$U_q(\mathfrak{sl}_3)$-Tilting Modules

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Contents

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1 Introduction & Recollection

1.1 Introduction

In this master thesis we study tilting modules of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ and $q \in \mathbb{C}\backslash\{0\}$ a root of unity. Of key interest for us is the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and especially the examples $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

Quantum groups or quantized universal enveloping algebras play an important role in representation theory. Starting from Lie theory, one may view the quantum group $U_q(\mathfrak{g})$ as a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Via the so-called $\mathcal{A}$-form or classical limit, where one specializes $q \mapsto 1$, we get the universal enveloping algebra back. Further, they are a standard example of a non-cocommutative Hopf algebra. There are many different definitions, depending on which setup one is interested in (Kac-Moody algebras or Lie algebras, q an indeterminate or $q \in k\backslash\{0\}$ with $q^2 \neq 1$, ...).

Tilting modules are a class of modules over a quasi-hereditary algebra. They were first introduced as a quotient of a projective module by another projective module with some further properties. In this thesis, we will construct them (in the category of integrable modules over the quantum group) with the extension functor. Tilting modules in contrast to e.g. simple modules might be harder to construct, but they have some nice properties, e.g. they are self dual and they are closed under direct sums and tensor products. In particular, the full subcategory of tilting modules is semisimple.

It is a known fact, that the category of finite dimensional $U_q(\mathfrak{g})$-modules for q not a root of unity is semisimple. This makes its theory about tilting modules not as interesting, since then every module is tilting. In contrast, for q a root of unity the category is not semisimple. In particular, we have indecomposable modules which are not necessarily simple. Important examples are the so-called Weyl modules, dual Weyl modules and indecomposable tilting modules.

Here, we will consider complete sets of pairwise non-isomorphic indecomposable modules, which are classified by their highest weight, where we will denote by X^+ the dominant integral weights:

$$\begin{tikzcd}
 & \left\{L(\lambda) \mid \lambda \in X^+\right\} & \\
\left\{\nabla(\lambda) \mid \lambda \in X^+\right\} \arrow[ur, "\text{The dual Weyl module } \nabla(\lambda) \text{ has simple socle } L(\lambda)."] \arrow[rr, leftrightarrow, "\nabla(\lambda) \text{ and } \Delta(\lambda) \text{ are}", "\text{the twisted dual of each other.}"'] & & \left\{\Delta(\lambda) \mid \lambda \in X^+\right\} \arrow[ul, "\text{The Weyl module } \Delta(\lambda) \text{ has simple head } L(\lambda)."'] \\
 & \left\{T(\lambda) \mid \lambda \in X^+\right\} \arrow[ul, "\text{The tilting module } T(\lambda) \text{ has a dual Weyl filtration.}"] \arrow[ur, "\text{The tilting module } T(\lambda) \text{ has a Weyl filtration.}"'] &
\end{tikzcd}$$

In the rest of this section we give a short recollection of some facts about the Lie algebra $\mathfrak{sl}_{n+1}$, which is our main example, in particular $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

In the second section we give the definition of the quantum group $U_q(\mathfrak{g})$ and the category of integrable modules $\mathcal{C}$. Furthermore, we will define the above mentioned modules and prove some properties. In particular, we construct the indecomposable tilting modules.

In the third section, we introduce the "quotient" category $\mathcal{C}_\ell^-$. This category is in some sense a quotient of the category of all tilting modules by the so-called negligible modules. These modules can be characterized in two ways: They are in the span of fixed modules by the action of simple reflections, and they have quantum dimension zero. By the second fact, one can easily deduce that they form an ideal in the Grothendieck ring, so we also have a new tensor product in $\mathcal{C}_\ell^-$. Another property of this category is, that now we have a finite number of weights (namely the weights in the fundamental alcove of the affine Weyl group) and that our indecomposable tilting modules coincide with the simple modules.

In the last section, we will give a combinatorial description for the category $\mathcal{C}_\ell^-$ for $\mathfrak{g} = \mathfrak{gl}_{n+1}$. But by taking a quotient, we will again be in the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$. At the very end, we do brief introduction to the combinatorial description in Lie type C.

In the whole thesis, we only work over the complex numbers $\mathbb{C}$, $q \in \mathbb{C}$ will be a primitive ℓ^{th} root of unity, $\ell \in \mathbb{N}$ odd and $\ell > h$, where h is the Coxeter number of $\mathfrak{g}$. Here the natural numbers $\mathbb{N}$ are meant as being the strictly positive integers, i.e. $\mathbb{N} = \{1, 2, 3, 4, ...\}$. If we want to include 0, we denote this set by $\mathbb{Z}_{\geq 0}$.

1.2 The Lie Algebra $\mathfrak{sl}_{n+1}$

Before we start with the general definition of the quantum group $U_q(\mathfrak{g})$ for a complex semisimple Lie algebra $\mathfrak{g}$, we introduce (very briefly) the Lie algebra $\mathfrak{sl}_{n+1}$ over the complex numbers $\mathbb{C}$. This is to recollect and to fix notation. For more details see e.g. [10].

Let $n \in \mathbb{N}$. Then the Lie algebra $\mathfrak{gl}_{n+1}$ consists of all $(n+1) \times (n+1)$-matrices over $\mathbb{C}$ with the Lie bracket being the standard matrix commutator, i.e. for $B, C \in \mathfrak{gl}_{n+1}$:

$$[B, C] = BC - CB,$$

where BC and CB are the ordinary matrix products.

The Lie algebra $\mathfrak{sl}_{n+1}$ is the Lie subalgebra of $\mathfrak{gl}_{n+1}$ consisting of matrices with trace 0, i.e.

$$\mathfrak{sl}_{n+1} = \left\{ B \in \mathfrak{gl}_{n+1} \mid \operatorname{tr}(B) = 0 \right\}.$$

In other words, the Lie algebra $\mathfrak{sl}_{n+1}$ consists of $(n+1) \times (n+1)$-matrices over $\mathbb{C}$, whose diagonal entries add up to 0, with the Lie bracket being the standard matrix commutator.

For $i, j \in \{1, ..., n+1\}$, we denote by $E_{i,j}$ the $(n+1) \times (n+1)$-matrix, which has as a single non-zero entry 1 at the $(i, j)^{\text{th}}$ spot. Then the following elements generate $\mathfrak{sl}_{n+1}$:

$$f_i := E_{i+1,i}, \qquad h_i := E_{i,i} - E_{i+1,i+1}, \qquad e_i := E_{i,i+1},$$

for $i \in \{1, ..., n\}$.

Further, we fix the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_{n+1}$, namely the diagonal matrices contained in $\mathfrak{sl}_{n+1}$, i.e.

$$\mathfrak{h} = \operatorname{span}_{\mathbb{C}}(h_1, ..., h_n),$$

and the standard Borel subalgebra $\mathfrak{b}$, the upper triangular matrices contained in $\mathfrak{sl}_{n+1}$, i.e. $\mathfrak{b}$ is generated by the elements $\{h_i, e_i\}_{i \in \{1,...,n\}}$ as a Lie algebra.

We denote by $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ the set of simple roots. They satisfy for $i, j \in \{1, ..., n\}$:

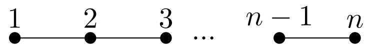
$$\alpha_j(h_i) = a_{ij},$$

where the a_{ij} are the entries of the Cartan matrix

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & & -1 & 2 \end{pmatrix} \in M_{n \times n}(\mathbb{C}).$$

So the h_i's are our co-roots, which we from now on denote by $\alpha_i^\vee = h_i$ for $i \in \{1, ..., n\}$.

Further, we have the following corresponding Dynkin diagram:



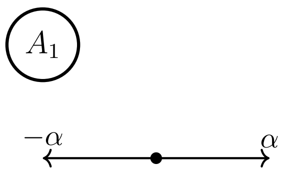
If we take the standard basis of the dual D_{n+1}^* of the diagonal matrices, i.e. $(\varepsilon_j)_{j=1}^{n+1}$ where ε_j of a diagonal matrix is its j^{th} diagonal entry, we may write

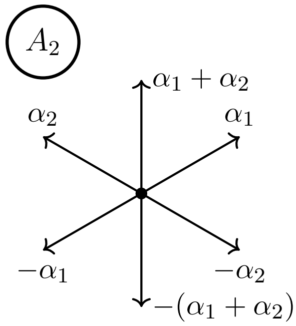
$$\alpha_j = \varepsilon_j - \varepsilon_{j+1}.$$

Also note, that the simple roots form a basis of the dual of the Cartan subalgebra $\mathfrak{h}^*$.

We will denote by Φ the set of roots, and by Φ^+, respectively Φ^-, the positive roots, respectively negative roots.

Example 1.2.1 For the cases $n = 2$ and $n = 3$ we can depict the root systems as follows:





where the root system of A_1 is in $\mathbb{R}$ and the root system of A_2 in $\mathbb{R}^2$.

The fundamental weights $\omega_1, ..., \omega_n \in \mathfrak{h}^*$ are characterized by:

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j},$$

for $i, j \in \{1, ..., n\}$.

Example 1.2.2 We have for our most important examples $n = 1$ and $n = 2$:

- For $n = 1$ we have
$$\omega_1 = \frac{1}{2} \cdot \alpha_1.$$
- For $n = 2$ we have
$$\omega_1 = \frac{1}{3} \cdot (2\alpha_1 + \alpha_2), \qquad \omega_2 = \frac{1}{3} \cdot (\alpha_1 + 2\alpha_2),$$
or alternatively
$$\alpha_1 = 2\omega_1 - \omega_2, \qquad \alpha_2 = -\omega_1 + 2\omega_2.$$

For $\alpha_i \in \Pi$, we define the simple reflection $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ via

$$\forall \lambda \in \mathfrak{h}^* : \quad s_i(\lambda) := \lambda - \lambda(h_i)\alpha_i. \tag{1}$$

Graphically, the simple reflection s_i is the reflection along the hyperplane orthogonal to α_i with respect to the symmetric bilinear form given by $(\alpha_k, \alpha_l) = a_{kl}$ for $k, l \in \{1, ..., n\}$.

The Weyl group $\mathcal{W}$ of $\mathfrak{sl}_{n+1}$ is generated by the simple reflections s_i for $i \in \{1, ..., n\}$. It can be identified with the symmetric group $\mathfrak{S}_{n+1}$, where the s_i are identified with the transposition $(i, i+1)$. Hence written as a Coxeter group, we have:

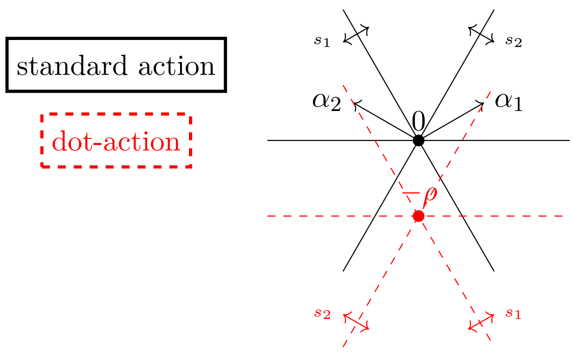
$$\mathcal{W} \quad = \quad \left\langle \; s_1, ..., s_n \; \middle| \; \begin{array}{l} \forall i \in \{1, ..., n\}, j \in \{1, ..., n-1\}, |i-j| > 1 : \\ s_i^2 = \mathrm{id}, \;\; (s_j s_{j+1})^3 = \mathrm{id}, \;\; (s_i s_j)^2 = \mathrm{id} \end{array} \right\rangle$$

There are two important actions of the Weyl group on $\mathfrak{h}^*$: the standard action, which is given by the definition of the simple reflections as in (1), and the dot-action. The dot-action is defined as a shift of the standard action by $\rho := \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, the half-sum of positive roots, namely:

$$\forall i \in \{1, ..., n\}, \lambda \in \mathfrak{h}^* : \quad s_i \bullet \lambda = s_i(\lambda + \rho) - \rho$$

So under the standard action 0 is a fixed point, under the dot-action $-\rho$ is a fixed point.

Example 1.2.3 For $n = 2$ we can depict the actions of $\mathcal{W}$ on $\mathfrak{h}^*$:



The connected components of $\mathfrak{h}^* \setminus \bigcup_{i \in \{1,...,n\}} H_{0,\alpha_i}$, where H_{0,α_i} is the hyperplane along which reflects s_i, i.e. $H_{0,\alpha_i} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$, are called alcoves. In the rest of this thesis, by alcoves we mean the alcoves corresponding to the dot-action, i.e. they are shifted by $-\rho$.

In Example 1.2.3, one can see that there are six alcoves for $\mathfrak{g} = \mathfrak{sl}_3$.

2 The Quantum Group $U_q(\mathfrak{g})$

We will now define the quantum group (or quantized universal enveloping algebra) of a complex semisimple Lie algebra $\mathfrak{g}$. There are more general definitions (for example for infinite dimensional Lie algebras), but those are not of interest for us right now. For a more detailed introduction to quantized universal enveloping algebras we refer for instance to [12].

2.1 Quantum Numbers

To define the quantum group we need some "q-calculus".

First we consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials with integer coefficients and formal parameter v.

For $r \in \mathbb{Z}$, we define the v-integer or quantum number $[r]_v \in \mathcal{A}$ by

$$[r]_v := \frac{v^r - v^{-r}}{v - v^{-1}} = \begin{cases} v^{r-1} + v^{r-3} + ... + v^{1-r}, & \text{if } r \in \mathbb{N}, \\ 0, & \text{if } r = 0, \\ -v^{r+1} - v^{r+3} - .... - v^{-r-1}, & \text{else.} \end{cases}$$

For $r \in \mathbb{N}$ we define

$$[r]_v! := [r]_v \cdot [r-1]_v \cdot ... \cdot [1]_v, \quad [0]_v! := 1,$$

and for $r \geq s \geq 0$, $r, s \in \mathbb{Z}$ the v-binomial coefficient

$$\begin{bmatrix} r \\ s \end{bmatrix}_v := \frac{[r]_v!}{[s]_v! \cdot [r-s]_v!}.$$

Later on, we use the same notation with a different index, e.g. $[r]_q$ for some $q \in \mathbb{C}\backslash\{0\}$, by inserting q in the place of v into the formulas. But we have to be careful, since the binomial coefficient might not be well defined (the quantum number $[r]_q$ is 0 for some cases). In particular this happens in the case for some roots of unity.

We obtain the following:

Lemma 2.1.1 *Let $r \geq s + 1 \in \mathbb{N}$. It holds:*

(i) $\begin{bmatrix} r+1 \\ s \end{bmatrix}_v = v^s \begin{bmatrix} r \\ s \end{bmatrix}_v + v^{s-r-1} \begin{bmatrix} r \\ s-1 \end{bmatrix}_v$,

(ii) $\begin{bmatrix} r \\ s \end{bmatrix}_v \in \mathcal{A}$,

(iii) $[r]_v \mapsto r$ *and* $\begin{bmatrix} r \\ s \end{bmatrix}_v \mapsto \binom{r}{s}$ *as* $v \mapsto 1$.

Proof. (i) Direct calculation.

(ii) Follows from (i) by induction.

(iii) Follows from $[r]_v = v^{r-1} + v^{r-3} + ... + v^{1-r}$.

□

2.2 Definition

To define the quantum group, we use the following setup.

Given a complex semisimple Lie algebra $\mathfrak{g}$, we denote by Φ the corresponding set of roots in some Euclidean space E, by $\Phi^+ \subset \Phi$ a fixed set of positive roots and by $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset \Phi^+$ the set of simple roots for some $n \in \mathbb{N}$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding co-root and by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ the half-sum of all positive roots. Further, let $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ for $i, j \in \{1, 2, ..., n\}$ and $A = (a_{ij})_{i,j=1}^n$ be the Cartan matrix. Let $D = \operatorname{diag}(d_1, d_2, ..., d_n)$ be the diagonal matrix with all $d_i \in \mathbb{N}$ minimal, such that $DA = (d_i a_{ij})_{i,j=1}^n$ is symmetric.

For $i \in \{1, ..., n\}$, we denote $v_i := v^{d_i}$ (and similarly later on for $q_i := q^{d_i}$).

In the special case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ we have seen in the first section that $d_i = 1$ for all $i \in \{1, ..., n\}$. Hence, for $\mathfrak{sl}_{n+1}$ we do not need to distinguish between v and the v_i's or q and the q_i's.

Furthermore, let X be the set of (integral) weights, i.e.

$$X = \{\lambda \in E \mid \forall \alpha_i \in \Pi : \ \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\},$$

and let $X^+ \subset X$ be the set of dominant (integral) weights, i.e.

$$X^+ = \{\lambda \in X \mid \forall \alpha_i \in \Pi : \langle \lambda, \alpha_i^\vee \rangle \geq 0\}.$$

We have a partial order on X defined for $\lambda, \mu \in X$ as follows

$$\mu \leq \lambda \qquad \Longleftrightarrow \qquad \lambda - \mu = \sum_{i=1}^{n} c_i \alpha_i \text{ for some } c_i \in \mathbb{Z}_{\geq 0}.$$

The fundamental weights $\omega_i \in X$ for $i \in \{1, 2, ..., n\}$ are characterized by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j} \quad \text{for } j \in \{1, ..., n\}.$$

Definition 2.2.1 The quantum group $U_v(\mathfrak{g})$ is the associative unital algebra over $\mathbb{Q}(v)$ with the following generators and relations:

- generators:
$$\left\{E_i, F_i, K_i, K_i^{-1} \mid i \in \{1, ..., n\}\right\}$$
- relations:
  (1) $\forall i, j \in \{1, ..., n\} : \ K_i K_i^{-1} = 1 = K_i^{-1} K_i, \ K_i K_j = K_j K_i$
  (2) $\forall i, j \in \{1, ..., n\} : \ K_i E_j K_i^{-1} = v_i^{a_{ij}} E_j$
  (3) $\forall i, j \in \{1, ..., n\} : \ K_i F_j K_i^{-1} = v_i^{-a_{ij}} F_j$
  (4) $\forall i, j \in \{1, ..., n\} : \ E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$
  (5) $\forall i \neq j \in \{1, ..., n\} : \ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{v_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0$

(6) $\forall i \neq j \in \{1,...,n\} : \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{v_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0$

where the a_{ij} are the entries of the Cartan matrix A.

Remark 2.2.2
- The relations (5) and (6) are called quantum Serre relations.
- The generators E_i and F_i for $i \in \{1,...,n\}$ are called Chevalley generators.

Since we want to consider quantum groups at roots of unity, we need to specialize $v \mapsto q$ for $q \in \mathbb{C}$ some root of unity. This we can do via the so-called $\mathcal{A}$-form $U_{\mathcal{A}}(\mathfrak{g})$:

Definition 2.2.3 For $r \in \mathbb{N}$, $i \in \{1,...,n\}$ we define the divided power elements:

$$E_i^{(r)} = \frac{1}{[r]_{v_i}!} E_i^r, \qquad F_i^{(r)} = \frac{1}{[r]_{v_i}!} F_i^r.$$

The $\mathcal{A}$-subalgebra $U_{\mathcal{A}}(\mathfrak{g})$ of $U_v(\mathfrak{g})$ is defined as the $\mathcal{A}$-subalgebra generated by the elements $\left\{E_i^{(r)}, F_i^{(r)}, K_i, K_i^{-1}\right\}_{i \in \{1,...,n\}, r \in \mathbb{N}}$.

Remark 2.2.4 Note, that we changed the generators. Later we specialize $v \mapsto q$ to some root of unity, such that the quantum numbers sometimes are 0 and therefore the quantum binomial coefficients are not well-defined. In particular, the quantum Serre relations in the definition of $U_v(\mathfrak{g})$ would not be well-defined. However, by Lemma 2.1.1 one can still use the quantum binomial coefficients in formulas.

We rewrite the quantum Serre relations in a way with the divided power generators, such that we do not need the quantum binomial coefficients, e.g. we have:

$$\forall i \neq j \in \{1,...,n\} : \sum_{k=0}^{1-a_{ij}} (-1)^k \left[1 - a_{ij}\right]_{v_i}! \ E_i^{(1-a_{ij}-k)} E_j E_i^{(k)} = 0$$

Now to specialize $v \mapsto q$, for any $q \in \mathbb{C}\backslash\{0\}$ we can consider $\mathbb{C}$ as $\mathcal{A}$-module by letting $v \in \mathcal{A}$ act as multiplication by q.

Definition 2.2.5 Let $\ell \in \mathbb{N}$, ℓ odd, $\ell > h$ and $q \in \mathbb{C}$ a primitive ℓ^{th}-root of unity. (Here h denotes the Coxeter number. For example in type A_n we have $h = n + 1$.)

We define the quantum group $U_q(\mathfrak{g})$ as follows:

$$U_q(\mathfrak{g}) := U_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{C}$$

For $u \in U_{\mathcal{A}}(\mathfrak{g})$, we abbreviate $u \otimes_{\mathcal{A}} 1 \in U_q(\mathfrak{g})$ for simplicity by u and omit the tensor symbol.

Remark 2.2.6 We want these assumptions on ℓ (and therefore q), since for $\ell \leq h$ the fundamental alcove would not include the fundamental weights (see Section 3) and q^2

should also be a primitive ℓ^{th} root of unity (otherwise some calculations later on would change).

Remark 2.2.7 In $U_{\mathcal{A}}(\mathfrak{g})$ and $U_q(\mathfrak{g})$ we have some additional relations to the relations in the definition of $U_v(\mathfrak{g})$ (Definition 2.2.1):

- In $U_{\mathcal{A}}(\mathfrak{g})$, we have the relations:

$$\begin{aligned} F_i^{(r)}F_i^{(s)} &= \frac{F_i^r}{[r]_{v_i}!}\frac{F_i^s}{[s]_{v_i}!} = \frac{F_i^{r+s}}{[r]_{v_i}![s]_{v_i}!} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{v_i} \frac{F_i^{r+s}}{[r+s]_{v_i}!} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{v_i} F_i^{(r+s)}, \\ E_i^{(r)}E_i^{(s)} &= \begin{bmatrix} r+s \\ r \end{bmatrix}_{v_i} E_i^{(r+s)}, \end{aligned}$$

  for $r, s \in \mathbb{N}$, $i \in \{1, .., n\}$.

- The quantum number $[\ell]_v \in \mathcal{A}$ acts as 0 on $\mathbb{C}$:

$$[\ell]_v.1 = \left(v^{\ell-1} + v^{l-3} + ... + v^{1-\ell}\right).1 = \sum_{i=1}^{\ell} q^i = 0,$$

  so in particular we have in $U_q(\mathfrak{g})$ for all $i \in \{1, ..., n\}$, $r \geq \ell$:

$$E_i^r = [r]_q!E_i^{(r)} = 0 = [r]_q!F_i^{(r)} = F_i^r.$$

We have in each of the following cases a triangular composition:

$$U_v(\mathfrak{g}) = U_v^- U_v^0 U_v^+, \qquad U_{\mathcal{A}}(\mathfrak{g}) = U_{\mathcal{A}}^- U_{\mathcal{A}}^0 U_{\mathcal{A}}^+, \qquad U_q(\mathfrak{g}) = U_q^- U_q^0 U_q^+,$$

where:

- the subalgebra U_v^- of $U_v(\mathfrak{g})$ is generated by the set $\{F_i\}_{i\in\{1,...,n\}}$, U_v^0 by the set $\left\{K_i, K_i^{-1}\right\}_{i\in\{1,...,n\}}$ and U_v^+ by the set $\{E_i\}_{i\in\{1,...,n\}}$.

