Diagrammatics for singular Soergel bimodules

Tim Blödtner

Born 15.07.1997 in Hohenmölsen, Germany

Master's Thesis Mathematics

Advisor: Prof. Dr. Catharina Stroppel

Second Advisor: Dr. Daniel Tubbenhauer

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

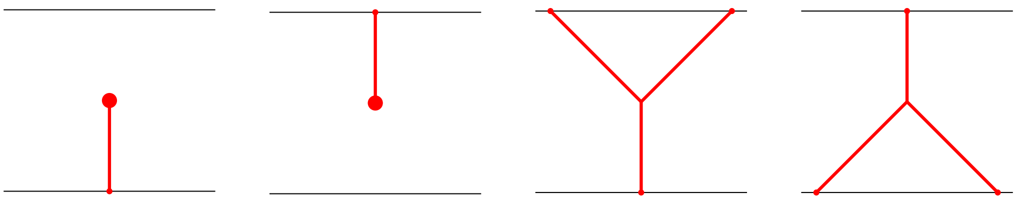
Rheinischen Friedrich-Wilhelms-Universität Bonn

Contents

1 Introduction

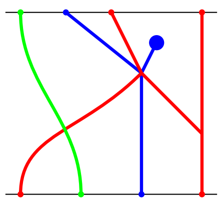
To a Coxeter group (W, S) one can define the Hecke algebra $\mathcal{H}$ which is a deformation of the group algebra of W. One usually considers two bases in this algebra, the standard basis and the Kazhdan–Lusztig basis. The coefficients of the base change matrix between these two bases are known as the Kazhdan–Lusztig polynomials. Kazhdan and Lusztig conjectured [KL79] that these polynomials can be used to describe characters of simple highest weight modules over complex semisimple Lie algebras and this was later proven by Beilinson–Bernstein [BB81] and Brylinski–Kashiwara [BK81] in 1981. This justifies the importance of the Kazhdan–Lusztig polynomials.
A consequence of these results is, that if W is a Weyl group, the sum of all coefficients in a given Kazhdan–Lusztig polynomial is a non-negative number, since it can be interpreted as a certain Jordan–Hölder multiplicity in Lie theory. The Kazhdan–Lusztig positivity conjecture states that all coefficients of these polynomials (for arbitrary Coxeter groups) are positive. In order to prove this conjecture Soergel considered a certain category $\mathbb{S}\mathrm{Bim}$ of special bimodules attached to a Coxeter system which are nowadays called Soergel bimodules. He proved [Soe92, Soe07] that this monoidal category categorifies the Hecke algebra $\mathcal{H}$ and he also proved that the indecomposable bimodules are classified by the elements of the Coxeter group W. Indecomposable Soergel bimodules are exactly direct summands of the so-called Bott–Samelson bimodules which are much easier to describe. They categorify monomials in the Kazhdan–Lusztig generators of $\mathcal{H}$. It is the passage to direct summands which makes the category of Soergel bimodules extremely hard to understand.
Soergel conjectured that under his categorification these indecomposable bimodules correspond to the Kazhdan–Lusztig basis of $\mathcal{H}$. Assuming this conjecture he was able to prove the Kazhdan–Lusztig positivity conjecture by relating the coefficients of the Kazhdan–Lusztig polynomials to dimensions of certain homomorphism spaces [Soe07]. However, Soergel could only prove his conjecture for some Coxeter groups (in particular Weyl groups) [Soe92].
Soergel's conjecture yields more far reaching consequences than just a proof of the Kazhdan–Lusztig positivity conjecture. For instance it provides a natural "geometry" for arbitrary Coxeter groups. Soergel (bi)modules were originally introduced by Soergel to better understand category $\mathcal{O}$ and Harish-Chandra bimodules. In particular Soergel's conjecture also implies the Kazhdan–Lusztig conjecture on characters of simple highest weight modules. The recent courses [EMTW20] and [Str20b] give an overview about such details.
Soergel's conjecture was proven for arbitrary Coxeter groups by Elias and Williamson [EW14]. The catalyst to this advancement was their diagrammatic theory for Soergel bimodules. They introduced a diagrammatic category by generators and relations and

proved that this category is equivalent to Soergel bimodules (at least under some technical assumptions for the general case). This was done in [EW16] which is also the main source for this thesis. Objects in this category are sequences of points on a line which are labelled or "coloured" by elements of S. The morphisms encode all the information and are coloured graphs between two such sequences. They are built out of certain generators including



(end)dot (start)dot split merge

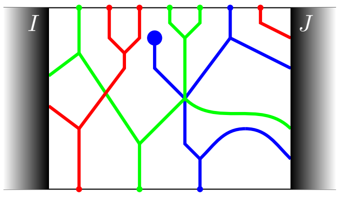
(see Definition 4.1) and could for example look as following.



