On Hochschild Cohomology, Koszul Duality and DG Categories

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Contents

1. Introduction 2
2. Hochschild Cohomology 7
2.1. Hochschild Cohomology for Associative Algebras 7
2.2. Koszul Algebras 13
3. The DG World 19
3.1. Graded Categories and DG Categories 19
3.2. Homotopy Categories and Triangulated Structure 27
3.3. The Derived Category 34
3.4. Hochschild Cohomology for DG Algebras 36
3.5. Hochschild Cohomology for DG Categories 47
4. Derived Functors 52
4.1. Categories of H-Projectives and H-Injectives 52
4.2. Existence of Derived Functors 52
5. Keller's Theorem(s) 54
5.1. Keller Triples for DG Algebras 54
5.2. Keller Triples for DG Categories 57
6. The DG Koszul Dual 64
6.1. Slightly New Definitions 64
6.2. An Isomorphism of Categories 65
7. Appendix A: The Verdier Quotient in Triangulated Categories 67
7.1. Triangulated Categories 67
7.2. The Verdier Quotient 69
References 75

1. Introduction

Let k be a field and A and B unital, associative, k-algebras and $\mathcal{D}A$ and $\mathcal{D}B$ their respective unbounded derived categories. In analogy to the classical theorem of Morita, [Hap] and [Ric] proved that there is triangulated equivalence of unbounded derived categories

$$\mathcal{D}A \longrightarrow \mathcal{D}B, \tag{0.1}$$

if and only if there exists an A-B-bimodule X such that

$$? \otimes^{\mathbb{L}}_A X : \mathcal{D}A \longrightarrow \mathcal{D}B,$$

is a triangulated equivalence. Two k-algebras satisfying equation (0.1) are called derived Morita equivalent, and it is an immediate consequence that their Hochschild cohomology groups are isomorphic as graded algebras, see Section 5, via the induced map on the morphism spaces of the derived category.

Two interesting and important examples of algebras that don't quite (but almost) fit into the Mortia framework are Koszul algebras and their Koszul duals, see [Pri]. A Koszul algebra, A, is a positively graded algebra with some nice homological properties, see Section 2.2 for a precise defintiion. For a Koszul algebra A, it's so called Koszul dual is the k-algebra, $A^! = \mathrm{Ext}^*_A(A_0, A_0)$. It was shown in [BGS], and see [MOS] for a slightly more generalized statement, that there is an equivalence of certain bounded derived categories

$$K : \mathcal{D}^{\downarrow}A \longrightarrow \mathcal{D}^{\uparrow}A^!, \tag{0.2}$$

Where K is the so called Koszul duality functor. However, unlike in the Morita case K messes up the grading on the Hom spaces, because a shift in the internal (adams) degree becomes a cohomological shift under the the Koszul duality functor, ([BGS], [MOS]) and we no longer have an isomorphism of graded algebras between the Hochschild cohomology of A and $A^!$.

Indeed in Section 2.2 we will compute the Hochschild cohomology of a pair $A = \mathbb{C}[x]$ and $A^! = \mathbb{C}[\zeta]/(\zeta^2)$ and see this explicitly. One notes that these two algebras are obviously not derived equivalent since for example their centers are not isomorphic, [Ric]. Using the original definition of Hochschild cohomology given by Hochschild himself, [Hoch], it is in general difficult to compute it. In Section 2.1 we will interpret the Hochschild cohomology of an associative k-algebra as

$$\mathrm{HH}^i(A) = (\mathcal{D}A^e)(A, [i]A), \tag{0.3}$$

Where A^e is the enveloping algebra of A. This immediately suggests that if we can find a nicer A^e-resolution of A then it will be easier to compute the Hochschild cohomology. To make these computations for $\mathbb{C}[x]$ and $\mathbb{C}[\zeta]/(\zeta^2)$ we will use an idea that we learned from the paper [VdB] of Van den Bergh. Roughly, the idea is to exploit the fact that the Koszul complex, K, of a Koszul algebra is a graded linear projective resolution of A_{0A} (where we view A_0 as a right A module) and taking twisted tensor products on the left and right of K by A gives a graded bimodule resolution of A.

In order to fit a Koszul algebra A and its dual $A^!$ into the Morita framework, it turns out we must work with two gradings. i.e. We must view A and $A^!$ as (adams) graded-differentially graded-algebras. If we make this change of view point it was shown in the thesis of Lefèvre, [Lef, Chapter 2], that indeed we get an equivalence of unbounded derived and coderived categories. We will show, that if one views A as an (adams)graded-dg-algebra concentrated in cohomological degree 0, and $A^!$ as an (adams)graded-dg-algebra, which we will denote as $A^{!^{dg}}$, that sits diagonally between the adams grading and the cohomological grading then one recovers the graded isomorphism

$$\mathrm{HH}^*(A) \longrightarrow \mathrm{HH}^*(A^{!^{dg}}), \tag{0.4}$$

of Hochschild cohomology. Equation (0.4) first appeared in print in [Kel2], however with few details. One of the goals of this thesis is to try and clarify how one achieves the isomorphism (0.4). Moreover, in Section 3.4 we will illustrate how it works in the case of $\mathbb{C}[x]$ and $\mathbb{C}[x]^{!^{dg}}$ and $\mathbb{C}[\zeta]/(\zeta^2)$ and $\mathbb{C}[\zeta]/(\zeta^2)^{!^{dg}}$.

With our sights on proving equation (0.4) we first need to understand what Hochschild cohomology is for differentially graded algebras, i.e. graded algebras with a differential. This leads us to studying differential graded categories which are k-linear categories enriched over chain complexes, and modules over dg-categories. In Section 3 we develop the theory of modules over dg-categories and the derived category of modules over dg-categories. A dg-algebra can be thought of as a special case of a dg-category with one object (we will explain this in more detail in 3.1), so in particular the the theory of dg-categories covers the case of a dg-algebra. One issue we will have to come to terms with in the theory is that for a module M over some dg-algebra A, a free resolution, see 3.4, may not necessarily exist because as we will see the differentials mess things up. This will force us to introduce the notion of a semi-free object and we will show in detail that there are enough semi-free resolutions in the derived category of modules over a dg-algebra, see 3.4. Most of the work we do can then be generalized in an obvious way to the derived category of modules over a dg-category. We will then interpret the Hochschild cohomology of a dg-category (and hence a dg-algebra), $\mathscr{A}$, as

$$\mathrm{HH}^i(\mathscr{A}) = (\mathcal{D}\mathscr{A}^e)(\mathcal{I}_{\mathscr{A}}, [i]\mathcal{I}_{\mathscr{A}}), \tag{0.5}$$

where $\mathscr{A}^e$ is the dg-category $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ and $\mathcal{I}_{\mathscr{A}}$ is the $\mathscr{A}^e$ module that takes the pair of objects $(a, a') \in \mathscr{A}^e$ to $\mathscr{A}(a, a')$. One can see that if we take the special case of $\mathscr{A} = A$, then $\mathcal{I}_{\mathscr{A}}$ is nothing more than A with the its canonical bimodule structure. Thus one can think of $\mathcal{I}_{\mathscr{A}}$ as the "diagonal $\mathscr{A}$-bimodule". Most of the theory in Section 3 was developed in the seminal papers [Kel1] and [Drin].

The last ingredient we need in proving (0.4), is a very deep result of Keller's, from [Kel2], which in some sense can be viewed as saying that the Hochschild cochain complex of a dg-category is functorial. Section 4 is devoted completely to establishing this result. In order to state it, we introduce the notion of a Keller triple, which is a triple $(\mathscr{A}, X, \mathscr{B})$ where $\mathscr{A}$ and $\mathscr{B}$ are dg-categories and X an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module. We will call a triple left (resp. right) admissible, if roughly, the left (resp. right) action of $\mathscr{A}$ (resp. $\mathscr{B}$) induces a quasi-isomorphism. We will show that for a left admissible triple there is canonical map

$$\varphi_X : C^*(\mathscr{B}) \longrightarrow C^*(\mathscr{A}), \tag{0.6}$$

where $C^*(?)$ denotes the Hochschild cochain complex. In fact, the map φ_X is a morphism of B_∞-algebras (we will elaborate more on this in a moment). Similarly we will show that if $(\mathscr{A}, X, \mathscr{B})$ is right admissible then there is a canonical map

$$\tilde{\varphi}_X : C^*(\mathscr{A}) \longrightarrow C^*(\mathscr{B}), \tag{0.7}$$

of B_∞-algebras. Keller's theorem can now be stated as:

Theorem 1.0.1. *Suppose that $(\mathscr{A}, X, \mathscr{B})$ is a left admissible Keller triple. Then the following are true,*

(1) φ_X depends only on the isomorphism class of $X \in \mathcal{D}\ \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$.

(2) If $(\mathscr{A}, X, \mathscr{B})$ is right admissible then φ_X is an isomorphism in $\mathrm{Ho}(\mathrm{B}_\infty)$*. In particular if φ_X is an isomorphism if $? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathcal{D}\mathscr{A} \longrightarrow \mathcal{D}\mathscr{B}$ is a triangulated equivalence.*

(3) Suppose that $F : \mathscr{A} \to \mathscr{B}$ is fully faithful. Then $\varphi_{X_F} = F^*$. In particular for $(\mathscr{A}, \mathcal{I}_{\mathscr{A}}, \mathscr{A})$, the morphism $\varphi_{\mathcal{I}_{\mathscr{A}}}$ is the identity.

(4) Suppose that $(\mathscr{B}, Y, \mathscr{C})$ and $(\mathscr{A},\ Z := X \otimes^{\mathbb{L}}_{\mathscr{B}} Y,\ \mathscr{C})$ are both left admissible Keller triples then $\varphi_Z = \varphi_X \circ \varphi_Y$.

The way we will prove (0.4) is that we will realize that the theory of (adams)graded-dg-modules is contained in the theory of dg-categories . More precisely we will show that there is an equivalence of categories between the category of graded-dg-modules and modules over a suitably chosen dg-category. Then, using this equivalence, to a triple $(A^{!^{dg}}, K^{dg}, A)$ we will associate a Keller triple $(\mathscr{A}, \mathcal{K}, \mathscr{B})$ and show that this triple is a left and right admissible and hence by ii) of Theorem 1.0.1 it will follow that the hochschild cochains of A and $A^{!^{dg}}$ are isomorphic in the homotopy category B_∞-algebras and hence their Hochschild cohomology is isomorphic as graded algebras.

We now would like to say a few words about B_∞-algebras. Inspired by the earlier of work of Baues, B_∞-algebras were first explicitly described by Getzler and Jones in [GJ]. Roughly speaking a chain complex (V, d) has the structure of a B_∞-algebra if the tensor coalgebra, $T([1]V)$ has the structure of a dg-bialgebra. It was sketched in [GJ] that the Hochschild cohomology cochains of a dg(they even showed A_∞)-algebra A has the structure of a B_∞-algebra and that this structure induces the Gerstenhaber bracket on the Hochschild cohomology (see Section 2.1). In [Kel3], Keller showed that the isomorphism on Hochschild cohomology induced by a derived Morita equivalence respected the Gerstenhaber bracket and later showed

Theorem 1.0.2. *If $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a fully faithful dg-functor between dg-categories. Then the induced map*

$$F^* : C^*(\mathscr{B}) \longrightarrow C^*(\mathscr{A}), \tag{0.8}$$

on Hochschild cochains, is a morphism of B_∞-algebras.

Theorem 1.0.2 has been been stated in many places, and has been sketched in a few of those places: [Shoi], [Kel2], [Low]. But as far as we are aware, no detailed proofs exist in the literature. Theorem 1.0.2, as we will see, plays a critical role in the proof of Theorem 1.0.1.

Now we say a little bit about notation for the rest of this thesis. throughout this work k will be assumed to be a field unless otherwise stated. Tensor products without decorations will always be over the ground field k. We will always take the word "graded" to mean $\mathbb{Z}$-graded and the degree of a homogeneous element $a \in M$ where M is a graded k-module will be denoted as $|a|$. The spaces of homomorphisms between objects a, b in a category $\mathscr{A}$ will be denoted as $\mathscr{A}(a, b)$. All categories are assumed to be small (i.e. the collection of objects and morphisms are sets) and all modules, whether they be over a category or a ring, are assumed to be right modules unless otherwise stated.

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2. Hochschild Cohomology

Hochschild cohomology was first introduced by G. Hochschild in [Hoch]. The construction in [Hoch] goes roughly as follows, given a finite dimensional associative algebra A over a field k and an A-A-bimodule X one can define the "cohomology of A with coefficients in X", $\mathrm{HH}^*(A, X)$. Hochschild showed that A is separable if and only if $\mathrm{HH}^1(A, X)$ vanishes for all bimodules X. Later on in the late 80's and early 90's it was shown that $\mathrm{HH}^*(A, X)$ is in some sense a derived invariant (see [Ric] or [Hap]) and controls, for instance, deformations of X. We will first come up with a definition in terms of derived categories then we will obtain the original definition due to Hochschild from it. The upshot, is that putting Hochschild's definition in the framework of derived categories will give us more flexibility in computing the cohomology explicitly and will also tell us how to generalize to the case of dg-algebras and then dg-categories. We will assume that k is a field and that all our algebras are associative over k and unital. Also in this section we will compute Hochschild cohomology of $\mathbb{C}[x]$ and $\mathbb{C}[\zeta]/(\zeta^2)$ (Section 2.2) explicitly and the theory that we build as well as the results will point to general phenomena that will all come together in Section 6.

2.1. **Hochschild Cohomology for Associative Algebras.**

We would like to associate a sequence of cohomology groups to A, an arbitrary k-algebra. To do this, we need a resolution of A. There is always a very nice free resolution of A as an A-A-bimodule known as the bar resolution.

Definition 2.1.1. *Let A be an k-algebra. The bar complex of A, $(\mathrm{B}^{-i}(A), b^{-i})$ for $i \in \mathbb{Z}_{0\geq}$ is the complex*

$$\cdots \xrightarrow{b^{-3}} A \otimes A \otimes A \otimes A \xrightarrow{b^{-2}} A \otimes A \otimes A \xrightarrow{b^{-1}} A \otimes A \tag{1.1}$$

where $B^{-i}(A) := A \otimes A^{\otimes i} \otimes A$. For $i > 0$ we define $b^{-i} : A \otimes A^{\otimes i} \otimes A \to A \otimes A^{\otimes(i-1)} \otimes A$ as

$$\begin{aligned} b^{-i}(a_1 \otimes \cdots \otimes a_{i+2}) &= a_1 \cdot a_2 \otimes \cdots \otimes a_{i+2} \\ &+ \sum_{j=2}^{i+1} (-1)^{j+1} a_1 \otimes \cdots \otimes a_j \cdot a_{j+1} \otimes \cdots \otimes a_{i+2} \end{aligned}$$

and for $i = 0$, $b^0 = 0$. We will sometimes write $B(A)$ for the complex as a whole.

One can check directly by computation that $b^{-i+1} \circ b^{-i} = 0$. Thus $B(A)$ is a chain complex. Also note that the differentials are homomorphisms of A-A-bimodules and that each $\mathrm{B}^{-i}(A)$ is a free A-A-bimodule.

Proposition 2.1.2. *Let $\epsilon : A \otimes A \longrightarrow A$ such that $\epsilon(a \otimes a') = aa'$ (note that ϵ is a bimodule homomorphism). Then*

$$\cdots \xrightarrow{b^{-2}} A \otimes A \otimes A \xrightarrow{b^{-1}} A \otimes A \xrightarrow{\epsilon} A \longrightarrow 0 \tag{1.2}$$

is a chain complex and is acyclic. Hence $\mathrm{B}(A)$ is a free resolution of A as an A-A-bimodule.

Proof. Since $\epsilon(a_1 \cdot a_2 \otimes a_3 - a_1 \otimes a_2 \cdot a_3) = 0$ it follows that (1.2) is a chain complex. One can construct a chain homotopy s:

$$\begin{array}{rcl} s^{-i} : A^{\otimes i} & \longrightarrow & A^{\otimes(i+1)} \\ a_1 \otimes \cdots \otimes a_i & \mapsto & 1 \otimes a_1 \otimes \cdots a_i \end{array}$$

and check that $b \circ s + s \circ b = \mathrm{id}$, $\epsilon \circ s + s \circ \epsilon = \mathrm{id}$ and $\epsilon \circ s^{-1} = \mathrm{id}$. Thus 1.2 is acyclic. $\square$

In the sequel, for A an algebra, Mod-A will denote the category of right A modules, and we write $\mathrm{Hom}_A(M, N)$ for the space of morphisms between two objects in this category, this breaks from our convention of writing hom-spaces in general but here it feels like a more natural notation.

For an A-A-bimodule X we could now define our cohomology groups as the cohomology groups of of the complex

$$(\mathrm{Hom}_{A\text{-}A}(\mathrm{B}^{-i}(A), X), b^{-i^*})_{i \in \mathbb{Z}_{0\geq}} \tag{1.3}$$

Thus we are lead to the following definition.

Definition 2.1.3. *The Hochschild cohomology of A with coefficients in X is*

$$\mathrm{HH}^i(A, X) := \mathrm{Ext}_{\mathrm{A\text{-}A}}{}^i(A, X) \tag{1.4}$$

This paragraph is for people who already know a little about dg-categories, but can also act a preview of things to come. Recall that we can view an A-A-bimodule as a A^e-module where $A^e = A^{op} \otimes A$. We can think of A^e as a dg-algebra concentrated in degree 0 with trivial differential and thus think of it as a dg-category as in Example 3.1.4. So, $\mathcal{D}A^e$ in the sense of Definition 3.3.1 is the derived category of chain complexes over A^e. We now recast Definition 2.1.3.

Definition 2.1.4. *The Hochschild cohomology of A with coefficients in X is*

$$\mathrm{HH}^i(A, X) := (\mathcal{D}A^e)(A, [i]X). \tag{1.5}$$

Whenever $X = A$ we write $\mathrm{HH}^*(A) := \mathrm{HH}^*(A, A)$.

This is indeed a reformulation of 2.1.3 since

$$\begin{array}{rcl} (\mathcal{D}A^e)(A, [i]X) & \cong & (\mathcal{D}A^e)(B(A), [i]X) \\ & \cong & (\mathcal{H}A^e)(B(A), [i]X) \\ & \cong & H^i(\mathrm{Hom}_{A^e}(B(A), X)) \\ & \cong & \mathrm{Ext}_{\mathrm{A\text{-}A}}{}^i(A, X). \end{array}$$

The first isomorphism is from Proposition 2.1.2. The second is standard in the theory of chain complexes over some algebra but it will follow from the general theory we establish for dg-categories, see Lemma 3.3.2. The third and fourth follow directly from definitions. Note that for an adams graded algebra, A, $\mathrm{HH}^i(A, X)$ inherits a grading.

Definition 2.1.4 has a couple of advantages. Firstly it lends itself nicely to generalization. Secondly, it allows us to compute Hochschild cohomology with any resolution of A we like. For example, if we are clever enough to find a finite resolution or a periodic resolution it is much easier to compute cohomology than with the bar resolution. We will now show how one derives Hochschild's original definition from the one we initially chose. The first lemma is well known.

Lemma 2.1.5. *Let A be a k-algebra and M an A^e-module. Then*

$$\begin{tikzcd} \mathit{Mod}\text{-}k \arrow[r, bend left, "?\otimes_k A^e"] & \mathit{Mod}\text{-}A^e \arrow[l, bend left, "Res"] \end{tikzcd} \tag{1.6}$$

is an adjoint pair $(? \otimes A^e, Res)$ of functors.

However, it has the following relevant corollary:

Corollary 2.1.6. *For $i \in \mathbb{Z}_{0\geq}$*

$$\mathrm{Hom}_{A^e}(\mathrm{B}^{-i}(A), M) \cong \mathrm{Hom}_k(A^{\otimes i}, X)$$

Proof. Follows directly from Lemma 2.1.5. □

Which, in turn is used in stating the following lemma:

Lemma 2.1.7. *The morphism d^i making the diagram:*

$$\begin{tikzcd} \mathrm{Hom}_k(A^{\otimes i}, A) \arrow[rr, "d^i"] \arrow[d, "\phi"', "\cong"] & & \mathrm{Hom}_k(A^{\otimes i+1}, A) \\ \mathrm{Hom}_{A^e}(A \otimes A^{\otimes i} \otimes A, A) \arrow[rr, "b^{-i-1*}"] & & \mathrm{Hom}_{A^e}(A \otimes A^{\otimes i+1} \otimes A, A) \arrow[u, "\cong", "\psi"'] \end{tikzcd}$$

commute can be described as

$$\begin{aligned} d^i(f)(a_1 \otimes \cdots \otimes a_{i+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ &\quad + \sum_{j=1}^{i} f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ &\quad + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}. \end{aligned} \tag{1.7}$$

Where the vertical maps come from the adjunction 2.1.5*, and b^{-i-1*} from Definition* 1.3*.*

Proof. We have

$$\begin{aligned} d^i(f) &= (\psi \circ b^{-i-1*} \circ \phi)(f) \\ &= (\psi \circ b^{-i-1*})(\mathrm{mult} \circ \mathrm{id}_A \otimes f \otimes \mathrm{id}_A) \\ &= \mathrm{mult} \circ \mathrm{id} \otimes f \otimes \mathrm{id} \circ b^{-i-1} \circ \iota_A \quad (*) \end{aligned}$$

where $\iota_A : a_1 \otimes \cdots \otimes a_{i+1} \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_{i+1} \otimes 1$. Then one easily verifies the lemma by applying $(*)$ to $a_1 \otimes \cdots a_{i+1}$ □

Corollary 2.1.6 and Lemma 2.1.7 lead us to the definition of the Hochschild cohomology that Hochschild first gave.

Definition 2.1.8. *(Hochschild) Let A be an algebra over k and let X be an A-A-bimodule. Then the Hochschild cochain complex, $(C^*(A, X), d^*)$ is defined* *to be*

$$\mathrm{Hom}_k(k,X) \xrightarrow{d^0} \mathrm{Hom}_k(A,X) \xrightarrow{d^1} \mathrm{Hom}_k(A\otimes A,X) \xrightarrow{d^2} \cdots \tag{1.8}$$

$$\cdots \xrightarrow{d^{i-1}} \mathrm{Hom}_k(A^{\otimes i},X) \xrightarrow{d^i} \cdots$$

The differential $d^i : \mathrm{Hom}_k(A^{\otimes i},A) \longrightarrow \mathrm{Hom}_k(A^{\otimes i+1},A)$ *is given by*

$$\begin{aligned}(d^i f)(a_1\otimes\cdots\otimes a_{i+1}) &= a_1 f(a_2\otimes\cdots\otimes a_{i+1})\\ &+\sum_{j=1}^{i}(-1)^j f(a_1\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_{i+1})\\ &+(-1)^{i+1} f(a_1\otimes\cdots\otimes a_i)a_{i+1}\end{aligned} \tag{1.9}$$

for $f\in\mathrm{Hom}_k(A^{\otimes i},A)$ *and* $a_1,\cdots,a_{i+1}\in A$*. The* i^{th} *Hochschild cohomology of* A *with coefficients in* X *is*

$$\mathrm{HH}^i(A,X) := \mathrm{Ker}(d^i)/\,\mathrm{Im}(d^{i-1}) \tag{1.10}$$

Now that we have seen a few ways of defining Hochschild cohomology, an obvious question to ask is: What information does it tell us? In general this seems to be a difficult question, but there are some very nice and straightforward descriptions of the low dimensional cohomology groups, which we sum up in the following proposition.

Proposition 2.1.9. *Let* A *be an associative algebra, then*

i) $\mathrm{HH}^0(A) = Z(A)$ *where* $Z(A)$ *is the center of the algebra.*

ii) $\mathrm{HH}^1(A) = Der(A)/Inn(A)$*.*

Here, $\mathrm{Der}(A)$ is the set of derivations on A. i.e. k-linear maps $f : A \to A$ such that $f(aa') = f(a)\cdot a' + a\cdot f(a')$, and $\mathrm{Inn}(A)$ is the set of inner derivations i.e. derivations f such that $f(a) = a\cdot x - x\cdot a$ for $x\in A$. (In particular $\mathrm{Inn}(A) = 0$ if A is commutative)

Proof. i) Using 2.1.8 and equation 1.9 we see that

$$\mathrm{Ker}(d^0) = \{f\in\mathrm{Hom}_k(k,A)|\ a\cdot f(1) = f(1)\cdot a \text{ for all } a\in A\}$$

ii) This follows from Equation 1.9 . □

We now show two examples demonstrating Lemma 2.1.9.

Example 2.1.10. Consider the associative algebra $\mathbb{C}[\zeta]/(\zeta^2)$ with ζ in (adams) degree 1. Since $\mathbb{C}[\zeta]/(\zeta^2)$ is commutative it follows that

$$\mathrm{HH}^0(\mathbb{C}[\zeta]/(\zeta^2)) = Z(\mathbb{C}[\zeta]/(\zeta^2)) \cong \mathbb{C}[\zeta]/(\zeta^2).$$

To calculate HH^1, let

$$f : \mathbb{C}[\zeta]/(\zeta^2) \longrightarrow \mathbb{C}[\zeta]/(\zeta^2),$$

be a derivation. Since $f(1) = f(1 \cdot 1) = 2f(1)$ it follows that $f(1) = 0$. Suppose that $f(\zeta) = \alpha \cdot 1 + \beta \cdot \zeta$ for some $\alpha, \beta \in \mathbb{C}$. Then

$$f(0) = f(\zeta^2) = \zeta f(\zeta) + f(\zeta)\zeta = (2\alpha + 2\beta\zeta)\zeta.$$

Thus $f(\zeta) = \beta\zeta$. It follows that $\mathrm{Der}(\mathbb{C}[\zeta]/\zeta)$ is spanned by the map g that sends $1 \mapsto 0$ and $\zeta \mapsto \zeta$. Since there are no inner derivations, we see that

$$\mathrm{HH}^1(\mathbb{C}[\zeta]/(\zeta^2)) = \mathrm{Der}(\mathbb{C}[\zeta]/(\zeta^2)) \cong \langle 1 \rangle \mathbb{C}$$

where $\langle 1 \rangle$ shifts the grading down by 1.

Example 2.1.11. Let $\mathbb{C}[x]$ be the associative polynomial algebra with x in (adams) degree 1. Since $\mathbb{C}[x]$ is commutative it follows that

$$\mathrm{HH}^0(\mathbb{C}[x]) = Z(\mathbb{C}[x]) \cong \mathbb{C}[x].$$

To calculate HH^1, let

$$f : \mathbb{C}[x] \longrightarrow \mathbb{C}[x],$$

be a derivation. We claim that $f = p\frac{\partial}{\partial x}$ for some $p \in \mathbb{C}[x]$. Set $p := f(x)$. Then since f is $\mathbb{C}$ linear we only have to check that

$$f(x^a) = p\frac{\partial}{\partial x}(x^a) \text{ for } a \geq 0. \tag{1.11}$$

Clearly Equation (1.11) is true for $a = 0, 1$, since f is a derivation and from the definition of p. Then by induction it follows that

$$\begin{aligned} f(x^a) &= f(x^{a-1}) \cdot x + x^{a-1} \cdot f(x) \\ &= (a-1)p \cdot x^{a-2} \cdot x + x^{a-1} \cdot p \\ &= ap \cdot x^{a-1} \\ &= p\frac{\partial}{\partial x}(x^a), \end{aligned}$$

for $a > 0$. Thus,

$$\mathrm{HH}^1(\mathbb{C}[x]) = \mathrm{Der}(\mathbb{C}[x]) \cong \langle 1 \rangle \mathbb{C}[x],$$

where $\langle 1 \rangle$ shifts the grading down by 1.

Remark 2.1.12. Clearly $\mathrm{Der}(A)$ has a Lie algebra structure, via taking the commutator. It is also easy to see that $\mathrm{Inn}(A)$ is a Lie ideal, hence $\mathrm{HH}^1(A)$ naturally carries a Lie algebra structure.

Things get more interesting when one looks at the Hochschild groups for $n = 2$ and $n = 3$ as the next remark will point out.

Remark 2.1.13. $\mathrm{HH}^2(A)$ controls infinitesimal deformations of the (graded) algebra A, where as $\mathrm{HH}^3(A)$ controls obstructions to higher level or formal deformations. See for instance [MS, Proposition 28] and also the reference [BeGi] given there.

With the description of the Hochschild cohomology in terms of Ext groups it is more or less clear that there is a graded associative algebra structure coming from the Yoneda product on Ext. More interestingly, Gerstenhaber showed that there is a product for cochains $c_1 \in C^p(A)$ and $c_2 \in C^q(A)$

$$c_1 \bullet c_2 = \sum_{i=0}^{p-1} (-1)^{i(q-1)} c_1 \bullet_i c_2 \tag{1.12}$$

where,

$c_1 \bullet_i c_2(a_1 \otimes \cdots \otimes a_{p+q-1}) = c_1(a_1 \otimes \cdots a_i \otimes c_2(a_{i+1} \otimes \cdots \otimes a_{i+q}) \otimes a_{i+q+1} \otimes \cdots \otimes a_{p+q-1})$. In [Ger] Gerstenhaber proved the following nice result.

