On KLR and quiver Schur algebras

Florian Seiffarth

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Advisor: Prof. Dr. Catharina Stroppel

Second Advisor: Prof. Dr. Jan Schröer

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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1 Introduction

Khovanov and Lauda introduced in [KL09] the remarkable KLR algebra (named after them and Rouquier who independently discovered it in [Rou08]) which is also known as quiver Hecke algebra. Their goal was to categorify the negative half of the quantized universal enveloping algebra $\mathcal{U}_q^-(\mathfrak{g})$ for any simply-laced Kac-Moody algebra $\mathfrak{g}$. Starting with the generalized Cartan matrix for $\mathfrak{g}$ or equivalently its Dynkin diagram Γ they diagrammatically defined a new family of algebras $R(\nu)$, $\nu \in \mathbb{N}[\mathbb{V}]$, where $\mathbb{V}$ is the set of vertices of Γ. Hence to each element $\nu = \sum_{i\in\mathbb{V}} \nu_i \cdot i$ for $\nu_i \in \mathbb{N}$ they attached an algebra $R(\nu)$. Sending a word $\theta_{i_1} \dots \theta_{i_k}$ in the standard Lusztig generators $\theta_{\mathbf{i}}$ of the quantum group $\mathcal{U}_q^-(\mathfrak{g})$ to the projective module labelled by $(i_1, \dots, i_k)$ defines a $\mathbb{Z}[q, q^{-1}]$-linear map

$$\gamma : \mathcal{U}_q^-(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{\nu\in\mathbb{N}[\mathbb{V}]} \mathrm{K}(R(\nu)) \tag{1.1}$$

which is an isomorphism by [VV11, §4.1, Thm. 4.5]. Here $\mathrm{K}(R(\nu))$ is the Grothendieck group of the category of finitely generated graded projective $R(\nu)$-modules. Under this isomorphism the summand $R(\nu)$ with $\nu = \sum_{i\in\mathbb{V}} \nu_i \cdot i$ corresponds precisely to the weight space of weight $-\sum_{i\in\mathbb{V}} \nu_i \cdot \alpha_i$.

While Khovanov's and Lauda's approach was purely combinatorial and algebraic, and Rouquier's more categorical and algebraic, Vasserot and Varagnolo [VV11] used geometry.

Based on the geometric approach for the affine Lie algebra $\hat{\mathfrak{sl}}_e$ (i.e. graphs given by the affine Dynkin diagram of type A) Stroppel and Webster gave a generalization of KLR algebras extending the isomorphism in (1.1) for $\mathfrak{g} = \hat{\mathfrak{sl}}_e$ to $\hat{\mathfrak{gl}}_e$, see [SW11]. Their aim was to answer the question if there is a natural graded version of the cyclotomic q-Schur algebra introduced by Dipper, James and Mathas [DJM98]. The so-called **quiver Schur algebra** is based on the quiver Γ_e corresponding to the affine Dynkin diagram of type A with the cyclic orientation, see figure 1. The category

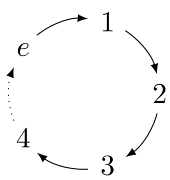
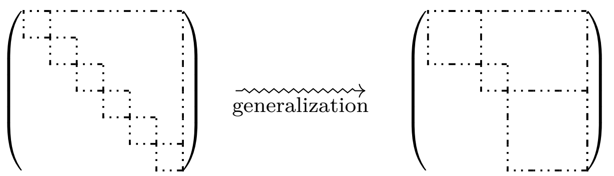


Figure 1: The quiver of the affine Dynkin diagram of $\hat{\mathfrak{sl}}_e$

of nilpotent representations of this quiver plays an important role in representation theory, see e.g. [Sch12] and is well-understood. For the definition of the quiver Schur algebra, Stroppel and Webster considered nilpotent representations of this quiver additionally equipped with compatible partial flags (that is for each such representation the data of a filtration with semisimple subquotients), generalizing the geometric construction of the KLR algebras, see figure 6.

The generalization from KLR algebras to quiver Schur algebras can be imagined as going from the special case of complete flags to the general case of arbitrary flags, or more geometrically, going from upper triangular matrices to block upper triangular matrices



For a fixed dimension vector $\mathbf{d} = (d_1, \dots, d_e)$ let $\mathcal{Q}(\hat{\lambda})$ be the variety of all compatible flags of type $\hat{\lambda}$ with $d(\hat{\lambda}) = \mathbf{d}$, see Section 4.1. It has the obvious action of the group $G := \mathrm{GL}(d_1) \times \ldots \times \mathrm{GL}(d_e)$ given by conjugation. Let $\mathrm{Rep}_{\mathbf{d}}$ be the set of all possible quiver representations of the quiver Γ_e with dimension vector $\mathbf{d}$. The "Steinberg variety"

$$\mathcal{Z}(\hat{\lambda}, \hat{\mu}) = \mathcal{Q}(\hat{\lambda}) \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}(\hat{\mu}) \tag{1.2}$$

can be viewed as an analogue of the classical Steinberg variety, see [CG97, §3.3], an important player in geometric representation theory. The quiver Schur algebra $A_{\mathbf{d}}$ is then the vector space

$$A_{\mathbf{d}} := \bigoplus_{(\hat{\lambda}, \hat{\mu})} H_*^{\mathrm{BM},G}(\mathcal{Z}(\hat{\lambda}, \hat{\mu}))$$

with the algebra structure given by the convolution product on the G-equivariant Borel-Moore homology where G acts diagonally on the "Steinberg variety" (1.2), see Section 4.2. The sum runs over all ordered pairs $(\hat{\lambda}, \hat{\mu})$ with dimension vector $\mathbf{d}$.

There also is a diagrammatic approach of the quiver Schur algebras $A_{\mathbf{d}}$ together with a faithful action on direct sums of invariant rings

$$\bigoplus_{\hat{\lambda}} \Bbbk[x_1, \dots, x_n]^{S_{\hat{\lambda}}} \tag{1.3}$$

which allows to make explicit calculations. In this thesis we will consider this action in more detail with the goal to get a better insight into the algebra $A_{\mathbf{d}}$.

One of the main tools hereby are the Demazure operators already introduced in 1973 by Demazure [Dem73] to understand cohomology rings of flag varieties. Hence it is not surprising that they appear here in the context of representations of quivers with compatible flags. The faithful action on (1.3) is completely determined by Demazure operators and multiplication with polynomials, see Section 4.3 and Section 4.4.

An example of a Demazure operator is given by the endomorphism

$$\Delta_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$$

where s_i is the element of the symmetric group which permutes the variables x_i and x_{i+1}, i.e. the image of Δ_i is contained in the subring of s_i-invariant polynomials.

For the quiver Schur algebra $A_{\mathbf{d}}$ an explicit generating set and a basis are known, see [SW11, Thm. 3.11].

A full set of defining relations is however not known in general. It is difficult to compute a full list of relations even for special cases.

Main result. The main result of the thesis is the description of complete lists of relations for some special dimension vectors and the consequences for the general case.

In **Proposition 5.1.2** we give a full list of the relations of the quiver Schur algebra $A_{\mathbf{d}}$ with dimension vector $\mathbf{d} = (1, 1, 0)$. For the special case that $\mathbf{d} = d \cdot \alpha_i$, $d \in \mathbb{Z}_{>0}$, is some multiple of a unit vector, we get by **Theorem 5.3.1** that the list of relations given by **Proposition 5.2.1** and **Proposition 5.2.3** is complete. The proof uses connections between the special quiver Schur algebra and the theory of web categories, see [TVW15], [CKM14].

Subsequently we ask the question, how exchanging special labels by general ones affects the relations in Proposition 5.2.1. We follow two strategies to address this question. The first is “coarser”, considering only highest order terms (in the sense of Definition 4.6.7). In **Theorem 6.1.3** we show that the special relations from Proposition 5.2.1 in general hold up to lower order terms (i.e. the number of Demazure operators acting). The second approach is “finer”, taking lower order terms into consideration. In **Theorem 6.3.1** we give explicit relations of ladders where the diagonal strands are labelled by the special vectors α_i.

Quick summary. The thesis can roughly be divided into two parts. The first part recalls the definition and basic properties of KLR algebras, while the second focuses on the quiver Schur algebras.

Part I: In Section 3.2 we recall the definition of KLR algebras from [KL09] and the slight modification from [KL11]. In contrast to the diagrammatic approach, another approach deals with algebraic generators and their relations, see Definition 3.2.7. Afterwards we introduce the Demazure operator as in [Dem73] and shortly recall the theory of root systems and Weyl groups, see Section 3.4. The section dealing with Demazure operators prepares for the combinatorics, which become important later on. To motivate the KLR basis theorem, see Theorem 3.9.4, and to illustrate how the Demazure operator appears naturally in the setting of KLR algebras we consider the nil Hecke ring in Section 3.6. The nil Hecke ring is a free module with basis given by certain products of Demazure operators, see Theorem 3.6.5. The faithful representation from Section 3.8 is helpful to do calculations with the KLR diagrams. Throughout the whole section we provide examples for the definitions and constructions.

Part II: We start in Section 4.1 with the definition of the quiver Schur algebras illustrated by several examples, see in particular figure 6 and Example 4.1.12. It involves a variety of combinatorial tools which are introduced. The diagrammatic approach is displayed in detail in Section 4.4. Subsequently we concentrate on this approach. By connecting both parts we show in detail that the KLR algebra is a subalgebra of the quiver Schur algebra considering the affine Lie algebra $\hat{\mathfrak{sl}}_e$, see Proposition 4.5.1. We recall the basis theorem from [SW11, Thm. 3.11.], see Section 4.7, and introduce some very helpful tools, see Section 4.6.

The above mentioned parts collect known results from various sources in the literature. The main focus for the remainder is to describe concrete relations and even presentations.

We are able to give presentations of the quiver Schur algebras $A_{\mathbf{d}}$ for special dimension vectors $\mathbf{d} = (1, 1, 0)$ and $\mathbf{d} = d \cdot \alpha_i$, $d \in \mathbb{Z}_{>0}$, see the main results from above and Section 5.

In Section 6 we generalize the relations gained from the special case, proving the propositions and theorems from above.

The computations of the action of Demazure operators on polynomials are quite involved and barely manageable to do by hand, even in small cases. We therefore wrote a computer program using Python code based on SageMath [Sage] which computes the action of the generators of the algebra (merge, splits) and of the Demazure operators on the representation (1.3), see Appendix D.

Acknowledgements. First of all, I would like to thank my advisor Prof. Catharina Stroppel for her continuous and tireless support and her numerous useful remarks. Furthermore, I was delighted about the carefully reading and the proposed corrections of Heike Herr and Simon Omlor. I would especially like to thank my parents, Antje Seiffarth and Frank Seiffarth, for the support from home, as well as Tatjana Weidemann and all my friends in Bonn for the wonderful years of study.

2 Notations

Throughout the thesis we will use the following notations:

- $\mathbb{Z}$: the set of integers,
- $\mathbb{N}$: the set of non negative integers,
- $\mathbb{R}$: the set of real numbers,
- $\mathbb{C}$: the set of complex numbers,
- $\Bbbk$: field of characteristic 0,
- $\Bbbk[\mathbb{Y}_n]$: The polynomial ring $\Bbbk[y_1, \dots, y_n]$ in the variables $y_1, \dots, y_n$ over the field $\Bbbk$, and the same for $\Bbbk[\mathbb{X}_n]$ and $x_1, \dots, x_n$,
- $\mathrm{Mat}(n, \Bbbk)$: $n \times n$ matrices over the field $\Bbbk$,
- $\mathrm{Diag}(n, \Bbbk)$: diagonal matrices in $\mathrm{Mat}(n, \Bbbk)$,
- Γ : graph or quiver,
- Γ_e : quiver corresponding to the affine Dynkin diagram of type A with e vertices,
- $\mathbb{V}$: vertex set of a graph or quiver,
- ν : element in $\mathbb{N}[\mathbb{V}]$,
- $R(\nu)$: KLR algebra corresponding to ν,
- V, U : vector spaces,
- e : number of vertices in the affine quiver of type A,
- $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$: nonzero vectors in $\mathbb{Z}^e_{\geq 0}$,
- α_i : special vector in $\mathbb{Z}^e_{\geq 0}$ with 1 at position i and 0 otherwise,
- S_n : the symmetric group on the finite set of $\{1, \dots, n\}$ generated be the simple reflections $s_1 = (1\ 2), \dots, s_{n-1} = (n-1\ n)$,
- R : root system,
- W : Weyl group,
- λ, μ : compositions of a natural number,
- $\hat{\lambda}, \hat{\mu}$: vector compositions,
- $\check{\lambda}, \check{\mu}$: transposed compositions,
- $A_{\mathbf{d}}$: quiver Schur algebra corresponding to the dimension vector $\mathbf{d}$,
- $e_{\hat{\lambda}}$, $s^k_{\hat{\lambda}}$, $m^k_{\hat{\lambda}}$, $x^k_{\hat{\lambda}}(P)$: generating morphisms of the quiver Schur algebra, namely idempotent, split at position k, merge at position k, polynomial at position k,
- D : diagrams of the quiver Schur or KLR algebra,
- $\mathrm{HC}(D)$, $\mathrm{HO}(D)$: highest coefficient and highest order of a quiver Schur diagram.

3 KLR algebras

As mentioned in the introduction the KLR algebra was first introduced by Khovanov and Lauda in [KL09]. Their goal was to categorify $\mathrm{U}_q^-(\mathfrak{g})$ and its finite dimensional representations, see [KL09, §3.4] and [VV11, Thm. 4.1]. We will now recall the diagrammatic description of this algebra starting with the definition from [KL09] slightly modified analogous to [KL11]. Afterwards we discuss the main basic properties.

3.1 Basic definitions

Let Γ be an unoriented graph (not necessarily finite) with vertex set $\mathbb{V}_\Gamma$ and edge set $\mathbb{E}_\Gamma$. It should be simply laced, i.e. no loops and no multiple edges are allowed. In the following we abbreviate $\mathbb{V} := \mathbb{V}_\Gamma$ and $\mathbb{E} := \mathbb{E}_\Gamma$ where it is already apparent from the context. Denote by

$$\mathbb{Z}[\mathbb{V}] := \left\{ \sum_{i \in \mathbb{V}} \nu_i \cdot i \;\middle|\; \nu_i \in \mathbb{Z},\ \nu_i = 0 \text{ for almost all } i \right\}$$

the group freely generated by the vertex set $\mathbb{V}$ with coefficients in $\mathbb{Z}$. On $\mathbb{Z}[\mathbb{V}]$ we can define a bilinear form $(\cdot,\cdot)$ which is given by

$$i \cdot j = (i,j) := \begin{cases} 0 & \text{if there is no edge between } i \text{ and } j,\ i \neq j\ , \\ -1 & \text{if there is an edge between } i \text{ and } j\ , \\ 2 & \text{if } i = j. \end{cases}$$

on the set of generators $i, j \in \mathbb{V}$. Concerning finite graphs the matrix M with $M_{ij} = i \cdot j$ given by the bilinear form is a generalized Cartan matrix in the sense of [Kac90, Ch. 4]. It defines a so-called Kac-Moody Lie algebra $\mathfrak{g}$ as in [Kac90, §1.3] and we get back the graph Γ by taking its Dynkin diagram.

There is a complete classification of Dynkin diagrams of finite and affine type in [Kac90, pp. 53ff.]. We consider here the finite type A_n case (figure 2) and the affine type A case (figure 1).

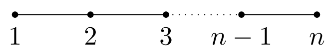


Figure 2: Dynkin diagram of type A_n

Let us fix now the quiver with vertex set $\mathbb{V}$. For an element

$$\nu = \sum_{i \in \mathbb{V}} \nu_i \cdot i \in \mathbb{N}[\mathbb{V}]$$

define

$$|\nu| := \sum_{i \in \mathbb{V}} \nu_i \in \mathbb{N}$$

as the **length** of ν. A **sequence of vertices** from $\mathbb{V}$ is defined as an expression $\mathbf{i} = (i_1 i_2 i_3 \dots i_n)$, where $i_j \in \mathbb{V}$ for $j = 1, \dots, n$. The symmetric group S_n acts on such an expression. For a simple reflection $s_k = (k\ k+1)$ the sequence $s_k(\mathbf{i})$ is the sequence with the k-th and $(k+1)$-th entry of $\mathbf{i}$ swapped.

For $\nu \in \mathbb{N}[\mathbb{V}]$ with $|\nu| = n$ define

$$\operatorname{Seq}(\nu) := \left\{ \mathbf{i} = (i_1 \dots i_n) \;\middle|\; \sum_{k=1}^{n} \delta_{j,i_k} = \nu_j \text{ for all } j \in \mathbb{V} \right\}$$

with

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } i, j \in \mathbb{V}$$

as the set of all sequences such that all vertices $i \in \mathbb{V}$ appear ν_i times in the sequence.

Example 3.1.1. Let $\mathbb{V} = \{i, j\}$ be the vertex set and $\nu = 3 \cdot i + 2 \cdot j$. Then the length of ν is equal to 5 and all possible sequences in $\operatorname{Seq}(\nu)$ are the ones where i and j appear three respectively two times, i.e.

$$\begin{aligned} \operatorname{Seq}(\nu) = \{&(iiijj), (iijij), (ijiij), (jiiij), (iijji), \\ &(ijiji), (jiiji), (ijjii), (jijii), (jjiii)\}. \end{aligned}$$

3.2 Diagrammatic approach

Fix some $\nu \in \mathbb{N}[\mathbb{V}]$ and let $n := |\nu|$ be the length of ν. We now define certain planar diagrams D attached to a pair of sequences in $\operatorname{Seq}(\nu)$. Assume we are given two sequences from $\operatorname{Seq}(\nu)$, one called bot(D) and the other one top(D). We attach the components of $\operatorname{bot}(D)$ to the points $(1, 0), \dots, (n, 0) \in \mathbb{R}^2$ and the components of $\operatorname{top}(D)$ to the points $(1, 1), \dots, (n, 1) \in \mathbb{R}^2$ and call them labels. Then a diagram with bottom labels given by the sequence $\operatorname{bot}(D)$ and top labels given by the sequence $\operatorname{top}(D)$ is a collection of arcs (also called strands) in the plane connecting each point of the form $(i, 0)$ with exactly one point of the form $(j, 1)$ such that their labels match. Only two arcs of a diagram are allowed to cross in one point. Additionally every arc may carry some dots which do not lie on a crossing. Following these rules we have constructed a diagram D for a given element ν and sequences $\text{top} := \operatorname{top}(D),\ \text{bot} := \operatorname{bot}(D)$. For example if we fix an element $\nu = 2i + j$ then some possible diagrams with $\operatorname{top}(D) = (iji)$ and $\operatorname{bot}(D) = (iji)$ can be depicted as in figure 3.

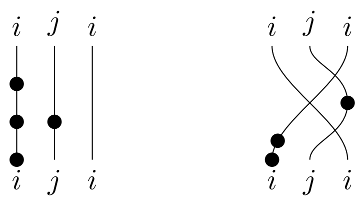


Figure 3: Example diagrams

We say that two diagrams are equivalent if they differ only by a finite sequence of the following moves

i) by an isotopy in the plane which does not change the combinatorial type,

ii) by sliding a dot along a strand (not passing through a crossing),

iii) by a height move.

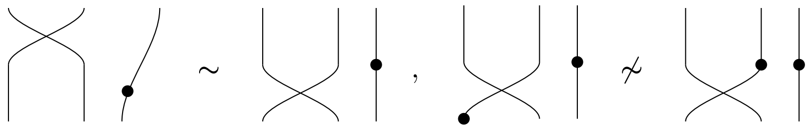


Figure 4: Equivalences of diagrams

In figure 4 we see examples of equivalent and non equivalent diagrams.

Two diagrams can be multiplied by **vertically stacking of diagrams**. Explicitly for two diagrams D and D' the vertically stacking $D'' := D \cdot D'$ is defined as the stacking (see figure 5) of D on top of D' if $\mathrm{bot}(D) = \mathrm{top}(D')$ and zero otherwise. By the equivalence conditions from above the diagram D'' is again a diagram as defined before with $\mathrm{bot}(D'') = \mathrm{bot}(D')$ and $\mathrm{top}(D'') = \mathrm{top}(D)$.

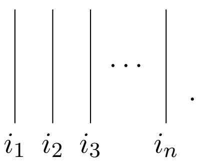
We now consider the $\mathbb{Z}$-module $D(\nu)$ defined by all $\mathbb{Z}$-linear combinations of equivalence classes of diagrams D with $\mathrm{top}(D), \mathrm{bot}(D) \in \mathrm{Seq}(\nu)$. Additionally by the vertically stacking $D \cdot D'$ of two diagrams D and D' we can define an algebra structure on $D(\nu)$ by $\mathbb{Z}$-linear extension. Hence we have the following statement.

Lemma 3.2.1. *The assignment*

$$\begin{array}{ccc} D(\nu) \times D(\nu) & \longrightarrow & D(\nu) \\ (D, D') & \longmapsto & D \cdot D' \end{array}$$

defines on $D(\nu)$ an associative and unital algebra structure.

Proof. The multiplication is clearly associative, because the order of stacking together the diagrams is associative. For a sequence $\mathbf{i} \in \mathrm{Seq}(\nu)$ where $\mathbf{i} = (i_1 i_2 \dots i_n)$ there is the diagram

$1_{\mathbf{i}} :=$ 

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Let $\mathbf{j}$, $\mathbf{k} \in \mathrm{Seq}(\nu)$. Define by $D(\nu)^{\mathbf{j}}_{\mathbf{k}}$ the set of all $\mathbb{Z}$-linear combinations of diagrams in $D(\nu)$ such that $\mathrm{top}(D) = \mathbf{j}$ and $\mathrm{bot}(D) = \mathbf{k}$. For any $\mathbb{Z}$-linear combination of diagrams $X \in D(\nu)^{\mathbf{j}}_{\mathbf{k}}$ we get

$$1_{\mathbf{i}} \cdot X = \begin{cases} X & \text{if } \mathbf{i} = \mathbf{j}, \\ 0 & \text{if } \mathbf{i} \neq \mathbf{j}, \end{cases}$$

by vertically stacking of diagrams and the equivalence rules from above. By analogous reasons we get

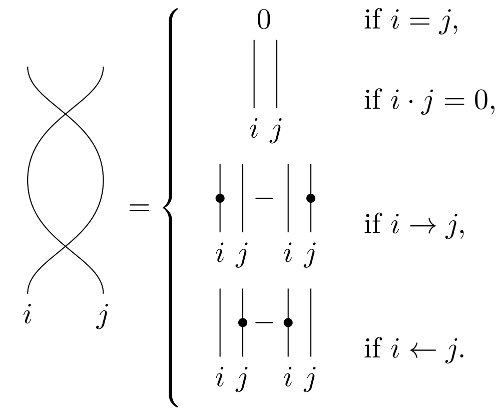
$$X \cdot 1_{\mathbf{i}} = \begin{cases} X & \text{if } \mathbf{i} = \mathbf{k}, \\ 0 & \text{if } \mathbf{i} \neq \mathbf{k}. \end{cases}$$

Hence adding the $1_{\mathbf{i}}'s$ over all possible sequences $\mathbf{i} \in \mathrm{Seq}(\nu)$ gives the unit in $D(\nu)$, i.e. $1_{D(\nu)} = \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} 1_{\mathbf{i}}$. □

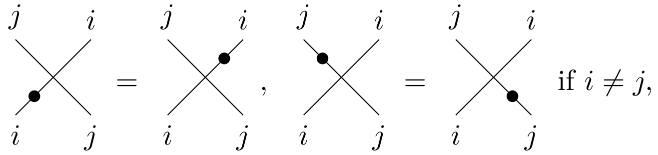
For the following definition choose an orientation of the edges of the underlying graph Γ. For two vertices $i, j \in \mathbb{V}$ write $i \to j$ if there is an oriented edge from vertex i to vertex j. For unoriented graphs there is a slightly different definition of KLR algebras introduced in [KL09, pp. 5ff.], but as we can see in Remark 3.2.6 the following definition of KLR algebras does not depend (up to isomorphism) on the chosen orientation. For example an orientation of the affine Dynkin diagram of type A can be chosen as in figure 1.

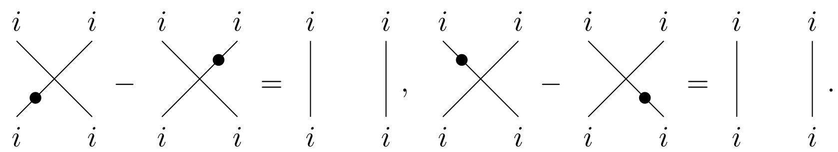
Definition 3.2.2. The **KLR algebra** or **quiver Hecke algebra** $R(\nu)$ is the quotient of $D(\nu)$ modulo the following relations.

($R1$)

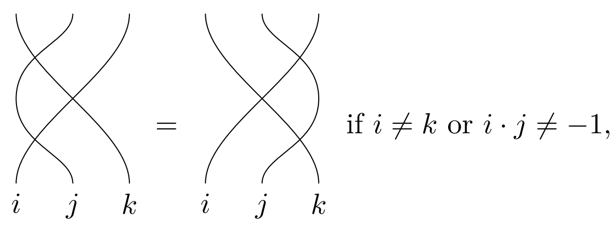


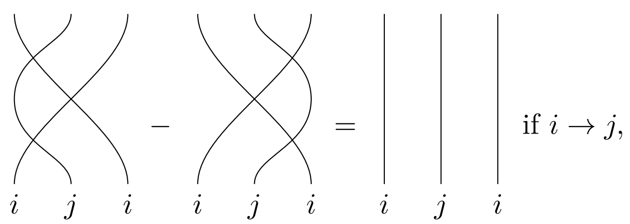
($R2$)

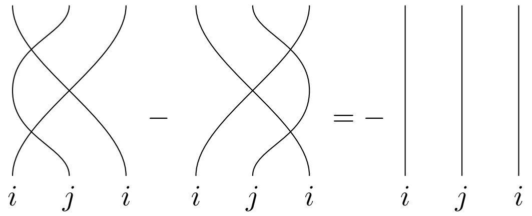




($R3$)







if $i \leftarrow j$.

From Lemma 3.2.1 it follows directly

Proposition 3.2.3. *The KLR algebra $R(\nu)$ is an associative and unital algebra.*

Example 3.2.4. In figure 5 we consider an explicit example of the multiplication "vertically stacking" of two diagrams $D \cdot D'$ where $\mathrm{top}(D') = (jijkik) = \mathrm{bot}(D)$. Here $\nu = 2 \cdot i + 2 \cdot j + 2 \cdot k$. The product of the two diagrams vanishes in the KLR

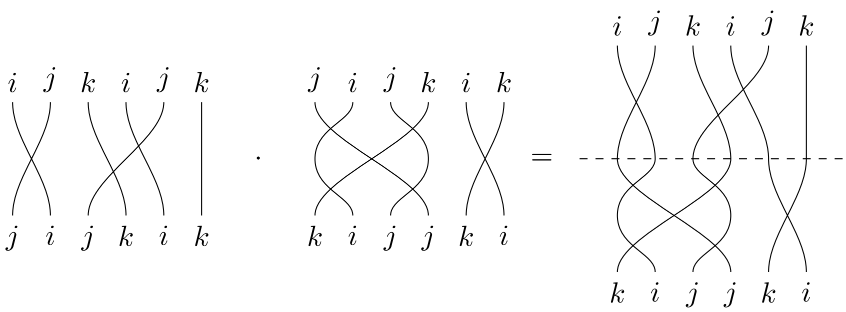


Figure 5: Multiplication of diagrams

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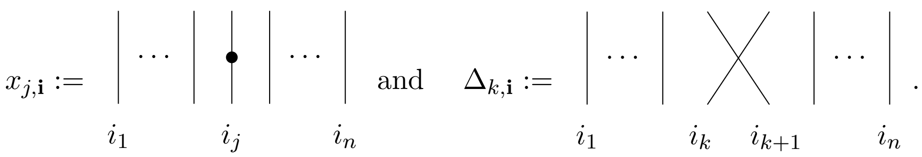
algebra $R(\nu)$ if $i = j$ or $j = k$, because then two neighboured strands cross twice and this is zero by relation $(R1)$.

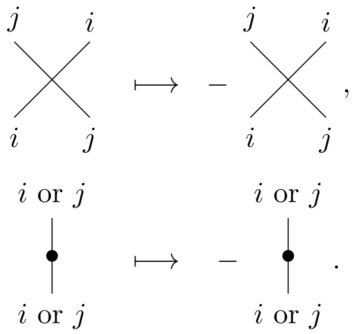
Now we look at some special diagrams in $R(\nu)$, either with only one dot on the j-th arc or one crossing of the arcs at positions k and $k+1$ of the bottom sequence and no dots on the arcs. For a given sequence $\mathbf{i} = (i_1 i_2 \dots i_n) \in \mathrm{Seq}(\nu)$ define these elements by



Lemma 3.2.5. *The elements $1_{\mathbf{i}}$, $x_{j,\mathbf{i}}$ and $\Delta_{k,\mathbf{i}}$, with $\mathbf{i} \in \mathrm{Seq}(\nu)$, $j \in \{1, \dots, |\nu|\}$ and $k \in \{1, \dots, |\nu| - 1\}$, generate the KLR algebra $R(\nu)$.*

Proof. By multiplication of these elements (vertically stacking) we can generate arbitrary many dots at any position by the generators $x_{j,\mathbf{i}}$ and arbitrary crossings at any position by the generators $\Delta_{k,\mathbf{i}}$. $\square$

Remark 3.2.6. The orientation change of an edge $i \longrightarrow j$ to $i \longleftarrow j$ gives an isomorphism between KLR algebras by the following assignment on generators



This can be easily checked by looking at the relations appearing in Definition 3.2.2. For example consider an edge $k \to l$ and the relation $(R1)$. Then the left hand side does not change, changing the orientation to $k \leftarrow l$. On the right hand side we change the edge orientations, but also the diagrams change because of the minus sign in the assignment.

The algebra $R(\nu)$ admits a grading by defining the degrees of the algebra generators by

$$\deg\left(x_{j,\mathbf{i}}\right) = 2, \quad \deg\left(\Delta_{k,\mathbf{i}}\right) = -i_k \cdot i_{k+1}.$$

The grading is well-defined, since the relations $(R1)$ - $(R3)$ are homogeneous.

We can alternatively define the KLR algebra or quiver Hecke algebra algebraically as follows.

Definition 3.2.7. The **KLR algebra** or **quiver Hecke algebra** $R(\nu)$ relative to ν with $|\nu| = n$ is the algebra freely generated by

$$1_{\mathbf{i}},\ x_{j,\mathbf{i}},\ \Delta_{k,\mathbf{i}} \text{ for } \mathbf{i} \in \mathrm{Seq}(\nu),\ j \in \{1,\dots,n\} \text{ and } k \in \{1,\dots,n-1\}$$

subject to the following relations

$(\overline{R0})$

$$\begin{aligned} 1_{\mathbf{i}} \cdot 1_{\mathbf{i}'} &= \delta_{\mathbf{i}\mathbf{i}'} \cdot 1_{\mathbf{i}}, \\ 1_{\mathbf{i}} \cdot x_{j,\mathbf{i}} &= x_{j,\mathbf{i}} = x_{j,\mathbf{i}} \cdot 1_{\mathbf{i}}, \\ 1_{s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} &= \Delta_{k,\mathbf{i}} = \Delta_{k,\mathbf{i}} \cdot 1_{\mathbf{i}}, \\ x_{j,\mathbf{i}} \cdot x_{j',\mathbf{i}} &= x_{j',\mathbf{i}} \cdot x_{j,\mathbf{i}}, \\ \Delta_{k,s_{k'}(\mathbf{i})} \cdot \Delta_{k',\mathbf{i}} &= \Delta_{k',s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} \text{ for } |k-k'| > 1, \end{aligned}$$

for $\mathbf{i}' \in \mathrm{Seq}(\nu)$, $j' \in \{1,\dots,n\}$ and $k' \in \{1,\dots,n-1\}$.

$(\overline{R1})$

$$\Delta_{k,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} = \begin{cases} 0 & \text{if } i_k = i_{k+1}, \\ 1_{\mathbf{i}} & \text{if } i_k \cdot i_{k+1} = 0, \\ x_{k,\mathbf{i}} - x_{k+1,\mathbf{i}} & \text{if } i_k \to i_{k+1}, \\ x_{k+1,\mathbf{i}} - x_{k,\mathbf{i}} & \text{if } i_k \leftarrow i_{k+1}. \end{cases}$$

($\overline{R2}$) If $i_k \neq i_{k+1}$ then

$$\begin{array}{rcl} \Delta_{k,\mathbf{i}} \cdot x_{k,\mathbf{i}} & = & x_{k+1,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}}, \\ x_{k,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} & = & \Delta_{k,\mathbf{i}} \cdot x_{k+1,\mathbf{i}}. \end{array}$$

If $i_k = i_{k+1}$ then

$$\begin{array}{rcrcl} \Delta_{k,\mathbf{i}} \cdot x_{k,\mathbf{i}} & - & x_{k+1,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} & = & 1_{\mathbf{i}}, \\ x_{k,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} & - & \Delta_{k,\mathbf{i}} \cdot x_{k+1,\mathbf{i}} & = & 1_{\mathbf{i}}. \end{array}$$

($\overline{R3}$) If $i_k \neq i_{k+2}$ or $i_k \cdot i_{k+1} \neq -1$ then

$$\Delta_{k,s_{k+1}s_k(\mathbf{i})} \cdot \Delta_{k+1,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} = \Delta_{k+1,s_k s_{k+1}(\mathbf{i})} \cdot \Delta_{k,s_{k+1}(\mathbf{i})} \cdot \Delta_{k+1,\mathbf{i}}.$$

If $i_k = i_{k+2}$ and $i_k \to i_{k+1}$ then

$$\Delta_{k,s_{k+1}s_k(\mathbf{i})} \cdot \Delta_{k+1,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} - \Delta_{k+1,s_k s_{k+1}(\mathbf{i})} \cdot \Delta_{k,s_{k+1}(\mathbf{i})} \cdot \Delta_{k+1,\mathbf{i}} = 1_{\mathbf{i}}.$$

If $i_k = i_{k+2}$ and $i_k \leftarrow i_{k+1}$ then

$$\Delta_{k,s_{k+1}s_k(\mathbf{i})} \cdot \Delta_{k+1,s_k(\mathbf{i})} \cdot \Delta_{k,\mathbf{i}} - \Delta_{k+1,s_k s_{k+1}(\mathbf{i})} \cdot \Delta_{k,s_{k+1}(\mathbf{i})} \cdot \Delta_{k+1,\mathbf{i}} = -1_{\mathbf{i}}.$$

Here $s_k(\mathbf{i})$ denotes the sequence where the labels of the positions k and $k+1$ are swapped. By the first relation it follows that the product of two generators is zero if the sequences do not fit together.