- the subalgebra $U_{\mathcal{A}}^-$ of $U_{\mathcal{A}}(\mathfrak{g})$ is generated as an $\mathcal{A}$-algebra by the set $\left\{F_i^{(r)}\right\}_{i\in\{1,...,n\},r\in\mathbb{N}}$, $U_{\mathcal{A}}^+$ by the set $\left\{E_i^{(r)}\right\}_{i\in\{1,...,n\},r\in\mathbb{N}}$ and $U_{\mathcal{A}}^0$ by the set $\left\{K_i, K_i^{-1}, \tilde{K}_{i,t} \;\middle|\; t \in \mathbb{N}, i \in \{1, ..., n\}\right\}$, where

$$\tilde{K}_{i,t} = \begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v_i^{(1-s)} - K_i^{-1} v_i^{-(1-s)}}{v_i^s - v_i^{-s}},$$

  and similarly for U_q^-, U_q^0 and U_q^+.

Definition 2.2.8 We denote $U_q^{\leq 0} := U_q^- U_q^0$ and $U_q^{\geq 0} := U_q^0 U_q^+$.

Further, $U_q(\mathfrak{g})$ also admits a grading:

Proposition 2.2.9 *The function* $\deg : U_q(\mathfrak{g}) \to X$ *defined on the generators by*

$$\deg\left(E_i^{(r)}\right) = r\alpha_i, \qquad \deg\left(K_i^{\pm 1}\right) = 0, \qquad \deg\left(F_i^{(r)}\right) = -r\alpha_i,$$

for $i \in \{1, ..., n\}$ *and* $r \in \mathbb{N}$ *defines a* X*-grading on* $U_q(\mathfrak{g})$*. It is given by conjugation by the* K_i*'s.*

Proof. Note, that by definition it holds for $i, j \in \{1, ..., n\}$ and $r \in \mathbb{N}$:

$$K_j E_i^{(r)} K_j^{-1} = q_j^{r\langle \alpha_i, \alpha_j^\vee \rangle} E_i^{(r)}, \quad K_j K_i K_j^{-1} = K_i, \quad K_j F_i^{(r)} K_j^{-1} = q_j^{-r\langle \alpha_i, \alpha_j^\vee \rangle} F_i^{(r)}.$$

In particular, the relations in Definition 2.2.1 and the additional relations are homogenous. Hence, the grading given by deg is well-defined. □

So, we may decompose $U_q(\mathfrak{g}) = \bigoplus_{\gamma \in X} U_q(\mathfrak{g})_\gamma$ as a vector space, where $U_q(\mathfrak{g})_\gamma = \{u \in U_q(\mathfrak{g}) \mid \deg(u) = \gamma\}$ is the subspace of degree γ.

Furthermore, there are several Hopf algebra structures on $U_q(\mathfrak{g})$, for instance we use the one given by the following co-multiplication Δ, co-unit ε and antipode S.

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -K_i^{-1} E_i, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_i K_i, \\
\Delta(K_i) &= K_i \otimes K_i, & \varepsilon(K_i) &= 1, & S(K_i) &= K_i^{-1}.
\end{aligned}$$

In particular, we can tensor two $U_q(\mathfrak{g})$-modules and we have an induced action on the dual vector spaces of $U_q(\mathfrak{g})$-modules.

Remark 2.2.10 In fact, $U_q(\mathfrak{g})$-modules form a tensor category. For more information about tensor categories see [14, Section X1.2].

2.3 The Category of Integrable $U_q(\mathfrak{g})$-Modules $\mathcal{C}$

The category of all $U_q(\mathfrak{g})$-modules is too big. We only want to consider modules which are integrable. That means, similar as in the non-quantized case $U(\mathfrak{g})$ in classical Lie theory we want to have e.g. a weight space decomposition, so we can talk about highest weight modules. The following definition can be found in [3, Section 3.6].

Definition 2.3.1 A $U_q(\mathfrak{g})$-module M has a weight space decomposition if

$$M = \bigoplus_{\lambda \in X} M_\lambda,$$

where $M_\lambda = \left\{ m \in M \mid \forall i \in \{1, ..., n\} : \ K_i.m = q^{\langle \lambda, \alpha_i^\vee \rangle} m \right\}$.

A vector in $M_\lambda \setminus \{0\}$ is called a weight vector of weight λ.

Remark 2.3.2 We only consider so-called modules of type 1.

In general, one can have a more general weight space decomposition:

$$M = \bigoplus_{\varepsilon \in \{-1,+1\}^n, \lambda \in X} M_{\varepsilon,\lambda},$$

where

$$M_{\varepsilon,\lambda} = \left\{ m \in M \;\mid\; \forall i \in \{1,...,n\} :\; K_i.m = \varepsilon_i \cdot q^{\langle \lambda, \alpha_i^\vee \rangle} m \right\}.$$

Since the categories of different ε's are equivalent , it is enough to just consider modules of type 1, i.e. $\varepsilon = (1,1,...,1)$ (see e.g. [12, Section 5.2]).

Definition 2.3.3 The category of integrable $U_q(\mathfrak{g})$-modules $\mathcal{C}$ is defined to be the full subcategory of $U_q(\mathfrak{g})$-modules consisting of all finitely generated modules which have a weight space decomposition and where the $F_i^{(r)}$ and $E_i^{(r)}$ act locally as zero for large enough r, i.e. a $U_q(\mathfrak{g})$-module M is by definition an object of $\mathcal{C}$ if:

- M is finitely generated as an $U_q(\mathfrak{g})$-module.
- $M = \bigoplus_{\lambda \in X} M_\lambda$.
- There exists $N \in \mathbb{N}$, such that:

$$\forall r > N,\; i \in \{1,...,n\},\; m \in M :\; E_i^{(r)}.m = 0 = F_i^{(r)}.m.$$

Remark 2.3.4 The first and last condition imply that our objects are finite dimensional: For each $M \in \mathrm{Ob}(\mathcal{C})$ we have $M = U_q(\mathfrak{g}).m_1 + ... + U_q(\mathfrak{g}).m_r$ and each summand $U_q(\mathfrak{g}).m_i$ is finite dimensional.

Lemma 2.3.5 *Let $M \in \mathrm{Ob}(\mathcal{C})$. Then it holds for $i \in \{1,...,n\}$, $r \in \mathbb{N}$, $\lambda \in X$ and $m \in M_\lambda$:*

$$E_i^{(r)}.m \in M_{\lambda + r\alpha_i}, \qquad\qquad F_i^{(r)}.m \in M_{\lambda - r\alpha_i} \tag{2}$$

Proof. We only proof the first equation, the second follows analogously. Let $j \in \{1,...,n\}$.

$$\begin{aligned} K_j.\left(E_i^{(r)}.m\right) &= \left(K_j E_i^{(r)}\right).m = \left(q_j^{r\cdot a_{ji}} E_i^{(r)} K_j\right).m = q_j^{r\cdot\langle \alpha_i, \alpha_j^\vee \rangle} E_i^{(r)}.\left(q_j^{\langle \lambda, \alpha_j^\vee \rangle} m\right) \\ &= q_j^{\langle r\alpha_i + \lambda, \alpha_j^\vee \rangle} \cdot \left(E_i^{(r)}.m\right) \end{aligned}$$

This proves the claim. □

Remark 2.3.6 The category $\mathcal{C}$ is a Krull–Schmidt category, in particular submodules and quotients have a weight space decomposition. The argument (here for the category $\mathcal{O}$) one can find e.g. in [11, Section 1.1 and 1.2].

2.3.1 Dual Weyl Modules

Since our main objective is to understand finite dimensional modules, we go an unusual way of defining the Weyl modules and dual Weyl modules. (The ordinary construction of Verma modules $U_q(\mathfrak{g}) \otimes_{U_q^{\geq 0}} \mathbb{C}_\lambda$ is infinite dimensional and the $F^{(r)}$ would not act locally nilpotently!) We follow therefore the construction in [6, Subsections 2.2 and 2.3].

Definition 2.3.7 For $\lambda \in X^+$, we define the one-dimensional $U_q^{\leq 0}$-module $\mathbb{C}^\lambda$ as follows: As a vector space $\mathbb{C}^\lambda$ is $\mathbb{C}$ with basis vector 1_λ, where U_q^0 acts via

$$\forall i \in \{1, ..., n\}, t \in \mathbb{N}: \qquad K_i.1_\lambda = q_i^{\langle \lambda, \alpha_i^\vee \rangle} \cdot 1_\lambda, \qquad \tilde{K}_{i,t}.1_\lambda = \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle \\ t \end{bmatrix}_{q_i} \cdot 1_\lambda.$$

Further, we let the divided power elements $F_i^{(r)}$ act as 0.

Note: This is the only possible definition, since the $F_i^{(r)}$'s have to change the weight spaces (see Equation (2)).

Definition 2.3.8 Now we recall the induction functor $\mathrm{Ind}_{U_q^{\leq 0}}^{U_q}$ to get a finite dimensional $U_q(\mathfrak{g})$-module:

$$\begin{aligned} \mathrm{Ind}_{U_q^{\leq 0}}^{U_q}: \qquad U_q^{\leq 0}\text{-Mod} &\to U_q(\mathfrak{g})\text{-Mod}, \\ M &\mapsto \mathcal{F}\left(\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), M)\right), \end{aligned}$$

where:

- $U_q(\mathfrak{g})$ is a $U_q^{\leq 0}$-module via (the embedding and) left-multiplication.
- For $M \in U_q^{\leq 0}$-Mod, $\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), M)$ becomes a $U_q(\mathfrak{g})$-module via:
  For $u, \bar{u} \in U_q(\mathfrak{g}), f \in \mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), M)$:
  $$(u.f)(\bar{u}) = f(\bar{u}u)$$
- The functor $\mathcal{F}: \ U_q(\mathfrak{g})$-Mod $\to U_q(\mathfrak{g})$-Mod assigns to a $U_q(\mathfrak{g})$-module M' the submodule:
  $$\mathcal{F}(M') := \left\{ m \in \bigoplus_{\lambda \in X} M'_\lambda \;\middle|\; \forall i \in \{1, ..., n\}, r \gg 0: \ E_i^{(r)}.m = 0 = F_i^{(r)}.m \right\}$$

The dual Weyl module (or co-standard module) of highest weight $\lambda \in X^+$ is defined as:

$$\nabla(\lambda) = \mathrm{Ind}_{U_q^{\leq 0}}^{U_q}(\mathbb{C}^\lambda).$$

Remark 2.3.9 One should note, that the functor $\mathcal{F}$ sends a finitely generated module M' to its maximal submodule which is an object in our category $\mathcal{C}$.

Remark 2.3.10
- For $\lambda \in X^+$ we have:

$$\begin{aligned}\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda) &= \mathrm{Hom}_{U_q^{\leq 0}}(U_q^- U_q^0 U_q^+, \mathbb{C}^\lambda) \\ &\overset{\text{as vector spaces}}{\cong} \left(U_q^+\right)^*.\end{aligned}$$

  Or alternatively, let $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)$, and $u \in U_q(\mathfrak{g})$. Write $u = u_- u_0 u_+$ where $u_- \in U_q^-$, $u_0 \in U_q^0$ and $u_+ \in U_q^+$. It holds

$$\varphi(u) = \varphi(u_- u_0 u_+) = u_- u_0(\varphi(u_+)).$$

  So each element in $\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)$ is determined by its action on U_q^+.

- Let $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)_\mu$ be a weight vector. Then there exists $\gamma \in X$ such that

$$\varphi(u) \neq 0 \quad \Longrightarrow \quad u \in U_q(\mathfrak{g})_\gamma.$$

  Further, it holds $\mu = \lambda - \gamma$.

  Proof. By the decomposition into homogenous subspaces of $U_q(\mathfrak{g})$, there exists $\gamma \in X$ and $u \in U_q(\mathfrak{g})_\gamma$ with $\varphi(u) \neq 0$. Then it holds for $i \in \{1, ..., n\}$:

$$\begin{aligned} q_i^{\langle \mu, \alpha_i^\vee \rangle} \varphi(u) &= (K_i.\varphi)(u) = \varphi(u K_i) = \varphi\left(q_i^{-\langle \gamma, \alpha_i^\vee \rangle} K_i u\right) = q_i^{\langle -\gamma, \alpha_i^\vee \rangle} K_i.(\varphi(u)) \\ &= q_i^{\langle \lambda - \gamma, \alpha_i^\vee \rangle} \varphi(u). \end{aligned}$$

  Hence we have $\mu = \lambda - \gamma$ and this is the only possible value for γ. $\square$

- A direct consequence of the above calculations is $\dim \mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)_\lambda = 1$, since $\varphi \in \mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)_\lambda$ is determined by its action on U_q^+, and the only possible one-dimensional space of $U_q(\mathfrak{g})_0$, which is also in U_q^+, is $\mathbb{C} \cdot 1$.

  In particular, λ is the maximal weight of $\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), \mathbb{C}^\lambda)$, since U_q^+ only has degrees in X^+.

- With further calculations, one can prove $\dim \nabla(\lambda)_\lambda = 1 = \dim \nabla(\lambda)_{-\lambda}$ and the dual Weyl module $\nabla(\lambda)$ is generated by a weight vector in $\nabla(\lambda)_{-\lambda}$.

Also, the dual Weyl modules have the following property (see [4, Corollary 6.2 and Proposition 6.3]):

Proposition 2.3.11 *For $\lambda \in X^+$, the dual Weyl module $\nabla(\lambda)$ contains a unique simple $U_q(\mathfrak{g})$-module $L(\lambda)$. It has highest weight λ.*

Further, any simple module $S \in \mathrm{Ob}(\mathcal{C})$ is isomorphic to some $L(\lambda)$ for some $\lambda \in X^+$.

So we have a complete set of pairwise non-isomorphic simple modules $\left\{L(\lambda) \mid \lambda \in X^+\right\}$. They are characterized by their highest weight.

2.3.2 Weyl Modules

The dual Weyl module of highest weight $\lambda \in X^+$ is, like the name suggests, the dual of the Weyl module of highest weight $\lambda \in X^+$. However, to define the Weyl module we cannot just take the dual space, since that would flip our weights, so we also have to twist it.

We define a homomorphism of algebras $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$,

$$K_i^{\pm 1} \mapsto K_i^{\mp 1}, \qquad E_i^{(r)} \mapsto F_i^{(r)}, \qquad F_i^{(r)} \mapsto E_i^{(r)}$$

for $i \in \{1, ..., n\}, r \in \mathbb{N}$. One can check, that this defines an automorphism. Further, it holds $\omega^2 = id$, so it is self-inverse (i.e. ω is an involution).

For a $U_q(\mathfrak{g})$-module M, we define the twisted module ${}^{\omega}M$ to be the vector space M with the twisted action:

$$\forall u \in U_q(\mathfrak{g}), m \in {}^{\omega}M : \ u.m := \omega(u).m',$$

where m' is the corresponding vector in the untwisted module M.

Definition 2.3.12 For $M \in \mathrm{Ob}(\mathcal{C})$, we define the <u>twisted dual</u> $M^\star$ to be:

$$M^\star := \bigoplus_{\lambda \in X} {}^{\omega}((M_\lambda)^*),$$

so the action is given by: for $\lambda \in X, f \in {}^{\omega}((M_\lambda)^*), m \in M_\lambda, u \in U_q(\mathfrak{g})$:

$$(u.f)(m) = f(\omega(S(u)).m),$$

where the action in the argument of f on the right-hand side is the action of the unchanged, original module M.

One should note, that the weight spaces M_λ are finite dimensional and therefore the dual spaces M_λ^* as well. Also the dual weight space M_λ^* is now of weight $-\lambda$, so as mentioned above, we flip the weights and with the twist with ω we flip it back.

Definition 2.3.13 For $\lambda \in X^+$, we define the <u>Weyl module</u> (or <u>standard module</u>) of weight λ to be the twisted dual of the dual Weyl module of weight λ, i.e.:

$$\Delta(\lambda) := (\nabla(\lambda))^\star.$$

Remark 2.3.14 The Weyl module of weight $\lambda \in X^+$ can also be defined similar to the more common construction of Verma modules:

Let $\mathbb{C}_\lambda$ be the one-dimensional $U_q^{\geq 0}$-module by letting $K_i^{\pm 1}$ act by multiplication with $q^{\pm\langle \lambda, \alpha_i^\vee \rangle}$ (i.e. it is of weight λ) and $E_i^{(r)}$ acts as 0 for $i \in \{1, ..., n\}$, $r \in \mathbb{N}$. Then we may define

$$\Delta(\lambda) = \mathcal{G}\left(U_q(\mathfrak{g}) \otimes_{U_q^{\geq 0}} \mathbb{C}_\lambda\right),$$

where $\mathcal{G}$ is the functor taking a (finitely generated) $U_q(\mathfrak{g})$-module to its maximal finite dimensional quotient.

Remark 2.3.15 The Weyl module $\Delta(\lambda)$ has the following properties:

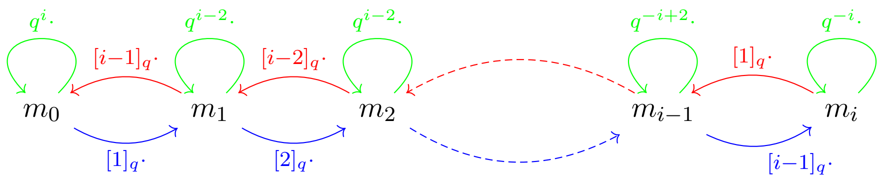
- it is a highest weight module of weight λ.
- $\dim \Delta(\lambda)_\lambda = 1$.
- it has a unique simple head $L(\lambda)$ isomorphic to the unique simple socle of $\nabla(\lambda)$.

Example 2.3.16 Taking the description of the standard modules of $U_q(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_2$ from [1, Definition 2.5], we have for the i^{th} Weyl module $\Delta(i \cdot \omega)$ a basis given by $\{m_0, m_1, ..., m_i\}$ as $\mathbb{C}$-vector space and the actions: for $k \in \{0, ..., i\}, r \in \mathbb{N}$:

$$K.m_k = q^{i-2k} \cdot m_k, \quad E^{(r)}.m_k = \begin{bmatrix} i-k+r \\ r \end{bmatrix}_q \cdot m_{k-r}, \quad F^{(r)}.m_k = \begin{bmatrix} k+r \\ r \end{bmatrix}_q \cdot m_{k+r},$$

where we set $m_j = 0$ for $j < 0$ and $j > i$.

Graphically the modules have the form:



where the red arrows to the left show the action of E, the blue arrows the right the action of F and the green arrows the action of K. (There are more arrows to consider since we left out the higher divided power generators!)

Back to our general Weyl modules $\Delta(\lambda)$, they have further the following universal property:

Proposition 2.3.17 (Universal property of Weyl modules) *Let $M \in \text{Ob}(\mathcal{C})$ be a highest weight module of weight $\lambda \in X^+$, i.e. there is a vector $m \in M \setminus \{0\}$ with the properties $M = U_q(\mathfrak{g}).m$, $E_i^{(r)}.m = 0$ for all $i \in \{1, ..., n\}, r \in \mathbb{N}$ and $m \in M_\lambda$. Then there exists a (up to scalars) unique non-zero morphism $\eta_\lambda : \Delta(\lambda) \to M$. Further, the morphism η_λ is surjective.*

Proof. By Remark 2.3.14 the Weyl module $\Delta(\lambda)$ is the maximal quotient of the Verma module $V(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q^{\geq 0}} \mathbb{C}_\lambda$ in the category $\mathcal{C}$. Let $1_\lambda \in \Delta(\lambda)_\lambda$ be a weight vector. Then it holds $U_q(\mathfrak{g}).1_\lambda = \Delta(\lambda)$. We define the morphism $\eta_\lambda : \Delta(\lambda) \to M$ by $1_\lambda \mapsto m$. This is a $U_q(\mathfrak{g})$-morphism, since the Weyl module $\Delta(\lambda)$ is the maximal quotient of the Verma module $V(\lambda)$: By the universal property of Verma modules M is a quotient of $V(\lambda)$. Since $\Delta(\lambda)$ is the maximal quotient in the category $\mathcal{C}$, the morphism η_λ must be surjective.

The morphism η_λ is unique up to scalar, since $\dim \Delta(\lambda)_\lambda = 1$.

This finishes the proof. $\square$

Corollary 2.3.18 *For $\lambda \in X^+$, $M \in \mathrm{Ob}(\mathcal{C})$ it holds:*

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(\Delta(\lambda), M) \quad \xleftrightarrow{1:1} \quad \left\{ m \in M_\lambda \mid \forall i \in \{1, ..., n\}, r \in \mathbb{N} : E_i^{(r)}.m \right\}.$$

So we have for $\lambda \in X^+$ a unique (up to scalars) morphism $c^\lambda : \Delta(\lambda) \to \nabla(\lambda)$, which in particular sends the simple head of $\Delta(\lambda)$ to the simple socle of $\nabla(\lambda)$:

$$c^\lambda : \ \Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda).$$

Remark 2.3.19 Another way to see this is via the (q-version) of the Frobenius reciprocity, which is in a sense the generalized dual version of the universal property of the Weyl modules (see [4, Section 2.12]):

For a module $M \in \mathrm{Ob}(\mathcal{C})$ and $\lambda \in X^+$ it holds

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M, \nabla(\lambda)) \cong \mathrm{Hom}_{U_q^{\leq 0}}(M, \mathbb{C}^\lambda).$$

In particular:

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(\Delta(\lambda), \nabla(\lambda)) \cong \mathrm{Hom}_{U_q^{\leq 0}}(\Delta(\lambda), \mathbb{C}^\lambda),$$

and by using $\Delta(\lambda)_\lambda \cong \mathbb{C}$, we get $\dim \mathrm{Hom}_{U_q(\mathfrak{g})}(\Delta(\lambda), \nabla(\lambda)) = 1$.

Remark 2.3.20 In the following, we will often utilize the $\mathrm{Ext}^i_{\mathcal{C}}$-functor, but one should be careful. Our category $\mathcal{C}$ does not have enough injectives, so one needs to go to its injective completion. One can find the definition of it in [13, Section 6.1] (here it is called indization) and in [13, Section 15.3] it is shown, that the extension functor (in e.g. our setup) is the same.

There are further properties of Weyl und dual Weyl modules. For example instead of just looking at the morphisms between those, one can consider the extension functor (see [6, Theorem 3.1]):

Theorem 2.3.21 (Ext-vanishing) *Let $\lambda, \mu \in X^+$. It holds:*

$$\mathrm{Ext}^i_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) \cong \begin{cases} \mathbb{C}c^\lambda, & \text{if } i = 0, \lambda = \mu, \\ 0, & \text{else.} \end{cases}$$

An important fact about the category $\mathcal{C}$ is the following (and without which we cannot do the theory of tilting modules):

Theorem 2.3.22 *The category of integrable $U_q(\mathfrak{g})$-modules $\mathcal{C}$ is a highest weight category.*

For details about highest weight categories see e.g. [9, Appendix].