This diagrammatic category can be considered independently of Soergel bimodules. The definition is much more elementary, it is better suited for generalisations and specialisations and allows to make explicit calculations which are even harder in the algebraic setting of Soergel bimodules. Moreover, one can use it as a categorification of the Hecke algebra in the same way as Soergel bimodules, but it works already under very weak assumptions. The diagrammatic theory also led to many more advancements than just the proof of Soergel's conjecture. In fact the diagrammatic category is a strictification of the monoidal category of Soergel bimodules and is therefore much more rigid and easier to handle, in particular in view of higher categories, and extremely useful in terms of categorification.

Hecke algebras arise naturally in representation theory, but even nicer is an enlargement, the so-called Hecke algebroid, and the Schur algebras sitting inside there. They arise for instance naturally from the representation theory of the general linear group. Based on works of Soergel [Soe92] and Stroppel [Str04] who introduced singular Soergel bimodules which are a generalization of Soergel bimodules Williamson introduced [Wil11] the 2-category of singular Soergel bimodules. He proved that this 2-category categorifies the Hecke algebroid in a similar fashion as Soergel bimodules categorify the Hecke algebra. Since the diagrammatic theory helped significantly to understand Soergel bimodules it is now natural to ask whether it is possible to generalize the diagrammatic theory to singular Soergel bimodules. In this thesis we will investigate this task for the symmetric group $W = S_n$.

We will start with the diagrammatical Soergel calculus of Elias and Williamson [EW16] and try to improve it step by step to fit it into the setup of singular Soergel bimodules. While Soergel bimodules are certain (R, R)-bimodules for a certain ring R depending on W, singular Soergel bimodules are certain (R^I, R^J)-bimodules where R^I, R^J are subrings of invariants for varying parabolic subsets $I, J \subseteq S$. To develop a suitable diagrammatic approach we first need to incorporate the (R^I, R^J)-bimodule structure into the setup. This will be done in Definition 6.1 where we fix some $I, J \subseteq S$ and generalize an idea of Elias [Eli16, Section 5] to define a new diagrammatic category ${}_I\mathcal{T}_J$. The objects will be the same as before. To describe the morphisms we follow an idea of Elias for one-sided Soergel bimodules. Namely, the restriction to the action to invariants is encoded by including a (black/grey) membrane on one side. We will do this now and will also include a membrane on the other side, and thus the main difference in the pictures will be two membranes. A morphism then looks for instance as follows.

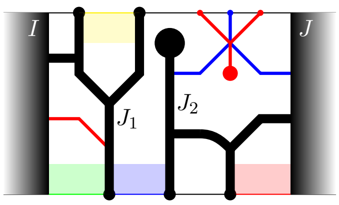


As a slight generalization of [Eli16, Theorem 5.6] we obtain the first result which connects ${}_I\mathcal{T}_J$ to a subcategory of singular Soergel bimodules.

Theorem 6.10. *There is an equivalence of categories* ${}_I\mathcal{F}_J : {}_I\mathcal{T}_J \longrightarrow {}_I\mathbb{BS}\mathrm{Bim}_J$*, where* ${}_I\mathbb{BS}\mathrm{Bim}_J$ *is the category of Bott–Samelson bimodules viewed as* (R^I, R^J)*-bimodules.*

Singular Soergel bimodules form a 2-category with objects parabolic subsets (I, J, etc.) of S, 1-morphisms the bimodules and 2-morphisms the bimodule morphisms. We will therefore similarly also collect all the categories ${}_I\mathcal{T}_J$ (for all choices of $I, J \subseteq S$) together into a 2-category $\mathcal{T}$. Then we will incorporate the analogue of passing from Bott–Samelson bimodules to Soergel bimodules by using the concept of partial idempotent completion. This basically means that we add some direct summands to define a new diagrammatic 2-category $s\mathcal{T}$ (Definition 6.21).