Lemma 2.1.14. *Endowed with the bracket defined by*

$$[c_1, c_2]_G = c_1 \bullet c_2 - (-1)^{(p-1)(q-1)} c_2 \bullet c_1, \quad c_1 \in C^p(A), \ c_2 \in C^q(A)$$

$[1]C^*(A)$ *becomes a differential graded Lie algebra. In particular* $[1]\,\mathrm{HH}^*(A)$ *becomes a graded super Lie algebra (i.e. the commutator satisfies the super Jacobi identity)*

The shift [1] appearing in Lemma 2.1.14 shifts the cohomological degree down by 1 and also appropriately modifies the differential, note that if the differential is zero $\langle 1 \rangle$ and [1] are the same. The next example shows in the case of the polynomial ring, how the Gerstenhaber bracket extends the Lie algebra structure on HH^1. Indeed it is the case in general that for any algebra A the Gerstenhaber bracket extends the Lie algebra structure on $\mathrm{HH}^1(A)$.

Example 2.1.15. Consider $\mathbb{C}[x]$ with x in (adams) degree 1. The Gerstenhaber bracket for cocycles $f \in C^1(\mathbb{C}[x])$ and $g \in C^1(\mathbb{C}[x])$ is

$$\begin{aligned}
[f,g]_G(q) &= f\frac{\partial}{\partial x}(g\frac{\partial}{\partial x}(q)) - g\frac{\partial}{\partial x}(f\frac{\partial}{\partial x}(q)) \\
&= f\frac{\partial}{\partial x}(g\frac{\partial q}{\partial x}) - g\frac{\partial}{\partial x}(f\frac{\partial q}{\partial x}) \\
&= f\frac{\partial g}{\partial x}\frac{\partial q}{\partial x} + fg\frac{\partial^2 q}{\partial x^2} - g\frac{\partial f}{\partial x}\frac{\partial q}{\partial x} - gf\frac{\partial^2 q}{\partial x^2} \\
&= f\frac{\partial g}{\partial x}\frac{\partial q}{\partial x} - g\frac{\partial f}{\partial x}\frac{\partial q}{\partial x}.
\end{aligned}$$

In particular we see that the Gerstenhaber bracket when restricted to 1-cochains is precisely the Lie bracket on $\mathrm{Der}(\mathbb{C}[x])$ (i.e. the commutator).

It was shown by Getzler and Jones in [GJ, Section 5.2] that the bracket in Lemma 2.1.14 is part of a more general algebraic picture on $C^*(A)$. In their language this means that $C^*(A)$ is a B_∞-algebra.

Definition 2.1.16. *A* B_∞*-algebra structure on a chain complex* V *is the structure of a dg-bialgebra on the coalgebra* $T([1]V) = \bigoplus_{k \geq 0}([1]V)^k$.

We quickly recall what a dg-bialgebra is.

Definition 2.1.17. *A dg-bialgebra* (C, d, μ, Δ)*, consists of a graded* k*-module* C*, such that* (C, d, Δ) *is a dg-coalgebra (i.e.* d *cocommutes with* Δ*) with a morphism of dg-coalgebras* $\mu : C \otimes C \longrightarrow C$ *turning* (C, d, μ) *into a dg-algebra.*

A reader that is familiar with the theory of A_∞-algebras may recall that one description of an A_∞-algebra structure on a graded vector space V is that $T([1]V)$ has the structure of a dg-coalgebra. Thus, V having a B_∞-algebra structure means it also has an A_∞-structure.

2.2. **Koszul Algebras.**

In this subsection we will compute the Hochschild cohomology of $\mathbb{C}[x]$ and $\mathbb{C}[\zeta]/(\zeta^2)$, as they were defined in Section 2.1 (i.e. with only an adams grading). These two algebras were not chosen at random. $\mathbb{C}[x]$ is a Koszul algebra and $\mathbb{C}[\zeta]/(\zeta^2)$ is its Koszul dual. To compute the Hochschild cohomology of both algebras we will use an idea described by Van den Bergh [VdB, Section 3] which is roughly, that if A is a Koszul algebra then one can build a bimodule resolution of A from it's Koszul complex. First we will recall some basic definitions and facts about Koszul algebras.

Definition 2.2.1. *Let k be a field and suppose that, $A = \bigoplus_{i\geq 0} A_i$, is an adams graded algebra such that $A_0 = k$ and each A_i is finite dimensional over k. We say that A is Koszul if there exists a graded linear projective resolution of ${A_0}_A$. i.e.*

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A_0 \tag{2.1}$$

Such that $P^{-i} = (P^{-i})_i A$. See [BGS] *and* [MOS].

As the next Lemma will show, our two algebras $\mathbb{C}[x]$ and $\mathbb{C}[\zeta]/(\zeta^2)$ are indeed examples of Koszul algebras.

Lemma 2.2.2. *a) $\mathbb{C}[x]$ is a Koszul algebra.*

b) $\mathbb{C}[\zeta]/(\zeta^2)$ is a Koszul algebra.

Proof. a) One can build the following resolution:

$$0 \longrightarrow \langle -1\rangle \mathbb{C}[x] \xrightarrow{x\cdot} \mathbb{C}[x] \longrightarrow \mathbb{C} \tag{2.2}$$

b) One can build the following infinite resolution:

$$\cdots \xrightarrow{\zeta\cdot} \langle -2\rangle \mathbb{C}[\zeta]/(\zeta^2) \xrightarrow{\zeta\cdot} \langle -1\rangle \mathbb{C}[\zeta]/(\zeta^2) \xrightarrow{\zeta\cdot} \mathbb{C}[\zeta]/(\zeta^2) \longrightarrow \mathbb{C} \tag{2.3}$$

both of which are obviously linear. $\square$

We would like to point out that not all of the results in [BGS] hold for $\mathbb{C}[x]$, since it is infinite dimensional over $\mathbb{C}$. In particular the results of Section 2.12 in [BGS] require that the Koszul algebra, A, is finitely generated over A_0. However, [MOS] get rid of this assumption (See definition of positively graded algebra there).

Next we introduce the notion of quadratic algebra. This notion as we will see will allow us to really get our hands on Koszul algebras.

Definition 2.2.3. *A quadratic algebra is an (adams) graded algebra $A = \bigoplus_{i\geq 0} A_i$ such that $A_0 = k$ and A is generated over A_0 by A_1 with relations in degree two. All this to say A is of the form $T(V)/(R)$ where $T(V)$ is the tensor algebra of some k-vector space and (R) is an ideal generated by relations of degree 2.*

The next lemma while perhaps trivial, is an illustration of the general phenomenon that all Koszul algebras are quadratic.

Lemma 2.2.4. *a)* $\mathbb{C}[x]$ *is a quadratic algebra.*

b) $\mathbb{C}[\zeta]/(\zeta^2)$ *is a quadratic algebra.*

Proof. a) Indeed $\mathbb{C}[x]$ is generated over $\mathbb{C}$ by x and the quadratic relation is just the trivial one.

b) Clearly $\mathbb{C}[\zeta]/(\zeta^2)$ is generated over $\mathbb{C}$ by ζ, and the relation $\zeta^2 = 0$ is quadratic. □

The following result can be found in [BGS, Proposition 2.9.1].

Theorem 2.2.5. *Suppose A is a Koszul algebra then it is also quadratic.*

With Theorem 2.2.5 in mind, it is natural to ask: Is there a characterization of which quadratic algebras are Koszul? Mazorchuk and Stroppel have given such a characterization in [MS, Theorem 30].

Next we introduce the notion of quadratic dual, it is a relatively straight forward construction with the upshot that the so called Koszul dual algebra of a Koszul algebra can be described as its quadratic dual.

Definition 2.2.6. *Let be a $A = T(V)/(R)$ be a quadratic algebra. We define $A^! = T(V^*)/(R^\perp)$ where $R^\perp := \{f \in V^* \otimes V^* | \ f(R) = 0\}$ to be the quadratic to dual of A.*

Remark 2.2.7. Alternatively (as is done in [MOS]), one can define the quadratic dual more generally as (if dim $A_1 < \infty$), $A^! = T(A_1^*)/\mathrm{Im}(m^*)$, where

$$\begin{array}{rcl} m : A_1 \otimes A_1 & \longrightarrow & A_2 \\ a \otimes a' & \mapsto & aa'. \end{array}$$

In particular for a quadratic algebra it is immediate from the definition that $A^{!^!} = A$. The next remark will show that if we use the general definition described in Remark 2.2.7 for algebras that are not quadratic, then taking the quadratic dual twice will not necessarily give back the original algebra.

Remark 2.2.8. Let $A = \bigoplus_{i\geq 0} A_i$ (not necessarily quadratic!) such that dim $A_1 < \infty$, using the definition of quadratic dual in Remark 2.2.7, $A^! = T(A_1^*)/\mathrm{Im}(m^*)$. It could happen that the two algebras have the same quadratic dual. For example let, $C_n = \mathbb{C}[x]/(x^n)$ for $n \geq 3$, then $C_n^! = \mathbb{C}[y]/\mathrm{Im}(m^*)$ where $y = x^*$. But then

$$\begin{array}{rcl} m^*(y)(1 \otimes 1) & = & 0 \\ m^*(y)(x \otimes 1) & = & 1 \\ m^*(y)(1 \otimes x) & = & 1 \\ m^*(y^2)(x \otimes x) & = & 1. \end{array}$$

In particular $y \otimes 1 - 1 \otimes y = 0$. It follows thats $\mathbb{C}[y]/\mathrm{Im}(m^*) \cong \mathbb{C}[y]/(y^2)$ for all $n \geq 3$.

The next lemma shows that our two Koszul algebras $\mathbb{C}[x]$ and $\mathbb{C}[\zeta]/(\zeta^2)$ are quadratic dual to each other.

Lemma 2.2.9. *a)* $\mathbb{C}[x]^! = \mathbb{C}[\zeta]/(\zeta^2)$.

b) $\mathbb{C}[\zeta]/(\zeta^2)^! = C[x]$.

Proof. a) We set $\zeta = x^*$ and since $\mathbb{C}[x]$ is a quadratic algebra with the trivial relation then $R^\perp := \{\alpha \cdot \zeta \otimes \zeta | \ \alpha \in \mathbb{C}\}$.

b) Follows from the definition of quadratic dual. □

We now define the so called Koszul complex associated to a Koszul algebra A. It will give us a linear projective resolution of A_0, and as we will see, turns out to be a construction of incredible importance.

Definition 2.2.10. *Suppose A is a Koszul algebra, the Koszul complex of A is*

$$\cdots \xrightarrow{e\cdot} (A^!_2)^* \otimes A \xrightarrow{e\cdot} (A^!_1)^* \otimes A \xrightarrow{e\cdot} (A^!_0)^* \otimes A \tag{2.4}$$

Where

$$e := \sum_{i=1}^{n} \eta_i \otimes x_i \in A^! \otimes A \tag{2.5}$$

where $\{x_i\}_{i=1}^n$ is a basis for A_1 and $\{\eta_i\}_{i=1}^n$ the dual basis of A_1^*.

Explicitly what is going on is, for $z^* \otimes y \in (A^!_j)^* \otimes A$

$$\begin{aligned} e \cdot (z^* \otimes y) &:= \sum_{i=1}^{n} \eta_i \cdot z^* \otimes x_i \cdot y \\ &:= \sum_{i=1}^{n} z^*(? \cdot \eta_i) \otimes x_i y \in (A^!_{j-1})^* \otimes A \end{aligned} \tag{2.6}$$

It is shown in [BGS, Theorem 2.6.1] that (2.4) is a graded linear projective resolution of k_A and is the main ingredient in the proof of the following theorem.

Theorem 2.2.11. [BGS, Theorem 2.10.1] *If A is a Koszul algebra then $A^!$ is also a Koszul algebra.*

For a Koszul algebra A, we call the algebra $\mathrm{Ext_A}^*(A_0, A_0)$ (Where Ext is computed in the ungraded category) the Koszul dual of A. The next important theorem describes the relationship between the Koszul dual of A and its quadratic dual.

Theorem 2.2.12. *Let A be a Koszul algebra, then $A^! \cong \mathrm{Ext_A}^*(A_0, A_0)$*

Proof. Full proofs can be found in either [BGS] or [MS, Prop 17] but the idea is to show that the left action of $A^!$ on the Koszul complex, K, induces a quasi-isomorphism between $A^!$ and $\mathrm{End}_A(K)$. □

We now turn our attention to building a graded A-A-bimodule resolution out of the Koszul complex. Following [VdB] we define the following two complexes

$(K'(A), d_L)$ and $(K'(A), d_R)$, with the same objects but with two different differentials.

$$\cdots \underset{d_L}{\overset{d_R}{\rightrightarrows}} A \otimes (A^!_2)^* \otimes A \underset{d_L}{\overset{d_R}{\rightrightarrows}} A \otimes (A^!_1)^* \otimes A \underset{d_L}{\overset{d_R}{\rightrightarrows}} A \otimes (A^!_0)^* \otimes A \tag{2.7}$$

Where

$$\begin{aligned} d_R(r \otimes f \otimes s) &= \sum_{i=1}^{n} x_i \cdot r \otimes \eta_i \cdot f \otimes s = \sum_{i=1}^{n} x_i r \otimes f(? \cdot \eta_i) \otimes s \\ d_L(r \otimes f \otimes s) &= \sum_{i=1}^{n} r \otimes f \cdot x_i \otimes s \cdot \eta_i = \sum_{i=1}^{n} r \otimes f(\eta_i \cdot ?) \otimes s x_i \end{aligned} \tag{2.8}$$

We check that $d_R^2 = 0$.

$$d_R(\sum_{i=1}^{n} x_i \cdot r \otimes \eta_i \cdot f \otimes s) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_j \cdot x_i \cdot r \otimes \eta_j \cdot \eta_i \cdot f \otimes s \right) = 0 \tag{2.9}$$

holds because, when $i \neq j$ the, $\eta_i \cdot \eta_j = 0$ and when $i = j$, then either $x_i^2 = 0$ or $\eta_i^2 = 0$ by definition of the quadratic dual and so $d_R^2 = 0$. In a similar way one sees that $d_L^2 = 0$. Define a new differential

$$d^i := d_R + (-1)^i d_L.$$

Since

$$d_R d_L = d_L d_R,$$

and

$$d_R^2 = 0 = d_L^2.$$

It follows that

$$\cdots \xrightarrow{d} A \otimes (A^!_2)^* \otimes A \xrightarrow{d} A \otimes (A^!_1)^* \otimes A \xrightarrow{d} A \otimes (A^!_0)^* \otimes A \tag{2.10}$$

is a chain complex which we denote as $(K'(A), d)$.

One sees that $(K'(A), d_R)$ is an A^e-resolution of $k \otimes_A A$ and $(K'(A), d_L)$ is an A^e-resolution of $A \otimes_A k$ and by totalizing the differentials we obtain an A^e-resolution A with the its canonical diagonal bimodule structure.

Noting that $A^!_0 = k$ we define $\epsilon : A \otimes A \to A$ such that $\epsilon(a \otimes a') = aa'$

Proposition 2.2.13. *Suppose that A is Koszul. Then the complex*

$$\cdots \xrightarrow{d} A \otimes (A^!_1)^* \otimes A \xrightarrow{d} A \otimes (A^!_0)^* \otimes A \xrightarrow{\epsilon} A \longrightarrow 0 \tag{2.11}$$

is exact. Hence $(K'(A), d)$ is a free A^e-resolution of A.

Proof. The complex (2.11) is a complex of free left A modules and by tensoring with $k_A \otimes_A ?$ one recovers (2.4) which is exact. Then it follows from the graded version of Nakayama's lemma that (2.11) is exact as well. □

Remark 2.2.14. It follows that the complex (2.11) is quasi-isomorphic to the bar resolution, since both are quasi-isomorphic to A. But in fact one can show that (2.11) can be embedded as a subcomplex in the bar resolution, see for example [Pri].

Recalling Definition 2.1.3, we now would like to compute $\mathrm{HH}^*(\mathbb{C}[x])$ and $\mathrm{HH}^*(\mathbb{C}[\zeta]/(\zeta^2))$ using the resolution provided in Proposition 2.2.13. We will see in the case of $\mathbb{C}[x]$, it is finite and in the case of $\mathbb{C}[\zeta]/(\zeta^2)$, it is periodic.

Lemma 2.2.15. *The Hochschild cohomology for* $\mathbb{C}[x]$ *is*

$$\mathrm{HH}^i(\mathbb{C}[x]) \quad \cong \quad \begin{cases} \langle i\rangle\mathbb{C}[x] & for\ i=0,1 \\ 0 & otherwise \end{cases} \tag{2.12}$$

In particular, all the Hochschild cohomology groups are already determined by Example 2.1.11.

Proof. Since $\mathbb{C}[x]$ is Koszul by Lemma 2.2.2, Proposition 2.2.13 gives us

$$0 \longrightarrow \mathbb{C}[x]\otimes\mathbb{C}\langle\zeta\rangle^*\otimes\mathbb{C}[x] \xrightarrow{d} \mathbb{C}[x]\otimes\mathbb{C}[x] \xrightarrow{\epsilon} \mathbb{C}[x]$$

Where $\mathbb{C}\langle\zeta\rangle$ is the $\mathbb{C}$-vector space spanned by ζ. Notice now that

$$\begin{aligned} d(r\otimes s\otimes f) &= d_R(r\otimes s\otimes f) - d_L(r\otimes s\otimes f) \\ &= xr\otimes\zeta s\otimes f - r\otimes sx\otimes f \end{aligned} \tag{2.13}$$

Now applying $\mathrm{Hom}_{\mathbb{C}[x]^e}(?,\mathbb{C}[x])$ we have the complex

$$\mathrm{Hom}_{\mathbb{C}[x]^e}(\mathbb{C}[x]\otimes\mathbb{C}[x],\mathbb{C}[x]) \xrightarrow{d^*} \mathrm{Hom}_{\mathbb{C}[x]^e}(\mathbb{C}[x]\otimes\mathbb{C}\langle\zeta\rangle^*\otimes\mathbb{C}[x],\mathbb{C}[x]) \longrightarrow 0 \tag{2.14}$$

It follows from 2.13 and the fact that $\mathbb{C}[x]\otimes\mathbb{C}[x]$ is a commutative algebra that $d^*=0$. Then using the adjunction between restriction and extension of scalars (2.1.5) it follows that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}[x]^e}(\mathbb{C}[x]\otimes\mathbb{C}[x],\mathbb{C}[x]) &\cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}[x]) \cong \mathbb{C}[x], \\ \mathrm{Hom}_{\mathbb{C}[x]^e}(\mathbb{C}[x]\otimes\mathbb{C}\langle\zeta\rangle^*\otimes\mathbb{C}[x],\mathbb{C}[x]) &\cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\langle\zeta\rangle^*,\mathbb{C}[x]) \cong \langle 1\rangle\mathbb{C}[x]. \end{aligned}$$

The claim follows. ☐

Remark 2.2.16. There are many other ways to compute the Hochschild cohomology of $\mathbb{C}[x]$. One possible way is to use the Koszul complex in the sense of [Wei] from algebraic geometry. Also, one can use the Künneth theorem to compute the Hochschild cohomology of $\mathbb{C}[x_1,\ldots,x_n]$ inductively once the result has been established for $\mathbb{C}[x]$. i.e. it gives

$$\mathrm{HH}^i(\mathbb{C}[x_1,\ldots,x_n]) \cong \bigoplus_{a+b=i} \mathrm{HH}^a(\mathbb{C}[x_1,\cdots,x_{n-1}])\otimes\mathrm{HH}^b(\mathbb{C}[x_n]).$$

Lastly, there is the Hochschild, Kostant, Rosenberg Theorem, see [Gin], that says for a smooth algebra A, $\mathrm{HH}^i(A)\cong\Lambda^i_A\mathrm{Der}(A)$.

Lemma 2.2.17. *The Hochschild cohomology for* $\mathbb{C}[\zeta]/(\zeta^2)$ *is*

$$\mathrm{HH}^i(\mathbb{C}[\zeta]/(\zeta^2)) \quad \cong \quad \begin{cases} \mathbb{C}[\zeta]/(\zeta^2), & i=0 \\ \langle i\rangle\mathbb{C}, & i>0 \end{cases} \tag{2.15}$$

Proof. It follows from Lemma 2.2.2 that $\mathbb{C}[\zeta]/(\zeta^2)$ is a Koszul algebra so by 2.2.13 we have a resolution

$$\cdots \xrightarrow{d} \mathbb{C}[\zeta]/(\zeta^2)\otimes\mathbb{C}\langle x^i\rangle\otimes\mathbb{C}[\zeta]/(\zeta^2) \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{C}[\zeta]/(\zeta^2)\otimes\mathbb{C}[\zeta]/(\zeta^2)$$

Applying $\mathrm{Hom}_{\mathbb{C}[\zeta]/(\zeta^2)^e}(?, \mathbb{C}[\zeta]/(\zeta^2))$ gives

$$\mathrm{Hom}_{\mathbb{C}[\zeta]/(\zeta^2)^e}(\mathbb{C}[\zeta]/(\zeta^2) \otimes \mathbb{C}[\zeta]/(\zeta^2), \mathbb{C}[\zeta]/(\zeta^2)) \xrightarrow{d^*} \cdots$$

$$\xrightarrow{d^*} \mathrm{Hom}_{\mathbb{C}[\zeta]/(\zeta^2)^e}(\mathbb{C}[\zeta]/(\zeta^2) \otimes \mathbb{C}\langle x^i \rangle \otimes \mathbb{C}[\zeta]/(\zeta^2), \mathbb{C}[\zeta]/(\zeta^2)) \xrightarrow{d^*} \cdots$$

We see that $d^i = 0$ for i odd and by the adjointess between restriction and extension of scalars we see that for all i

$$\begin{aligned} & \mathrm{Hom}_{\mathbb{C}[\zeta]/(\zeta^2)^e}(\mathbb{C}[\zeta]/(\zeta^2) \otimes \mathbb{C}\langle x^i \rangle \otimes \mathbb{C}[\zeta]/(\zeta^2), \mathbb{C}[\zeta]/(\zeta^2)) \\ \cong \ & \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\langle x^i \rangle, \mathbb{C}[\zeta]/(\zeta^2)) \\ \cong \ & \langle i \rangle \mathbb{C}[\zeta]/(\zeta^2) \end{aligned}$$

Thus via adjointess we see that $d^i = 2\zeta$ for i even. $\square$

As we have seen from Lemma 2.2.15 and Lemma 2.2.17, $\mathrm{HH}^*(A)$ and $\mathrm{HH}^*(A^!)$ are in general not isomorphic. But, if we work in the dg-setting and define the so called dg-Koszul dual, we will then obtain one of the main theorems of this thesis which is due to Keller that

$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(A^{!^{dg}})$$

as graded algebras. We will therefore now pass to the dg-setting.

3. The DG World

According to [Drin], the notion of dg-category was introduced at least as early as 1964. One of the reasons that dg-categories have become popular is that by passing to the homotopy category of chain complexes one loses a lot information and after the work of Bondal and Kapranov many people believe, one should work with so called pre-triangulated dg-categories instead of triangulated categories when defining derived categories and taking quotients. We will not say any more about this, but the reader is encouraged to take look at [Drin].

More recently, another place where dg-categories play a fundamental role is in the theory of noncommutative motives in the sense of Kontsevich and Tabuada, see [Tab]. Roughly, one description of the category of noncommutative motives is that it is a category whose objects are dg-categories and whose morphism spaces between two objects are obtained by taking the Grothendieck group of a triangulated category coming from bimodules over both objects. Again, we will not say anymore than this, but the reader is encouraged to look at [Tab] for a readable introduction to this topic.

Coming back down to earth, a thought that is useful to keep in mind while reading this section that dg-categories are simply k-linear categories enriched over chain complexes.

3.1. **Graded Categories and DG Categories.**

We will first introduce the notion of a graded category. Most of the definitions and lemmas stated in this section can be found in [Kel1]. Recall that a graded k-algebra, A, is a k-algebra which is graded as a k-module such that $A^iA^j \subseteq A^{i+j}$. A graded category, as we will see, can be viewed as a generalization of a graded algebra i.e. a graded algebra with 'many objects'.

Definition 3.1.1. *A $\mathscr{A}$ be a k-linear category. We say that $\mathscr{A}$ is a graded category if $\mathscr{A}(a,b)$ is a graded k-module for all $a,b \in \mathscr{A}$ and the composition of morphisms is given by a k-linear map of degree zero:*

$$\circ : \mathscr{A}(b,c) \otimes \mathscr{A}(a,b) \longrightarrow \mathscr{A}(a,c) \tag{1.1}$$

$$f \otimes g \mapsto f \circ g \tag{1.2}$$

where degree zero means, $\circ : \mathscr{A}(b,c)^i \otimes \mathscr{A}(a,b)^j \longrightarrow \mathscr{A}(a,c)^{i+j}$.

Definition 3.1.2. *Let* Gra k *be the category whose objects are graded k-modules and morphism space for two objects U, V is*

$$(\text{Gra } k)(V,W) = \bigoplus_{i\in\mathbb{Z}} (\text{Gra } k)(V,W)^i \tag{1.3}$$

where

$$(\text{Gra } k)(V,W)^i = \{f \in \text{Hom}_k(V,W) | f(V^j) \subset W^{i+j}\} \tag{1.4}$$

Thus the morphism space for two objects V, W is a graded k-module and composition of morphisms is just the composition of maps.

Example 3.1.3. It follows from the definition that Gra k is a graded category.

The next example shows precisely in what sense graded categories generalize graded algebras.

Example 3.1.4. Suppose A is a graded k-algebra. Then we can view A as a graded category, which we will denote as $\mathcal{A}$, with one object $\bullet$ such that $\mathcal{A}(\bullet, \bullet) = A$. The composition law is given by multiplication in the algebra A. Conversely any graded category with one object can be viewed as a graded algebra where the composition law induces and algebra structure on the endomorphism space of the lone object.

The next example shows that for graded categories, taking the opposite category also produces a graded category.

Example 3.1.5. Given a graded category $\mathscr{A}$ then its graded opposite, $\mathscr{A}^{\mathrm{op}}$ is a graded category where the composition becomes

$$\begin{aligned} \circ_{\mathrm{op}} : \mathscr{A}^{\mathrm{op}}(b,c)^i \otimes \mathscr{A}^{\mathrm{op}}(a,b)^j &\longrightarrow \mathscr{A}^{\mathrm{op}}(a,c)^{i+j} \\ g \otimes f &\longmapsto g \circ_{\mathrm{op}} f = f \circ g \end{aligned} \tag{1.5}$$

which is easily seen to be a graded homogeneous map of degree 0.

When we are working with just (adams) graded objects Example 3.1.5 will suffice. However if we are working with differentially graded categories we have to use a slightly modified version of Example 3.1.5. The reason for this is best illustrated by thinking about chain complexes. Suppose (V, d_V) and (W, d_W) are chain complexes. The tensor product of these two complexes is a totalization of a bicomplex and thus introduces signs into the formula for the differential on $V \otimes W$ (resp. $W \otimes V$). In particular the graded map

$$\tilde{\tau} : V \otimes W \to W \otimes V, \tag{1.6}$$

that sends $v \otimes w$ to $w \otimes v$ in general wont respect the differential and we see that if we make the following modification

$$\begin{aligned} \tau : V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\mapsto (-1)^{|v|\,|w|} w \otimes v, \end{aligned} \tag{1.7}$$

then this map is indeed a morphism of chain complexes and this leads us to very naturally define the opposite dg-category as

Example 3.1.6. Given a graded category $\mathscr{A}$ then its dg-opposite $\mathscr{A}^{op}$ is a graded category, where the composition law naturally becomes

$$\begin{aligned} \circ_{\mathrm{op}} : \mathscr{A}^{op}(b,c)^i \otimes \mathscr{A}^{op}(a,b)^j &\longrightarrow \mathscr{A}^{op}(a,c)^{i+j} \\ g \otimes f &\mapsto (-1)^{|f|\cdot|g|} g \circ_{\mathrm{op}} f = (-1)^{|f|\cdot|g|} g \circ f. \end{aligned}$$

For now we can use which ever definition of $\mathscr{A}^{\mathrm{op}}$ we wish but after we define what a dg-category is we will always mean the dg-opposite and not the graded one.

Definition 3.1.7. *A graded functor is a functor, $F : \mathscr{A} \to \mathscr{B}$, between graded categories such that*

$$F(a,b) : \mathscr{A}(a,b) \to \mathscr{B}(F(a), F(b))$$

is k-linear and homogeneous of degree zero for all $a, b \in \mathscr{A}$.

The information of a module M over a ring R can be encoded as a functor from the category with one object whose endomorphism space is just R to the category of abelian groups. We use this idea to define what a module over a (graded, dg) category should be.

Definition 3.1.8. *A (right) graded $\mathscr{A}$-module is a graded functor $M : \mathscr{A}^{op} \to$ Gra k. A left graded $\mathscr{A}$-module is a graded functor $M : \mathscr{A} \to$ Gra k.*

Definition 3.1.9. *Let M and N be graded modules over $\mathscr{A}$. We say that $N \subseteq M$ is a submodule if $N(a) \subseteq M(a)$ is a submodule for all $a \in \mathscr{A}$.*

Example 3.1.10. Let A be a graded k-algebra. Suppose that M is a graded module over A. We can view M as a graded module over $\mathcal{A}$ as in 3.1.4. As follows: define $\mathcal{M} : \mathcal{A}^{op} \to$ Gra k, such that $\bullet \mapsto M_k$ where M_k is the underlying graded k-module of M and $\mathcal{M}(a)(m) = m \cdot a$ for $a \in \mathcal{A}(\bullet, \bullet)$, $m \in M_k$. Conversely if $\mathscr{A}$ is a graded category with one object and N is a graded $\mathscr{A}$-module then we can view N as a graded module over the algebra $\mathscr{A}(\bullet, \bullet)$.

