf{1} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \right|^2,$$

implying

$$\sum_r \left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r} \right|^2 = \frac{1}{|\mathrm{ad}|} \left(\frac{1}{|\mu|} - |\mathbf{1}| \left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathbf{1} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \right|^2 \right).$$

Call $M = \mathrm{m}^{\mathrm{ad}}_{\lambda \otimes \bar{\lambda}}$, then r runs from 0 to $M-1$. We choose a multiplicity separation scheme where we set:

$$\left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{000r} \right|^2 = 0 \quad \text{for} \quad r = 1, \ldots, M-1$$

and therefore have:

$$\left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathrm{ad} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \right|^2 = \frac{1}{|\mathrm{ad}|} \left(\frac{1}{|\mu|} - |\mathbf{1}| \left| \begin{Bmatrix} \lambda & \bar{\lambda} & \mathbf{1} \\ \epsilon & \epsilon & \mu \end{Bmatrix}_{0000} \right|^2 \right).$$

The explanation for this process is the same we have given in CASE 5 of Corollary 2.3.5 but generalized to $M \geq 2$.
CASE 3: $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{0000}$. Let us make explicit the strategy suggested in

[GJ15]. By Lemma 2.3.5, we have either one or two choices for μ when we consider the other five representations as fixed. In the first case, by the unitarity symmetry we have:

$$\left|\begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \mu \end{Bmatrix}_{0000}\right|^2 = \frac{1}{|\beta|\,|\mu|}.$$

In the second case, call μ_1, μ_2 the only two irreducible representations that can fit the position $(2,3)$ and call $X_i := \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \mu_i \end{Bmatrix}_{0000}$. The Racah-backcoupling rule establishes a linear relation between X_1 and X_2. Indeed, denoting by $\#$ the product of some specific known coefficients, (1.161) reads:

$$\begin{aligned} X_i =& \sum_{\nu} \# \, |\nu| \begin{Bmatrix} \bar{\epsilon} & \nu & \lambda \\ \epsilon & \mu_i & \alpha \end{Bmatrix}_{0000} \begin{Bmatrix} \lambda & \alpha & \beta \\ \bar{\epsilon} & \epsilon & \nu \end{Bmatrix}_{0000} \\ =& \# \, |\mu_1| \begin{Bmatrix} \bar{\epsilon} & \mu_1 & \lambda \\ \epsilon & \mu_i & \alpha \end{Bmatrix}_{0000} X_1 + \# \, |\mu_2| \begin{Bmatrix} \bar{\epsilon} & \mu_2 & \lambda \\ \epsilon & \mu_i & \alpha \end{Bmatrix}_{0000} X_2, \end{aligned}$$

which allows us to express X_2 in terms of X_1 and Type IV $6j$-symbols (recall that Type IV $6j$-symbols are computable as shown in Subsection 2.3.4). We write X_2 as $X_2(X_1)$ to emphasize when X_2 is expressed via X_1 as just explained. By the unitarity symmetry, we have:

$$\frac{1}{|\beta|} = |\mu_1|\,|X_1|^2 + |\mu_2|\,|X_2|^2 = |\mu_1|\,|X_1|^2 + |\mu_2|\,|X_2(X_1)|^2 \,,$$

which solves the square module of X_1. We then solve the square module of X_2 through the linear relation between X_1 and X_2. □

2.3.6 Subcase (2.6) - Type II 6*j*-symbols

Let us now focus on $6j$-symbols like in (2.6). Call:

$$\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \epsilon & \lambda' \end{Bmatrix}_{0r0s} \tag{2.55}$$

where $p(\lambda) \geq p(\alpha) \geq p(\beta) \geq 2$ and $p(\beta'), p(\lambda') \geq 2$. Call $A := (\lambda\alpha\beta)$, $B := (\beta'\alpha\bar{\lambda}')$, then we have also $A \geq B$.

Lemma 2.3.6. *Assume $N \geq 8$. Then we have that $p(\alpha) \geq 3$. As a consequence, $p(\lambda) \geq 3$ as well.*

Proof. Assume $p(\alpha) = 2$.
The condition $2 = p(\alpha) \geq p(\beta) \geq 2$ forces $p(\beta)$ to be 2.
A triad of the type $(2_p 2_p \varepsilon)$ does not exist for $N \geq 4$, therefore $(\bar{\beta}' \epsilon \beta)$ being a triad implies $p(\beta') = 3$.
If $p(\lambda') \geq 3$, then the following is the standard order for B: $(\bar{\lambda}' \beta' \alpha)$. The standard order of A is $(\lambda \alpha \beta)$. Since $p(\alpha) = p(\beta)$ and $p(\beta') > p(\alpha)$, we conclude $B > A$, against the initial condition $A \geq B$. We therefore conclude that $p(\lambda') = 2$.
Now, B is a triad of the type $(3_p 2_p 2_p)$, but such a triad does not exist for $N \geq 8$. □

Lemma 2.3.7. *Assume $N \geq 8$. If $p(\beta) = 2$ then $p(\lambda) = p(\alpha) = p(\beta') = 3$, $p(\lambda') = 2$, i.e. $\mathcal{S} = \begin{Bmatrix} 3_p & 3_p & 2_p \\ 3_p & \epsilon & 2_p \end{Bmatrix}_{0r0s}$.*

Proof. Assume $p(\beta) = 2$.
A triad of the type $(2_p 2_p \varepsilon)$ does not exist for $N \geq 4$, therefore $(\bar{\beta}' \epsilon \beta)$ being a triad implies $p(\beta') = 3$ (recall $p(\beta') \geq 2$).
Let us now prove $p(\lambda') = 2$. Assume by contradiction $p(\lambda') \geq 3$. Then α must have the smallest power within the representations in the triad B (indeed, $p(\beta') = 3$ and $p(\lambda') \geq 3$ tell us that the possible standard orders for B are $(\alpha \bar{\lambda}' \beta')$, $(\bar{\lambda}' \alpha \beta')$, $(\bar{\lambda}' \beta' \alpha)$, but in the first two cases the condition $A \geq B$ would imply that $p(\beta) \geq p(\beta')$ which is a contradiction). Then $A \geq B$ implies $p(\alpha) \leq p(\beta) = 2$ namely $p(\alpha) = 2$, against Lemma 2.3.6.
A triad of the type $(2_p 2_p \varepsilon)$ does not exist for $N \geq 4$, therefore $(\lambda \bar{\epsilon} \lambda')$ being a triad implies $p(\lambda) = 3$.
Finally, we have $3 = p(\lambda) \geq p(\alpha) \geq 3$ (where the last inequality follows by Lemma 2.3.6), implying $p(\alpha) = 3$. □

Lemma 2.3.8. *If $N = 8, 9$ then assuming $p(\lambda') = 2$ and $p(\beta) > 2$ we get that $p(\lambda) = p(\alpha) = p(\beta) = 3$ and $p(\beta') = 4$, i.e. $\mathcal{S} = \begin{Bmatrix} 3_p & 3_p & 3_p \\ 4_p & \epsilon & 2_p \end{Bmatrix}_{0r0s}$. If $N \geq 10$, the case in which $p(\lambda') = 2$ and $p(\beta) > 2$ does not occur.*