The objects will now be sequences of dots labelled by subsets of S and the spaces in-between are also labelled by subsets of S (under some conditions). We have to include thicker lines into the morphisms which are similar to the two membranes and were also introduced by Elias [Eli16]. They capture the transition from elements of S to subsets of S in the labelling of the dots. Moreover, we introduce coloured areas into the pictures in order to capture the labelling of the spaces in-between the dots. This is all mirroring the transition from simple reflections to parabolic subsets in the definitions of regular and singular Soergel bimodules. A morphisms in $s\mathcal{T}$ will then look as follows.



Our first main result is then an equivalence between $s\mathcal{T}$ and singular Bott–Samelson bimodules (whose Karoubian closure are singular Soergel bimodules).

Theorem 6.27. *There is an equivalence of* 2*-categories* $s\mathcal{F} : s\mathcal{T} \longrightarrow \mathrm{s}\mathbb{BS}\mathrm{Bim}$.

Partial idempotent completions allow us to construct more complicated categories like $s\mathcal{T}$. However, to understand this category we secretly use a trick which transfers calculations to the original category plus the knowledge of idempotents. This is quite convenient for abstract arguments, but in practice the idempotents are hopeless to compute. Our dream would be a complete understanding of all idempotents and their interactions. This is a hard problem. We solve it completely at least for the case $W = S_3$ where we define another 2-category $s\mathfrak{T}$ by generators and relations (Definition 7.8) and prove

Theorem 7.13. *The* 2*-category* $s\mathfrak{T}$ *is equivalent to* $s\mathcal{T}$*, and hence gives a presentation of* $s\mathcal{T}$*.*

We will now give a short summary of each chapter of this thesis.

- In Chapter 2 we will recall some basic notions which are fundamental for all upcoming chapters. We recall the definitions of Coxeter groups (W, S) and the Hecke algebra $\mathcal{H}$ and some basic properties. We continue by recalling the definition of the Hecke algebroid and calculate some examples. We finish this chapter with the definition of graded bimodules and graded categories and collect some basic facts about them. For proofs we refer to the literature.

- We recall the concept of a realization $\mathfrak{h}$ of (W, S) in Chapter 3 which allows us to define the ring $R = S(\mathfrak{h})$ on which W acts naturally. Then the structure of R as a module over the rings of invariants R^J is will be examined where $J \subset S$. First, we do this for general Coxeter systems (W, S) and then we construct an explicit basis in the case $W = S_n$.
After that we finally define the category of Soergel bimodules and state the main theorems for them. Afterwards the same is done for singular Soergel bimodules.

- Chapter 4 is an introduction to the diagrammatics of Elias and Williamson [EW16]. We begin with defining the diagrammatic category $\mathcal{D}$ for $W = S_n$ and then explain what changes in the general case. In this chapter we only recollect statements and results from [EW16]. In the second part we present results of Elias [Eli16]. He generalized the diagrammatics to a category $g\mathcal{D}$ by using partial idempotent completion.

- In Chapter 5 we step away from the diagrammatics to do some calculations on the algebraic side. We give a complete description of the 2-category of singular Soergel bimodules for S_3. More precisely, we classify all indecomposable bimodules and explain how every bimodule decomposes into them. Then we compute all the homomorphism spaces between any pair of indecomposable bimodules.

- Chapter 6 contains the main results of this thesis. In the first section we generalize the ideas of Elias [Eli16, section 5] to get the diagrammatic category ${}_I\mathcal{T}_J$ and prove that it is equivalent to a category of Bott–Samelson bimodules ${}_I\mathbb{BS}\mathrm{Bim}_J$.
In the second section we use the concept of partial idempotent completion to define the 2-category $s\mathcal{T}$ which is a generalisation of ${}_I\mathcal{T}_J$. We identify morphisms in $s\mathcal{T}$ with new pictures and present some new relations for these. Moreover, we prove the equivalence between $s\mathcal{T}$ and the category of singular Bott–Samelson bimodules.

- In Chapter 7 we restrict ourselves again to the case $W = S_3$. First we give a description for $g\mathcal{D}$ by generators and relations (without complicated idempotent relations and inclusion or projection morphisms) and prove Theorem 7.5. In the second part we give a description for $s\mathcal{T}$ by generators and relations and prove Theorem 7.13.