Definition 3.1.11. *For each $a \in ob(\mathscr{A})$ we define the right module represented by a as*

$$\hat{a} := \mathscr{A}(-, a).$$

Sometimes $\hat{a}$ is referred to as an indecomposable projective module see for example [MOS, 2.3.4] or free module see [Kel1, 1.2].

Definition 3.1.12. *Let M and N be two graded $\mathscr{A}$-modules, a graded homomorphism from M to N over $\mathscr{A}$ is defined to be a graded natural transformation, τ, from M to N, i.e. a family of degree zero k-linear maps $\{\tau_a\}_{a \in \mathscr{A}^{op}}$ such that for any objects $a, a' \in \mathscr{A}^{op}$ and morphism $f \in \mathscr{A}^{op}(a, a')$ the diagram*

$$\begin{tikzcd} M(a) \arrow[r, "\tau_a"] \arrow[d, "M(f)"] & N(a) \arrow[d, "N(f)"] \\ M(a') \arrow[r, "\tau_{a'}"] & N(a') \end{tikzcd}$$

commutes.

One defines a morphism of left graded $\mathscr{A}$-modules in an analogous way.

Example 3.1.13. *Suppose that $f : M \to N$ is a graded homomorphism (i.e. k-linear and of degree zero) of A-modules. Then we can view it as an $\mathcal{A}$-module homomorphism from $\mathcal{M}$ to $\mathcal{N}$ where $\mathcal{A}$ is in the sense of* 3.1.4 *and $\mathcal{M}$ and $\mathcal{N}$ are defined in* 3.1.10. *Vice versa, if we have a category with one object then a graded homomorphism of modules over this category can be viewed as a graded homomorphism of modules over the algebra associated to the set of endomorphism of the lone object.*

Definition 3.1.14. *Let $\mathcal{G}\mathscr{A}$ be the category whose objects are graded $\mathscr{A}$-modules and whose morphism spaces $(\mathcal{G}\mathscr{A})(M, N)$ are graded $\mathscr{A}$-module homomorphisms as defined in* 3.1.12.

We endow $\mathcal{G}\mathscr{A}$ with a shift functor $M \mapsto \langle 1 \rangle M$, where $(\langle 1 \rangle M(a))^p = (M(a))^{p+1}$. For a morphism $f : M \to N$,

$$\langle 1 \rangle f_a : \langle 1 \rangle M(a) \longrightarrow \langle 1 \rangle N(a)$$

explicitly, $\langle 1\rangle f_a^p = f_a^{p+1}$ (where $f_a^p : M(a)^p \to N(a)^p$).

Definition 3.1.15. *We define the category* Gra $\mathscr{A}$ *to be the category with the same objects as* $\mathcal{G}\mathscr{A}$ *and with morphism spaces:*

$$(\text{Gra } \mathscr{A})(M, N) = \textstyle\coprod_{p\in\mathbb{Z}} (\mathcal{G}\mathscr{A})(M, \langle p\rangle N)$$

where composition of morphisms produced by $f : M \to \langle q\rangle N$ *and* $g : L \to \langle p\rangle M$ *is given by* $\langle p\rangle f \circ g$.

Lemma 3.1.16. *Suppose* $\mathscr{A}$ *is a graded category. Then for every* $a \in \mathscr{A}$ *and* $M \in$ Gra $\mathscr{A}$ *there is a canonical isomorphism*

$$(\text{Gra } \mathscr{A})(\hat{a}, M) \cong M(a), f \mapsto (f_a)(id_a) \tag{1.8}$$

of graded k*-modules.*

Proof. This is a direct consequence of the Yoneda lemma. $\square$

Remark 3.1.17. Since the shift functor on $\mathcal{G}\mathscr{A}$ can be extended to Gra $\mathscr{A}$. It follows that Gra $\mathscr{A}$ is a $\mathbb{Z}$-category in the sense of [AJS, Appendix E].

Recall that a dg-algebra is a graded algebra with a differential, d, which is a k-linear map of degree 1 such that $d^2 = 0$ and satisfies the graded Leibniz rule (i.e. $d(a \cdot a') = d(a) \cdot a' + (-1)^{|a|} a \cdot d(a')$). A dg-category is a generalization of this as we will see.

Definition 3.1.18. *A dg-category,* $\mathscr{A}$*, is a graded category whose morphism spaces are endowed with differentials,* d*, which are* k*-linear maps of degree 1 such that* $d^2 = 0$*. We also require that the differentials satisfy the graded Leibniz rule. i.e.*

$$d(fg) = d(f)g + (-1)^p f d(g) \textit{ for all } f \in \mathscr{A}(B, C)^p \textit{ and } g \in \mathscr{A}(A, B)$$

Definition 3.1.19. *Let* Dif k *be the category whose objects are dg* k*-modules i.e. chain complexes of* k*-modules, where the morphism spaces are* (Dif $k)(V, W) =$ (Gra $k)(V, W)$ *and are endowed with the differential such that for a homogeneous element* $f \in$ (Gra $k)(V, W)$

$$d(f) = d_V f - (-1)^{|f|} f d_W \tag{1.9}$$

Note that this is not the category of chain complexes since the morphisms don't have to commute with the differentials and we take maps of all degrees. Definition 3.1.19 however, gives our favorite example of a dg-category:

Lemma 3.1.20. Dif k *as defined in* 3.1.19*, is a dg-category.*

Proof. First we check that $d^2 = 0$. Applying Equation 1.9 twice to a homogeneous morphism f gives

$$\begin{aligned} d^2(f) &= d(d_V f - (-1)^{|f|} f d_W) \\ &= d_V^2 f - (-1)^{|f|+1} d_V f d_W + (-1)^{|f|+1}(d_V f d_W - (-1)^{|f|+1} f d_W^2) \\ &= 0 \end{aligned}$$

Then the only slightly non obvious thing left to check is that the differential satisfies the graded Leibniz rule in Definition 3.1.18. Applying Equation (1.9) to fg gives the formula

$$d(fg) = d_V fg - (-1)^{|f|\,|g|} fg d_W \tag{1.10}$$

then making the the following substitutions

$$d_V f = d(f) + (-1)^{|f|} f d_W \tag{1.11}$$

$$g d_W = (-1)^{|g|}(d(g) - d_V g)$$

into Equation (1.10) gives the graded Leibniz rule. □

Example 3.1.21. Suppose that $\mathscr{A}$ is a dg-category. Then taking the dg-opposite $\mathscr{A}^{op}$ naturally becomes a dg-category with the same differential as $\mathscr{A}$. This is precisely because of the sign rule introduced in the definition of the dg-opposite.

We want to stress that from now on unless stated otherwise we will always mean by opposite the dg-opposite.

Remark 3.1.22. Note that if we put everything in cohomological degree 0 and set the differential $d = 0$ the two notions of opposite are the same.

As an analogue to 3.1.4 a dg-algebra can be viewed as a dg-category with one object, where the differential comes from the differential on A. Conversely a dg-category with one object can be viewed as a dg-algebra. Next we define the notion of a dg-functor between dg-categories which will allow us to define dg-modules over dg-categories.

Definition 3.1.23. *Let $\mathscr{A}$ and $\mathscr{B}$ be dg-categories, a dg-functor $F : \mathscr{A} \to \mathscr{B}$ is a graded functor that commutes with the differential i.e. $F(df) = dF(f)$ for all morphisms $f \in \mathscr{A}$.*

Thus, just as in the case of graded categories we have the following definition of a module over a dg-category.

Definition 3.1.24. *a) A (right) dg-$\mathscr{A}$-module is a dg-functor $M : \mathscr{A}^{op} \to \mathrm{Dif}\ k$*
b) A left dg-$\mathscr{A}$-module is a dg-functor $M : \mathscr{A} \to \mathrm{Dif}\ k$.

With the notion of dg-module in hand it is very natural to define what a submodule should be.

Definition 3.1.25. *Let M and N be dg-modules over $\mathscr{A}$. We say that N is a submodule of M, if $N(a) \subseteq M(a)$ is a submodule for all $a \in \mathscr{A}$.*

There is also a differential graded version of Example 3.1.10 which will be a main protagonist in much of what we do later.

Definition 3.1.26. *Let $\mathrm{Dif}\ \mathscr{A}$ be the dg-category whose objects are $\mathscr{A}$ modules and whose morphism spaces are:*

$$(\mathrm{Dif}\ \mathscr{A})(M, N) = (\mathrm{Gra}\ \mathscr{A})(M, N)$$

with differential given by:

$$df = d_N \cdot f - (-1)^{|f|} f \cdot d_M \tag{1.12}$$

The category $\mathrm{Dif}\ \mathscr{A}$ has a dg-shift i.e. auto-equivalence which we shall denote as

$$[1] : \mathrm{Dif}\ \mathscr{A} \longrightarrow \mathrm{Dif}\ \mathscr{A} \tag{1.13}$$

such that $([1]M(a))^i = (M(a))^{i+1}$, $d_{[1]M} = -d_M$ and for a morphism, $f : M \longrightarrow N$, of $\mathscr{A}$-modules

$$[1]f_a : [1]M(a) \longrightarrow N(a) \tag{1.14}$$

Where $[1]f_a^p = f_a^{p+1}$. It is with this shift that the derived category of modules over $\mathscr{A}$ becomes a triangulated category. In the sequel we will use this notation also for the shift functor in an abstract triangulated category and we hope that it will not cause the reader confusion.

There is a dg analogue of lemma 3.1.16.

Lemma 3.1.27. *Let $\mathscr{A}$ be a dg-category, for $a \in \mathscr{A}$ and M an $\mathscr{A}$-module. There is a natural isomorphism of dg-k-modules*

$$(\text{Dif } \mathscr{A})(\hat{a}, M) \cong M(a),\ f \mapsto f_a(id_a)$$

Proof. This is a direct consequence of the Yoneda lemma. $\square$

We would like to introduce the notion of a bimodule. In the case of modules over an algebra, an R'-R-bimodule structure on X can naturally be given the structure of a right $R'^{op} \otimes R$ module. The situation can be a slightly subtle in the dg setting, so we will ruminate on it for a little while. Suppose that X is a dg-A'-A-bimodule for dg-algebras A' and A. Then the corresponding right action of $A'^{\text{op}} \otimes A$ is induced by the composition

$$\begin{array}{ccccccc} X \otimes A'^{\text{op}} \otimes A & \xrightarrow{\tau \otimes \text{id}} & A' \otimes X \otimes A & \xrightarrow{\lambda \otimes \text{id}} & X \otimes A & \xrightarrow{\text{id} \otimes \rho} & X \\ & & & & & & \\ x \otimes a' \otimes a & \longmapsto & (-1)^{|x|\ |a'|} a' \otimes x \otimes a & \longmapsto & (-1)^{|x|\ |a'|} a' x \otimes a & \longmapsto & (-1)^{|x|\ |a'|} a' x a \end{array} \tag{1.15}$$

Where τ is the map from (1.7) and λ (resp. ρ) is the left (resp. right) action of A' (resp. A) on X. Taking $B := A'^{\text{op}} \otimes A$ and by abuse of notation ρ to be the composition of morphism defined in (1.15), one can then check that the diagram

$$\begin{tikzcd} X \otimes B \otimes B \arrow[r, "\rho \otimes \text{id}"] \arrow[d, "\text{id} \otimes m_B"'] & X \otimes B \arrow[d, "\rho"] \\ X \otimes B \arrow[r, "\rho"] & X \end{tikzcd}$$

commutes (don't forget all the signs!). Another thing that we would like to point out is that, while our dg-algebras A and A' might be commutative, in general it is not true $A'^{\text{op}} \otimes A$ must be commutative. We will see an instance of this in Example 3.1.29.

Example 3.1.28. Let $\Lambda(x)$ be the ring $\mathbb{C}[x]/(x^2)$, where we view it as an adams graded dg-algebra by putting x in adams degree -1 and homological degree 1 and taking the differential $d = 0$. Notice first that $\Lambda(x)^{\text{op}} = \Lambda(x)$ since $x \cdot_{\text{op}} x = 0 = x \cdot x$. Thus the natural bimodule structure on $\Lambda(x)$ corresponds to a right $\Lambda(x) \otimes \Lambda(x)$-module. By abuse of notation if we set y equal to $x \otimes 1$ and x equal to $1 \otimes x$, then $\Lambda(x) \otimes \Lambda(x)$ is just the (adams graded) dg-algebra $\mathbb{C}[x, y]/(x^2, y^2)$ where x and y have adams degree -1 and homological degree $+1$ and the differential is 0. In the sequel we will write this algebra as $\Lambda(x, y)$.

Example 3.1.29. Let $\mathbb{C}[x]^{dg}$ be the polynomial algebra in one variable such that x has adams degree -1, homological degree 1 and set $d = 0$. In contrast with Example 3.1.28, $\mathbb{C}[x]^{dg}$ is not equal to its opposite since $x \cdot_{\mathrm{op}} x = (-1)x \cdot x$. It is interesting to look at how the right $\mathbb{C}[x]^{dg^{\mathrm{op}}} \otimes \mathbb{C}[x]^{dg}$-module structure associated to the canonical $\mathbb{C}[x]^{dg}$-$\mathbb{C}[x]^{dg}$-bimodule structure on $\mathbb{C}[x]^{dg}$ acts. If we abuse notation and take y^i to be $x^i \otimes 1$ and x^i to be $1 \otimes x^i$ in $\mathbb{C}[x]^{dg^{\mathrm{op}}} \otimes \mathbb{C}[x]^{dg}$ then we have the following set of formulas that describe the the right action:

$$\begin{aligned} x^a \cdot y^b &= \begin{cases} (-1)^a x^{a+b}, & \text{if b is odd,} \\ x^{a+b}, & \text{if b is even,} \end{cases} \\ x^a \cdot x^b &= \qquad\quad x^{a+b}. \end{aligned}$$

These relations are immediate from the definition of the map (1.15).

We now try and model the correspondence between bimodules and right modules by saying that for two dg-categories $\mathscr{A}$ and $\mathscr{B}$, an $\mathscr{A}$-$\mathscr{B}$-bimodule is a right $\mathscr{A}^{op} \otimes \mathscr{B}$-module. For this definition to make sense we have to define what a tensor product of two dg-categories is.

Definition 3.1.30. *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are two dg-categories. we define the category $\mathscr{A} \otimes \mathscr{B}$ to be the dg-category whose set of objects is $ob(\mathscr{A}) \times ob(\mathscr{B})$ and such that the set of morphisms between to objects (a, b) and $(a', b,)$ is given by*

$$\mathscr{A} \otimes \mathscr{B}((a, b), (a', b')) := \mathscr{A}(a, a',) \otimes \mathscr{B}(b, b'), \tag{1.16}$$

the composition of two morphisms is given by

$$(f' \otimes g')(f \otimes g) = (-1)^{|f|\,|g'|} f'f \otimes g'g. \tag{1.17}$$

The differential is given by

$$d = d_{\mathscr{A}} \otimes id_{\mathscr{B}} + id_{\mathscr{A}} \otimes d_{\mathscr{B}}. \tag{1.18}$$

Definition 3.1.31. *A dg-$\mathscr{A}$-$\mathscr{B}$-bimodule is a (right) $\mathscr{A}^{op} \otimes \mathscr{B}$ module.*

A dg-A-B-bimodule over dg-algebras A and B can be realized as a dg-$\mathcal{A}$-$\mathcal{B}$-bimodule where $\mathcal{A}$ and $\mathcal{B}$ are defined as in 3.1.4. Also any dg-bimodule over two dg-categories with one object each, can be realized as a dg-bimodule over two dg-algebras.

We now, state a very useful Tensor-Hom adjunction in the dg-category setting that will be very useful for us in the sequel.

Lemma 3.1.32. *(Tensor-Hom adjunction) Let $\mathscr{A}$ and $\mathscr{B}$ be two dg-categories and X an $\mathscr{A}$-$\mathscr{B}$-bimodule. There is an adjunction $(? \otimes_{\mathscr{A}} X, (\mathrm{Dif}\mathscr{B})(X, ?))$ of functors*

$$\begin{tikzcd} \mathrm{Dif}\ \mathscr{A} \arrow[r, bend left, "? \otimes_{\mathscr{A}} X"] & \mathrm{Dif}\ \mathscr{B} \arrow[l, bend left, "(\mathrm{Dif}\ \mathscr{B})(X,?)"] \end{tikzcd} \tag{1.19}$$

Where for $M \in \mathrm{Dif}\ \mathscr{B}$ and $a \in \mathscr{A}^{op}$

$$(\mathrm{Dif}\ \mathscr{B})(X, M)(a) := (\mathrm{Dif}\ \mathscr{B})(X(?, a), M) \tag{1.20}$$

and for $N \in \mathrm{Dif}\ \mathscr{A}$ *and* $b \in \mathscr{B}$

$$(N \otimes_{\mathscr{A}} X)(b) := \mathrm{Coker}\left(\coprod_{a,a' \in \mathscr{A}} N(a') \otimes \mathscr{A}(a,a') \otimes X(b,a) \xrightarrow{\nu} \coprod_{a \in \mathscr{A}} N(a) \otimes X(b,a) \right) \tag{1.21}$$

where ν *is the map*

$$\nu(n \otimes f \otimes x) = N(f)(n) \otimes x - n \otimes X(b,f)(x). \tag{1.22}$$

With Lemma 3.1.32 in mind, for X an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module and T an $\mathscr{B}$-module, it is straightforward to make sense of $X \otimes_{\mathscr{B}} T$ as an $\mathscr{A}^{\mathrm{op}}$-module and for Y an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module and then $(\mathrm{Dif}\ \mathscr{B})(X,Y)$ has the obvious structure of an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$-module. There are also straightforward tensor-hom adjunctions in these cases. The next lemma gives a canonical description of $\hat{a} \otimes_{\mathscr{A}} X$

Lemma 3.1.33. *Let* X *be a dg-*$\mathscr{A}$*-*$\mathscr{B}$*-bimodule and* $a \in \mathscr{A}$ *then there is an isomorphism in* $\mathrm{Dif}\ \mathscr{B}$

$$\hat{a} \otimes_{\mathscr{A}} X \cong X(?,a) \tag{1.23}$$

that is natural in a *and* X*.*

Proof. Let $M \in \mathrm{Dif}\ \mathscr{B}$ then via 3.1.32 there is an isomorphism

$$(\mathrm{Dif}\ \mathscr{B})(\hat{a} \otimes_{\mathscr{A}} X, M) \cong (\mathrm{Dif}\ \mathscr{A})(\hat{a}, (\mathrm{Dif}\ \mathscr{B})(X,M))$$

Then using 3.1.27 there is an isomorphism

$$(\mathrm{Dif}\ \mathscr{A})(\hat{a}, (\mathrm{Dif}\ \mathscr{B})(X,M)) \cong (\mathrm{Dif}\ \mathscr{B})(X(?,a), M)$$

The lemma follows. □

The next easy lemma illustrates the the relationship between the functor $? \otimes_{\mathscr{A}} X$ and the left action of $\mathscr{A}$ on X. We note that there is also a similar statement for $X \otimes_{\mathscr{B}} ?$ and the right action of $\mathscr{B}$ on X.

Lemma 3.1.34. *Let* X *be an* $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$*-module. The following diagram commutes for all and* $a, a' \in \mathscr{A}$

$$\begin{tikzcd} (\mathrm{Dif}\ \mathscr{A})(\hat{a}, \hat{a}') \arrow[r] \arrow[d] & (\mathrm{Dif}\ \mathscr{B})(\hat{a} \otimes_{\mathscr{A}} X, \hat{a}' \otimes_{\mathscr{A}} X) \\ \mathscr{A}(a,a') \arrow[r, "\lambda"] & (\mathrm{Dif}\ \mathscr{B})(X(?,a), X(?,a')) \arrow[u] \end{tikzcd} \tag{1.24}$$

Where the vertical maps are given by the canonical isomorphisms and $\lambda(f) := X(?,f)$ *is called the left action map.*

Proof. One just has to note that for some $f : \hat{a} \to \hat{a}'$, the morphism $f \otimes_{\mathscr{A}} X : \hat{a} \otimes_{\mathscr{A}} X \to \hat{a}' \otimes_{\mathscr{A}} X$ is induced by f and the universal property of the cokernel. Which in turn induces the map $X(?,f) : X(?,a) \to X(?,a')$. □

3.2. **Homotopy Categories and Triangulated Structure.**

In this subsection we want to associate to Dif $\mathscr{A}$ a homotopy category that generalizes the situation in the case of complexes (i.e. complexes of modules over some R). Perhaps not surprisingly, we will see that our generalization has the structure of a triangulated category that generalizes the case of complexes. We will also state and prove a few facts about so called thick triangulated subcategories generated by some set of objects in a triangulated category. This is important since what we will see is that in many situations knowing what happens to the generating objects is enough to know the general phenomena on the subcategory they generate.

Definition 3.2.1. *Let $\mathscr{A}$ be a dg-category. The categories $\mathcal{C}\mathscr{A}$ and $\mathcal{H}\mathscr{A}$ have the same objects as* Dif $\mathscr{A}$ *but with morphism spaces*

$$\begin{aligned} \mathcal{C}\mathscr{A}(M,N) &= Z^0(\mathrm{Dif}\ \mathscr{A})(M,N) \\ \mathcal{H}\mathscr{A}(M,N) &= H^0(\mathrm{Dif}\ \mathscr{A})(M,N) \end{aligned}$$

respectively, where $Z^0(\mathrm{Dif}\ \mathscr{A})(M,N)$ is set of all the morphisms $f \in (\mathrm{Dif}\ \mathscr{A})(M,N)^0$ such that $d(f) = 0$, and $H^0(\mathrm{Dif}\ \mathscr{A})(M,N) = Z^0(\mathrm{Dif}\ \mathscr{A})(M,N)/\operatorname{Im}(d^{-1}_{M,N})$, where $d^{-1}_{M,N} : (\mathrm{Dif}\ \mathscr{A})(M,N)^{-1} \longrightarrow (\mathrm{Dif}\ \mathscr{A})(M,N)^0$.

We call $\mathcal{H}\mathscr{A}$ the homotopy category. What is happening in Definition 3.2.1 is that in $\mathcal{C}\mathscr{A}$ we are now only looking at degree zero morphisms that commute with the differential and in $\mathcal{H}\mathscr{A}$ we are looking at degree zero morphisms that commute with the differential but additionally we impose the equivalence relation that two maps are the same if they are homotopic. If we take $\mathscr{A} = k$, then $\mathcal{C}k$ is the category of chain complexes of modules over k and $\mathcal{H}k$ is the corresponding homotopy category.

Example 3.2.2. $Z^0(\mathrm{Dif}\ k)$ is just the usual category, of chain complexes modules over k, and $H^0(\mathrm{Dif}\ k)$ is its homotopy category.

Lemma 3.2.3. *Let $\mathscr{A}$ be a dg-category and M an $\mathscr{A}$-module. Then we have the canonical isomorphisms*

$$(\mathcal{H}\mathscr{A})(\hat{a}, [n]M) \cong H^n((\mathrm{Dif}\ \mathscr{A})(\hat{a}, M)) \cong H^n M(a) \tag{2.1}$$

Proof. This is a direct consequence of Lemma 3.1.27. □

What we want to do next is to generalize the notion of a quasi-isomorphism of chain complexes to modules over dg-categories.

Definition 3.2.4. *Two objects $M, N \in \mathcal{C}\mathscr{A}$ or $\mathcal{H}\mathscr{A}$ are said to be quasi-isomorphic if there is a morphism $f : M \to N$ such that $f_a : M(a) \to N(a)$ is a quasi-isomorphism for all $a \in \mathscr{A}$*

In the case of $\mathscr{A} = k$ we see that this is just quasi-isomorphism in the usual sense of complexes. The shift functor on Dif $\mathscr{A}$ naturally descends to $\mathcal{C}\mathscr{A}$ and $\mathcal{H}\mathscr{A}$, i.e. they are both $\mathbb{Z}$-categories in the sense of [AJS, Appendix E] . Thus given an morphism $f : M \to N$ in $\mathcal{C}\mathscr{A}$ we can define *cone*(f) as the dg-module whose underlying graded structure is $[1]M \oplus N$ and whose differential is

$$\begin{pmatrix} [1]d_M & 0 \\ [1]f & d_N \end{pmatrix}. \tag{2.2}$$

This is clearly seen to be a generalization of the case of chain complexes and just as in the case of chain complexes one can show that f is a quasi-isomorphism if and only if $cone(f)$ is acyclic.

Theorem 3.2.5. *Suppose $\mathscr{A}$ is a dg-category. Then, $\mathcal{H}\mathscr{A}$ has the structure of a triangulated category where distinguished triangles are those triangles that are isomorphic to images of triangles of the form:*

$$M \xrightarrow{f} N \longrightarrow cone(f) \longrightarrow [1]M, \tag{2.3}$$

under the embedding $\iota : \mathcal{C}\mathscr{A} \hookrightarrow \mathcal{H}\mathscr{A}$, for some $f \in \mathcal{C}\mathscr{A}$.

Proof. See [Kel1] and [Pos]. □

Definition 3.2.6. *An $\mathscr{A}$-module M is called acyclic if $H^*M(a) = 0$ for all $a \in \mathscr{A}$.*

Definition 3.2.7. *Let P be a dg-module over $\mathscr{A}$, then P is called homotopy projective (h-projective) if $(\mathcal{H}\mathscr{A})(P, N) = 0$ for all N acyclic (see* 3.2.6*).*

Let $\mathcal{P}$ be the full subcategory of $\mathcal{H}\mathscr{A}$ whose set of objects consists of the h-projectives. The category $\mathcal{P}$ is seen to be triangulated since it is closed under shifts and taking mapping cones. To spell out the last point, consider a morphism $f : X \to Y$, applying the cohomological functor $(\mathcal{H}\mathscr{A})(?, N)$, with N acyclic, to the triangle

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow [1]X \tag{2.4}$$

Gives a long exact sequence

$$\cdots \longleftarrow 0 \longleftarrow (\mathcal{H}\ \mathscr{A})(\mathrm{cone}(f), N) \longleftarrow 0 \longleftarrow \cdots \tag{2.5}$$

Hence cone(f) is h-projective. We will see in Section 3, that $\mathcal{P}$ is in fact (triangulated)equivalent to the derived category of $\mathscr{A}$.

Definition 3.2.8. *Let I be a dg-module over $\mathscr{A}$, I is called homotopy injective (h-injective) if $(\mathcal{H}\mathscr{A})(N, I) = 0$ for all N acyclic.*

Example 3.2.9. It follows directly from the definition that h-projectives in $\mathcal{H}\mathscr{A}$ are h-injectives in $(\mathcal{H}\mathscr{A})^{\mathrm{op}}$ and vice versa. Note the completely analogous situation where objects that are projective in k-Mod are injective in k-Mod$^{\mathrm{op}}$

Dually it is also true that the full subcategory of $\mathcal{H}\mathscr{A}$ of h-injectives, $\mathcal{I}$, is triangulated and will be seen to be equivalent to the derived category of $\mathscr{A}$.

The h-projectives and h-injectives allow us to handle the derived category in a concrete way. Now we give a couple more examples.

Example 3.2.10. a) In $\mathcal{H}k$, all right bounded complexes of projectives are h-projective. This follows from the the fact that if P is a right bounded complex of projectives, then

$$(\mathcal{D}k)(P, X) \cong (\mathcal{H}k)(P, X).$$

for all $X \in \mathcal{D}k$ (the derived category of chain complexes of k-modules). We will show in the next subsection that this is equivalent to P being h-projective.

b) In $\mathcal{H}\ k$ all left bounded complexes of injectives are h-injective. The explanation for this is dual to part a).

Lemma 3.2.11. *Let $\mathscr{A}$ be a dg-category and $a \in \mathscr{A}$ and $n \in \mathbb{Z}$, then $[n]\hat{a}$ is h-projective.*

Proof. Let N be acyclic, using Lemma 3.2.3

$$(\mathcal{H}\mathscr{A})([n]\hat{a}, N) \cong H^{-n}N(a) \cong 0 \tag{2.6}$$

and the claim follows. □

We now move on to our second goal for this subsection and discuss some results and definitions about general triangulated categories that will be very important to us later on. Most of the definitions can be found in [Nee] or [Mur2].

Definition 3.2.12. *A subcategory $\mathscr{C}$ of $\mathscr{T}$ is said to be strict if for each $c \in \mathscr{C}$ all objects isomorphic to c in $\mathscr{T}$ are also in $\mathscr{C}$.*

Definition 3.2.13. *Let $\mathscr{T}$ be a triangulated category. A triangulated subcategory $\mathscr{C}$ of $\mathscr{T}$ is a full, and additive subcategory such that $[1]\mathscr{C} = \mathscr{C}$, and if for any distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow [1]X \tag{2.7}$$

such that $X, Y \in \mathscr{C}$ then $Z \in \mathscr{C}$.

Definition 3.2.14. *A subcategory category $\mathcal{C}$ of a triangulated category $\mathscr{T}$ is called thick if it is triangulated, strict, and contains all direct summands of its objects.*

One of the reasons that thick subcategories are so important is that they allow us to express the universal property of the Verdier quotient, in this sense they play a similar role to that of a normal subgroup when taking the quotients of groups. Also, in Section 4 it will be critical that the full subcategory of acyclic modules is thick to ensure that both left and right derived functors exist. We will see that for thick subcategories generated by a set of objects many properties can be often just checked on generating objects . An important example of a thick subcategory is the full subcategory of acyclic chain complexes in the homotopy category of chain complexes. Indeed it is triangulated since H^0 is a cohomological functor (i.e. it takes triangles to exact sequences), and the fact that it is strict and closed under taking direct summands is more or less obvious.