Remark 3.2.8. The Definitions 3.2.2 and 3.2.7 describe the same algebra, since the relations ($R1$)-($R3$) turn into ($\overline{R1}$)-($\overline{R3}$). And ($\overline{R0}$) describes the equivalence under isotopy which does not change the combinatorial type. Hence we have a diagrammatic and an algebraic description of the KLR algebra.

We will pause now for a moment and look at the special case of the KLR algebra where $\nu = n \cdot i$ for a vertex i and $n \in \mathbb{Z}_{>0}$. In Section 3.7 we see that this special case is related to the nil Hecke ring. The crossings are naturally related to the so-called Demazure operators which will be an important tool to understand the KLR algebra in a better way. These operators were originally introduced by Demazure in [Dem73] to study invariant rings and coinvariant rings for the action of reflection groups and Weyl groups. We therefore first recall basics of root systems and Weyl groups (see also [Hum72, Ch. III]) before we introduce the Demazure operator in Definition 3.4.1. Later, in Section 3.8, we continue with the general theory about KLR algebras.

3.3 Root systems

In this section let V be a finite dimensional Euclidean vector space, i.e. a finite dimensional $\mathbb{R}$-vector space with a scalar product $(\cdot,\cdot) : V \times V \to \mathbb{R}$.

Definition 3.3.1. A **root system** is a finite set of vectors $R \subset V$ such that:

(1) R spans V as a vector space, $0 \notin R$.

(2) If $\alpha, \beta \in R$ then $s_\alpha(\beta) := \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta - \langle \alpha^\vee, \beta \rangle \alpha \in R$ where $\alpha^\vee \in V^*$ with $\langle \alpha^\vee, \cdot \rangle = 2\frac{(\alpha,\cdot)}{(\alpha,\alpha)}$.

(3) For all $\alpha, \beta \in R$ it follows that $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.

We want to restrict to **reduced root systems**, where additionally it holds

(4) If $\alpha \in R$, then $\lambda\alpha \in R$ if and only if $\lambda = \pm 1$.

Remark 3.3.2. The elements in R are called **roots**. The dimension of V is the **rank** of the root system. The element $\alpha^\vee$ is called **coroot** of α.

A **basis** of a root system $R \subset V$ is a set $\mathcal{B} \subset R$ of roots which is a basis of V with the additional property that for all roots $\alpha \in R$ and

$$\alpha = \sum_{\beta_i \in \mathcal{B}} c_i \cdot \beta_i$$

either all coefficients $c_i \in \mathbb{Z}_{\geq 0}$ or all $c_i \in \mathbb{Z}_{\leq 0}$. By [Hum72, Ch. 10.1] all root systems $R \subset V$ have a basis. The roots in $\mathcal{B}$ are called **simple roots**. For a fixed basis $\mathcal{B}$ denote by R^+ (resp. $-R^+$) the set of **positive** (resp. **negative** roots) where

$$R^+ := \left\{ \alpha \in R \;\middle|\; \alpha = \sum_{\beta_i \in \mathcal{B}} c_i \cdot \beta_i,\ c_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Definition 3.3.3. Let $R \subset V$ be a root system. Then the group

$$W := W(R, V) := \langle s_\alpha \mid \alpha \in R \rangle$$

generated by the reflections s_α for $\alpha \in R$ is called **Weyl group** of the root system $R \subset V$.

For a fixed basis $\mathcal{B}$ of R the Weyl group is a Coxeter group with generators s_α for simple roots $\alpha \in \mathcal{B}$, see [Hum72, Ch. 10.3]. Therefore we can talk about reduced expressions for elements $w \in W$. The expression $s_{\alpha_1} \dots s_{\alpha_r}$ with $\alpha_1, \dots, \alpha_r \in \mathcal{B}$ is a reduced expression of w if $w = s_{\alpha_1} \dots s_{\alpha_r}$ and r is minimal with this property.

Remark 3.3.4.

- By definition the Weyl group is a subset of the group of permutations of the set R. The fact that R is finite implies that the Weyl group of R is finite.
- There is no unique basis, so positive roots always depend on the choice of a basis. In fact there are exactly $|W|$ many bases, see [Hum72, Ch. 10].

Example 3.3.5 (Root system of type A_{n-1})**.** We will now consider the root system corresponding to the semisimple Lie algebra

$$\mathfrak{sl}_n(\mathbb{C}) = \left\{ A \in \mathrm{Mat}(n, \mathbb{C}) \;\middle|\; \sum_{i=1}^{n} A_{ii} = 0 \right\}.$$

Let

$$\mathfrak{h} = \{A \in \mathfrak{sl}_n(\mathbb{C}) \cap \mathrm{Diag}(n, \mathbb{C})\}$$

be the standard Cartan subalgebra of $\mathfrak{sl}_n$. Then $\dim(\mathfrak{h}^*) = n - 1$ with basis

$$\mathcal{B} = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\},$$

where $\varepsilon_i(A) = A_{ii}$ returns the i-th diagonal entry of the matrix. Then

$$R = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n\}$$

forms a root system of rank $n-1$. A basis is given by $\mathcal{B}$ with corresponding positive roots $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$. The Weyl group $W = S_n$ acts on the set of roots by permuting the ε_i's.

To simplify the combinatorics we often consider the reductive Lie algebra

$$\mathfrak{gl}_n = \{A \in \mathrm{Mat}(n, \mathbb{C})\} = \mathfrak{sl}_n \oplus \mathbb{C} \cdot \mathrm{Id}$$

instead of $\mathfrak{sl}_n$. The standard Cartan subalgebra is

$$\mathfrak{h} = \{A \in \mathrm{Diag}(n, \mathbb{C})\}$$

and we have $\dim(\mathfrak{h}^*) = n$. Choose a basis $\mathcal{B} = \{\varepsilon_i \mid 1 \leq i \leq n\}$ where $\varepsilon_i(A) = A_{ii}$ is again the i-th diagonal entry and the Weyl group S_n acts by permutation of the ε_i's.

For a $\Bbbk$-vector space V consider the **symmetric algebra** $\mathrm{S}(V)$ which is the quotient of the **tensor algebra**

$$\mathrm{T}(V) := \Bbbk \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

by the ideal generated by $v \otimes w - w \otimes v$ for vectors $v, w \in V$. The action of the Weyl group $W := W(R, V)$ on V extends to an action on the symmetric algebra $\mathrm{S}(V)$. Let $\alpha_1, \dots, \alpha_n$ be a basis of the root system $R \subset V$ and hence a $\Bbbk$-basis of the Euclidean vector space V. Then $\mathrm{S}(V)$ is a polynomial ring $\Bbbk[\mathbb{Y}_n] := \Bbbk[y_1, \dots, y_n]$ in variables $y_1, \dots, y_n$ by mapping the basis of the root system α_i to the variables y_i.

If we now speak about roots we always regard them as elements in the symmetric algebra or under the isomorphism as elements in the polynomial ring. The action of the Weyl group on the polynomial ring is given by the isomorphism, namely

$$w(y_{i_1} \cdot \ldots \cdot y_{i_r}) = w(y_{i_1}) \cdot \ldots \cdot w(y_{i_r})$$

for $w \in W$, $1 \leq i_1, \dots, i_r \leq n$. The action of W on R even extends to the field of rational functions $\Bbbk(\mathbb{Y}_n)$ by

$$w\left(\frac{f}{g}\right) = \frac{w(f)}{w(g)}$$

for $f, g \in \Bbbk[\mathbb{Y}_n],\ g \neq 0$.

The **ring of W-invariants** is then defined as

$$\Bbbk[\mathbb{Y}_n]^W := \{f \in \Bbbk[\mathbb{Y}_n] \mid w(f) = f \text{ for all } w \in W\}.$$

Example 3.3.6. Now we apply this theory to the root system A_{n-1}. We denote the ε_i's from Example 3.3.5 by x_i and consider the polynomial ring $\Bbbk[\alpha_1, \dots, \alpha_{n-1}]$ with $\alpha_i = x_i - x_{i+1}$. We have the inclusion

$$\Bbbk[\alpha_1, \dots, \alpha_{n-1}] \hookrightarrow \Bbbk[x_1, \dots, x_n]$$

of algebras sending α_i to $x_i - x_{i+1}$.

By the fundamental theorem of symmetric polynomials we can describe the ring of $W = S_n$ invariants $\Bbbk[\mathbb{X}_n]^{S_n}$, namely

$$\Bbbk[\mathbb{X}_n]^{S_n} = \Bbbk[e_1, \dots, e_n]$$

where

$$e_i := \sum_{1 \le k_1 < \ldots < k_i \le n} x_{k_1} \ldots x_{k_i}$$

are the elementary symmetric polynomials. For $n = 2$ we obtain the invariant ring $\Bbbk[x_1, x_2]^{S_2} = (x_1 + x_2, x_1 \cdot x_2) \subset \Bbbk[x_1, x_2]$.

3.4 Demazure operators

The Demazure operators were introduced in [Dem73] to study symmetric invariant rings and especially quotients by invariant polynomials. We will shortly discuss one of his main results which should illustrate the important role of the Demazure operators. Let $R \subset V$ be a root system and $\Bbbk[\mathbb{Y}_n]$ the corresponding polynomial ring as before. Define by $\Bbbk[\mathbb{Y}_n]_+^W$ all W-invariant polynomials without a constant term. Denote by I the ideal inside $\Bbbk[\mathbb{Y}_n]$ generated by $\Bbbk[\mathbb{Y}_n]_+^W$. Then the quotient

$$\Bbbk[\mathbb{Y}_n] \big/ I \tag{3.1}$$

appears in invariant theory [Kan01, Ch. 18, Ch. 23] as ring of covariants and also as cohomology of the flag variety (see for type A [Ful96, Prop. 3]). As a result of [Dem73, Thm. 2b)] the quotient (3.1) can easily be calculated by the Demazure operators which will be introduced below. For each reduced expression $w \in W$, $w = s_{\alpha_1} \ldots s_{\alpha_r}$ we get a basis element $\Delta_w(a) = \Delta_{\alpha_1} \ldots \Delta_{\alpha_r}(a)$ where Δ_{α_i} is a Demazure operator and a is the product of all positive roots which is an element in $\Bbbk[\mathbb{Y}_n] \big/ I$.

For our purposes and also for Demazures result it is important that Δ_w is independent of the chosen reduced expression, see Section 3.5.

Definition 3.4.1. Let $\alpha \in R$ be a root. Define the **Demazure operator** corresponding to the root α as

$$\Delta_\alpha(f) := \frac{f - s_\alpha(f)}{\alpha} \in \Bbbk[\mathbb{Y}_n] \tag{3.2}$$

for $f \in \Bbbk[\mathbb{Y}_n]$.

Remark 3.4.2. The Demazure operator is a $\Bbbk$-linear endomorphism of the polynomial ring $\Bbbk[\mathbb{Y}_n]$. Indeed, since for an element $y_{i_1} \cdot \ldots \cdot y_{i_r} \in \Bbbk[\mathbb{Y}_n]$ it holds

$$\begin{aligned} & y_{i_1} \cdot \ldots \cdot y_{i_r} - s_\alpha(y_{i_1} \cdot \ldots \cdot y_{i_r}) \\ & \quad = y_{i_1} \cdot \ldots \cdot y_{i_r} - s_\alpha(y_{i_1}) \cdot \ldots \cdot s_\alpha(y_{i_r}) \\ & \quad = y_{i_1} \cdot \ldots \cdot y_{i_r} - (y_{i_1} - \langle \alpha^\vee, y_{i_1} \rangle \alpha) \cdot \ldots \cdot (y_{i_r} - \langle \alpha^\vee, y_{i_r} \rangle \alpha) \\ & \quad = C \cdot \alpha \end{aligned}$$

for some $C \in \Bbbk[\mathbb{Y}_n]$ which depends on $y_{i_1} \cdot \ldots \cdot y_{i_r}$. Hence it follows that for $f \in \Bbbk[\mathbb{Y}_n]$

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha} = \frac{f - (f - C(f)\alpha)}{\alpha} = C(f) \in \Bbbk[\mathbb{Y}_n],$$

i.e. the result is again a polynomial and so the Demazure operator is well-defined.

Remark 3.4.3. Note that by definition every Demazure operator reduces the degree of a polynomial at least by one.

In the following we will often consider type A_{n-1} and work with $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$, hence with $\Bbbk[x_1,\dots,x_n] = \mathrm{S}(\mathfrak{h}_{\mathfrak{gl}_n})$ instead of $\Bbbk[\mathbb{Y}_{n-1}] = \mathrm{S}(\mathfrak{h}_{\mathfrak{sl}_n})$.

Example 3.4.4 (Type A_1)**.** Let us consider the quotient

$$\Bbbk[x_1,x_2]\,/_{I}$$

with

$$I = \Bbbk[x_1,x_2]\cdot\Bbbk[x_1,x_2]_+^{S_2} = (x_1+x_2, x_1\cdot x_2).$$

Hence

$$\Bbbk[x_1,x_2]\,/_{I} \cong \langle 1, x_1\rangle_{\Bbbk}$$

as vector spaces and we see that $x_1, \Delta_1(x_1) = \frac{x_1 - s_1(x_1)}{x_1 - x_2} = 1$ form a basis.

Lemma 3.4.5. *The Demazure operators satisfy the following properties:*

1) The s_α-invariance

$$s_\alpha(\Delta_\alpha(f)) = \Delta_\alpha(f). \tag{3.3}$$

2) The 0-property

$$\Delta_\alpha^2(f) = 0. \tag{3.4}$$

3) The derivation property or Leibniz rule

$$\begin{aligned} \Delta_\alpha(f\cdot g) &= \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g) && (3.5)\\ &= \Delta_\alpha(f)g + f\Delta_\alpha(g) - \alpha\Delta_\alpha(f)\Delta_\alpha(g). && (3.6)\end{aligned}$$

4) The braid relation

$$\Delta_\alpha\Delta_\beta\Delta_\alpha\Delta_\beta\ldots = \Delta_\beta\Delta_\alpha\Delta_\beta\Delta_\alpha\ldots,\ \textit{if } s_\alpha s_\beta s_\alpha s_\beta\ldots = s_\beta s_\alpha s_\beta s_\alpha\ldots. \tag{3.7}$$

Proof. The equation (3.3) follows by

$$s_\alpha(\Delta_\alpha(f)) = \frac{s_\alpha(f) - s_\alpha(s_\alpha(f))}{s_\alpha(\alpha)} = \frac{s_\alpha(f) - f}{-\alpha} = \Delta_\alpha(f).$$

Hence it follows that $\Delta_\alpha(f)$ is invariant under the s_α action and therefore (3.4) follows directly from (3.3). In particular if f is invariant under s_α then $\Delta_\alpha(f) = 0$. The equation (3.5) follows with

$$\begin{aligned}\Delta_\alpha(f\cdot g) &= \frac{f\cdot g - s_\alpha(f\cdot g)}{\alpha} = \frac{f\cdot g - s_\alpha(f)\cdot s_\alpha(g)}{\alpha}\\ &= \frac{(f - s_\alpha(f))g}{\alpha} + \frac{s_\alpha(f)(g - s_\alpha(g))}{\alpha} = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g).\end{aligned}$$

And (3.6) can be followed from (3.5) by

$$\begin{aligned}\Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g) &= \Delta_\alpha(f)g + f\Delta_\alpha(g) - \alpha\cdot\frac{f\Delta_\alpha(g) - s_\alpha(f)\Delta_\alpha(g)}{\alpha}\\ &= \Delta_\alpha(f)g + f\Delta_\alpha(g) - \alpha\Delta_\alpha(f)\Delta_\alpha(g).\end{aligned}$$

For equation (3.7) see the proof of Theorem 3.5.7 below. □

Example 3.4.6 (Demazure operators for type A systems)**.** Looking at the polynomial ring $\Bbbk[\mathbb{X}_n]$ the roots are given by $\alpha_{ij} = x_i - x_j$ for generators x_i, x_j. Hence in this case the Demazure operators are given by

$$\Delta_{\alpha_{ij}}(f) = \frac{f - s_{ij}(f)}{x_i - x_j} \in \Bbbk[\mathbb{X}_n]$$

for a polynomial $f \in \Bbbk[\mathbb{X}_n]$ where s_{ij} is the element of the Weyl group S_n which permutes the variables x_i and x_j.

Now we can explicitly calculate some Demazure operators for the root system A_2 and roots $\alpha = x_1 - x_2$ and $\beta = x_2 - x_3$. Let us look at some example polynomial $f = x_1^2 - 2x_2 + x_3 x_2^2 \in \mathbb{R}[x_1, x_2, x_3]$. Then we can calculate

$$\begin{aligned}
\Delta_\beta \Delta_\alpha(f) &= \Delta_\beta \left(\frac{x_1^2 - 2x_2 + x_3 x_2^2 - (x_2^2 - 2x_1 + x_3 x_1^2)}{x_1 - x_2} \right) \\
&= \Delta_\beta(2 + (x_1 + x_2) - x_3(x_1 + x_2)) \\
&= \frac{2 + (x_1 + x_2) - x_3(x_1 + x_2) - (2 + (x_1 + x_3) - x_2(x_1 + x_3))}{x_2 - x_3} \\
&= 1 + x_1.
\end{aligned}$$

Notice that the degree of the polynomial is reduced by two.

3.5 Operators of reduced expressions

From now on fix a basis $\mathcal{B}$ of a root system $R \subset V$. For an element $w \in W$ denote by $l(w) := \min\{r \mid w = s_{\alpha_1} \dots s_{\alpha_r}, \alpha_1, \dots, \alpha_r \in \mathcal{B}\}$ the length of w. Then we call $w = s_{\alpha_1} \dots s_{\alpha_{l(w)}}$ a reduced expression. Let R^+ be the set of positive roots according to $\mathcal{B}$.

We can ask whether the operator $\Delta_{\alpha_1} \dots \Delta_{\alpha_r}$ corresponding to a reduced expression $w = s_{\alpha_1} \dots s_{\alpha_r} \in W$ is independent of the chosen reduced expression. This is indeed true as we will see in Theorem 3.5.7. Also the lemmas we need to prove the theorem provide interesting results. This section is based on [Dem73, §4].

Remark 3.5.1. The Demazure operator can be considered as an element in the algebra

$$\Bbbk(\mathbb{Y}_n)[W] := \left\{ \sum_{w \in W} f_w \cdot w \;\middle|\; f_w \in \Bbbk(\mathbb{Y}_n) \right\}$$

freely generated by W with coefficients in $\Bbbk(\mathbb{Y}_n)$, the field of rational functions. We denote by

$$\frac{w}{y}$$

the element

$$\frac{1}{y} \cdot w \in \Bbbk(\mathbb{Y}_n)[W]$$

for $w \in W$, $y \in \Bbbk[\mathbb{Y}_n] \backslash \{0\}$.

Lemma 3.5.2. *Let $w \in W$ and $w = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced expression. Then for $R(w) := R^+ \cap w(-R^+)$ it holds that*

$$R(w) = \{\alpha_1, s_1(\alpha_2), s_1 s_2(\alpha_3) \dots, s_1 \dots s_{r-1}(\alpha_r)\}$$

where $s_i := s_{\alpha_i}$.

Proof. By [Bou08, Ch. VI, Cor. 2] it follows that the roots

$$s_r \dots s_{i+1}(\alpha_i) \text{ for } 1 \leq i \leq r-1$$

are exactly the positive roots such that $w(s_r \dots s_{i+1}(\alpha_i))$ is negative. Hence we have that

$$-s_r \dots s_{i+1}(\alpha_i) = s_r \dots s_{i+1} s_i(\alpha_i)$$

are exactly all negative roots such that

$$w(s_r \dots s_{i+1} s_i(\alpha_i)) = s_1 \dots s_{i-1}(\alpha_i)$$

is positive. Hence the claim follows. □

Lemma 3.5.3. *Let $w \in W$ and $w = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced expression. The set $R(w) := R^+ \cap w(-R^+)$ is defined as before. Then it holds*

$$\left(\prod_{\alpha \in R(w)} \alpha\right) \Delta_{\alpha_1} \dots \Delta_{\alpha_r} = (-1)^r w + \sum_{l(w')<l(w)} f_{w'} w' \tag{3.8}$$

with $f_{w'} \in \Bbbk(\mathbb{Y}_n)$.

Before proving this lemma let us make an example for $\mathfrak{sl}_2$.

Example 3.5.4 (Example for $\mathfrak{sl}_2$)**.** By Example 3.3.5 the set of roots is given by $R = \{x_1 - x_2, x_2 - x_1\}$. Chose $R^+ := \{x_1 - x_2\}$ to be the set of positive roots. The Weyl group W is given by $S_2 = \{\mathrm{id}, s_1\}$. For the simple reflection $w = s_1$ it follows $R(w) = \{x_1 - x_2\} \cap \{s_1(x_2 - x_1)\} = R^+$ and hence the lemma leads to

$$(x_1 - x_2) \cdot \Delta_1 = -s_1 + \mathrm{Id}\,.$$

For $w = \mathrm{id}$ it holds $R(w) = \emptyset$ and hence $\Delta_{\mathrm{id}} = \mathrm{Id}$.

Proof of Lemma 3.5.3. Define $\Delta := \Delta_{\alpha_1} \dots \Delta_{\alpha_r}$ and write $s_i := s_{\alpha_i}$. Then the definition of the Demazure operator gives

$$\begin{aligned} \Delta &= \alpha_1^{-1}(\mathrm{Id} - s_1)\alpha_2^{-1}(\mathrm{Id} - s_2) \dots \alpha_r^{-1}(\mathrm{Id} - s_r) \\ &= (-1)^r \alpha_1^{-1} s_1 \alpha_2^{-1} s_2 \dots \alpha_r^{-1} s_r + \sum_{l(w')<l(w)} g_{w'} w' \end{aligned} \tag{3.9}$$

for some $g_{w'} \in \Bbbk(\mathbb{Y}_n)$. We want to rewrite the first summand of (3.9) and claim that

$$\alpha_1^{-1} s_1 \alpha_2^{-1} s_2 \dots \alpha_r^{-1} s_r = \left(\prod_{i=1}^{r} s_1 \dots s_{i-1}(\alpha_i^{-1})\right) s_1 \dots s_r. \tag{3.10}$$

For $r = 2$ and some $f \in \Bbbk[\mathbb{Y}_n]$ it holds

$$\begin{aligned} (\alpha_1^{-1} s_1 \alpha_2^{-1} s_2)(f) &= \alpha_1^{-1} \cdot s_1 \left(\alpha_2^{-1} \cdot s_2(f)\right) \\ &= \alpha_1^{-1} \cdot s_1(\alpha_2^{-1}) \cdot s_1 s_2(f). \end{aligned}$$

Assume (3.10) holds for $r-1$ then

$$\begin{aligned}(\alpha_1^{-1}s_1\alpha_2^{-1}s_2\dots\alpha_r^{-1}s_r)(f) &= \alpha_1^{-1}s_1\alpha_2^{-1}s_2\dots\alpha_{r-1}^{-1}s_{r-1}(\alpha_r^{-1}s_r(f))\\ &= \alpha_1^{-1}\cdot s_1(\alpha_2^{-1})\cdot s_1s_2(\alpha_3^{-1})\cdot\ldots\cdot s_1\dots s_{r-1}(\alpha_r^{-1}s_r(f))\\ &= \alpha_1^{-1}\cdot s_1(\alpha_2^{-1})\cdot s_1s_2(\alpha_3^{-1})\cdot\ldots\cdot s_1\dots s_{r-1}(\alpha_r^{-1})\cdot s_1\dots s_r(f)\\ &= \left(\prod_{i=1}^{r} s_1\dots s_{i-1}(\alpha_i^{-1})\right) s_1\dots s_r(f)\end{aligned}$$

implies that (3.10) holds for all r. Substitution of (3.10) into (3.9) leads to

$$\begin{aligned}\Delta &= (-1)^r\left(\prod_{i=1}^{r} s_1\dots s_{i-1}(\alpha_i^{-1})\right) s_1\dots s_r + \sum_{l(w')<l(w)} g_{w'}w'\\ &= (-1)^r\left(\prod_{\alpha\in R(w)}\alpha^{-1}\right) w + \sum_{l(w')<l(w)} g_{w'}w'\end{aligned}$$

using the fact from Lemma 3.5.2 that all roots in the set $R(w)$ are exactly of the form $s_1\dots s_{i-1}(\alpha_i)$ for $1\le i\le r$. $\square$

We denote by

$$e_{\mathrm{sgn}} := \sum_{w\in W}(-1)^{l(w)}w \tag{3.11}$$

the **signed sum** over all elements in the Weyl group which is an endomorphism of $\Bbbk[\mathbb{Y}_n]$. By

$$d := \prod_{\alpha\in R^+}\alpha \tag{3.12}$$

we denote the **product of all positive roots**.

Lemma 3.5.5. *Let L be a $\Bbbk[\mathbb{Y}_n]^W$-linear endomorphism of $\Bbbk[\mathbb{Y}_n]$ such that $L(g)=0$ for all $g\in\Bbbk[\mathbb{Y}_n]$ with $\deg(g)<|R^+|$, the number of positive roots. Then for all $f\in\Bbbk[\mathbb{Y}_n]$ it holds*

$$|W|\cdot d\cdot L(f) = L(d)\cdot e_{\mathrm{sgn}}(f),$$

in particular

$$d\cdot L = \frac{L(d)}{|W|}\cdot e_{\mathrm{sgn}} =: \lambda\cdot e_{\mathrm{sgn}}.$$

Proof. By [Dem73, Lem. 1, Prop. 1] every element $f\in\Bbbk[\mathbb{Y}_n]$ is of the form

$$f=\sum h_i\cdot g_i + d\cdot p$$

where $g_i, p\in\Bbbk[\mathbb{Y}_n]^W$ and all the h_i are homogeneous polynomials of degree less than $|R^+|$. Then the assumptions on L give

$$L(f) = \sum g_i\cdot\underbrace{L(h_i)}_{=0} + p\cdot L(d) = p\cdot L(d). \tag{3.13}$$

The endomorphism e_{sgn} also fulfills the assumptions required in this lemma (see for example the theory about skew invariants in [Kan01, §20]) hence it follows

$$e_{\mathrm{sgn}}(f) = p\cdot e_{\mathrm{sgn}}(d). \tag{3.14}$$

Using that $e_{\mathrm{sgn}}(d) = |W| \cdot d$, we get by comparing (3.13) and (3.14) that

$$\frac{L(f)}{L(d)} = \frac{e_{\mathrm{sgn}}(f)}{|W| \cdot d}$$

and the claim follows. □

Lemma 3.5.6. *Let $w_0 \in W$ be the longest element and $w_0 = s_{\alpha_1} \dots s_{\alpha_r}$ a reduced expression. Then for all $f \in \Bbbk[\mathbb{Y}_n]$ it holds*

$$\Delta_{\alpha_1} \dots \Delta_{\alpha_r}(f) = \frac{e_{\mathrm{sgn}}(f)}{d}. \tag{3.15}$$

Proof. For the longest element w_0 of W it holds $l(w_0) = |R^+|$, see [Hum90, p. 16]. Hence it holds $R(w_0) = R^+$ and thus $\prod_{\alpha \in R(w_0)} \alpha = d$. We denote by Δ the operator $\Delta_{\alpha_1} \dots \Delta_{\alpha_r}$. Then on the one side by Lemma 3.5.3 we have

$$d \cdot \Delta = (-1)^r w_0 + \sum_{l(w')<l(w_0)} f_{w'} w'. \tag{3.16}$$

On the other side Δ is a $\Bbbk[\mathbb{Y}_n]^W$-linear endomorphism and by Remark 3.4.3 the Demazure operators reduce the degree of a polynomial at least by one hence it holds that $\Delta(g) = 0$ for all $g \in \Bbbk[\mathbb{Y}_n]$ such that $\deg(g) < |R^+|$. Therefore by Lemma 3.5.5 there is a $\lambda \in \Bbbk(\mathbb{Y}_n)$ such that

$$d \cdot \Delta = \lambda \cdot e_{\mathrm{sgn}} \overset{(3.11)}{=} \lambda(-1)^r w_0 + \lambda \sum_{l(w')<l(w_0)} (-1)^{l(w')} w'. \tag{3.17}$$

The Theorem of Dedekind [Jac64, p. 25] implies that the elements $w \in W$ are independent over $\Bbbk(\mathbb{Y}_n)$. Therefore, comparing coefficients of (3.16) and (3.17) we get that $\lambda = 1$ and the claim follows. □

Theorem 3.5.7. *For two reduced expressions $s_{\alpha_1} \dots s_{\alpha_r}$ and $s_{\beta_1} \dots s_{\beta_r}$ of the same element $w \in W$ it holds that $\Delta_{\alpha_1} \dots \Delta_{\alpha_r} = \Delta_{\beta_1} \dots \Delta_{\beta_r}$.*

Proof. Since all the Weyl groups W are Coxeter groups, reduced expressions of the same element can be converted into each other by the given braid relations, see the Theorem of Matsumoto [Mat64]. Therefore it suffices to show that for all roots $\alpha, \beta \in \mathcal{B}$ with $s_\alpha s_\beta$ of order m (i.e. $(s_\alpha s_\beta)^m = 1$) it holds

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots$$

with m factors on each side. But the elements $s_\alpha s_\beta s_\alpha \dots$ and $s_\beta s_\alpha s_\beta \dots$ are two different reduced expressions of the longest element of the Weyl group generated by s_α and s_β. Hence by Lemma 3.5.6 it follows directly that

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots .$$

□

Definition 3.5.8. Let $w \in W$ and $w = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced expression. Then we define

$$\Delta_w := \Delta_{\alpha_1} \dots \Delta_{\alpha_r} \tag{3.18}$$

the Demazure operator corresponding to w. For $w = \mathrm{id} \in W$ we define $\Delta_{\mathrm{id}} = \mathrm{Id}$.

Corollary 3.5.9. *For all $w, w' \in W$ it holds*

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w'), \\ 0 & \text{otherwise }. \end{cases}$$

Proof. The first case is clear by the Theorem 3.5.7, because if $l(ww') = l(w) + l(w')$ then we have for $w = s_{\alpha_1} \dots s_{\alpha_{l(w)}}$ and $w' = s_{\beta_1} \dots s_{\beta_{l(w')}}$ two reduced expressions, that $ww' = s_{\alpha_1} \dots s_{\alpha_{l(w)}} s_{\beta_1} \dots s_{\beta_{l(w')}}$ is also a reduced expression.

Now let $l(w) = 1$, $w = s_\alpha$ for some $\alpha \in \mathcal{B}$ and $l(ww') \neq l(w') + 1$. Hence by [Hum90, §1.6, §1.7] it follows that $l(ww') = l(w') - 1$ and there is a reduced expression $w'' = s_{\beta_1} \dots s_{\beta_r}$ such that $w' = s_\alpha w''$ with $l(w') = l(s_\alpha) + l(w'')$. Applying the first part it holds $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{w''}$ which implies $\Delta_w \Delta_{w'} = \Delta_{s_\alpha}^2 \Delta_{w''} = 0$. Hence induction on the length of w gives the claim. □

For a reduced expression $w = s_{\alpha_1} \dots s_{\alpha_r}$ and the corresponding product of Demazure operators $\Delta_w = \Delta_{\alpha_1} \dots \Delta_{\alpha_r}$ the derivation property (3.5) can be extended to a product of Demazure operators, applying it successively, hence we get

$$\Delta_w(f \cdot g) = \Delta_{\alpha_1} \dots \Delta_{\alpha_r}(f \cdot g) = \sum X_{\alpha_1} \dots X_{\alpha_r}(f) Y_{\alpha_1} \dots Y_{\alpha_r}(g) \tag{3.19}$$

and the sum runs over all possible combinations where either X_{α_i} is Δ_{α_i} and Y_{α_i} is the identity or X_{α_i} is s_{α_i} and Y_{α_i} is Δ_{α_i}.

3.6 Nil Hecke ring

Let $R \subset V$ be a root system of rank n as before and W its Weyl group acting on $\Bbbk[\mathbb{Y}_n]$. Choose a basis $\mathcal{B}$ of the root system.

Definition 3.6.1. The **nil Hecke ring** NH_n is the ring generated by the following endomorphisms of $\Bbbk[\mathbb{Y}_n]$

i) the identity Id on $\Bbbk[\mathbb{Y}_n]$,

ii) the multiplication by a variable y_j where $y_j(f) = y_j \cdot f$ for $1 \leq j \leq n$,

iii) the Demazure operators Δ_α, where $\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha} \in \Bbbk[\mathbb{Y}_n]$, $\alpha \in R$,

for $f \in \Bbbk[\mathbb{Y}_n]$.

Remark 3.6.2. We will consider the roots α as elements in the polynomial ring $\Bbbk[\mathbb{Y}_n]$ under the identification of simple roots and variables as seen before.