Acknowledgements

First and foremost, I would like to thank Prof. Dr. Catharina Stroppel for suggesting this topic to me and for her extraordinary supervision of this thesis. Thank you for the many long conversations, helpful suggestions and instructive explanations which greatly enhanced my understanding of the topic.
I would also like to thank Dr. Daniel Tubbenhauer and Christian Nöbel for reading through an earlier draft of this thesis and giving helpful feedback.
A special thanks goes to my parents who supported me throughout all my studies, and to my friends with whom I could talk about my thesis even if they did not understand anything.

2 Basics

2.1 Coxeter groups

In this section we will give the definition of Coxeter groups and state some standard facts about them. Standard references are [Bou81] and [Hum90].

Definition 2.1. A pair (W, S) of a group W and a finite subset $S \subset W$ is called *Coxeter system* if there are $m_{st} \in \mathbb{N} \cup \{\infty\}$ for all $s, t \in S$ such that

1. $m_{ss} = 1$ for $s \in S$;
2. $m_{st} \geqslant 2$ if $s \neq t \in S$;
3. $W = \langle s \in S \mid (st)^{m_{st}} = e \rangle$ (in particular S generates W), where $e \in W$ is the neutral element.

The condition $m_{st} = \infty$ means that no relation of the form $(st)^m = e$ should be imposed. The group W is then called *Coxeter group* with set of *generators* (or *simple reflections*) S. ♢

Remark 2.2. Note that for $s, t \in S$ we have $m_{st} = m_{ts}$, since $ts = (st)^{-1}$ and st need to have the same order.
Since S is a finite set we will identify it with the set $\{1, \dots, |S|\}$, i.e. we fix a map $S \xrightarrow{\cong} \{1, \dots, |S|\}$. We will write the elements of S as $s_1, s_2, \dots, s_{|S|}$ via this identification and sometimes write $i \in S$ for a natural number i by which we mean s_i. ♢

Example 2.3. Our main example for a Coxeter group will be S_n. We know that $W = S_n$ becomes a Coxeter system (W, S) via the following choice of S

$$S = \{\text{simple transpositions}\} = \{(i, i+1) \in W \mid 1 \leqslant i \leqslant n-1\}.$$

We have an obvious identification $S \cong \{1, \dots, n-1\}$ via $s_i = (i, i+1)$. Now the numbers $m_{ij} = m_{s_i s_j}$ are given as follows:

- $m_{ii} = 1$ for $1 \leqslant i \leqslant n-1$;
- $m_{ij} = 2$ for $|i - j| > 1$;
- $m_{i,i+1} = 3$ for $1 \leqslant i \leqslant n-2$.

Definition 2.4. Let $w \in W$ and write $w = s_{i_1} \cdots s_{i_d}$. We call $(s_{i_1}, \dots, s_{i_d})$ an *expression* for w. We call an expression $(s_{i_1}, \dots, s_{i_d})$ *reduced* if there is no expression $(s_{j_1}, \dots, s_{j_{d'}})$ for w with $d' < d$.
We define the *length function* $\ell : W \longrightarrow \mathbb{N}_0$ by $\ell(w) = d$ if there is a reduced expression $(s_{i_1}, \dots, s_{i_d})$ for w (including the empty expression for e). ♢

Remark 2.5. Note that for $w \in W$ we have $\ell(w) = 0 \iff w = e$ and $\ell(w) = 1 \iff w \in S$. Moreover, one can check that $\ell(w^{-1}) = \ell(w)$ for all $w \in W$. Indeed, if $w = s_{i_1} \cdots s_{i_d}$ is a reduced expression, then $w^{-1} = s_{i_d} \cdots s_{i_1}$, and thus $\ell(w^{-1}) \leqslant \ell(w)$. This implies that $\ell(w) = \ell\left(\left(w^{-1}\right)^{-1}\right) \leqslant \ell(w^{-1})$, and hence $\ell(w^{-1}) = \ell(w)$.
In the definition we distinguished between the expression $(s_{i_1}, \ldots, s_{i_d})$ and the element $s_{i_1} \cdots s_{i_d} \in W$ which is necessary, since $w = s_{i_1} \cdots s_{i_d}$ might have many expressions. However, we won't be so precise from now on. Instead we will often write "let $w = s_{i_1} \cdots s_{i_d}$ be an (reduced) expression" and mean by it that $(s_{i_1}, \ldots, s_{i_d})$ is an (reduced) expression for w. ♢

The following is a result of Matsumoto [Mat64].