Definition 3.2.15. *Let $\mathscr{T}$ be a triangulated category. Suppose that $S \subset ob(\mathscr{T})$ we define Thick$\langle S \rangle$ to be the smallest (with respect to the inclusion) thick subcategory containing S.*

Concretely, Thick$\langle S \rangle$ contains S, is closed under shifts, cones and taking direct summands.

The next three lemmas will be of utmost importance in what follows in the sequel. Similar statements in less generality can be found in [Kel1].

Lemma 3.2.16. *Let $\mathscr{U}$ be a triangulated category and $S \subset ob(\mathscr{U})$. Suppose that $F : \mathscr{U} \longrightarrow \mathscr{T}$ is a triangulated functor between triangulated categories. Then, F induces an isomorphism*

$$\mathscr{U}(s, [n]t) \xrightarrow{\cong} \mathscr{T}(F(s), [n]F(t)) \tag{2.8}$$

for all $s, t \in S$ and $n \in \mathbb{Z}$ if and only if $F|_{Thick\langle S\rangle}$ is fully faithful.

Proof. Fix $t \in S$. Let $\mathscr{C}$ be a full subcategory of Thick$\langle S\rangle$ (remember that Thick$\langle S\rangle$ itself is a full subcategory of $\mathscr{U}$) consisting of objects x such that the induced map

$$\mathscr{U}(x, [n]t) \xrightarrow{\cong} \mathscr{T}(F(x), [n]F(t)) \tag{2.9}$$

is an isomorphism for all $n \in \mathbb{Z}$. Clearly $S \subset \mathscr{C}$ and it follows easily that $\mathscr{C}$ is strict, triangulated. We claim that it is also closed under taking direct summands.

Suppose that $x \in \mathscr{C}$ and $x_1, x_2 \in$ Thick$\langle S\rangle$ such that

$$x \cong x_1 \oplus x_2 \tag{2.10}$$

We will show that $x_1, x_2 \in \mathscr{C}$. We have the following system of maps

$$\begin{tikzcd} x_1 \arrow[r, shift left, "i"] & x \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & x_2 \arrow[l, shift left, "j"] \end{tikzcd} \tag{2.11}$$

such that $p \circ i = id_{x_1}$, $q \circ j = id_{x_2}$, $i \circ p + j \circ q = id_x$. We have the following natural transformation for each $n \in \mathbb{Z}$

$$\tau^n : G := \mathscr{U}(?, [n]t) \longrightarrow G' := \mathscr{T}(F(?), [n]F(t)). \tag{2.12}$$

For $x \in \mathscr{C}$ clearly τ^n_x is an isomorphism and the following diagram

$$\begin{tikzcd} G(x) \arrow[r, "\cong"] \arrow[d, "\tau^n_x"'] & G(x_1) \oplus G(x_2) \arrow[d, "\tau^n_{x_1} \oplus \tau^n_{x_2}"] \\ G'(x) \arrow[r, "\cong"] & G'(x_1) \oplus G'(x_2) \end{tikzcd} \tag{2.13}$$

where the horizontal maps are $\begin{pmatrix} G(i) & G(j) \end{pmatrix}$ and $\begin{pmatrix} G'(i) & G'(j) \end{pmatrix}$ respectively commutes. It now follows that $\tau^n_{x_1}$ and $\tau^n_{x_2}$ are isomorphisms and thus $x_1, x_2 \in \mathscr{C}$. Hence $\mathscr{C}$ is a thick subcategory containing S. Thus it follows that $\mathscr{C} =$ Thick$\langle S\rangle$.

Again, we now fix some $t \in S$ and define $\mathscr{D}$ to be the full subcategory of Thick$\langle S\rangle$ consisting of objects z such that:

$$\mathscr{U}([-n]t, z) \xrightarrow{\sim} \mathscr{T}([-n]F(t), F(z)) \tag{2.14}$$

Then we see that $S \subset \mathscr{D}$ and $\mathscr{D}$ is strict, triangulated and closed under taking direct summands. Hence $\mathscr{D} =$ Thick$\langle S\rangle$. What we have shown is that $\mathscr{C} =$ Thick$\langle S\rangle = \mathscr{D}$ and the statement follows. $\square$

Lemma 3.2.17. *Suppose $F, G : \mathscr{U} \to \mathscr{T}$ are triangulated functors between triangulated categories and $S \subset ob(\mathscr{U})$. Let F' and G' denote the restrictions of F and G to Thick$\langle S\rangle$. Suppose that*

$$\tau : F' \longrightarrow G' \tag{2.15}$$

is a triangulated natural transformation. Then τ is an isomorphism if and only if τ_s is invertible for all $s \in S$.

Proof. The forward implication is clear. To prove the reverse implication, let $\mathscr{D}$ be the full subcategory of $\mathrm{Thick}\langle S\rangle$ such that $d \in \mathscr{D}$ if τ_d is invertible. Then one sees that $\mathscr{D}$ is thick and hence $\mathscr{D} = \mathrm{Thick}\langle S\rangle$ thus τ is an isomorphism. □

Definition 3.2.18. *A triangulated category $\mathscr{C}$ is said to be idempotent complete (or sometimes Karoubian), if for any $X \in \mathscr{C}$ and $e \in End_{\mathscr{C}}(X)$ an idempotent (i.e. $e^2 = e$), there exist morphisms f, g:*

$$X \xrightarrow{f} Y \xrightarrow{g} X \tag{2.16}$$

such that $e = g \circ f$ and $id_Y = f \circ g$.

Theorem 3.2.19. *Suppose $\mathscr{C}$ is a triangulated category admitting countable coproducts then $\mathscr{C}$ is idempotent complete.*

Proof. A proof can be found in [Nee, Proposition 1.6.8.] using the construction of homotopy colimit. □

Corollary 3.2.20. *Let $\mathscr{A}$ be a dg-category, then $\mathcal{D}\mathscr{A}$ (the derived category of $\mathscr{A}$ see Definition 3.3.1) is idempotent complete.*

Proof. Since $\mathcal{D}\mathscr{A}$ admits arbitrary coproducts, [Kel4], it follows from Theorem 3.2.19, that $\mathcal{D}\mathscr{A}$ is idempotent complete. □

Lemma 3.2.21. *Suppose we have a functor, $F : \mathscr{P} \to \mathscr{T}$, between two triangulated categories and a set $S \subset ob(\mathscr{P})$. Then the following holds:*

i) Restricting F induces a functor $F' : Thick\langle S\rangle \to Thick\langle F(S)\rangle$.

ii) If $Thick\langle S\rangle$ is idempotent complete and F' is fully faithful then F' is an equivalence.

Proof. i) For $x \in \mathrm{Thick}\langle S\rangle$, let $\mathscr{D}$ be the full subcategory of $\mathrm{Thick}\langle S\rangle$ consisting of objects x such that $F(x) \in \mathrm{Thick}\langle F(S)\rangle$. Then we see that $\mathscr{D}$ is strict, triangulated, and closed under taking direct summands hence $\mathscr{D} = \mathrm{Thick}\langle S\rangle$. It follows that $F'(x) \in \mathrm{Thick}\langle F(S)\rangle$.

ii) We will show that F' is essentially surjective. Let $\mathscr{C}$ be the full subcategory of $\mathrm{Thick}\langle FS\rangle$ where

$$\mathscr{C} := \{y \in \mathrm{Thick}\langle FS\rangle \mid \text{there exists an } x \in \mathrm{Thick}\langle S\rangle,\ F'(x) \cong y\}.$$

Clearly $\mathscr{C}$ is strict. Moreover, it follows that $\mathscr{C}$ is triangulated, since it is clearly is closed under shifts, since F' is a triangulated functor. We claim that $\mathscr{C}$ is closed under taking cones:

$$x' \xrightarrow{f'} y' \longrightarrow z' \longrightarrow [1]x$$

in $\mathscr{T}$ such that $x', y' \in \mathscr{C}$. Then there is a map $\tilde{f}'$, such that the diagram

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "\cong"] & y' \arrow[d, "\cong"] \\ F'(x) \arrow[r, "\tilde{f}'"] & F'(y) \end{tikzcd}$$

commutes. Using the assumption that F' is fully faithful it follows that there is a unique $f \in \mathscr{P}(x, y)$ such that $F'(f) = \tilde{f}' \in \mathscr{T}(F'(x), F'(y))$. Thus there is a triangle

$$x \xrightarrow{f} y \longrightarrow z \longrightarrow [1]x$$

in Thick$\langle S \rangle$. Then since F' is a triangulated functor,

$$F'(x) \xrightarrow{\tilde{f}'} F'(y) \longrightarrow F'(z) \longrightarrow [1]F'(x)$$

is a triangle in Thick$\langle FS \rangle$. By the definition of triangulated categories. By TR03, see Appendix A, there is a morphism of triangles

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "\cong"] & y' \arrow[r] \arrow[d, "\cong"] & z' \arrow[r] \arrow[d] & {[1]x'} \arrow[d, "\cong"] \\ F'(x) \arrow[r, "\tilde{f}'"] & F'(y) \arrow[r] & F'(z) \arrow[r] & {[1]F'(x)} \end{tikzcd}$$

It follows from [Nee] that the middle morphism is an isomorphism. Hence $z' \in \mathscr{C}$ and we see that $\mathscr{C}$ is closed under taking cones.

We finally show that $\mathscr{C}$ is closed under taking direct summands. Suppose that $x' \in \mathscr{C}$ such that there are $y', z' \in$ Thick$\langle FS \rangle$

$$x' \cong y' \oplus z'$$

in Thick$\langle FS \rangle$. There are maps

$$y' \xrightarrow{i} y' \oplus z' \xrightarrow{p} y'$$

such that $p \circ i = \mathrm{id}_{y'}$. Define $e := i \circ p$. Then e is an idempotent on $y' \oplus z'$. Since $x' \in \mathscr{C}$ there exists $x \in$ Thick$\langle S \rangle$ such that we have the following commutative diagram

$$\begin{tikzcd} F'(x) \arrow[r, "\cong", "\epsilon"'] \arrow[d, "e''"] & x' \arrow[r, "\cong", "h"'] \arrow[d, "e'"] & y' \oplus z' \arrow[d, "e"] \\ F'(x) \arrow[r, "\cong", "\epsilon"'] & x' \arrow[r, "\cong", "h"'] & y' \oplus z' \end{tikzcd}$$

where $e' := h^{-1} \circ e \circ h$ and $e'' := \epsilon^{-1} \circ e' \circ \epsilon$. Since F' is fully faithful, there is a unique $f : x \longrightarrow x$, such that $F'(f) = e''$. It follows that f is an idempotent. Since we have assumed that Thick$\langle S \rangle$ is idempotent complete it follows that there are morphisms ϕ, ψ

$$x \xrightarrow{\phi} y \xrightarrow{\psi} x$$

such that $\psi\phi = f$ and $\phi\psi = \mathrm{id}_y$. Consider the following diagram

$$\begin{tikzcd}
F'(y) \arrow[r, "F'(\psi)"] & F'(x) \arrow[r, "F'(\phi)"] \arrow[d, "h\circ\epsilon"', "\cong"] & F'(y) \arrow[r, "F'(\psi)"] & F'(x) \arrow[r, "F'(\phi)"] \arrow[d, "h\circ\epsilon"', "\cong"] & F'(y) \\
y' \arrow[r, "i"] & y' \oplus z' \arrow[r, "p"] & y' \arrow[r, "i"] & y' \oplus z' \arrow[r, "p"] & y'
\end{tikzcd}$$

and define $g := F'(\phi) \circ (h \circ \epsilon)^{-1} \circ i$ and $g^{-1} := p \circ (h \circ \epsilon) \circ F'(\psi)$. Then

$$\begin{aligned}
g \circ g^{-1} &= F'(\phi) \circ (h \circ \epsilon)^{-1} \circ i \circ p \circ (h \circ \epsilon) \circ F'(\psi) \\
&= F'(\phi) \circ e'' \circ F'(\psi) \\
&= F'(\phi) \circ F'(f) \circ F'(\psi) \\
&= F'(\phi \circ \psi \circ \phi \circ \psi) \\
&= \mathrm{id}_{F'(y)}
\end{aligned}$$

One does a similar computation for $g^{-1} \circ g$ and concludes that $y' \cong F'(y)$ thus $y \in \mathscr{C}$ thus $\mathscr{C}$ is closed under taking direct summands so altogether $\mathscr{C}$ is a thick subcategory of $\mathrm{Thick}\langle FS \rangle$ containing FS, hence $\mathscr{C} = \mathrm{Thick}\langle FS \rangle$. So we see that F' is essentially surjective. □

The next definition and lemmas are easy but technical results that can be skipped on a first reading. They will be used in Remark 5.1.5 and Remark 5.2.7.

Definition 3.2.22. *We say that a commutative square*

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "g"] & V \arrow[d, "h"] \\
W \arrow[r, "r"] & Z
\end{tikzcd} \tag{2.17}$$

is homotopy cartesian if there exists a distinguished triangle:

$$U \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} V \oplus W \xrightarrow{\begin{pmatrix} -h & r \end{pmatrix}} Z \xrightarrow{l} [1]U \tag{2.18}$$

for some morphism $l : Z \to [1]U$.

Lemma 3.2.23. *Let*

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "g"] & V \arrow[d, "h"] \\
W \arrow[r, "r"] & Z
\end{tikzcd}$$

be a homotopy cartesian square. If

$$U \xrightarrow{g} W \longrightarrow W' \longrightarrow [1]U$$

is a triangle. Then there is a triangle

$$V \xrightarrow{h} Z \longrightarrow W' \longrightarrow [1]V$$

such that the square is completed to a morphism of triangles:

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "f"] & W \arrow[r] \arrow[d, "r"] & W' \arrow[r] \arrow[d, "id"] & {[1]U} \arrow[d] \\ V \arrow[r, "h"] & Z \arrow[r] & W' \arrow[r] & {[1]V} \end{tikzcd}$$

Proof. [Nee, Lemma 1.4.4.] □

Lemma 3.2.24. *Suppose that we have a homotopy cartesian square:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"] & V \arrow[d, "h"] \\ W \arrow[r, "r"] & Z \end{tikzcd} \tag{2.19}$$

Then,
i) f is invertible if and only if r is invertible.
ii) h is invertible if and only if g is invertible.

Proof. Both i) and ii) follow from the previous lemma and the observation that if you have a homotopy cartesian square

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"] & V \arrow[d, "h"] \\ W \arrow[r, "r"] & Z \end{tikzcd}$$

then

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "f"] & W \arrow[d, "r"] \\ V \arrow[r, "h"] & Z \end{tikzcd}$$

is also again a homotopy cartesian square because you can use the same connecting morphism. □

3.3. **The Derived Category.**

We are now ready to define the derived category of modules over a dg-category $\mathscr{A}$. The definition will be very much the same as in the case of complexes. Indeed, if we take $\mathscr{A} = k$, $\mathcal{D}\mathscr{A}$ is just the unbounded derived category of chain complexes over k in the usual sense. Our definition will make use of the construction that is known as the Verdier quotient, we refer the interested reader to the Appendix A for a brief description of it.

Definition 3.3.1. *The derived category $\mathcal{D}\mathscr{A}$ is obtained by taking the Verdier quotient of $\mathcal{H}\mathscr{A}$ by the thick subcategory of all acyclic dg-$\mathscr{A}$-modules. (note in particular it is triangulated.)*

With Definition 3.3.1 it could be that in general the morphism space between two objects in $\mathcal{D}\mathscr{A}$ could be much larger than then the space of morphisms between the same objects in say $\mathcal{H}\mathscr{A}$. The next two lemmas will show that in some cases the morphism space between two objects is the same as the morphism space between the same objects in $\mathcal{H}\mathscr{A}$.

Lemma 3.3.2. *Let $P \in ob(\mathcal{D}\mathscr{A})$. Then, P is h-projective if and only if*

$$(\mathcal{D}\mathscr{A})(P, M) = (\mathcal{H}\mathscr{A})(P, M) \tag{3.1}$$

for all $M \in ob(\mathcal{D}\mathscr{A})$.

Proof. Suppose first that P is h-projective. Let $[g, f]$ be a morphism in $(\mathcal{D}\mathscr{A})(P, M)$. It can be represented as a diagram

$$\begin{tikzcd} & N & \\ P \arrow[ur, "g"] & & M \arrow[ul, "f"'] \end{tikzcd} \tag{3.2}$$

where f is a quasi-isomorphism. Thus cone(f) is acyclic and there an isomorphism

$$(\mathcal{H}\mathscr{A})(P, M) \xrightarrow{f_*} (\mathcal{H}\mathscr{A})(P, N) \tag{3.3}$$

Thus there exists an $s \in (\mathcal{H}\ \mathscr{A})(P, M)$ such that $f \circ s = g$ and we see that the diagram

$$\begin{tikzcd} & N \arrow[d, "\mathrm{id}"] & \\ P \arrow[ur, "g"] \arrow[r, "g"] \arrow[dr, "s"'] & N & M \arrow[ul, "f"'] \arrow[l, "f"'] \arrow[dl, "\mathrm{id}"] \\ & M \arrow[u, "f"'] & \end{tikzcd} \tag{3.4}$$

commutes. Hence we have a surjection.

If there are two morphisms $\phi, \psi \in (\mathcal{H}\mathscr{A})(P, M)$ that get identified in $(\mathcal{D}\mathscr{A})(P, M)$ this means that there is a quasi-isomorphism $f : N \to P$, such that $\phi \circ f = \psi \circ f$. i.e. the diagram

$$\begin{tikzcd} & P \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\phi"] & \\ P & N \arrow[u, "f"'] \arrow[l, "f"'] \arrow[r] \arrow[d, "f"] & M \\ & P \arrow[ul, "\mathrm{id}"] \arrow[ur, "\psi"'] & \end{tikzcd} \tag{3.5}$$

commutes.We will show that there exists a $g \in (\mathcal{H}\mathscr{A})(P, N)$ such that $f \circ g = \mathrm{id}_P$. From this it will follows that $\phi = \phi \circ f \circ g = \psi \circ f \circ g = \psi \in (\mathcal{H}\mathscr{A})(P, M)$.

Let, $f \in (\mathcal{H}\mathscr{A})(N, P)$ a quasi-isomorphism. Then we form the distinguished triangle

$$N \xrightarrow{f} P \longrightarrow \mathrm{cone}(f) \tag{3.6}$$

in $(\mathcal{H}\ \mathscr{A})$. Since f is a quasi-isomorphism it follows that cone(f) is acyclic. Then since $(\mathcal{H}\mathscr{A})(P, -)$ is a homological functor and $(\mathcal{H}\ \mathscr{A})(P, \mathrm{cone}(f)) = 0$ it follows that

$$(\mathcal{H}\mathscr{A})(P, N) \xrightarrow{f_*} (\mathcal{H}\mathscr{A})(P, P) \tag{3.7}$$

is an isomorphism. Thus, there is a $g \in (\mathcal{H}\mathscr{A})(P, N)$ such that $f \circ g = \mathrm{id}_P$.

On the other hand if N is acyclic then by assumption $(\mathcal{H}\mathscr{A})(P, N) = (\mathcal{D}\mathscr{A})(P, N)$. But $(\mathcal{D}\mathscr{A})(P, N) = 0$ thus P is h-projective. $\square$

Lemma 3.3.3. *Let $I \in ob(\mathcal{D}\mathscr{A})$. Then, I is h-injective if and only if*

$$(\mathcal{D}\mathscr{A})(M, I) = (\mathcal{H}\mathscr{A})(M, I) \tag{3.8}$$

for all $M \in ob(\mathcal{D}\mathscr{A})$.

Proof. The proof is dual to Lemma 3.3.2. □

One should note that the situation here is very similar to the case of the derived category of complexes, for example see [Wei, Corollary 10.4.7.]. Also, the proofs in chain complexes are almost exactly the same as the ones we have just given in the dg-category realm.

Definition 3.3.4. *Let $S = \{\hat{a} \mid a \in \mathscr{A}\}$. Define the subcategory of perfect objects,* $\mathrm{Perf}(\mathscr{A}) := \mathit{Thick}\langle S\rangle$ *of $\mathcal{D}\mathscr{A}$.*

Remark 3.3.5. Since $\mathcal{D}\mathscr{A}$ is idempotent complete then so is $\mathrm{Perf}(\mathscr{A})$

If we take $\mathscr{A} = k$ then it is straightforward to see that the objects of $\mathrm{Perf}(k)$ are the so called perfect chain complexes. i.e. bounded chain complexes of finitely generated projective modules. It is a standard result in homological algebra that $\mathrm{Perf}(k)$ is the full subcategory of compact objects. This result can be extended to the case of dg-categories as the next remark points out.

Remark 3.3.6. An alternate description of $\mathrm{Perf}(\mathscr{A})$ is the full subcategory of $\mathcal{D}\mathscr{A}$ of compact objects. i.e objects $a \in \mathcal{D}\mathscr{A}$ such that the functor

$$(\mathcal{D}\mathscr{A})(a, ?) : \mathcal{D}\mathscr{A} \longrightarrow \text{Mod-}k,$$

commutes with infinite direct sums, see [Kel1] and [Kel4].

Finally, we would like to make one last remark so that the reader might see a bigger picture.

Remark 3.3.7. For the reader that is comfortable with model categories, we remark that $\mathcal{D}\mathscr{A}$ has a projective model structure as well as an injective model structure. In fact one can show that that the cofibrant (resp. fibrant) objects are h-projective (resp. h-injective). See, [LS, Section 2.4].

3.4. **Hochschild Cohomology for DG Algebras.**

All algebras in the section will be considered to be differentially graded algebras unless otherwise stated. We make the following definition generalizing Definition 2.1.4.

Definition 3.4.1. *Let A be a dg-algebra then*

$$\mathrm{HH}^i(A, X) = (\mathcal{D}A^e)(A, [i]X). \tag{4.1}$$

In the sequel the results we will be concerned with are about cochain complexes and we would like to work rather explicitly with them. This is problematic because there could be many resolutions to choose from or perhaps even none all. What we would like is to have one explicit resolution that exists for all dg-algebras, A, since it would allow us then to concretely talk about the Hochschild cochain complex for an arbitrary dg-algebra. Just as in the associative algebra case the bar resolution provides the answer but there are a couple of issues. Firstly, h-projective resolutions in $\mathcal{D}A^e$ are single dg-modules over A and not complexes of dg-modules. Secondly, the entries in the bar resolution are no longer free. As we will see, totalizing the bar resolution will take care of the first problem. The second problem will be resolved with the notion of semi-free.

Definition 3.4.2. *Let M be a dg-module over a dg-algebra A. We say that M is free if it isomorphic to a direct sum of dg-modules of the form $[n]A$ for $n \in \mathbb{Z}$.*

Definition 3.4.3. *A dg-module M is semi-free if there is a (possibly infinite) filtration*

$$0 = M(0) \subseteq M(1) \subseteq \cdots \subseteq M(i) \subseteq M(i+1) \subseteq \cdots \subseteq M \tag{4.2}$$

(i.e. bounded below) such that $\bigcup_{i \geq 0} M(i) = M$ (i.e. an exhaustive filtration) and $M(i+1)/M(i)$ free dg-module for all $i \geq 0$.

Remark 3.4.4. It is not difficult to show that if M is a dg-module that has an exhaustive bounded below filtration such that $M(i+1)/M(i)$ is isomorphic to a semi-free module, then M is semi-free.

The usual construction of free resolutions of modules over a ring doesn't work in the dg-setting since the map

$$\begin{array}{rcl} \epsilon : \bigoplus_{i \in I} [n_i]A & \longrightarrow & M \\ (0, \ldots, 0, 1, 0, \ldots) & \mapsto & m_i, \end{array}$$

where m_i is a generator of degree n_i, is not a morphism of dg-modules if the m_i are not cocycles.

Definition 3.4.5. *Let M be a dg-module over A. The pair $(\bar{M}, f)$ where $\bar{M}$ is a semi-free dg-module and $f : \bar{M} \to M$ is a morphism of dg A modules that is a quasi-isomorphism (i.e. induces isomorphism in cohomology) is called a semi-free resolution of M.*

The next theorem appears in [Drin], and shows that there are enough semi-free's in $\mathcal{D}A$. For the convenience of the reader we give a somewhat more detailed proof than what appears there.

Theorem 3.4.6. *Let M be a dg-module over A. Then M has a semi-free resolution.*

Proof. <u>Step 1:</u> Let $Z(M) =$ cocycles in $M = \{m \in M |\ d(m) = 0\}$. Then we define

$$\bar{M}(1) := \bigoplus_{i \in I} [n_i]A \cong Z(M) \otimes_k A$$

where $Z(M)$ has homogeneous basis elements z_i, $i \in I$ of degree n_i. Clearly $\bar{M}(1)$ is a free dg-module and is thus the first step in our filtration. Next we define a morphism

$$\begin{array}{rcl} f_1 : \bar{M}(1) & \longrightarrow & M \\ e_i := (0, \ldots, 0, 1, 0, \ldots) & \mapsto & z_i. \end{array} \tag{4.3}$$

Indeed f_1 is a morphism of dg-A-modules since

$$\begin{aligned} f_1(d(e_i)) &= 0 \\ &= d(z_i) \\ &= d(f_1(e_i)). \end{aligned}$$

<u>Step 2:</u> Now we construct $\bar{M}(2)$, such that $\bar{M}(2)/\bar{M}(1)$ is free and a morphism $f_2 : \bar{M}(2) \longrightarrow M$ such that $f_2|_{\bar{M}(1)} = f_1$. Let F_2 be the free module isomorphic to $Z(\mathrm{Ker}(f_1)) \otimes_k A$, and let

$$\tilde{f} : F_2 \longrightarrow \mathrm{Ker}(f_1) \subseteq \bar{M}(1),$$

be the map defined in the same way as in equation (4.3). Then set

$$f := \iota \circ \tilde{f} : F_2 \longrightarrow \bar{M}(1),$$

and $\bar{M}(2) := \mathit{cone}(f)$. Then the map,

$$\tilde{\pi} : \bar{M}(2)/\bar{M}(1) \longrightarrow [1]F_2$$

induced by the projection $\pi : \mathit{cone}(f) \longrightarrow [1]F_2$ is A-linear and compatible with the differential since

$$\begin{aligned} \pi(d_{cone(f)}(m,n)) &= \pi(\begin{pmatrix} -d_{F_2} & 0 \\ f_2 & d_{\bar{M}(1)} \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}) \\ &= \pi(-d_{F_2}(m), f(m) + d_{\bar{M}(1)}(n)) \\ &= \pi(d_{[1]F_2}(m), f(m) + d_{\bar{M}(1)}(n)) \\ &= d_{[1]F_2}\pi((m,n)). \end{aligned}$$

Also, $\tilde{\pi}$ is clearly an isomorphism. Thus we constructed $\bar{M}(2)$ such that $\bar{M}(2)/\bar{M}(1)$ is free and we define

$$f_2 := f_1 \circ f,$$

clearly $f_2|_{\bar{M}(1)} = f_1$.

Repeating this argument defines $\bar{M}(i)$'s such that $\bar{M}(i)/\bar{M}(i-1)$ is free and maps $f_i : \bar{M}(i) \longrightarrow M$ such that $f_i|_{\bar{M}(i-1)} = f_{i-1}$. We define the semi-free resolution of M to be $(\mathrm{colim}\, \bar{M}(i), \mathrm{colim}\, f_i)$. Clearly $\bar{M} := \mathrm{colim}\, \bar{M}(i) = \bigcup \bar{M}(i)$. It follows that $f := \mathrm{colim}\, f_i$ is a quasi-isomorphism since the map $\mathit{cone}(f_{i-1}) \longrightarrow \mathit{cone}(f_i)$ induces the zero map in cohomology and hence $\mathit{cone}(f)$ is acyclic. $\square$

Now we show that semi-free objects are in fact h-projective and hence there are enough h-projectives in $\mathcal{D}A$.

Lemma 3.4.7. *If M is a semi-free dg-A-module then M is homotopy projective.*

Proof. Suppose M is semi-free. Then it has a filtration as in 4.2. Since the inclusions are split inclusions of the underlying graded structure it follows that the following diagram is a triangle in $\mathcal{H}\mathscr{A}$

$$\coprod_{i\geq 0} M(i) \xrightarrow{\Phi} \coprod_{i\geq 0} M(i) \longrightarrow M. \tag{4.4}$$

Where the graded components of Φ are

$$M(i) \xrightarrow{\begin{pmatrix} 1 \\ -\iota \end{pmatrix}} M(i) \oplus M(i+1) \longrightarrow M. \tag{4.5}$$

<u>Claim:</u> $M(i)$ is h-projective for $i \geq 0$.

By 3.2.11 it follows that free dg-modules are h-projective. Hence $M(1)$ is h-projective.

Suppose that $M(j)$ is h-projective for some $j > 1$ and consider the short exact sequence in $\mathcal{H}\mathscr{A}$

$$0 \longrightarrow M(j) \xrightarrow{\iota} M(j+1) \longrightarrow M(j+1)/M(j) \longrightarrow 0$$

Then for N an acylic dg A-module the sequence

$$(\mathcal{H}\mathscr{A})(M(j+1)/M(j), N) \longrightarrow (\mathcal{H}\mathscr{A})(M(j+1), N) \longrightarrow (\mathcal{H}\mathscr{A})(M(j), N) \longrightarrow 0$$

is exact. Since $M(j)$ is h-projective by assumption and $M(j+1)/M(j)$ is h-projective by 3.2.11, it follows that $M(j+1)$ is h-projective. Hence for all i, $M(i)$ is h-projective.