Lemma 3.6.3. *The nil Hecke ring NH_n is already generated by the identity, the multiplications and the Demazure operators Δ_α for $\alpha \in \mathcal{B}$.*

Proof. We have to show that we can generate Δ_β for $\beta \in R$. First note that $s_\alpha = \mathrm{Id} - \alpha \Delta_\alpha$ for $\alpha \in \mathcal{B}$ can be generated. Hence the Weyl group W can be generated. For every root $\beta \in R$ we find an element $w \in W$ such that $w(\beta) \in \mathcal{B}$. Let f be a polynomial in $\Bbbk[\mathbb{Y}_n]$ and consider

$$\begin{aligned} (w^{-1} \Delta_{w(\beta)} w)(f) &= w^{-1} \left(\frac{w(f) - s_{w(\beta)}(w(f))}{w(\beta)} \right) \\ &= \frac{f - s_\beta(f)}{\beta} = \Delta_\beta(f). \end{aligned}$$

Hence the claim follows. □

Proposition 3.6.4. *All elements of the nil Hecke ring* NH_n *are generated as linear combinations over* $\Bbbk$ *by elements of the form*

$$y_1^{m_1}\dots y_n^{m_n}\Delta_w$$

where $m_1,\dots,m_n\in\mathbb{N}$ *and* $w\in W$.

Proof. By Lemma 3.6.3 it suffices to look at Demazure operators Δ_α where $\alpha\in\mathcal{B}$. By Corollary 3.5.9 it holds that the product of Δ_w and $\Delta_{w'}$ for $w,w'\in W$ is either 0 or $\Delta_{ww'}$ if ww' is reduced hence it suffices to consider reduced expressions. If there is an y_i to the right of a Δ_α then one can swap y_i and Δ_α and it holds

$$R_1\Delta_\alpha y_i R_2 = R_1\Delta_\alpha(y_i)R_2 + R_1 s_\alpha(y_i)\Delta_\alpha R_2$$

by the derivation property of the Demazure operator, see (3.5). Here R_1 and R_2 are arbitrary combinations of Demazure operators and variables. Inductively it follows that every element can be written as a sum as above. □

The nil Hecke ring is a $\Bbbk[\mathbb{Y}_n]$-module defined by the action

$$\begin{array}{ccc} \Bbbk[\mathbb{Y}_n]\times\mathrm{NH}_n & \longrightarrow & \mathrm{NH}_n \\ (f,D) & \longmapsto & f\cdot D \end{array}$$

considering polynomials as endomorphisms in NH_n.

Theorem 3.6.5. *The nil Hecke ring* NH_n *is a free* $\Bbbk[\mathbb{Y}_n]$*-module with basis given by the operators* $(\Delta_w)_{w\in W}$.

Proof. By Proposition 3.6.4 we know that elements of the form $y_1^{m_1}\dots y_n^{m_n}\Delta_w$ generate the nil Hecke ring, i.e. it suffices to show that the Δ_w are linearly independent over $\Bbbk[\mathbb{Y}_n]$. Let $a_w\in\Bbbk[\mathbb{Y}_n]$ such that

$$\sum_{w\in W} a_w\Delta_w = 0$$

and l be maximal with the property such that there exists an element $w''\in W$ with $l=l(w'')$ and $a_{w''}\neq 0$. Then by Lemma 3.5.3 we may write

$$\begin{aligned} 0=\sum_{w\in W} a_w\Delta_w &= \sum_{w\in W}\left(\prod_{\alpha\in R(w)}\alpha^{-1}\cdot a_w\cdot(-1)^{l(w)}w + \sum_{l(w')<l(w)} f_{w'}w'\right)\\ &= \sum_{l(w'')=l}\frac{(-1)^{l(w'')}}{\prod_{\alpha\in R(w'')}\alpha}a_{w''}w'' + \sum_{l(w')<l} g_{w'}w' \end{aligned}$$

for some $f_{w'},g_{w'}\in\Bbbk(\mathbb{Y}_n)$.

The elements $w\in W$ are linear independent over $\Bbbk(\mathbb{Y}_n)$ hence $a_{w''}=0$ for all w'' with $l(w'')=l$ which is a contradiction to the choice of $a_{w''}$, i.e. all the a_w vanish and hence the Δ_w are linearly independent over $\Bbbk[\mathbb{Y}_n]$. □

Remark 3.6.6. All this can also be done with elements of the form $\Delta_w y_1^{m_1}\dots y_n^{m_n}$ by using the derivation property (3.5) in the other direction.

Proposition 3.6.7. *The center* $Z(\mathrm{NH}_n)$ *of the nil Hecke ring is isomorphic to the invariant ring* $\Bbbk[\mathbb{Y}_n]^W$ *where polynomials are identified with the corresponding endomorphisms in* NH_n.

Proof. Let α be a simple root and $f, g \in \Bbbk[\mathbb{Y}_n]$ two polynomials. The derivation property (3.5) implies

$$(3.20)\qquad \begin{aligned}(\Delta_\alpha \cdot f)(g) = \Delta_\alpha(f \cdot g) &= \Delta_\alpha(f)\cdot g + s_\alpha(f)\cdot\Delta_\alpha(g)\\ &= \Delta_\alpha(f)\cdot g + (s_\alpha(f)\cdot\Delta_\alpha)(g).\end{aligned}$$

Hence $\Delta_\alpha \notin Z(\mathrm{NH}_n)$ for all simple roots α. By (3.20), for a polynomial f commuting with all Demazure operators is equivalent to being s_α-invariant for all simple roots α. Hence $Z(\mathrm{NH}_n) \cong \Bbbk[\mathbb{Y}_n]^W$. □

3.7 Nil Hecke ring for type A systems

Recalling Example 3.3.5 we get the root system $R = \{x_i - x_j \mid 1 \le i, j \le n,\ i \ne j\}$ for the underlying Dynkin diagram A_{n-1}. The Weyl group $W = S_n$ acts on R permuting the x_i. A basis is given by $\mathcal{B} = \{x_i - x_{i+1} \mid 1 \le i \le n-1\}$. To simplify the combinatorics we again look at the polynomial ring $\Bbbk[\mathbb{X}_n] := \Bbbk[x_1, \dots, x_n]$ instead of $\Bbbk[\alpha_1, \dots, \alpha_{n-1}]$ with $\alpha_i = x_i - x_{i+1}$. The corresponding nil Hecke ring will be modified by looking at endomorphisms of $\Bbbk[\mathbb{X}_n]$ instead of the original polynomial ring, i.e. the modified nil Hecke ring NH_n is generated by the endomorphisms of $\Bbbk[\mathbb{X}_n]$ which are given by

i) the identity Id on $\Bbbk[\mathbb{X}_n]$,

ii) Demazure operators Δ_i, where $\Delta_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$, $\quad 1 \le i \le n-1$ for some $f \in \Bbbk[\mathbb{X}_n]$ where s_i permutes x_i and x_{i+1},

iii) multiplications by x_j, $1 \le j \le n$ where $x_j(f) = x_j \cdot f$.

Interestingly, the nil Hecke ring of type A systems connects the theory of Demazure operators and the KLR algebras of special type $R(n \cdot i)$. We will see that crossings correspond to Demazure operators and dots to a multiplication by a variable.

Remark 3.7.1. All the statements proved for the nil Hecke ring also hold in the modified version, because we have never used special properties about polynomial rings except the fact that there is a W action given. But we have that the inclusion

$$\begin{array}{ccc}\Bbbk[\alpha_1, \dots, \alpha_{n-1}] & \hookrightarrow & \Bbbk[\mathbb{X}_n]\\ \alpha_i & \mapsto & x_i - x_{i+1}\end{array}$$

is invariant under the action of S_n.

Lemma 3.7.2. *In the modified nil Hecke* NH_n *ring the following relations hold*

0) $\mathrm{Id}\cdot x_k = x_k = x_k \cdot \mathrm{Id}$,
$\mathrm{Id}\cdot\Delta_j = \Delta_j = \Delta_j \cdot \mathrm{Id}$,
$x_k x_{k'} = x_{k'} x_k$,
$\Delta_j \Delta_{j'} = \Delta_{j'} \Delta_j$ *if* $|j - j'| > 1$,

1) $\Delta_j^2 = 0$,

2) $\Delta_j x_k = \Delta_j(x_k)\cdot\mathrm{Id} + s_j(x_k)\Delta_j = \begin{cases} \mathrm{Id} + x_{j+1}\Delta_j & \textit{if } k = j,\\ -\mathrm{Id} + x_j\Delta_j & \textit{if } k = j+1,\\ x_k\Delta_j & \textit{otherwise}\ , \end{cases}$

3) $\Delta_j \Delta_{j+1} \Delta_j = \Delta_{j+1} \Delta_j \Delta_{j+1}$ *for* $j \in \{1, \dots, n-2\}$,

for all $k, k' \in \{1, \dots, n\}$ *and* $j, j' \in \{1, \dots, n-1\}$.

Proof. The claim follows by direct calculations. □

Remark 3.7.3. At various parts of this thesis we use the *ideal generated by the given relations*. Assume we have some relations $x_i = y_i$, then by the *ideal generated by the given relations* we mean the ideal generated by the elements $x_i - y_i$.

Lemma 3.7.4. *The relations given in Lemma 3.7.2 are the defining relations of the modified nil Hecke ring* NH_n*.*

Proof. Consider the free algebra $\Bbbk \langle \Delta_1, \dots, \Delta_{n-1}, x_1, \dots, x_n \rangle$ generated by the elements $\Delta_1, \dots, \Delta_{n-1}, x_1, \dots, x_n$. Let I be the ideal generated by the relations given in Lemma 3.7.2. Now consider the quotient

$$\Bbbk \langle \Delta_1, \dots, \Delta_{n-1}, x_1, \dots, x_n \rangle / I .$$

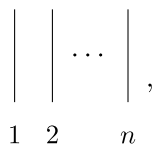
By Lemma 3.7.2 it exists an algebra homomorphism

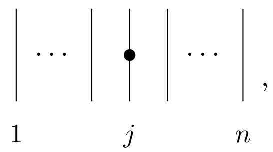
$$\Bbbk \langle \Delta_1, \dots, \Delta_{n-1}, x_1, \dots, x_n \rangle / I \longrightarrow \mathrm{NH}_n \tag{3.21}$$

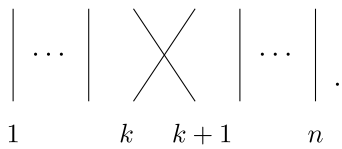
which is surjective, because all generators in NH_n lie in the image of the map. Furthermore, we can write all elements on the left side as a sum of elements of the form $x_1^{m_1} \dots x_n^{m_n} \Delta_w$ for some numbers $m_1, \dots, m_n \in \mathbb{N}$ and $w \in W$ by applying the relations. Hence by Theorem 3.6.5 the homomorphism in (3.21) is an isomorphism and it follows that the relations of I are the defining relations. □

Proposition 3.7.5. *There is an isomorphism of algebras given by the mapping*

$\mathrm{NH}_n \longrightarrow R(n \cdot i),$

$\mathrm{Id} \longmapsto$  ,

$x_j \longmapsto$  ,

$\Delta_k \longmapsto$  .

Here all strands are labelled by i and the numbers give the positions of the strands.

Proof. Define $\mathbf{i} = (i \dots i)$ to be the sequence consisting of n-times the vertex i. Then we know that $1_{\mathbf{i}}$, $x_{j,\mathbf{i}}$ and $\Delta_{k,\mathbf{i}}$ for $1 \leq j \leq n$ and $1 \leq k \leq n-1$ are the algebra generators of the algebra $R(n \cdot i)$. Therefore the map is surjective. To show that the map given above is an isomorphism of algebras it suffices to check that the generators of the nil Hecke ring fulfill the same relations as the ones of the KLR algebra. For the nil Hecke ring we showed in Lemma 3.7.4 that a list of all relations is given by Lemma 3.7.2. Therefore the claim follows, because the defining relations 0)-3) of Lemma 3.7.2 correspond exactly to the relations ($\overline{R0}$)-($\overline{R3}$) of the KLR algebra. □

The facts of the nil Hecke ring proven above now hold as well for our special KLR algebra. The Theorem 3.6.5 gives a basis for the nil Hecke ring. Together with the isomorphism before one gets a basis of $R(n \cdot i)$. In diagrams this means that $R(n \cdot i)$ is generated linearly as a vector space by diagrams where there are first some crossings according to a reduced expression of an element of the symmetric group and then some dots on the strands. This approach can be generalized for the KLR algebra.

3.8 Faithful Representation of the KLR algebra

We will replace the integers used in Section 3.2 by a field $\Bbbk$ of characteristic 0. This does not change anything except the fact that we consider $\Bbbk$-linear combinations of diagrams instead of $\mathbb{Z}$-linear ones.

It is much easier to handle the KLR algebra and compare diagrams if we can compare their actions on a faithful representation, e.g. if the action is the same then the faithfulness immediately implies that the acting elements are the same. For this we define a sum of certain polynomial rings. For an element $\nu \in \mathbb{N}[\mathbb{V}]$ define the space

$$V_\nu := \bigoplus_{\mathbf{i} \in \mathrm{Seq}(\nu)} \Lambda(\mathbf{i}) \tag{3.22}$$

where $\Lambda(\mathbf{i}) := \Bbbk[x_1^{\mathbf{i}}, \dots, x_n^{\mathbf{i}}]$ for a sequence $\mathbf{i} = (i_1 \dots i_n) \in \mathrm{Seq}(\nu)$ with $n := |\nu|$.

The symmetric group S_n acts on V_ν componentwise by

$$\begin{array}{rccc} w: & \Lambda(\mathbf{i}) & \longrightarrow & \Lambda(w(\mathbf{i})), \\ & x_k^{\mathbf{i}} & \mapsto & x_{w(k)}^{w(\mathbf{i})} \end{array}$$

for $w \in S_n$ where $w(\mathbf{i})$ is the usual action on sequences and $w(k)$ is the usual action on the set $\{1, \dots, n\}$.

Lemma 3.8.1. *The action of $R(\nu)$ on V_ν, defined componentwise on $f \in \Lambda(\mathbf{j})$ for all $\mathbf{j} \in \mathrm{Seq}(\nu)$ by*

1) $1_{\mathbf{i}}(f) = f$,

2) $x_{j,\mathbf{i}}(f) = x_j^{\mathbf{i}} \cdot f$,

3) $\Delta_{k,\mathbf{i}}(f) = \begin{cases} s_k(f) & \text{if } i_k \cdot i_{k+1} = 0 \text{ or } i_k \leftarrow i_{k+1}, \\ \Delta_k^{\mathbf{i}}(f) & \text{if } i_k = i_{k+1}, \\ (x_{k+1}^{s_k(\mathbf{i})} - x_k^{s_k(\mathbf{i})}) \cdot s_k(f) & \text{if } i_k \to i_{k+1}. \end{cases}$

for $\mathbf{i} = \mathbf{j}$ and zero otherwise, is faithful. Here $\Delta_k^{\mathbf{i}}(f) = \frac{f - s_k(f)}{x_k^{\mathbf{i}} - x_{k+1}^{\mathbf{i}}}$ is a Demazure operator. The arrows denote the direction of the edges.

Proof. We omit to check that the definition of the action respects the relations of the KLR algebra. For the calculation see [KL09, Prop. 2.3]. □

Remark 3.8.2. For the special KLR algebra $R(n \cdot i)$ the faithful action agrees with the polynomial action of the modified nil Hecke ring.

3.9 Basis of the KLR algebra

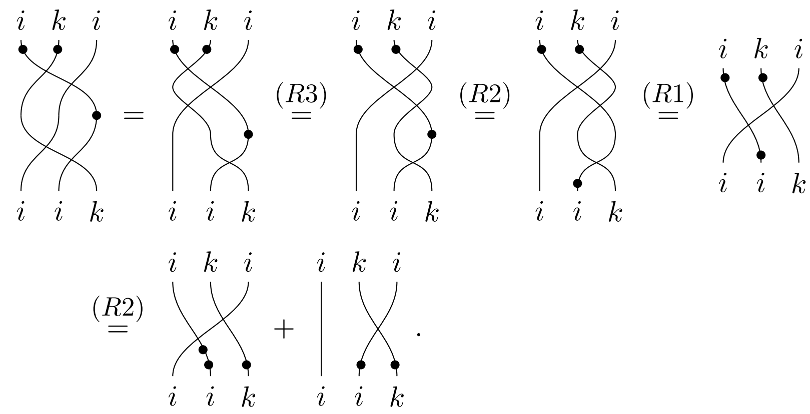
The main idea how to write down a basis of the KLR algebra is to rewrite diagrams as sums of diagrams with “lower order”, i.e. fewer crossings, and “moving the dots trough the crossings” to the bottom of the diagrams. There are two reduction steps.

1. A diagram containing two arcs which intersect more than one time can be written by $(R1)$-$(R3)$ as a linear combinations of diagrams where the two arcs intersect once less.

2. If in a diagram there is a dot above a crossing then with the help of $(R2)$ the dot can be “moved through the crossing” generating an additional term with one crossing fewer. Here no additional crossings appear hence the first step is not touched.

Applying these reduction steps on an arbitrary diagram we can write this as sum of diagrams where two arcs do not cross more than one time and all dots are below the crossings. Hence the set of all diagrams, where two arcs do not cross more than once and all the dots are at the bottom of the diagram, spans the KLR algebra as a $\Bbbk$-vector space. This means a diagram of the spanning set with n arcs can be completely described by the bottom sequence $\mathrm{bot}(D)$, a reduced expression of an element from the symmetric group S_n representing the crossings ($s_k \in S_n$ stands for the crossing of the arcs k and $k+1$) and a tuple $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n_{\geq 0}$ denoting the number of dots on the respective arcs.

Example 3.9.1. We use the steps from above to rewrite the following example diagram as sum of diagrams where the dots are below the crossings and no arcs cross more than one time. Assume $i \neq k$ and $i \cdot k = 0$ then



Remark 3.9.2. Two diagrams from the spanning set with the same bottom sequences and the same dot tuple but different reduced expressions of the same element in the symmetric group differ only by a sum of diagrams with fewer crossings using the braid relation $(R3)$.

For each element $w \in S_n$ fix a reduced expression $\mathrm{red}_w := s_{i_1} \ldots s_{i_r}$ of w. For sequences $\mathbf{i}, \mathbf{j} \in \mathrm{Seq}(\nu)$ define the subset $S_{\mathbf{i}}^{\mathbf{j}} \subset S_n$ of elements which send $\mathbf{i}$ to $\mathbf{j}$ by acting on the sequence. Define by

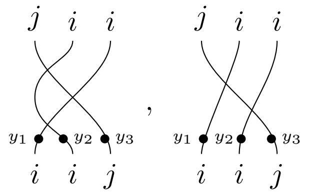
$$\mathrm{red}_{S_{\mathbf{i}}^{\mathbf{j}}} := \bigcup_{w \in S_{\mathbf{i}}^{\mathbf{j}}} \mathrm{red}_w$$

the set of all fixed reduced expressions of elements in $S_{\mathbf{i}}^{\mathbf{j}}$.

Denote by $B_{\mathbf{i}}^{\mathbf{j}}$ the set of all diagrams D from the spanning set with $\mathrm{bot}(D) = \mathbf{i}$ and $\mathrm{top}(D) = \mathbf{j}$. The crossings are given according to the reduced expressions $\mathrm{red}_w \in \mathrm{red}_{S_{\mathbf{i}}^{\mathbf{j}}}$ and $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n_{\geq 0}$.

Example 3.9.3. Let $\nu = 2i + j$ and $\mathbf{i} = (iij), \mathbf{j} = (jii) \in \mathrm{Seq}(\nu)$. Then fix for each element $w \in S_3$ a reduced expression, e.g. fix $s_1s_2s_1$ (instead of $s_2s_1s_2$) all the other reduced expressions are unique. Then $\mathrm{red}_{S_{\mathbf{i}}^{\mathbf{j}}} = \{s_1s_2s_1, s_1s_2\}$. Hence

$B_{\mathbf{i}}^{\mathbf{j}} = \Big\{$



$\Big| \; (y_1, y_2, y_3) \in \mathbb{Z}^3_{\geq 0} \Big\}$

is a spanning set where $(y_1, y_2, y_3) \in \mathbb{Z}^3_{\geq 0}$ denotes the number of dots on the strands. By Theorem 3.9.4 it follows that $B_{\mathbf{i}}^{\mathbf{j}}$ is a basis of $R(\nu)_{\mathbf{i}}^{\mathbf{j}}$ which is the set of all diagrams from $R(\nu)$ such that the top is labelled by $\mathbf{j}$ and the bottom by $\mathbf{i}$.

Theorem 3.9.4. *[KL09, Thm. 2.5.] The set $B_{\mathbf{i}}^{\mathbf{j}}$ is a homogeneous basis of the free graded abelian group $R(\nu)_{\mathbf{i}}^{\mathbf{j}}$.*

Proof. We already know that $B_{\mathbf{i}}^{\mathbf{j}}$ is a spanning set. Hence it suffices to show the linear independence of the spanning set. For this the main idea is checking that the elements act on the faithful representation by linear independent operators, see [KL09, Thm. 2.5.] for details. $\square$

Corollary 3.9.5. *A basis of the KLR algebra $R(\nu)$ is given by*

$$B(\nu) := \bigcup_{\mathbf{i},\mathbf{j} \in \mathrm{Seq}(\nu)} B_{\mathbf{i}}^{\mathbf{j}}.$$

This brings us to the end of the section about KLR algebras and we now turn to the introduction of the quiver Schur algebras.

4 Quiver Schur algebras

The goal of this section is to introduce a generalization of the KLR algebras by looking at quiver representations and quiver flag varieties following [SW11], see figure 6. In Proposition 4.5.1 we show in detail that the KLR algebra given the graph of the Dynkin diagram of affine type A is a subalgebra of the so-called quiver Schur algebra. First of all we present the motivation where the idea of the quiver Schur algebras comes from. Afterwards we go more into detail and prepare for its definition, see Section 4.4. The basics of calculating with quiver Schur diagrams are introduced in Section 4.6. In Section 4.7 we recall the basis theorem of the quiver Schur algebras, see [SW11, Thm. 3.11], in the notation introduced in Section 4.4.

4.1 Basic definitions

Definition 4.1.1. A **quiver** Γ consists of a set of vertices $\mathbb{V}$ and a set of edges $\mathbb{E}$. The edges should be directed, i.e. we have two functions $h : \mathbb{E} \to \mathbb{V}$ and $t : \mathbb{E} \to \mathbb{V}$ where h denotes the head of the edge and t the tail. The edges are directed from tail to head.

We are interested in the quivers Γ_e where the vertices and edges are given by affine Dynkin diagrams of type A corresponding to the affine Lie algebra $\hat{\mathfrak{sl}}_e$ with cyclic orientation of the edges, see figure 1 in the introduction. Therefore, we fix some $e \geq 3$ and denote by $\mathbb{V} = \{1, \ldots, e\}$ the vertex set of the quiver Γ_e. The special case $e = \infty$ is also possible with $\mathbb{V} = \mathbb{Z}$ for the infinite quiver.

Definition 4.1.2. A (finite dimensional) **representation** (V, f) **of** Γ_e over a field $\Bbbk$ consists of

i) $\Bbbk$-vector spaces V_i for each vertex $i \in \mathbb{V}$ such that $\sum_{i \in \mathbb{V}} \dim V_i < \infty$,

ii) $\Bbbk$-linear maps $f_i : V_i \to V_{i+1}$ for $i \in \{1, \ldots, e\}$ and $e + 1 := 1$.

Definition 4.1.3. A **subrepresentation** (U, f) of a representation (V, f) of Γ_e is a collection of vector subspaces $U_i \subset V_i$ such that for all $f_i : V_i \to V_{i+1}$ the map f_i restricts to a map on the subspaces, i.e. $f_i(U_i) \subset U_{i+1}$.

For every finite dimensional representation (V, f) of Γ_e we can define the corresponding **dimension vector** which is given by the tuple $\mathbf{d} = (d_1, d_2, \ldots, d_e)$, where $d_i := \dim V_i$. The finiteness condition implies that there are only finitely many $d_i \neq 0$. We denote by $d = \sum_{i \in \mathbb{V}} d_i$ the **total dimension**, which is finite. For some $j \in \{1, \ldots, e\}$ we define the special dimension vector $\alpha_j = (d_1, d_2, \ldots, d_e)$ with $d_i = \delta_{ij}$ for $1 \leq i \leq e$. This special vector will play an important role in Proposition 4.5.1 and Section 5.

We will now work with representations over the field of complex numbers $\mathbb{C}$.

Definition 4.1.4. The affine **space of representations** of Γ_e over $\mathbb{C}$ for a fixed dimension vector $\mathbf{d}$ is denoted by

$$\mathrm{Rep}_{\mathbf{d}} := \bigoplus_{i \in \mathbb{V}} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_{i+1}}). \tag{4.1}$$

Connecting the diagrammatic approach of the KLR algebras and the approach with quiver representations we need to add flags to the setting. Every vertex in the quiver which corresponds to a vector space in the representation carries a flag, see figure 6. The different flags should be compatible with the quiver representation, i.e. the restriction of the linear maps onto the subspaces should fulfill certain properties which are introduced in Definition 4.1.14. We will now formalise the construction in figure 6. We note that a flag

$$\{0\} = V^0 \subset V^1 \subset V^2 \subset \ldots \subset V^r$$

for vector spaces $V^1, V^2, \ldots, V^r$ can be described by the dimension steps from one subspace to the next. We get a sequence of non negative integer numbers by looking at $\dim V^{i+1} - \dim V^i$ for $i \in \{0, \ldots, r-1\}$. The sum of all numbers in this sequence denotes the dimension of the whole space. This fits the next definition dealing with compositions.

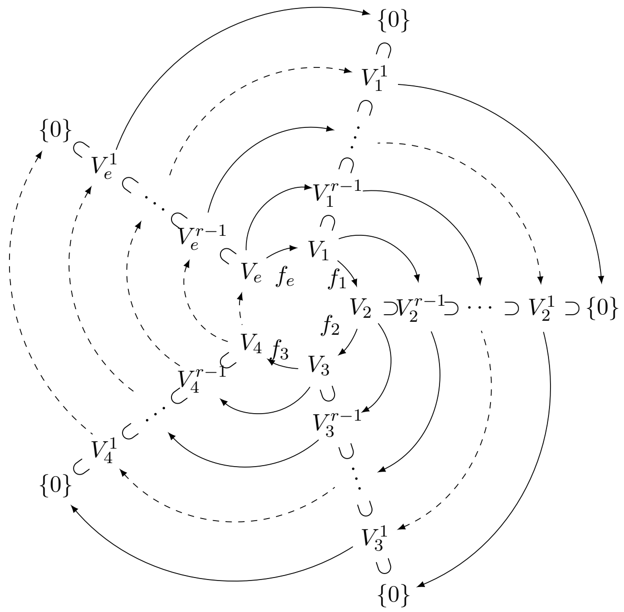


Figure 6: Quiver representation with compatible flags

Remark 4.1.5. Here the meaning of flags is a little bit different from the standard definition, because also non proper inclusions with dimension step zero are allowed. This does not change the properties of a flag but is needed for the compatibility of flags and representations, see Definition 4.1.14.

Definition 4.1.6. A **composition** μ of $n \in \mathbb{Z}_{>0}$ of length $r = r(\mu)$ is a tuple $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{\geq 0}$ such that $\sum_{i=1}^r \mu_i = n$.

Definition 4.1.7. A vector $\mathbf{a} \in \mathbb{Z}^m_{\geq 0}$ for $m > 0$ is called **vector of type** m.

Definition 4.1.8. A **vector composition** of type $m \in \mathbb{Z}_{>0}$ and length $r = r(\hat{\mu})$ is a tuple $\hat{\mu} = (\hat{\mu}^{(1)}, \ldots, \hat{\mu}^{(r)})$ of nonzero vectors of type m. The **dimension vector** of such a vector composition is given by $\mathbf{d} = d(\hat{\mu}) = \sum_{1 \leq i \leq r} \hat{\mu}^{(i)}$.

For $m = e$ the vector composition is called **residue data** and VComp_e is the set of all possible residue data. Denote by

$$\mathrm{VComp}_e(\mathbf{d}) := \{\hat{\mu} \in \mathrm{VComp}_e \mid d(\hat{\mu}) = \mathbf{d}\}$$

the set of all residue data for a given dimension vector $\mathbf{d}$. We can view a vector composition of type m and length r as a $r \times m$ matrix $M(\hat{\mu})$ called **composition matrix** where the i-th row of the matrix is given by the i-th vector in the vector composition, i.e. $(M(\hat{\mu}))_{i,j} = \hat{\mu}_j^{(i)}$. Then going through the columns gives the so-called **flag type sequence** $t(\hat{\mu})$.

A vector composition $\hat{\mu}$ has **complete flag type** if every vector in the composition is a unit vector with exactly one non zero entry.

Definition 4.1.9. The **transposed composition of** $\hat{\mu}$ of type $r(\hat{\mu})$ and length m is the tuple $\check{\mu} = (\check{\mu}^{(1)}, \ldots, \check{\mu}^{(m)})$ where $\check{\mu}_i^{(j)} = (M(\hat{\mu}))_{i,j}$, i.e. $\check{\mu}^{(j)}$ is the j-th column of $M(\hat{\mu})$. For $m = e$ this is called **flag data**.

A transposed composition of type r and length $m = e$ can be seen as a collection of e compositions of length r. As mentioned above a composition of length r of n corresponds to flags of length r where the dimension of the last space equals n and the dimension steps are given by the composition. Hence a transposed vector composition will correspond to flags for every vertex in the quiver.

In figure 6, given a vector composition $\hat{\mu}$, then the dimensions of the spaces V_j^i are given by $\sum_{k=1}^{i} \check{\mu}_k^{(j)}$.

Definition 4.1.10. Let μ be a composition of n of length r. Then $\mathcal{F}(\mu)$ is the **variety of flags of type** μ with

$$\{0\} \subset F^1(\mu) \subset F^2(\mu) \subset \ldots \subset F^r(\mu) = \mathbb{C}^n$$

and $\dim F^i(\mu) = \sum_{j=1}^{i} \mu_j$. The vectors of the flag data of a residue data $\hat{\mu}$ give compositions $\check{\mu}^{(1)}, \ldots, \check{\mu}^{(e)}$ of $d_1, \ldots, d_e$ each of length r. Hence one can define the variety of partial flags for a residue data $\hat{\mu}$ by a product of flag varieties. Define

$$\mathcal{F}(\hat{\mu}) := \prod_{i=1}^{e} \mathcal{F}(\check{\mu}^{(i)})$$

as the product of partial flags inside $\mathbb{C}^d = \prod_{i=1}^{e} \mathbb{C}^{d_i}$ corresponding to the dimension vector $\mathbf{d} = (d_1, \ldots, d_e) = \sum_{i=1}^{r} \hat{\mu}^{(i)}$ with total dimension $d = \sum_{i=1}^{e} d_i$.

Definition 4.1.11. Let $\hat{\mu}$ be a residue data of length r. Then the **residue sequence** $\operatorname{res}(\hat{\mu})$ is defined by

$$\underbrace{1 \ldots 1}_{\hat{\mu}_1^{(1)}} \; \underbrace{2 \ldots 2}_{\hat{\mu}_2^{(1)}} \cdots \underbrace{e \ldots e}_{\hat{\mu}_e^{(1)}} | \underbrace{1 \ldots 1}_{\hat{\mu}_1^{(2)}} \cdots | \underbrace{1 \ldots 1}_{\hat{\mu}_1^{(r)}} \cdots \underbrace{e \ldots e}_{\hat{\mu}_e^{(r)}}. \tag{4.2}$$

The parts separated by vertical lines are called **blocks of** $\operatorname{res}(\hat{\mu})$.

We can interpret the residue sequences in the context of figure 6. All numbers of the same value in the residue sequence correspond to the same flag and all elements in the same block correspond to the same level of the flags. Residue sequences are important to construct a basis of the quiver Schur algebras, see Theorem 4.7.5.

To get a better feeling for the definitions we give an example.

Example 4.1.12. Fix $e = m = 3$, $r = 4$ and let

$$\hat{\mu} = ((1,2,1),(0,0,1),(2,1,0),(0,3,2)) \in \mathrm{VComp}_e$$

be a residue data. Then the dimension vector $\mathbf{d}$ is given by $\mathbf{d} = (3,6,4)$ and the total dimension is given by $d = 13$. The composition matrix is

$$M(\hat{\mu}) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

The flag data can be easily written down by going through the columns of the composition matrix

$$\check{\mu} = \{(1,0,2,0),(2,0,1,3),(1,1,0,2)\}.$$

After all we get the residue sequence

$$\operatorname{res}(\hat{\mu}) = 1\ 22\ 3|3|11\ 2|222\ 33.$$

Let (V, f) be a representation of the quiver Γ_e with $e = 3$ over $\mathbb{C}$ and the vector spaces at the vertices are given by $V_1 = \mathbb{C}^3, V_2 = \mathbb{C}^6, V_3 = \mathbb{C}^4$. A possible product of partial flags inside $\mathbb{C}^3 \times \mathbb{C}^6 \times \mathbb{C}^4 = \mathbb{C}^{13}$ corresponding to $\hat{\mu}$ is given by

$$(\mathbb{C}^1 \subset \mathbb{C}^1 \subset \mathbb{C}^3 \subset \mathbb{C}^3) \times (\mathbb{C}^2 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^6) \times (\mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^2 \subset \mathbb{C}^4).$$

Now we want to include the linear maps of the quiver representation into this model.

Definition 4.1.13. A representation (V, f) of the quiver Γ_e is called **nilpotent** if the map $f_e \ldots f_2 f_1 : V_1 \to V_1$ is nilpotent. For $e = \infty$ all representations are called nilpotent.