Lemma 2.6. *Let $w = s_{i_1} \cdots s_{i_d} = s_{j_1} \cdots s_{j_d}$ be two reduced expressions for an element $w \in W$. Then one can transform $s_{i_1} \cdots s_{i_d}$ to $s_{j_1} \cdots s_{j_d}$ by repeatedly applying so-called braid moves which transform*

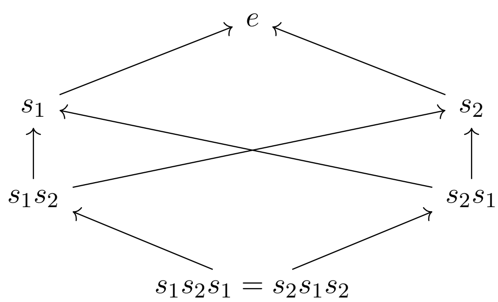
$$\underbrace{sts\cdots}_{m_{st}\text{ factors}} \quad \text{to} \quad \underbrace{tst\cdots}_{m_{st}\text{ factors}}$$

for some $s, t \in S$. These braid moves are allowed by the relation $(st)^{m_{st}} = e$.

Definition 2.7. Let (W, S) be a Coxeter system. Let $s_{i_1} \cdots s_{i_d}$ and $s_{j_1} \cdots s_{j_{d'}}$ be two expressions. We call $s_{j_1} \cdots s_{j_{d'}}$ a *subexpression* of $s_{i_1} \cdots s_{i_d}$ if there is a strictly increasing function $\varphi : \{1, \ldots, d'\} \longrightarrow \{1, \ldots, d\}$ such that $s_{j_k} = s_{i_{\varphi(k)}}$ for all $k = 1, \ldots, d'$. ♢

Definition 2.8. We define a partial ordering on the elements of W, called *Bruhat order.* For $w, u \in W$ we write $u \leqslant w$ if there are reduced expressions $w = s_{i_1} \cdots s_{i_d}$ and $u = s_{j_1} \cdots s_{j_{d'}}$ such that $s_{j_1} \cdots s_{j_{d'}}$ is a subexpression of $s_{i_1} \cdots s_{i_d}$. ♢

Example 2.9. We consider $W = S_3$ with the set generators $S = \{s_1, s_2\}$ where $s_1 = (1,2), s_2 = (2,3)$ are the simple transpositions. Now we can write down the Bruhat order for this Coxeter system as follows.



An arrow means that the element at the source of the arrow is greater than the element at the target of the arrow in the Bruhat order. This picture together with transitivity then give the complete Bruhat order.

Remark 2.10. One can show that for $w, u \in W$ one has $u \leqslant w$ if and only if for any reduced expression $w = s_{i_1} \cdots s_{i_d}$ there is a reduced expression $u = s_{j_1} \cdots s_{j_{d'}}$ such that $s_{j_1} \cdots s_{j_{d'}}$ is a subexpression of $s_{i_1} \cdots s_{i_d}$. ♢

Theorem 2.11 (Strong exchange condition)**.** *Let $w = s_{i_1} \cdots s_{i_d}$ be an expression (not necessarily reduced) for $w \in W$. Let t be a reflection, i.e. $t = usu^{-1}$ for some $s \in S, u \in W$. Suppose $\ell(wt) < \ell(w)$, then there is an index $1 \leqslant k \leqslant d$ for which $wt = s_{i_1} \cdots s_{i_{k-1}} \widehat{s_{i_k}} s_{i_{k+1}} \cdots s_{i_d}$ (where the hat means that this factor has been omitted). If the expression for w is reduced, then k is unique.*

Corollary 2.12 (Deletion property)**.** *Let $w = s_{i_1} \cdots s_{i_d}$ be an expression for $w \in W$ such that $\ell(w) < d$. Then there exist $1 \leqslant l < k \leqslant d$ such that $w = s_{i_1} \cdots \widehat{s_{i_l}} \cdots \widehat{s_{i_k}} \cdots s_{i_d}$.*

Lemma 2.13. *Let (W, S) be a Coxeter system such that W is finite. Then W has a unique longest element w_0 with respect to the length function ℓ. This element is self-inverse and is greater than any other element of W in the Bruhat order. Moreover, for $w \in W$ we have $\ell(ww_0) = \ell(w_0 w) = \ell(w_0) - \ell(w)$.*

The longest element in S_n is the permutation that reverses the order of $1, \ldots, n$.