2.3.3 Tilting Modules

Definition 2.3.23 (i) An object $M \in \mathrm{Ob}(\mathcal{C})$ has a Weyl filtration (or Δ-filtration), if there exists a flag of submodules

$$0 = M_0 \subset M_1 \subset ... \subset M_{r-1} \subset M_r = M$$

for some $r \in \mathbb{N}$, such that

$$\forall i \in \{1, ..., r\} : \; M_i/M_{i-1} \cong \Delta(\lambda_i)$$

for some $\lambda_i \in X^+$.

(ii) Similarly, an object $M \in \mathrm{Ob}(\mathcal{C})$ has a dual Weyl filtration (or ∇-filtration), if there exists a flag of submodules

$$0 = M_0 \subset M_1 \subset ... \subset M_{r-1} \subset M_r = M$$

for some $r \in \mathbb{N}$, such that

$$\forall i \in \{1, ..., r\} : \; M_i/M_{i-1} \cong \nabla(\lambda_i)$$

for some $\lambda_i \in X^+$.

(iii) An object $M \in \mathrm{Ob}(\mathcal{C})$ is called a tilting module, if M has a Weyl filtration as well as a dual Weyl filtration.

Remark 2.3.24 Note that by taking the twisted dual one takes a Weyl filtration to a dual Weyl filtration and vice versa. In particular we have for a module $M \in \mathrm{Ob}(\mathcal{C})$:

$$M \text{ has a } \Delta\text{-filtration} \quad \Longleftrightarrow \quad M^\star \text{ has a } \nabla\text{-filtration}$$

Further, we can say more about the appearing weights λ_i in the filtrations, which is a corollary of the Ext-vanishing theorem (Theorem 2.3.21) (see [6, Corollary 3.4]):

Corollary 2.3.25 *Let $M \in \mathrm{Ob}(\mathcal{C})$ with a Δ-filtation, $N \in \mathrm{Ob}(\mathcal{C})$ with a ∇-filtration and $\lambda \in X^+$. It holds:*

(i) $\dim \mathrm{Hom}_{U_q(\mathfrak{g})}(M, \nabla(\lambda)) = |\{i \mid \lambda_i = \lambda\}|$,

(ii) $\dim \operatorname{Hom}_{U_q(\mathfrak{g})}(\Delta(\lambda), N) = |\{i \mid \lambda_i = \lambda\}|$,

where the λ_i are from the respective filtrations ($M_i/M_{i-1} \cong \Delta(\lambda_i)$ respectively $N_i/N_{i-1} \cong \nabla(\lambda_i)$). In particular, the multiplicities with the respect to a filtration is independent of the choice of the filtration.

Proof. We will show (i), and (ii) follows by duality.

Let $r \in \mathbb{N}$ be the length of the Δ-filtration of M. We will use induction on r.

So if $r = 1$, then we have $M = \Delta(\mu)$ for some $\mu \in X^+$ and

$$\dim(\operatorname{Hom}_{U_q(\mathfrak{g})}(M, \nabla(\lambda))) = (M : \Delta(\lambda))$$

is given by the Ext-vanishing theorem.

Now let $r > 1$. Then we have a short exact sequence

$$0 \longrightarrow M_{r-1} \hookrightarrow M \twoheadrightarrow \Delta(\lambda_r) \longrightarrow 0$$

and we know that $\dim(\operatorname{Hom}_{U_q(\mathfrak{g})}(M, \nabla(\lambda)))$ and $(M : \Delta(\lambda))$ are additive regarding short exact sequences, which gives us the claim by applying induction to M_{r-1}. □

Also we have another consequence of the Ext-vanishing theorem, which gives us a criteria to identify modules which have a Δ- or ∇-filtration. In particular we can identify tilting modules using the extension functor (see [6, Proposition 3.5]):

Proposition 2.3.26 (Ext-criteria) *Let $M, N \in \operatorname{Ob}(\mathcal{C})$. Then the following are equivalent:*

(i) The $U_q(\mathfrak{g})$-module M has a Δ-filtration (respectively N has a ∇-filtration).

(ii) It holds: $\forall \lambda \in X^+, i > 0 : \operatorname{Ext}^i_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ (respectively $\operatorname{Ext}^i_{\mathcal{C}}(\Delta(\lambda), N) = 0$).

(iii) It holds: $\forall \lambda \in X^+$: $\operatorname{Ext}^1_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ (respectively $\operatorname{Ext}^1_{\mathcal{C}}(\Delta(\lambda), N) = 0$).

So we have an alternative characterization for tilting modules:

Corollary 2.3.27 *A $U_q(\mathfrak{g})$-module T is a tilting module, if and only if:*

$$\forall \lambda \in X^+ : \qquad \operatorname{Ext}^1_{\mathcal{C}}(T, \nabla(\lambda)) = 0 = \operatorname{Ext}^1_{\mathcal{C}}(\Delta(\lambda), T)$$

With this characterization we can easily deduce:

Corollary 2.3.28 *Given non-zero modules $D_1, D_2, D \in \operatorname{Ob}(\mathcal{C})$ such that $D = D_1 \oplus D_2$. Then it holds:*

D is a tilting module if and only if D_1 and D_2 are tilting modules.

Proof. For $\lambda \in X^+$ we have:

$$\mathrm{Ext}^1_{\mathcal{C}}(D, \nabla(\lambda)) = \mathrm{Ext}^1_{\mathcal{C}}(D_1 \oplus D_2, \nabla(\lambda)) = \mathrm{Ext}^1_{\mathcal{C}}(D_1, \nabla(\lambda)) \oplus \mathrm{Ext}^1_{\mathcal{C}}(D_2, \nabla(\lambda))$$

and

$$\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), D) = \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), D_1 \oplus D_2) = \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), D_1) \oplus \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), D_2).$$

So if one side is zero, then the other side is zero, too. □

There are some additional facts about highest weight categories. In particular a version of Ringel's theorem (see [6, Proposition 3.11]):

Theorem 2.3.29 *There is a complete set of indecomposable, non-isomorphic tilting modules* $\left\{T(\lambda) \mid \lambda \in X^+\right\}$ *such that for* $\lambda \in X^+$*:*

(i) every weight μ *of* $T(\lambda)$ *satisfies* $\mu \leq \lambda$*,*

(ii) $\dim T(\lambda)_\lambda = 1$*,*

and every tilting module $D \in \mathrm{Ob}(\mathcal{C})$ *can be written as a direct sum of these, i.e.:*

$$\forall \lambda \in X^+ : \ \exists a^D_\lambda \in \mathbb{Z}_{\geq 0} : \quad D = \bigoplus_{\lambda \in X^+} T(\lambda)^{a^D_\lambda}$$

Proof. First we construct the indecomposable tilting modules $T(\lambda)$. So let $\lambda \in X^+$ be fixed.

If $\Delta(\lambda)$ is a tilting module, we set $T(\lambda) := \Delta(\lambda)$.

Otherwise $\Delta(\lambda)$ has no ∇-filtration, so by Corollary 2.3.27 there exists $\mu \in X^+$ with the property $\dim(\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), \Delta(\lambda))) \neq 0$. So let $\mu_2 \in X^+$ be minimal with that property and set $m_2 = \dim(\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu_2), \Delta(\lambda))) \neq 0$.

Note that we have $\mu_2 < \lambda$, since $m_2 \neq 0$ implies $\mathrm{Hom}(\Delta(\mu_2), \Delta(\lambda)) \neq 0$ and therefore by Corollary 2.3.18 the μ_2 weight space in $\Delta(\lambda)$ is not 0 (and all weights μ of $\Delta(\lambda)$ have the property $\mu \leq \lambda$).

By the properties of the Ext^1-functor we get a non-splitting extension:

$$0 \longrightarrow \Delta(\lambda) \hookrightarrow M_2 \twoheadrightarrow \Delta(\mu_2)^{m_2} \longrightarrow 0.$$

If M_2 is a tilting module, we set $T(\lambda) = M_2$. Otherwise we choose (again by Corollary 2.3.27) $\mu_3 \in X^+$ minimal with the property $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu_3), M_2) = m_3 \neq 0$ (since, by construction, M_2 has a Δ-filtration). Again we have $\mu_3 < \lambda$ and $\mu_3 < \mu_2$. We get a non-splitting extension:

$$0 \longrightarrow M_2 \hookrightarrow M_3 \twoheadrightarrow \Delta(\mu_3)^{m_3} \longrightarrow 0.$$

If M_3 is a tilting module, we set $T(\lambda) := M_3$. Otherwise we continue in this fashion and get a filtration of the form:

$$0 = M_0 \subset \Delta(\lambda) =: M_1 \subset M_2 \subset M_3 \subset ...$$

with the property $M_{k+1}/M_k \cong \Delta(\mu_{k+1})^{m_{k+1}}$ and $\mu_{k+1} < \mu_k < ... < \mu_2 < \lambda$ for $k \in \{0,1,...\}$. Since there are only finitely many weights μ in X^+ with the property $\mu < \lambda$, this process stops at some point, say at M_r. So we have a ∇-filtration (since we cannot find $\mu \in X^+$ with $\dim(\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), M_r)) \neq 0$) and by construction we have a Δ-filtration, i.e. M_r is a tilting module. Also by construction we have $\dim(M_r)_\lambda = 1$ and it is indecomposable. So we can set $T(\lambda) = M_r$.

Now suppose $D \in \mathrm{Ob}(\mathcal{C})$ is an indecomposable tilting module. We want to show $D \cong T(\lambda)$ for some $\lambda \in X^+$.

Let λ be a maximal weight in D. Then we get by the universal property of $\Delta(\lambda)$ a non-zero $U_q(\mathfrak{g})$-homomorphism $\varphi : \Delta(\lambda) \to D$. Also by duality we get a $U_q(\mathfrak{g})$-homomorphism $\psi : D \to \nabla(\lambda)$ such that $\psi \circ \varphi \neq 0$.

By definition of the indecomposable tilting module $T(\lambda)$ we have an inclusion $\iota : \Delta(\lambda) \hookrightarrow T(\lambda)$ and a surjection $\pi : T(\lambda) \twoheadrightarrow \nabla(\lambda)$. Consider the diagram:

$$\begin{tikzcd}
\Delta(\lambda) \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & T(\lambda) \arrow[r, two heads, "\pi"] & \nabla(\lambda) \\
& D \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Both $\psi \circ \varphi$ and $\pi \circ \iota$ are non-zero, so we can scale one by a non-zero scalar, such that the diagram commutes.

Further, since D is a tilting module, it holds:

$$\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), D) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(D, \nabla(\lambda)).$$

In particular, we get:

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{coker}(\iota), D) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(D, \ker(\pi)).$$

So φ extends to an $U_q(\mathfrak{g})$-homomorphism $\bar{\varphi} : T(\lambda) \to D$ and ψ factors through D via $\bar{\psi} : D \to T(\lambda)$. Further, $\bar{\psi} \circ \bar{\varphi}$ is an isomorphism on $T(\lambda)_\lambda$, hence it is an isomorphism on the whole of $T(\lambda)$. In particular, $T(\lambda)$ embeds into D and therefore is a summand of D. Since we assumed D to be indecomposable, it follows $D \cong T(\lambda)$.

By the Krull–Schmidt property, it follows that we can decompose every tilting module into a direct sum of the constructed indecomposable tilting modules $T(\lambda)$. This shows the theorem. □

Remark 2.3.30 The indecomposable tilting modules $T(\lambda)$ have the property, that they are selfdual, since taking the twisted dual sends a Δ-filtration to a ∇-filtration and a ∇-filtration to a Δ-filtration. So the twisted dual $T(\lambda)^\star$ is still a tilting module of the same dimension and highest weight λ, which is, by the characterization above, isomorphic to $T(\lambda)$.

The following fact about tilting modules is important, but since the proof is quite lengthy, we will only state it. For the proof in type A see [6, Proposition 3.10]. In the general case, the proof relies on Lusztig's crystal bases (see [16]).

Theorem 2.3.31 *For two tilting modules D, D' it holds: $D \otimes D'$ is a tilting module.*

Remark 2.3.32 If one only considers tilting modules, one can easily deduce that the tensor product is symmetric. We know that the character of a tilting module D $\operatorname{ch} D = \sum_{\lambda \in X} \dim D_\lambda \cdot e^\lambda \in \mathbb{Z}(X)$ is multiplicative, i.e. $\operatorname{ch}(M \otimes N) = \operatorname{ch} M \cdot \operatorname{ch} N$, and by Theorem 2.3.29 we can deduce the decomposition of tilting modules into indecomposable tilting modules from their character (starting by a maximal weight λ we can "subtract" $T(\lambda)$ and get a tilting module with a smaller dimension and so on). In particular we have $a^\nu_{\lambda,\mu} = a^\nu_{\mu,\lambda}$, where $a^\nu_{\lambda,\mu}$ is the multiplicity of $T(\nu)$ in $T(\lambda) \otimes T(\mu)$.

2.3.4 The Grothendieck Ring $\mathcal{R}$

As in [3, Subsection 3.19], we now consider the Grothendieck ring of the category of integrable $U_q(\mathfrak{g})$-modules $\mathcal{C}$.

Definition 2.3.33 The Grothendieck group $\mathcal{R} = \mathcal{K}_0(\mathcal{C})$ of the category $\mathcal{C}$ is generated by the set $\{[M] \mid M \in \operatorname{Ob}(\mathcal{C})\}$ and the relations:

For every short exact sequence $0 \to M \hookrightarrow M' \twoheadrightarrow M'' \to 0$ in $\mathcal{C}$ we have the relation $[M'] = [M] + [M'']$.

We can define a multiplication on $\mathcal{R}$ turning it into a ring via

$$[M] \cdot [N] := [M \otimes N] \qquad \text{for } M, N \in \operatorname{Ob}(\mathcal{C}).$$

Then we have for $\lambda, \mu \in X^+$ by Theorem 2.3.31 and Remark 2.3.32:

$$\forall \nu \in X^+ \; \exists a^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0} :$$

$$[T(\lambda)] \cdot [T(\mu)] = [T(\lambda) \otimes T(\mu)] = \left[\bigoplus_{\nu \in X^+} T(\nu)^{a^\nu_{\lambda,\mu}} \right] = \sum_{\nu \in X^+} a^\nu_{\lambda,\mu} \, [T(\nu)]$$

So it follows directly:

Corollary 2.3.34 *The subset $\mathcal{R}^t = \operatorname{span}_{\mathbb{Z}} \left\{ [T(\lambda)] \mid \lambda \in X^+ \right\}$ of the Grothendieck ring $\mathcal{R}$ forms a commutative subring.*

One should also note an important fact: The Grothendieck ring $\mathcal{R}$ has the following $\mathbb{Z}$-bases:

- $\left\{ [L(\lambda)] \mid \lambda \in X^+ \right\}$
- $\left\{ [\Delta(\lambda)] \mid \lambda \in X^+ \right\}$

The first follows by the definition of the Grothendieck ring, the second by the fact, that our Weyl modules $\Delta(\lambda)$ have unique simple heads $L(\lambda)$ and a finite filtration of simple modules with smaller highest weights.

3 The Category $\mathcal{C}_\ell^-$

We only want to consider the tilting modules which have summands only of the form $T(\lambda)$ with $\lambda \in \Lambda_\ell$, where $\Lambda_\ell = \left\{\lambda \in X^+ \mid \forall \alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle < \ell \right\}$. Note that Λ_ℓ consists of the integrable weights in the fundamental alcove C of the affine Weyl group of level $k = \ell - h$, where h is the Coxeter number.

Example 3.0.1 In the (for us) most important case $\mathfrak{g} = \mathfrak{sl}_3$ it holds:

$$X^+ = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}_{\geq 0}\}.$$

Then we have for $\ell \in \mathbb{N}$:

$$\begin{aligned} \Lambda_\ell &= \left\{\lambda \in X^+ \mid \forall \alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle < \ell \right\} \\ &= \left\{\lambda \in X^+ \mid \forall \alpha \in \Phi^+ : \langle \lambda, \alpha^\vee \rangle \leq k \right\} \\ &= \left\{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}_{\geq 0} : 0 \leq \langle m_1\omega_1 + m_2\omega_2, \alpha_1^\vee + \alpha_2^\vee \rangle \leq k \right\} \\ &= \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}_{\geq 0} : 0 \leq m_1 + m_2 \leq k\}. \end{aligned}$$

To this end, we have to change our category, otherwise our tensor products would have too many summands (i.e. they could include summands of the form $T(\lambda)$ with $\lambda \in X^+ \backslash \Lambda_\ell$). The idea is to quotient out the so-called negligible modules, which are given by the following definition.

Definition 3.0.2 A $U_q(\mathfrak{g})$-module $M \in \mathrm{Ob}(\mathcal{C})$ is called negligible, if it may be written in the form

$$M = \bigoplus_{\lambda \in X^+ \backslash \Lambda_\ell} T(\lambda)^{a_\lambda^M}.$$

In words, a tilting module M is negligible if $a_\lambda^M = 0$ for all $\lambda \in \Lambda_\ell$ (using the notation in Theorem 2.3.29).

First, we recall what it means for a weight to be ℓ-regular or ℓ-singular (see [2, Section 3]):

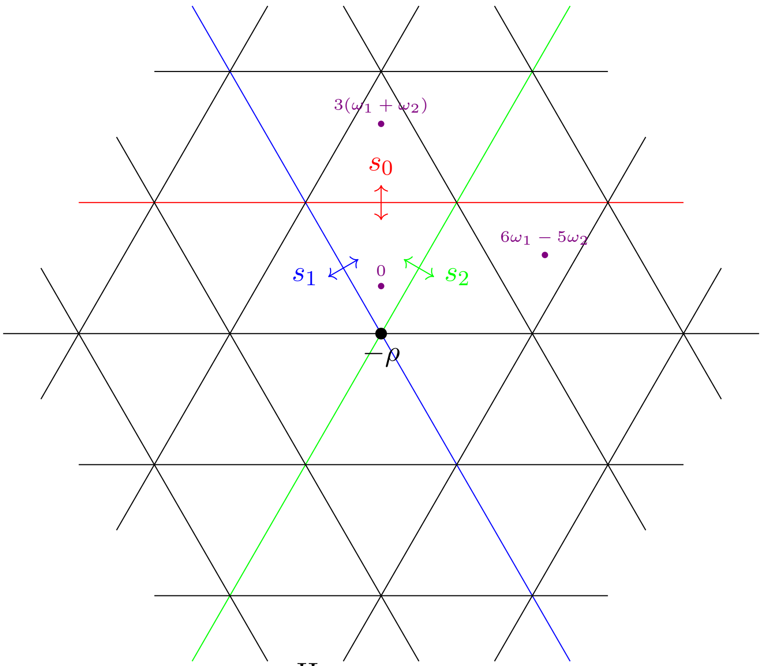
Definition 3.0.3 A weight $\lambda \in X$ is called ℓ-singular, if there exists a root $\alpha \in \Phi$ such that $\langle \lambda + \rho, \alpha^\vee \rangle$ is divisible by ℓ. Otherwise λ is called ℓ-regular.

So in particular, the weights on the border of Λ_ℓ are ℓ-singular and the weights in Λ_ℓ are ℓ-regular.

3.1 The Affine Weyl Group $\mathcal{W}_\ell$ of $\mathfrak{sl}_3$

The affine Weyl group $\mathcal{W}_\ell$ is generated by the simple reflection $\{s_i \mid i \in \{0, 1, ..., n\}\}$. Here we have one new generator s_0 in comparison to the ordinary Weyl group $\mathcal{W}$, which reflects on the hyperplane $H_{k+1,\theta} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \theta^\vee \rangle = k+1\}$, where θ is the highest weight of the adjoint representation of $\mathfrak{g}$ or alternatively the highest root. In the case of $\mathfrak{g} = \mathfrak{sl}_3$, we just have $\theta = \rho = \alpha_1 + \alpha_2$.

This new reflections makes the affine Weyl group $\mathcal{W}_\ell$ infinite. So instead having six alcoves, we now have infinitely many:



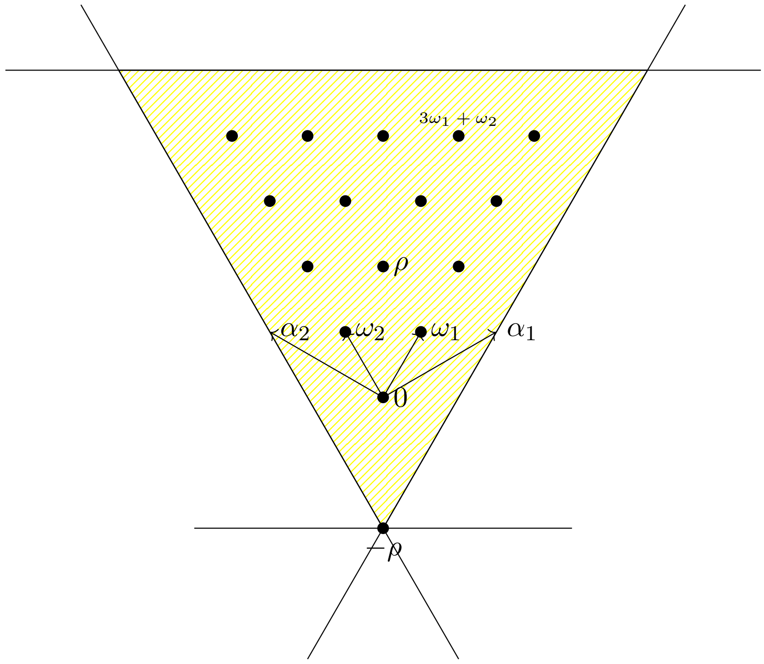
Here you see,
how the simple reflections act under the ordinary dot-action
and some explicit weights for $k = 2$ and $\ell = 5$.

Remark 3.1.1 It is a known fact, that the affine Weyl group $\mathcal{W}_\ell$ acts simply transitive on the alcoves. In particular we have a bijection

$$\{\text{alcoves}\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{elements in } \mathcal{W}_\ell\}$$

Note that the alcove corresponding to the identity in the Weyl group $\mathcal{W}_\ell$ is the fundamental alcove C.

For $k = 4$ one can depict the fundamental alcove (under the dot-action) C, where the dots (except $-\rho$) represent elements in Λ_ℓ:



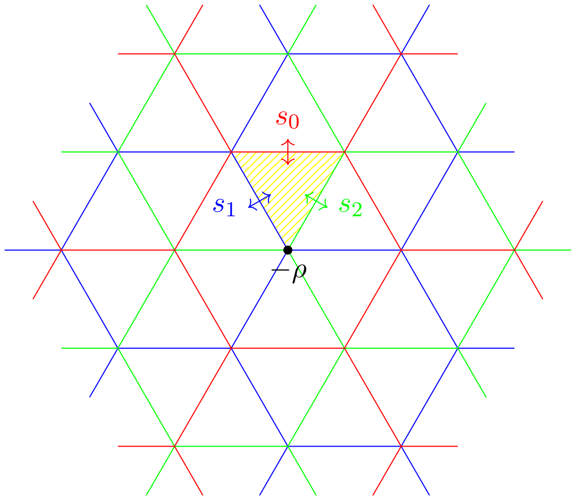
Remark 3.1.2 One should note, that a weight is ℓ-singular if and only if it lies on an alcove wall. This follows easily, by visualizing the alcove walls by shifts by multiples of ℓ in the normalized orthogonal (i.e. α_1, α_2 or $\alpha_1 + \alpha_2$) direction to the original alcove walls of the non-affine Weyl group $\mathcal{W}$.