Definition 4.1.14. For a vector composition $\hat{\mu} \in \mathrm{VComp}_e(\mathbf{d})$ of length r a **representation with compatible flags of type** $\check{\mu}$ is a nilpotent representation (V, f) of Γ_e together with flags

$$F(i) := \left(\{0\} = F(i)^0 \subset F(i)^1 \subset \ldots \subset F(i)^{r-1} \subset F(i)^r = V_i\right)$$

of type $\check{\mu}^{(i)}$ inside V_i for each $1 \leq i \leq e$ such that

$$\dim F(i)^l = \sum_{k=1}^{l} \check{\mu}_k^{(i)} \text{ and } f_i(F(i)^j) \subset F(i+1)^{j-1} \text{ for } 1 \leq j \leq r.$$

The subset of $\mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\mu})$ which consists of all representations with compatible flags will be denoted by

$$\mathcal{Q}(\hat{\mu}) := \{(V, f, F) \in \mathrm{Rep}_{\mathbf{d}} \times \mathcal{F}(\hat{\mu}) \mid f_i(F(i)^j) \subset F(i+1)^{j-1} \text{ for all } i, j\}.$$

The set $\mathcal{Q}(\hat{\mu})$ is equipped with the conjugation action of the group

$$G := G_{\mathbf{d}} = \mathrm{GL}_{d_1} \times \ldots \times \mathrm{GL}_{d_e}$$

which acts by change of basis.

Remark 4.1.15. The elements of $\mathcal{Q}(\hat{\mu})$ can be presented as in figure 6 where the inner circle is an element of $\mathrm{Rep}_{\mathbf{d}}$. Adding the flag variety $\mathcal{F}(\hat{\mu})$ (here the flags are denoted by V_i instead of $F(i)$) we get the flags of the picture. The maps going outwardly should represent the inner maps restricted to the subspaces and their image lies by compatibility of the flags in the next outer subspace.

Example 4.1.16. Continuing Example 4.1.12 the representation has compatible flags if the maps f_1, f_2 and f_3 restricted to the subspaces of the flags lie in the corresponding subspace in the next flag as shown in figure 10, Appendix A.1.

4.2 Merges, splits and geometric approach

Definition 4.2.1. Let $\hat{\lambda} = (\hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(r)})$ and $\hat{\lambda}_k$ be vector compositions. Then $\hat{\lambda}_k$ is a **merge** of $\hat{\lambda}$ and hence $\hat{\lambda}$ a **split** of $\hat{\lambda}_k$ at the index k if

$$\hat{\lambda}_k = \left(\hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(k)} + \hat{\lambda}^{(k+1)}, \ldots, \hat{\lambda}^{(r)}\right).$$

If $\hat{\lambda}_k$ is a merge of $\hat{\lambda}$ one can define the quivers equipped with flags which are compatible with respect to the merge, i.e.

$$\mathcal{Q}(\hat{\lambda}, k) := \left\{(V, f, k) \in \mathcal{Q}(\hat{\lambda}) | \ f_i \left(F(i)^{k+1}\right) \subset F(i+1)^{k-1}\right\}.$$

There are the maps $\mathcal{Q}(\hat{\lambda}, k) \hookrightarrow \mathcal{Q}(\hat{\lambda})$ and $\mathcal{Q}(\hat{\lambda}, k) \to \mathcal{Q}(\hat{\lambda}_k)$ where the first map is given by inclusion and the second by deleting the k-th flag subspace in every flag, see figure 11, Appendix A.2.

Considering quiver representations together with compatible flags there are the $G_{\mathbf{d}}$ equivariant morphisms

$$\begin{array}{rcl} p & : & \mathcal{Q}(\hat{\lambda}) \to \mathrm{Rep}_{\mathbf{d}} \quad , \quad (V, f, \mathcal{F}) \to (V, f) \\ \pi & : & \mathcal{Q}(\hat{\lambda}) \to \mathcal{F}(\hat{\lambda}) \quad , \quad (V, f, \mathcal{F}) \to \mathcal{F} \end{array} \tag{4.3}$$

one forgetting the flags and the other one forgetting the representation. The fibers of p are called the **quiver partial flag varieties**.

Definition 4.2.2. Let $\hat{\lambda}, \hat{\mu} \in \mathrm{VComp}_e(\mathbf{d})$ be two vector compositions with the same dimension vector $\mathbf{d}$. Then the space

$$Z(\hat{\lambda}, \hat{\mu}) = \mathcal{Q}(\hat{\lambda}) \times_{\mathrm{Rep}_{\mathbf{d}}} \mathcal{Q}(\hat{\mu}) \tag{4.4}$$

is called **"Steinberg variety"** in the sense of the classical Steinberg varieties, see [CG97, §3.3]. The group action of $G_{\mathbf{d}}$ from before extends to a diagonal action on the Steinberg variety. We define

$$\mathcal{H}(\hat{\lambda}, \hat{\mu}) := {Z(\hat{\lambda}, \hat{\mu})} \big/ {G_{\mathbf{d}}}. \tag{4.5}$$

From this point on there are two possible approaches to define the quiver Schur algebra.

The first one is geometric and uses the space $\mathcal{H}(\hat{\lambda}, \hat{\mu})$. Then the quiver Schur algebra $A_{\mathbf{d}}$ is the space

$$\bigoplus_{(\hat{\lambda}, \hat{\mu})} H_*^{\mathrm{BM}}(\mathcal{H}(\hat{\lambda}, \hat{\mu}))$$

where $(\hat{\lambda}, \hat{\mu})$ runs over all ordered pairs of vector compositions in $\mathrm{VComp}_e(\mathbf{d})$ and H_*^{BM} denotes the Borel-Moore cohomology, see [SW11, §2.2] and [CG97] for details. The convolution product of the equivariant Borel-Moore cohomology gives an associative non-unital graded algebra structure, see [SW11, (2.9)]. The algebra $A_{\mathbf{d}}$ acts faithfully on the space

$$\bigoplus_{\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})} H_*^{\mathrm{BM}} \left({\mathcal{Q}(\hat{\lambda})} \big/ {G_{\mathbf{d}}} \right) \tag{4.6}$$

and the action is natural on the sums of cohomologies, see [SW11, §2.3]. The identification of (4.6) with some invariant space $V_{\mathbf{d}}$ which is defined in the next section leads to the second one, the algebraic approach.

We will now concentrate on the algebraic approach and introduce the quiver Schur algebra as a subalgebra of the endomorphism ring of the space $V_{\mathbf{d}}$, Section 4.4. Then the algebra can be described by certain diagrams similarly to the KLR case but generalizing it. Due to [SW11, §3] both approaches lead to the same result.

4.3 The space $V_{\mathbf{d}}$

Similarly to the faithful polynomial representation of the KLR algebras, see Section 3.8, we can define certain polynomial rings depending on some dimension vector $\mathbf{d} = (d_1, \dots, d_e)$. Define for each $i \in \{1, \dots, e\}$ a polynomial ring $\Bbbk[x_{i,1}, \dots, x_{i,d_i}]$ in d_i variables. The polynomial ring $R(\mathbf{d})$ is given by tensoring up these polynomial rings, i.e.

$$R(\mathbf{d}) := \bigotimes_{i=1}^{e} \Bbbk[x_{i,1}, \dots, x_{i,d_i}] \cong \Bbbk[x_{1,1}, \dots, x_{1,d_1}, \dots, x_{e,1}, \dots, x_{e,d_e}]. \tag{4.7}$$

There is an action of the symmetric group $S_{\mathbf{d}} := S_{d_1} \times \ldots \times S_{d_e}$ given by permuting the variables of the tensor factors, i.e. for $1 \leq i \leq e$ the symmetric group S_{d_i} acts on the variables $x_{i,j}$, $1 \leq j \leq d_i$.

Let $\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})$ be a vector composition with dimension vector $\mathbf{d} = d(\hat{\lambda})$. Then define the invariant ring $\Lambda(\hat{\lambda})$ by

$$\Lambda(\hat{\lambda}) := R(\mathbf{d})^{S_{\hat{\lambda}}} \tag{4.8}$$

where

$$S_{\hat{\lambda}} := S_{\hat{\lambda}_1^{(1)}} \times S_{\hat{\lambda}_1^{(2)}} \times \ldots \times S_{\hat{\lambda}_1^{(r)}} \times S_{\hat{\lambda}_2^{(1)}} \times \ldots \times S_{\hat{\lambda}_e^{(r)}} \subset S_{\mathbf{d}}. \tag{4.9}$$

Summing up over all possible vector compositions $\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})$ we get the space

$$V_{\mathbf{d}} := \bigoplus_{\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})} \Lambda(\hat{\lambda}). \tag{4.10}$$

Remark 4.3.1. By [SW11, (2.10)] we have the identification

$$V_{\mathbf{d}} \cong \bigoplus_{\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})} H_*^{\mathrm{BM}} \left(\mathcal{Q}(\hat{\lambda}) \big/ G_{\mathbf{d}} \right).$$

We calculate the invariant spaces for some examples of vector compositions.

Example 4.3.2. 1) Let $e = 3$ and $m = r = 2$. Consider the vector composition

$$\hat{\lambda} = ((1,1,0), (2,1,0))$$

with dimension vector $\mathbf{d} = (3,2,0)$. Then

$$R(\mathbf{d}) = \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}] \otimes \Bbbk[x_{2,1}, x_{2,2}].$$

The group $S_{\mathbf{d}} = S_3 \times S_2$ acts on $R(\mathbf{d})$ by permuting variables. We have $S_{\hat{\lambda}} = S_1 \times S_2 \times S_1 \times S_1 \subset S_3 \times S_2 = S_{\mathbf{d}}$ and hence for the invariant space we get that

$$\Lambda(\hat{\lambda}) = R(\mathbf{d})^{S_{\hat{\lambda}}} = (x_{1,1}, x_{1,2} + x_{1,3}, x_{1,2} \cdot x_{1,3}, x_{2,1}, x_{2,2}).$$

2) Let $\hat{\lambda} = (\alpha_j, \dots, \alpha_j)$ be a vector composition with dimension vector $\mathbf{d} = r \cdot \alpha_j$ for the special vector α_j. Then $\Lambda(\hat{\lambda}) = R(\mathbf{d})^{S_{\hat{\lambda}}} = R(\mathbf{d})$, because the group $S_{\hat{\lambda}} = S_1 \times \ldots \times S_1$ (r-times) is trivial.

3) For $d_1, \dots, d_e \in \mathbb{Z}_{\geq 0}$ let $\hat{\lambda} = (d_1 \cdot \alpha_1, \dots, d_e \cdot \alpha_e)$ be a vector composition with dimension vector $\mathbf{d} = (d_1, d_2, \dots, d_e)$. Hence $S_{\hat{\lambda}} = S_{\mathbf{d}}$ and $\Lambda(\hat{\lambda}) = R(\mathbf{d})^{S_{\mathbf{d}}}$ is the ring of total invariants.

The last two examples are extreme cases, where the results are the rings with no invariance restriction and with maximal invariance restriction.

We will introduce some important notations concerning the action of symmetric groups and the Demazure operators on the polynomial ring $R(\mathbf{d})$ and define what is meant by "shifting a polynomial by a vector".

Definition 4.3.3. Let $R(\mathbf{d}) = \bigotimes_{i=1}^{e} \Bbbk[x_{i,1}, \dots, x_{i,d_i}]$ be the polynomial ring as above. We define for $i \in \{1, \dots, e\}$

1) the generators $s_k^{(i)}$, $k \in \{1, \dots, d_i - 1\}$ of S_{d_i} which act on $R(\mathbf{d})$ by permuting the variables $x_{i,k}$ and $x_{i,k+1}$.

2) the Demazure operators $\Delta_k^{(i)}$ by

$$\Delta_k^{(i)}(f) := \frac{f - s_k^{(i)}(f)}{x_{i,k} - x_{i,k+1}} \text{ for } 1 \leq k \leq d_i - 1$$

for some $f \in R(\mathbf{d})$.

3) the operator

$$\Delta_w^{(i)} := \Delta_{k_1}^{(i)} \dots \Delta_{k_1}^{(i)}$$

for a reduced expression $w = s_{k_1}^{(i)} \dots s_{k_r}^{(i)} \in S_{d_i}$.

Let $P \in R(\mathbf{d})$ and $\mathbf{c} = (c_1, \dots, c_e) \in \mathbb{Z}_{\geq 0}^{e}$. The polynomial P **shifted by the vector c** is the polynomial $P_{\mathbf{c}}$ which results from changing all the variables $x_{i,j}$ in P into $x_{i,j+c_i}$. Here the vector $\mathbf{c}$ has to be chosen such that all variables exist.

We are now prepared to define the quiver Schur algebra diagrammatically.

4.4 Diagrammatic quiver Schur algebras

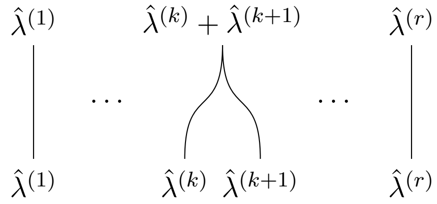
In the following all the vector compositions which will be used are residue data, i.e. of type e, and also all vectors are of type e. For a fixed dimension vector $\mathbf{d}$ the **quiver Schur algebra** $A_{\mathbf{d}}$ can be defined as a subalgebra of the endomorphism ring of $V_{\mathbf{d}}$ generated by the following endomorphisms and their corresponding diagrams. In Definition 4.4.4 the explicit action of the generators on $V_{\mathbf{d}}$ is written down. This corresponds to the convolution product defined in [SW11, §2.2].

Definition 4.4.1. Let $\mathbf{d}$ be a vector of type e. The quiver Schur algebra $A_{\mathbf{d}}$ is the algebra generated by the following endomorphisms (and their corresponding diagrams) defined on the components $\Lambda(\hat{\lambda})$ of $V_{\mathbf{d}}$ where the vector composition $\hat{\lambda} = (\hat{\lambda}^{(1)}, \dots, \hat{\lambda}^{(r)})$ runs over all elements in $\mathrm{VComp}_e(\mathbf{d})$:

1) **Idempotents:** $e_{\hat{\lambda}} : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda})$

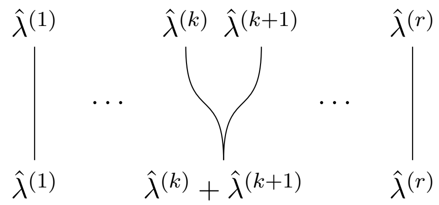
$$e_{\hat{\lambda}} = \begin{array}{ccccc} \hat{\lambda}^{(1)} & \hat{\lambda}^{(2)} & \hat{\lambda}^{(3)} & & \hat{\lambda}^{(r)} \\ \Big| & \Big| & \Big| & \cdots & \Big| \\ \hat{\lambda}^{(1)} & \hat{\lambda}^{(2)} & \hat{\lambda}^{(3)} & & \hat{\lambda}^{(r)} \end{array} .$$

2) **Merges:** $m^k_{\hat{\lambda}} : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda}_k)$ where $\hat{\lambda}_k$ is the merge of $\hat{\lambda}$ at position k for $1 \leq k \leq r-1$.

$m^k_{\hat{\lambda}} =$ 

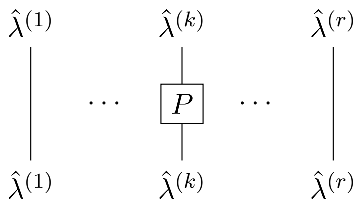
.

3) **Splits:** $s^k_{\hat{\lambda}} : \Lambda(\hat{\lambda}_k) \longrightarrow \Lambda(\hat{\lambda})$ where $\hat{\lambda}_k$ is the merge of $\hat{\lambda}$ at position k for $1 \leq k \leq r-1$.

$s^k_{\hat{\lambda}} =$ 

.

4) **Polynomials:** $x^k_{\hat{\lambda}}(P) : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda})$ where P is an arbitrary polynomial in $\Lambda(\hat{\lambda}^{(k)})$ shifted by the vector $\sum_{i=1}^{k-1} \hat{\lambda}^{(i)}$ if $2 \leq k \leq r$.

$x^k_{\hat{\lambda}}(P) =$ 

.

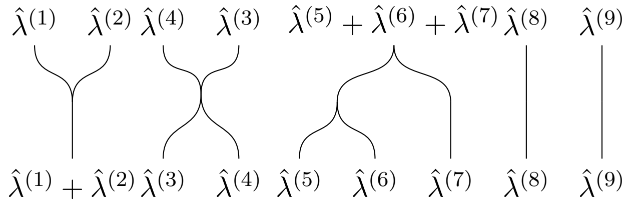
The multiplicative structure corresponds to the horizontally stacking of diagrams as we had for the KLR diagrams before. If the labels of two diagrams which are stacked together do not fit together then their product is defined to be zero. The endomorphism action is explicitly given in Definition 4.4.4. It provides that height moves inside a diagram are allowed unless the combinatorial type is not changed, see Example 4.4.3. We call a morphism $\Lambda(\hat{\lambda}) \to \Lambda(\hat{\mu})$ generated by idempotents, merges, splits and polynomials a (quiver Schur) diagram with bottom labels given by $\hat{\lambda}$ and top labels given by $\hat{\mu}$.

Remark 4.4.2. As vector spaces we have the decomposition

$$A_{\mathbf{d}} \cong \bigoplus_{(\hat{\lambda},\hat{\mu})} A^{\hat{\mu}}_{\hat{\lambda}} \tag{4.11}$$

for $(\hat{\lambda}, \hat{\mu}) \in \mathrm{VComp}_e(\mathbf{d}) \times \mathrm{VComp}_e(\mathbf{d})$ and $A^{\hat{\mu}}_{\hat{\lambda}}$ is the set of all $\Bbbk$-linear combinations of diagrams D such that the bottom labels of D are given by $\hat{\lambda}$ and the top labels by $\hat{\mu}$.

Example 4.4.3. For the vector compositions $\hat{\lambda} = (\hat{\lambda}^{(1)} + \hat{\lambda}^{(2)}, \hat{\lambda}^{(3)}, \dots, \hat{\lambda}^{(9)})$ and $\hat{\mu} = (\hat{\lambda}^{(1)}, \hat{\lambda}^{(2)}, \hat{\lambda}^{(4)}, \hat{\lambda}^{(3)}, \hat{\lambda}^{(5)} + \hat{\lambda}^{(6)} + \hat{\lambda}^{(7)}, \hat{\lambda}^{(8)}, \hat{\lambda}^{(9)})$ with dimension vector $\mathbf{d} = d(\hat{\lambda})$ the diagram



is an element of $A_{\hat{\lambda}}^{\hat{\mu}} \subset A_{\mathbf{d}}$.

For a vector composition $\hat{\lambda}$ we call the morphism $\mathrm{cr}_{\hat{\lambda}}^k := s_{\hat{\lambda}^k}^k m_{\hat{\lambda}}^k$ a **crossing** at position k where $\hat{\lambda}^k$ is the vector composition $\hat{\lambda}$ but with swapped vectors at positions k and $k+1$, i.e. a merge followed by a split with swapped vectors, see figure 7.

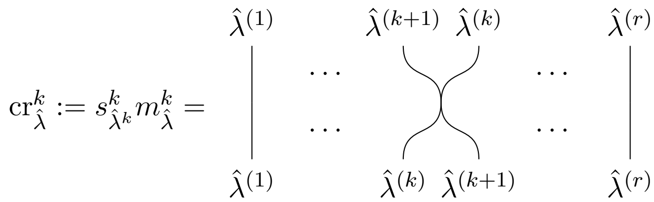
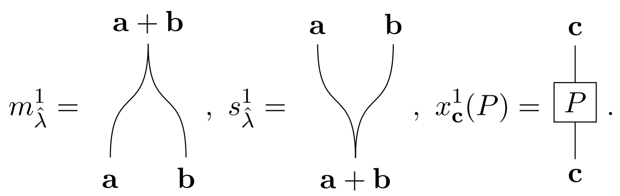


Figure 7: Crossing of two strands

Now we give the concrete action of the generators which arises from the geometric approach, see [SW11, Prop. 3.4.]. We start with the simple case where $\hat{\lambda} = (\mathbf{a}, \mathbf{b})$ is some vector composition, $\mathbf{c}$ some vector and $P \in \Lambda(\mathbf{c})$ is some polynomial. Then we have the corresponding generators



By height moves each diagram can be seen as a vertical composition of the generators of the quiver Schur algebra. In particular we have a monoidal category by horizontally and vertically stacking. Hence it suffices to know the action of the simple generators from above and the idempotents to calculate actions of arbitrary diagrams. The action of a general merge, split or polynomial diagram will arise by calculating the simple ones plus shifting the variables by the right vector, see Remark 4.4.5.

Definition 4.4.4. The action of the generators

$$e_{\hat{\mu}}, \ m_{\hat{\lambda}} := m_{\hat{\lambda}}^1, \ s_{\hat{\lambda}} := s_{\hat{\lambda}}^1, \ x_{\mathbf{c}}(P) := x_{\mathbf{c}}^1(P)$$

for an arbitrary vector composition $\hat{\mu} \in \mathrm{VComp}_e$, a vector composition $\hat{\lambda} = (\mathbf{a}, \mathbf{b})$ and a vector $\mathbf{c}$ is defined as follows

1) $e_{\hat{\mu}}(f) = f \in \Lambda(\hat{\mu})$ for all $f \in \Lambda(\hat{\mu})$.

2) $m_{\hat{\lambda}}(f) = \sum_{w \in S_{\mathbf{a}+\mathbf{b}}} (-1)^{l(w)} w(f) \prod_{i=1}^{e} \frac{1}{a_i! b_i!} R(w, a_i, b_i) \in \Lambda((\mathbf{a}+\mathbf{b}))$
with

$$R(w, a_i, b_i) = \frac{w\left(\prod_{1 \le j < k \le a_i} (x_{i,j} - x_{i,k}) \prod_{a_i < l < m \le a_i + b_i} (x_{i,l} - x_{i,m})\right)}{\prod_{1 \le j < k \le a_i + b_i} (x_{i,j} - x_{i,k})}$$

for all $f \in \Lambda((\mathbf{a}, \mathbf{b}))$.

3) $s_{\hat{\lambda}}(f) = \prod_{i=1}^{e} \prod_{j=1}^{a_{i+1}} \prod_{k=a_i+1}^{a_i+b_i} (x_{i+1,j} - x_{i,k}) \cdot f \in \Lambda((\mathbf{a}, \mathbf{b}))$ for $f \in \Lambda((\mathbf{a}+\mathbf{b}))$.

4) $x_{\mathbf{c}}(P)(f) = P \cdot f \in \Lambda((\mathbf{c}))$ for all $f \in \Lambda((\mathbf{c}))$

The indices in Definition 4.4.4 are cyclic in e, i.e. we have that $e+1 := 1$ for the split action. By convention a product over the empty set is equal to 1.

Remark 4.4.5. If there are some arcs labelled by $\hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(k-1)}$ for $k \in \mathbb{Z}_{\ge 2}$ and at the k-th position there is some merge, split or polynomial then the action can be computed by first applying Definition 4.4.4 to the merge split or polynomial and then shifting all the variables by the vector $\sum_{i=1}^{k-1} \hat{\lambda}^{(i)}$, see Example 4.4.6.

In interest of readability we use column vectors instead of row vectors in all explicit examples with diagrams.

Example 4.4.6 (Diagram action)**.** We consider the dimension vector $\mathbf{d} = (3, 3, 2)$ and vector compositions $\hat{\lambda} = ((1, 2, 1), (2, 1, 1))$ and $\hat{\mu} = ((1, 2, 1), (1, 0, 1), (1, 1, 0))$ with $d(\hat{\lambda}) = d(\hat{\mu}) = \mathbf{d}$. We get the ring

$$R(\mathbf{d}) \cong \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}] \otimes \Bbbk[x_{2,1}, x_{2,2}, x_{2,3}] \otimes \Bbbk[x_{3,1}, x_{3,2}]$$

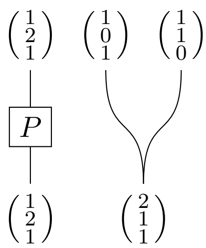
and hence the invariant rings

$$\Lambda(\hat{\lambda}) \cong \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}]^{S_1 \times S_2} \otimes \Bbbk[x_{2,1}, x_{2,2}, x_{2,3}]^{S_2 \times S_1} \otimes \Bbbk[x_{3,1}, x_{3,2}]^{S_1 \times S_1}$$

and

$$\Lambda(\hat{\mu}) \cong \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}]^{S_1 \times S_1 \times S_1} \otimes \Bbbk[x_{2,1}, x_{2,2}, x_{2,3}]^{S_2 \times S_1} \otimes \Bbbk[x_{3,1}, x_{3,2}]^{S_1 \times S_1}.$$

Let $P \in \Lambda((1, 2, 1))$. Then the diagram

$D :=$ 

corresponds to an endomorphism

$$D : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\mu}).$$

with the following action on polynomials $f \in \Lambda(\hat{\lambda})$. Ignoring the first strand and applying the split formula from Definition 4.4.4 we get that the split acts by multiplication with the polynomial $(x_{3,1} - x_{2,1})$. Shifting this by the vector $(1, 2, 1)$ we get that

$$D(f) = P \cdot (x_{3,2} - x_{2,3}) \cdot f \in \Lambda(\hat{\mu}).$$

Remark 4.4.7. The quiver Schur algebra is graded, see [SW11, Prop. 3.3]. Let $\hat{\lambda}'$ be a merge or split of $\hat{\lambda}$ at the k-th position. Then the merge or split diagram is homogeneous of degree

$$-\{\hat{\lambda}^{(k+1)}, \hat{\lambda}^{(k)}\} := \sum_{i=1}^{e} \hat{\lambda}_i^{(k)} \left(\hat{\lambda}_{i-1}^{(k+1)} - \hat{\lambda}_i^{(k+1)}\right).$$

As we will see by several examples it is extremely hard to deduce relations from the very combinatorial formulas of Definition 4.4.4 and hence there is no complete list of relations known. But for some special cases we will show that it is possible to give a full presentation of the quiver Schur algebra, see Section 5.3. For this we look at some general facts and techniques in Section 4.6. We also have that the KLR algebra is a subalgebra of the quiver Schur algebra, namely if we only consider vector compositions of complete flag type. This leads to the following.

4.5 KLR and quiver Schur algebra

We will see in Proposition 4.5.1 that the KLR algebra for the fixed graph Γ_e given by the directed Dynkin diagram of affine type A_e arises as a subalgebra of the quiver Schur algebra if only the special vectors α_i are allowed as labels. Then the label α_i in the quiver Schur algebra corresponds to the label i in the KLR algebra.

Proposition 4.5.1. *Let $e > 2$ and Γ_e be the affine Dynkin diagram of type A_e with cyclic orientation and vertex set $\mathbb{V} = \{v_1, \ldots, v_e\}$. Fix some $\nu = \sum_{i=1}^{e} \nu_i \cdot v_i \in \mathbb{N}[\mathbb{V}]$ with $r := |\nu|$. Then the KLR algebra $R(\nu)$ based on the quiver Γ_e is a subalgebra of the quiver Schur algebra $A_{\mathbf{d}}$ with dimension vector $\mathbf{d} = (\nu_1, \ldots, \nu_e)$.*

Proof. For a sequence $\mathbf{i} = (v_{i_1} \ldots v_{i_r}) \in \mathrm{Seq}(\nu)$ consider the corresponding vector composition $\hat{\lambda}(\mathbf{i}) = (\alpha_{i_1}, \ldots, \alpha_{i_r})$ of type e. Hence its dimension vector is given by $\mathbf{d} = d(\hat{\lambda}(\mathbf{i})) = (\nu_1, \ldots, \nu_e)$. We define the map

$$\chi_{\mathbf{i}} : \{1, \ldots, r\} \longrightarrow \mathbb{N}$$

such that

$$\chi_{\mathbf{i}}(j) = |\{k \mid 1 \le k \le j \text{ and } i_k = i_j\}|$$

is the number how often the vertex v_{i_j} appears in the sequence $(v_{i_1} \ldots v_{i_r})$ in the first j positions, see Example 4.5.2.

Example 4.5.2. Consider an example for $e = 3$, i.e. working with the vertex set v_1, v_2, v_3 and let $\nu := v_1 + 2 \cdot v_2 + v_3$. Choose the sequence $\mathbf{i} = (v_1 v_2 v_2 v_3) \in \mathrm{Seq}(\nu)$ with $i_1 = 1$, $i_2 = 2$, $i_3 = 2$, $i_4 = 3$. Hence $\chi_{\mathbf{i}}(1) = 1$, $\chi_{\mathbf{i}}(2) = 1$, $\chi_{\mathbf{i}}(3) = 2$ and $\chi_{\mathbf{i}}(4) = 1$ and $\hat{\lambda}(\mathbf{i}) = ((1, 0, 0), (0, 1, 0), (0, 1, 0), (0, 0, 1))$.

We will show that the actions of the KLR algebra generators $1_{\mathbf{i}}$, $\Delta_{k,\mathbf{i}}$, $x_{k,\mathbf{i}}$ correspond to the actions of the elements $e_{\hat{\lambda}}$, $cr^k_{\hat{\lambda}} = s^k_{\hat{\lambda}^k} m^k_{\hat{\lambda}}$, $x^k_{\hat{\lambda}}(x_{i_k,\chi_{\mathbf{i}}(k)})$ of the quiver Schur algebra $A_{\mathbf{d}}$ for $\hat{\lambda} := \hat{\lambda}(\mathbf{i})$. Note that for this special vector composition the vector space on which the KLR algebra acts coincides with the space on which the quiver Schur elements act, because we have

$$V_\nu = \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} \Lambda(\mathbf{i}) = \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} \mathbb{k}[x_1^{\mathbf{i}},\dots,x_r^{\mathbf{i}}] \tag{4.12}$$

$$\cong \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} \mathbb{k}[x_{1,1},\dots,x_{1,\nu_1},x_{2,1},\dots,x_{e,\nu_e}] \tag{4.13}$$

$$= \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} R(\mathbf{d}) = \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} R(\mathbf{d})^{S_{\hat{\lambda}(\mathbf{i})}} \tag{4.14}$$

$$= \bigoplus_{\mathbf{i}\in\mathrm{Seq}(\nu)} \Lambda(\hat{\lambda}(\mathbf{i})). \tag{4.15}$$

For the summand corresponding to the sequence $\mathbf{i} = (v_{i_1}\dots v_{i_r})$ the isomorphism in (4.13) is given by

$$x_j^{\mathbf{i}} \longmapsto x_{i_j,\chi_{\mathbf{i}}(j)}$$

for $j \in \{1,\dots,r\}$.

The equality (4.14) follows analogous to Example 4.3.2 3), because the group $S_{\hat{\lambda}(\mathbf{i})} = S_1 \times S_1 \dots \times S_1$ (r-times) is trivial.

Therefore it suffices to compute the actions of $e_{\hat{\lambda}}$, $cr^k_{\hat{\lambda}} = s^k_{\hat{\lambda}^k} m^k_{\hat{\lambda}}$, $x^k_{\hat{\lambda}}(x_{i_k,\chi_{\mathbf{i}}(k)})$ on $\Lambda(\hat{\lambda}(\mathbf{i}))$ and compare this with the faithful action of $1_{\mathbf{i}}, \Delta_{k,\mathbf{i}}, x_{k,\mathbf{i}}$ on $\Lambda(\mathbf{i})$. For the idempotents and polynomials it follows directly that the action defined in Definition 4.4.4 is the same as the action defined in Lemma 3.8.1. Now consider the crossings of two strands at position k. Say the k-th strand is labelled by α_i and the $(k+1)$-th by α_j where the k-th strand is the a-th strand with label α_i in the vector composition and the $(k+1)$-th the b-th strand with label α_j.

Considering the merge action 4.4.4 2) we get

$$\begin{array}{rccl} m^k_{\hat{\lambda}}: & \Lambda(\dots,\alpha_i,\alpha_j,\dots) & \longrightarrow & \Lambda(\dots,\alpha_i+\alpha_j,\dots) \\ & f & \longmapsto & \begin{cases} \Delta_a^{(i)}(f) = \frac{f - s_a^{(i)}(f)}{x_{i,a}-x_{i,a+1}} & \text{if } i=j, \\ f & \text{otherwise.} \end{cases} \end{array} \tag{4.16}$$

The split action 4.4.4 3) implies

$$\begin{array}{rccl} s^k_{\hat{\lambda}^k}: & \Lambda(\dots,\alpha_j+\alpha_i,\dots) & \longrightarrow & \Lambda(\dots,\alpha_j,\alpha_i,\dots) \\ & f & \longmapsto & \begin{cases} (x_{i+1,b}-x_{i,a})f & \text{if } i=j-1, \\ f & \text{otherwise.} \end{cases} \end{array} \tag{4.17}$$

Hence altogether a crossing with these labels corresponds to the morphism

$$\begin{array}{rccl} cr^k_{\hat{\lambda}} = s^k_{\hat{\lambda}^k} m^k_{\hat{\lambda}}: & \Lambda(\dots,\alpha_i,\alpha_j,\dots) & \longrightarrow & \Lambda(\dots,\alpha_j,\alpha_i,\dots) \\ & f & \longmapsto & \begin{cases} \Delta_a^{(i)}(f) & \text{if } i=j, \\ (x_{i+1,b}-x_{i,a})f & \text{if } i=j-1, \\ f & \text{otherwise.} \end{cases} \end{array} \tag{4.18}$$

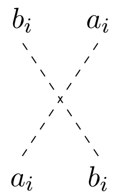
Comparing (4.18) with Lemma 3.8.1 3) under the isomorphism of (4.13) this is exactly the faithful action of $\Delta_{k,\mathbf{i}}$ on $\Lambda(\mathbf{i})$. Hence the KLR algebra is a subalgebra of the quiver Schur algebra. □

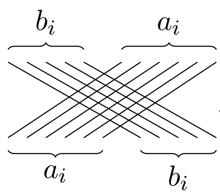
4.6 Diagram calculations

As mentioned before it is extremely hard to deduce relations from the action of merges and splits. That is the reason why we look closer at the formulas of Definition 4.4.4 and introduce some diagram tools to interpret them. We will see in Proposition 4.6.5 that the merge formula is in fact a product of certain Demazure operators which is quite useful, because we can apply the derivation property of Demazure operators (3.5). In Lemma 4.6.11 we will show that the split formula reduces to the identity in some special cases which we will look more closely at in the next section. We also introduce the meaning of highest coefficients of diagrams, see Definition 4.6.7.