Proof. Assume $N = 8, 9$. Assume $p(\lambda') = 2$ and $p(\beta) > 2$. The triad $(\lambda \bar{\epsilon} \lambda')$ implies $p(\lambda) \leq p(\lambda') + 1 = 3$, therefore by Lemma 2.3.6 we get $p(\lambda) = p(\alpha) = 3$. By the hypothesis, we then have $p(\beta) = 3$ as well. Triads of the type $(3_p 2_p 2_p)$ and $(3_p 3_p \varepsilon)$ do not exist for $N \geq 8$, therefore $(\alpha \beta' \bar{\lambda}')$ and $(\bar{\beta}' \beta \epsilon)$ being triads imply $p(\beta') \neq 2, 3$. Then $p(\beta') = 4$, since $2 = p(\beta) - 1 \leq p(\beta') \leq p(\beta) + 1 = 4$ due to $(\bar{\beta}' \beta \epsilon)$ being a triad. Hence, we conclude that $\mathcal{S}$ must be of the type $\mathcal{S} = \begin{Bmatrix} 3_p & 3_p & 3_p \\ 4_p & \epsilon & 2_p \end{Bmatrix}_{0r0s}$.
Assume now $N \geq 10$. For such N, a triad of the type $(3_p 3_p 3_p)$ does not exist, so it is not possible to have a $6j$-symbol of the type $\mathcal{S} = \begin{Bmatrix} 3_p & 3_p & 3_p \\ 4_p & \epsilon & 2_p \end{Bmatrix}_{0r0s}$, which would

be the only possibility if we require $p(\lambda')=2$ and $p(\beta)>2$, as shown above. We conclude that for $N\geq 10$ the case $p(\lambda')=2$ and $p(\beta)>2$ does not occur. ☐

Lemma 2.3.9. *Assume $N\geq 10$. Then $p(\lambda)=3$ implies $p(\beta)=2$.*

Proof. Assume $p(\lambda)=3$. We know that $p(\lambda)\geq p(\alpha)\geq 3$ by $\mathcal{S}$ being of Type II and Lemma 2.3.6, hence $p(\alpha)=3$. For $N\geq 10$, there is no triad of the type $(3_p3_p3_p)$, so $p(\beta)$ cannot be 3. The only case left is $p(\beta)=2$. ☐

Corollary 2.3.9. *Assume $N\geq 10$. The following four conditions are all equivalent: $p(\beta)=2$, $p(\lambda')=2$, $p(\lambda)=3$, $\mathcal{S}=\begin{Bmatrix}3_p & 3_p & 2_p\\ 3_p & \epsilon & 2_p\end{Bmatrix}_{0r0s}$.*

Proposition 2.3.5. *Let $N\geq 10$. Assume $\mathcal{S}$ to be of the following particular shape: $\mathcal{S}=\begin{Bmatrix}3_p & 3_p & 2_p\\ 3_p & \epsilon & 2_p\end{Bmatrix}_{0r0s}$. Then $\mathcal{S}$ is computable.*

Proof. Denoting by # the product of some specific known coefficients, the version of the pentagon relation given by (1.158) makes $\mathcal{S}$ coincide with:

$$\sum_{\xi\nu_3 etz} \# \begin{Bmatrix}\lambda & \nu_3 & \bar{\varepsilon}_2\\ \xi & \epsilon & \lambda'\end{Bmatrix}_{0e00} \begin{Bmatrix}\nu_3 & \bar{\lambda}' & \xi\\ \beta' & \varepsilon_1 & \bar{\alpha}\end{Bmatrix}_{0t0z} \begin{Bmatrix}\varepsilon_1 & \bar{\beta}' & \xi\\ \epsilon & \varepsilon_2 & \bar{\beta}\end{Bmatrix}_{0000} \begin{Bmatrix}\lambda & \alpha & \beta\\ \varepsilon_1 & \varepsilon_2 & \nu_3\end{Bmatrix}_{000s},$$

where we have chosen ν_1, ν_2 to be primitive: $\nu_1=\varepsilon_1$ and $\nu_2=\varepsilon_2$. This choice is legitimate since $p(\beta)=2$, $p(\beta')=p(\lambda)=p(\alpha)=3$. Notice that $p(\nu_3)=2,4$. Let us now examine the terms in the sum above. The symbols $\begin{Bmatrix}\lambda & \nu_3 & \bar{\varepsilon}_2\\ \xi & \epsilon & \lambda'\end{Bmatrix}_{0e00}$, $\begin{Bmatrix}\varepsilon_1 & \bar{\beta}' & \xi\\ \epsilon & \varepsilon_2 & \bar{\beta}\end{Bmatrix}_{0000}$, $\begin{Bmatrix}\lambda & \alpha & \beta\\ \varepsilon_1 & \varepsilon_2 & \nu_3\end{Bmatrix}_{000s}$ have at least three primitive triads: they have already been shown the computable in the $SU(N)$ case in the previous subsections. The remaining symbol to analyze is $\mathcal{U}=\begin{Bmatrix}\nu_3 & \bar{\lambda}' & \xi\\ \beta' & \varepsilon_1 & \bar{\alpha}\end{Bmatrix}_{0t0z}$. Looking at the symbol $\begin{Bmatrix}\lambda & \nu_3 & \bar{\varepsilon}_2\\ \xi & \epsilon & \lambda'\end{Bmatrix}_{0e00}$, we see that ξ is involved in the triad $(\bar{\xi}\epsilon\bar{\varepsilon}_2)$, implying $p(\xi)=0,2$. If $p(\xi)=0$ then $\mathcal{U}$ is trivial, therefore computable. Assume now $p(\xi)=2$. Then, $\mathcal{U}=\begin{Bmatrix}\nu_3 & 2_p & 2_p\\ 3_p & \varepsilon_1 & 3_p\end{Bmatrix}_{0t0z}$ so $\mathcal{U}$ is related to the symbol $\mathcal{T}=\begin{Bmatrix}\bar{\alpha} & \bar{\beta}' & \lambda'\\ \xi & \bar{\nu}_3 & \bar{\varepsilon}_1\end{Bmatrix}_{00t'z'}$ after an exchange of rows in the first and second column and a cyclic permutation of the columns. Notice that $\mathcal{T}=\begin{Bmatrix}3_p & 3_p & 2_p\\ 2_p & 2_p/4_p & \bar{\varepsilon}_1\end{Bmatrix}_{00t'z'}$ is a Type III $6j$-symbol. We apply again the pentagon relation, getting $\mathcal{T}$ equal to:

$$\sum_{\psi\eta_3} \# \begin{Bmatrix}\bar{\alpha} & \eta_3 & \bar{\varepsilon}_4\\ \psi & \nu_3 & \varepsilon_1\end{Bmatrix}_{0000} \begin{Bmatrix}\eta_3 & \bar{\varepsilon}_1 & \psi\\ \xi & \varepsilon_3 & \beta'\end{Bmatrix}_{0000} \begin{Bmatrix}\varepsilon_3 & \bar{\xi} & \psi\\ \nu_3 & \varepsilon_4 & \bar{\lambda}'\end{Bmatrix}_{0t'00} \begin{Bmatrix}\bar{\alpha} & \bar{\beta}' & \lambda'\\ \varepsilon_2 & \varepsilon_4 & \eta_3\end{Bmatrix}_{000z'}$$

after choosing the two free irreducible representations to be primitive, i.e. to be ε_3 and ε_4. All the four symbols in the latter sum have at least three primitive triads and are therefore computable, making $\mathcal{T}$ (and so $\mathcal{U}$) computable as well. □

Lemma 2.3.10. *Assume $N \geq 10$. If $p(\beta') = 2$ then $p(\beta) = 3$, implying $p(\lambda') \geq 3$, $p(\lambda) \geq 4$ as well.*

Proof. Assume $p(\beta') = 2$. At the beginning of this section we recalled $p(\beta) \geq 2$. The triad $(\bar{\beta}'\epsilon\beta)$ of $\mathcal{S}$ cannot be of the type $(2_p 2_p \epsilon)$ since such a triad does not exist for $N \geq 4$, therefore $p(\beta) \geq 3$. By 3.) of Fact 2.1.1, $p(\beta) \leq p(\beta') + 1 = 3$. We therefore conclude $p(\beta) = 3$.
By Corollary 2.3.9, $p(\beta) = 3$ implies then that $p(\lambda') \geq 3$, $p(\lambda) \geq 4$. □