Let N be any acyclic dg A-module. The functor $(\mathcal{H}\mathscr{A})(?, N)$ is a cohomoloical functor, thus there is a long exact sequence

$$\cdots \longrightarrow (\mathcal{H}\mathscr{A})(\coprod_{i \geq 0} M(i), N) \longrightarrow (\mathcal{H}\mathscr{A})(M, N) \longrightarrow [1](\mathcal{H}\mathscr{A})(\coprod_{i \geq 0} M(i), N) \longrightarrow \cdots$$

but since $[l](\mathcal{H}\mathscr{A})(\coprod_{i \geq 0} M(i), N) = 0$ for all $l \in \mathbb{Z}$, it follows that $(\mathcal{H}\mathscr{A})(M, N) = 0$, thus M is h-projective. $\square$

Recall the bar resolution $(B^{-i}(A), b^{-}i)$ for an associative algebra. Let us now replace the associative algebra with a dg-algebra (the tensor products now become tensor products in the dg sense!). One notices from the formula for the differential that it is a degree zero A^e-map hence $b^{-i} \in \mathcal{C}A^e$. Therefore we have a bar complex for any dg-algebra A.

Definition 3.4.8. *Let A be a dg-algebra and $(B^{-i}(A), b^{-}i)$ its bar complex. The coproduct totalization of the bar complex is denoted $B(A)$ and called the bar resolution of A; i.e. $B(A)$ is the dg-module whose underlying graded module structure is given by*

$$\coprod_{i \in \mathbb{Z}} [-i] \, \mathrm{B}^i(A),$$

where the i^{th} differential is defined by

$$d^i = d_{[-i] \, \mathrm{B}^i(A)} + b^i.$$

Now, we must justify the name Bar resolution. We see that $\mathrm{B}(A)$ is naturally an A^e-module since $[i] \, \mathrm{B}^{-i}(A) = A \otimes ([1]A)^{\otimes i} \otimes A$. Let $\epsilon : \mathrm{B}(A) \to A$ be the map such that $\epsilon(a \otimes a') = a \cdot a'$ and is zero otherwise.

Lemma 3.4.9. *Let A be a dg-algebra then $(\mathrm{B}(A), \epsilon)$ is a semi-free resolution of A as an A^e module.*

Proof. We first claim that ϵ is a quasi-isomorphism. For this consider the bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
\cdots \arrow[r] & (A^{\otimes 3})^{j+1} \arrow[r] \arrow[u] & (A^{\otimes 2})^{j+1} \arrow[r, "\epsilon"] \arrow[u] & A^{j+1} \arrow[u] \\
\cdots \arrow[r] & (A^{\otimes 3})^{j} \arrow[r] \arrow[u] & (A^{\otimes 2})^{j} \arrow[r, "\epsilon"] \arrow[u] & A^{j} \arrow[u] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd} \tag{4.6}$$

Then we see via the argument given in Lemma 2.1.2 that each row is acyclic and then via the acyclic assembly lemma [Wei, Lemma 2.7.3] it follows that the coproduct totalization of this bicomplex is acyclic. Thus it follows that ϵ is a quasi-isomorphism.

Since k is a field for $i \geq 0$, $([1]A)^{\otimes i}$ is a semi-free dg k-module. Thus for each $i \geq 0$ there is a filtration

$$0 = F(0) \subseteq F(1) \subseteq \cdots \subseteq F(j) \subseteq F(j+1) \subseteq \cdots \subseteq ([1]A)^{\otimes i}$$

such that $F(j+1)/F(j)$ is a free dg k-module for all $j \geq 0$. Then by tensoring with A^e we get a filtration

$$0 \subseteq F(1) \otimes A^e \subseteq \cdots \subseteq F(j) \otimes A^e \subseteq F(j+1) \otimes A^e \subseteq \cdots \subseteq ([1]A)^{\otimes i} \otimes A^e.$$

We see that $F(j+1) \otimes A^e / F(j) \otimes A^e \cong F(j+1)/F(j) \otimes A^e$ for all $j \geq 0$ thus each $A \otimes ([1]A)^{\otimes i} \otimes A$ is a semi-free A^e-module. Then for $j = 0$ let $G(j) := 0$ and for $j \geq 1$ let

$$G(j) := \mathrm{Tot}^{\coprod}(\cdots \longrightarrow 0 \longrightarrow \mathrm{B}^{-j+1}(A) \longrightarrow B^{-j+2}(A) \longrightarrow \cdots \longrightarrow \mathrm{B}^0(A))$$

so we have a filtration

$$0 = G(0) \subseteq \cdots \subseteq G(j) \subseteq G(j+1) \subseteq \cdots \subseteq \mathrm{B}(A)$$

such that $G(j+1)/G(j) \cong [j+1]\,\mathrm{B}^{-j-1}(A)$ for all $j \geq 0$. Then it follows from Remark 3.5.5 that $\mathrm{B}(A)$ is semi-free. □

We are now in a position to make a reasonable definition of the Hochschild cohomology cochain complex in analogy with the associative case, see Definition 2.1.4.

Definition 3.4.10. *Let A be a dg-algebra and X a dg-A^e-module, define for $i \geq 0$:*

$$C^i(A, X) := (\mathrm{Dif}\ A^e)(\mathrm{B}(A), X)^i \tag{4.7}$$

i.e the space of degree i morphisms. We define the differential to be the natural differential on $(\mathrm{Dif}\ A^e)(\mathrm{B}(A), X)$ induced by the differentials on $\mathrm{B}(A)$ and X respectively.

Indeed taking cohomology of 3.4.10 gives

$$H^i(\text{Dif } A^e)(\mathrm{B}(A), X) \cong (\mathcal{H}A^e)(\mathrm{B}(A), [i]X).$$

By Lemma 3.4.7, we know $\mathrm{B}(A)$ is h-projective hence

$$(\mathcal{H}A^e)(\mathrm{B}(A), [i]X) \cong (\mathcal{D}A^e)(\mathrm{B}(A), [i]X),$$

and by Lemma 3.4.9, $\mathrm{B}(A)$ is quasi-isomorphic to A thus,

$$(\mathcal{D}A^e)(\mathrm{B}(A), [i]X) \cong (\mathcal{D}A^e)(A, [i]X).$$

But for our purposes we would like to work over the ground field k. To do this we need a dg version of Lemma 2.1.5.

Lemma 3.4.11. *Suppose we have two dg-k-algebras A and B. Then we have the following pair of adjoint functors*

$$\begin{tikzcd} (\text{Dif } k) \arrow[r, bend left, "A\otimes ?\otimes B"] & (\text{Dif } A^{op}\otimes B) \arrow[l, bend left, "\text{res}"] \end{tikzcd} \tag{4.8}$$

In particular we have for all $M \in \text{Dif } A^{op}\otimes B$ and $N \in \text{Dif} k$ the following pair of inverse isomorphisms dg k-modules

$$\begin{tikzcd} (\text{Dif } A^{op}\otimes B)(A\otimes N\otimes B, M) \arrow[r, bend left, "\psi_{N,M}"] & (\text{Dif } k)\ (N, \text{res}(M)) \arrow[l, bend left, "\phi_{N,M}"] \end{tikzcd} \tag{4.9}$$

where for $f \in (\text{Dif } k)(N, \text{res}(M))$ we have

$$\phi_{N,M}(f) = \text{mult} \circ (id_A \otimes f \otimes id_B), \tag{4.10}$$

and for $g \in (\text{Dif } A^{op}\otimes B)(A\otimes N\otimes B, M)$

$$\psi_{N,M}(g) = g \circ \iota_N, \tag{4.11}$$

where $\iota_N : N \longrightarrow A\otimes N\otimes B$ such that $n \mapsto 1_A \otimes n \otimes 1_B$.

Proof. The proof is the same as the associative case but now there are many more signs to keep track of that are in the tensor products. $\square$

Using 3.4.11 we define maps, δ^{i^*} to be maps such that

$$\begin{tikzcd} (\text{Dif } A^e)(A\otimes A^{\otimes i+1}\otimes A, X) \arrow[d, "\psi_{A^{\otimes i},X}"] & (\text{Dif } A^e)(A\otimes A^{\otimes i}\otimes A, X) \arrow[l, "b^{-i-1*}"'] \\ (\text{Dif } k)(A^{\otimes i+1}, X) & (\text{Dif } k)(A^{\otimes i}, X) \arrow[l, "\delta^{i^*}"] \arrow[u, "\phi_{A^{\otimes i},X}"'] \end{tikzcd} \tag{4.12}$$

In formulas, for any homogeneous $f \in \text{Dif } k\ (A^{\otimes i}, A)$ we have

$$\begin{aligned} \delta^{i^*}(f)(a_1\otimes\cdots\otimes a_{i+1}) &= (-1)^{|a_1|\,|f|} a_1 \cdot f(a_2\otimes\cdots\otimes a_{i+1}) \\ &+ \sum_{j=1}^{j=i} (-1)^j f(a_1\otimes\cdots a_j a_{j+1}\cdots\otimes a_{i+1}) \\ &+ (-1)^{i+1} f(a_1\otimes\cdots\otimes a_i)\cdot a_{i+1} \end{aligned} \tag{4.13}$$

Thus we now can define the Hochschild complex of A with coefficients in X over k.

Definition 3.4.12. *For X an $A^{\mathrm{op}} \otimes A$-module the Hochschild complex, $(C^i(A,X), d^i)_{i\in\mathbb{Z}}$, is defined for $i \geq 0$*

$$C^i(A,X) = \mathrm{Tot}^{\prod}(Z)^i. \tag{4.14}$$

Where Z is the bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(\mathrm{Dif}\ k)(k,X)^1 \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A,X)^1 \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A^{\otimes 2},X)^1 \arrow[r] \arrow[u] & \cdots \\
(\mathrm{Dif}\ k)(k,X)^0 \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A,X)^0 \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A^{\otimes 2},X)^0 \arrow[r] \arrow[u] & \cdots \\
(\mathrm{Dif}\ k)(k,X)^{-1} \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A,X)^{-1} \arrow[r] \arrow[u] & (\mathrm{Dif}\ k)(A^{\otimes 2},X)^{-1} \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{4.15}$$

Where the horizontal arrows are given by formula (4.13)*, and the vertical arrows are the normal differentials on the Hom space of two dg-modules. The differentials d^i those on $\mathrm{Tot}^{\prod}(Z)$.*

Lemma 3.4.13. *Let $\mathrm{Ch}(\mathcal{C}(A))$ denote the category of chain complexes of objects in $\mathcal{C}(A)$ and X and dg A module. Then the diagram*

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{C}(A)) \arrow[r, "\mathrm{Tot}^{\coprod}"] \arrow[d, "(\mathrm{Dif}\ A)(?,X)"'] & \mathcal{C}(A) \arrow[d, "(\mathrm{Dif}\ A)(?,X)"] \\
\mathrm{Ch}(\mathcal{C}(k)) \arrow[r, "\mathrm{Tot}^{\prod}"] & \mathcal{C}(k)
\end{tikzcd} \tag{4.16}$$

commutes up to isomorphism.

Proof. Let $M_\bullet \in \mathrm{Ch}(\mathcal{C}(A))$ then one shows by computation that the following map is an isomorphism.

$$(\mathrm{Dif}\ A)(\mathrm{Tot}^{\coprod}(M_\bullet), X) \xrightarrow{\varphi} \mathrm{Tot}^{\prod}(\mathrm{Dif}\ A)(M_\bullet, X) \tag{4.17}$$

Where for homogeneous $f \in (\mathrm{Dif}\ A)(\mathrm{Tot}^{\coprod}(M_\bullet), X)$

$$\varphi : f \longmapsto (f \circ \mathrm{Tot}^{\coprod}(\iota_{M_i}))_{i\in\mathbb{Z}} \tag{4.18}$$

where ι_{M_i} embeds the chain complex that has M_i in the i^{th} position and zeros everywhere else into $M_\bullet$. $\square$

Corollary 3.4.14. *The complexes of* 3.4.10 *and* 3.4.12 *are isomorphic.*

Remark 3.4.15. One can show that for A a dg-k-algebra, $C^*(A)$ has the structure of a B_∞-algebra. See, [GJ].

Recalling the notation in Example 3.1.28, let $\Lambda(x)^{dg} = \mathbb{C}[x]/(x^2)$ with x in adams degree -1. We write the superscript dg, here the emphasize the diagonal grading. We give $\Lambda(x)^{dg}$ the structure of a dg-algebra with x in homological degree 1 and differential $d = 0$. So we are thinking of $\Lambda(x)^{dg}$ as an adams graded dg-algebra that sits "diagonally". In the next Lemma we calculate the Hochschild cohomology of $\Lambda(x)^{dg}$, note that the adams grading descends to the Hochschild groups.

Lemma 3.4.16. *The Hochschild cohomology of* $\Lambda(x)^{dg}$ *is*

$$\mathrm{HH}^i(\Lambda(x)^{dg})_j \quad \cong \quad \begin{cases} \mathbb{C}, & i = 0,\ j \geq 0 \\ \mathbb{C}, & i = 1,\ j \geq -1 \\ 0, & \textit{otherwise} \end{cases}$$

Proof. We have the resolution of $\Lambda(x)^{dg}$

$$\cdots \xrightarrow{d} [-2]\langle 2\rangle\Lambda(x,y)^{dg} \xrightarrow{d} [-1]\langle 1\rangle\Lambda(x,y)^{dg} \xrightarrow{d} \Lambda(x,y)^{dg} \xrightarrow{\epsilon} \Lambda(x)^{dg}. \tag{4.19}$$

Where

$$\epsilon: \qquad 1 \mapsto 1, \qquad x \mapsto x, \qquad y \mapsto x, \qquad xy \mapsto 0,$$

and d is given by left multiplication by $(x - y)$. Applying $(\mathrm{Dif}\ \Lambda(x,y)^{dg})(?, \Lambda(x)^{dg})$ to the Complex (4.19) gives the following chain complex

$$(\mathrm{Dif}\ \Lambda(x,y)^{dg})(\Lambda(x,y)^{dg}, \Lambda(x)^{dg}) \xrightarrow{0} (\mathrm{Dif}\ \Lambda(x,y)^{dg})([-1]\langle -1\rangle\Lambda(x,y)^{dg}, \Lambda(x)^{dg}) \xrightarrow{0} \cdots, \tag{4.20}$$

which is isomorphic via adjunction to the complex

$$\Lambda(x)^{dg} \xrightarrow{0} [1]\langle -1\rangle\Lambda(x)^{dg} \xrightarrow{0} [2]\langle -2\rangle\Lambda(x)^{dg} \xrightarrow{0} \cdots, \tag{4.21}$$

which is the adams graded bicomplex whose arrows are all the 0 map and whose only nonzero elements look like

$$\begin{tikzcd}
x & & & \\
1 \arrow[u] \arrow[r] & \langle -1\rangle x & & \\
& \langle -1\rangle 1 \arrow[u] \arrow[r] & \langle -2\rangle x & \\
& & \langle -2\rangle 1 \arrow[u] \arrow[r] & \langle -3\rangle x \\
& & & & \ddots,
\end{tikzcd} \tag{4.22}$$

which we totalize with respect to the product. It follows that

$$\begin{aligned}\mathrm{HH}^0(\Lambda(x)^{dg})_j &\cong \mathbb{C}, \quad j \geq 0\\ \mathrm{HH}^0(\Lambda(x)^{dg})_j &\cong 0, \quad j < 0,\end{aligned}$$

and,

$$\begin{aligned}\mathrm{HH}^1(\Lambda(x)^{dg})_j &\cong \mathbb{C}, \quad j \geq -1\\ \mathrm{HH}^1(\Lambda(x)^{dg})_j &\cong 0, \quad j < -2.\end{aligned}$$

□

Corollary 3.4.17. *For all $i, j \in \mathbb{Z}$*

$$\mathrm{HH}^i(\mathbb{C}[x])_j \cong H^i(\Lambda(x)^{dg})_j$$

Proof. Is immediate from comparing Lemmas 3.4.16 and 2.2.15 □

Corollary 3.4.17 is part of a general phenomena (that we have alluded to before) that if A is a Koszul algebra , then there is a graded isomorphism of algebras between the Hochschild cohomology of A and its dg-Koszul dual $A^{!^{dg}}$ where now we view $A^!$ as an algebra that sits diagonally between the homological grading and adams grading. To illustrate this point further we calculate one more example.

Let $\mathbb{C}[x]^{dg}$ be an adams graded dg algebra where we put x in adams degree -1 and in homological degree 1 and the differential 0.

Lemma 3.4.18. *The Hochschild cohomology of $\mathbb{C}[x]^{dg}$ is*

$$\mathrm{HH}^i(\mathbb{C}[x]^{dg})_j \quad \cong \quad \begin{cases} \mathbb{C}, & \textit{if } i = 0,\ j = 1, 2,\\ \mathbb{C}, & \textit{if } i > 0,\ j = -i,\\ 0, & \textit{otherwise}\end{cases}$$

Proof. we have the following $B := \mathbb{C}[x]^{dg^{\mathrm{op}}} \otimes \mathbb{C}[x]^{dg}$ resolution of $\mathbb{C}[x]^{dg}$ (we will write abbreviate $1 \otimes x$ as y and $x \otimes 1$ as x).

$$0 \longrightarrow [-1]\langle 1\rangle B \xrightarrow{d} B \xrightarrow{\epsilon} \mathbb{C}[x]^{dg} \tag{4.23}$$

where ϵ is defined as the map

$$y^a x^b \mapsto x^{a+b} \tag{4.24}$$

and d is left multiplication by $(x - y)$.

<u>Claim 1.</u> The map ϵ is a morphism of B-modules.

We see that

$$\begin{aligned}(y^a x^b) \cdot x = y^a x^{b+1} &\quad\mapsto\quad x^{a+b+1} = x^{a+b} \cdot x\\ (y^a x^b) \cdot y = (-1)^{a+b} y^{a+1} x^b &\quad\mapsto\quad (-1)^{a+b} y^{a+b+1} = x^{a+b} \cdot y,\end{aligned}$$

and the claim follows.

<u>Claim 2.</u> The Kernel of ϵ is $\langle x - y\rangle$ where $\langle x - y\rangle$ denotes the submodule of B generated by $(x - y)$.

Clearly $\langle x-y\rangle \subseteq \mathrm{Ker}(\epsilon)$. Assume that $\sum_{i=0}^{n} \alpha_i y^i x^{n-i} \in \mathrm{Ker}(\epsilon)$, then

$$0 = \epsilon(\sum_{i=0}^{n} \alpha_i y^i x^{n-i}) = \sum_{i=0}^{n} \alpha_i x^n,$$

and hence $\sum_{i=0}^{n} \alpha_i = 0$. So, if we can show that

$$y^i x^{n-i} - y^j x^{n-j} \in \langle x-y\rangle,$$

for $i > j$ the claim will follow. First we establish the equation

$$y^a - x^a = (y-x)\cdot \sum_{i=0}^{a-1} (-1)^i y^i x^{a-i-1}, \tag{4.25}$$

for $a \geq 0$. We will study the coefficients of the $y^j x^{a-j}$ on the right hand side of Equation (4.25). When $j = 0, a$ we have

$$\begin{aligned} (-1)x\cdot x^{a-1} &= -x^a \\ (-1)^{a-1} y\cdot y^{a-1} &= y^a. \end{aligned}$$

For $0 < j < a$ we have

$$\begin{aligned} y\cdot(-1)^{j-1} y^{j-1} x^{a-j} - x\cdot(-1)^j y^j x^{a-j-1} &= (-1)^{j-1}(-1)^{j-1} y^j x^{a-j} - (-1)^j(-1)^j y^j x^{a-j} \\ &= y^j x^{a-j} - y^j x^{a-j} \\ &= 0. \end{aligned}$$

This establishes Equation (4.25). The claim now follows since

$$\begin{aligned} y^i x^{n-i} - y^j x^{n-j} &= ((-1)^{j(i-j)} y^{i-j} - (-1)^{i(i-j)} x^{i-j} y^j)\cdot y^j x^{n-i} \\ &= (-1)^{j(i-j)}(y^{i-j} - x^{i-j})\cdot y^j x^{n-i}. \end{aligned}$$

Indeed it follows now that the Complex 4.23 is exact.

Applying $(\mathrm{Dif}\ B)(?, \mathbb{C}[x]^{dg})$ to the resolution in 4.23 gives the complex

$$(\mathrm{Dif}\ B)(B, \mathbb{C}[x]^{dg}) \xrightarrow{d^*} (\mathrm{Dif}\ B)([-1]\langle 1\rangle B, \mathbb{C}[x]^{dg}) \longrightarrow 0 \longrightarrow \cdots \tag{4.26}$$

Which in turn is isomorphic to complex

$$\mathbb{C}[x]^{dg} \xrightarrow{\tilde{d}} [1]\langle -1\rangle \mathbb{C}[x]^{dg} \longrightarrow 0 \longrightarrow \cdots \tag{4.27}$$

The differential $\tilde{d}$ is induced from d^* and has the following description

$$x^a \mapsto 0 \qquad a \text{ is even} \tag{4.28}$$

$$x^a \mapsto 2x^{a+1} \qquad a \text{ is odd.}$$

Complex (4.27) can be realized as the bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
x^3 \arrow[u] \arrow[r, "2x^2"] & \langle -1\rangle x^4 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
x^2 \arrow[u] \arrow[r, "0"] & \langle -1\rangle x^3 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
x \arrow[u] \arrow[r, "2x"] & \langle -1\rangle x^2 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
1 \arrow[u] \arrow[r, "0"] & \langle -1\rangle x \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
0 \arrow[u] \arrow[r] & \langle -1\rangle 1 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots
\end{tikzcd} \tag{4.29}$$

Where all the unlabeled arrows are 0. Totalizing with respect to the product gives us the chain complex

$$\mathbb{C}1 \oplus \langle -1\rangle \mathbb{C}1 \xrightarrow{\partial^0} \mathbb{C}x \oplus \langle -1\rangle \mathbb{C}x \xrightarrow{\partial^1} \mathbb{C}x^2 \oplus \langle -1\rangle \mathbb{C}x^2 \xrightarrow{\partial^2} \cdots \tag{4.30}$$

Where the differentials are

$$\partial^i = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & i \text{ even} \\ \\ \begin{pmatrix} 0 & 0 \\ 2x^i & 0 \end{pmatrix}, & i \text{ odd} \end{cases} \tag{4.31}$$

Thus, for i even,

$$\mathrm{Ker}(\partial^i) = \mathbb{C}x^i \oplus \langle -1\rangle \mathbb{C}x^i \tag{4.32}$$

and for i odd,

$$\mathrm{Ker}(\partial^i) = \langle -1\rangle \mathbb{C}x^i \qquad \mathrm{Im}(\partial^i) = \langle -1\rangle \mathbb{C}x^{i+1} \tag{4.33}$$

So taking homology of the complex (4.30) at $i = 0$ gives

$$\begin{aligned} \mathrm{HH}^0(\mathbb{C}[x]^{dg})_j &\cong \mathbb{C}, \quad j = 1, 2, \\ \mathrm{HH}^0(\mathbb{C}[x]^{dg})_j &= 0, \quad \text{otherwise}, \end{aligned}$$

and for $i > 0$

$$\begin{aligned} \mathrm{HH}^i(\mathbb{C}[x]^{dg})_j &\cong \mathbb{C}, \quad i = -j, \\ \mathrm{HH}^i(\mathbb{C}[x]^{dg})_j &= 0, \quad \text{otherwise}, \end{aligned}$$

$\square$

Corollary 3.4.19. *For all $i, j \in \mathbb{Z}$*

$$\mathrm{HH}^i(\mathbb{C}[\zeta]/(\zeta^2))_j \cong \mathrm{HH}^i(\mathbb{C}[x]^{dg})_j$$

Proof. Follows from Lemmas 3.4.18 and 2.2.17. $\square$

3.5. **Hochschild Cohomology for DG Categories.**

We now will generalize the work that we did in the previous subsection to the case of modules over some dg-category. In this subsection when we talk about a dg-module, we will always mean a dg-module over some dg-category $\mathscr{A}$ unless otherwise stated. We will know how to define Hochschild cohomology for a dg-category $\mathscr{A}$ with coefficients in X, an $\mathscr{A}^e$-module, if we have a module to play the role of A. Luckily there is an obvious choice. Most of the definitions in the previous section generalize in the obvious way.

Definition 3.5.1. *Let $\mathscr{A}$ be a dg-category then let $\mathcal{I}_{\mathscr{A}}$ be the dg $\mathscr{A}^e$-module such that for $a, a' \in \mathscr{A}$*

$$\mathcal{I}_{\mathscr{A}}(a, a') = \mathscr{A}(a, a') \tag{5.1}$$

Definition 3.5.2. *Let $\mathscr{A}$ be a dg category and X an $\mathscr{A}^e$-module. Then the Hochschild cohomology of $\mathscr{A}$ is*

$$\mathrm{HH}^i(\mathscr{A}, X) := (\mathcal{D}\mathscr{A})(\mathcal{I}_{\mathscr{A}}, [i]X) \tag{5.2}$$

Now just as in Section 2.3, we need to find an $\mathscr{A}^e$-resolution of $\mathcal{I}_{\mathscr{A}}$ that will exist for any dg-category $\mathscr{A}$ and we would also like an explicit description over the ground field k. As we will see, we can transfer all that we have done in the dg-algebra case to the dg-category world. The first thing we need is a notion of semi-free.

Definition 3.5.3. *Let M be a dg module over a dg-category $\mathscr{A}$. We call M free if it is isomorphic to the direct sum of dg-modules of the form $[n]\hat{a}$ for $n \in \mathbb{Z}$ and $a \in \mathscr{A}$.*

Definition 3.5.4. *A dg-module M is semi-free if there is a (possibily infinite) filtration*

$$0 = M(0) \subseteq M(1) \subseteq \cdots \subseteq M(i) \subseteq M(i+1) \subseteq \cdots \subseteq M \tag{5.3}$$

(i.e. bounded below) such that $\bigcup_{i \geq 0} M(i) = M$ (i.e. exhaustive) and $M(j+1)/M(j)$ is free in the sense of 3.5.3.

Remark 3.5.5. if M is a dg module that has an exhaustive bounded below filtration such that $M(i+1)/M(i)$ is isomorphic to a semi-free module. Then M is semi-free.

Definition 3.5.6. *Let M be a dg module over $\mathscr{A}$. The pair $(\bar{M}, f)$ where $\bar{M}$ is a semi-free dg module and $f : \bar{M} \to M$ is a morphism of dg A modules that is a quasi-isomorphism (i.e. induces isomorphism in cohomology) is called a semi-free resolution of M.*

Just as in the case of dg-algebras, there are enough semi-free objects in $\mathcal{D}\mathscr{A}$. However, while the proof is philosophically the same there are some subtleties, for which the reader should consult [Kel1].

Theorem 3.5.7. *Let M be a dg module over $\mathscr{A}$. Then M has a semi-free resolution.*

Proof. See [Kel1, Section 3.5] □

It turns out that the semi-free objects in the case of dg-categories are again h-projective.

Lemma 3.5.8. *If M be a semi-free dg-$\mathscr{A}$-module, it is h-projective.*

Proof. The proof is exactly the same as 3.4.7. □

Now we have to redefine the bar resolution. The following is a natural definition.

Definition 3.5.9. *Let $\mathscr{A}$ be a dg-category the define* $\mathrm{B}(\mathscr{A})$ *to be the product totalizaiton of the complex*

$$\cdots \longrightarrow \mathrm{B}^{-i}(\mathscr{A}) \longrightarrow \cdots \longrightarrow \mathrm{B}^{0}(\mathscr{A}) \tag{5.4}$$

where $B^{-i}(\mathscr{A})$ is the $\mathscr{A}^e$ module

$$\bigoplus \mathscr{A}(a_i, ?) \otimes \mathscr{A}(a_{i-1}, a_i) \otimes \cdots \otimes \mathscr{A}(a_0, a_1) \otimes \mathscr{A}(?, a_0) \tag{5.5}$$

Where the sum is over all i-tuples of objects in $\mathscr{A}$. For $\phi_i \otimes \cdots \otimes \phi_1 \in \mathscr{A}(a_i, a) \otimes \cdots \otimes \mathscr{A}(a_0, a')$ the differential is given by

$$\begin{aligned} b^{-i}(\phi_i \otimes \cdots \otimes \phi_1) &= \phi_i \circ \phi_{i-1} \otimes \cdots \otimes \phi_1 \\ &+ \sum_{j=2}^{i-1} (-1)^{j-1} \phi_i \otimes \cdots \otimes \phi_{i-j+1} \circ \phi_{i-j} \otimes \cdots \otimes \phi_1 \end{aligned} \tag{5.6}$$

Let $\epsilon : \mathrm{B}(\mathscr{A}) \to \mathcal{I}_{\mathscr{A}}$ such that

$$\begin{aligned} \epsilon_{a,a'} : \bigoplus_{a_0 \in \mathscr{A}} \mathscr{A}(a_0, a') \otimes \mathscr{A}(a, a_0) &\longrightarrow \mathscr{A}(a, a') \\ \phi \otimes \psi &\mapsto \phi \circ \psi \end{aligned}$$

and is zero otherwise.