So in particular, the set of all ℓ-singular weights is the union of fixed points under the dot-action of the affine Weyl group $\mathcal{W}_\ell$.

3.1.1 The Right Action

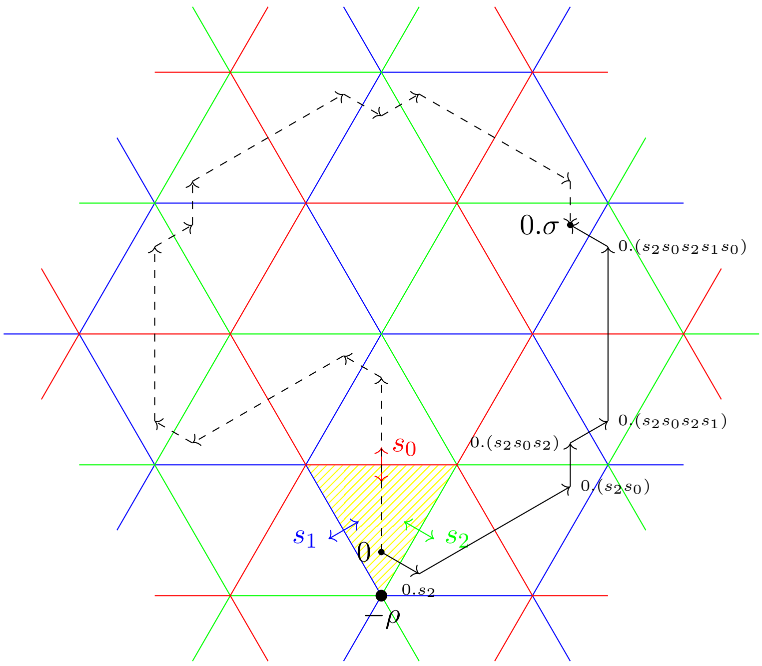
Now we want to introduce a right action of the affine Weyl group $\mathcal{W}_\ell$. Instead of acting by a reflection on a whole hyperplane, the action depends on in which alcove the element the simple reflection acts on is located.

Each alcove has three walls. The idea is the following: depending on which simple reflection acts the element will be reflected on one of the walls into a neighbored alcove. In the fundamental alcove the actions of the simple reflections look the same, outside you color the walls by reflecting the already colored ones on the hyperplanes, which would look like this:



So starting in the fundamental alcove (shaded in yellow above), we get paths where crossing over one of the walls is the action of a simple reflection. Note that we can read off the length of an element of the Weyl group from the alcove picture: Given an element $\tau \in \mathcal{W}_\ell$, the length $l(\tau)$ of τ is the minimal number of alcove walls a path crosses if it starts in the fundamental alcove (e.g. 0) and ends at $(0.\tau)$.

Example 3.1.3 Here we have the actions for the words $(s_2 s_0 s_2 s_1 s_0 s_1)$ (solid path) and $(s_0 s_1 s_0 s_2 s_0 s_2 s_1 s_0 s_1 s_2 s_0 s_2)$ (dashed path) applied to 0. Note that we defined this as a right action, so we read the action from left to right, i.e. $s_i s_j$ acts first by s_i and then by s_j.



One can easily check, that they are the same element σ and have the reduced expression $(s_0s_2s_1s_0)$, which would be the shortest path to the point $0.\sigma$:

$$\begin{aligned} s_2s_0s_2s_1s_0s_1 &= s_0s_2s_0s_0s_1s_0 \\ &= s_0s_2s_1s_0, \\ s_0s_1s_0s_2s_0s_2s_1s_0s_1s_2s_0s_2 &= s_0s_1s_0s_0s_2s_0s_0s_1s_0s_0s_2s_0 \\ &= s_0s_1s_2s_1s_2s_0 \\ &= s_0s_2s_1s_2s_2s_0 \\ &= s_0s_2s_1s_0. \end{aligned}$$

So we have for the length $l(\sigma) = 4$, which is also the minimal number of alcove walls one has to cross from 0 to $0.\sigma$.

Remark 3.1.4 Note that the $\mathcal{W}_\ell$-orbits coincide for the right and left action.

Now we define the action on the Grothendieck group $\mathcal{R}$ of our category $\mathcal{C}$. By definition, the affine Weyl group $\mathcal{W}_\ell$ acts on the set of alcoves, and hence we cannot restrict the action to the set of integral weights. Thus, we need to introduce new elements in $\mathcal{R}$, where we will use the induction functor $\mathrm{Ind}_{U_q^{\leq 0}}^{U_q} : U_q^{\leq 0}\text{-Mod} \to U_q(\mathfrak{g})\text{-Mod}$ (see Definition 2.3.8).

Note, that the induction functor $\mathrm{Ind}_{U_q^{\leq 0}}^{U_q}$ is left exact but not right exact, since $\mathrm{Hom}_{U_q^{\leq 0}}(U_q(\mathfrak{g}), -)$ is left exact but not right exact. Therefore, we may use its right derivation.

Definition 3.1.5 For $\lambda \in X$ we set

$$\chi(\lambda) := \sum_{k\geq 0} (-1)^k [H^k(\mathbb{C}^\lambda)],$$

where H^k is the k^{th} right derived functor of $H^0 = \mathrm{Ind}_{U_q^{\leq 0}}^{U_q}$ and for $M \in \mathrm{Ob}(\mathcal{C})$ [M] denotes again the equivalence class of M in $\mathcal{R}$ (see Definition 2.3.33).

For us, we have the equality for $\lambda \in X^+$: $\chi(\lambda) = [\Delta(\lambda)]$.

Definition 3.1.6 Now, for all $\lambda \in X$, $\tau \in \mathcal{W}_\ell$ we define the right action of the affine Weyl group $\mathcal{W}_\ell$ by

$$\chi(\lambda).\tau := \chi(\lambda.\tau),$$

where on the right-hand side we have the just in this section explained right action of the affine Weyl group $\mathcal{W}_\ell$ on the weight lattice X.

In particular, for $\tau \in \mathcal{W}_\ell$, $\lambda \in X^+$ with $(\lambda.\tau) \in X^+$ it holds

$$[\Delta(\lambda)].\tau = [\Delta(\lambda.\tau)].$$

3.1.2 The Linkage Principle

In Lie theory, the linkage principle gives a condition for the appearing weights in the Jordan-Hölder composition series of a highest weight module. Namely, that if M is a highest weight module of weight $\lambda \in X$, then the only weights occurring as highest weight of composition factors are linked to λ, i.e. they are in the $\mathcal{W}$-orbit of λ under the dot-action (see [11, Section 1.8]).

The category of integrable modules $\mathcal{C}$ also has a form of the linkage principle for quantum groups at roots of unity (see [2, Equation 1.2] and [4, Theorem 8.1]):

Definition 3.1.7 Let $\lambda, \mu \in X$. Then we say μ is <u>strongly linked</u> to λ, if there exist $\lambda_1, \lambda_2, ..., \lambda_r \in X$, $\beta_1, ..., \beta_{r-1} \in \Phi^+$ and $m_1, ..., m_{r-1} \in \mathbb{Z}_{\geq 0}$ such that

$$\mu = \lambda_1 \leq s_{\beta_1 \bullet}\lambda_1 + m_1 \ell \beta_1 = \lambda_2 \leq ... \leq s_{\beta_{r-1}\bullet}\lambda_{r-1} + m_{r-1}\ell\beta_{r-1} = \lambda_r = \lambda,$$

where s_{β_j} is the Weyl group element given by $s_{\beta_j} : \mathfrak{h}^* \to \mathfrak{h}^*$, $\lambda \mapsto \lambda - \langle \lambda, \beta_j^\vee \rangle \beta_j$.

Proposition 3.1.8 *If $\lambda, \mu \in X^+$ and $L(\mu)$ is a composition factor of $\nabla(\lambda)$, then μ is strongly linked to λ.*

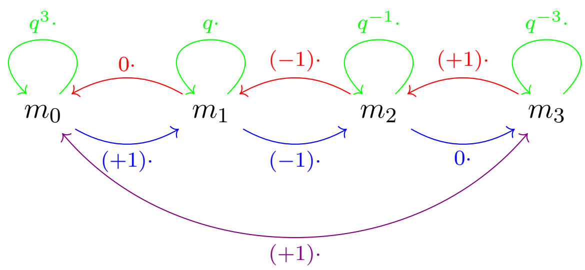
Remark 3.1.9 Note, that there exists an element of the affine Weyl group $\tau \in \mathcal{W}_\ell$ with $\tau_\bullet \lambda_i = s_{\beta_i \bullet}\lambda_i + m_i \ell \beta_i$.

This gives us the nice consequence:

Proposition 3.1.10 *Let $\lambda, \mu \in X^+$. If $L(\lambda)$ and $L(\mu)$ both are composition factors of an indecomposable module $M \in \mathcal{C}$, then $\lambda \in \mathcal{W}_{\ell\bullet}\mu$, i.e. they are in the same $\mathcal{W}_\ell$-orbit under the dot-action.*

Example 3.1.11

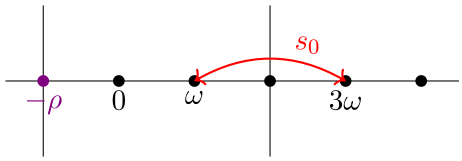
- The smallest non-trivial example is for $\mathfrak{g} = \mathfrak{sl}_2$ and $\ell = 3$. One can also find this example in [6, Example 2.13]. As in Example 2.3.16, the module $\Delta(3\omega)$ can be visualized by



where the red arrows to the left show the action of E, the blue arrows to the right the action of F, the green arrows the action of K and the violet arrow to the left the action of $E^{(3)}$, respectively to the right of $F^{(3)}$.

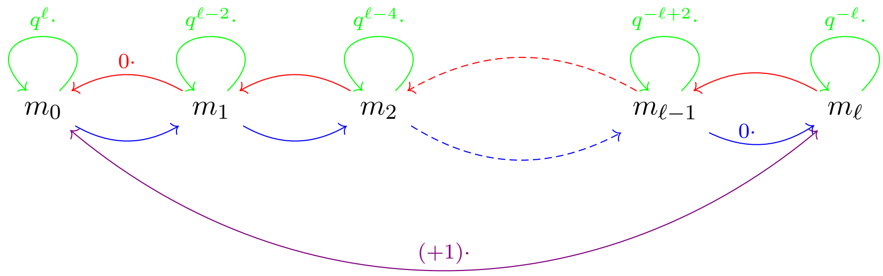
From this picture, we can see that the subspace spanned by m_1 and m_2 are stable under the action of $U_q(\mathfrak{sl}_2)$. This submodule is isomorphic to the simple module $L(\omega)$. In particular $\Delta(3\omega)$ has the composition series $0 \subset L(\omega) \subset \Delta(3\omega)$ and the composition factors $L(\omega)$ and $L(3\omega)$.

Note, that we have $s_{0\bullet}\omega = 3\omega$ for $\ell = 3$:



So like in the above proposition ω is in the $\mathcal{W}_\ell$-dot-orbit of 3ω.

- One can even generalize this for a general ℓ, which satisfies our assumptions. Namely, $\Delta(\ell\omega)$ has the form



Note, that we have for $r \in \{1, ..., \ell - 1\}$:

$$E^{(r)}.m_r = \begin{bmatrix} \ell - r + r \\ r \end{bmatrix}_q \cdot m_0 = \frac{[\ell]_q!}{[r]_q![\ell - r]_q!} \cdot m_0 = 0,$$
$$F^{(r)}.m_{(\ell-r)} = \begin{bmatrix} \ell - r + r \\ r \end{bmatrix}_q \cdot m_\ell = 0.$$

In particular, the subspace spanned by $\{m_1, m_2, ..., m_{\ell-1}\}$ is stable under the action of $U_q(\mathfrak{sl}_2)$. Further, one can easily check that this submodule of $\Delta(\ell\omega)$ is isomorphic to the simple module $L((\ell - 2)\omega)$. So the module $\Delta(\ell\omega)$ has composition factors $L(\ell\omega)$ and $L((\ell - 2)\omega)$. The weights satisfy $s_{0\bullet}\ell\omega = (\ell - 2)\omega$, so they are in the same $\mathcal{W}_\ell$-orbit under the dot-action.

Also, the above proposition gives us the following consequence for weights in the fundamental alcove:

Corollary 3.1.12 *Let $\lambda \in \Lambda_\ell$. Then we have $T(\lambda) = \Delta(\lambda) = \nabla(\lambda) = L(\lambda)$.*

Proof. We only have to note (see Theorem 2.3.29), that $T(\lambda)$ is indecomposable and $L(\lambda)$ is by definition a composition factor, and all other possible weights are in the fundamental alcove (since $T(\lambda)$ has highest weight λ, so all occuring weights μ fulfill $\mu \leq \lambda$), but also all other weights in Λ_ℓ are in a different $\mathcal{W}_\ell$-orbit. Which gives us that $L(\lambda)$ is the only composition factor. It only occurs once, since $\dim T(\lambda)_\lambda = 1$. □

3.1.3 The Subspace $\mathcal{R}'$ Generated by Negligible Tilting Modules

We now will consider the subset $\mathcal{R}' := \operatorname{span}_{\mathbb{Z}} \left\{ [T(\lambda)] \mid \lambda \in X^+ \backslash \Lambda_\ell \right\}$ of the Grothendieck ring $\mathcal{R}$. Then following [3, Proposition 2.8], we obtain the property:

Proposition 3.1.13 *It holds:*

$$\mathcal{R}' \subset \operatorname{span}_{\mathbb{Z}} \{[M] \in \mathcal{R} \mid [M] \cdot s = [M] \textit{ for some simple reflection } s \in \mathcal{W}_\ell\} =: \overline{\mathcal{R}}$$

To prove this, we need the following lemma (see [3, Subsection 2.7]):

Lemma 3.1.14 *For $\lambda \in \Lambda_\ell$, $\tau, \omega \in \mathcal{W}_\ell$ and $s \in \mathcal{W}_\ell$ a simple reflection such that $\omega s_\bullet \lambda, \omega_\bullet \lambda, \tau_\bullet \lambda \in X^+$ and $\omega \tau_\bullet \lambda < \omega_\bullet \lambda$ we have:*

$$[T(\omega_\bullet \lambda) : \Delta(\tau_\bullet \lambda)] = \begin{cases} [T(\omega_\bullet \lambda) : \Delta(\tau s_\bullet \lambda)], & \textit{if } \tau s_\bullet \lambda \in X^+, \\ 0, & \textit{else.} \end{cases}$$

Remark 3.1.15 We have the following property for the left and right action of the affine Weyl group on the fundamental weights: For $\lambda \in \Lambda_\ell$, $\omega, \tau \in \mathcal{W}_\ell$:

$$(\omega_\bullet \lambda).\tau = (\omega\tau)_\bullet \lambda = \omega_\bullet(\lambda.\tau)$$

In particular, we can also formulate Lemma 3.1.14 as follows: For $\lambda > \mu \in X^+$, and $s \in \mathcal{W}_\ell$ being a simple reflection:

$$[T(\lambda) : \Delta(\mu)] = \begin{cases} [T(\lambda) : \Delta(\mu.s)], & \text{if } \mu.s \in X^+, \mu \in \mathcal{W}_{\ell\bullet}\lambda, \\ 0, & \text{else.} \end{cases}$$

Now we prove Proposition 3.1.13.

Proof of Proposition 3.1.13. Let $\lambda \in X^+ \backslash \Lambda_\ell$. If λ is ℓ-singular, then we have:

$$[T(\lambda)] = \sum_{\mu \in X^+,\ \ell\text{-singular}} a_\mu [\Delta(\mu)],$$

for some $a_\mu \in \mathbb{Z}$, since the linkage principle tells us, that all possible weights must be in the $\mathcal{W}_\ell$-orbit of λ, which (in this case) does not contain any ℓ-regular weight. Thus we directly obtain:

$$[T(\lambda)] \in \operatorname{span}_{\mathbb{Z}} \{[M] \in \mathcal{R} \mid [M] \cdot s = [M] \text{ for some simple reflection } s \in \mathcal{W}_\ell\}$$

On the other hand, if λ is ℓ-regular, then by Lemma 3.1.14 we may write: For $s \in \mathcal{W}_\ell$ a fixed simple reflection with $\lambda.s \in X^+$:

$$\begin{aligned}
[T(\lambda)] &= \sum_{\mu \in \mathcal{W}_{\ell \bullet} \lambda \cap X^+} a_\mu [\Delta(\mu)] \\
&= \sum_{\substack{\mu \in \mathcal{W}_{\ell \bullet} \lambda \cap X^+: \\ \mu.s \in X^+, \mu < \mu.s}} a_\mu [\Delta(\mu)] + \sum_{\substack{\mu \in \mathcal{W}_{\ell \bullet} \lambda \cap X^+: \\ \mu.s \in X^+, \mu < \mu.s}} a_{\mu.s} [\Delta(\mu.s)] \\
&= \sum_{\substack{\mu \in \mathcal{W}_{\ell \bullet} \lambda \cap X^+: \\ \mu.s \in X^+, \mu < \mu.s}} a_\mu ([\Delta(\mu)] + [\Delta(\mu.s)])
\end{aligned}$$

Note that s fixes the summand $[\Delta(\mu)] + [\Delta(\mu.s)]$. So we have written the generators of $\mathcal{R}'$ as sums of elements in $\mathcal{R}$, which are fixed under some simple reflection in $\mathcal{W}_\ell$, which shows the claim. □

If $\lambda \in X^+$ and $\omega \in \mathcal{W}_\ell$ with $\lambda.\omega \in X^+$, we get by induction on the length of ω:

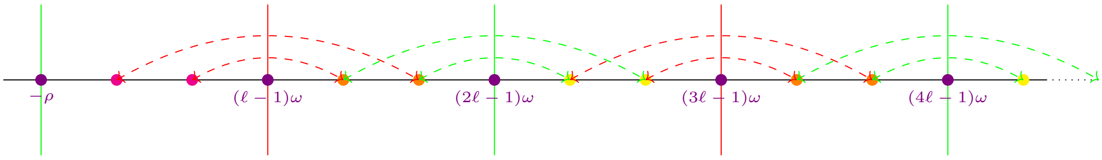
$$[\Delta(\lambda)] + (-1)^{l(\omega)+1} [\Delta(\lambda.\omega)] \in \overline{\mathcal{R}},$$

which actually gives us a basis for $\overline{\mathcal{R}}$, since the $\mathbb{Z}$-span of

$$\begin{gathered}
\left\{ [\Delta(\lambda)] \mid \lambda \in X^+ \ \ell\text{-singular} \right\} \\
\cup \left\{ [\Delta(\lambda)] + (-1)^{l(\omega)+1} [\Delta(\lambda.\omega)] \mid \lambda \in \Lambda_\ell, \omega \in \mathcal{W}_\ell : \ \lambda.\omega \in X^+ \right\}
\end{gathered}$$

is in $\overline{\mathcal{R}}$, the above set is clearly linearly independent and the $\mathbb{Z}$-span of $\{[\Delta(\lambda)] \mid \lambda \in \Lambda_\ell\}$ intersects with $\overline{\mathcal{R}}$ in zero.

Example 3.1.16 In the case $\mathfrak{g} = \mathfrak{sl}_2$, we have as $\mathbb{Z}$-basis $\{\Delta(\lambda) \mid \lambda \in X^+\}$ of $\mathcal{R}$. Then we can illustrate X^+:



where the violet weights are ℓ-singular, the magenta ones are in Λ_ℓ, the red dashed arrows illustrate the action of s_0 and the green dashed arrows the action of s_1.

For the ℓ-singular weights in X^+ we may write $\{(m\ell - 1)\omega \mid m \in \mathbb{Z}_{\geq 0}\}$.

For the second set we obtain

$$\begin{gathered}
\{[\Delta(m\omega)] + [\Delta((m\omega).(s_0 s_1)^r)] \mid 0 \leq m \leq \ell - 2, r \in \mathbb{Z}_{\geq 0}\} \\
\cup \{[\Delta(m\omega)] - [\Delta((m\omega).((s_0 s_1)^r s_0))] \mid 0 \leq m \leq \ell - 2, r \in \mathbb{Z}_{\geq 0}\}
\end{gathered}$$

The weights of the form $(m\omega).(s_0s_1)^r$ are depicted above in yellow (for $r > 0$), the weights of the form $(m\omega).((s_0s_1)^r s_0)$ in orange. As one can see, for the yellow weights one has to cross an even number of alcove walls (starting in the fundamental alcove) and for the orange ones one has to cross an odd number of alcove walls.

Further, we need the following lemma, because the corresponding corollary is interesting for our purposes (for a proof see [3, Subsection 2.9])

Lemma 3.1.17 *For any $[M] \in \overline{\mathcal{R}}, \lambda \in X^+$ and $\omega \in \mathcal{W}_\ell$ we have:*

$$[L(\lambda)] \cdot ([M].\omega) \equiv ([L(\lambda)] \cdot [M]).\omega \bmod \overline{\mathcal{R}}$$

In particular, we have:

Corollary 3.1.18 *The subset $\mathcal{R}'$ forms an ideal in the Grothendieck subring $\mathcal{R}^t$.*

Proof. This follows directly from the Lemma 3.1.17, since for the basis $\left\{L(\lambda) \mid \lambda \in X^+\right\}$ of $\mathcal{R}$ and generators $[M] \in \mathcal{R}'$ with $[M].s_i = [M]$ for some simple reflection $s_i \in \mathcal{W}_\ell$ we have:

$$\begin{aligned} ([L(\lambda)] \cdot [M]).s_i &\equiv [L(\lambda)] \cdot ([M].s_i) \bmod \overline{\mathcal{R}} \\ &= [L(\lambda)] \cdot [M] \bmod \overline{\mathcal{R}} \end{aligned}$$

So in particular, we have $[L(\lambda)] \cdot [M] \in \overline{\mathcal{R}}$ by the definition of $\overline{\mathcal{R}}$ (in Proposition 3.1.13). Further, by restricting to the subring $\mathcal{R}^t$ we have proven the corollary. ☐

3.2 The Definition of $\mathcal{C}_\ell^-$

For $\lambda, \mu \in X^+$ we may write:

$$[T(\lambda) \otimes T(\mu)] = \left(\sum_{\nu \in \Lambda_\ell} a_{\lambda,\mu}^\nu [T(\nu)] \right) + P,$$

for some rest $P \in \mathcal{R}'$. With these structure constants $a_{\lambda,\mu}^\nu$ we define:

Definition 3.2.1 The category $\mathcal{C}_\ell^-$ is defined as the full subcategory of $\mathcal{C}$ with objects whose maximal weights are only in the fundamental alcove, i.e. in Λ_ℓ. We make $\mathcal{C}_\ell^-$ into a tensor category via:

$$\Delta(\lambda) \bar{\otimes} \Delta(\mu) := \bigoplus_{\nu \in \Lambda_\ell} \Delta(\nu)^{a_{\lambda,\mu}^\nu} \qquad \text{for } \lambda, \mu \in \Lambda_\ell$$

Remark 3.2.2 Note, that by Corollary 3.1.12 we have $L(\lambda) = \Delta(\lambda) = T(\lambda)$ for $\lambda \in \Lambda_\ell$. In particular, every object in $\mathcal{C}_\ell^-$ is tilting and may be written as a direct sum of objects of the form $T(\lambda) = \Delta(\lambda)$ for $\lambda \in \Lambda_\ell$, so the above definition of the tensor product suffices:

For general objects $\left(\bigoplus_{\lambda\in\Lambda_\ell} \Delta(\lambda)^{b_\lambda}\right)$, $\left(\bigoplus_{\mu\in\Lambda_\ell} \Delta(\mu)^{c_\mu}\right) \in \mathrm{Ob}(\mathcal{C}_\ell^-)$ it holds

$$\left(\bigoplus_{\lambda\in\Lambda_\ell} \Delta(\lambda)^{b_\lambda}\right) \bar{\otimes} \left(\bigoplus_{\mu\in\Lambda_\ell} \Delta(\mu)^{c_\mu}\right) = \bigoplus_{\nu\in\Lambda_\ell} \Delta(\nu)^{\sum_{\lambda,\mu\in\Lambda_\ell} b_\lambda c_\mu a^\nu_{\lambda,\mu}}.$$

We denote the Grothendieck group (with the just defined multiplication turning it into a ring) of the category $\mathcal{C}_\ell^-$ by $\mathcal{F}$.