Lemma 2.3.11. *Assume $N \geq 12$. If $p(\beta') = 2$ and $p(\alpha) = 3$, then the following are all the allowed Type II $6j$-symbols:*

$$\begin{Bmatrix} 4_p & 3_p & 3_p \\ 2_p & \epsilon & 3_p \end{Bmatrix}_{0r0s}, \qquad \begin{Bmatrix} 4_p & 3_p & 3_p \\ 2_p & \epsilon & 5_p \end{Bmatrix}_{0r0s}, \qquad \begin{Bmatrix} 6_p & 3_p & 3_p \\ 2_p & \epsilon & 5_p \end{Bmatrix}_{0r0s}. \tag{2.56}$$

Proof. Assume $p(\beta') = 2$ and $p(\alpha) = 3$. By Lemma 2.3.10, we have that $p(\beta) = 3$, $p(\lambda') \geq 3$, $p(\lambda) \geq 4$. By 2.) of Fact 2.1.1, $p(\lambda') \leq p(\alpha) + p(\beta') = 5$. Since a triad of the type $(4_p 3_p 2_p)$ does not exist for $N \geq 10$, the values that $p(\lambda')$ can assume are 3 and 5.
In the case $p(\lambda') = 3$, the only possibility for $p(\lambda)$ is to be equal to 4, since $4 \leq p(\lambda) \leq p(\lambda') + 1 = 4$ (using 3.) of Fact 2.1.1).
In the case $p(\lambda') = 5$, we have $4 \leq p(\lambda) \leq p(\lambda') + 1 = 6$. Here, $p(\lambda) = 5$ does not occur for $N \geq 12$, since in this case a triad of the type $(5_p 5_p \varepsilon)$ does not exist. □

Remark 2.3.2. Consider $N \geq 12$.
We saw in Corollary 2.3.9 that if we start by imposing either the condition $p(\lambda) = 3$ only, or $p(\beta) = 2$ only, or $p(\lambda') = 2$ only, then all the other representations involved in $\mathcal{S}$ have only one possible value for their power. These are the minimal values for the power of λ, β, λ'.
We now wonder what happens if we start by imposing a condition on $p(\beta')$ instead, e.g. the minimal value for the power of β' which is $p(\beta') = 2$. If we add the requirement of $p(\alpha) = 3$, then by Lemma 2.3.11 we have finitely many cases for the power of the other representations of $\mathcal{S}$. If we do not impose a limit on the power of α, then the problem is that the condition $p(\beta') = 2$ alone is not enough to have a bound on the power of the other representations. For instance, the following are all allowed Type II $6j$-symbols with $p(\beta') = 2$ and $p(\alpha) = 4$:

$$\begin{Bmatrix} 5_p & 4_p & 3_p \\ 2_p & \epsilon & 4_p \end{Bmatrix}_{0r0s}, \qquad \begin{Bmatrix} 5_p & 4_p & 3_p \\ 2_p & \epsilon & 6_p \end{Bmatrix}_{0r0s}, \qquad \begin{Bmatrix} 7_p & 4_p & 3_p \\ 2_p & \epsilon & 6_p \end{Bmatrix}_{0r0s}. \tag{2.57}$$

In general, for $p(\beta') = 2$ and $p(\alpha) = n \geq 3$ with N high enough, the following are all the allowed Type II $6j$-symbols by Lemma A.4.6:

$$\begin{Bmatrix} (n+1)_p & n_p & 3_p \\ 2_p & \epsilon & n_p \end{Bmatrix}_{0r0s}, \quad \begin{Bmatrix} (n+1)_p & n_p & 3_p \\ 2_p & \epsilon & (n+2)_p \end{Bmatrix}_{0r0s},$$
$$\begin{Bmatrix} (n+3)_p & n_p & 3_p \\ 2_p & \epsilon & (n+2)_p \end{Bmatrix}_{0r0s}. \tag{2.58}$$

We conclude the chapter with the following remark, in which we highlight the main issues in computing $6j$-symbols involving representations of high power.

Remark 2.3.3. Consider $\mathcal{S} = \begin{Bmatrix} \lambda & \alpha & \beta \\ \beta' & \epsilon & \lambda' \end{Bmatrix}_{0r0s} = \begin{Bmatrix} 4_p & 3_p & 3_p \\ 2_p & \epsilon & 3_p \end{Bmatrix}_{0r0s}$.
One option to compute $\mathcal{S}$ is to use the unitarity symmetry. The only representations of $\mathcal{S}$ which we can sum over and obtain representations of lower power are λ and β. For instance, summing over λ we would have representations of power 2 at position $(1,1)$ (giving us a symbol which is computable by Proposition 2.3.5 after applying some tetrahedral symmetries) and all the other representations of power 4 which can fit position $(1,1)$: in general we have more than one of such $6j$-symbols with different λ's of power 4. Hence, the unitarity symmetry alone can help us only in the case where λ is uniquely determined, i.e when the Young diagram of λ' is made of three boxes in one row or one column only.
Another option is to apply the pentagon relation via (1.158). In this case we obtain sums of factors involving four $6j$-symbols. At least one of these, call it $\mathcal{T}$, is either of Type I or of Type III. The idea is then to follow the proofs of Proposition 2.2.5 and Proposition 2.2.6 to compute $\mathcal{T}$. The problem then lies in the fact that along this process we utilize again the Biedenharn-Elliot sum rule, facing other Type II $6j$-symbols of the exact same nature of $\mathcal{S}$, constituting an obstacle to an induction argument on the power of representations.
A third attempt would be then to apply the strategies above (eventually together with the Racah-backcoupling rule) and check if it is possible to reach the right amount of independent equations to indeed show the computability of $\mathcal{S}$.

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Appendices

Appendix A

Elements of Representation Theory

In this chapter we present some peculiar elements of Representation Theory which are needed to study and compute $6j$-symbols as defined in Chapter 1. For this purpose, we fix a compact Lie group G.

A.1 General Remarks

In this section we highlight some general remarks about Representation Theory.

Remark A.1.1. If $\lambda\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G and $\mathcal{B}$ is an arbitrary basis of V, then for any g in G we have that:

$$\mathcal{M}_{\mathcal{B}^\vee}\left(\bar{\lambda}(g)\right) = {}^{\mathrm{t}}\mathcal{M}_{\mathcal{B}}(\lambda(g))^{-1}. \tag{A.1}$$

Remark A.1.2. Let $\lambda\colon G \to \mathrm{GL}(V)$ and $\sigma\colon G \to \mathrm{GL}(W)$ be two equivalent representations of G. This means that there exists an isomorphisms of modules $\varphi\colon V \to W$. Denote the conjugation by φ with γ_φ, namely:

$$\gamma_\varphi\colon \mathrm{GL}(V) \to \mathrm{GL}(W); \quad \gamma_\varphi(f) \coloneqq \varphi \circ f \circ \varphi^{-1}.$$

We then have that the following diagram commutes:

$$\begin{array}{ccc} & & \mathrm{GL}(V) \\ & \overset{\lambda}{\nearrow} & \downarrow \gamma_\varphi \\ G & & \\ & \underset{\sigma}{\searrow} & \\ & & \mathrm{GL}(W) \end{array} .$$

In terms of matrices, for any g in G we get:

$$\mathcal{M}_{\mathcal{C}}(\sigma(g)) = \mathcal{M}_{\mathcal{CB}}(\varphi) \cdot \mathcal{M}_{\mathcal{B}}(\lambda(g)) \cdot \left(\mathcal{M}_{\mathcal{CB}}(\varphi)\right)^{-1}. \tag{A.2}$$

Let us see now a description of the one-dimensional trivial G-module seen as a submodule of the tensor product of a finite-dimensional G-module with its dual:

Proposition A.1.1. *Let $\lambda\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G and $\bar{\lambda}\colon G \to \mathrm{GL}\left(V^\vee\right)$ its dual. Let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis of V and $\mathcal{B}^\vee = (\theta_1, \ldots, \theta_n)$ its dual basis. Consider the following vector in $V \otimes V^\vee$:*

$$z \coloneqq v_1 \otimes \theta_1 + \ldots + v_n \otimes \theta_n. \tag{A.3}$$

Then $(\lambda \otimes \bar{\lambda})(g)(z) = z$ for any g in G, i.e. the vector z spans a one-dimensional subspace of $V \otimes V^\vee$ which is G-isomorphic to the trivial G-module. Furthermore, z is independent of $\mathcal{B}$.