Definition 4.6.1. Let $a_i, b_i \in \mathbb{Z}_{>0}$ be positive integers. Define $w_0^{a_i,b_i} \in S_{a_i+b_i}$ as the minimal length coset representative of the longest element w_0 in $S_{a_i+b_i} \big/ S_{a_i} \times S_{b_i}$.

The symmetric group generators of such an element can easily be determined by the following permutation diagram corresponding to the representative $w_0^{a_i,b_i}$.

(4.19) $w_0^{a_i,b_i} \cong$ 

$:=$ 

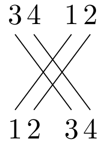
.

This diagram clearly corresponds to the element $w_0^{a_i,b_i}$, because

1. there are no crossings between the first a_i and last b_i strands and two strands do not cross more than once, i.e. the element is a minimal length representative
2. the number of crossings is maximal with these properties, i.e. the element is a minimal length coset representative of the longest element.

We can immediately determine a reduced expression $w_0^{a_i,b_i} = s_{i_1} \dots s_{i_{a_i b_i}}$ with generators $s_{i_1}, \dots, s_{i_{a_i b_i}}$ from the diagram. Here a crossing at position $k, k+1$ stands for the generator s_k in the group $S_{a_i+b_i}$. Hence by going through all crossings from bottom right to top left along the strands b_i we identify the generators of $w_0^{a_i,b_i}$.

Example 4.6.2. Let $a_i = b_i = 2$. Then the element $w_0^{2,2}$ can be displayed by the following permutation diagram

$w_0^{2,2} \cong$ 

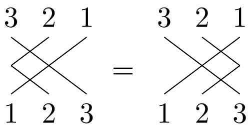
.

We can read off a reduced expression $w_0^{2,2} = s_2 s_3 s_1 s_2 \in S_4$.

The second thing we can read off the diagram is how the element $w_0^{a_i,b_i}$ acts on the set $\{1, \dots, a_i + b_i\}$, namely by looking at the position of the endpoints of the strands. Later on we use such permutation diagrams for arbitrary elements of $S_{a_i+b_i}$. The multiplication in the symmetric group corresponds to horizontally stacking of two diagrams. Two permutation diagrams are equal if the corresponding elements of the symmetric group are equal, i.e. if the action on the set $\{1, \ldots, a_i + b_i\}$ is the same. In diagrams this means if the strands are labelled by $1, \ldots, a_i + b_i$, then two diagrams equal if the labels at the top of the diagrams equal.

Example 4.6.3. In the symmetric group S_3 we have two different reduced expressions of the element which permutes 1 and 3

$$s_1 s_2 s_1 = s_2 s_1 s_2 \leftrightarrow$$



which is easily checked by comparing the top sequences.

We can express the merge action by a product of Demazure operators, namely:

Lemma 4.6.4. *[SW11, Prop. 3.6] Let $a_i, b_i \in \mathbb{Z}_{>0}$ and $\hat{\lambda} = (a_i \cdot \alpha_i, b_i \cdot \alpha_i)$ be a vector composition for some $1 \leq i \leq e$, then the merge action, see Definition 4.4.4 2), is given by*

$$m^1_{\hat{\lambda}}(f) = \Delta^{(i)}_{w_0^{a_i,b_i}}(f)$$

for $f \in \Lambda(\hat{\lambda})$.

Proposition 4.6.5. *[SW11, Prop. 3.6] Let $m_{(\mathbf{a},\mathbf{b})}$ be the merge of the two vectors $\mathbf{a} = (a_1, \ldots, a_e)$ and $\mathbf{b} = (b_1, \ldots, b_e)$ of type e. Then the merge action on polynomials $f \in \Lambda((\mathbf{a}, \mathbf{b}))$, see 4.4.4 2), is given by*

$$m_{(\mathbf{a},\mathbf{b})}(f) = \Delta^{(e)}_{w_0^{a_e,b_e}} \ldots \Delta^{(1)}_{w_0^{a_1,b_1}}(f) \in \Lambda((\mathbf{a}+\mathbf{b})).$$

Remark 4.6.6. By the previous proposition we see immediately that the merges are associative, because Demazure operators with different superscripts commute. The associativity of splits is clear by the definition of its action.

As we will see by several examples the Proposition 4.6.5 is very useful for explicit calculations and it has the big advantage over the formula from Definition 4.4.4 that we can apply the derivation rule (3.5) for Demazure operators.

Applying these facts to a quiver Schur diagram D of $A_{\mathbf{d}}$ with bottom labels $\hat{\lambda}$ and top labels $\hat{\mu}$, we can speak about highest coefficients and highest terms of D. Namely, let Δ be the product of all Demazure operators which are given by all merge morphisms of the diagram D according to Proposition 4.6.5 and $w_\Delta \in S_{\mathbf{d}}$ its corresponding symmetric group element. By the previous observations (Proposition 4.6.5 and the extended derivation property (3.19)) the diagram D acts on $\Lambda(\hat{\lambda})$ by

$$P \cdot \Delta + R \tag{4.20}$$

for some polynomial P and a rest term R.

Definition 4.6.7. Let D, P, Δ and R as in (4.20). We define R to be the **terms of lower order**, regarding the number of Demazure operators which act.

We define by

$$\mathrm{HC(D)} := \mathrm{P}$$

the **highest coefficient of** D and by

$$\mathrm{HO}(D) := \begin{cases} \Delta & \text{if } \Delta \text{ does not act by zero }, \\ 0 & \text{otherwise }, \end{cases}$$

the **highest order of** D.

Let us make an example for an explicit quiver Schur diagram.

Example 4.6.8 (Highest coefficients)**.** Consider the dimension vector $\mathbf{d} = (3, 2, 0)$ and the vector compositions $\hat{\lambda} = ((2, 1, 0), (1, 1, 0))$ and $\hat{\mu} = ((1, 2, 0), (2, 0, 0))$ with $d(\hat{\lambda}) = d(\hat{\mu}) = \mathbf{d}$. We get

$$R(d) = \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}] \otimes \Bbbk[x_{2,1}, x_{2,2}]$$

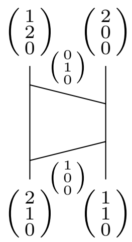
and hence the invariant rings

$$\Lambda(\hat{\lambda}) = \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}]^{S_2 \times S_1} \otimes \Bbbk[x_{2,1}, x_{2,2}]^{S_1 \times S_1}$$

and

$$\Lambda(\hat{\mu}) = \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}]^{S_1 \times S_2} \otimes \Bbbk[x_{2,1}, x_{2,2}]^{S_2}.$$

The left ladder diagram (see Appendix C)

$D =$ 

corresponds to a morphism $\Lambda(\hat{\lambda}) \to \Lambda(\hat{\mu})$ given by a split of the leftmost strand

$$\mathcal{S}_1 := s^1_{\left(\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\right)} = (x_{2,1} - x_{1,2}).$$

Then by a merge of the second and third strand

$$\Delta_2^{(1)} = m^2_{\left(\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\right)},$$

followed by another split of the rightmost strand

$$\mathcal{S}_2 := s^2_{\left(\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\0\\0\end{smallmatrix}\right)\right)} = (x_{2,2} - x_{1,2}) \cdot (x_{2,2} - x_{1,3})$$

and another merge of the first two strands

$$\Delta_1^{(2)} = m^1_{\left(\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\0\\0\end{smallmatrix}\right)\right)}.$$

Hence we get the following action of D on $\Lambda(\hat{\lambda})$

$$\mathcal{D} := \Delta_1^{(2)} \cdot \overbrace{(x_{2,2} - x_{1,2}) \cdot (x_{2,2} - x_{1,3})}^{=\mathcal{S}_2} \cdot \Delta_2^{(1)} \cdot \overbrace{(x_{2,1} - x_{1,2})}^{=\mathcal{S}_1}.$$

Applying the derivation property of the Demazure operators we get

$$\begin{aligned}
\mathcal{D} &= \Delta_1^{(2)} \cdot \mathcal{S}_2 \cdot s_2^{(1)}(\mathcal{S}_1) \cdot \Delta_2^{(1)} + \Delta_1^{(2)} \cdot \mathcal{S}_2 \cdot \Delta_2^{(1)}(\mathcal{S}_1) \\
&= s_1^{(2)} \left(\mathcal{S}_2 \cdot s_2^{(1)}(\mathcal{S}_1)\right) \cdot \Delta_1^{(2)}\Delta_2^{(1)} + \Delta_1^{(2)} \left(\mathcal{S}_2 \cdot s_2^{(1)}(\mathcal{S}_1)\right) \cdot \Delta_2^{(1)} \\
&\quad + s_1^{(2)} \left(\mathcal{S}_2 \cdot \Delta_2^{(1)}(\mathcal{S}_1)\right) \cdot \Delta_1^{(2)} + \Delta_1^{(2)} \left(\mathcal{S}_2 \cdot \Delta_2^{(1)}(\mathcal{S}_1)\right) \\
&= s_1^{(2)} \left(\mathcal{S}_2 \cdot s_2^{(1)}(\mathcal{S}_1)\right) \cdot \Delta_1^{(2)}\Delta_2^{(1)} + \text{ terms of lower order } .
\end{aligned}$$

We obtain the highest coefficient of D

$$\mathrm{HC}(D) = (x_{2,1} - x_{1,2}) \cdot (x_{2,1} - x_{1,3}) \cdot (x_{2,2} - x_{1,1})$$

and the highest order of D

$$\mathrm{HO}(D) = \Delta_1^{(2)}\Delta_2^{(1)}.$$

Lemma 4.6.9. *Let D, D', E, E' be quiver Schur diagrams. If it holds*

$$\mathrm{HC}(D) \cdot \mathrm{HO}(D) = \mathrm{HC}(E) \cdot \mathrm{HO}(E) \text{ and } \mathrm{HC}(D') \cdot \mathrm{HO}(D') = \mathrm{HC}(E') \cdot \mathrm{HO}(E'),$$

then also

$$\mathrm{HC}(D \cdot D') \cdot \mathrm{HO}(D \cdot D') = \mathrm{HC}(E \cdot E') \cdot \mathrm{HO}(E \cdot E').$$

Proof. Let $\mathrm{HO}(D) = \Delta \neq 0$. Then by the derivation property of Demazure operators we have that

$$\mathrm{HC}(D \cdot D') = \mathrm{HC}(D) \cdot w_\Delta(\mathrm{HC}(D')) = \mathrm{HC}(E) \cdot w_\Delta(\mathrm{HC}(E')) = \mathrm{HC}(E \cdot E')$$

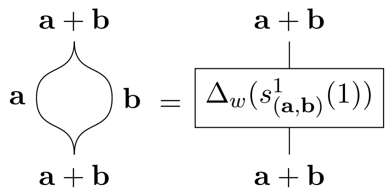
and by the definition of the highest order it holds

$$\mathrm{HO}(D \cdot D') = \mathrm{HO}(D) \cdot \mathrm{HO}(D') = \mathrm{HO}(E) \cdot \mathrm{HO}(E') = \mathrm{HO}(E \cdot E'),$$

hence the claim follows. $\square$

One of the few relations known in general is the next one, see [SW11, Prop. 3.8].

Lemma 4.6.10. *Let $\mathbf{a} = (a_1, \ldots, a_e)$ and $\mathbf{b} = (b_1, \ldots, b_e)$ be vectors of type e. Denote by $\Delta_w := \Delta^{(e)}_{w_0^{a_e,b_e}} \ldots \Delta^{(1)}_{w_0^{a_1,b_1}}$ the merge action of $m^1_{(\mathbf{a},\mathbf{b})}$ and by $s^1_{(\mathbf{a},\mathbf{b})}$ the split from $\Lambda((\mathbf{a}+\mathbf{b}))$ to $\Lambda((\mathbf{a},\mathbf{b}))$. Then we get the relation*



of quiver Schur diagrams. In particular the relation implies that a double crossing of two strands acts like the crossing multiplied by the polynomial $\Delta_w(s^1_{(\mathbf{a},\mathbf{b})}(1))$.

Proof. For a polynomial $f \in \Lambda((\mathbf{a}+\mathbf{b}))$ we have that the left diagram corresponds to the action of a merge after a split

$$m^1_{(\mathbf{a},\mathbf{b})} s^1_{(\mathbf{a},\mathbf{b})}(f) = \Delta^{(e)}_{w_0^{a_e,b_e}} \dots \Delta^{(1)}_{w_0^{a_1,b_1}} (s^1_{(\mathbf{a},\mathbf{b})}(1) \cdot f) \tag{4.21}$$

$$= \Delta^{(e)}_{w_0^{a_e,b_e}} \dots \Delta^{(1)}_{w_0^{a_1,b_1}} (s^1_{(\mathbf{a},\mathbf{b})}(1)) \cdot f. \tag{4.22}$$

This follows by applying the extended derivation rule (3.19) and the fact that f is invariant under all permutations hence the Demazures applied to f vanish. The first claim follows, because the right side acts as (4.22). Adding a merge of $\mathbf{b}, \mathbf{a}$ at the bottom and a split into $\mathbf{b}, \mathbf{a}$ at the top we get a double crossing and the second assumption follows. □

Using the permutation diagrams from before, see (4.19), and Theorem 3.5.7 the question if two products of Demazure operators are the same reduces to the question if the permutation diagrams of the corresponding elements from the symmetric group are the same, which is much easier to see. In particular if we are in some special case, where all the splits act by the identity and there are no polynomial generators, then all the diagrams act by a product of Demazure operators and we can use the diagram computation from (4.19) to get relations between quiver Schur diagrams.

For this let us look at these cases where the split acts by the identity.

Lemma 4.6.11. *Let $\hat{\lambda} = (\mathbf{a} + \mathbf{b})$ be a vector composition. If $a_{i+1}b_i = 0$ for all $1 \leq i \leq e$ then it holds that the split action is given by the identity, i.e.*

$$s_{\hat{\lambda}}(f) = f \text{ for all } f \in \Lambda(\mathbf{a} + \mathbf{b}).$$

This is in particular the case if $\mathbf{a} + \mathbf{b} = (a_i + b_i) \cdot \alpha_i$.

Proof. If $a_{i+1}b_i = 0$ for all $1 \leq i \leq e$ then either $a_{i+1} = 0$ or $b_i = 0$. Considering the split formula, see 4.4.4 3), we obtain that if $a_{i+1} = 0$ then the second product runs over the empty set and if $b_i = 0$ then the third product runs over the empty set. Hence the claim follows. □

These facts we use in Section 5 to derive relations between diagrams of quiver Schur algebras with special dimension vectors.

4.7 Basis of the quiver Schur algebra

We will now recall the construction of a basis for the quiver Schur algebra $A_{\mathbf{d}}$ following [SW11, §3.4] in terms of the notation from Section 4.4.

Starting with two vector compositions $\hat{\lambda}, \hat{\mu}$ of the same dimension vector $\mathbf{d}$ we will give a basis of all morphisms

$$\Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\mu})$$

of the quiver Schur algebra $A_{\mathbf{d}}$, i.e. of all diagrams with bottom label $\hat{\lambda}$ and top label $\hat{\mu}$. The idea will be to consider the residue sequences of $\hat{\lambda}$ and $\hat{\mu}$ and the sequences arising from splitting up the blocks of both sequences until they become block permutations of each other. Hence in the end we build a basis element out of splits followed by multiplication with polynomials followed by crossings (merge-splits) followed by merges.

For a fixed dimension vector $\mathbf{d}$ let $\hat{\lambda}, \hat{\mu} \in \mathrm{VComp}_e(\mathbf{d})$ be two residue data and their corresponding residue sequences are given by $\mathrm{res}(\hat{\lambda})$ and $\mathrm{res}(\hat{\mu})$. Denote by

$$S^{\hat{\lambda}} := S_{\lambda_1^{(1)}} \times \ldots \times S_{\lambda_e^{(1)}} \times S_{\lambda_1^{(2)}} \times \ldots \times S_{\lambda_e^{(r)}}$$

(resp. $S^{\hat{\mu}}$) the symmetric group which preserves the blocks of the corresponding residue sequence. They can be embedded into $S_{\mathbf{d}}$ and one can consider the set of double cosets

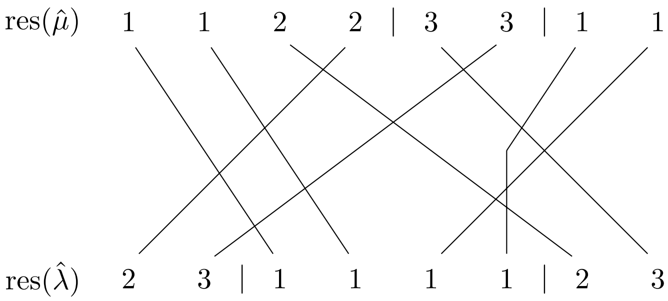
$$S^{\hat{\mu}} \backslash S_{\mathbf{d}} / S^{\hat{\lambda}} \,. \tag{4.23}$$

Now fix a permutation $\pi \in S_{\mathbf{d}}$ which sends $\mathrm{res}(\hat{\lambda})$ to $\mathrm{res}(\hat{\mu})$ by permuting the numbers of the residue sequence. Let $\tilde{\pi}$ be the shortest double coset representative of π in (4.23). Such a permutation of residue sequences and its shortest double coset representative can be expressed graphically considering the following example.

Example 4.7.1. Consider the two vector compositions

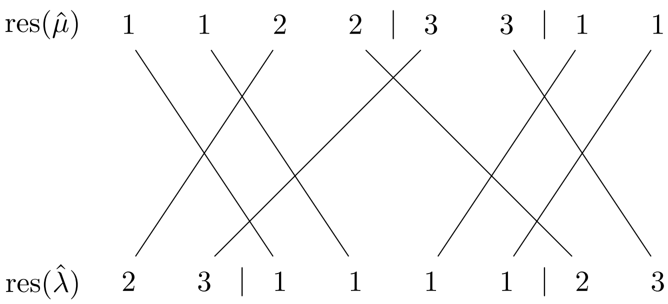
$$\hat{\lambda} = ((0,1,1),(4,0,0),(0,1,1))\,,\ \hat{\mu} = ((2,2,0),(0,0,2),(2,0,0))$$

and a permutation $\pi = (34)(56)(78) \in S_4 \times S_2 \times S_2 = S_{\mathbf{d}}$, i.e. $\pi(\mathrm{res}(\hat{\lambda})) = \mathrm{res}(\hat{\mu})$. Then we get the following pictures.



$= \pi.$

The shortest double coset representative is given by



$= \tilde{\pi}.$

In general a shortest double coset representative means graphically, that strands labelled by the same number and ending in the same block do not cross. This follows immediately by the definition of $S^{\hat{\lambda}}$ and $S^{\hat{\mu}}$ which consist of all permutations of strands of one colour in a block. So clearly idempotents, splits and merges presented as such diagrams correspond to shortest representations.

Now define the refinement sequences $\mathrm{res}(\hat{\lambda}')$, $\mathrm{res}(\hat{\mu}')$ of $\mathrm{res}(\hat{\lambda})$ resp. $\mathrm{res}(\hat{\mu})$ by refining the blocks of the original sequences such that the permutation $\tilde{\pi}$ sends all elements of a block from $\operatorname{res}(\hat{\lambda}')$ to one block in $\operatorname{res}(\hat{\mu}')$ and the other way round. Additionally if a block is split up we reorder the new blocks in the old one such that each two strands from different new blocks in one old block do not cross, see Example 4.7.2. This kind of "artificial" requirement can be made, because we are only interested in basis diagrams. The later construction, Lemma 4.6.10, and the associativity of splits and merges, will imply that all the other diagrams which arise through non reordered blocks are generated by the diagrams with reordered blocks.

The refinement from $\operatorname{res}(\hat{\lambda})$ to $\operatorname{res}(\hat{\lambda}')$ corresponds to a product of split morphisms

$$s_{\hat{\lambda},\hat{\mu}}(\pi) := s_{\hat{\tau}_i}^{k_i} \dots s_{\hat{\tau}_1}^{k_1} : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda}') \tag{4.24}$$

where the vector compositions $\hat{\tau}_j,\ j \in \{1,\dots,i\},\ i \in \mathbb{Z}_{>0}, k_j \in \{1,\dots,r(\hat{\tau}_j)-1\}$ are determined by the above construction and especially the permutation π. Note that for example $\hat{\tau}_i = \hat{\lambda}'$. The refinement from $\operatorname{res}(\hat{\mu})$ to $\operatorname{res}(\hat{\mu}')$ corresponds to a product of merge morphisms

$$m_{\hat{\lambda},\hat{\mu}}(\pi) := m_{\hat{\xi}_i}^{k_i} \dots m_{\hat{\xi}_1}^{k_1} : \Lambda(\hat{\mu}') \longrightarrow \Lambda(\hat{\mu}) \tag{4.25}$$

where the vector compositions $\hat{\xi}_j,\ j \in \{1,\dots,i\},\ i \in \mathbb{Z}_{>0}, k_j \in \{1,\dots,r(\hat{\xi}_j)-1\}$ are determined by the above construction and the permutation π. Note that for example $\hat{\xi}_1 = \hat{\mu}'$.

Example 4.7.2. In figure 8 we continue Example 4.7.1 and construct the refined residue sequences $\operatorname{res}(\hat{\lambda}')$ and $\operatorname{res}(\hat{\mu}')$.

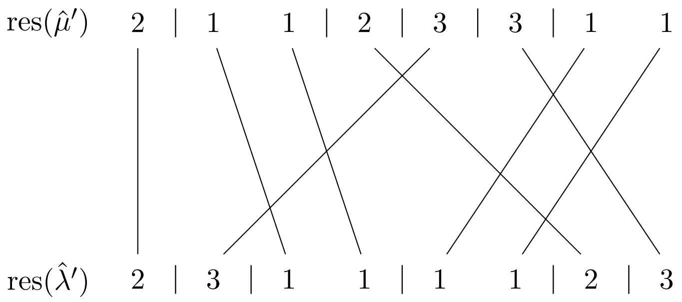


Figure 8: Refinement of residue sequences

After the refining step we get the resulting vector compositions $\hat{\lambda}'$ and $\hat{\mu}'$ of same block type, i.e. there is a bijection between the blocks. Hence there is an element $\sigma \in S_l$, where l is the number of blocks which permutes the blocks of $\operatorname{res}\hat{\lambda}'$ into the ones of $\operatorname{res}\hat{\mu}'$.

In Example 4.7.2 it holds that $l = 6$ and $\sigma = (56)(34)(45)(23)$. Now fix a reduced expression $\sigma = s_{i_1} \dots s_{i_r}$. Then we can associate the reduced expression (the permutation of blocks) with a product of crossings in the diagram. The crossings depending on the reduced expression give a morphism

$$\operatorname{cr}_{\hat{\lambda},\hat{\mu}}(\pi) := \operatorname{cr}_{\hat{\nu}_i}^{k_i} \dots \operatorname{cr}_{\hat{\nu}_1}^{k_1} : \Lambda(\hat{\lambda}') \longrightarrow \Lambda(\hat{\mu}') \tag{4.26}$$

where the vector compositions $\hat{\nu}_j,\ j \in \{1,\dots,i\},\ i \in \mathbb{Z}_{>0}, k_j \in \{1,\dots,r(\hat{\nu}_i)-1\}$ are determined by the reduced expression.

Now the morphisms (4.24), (4.26) and (4.25) can be put together to obtain a morphism from $\Lambda(\hat{\lambda})$ to $\Lambda(\hat{\mu})$. We denote the resulting morphism by

$$b^1_{\hat{\lambda},\hat{\mu}}(\pi) := m_{\hat{\lambda},\hat{\mu}}(\pi)\,\mathrm{cr}_{\hat{\lambda},\hat{\mu}}(\pi) s_{\hat{\lambda},\hat{\mu}}(\pi) : \Lambda(\hat{\lambda}) \to \Lambda(\hat{\mu}). \tag{4.27}$$

Note that this definition depends on the reduced expression of σ. Additionally we can add polynomials $P \in \Lambda(\hat{\lambda}')$ to the setting and get the morphism

$$b^P_{\hat{\lambda},\hat{\mu}}(\pi) := m_{\hat{\lambda},\hat{\mu}}(\pi)\,\mathrm{cr}_{\hat{\lambda},\hat{\mu}}(\pi) P s_{\hat{\lambda},\hat{\mu}}(\pi) : \Lambda(\hat{\lambda}) \to \Lambda(\hat{\mu}) \tag{4.28}$$

where P is the morphism given by multiplication with the polynomial P.

Example 4.7.3. The morphism $b^1_{\hat{\lambda},\hat{\mu}}(\pi)$ for $\sigma = (56)(34)(45)(23)$ corresponding to Example 4.7.1 looks like the one shown in figure 9.

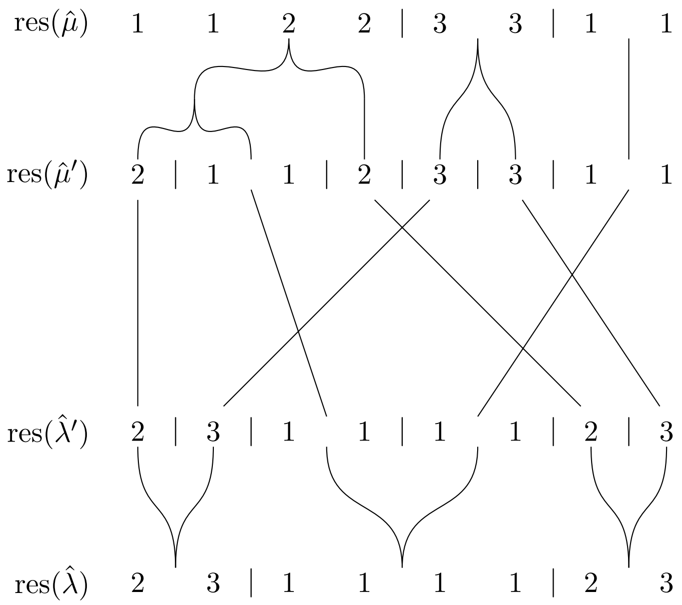


Figure 9: Example morphism for given vector compositions $\hat{\lambda}$ and $\hat{\mu}$

Remark 4.7.4. If we choose two different reduced expressions of σ we get different diagrams. But by Remark 6.1.4 they only differ by a linear combination of morphisms corresponding to double coset representatives of shorter length. Even non reduced expressions can be written by Lemma 4.6.10 by a reduced one multiplied by a polynomial.

Theorem 4.7.5. *[SW11, Thm. 3.11] The morphisms*

$$b^P_{\hat{\lambda},\hat{\mu}}(\pi)$$

running over all vector compositions $\hat{\lambda}, \hat{\mu} \in \mathrm{VComp}_e(\mathbf{d})$, over all permutations π which send $\mathrm{res}(\hat{\lambda})$ to $\mathrm{res}(\hat{\mu})$ and over all polynomials $P \in \Lambda(\hat{\lambda}')$ generate the quiver Schur algebra $A_{\mathbf{d}}$ as a $\Bbbk$-vector space. If the indices run over all

i) ordered pairs $(\hat{\lambda}, \hat{\mu})$ of vector compositions $\hat{\lambda}, \hat{\mu} \in \mathrm{VComp}_e(\mathbf{d})$

ii) π which are minimal coset representatives of elements in $S^{\hat{\mu}}\Big\backslash S_{\mathbf{d}}\Big/ S^{\hat{\lambda}}$

iii) basis elements P of $\Lambda(\hat{\lambda}')$

then the morphisms $b^P_{\hat{\lambda},\hat{\mu}}(\pi)$ form a $\Bbbk$-basis of $A_{\mathbf{d}}$.

Remark 4.7.6. By Theorem 4.7.5 it follows that every diagram of a the quiver Schur algebra $A_{\mathbf{d}}$ can be written as a sum of diagrams of the form that there are first the splits then the polynomials, then crossings and at the end the merges.

Leaving out the polynomials, the number of different basis diagrams $b^1_{\hat{\lambda},\hat{\mu}}(\pi)$ for fixed vector compositions $\hat{\lambda}$ and $\hat{\mu}$ can be determined by the set of minimal coset representatives π of elements in $S^{\hat{\mu}}\Big\backslash S_{\mathbf{d}}\Big/ S^{\hat{\lambda}}$. Note the following interesting fact by [DJ86, Lem. 1.7].

For compositions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ of $n \in \mathbb{Z}_{>0}$ there is a one to one correspondence between the sets of

(4.29)
$$\left\{ \begin{array}{l} \text{minimal coset representatives} \\ \text{of elements in } \; S_\mu \backslash S_n / S_\lambda \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{row-semistandard } \lambda\text{-tableaux} \\ \text{of type } \mu \end{array} \right\}.$$

Here row-semistandard tableau means that the numbers in the tableau along rows are non decreasing.

Hence the basis diagrams are up to the polynomial morphisms in one to one correspondence to certain row-semistandard tableaux, see Example 4.7.7.

Example 4.7.7. Let $\hat{\lambda} = ((3,2,0),(2,2,0))$ and $\hat{\mu} = ((2,2,0),(3,2,0))$ be vector compositions. The corresponding compositions are $\lambda_1 = (3,2)$, $\mu_1 = (2,3)$ and $\lambda_2 = (2,2)$, $\mu_2 = (2,2)$. The possible row-semistandard $(3,2)$-tableaux of type $(2,3)$ are

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & 1 \\ \cline{1-2}\end{array}\;.$$

The possible row-semistandard $(2,2)$-tableaux of type $(2,2)$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\;, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline\end{array}\;, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}\;.$$

Hence there are (up to polynomial morphisms) nine basis diagrams with bottom label $\hat{\lambda}$ and top label $\hat{\mu}$ which can be described in more detail writing down the residue sequences of $\hat{\lambda}$ and $\hat{\mu}$, see Appendix B.1.

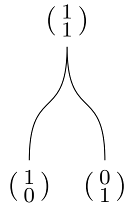
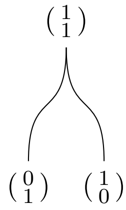
5 Defining relations in special examples

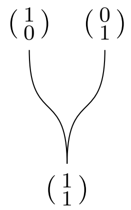
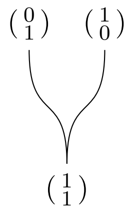
We will give full presentations of the quiver Schur algebra $A_{\mathbf{d}}$ (in terms of generators and relations) for the dimension vector $\mathbf{d} = (1,1,0)$ and for the special case where $\mathbf{d} = d \cdot \alpha_i$ for some $d \in \mathbb{Z}_{>0}$.

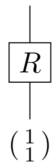
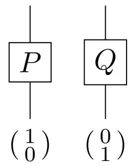
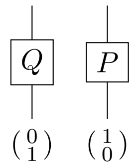
5.1 Defining relations for $\mathbf{d} = (1,1,0)$

Example 5.1.1. Consider the dimension vector $\mathbf{d} = (1,1,0)$ and the corresponding polynomial ring $R(\mathbf{d}) = \Bbbk[x_{1,1}] \otimes \Bbbk[x_{2,1}]$ which is isomorphic to $\Lambda(\hat{\lambda})$ for all vector compositions $\hat{\lambda}$ of $\mathbf{d}$.

In the following we keep in mind that $e = 3$ but we will omit the 0's in the third position of the vectors. The generators of the quiver Schur algebra $A_{\mathbf{d}}$ are given by

$m_1 := m^1_{(\binom{1}{0},\binom{0}{1})} =$  , $m_2 := m^1_{(\binom{0}{1},\binom{1}{0})} =$ 

$s_1 := s^1_{(\binom{1}{0},\binom{0}{1})} =$  , $s_2 := s^1_{(\binom{0}{1},\binom{1}{0})} =$ 

$p_0(R) := Re_0 =$  , $p_1(R) := Re_1 =$  , $p_2(R) := Re_2 =$ 

with $R := P \cdot Q$ running over all possible choices for

$$P = x_{1,1}^a \in \Bbbk[x_{1,1}], a \in \mathbb{Z}_{\geq 0} \text{ and } Q = x_{2,1}^b \in \Bbbk[x_{2,1}], b \in \mathbb{Z}_{\geq 0}.$$

We have the idempotents $e_0 = p_0(1)$, $e_1 = p_1(1)$ and $e_2 = p_2(1)$. For this algebra we can write down a full list of generators, similar to [MS16, Thm. 10.1].

Proposition 5.1.2. *The following list of relations is complete.*

- $(E1)$ $p_i(R)p_i(R') = p_i(R \cdot R')$ *for* $i \in \{0,1,2\}$,
- $(E2)$ $s_1 m_1 = e_1$,
  $s_2 m_2 = (x_{2,1} - x_{1,1})e_2$,
- $(E3)$ $m_1 p_1(R) s_1 = Re_0$,
  $m_2 p_2(R) s_2 = R(x_{2,1} - x_{1,1})e_0$,
- $(E4)$ $p_0(R)m_i = m_i p_i(R)$ *for* $i \in \{1,2\}$,
- $(E5)$ $s_i p_0(R) = p_i(R)s_i$ *for* $i \in \{1,2\}$,
- $(E6)$ $p_1(R)s_1 m_2 = s_1 m_2 p_2(R)$,
  $p_2(R)s_2 m_1 = s_2 m_1 p_1(R)$.