The next Lemma shows that $\mathrm{B}(\mathscr{A})$ provides us with a semi-free and hence an h-projective resolution for $\mathcal{I}_{\mathscr{A}}$. The reader should compare this to the situation in the case of associative algebras and dg-algebras. Indeed, restricting to the case associative algebra, A, $\mathrm{B}(A)$ gives the complex 2.1.1 and similarly in the dg-algebra case.

Lemma 3.5.10. $\mathrm{B}(\mathscr{A})$ *is semi-free and* $\epsilon : \mathrm{B}(\mathscr{A}) \to \mathcal{I}_{\mathscr{A}}$ *is a quasi-isomorphism of dg $\mathscr{A}^e$-modules.*

Proof. In order to check that ϵ is a quasi-isomorphism we just have to check that it is a quasi-isomorphism on all pairs of objects (a, a') where $a, a' \in \mathscr{A}$. With this in mind for any pair (a, a') such that $a, a' \in \mathscr{A}$ we have the following bicomplex

$$\begin{tikzcd}
 \vdots & \vdots & \vdots \\
\cdots (\bigoplus \mathscr{A}(a_1,a') \otimes \mathscr{A}(a_0,a_1) \otimes \mathscr{A}(a,a_0))^i \arrow[u] \arrow[r] & (\bigoplus \mathscr{A}(a_0,a') \otimes \mathscr{A}(a,a_0))^i \arrow[u] \arrow[r, "\epsilon"] & \mathscr{A}^i(a,a') \arrow[u] \\
\cdots (\bigoplus \mathscr{A}(a_1,a') \otimes \mathscr{A}(a_0,a_1) \otimes \mathscr{A}(a,a_0))^{i-1} \arrow[u] \arrow[r] & (\bigoplus \mathscr{A}(a_0,a') \otimes \mathscr{A}(a,a_0))^{i-1} \arrow[u] \arrow[r, "\epsilon"] & \mathscr{A}^{i-1}(a,a') \arrow[u] \\
 \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

We see that the rows of this bicomplex are acyclic via the splitting maps, the first of which is given. One sees then how to define the others

$$\begin{aligned} s : \bigoplus \mathscr{A}(a_{i+1},a') \otimes \cdots \otimes \mathscr{A}(a,a_0) \quad &\longrightarrow \quad \bigoplus \mathscr{A}(a_i,a') \otimes \cdots \otimes \mathscr{A}(a,a_0) \\ \phi \quad &\mapsto \quad \phi \otimes \mathrm{id}_a^{\otimes i} \end{aligned}$$

Then using the acyclic assembly lemma it follows that the product totalization of this complex is acyclic and hence $\epsilon_{a,a'}$ is a quasi-isomorphism.

To see that $\mathrm{B}(\mathscr{A})$ is quasi-isomorphism we essentially follow the same proof used in Lemma 3.4.9. We first show that each $B^{-i}(\mathscr{A})$ is a semi-free $\mathscr{A}^e$ module. First fix an i-tuple $a_0,\dots,a_i$. Then we see that

$$\mathscr{A}(a_{i-1},a_i) \otimes \cdots \otimes \mathscr{A}(a_0,a_1) \tag{5.7}$$

is a semi-free dg-k-module. Thus it admits a filtration

$$0 = F(0) \subseteq \cdots \subseteq \mathscr{A}(a_{i-1},a_i) \otimes \cdots \otimes \mathscr{A}(a_0,a_1) \tag{5.8}$$

such that $F(j+1)/F(j)$ for all $j \geq 0$ are free dg-k-modules. Then then setting $F'(j) = \mathscr{A}(a_i,?) \otimes F(j) \otimes \mathscr{A}(?,a_0)$ we get a new filtration

$$0 = F'(0) \subseteq \cdots \subseteq \mathscr{A}(a_i,?) \otimes \mathscr{A}(a_{i-1},a_i) \otimes \cdots \otimes \mathscr{A}(a_0,a_1) \otimes \mathscr{A}(?,a_0) \tag{5.9}$$

such that for each $j \geq 0$, $F'(j+1)/F'(j)$ is isomorphic to a direct sum of shifts of $\widehat{(a_0,a_i)}$. It follows that $B^{-i}(A)$ is a semi-free dg-$\mathscr{A}^e$-module, since direct sums of semi-free modules are again semi-free. Now we have a bounded above complex of dg-$\mathscr{A}^e$-modules

$$\cdots \longrightarrow \mathrm{B}^{-i}(\mathscr{A}) \longrightarrow \cdots \longrightarrow \mathrm{B}^0(\mathscr{A}) \tag{5.10}$$

and for $j = 0$ we set $G(0) := 0$ and for $j \geq 1$

$$G(j) := \mathrm{Tot}^{\coprod}(\cdots \longrightarrow 0 \longrightarrow \mathrm{B}^{-j+1}(\mathscr{A}) \longrightarrow \cdots \longrightarrow B^0(\mathscr{A})) \tag{5.11}$$

This defines for us an exhaustive bounded below filtration whose quotients are semi-free $\mathscr{A}^e$-modules hence $\mathrm{B}(\mathscr{A})$ is a semi-free $\mathscr{A}^e$-module. $\square$

Now, while not surprisingly but still very importantly we can make the following definition for the Hochschild cohomology of a dg-category. By now it should be clear that this is indeed equivalent to definition 3.5.2.

Definition 3.5.11. *The Hochschild cochain complex of $\mathscr{A}$ with coefficients in X, for X an $\mathscr{A}^e$-module is*

$$(\mathrm{Dif}\ \mathscr{A}^e)(\mathrm{B}(\mathscr{A}), X) \tag{5.12}$$

We now state state the following very general adjointness result that can be found in [Drin] without giving a proof, but we explain a special case of it which is interesting to us. This result will be used so that we can work over the ground field k.

Proposition 3.5.12. *Suppose that $F : \mathscr{C} \to \mathscr{D}$ is a dg-functor, let $X_F = \mathscr{D}(?, F(?))$. Then there is an adjunction $(? \otimes X_F, (\mathrm{Dif}\ \mathscr{C}^{\mathrm{op}} \otimes \mathscr{D})(X_F, ?))$*

$$\begin{tikzcd} (\mathrm{Dif}\ \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}) \arrow[r, bend left, "? \otimes X_F"] & (\mathrm{Dif}\ \mathscr{C}^{op} \otimes \mathscr{D}) \arrow[l, bend left, "(\mathrm{Dif}\ \mathscr{C}^{op} \otimes \mathscr{D})(X_F, ?)"] \end{tikzcd} \tag{5.13}$$

Proof. The proof is essentially the same as in the case of dg-algebras, see [Drin]. $\square$

Taking a special case of Lemma 3.5.12 when $\mathscr{C} = k$ and $\mathscr{D} = \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ and $F_{a_0,a_p} : k \to \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ that sends $\bullet$ to $\widehat{(a_0, a_p)}$ gives the isomorphism

$$\begin{aligned} &(\mathrm{Dif}\ \mathscr{A}^e)(\mathscr{A}(a_p, ?) \otimes \mathscr{A}(a_{p-1}, a_p) \otimes \cdots \otimes \mathscr{A}(?, a_0), X) \\ &\quad \cong (\mathrm{Dif}\ k)(\mathscr{A}(a_{p-1}, a_p) \otimes \cdots \otimes \mathscr{A}(a_0, a_1), X(a_0, a_p)) \end{aligned}$$

Then for every $p \geq 0$, the horizontal differentials are defined such that the following diagram commutes

$$\begin{tikzcd} \prod(\mathrm{Dif}\ \mathscr{A}^e)(\mathscr{A}(a_p, ?) \otimes \cdots \mathscr{A}(?, a_0), X)^i \arrow[r] \arrow[d] & \prod(\mathrm{Dif}\ \mathscr{A}^e)(\mathscr{A}(a_p, ?) \otimes \cdots \mathscr{A}(?, a_0), X)^{i+1} \\ \prod(\mathrm{Dif}\ k)(\mathscr{A}(a_p, a_{p-1}) \otimes \cdots \mathscr{A}(a_0, a_1), X(a_0, a_p))^i \arrow[r, "\delta"] & \prod(\mathrm{Dif}\ k)(\mathscr{A}(a_p, a_{p-1}) \otimes \cdots \mathscr{A}(a_0, a_1), X(a_0, a_p))^{i+1} \arrow[u] \end{tikzcd} \tag{5.14}$$

Where the products are taken over all p-tuples of objects. Explicitly δ is given by

$$\begin{aligned} & df(\phi_{p+1} \otimes \cdots \otimes \phi_1) \\ = \ & (-1)^{|\phi_{p+1}|\,|f|}\phi_{i+1} \circ f(\phi_p \otimes \cdots \otimes \phi_1) \\ & + \sum_{j=1}^{p} (-1)^j f(\phi_{p+1} \otimes \cdots \otimes \phi_{p-j+2} \circ \phi_{p-j+1} \cdots \otimes \ \otimes \phi_1) \\ & + (-1)^{p+1} f(\phi_{p+1} \otimes \cdots \otimes \phi_2) \circ \phi_1. \end{aligned} \tag{5.15}$$

This leads us to make the following definition of the Hochschild complex over k.

Definition 3.5.13. *Let $\mathscr{A}$ be a dg-category, and X an $\mathscr{A}^e$-module. Define the Hochschild complex $C^*(\mathscr{A})$ to be the product totalization of the bicomplex whose $(p, j)^{th}$ entry is:*

$$\prod_{a_0, \ldots, a_p} (\mathrm{Dif}\ k)(\mathscr{A}(a_{p-1}, a_p) \otimes \cdots \otimes \mathscr{A}(a_0, a_1), X(a_0, a_p))^j,$$

where the product is take over all p-tuples of objects. The vertical maps are the differentials on:

$$\prod_{a_0,\ldots,a_p} (\mathrm{Dif}\ k)(\mathscr{A}(a_{p-1},a_p)\otimes\cdots\otimes\mathscr{A}(a_0,a_1), X(a_0,a_p))$$

*and the horizontal maps are given by Formula (*5.14 *).*

Lemma 3.5.14. *Let* $\mathrm{Ch}(\mathcal{C}(\mathscr{A}))$ *denote the category of chain complexes of objects in* $\mathcal{C}(\mathscr{A})$ *and* X *a dg* $\mathscr{A}$ *module. Then the diagram*

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{C}(\mathscr{A})) \arrow[r, "\mathrm{Tot}\Pi"] \arrow[d, "(\mathrm{Dif}\ \mathscr{A})(?,X)"'] & \mathcal{C}(\mathscr{A}) \arrow[d, "(\mathrm{Dif}\ \mathscr{A})(?,X)"] \\ \mathrm{Ch}(\mathcal{C}(k)) \arrow[r, "\mathrm{Tot}\Pi"] & \mathcal{C}(k) \end{tikzcd} \tag{5.16}$$

commutes up to natural isomorphism.

Proof. Works the same was as in the dg-algebra case. □

Corollary 3.5.15. *The chain complexes defined in Definition* 3.5.11 *and in Definition* 3.5.13 *are isomorphic.*

Remark 3.5.16. One can show that that $C^*(\mathscr{A})$ has the structure of a B_∞ algebra using a similar argument to the one showing that $C^*(A)$ is a B_∞-algebra for A a dg-algebra. Alternatively one can use the language of graded quivers, [Low].

Remark 3.5.17. It is a theorem of Keller's that if $F : \mathscr{A} \to \mathscr{B}$ is a fully faithful dg-functor then map induced on the Hochschild cochains given by restricting along F induces a morphisms of B_∞-algebras. A morphism of B_∞-algebras is a morphism of the underlying chain complex such that the map induced on the shifted tensor algebras is a morphism of dg-bialgebras. We will take this theorem as a black box and not attempt a proof. As pointed out in the introduction, sketches can be found in [Low] and [Shoi].

4. Derived Functors

In this section we will introduce derived functors which attempt to answer the question: Given a triangulated functor $F : \mathscr{D} \to \mathscr{T}$ between triangulated categories, when can we extend F to the Verdier quotient $\mathscr{D}/\mathscr{C}$?

4.1. **Categories of H-Projectives and H-Injectives.**

In the previous section we saw that $\mathcal{D}\mathscr{A}$ has enough h-projective objects, since the semi-free resolution of each object is h-projective. In particular, we have the following fact.

Proposition 4.1.1. *Let $\mathcal{P}$ be the triangulated subcategory of h-projectives in $\mathcal{H}\mathscr{A}$. Then $\iota : \mathcal{P} \hookrightarrow \mathcal{D}\mathscr{A}$ is a triangulated equivalence.*

Proof : It follows from Theorem 3.5.7 that ι is essentially surjective and from Lemma 3.3.2, that ι is fully faithful. □

Dually, one can show that $\mathcal{D}\mathscr{A}$ has enough h-injectives (see [Kel1]). This gives us a dual version of Proposition 4.1.1

Lemma 4.1.2. *Let $\mathcal{I}$ be the triangulated subcategory of h-injectives in $\mathcal{H}\mathscr{A}$. Then $\iota : \mathcal{I} \hookrightarrow \mathcal{D}\mathscr{A}$ is a triangulated equivalence.*

For $X \in \mathcal{D}\mathscr{A}$, $\mathrm{p}\,X$ (resp. $\mathrm{i}\,X$) will denote h-porjective resolutions of X (resp. h-injective resolution).

4.2. **Existence of Derived Functors.**

We now give the definition of left and right derived functors and collect two existence theorems that we will state but not prove.

Definition 4.2.1. *Let $F : \mathscr{D} \to \mathscr{T}$ be a triangulated functor between triangulated categories and $\mathscr{C}$ a triangulated subcategory of $\mathscr{D}$. The left derived functor of F is the pair $(\mathbb{L}F, \eta)$ where $\mathbb{L}F : \mathscr{D}/\mathscr{C} \to \mathscr{T}$ is a triangulated functor and $\eta : \mathbb{L}F \circ q \to F$ is a triangulated natural transformation. Satisfying the universal property that if (G, ζ) is another pair, there is a triangulated natural transformation $\nu : G \to \mathbb{L}F$ such that the diagram*

$$\begin{tikzcd} & F(x) & \\ \mathbb{L}F \circ q(x) \arrow[ur, "\eta_x"] & & G \circ q(x) \arrow[ul, "\zeta_x"'] \arrow[ll, "\nu_{q(x)}"'] \end{tikzcd} \tag{2.1}$$

commutes for all $x \in \mathscr{D}$.

Definition 4.2.2. *Let $F : \mathscr{D} \to \mathscr{T}$ be a triangulated functor between triangulated categories and $\mathscr{C}$ a triangulated subcategory of $\mathscr{D}$. The right derived functor of F is the pair $(\mathbb{R}F, \epsilon)$ where $\mathbb{R}F : \mathscr{D}/\mathscr{C} \to \mathscr{T}$ is a triangulated functor and $\epsilon : F \to \mathbb{R}F \circ q$ is a triangulated natural transformation. Satisfying the universal property that if (G, φ) is another pair, there is a triangulated natural* *transformation* $\psi : \mathbb{R}F \to G$ *such that the diagram*

$$\begin{tikzcd} & F(x) \arrow[dl, "\epsilon_x"'] \arrow[dr, "\varphi_x"] & \\ \mathbb{R}F \circ q(x) \arrow[rr, "\psi_{q(x)}"] & & G \circ q(x) \end{tikzcd} \tag{2.2}$$

commutes for all $x \in \mathscr{D}$.

The answer(s) to the question that was posed at the beginning of the section are given in the following two theorems. Whose rather lengthy proofs can be found in [Mur1].

Theorem 4.2.3. *Let* $F : \mathscr{D} \to \mathscr{T}$ *be a triangulated functor between triangulated categories and suppose that* $\mathscr{C}$ *is a thick triangulated subcategory of* $\mathscr{D}$. *Suppose that for each* $x \in \mathscr{D}$ *there is an* $\eta_x : p_x \to x$ *in* $\mathrm{Morph}_{\mathscr{C}}$ *such that* p_x *is left* F*-acyclic. Then* F *has a left derived functor* $(\mathbb{L}F, \zeta)$ *such that*

i) $\mathbb{L}F(x) = F(p_x)$ *and* $\zeta_x = F(\eta_x)$

ii) $x \in \mathscr{D}$ *is left* F*-acyclic if and only if* ζ_x *is an isomorphism.*

Theorem 4.2.4. *Let* $F : \mathscr{D} \to \mathscr{T}$ *be a triangulated functor between triangulated categories and suppose that* $\mathscr{C}$ *is a thick triangulated subcategory of* $\mathscr{D}$. *Suppose that for each* $x \in \mathscr{D}$ *there is an* $\nu_x : x \to i_x$ *in* $\mathrm{Morph}_{\mathscr{C}}$ *such that* i_x *is right* F*-acyclic. Then* F *has a right derived functor* $(\mathbb{R}F, \psi)$ *such that*

i) $\mathbb{R}F(x) = F(i_x)$ *and* $\psi_x = F(\nu_x)$

ii) $x \in \mathscr{D}$ *is right* F*-acyclic if and only if* ψ_x *is an isomorphism.*

Now in both theorems there is the new terminology of left (resp. right) F-acyclic. The interested reader can find them in [Mur1] and [Mur2], but it will not matter because we are interested in the case where $\mathscr{D} = \mathcal{H}\mathscr{A}$ and $\mathscr{T} := \mathcal{D}\mathscr{B}$ and $\mathscr{C} := \{\text{Acyclic modules}\}$, for $\mathscr{A}$ and $\mathscr{B}$ dg categories. In this case, the h-projectives (resp. h-injectives) are the left (resp. right) F-acyclics, for any F (see [Mur1]). In particular, for any triangulated functor $F : \mathcal{H}\mathscr{A} \to \mathcal{D}\mathscr{B}$ Theorems 4.2.3 and 4.2.4 tell us not only that F has left and right derived functors but also how to compute them.

One might be concerned that way we compute derived functor depends on our choices of h-projective (resp. h-injective) resolutions. As the next remark shows, our choices wont matter up to canonical isomorphism.

Remark 4.2.5. *Suppose* $\mathcal{U}$ *and* $\mathcal{U}'$ *are each collections of h-projective resolutions in* $\mathcal{H}\mathscr{A}$, *and* $F : \mathcal{H}\mathscr{A} \to \mathcal{D}\mathscr{B}$ *a triangulated functor then it follows from the universal property of left derived functors and ii) of Theorem* 4.2.3 *that* $\mathbb{L}F_{\mathcal{U}}(M) \cong \mathbb{L}F_{\mathcal{U}'}(M)$ *canonically. Dually,* $\mathbb{R}F$ *is also invariant up to canonical isomorphism, under choice of h-injective resolution.*

5. Keller's Theorem(s)

Suppose A and B are both k-algebras such that

$$\mathcal{D}A \xrightarrow{\simeq} \mathcal{D}B,$$

is an equivalence of triangulated categories. As noted in the introduction, it follows from the work of Rickard and Happel, see [Ric] and [Hap], that there is an $A^{\mathrm{op}} \otimes B$-module, called a tilting bimodule, such that

$$? \otimes^{\mathbb{L}}_A X : \mathcal{D}A \xrightarrow{\simeq} \mathcal{D}B.$$

It follows then,

$$\begin{aligned} \mathrm{HH}^i(A) = (\mathcal{D}A^{\mathrm{op}} \otimes A)(A, [i]A) &\longrightarrow (\mathcal{D}A^{\mathrm{op}} \otimes B)(A \otimes^{\mathbb{L}}_A X, [i]A \otimes^{\mathbb{L}}_A X) \\ &= (\mathcal{D}A^{\mathrm{op}} \otimes B)(X, [i]X), \end{aligned}$$

is an isomorphism and,

$$\begin{aligned} \mathrm{HH}^i(B) = (\mathcal{D}B^{\mathrm{op}} \otimes B)(B, [i]B) &\longrightarrow (\mathcal{D}A^{\mathrm{op}} \otimes B)(X \otimes^{\mathbb{L}}_B B, [i]X \otimes^{\mathbb{L}}_B B) \\ &= (\mathcal{D}A^{\mathrm{op}} \otimes B)(X, [i]X), \end{aligned}$$

is also an isomorphism.

Thus there is an isomorphism

$$\phi_X : \mathrm{HH}^*(B) \to \mathrm{HH}^*(A), \tag{0.3}$$

of graded algebras. It was shown in [Kel3] , that this map respects the Gerstenhaber bracket. As we noted before, Koszul duality in the sense of [BGS] and [MOS] doesn't fit into this because it messes up gradings does not give a graded isomorphism of Hochschild cohomology. The goal of this section is to show that one can lift ϕ_X to an isomorphism $\varphi_X : C^*(B) \to C^*(A)$ in the homotopy category of B_∞ algebras, $\mathrm{Ho}(\mathrm{B}_\infty)$. i.e. The the category of B_∞-algebras obtained by formally inverting all all morphisms which induce quasi-isomorphisms on the underlying chain complexes. We will prove an even more general result: if $? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathrm{Perf}(\mathscr{A}) \to \mathcal{D}\mathscr{B}$ is fully faithful, then it induces an isomorphism $\varphi_X : C^*(\mathscr{B}) \to C^*(\mathscr{A})$ in $\mathrm{Ho}(\mathrm{B}_\infty)$. All the results in this section are due to Keller.

5.1. **Keller Triples for DG Algebras.**

Suppose that X is an $A^{\mathrm{op}} \otimes B$-module.

Lemma 5.1.1.

$$? \otimes^{L}_{\mathscr{A}} X : \mathrm{Perf}(A) \longrightarrow \mathcal{D}B \tag{1.1}$$

is fully faithful if and only if

$$H^n(A) \longrightarrow (\mathcal{D}B)(X, [n]X) \tag{1.2}$$

is an isomorphism for all $n \in \mathbb{Z}$ *if and only if*

$$\lambda : A \longrightarrow (\mathrm{Dif}\ B)(\mathrm{p}\,X, \mathrm{p}\,X) \tag{1.3}$$

is an isomorphism in $\mathcal{D}k$ *and* $\mathrm{p}\,X$ *is an h-projective resolution of* $X \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$.

Proof. The first if and only if statement is a direct consequence of Lemma 3.2.16. The second if and only if statement is a direct consequence of Lemma 3.1.34. □

Since $(\mathrm{Dif}\ B)(\mathrm{p}\,X, \mathrm{p}\,X) = \mathbb{R}\mathrm{Hom}_B(X, \mathrm{p}\,X) \cong \mathbb{R}\mathrm{Hom}_B(X, X)$ we will by abuse of notation write λ for the map

$$A \to (\mathrm{Dif}\ B)(\mathrm{p}\,X, \mathrm{p}\,X) \to \mathbb{R}\mathrm{Hom}_B(X, X) \tag{1.4}$$

which is given by composing λ from Lemma 5.1.1, with $\pi_X : \mathrm{p}\,X \to X$. Now we come to the definition of the central object of this section.

Definition 5.1.2. *(1) We call a triple (A, X, B) where A and B are two dg k-algebras and X an $A^{op} \otimes B$-module a Keller triple.*

(2) A Keller triple (A, X, B) is said to be left admissible if the map λ appearing in equation 1.4 is an isomorphism in $\mathcal{D}k$. We will call λ the derived left action of A.

(3) A Keller triple (A, X, B) is said to be right admissible if the map $\rho : B^{\mathrm{op}} \to \mathbb{R}Hom_{A^{\mathrm{op}}}(X, X)$ is an isomorphism in $\mathcal{D}k$. We will refer to ρ as the derived right action of B.

(4) A Keller triple (A, X, B) is said to be admissible if it is both left and right admissible.

Definition 5.1.3. *Given a Keller triple (A, X, B) we define its Keller category $\mathcal{G}(A, X, B)$, to be the dg category with whose set of objects is $\{a, b\}$ and whose morphism spaces are given by*

$$\mathcal{G}(a, a) = A, \qquad \mathcal{G}(b, b) = B, \qquad \mathcal{G}(b, a) = X, \qquad \mathcal{G}(a, b) = 0, \tag{1.5}$$

We will sometimes write $\mathcal{G}$ instead of $\mathcal{G}(A, X, B)$ when there is no room for confusion.

Given a Keller triple (A, X, B) the inclusion functor $\iota_A : A \hookrightarrow \mathcal{G}(A, X, B)$ (resp. ι_B) is fully faithful, pulling back along ι_A (resp. ι_B) we get a morphism of B_∞-algebras

$$\iota_A^* : C^*(A) \to C^*(\mathcal{G}) \tag{1.6}$$

(resp. ι_B^*).

We now take a closer look at $C^*(\mathcal{G})$. As a graded vector space it looks like

$$C^*(\mathcal{G}) = C^*(A) \oplus [1]C^*(A, X, B) \oplus C^*(B). \tag{1.7}$$

Where $C^*(A, X, B)$ is the chain complex that computes $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}} \otimes B}(X, X)$. More explicitly, $C^*(A, X, B)$ is the product totalization of of the bicomplex whose $(i, j)^{\mathrm{th}}$ component is

$$\prod_{m+n=i} \mathrm{Hom}(A^{\otimes^m} \otimes X \otimes B^{\otimes n}, X)^j. \tag{1.8}$$

Schematically the complex $C^*(\mathcal{G})$ can be described as

$$\begin{tikzcd} C^{i+1}(A) & C^i(A,X,B) & C^{i+1}(B) \\ C^i(A) \arrow[u, "d"] \arrow[ur, "d_{A,X}"] & C^{i-1}(A,X,B) \arrow[u, "d"] & C^i(B). \arrow[u, "d"'] \arrow[ul, "d_{X,B}"'] \end{tikzcd} \tag{1.9}$$

Where $d_{A,X}$ (resp. $d_{X,B}$) is the map induced by λ_* (resp. ρ_*) i.e. the following diagram of chain complexes commutes.

$$\begin{tikzcd} \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}\otimes A}(A,A) \arrow[r, "\lambda_*"] \arrow[dd, "①"', "\cong"] & \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}\otimes A}(A, \mathbb{R}\mathrm{Hom}_B(X,X)) \arrow[d, "②"', "\cong"] \\ & \mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}\otimes B}(X,X) \arrow[d, "③"', "\cong"] \\ C^*(A) \arrow[r, "d_{A,X}"] & C^*(A,X,B). \end{tikzcd} \tag{1.10}$$

Where the ①, ③ are given by the isomorphism (4.17) and ② by derived hom-tensor adjointness.

The next theorem is from [Kel2, Section 4.5] although there it appears in a slightly different form.

Theorem 5.1.4. *Let (A,X,B) be a Keller triple and $\mathcal{G}$ its associated Keller category*

i) If (A,X,B) is left admissible then ι_B^* is an isomorphism in $\mathcal{D}k$.

ii) If (A,X,B) is right admissible then ι_A^* is an isomorphism in $\mathcal{D}k$.

Proof. We will just prove i), the argument for ii) will be symmetric. In order to show that ι_B^* is a quasi-isomorphism it is enough to show that $cone(\iota_B^*)$ is acyclic. Taking the cone of a morphism of chain complexes is the same as totalizing a bicomplex with two vertical columns. The spectral sequence associated to this bicomplex stablizes at the E_2 page. The E_1 page is $\mathrm{Ker}(\iota_B^*)$ which as a graded vector space is just $C^*(A)\oplus[1]C^*(A,X,B)$ but the differential is

$$\begin{pmatrix} d_{[1]C^*(A)} & 0 \\ \lambda_* & d_{[1]C^*(A,X,B)} \end{pmatrix} \tag{1.11}$$

Thus $\mathrm{Ker}(\iota_B^*)$ is the $cone(\lambda_*)$, but by assumption λ is invertible hence so is λ_* thus $cone(\lambda_*)$ is acyclic and the E_2 page vanishes. So we see that $cone(\iota_B^*)$ is acyclic. $\square$

Remark 5.1.5. *Theorem also follows from* 5.1.4 *follows from showing that* $[-1]cone(\begin{pmatrix} d_{A,X} & d_{X,B} \end{pmatrix}) \cong C^*(\mathcal{G})$. *In particular this says that there is a homotopy cartesian square*

$$\begin{tikzcd} C^*(\mathcal{G}) \arrow[r, "\iota_B^*"] \arrow[d, "\iota_B^*"'] & C^*(A) \arrow[d, "d_{A,X}"] \\ C^*(B) \arrow[r, "d_{X,B}"'] & C^*(A,X,B) \end{tikzcd} \tag{1.12}$$

in $\mathcal{D}k$. Then assuming that (A,X,B) is left (resp. right) admissible, applying Lemma 3.2.24 *shows that ι_B^* (resp. ι_A^*) is invertible.*

5.2. **Keller Triples for DG Categories.**

In this subsection $\mathscr{A}$ and $\mathscr{B}$ are dg-categories and X is an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module.

Lemma 5.2.1. *The functor*

$$? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathrm{Perf}(\mathscr{A}) \to \mathcal{D}\mathscr{B} \tag{2.1}$$

is fully faithful if and only if,

$$(\mathcal{D}\mathscr{A})(\hat{a}, [i]\hat{a'}) \to (\mathcal{D}\mathscr{B})(X(?, a), [i]X(?, a')) \tag{2.2}$$

is an isomorphism for all $a, a' \in \mathscr{A}$ and $i \in \mathbb{Z}$ if and only if,

$$\lambda : \mathcal{I}_{\mathscr{A}} \to (\mathrm{Dif}\ \mathscr{B})(\mathrm{p}\, X, \mathrm{p}\, X) \tag{2.3}$$

is an isomorphism in $\mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ where $\mathrm{p}\, X$ is an h-projective resolution of X in $\mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$.