Example 3.2.3 Let us consider $\mathfrak{g} = \mathfrak{sl}_2$ (instead of $\mathfrak{sl}_3$) and $\ell = 5$. Then we have

$$\Lambda_5 = \left\{m\omega \mid m \in \mathbb{Z}_{\geq 0}, 0 < \langle (m+1)\omega, \alpha^\vee\rangle < \ell\right\} = \{0, \omega, 2\omega, 3\omega\}.$$

In the category $\mathcal{C}$ we have

$$\Delta(2\omega) \otimes \Delta(3\omega) = T(2\omega) \otimes T(3\omega) = T(5\omega) \oplus T(\omega),$$

but in contrast in $\mathcal{C}_5^-$ we have

$$\Delta(2\omega)\bar{\otimes}\Delta(3\omega) = T(\omega) = \Delta(\omega), \tag{3}$$

since $5\omega \notin \Lambda_\ell$.

3.3 The Quantum Trace

Following [7, Definition 2.3.3], we now define the quantum trace of an endomorphism in a ribbon category (in the reference it is just called "trace"). Our goal is to characterize the negligible tilting modules with the quantum dimension, namely we will prove the following result (see [2, Theorem 3.4]):

Theorem *The quantum dimension of $T(\lambda)$ is zero for all $\lambda \in X^+ \backslash \Lambda_\ell$.*

Recall 3.3.1 A ribbon category is a rigid and braided tensor category with functorial isomorphismn $\delta_V : V \to V^{**}$ satisfying some properties (see e.g. [7, Definition 2.2.1]).

Definition 3.3.2 Given a ribbon category, the quantum trace $\mathrm{qtr}(f)$ of an endomorphism $f : V \to V$ is the composition of morphisms given by the definition of ribbon category and the endomorphism f:

$$\mathrm{qtr}(f): \quad 1 \xrightarrow{i_V} V \otimes V^* \xrightarrow{f\otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{\delta_V \otimes \mathrm{id}_{V^*}} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} 1$$

In general for an endomorphism f, the quantum trace $\mathrm{qtr}(f)$ is an element in $\mathrm{End}(1)$. Note that, in our setup of $U_q(\mathfrak{g})$-modules, 1 is the trivial $U_q(\mathfrak{g})$-module $\mathbb{C}$ of highest weight 0 and thus, $\mathrm{End}(1)$ is the ground field $\mathbb{C}$.

Definition 3.3.3 The quantum dimension of an object V in a ribbon category is the quantum trace of the identity morphism of V, i.e.:

$$\mathrm{qdim}(V) = \mathrm{qtr}(\mathrm{id}_V)$$

Example 3.3.4 In the category of finite dimensional vector spaces over a field k, given a vector space V and a fixed basis $(v_1, ..., v_r)$ of V, let $(v_1^*, ..., v_r^*)$ be the corresponding dual basis of V^* and $(\delta_1, ..., \delta_r)$ the corresponding evaluation maps in V^{**}, i.e. for $i, j \in \{1, ..., r\}: \ \delta_i(v_j^*) = v_j^*(v_i) = \delta_{i,j}$.

Then the morphisms i_V, δ_V and ev_{V^*} are given by

$$\begin{array}{llll} i_V: & k \to V \otimes V^*, & 1 \mapsto \sum_{i=1}^r v_i \otimes v_i^*, & \\ \delta_V: & V \to V^{**}, & v_i \mapsto \delta_i & \text{for } i \in \{1, ...r\}, \\ \mathrm{ev}_{V^*}: & V^{**} \otimes V^* \to k, & \delta_i \otimes v_j^* \mapsto \delta_i(v_j^*) & \text{for } i, j \in \{1, ..., r\}. \end{array}$$

In particular we have for $f = \mathrm{id}_V$:

$$1 \mapsto \sum_{i=1}^r v_i \otimes v_i^* \mapsto \sum_{i=1}^r \delta_i \otimes v_i^* \mapsto \sum_{i=1}^r \delta_i(v_i^*) = r = \dim_k(V).$$

Hence, the quantum dimension of V is just the ordinary dimension of V over k. Analogously, the quantum trace of an endomorphism is the ordinary trace.

Remark 3.3.5 Back to $U_q(\mathfrak{g})$-modules, for $u \in U_q(\mathfrak{g})$ it holds:

$$S^2(u) = K_{2\rho}^{-1} u K_{2\rho}.$$

In words, S^2 is the conjugation by $K_{2\rho}$, and the morphism $\delta_V : V \to V^{**}$ is given by

$$m \longmapsto (f \mapsto f(K_{2\rho}^{-1}.m)).$$

With those we can calculate an explicit formula for $U_q(\mathfrak{g})$-modules (see [7, Exercise 2.3.4] or by definition in [2, Definition 3.1]):

Proposition 3.3.6 *In the category of integral $U_q(\mathfrak{g})$-modules $\mathcal{C}$, the quantum trace of an endomorphism $f : V \to V$ and an object $V \in \mathrm{Ob}(\mathcal{C})$ is given by*

$$\mathrm{qtr}(f) = \mathrm{tr}_V(K_{2\rho} f), \qquad \mathrm{qdim}(V) = \mathrm{tr}_V(K_{2\rho}),$$

where tr_V is the ordinary trace function of an endomorphism and $K_{2\rho} = \prod_{\alpha \in \Phi^+} K_\alpha$.

We use the notation: : $K_\alpha = \prod_{i=1}^n K_i^{c_{\alpha,i}}$ for $\alpha \in \Phi^+$, $c_{\alpha,i} \in \mathbb{N}$ with $\alpha = \sum_{i=1}^n c_{\alpha,i} \cdot \alpha_i$.

In particular, since we have a weight space decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$, we get for the quantum trace:

$$\begin{aligned} \operatorname{qdim}(V) &= \operatorname{tr}_V(K_{2\rho}) \\ &= \sum_{\lambda \in X} \dim V_\lambda \cdot q^{2\langle \lambda, \rho^\vee \rangle} \\ &= \sum_{\lambda \in X} \dim V_\lambda \cdot q^{\sum_{\alpha \in \Phi^\vee} \langle \lambda, \alpha^\vee \rangle}. \end{aligned}$$

Proof. Let $V \in \operatorname{Ob}(\mathcal{C})$ with weight space decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$. Let $(m_{i,\lambda})_{i=1}^{\dim V_\lambda}$ be a basis of the respective weight space V_λ for $\lambda \in X$ and $(m_{i,\lambda}^*)_{i=1}^{\dim V_\lambda}$ be the corresponding dual basis of $V_\lambda^* \subset V^*$. Further, let f be an endomorphism of V.

Then it holds:

$$\begin{aligned} \operatorname{qtr}(f):\ 1 &\mapsto \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} m_{i,\lambda} \otimes m_{i,\lambda}^* \mapsto \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} f(m_{i,\lambda}) \otimes m_{i,\lambda}^* \\ &\mapsto \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} \delta_V(f(m_{i,\lambda})) \otimes m_{i,\lambda}^* \mapsto \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} \delta_V(f(m_{i,\lambda}))(m_{i,\lambda}^*), \end{aligned}$$

and we have

$$\begin{aligned} \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} \delta_V(f(m_{i,\lambda}))(m_{i,\lambda}^*) &= \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} m_{i,\lambda}^*(K_{2\rho}^{-1}.(f(m_{i,\lambda}))) \\ &\overset{(*)}{=} \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} m_{i,\lambda}^*((K_{2\rho} f)(m_{i,\lambda})) \\ &= \operatorname{tr}_V(K_{2\rho} f), \end{aligned}$$

where we use:

(*) the symmetry of weight spaces ($\dim V_\lambda = \dim V_{-\lambda}$) and f preserves weight spaces.

This proves the claim for the quantum trace. In particular, for $f = \mathrm{id}_V$, we have

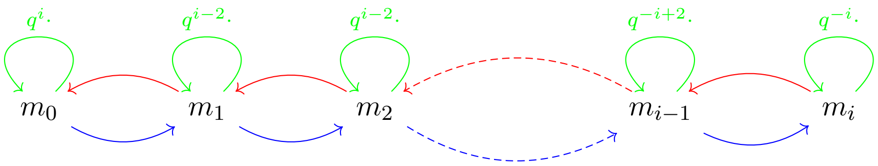
$$\begin{aligned} \operatorname{tr}_V(K_{2\rho} \mathrm{id}_V) &= \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} m_{i,\lambda}^*(K_{2\rho}.m_{i,\lambda}) \\ &= \sum_{\lambda \in X} \sum_{i=1}^{\dim V_\lambda} q^{\sum_{\alpha \in \Phi^+} \langle \lambda, \alpha^\vee \rangle} \underbrace{m_{i,\lambda}^*(m_{i,\lambda})}_{=1} \\ &= \sum_{\lambda \in X} \dim V_\lambda \cdot q^{\sum_{\alpha \in \Phi^+} \langle \lambda, \alpha^\vee \rangle}. \end{aligned}$$

This finishes the proof of the proposition. $\square$

In particular, if we know the character of a module M, which is defined by $\operatorname{ch} M = \sum_{\lambda \in X} \dim M_\lambda \cdot e^\lambda \in \mathbb{Z}(X)$, then we only have to substitute e^λ by $q^{2\langle \lambda, \rho^\vee \rangle}$ to get the quantum dimension of M.

Remark 3.3.7 Note that for $q \mapsto 1$ we get the ordinary dimension back, i.e. $\operatorname{qdim} V \mapsto \dim V$.

Example 3.3.8 Going back to our example $\mathfrak{g} = \mathfrak{sl}_2$ (see Example 2.3.16). Recall that the i^{th} Weyl module $\Delta(i\omega)$ has the form:



where the red arrows to the left show the action of E (up to a scalar), the blue arrows the right the action of F (up to a scalar) and the green arrows the action of K (and leaving out the action of the other generators).

So we get an easy formula for the quantum dimensions (noting $K_{2\rho} = K$):

$$\operatorname{qdim}(\Delta(i \cdot \omega)) = \sum_{k=0}^{i} q^{i-2k}$$

One should observe, that since q is a root of unity, the quantum dimension sometimes vanishes:

For example, take $q = e^{i\frac{2\pi}{5}}$ (i.e. $\ell = 5$), $i = 4$, then

$$\operatorname{qdim}(\Delta(4\omega)) = q^4 + q^2 + 1 + q^{-2} + q^{-4} = 0.$$

Lemma 3.3.9 *The quantum trace is additive and multiplicative, in the sense that we have for $M, N \in \operatorname{Ob}(\mathcal{C})$:*

$$\operatorname{qdim}(M \oplus N) = \operatorname{qdim}(M) + \operatorname{qdim}(N), \qquad \operatorname{qdim}(M \otimes N) = \operatorname{qdim}(M) \cdot \operatorname{qdim}(N).$$

In particular, the quantum dimension behaves quite similarly to the ordinary dimension.

Proof. The additivity is clear.

Regarding the multiplicativity, it is easy to check in our category $\mathcal{C}$, using the fact that the dimension of weight spaces are given by $\dim(M \otimes N)_\lambda = \sum_{\mu \in X} \dim M_\mu \cdot \dim N_{\lambda - \mu}$.

Then it follows directly by Proposition 3.3.6:

$$\begin{aligned}
\operatorname{qdim}(M\otimes N) &= \sum_{\lambda\in X}\dim(M\otimes N)_\lambda\cdot q^{2\langle\lambda,\rho^\vee\rangle}\\
&= \sum_{\lambda\in X}\sum_{\mu\in X}\dim M_\mu\cdot\dim N_{\lambda-\mu}\cdot q^{2\langle\mu,\rho^\vee\rangle}\cdot q^{2\langle\lambda-\mu,\rho^\vee\rangle}\\
&= \left(\sum_{\mu\in X}\dim M_\mu\cdot q^{2\langle\mu,\rho^\vee\rangle}\right)\cdot\left(\sum_{\nu\in X}\dim N_\nu\cdot q^{2\langle\nu,\rho^\vee\rangle}\right)\\
&= \operatorname{qdim} M\cdot\operatorname{qdim} N
\end{aligned}$$

Alternatively, one can use the fact that the characters are multiplicative. □

Corollary 3.3.10 *The $\mathbb{Z}$-span of equivalence classes of modules with vanishing quantum dimension form an ideal in the Grothendieck ring $\mathcal{R}$.*

Example 3.3.11 If we look back at Example 3.2.3, so $\ell = 5$ and $\mathfrak{g} = \mathfrak{sl}_2$, then it holds:

$$\begin{aligned}
\operatorname{qdim}\left(\Delta(2\omega)\otimes\Delta(3\omega)\right) &= (q^2+1+q^{-2})(q^3+q+q^{-1}+q^{-3})\\
&= q^5+2q^3+3q+3q^{-1}+2q^{-3}+q^{-5}\\
&= \underbrace{q^5+2q^3+2q+2q^{-1}+2q^{-3}+q^{-5}}_{=0=\operatorname{qdim}T(5\omega)}+q+q^{-1}\\
&= \operatorname{qdim}\Delta(\omega)
\end{aligned}$$

This of course is predicted by Equation (3).

Remark 3.3.12 Also recall the Weyl's character formula for $\lambda \in X^+$ (see [10, Section 24.3]). There is a non-trivial fact that the characters of the dual Weyl modules (and by duality of the Weyl modules) are given by Weyl's character formula (for references see [6, Section 2]), i.e.:

$$\operatorname{ch}\Delta(\lambda) = \frac{\sum_{\omega\in\mathcal{W}}(-1)^{l(\omega)}e^{\omega(\lambda+\rho)}}{\sum_{\omega\in\mathcal{W}}(-1)^{l(\omega)}e^{\omega(\rho)}}.$$

In particular, we have:

$$\operatorname{qdim}\Delta(\lambda) = \prod_{\alpha\in\Phi^+}\frac{q^{\langle\lambda+\rho,\alpha^\vee\rangle}-q^{-\langle\lambda+\rho,\alpha^\vee\rangle}}{q^{\langle\rho,\alpha^\vee\rangle}-q^{-\langle\rho,\alpha^\vee\rangle}}.$$

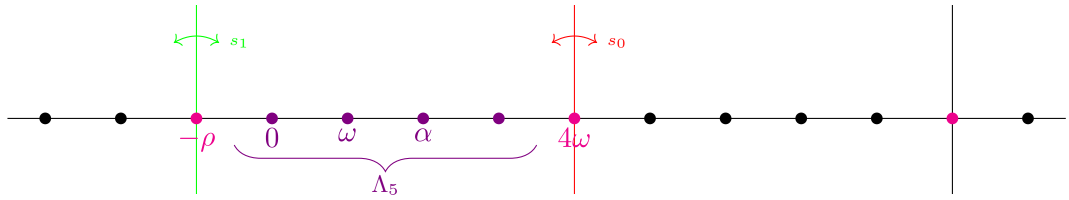
The product is only 0 if one factor is 0 and this is only the case (for ℓ odd) if λ is ℓ-singular:

$$q^{\langle\lambda+\rho,\alpha^\vee\rangle}-q^{-\langle\lambda+\rho,\alpha^\vee\rangle}=0 \quad\Leftrightarrow\quad q^{\langle\lambda+\rho,\alpha^\vee\rangle}=q^{-\langle\lambda+\rho,\alpha^\vee\rangle} \quad\Leftrightarrow\quad q^{2\langle\lambda+\rho,\alpha^\vee\rangle}=1$$

Hence, we get

$$\operatorname{qdim}\Delta(\lambda)=0 \qquad\Longleftrightarrow\qquad \lambda \text{ is } \ell\text{-singular} \tag{4}$$

Example 3.3.13 Just in the Example 3.3.8, we have seen, that for $\ell = 5$ and $\mathfrak{g} = \mathfrak{sl}_2$ it holds $\operatorname{qdim} \Delta(4\omega) = 0$ and 4ω is ℓ-singular (it holds $\langle 4\omega + \rho, \alpha^\vee \rangle = 5$):



Here the dots represents elements in X and the vertical lines the alcove walls as before.

In our case, $\mathfrak{g} = \mathfrak{sl}_3$ one can even simplify the formula above:
Recall, that in the case $\mathfrak{g} = \mathfrak{sl}_3$ we can write $\lambda \in X$ as $\lambda = m_1\omega_1 + m_2\omega_2$ for $m_1, m_2 \in \mathbb{Z}$ and if $\lambda \in X^+$ we have $m_1, m_2 \geq 0$. In this setup, there is an even easier formula to calculate the quantum dimension of $L(\lambda) = \Delta(\lambda)$ for $\lambda \in \Lambda_\ell$ (see [7] and [19]):

Theorem 3.3.14 (Quantum Weyl dimension formula for $\mathfrak{sl}_3$) *For all* $(m_1\omega_1 + m_2\omega_2) \in \Lambda_\ell$ *we have:*

$$\operatorname{qdim}(L(m_1\omega_1 + m_2\omega_2)) = \frac{1}{[2]_q}[m_1+1]_q[m_2+1]_q[m_1+m_2+2]_q.$$

Proof. This follows directly by applying the Weyl character formula:

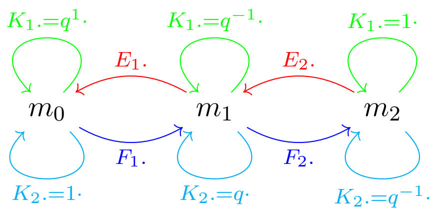
$$\begin{aligned}\operatorname{qdim} L(\lambda) &= \prod_{\alpha\in\Phi^+} \frac{q^{\langle\lambda+\rho,\alpha^\vee\rangle} - q^{-\langle\lambda+\rho,\alpha^\vee\rangle}}{q^{\langle\rho,\alpha^\vee\rangle} - q^{-\langle\rho,\alpha^\vee\rangle}} \\ &= \frac{[\langle\lambda+\rho,\alpha_1^\vee\rangle]_q}{[\langle\rho,\alpha_1\rangle]_q} \cdot \frac{[\langle\lambda+\rho,\alpha_2^\vee\rangle]_q}{[\langle\rho,\alpha_2\rangle]_q} \cdot \frac{[\langle\lambda+\rho,\alpha_1^\vee+\alpha_2^\vee\rangle]_q}{[\langle\rho,\alpha_1+\alpha_2^\vee\rangle]_q}.\end{aligned}$$

Inserting $\lambda = m_1\omega_1 + m_2\omega_2$ and $\rho = \omega_1 + \omega_2$ proves the theorem. $\square$

Example 3.3.15 For $\mathfrak{g} = \mathfrak{sl}_3$, the two easiest examples are $L(0)$ and $L(\omega_1)$:
In the case $\lambda = 0$, $L(\lambda)$ is one dimensional and K_1 and K_2 act as 1, E_1, E_2, F_1 and F_2 as 0. So we have $\operatorname{qdim} L(0) = 1$ and on the other hand the quantum Weyl dimension formula tells us

$$\operatorname{qdim} L(0) = \frac{[1]_q[1]_q[2]_q}{[2]_q} = 1.$$

The module $L(\omega_1)$ has a less trivial form, depicted in the following diagram



where we illustrate the action of the generators. If there is no arrow for the operators starting at a basis vector, they act as 0. We have $K_{2\rho} = K_1^2 K_2^2$, and we obtain for the quantum dimension: $\operatorname{qdim}(L(\omega_1)) = q^2 + 1 + q^{-2}$.

On the other hand the quantum Weyl dimension formula tells us:

$$\operatorname{qdim}(L(\omega_1)) = \frac{[2]_q[1]_q[3]_q}{[2]_q} = [3]_q = q^2 + 1 + q^{-2}.$$

Here, the quantum dimension could only be 0 if $[3]_q = 0$, i.e. $\ell = 3$, which is not allowed for $\mathfrak{g} = \mathfrak{sl}_3$ (since $\ell > h = 3$ by assumption).

Now back to our tilting modules: The negligible modules have the following property (see [2, Theorem 3.4]):

Theorem 3.3.16 *Let $M \in \operatorname{Ob}(\mathcal{C})$ be tilting module with no connected component of the form $T(\lambda)$ with $\lambda \in \Lambda_\ell$. Then for any endomorphism f of M the quantum trace of f vanishes.*

Proof. By Theorem 2.3.29, it suffices to prove the theorem for $M = T(\lambda)$ for $\lambda \in X^+ \backslash \Lambda_\ell$. Further, if the endomorphism f is nilpotent, so is $K_{2\rho}.f$ (since we have a weight space decomposition and f preserves the weights). So in particular, we have $\operatorname{qtr}(f) = 0$.

Since $T(\lambda)$ is indecomposable, we know that any endomorphism f of $T(\lambda)$ is equal to a multiple of the identity plus a nilpotent endomorphism (f is nilpotent if it is 0 on $T(\lambda)_\lambda$). So the theorem reduces to the following proposition. □

Proposition 3.3.17 *Let $\lambda \in X^+$. It holds:*

$$\operatorname{qdim} T(\lambda) = 0 \qquad \Longleftrightarrow \qquad \lambda \notin \Lambda_\ell$$

So we have an alternative characterization for the ideal $\mathcal{R}' \subset \mathcal{R}^t$:

Corollary 3.3.18 *We may characterize $\mathcal{R}' := \operatorname{span}_{\mathbb{Z}} \left\{ [T(\lambda)] \mid \lambda \in X^+ \backslash \Lambda_\ell \right\}$ as follows:*

$$\mathcal{R}' = \operatorname{span}_{\mathbb{Z}} \left\{ [M] \in \mathcal{R}^t \mid M \in \operatorname{Ob}(\mathcal{C}),\ \operatorname{qdim}(M) = 0 \right\}.$$

By Corollary 3.3.10, we have shown: $\mathcal{R}'$ forms an ideal in $\mathcal{R}^t$.