Proof. Let $g \in G$. Call $A = (A_{ij})_{i,j} \coloneqq \mathcal{M}_{\mathcal{B}}(\lambda(g))$ and $B = (B_{ij})_{i,j} \coloneqq \mathcal{M}_{\mathcal{B}^\vee}(\bar{\lambda}(g))$. Recall that $B = {}^{\mathrm{t}}A^{-1}$. Then we have that:

$$\begin{aligned}
(\lambda \otimes \bar{\lambda})(g)(z) &= \sum_{k=1}^{n} (\lambda \otimes \bar{\lambda})(g)(v_k \otimes \theta_k) = \sum_{k=1}^{n} \lambda(g)(v_k) \otimes \bar{\lambda}(g)(\theta_k) = \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} A_{ik} v_i \right) \otimes \left(\sum_{j=1}^{n} B_{jk} \theta_j \right) = \sum_{i,j,k=1}^{n} A_{ik} B_{jk} v_i \otimes \theta_j = \\
&= \sum_{i,j,k=1}^{n} A_{ik} \left({}^{\mathrm{t}}A^{-1}\right)_{jk} v_i \otimes \theta_j = \sum_{i,j,k=1}^{n} A_{ik} \left(A^{-1}\right)_{kj} v_i \otimes \theta_j = \\
&= \sum_{i,j,k=1}^{n} (A \cdot A^{-1})_{ij} v_i \otimes \theta_j = \sum_{i,j=1}^{n} \delta_{ij} v_i \otimes \theta_j = \sum_{i=1}^{n} v_i \otimes \theta_i = z.
\end{aligned}$$

To show that z is independent of $\mathcal{B}$, pick another basis $\mathcal{C} = (w_1, \ldots, w_n)$ and its dual $\mathcal{C}^\vee = (\eta_1, \ldots \eta_n)$. Call now $A = (A_{ij})_{i,j} \coloneqq \mathcal{M}_{\mathcal{B}\mathcal{C}}(\mathrm{id}_V)$ and $B = (B_{ij})_{i,j} \coloneqq \mathcal{M}_{\mathcal{B}^\vee \mathcal{C}^\vee}(\mathrm{id}_{V^\vee})$. Notice that $B = {}^{\mathrm{t}}A^{-1}$. We then have that:

$$\begin{aligned}
\sum_{k=1}^{n} w_k \otimes \eta_k &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} A_{ik} v_i \right) \otimes \left(\sum_{j=1}^{n} B_{jk} \theta_j \right) = \sum_{i,j,k=1}^{n} A_{ik} B_{jk} v_i \otimes \theta_j = \\
&= \sum_{i,j,k=1}^{n} A_{ik} \left({}^{\mathrm{t}}A^{-1}\right)_{jk} v_i \otimes \theta_j = \sum_{i,j,k=1}^{n} A_{ik} \left(A^{-1}\right)_{kj} v_i \otimes \theta_j = \\
&= \sum_{i,j,k=1}^{n} (A \cdot A^{-1})_{ij} v_i \otimes \theta_j = \sum_{i,j=1}^{n} \delta_{ij} v_i \otimes \theta_j = \sum_{i=1}^{n} v_i \otimes \theta_i = z.
\end{aligned}$$

□

A.2 Unitary Representations

Let us study what happens when we introduce more structure on a G-module.

Definition A.2.1. Let $\lambda\colon G \to \mathrm{GL}(V)$ be a representation of G. An (Hermitian) inner product $h\colon V \times V \to \mathbb{C}$ is called ***G*-invariant** when $h(\lambda(g)(v), \lambda(g)(w)) = h(v, w)$ for all g in G and v, w in V. A G-module together with a G-invariant inner product is called **unitary**, as well as its associated representation.

We will consider an inner product $h(\cdot, \cdot)$ to be antilinear in the first argument and linear in the second one.

Definition A.2.2. Let $\lambda\colon G \to \mathrm{GL}(V)$ and $\sigma\colon G \to GL(W)$ be representations of G. Let h be a G-invariant inner product on V and k a G-invariant inner product on W. An isomorphism of modules $\varphi\colon V \to W$ is called an **isometry of modules** or a ***G*-isometry** when $h(u, v) = k(\varphi(u), \varphi(v))$ for any u, v in V. When such an isometry of modules exists, we say that V and W are ***G*-isometric**.

Fact A.2.1. *Let V be a finite-dimensional complex vector space equipped with a Hermitian product $h(\cdot, \cdot)$. Let $f\colon V \to V$ be a linear map and let $\mathcal{B}$, $\mathcal{C}$ be arbitrary orthonormal bases of V. We then have that f is an isometry, i.e. $h(f(v), f(w)) = h(v, w)$ for all v, w in V, if and only if $\mathcal{M}_{\mathcal{CB}}(f)$ is a unitary matrix.*

Corollary A.2.1. *Assume $\lambda\colon G \to \mathrm{GL}(V)$ to be a finite-dimensional unitary representation with a G-invariant inner product h on V. Let $\mathcal{B}$, $\mathcal{C}$ be arbitrary orthonormal bases of V with respect to h. Then $\mathcal{M}_{\mathcal{CB}}(\lambda(g))$ is a unitary matrix for any g in G.*

Proof. Since λ is unitary by hypothesis, we have that $\lambda(g)$ is an isometry for any g in G by definition of unitary representation. The statement then follows by Fact A.2.1. □

Remark A.2.1. Assume $\lambda\colon G \to \mathrm{GL}(V)$ to be a finite-dimensional unitary representation with a G-invariant inner product h on V. Let $\mathcal{B}$ be an orthonormal basis of V. We know that $\mathcal{M}_{\mathcal{B}}(\lambda(g))$ is a unitary matrix for any g in G by Corollary A.2.1. Therefore, for any g in G we have that:

$$\mathcal{M}_{\mathcal{B}^\vee}\left(\bar{\lambda}(g)\right) = {}^{\mathrm{t}}\mathcal{M}_{\mathcal{B}}(\lambda(g))^{-1} = {}^{\mathrm{t}}\mathcal{M}_{\mathcal{B}}(\lambda(g))^{\dagger} = {}^{\mathrm{t}}\left({}^{\mathrm{t}}\mathcal{M}_{\mathcal{B}}(\lambda(g))^{*}\right) = \mathcal{M}_{\mathcal{B}}(\lambda(g))^{*}, \tag{A.4}$$

where we have used (A.1).