Proof. The proof is the same as the dg-algebra case. □

Just as was done in the dg-algebra case, there is a notion of of Keller triple for dg-categories which we now describe.

Definition 5.2.2. *(1) A triple $(\mathscr{A}, X, \mathscr{B})$ is called a Keller triple, where $\mathscr{A}$ and $\mathscr{B}$ are dg-categories and X an $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module.*

(2) A Keller triple is called left (resp. right) admissible if λ (resp. ρ) is an isomorphism in $\mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ (resp. $\mathcal{D}\mathscr{B}^{\mathrm{op}} \otimes \mathscr{B}$).

(3) A Keller triple that is both left and right admissible will be called admissible.

The next Lemma is just a reformulation of Lemma 5.2.1 into the language Keller triples.

Lemma 5.2.3. *Suppose that $(\mathscr{A}, X, \mathscr{B})$ is a Keller triple then*

(1) The functor

$$? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathrm{Perf}(\mathscr{A}) \to \mathcal{D}\mathscr{B}$$

is fully faithful if and only if the Keller triple is left admissible.

(2) The functor

$$X \otimes^{\mathbb{L}}_{\mathscr{B}} ? : \mathrm{Perf}(\mathscr{B}^{\mathrm{op}}) \to \mathcal{D}(\mathscr{A}^{\mathrm{op}})$$

is fully faithful if and only if the Keller triple is right admissible.

Proof. Follows immediately from definitions. □

We now give a sufficient condition for a Keller triple to admissible.

Lemma 5.2.4. *Let $(\mathscr{A}, X, \mathscr{B})$ be a Keller triple and suppose*

$$? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathcal{D}\mathscr{A} \to \mathcal{D}\mathscr{B} \tag{2.4}$$

is an equivalence of triangulated categories then the Keller triple is admissible.

Proof. Clearly $(\mathscr{A}, X, \mathscr{B})$ is left admissible. There is a functor

$$\mathrm{Hom}_{\mathscr{B}}(?, \mathcal{I}_{\mathscr{B}}) : \mathrm{Dif}\ \mathscr{B} \longrightarrow \mathrm{Dif}\ (\mathscr{B}^{\mathrm{op}})^{\mathrm{op}},$$

where $\mathrm{Hom}_{\mathscr{B}}(M, \mathcal{I}_{\mathscr{B}})(b) := \mathrm{Hom}_{\mathscr{B}}(M, \mathscr{B}(?, b))$. Right deriving this functor gives

$$Tr_{\mathscr{B}} := \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(?, \mathcal{I}_{\mathscr{B}}) : \mathcal{D}\mathscr{B} \to \mathcal{D}(\mathscr{B}^{\mathrm{op}})^{\mathrm{op}}.$$

Let $N \in \mathrm{Perf}\,\mathscr{B}$, we claim that

$$X \otimes^{\mathbb{L}}_{\mathscr{B}} \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(N, \mathcal{I}_{\mathscr{B}}) \xrightarrow{\cong} \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(N, X) \tag{2.5}$$

is a natural isomorphism. This is true because (2.5) is true for $N = \hat{b}$ where $b \in \mathscr{B}$, and it follows from Lemma 3.2.17 that (2.5) is an isomorphism for all $N \in \mathrm{Perf}\,\mathscr{B}$.

Since $? \otimes^{\mathbb{L}}_{\mathscr{A}} X$ is an equivalence its adjoint $\mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, ?)$ is also an equivalence. Thus, we have the following isomorphism in $\mathcal{D}k$

$$\mathbb{R}\mathrm{Hom}_{\mathscr{B}}(N, X) \xrightarrow{\cong} \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, N), \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, X)).$$

It follows from our assumption that $\lambda : \mathcal{I}_{\mathscr{A}} \to \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(X, X)$ is an isomorphism in $\mathcal{D}\ \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ that there is a canonical isomorphism

$$\mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, N), \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, X)) \xrightarrow{\cong} \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, N), \mathcal{I}_{\mathscr{A}}).$$

Hence by Lemma 3.2.21, the functor $Tr_{\mathscr{B}}$ induces an equivalence

$$\mathrm{Perf}\,\mathscr{B} \xrightarrow{\cong} \mathrm{Perf}\ (\mathscr{B}^{\mathrm{op}})^{\mathrm{op}},$$

and so there is a natural isomorphism

$$(X \otimes^{\mathbb{L}}_{\mathscr{B}} ?) \circ Tr_{\mathscr{B}} \xrightarrow{\cong} Tr_{\mathscr{A}} \circ \mathbb{R}\mathrm{Hom}_{\mathscr{B}}(X, ?)$$

of functors from $\mathrm{Perf}\,\mathscr{B} \to \mathcal{D}\ (\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$. It follows that

$$X \otimes^{\mathbb{L}}_{\mathscr{B}} ? : \mathrm{Perf}(\mathscr{B}^{\mathrm{op}}) \to \mathcal{D}\ (\mathscr{A}^{\mathrm{op}})$$

is fully faithful and applying Lemma 5.2.3, we see that $(\mathscr{A}, X, \mathscr{B})$ is right admissible. So $(\mathscr{A}, X, \mathscr{B})$ is an admissible Keller triple . $\square$

Now we extend, the notion of Keller category to dg-categories. Not surprisingly, the definition is the obvious one.

Definition 5.2.5. *The Keller category, $\mathcal{G}(\mathscr{A}, X, \mathscr{B})$, associated to a Keller triple $(\mathscr{A}, X, \mathscr{B})$, is the dg-category whose set of objects is the disjoint union of the objects of $\mathscr{A}$ and $\mathscr{B}$ and whose morphism spaces are*

$$\mathcal{G}(a, a') = \mathscr{A}(a, a'), \qquad \mathcal{G}(b, b') = \mathscr{B}(b, b'),$$
$$\mathcal{G}(b, a) = X(b, a), \qquad \mathcal{G}(a, b) = 0.$$

In the case that $\mathscr{A}$ and $\mathscr{B}$ are dg-categories that each have one object (i.e. dg algebras), then pictorially one can think of $\mathcal{G}$ as

$$\begin{array}{cc} A & B \\ & \\ a \xleftarrow{X} & b. \end{array}$$

In this case one can also think of $\mathcal{G}$ as an upper-triangular algebra

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix},$$

But we prefer the first picture.

Given a Keller triple $(\mathscr{A}, X, \mathscr{B})$ the cannonical inclusion functors $\iota_{\mathscr{A}}$ (resp. $\iota_{\mathscr{B}}$) are fully faithful. Thus pulling back along $\iota_{\mathscr{A}}$ (resp. $\iota_{\mathscr{B}}$) induces morphisms of B_{∞}-algebras

$$\iota^*_{\mathscr{A}} : C^*(\mathscr{A}) \to C^*(\mathcal{G}) \tag{2.6}$$

(resp $\iota^*_{\mathscr{B}}$).

The next theorem is a dg-category version of Theorem 5.1.4.

Theorem 5.2.6. [Kel2] *Given a Keller triple $(\mathscr{A}, X, \mathscr{B})$, then*

(1) If it is left admissible then $\iota^*_{\mathscr{B}}$ is invertible in $\mathcal{D}k$.

(2) If it is right admissible then $\iota^*_{\mathscr{A}}$ is invertible in $\mathcal{D}k$.

Proof. The proof is exactly the same as in 5.1.4 □

Remark 5.2.7. One can also show that $C^*(\mathcal{G}) \cong [-1]cone\big(\big(d_{\mathscr{A},X}, \quad d_{X,\mathscr{B}}\big)$. In particular this shows that

$$\begin{tikzcd} C^*(\mathcal{G}) \arrow[r] \arrow[d] & C^*(\mathscr{A}) \arrow[d] \\ C^*(\mathscr{B}) \arrow[r] & C^*(\mathscr{A}, X, \mathscr{B}) \end{tikzcd} \tag{2.7}$$

is a homotopy cartesian square in $\mathcal{D}k$ and Theorem 5.2.6 follows.

As an immediate consequence of Theorem 5.2.6 we have the following corollaries:

Corollary 5.2.8. *For any left admissible Keller triple There is a canonical morphism in* $\mathrm{Ho}(\mathrm{B}_{\infty})$

$$\varphi_X : C^*(\mathscr{B}) \to C^*(\mathscr{A}), \tag{2.8}$$

where $\varphi_X := (\iota^*_{\mathscr{A}}) \circ (\iota^*_{\mathscr{B}})^{-1}$

Corollary 5.2.9. *For any right admissible Keller triple there is a canonical morphism of* $\mathrm{Ho}(\mathrm{B}_{\infty})$

$$\tilde{\varphi}_X : C^*(\mathscr{A}) \to C^*(\mathscr{B}), \tag{2.9}$$

where $\tilde{\varphi}_X := (\iota^*_{\mathscr{B}}) \circ (\iota^*_{\mathscr{A}})^{-1}$

Corollary 5.2.10. *For any admissible Keller triple the canonical morphisms* (2.8) *and* (2.9) *are inverses to one another and thus define an isomorphism in* $\mathrm{Ho}(\mathrm{B}_{\infty})$.

Corollary 5.2.11. *Let $(\mathscr{A}, X, \mathscr{B})$ be a Keller triple and suppose*

$$? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathcal{D}\mathscr{A} \to \mathcal{D}\mathscr{B} \tag{2.10}$$

is a triangulated equivalence. Then

$$\varphi_X : C^*(\mathscr{B}) \to C^*(\mathscr{A}) \tag{2.11}$$

is an isomorphism in $\mathrm{Ho}(\mathrm{B}_{\infty})$.

Proof. By Lemma 5.2.4 the Keller triple is admissible and from Corollaries 5.2.9 and 5.2.10 it follows that φ is an isomorphism. $\square$

The next Proposition gives a composition rule for Keller triples and will be strengthened in Theorem 5.2.17.

Proposition 5.2.12. *Let $(\mathscr{B}, Y, \mathscr{C}), (\mathscr{A}, X, \mathscr{B})$, and $(\mathscr{A}, Z := X \otimes^{\mathbb{L}}_{\mathscr{B}} Y, \mathscr{C})$ be Keller triples. Suppose that X is h-projective over $\mathscr{B}$. If $(\mathscr{B}, Y, \mathscr{C}), (\mathscr{A}, X, \mathscr{B}), (\mathscr{A}, Z, \mathscr{C})$ are all left admissible, then $\varphi_Z = \varphi_X \circ \varphi_Y$*

Proof. Since X is h-projective over $\mathscr{B}$, by the existence theorem of left derived functors $Z = X \otimes_{\mathscr{B}} Y$. We have the following diagram of fully faithful functors:

$$\begin{tikzcd}
\mathcal{G}(\mathscr{A}, Z, \mathscr{C}) \arrow[dr] & & & \\
& \mathscr{U} & \mathcal{G}(\mathscr{B}, Y, \mathscr{C}) \arrow[l] & \mathscr{C} \arrow[l] \arrow[ulll, bend right] \\
& \mathcal{G}(\mathscr{A}, X, \mathscr{B}) \arrow[u] & \mathscr{B} \arrow[l] \arrow[u] & \\
& \mathscr{A} \arrow[u] \arrow[uuul, bend left] & &
\end{tikzcd} \tag{2.12}$$

where $\mathscr{U}$ is the category whose set of objects is $ob(\mathscr{A}) \coprod ob(\mathscr{B}) \coprod ob(\mathscr{C})$ and whose morphism spaces are

$$\begin{array}{lll}
\mathscr{U}(a, a') = \mathscr{A}(a, a'), & \mathscr{U}(b, b') = \mathscr{B}(b, b'), & \mathscr{U}(c, c') = \mathscr{C}(c, c'), \\
\mathscr{U}(b, a) = X(b, a), & \mathscr{U}(c, b) = Y(c, b), & \mathscr{U}(c, a) = Z(c, a), \\
\mathscr{U}(a, b) = 0, & \mathscr{U}(b, c) = 0, & \mathscr{U}(a, c) = 0.
\end{array} \tag{2.13}$$

Then the following diagram commutes in $\mathrm{Ho}(\mathrm{B}_\infty)$

$$\begin{tikzcd}
C^*(\mathcal{G}(\mathscr{A}, Z, \mathscr{C})) \arrow[drrr, bend left=10, "\cong"] \arrow[ddr, bend right] & & & \\
& C^*(\mathscr{U}) \arrow[ul] \arrow[r, "①"] \arrow[d] & C^*(\mathcal{G}(\mathscr{B}, Y, \mathscr{C})) \arrow[r, "\cong"'] \arrow[d] & C^*(\mathscr{C}) \\
& C^*(\mathcal{G}(\mathscr{A}, X, \mathscr{B})) \arrow[r, "\cong"] \arrow[d] & C^*(\mathscr{B}) & \\
& C^*(\mathscr{A}) & &
\end{tikzcd} \tag{2.14}$$

The theorem will follow if we can show that ① is an isomorphism. But we can view $\mathscr{U}$ as the Keller category associated to the triple $(\mathscr{A}, U, \mathcal{G}(\mathscr{B}, Y, \mathscr{C}))$. Where U is

$$U(b, a) = X(b, a), \qquad U(c, a) = Z(c, b). \tag{2.15}$$

Moreover, $(\mathscr{A}, U, \mathcal{G}(\mathscr{B}, Y, \mathscr{C}))$ is left admissible since

$$(\mathcal{D}\mathscr{A})(\hat{a}, [i]\hat{a'}) \longrightarrow (\mathcal{D}\ \mathcal{G}(\mathscr{B}, Y, \mathscr{C}))(U(?, a), [i]U(?, a')), \tag{2.16}$$

is an isomorphism for all $a, a' \in \mathscr{A}$ and $i \in \mathbb{Z}$. Thus it follows that ① is an isomorphism. $\square$

Proposition 5.2.13. *If $(\mathscr{A},\mathcal{I}_{\mathscr{A}},\mathscr{A})$ is a Keller triple then it is admissible and φ_X is the identity.*

Proof. Clearly $(\mathscr{A},\mathcal{I}_{\mathscr{A}},\mathscr{A})$ is admissible. Then using Lemma 5.2.12 where we take $\mathscr{A}=\mathscr{B}=\mathscr{C}$ and $X=Y=\mathcal{I}_{\mathscr{A}}$, it follows that

$$\varphi_{\mathcal{I}_{\mathscr{A}}}=\varphi_{\mathcal{I}_{\mathscr{A}}\otimes^{\mathbb{L}}_{\mathscr{A}}\mathcal{I}_{\mathscr{A}}}=\varphi_{\mathcal{I}_{\mathscr{A}}}\circ\varphi_{\mathcal{I}_{\mathscr{A}}}. \tag{2.17}$$

But $\phi_{\mathcal{I}_{\mathscr{A}}}$ is invertible so it follows that $\phi_{\mathcal{I}_{\mathscr{A}}}=\mathrm{id}$. ☐

Lemma 5.2.14. *If $F:\mathscr{A}\to\mathscr{B}$ is a fully faithful dg-functor and X_F is the bimodule defined by*

$$X(b,a)=\mathscr{B}(b,F(a)),\ a\in\mathscr{A},\ b\in\mathscr{B}$$

Then $\varphi_{X_F}=F^*$ in $\mathrm{Ho}(B_\infty)$.

Proof. We have the following diagram of fully faithful dg-functors

$$\begin{tikzcd}
\mathcal{G}(\mathscr{A},X_F,\mathscr{B}) & & \mathscr{B} \arrow[ll] & \\
\mathscr{A} \arrow[u] & & \mathcal{G}(\mathscr{A},\mathcal{I}_{\mathscr{A}},\mathscr{A}) \arrow[ull, "{(\mathrm{id},F)}"'] & \mathscr{A} \arrow[l] \arrow[ul, "F"'] \\
& & \mathscr{A} \arrow[ull, "\mathrm{id}"] \arrow[u] &
\end{tikzcd}$$

This gives rise to the following commutative diagram in $\mathrm{Ho}(\mathrm{B}_\infty)$ where we set $\mathcal{G}:=\mathcal{G}(\mathscr{A},X_F,\mathscr{B})$ and $\mathcal{G}':=\mathcal{G}(\mathscr{A},\mathcal{I}_{\mathscr{A}},\mathscr{A})$

$$\begin{tikzcd}
C^*(\mathcal{G}) \arrow[r] \arrow[d] \arrow[dr] & C^*(\mathscr{B}) \arrow[dr, "F^*"] & \\
C^*(\mathscr{A}) \arrow[dr, "\mathrm{id}"'] & C^*(\mathcal{G}') \arrow[r] \arrow[d] & C^*(\mathscr{A}) \\
& C^*(\mathscr{A}) &
\end{tikzcd}$$

It follows from Proposition 5.2.13 that $\varphi_{\mathcal{I}_{\mathscr{A}}}$ is the identity and hence it follows that $\varphi_{X_F}=F^*$. ☐

Remark 5.2.15. Suppose we have a dg-functor $F:\mathscr{A}\to\mathscr{B}$ which is not necessarily fully faithful. We could still have defined a bimodule X_F as in Lemma 5.2.14, but in general $(\mathscr{A},X_F,\mathscr{B})$ might not be left admissible. However, suppose that F induces a fully faithful functor $H^*(F):H^*(\mathscr{A})\to H^*(\mathscr{B})$ then it follows that

$$?\otimes^{\mathbb{L}}_{\mathscr{A}}X_F:\mathrm{Perf}\,\mathscr{A}\to\mathcal{D}\,\mathscr{B} \tag{2.18}$$

is fully faithful using that the diagram

$$\begin{tikzcd}
(\mathcal{H}\,\mathscr{A})(\hat{a},[i]\hat{a'}) \arrow[r] \arrow[d, "\cong"] & (\mathcal{H}\,\mathscr{B})(\hat{a}\otimes X_F,[i]\hat{a'}\otimes X_F) \\
H^i\mathscr{A}(a,a') \arrow[r, "H^i(F)", "\cong"'] & H^i\mathscr{B}(F(a),F(a')) \arrow[u, "\cong"]
\end{tikzcd} \tag{2.19}$$

commutes for all $a,a'\in\mathscr{A}$ and $i\in\mathbb{Z}$ and Lemma 3.2.16. Thus it follows that $(\mathscr{A},X_F,\mathscr{B})$ is left admissible and hence there is a morphism $\varphi_F:=\varphi_{X_F}$

in $\mathrm{Ho}(\mathrm{B}_\infty)$. Suppose now that $G : \mathscr{B} \to \mathscr{C}$ is another dg-functor such that $(\mathscr{B}, X_G, \mathscr{C})$ is left admissible. Then since X_G is clearly h-projective over $\mathscr{C}$, and $X_F \otimes_{\mathscr{A}} X_G \cong X_{G\circ F}$, it follows from Lemma 5.2.12 that

$$\varphi_{X_{G\circ F}} = \varphi_{X_F \otimes_{\mathscr{B}} X_G} = \varphi_F \circ \varphi_G \tag{2.20}$$

What one should take away from this remark is that whenever $F : \mathscr{A} \to \mathscr{B}$ is a dg-functor that induces a fully faithful functor in homology, then φ_F is defined and functorial.

Remark 5.2.15 will be crucial in proving the next theorem which establishes the uniqueness of φ_X.

Theorem 5.2.16. *Let $(\mathscr{A}, X, \mathscr{B})$ be a left admissible Keller triple. Then φ_X depends only on the isomorphism class of $X \in \mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$*

Proof. Suppose that $f : X \to X'$ is an isomorphism in $\mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$. Then there is an induced dg-functor, F, on their respective Keller categories

$$F : \mathcal{G}(\mathscr{A}, X, \mathscr{B}) \to \mathcal{G}(\mathscr{A}, X', \mathscr{B}) \tag{2.21}$$

F in (2.21) may not be fully faithful but it induces a fully faithful functor in homology. Thus Remark 5.2.15 applies and there is a commutative diagram

$$\begin{tikzcd} & C^*(\mathcal{G}) \arrow[dl] \arrow[dr] & \\ C^*(\mathscr{A}) & & C^*(\mathscr{B}) \\ & C^*(\mathcal{G}') \arrow[ul] \arrow[ur] \arrow[uu, "\varphi_F"'] & \end{tikzcd} \tag{2.22}$$

in $\mathrm{Ho}(\mathrm{B}_\infty)$, where $\mathcal{G} := \mathcal{G}(\mathscr{A}, X, \mathscr{B})$ and $\mathcal{G}' := \mathcal{G}(\mathscr{A}, X', \mathscr{B})$. But the two right-hand arrows of diagram (2.22) are invertible hence so is φ_F.So it follows that $\varphi_X = \varphi_{X'}$. ∎

We now collect all the lemmas of this subsection and generalize Lemma 5.2.12.

Theorem 5.2.17. [Kel2] *Suppose that $(\mathscr{A}, X, \mathscr{B})$ is a left admissible Keller triple. Then the following are true,*

(1) φ_X depends only on the isomorphism class of $X \in \mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$.

(2) If $(\mathscr{A}, X, \mathscr{B})$ is right admissible then φ_X is an isomorphism in $\mathrm{Ho}(\mathrm{B}_\infty)$. *In particular if φ_X is an isomorphism if $? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathcal{D}\mathscr{A} \longrightarrow \mathcal{D}\mathscr{B}$ is a triangulated equivalence.*

(3) Suppose that $F : \mathscr{A} \to \mathscr{B}$ is fully faithful. Then $\varphi_{X_F} = F^*$. In particular for $(\mathscr{A}, \mathcal{I}_{\mathscr{A}}, \mathscr{A})$, the morphism $\varphi_{\mathcal{I}_{\mathscr{A}}}$ is the identity.

(4) Suppose that $(\mathscr{B}, Y, \mathscr{C})$ and $(\mathscr{A},\ Z := X \otimes^{\mathbb{L}}_{\mathscr{B}} Y,\ \mathscr{C})$ are both left admissible Keller triples then $\varphi_Z = \varphi_X \circ \varphi_Y$.

Proof. (1) is proved in Theorem 5.2.16. (2) is proved in Corollary 5.2.10 and Lemma 5.2.4. (3) is proved in Proposition 5.2.13and Lemma 5.2.14 . (4) follows immediately from (1) combined with Lemma 5.2.12. $\square$

The next paragraph is sketchy but is meant to point to the larger framework that Theorem 5.2.17 fits into. All of the definitions can be found throughout [Kel1] and in particular [Kel4, 5.4].

Let dgcat_k denote the category of (small)dg-categories, and $\mathrm{rep}(\mathscr{A}, \mathscr{B})$ be the full subcategory of $\mathcal{D}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$ formed by bimodules X such that the functor

$$? \otimes^{\mathbb{L}} X : \mathcal{D}\mathscr{A} \longrightarrow \mathcal{D}\mathscr{B},$$

takes the representable $\mathscr{A}$-modules (i.e. the free modules) to objects which are isomorphic to representable $\mathscr{B}$-modules. Elements of $\mathrm{rep}(\mathscr{A}, \mathscr{B})$ are known as quasi-functors. A dg-functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is called a Morita morphism if it induces an equivalence $? \otimes^{\mathbb{L}} X_F : \mathcal{D}\mathscr{A} \longrightarrow \mathcal{D}\mathscr{B}$ with X_F defined as in Remark 5.2.15.

Theorem 5.2.17 shows that in some sense taking Hochschild cochains is functorial. Precisely it shows that the Hochschild complex is a functor

$$C^* : \mathrm{Hmo}_{\mathrm{ff}}^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathrm{B}_\infty),$$

where $\mathrm{Hmo}_{\mathrm{ff}}$ is the (non-full) subcategory of Hmo, the localization of dgcat_k with respect to the Morita morphisms, whose morphisms are quasi-functors $X \in \mathrm{rep}(\mathscr{A}, \mathscr{B})$ such that $? \otimes^{\mathbb{L}}_{\mathscr{A}} X : \mathrm{Perf}(\mathscr{A}) \longrightarrow \mathcal{D}(\mathscr{B})$, is fully faithful.

6. The DG Koszul Dual

In this section we apply the results of Section 5 to the case when A is a Koszul algebra in the sense of Section 2.2. As we pointed out in Section 2.2, in general

$$\mathrm{HH}^*(A) \ncong \mathrm{HH}^*(A^!), \tag{0.23}$$

for (adams) graded associative algebras. We will however get an isomorphism if we replace $A^!$ with $A^{!^{dg}}$ which we will describe in the sequel. In this section we will work with adams graded dg-modules. Our notation is the following, for an graded dg-module $M = \bigoplus_{i,j} M_j^i$, the superscript i denotes the cohomological grading and the subscript j denotes the adams grading. We will denote the shift down by $p \in \mathbb{Z}$ in the adams grading by $\langle p \rangle$, i.e.

$$\langle p \rangle M_j^i = M_{j+p}^i,$$

and we will denote the shift down by $q \in Z$ in the differential grading by

$$[q] M_j^i = M_j^{i+q}.$$

6.1. **Slightly New Definitions.**

We first start with defining the dg-quadratic dual of an (adams graded) quadratic algebra.

Definition 6.1.1. *Let $A = T(V)/(R)$ be a quadratic algebra. We define the dg-quadratic dual of A as the adams graded dg-algebra $A^{!^{dg}} := T(\langle 1 \rangle V^*)/(R^{\perp^{dg}})$ where $R^{\perp^{dg}} = \{f \in \langle 1 \rangle V^* \otimes \langle 1 \rangle V^* |\ f(\langle 2 \rangle R)\}$.*

we have already seen two examples of this dg-quadratic dual. Namely, $\mathbb{C}[x]^{!^{dg}} = \Lambda(x^*)$ and $\mathbb{C}[\zeta]/(\zeta^2)^{!^{dg}} = \mathbb{C}[\zeta^*]$, where $\mathbb{C}[\zeta^*]$ is the adams graded dg-algebra with ζ^* is cohomological degree 1 and adams degree -1. One can define the Koszul complex in this case as well:

Definition 6.1.2. *Suppose A is a Koszul algebra. the dg-Koszul complex, K^{dg}, of A is*

$$\cdots \xrightarrow{e\cdot} (A^{!^{dg}}_2)^* \otimes A \xrightarrow{e\cdot} (A^{!^{dg}}_1)^* \otimes A \xrightarrow{e\cdot} (A^{!^{dg}}_0)^* \otimes A, \tag{1.1}$$

with,

$$e := \sum_{i=1}^{n} \eta_i \otimes x_i \in A^{!^{dg}} \otimes A, \tag{1.2}$$

where $\{x_i\}$ is a basis for A_1 and $\{\eta_i\}$ is a basis of $\langle 1 \rangle A_1^*$.

We see that Koszul complex K^{dg} is an adams graded dg-$A^{!^{dg^{\mathrm{op}}}} \otimes A$-module. If we just consider its A-module structure it gives us a graded resolution of A_0. Next we define the so called dg-Koszul dual.

Definition 6.1.3. *Let A be a Koszul algebra, in the sense of Section 2.2. The dg-Koszul dual of A is a graded dg-algebra with zero differential defined by*

$$(E(A))^q_{-q} := ext_A(A_0, \langle -q \rangle A_0) \tag{1.3}$$

and zero otherwise. Where $ext_A(?,?)$ is in the category of graded A-modules.

One should compare the next Lemma with Theorem 2.2.12.

Lemma 6.1.4. *Let A be a Koszul algebra, then $E(A) \cong A^{!^{dg}}$ as adams graded dg-algebras.*

Proof. The proof is exactly the same as in the adams graded associative case, but now one uses the dg-Koszul complex, K^{dg}. □

6.2. **An Isomorphism of Categories.**

Definition 6.2.1. *Suppose we have an graded dg-algebra A. Let* ggrMod-A *denote the category of right adams graded dg-modules over A. Whose objects are of the form $M = \bigoplus_{i,j\in\mathbb{Z}} M_i^j$ and whose morphisms are morphisms that respect both gradings and commute with the differential.*

Definition 6.2.2. *Suppose, A, is a graded dg-algebra, we define $\mathscr{A}$, to be the dg-category whose objects are integers and whose morphism space for two objects i, j is $\mathscr{A}(i,j) = \bigoplus_{l\in\mathbb{Z}} A^l_{j-i}$.*

Let F, be the the functor $F : \text{ggrMod-}A \longrightarrow \mathcal{C}\mathscr{A}$ such that on the objects

$$F : M \mapsto F(M),$$

where $F(M) : \mathscr{A}^{\text{op}} \longrightarrow \text{Dif } k$ such that $F(M(i)) = \bigoplus_{l\in\mathbb{Z}} M^l_{-i}$, and for a morphism $f : M \longrightarrow N$, $F(f) : F(M) \longrightarrow F(N)$ such that $F(f)_i : \bigoplus_{l\in\mathbb{Z}} M^l_{-i} \longrightarrow \bigoplus_{l\in\mathbb{Z}} N^l_{-i}$ is the morphism of dg-A-modules induced by f.

Now, we define a functor, G, going in the opposite direction. That is, $G : \mathcal{C}\mathscr{A} \longrightarrow \text{ggrMod-}A$, such that on objects,

$$G : M \mapsto G(M) := \bigoplus_i M(-i),$$

where $G(M)^i_j = M(-i)^j$, one must keep in mind that $M(-i) \in \text{Dif } k$ thus has cohomological grading. On morphisms, G does the obvious thing. We have the following lemma.

Lemma 6.2.3. *The functors F and G,*

$$\text{ggrMod -}A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}\mathscr{A}, \tag{2.1}$$

define an isomorphism of categories.