Proof. Let A be the algebra freely generated by the elements

$$m_1, m_2, s_1, s_2, p_0(R), p_1(R), p_2(R)$$

and I the ideal generated by the relations $(E1)$-$(E6)$. Consider the homomorphism

$$\gamma : {}^{A}/_{I} \longrightarrow A_{\mathbf{d}}$$

sending generators to generators. It is well-defined, because the relations $(E1)$-$(E6)$ hold in $A_{\mathbf{d}}$ and surjective, because all generators of $A_{\mathbf{d}}$ lie in the image. We want to show that the homomorphism is an isomorphism and hence I is a full list of relations. By Theorem 4.7.5 a basis of $A_{\mathbf{d}}$ is known, given by the elements

i) $p_0(R),\ p_1(R),\ p_2(R)$,

ii) $s_2 m_1 p_1(R),\ s_1 m_2 p_2(R)$,

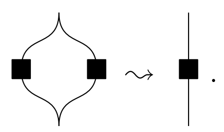
iii) $p_1(R)s_1,\ m_1 p_1(R),\ p_2(R)s_2,\ m_2 p_2(R)$.

Hence for the injectivity of γ it suffices to show that using the relations from I it is possible to write arbitrary elements from ${}^{A}/_{I}$ as a sum of basis elements from $A_{\mathbf{d}}$.

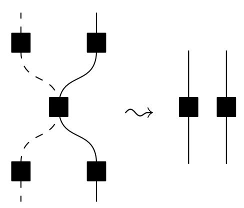
For a better readability we replace polynomials on a strand not separated by a split or merge by black boxes which is possible due to relation $(E1)$. Additionally all the labels are omitted. To distinguish between crossings and other types of diagrams we use dashed lines. By the notation $X \rightsquigarrow Y$ we mean that we can write diagrams of the form X as sums of diagrams of the form Y.

The first claim is that every part of an arbitrary diagram which is not a crossing, a single split or a single merge can be replaced by sums of idempotents and polynomials.

By the relation $(E3)$ we get

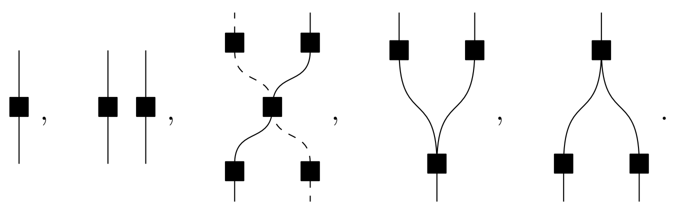


First using $(E4)$ and then $(E2)$ we have the reduction



for all parts not being a crossing.

Hence every diagram is a sum of diagrams of the form



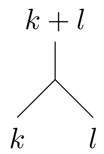
The first two diagrams are basis diagrams, see i). The third one, using relations $(E4)$ and $(E6)$, is a sum of basis diagrams from ii). The last two diagrams can be written by relations $(E4)$ and $(E5)$ as a sum of basis diagrams from iii). Hence we can write every element as a sum of basis elements from $A_{\mathbf{d}}$, i.e. γ is an isomorphism and the claim follows. $\square$

5.2 Defining relations for $\mathbf{d} = d \cdot \alpha_i$

In this section we consider the special quiver Schur algebras $A_{\mathbf{d}}$ where the dimension vector is given by $\mathbf{d} = d \cdot \alpha_i = (0, \dots, 0, d, 0, \dots, 0)$ for some fixed $i \in \{1, \dots, e\}$ and $d \in \mathbb{Z}_{>0}$. Hence all labels are of the form $k \cdot \alpha_i$ with $k \in \mathbb{Z}_{>0}$ and we write k instead of $k \cdot \alpha_i$. The algebra acts on polynomials of some invariant rings inside $R(\mathbf{d}) = \Bbbk[x_{i,1}, \dots, x_{i,d}]$. We write $s_j := s_j^{(i)}$ for the symmetric group generators, $\Delta_j := \Delta_j^{(i)}$ for the Demazure operators and $x_j := x_{i,j}$ for the variables. We use the simplified abbreviations defined in Appendix C for the diagrams.

Note that for this special case we have by Lemma 4.6.11 that all splits act by the identity. Let $k, l \in \mathbb{Z}_{>0}$ be labels. Then the merge

(5.1)



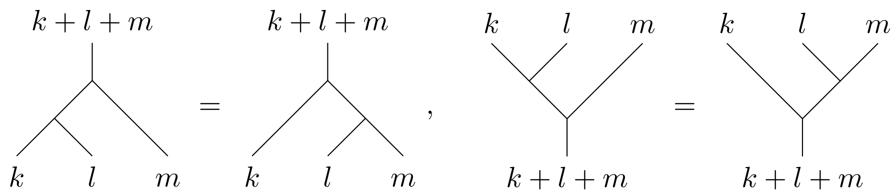
acts on the invariant ring $\Lambda(k,l) := \Bbbk[x_1, \dots, x_{k+l}]^{S_k \times S_l}$ by $\Delta_{w_0^{k,l}}$, see Lemma 4.6.4. The operator $\Delta_{w_0^{k,l}}$ can be explicitly written down by looking at the permutation diagram (4.19) with k, l instead of a_i, b_i. Reading the crossings out from the diagram from bottom to top and right to left gives a reduced expression of $w_0^{k,l}$. Hence we get in terms of Demazure operators

$$\Delta_{w_0^{k,l}} = \Delta_l \dots \Delta_{k+l-1} \dots \Delta_3 \dots \Delta_{k+2} \Delta_2 \dots \Delta_{k+1} \Delta_1 \dots \Delta_k \tag{5.2}$$

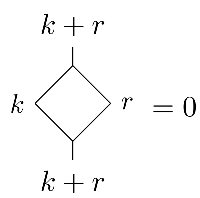
for the action of (5.1). The observations from Section 4.6 induce that every diagram of this special quiver Schur algebra generated by merges and splits corresponds to some permutation diagram of an element of a symmetric group. Hence we can test relations by looking only at the start and endpoints of the strands in these permutation diagrams. This leads to the following proposition.

Proposition 5.2.1. *Let $k, l, m, r, s \in \mathbb{Z}_{\geq 0}$ be labels. Then the following relations hold.*

($\mathcal{R}1$) *Associativity of merge and split:*

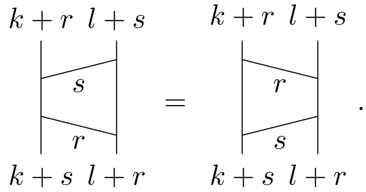


($\mathcal{R}2$) *Deleting the hole:*

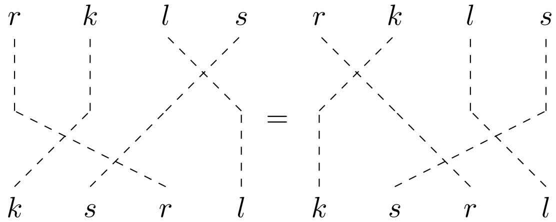


for all $r, k > 0$.

($\mathcal{R}3$) *Ladder relation:*



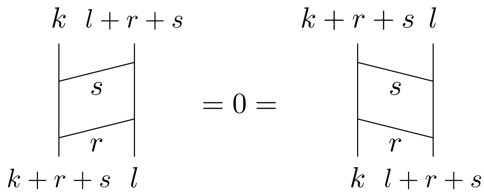
Proof. Of course the relation ($\mathcal{R}1$) holds in general. The "Deleting the hole" relation ($\mathcal{R}2$) follows from Lemma 4.6.10. For the "Ladder relation" ($\mathcal{R}3$) we use the diagram calculus from above to test if the permutation diagrams corresponding to the merges equal. Hence the equality of right and left ladder in ($\mathcal{R}3$) follows from



where the left hand permutation diagram corresponds to the the left hand ladder diagram and the right hand permutation diagram corresponds to the right hand ladder diagram. □

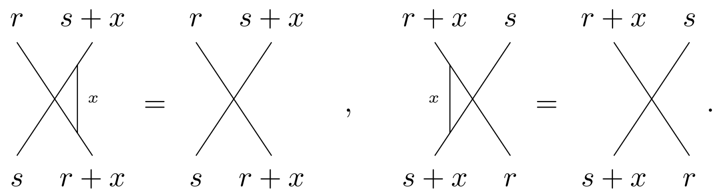
Corollary 5.2.2. *Let $k, l, m, r, s, x \in \mathbb{Z}_{\geq 0}$ be labels. The relations from Proposition 5.2.1 imply the following relations.*

($\mathcal{R}4$) *Deleting the big hole:*

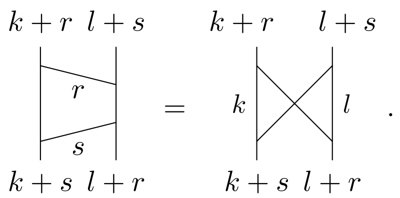


for $r, s > 0$.

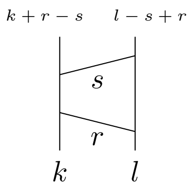
($\mathcal{R}5$) *Opening of an edge:*



($\mathcal{R}6$) *Ladder relation II:*



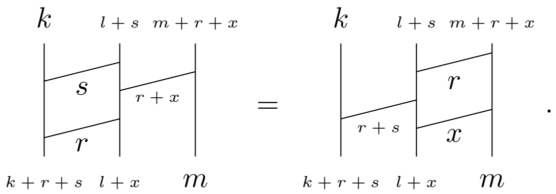
($\mathcal{R}7$) *Ladder relation III:*



$= 0$

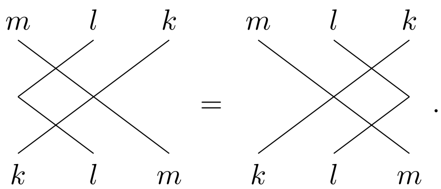
for $s > k$ and $s < k + r$.

($\mathcal{R}8$) *Move the ladder:*



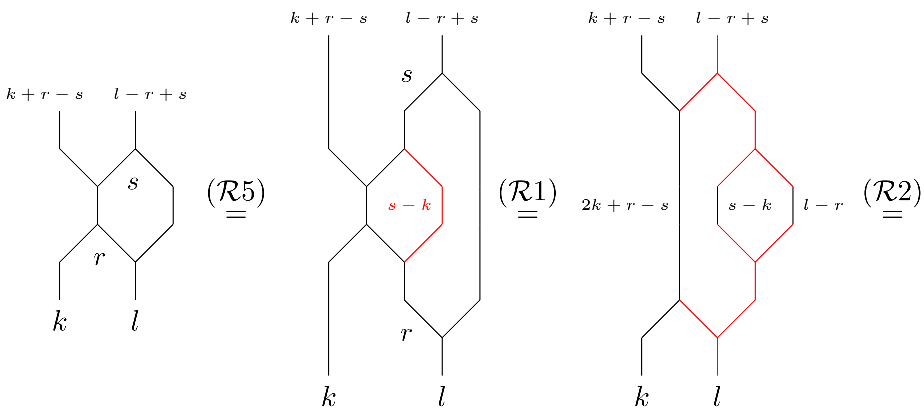
.

($\mathcal{R}9$) *Braid relation:*



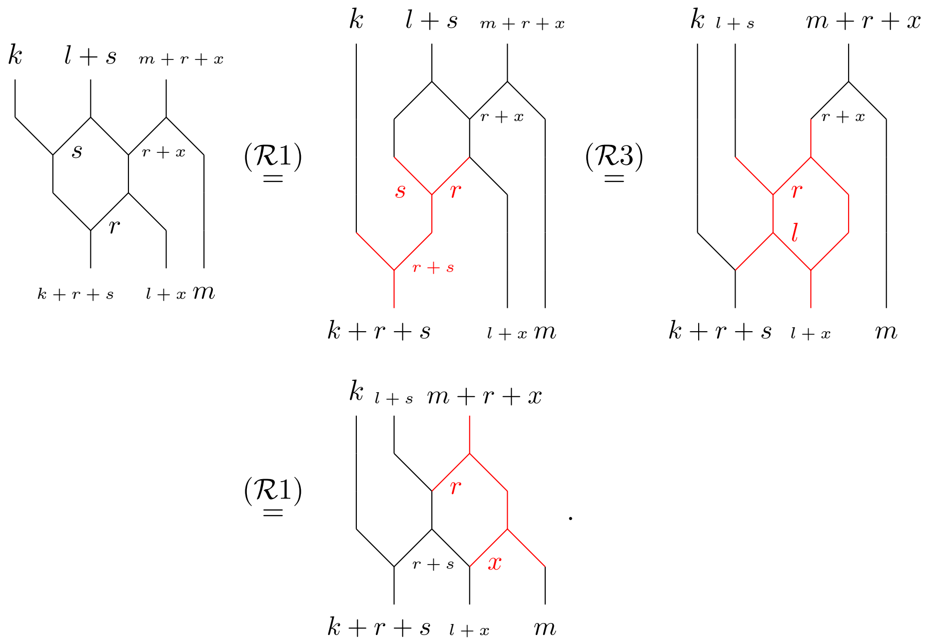
.

Proof. The relation “Deleting the big hole” ($\mathcal{R}4$) is showed in more generality in (6.1). It is the only relation which is not implied by the “Ladder relation” ($\mathcal{R}3$). The “Opening of an edge” relations ($\mathcal{R}5$) are special cases of ($\mathcal{R}3$) with $k = 0$ and $l = x$ respectively $l = 0$ and $k = x$. For the “Ladder relation II” ($\mathcal{R}6$) note that the right hand diagram without the left strand reduces to the first case of ($\mathcal{R}5$) for $x = l$ and we get the left ladder diagram. The remaining relations we proof by looking at the diagrams, displaying the simple splits and merges and using the relations from before. We mark the changes given by the previous relations in red. The “Ladder relation III” ($\mathcal{R}7$) is implied by the “Opening of an edge” relations ($\mathcal{R}5$), “Associativity of split and merge” ($\mathcal{R}1$) and the “Deleting the hole” relation ($\mathcal{R}2$)

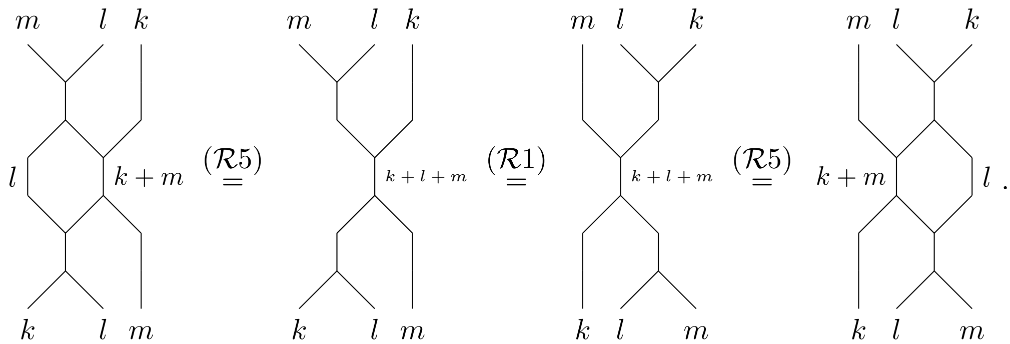


$0.$

We obtain the "Move the ladder" relation ($\mathcal{R}8$) by first applying associativity of splits, then the "Ladder relation" ($\mathcal{R}3$) and finally the associativity of merges



The "Braid relation" ($\mathcal{R}9$) is implied by first an "Opening(closing) of an edge" ($\mathcal{R}5$), then associativity of split and merge and then again an "Opening of an edge"

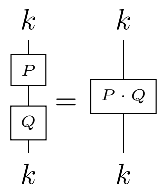


$\square$

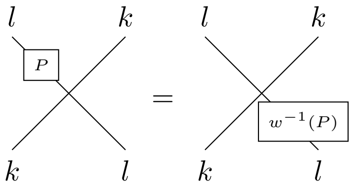
Now we look at relations including polynomial generators.

Proposition 5.2.3. *Let $k, l \in \mathbb{Z}_{>0}$ be labels. Then there are the following relations.*

($\mathcal{P}1$) *Let $P, Q \in \Bbbk[x_1, \ldots, x_k]^{S_k}$ be invariant polynomials. Then it holds that*



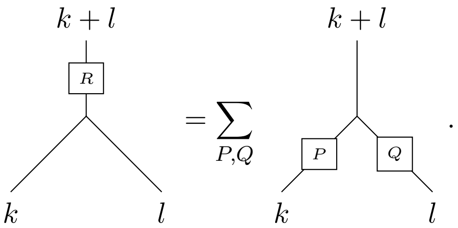
($\mathcal{P}2$) *Let $P \in \Bbbk[x_1, \ldots, x_l]^{S_l}$. Then it holds that*



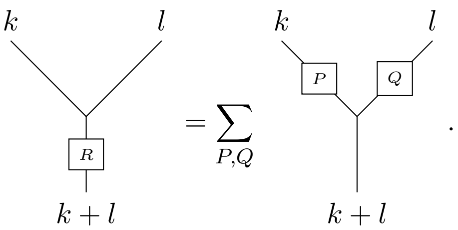
$+$ *rest terms of lower order.*

where $w := w_0^{k,l}$ is the element corresponding to the merge of the strands labelled by k and l. By lower order we mean that the length of the product of Demazure operators corresponding to the diagram action is smaller. The same holds for the mirrored relations.

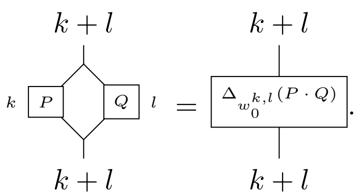
($\mathcal{P}3$) *Let $R \in \Bbbk[x_1, \ldots, x_{k+l}]^{S_{k+l}}$. Then there are polynomials $P \in \Bbbk[x_1, \ldots, x_k]^{S_k}$ and $Q \in \Bbbk[x_{k+1}, \ldots, x_{k+l}]^{S_l}$ such that*



($\mathcal{P}4$) *Let $R \in \Bbbk[x_1, \ldots, x_{k+l}]^{S_{k+l}}$. Then there are polynomials $P \in \Bbbk[x_1, \ldots, x_k]^{S_k}$ and $Q \in \Bbbk[x_{k+1}, \ldots, x_{k+l}]^{S_l}$ such that*



($\mathcal{P}5$) *Let $P \in \Bbbk[x_1, \ldots, x_k]^{S_k}$ and $Q \in \Bbbk[x_{k+1}, \ldots, x_{k+l}]^{S_l}$. Then it holds that*



Proof. The relation ($\mathcal{P}1$) immediately follows by the definition of the polynomial generators. Calculating the action of the right hand diagram in ($\mathcal{P}2$) for some $f \in \Lambda(k, l)$ we get by the derivation property (3.5) that

$$\Delta_{w_0^{k,l}}((w_0^{k,l})^{-1}(P) \cdot f) = P \cdot \Delta_{w_0^{k,l}}(f) + \text{ terms of lower order.} \tag{5.3}$$

and the diagrams are equal up to terms of lower order. The existence of polynomials P, Q such that ($\mathcal{P}3$) and ($\mathcal{P}4$) hold immediately follows from the basis Theorem 4.7.5

writing the diagrams on the left hand side as a sum of basis diagrams. The polynomials P, Q can be determined by the following.

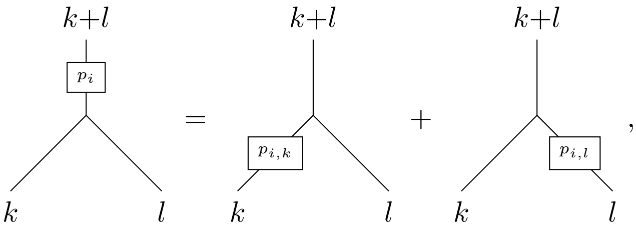
We know by [Mac03, Ch. I] that

$$\Bbbk[x_1, \ldots, x_{k+l}]^{S_{k+l}} = \Bbbk[p_1, \ldots, p_{k+l}]$$

where

$$p_i = \sum_{j=1}^{k+l} x_j^i$$

is the i-th power sum. Hence by the first relation ($\mathcal{P}1$) we can assume that $R = p_i$ for some $i \in \{1, \ \ldots, k+l\}$. Let $p_{i,k} := \sum_{j=1}^{k} x_j^i$ and $p_{i,l} := \sum_{j=1}^{l} x_{k+j}^i$. Then we get

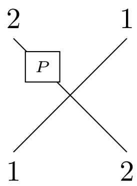


because by the extended derivation property (3.19) it holds

$$p_i \cdot \Delta_{w_0^{k,l}} = \underbrace{\Delta_{w_0^{k,l}}(p_i)}_{=0} + \Delta_{w_0^{k,l}} \cdot \underbrace{w_0^{k,l}(p_i)}_{=p_i}.$$

The same holds for the split relation ($\mathcal{P}4$). The relation ($\mathcal{P}5$) follows directly from Lemma 4.6.10 and is a generalization of ($\mathcal{R}2$). $\square$

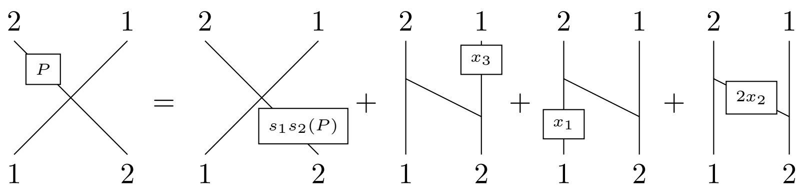
Example 5.2.4 (Lower order terms of a crossing)**.** Let $P := (x_1+x_2)^2 \in \Bbbk[x_1, x_2]^{S_2}$. Consider the diagram



which acts on polynomials $f \in \Bbbk[x_1] \otimes \Bbbk[x_2, x_3]^{S_1 \times S_2}$. We want to write this as a sum of basis diagrams with basis elements given as in Theorem 4.7.5. For this consider its action

$$\begin{aligned} \Delta_2\Delta_1(s_1s_2(P) \cdot f) &= \Delta_2(\Delta_1(s_1s_2(P)) \cdot f + s_2(P)\Delta_1(f)) \\ &= P \cdot \Delta_2\Delta_1(f) + \Delta_2(s_s(P)) \cdot \Delta_1(f) + \Delta_2\Delta_1(s_1s_2(P)) \cdot f \\ &= (x_1 + x_2)^2 \cdot \Delta_2\Delta_1(f) - (2x_1 + x_2 + x_3) \cdot \Delta_1(f) + 1 \cdot f \end{aligned}$$

Hence we get in diagrams



Note that even in this special case it is not easy to compute the diagrams of lower order terms.

5.3 Main theorem

One of the main goals of this thesis is to give a complete presentation, i.e. generators and relations of the quiver Schur algebra with special dimension vector $\mathbf{d} = d \cdot \alpha_i$ as introduced in the beginning of section Section 5.2. Although the special case might, at first glance, looks simple we need several non-trivial statements to achieve the goal.

We will use the standard generators (idempotents, splits, merges, polynomials) modulo the relations introduced in Proposition 5.2.1 and Proposition 5.2.3 and claim the following

Theorem 5.3.1. *Let $\mathbf{d} = d \cdot \alpha_i$ for $d \in \mathbb{Z}_{>0}$. Then the list of relations given by the Propositions 5.2.1 and 5.2.3 is complete, i.e. the algebra $A_{\mathbf{d}}$ is freely generated by $e^k_{\hat{\lambda}}$, $m^k_{\hat{\lambda}}$, $s^k_{\hat{\lambda}}$ and $x^k_{\hat{\lambda}}(P)$ (idempotents, splits, merges and polynomials) such that $\hat{\lambda}$ runs over all vector compositions with $d(\hat{\lambda}) = \mathbf{d}$ modulo the given relations $(\mathcal{R}1)$-$(\mathcal{R}3)$ and $(\mathcal{P}1)$-$(\mathcal{P}5)$.*

Proof. Let $A^f_{\mathbf{d}}$ be the algebra freely generated by $e^k_{\hat{\lambda}}$, $m^k_{\hat{\lambda}}$, $s^k_{\hat{\lambda}}$ and $x^k_{\hat{\lambda}}(P)$. Let I be the ideal generated by the relations from Proposition 5.2.1 and Proposition 5.2.3. We will show that the map

$$\gamma : A_I := A^f_{\mathbf{d}} / I \longrightarrow A_{\mathbf{d}} \tag{5.4}$$

which sends generators to generators is an isomorphism of algebras. It is well-defined, because all the relations given by I hold in the quiver Schur algebra and it is surjective, because all generators of $A_{\mathbf{d}}$ lie in the image of γ. For injectivity it suffices to show that every element of A_I can be written as a sum of basis elements of $A_{\mathbf{d}}$. By Proposition 5.3.2 we can assume that this is known for all diagrams without polynomial generators. We will show this later on.

In the following we will distinguish four types of morphisms, namely merges (m), splits (s), polynomials (p) and crossings (cr). A box

$$\boxed{m, s, p}$$

stands for an arbitrary quiver Schur diagram given by the types of morphisms displayed in the box (idempotents are always allowed). We stack and read the boxes as quiver Schur diagrams. An arrow

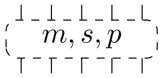
$$A \rightsquigarrow B$$

between two box representations A and B means that we can write diagrams of the form A as sums of diagrams of the form B.

By the braid relation $(\mathcal{R}9)$, associativity of split and merge $(\mathcal{R}1)$, $(\mathcal{P}1)$ and $(\mathcal{P}5)$ we can write every diagram of the form

$$\begin{array}{c} \boxed{m} \\ \boxed{\mathrm{cr}} \\ \boxed{p} \\ \boxed{s} \end{array} \tag{5.5}$$

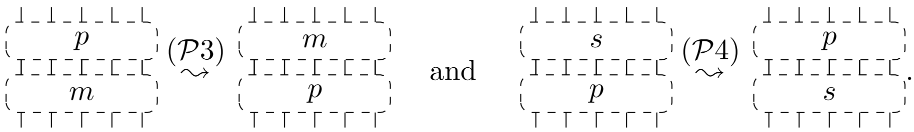
as a sum of basis diagrams given by the basis Theorem 4.7.5. Hence we will now show that we can write any arbitrary diagram



as a sum of diagrams of the form (5.5).

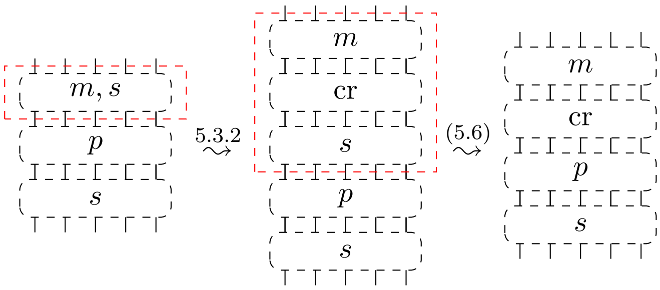
By the relations ($\mathcal{P}3$) and ($\mathcal{P}4$) we have

(5.6)



Now let P be the bottommost polynomial morphism such that there is a merge below P. If such a P does not exist we are in the case

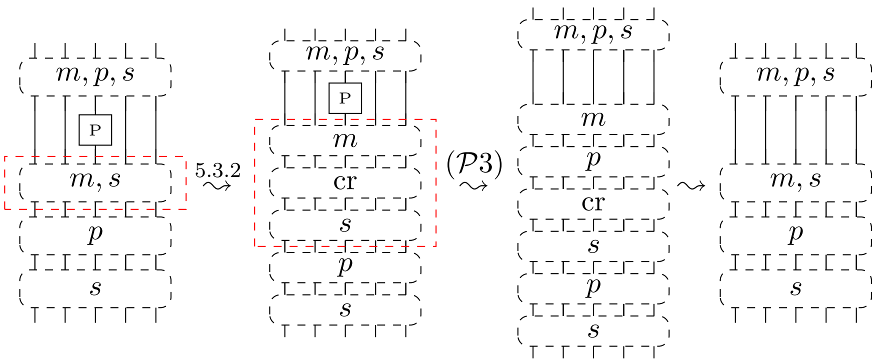
(5.7)



and can apply Proposition 5.3.2, see red boxes, and use (5.6) to get sums of the form (5.5).

Otherwise, we are in the case

(5.8)



and can apply Proposition 5.3.2, see red boxes, to write the diagram between P and the next polynomial morphism below P as a sum of basis diagrams. Using relation ($\mathcal{P}3$) we can pull P through the merges. By ($\mathcal{P}2$) we can pull the resulting polynomials through the crossings getting terms of lower order.

Repeating (5.8) while there is a polynomial above a merge morphisms leads to the first case and the claim follows. $\square$

Before proving Proposition 5.3.2 we will need some preparation and notations. Denote by A'_I the subalgebra of $A_I = A^f_{\mathbf{d}}/I$ generated by idempotents, splits and merges (without polynomial generators). Throughout the proof let $m, n \in \mathbb{Z}_{>0}$ and $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{l} = (l_1, \dots, l_n)$ be compositions of d, i.e. $\sum_{i=1}^m k_i = \sum_{j=1}^n l_j = d$.

We will sometimes consider instead of the whole algebra A'_I only the subspace $(A'_I)^{\mathbf{l}}_{\mathbf{k}}$ consisting of linear combinations of diagrams with bottom labels given by $\mathbf{k}$ and top labels given by $\mathbf{l}$.

Proposition 5.3.2. *Let $A'_{\mathbf{d}}$ be the subalgebra of the quiver Schur algebra $A_{\mathbf{d}}$ generated only by the idempotents, merges and splits (without polynomial generators). Then there is an isomorphism of algebras*

$$\gamma' : A'_I \xrightarrow{\sim} A'_{\mathbf{d}} \tag{5.9}$$

sending generators to generators.

Proof. As in the proof of Theorem 5.3.1 we have that γ' is well-defined and surjective because the non-polynomial relations given by I also hold in the algebra $A'_{\mathbf{d}}$ and the generators of $A'_{\mathbf{d}}$ lie in the image of γ'. We will show that $\dim A'_I = \dim A'_{\mathbf{d}}$ by the following three steps.

Step 1: We reduce A'_I to the green-red webs of [TVW15, §2.1].

Step 2: $\dim(A'_I)^{\mathbf{l}}_{\mathbf{k}} = \dim \operatorname{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right)$.

Step 3: $\dim \operatorname{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right) = |\ S_{\mathbf{l}}\backslash S_d/S_{\mathbf{k}}\ |$.

The last two steps imply that $\dim(A'_I)^{\mathbf{l}}_{\mathbf{k}} = |\ S_{\mathbf{l}}\backslash S_d/S_{\mathbf{k}}\ |$. On the other hand we know by the basis Theorem 4.7.5 that the dimension of

$$(A'_{\mathbf{d}})^{\mathbf{l}}_{\mathbf{k}} \subset A'_{\mathbf{d}},$$

the space generated by all quiver Schur diagrams without polynomial generators and bottom labels given by $\mathbf{k}$ and top labels given by $\mathbf{l}$, is also given by $|\ S_{\mathbf{l}}\backslash S_d/S_{\mathbf{k}}\ |$. Hence using the decomposition of the quiver Schur algebra, see Remark 4.4.2, this implies that $\dim A'_I = \dim A'_{\mathbf{d}}$ and hence γ' is an isomorphism. ☐

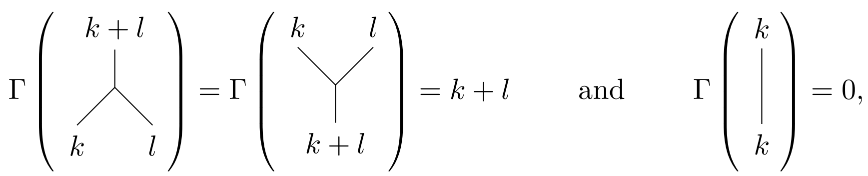
Let us now look at the three steps from above.

ad Step 1: The idea is that instead of looking at the algebra A'_I we consider morphisms of a subcategory of a web category. These web categories have their origins in [Kup96] and [CKM14].

In this case we consider subcategories of the free green-red web category defined in [TVW15, Def. 2.1]. Define by $\mathcal{C}$ the full subcategory of the category N-$\mathbf{Web}_g$, see [TVW15, Def. 2.7] with objects $\mathrm{Obj}(\mathcal{C})$ given by all sequences $\mathbf{k} = (k_1, \ldots, k_m)$ with $k_1, \ldots, k_m \in \mathbb{Z}_{>0}$ and $\sum_{i=1}^m k_i = d$.

Remark 5.3.3. The morphisms in C between objects $\mathbf{k}$ and $\mathbf{l}$ are given by webs which are defined as the diagrams of our special quiver Schur algebra, generated by idempotents, merges and splits but with different relations, see [TVW15, Def. 2.3].

In the context of [TVW15] we only use black and green coloured strands, hence in particular we do not need the colours and will omit them. We define the map Γ on the generators of the webs as follows

(5.10) 

for all $k, l \in \mathbb{Z}_{>0}$. For two webs D, D' it holds

$$\Gamma(D \cdot D') = \Gamma(D) + \Gamma(D'). \tag{5.11}$$

This means the map counts the number of merges and splits of a web weighted by the "thickness" of the strands (if a label is bigger we call the strand thicker).

Using the map Γ we can define a filtration on the space of morphism

$$\mathrm{Mor}(\mathcal{C}) := \bigoplus_{\mathbf{k},\mathbf{l} \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, \mathbf{l})$$

of $\mathcal{C}$ by

$$\mathrm{Mor}(\mathcal{C}) = \bigcup_{i \in \mathbb{N}} \mathrm{Mor}(\mathcal{C})_i \tag{5.12}$$

where $\mathrm{Mor}(\mathcal{C})_i \subset \mathrm{Mor}(\mathcal{C})$ is the set of all webs D such that $\Gamma(D) \leq i$.