To prove the proposition, we will use a few lemmas:

Lemma 3.3.19 *Let $\lambda \in X^+$ be ℓ-singular. Then the quantum dimension of $T(\lambda)$ vanishes.*

Proof. This follows easily from the linkage principle (3.1.10). Since λ is ℓ-singular, so is each μ, which appears in the Δ-filtration of $T(\lambda)$. We also know by the application of the Weyl character formula (4), that every appearing $\Delta(\mu)$ has quantum dimension 0. In particular, so does $T(\lambda)$. □

Now we have proven the claim of the proposition for the ℓ-singular case. For the ℓ-regular one, we need a few more steps:

Lemma 3.3.20 *Let $E, M \in \mathrm{Ob}(\mathcal{C})$ and suppose the quantum trace vanishes for all endomorphisms of M, i.e.*

$$\forall f \in \mathrm{End}_{U_q(\mathfrak{g})}(M): \ \mathrm{qtr}(f) = 0.$$

Then it holds:

$$\forall \varphi \in \mathrm{End}_{U_q(\mathfrak{g})}(E \otimes M): \ \mathrm{qtr}(\varphi) = 0$$

In particular, we have $\mathrm{qdim}(N) = 0$ *for all summands of $E \otimes M$.*

Remark 3.3.21 The conclusion $\mathrm{qdim}(N) = 0$ for all summands of $E \otimes M$ follows directly, since one may consider $f = p_N$ the projection onto the summand N. Then it holds $0 = \mathrm{qtr}(p_N) = \mathrm{qdim}(N)$.

Proof. Let $(e_1, ..., e_r)$ be a basis for E and denote by $(e_1^*, ..., e_r^*)$ (respectively $(e_1^{**}, ..., e_r^{**})$) the dual (resprectively double dual) basis of E^* (respectively E^{**}). Recall, we have an isomorphism of $U_q(\mathfrak{g})$-modules: $\delta_E: \ E \to E^{**}, \ e \longmapsto (f \mapsto f(K_{2\rho}^{-1}.e))$ and we also have $U_q(\mathfrak{g})$-homomorphisms:

$$\begin{aligned} i_{E^*}: \ \mathbb{C} &\to E^* \otimes E^{**}, & \mathrm{ev}_E: \ E^* \otimes E &\to \mathbb{C}, \\ 1 &\mapsto \sum_{k=1}^r e_k^* \otimes e_k^{**}, & e_i^* \otimes e_j &\mapsto \delta_{i,j}. \end{aligned}$$

So for $\varphi \in \mathrm{End}_{U_q(\mathfrak{g})}(E \otimes M)$ we define $\bar{\varphi} \in \mathrm{End}_{U_q(\mathfrak{g})}(M)$ as the following composition:

$$\bar{\varphi}: M \xrightarrow{i_{E^*} \otimes \mathrm{id}_M} E^* \otimes E^{**} \otimes M \xrightarrow{\mathrm{id}_{E^*} \otimes \delta_E^{-1} \otimes \mathrm{id}_M} E^* \otimes E \otimes M \xrightarrow{\mathrm{id}_{E^*} \otimes \varphi} E^* \otimes E \otimes M \xrightarrow{\mathrm{ev}_E \otimes \mathrm{id}_M} M.$$

Claim: $\mathrm{qtr}(\varphi) = \mathrm{qtr}(\bar{\varphi})$

The lemma follows from the claim, since by assumption $\mathrm{qtr}(\bar{\varphi}) = 0$.

Now to prove the claim, for $m \in M$ it holds:

$$\bar{\varphi}(m) = \mathrm{ev}_E \otimes \mathrm{id}_M \left(\sum_{k=1}^r e_k^* \otimes \varphi \left(\delta_E^{-1}(e_k^{**}) \otimes m \right) \right).$$

Note: $\delta_E^{-1}(e_k^{**}) = K_{2\rho}.e_k$

So we get:

$$\begin{aligned} K_{2\rho}\bar{\varphi}(m) &= \bar{\varphi}(K_{2\rho}.m) \\ &= \mathrm{ev}_E \otimes \mathrm{id}_M \left(\sum_{k=1}^r e_k^* \otimes \varphi \left(K_{2\rho}.e_k \otimes K_{2\rho}.m \right) \right) \\ &= \mathrm{ev}_E \otimes \mathrm{id}_M \left(\sum_{k=1}^r e_k^* \otimes \left(K_{2\rho}\varphi \right) \left(e_k \otimes m \right) \right). \end{aligned}$$

From this we can easily see the claim, since given a basis $(m_i)_i$ of M:

$$\begin{aligned}\operatorname{tr}(K_{2\rho}\bar{\varphi}) &= \sum_i m_i^* \left(\operatorname{ev}_E \otimes \operatorname{id}_M \left(\sum_{k=1}^{r} e_k^* \otimes (K_{2\rho}\varphi)(e_k \otimes m_i) \right) \right) \\ &= \sum_{i,k} (e_k^* \otimes m_i^*) \left(K_{2\rho}.\varphi(e_k \otimes m_i) \right) \\ &= \operatorname{tr}(K_{2\rho}\varphi),\end{aligned}$$

which is the formula for the quantum trace by Proposition 3.3.6. This finishes the proof. □

Before the final step of our proof of the proposition, we need to introduce the translation functors $\mathcal{T}_\mu^\lambda$.

3.4 Translation Functors

The idea of a translation functor is to make a highest weight module of say weight λ into a highest weight module of weight μ.

Definition 3.4.1 Let $\mu \in \Lambda_\ell$ and $M \in \operatorname{Ob}(\mathcal{C})$. We set $p_\mu(M)$ to be the maximal submodule of M whose composition factors have highest weight in $\mathcal{W}_{\ell \bullet}\mu$.

Now let $\mu, \lambda \in \bar{\Lambda}_\ell$ (where $\bar{\Lambda}_l$ is the union of Λ_ℓ and all integral weights on the alcove walls of the fundamental alcove). Further, let $\omega \in \mathcal{W}$ such that $\omega_\bullet(\mu - \lambda) \in X^+$. Then we define the translation functor $\mathcal{T}_\lambda^\mu : \mathcal{C} \to \mathcal{C}$ as follows for $M \in \operatorname{Ob}(\mathcal{C})$:

$$\mathcal{T}_\lambda^\mu(M) := p_\mu \left(M \otimes T(\omega_\bullet(\mu - \lambda)) \right),$$

and for morphisms $f : M \to N$

$$\mathcal{T}_\lambda^\mu(f) := \left(f \otimes \operatorname{id}_{T(\omega_\bullet(\mu-\lambda))} \right) \Big|_{\mathcal{T}_\lambda^\mu(M)}$$

Remark 3.4.2

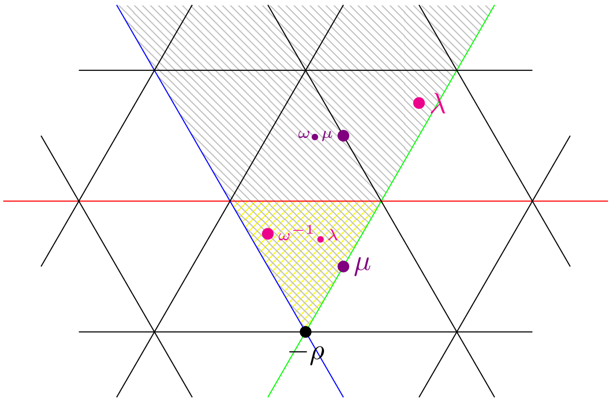
- Note, that $\omega \in \mathcal{W}$ exists and is unique, since it is an element in the non-affine Weyl group.
- $\mathcal{T}_\lambda^\mu(f)$ is well-defined, since f preserves weight spaces.
- The translation functors $\mathcal{T}_\lambda^\mu$ take tilting modules to tilting modules by Theorem 2.3.31.

Now to the proof of Proposition 3.3.17:

Proof of Proposition 3.3.17. We have already proven the claim for ℓ-singular weights (see Lemma 3.3.19).

Now let $\lambda \in X^+ \setminus \Lambda_\ell$ be ℓ-regular. Then there exists a unique $\omega \in \mathcal{W}_\ell \setminus \{\operatorname{id}\}$ such that ${\omega^{-1}}_\bullet\lambda \in \Lambda_\ell$. Further, we choose $\mu \in \bar{\Lambda}_\ell$ such that $\operatorname{stab}_{\mathcal{W}_\ell}(\mu) = \{\operatorname{id}, s_i\}$ and $\lambda.s_i < \lambda$

for some simple reflection s_i. This way we have that $\omega_\bullet\mu$ is on a lower alcove wall of the alcove containing λ.



Here you see an example for λ and μ in $\overline{X^+}$.

We already know by the ℓ-singular case, that $\operatorname{qdim} T(\omega_\bullet\mu) = 0$ and therefore by Lemma 3.3.20, that $\mathcal{T}_\mu^{\omega^{-1}\bullet\lambda}(T(\omega_\bullet\mu)) = p_{\omega^{-1}{}_\bullet\lambda}(T(\omega_\bullet\mu) \otimes T(\tau(\omega^{-1}{}_\bullet\lambda - \mu)))$ (for some $\tau \in \mathcal{W}$) has quantum dimension 0 and also all its summands.

Now it is only left to note, that $T(\omega_\bullet\mu) \otimes T(\tau(\omega^{-1}{}_\bullet\lambda - \mu))$ has maximal weight λ, and therefore also $\mathcal{T}_\mu^{\omega^{-1}\bullet\lambda}(T(\omega_\bullet\mu))$. So by our classification of tilting modules (Theorem 2.3.29), $T(\lambda)$ is a direct summand of $\mathcal{T}_\mu^{\omega^{-1}\bullet\lambda}(T(\omega_\bullet\mu))$. So in particular $\operatorname{qdim} T(\lambda) = 0$.

This finishes the proof of Proposition 3.3.17 and therefore also the proof of Theorem 3.3.16. □

3.5 Associativity of the Tensor Product

In the definition of the category $\mathcal{C}_\ell^-$, we have also defined a tensor product: Given $\left(a_{\lambda,\mu}^\nu\right)_{\lambda,\mu,\nu\in X^+}$ such that

$$T(\lambda) \otimes T(\mu) = \bigoplus_{\nu\in X^+} T(\nu)^{a_{\lambda,\mu}^\nu},$$

then we define:

$$\Delta(\lambda)\bar{\otimes}\Delta(\mu) = \bigoplus_{\nu\in\Lambda_\ell} \Delta(\nu)^{a_{\lambda,\mu}^\nu}$$

Now we can show, that this tensor product is associative (up to natural isomorphisms) (see [2, Section 4]), but we will use a bit more general definition.

We have the following corollary:

Corollary 3.5.1 *Let $\mu_1, ..., \mu_r \in \Lambda_l$. Then:*

$$\Delta(\mu_1) \otimes \Delta(\mu_2) \otimes ... \otimes \Delta(\mu_r) = Z \oplus \left(\bigoplus_{\lambda\in\Lambda_\ell} \Delta(\lambda)^{a_\lambda} \right)$$

for some $a_\lambda \in \mathbb{Z}_{\geq 0}$ and some $U_q(\mathfrak{g})$-module Z with the property that the quantum trace vanishes for all endomorphisms of Z.

Proof. By the linkage principle (especially Corollary 3.1.12), we know that the $\Delta(\mu_k)$'s are tilting and therefore by Theorem 2.3.31 $\Delta(\mu_1) \otimes \Delta(\mu_2) \otimes ... \otimes \Delta(\mu_r)$ is a tilting module. So we may write by the characterization of tilting modules (Theorem 2.3.29):

$$\Delta(\mu_1) \otimes \Delta(\mu_2) \otimes ... \otimes \Delta(\mu_r) = \left(\bigoplus_{\lambda \notin \Lambda_\ell} T(\lambda)^{a_\lambda} \right) \oplus \left(\bigoplus_{\lambda \in \Lambda_\ell} T(\lambda)^{a_\lambda} \right)$$

So we set $Z = \bigoplus_{\lambda \notin \Lambda_\ell} T(\lambda)^{a_\lambda}$ and by Theorem 3.3.16 every endomorphism of Z has quantum trace 0. $\square$

Further, we define for a tilting module D the module $\bar{D}$ as the maximal submodule of D which is also an object in $\mathcal{C}_l^-$, i.e. (with the characterization in Theorem 2.3.29):

$$\bar{D} = \bigoplus_{\lambda \in \Lambda_\ell} T(\lambda)^{a_\lambda^D}$$

We define the reduced tensor product $\bar{\otimes}$ as follows:

$$D_1 \bar{\otimes} D_2 := \overline{D_1 \otimes D_2}$$

for two tilting modules D_1 and D_2. (Note, that this definition coincides with the previous one in $\mathcal{C}_\ell^-$ if both modules M_1 and M_2 are objects in $\mathcal{C}_\ell^-$.)

Proposition 3.5.2 *The reduced tensor product $\bar{\otimes}$ is associative, i.e. given tilting modules D_1, D_2 and D_3 it holds:*

$$(D_1 \bar{\otimes} D_2) \bar{\otimes} D_3 \cong D_1 \bar{\otimes} (D_2 \bar{\otimes} D_3)$$

Proof. By definition of the reduced tensor product we may write $D_1 \otimes D_2 = (D_1 \bar{\otimes} D_2) \oplus Z_{1,2}$ and $D_2 \otimes D_3 = (D_2 \bar{\otimes} D_3) \oplus Z_{2.3}$. Further, $Z_{1,2}$ and $Z_{2,3}$ have the property that all endomorphism have quantum trace 0 (by Theorem 3.3.16). So by Lemma 3.3.20 we have: The quantum trace of every endomorphism of $D_3 \otimes Z_{1,2}$ (respectively $D_1 \otimes Z_{2,3}$) vanishes. In particular:

$$D_3 \bar{\otimes} Z_{1,2} = 0 = D_1 \bar{\otimes} Z_{2,3}.$$

Hence we get:

$$\begin{aligned} (D_1 \bar{\otimes} D_2) \bar{\otimes} D_3 &\cong ((D_1 \bar{\otimes} D_2) \bar{\otimes} D_3) \oplus \underbrace{(Z_{1,2} \bar{\otimes} D_3)}_{=0} \\ &\cong (D_1 \otimes D_2) \bar{\otimes} D_3 \\ &\cong \overline{(D_1 \otimes D_2) \otimes D_3} \\ &\cong \overline{D_1 \otimes (D_2 \otimes D_3)} \\ &\cong D_1 \bar{\otimes} (D_2 \otimes D_3) \\ &\cong D_1 \bar{\otimes} (D_2 \bar{\otimes} D_3). \end{aligned}$$

This proves the associativity of the tensor product. □

Remark 3.5.3 In some sense we can say that the category $\mathcal{C}_\ell^-$ is a quotient of the full subcategory $\mathcal{C}^t$ of the category $\mathcal{C}$, which consists of all tilting modules.

3.6 Quantum Racah Formula for $\mathfrak{sl}_3$

In this section, we only consider $\mathfrak{g} = \mathfrak{sl}_3$. With the quantum Racah formula we can compute the multiplicities of Weyl modules in the tensor product of two Weyl modules. (For the proof see [18].)

Theorem 3.6.1 (Quantum Racah formula) *For $\lambda, \gamma, \nu \in \Lambda_\ell$ it holds:*
The constant $a^\nu_{\lambda,\gamma}$ (from Definition 3.2.1) is given by

$$a^\nu_{\lambda,\gamma} = \sum_{\tau \in \mathcal{W}_\ell} (-1)^{l(\tau)} m_\gamma(\tau_\bullet \nu - \lambda),$$

where $l(\tau)$ is the length of a reduced expression of $\tau \in \mathcal{W}_\ell$ in terms of s_0, s_1, s_2 and $m_\gamma(\mu)$ is the dimension of the μ-weight space in the classical representation (i.e. representation of $U(\mathfrak{sl}_3)$, i.e. non-quantized) of highest weight λ.

Example 3.6.2 For $\ell = 5$, one has

$$\Lambda_5 = \{0, \omega_1, 2\omega_1, (\omega_1 + \omega_2), \omega_2, 2\omega_2\}.$$

- Starting with the easiest possible example: $\Delta(0) \otimes \Delta(0)$

  We have
  $$m_0(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{else.} \end{cases}$$
  So each summand is 0, unless $\tau_\bullet \nu = 0$. Since $\nu \in \Lambda_\ell$, this is only the case for $\tau = \mathrm{id}$ and $\nu = 0$, i.e.:
  $$a^\nu_{0,0} = \begin{cases} 1, & \text{if } \nu = 0, \\ 0, & \text{else.} \end{cases}$$

- Consider now the example $\Delta(\omega_1) \otimes \Delta(\omega_1)$.

  It holds
  $$m_{\omega_1}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \omega_1 \text{ or } \lambda = -\omega_1 + \omega_2 \text{ or } \lambda = -\omega_2, \\ 0, & \text{else.} \end{cases} \tag{5}$$
  So $m_{\omega_1}(\tau_\bullet \nu - \omega_1)$ is unequal zero if $\tau_\bullet \nu$ is one of the three values: $2\omega_1$, ω_2 and $\omega_1 - \omega_2$

  Note that $\tau_\bullet \nu$ never takes the value $\omega_1 - \omega_2$ for $\nu \in \Lambda_5$, since it is on the wall corresponding to s_2 of the fundamental alcove. On the other hand, $2\omega_1$ and ω_2

are in the fundamental alcove, so the only possible value for τ is the identity and therefore for ν only $2\omega_1$ and ω_2, i.e.:

$$a^{\nu}_{\omega_1,\omega_1} = \begin{cases} 1, & \text{if } \nu = 2\omega_1 \text{ or } \nu = \omega_2, \\ 0, & \text{else.} \end{cases}$$

- For a bit more complicated example, let us consider $\Delta(2\omega_1) \otimes \Delta(\omega_1 + \omega_2)$.

  (Note that the classical representation of weight $(\omega_1 + \omega_2) = (\alpha_1 + \alpha_2)$ is actually the adjoint representation of $\mathfrak{sl}_3$.)

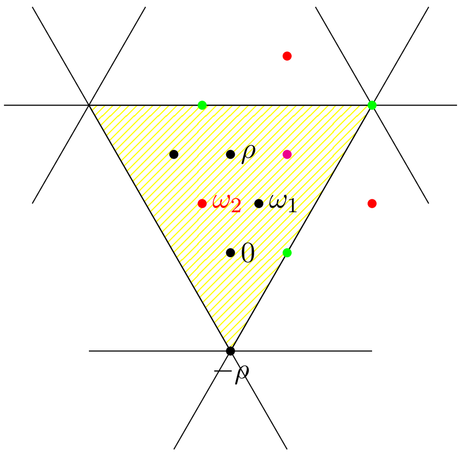
  We have

$$m_{(\omega_1+\omega_2)}(\lambda) = \begin{cases} 2, & \text{if } \lambda = 0, \\ 1, & \text{if } \lambda = \pm(\omega_1 + \omega_2) \text{ or } \lambda = \pm(-\omega_1 + 2\omega_2) \text{ or } \lambda = \pm(2\omega_1 - \omega_2), \\ 0, & \text{else.} \end{cases}$$

  So for $m_{(\omega_1-\omega_2)}(\tau_\bullet \nu - 2\omega_1) \neq 0$, it holds:

$$\tau_\bullet \nu \in \left\{(3\omega_1 + \omega_2), (\omega_1 + 2\omega_2), (4\omega_1 - \omega_2), \omega_2, (\omega_1 - \omega_2), (3\omega_1 - 2\omega_2), 2\omega_1\right\},$$

  where the last value has multiplicity 2.



  We should stress that the values $(\omega_1 + 2\omega_2)$, $(4\omega_1 - \omega_2)$ and $(\omega_1 - \omega_2)$ lie on alcove walls and are not possible for $\nu \in \Lambda_5$. Further, we can see for $\nu = \omega_2$ that we have only one summand:

$$a^{\omega_2}_{2\omega_1,(\omega_1+\omega_2)} = (-1)^{l(\mathrm{id})} m_{(\omega_1+\omega_2)}(\omega_2 - 2\omega_1) = 1.$$

The most interesting value for ν is $2\omega_1$ (the others have only summands which are 0):

$$\begin{aligned} a^{2\omega_1}_{2\omega_1,(\omega_1+\omega_2)} &= \sum_{\tau\in\mathcal{W}_5} (-1)^{l(\tau)} m_{(\omega_1+\omega_2)}(\tau_\bullet(2\omega_1)-2\omega_1) \\ &= (-1)^{l(\mathrm{id})} m_{(\omega_1+\omega_2)}(2\omega_2-2\omega_2) \\ &\quad + (-1)^{l(s_0)} m_{(\omega_1+\omega_2)}(3\omega_1+\omega_2-2\omega_2) \\ &\quad + (-1)^{l(s_2)} m_{(\omega_1+\omega_2)}(3\omega_1-2\omega_2-2\omega_2) \\ &= 2-1-1 \\ &= 0 \end{aligned}$$

So we have:

$$a^{\nu}_{2\omega_1,(\omega_1+\omega_2)} = \begin{cases} 1, & \text{if } \nu=\omega_2, \\ 0, & \text{else.} \end{cases}$$

We can check that the quantum dimensions of $L(2\omega_1)\otimes L(\omega_1+\omega_2)$ and $L(\omega_2)$ really coincide for q a 5^{th} root of unity, namely:

$$\begin{aligned} \mathrm{qdim}\ & \big(L(2\omega_1)\otimes L(\omega_1+\omega_2)\big) \\ &= \mathrm{qdim}(L(2\omega_1))\cdot \mathrm{qdim}(L(\omega_1+\omega_2)) \\ &= (q^{-4}+q^{-2}+2+q^2+q^4)\cdot(q^{-4}+2q^{-2}+2+2q^2+q^4) \\ &= \underbrace{q^{-8}}_{=q^2}+\underbrace{3q^{-6}}_{=3q^4}+6q^{-4}+9q^{-2}+10+9q^2+6q^4+\underbrace{3q^6}_{=3q^{-4}}+\underbrace{q^8}_{=q^{-2}} \\ &= q^{-2}+1+q^2+\underbrace{(9q^{-4}+9q^{-2}+9+9q^2+9q^4)}_{=0} \\ &= \mathrm{qdim}(L(\omega_2)). \end{aligned}$$

Example 3.6.3 Now, let us do an example for a general odd $\ell > h$. We want to compute $\Delta(\lambda)\bar{\otimes}\Delta(\omega_1)$ for any $\lambda\in\Lambda_\ell$.

Again, as in Equation (5) we have

$$\begin{aligned} m_{\omega_1}(\tau_\bullet\nu-\lambda) &= \begin{cases} 1, & \text{if } \tau_\bullet\nu-\lambda\in\{\omega_1,-\omega_1+\omega_2,-\omega_2\}, \\ 0, & \text{else,} \end{cases} \\ &= \begin{cases} 1, & \text{if } \tau_\bullet\nu\in\{\omega_1+\lambda,-\omega_1+\omega_2+\lambda,-\omega_2+\lambda\}, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Further, we note that for any $\lambda\in\Lambda_\ell$ the weights $\omega_1+\lambda, -\omega_1+\omega_2+\lambda$ and $-\omega_2+\lambda$ lie in the closure of the fundamental alcove. Hence, the only possibility for a non-zero summand in the quantum Racah formula is $\tau=\mathrm{id}$.