Theorem A.2.1. *Let V be a finite-dimensional G-module. Then V possesses a G-invariant inner product.*

Proof. See [BtD85, Theorem (1.7)]. □

Theorem A.2.2. *Let V be a finite-dimensional G-module and W a submodule of V. Then there exists a submodule U of V such that $V = W \oplus U$. In particular, any finite-dimensional G-module is a direct sum of irreducible submodules.*

Proof. See [BtD85, Proposition (1.9)]. □

Lemma A.2.1 (Schur's Lemma)**.** *Let V and W be irreducible G-modules. Then:*

1. *a morphism of modules $\varphi\colon V \to W$ is either an isomorphism or the null map;*
2. *the dimension of $\mathrm{Hom}_G(V, W)$ is either 1 when $V \cong W$ or 0 otherwise;*
3. *if φ is an isomorphism of modules from V to V, then φ is a scalar multiple of the identity map.*

Proof. See [BtD85, Theorem (1.10)]. □

Let us see some applications of Schur's Lemma:

Proposition A.2.1. *Let V be a finite-dimensional irreducible G-module. Then any two G-invariant inner products on V differ by a real positive constant factor.*

Proof. Let h and k be two G-invariant inner products on the finite-dimensional irreducible module V. Define two maps $H, K\colon V \to V^\vee$ such that:

$$H(v) := h(v, \cdot) \qquad \text{and} \qquad K(v) := k(v, \cdot)$$

for any v in V. It is easy to see that both H and K are injective antilinear morphisms of modules, which are also surjective due to the finite-dimensionality of V. Hence, $K^{-1} \circ H\colon V \to V$ is an isomorphisms of modules. Since V is irreducible by hypothesis, by Schur's Lemma we have that $K^{-1} \circ H = \phi\, \mathrm{id}_V$ for some $\phi \in \mathbb{C}$, i.e. $H = \phi K$. We therefore conclude that $h = \phi k$. Furthermore, $\phi k(v, v) = h(v, v) \in \mathbb{R}_+$ and $k(v, v) \in \mathbb{R}_+$ for any v in V, hence $\phi \in \mathbb{R}_+$. □

Proposition A.2.2. *Let V and W be finite-dimensional unitary G-modules. If V and W are G-isomorphic then they are G-isometric.*

Proof. Let h and k be G-invariant inner products on V and W respectively. Assume V to be irreducible. Let $\Psi\colon V \to W$ be an isomorphism of modules (so W is irreducible as well). Define $h'\colon W \times W \to \mathbb{C}$ such that:

$$h'(u, v) := k(\Psi(u), \Psi(v)) \quad \forall u, v \in V.$$

It is easy to check that h' is a G-invariant inner product on V, therefore $h = \phi h'$ for some $\phi \in \mathbb{R}_+$ by Proposition A.2.1. Define $\Phi := \sqrt{\phi}\Psi$, which is still an isomorphism of modules. We have that Φ is an isometry, indeed for any u, v in V we get:

$$k(\Phi(u), \Phi(v)) = k\left(\sqrt{\phi}\Psi(u), \sqrt{\phi}\Psi(v)\right) = \phi k(\Psi(u), \Psi(v)) = \phi h'(u, v) = h(u, v).$$

In the case V and W are not irreducible, we consider a decomposition into irreducible submodules guaranteed by Theorem A.2.2 and apply the statement to each single irreducible summand. □

Before to present next fact, recall the notions of multiplicity and coupling between representations outlined in Definitions 1.1.1 and 1.1.2.

Fact A.2.2. *Let $\lambda_1, \lambda_2, \lambda_3$ be finite-dimensional irreducible representations of G. Then the following statements hold:*

1. *λ_1, λ_2 couple to $\mathbf{1} \Leftrightarrow \lambda_2 \cong \bar{\lambda}_1$;*
2. *the multiplicity of $\mathbf{1}$ in $\lambda_1 \otimes \bar{\lambda}_1$ is 1;*
3. *$\lambda_1, \lambda_2, \lambda_3$ couple to $\mathbf{1} \Leftrightarrow \lambda_1, \lambda_2$ couple to $\bar{\lambda}_3 \Leftrightarrow \lambda_1, \lambda_3$ couple to $\bar{\lambda}_2 \Leftrightarrow \lambda_2, \lambda_3$ couple to $\bar{\lambda}_1$. Furthermore, $\mathrm{m}^{\mathbf{1}}_{\lambda_1 \otimes \lambda_2 \otimes \lambda_3} = \mathrm{m}^{\bar{\lambda}_3}_{\lambda_1 \otimes \lambda_2} = \mathrm{m}^{\bar{\lambda}_1}_{\lambda_2 \otimes \lambda_3} = \mathrm{m}^{\bar{\lambda}_2}_{\lambda_1 \otimes \lambda_3}$.*

Proof. 1., 2. Since $\mathrm{Hom}_G(V_{\lambda_1} \otimes V_{\lambda_2}, \mathbb{C}) \cong \mathrm{Hom}_G(V_{\lambda_1}, V_{\bar{\lambda}_2})$, we have that:

$$\mathrm{m}^{\mathbf{1}}_{\lambda_1 \otimes \lambda_2} = \dim \mathrm{Hom}_G(V_{\lambda_1} \otimes V_{\lambda_2}, \mathbb{C}) = \dim \mathrm{Hom}_G(V_{\lambda_1}, V_{\bar{\lambda}_2}),$$

where the latter is 1 in the case V_{λ_1} and $V_{\bar{\lambda}_1}$ are G-isomorphic and 0 otherwise by Schur's Lemma.

3. Consider the following isomorphic modules:

$$\begin{aligned}
\mathrm{Hom}_G((V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3}, \mathbb{C}) &\cong \mathrm{Hom}_G(V_{\lambda_1} \otimes V_{\lambda_2}, V_{\bar{\lambda}_3}),\\
\mathrm{Hom}_G((V_{\lambda_2} \otimes V_{\lambda_3}) \otimes V_{\lambda_1}, \mathbb{C}) &\cong \mathrm{Hom}_G(V_{\lambda_2} \otimes V_{\lambda_3}, V_{\bar{\lambda}_1}),\\
\mathrm{Hom}_G((V_{\lambda_1} \otimes V_{\lambda_3}) \otimes V_{\lambda_2}, \mathbb{C}) &\cong \mathrm{Hom}_G(V_{\lambda_1} \otimes V_{\lambda_3}, V_{\bar{\lambda}_2}).
\end{aligned}$$

Since $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \cong V_{\lambda_i} \otimes V_{\lambda_j} \otimes V_{\lambda_k}$ for $\{i, j, k\} = \{1, 2, 3\}$, the dimension of all the spaces above is the same, getting the claim by definition of multiplicity.

□

Definition A.2.3. Let λ be a finite-dimensional representation of G. We define the representation λ and the module V_λ to be:

- **real** when there exists an antilinear map $J \colon V_\lambda \to V_\lambda$ such that $J^2 = \mathrm{id}_{V_\lambda}$ and $J \circ \lambda(g) = \lambda(g) \circ J \ \forall g \in G$;

- **quaternionic** when there exists an antilinear map $J\colon V_\lambda \to V_\lambda$ such that $J^2 = -\operatorname{id}_{V_\lambda}$ and $J \circ \lambda(g) = \lambda(g) \circ J\ \forall g \in G$;
- **self-dual** when λ is equivalent to $\bar{\lambda}$;
- **complex** when λ is not self-dual.

Definition A.2.4. Let λ be a representation of G and $H\colon V_\lambda \times V_\lambda \to \mathbb{C}$ a bilinear form on V_λ. We say that H is ***G*-invariant** when $H(\lambda(g)(v), \lambda(g)(w)) = H(v, w)$ $\forall v, w \in V_\lambda$.

Proposition A.2.3. *Let λ be a finite-dimensional representation of G. Then:*

1. *λ is real if and only if there exists a G-invariant symmetric non-degenerate bilinear form on V_λ;*
2. *λ is quaternionic if and only if there exists a G-invariant skew-symmetric non-degenerate bilinear form on V_λ.*

Proof. See [BtD85, Proposition (6.4)]. □

Proposition A.2.4. *Let λ be an irreducible finite-dimensional representation of G. Then λ is one and only one of the following: real, quaternionic, complex.*

Proof. See [BtD85, Proposition (6.5)]. □

Definition A.2.5. Let λ be a representation of G. Define the following map:

$$\chi_\lambda\colon G \to \mathbb{C}; \quad g \mapsto \operatorname{Trace}(\lambda(g)). \tag{A.5}$$

Then χ_λ is called the **character associated with $\boldsymbol{\lambda}$**.