By definition of Γ this is indeed a filtration, because (5.11) provides that

$$\mathrm{Mor}(\mathcal{C})_i \cdot \mathrm{Mor}(\mathcal{C})_j \subset \mathrm{Mor}(\mathcal{C})_{i+j}.$$

We use this filtration in the following Lemma 5.3.4 to show that the associated graded algebra

$$\mathrm{gr}\, \mathrm{Mor}(\mathcal{C}) := \bigoplus_{i \in \mathbb{N}} \mathrm{Mor}(\mathcal{C})_i \big/ \mathrm{Mor}(\mathcal{C})_{i-1},$$

with $\mathrm{Mor}(\mathcal{C})_{-1} := 0$, is isomorphic to the algebra A'_I. Additionally, with

$$\dim \mathrm{gr}\, \mathrm{Mor}(\mathcal{C}) = \dim \mathrm{Mor}(\mathcal{C})$$

it follows that

$$\dim \mathrm{Mor}(\mathcal{C}) = \dim A'_I.$$

Hence we can use the results of [TVW15] concerning the web categories for our purposes to show the next step. In particular we use that

$$\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, \mathbf{l}) = \dim (A'_I)^{\mathbf{l}}_{\mathbf{k}}. \tag{5.13}$$

Lemma 5.3.4. *There is an isomorphism of algebras denoted by*

$$\begin{array}{rcccl} \Phi : \mathrm{gr}\, \mathrm{Mor}(\mathcal{C}) & := & \bigoplus_{i \in \mathbb{N}} \mathrm{Mor}(\mathcal{C})_i \big/ \mathrm{Mor}(\mathcal{C})_{i-1} & \xrightarrow{\sim} & A'_I \\ & & D + \mathrm{Mor}(\mathcal{C})_{i-1} & \longmapsto & D \end{array} \tag{5.14}$$

sending a coset $D + \mathrm{Mor}(\mathcal{C})_{i-1} \in \mathrm{Mor}(\mathcal{C})_i \big/ \mathrm{Mor}(\mathcal{C})_{i-1}$ *to the corresponding quiver Schur diagram* D.

Proof. We will use that the relations between the webs, see the relations (7)-(9) in [TVW15, Def. 2.3], coincide under the quotient $\mathrm{Mor}(\mathcal{C})_{i+1} \big/ \mathrm{Mor}(\mathcal{C})_i$ with the relations given by Proposition 5.2.1. We have that the relations (7) and ($\mathcal{R}1$) are the same. The relation (8) equals to ($\mathcal{R}2$) respectively (9) equals to ($\mathcal{R}3$) in the associated graded algebra. Hence the map Φ is well-defined. It is surjective, because the generators of A'_I lie in the image of Φ. We can define the inverse map by

$$D \longmapsto D + \mathrm{Mor}(\mathcal{C})_{\Gamma(D)-1} \in \mathrm{Mor}(\mathcal{C})_{\Gamma(D)} \big/ \mathrm{Mor}(\mathcal{C})_{\Gamma(D)-1}$$

sending a quiver Schur diagram to the corresponding web coset in the associated graded algebra. This is also well-defined because of the connections between the relations mentioned before. Hence Φ is an isomorphism of algebras. $\square$

Proof of Step 2: We use the following results from [TVW15].

Theorem 5.3.5. *[TVW15, Thm. 3.20] Let $N \in \mathbb{N}$, then there is a functor*

$$F : \mathcal{C} \longrightarrow \mathrm{Fund}(\mathfrak{gl}_N) \tag{5.15}$$

which is full, i.e. surjective on morphism spaces.

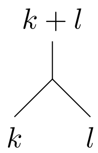
Here the category $\mathrm{Fund}(\mathfrak{gl}_N)$ has as objects tensor products

$$\bigwedge^{\mathbf{k}} V := \bigwedge^{k_1} V \otimes \ldots \otimes \bigwedge^{k_m} V$$

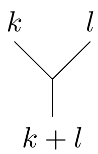
of the fundamental representations $\bigwedge^{k_i} V$ of $\mathfrak{gl}_N$ and $\mathbf{k}$ is a composition of d. The morphisms are the morphisms between these tensor products. The functor is defined on

- objects by
$$\mathbf{k} \longmapsto \bigwedge^{\mathbf{k}} V$$
with $\bigwedge^m V = 0$ if $m > N$ and $\bigwedge^0 V = \mathbb{C}$.

- morphisms by



$$\longmapsto \quad \bigwedge^k V \otimes \bigwedge^l V \longrightarrow \bigwedge^{k+l} V$$
$$v \otimes w \mapsto v \wedge w$$



$$\longmapsto \quad \bigwedge^{k+l} V \longrightarrow \bigwedge^k V \otimes \bigwedge^l V$$
$$w \mapsto \text{"split up in all possible ways"}$$

Theorem 5.3.6. *Let $N, d \in \mathbb{Z}_{>0}$, $N \geq d$ and $\mathbf{k}, \mathbf{l}$ compositions of d. Then the functor F from above induces an isomorphism on vector spaces*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, \mathbf{l}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right).$$

Proof. By [TVW15, Def. 2.5, Thm. 3.20] the kernel of F on morphisms is described, namely it is given by all webs with a strand labelled by a number bigger than N. But all labels are $\leq d$ and $N \geq d$ hence F induces an inclusion

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, \mathbf{l}) \hookrightarrow \mathrm{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right)$$

on the morphisms and therefore by Theorem 5.3.5 it induces an isomorphism. $\square$

Proof of Step 3: Let us first consider the special case $\mathbf{k} = \mathbf{l} = (1, \ldots, 1)$. In this case we have

$$\mathrm{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right) = \mathrm{End}_{\mathfrak{gl}_N}(V^{\otimes d}).$$

By the Schur-Weyl duality, see [GW09, Thm. 9.1.2], it follows

$$\mathrm{End}_{\mathfrak{gl}_N}(V^{\otimes d}) \cong \mathbb{C}[S_d].$$

Hence we get

$$\dim \mathrm{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right) = \dim \mathbb{C}[S_d] = |S_d| = |\ S_{\mathbf{l}} \backslash S_d / S_{\mathbf{k}}\ |$$

because $S_{\mathbf{k}}$ and $S_{\mathbf{l}}$ are the trivial groups.

Now we want to generalize this by viewing $\bigwedge^{\mathbf{k}} V$ as a quotient of $V^{\otimes d}$ via the map

$$v_1 \otimes \ldots \otimes v_d \mapsto v_{k_1} \wedge \ldots \wedge v_{k_1} \otimes \ldots \otimes \ldots \wedge v_d.$$

By the Theorem of Weyl, see [Hum72, §6.3], it follows that $\bigwedge^{\mathbf{k}} V$ is then also a direct summand of $V^{\otimes d}$.

If we consider $V^{\otimes d}$ as an S_d-module by permuting the factors, we can say that $\bigwedge^{\mathbf{k}} V$ are precisely the antiinvariants for the group $S_{\mathbf{k}}$, i.e.

$$\bigwedge^{\mathbf{k}} V \cong \{x \in V^{\otimes d} \mid s_i(x) = -x \text{ for all simple transpositions } s_i \in S_{\mathbf{k}}\}.$$

Hence the projection of $V^{\otimes d}$ onto $\bigwedge^{\mathbf{k}} V$ is given by the idempotent

$$e_{\mathbf{k}} := \frac{1}{|S_{\mathbf{k}}|} \sum_{w \in S_{\mathbf{k}}} (-1)^{l(w)} w. \tag{5.16}$$

It follows that

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{gl}_N}\left(\bigwedge^{\mathbf{k}} V, \bigwedge^{\mathbf{l}} V\right) &\cong \mathrm{Hom}_{\mathfrak{gl}_N}\left(e_{\mathbf{k}}(V^{\otimes d}), e_{\mathbf{l}}(V^{\otimes d})\right) \\
&\cong \mathrm{Hom}_{\mathbb{C}[S_d]}\left(e_{\mathbf{k}}\mathbb{C}[S_d], e_{\mathbf{l}}\mathbb{C}[S_d]\right) \\
&\cong e_{\mathbf{l}}\mathbb{C}[S_d]e_{\mathbf{k}}.
\end{aligned}$$

Here the last isomorphism is given by the map

$$f \longmapsto f(e_{\mathbf{k}}).$$

It is well-defined because $f(e_{\mathbf{k}}) \in e_{\mathbf{l}}\mathbb{C}[S_d]$ by definition and furthermore we have

$$f(e_{\mathbf{k}}) = f(e_{\mathbf{k}}^2) = f(e_{\mathbf{k}})e_{\mathbf{k}} \in e_{\mathbf{l}}\mathbb{C}[S_d]e_{\mathbf{k}}.$$

The inverse of the map is given by

$$e_{\mathbf{l}}xe_{\mathbf{k}} \longmapsto f_x$$

with $f_x(e_{\mathbf{k}}y) = e_{\mathbf{l}}xe_{\mathbf{k}}y$ for $y \in \mathbb{C}[S_d]$. It remains to show the following.

Lemma 5.3.7. *It holds*

$$\dim e_{\mathbf{l}}\mathbb{C}[S_d]e_{\mathbf{k}} = \mid S_{\mathbf{l}}\backslash S_d/S_{\mathbf{k}} \mid.$$

Proof. For all elements $w \in S_{\mathbf{k}}$ respectively $w' \in S_{\mathbf{l}}$ it holds that

$$w \cdot e_{\mathbf{k}} = (-1)^{l(w)} e_{\mathbf{k}} \text{ resp. } e_{\mathbf{l}} \cdot w' = (-1)^{l(w)} e_{\mathbf{l}}.$$

Hence, because the elements $w \in \mathbb{C}[S_d]$ form a basis of $\mathbb{C}[S_d]$, the minimal length representatives of elements of $S_{\mathbf{l}}\backslash S_d/S_{\mathbf{k}}$ span $e_{\mathbf{l}}\mathbb{C}[S_d]e_{\mathbf{k}}$.

Let J be the set of minimal length representatives. We will show that they are linear independent. Assume

$$\sum_{j \in J} a_j e_{\mathbf{l}} j e_{\mathbf{k}} = 0 \text{ with } a_j \in \mathbb{C}.$$

We can rewrite the left side by the definition of $e_{\mathbf{l}}$ and hence

$$\sum_{j \in J} a_j j + R = 0$$

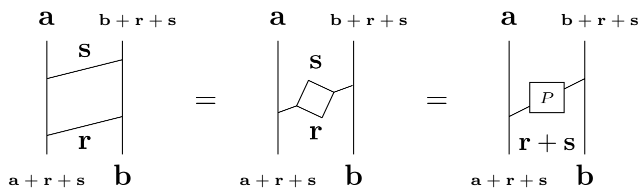
where R is a linear combination of elements $w \in S_d$ and $w \notin J$. Therefore, $a_j = 0$ for all $j \in J$, since the elements $w \in S_d$ form a basis of $\mathbb{C}[S_d]$ and the claim follows. □

6 Explicit relations for the general case

In the next step we will look for consequences of the special case relations from Section 5.2.

We replace the special labels in the relations of Proposition 5.2.1 by general labels (same special labels by same general labels). The "Associativity of split and merge" ($\mathcal{R}1$) of course holds in general. The relation "Deleting the hole" ($\mathcal{R}2$) turns into Lemma 4.6.10. Applying first the associativity of splits and merges and then Lemma 4.6.10 the relation "Deleting the big hole" ($\mathcal{R}4$) turns into the following relation

(6.1)



for general labels $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r}$, $\mathbf{s}$ and the polynomial $P = \Delta^{(e)}_{w_0^{r_e,s_e}} \dots \Delta^{(1)}_{w_0^{r_1,s_1}}(s^1_{(\mathbf{s},\mathbf{r})}(1))$ shifted by the vector $\mathbf{a}$.

The hard part is the ladder relation ($\mathcal{R}3$) which we will look closer at in the next sections.

We will introduce two different strategies to generalize the special relations from Section 5.2. One possibility will be to look only at highest coefficients. This leads to Proposition 6.1.2 and the statement that the generalized relations up to terms of lower order correspond to the special relations.

On the other hand we can go more into detail and calculate terms of lower order, see Section 6.3.

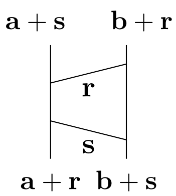
6.1 Highest coefficients of generalized diagrams

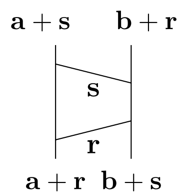
At first, we will generalize the special relations from Section 5.2 up to terms of lower order. By the general merge formula from Proposition 4.6.5 and the fact that Demazure operators with different superscripts commute we get that in general (general label vectors) the relations hold if we look at the highest order term (in the sense of Definition 4.6.7) up to coefficients given by the splits. In Proposition 6.1.2 we will show that these highest order coefficients are equal given the relations from Proposition 5.2.1, i.e. the relations hold up to terms of lower order.

Remark 6.1.1. Let D be some quiver Schur diagram which acts by $\Delta_2 \cdot \mathcal{S}_2 \cdot \Delta_1 \cdot \mathcal{S}_1$ on a polynomial where Δ_1, Δ_2 are products of Demazure operators (i.e. some merges) and $\mathcal{S}_1, \mathcal{S}_2$ some polynomials (e.g. some split actions). Using the derivation property extended to a product of Demazure operators (3.19) we get that the coefficient of the highest order term $\mathrm{HO}(D) = \Delta_2\Delta_1$ is given by $\mathrm{HC}(D) = w_2(\mathcal{S}_2) \cdot w_2 w_1(\mathcal{S}_1)$. Here w_1, w_2 are the elements of the symmetric group corresponding to the products of Demazure operators Δ_1, Δ_2.

Proposition 6.1.2. *If we replace in Proposition 5.2.1 the special label vectors of the relations* ($\mathcal{R}1$)*,* ($\mathcal{R}2$) *and* ($\mathcal{R}3$) *by general label vectors (same special labels by same general labels), then the equalities turn into equalities up to terms of lower order (in the sense of Definition 4.6.7). In particular the highest coefficients equal if the highest order does not vanish.*

Proof. The relations of the special case diagrams imply the equality of the highest orders (i.e. products of Demazures) in the general case. Hence it suffices to show that the highest coefficients equal. We know that ($\mathcal{R}1$) (“Associativity”) holds in general, i.e. the claim follows immediately and for ($\mathcal{R}2$) (“Deleting the hole”) the highest order vanishes, hence there is nothing to show. Therefore it suffices to look at the “Ladder relation” ($\mathcal{R}3$) and the action of the diagrams

(6.2) $D :=$ 

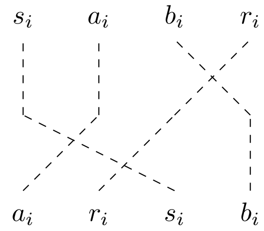
and $D' :=$ 

with arbitrary label vectors $\mathbf{a}, \mathbf{b}, \mathbf{r}, \mathbf{s} \in \mathbb{Z}^e_{\geq 0}$. We denote the action of the diagram D by $\Delta_2 \cdot \mathcal{S}_2 \cdot \Delta_1 \cdot \mathcal{S}_1$. Here $\mathcal{S}_1$ is the split action from $\mathbf{b}+\mathbf{s}$ into $(\mathbf{s}, \mathbf{b})$, Δ_1 is the the merge action from $(\mathbf{a}+\mathbf{r}, \mathbf{s})$ into $\mathbf{a}+\mathbf{r}+\mathbf{s}$, $\mathcal{S}_2$ the split action from $\mathbf{a}+\mathbf{r}+\mathbf{s}$ into $(\mathbf{a}+\mathbf{s}, \mathbf{r})$ and Δ_2 the merge action from $(\mathbf{r}, \mathbf{b})$ into $\mathbf{r}+\mathbf{b}$ (plus shifts by the correct vectors). Analogous D' acts by $\Delta'_2 \cdot \mathcal{S}'_2 \cdot \Delta'_1 \cdot \mathcal{S}'_1$. By Remark 6.1.1 the highest coefficients are

$$\mathrm{HC}(D) = w_2(\mathcal{S}_2) \cdot w_2 w_1(\mathcal{S}_1) \quad \text{and} \quad \mathrm{HC}(D') = w'_2(\mathcal{S}'_2) \cdot w'_2 w'_1(\mathcal{S}'_1)$$

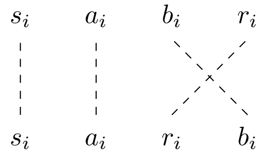
where the w’s are the symmetric group elements corresponding to the Demazure operators. The w’s are generated by some simple reflections $s^{(i)}_j$, $i \in \{1, \dots, e\}$ corresponding to the Demazure operators with i-th upper index. We denote the part of the w’s generated by the $s^{(i)}$’s by $w^{(i)}$. Using the permutation diagrams, see (4.19), we can display the symmetric group elements corresponding to the i-th upper index, namely

(6.3) $$w_2^{(i)} w_1^{(i)} = w_2'^{(i)} w_1'^{(i)} \cong$$



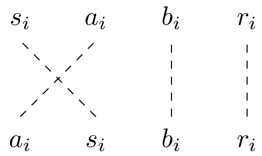
,

(6.4) $$w_2^{(i)} \cong$$



,

(6.5) $$w_2'^{(i)} \cong$$



.

Now we can explicitly write down the split actions given by Definition 4.4.4 and determine the highest coefficients by looking at the action of the permutation diagrams (6.3)-(6.5) on the splits given by permuting the indices of the variables. For the ladder diagram D we get that

$$\mathcal{S}_1 = \prod_{\substack{1\leq i\leq e \\ a_{i+1}+r_{i+1}+1\leq j\leq a_{i+1}+r_{i+1}+s_{i+1} \\ a_i+r_i+s_i+1\leq k\leq a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k})$$

and hence applying (6.3) leads to

$$w_2 w_1(\mathcal{S}_1) = \prod_{\substack{1\leq i\leq e \\ 1\leq j\leq s_{i+1} \\ a_i+s_i+1\leq k\leq a_i+b_i+s_i}} (x_{i+1,j} - x_{i,k}).$$

For the second split it follows

$$\mathcal{S}_2 = \prod_{\substack{1\leq i\leq e \\ 1\leq j\leq a_{i+1}+s_{i+1} \\ a_i+s_i+1\leq k\leq a_i+r_i+s_i}} (x_{i+1,j} - x_{i,k})$$

and by applying (6.4) we obtain

$$w_2(\mathcal{S}_2) = \prod_{\substack{1\leq i\leq e \\ 1\leq j\leq a_{i+1}+s_{i+1} \\ a_i+b_i+s_i+1\leq k\leq a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k}).$$

By analogous calculations for the ladder diagram D' we get that

$$\mathcal{S}_1' = \prod_{\substack{1\leq i\leq e \\ 1\leq j\leq a_{i+1} \\ a_i+1\leq k\leq a_i+r_i}} (x_{i+1,j} - x_{i,k})$$

and therefore with (6.3) it follows

$$w_2' w_1'(\mathcal{S}_1') = \prod_{\substack{1 \le i \le e \\ s_{i+1}+1 \le j \le a_{i+1}+s_{i+1} \\ a_i+b_i+s_i+1 \le k \le a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k}).$$

The split $\mathcal{S}_2'$ acts by

$$\mathcal{S}_2' = \prod_{\substack{1 \le i \le e \\ a_{i+1}+1 \le j \le a_{i+1}+s_{i+1} \\ a_i+s_i+1 \le k \le a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k})$$

and the action of (6.5) on the split gives

$$w_2'(\mathcal{S}_2') = \prod_{\substack{1 \le i \le e \\ 1 \le j \le s_{i+1} \\ a_i+s_i+1 \le k \le a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k}).$$

Comparing the product $w_2(\mathcal{S}_2)$ with $w_2' w_1'(\mathcal{S}_2')$ and the product $w_2 w_1(\mathcal{S}_1)$ with $w_2'(\mathcal{S}_2')$, we get

$$w_2(\mathcal{S}_2) = w_2' w_1'(\mathcal{S}_2') \cdot \prod_{\substack{1 \le i \le e \\ 1 \le j \le s_{i+1} \\ a_i+b_i+s_i+1 \le k \le a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k})$$

and

$$w_2 w_1(\mathcal{S}_1) \cdot \prod_{\substack{1 \le i \le e \\ 1 \le j \le s_{i+1} \\ a_i+b_i+s_i+1 \le k \le a_i+r_i+b_i+s_i}} (x_{i+1,j} - x_{i,k}) = w_2'(\mathcal{S}_2').$$

This implies

$$w_2(\mathcal{S}_2) \cdot w_2 w_1(\mathcal{S}_1) = w_2'(\mathcal{S}_2') \cdot w_2' w_1'(\mathcal{S}_1').$$

We hence showed that exchanging the labels in Proposition 5.2.1 leads to equalities up to terms of lower order. □

As a consequence it follows.

Theorem 6.1.3 (Equality up to lower order terms)**.** *Let D, D' be two diagrams of the special quiver Schur algebra $A_{\mathbf{d}}$ with $\mathbf{d} = d \cdot \alpha_i$, $d \in \mathbb{Z}_{>0}$, and without polynomial generators such that $D = D'$ modulo the relations from Proposition 5.2.1. Let D_{gen} respectively D'_{gen} be the diagrams consisting of the same strands as D respectively D' but with all special labels replaced by some general labels (same special labels by same general labels). Then it follows that* $\mathrm{HO}(D_{\text{gen}}) = \mathrm{HO}(D'_{\text{gen}})$ *and in particular* $\mathrm{HC}(D_{\text{gen}}) = \mathrm{HC}(D'_{\text{gen}})$*, if the highest order does not vanish, i.e. it holds*

$$D_{\text{gen}} \equiv D'_{\text{gen}} \textit{ (equality up to terms of lower order).}$$

Proof. By Proposition 6.1.2 the statement holds true for the relations from Proposition 5.2.1. But by Proposition 5.3.2 these relations imply all other relations without polynomials. Hence it suffices to show that the statement remains true if we multiply on both sides of the equation with diagrams for which the statement holds. This is a direct implication of Lemma 4.6.9. □

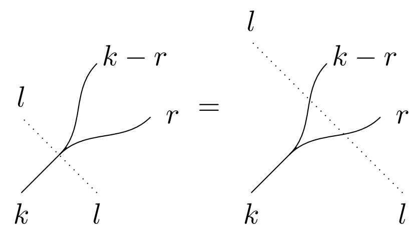
Remark 6.1.4. In particular the Theorem 6.1.3 immediately implies the equality up to terms of lower order for all relations from Corollary 5.2.2 where the special labels are replaced by general labels.

Before considering terms of lower order, we want to mention that there are relations of algebras related to the quiver Schur algebra. These can be translated into the quiver Schur settings but the exact calculations would exceed the scope of this thesis, see Section 6.2.

6.2 Adapted relations

There are several relations proven in the paper of [Sto16] in the context of thick calculus introduced by [Kho+12]. One of these relations is the Pitchfork lemma ([Sto16, Prop. 3]) which also holds for quiver Schur algebras.

Lemma 6.2.1 (Pitchfork lemma for quiver Schur algebras)**.** *Let $k \cdot \alpha_i,\ k \in \mathbb{Z}_{>0}$ and $l \cdot \alpha_j,\ l \in \mathbb{Z}_{>0},\ 1 \leq i \neq j \leq e$. Then it holds*



The strand with label k splits up into $(k-r)\cdot\alpha_i,\ r\cdot\alpha_i$. Here we omit the simple vectors α_i, α_j in the labelling. The dotted strand corresponds to the simple vector α_j and non-dotted to the simple vector α_i.

Proof. Merges of two strands of different type act by the identity, see Proposition 4.6.5. Splits of arcs with the same type act by the identity, see Lemma 4.6.11. There are no merges of the same type hence it suffices to look at the splits of different types. If $j \neq i+1$ then by Lemma 4.6.11 the split acts by the identity and hence both diagrams act by the identity. For $j = i+1$ we can compute the action of the split of the left hand diagram

$$\begin{aligned}\prod_{n=1}^{l}\prod_{m=1}^{k}(x_{j,n}-x_{i,m}) &= \prod_{n=1}^{l}\prod_{m=1}^{k-r}(x_{j,n}-x_{i,m}) \cdot \prod_{n=1}^{l}\prod_{m=k-r+1}^{k}(x_{j,n}-x_{i,m}) \\ &= \prod_{n=1}^{l}\prod_{m=1}^{k-r}(x_{j,n}-x_{i,m}) \cdot \prod_{n=1}^{l}\prod_{m=1}^{r}(x_{j,n}-x_{i,m+k-r})\end{aligned}$$

which corresponds to the action of the diagram on the right hand side. Hence the claim follows. □

Remark 6.2.2. If we replace the labels of the pitchfork diagrams by special label vectors, the left and right hand side equal using the "Opening of an edge" relation ($\mathcal{R}5$) and the associativity of splits. Hence applying Theorem 6.1.3 we get equality up to terms of lower order for general labels and in particular for the labels chosen above.

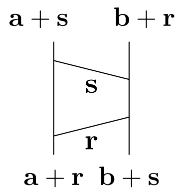
Furthermore, we claim that there are analogous equations of the Reidemeister moves [Sto16, Prop. 6] and [Sto16, Prop. 7] for the quiver Schur algebras.

6.3 Example on ladder relations

We will now come back to the generalized ladder relation. By the previous Proposition 6.1.2 we know that the ladder relation holds up to terms of lower order. But how do these terms of lower order look like?

We will give an algorithm how to calculate the terms of lower order of each diagram of the form

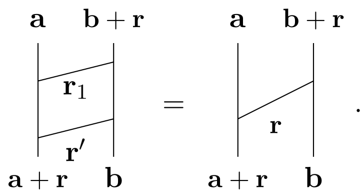
(6.6)



for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^e_{\geq 0}$, $\mathbf{r} = (r_1, \ldots, r_{e-1}, 0)$ and $\mathbf{s} = (s_1, \ldots, s_{e-1}, 0)$. The idea is to split up the diagonal strands, using (6.7) and (6.8) unless each diagonal strand is labelled by some simple label α_i. Using Theorem 6.3.1 for each simple labelled strand we can calculate the terms of lower order. We will make this explicit by an example, see Example 6.3.3.

Using the relation (6.1) for vectors $\mathbf{r} := (r_1, \ldots, r_{e-1}, 0)$, $\mathbf{r}_1 := (r_1, 0, \ldots, 0)$ and $\mathbf{r}' := (0, r_2, \ldots, r_{e-1}, 0)$ we obtain

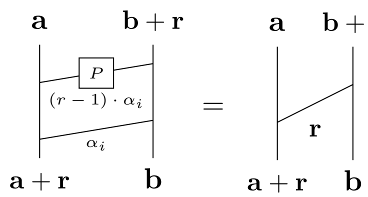
(6.7)



Hence we can inductively split up diagonal strands labelled by $\mathbf{r}$ into strands labelled by $r_i \cdot \alpha_i$.

Additionally for a vector $\mathbf{r} = r \cdot \alpha_i$, $r \in \mathbb{Z}_{>0}$ we get

(6.8)



with a polynomial P chosen such that $\Delta^{(i)}_{a_i+1} \cdots \Delta^{(i)}_{a_i+r-1}(P) = 1$. Hence altogether we can split up each diagonal strand into simple strands possibly with polynomials on it.

Now we give a ladder relation with simple diagonal strands. In the proof we will use the special properties of Demazure operators (see (3.5) and (3.19)). For this we recall the fact how to commute Demazure operators and polynomials following from the derivation property (3.5). For some polynomial Q and a Demazure operator $\Delta^{(i)}_j := \frac{\mathrm{Id} - s^{(i)}_j}{x_{i,j} - x_{i,j+1}}$ it follows

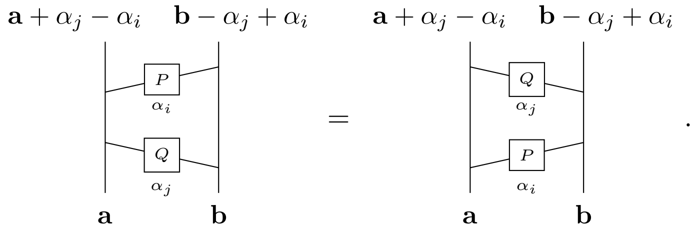
$$\Delta^{(i)}_j \cdot Q = \Delta^{(i)}_j(Q) \cdot \mathrm{Id} + s^{(i)}_j(Q) \cdot \Delta^{(i)}_j \tag{6.9}$$

and

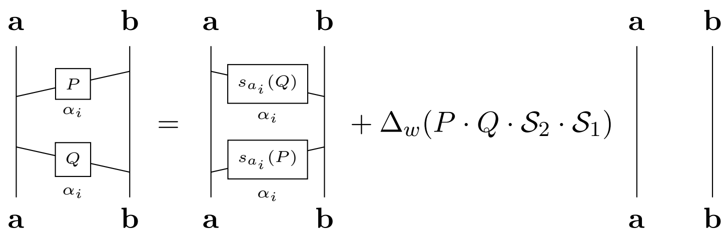
$$Q \cdot \Delta_j^{(i)} = \Delta_j^{(i)} \cdot s_j^{(i)}(Q) + \Delta_j^{(i)}(Q) \cdot \mathrm{Id}\,. \tag{6.10}$$

Theorem 6.3.1. *Let $\mathbf{a} = (a_1, \ldots, a_e)$ and $\mathbf{b} = (b_1, \ldots, b_e)$ be arbitrary label vectors such that $a_i \geq 1$ respectively $b_j \geq 1$ for some fixed $i, j \in \{1, \ldots, e\}$ and α_i, α_j the special vectors from before.*

1) For $i \neq j$ and polynomials $P \in \Bbbk[x_{i,a_i}]$ and $Q \in \Bbbk[x_{j,a_j+1}]$ we obtain



2) For $i = j$ and polynomials $P, Q \in \Bbbk[x_{i,a_i+1}]$ we obtain



where

$$\Delta_w := \Delta^{(i)}_{a_i+b_i-1} \ldots \Delta^{(i)}_{a_i+1} \Delta^{(i)}_1 \ldots \Delta^{(i)}_{a_i}$$

is the product of the Demazure operators corresponding to the action of the merges $m^1_{(\mathbf{a},\alpha_i,\mathbf{b}-\alpha_i)}$ and $m^2_{(\mathbf{a},\alpha_i,\mathbf{b}-\alpha_i)}$. The polynomials $\mathcal{S}_1 := s^2_{(\mathbf{a},\alpha_i,\mathbf{b}-\alpha_i)}(1)$ and $\mathcal{S}_2 := s^1_{(\mathbf{a},\alpha_i,\mathbf{b}-\alpha_i)}(1)$ are given by the split actions.

Proof. ad 1): We first consider the case $j \neq i+1$. Then the action of the left hand diagram in 1) is given by

(6.11)

$$\begin{aligned}\mathcal{D}_L := \overbrace{\Delta^{(i)}_{a_i+b_i-1} \ldots \Delta^{(i)}_{a_i}}^{=:\Delta^{(i)}_{[a_i+b_i-1,a_i]}} \cdot \overbrace{P \cdot (x_{i+1,1} - x_{i,a_i}) \cdot \ldots \cdot (x_{i+1,a_{i+1}} - x_{i,a_i})}^{=:A_2} \\ \cdot \underbrace{\Delta^{(j)}_1 \ldots \Delta^{(j)}_{a_j}}_{=:\Delta^{(j)}_{[1,a_j]}} \cdot \underbrace{Q \cdot (x_{j,a_j+1} - x_{j-1,a_{j-1}+1}) \cdot \ldots \cdot (x_{j,a_j+1} - x_{j-1,a_{j-1}+b_{j-1}})}_{=:A_1}.\end{aligned}$$

Using that $j \neq i$, $j \neq i+1$ and the property (6.9) of the Demazure operators we can commute $\Delta^{(j)}_{[1,a_j]}$ and A_2 and also $\Delta^{(i)}_{[a_i+b_i-1,a_i]}$ and A_1. Hence we get

$$\mathcal{D}_L = \Delta^{(i)}_{[a_i+b_i-1,a_i]} \Delta^{(j)}_{[1,a_j]} \cdot A_2 \cdot A_1 = \Delta^{(j)}_{[1,a_j]} \cdot A_1 \cdot \Delta^{(i)}_{[a_i+b_i-1,a_i]} \cdot A_2 = \mathcal{D}_R$$

where $\mathcal{D}_R$ is the action of the right hand diagram.

If $j = i+1$ then the product A_2 has to be replaced by

$$A_2' := (x_{i+1,1} - x_{i,a_i}) \cdot \ldots \cdot (x_{i+1,a_{i+1}+1} - x_{i,a_i})$$

and hence the same trick is possible, because of (6.10), for $1 \le k \le a_{i+1}$, it holds

$$P \cdot A_2' \cdot \Delta_k^{(i+1)} = \Delta_k^{(i+1)} \cdot s_k^{(i+1)}(P \cdot A_2') + \Delta_k^{(i+1)}(P \cdot A_2') \cdot \mathrm{Id}\,.$$

But the product $P \cdot A_2'$ is $s_k^{(i+1)}$ invariant and therefore

$$P \cdot A_2' \cdot \Delta_k^{(i+1)} = \Delta_k^{(i+1)} \cdot P \cdot A_2'$$

and the calculation runs as in the previous case.

ad 2): Concerning the merges and splits of the left hand diagram in 2) its action is given by

(6.12)

$$\begin{aligned}\mathcal{D}_L := \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot &\overbrace{(x_{i+1,1} - x_{i,a_i+1}) \cdot \ldots \cdot (x_{i+1,a_{i+1}} - x_{i,a_i+1})}^{=\mathcal{S}_2} \\ \cdot \Delta_1^{(i)} \ldots \Delta_{a_i}^{(i)} \cdot Q \cdot &\underbrace{(x_{i,a_i+1} - x_{i-1,a_{i-1}+1}) \cdot \ldots \cdot (x_{i,a_i+1} - x_{i-1,a_{i-1}+b_{i-1}})}_{=\mathcal{S}_1}\end{aligned}$$

with $\mathcal{S}_1, \mathcal{S}_2$ as in the assumption. Using the property (6.10), the braid relation of Demazure operators and the fact that $P \cdot \mathcal{S}_2$ is $s_1^{(i)}, \ldots, s_{a_i-1}^{(i)}$ invariant we can swap the operator $\Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)}$ to the first position, see (6.13). By (6.9) we can swap the operator $\Delta_{a_i}^{(i)}$ to the end, see (6.14).

$$\begin{aligned}
&(6.13) \quad \mathcal{D}_L = && \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)} \cdot Q \cdot \mathcal{S}_1 \\
&(6.14) \quad \phantom{\mathcal{D}_L} = && \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \\
& && + \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \cdot \Delta_{a_i}^{(i)} \\
&(6.15) \quad \phantom{\mathcal{D}_L} = && \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \\
& && + \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \cdot s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \cdot \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)}.
\end{aligned}$$

We can swap the polynomials $s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1)$ and $P \cdot \mathcal{S}_2$ in the second summand of (6.14). After that we use (6.9) and the fact that $s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1)$ is $s_{a_i+b_i-1}^{(i)}, \ldots, s_{a_i+1}^{(i)}$ invariant to commute $s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1)$ and $\Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)}$ to obtain (6.15).