Thus it holds:

$$\Delta(\lambda)\bar{\otimes}\Delta(\omega_1) = \bigoplus_{\nu\in\{\omega_1+\lambda,-\omega_1+\omega_2+\lambda,-\omega_2+\lambda\}\cap\Lambda_\ell} \Delta(\nu)$$

Example 3.6.4 The quantum Racah formula has a well-understood and explicit variant for $\mathfrak{sl}_2$ as well. Then going back to Example 3.2.3, so $\ell = 5$ and $\Lambda_\ell = \{0, \omega, 2\omega, 3\omega\}$ and we want to calculate $a^{r\omega}_{2\omega,3\omega}$ for $r \in \{0, 1, 2, 3\}$.

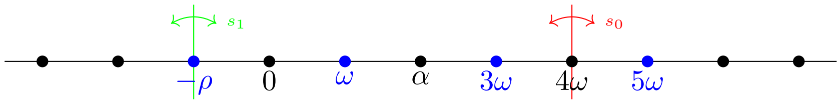
We have

$$m_{3\omega}(\mu) = \begin{cases} 1, & \text{if } \mu \in \{3\omega, \omega, -\omega, -3\omega\}, \\ 0, & \text{else.} \end{cases}$$

Hence, we have

$$m_{3\omega}(\tau.r\omega - 2\omega) \neq 0 \qquad \Longleftrightarrow \qquad \tau.r\omega \in \{5\omega, 3\omega, \omega, -\omega\}$$

Note $-\omega = -\rho$ is ℓ-singular, and thus not a possible value!



The only possible values for τ are id (here $r \in \{1, 3\}$ possible) and s_0 (here $r = 3$ possible). Thus, we get:

$$\begin{aligned} a^{3\omega}_{2\omega,3\omega} &= \underbrace{(-1)^{l(\mathrm{id})} m_{3\omega}(3\omega - 2\omega)}_{=1} + \underbrace{(-1)^{l(s_0)} m_{3\omega}(5\omega - 2\omega)}_{=-1} \\ &= 0, \\ a^{\omega}_{2\omega,3\omega} &= (-1)^{l(\mathrm{id})} m_{3\omega}(\omega - 2\omega) \\ &= 1. \end{aligned}$$

Hence, we have the same constants $a^{r\omega}_{2\omega,3\omega}$ as in Example 3.2.3.

4 A Combinatorial Description for $\mathfrak{g} = \mathfrak{gl}_{n+1}$

In this last section, we discuss a combinatorial description of the category $\mathcal{C}_\ell^-(\mathfrak{gl}_{n+1})$ for $\mathfrak{g} = \mathfrak{gl}_{n+1}$. By taking a quotient of this, one gets a description of the category $\mathcal{C}_\ell^-(\mathfrak{sl}_{n+1})$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$. For this last section, we follow [5, Section 3].

4.1 The Setup for $\mathfrak{gl}_{n+1}$

For $\mathfrak{gl}_{n+1}$, we have the standard Cartan subalgebra given by all diagonal matrices, i.e. $\mathfrak{h} = D_{n+1}$, and as in Section 1 we have the standard basis $(\varepsilon_j)_{j=1}^{n+1}$ of the dual D_{n+1}^*, where $\varepsilon_j(E_{i,i}) = \delta_{i,j}$. Via this basis, we can identify the weight lattice X with $\mathbb{Z}^{n+1}$ and we may write $\lambda \in X$ as $\lambda = \sum_{j=1}^{n+1} \lambda_j \varepsilon_j$ or alternatively $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n+1})$. Further, the positive roots Φ^+ for $\mathfrak{gl}_{n+1}$ are given by the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$ (and the simple roots again by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$). Thus, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n+1}) \in X$ is dominant integral if and only if $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n+1}$.

Remark 4.1.1 The main difference to the $\mathfrak{sl}_{n+1}$ case is that the dimension of the Cartan subalgebra increases by one. In particular, the dual Cartan subalgebra has as basis ε_1, ε_2, ..., ε_{n+1} instead of the differences $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, ..., $\varepsilon_n - \varepsilon_{n+1}$. Accordingly, the definition of the quantum group $U_q(\mathfrak{gl}_{n+1})$ has the generators $\left(D_i^{\pm 1}\right)_{i=1}^{n+1}$ instead of the generators $\left(K_i^{\pm 1}\right)_{i=1}^{n}$. Note however that $U_q(\mathfrak{sl}_{n+1})$ is a Hopf subalgebra of $U_q(\mathfrak{gl}_{n+1})$ generated by the usual Chevalley generators E_i, F_i and $K_i = D_i D_{i+1}^{-1}$ for $i \in \{1, ..., n\}$.

For a more detailed account see e.g. [8, Section 2].

Remark 4.1.2 Note, the theory about tilting modules in section 2 can be shown analogously for the case $\mathfrak{g} = \mathfrak{gl}_{n+1}$. So we assume the construction and statements hold for $U_q(\mathfrak{gl}_{n+1})$.

Further, the integral weights in the fundamental alcove Λ_ℓ are given by the set

$$\Lambda_\ell = \left\{\lambda \in X^+ \mid \lambda_1 - \lambda_{n+1} \leq \ell - n - 1 = k\right\},$$

and explicitly, the fundamental weights $\omega_1, \omega_2, ..., \omega_{n+1}$ are the following

$$\omega_i = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_i \quad \text{for } i \in \{1, ..., n+1\}.$$

Then the weights of the dual Weyl module $\nabla(\omega_i)$ are $\{\varepsilon_{j_1} + \varepsilon_{j_2} + ... + \varepsilon_{j_i} \mid 1 \leq j_1 < j_2 < .... < j_i \leq n+1\}$ where all occurring weights have multiplicity 1, since the ω_i's are minuscule.

Further, observe that for $\mu \in \Lambda_\ell$ it holds $\mu + \nu \in \bar{\Lambda}_\ell$ for all weights ν of $\nabla(\omega_i)$.

We have the following fact for such weights (see [5, Corollary 2.3]):

Lemma 4.1.3 *Suppose $\lambda, \mu \in \bar{\Lambda}_\ell$ are such that $\eta + \mu \in \bar{\Lambda}_\ell$ for all weights η of $\nabla(\lambda)$. Then for any $\nu \in \Lambda_\ell$ we have*

$$a_{\lambda,\mu}^{\nu} = \dim \nabla(\lambda)_{\nu-\mu}$$

In particular, here we get:

Corollary 4.1.4 *Let $i \in \{1, 2, ..., n+1\}$ and $\mu \in \Lambda_\ell$. Then it holds*

$$[\Delta(\omega_i)][\Delta(\mu)] = \sum [\Delta(\varepsilon_{j_1} + \varepsilon_{j_2} + ... + \varepsilon_{j_i} + \mu)], \tag{6}$$

where the sum runs over those i-tuples $1 \leq j_1 < j_2 < ... < j_i \leq n+1$ for which $\varepsilon_{j_1} + \varepsilon_{j_2} + ... + \varepsilon_{j_i} + \mu \in \Lambda_\ell$.

Corollary 4.1.5 *Let $\mu \in \Lambda_\ell$, $r \in \mathbb{Z}$. Then it holds*

$$[\Delta(r\omega_{n+1})][\Delta(\mu)] = [\Delta(\mu + r\omega_{n+1})].$$

Proof. It holds $[\Delta(\omega_{n+1})][\Delta(\lambda)] = [\Delta(\lambda + \omega_{n+1})]$ for all $\lambda \in \Lambda_\ell$ by Corollary 4.1.4. Hence we get by induction on r for $r > 1$:

$$\begin{aligned} [\Delta(r\omega_{n+1})][\Delta(\mu)] &= [\Delta(\omega_{n+1})][\Delta((r-1)\omega_{n+1})][\Delta(\mu)] \\ &= [\Delta(\omega_{n+1})][\Delta(\mu + (r-1)\omega_{n+1}] \\ &= [\Delta(\mu + r\omega_{n+1})]. \end{aligned}$$

We also note $[\Delta(\omega_{n+1})]$ has multiplicative inverse $[\Delta(-\omega_{n+1})]$. In particular, the claim also holds for $r < 0$. This finishes the proof. □

4.1.1 A Presentation of Weights

The action of the operators $\mathbf{a}_i$, which we define in the next subsection, can be realized graphically in two ways. But first let us discuss the presentation of weights by a formal factor $z^{\lambda_{n+1}}$ and a configuration.

Given a weight $\lambda \in \Lambda_\ell$, the differences $m_i = \lambda_i - \lambda_{i+1}$ and λ_{n+1} encode $\lambda = (\lambda_1, ..., \lambda_{n+1})$ in the basis of fundamental weights:

$$\lambda = \lambda_{n+1}\omega_{n+1} + \sum_{i=1}^{n} m_i \omega_i.$$

Further, it holds for a weight $\lambda \in X^+$ (i.e. $m_1, ..., m_n \geq 0$):

$$\lambda \in \Lambda_\ell \qquad \Longleftrightarrow \qquad m = \sum_{i=1}^{n} m_i \leq k.$$

Also we set $m_0 = k - m$.

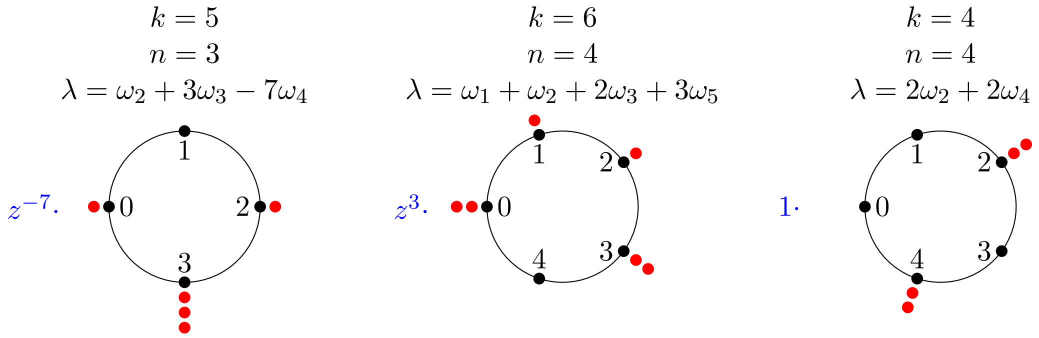
Example 4.1.6 Let $n = 4$ and $\lambda = (12, 10, 10, 8, 6)$. We can write λ as

$$\lambda = 6\omega_5 + 4\omega_1 + 0\omega_2 + 2\omega_3 + 2\omega_4,$$

and here it holds $m = 6$ and $m_0 = k - 6$.

A weight $\lambda \in \Lambda_\ell$ can be viewed as a configuration of k particles on a circle with $n+1$ marked points with m_i particles at place i and an additional formal factor $z^{\lambda_{n+1}}$, which we use to distinguish between multiples of ω_{n+1}. The circle can be viewed as an extended Dynkin diagram of type $\tilde{A}_n$.

Example 4.1.7 Here are some configurations with z-factor for some different values of k, n and λ.



4.2 The Operators $\mathbf{a}_i$

In the following we will consider the free $\mathbb{Z}$-module $\mathbb{Z}(\Lambda_\ell)$ with basis set Λ_ℓ. We denote its basis elements by e^λ for $\lambda \in \Lambda_\ell$.

Definition 4.2.1 Let $0 \le i \le n$. We define the linear operator $\mathbf{a}_i$ on $\mathbb{Z}(\Lambda_\ell)$ on the basis elements e^λ for $\lambda \in \Lambda_\ell$ via

$$\mathbf{a}_i(e^\lambda) = \begin{cases} e^{\lambda+\varepsilon_{i+1}}, & \text{if } \lambda + \varepsilon_{i+1} \in \Lambda_\ell, \\ 0, & \text{else.} \end{cases}$$

Remark 4.2.2 If we write $\lambda \in \Lambda_l$ in terms of the fundamental weights, then we have

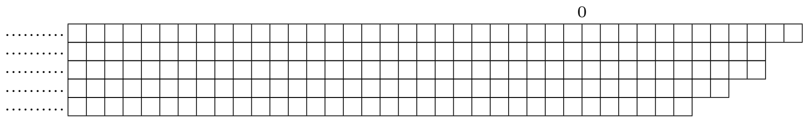
$$e^{\sum_{j=1}^{n+1} m_j\omega_j} \xmapsto{\mathbf{a}_i} e^{\sum_{j=1}^{i-1} m_j\omega_j + (m_i-1)\omega_i + (m_{i+1}+1)\omega_{i+1} + \sum_{j=i+2}^{n+1} m_j\omega_j},$$

if $m_i \ge 1$ and it is sent to 0 otherwise.

Now, we discuss two different ways to view the operators $\mathbf{a}_i$'s.

I. We may represent an arbitrary $\lambda = (\lambda_1, ..., \lambda_{n+1}) \in X$ by $n+1$ rows of boxes where the i^{th} row is infinite to the left and stops at number λ_i (which may be negative). The weights in Λ_ℓ are exactly the weights where the rows are non-increasing and there are at most k more boxes in the first row than in the last.

   Example 4.2.3 As in Example 4.1.6, let $n = 4$, $\lambda = (12, 10, 10, 8, 6)$.



In this presentation of the weights, the operator $\mathbf{a}_i$ adds a box in the $(i+1)^{\text{th}}$ row if after adding it is still an element in Λ_ℓ. Otherwise the weight is killed.

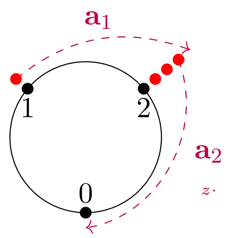
II. The second presentation of the operator $\mathbf{a}_i$ is on the set of configurations with formal z-factor (described in Subsubsection 4.1.1).

Then the operators $\mathbf{a}_i$ act on such a configuration by taking a particle at the i^{th} node to the $(i+1)^{th}$ node (identifying the $(n+1)^{\text{th}}$ node with the 0^{th} node) and if there is no particle at the i^{th} node, it kills the configuration. Further, the operator $\mathbf{a}_n$ acts additionally by multiplication by z to the z-factor. This way the operators can be viewed as "particle hopping" from one node to the next in a clockwise direction.

Remark 4.2.4 Starting at a particle and sending it with the operators $\mathbf{a}_i$'s clockwise around the circle does not change the configuration, but one would have added ω_{n+1} to λ. Hence, one cannot differentiate between adding multiples of ω_{n+1} by only looking at the configuration, but then we can distinguish them by the exponent of z.

Remark 4.2.5 In the reference [5] one should be careful, since they forgot to mention the factor z.

Example 4.2.6 Consider $k=4$, $n=2$ and $\lambda = \omega_1 + 3\omega_2$. Then the configuration corresponding to λ is the following:



where the operator $\mathbf{a}_0$ would act by 0, since there is no particle at the node 0.

Remark 4.2.7 Note, that the operators do not commute in general! One can easily see in the above example, namely $\mathbf{a}_2\mathbf{a}_0(\lambda) = 0$ but $\mathbf{a}_0\mathbf{a}_2(\lambda) = 2\omega_1 + 2\omega_2 + \omega_3$.

In general, it is easy to see that $\mathbf{a}_i$ and $\mathbf{a}_j$ commute if and only if i and j are not neighbors in the circle (i.e. $|i-j| \neq 1 \bmod n+1$).

More precisely, if one ignores the additional factor z of the action of $\mathbf{a}_n$, then they generate the affine local plactic algebra $\mathrm{Pl}(A)$ (see [15, Section 5.1]), which can be defined as the free algebra generated by the elements of $A = \{\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_n\}$ modulo the relations

$$\begin{aligned} \mathbf{a}_i\mathbf{a}_j - \mathbf{a}_j\mathbf{a}_i &= 0 \quad \text{if } i \neq j \pm 1, \\ \mathbf{a}_{i+1}\mathbf{a}_i^2 &= \mathbf{a}_i\mathbf{a}_{i+1}\mathbf{a}_i, \\ \mathbf{a}_{i+1}^2\mathbf{a}_i &= \mathbf{a}_{i+1}\mathbf{a}_i\mathbf{a}_{i+1} \end{aligned}$$

for $i, j \in \{0, 1, ..., n\}$ and read cyclically, i.e. $\mathrm{mod}(n+1)$.

4.3 The Combinatorial Fusion Ring $\mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$

The combinatorial fusion ring $\mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$ has as a $\mathbb{Z}$-basis the weights in Λ_ℓ, but for the multiplication we need to use non-commutative Schur polynomials with argument $\underline{\mathbf{a}} = (\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_n)$.

The following definitions can be found in [5, Section 3.1], which also uses references to [15, Section 5.3].

Definition 4.3.1 Set $\underline{\mathbf{a}} = (\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_n)$. Further, let $I \subsetneq \{0, 1, ..., n\}$. We define $\underline{\mathbf{a}}_I$ as the product over I of its elements in anticlockwise cyclical order, i.e.:

A monomial $\mathbf{a}_{j_1}\mathbf{a}_{j_2}...\mathbf{a}_{j_r}$ is in <u>anticlockwise cyclical order</u>, if for any two indices j_i and j_l with $j_i = j_l + 1 \bmod n+1$ the variable $\mathbf{a}_{j_l}$ occurs to the right of $\mathbf{a}_{j_i}$.

Further, we define the non-commutative <u>elementary symmetric polynomials</u> $\mathbf{e}_1(\underline{\mathbf{a}})$, $\mathbf{e}_2(\underline{\mathbf{a}})$, ..., $\mathbf{e}_n(\underline{\mathbf{a}})$ by

$$\mathbf{e}_r(\underline{\mathbf{a}}) = \sum_{\substack{I \subset \{0,1,\dots,n\}: \\ |I| = r}} \underline{\mathbf{a}}_I.$$

By convention we set $\mathbf{e}_r(\underline{\mathbf{a}}) = 0$ for $r < 0$ and $r > n+1$, $\mathbf{e}_0(\underline{\mathbf{a}}) = 1$ is defined as the identity and $\mathbf{e}_{n+1}(\underline{\mathbf{a}}) = z \cdot 1$ the multiplication by the indeterminate z.

Example 4.3.2

- Let $n = 2$. Then

$$\begin{aligned} \mathbf{e}_1(\underline{\mathbf{a}}) &= \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2, \\ \mathbf{e}_2(\underline{\mathbf{a}}) &= \mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_2\mathbf{a}_1 + \mathbf{a}_0\mathbf{a}_2. \end{aligned}$$

- Let $n = 3$. Then

$$\begin{aligned} \mathbf{e}_1(\underline{\mathbf{a}}) &= \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3, \\ \mathbf{e}_2(\underline{\mathbf{a}}) &= \mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_2\mathbf{a}_0 + \mathbf{a}_0\mathbf{a}_3 + \mathbf{a}_2\mathbf{a}_1 + \mathbf{a}_1\mathbf{a}_3 + \mathbf{a}_3\mathbf{a}_2, \\ \mathbf{e}_3(\underline{\mathbf{a}}) &= \mathbf{a}_2\mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_1\mathbf{a}_0\mathbf{a}_3 + \mathbf{a}_0\mathbf{a}_3\mathbf{a}_2 + \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1. \end{aligned}$$

- Let $n = 4$. Then

$$\begin{aligned} \mathbf{e}_3(\underline{\mathbf{a}}) = {} & \mathbf{a}_2\mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_1\mathbf{a}_0\mathbf{a}_3 + \mathbf{a}_1\mathbf{a}_0\mathbf{a}_4 + \mathbf{a}_0\mathbf{a}_3\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_0\mathbf{a}_4 + \mathbf{a}_0\mathbf{a}_4\mathbf{a}_3 \\ & + \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1 + \mathbf{a}_2\mathbf{a}_1\mathbf{a}_4 + \mathbf{a}_1\mathbf{a}_4\mathbf{a}_3 + \mathbf{a}_4\mathbf{a}_3\mathbf{a}_2. \end{aligned}$$

Even though the operators $\mathbf{a}_i$ and $\mathbf{a}_j$ for neighbored i and j do not commute, we have the following fact about the just defined polynomials (see [15, Corollary 5.14] or [17]).

Lemma 4.3.3 *The elementary symmetric polynomials pairwise commute.*

With the non-commutative elementary symmetric polynomials we can now define the non-commutative Schur polynomials, which are well-defined because of the above lemma.

Definition 4.3.4 Let $\lambda \in \Lambda_\ell$, m, m_1, ..., m_n as above. Then λ^t denotes the partition with m_i rows of length i for each $i \in \{1, ..., n\}$.

The non-commutative Schur polynomial of λ is given by

$$s_\lambda(\underline{\mathbf{a}}) = z^{\lambda_{n+1}} \cdot \det\left(\mathbf{e}_{\lambda_i^t - i + j}(\underline{\mathbf{a}})\right)_{i,j=1}^{m},$$

and if λ^t is the empty partition, we set $s_\lambda(\underline{\mathbf{a}}) = z^{\lambda_{n+1}}$.

Example 4.3.5

- Let $n \in \mathbb{N}$, $i \in \{1, ..., n\}$ and $\lambda = \omega_i$. Then we have $m = 1$ and $\lambda_1^t = i$. Hence,
$$s_\lambda(\underline{\mathbf{a}}) = \det(\mathbf{e}_i(\underline{\mathbf{a}})) = \mathbf{e}_i(\underline{\mathbf{a}}).$$
- Let $n = 3$, $\lambda = (4, 3, 2, 2)$. Then we have $m_1 = 1$, $m_2 = 1$, $m_3 = 0$ and $m = 2$. So we have $\lambda^t = (2, 1)$ and
$$\begin{aligned} s_\lambda(\underline{\mathbf{a}}) &= z^2 \cdot \det\begin{pmatrix} \mathbf{e}_{\lambda_1^t-1+1}(\underline{\mathbf{a}}) & \mathbf{e}_{\lambda_1^t-1+2}(\underline{\mathbf{a}}) \\ \mathbf{e}_{\lambda_2^t-2+1}(\underline{\mathbf{a}}) & \mathbf{e}_{\lambda_2^t-2+2}(\underline{\mathbf{a}}) \end{pmatrix} \\ &= z^2 \cdot \det\begin{pmatrix} \mathbf{e}_2(\underline{\mathbf{a}}) & \mathbf{e}_3(\underline{\mathbf{a}}) \\ \mathbf{e}_0(\underline{\mathbf{a}}) & \mathbf{e}_1(\underline{\mathbf{a}}) \end{pmatrix} \\ &= z^2 \cdot (\mathbf{e}_2(\underline{\mathbf{a}})\mathbf{e}_1(\underline{\mathbf{a}}) - \underbrace{\mathbf{e}_0(\underline{\mathbf{a}})}_{=1}\mathbf{e}_3(\underline{\mathbf{a}})). \end{aligned}$$

Using the non-commutative Schur polynomials we can now define the combinatorial fusion ring:

Definition 4.3.6 The combinatorial fusion ring $\mathcal{F}_\ell^c(\mathfrak{gl}_{n+1})$ is defined as the free $\mathbb{Z}$-module $\mathbb{Z}(\Lambda_\ell)$ with basis $(e^\lambda)_{\lambda \in \Lambda_\ell}$ equipped with the following multiplication for $\lambda, \mu \in \Lambda_\ell$

$$e^\lambda \star e^\mu = s_\lambda(\underline{\mathbf{a}})(e^\mu).$$

Example 4.3.7

- Let $n \in \mathbb{N}$, $\mu \in \Lambda_\ell$ and $0 \in \Lambda_\ell$ be the 0 weight. Then
$$e^0 \star \mu = \underbrace{s_0(\underline{\mathbf{a}})}_{=1} e^\mu = e^\mu.$$

Thus, $e^0 \in \Lambda_\ell$ is the multiplicative identity in $\mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$.