Proposition A.2.5. *Let λ be a finite-dimensional irreducible representation of G. Then:*

$$\int_{g\in G} \chi_\lambda(g^2)\, d\mu = \begin{cases} 1 & \lambda \textit{ is real,} \\ -1 & \lambda \textit{ is quaternionic,} \\ 0 & \lambda \textit{ is complex,} \end{cases} \tag{A.6}$$

where the symbol of integration refers to the (normalized) Haar-integral.

Proof. See [BtD85, Proposition (6.8)]. □

Definition A.2.6. Let λ be a finite-dimensional irreducible representation. Then the number $\int_{g\in G} \chi_\lambda(g^2)\, d\mu$ is called the **Frobenius-Schur indicator** of λ and we will denote it by ι_λ.

Fact A.2.3. *Let λ be a finite-dimensional unitary representation of G of dimension d. Let J be a unitary complex $d \times d$ matrix. Let $\mathcal{B}$ be an orthonormal basis of V_λ. If $J \cdot \mathcal{M}_\mathcal{B}(\lambda(g)) \cdot J^{-1} = \mathcal{M}_\mathcal{B}(\lambda(g))^*$ for any g in G, then J is either symmetric or skew-symmetric, i.e. ${}^{\mathrm{t}}J = \phi J$ where $\phi = \pm 1$. In such a case, λ is self-dual and ϕ is the Frobenius-Schur indicator of λ: if J is symmetric then λ is real and if J is skew-symmetric then λ is quaternionic.*

Proof. If $g \in G$, the condition $J \cdot \mathcal{M}_\mathcal{B}(\lambda(g)) \cdot J^{-1} = \mathcal{M}_\mathcal{B}(\lambda(g))^*$ implies both $J^{-1} \cdot \mathcal{M}_\mathcal{B}(\lambda(g))^* \cdot J = \mathcal{M}_\mathcal{B}(\lambda(g))$ and $J^* \cdot \mathcal{M}_\mathcal{B}(\lambda(g))^* \cdot (J^{-1})^* = \mathcal{M}_\mathcal{B}(\lambda(g))$. Putting these latter two expressions together, we get:

$$J^* \cdot \mathcal{M}_\mathcal{B}(\lambda(g))^* \cdot (J^{-1})^* = \mathcal{M}_\mathcal{B}(\lambda(g)) = J^{-1} \cdot \mathcal{M}_\mathcal{B}(\lambda(g))^* \cdot J,$$

namely

$$(J^*J) \cdot \mathcal{M}_\mathcal{B}(\lambda(g)) = \mathcal{M}_\mathcal{B}(\lambda(g)) \cdot (J^*J)$$

for any g in G. By Schur's Lemma, we get that $J^*J = \phi I_d$ for some $\phi \in \mathbb{C}$, where I_d is the $d \times d$ identity matrix. Since J is unitary, we have that $|\phi| = 1$ and:

$$\phi I_d = J^*J = ({}^{\mathrm{t}}J)^{-1}J, \qquad \text{namely} \qquad {}^{\mathrm{t}}J = \phi^* J.$$

Observe now the following:

$$\begin{aligned} J^*J &= ({}^{\mathrm{t}}J)^{-1}J = (\phi^*J)^{-1}J = \phi J^{-1}J = \phi I_d, \\ JJ^* &= J({}^{\mathrm{t}}J)^{-1} = J(\phi^*J)^{-1} = \phi JJ^{-1} = \phi I_d, \end{aligned}$$

showing that $J^*J = JJ^*$. We therefore conclude that:

$$\phi I_d = JJ^* = J^*J = (JJ^*)^* = (\phi I_d)^* = \phi^* I_d,$$

implying $\phi^* = \phi$, i.e. $\phi \in \mathbb{R}$. Hence, ϕ is a real number of module 1, i.e. $\phi = \pm 1$. Since $\mathcal{B}$ is orthonormal and λ is unitary, we have that $\mathcal{M}_\mathcal{B}(\lambda(g))$ is unitary for any g in G by Fact A.2.1, therefore the initial condition $J \cdot \mathcal{M}_\mathcal{B}(\lambda(g)) \cdot J^{-1} = \mathcal{M}_\mathcal{B}(\lambda(g))^*$ implies ${}^{\mathrm{t}}\mathcal{M}_\mathcal{B}(\lambda(g)) \cdot J \cdot \mathcal{M}_\mathcal{B}(\lambda(g)) = J$ for any g in G, proving that J defines a non-degenerate G-invariant bilinear form H on V_λ (where $H(v, w) := {}^{\mathrm{t}}[v]_\mathcal{B} \cdot J \cdot [w]_\mathcal{B}$ $\forall v, w \in V_\lambda$). By Proposition A.2.3, we have that:

$$\begin{aligned} \phi = 1 \;&\Leftrightarrow\; J \text{ is symmetric} \;\Leftrightarrow\; H \text{ is symmetric} \;\Rightarrow\; \lambda \text{ is real}, \\ \phi = -1 \;&\Leftrightarrow\; J \text{ is skew-symmetric} \;\Leftrightarrow\; H \text{ is skew-symmetric} \;\Rightarrow\; \lambda \text{ is quaternionic}. \end{aligned}$$

By the above, we see that ϕ coincides with the Frobenius-Schur indicator in the corresponding cases. □

Let V be a finite-dimensional G-module together with a G-invariant inner product h. Define the map:

$$\begin{array}{rccc} \mathcal{J}_V\colon & V & \to & V^\vee \\ & v & \mapsto & h(v,\cdot) \end{array} . \tag{A.7}$$

It is easy to see that $\mathcal{J}_V$ is well defined, antilinear and bijective. Define the map:

$$\begin{array}{rccc} h^\vee\colon & V^\vee \times V^\vee & \to & \mathbb{C} \\ & (\theta,\omega) & \mapsto & h\left(\mathcal{J}_V^{-1}(\theta), \mathcal{J}_V^{-1}(\omega)\right) \end{array} . \tag{A.8}$$

It is easy to see that $h^\vee$ is a G-invariant inner product on $V^\vee$. Define the map:

$$\begin{array}{rccc} \mathcal{J}_{V^\vee}\colon & V^\vee & \to & V^{\vee\vee} \\ & \theta & \mapsto & h^\vee(\theta,\cdot) \end{array} , \tag{A.9}$$

which again is well defined, antilinear and bijective. We define also:

$$\begin{array}{rccc} h^{\vee\vee}\colon & V^{\vee\vee} \times V^{\vee\vee} & \to & \mathbb{C} \\ & (\varphi,\psi) & \mapsto & h^\vee\left(\mathcal{J}_{V^\vee}^{-1}(\varphi), \mathcal{J}_{V^\vee}^{-1}(\psi)\right) \end{array} . \tag{A.10}$$

Again, $h^{\vee\vee}$ is a G-invariant inner product on $V^{\vee\vee}$. For any v, w in V, we have:

$$h(v,w) = h^{\vee\vee}\left(\left(\mathcal{J}_{V^\vee}\circ\mathcal{J}_V\right)(v), \left(\mathcal{J}_{V^\vee}\circ\mathcal{J}_V\right)(w)\right),$$

hence $\mathcal{J}_{V^\vee}\circ\mathcal{J}_V$ is a linear isometry between V and its bidual, when the latter is equipped with $h^{\vee\vee}$. For this reason, we consider V and $V^{\vee\vee}$ identified via $\mathcal{J}_{V^\vee}\circ\mathcal{J}_V$.

Remark **A.2.2.** If $\mathcal{B} = (v_1,\ldots,v_n)$ is an orthonormal basis of V and $\mathcal{B}^\vee = (\theta_1,\ldots,\theta_n)$ is its dual basis, notice that $\mathcal{J}_V(v_i) = \theta_i$ for any i and that $\mathcal{B}^\vee$ is orthonormal with respect to $h^\vee$.