Swapping the Demazure operator $\Delta_{a_i}^{(i)}$ and $P \cdot \mathcal{S}_2$ in the last line of (6.15) we get (6.16). Except for $\Delta_{a_i}^{(i)}$ all the other Demazure operators with upper index i act by zero on $\Lambda(\mathbf{a}, \mathbf{b})$. Hence using the action of a product of Demazure operators (3.19) we finally obtain (6.17).

$$\begin{aligned}
&(6.16) \quad \mathcal{D}_L = && \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \\
& && + \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \cdot s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \cdot \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \cdot \Delta_{a_i}^{(i)}(P \cdot \mathcal{S}_2) \\
& && + \underbrace{\Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \cdot s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \cdot \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \Delta_{a_i}^{(i)} \cdot s_{a_i}^{(i)}(P \cdot \mathcal{S}_2)}_{=:\mathcal{D}_R} \\
&(6.17) \quad \phantom{\mathcal{D}_L} = && \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \left(P \cdot \mathcal{S}_2 \cdot \Delta_{a_i}^{(i)}(Q \cdot \mathcal{S}_1)\right) \cdot \mathrm{Id} \\
& && + \Delta_1^{(i)} \ldots \Delta_{a_i-1}^{(i)} \left(s_{a_i}^{(i)}(Q \cdot \mathcal{S}_1) \cdot \Delta_{a_i+b_i-1}^{(i)} \ldots \Delta_{a_i+1}^{(i)} \Delta_{a_i}^{(i)}(P \cdot \mathcal{S}_2)\right) \cdot \mathrm{Id} \\
& && + \mathcal{D}_R.
\end{aligned}$$

Here the last summand $\mathcal{D}_R$ is equal to the action of the right hand ladder diagram. We again use that $s^{(i)}_{a_i}(Q\cdot\mathcal{S}_1)$ is $s^{(i)}_{a_i+b_i-1},\dots,s^{(i)}_{a_i+1}$ invariant and commute $s^{(i)}_{a_i}(Q\cdot\mathcal{S}_1)$ and $\Delta^{(i)}_{a_i+b_i-1}\dots\Delta^{(i)}_{a_i+1}$. Furthermore, we can summarize the first two summands of (6.17), see (6.18), and finish using once more the derivation property of $\Delta^{(i)}_{a_i}$, see (6.19).

$$\mathcal{D}_L = \Delta^{(i)}_1\dots\Delta^{(i)}_{a_i-1}\Delta^{(i)}_{a_i+b_i-1}\dots\Delta^{(i)}_{a_i+1}\Big(\Delta^{(i)}_{a_i}(Q\cdot\mathcal{S}_1)\cdot P\cdot\mathcal{S}_2 + s^{(i)}_{a_i}(Q\cdot\mathcal{S}_1)\cdot\Delta^{(i)}_{a_i}(P\cdot\mathcal{S}_2)\Big)\cdot\mathrm{Id}+\mathcal{D}_R \tag{6.18}$$

$$= \Delta^{(i)}_1\dots\Delta^{(i)}_{a_i-1}\Delta^{(i)}_{a_i+b_i-1}\dots\Delta^{(i)}_{a_i+1}\Delta^{(i)}_{a_i}\left(Q\cdot\mathcal{S}_1\cdot P\cdot\mathcal{S}_2\right)\cdot\mathrm{Id}+\mathcal{D}_R. \tag{6.19}$$

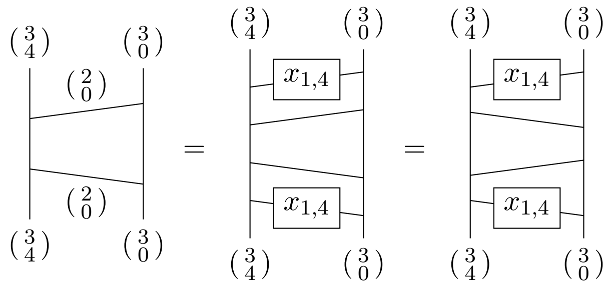
The claim follows using the braid relation of Demazure operators. $\square$

Remark 6.3.2. The number of Demazure operators in the product Δ_w in Theorem 6.3.1 is given by a_i+b_i-1.

Example 6.3.3 (Ladder relation)**.** We calculate the terms of lower order of one ladder diagram exemplary. For this take the ladder from (6.6) with $\mathbf{a}=(3,4,0)$, $\mathbf{b}=(3,0,0)$ and $\mathbf{r}=\mathbf{s}=(2,0,0)$. In the following we omit the 0's in the third entry of the vectors. All diagonal strands not explicitly labelled are of label $(1,0,0)$.

Using (6.7) and (6.8) unless each diagonal strand is labelled by some simple label α_i we obtain the first equation of (6.20), because $\Delta^{(1)}_4(x_{1,4})=1$.

(6.20) $D :=$



.

By Theorem 6.3.1 we can turn around the inner right ladder into a left ladder without getting a term of lower order, because the splits act by a polynomial of degree four but by Remark 6.3.2 we have five Demazure operators corresponding to the merges. For the relations (6.21) and (6.22) we use the previous Theorem 6.3.1 again. We calculate the polynomial given by the splits of the left hand diagrams of (6.21) and (6.22) below and get

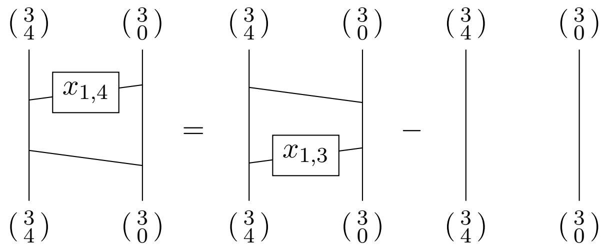
$$X := (x_{2,1}-x_{1,4})\cdot(x_{2,2}-x_{1,4})\cdot(x_{2,3}-x_{1,4})\cdot(x_{2,4}-x_{1,4}).$$

Hence the factor in front of the idempotent is given by

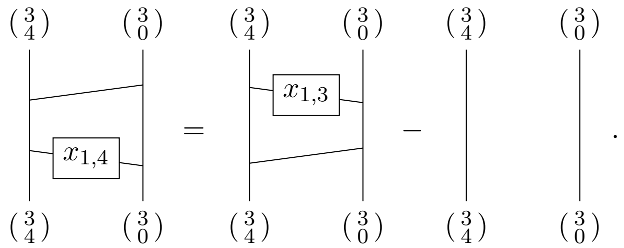
$$\Delta^{(1)}_5\Delta^{(1)}_4\Delta^{(1)}_1\Delta^{(1)}_2\Delta^{(1)}_3(x_{1,4}\cdot X) = -1.$$

This can be calculated by hand or by the little Python program, see Appendix D.3.

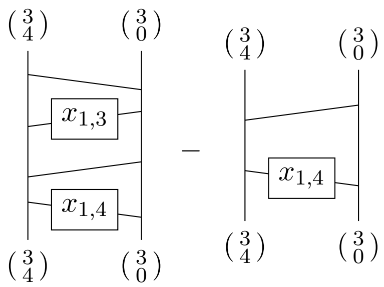
By this observations we get the relations

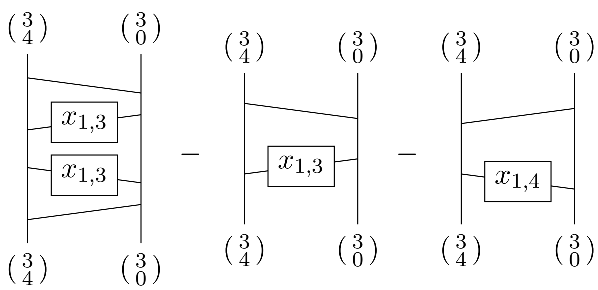
(6.21) 

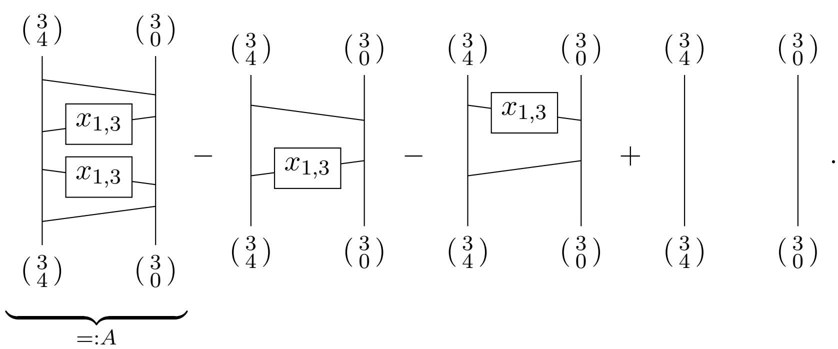
and

(6.22) 

Inserting (6.22) and (6.21) into (6.20) we obtain (6.23), (6.24) and (6.25)

(6.23) $D =$ 

(6.24) $=$ 

(6.25) $=$ 

The diagram A can be rewritten using Theorem 6.3.1 again. We denote the polynomial given by the split actions of the inner ladder diagram by

$$Y := (x_{2,1} - x_{1,3}) \cdot (x_{2,2} - x_{1,3}) \cdot (x_{2,3} - x_{1,3}) \cdot (x_{2,4} - x_{1,3}).$$

Hence we get for the factor in front of the idempotent

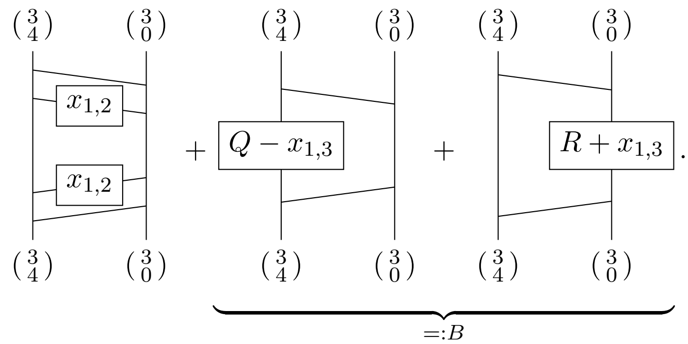
$$\begin{aligned} P &= \Delta_5^{(1)}\Delta_4^{(1)}\Delta_3^{(1)}\Delta_1^{(1)}\Delta_2^{(1)}(x_{1,3}^2 \cdot Y) \\ &= \sum_{i=1}^{6} x_{1,i} - \sum_{i=1}^{4} x_{2,i}. \end{aligned}$$

Again we calculate this with the Python program. The invariant polynomials

$$Q := \sum_{i=1}^{3} x_{1,i} - \sum_{i=1}^{4} x_{2,i} \in \Lambda((3,4,0)) \quad \text{and} \quad R := \sum_{i=4}^{6} x_{1,i} \in \Lambda',$$

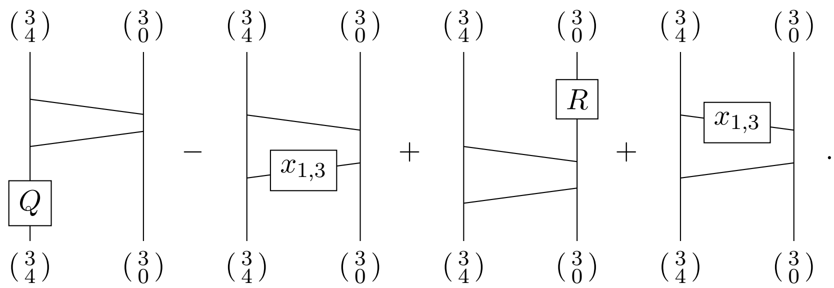
where Λ' is the invariant ring $\Lambda((3,0,0))$ shifted by $(3,4,0)$, are defined such that $P = Q + R$. We apply this to the diagram A and it holds

(6.26) $A =$

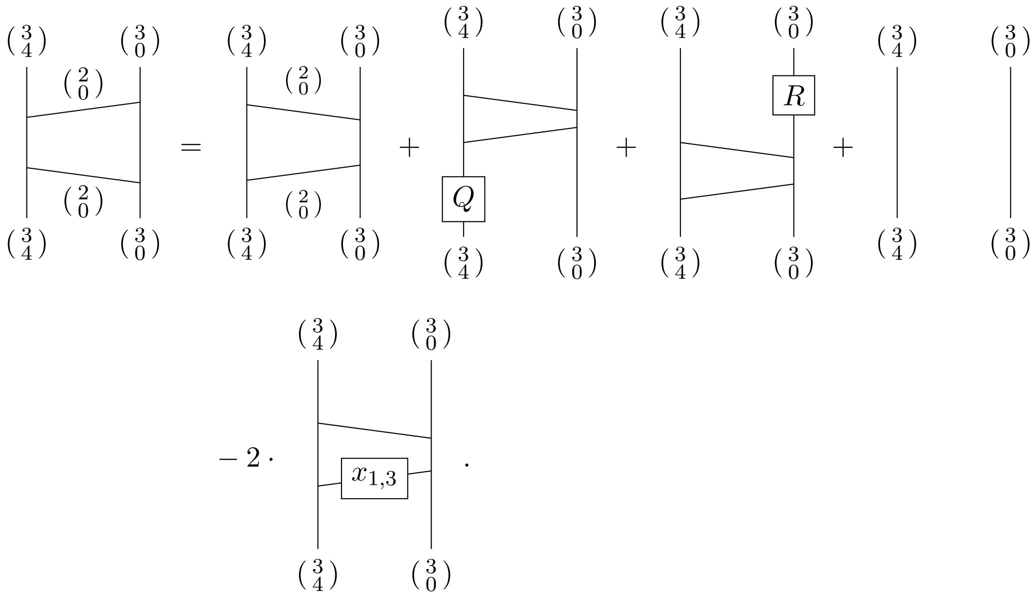


We can rewrite B as follows

(6.27) $B =$



Altogether, inserting (6.26) and (6.27) into (6.25) we get the "whole" relation also including all the terms of lower order.



6.4 Outlook

Of course the ultimate goal is to write down a complete list of relations for the quiver Schur algebra of an arbitrary dimension vector.

In this thesis we gave a complete list of the special cases $\mathbf{d} = d \cdot \alpha_i$. In the proof of this we saw connections to the theory of web categories [TVW15], [CKM14]. By the result of Theorem 6.1.3 we obtained general relations from the special ones by replacing the special label vectors by general label vectors. Hence it would be interesting if there are other relations than these.

The ladder relation is the most interesting one among the special relations, because we do not understand its terms of lower order in the general case. Hence it would be nice to extend Theorem 6.3.1 to get an explicit formula of the algorithm we gave in Section 6.3. By Theorem 5.3.1 and Theorem 6.1.3 the understanding of lower order terms of the ladder relations will have an impact on figuring out terms of lower order concerning relations which are implied by the ladder relation.

We could also ask if Theorem 6.1.3 generalizes to the following.

Conjecture. *Let $\mathbf{d} \in \mathbb{Z}^{\mathrm{e}}_{\geq 0}$ be some arbitrary dimension vector. Let D, D' be two diagrams in $A_{\mathbf{d}}$ with the same bottom and top labels $\hat{\lambda}$ respectively $\hat{\mu}$ such that there is no polynomial generator appearing in D and D' and it holds* $\mathrm{HO}(D) = \mathrm{HO}(D') \neq 0$ *(in the sense of Definition 4.6.7), i.e. the diagram D acts by*

$$P \cdot \Delta + \text{ lower terms}$$

and D' acts by

$$Q \cdot \Delta + \text{ lower terms}$$

for $P, Q \in R(\mathbf{d})$. Then it already holds that $P = Q$, i.e. $\mathrm{HC}(D) = \mathrm{HC}(D')$ *and hence*

$$D \equiv D' \text{ (equality up to terms of lower order).}$$

Appendices

A Figures

A.1 Example of a representation with compatible flag

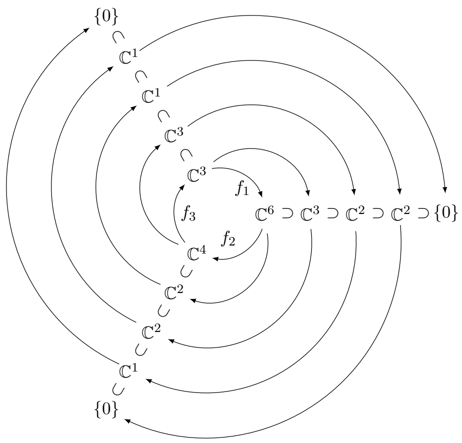


Figure 10: Example of compatible flags

A.2 Merge on flag representations

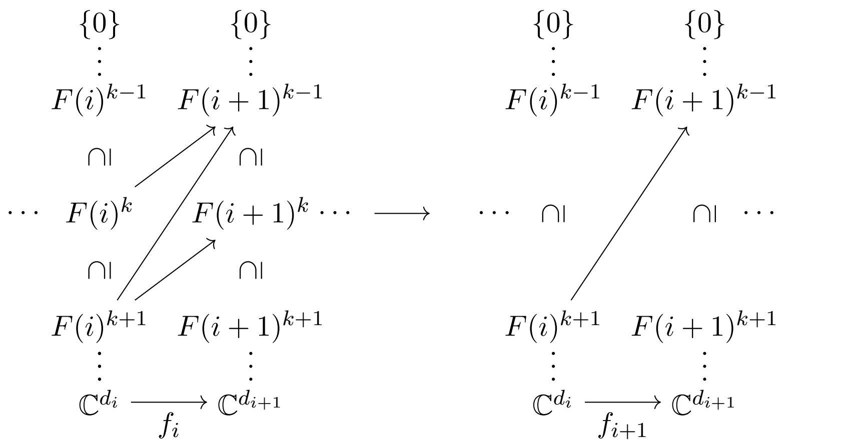
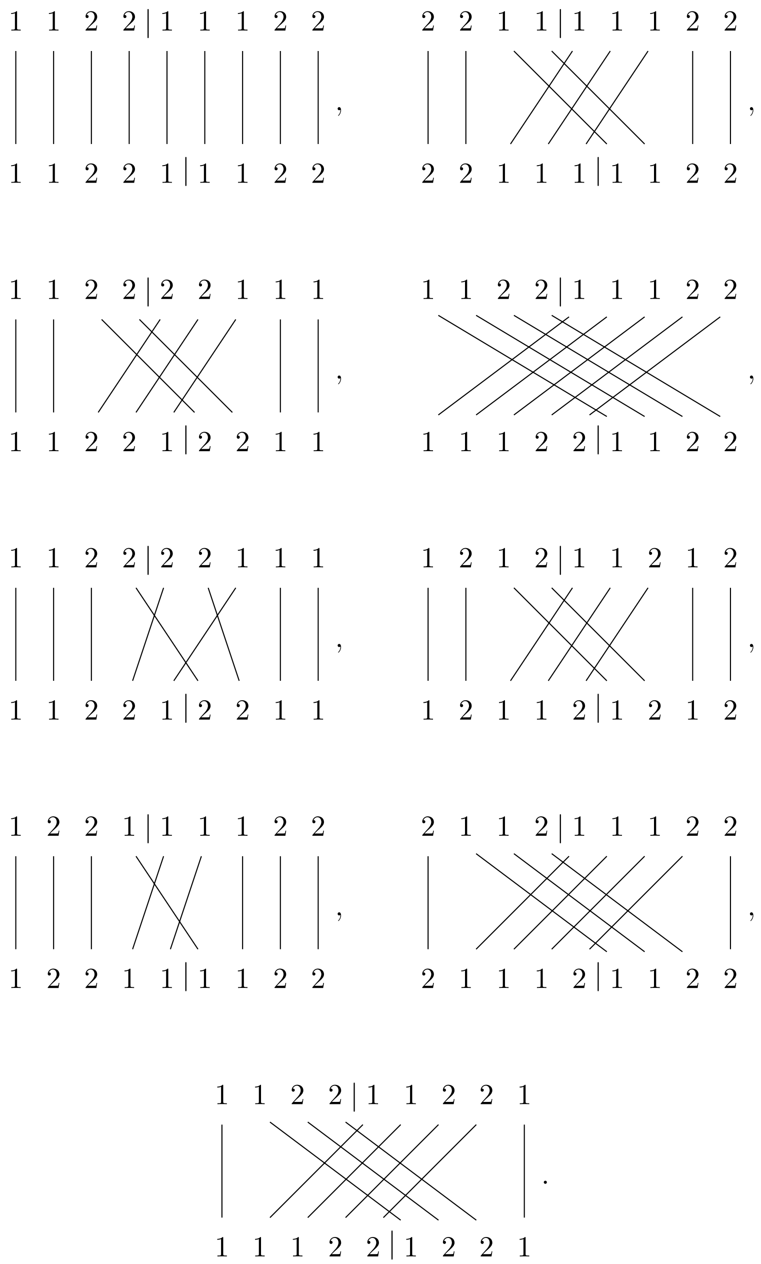


Figure 11: Merge on flag representations

B Examples

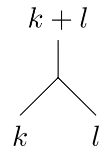
B.1 Quiver Schur basis example

A basis (up to the polynomial morphisms) for quiver Schur diagrams with bottom label $\hat{\lambda} = ((3,2,0),(2,2,0))$ and top label $\hat{\mu} = ((2,2,0),(3,2,0))$ is given by the diagrams



C Abbreviations of special diagrams

We define some abbreviations of special diagrams.



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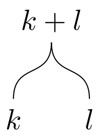
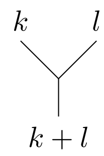


(merge)



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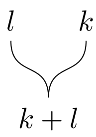
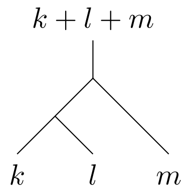


(split)



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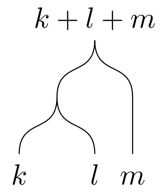
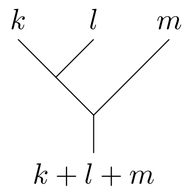


(double merge)



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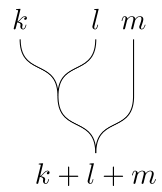
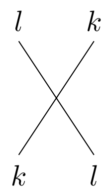


(double split)



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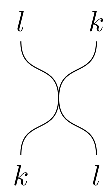
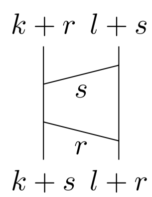


(crossing)



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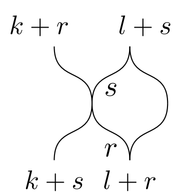
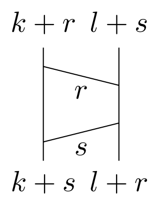


(right ladder)



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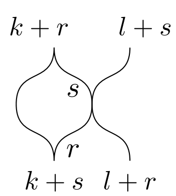
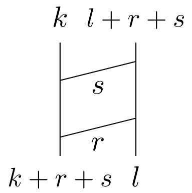


(left ladder)



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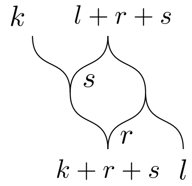
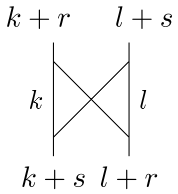
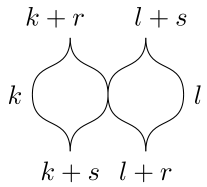
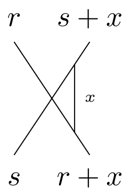


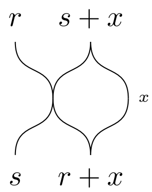
(big hole)



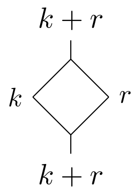
:=


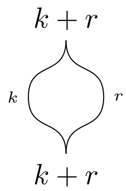
(cross ladder)



:=


(open up edge)



:=


(hole)

The small numbers at the strands stand for the label of this strands. All labels $k, l, m, r, s, x \in \mathbb{Z}_{\geq 0}$ are chosen such that no label is negative.

D Python code

D.1 QSA.py

The following python file *QSA.py* contains several functions computing merge and split actions, the action of Demazure operators and the action of simple reflections on polynomials. To use this, install SageMath and compile the main file by:

$ sage -python main.py

The main program *main.py* can be found below the *QSA.py*.

The main functions are

- `def merge(polynomial, vecA, vecB, shiftVect=[])`
- `def split(vecA, vecB, shiftVect=[])`
- `def d(index, polynomial)`
- `def s(index, polynomial)`

Here the *polynomial* is given by variables *xi_j* according to variables $x_{i,j}$ given by the dimension vector. The vectors *vecA*, *vecB* are given by lists, e.g.

$$[1, 2, 3] \cong (1, 2, 3).$$

The shift vector *shiftVect* shifts the actions of the merge and split if there are idempotents left of the merge or split.

The function **merge** calculates the merge action on *polynomial* merging together the vectors *vecA* and *vecB* together with some possible shift.

The function **split** calculates the split polynomial for the split from *vecA*+*vecB* into *vecA*, *vecB*. The *index* is some string 'i_j' such that the function from above *d('i_j', polynomial)* corresponds to the Demazure operator $\Delta_j^{(i)}$(*polynomial*). The function *s('i_j', polynomial)* corresponds to the element $s_j^{(i)}$ of the symmetric group acting on *polynomial*.

There is some short writing for Demazure operators and swaps for small indices (< 10) such that we can write *dij(polynomial)* = *d('i_j', polynomial)* and *sij(polynomial)* = *s('i_j', polynomial)*. In Appendix D.3 some example calculations can be found.

```
#Sage Symbolic Computation
from sage.all import *
from numpy import array
import math
import itertools

"""Compute split, merges, Demazure operators and swap of variables for polynomials"""

"""factorizes polynomial, no error for 0 polynomial."""
def safe_factor(polynomial):
    if polynomial == 0:
```

```
        return polynomial
    else:
        return factor(polynomial)

"""returns the dimension vector of two vectors"""
def dimVec(vecA, vecB):
    result=[]
    for i in range(len(max(vecA, vecB))):
        result.append(vecA[i]+vecB[i])
    return result

"""Action of the symmetric group on polynomials which swaps
↪ generatorA and generatorB (generators look like xi_j)"""
def swap(generatorA, generatorB, polynomial):
     if type(polynomial) is int:
        return polynomial
     else:
        return safe_factor(polynomial({generatorA:generatorB,
↪   generatorB:generatorA}))

"""short writing of swap polynomial + index given by 'a_b' or ab if a
↪ and b only one digit"""
def s(index, polynomial):
    index = str(index)
    if "_" in index:
        index = index.split("_")
        first_index = index[0]
        second_index = index[1]
        return swap(var('x' + first_index + "_" + second_index)
↪   ,var('x' + first_index + "_" + str(int(second_index) + 1)),
↪   polynomial)
    elif "x" in index:
        return swap(var('x' + index[1] + "_" + index[2]), var('x' +
↪   index[1] + "_" + str(int(index[2]) + 1)), polynomial)
    else:
        return swap(var('x' + index[0] + "_" + index[1]), var('x' +
↪   index[0] + "_" + str(int(index[1]) + 1)), polynomial)

"""Define the demazure operator on polynomials whith generatorA and
↪ generatorB given by xi_j"""
def DemOp(generatorA, generatorB, polynomial):
    #print(polynom)
    if type(polynomial) == int:
        return 0
    else:
        result = (polynomial - swap(generatorA, generatorB,
↪   polynomial)) / (generatorA-generatorB)
    return safe_factor(result)
```

```
"""short writing of the Demazure operator on polynomial if the index given by 'a_b' or ab if a and b only one digit"""
def d(index, polynomial):
    index = str(index)
    if "_" in index:
        index = index.split("_")
        first_index = index[0]
        second_index = index[1]
        return DemOp(var('x' + first_index + "_" + second_index), var('x' + first_index + "_" + str(int(second_index) + 1)),polynomial)
    else:
        return DemOp(var('x' + index[0] + "_" + index[1]), var('x' + index[0] + "_" + str(int(index[1]) + 1)),polynomial)

"""prints the index list of generators if merging together two vectors with a, b at position i plus a shift vector"""
def index_list(a, b, i, shift):
    result = []
    for j in range(b):
        for k in range(a, 0, -1):
            result.append(str(i) + "_" + str(shift + j + k))
    return result

"""Calculation of the merge action of two given vectors vecA, vecB into vecA+vecB with given shift vector by product of Demazures"""
def merge(polynomial, vecA, vecB, shiftVect=[]):
    dim=len(vecA)
    if shiftVect == []:
        for i in range(dim):
            shiftVect.append(0)
    result = polynomial
    for i in range(dim):
        indexList = index_list(vecA[i], vecB[i], i+1, shiftVect[i])
        for j in range(len(indexList)):
            result = d(indexList[j], result)
    return result

"""Calculation of the split action of the vectors vecA+vecB into vecA,vecB with given shift vector"""
def split(vecA, vecB, shiftVect=[]):
    dim=len(vecA)
    if shiftVect == []:
        for i in range(dim):
            shiftVect.append(0)
    if(len(vecA) == len(vecB)):
        result = 1
        for i in range(len(vecA)):
```

```
            if i == len(vecA)-1:
                for j in range(1, vecA[0] + 1):
                    for k in range(vecA[dim - 1] + 1, vecB[dim - 1] + vecA[dim - 1] + 1):
                        result *= (var('x' + str(1) + "_" + str(j + shiftVect[0])) - var('x' + str(dim) + "_" + str(k + shiftVect[dim-1])))
            else:
                for j in range(1, vecA[i+1] + 1):
                    for k in range(vecA[i] + 1, vecB[i] + vecA[i] + 1):
                        result *= (var('x' + str(i + 2) + "_" + str(j + shiftVect[i + 1])) - var('x' + str(i + 1) + "_" + str(k + shiftVect[i])))
        return result;
```

D.2 main.py

This file contains the main function. The important part is to define the dimension_vector. Then the variables corresponding to the vector are initialized and the computation can start, for examples see Appendix D.3.

```
from QSA import *
from sage.all import *
from numpy import array
import math
import itertools

"""defines the variables of the polynomial ring given by the dimension vector"""
def polyRing(dimVec):
    #Set the variable string "x1_1, x_1_2, ..."
    variables = ''
    for i in range(1,len(dimVec) + 1):
        for j in range(1,dimVec[i - 1] + 1):
            if i == 1 and j == 1:
                variables += 'x' + str(i) + "_" + str(j)
            else:
                variables += ',x' + str(i) + "_" + str(j)
    var(variables)  #Defines the symbolic variables
    #print(variables)

    #Calculates the polynomial ring out of the dimension vector, "variables" is the set of generators
    #Defines the polynomial ring in the variables "variables"
    R = QQ[variables]

    #Generators of the Ring
    z = R.gens()
    return R
```

```python

def main():
    #dimension vector for computation
    dimension_vector = [12,12,12]

    #initialize the varibles which correspond to the dimension vector such that we can use the variables xi_j, ...
    variables = polyRing(dimension_vector)

    #initializes in dependence of the variables functions di_j and si_j which are the Demazures resp. swappings of polynomials
    func_template = """def d%s(polynomial) : return d(%s, polynomial)"""
    for x in range(1, len(dimension_vector) + 1):
        for y in range(1, dimension_vector[x - 1]):
            exec(func_template % (str(x) + str(y), str(x) + str(y)))
    func_template1 = """def s%s(polynomial) : return s(%s, polynomial)"""
    for x in range(1, len(dimension_vector) + 1):
        for y in range(1, dimension_vector[x - 1]):
            exec(func_template1 % (str(x) + str(y), str(x) + str(y)))

    #do calculations here
    print(split([0,1,0],[2,0,0], [1,1,0]))
if __name__=='__main__':
    main()
```

D.3 Python example

This is an example how to calculate split and merge actions. We can calculate the polynomial

$$\Delta_5^{(1)}\Delta_4^{(1)}\Delta_3^{(1)}\Delta_1^{(1)}\Delta_2^{(1)}\left(x_{1,3}^2(x_{2,1}-x_{1,3})(x_{2,2}-x_{1,3})(x_{2,3}-x_{1,3})(x_{2,4}-x_{1,3})\right)$$

where $(x_{2,1}-x_{1,3})(x_{2,2}-x_{1,3})(x_{2,3}-x_{1,3})(x_{2,4}-x_{1,3})$ is the split action from $(3,4,0)$ to $((2,4,0),(1,0,0))$ with no shift vector by the program using the following code.

```python
S_1 = split([2,4,0], [1,0,0], [0,0,0])
d15(d14(d13(d11(d12(x1_3**2*S_1)))))
```

If we want to calculate some merge, say of two strands labelled by $(3,2,1)$ and $(1,2,3)$ and there are some idempotents left of the merge with label $(2,2,2)$ we can calculate the merge action on a polynomial f of this diagram by

```python
merge(f, [3,2,1], [1,2,3], [2,2,2])
```

Index

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