- Let $n = 3$, $\lambda = \omega_2$. Then by Example 4.3.5 we know $s_\lambda(\underline{\mathbf{a}}) = \mathbf{e}_2(\underline{\mathbf{a}})$. Then for $I = \{0, 2\}$ and $\mu \in \Lambda_\ell$ we have

$$\begin{aligned}\underline{\mathbf{a}}_I(e^\mu) = \mathbf{a}_2\mathbf{a}_0(e^\mu) &= \begin{cases}\mathbf{a}_2(e^{\varepsilon_1+\mu}), & \text{if } \varepsilon_1 + \mu \in \Lambda_\ell,\\ 0, & \text{else,}\end{cases}\\ &= \begin{cases}e^{\varepsilon_3+\varepsilon_1+\mu} & \text{if } \varepsilon_3 + \varepsilon_1 + \mu \in \Lambda_\ell,\\ 0, & \text{else.}\end{cases}\end{aligned}$$

Similarly, we get the analogous result for each summand of $\mathbf{e}_2(\underline{\mathbf{a}})(e^\mu)$. Hence, we get

$$e^\lambda \star e^\mu = \sum_{\substack{j_1<j_2:\\ \varepsilon_{j_1+1}+\varepsilon_{j_2}+\mu\in\Lambda_\ell}} e^{\varepsilon_{j_1+1}+\varepsilon_{j_2}+\mu},$$

where the sum is the formal sum of weights and not the sum in the weights lattice.

- More general, as in Example 4.3.5, consider $n \in \mathbb{N}$, $i \in \{1, ..., n\}$ and $\lambda = \omega_i$. Then for $\mu \in \Lambda_\ell$ we have by definition

$$e^\lambda \star e^\mu = s_\lambda(\underline{\mathbf{a}})(e^\mu) = \mathbf{e}_i(\underline{\mathbf{a}})(e^\mu).$$

Let $I = \{j_1 < j_2 < ... < j_i\} \subset \{0, 1, ..., n\}$. Then each consecutive row of $\mathbf{a}_{j_r}\mathbf{a}_{j_r-1}...\mathbf{a}_{j_s}$ (in anticlockwise cyclical order) adds $\varepsilon_{j_r} + ... + \varepsilon_{j_s}$ to μ if it is still a weight in Λ_ℓ and it does not change if another consecutive row acted first. Hence, in formulas we have

$$\underline{\mathbf{a}}_I(e^\mu) = \begin{cases}e^{\varepsilon_{j_1+1}+\varepsilon_{j_2+1}+...+\varepsilon_{j_i+1}+\mu}, & \text{if } \varepsilon_{j_1+1} + \varepsilon_{j_2+1} + ... + \varepsilon_{j_i+1} + \mu \in \Lambda_\ell,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, it holds

$$e^{\omega_i} \star e^\mu = \mathbf{e}_i(\underline{\mathbf{a}}(e^\mu)) = \sum_{\substack{j_1<j_2<....<j_i:\\ \varepsilon_{j_1}+...+\varepsilon_{j_i}+\mu\in\Lambda_\ell}} e^{\varepsilon_{j_1}+...+\varepsilon_{j_i}+\mu}.$$

Now that we defined the combinatorial fusion ring $\mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$, we want to identify it with the Grothendieck ring $\mathcal{F}$ of $\mathcal{C}^-_\ell(\mathfrak{gl}_{n+1})$ (see [5, Theorem 3.1]).

Theorem 4.3.8 *The map $\Xi : \mathcal{F} \to \mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$ taking each basis element $[\Delta(\lambda)] \in \mathcal{F}$ to the basis element $e^\lambda \in \mathcal{F}^c_\ell(\mathfrak{gl}_{n+1})$ is a ring isomorphism.*

Proof. We prove $\Xi([\Delta(\lambda)][\Delta(\mu)]) = e^\lambda \star e^\mu$ by induction on $m = \lambda_1 - \lambda_{n+1}$.

- The case $m = 0$ follows by Corollary 4.1.5, namely for $\lambda = r\omega_{n+1}$ it holds

  $$\Xi([\Delta(r\omega_{n+1})][\Delta(\mu)]) = \Xi([\Delta(r\omega_{n+1} + \mu)]) = e^{r\omega_{n+1}+\mu} = z^r \cdot e^\mu = e^{r\omega_{n+1}} \star e^\mu$$

  For the cases $m > 0$ we only need to consider $\lambda \in \Lambda_\ell$ with $\lambda_{n+1} = 0$. Otherwise write $[\Delta(\lambda_{n+1}\omega_{n+1} + \lambda')] = [\Delta(\lambda_{n+1}\omega_{n+1})][\Delta(\lambda')]$ and apply the case $m = 0$.

- Now let $m = 1$, i.e. $\lambda = \omega_i$ for some $i \in \{1, ..., n\}$. Then by Corollary 4.1.4 and Example 4.3.7 the summands appearing in each product are the same, i.e. $\Xi([\Delta(\omega_i)][\Delta(\mu)]) = e^{\omega_i} \star e^\mu$.

- Now let $m > 1$. Then we may write $\lambda = \lambda' + \omega_i$ for some $i \in \{1, ..., n\}$. Let $i \in \{1, ..., n\}$ be minimal with this property. We prove the claim by induction on i.

  - For $i = 1$, it holds

    $$[\Delta(\lambda')][\Delta(\omega_i)] = [\Delta(\lambda)] + \sum_\eta [\Delta(\eta)], \tag{7}$$

    for some $\eta \in \Lambda_\ell$, where each η satisfies $\eta_1 - \eta_{n+1} < m$ (since in equation (6) in Corollary 4.1.4 there is only one summand where $\eta_1 > \lambda'_1$, namely λ). Then by the induction hypothesis (of the induction on m) we have

    $$\begin{aligned}
    \Xi([\Delta(\lambda)][\Delta(\mu)]) &= \Xi([\Delta(\lambda')][\Delta(\omega_i)][\Delta(\mu)]) - \sum_\eta \Xi([\Delta(\eta)][\Delta(\mu)]) && (8)\\
    &= e^{\lambda'} \star \Xi([\Delta(\omega_i)][\Delta(\mu)]) - \sum_\eta e^\eta \star e^\mu && (9)\\
    &= e^{\lambda'} \star e^{\omega_i} \star e^\mu - \sum_\eta e^\eta \star e^\mu && (10)\\
    &= e^\lambda \star e^\mu. && (11)
    \end{aligned}$$

  - For $i > 1$, the equality in (7) still holds, but the $\eta \in \Lambda_\ell$ might also have the property $\eta_1 - \eta_{n+1} = m$. But these latter ones may be written as $\eta + \omega_j$ where $j < i$. Hence, using both induction hypotheses the claim follows by the same calculation as in (8).

This finishes the proof of the theorem. □

4.4 The Combinatorial Fusion Ring $\mathcal{F}_\ell^c(\mathfrak{sl}_{n+1})$

To distinguish between $\mathfrak{gl}_{n+1}$ and $\mathfrak{sl}_{n+1}$, we write $X(\mathfrak{gl}_{n+1})$, $X(\mathfrak{sl}_{n+1})$, $\mathcal{C}_\ell^-(\mathfrak{gl}_{n+1})$, ...

In this last part, we only give the idea and the one important statement without further details from [5, Section 3.2].

We may identify $X(\mathfrak{sl}_{n+1}) = \left\{\lambda \in X(\mathfrak{gl}_{n+1}) \mid \lambda_{n+1} = 0\right\}$. Then any $\lambda \in X(\mathfrak{gl}_{n+1})$ equals a unique element in $X(\mathfrak{sl}_{n+1})$ modulo a multiple of ω_{n+1}. Similarly, we have $X^+(\mathfrak{sl}_{n+1}) = X(\mathfrak{sl}_{n+1}) \cap X^+(\mathfrak{gl}_{n+1})$ and $\Lambda_\ell(\mathfrak{sl}_{n+1}) = X(\mathfrak{sl}_{n+1}) \cap \Lambda_\ell(\mathfrak{gl}_{n+1})$.

Note, that $\Lambda_\ell(\mathfrak{sl}_{n+1})$ is finite, even though $\Lambda_\ell(\mathfrak{gl}_{n+1})$ is not.

By restriction of $\mathfrak{gl}_{n+1}$ to $\mathfrak{sl}_{n+1}$, we get surjections of the Grothendieck rings of the categories $\mathcal{C}_\ell^-(\mathfrak{gl}_{n+1})$ and $\mathcal{C}_\ell^-(\mathfrak{sl}_{n+1})$, namely $\mathcal{F}(\mathfrak{gl}_{n+1}) \twoheadrightarrow \mathcal{F}(\mathfrak{sl}_{n+1})$ and of the combinatorial fusion rings $\mathcal{F}_\ell^c(\mathfrak{gl}_{n+1}) \twoheadrightarrow \mathcal{F}_\ell^c(\mathfrak{sl}_{n+1})$.

In particular, we obtain the following isomorphisms:

Theorem 4.4.1 *We have ring isomorphisms*

$$\mathcal{F}(\mathfrak{sl}_{n+1}) \cong \mathcal{F}(\mathfrak{gl}_{n+1})/([\Delta(\omega_{n+1})] - 1)$$

and

$$\mathcal{F}_\ell^c(\mathfrak{sl}_{n+1}) \cong \mathcal{F}_\ell^c(\mathfrak{gl}_{n+1})/(e^{\omega_{n+1}} - 1),$$

such that the isomorphism Ξ from Theorem 4.3.8 induces an isomorphism

$$\mathcal{F}(\mathfrak{sl}_{n+1}) \cong \mathcal{F}_\ell^c(\mathfrak{sl}_{n+1}).$$

Hence, to describe $\mathcal{F}(\mathfrak{sl}_{n+1})$ combinatorially with the configurations and the operators $\underline{\mathbf{a}}$, we do not need to keep track of the z-factor since we set $z = 1$ via the isomorphism. Thus, here the operator $\mathbf{a}_n$ has no additional multiplication by z. The $\mathbb{Z}$-basis is solely given by the configurations with k particles.

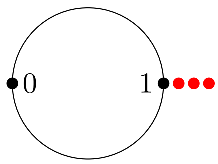
So instead of using the quantum Racah formula (Theorem 3.6.1), we can compute the multiplicities in the combinatorial fusion ring $\mathcal{F}_\ell^c(\mathfrak{sl}_{n+1})$.

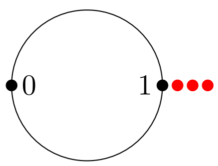
Example 4.4.2 • First, we may look at the Example 3.2.3/Example 3.6.4, i.e. $n = 1$ and $\ell = 5$. We want to calculate $e^{2\omega} \star e^{3\omega}$.

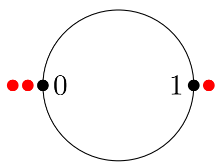
For $\lambda = 2\omega$ we have $m = m_1 = 2$. Hence, we get $\lambda^t = (1, 1)$ and

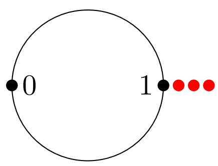
$$s_\lambda(\underline{\mathbf{a}}) = \det \begin{pmatrix} \mathbf{e}_1(\underline{\mathbf{a}}) & \mathbf{e}_2(\underline{\mathbf{a}}) \\ \mathbf{e}_0(\underline{\mathbf{a}}) & \mathbf{e}_1(\underline{\mathbf{a}}) \end{pmatrix} = (\mathbf{e}_1(\underline{\mathbf{a}}))^2 - 1.$$

Further, we have $\mathbf{e}_1(\underline{\mathbf{a}}) = \mathbf{a}_0 + \mathbf{a}_1$ and for $\lambda = 3\omega$ we have $m = m_1 = 3 = k$ and $m_0 = 0$, in particular $\mathbf{a}_0(e^{3\omega}) = 0$. Thus, it holds:

$$e^{2\omega} \star e^{3\omega} = (\mathbf{a}_0^2 + \mathbf{a}_0\mathbf{a}_1 + \mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_1^2 - 1)$$ 

$=$ 

$+$ 

$-$ 

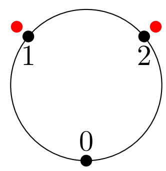
$= e^\omega.$

- Going back to Example 3.6.2, i.e. $n = 2$ and $\ell = 5$, we want to compute $e^{2\omega_1} \star e^{\omega_1+\omega_2}$. With the quantum Racah formula, the computation was rather lengthy.

  For $\lambda = 2\omega_1$, we have $m = m_1 = 2 = k$. Hence, as above we have $\lambda^t = (1, 1)$. It follows

$$\begin{aligned} s_\lambda &= \det \begin{pmatrix} \mathbf{e}_1(\underline{\mathbf{a}}) & \mathbf{e}_2(\underline{\mathbf{a}}) \\ \mathbf{e}_0(\underline{\mathbf{a}}) & \mathbf{e}_1(\underline{\mathbf{a}}) \end{pmatrix} = (\mathbf{e}_1(\underline{\mathbf{a}}))^2 - \mathbf{e}_2(\underline{\mathbf{a}}) \\ &= (\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2)^2 - \mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_2\mathbf{a}_1 + \mathbf{a}_0\mathbf{a}_2 \\ &= \mathbf{a}_0^2 + \mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_0\mathbf{a}_1 + \mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_0. \end{aligned}$$

  For $\lambda = \omega_1 + \omega_2$, we have $m_1 = m_2 = 1$ and $m_0 = 0$. Hence, the configuration corresponding to $\omega_1 + \omega_2$ is:

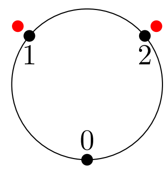


  Note:

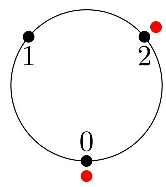
$$\mathbf{a}_0^2(e^{\omega_1+\omega_2}) = \mathbf{a}_1^2(e^{\omega_1+\omega_2}) = \mathbf{a}_2^2(e^{\omega_1+\omega_2}) = \mathbf{a}_0\mathbf{a}_1(e^{\omega_1+\omega_2}) = \mathbf{a}_2\mathbf{a}_0(e^{\omega_1+\omega_2}) = 0.$$

  Thus, it holds:

$$e^{2\omega_1} \star e^{\omega_1+\omega_2} = \mathbf{a}_1\mathbf{a}_2$$



$$=$$



$$= e^{\omega_2}.$$

Now we know two ways to compute the products in the fusion ring $\mathcal{F}(\mathfrak{sl}_{n+1})$ of the category $\mathcal{C}_\ell^-(\mathfrak{sl}_{n+1})$, namely with the quantum Racah formula and in the combinatorial fusion ring. In comparison, doing the computations in the combinatorial fusion ring $\mathcal{F}_\ell^c(\mathfrak{sl}_{n+1})$ is for higher n and ℓ a more straightforward and less complicated calculation. Further, we may always check our results by comparing the quantum dimensions, which are given by the Weyl character formula.

4.4.1 The Even Case in Type A

Up till now we worked with the assumption that ℓ is odd. In the case of ℓ even, we set $\ell' = \frac{\ell}{2}$. Then we may replace ℓ by ℓ' and the statements remain true, since in type A all roots have the same lengths. But we have to assume that $\ell \geq 2n$, since otherwise the set of weights in the fundamental alcove is empty. For further information see e.g. [3, Section 3] and [5, Subsection 3.2.1].

4.5 The Combinatorial Fusion Ring in Type C

We will finish by briefly discussing Fusion rings in Lie type C.

In the case $\mathfrak{g} = \mathfrak{sp}_{2n}$, one can do a similar construction with operators $\mathbf{a}_i$'s to define the combinatorial fusion ring $\mathcal{F}_\ell^c(\mathfrak{sp}_{2n})$ (see [5, Section 4]).

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we may again identify the integral weights X with $\mathbb{Z}^n$ with basis $(\varepsilon_i)_{i=1}^n$. The positive roots are given by the set $\{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \mid 1 \leq i \leq n\}$ and the simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in \{1, ..., n-1\}$ and $\alpha_n = 2\varepsilon_n$.

As in type A, the fundamental weights are given by $\omega_i = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_i$ for $i \in \{1, ..., n\}$. In particular, we have

$$\rho = \frac{1}{2}\left(\sum_{1\leq i<j\leq n} (\varepsilon_i - \varepsilon_j + \varepsilon_i + \varepsilon_j) + \sum_{i=1}^{n} 2\varepsilon_i\right) = \sum_{i=1}^{n}(n+1-i)\varepsilon_i = \sum_{i=1}^{n}\omega_i.$$

A weight $\lambda \in X$ may be expressed as a sum of the ε_i's, say $\lambda = \sum_{i=1}^n \lambda_i\varepsilon_i$, and as a sum of the fundamental weights, say $\lambda = \sum_{i=1}^n m_i\omega_i$, and again we have $m_i = \lambda_i - \lambda_{i+1}$ for $i \in \{1, ..., n\}$ (and setting $\lambda_{n+1} = 0$). In this notation, we may identify the dominant weights via

$$\lambda \in X^+ \quad \Longleftrightarrow \quad \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0 \quad \Longleftrightarrow \quad m_i \geq 0 \text{ for all } i \in \{1, ..., n\}.$$

The highest short and the highest long root are

$$\begin{aligned} \alpha_0 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{n-1} + \alpha_n = \varepsilon_1 + \varepsilon_2, \\ \beta_0 &= 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{n-1} + \alpha_n = 2\varepsilon_1. \end{aligned}$$

For ℓ odd we have

$$\begin{aligned} \Lambda_\ell &= \{\ \lambda \in X^+ \mid \langle \lambda + \rho, \alpha_0^\vee \rangle < \ell\} \\ &= \{\lambda \in X^+ \mid m_1 + 2m_2 + 2m_3 + ... + 2m_n < \ell - 2n + 1\}, \end{aligned}$$

and for ℓ even (set $\ell' = \frac{\ell}{2}$)

$$\begin{aligned} \Lambda_\ell &= \{\lambda \in X^+ \mid \langle \lambda + \rho, \beta_0^\vee \rangle < \ell'\} \\ &= \{\lambda \in X^+ \mid m_1 + m_2 + ... + m_n < \ell' - n\}. \end{aligned}$$

In both cases we need the assumption $\ell > 2n = h$ since otherwise $\Lambda_\ell = \emptyset$.

In this setup, we now may define the operators $\mathbf{a}_i$, $\mathbf{e}_i$ and s'_λ as follows.

Definition 4.5.1 Consider the free $\mathbb{Z}$-modules $\mathbb{Z}(X)$ and $\mathbb{Z}(\Lambda_\ell)$ with bases $(e^\lambda)_{\lambda\in X}$ respectively $(e^\lambda)_{\lambda\in\Lambda_\ell}$. We define the $\mathbb{Z}$-linear map $\pi_\ell : \mathbb{Z}(X) \to \mathbb{Z}(\Lambda_\ell)$ on the basis elements e^λ via

$$\pi_\ell(e^\lambda) = \begin{cases} (-1)^{l(\omega)} e^{\omega_\bullet\lambda}, & \text{if there exists } \omega \in \mathcal{W}_\ell \text{ with } \omega_\bullet\lambda \in \Lambda_\ell, \\ 0, & \text{else.} \end{cases}$$

To define the operators $\mathbf{a}_i$'s, we set $\varepsilon_{n+i} = -\varepsilon_i$ for $i \in \{1, ..., n\}$.

For $j \in \{1, 2, ..., 2n\}$ we define the $\mathbb{Z}$-linear operators $\mathbf{a}_j$ on $\mathbb{Z}(X)$ on the basis element e^λ by

$$\mathbf{a}_j(e^\lambda) = e^{\lambda+\varepsilon_j}.$$

For each subset $J = \{j_1 < j_2 < ... < j_i\} \subset \{1, 2, ..., 2n\}$ we set $\underline{\mathbf{a}}_J = \mathbf{a}_{j_1}\mathbf{a}_{j_2}...\mathbf{a}_{j_i}$. Then for $1 \leq i \leq n$ we define the $\mathbb{Z}$-linear operators $\mathbf{e}_i$ on $\mathbb{Z}(\Lambda_\ell)$ on the basis element e^λ by

$$\mathbf{e}_i(e^\lambda) = \sum_{J\subset\{1,...,2n\}:\ |J|=i} \pi_\ell(\underline{\mathbf{a}}_J(e^\lambda)).$$

Remark 4.5.2
- One may again view the operators as particle hopping on some extended Dynkin diagram (see [5, Subsection 4.3]), but this we will not discuss here any further.
- It holds again: The operators $\mathbf{e}_i$ and $\mathbf{e}_j$ commute for all $1 \leq i, j \leq n$ (see [5, Proposition 4.1]).

Definition 4.5.3 We set $\mathbf{e}'_0 = 1$, $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_i = \mathbf{e}_i - \mathbf{e}_{i-2}$ for $2 \leq i \leq n$ and $\mathbf{e}'_i = 0$ otherwise. Then analogously to Definition 4.3.4 we consider for $\lambda \in \Lambda_\ell$ the transposed partition λ^t consisting of m_i rows of lengths i and we define the operators s'_λ on $\mathbb{Z}(\Lambda_\ell)$ by

$$s'_\lambda = \det\left(\mathbf{e}'_{\lambda^t_i - i + j}\right)_{i,j=1}^m.$$

Definition 4.5.4 We define the <u>combinatorial fusion ring</u> $\mathcal{F}^c_\ell(\mathfrak{sp}_{2n})$ as the free $\mathbb{Z}$-module $\mathbb{Z}(\Lambda_\ell)$ with the following multiplication for the basis elements e^λ and e^μ

$$e^\lambda \star e^\mu = s'_\lambda(e^\mu).$$

We have the result (see [5, Theorem 4.2]):

Theorem 4.5.5 *For any $\ell > 2n$ there is an isomorphism of rings $\mathcal{F}(\mathfrak{sp}_{2n}) \to \mathcal{F}^c_\ell(\mathfrak{sp}_{2n})$ taking each basis element $[\Delta(\lambda)]$ to the basis element e^λ.*

The motivation for the operators $\mathbf{e}'_i$ is given by the definition of $s'_{\omega_i} = \mathbf{e}'_i$ and the Pieri rules for $[\Delta(\omega_i)][\Delta(\mu)]$, namely in type C we have

$$[\Delta(\omega_i)][\Delta(\mu)] = \begin{cases} \sum_{j=1}^{2n} [\Delta(\mu + \varepsilon_j)], & \text{if } i = 1, \\ \sum_{J:\ |J|=i} [\Delta(\mu + \varepsilon_J)] - \sum_{J:\ |J|=i-2} [\Delta(\mu + \varepsilon_J)], & \text{if } i \geq 2, \end{cases}$$

where all J's are subsets of $\{1, ..., 2n\}$ and we set $\varepsilon_J = \sum_{j \in J} \varepsilon_j$.

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