Consider a second finite-dimensional G-module W together with a G-invariant inner products k. Define the following maps:

$$\begin{array}{rccc} h\oplus k\colon & (V\oplus W)\times(V\oplus W) & \to & \mathbb{C} \\ & ((v_1,w_1),(v_2,w_2)) & \mapsto & h(v_1,v_2)+k(w_1,w_2) \end{array} , \tag{A.11}$$

$$\begin{array}{rccc} h\otimes k\colon & (V\otimes W)\times(V\otimes W) & \to & \mathbb{C} \\ & (v_1\otimes w_1, v_2\otimes w_2) & \mapsto & h(v_1,v_2)\cdot k(w_1,w_2) \end{array} \tag{A.12}$$

and extend them antilinearly on the first argument and linearly on the second argument. It is easy to check that $h\oplus k$ and $h\otimes k$ define G-invariant inner products on the (standard) G-modules $V\oplus W$ and $V\otimes W$ respectively.

Remark **A.2.3.** If $\mathcal{B}$ and $\mathcal{C}$ are orthonormal bases of V and W respectively, then $\mathcal{B}\oplus\mathcal{C}$ and $\mathcal{B}\otimes\mathcal{C}$ are orthonormal bases of $V\oplus W$ and $V\oplus W$ when equipped with $h\otimes k$ and $h\otimes k$ respectively.

All of the above easily generalizes to the case in which we consider direct sums and tensor products of more than two modules.

A.3 Faithful Representations

In this section we will analyze some sufficient conditions for our arbitrary compact Lie group G to have a faithful finite-dimensional irreducible representation. We will also see what such conditions allow.

Definition A.3.1. A representation $\lambda\colon G \to \mathrm{GL}(V)$ of G is called **faithful** when λ is injective. In this case, we call V a **faithful module**.

Recall the notation $V(k,l) = V^{\otimes k} \otimes \left(V^{\vee}\right)^{\otimes l}$ defined by (2.1) for a generic vector space V and for $k, l \in \mathbb{N}$.

Theorem A.3.1 (Peter-Weyl Theorem)**.** *The group G admits a finite-dimensional faithful representation. Furthermore, if V is a faithful G-module then every irreducible G-module is G-isomorphic to a submodule of $V(k,l)$ for some $k, l \in \mathbb{N}$.*

Proof. See [Fol15, Theorem (5.13)] and [BtD85, Theorem (4.4)]. □

Corollary A.3.1. *Every irreducible representation of G is finite-dimensional.*

Proof. Fix an arbitrary irreducible G-module W. By Theorem A.3.1, there exists a finite-dimensional faithful G-module V and W is isomorphic to a submodule of $V(k,l)$ for some $k, l \in \mathbb{N}$. Being V finite-dimensional, so is $V(k,l)$ and so is W. □

Proposition A.3.1. *An infinite compact Lie group has countably infinitely many non-equivalent irreducible representations.*

Proof. See [BtD85, Chapter 3, Section 4, Excercise 3]. □

Definition A.3.2. A faithful representation of G of minimal dimension is called **primitive**, as well as its associated module.

Proposition A.3.2. *The following holds:*

1. *if V is a primitve G-module, then it is finite-dimensional;*
2. *the group G admits a primitive representation.*

Proof.
1. Assume V to be a primitive G-module. We know by Theorem A.3.1 that G has a finite-dimensional faithful representations, therefore V must be finite-dimensional by the minimality of its dimension.
2. Let W be a finite-dimensional faithful G-module (we are allowed to do this by Theorem A.3.1). The set:

$$\{\, \dim V \mid V \text{ is a faithful } G\text{-module and } \dim V \leq \dim W \,\}$$

is finite and non-empty, hence its minimum exists.

□

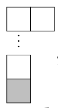
Remark A.3.1. The compact Lie group $S^1 \times S^1$ is Abelian, therefore all irreducible representations are one-dimensional (see [BtD85, Proposition (1.13)]), but in this case none of them is faithful, so no primitive representation of $S^1 \times S^1$ is irreducible. In particular, $S^1 \times S^1$ is not a simple Lie group. Hence, when we look for hypothesis for which primitive representations are irreducible, we need to ask the compact Lie group under consideration to be simple.

Remark A.3.2. We know that $SU(N)$ admits an N-dimensional irreducible representation that goes under the name of *fundamental representation*. Such representation is faithful of minimal dimension. It is natural then to ask ourselves the following question:

if G is a simple compact Lie group, does it admit a faithful irreducible representation?

The answer in general is no: the center of a simple Lie group of type D_{2l} is not cyclic, so no irreducible representation is faithful. For all the other types (A_l, B_l, C_l, E_6, E_7, E_8, F_4, G_2), such a representation exists.

A.4 Some properties about irreducible representations of $SU(N)$

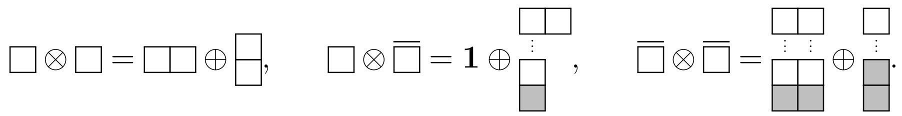
In this section we study some properties about finite-dimensional irreducible representations of the compact Lie group $SU(N)$ with $N \geq 3$.
We will make use of the tool of Young tableaux (see [Ful97] as a reference). Let us establish the following convention: when a box in a Young tableau T is coloured in gray, it highlights the fact that such box is missing from T. We use this notation to make clear how many boxes would be needed to reach the hight of N boxes in certain columns of a tableau. For instance, denoting the fundamental representation of $SU(N)$ by ϵ, we have that ϵ corresponds to the tableau $\square$ and its dual $\bar{\epsilon}$ corresponds to the tableau $\overline{\square} :=$ , where the gray box tells us that the column there has $N-1$ boxes. Another example is given by , which corresponds to ad, the adjoint representation of $SU(N)$. The reason why this convention turns out to be useful is the following: let λ be a finite-dimensional irreducible representation of $SU(N)$ and T its associated Young tableau with columns $T_1, \ldots, T_m$, then it is

not difficult to realize that the power of λ (see Definition 2.1.1) is:

$$p(\lambda) = \min \left\{ a + b \;\middle|\; \begin{array}{l} i = 1, \ldots, m;\ a \text{ is the total number of boxes needed} \\ \text{for the columns from } T_1 \text{ to } T_i \text{ to be of height } N;\ b \text{ is} \\ \text{the total number of boxes among the columns from} \\ T_{i+1} \text{ to } T_m \end{array} \right\}. \tag{A.13}$$

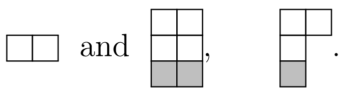
A.4.1 Irreducible representations of power 2

We have:

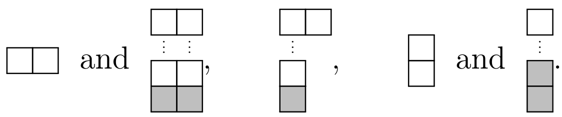


(A.14)

Therefore, for $N = 3$ the irreducible representations of power 2 are:



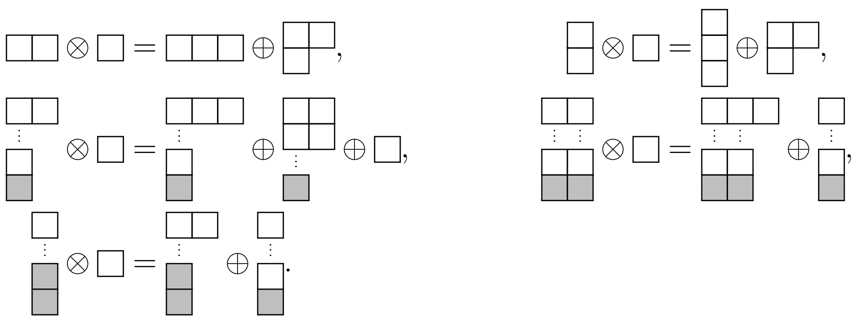
For $N \geq 4$, the irreducible representations of power 2 are:



(A.15)

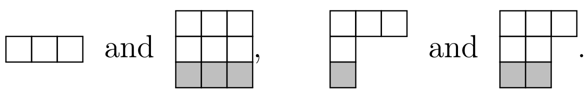
A.4.2 Irreducible representations of power 3

We have:

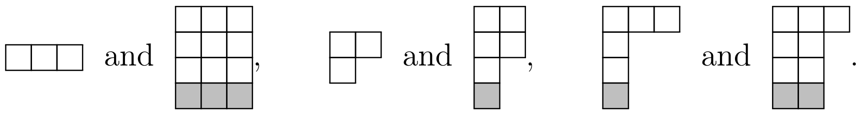


(A.16)

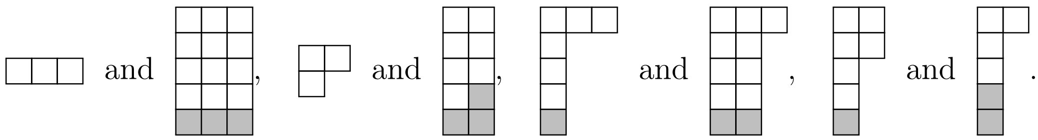
Therefore, for $N = 3$ the irreducible representations of power 3 are:



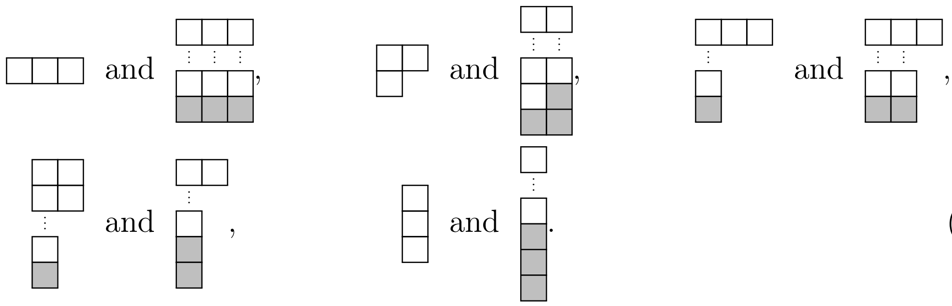
For $N = 4$, the irreducible representations of power 3 are:



For $N = 5$, the irreducible representations of power 3 are:



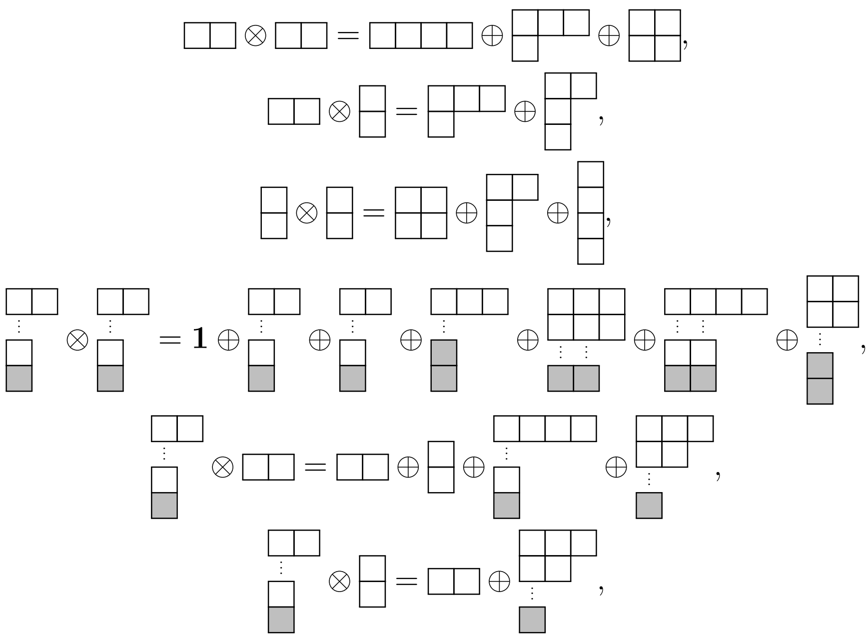
For $N \geq 6$, the irreducible representations of power 3 are:

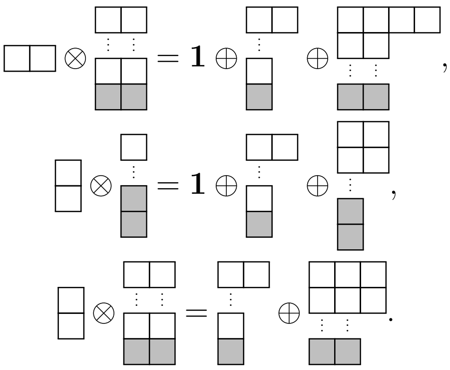


(A.17)

A.4.3 Products of irreducible representations of power 2

We have:

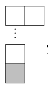




From all the above computations we can conclude the remaining products of irreducible representations of power 2 by simply taking the dual.

A.4.4 Some Results

Lemma A.4.1. *If λ is a finite-dimensional irreducible representation of $SU(N)$ then* ad $\in \lambda \otimes \bar{\lambda}$.

Proof. If we represent λ and $\bar{\lambda}$ as Young diagrams A and B, then B is complementary to A. When we perform $A \otimes B$, it is easy to see that one way of arrange the boxes gives rise to the tableau , which corresponds to ad. □

Lemma A.4.2. *If $N \geq 4$, then a primitive triad of the type $(\varepsilon_1\varepsilon_2\varepsilon_3)$ does not exist for $SU(N)$.*

Proof. We analyze all the products that can produce irreducible representations of power 2, namely $\epsilon \otimes \epsilon$, $\epsilon \otimes \bar{\epsilon}$, $\bar{\epsilon} \otimes \bar{\epsilon}$. As we see above, if $N > 3$ then neither ϵ nor $\bar{\epsilon}$ can be a summand in a decomposition of such tensor products. □

Lemma A.4.3. *If $N \geq 6$, then a primitive triad of the type $(2_p 2_p \varepsilon)$ does not exist for $SU(N)$.*

Proof. Above we saw all Young diagrams corresponding to irreducible representations of $SU(N)$ of power 2. By examining every product of the type $2_p \otimes \epsilon$, we see that no decomposition presents an irreducible representation of power 2 as a summand. □

Lemma A.4.4. *If $N \geq 8$, then a triad of the type $(3_p 2_p 2_p)$ does not exist for $SU(N)$.*

Proof. By the computations above, if $N \geq 8$ then no irreducible representation of power 3 can be a summand of a decomposition of the tensor product of two irreducible representations of power 2. □

Lemma A.4.5. *Let λ, α, β be irreducible representations of $SU(N)$ such that $\beta \in \lambda \otimes \alpha$. Assume $p(\lambda) = p(\alpha) = 2$. If $\lambda = \alpha = \beta = \mathrm{ad}$ then the multiplicity of β in $\lambda \otimes \alpha$ is 2, otherwise is 1.*

Proof. It follows from the computations above. □

Let us see now a (general but not optimal) condition regarding the admissibility of a triad depending on the parity of the power of its representations when N is high enough:

Lemma A.4.6. *Let $(\lambda\alpha\beta)$ be an ordered triad. If $N > 2(p(\lambda) + p(\alpha))$ then $p(\beta) \equiv_2 p(\lambda) + p(\alpha)$.*

Example A.4.1. Let $(\lambda\alpha\beta)$ be an ordered triad.

- If $(\lambda\alpha\beta) = (4_p 4_p n_p)$ and $N > 8$, then n must be even, i.e. $n \in \{0, 2, 4\}$.
- If $(\lambda\alpha\beta) = (4_p 3_p n_p)$ and $N > 7$, then n must be odd, i.e. $n \in \{1, 3\}$.

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