Proof. Straightforward. □

The next theorem gives finally the isomorphism between $C(A)$ and $C(A^{!^{dg}})$.

Remark 6.2.4. The isomorphism in Lemma 6.2.3 extends to an isomorphism of derived categories.

Theorem 6.2.5. [Kel2] *Let A be a Koszul algebra, then*

$$\phi_{K^{dg}} : C^*(A) \longrightarrow C^*(A^{!^{dg}}), \tag{2.2}$$

is an isomorphism in $\text{Ho}(B_\infty)$.

Proof. Let K^{dg} be the dg-Koszul complex of A. It is an $A^{!dg^{\mathrm{op}}} \otimes A$-module and as right A module is a resolution of A_0. In particular it follows from the definition of $A^{!dg}$ and the proof of Lemma 6.1.4 that the map

$$(A^{!dg})^q_{-q} \longrightarrow \mathbb{R}\mathrm{Hom}_A(K^{dg}, [q]\langle -q\rangle K^{dg}), \tag{2.3}$$

induced by left multiplication is a quasi-isomorphism.

Let $\mathscr{A}$ be as in Definition 6.2.2. We note that now, our A is concentrated in zeroth cohomological degree and only has adams grading, thus for $i, j \in \mathbb{Z}$, the morphism space $\mathscr{A}(i, j) = A_{j-i}$ is concentrated in cohomological degree 0.

We associate to $A^{!dg}$, the dg-category $\mathscr{B}$, whose objects are integers and for $i, j \in \mathbb{Z}$ the morphism spaces are given by,

$$\mathscr{B}(i, j) := (A^{!dg})_{j-i} = (A^{!dg})^{i-j}_{j-i}.$$

Note that $\mathcal{B}(i, j)$ sits in cohomological degree $i - j$. Given $i, j \in \mathbb{Z}$.

We now associate to K^{dg} the dg-$\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}$-module, $\mathcal{K}$ defined as,

$$\mathcal{K}(i, j) = (K^{dg})_{j-i}.$$

The way to visualize what, $\mathcal{K}$ does is to think that to each pair i, j it associates a cohomological slice of K^{dg} at adams degree $j - i$.

We must show that

$$(\mathcal{D}\mathscr{B})(\hat{i}, [n]\hat{j}) \longrightarrow (\mathcal{D}\mathscr{A})(\mathcal{K}(?, i), [n]\mathcal{K}(?, j)),$$

is an isomorphism for all $n \in \mathbb{Z}$. By Lemma 5.2.1 this is equivalent to showing that

$$\lambda : \mathscr{B} \longrightarrow \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathcal{K}, \mathcal{K})$$

is a quasi-isomorphism. However for each pair $i, j \in \mathbb{Z}$ we have

$$\lambda_{i,j} : A^{!dg}_{i-j} = (A^{!dg})^{j-i}_{i-j} \longrightarrow \mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathcal{K}(?, i), \mathcal{K}(?, j)).$$

Now by definition of the functor G,

$$G\mathcal{K}(-, i) = \bigoplus_{l \in \mathbb{Z}} G\mathcal{K}(-l, i) = \bigoplus_{l \in \mathbb{Z}} K^{dg}_{i+l} = \langle i\rangle K^{dg}.$$

It now follows from Lemma 6.2.3 that

$$\mathbb{R}\mathrm{Hom}_{\mathscr{A}}(\mathcal{K}(?, i), \mathcal{K}(?, j)) \cong \mathbb{R}\mathrm{Hom}_A(K^{dg}, [j - i]\langle i - j\rangle K^{dg}).$$

Thus, for all $i, j \in \mathbb{Z}$, the map $\lambda_{i,j}$ is a quasi-isomorphism since (2.3) is a quasi-isomorphism. What we have have shown is that $? \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{K}$ is fully faithful (and thus $\phi_{K^{dg}}$ actually exists!). A similar argument will show that $\mathcal{K} \otimes^{\mathbb{L}}_{\mathcal{A}} ?$ is fully faithful and hence we can apply Theorem 5.2.17 to conclude that $\phi_{K^{dg}}$ is an isomorphism in $\mathrm{Ho}(B_\infty)$. $\square$

7. Appendix A: The Verdier Quotient in Triangulated Categories

7.1. **Triangulated Categories.**

We begin by first recalling the definition of a triangulated category for the convenience of the reader. Let $\mathscr{T}$ be an additive category, and $[1] : \mathscr{T} \longrightarrow \mathscr{T}$ an autofunctor. We call a diagram in $\mathscr{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} [1]X \tag{1.1}$$

where $v \circ u = 0$, $w \circ v$, and $[1]u \circ w = 0$ a candidate triangle. A morphism of candidate triangles is a collection of maps (f, g, h) such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & {[1]X} \arrow[d, "{[1]f}"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & {[1]X',} \end{tikzcd} \tag{1.2}$$

commutes.

Definition 7.1.1. *A triangulated category is a triple $(\mathscr{T}, [1], S)$ (we will usually abuse notation and write $\mathscr{T}$ instead of the triple) where S is some collection of candidate triangles which we call (distinguished) triangles, together with the following 4 axioms:*

<u>*TR0:*</u> *The collection S is closed under isomorphism and for any $X \in \mathscr{T}$ the candidate triangle*

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow [1]X\,, \tag{1.3}$$

is distinguished.

<u>*TR1:*</u> *("S contains all cones.") For any morphism $f : X \longrightarrow Y$, in $\mathscr{T}$ there is a triangle*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow [1]X. \tag{1.4}$$

<u>*TR2:*</u> *("S is closed under rotating of triangles") If*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} [1]X \tag{1.5}$$

is a triangle. Then so is

$$Y \xrightarrow{-v} Z \xrightarrow{-w} [1]X \xrightarrow{-[1]u} [1]Y. \tag{1.6}$$

<u>*TR3:*</u> *For any commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & {[1]X} \arrow[d, "{[1]f}"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & {[1]X',} \end{tikzcd} \tag{1.7}$$

where the two rows are triangles, there exists a non-unique morphism $h : Z \longrightarrow Z'$ such that the diagram commutes.

<u>*TR4*</u> *(The octahedral axiom) Given triangles*

$$X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow [1]X, \tag{1.8}$$
$$Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow [1]Y,$$
$$X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow [1]X,$$

there exists a triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow [1]Z,' \tag{1.9}$$

such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "id"] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & {[1]X} \arrow[d, "id"] \\
X \arrow[r, "vu"] \arrow[d, "u"] & Z \arrow[r] \arrow[d, "id"] & Y' \arrow[r] \arrow[d] & {[1]X} \arrow[d, "{[1]u}"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "id"] & {[1]Y} \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & {[1]Z'},
\end{tikzcd} \tag{1.10}$$

commutes.

A category $\mathscr{T}$ satisfying axioms TR1-TR3, is sometimes called a pre-triangulated category. The octahedral axiom (TR4) has a reputation for being slightly esoteric to beginners but can be reformulated as, see for example [Nee]:
<u>TR4':</u> Given any diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & {[1]X} \arrow[d, "{[1]f}"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & {[1]X'},
\end{tikzcd} \tag{1.11}$$

then it is possible to chose a morphism $h : Z \longrightarrow Z'$, such that the "totalization of the the bicomplex in 1.11 , completed with the morphism h" is again a triangle. i.e. The diagram

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} [1]X \oplus Z' \xrightarrow{\begin{pmatrix} -[1]u & 0 \\ [1]f & w' \end{pmatrix}} [1]Y \oplus [1]X' \tag{1.12}$$

is a triangle.

The prototypical examples of a triangulated categories are the homotopy category of chain complexes, $\mathcal{HA}$, and its corresponding derived category, $\mathcal{DA}$, associated to an abelian category $\mathcal{A}$. In this case, the set of (distinguished) triangles is defined to be the all candidate triangles isomorphic to some diagram of the form

$$X \xrightarrow{f} Y \longrightarrow cone(f) \longrightarrow [1]X, \tag{1.13}$$

in the homotopy category. Note that in the category of chain complexes $\mathcal{CA}$, the diagram 1.13, in general, is not even a candidate triangle, because the composition of f and the inclusion into its mapping cone need not be zero.

We would also like to point out that in general $\mathcal{HA}$ is not an abelian category since in a triangulated category all monomorphisms split, see [Mur2]. In particular this shows that any abelian triangulated category is semi-simple.

7.2. **The Verdier Quotient.**

We now turn to the main goal of this appendix which is to sketch the construction of the Verdier quotient. In the main references for this section [Mur2] and [Nee], strictness is assumed in the definition of triangulated subcategory, we do not assume this and all the constructions we make go through with out it.

Theorem 7.2.1. *(Verdier) Let $\mathscr{T}$ be a triangulated category, and $\mathscr{C}$ a triangulated subcategory. The Verdier quotient is a pair $(\mathscr{T}/\mathscr{C}, q)$ where $\mathscr{T}/\mathscr{C}$ is a triangulated category and $q : \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{C}$ is a triangulated functor, such that $\mathscr{C}$ is a triangulated subcategory of* $\mathrm{Ker}(q)$. *With the universal property such that for any other triangulated functor $F : \mathscr{T} \longrightarrow \mathcal{D}$, with $\mathscr{C}$ a triangulated subcategory of* $\mathrm{Ker}(F)$, *there is is a unique triangulated functor $\tilde{F} : \mathscr{T}/\mathscr{C} \longrightarrow \mathscr{D}$ such that the diagram*

$$\begin{tikzcd} \mathscr{T} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{D} \\ \mathscr{T}/\mathscr{C} \arrow[ur, "\tilde{F}"'] & \end{tikzcd} \tag{2.1}$$

commutes. i.e. the functor F factors through the Verdier quotient.

Before getting to work, we would like to convince the reader that for any triangulated functor $F : \mathscr{A} \longrightarrow \mathscr{B}$, the gadget $\mathrm{Ker}(F)$ (i.e. the full subcategory of $\mathscr{A}$ whose set of objects is $\{X \in \mathscr{A} |\ F(X) \cong 0\}$, where 0 is the (unique)zero object in $\mathscr{B}$) is a thick subcategory of $\mathscr{A}$, in the sense of Definition 3.2.14. This is important because we will see that $\mathrm{Ker}(q)$ appearing in Theorem 7.2.1, is the smallest thick subcategory containing $\mathscr{C}$.

Lemma 7.2.2. *Let $F : \mathscr{A} \longrightarrow \mathscr{B}$, then* $\mathrm{Ker}(F)$ *is thick.*

Proof. Clearly $\mathrm{Ker}(F)$ is a strict full subcategory (not yet triangulated!) $\mathrm{Ker}(F)$ of $\mathscr{A}$ that is closed under [1]. To see it is triangulated, take any morphism $f : X \longrightarrow Y \in \mathrm{Ker}(F)$. There is a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow [1]X \tag{2.2}$$

in $\mathscr{A}$. Applying F to this triangle gives a triangle

$$F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(Z) \longrightarrow [1]F(X) \tag{2.3}$$

in $\mathscr{B}$. Then we can complete the following diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\cong"'] & F(Y) \arrow[r] \arrow[d, "\cong"'] & F(Z) \arrow[r] & {[1]F(X)} \arrow[d, "\cong"'] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd} \tag{2.4}$$

to a morphism of triangles and by [Nee, Proposition 1.1.20.], $F(Z) \cong 0$ and thus $Z \in \mathrm{Ker}(F)$. Being closed under taking direct summands follows easily since F is an additive functor. The claim follows. $\square$

The objects of $\mathscr{T}/\mathscr{C}$ are easily described, since they are just the objects of $\mathscr{T}$. What is more difficult is describing the morphisms. Indeed to do this we will develop a so called calculus of fractions similar to the case of the Gabriel-Zisman localization, see [Wei, Chapter 10].

Definition 7.2.3. *Let $Morph_{\mathscr{C}}$ be the collection of morphisms in $\mathscr{T}$ such that $f \in Morph_{\mathscr{C}}$ if and only if in some triangle*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow [1]X$$

the object Z lives in $\mathscr{C}$.

The next two lemmas, are technical results that help us to establish that there is a (not necessarily triangulated) subcategory of $\mathscr{T}$, which by abuse of notation we will call $\mathrm{Morph}_{\mathscr{C}}$ whose objects are the same as $\mathscr{T}$ and whose morphisms are those in $\mathrm{Morph}_{\mathscr{C}}$.

Lemma 7.2.4. *If $f : X \longrightarrow Y$ is an isomorphism in $\mathscr{T}$, then $f \in Morph_{\mathscr{C}}$*

Proof. If $f : X \longrightarrow Y$ is an isomorphism then there is a triangle

$$X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow [1]X.$$

Clearly $0 \in \mathscr{C}$, the claim follows. $\square$

Lemma 7.2.5. *(2 out of 3) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$ be two morphisms in $\mathscr{T}$. If any two of f, g and gf lie in $Morph_{\mathscr{C}}$, then so does the third.*

Proof. Follows immediately from TR4. $\square$

Proposition 7.2.6. *There is a (not necessarily triangulated) subcategory of $\mathscr{T}$, which we denote as $Morph_{\mathscr{C}}$ whose objects are the same as $\mathscr{T}$ and whose morphisms are those in $Morph_{\mathscr{C}}$.*

Proof. For $X \in \mathscr{C}$, the map $\mathrm{id} : X \longrightarrow X$ is an isomorphism and hence lies in $\mathrm{Morph}_{\mathscr{C}}$ by Lemma 7.2.4. It follows from Lemma 7.2.5 that the composition of two morphisms in $\mathrm{Morph}_{\mathscr{C}}$ is again in $\mathrm{Morph}_{\mathscr{C}}$. The claim follows. $\square$

Example 7.2.7. Let $\mathscr{C}$ be the full subcategory of $\mathcal{H}k$ whose objects consists of the acyclic complexes. It is straight forward to check that $\mathscr{C}$ is triangulated subcategory. We see $f \in \mathrm{Morph}_{\mathscr{C}}$ if and only if there is some triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow [1]X$$

in $\mathcal{H}k$, such that Z is acyclic or equivalently if f is a quasi-isomorphism.

For any two objects X, Y in $\mathscr{T}$ let $\alpha(X,Y)$ be the class of diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \tag{2.5}$$

such that $f \in \mathrm{Morph}_{\mathscr{C}}$. We now impose a relation, $R(X,Y)$, on $\alpha(X,Y)$ as follows: Two diagrams (Z, f, g) and (Z', f', g') belong to $R(X,Y)$ if and only if there is a diagram$(Z'', f'', g'') \in \alpha(X,Y)$ and morphisms $u : Z'' \longrightarrow Z$ and $v : Z'' \longrightarrow Z'$ such that the diagram

$$\begin{tikzcd} & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & Z'' \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[u, "v"] \arrow[d, "u"'] & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \tag{2.6}$$

the diagram commutes. One notices immediately that since $\mathrm{Morph}_{\mathscr{C}}$ is a category that u, v are also in $\mathrm{Morph}_{\mathscr{C}}$.

Lemma 7.2.8. *The relation $R(X,Y)$ defined in figure* (2.6) *is an equivalence relation.*

Proof. The only non-trivial thing to prove is transitivity, see [Nee, Lemma 2.1.14]. ☐

We are now in a position to define the morphism space between two objects in the Verdier quotient.

Definition 7.2.9. *Let $(\mathscr{T}/\mathscr{C})(X,Y)$ be the set $\alpha(X,Y)$ modulo the equivalence relation $R(X,Y)$. Since we mod out by $R(X,Y)$ our tuples no longer have to represent morphisms as triples, hence from now we will denote elements in $(\mathscr{T}/\mathscr{C})(X,Y)$ as $[f,g]$, where $f \in \mathrm{Morph}_{\mathscr{C}}$.*

There is still some clarification needed. Firstly, we have to say how to define the composition of two maps. Secondly, there are some set theoretic issues i.e. $(\mathscr{T}/\mathscr{C})(X,Y)$ might not be a set. We will discuss the first point in some detail and point the reader to [Mur2] for an in depth discussion of the second.

For diagrams $(W_1, f, g) \in \alpha(X,Y)$ and $(W_2, s, t) \in \alpha(Y,Z)$. We have the diagram

$$\begin{tikzcd} & W_1 \arrow[dl, "f"'] \arrow[dr, "g"] & & W_2 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y & & Z. \end{tikzcd} \tag{2.7}$$

Taking the so called "homotopy pullback" of the diagram

$$\begin{tikzcd} & W_2 \arrow[d, "s"] \\ W_1 \arrow[r, "g"] & Y, \end{tikzcd}$$

gives the homotopy cartesian square

$$\begin{tikzcd} W_3 \arrow[r, "g'"] \arrow[d, "s'"'] & W_2 \arrow[d, "s"] \\ W_1 \arrow[r, "g"] & Y. \end{tikzcd}$$

Then, it follows from applying Lemma 3.2.23 to (7.2) that $s \in \mathrm{Morph}_{\mathscr{C}}$ if and only if $s' \in \mathrm{Morph}_{\mathscr{C}}$. Thus $fs' \in \mathrm{Morph}_{\mathscr{C}}$. So the diagram

$$\begin{tikzcd} & W_3 \arrow[dl, "fs'"'] \arrow[dr, "tg'"] & \\ X & & Z \end{tikzcd} \tag{2.8}$$

is an element in $\alpha(X,Z)$. It is easy to check that the diagram 2.8 is independent of the choice of s' and g' modulo the equivalence relation $R(X,Z)$ on $\alpha(X,Z)$ so we get a map

$$\alpha(X,Y) \times \alpha(Y,Z) \longrightarrow (\mathscr{T}/\mathscr{C})(X,Z). \tag{2.9}$$

Lemma 7.2.10. *The map* 2.9 *descends to a well defined map*

$$(\mathscr{T}/\mathscr{C})(X,Y) \times (\mathscr{T}/\mathscr{C})(Y,Z) \longrightarrow (\mathscr{T}/\mathscr{C})(X,Z), \tag{2.10}$$

and is associative.

Proof. The proof is straightforward see [Nee, Lemma 2.1.19]. □

It is clear that since $\mathrm{id}_X \in (\mathscr{T})(X,X)$ is an isomorphism, then $\mathrm{id}_X \in \mathrm{Morph}_{\mathscr{C}}$. Thus $[\mathrm{id}_X, \mathrm{id}_X] \in (\mathscr{T}/\mathscr{C})(X,X)$ and plays the role of the identity morphism in the Verdier quotient.

To summarize our discussion so far, we have defined a category $\mathscr{T}/\mathscr{C}$ whose objects are the objects of $\mathscr{T}$, and for two objects $X, Y \in \mathscr{T}/\mathscr{C}$, $(\mathscr{T}/\mathscr{C})(X,Y) = \alpha(X,Y)/R(X,Y)$ and the composition of two morphisms is given by Figure 2.8.

We also have an obvious candidate for the universal functor $q : \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{C}$, namely:

$$\begin{array}{rcl} q : ob(\mathscr{T}) & \longrightarrow & ob(\mathscr{T}/\mathscr{C}) \\ X & \mapsto & X, \end{array} \tag{2.11}$$

and for $X, Y \in \mathscr{T}/\mathscr{S}$

$$\begin{array}{rcl} q : (\mathscr{T})(X,Y) & \longrightarrow & (\mathscr{T}/\mathscr{C})(X,Y) \\ f & \mapsto & [1, f]. \end{array} \tag{2.12}$$

Note that for $f \in \mathrm{Morph}_{\mathscr{C}}$, $q(f) = \mathrm{id}$ in $\mathscr{T}/\mathscr{C}$.

Detailed proofs that $\mathscr{T}/\mathscr{C}$ is an additive category and that q is an additive functor can be found in [Mur2] and [Nee]. We will focus on describing the universal property and the triangulated structure of $\mathscr{T}/\mathscr{C}$.

Proposition 7.2.11. *The functor $q : \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{C}$ is universal amongst all functors $F : \mathscr{T} \longrightarrow \mathscr{D}$ such that F sends $f \in Morph_{\mathscr{C}}$ to isomorphisms. i.e.*

there exists a unique functor $\tilde{F} : \mathscr{T}/\mathscr{C} \longrightarrow \mathscr{D}$ *such that the diagram*

$$\begin{tikzcd} \mathscr{T} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{D} \\ \mathscr{T}/\mathscr{C} \arrow[ur, "\tilde{F}"'] & \end{tikzcd} \tag{2.13}$$

commutes.

Proof. Suppose $F : \mathscr{T} \longrightarrow \mathscr{D}$ is a functor that sends morphisms in $\mathrm{Morph}_{\mathscr{C}}$ to isomorphisms. Since the objects of $\mathscr{T}$ and $\mathscr{T}/\mathscr{C}$ are the same, $\tilde{F}$ must be the same as F on the objects. For $X, Y \in \mathscr{T}/\mathscr{C}$ we define

$$\begin{aligned} \tilde{F} : (\mathscr{T}/\mathscr{C})(X, Y) &\longrightarrow (\mathscr{D})(F(X), F(Y)) \\ [f, g] &\mapsto F(g)F(f)^{-1}. \end{aligned} \tag{2.14}$$

It is easy to check that this construction is well defined and that $\tilde{F}$ is indeed a functor. Clearly, $\tilde{F}$ is unique and $\tilde{F} \circ q = F$ and the claim follows. $\square$

The autofunctor [1] on $\mathscr{T}$ descends to $\mathscr{T}/\mathscr{C}$ i.e where $[1][f, g] = [[1]f, [1]g]$ it is easy to see that this defines an autofunctor on $\mathscr{T}/\mathscr{C}$. We define a triangle in $\mathscr{T}/\mathscr{C}$ to be a diagram that is isomorphic to a diagram

$$q(X) \longrightarrow q(Y) \longrightarrow q(Z) \longrightarrow [1]q(Z), \tag{2.15}$$

where,

$$X \longrightarrow Y \longrightarrow Z \longrightarrow [1]X, \tag{2.16}$$

is a triangle in $\mathscr{T}$.

Theorem 7.2.12. *With the structure described in figure* 2.15*,* $\mathscr{T}/\mathscr{C}$ *becomes a triangulated category and* q *a triangulated functor.*

Proof. Detailed, but lengthy proofs can be found in [Mur2, Section 2] and [Nee, Chapter 2]. $\square$

The next set of arguments, aims to show that $\mathscr{C} \subseteq \mathrm{Ker}(q)$ and that q has the universal property described in Theorem 7.2.1.

Lemma 7.2.13. *Let* $f, g : X \longrightarrow Y$ *be morphisms in* $\mathscr{T}$*. The following are equivalent:*

1) $q(f) = q(g)$.

2) There exists an $\alpha : W \longrightarrow X \in \mathit{Morph}_{\mathscr{C}}$ *with* $f\alpha = g\alpha$.

3 $f - g : X \longrightarrow Y$ *factors through some object of* $\mathscr{C}$.

Proof. The proof uses arguments very similar to those of Lemma 3.3.2. $\square$

Lemma 7.2.14. *A morphism in* $\mathscr{T}/\mathscr{C}$ *of the form*

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

is an isomorphism if and only if there exists morphisms $f, h \in \mathscr{T}$ such that $gf, hg \in \mathrm{Morph}_{\mathscr{C}}$.

Proof. Suppose that $[\alpha, g] = [\mathrm{id}_W, g][\alpha, \mathrm{id}_W] = q(g)q(\alpha)^{-1}$ is an isomorphism in $\mathscr{T}/\mathscr{C}$. Then it must be that also $q(g)$ is also an isomorphism. Suppose the diagram

$$\begin{tikzcd} & M \arrow[dl, "\beta"'] \arrow[dr, "f"] & \\ Y & & W \end{tikzcd}$$

is a right inverse to $q(g)$. It follows that $[\beta, gf] = \mathrm{id}$ and so it follows that $gf \in \mathrm{Morph}_{\mathscr{C}}$ since by the 2 out of 3 Lemma for $\mathrm{Morph}_{\mathscr{C}}$. We can write the left inverse of $q(g)$ as $q(a)^{-1}q(h)$. Then $q(a)^{-1}q(h)q(g) = \mathrm{id}$ implies that $q(hg) = q(a)$ then by 7.2.13 there exists $t \in \mathrm{Morph}_{\mathscr{C}}$ such that $hgt = at$. Then it follows from the 2 out 3 lemma that $hg \in \mathrm{Morph}_{\mathscr{C}}$.

Conversely, suppose there exists morphisms f, h such that $gf, hg \in \mathrm{Morph}_{\mathscr{C}}$. Then $q(g)$ has a left and right inverse and hence so does $q(g)q(\alpha)^{-1}$. □

Proposition 7.2.15. *The zero morphism $g : X \longrightarrow 0$ in $\mathscr{T}$ becomes an isomorphism in $\mathscr{T}/\mathscr{C}$ if and only if there exists $Y \in \mathscr{T}$ with $X \oplus Y \in \mathscr{C}$.*

Proof. Suppose that $q(g)$ is an isomorphism, then Lemma 7.2.14 there exists $h : 0 \longrightarrow [1]Y$ so that the composition $hg \in \mathrm{Morph}_{\mathscr{C}}$. Taking the direct sum of the triangles

$$X \longrightarrow 0 \longrightarrow [1]X \longrightarrow [1]X$$
$$0 \longrightarrow [1]Y \longrightarrow [1]Y \longrightarrow 0,$$

gives the triangle

$$X \xrightarrow{0} [1]Y \longrightarrow [1](X \oplus Y) \longrightarrow [1]X.$$

Since $0 = hg : X \longrightarrow [1]Y$ is in $\mathrm{Morph}_{\mathscr{C}}$, it follows that $[1](X \oplus Y)$ must be in $\mathscr{C}$ and since $\mathscr{C}$ is triangulated it follows that $X \oplus Y$ is in $\mathscr{C}$.

Conversely, suppose that there is a $Y \in \mathscr{T}$ such that $X \oplus Y \in \mathscr{C}$. Let $h : 0 \longrightarrow [1]Y$ and $f : 0 \longrightarrow X$ be the zero maps. Then $gf : 0 \longrightarrow 0$ is an isomorphism and hence is in $\mathrm{Morph}_{\mathscr{C}}$. If we show that $hg \in \mathrm{Morph}_{\mathscr{C}}$ we are done since then it follows that g is an isomorphism in $\mathscr{T}/\mathscr{C}$. To this end, the 0 map $hg : X \longrightarrow [1]Y$ fits into the triangle

$$X \xrightarrow{0} [1]Y \longrightarrow [1](X \oplus Y) \longrightarrow [1]X,$$

with $[1](X \oplus Y) \in \mathscr{C}$. Hence $hg \in \mathrm{Morph}_{\mathscr{C}}$ and the claim follows. □

Corollary 7.2.16. *$\mathscr{C}$ is a triangulated subcategory of* $\mathrm{Ker}(q)$.

Proof. Let $X \in \mathscr{C}$. The category $\mathscr{C}$ is additive so for $0 \in \mathscr{T}$, $0 \oplus X \in \mathscr{C}$, and hence $q(X) \cong 0 \in \mathscr{T}/\mathscr{C}$ by Proposition 7.2.15. □

Corollary 7.2.17. $\mathrm{Ker}(q)$ *is the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{C}$*

Proof. Suppose $\mathscr{C}$ is a triangulated subcategory of M a thick subcategory. Let $X \in \mathrm{Ker}(q)$, then the zero map $X \longrightarrow 0$ becomes an isomorphism in $\mathscr{T}/\mathscr{C}$ hence by Proposition 7.2.15, there exits a $Y \in \mathscr{T}$ such that $X \oplus Y \in \mathscr{C} \subseteq M$. But since M is thick it follows that $X \in M$. The claim follows. □

Theorem 7.2.18. *For a triangulated functor $F : \mathscr{T} \longrightarrow \mathscr{D}$, such that $\mathscr{C}$ is a triangulated subcategory of* $\mathrm{Ker}(F)$*. There exists a triangulated functor $\tilde{F}$ such that the diagram*

$$\begin{tikzcd} \mathscr{T} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{D} \\ \mathscr{T}/\mathscr{C} \arrow[ur, "\tilde{F}"'] & \end{tikzcd} \tag{2.17}$$

commutes.

Proof. Since $\mathscr{C}$ is a triangulated subcategory of $\mathrm{Ker}(F)$, then F sends morphisms in $\mathrm{Morph}_{\mathscr{C}}$ to isomorphisms in $\mathscr{D}$. Since, for $f \in \mathrm{Morph}_{\mathscr{C}}$, there is a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow [1]X,$$

with $Z \in \mathscr{C}$. Applying the triangulated functor F gives a triangle

$$F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(Z) \longrightarrow [1]F(X), \tag{2.18}$$

in $\mathscr{D}$. But by assumption $F(Z) \cong 0$ and thus $F(f)$ is an isomorphism. Via Proposition 7.2.11, there is an additive functor $\tilde{F}$ such that $\tilde{F} \circ q = F$. We just have to show that $\tilde{F}$ is triangulated. Since F is a triangulated functor there is a natural equivalence $\psi : F[1] \longrightarrow [1]F$. Since $\tilde{F}(X) = F(X)$ for every $X \in \mathscr{T}/\mathscr{C}$ one can check that the isomorphisms ψ_X also give a natural equivalence $\tilde{F}[1] \longrightarrow [1]\tilde{F}$. The claim follows. □

It is now more or less obvious that in the case of the derived category of chain complexes that the Verdier quotient of $\mathcal{H}k$ by the thick subcategory of acyclic complexes is the same as the derived category one gets by Gabriel-Zisman localization that is most commonly seen in homological algebra books [Wei].

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