Corollary 2.14. *Let (W, S) be a Coxeter system such that W is finite. Let $s \in S$, then there are reduced expressions $w_0 = s_{i_1} \cdots s_{i_d}$ and $w_0 = s_{j_1} \cdots s_{j_d}$ such that $s = s_{i_1} = s_{j_d}$.*

Definition 2.15. We call a subset $I \subset S$ a *parabolic subset* and denote by W_I the subgroup of W generated by I. We call such subgroups of W *parabolic subgroups.* We call a parabolic subset $I \subset S$ *finitary* if W_I is finite. In this case we denote by w_I the longest element of W_I. ♢

Lemma 2.16. *Let $I \subset S$ be a parabolic subset, then (W_I, I) becomes a Coxeter system with the relations induced from (W, S). Moreover, the length functions of W and W_I agree on W_I.*

Remark 2.17. Let $W = S_n$ and let $J \subset S = \{\text{simple transpositions}\}$ be a parabolic subset. Then $W_J = S_{e_1} \times S_{e_2} \times \cdots \times S_{e_m}$ where for example $S_{e_1} \cong \langle s_1, \ldots, s_{e_1 - 1} \rangle$ and $s_{e_1} \notin J$. Then the longest element w_J of W_J can be written as $w_J = w_{e_1} w_{e_2} \cdots w_{e_m}$ where w_{e_i} are the longest elements of the S_{e_i} viewed as elements of S_n. Moreover, for $w = (w_1, \ldots, w_m)$ we have

$$\ell(w) = \sum_{k=1}^{m} \ell_k(w_k)$$

where ℓ_k is the length function on S_{e_k}. ♢

Definition 2.18. For a finitary subset $I \subset S$ we define

$$\pi(I) = v^{\ell(w_I)} \cdot \sum_{w \in W_I} v^{-2\ell(w)}.$$

and call it *Poincaré polynomial* of W_I. ♢

Example 2.19. Let $W = S_4$ with $S = \{\text{simple transpositions}\}$ and $I = \{s_1, s_2\}$. Then W_I has one element of length 0, two elements of length 1, two elements of length 2 and one (longest) element of length 3. This is due to the fact that W_I is isomorphic to S_3. Thus, the Poincaré polynomial is given by

$$\pi(I) = v^3 \cdot \left(1 + 2v^{-2} + 2v^{-4} + v^{-6}\right) = v^3 + 2v + 2v^{-1} + v^{-3}.$$

If we consider the parabolic subset $J = \{s_1, s_3\}$, then W_J consists of the elements $1, s_1, s_3, s_1s_3$, since s_1 and s_3 commute. Thus, it has one element of length 0, two elements of length 1 and one element of length 2. Hence, in this case the Poincaré polynomial is given by

$$\pi(J) = v^2 \cdot \left(1 + 2v^{-2} + v^{-4}\right) = v^2 + 2 + v^{-2}.$$

Definition 2.20. Let $I \subset S$ be a parabolic subset. We define

$$D_I = \{w \in W \mid ws > w \text{ for all } s \in I\} \quad \text{and} \quad {}_ID = (D_I)^{-1}.$$

If $I \subset S$ is finitary we define

$$D^I = \{w \in W \mid ws < w \text{ for all } s \in I\} \quad \text{and} \quad {}^ID = \left(D^I\right)^{-1}.$$

The elements of D_I and D^I (respectively ${}_ID$ and ${}^ID$) are called the *minimal* and *maximal left* (respectively *right*) *coset representatives.*
Given two subsets $I, J \subset S$ we define

$${}_ID_J = {}_ID \cap D_J$$

and if I and J are finitary we define

$${}^ID^J = {}^ID \cap D^J.$$

We call the elements of ${}_ID_J$ and ${}^ID^J$ *minimal* and *maximal double coset representatives* respectively. ◊

Proposition 2.21. *Let $I, J \subset S$ be two parabolic subsets. Every double coset $p = W_IxW_J$ (for some $x \in W$) contains a unique element of ${}_ID_J$ and this is the unique element of smallest length in p.*
If I and J are finitary p also contains a unique element of ${}^ID^J$, and this is the unique element of maximal length in p.

Proof. A proof can be found in [Str20a]. □

Example 2.22. Let us consider S_3 with simple transpositions s_1, s_2 again and choose $I = \{s_1\}, J = \{s_2\}$. Let us first compute the double coset p which contains e. We have that $s_1 = s_1e, s_2 = es_2$ and $s_1s_2 = s_1es_2$ are in p. Thus, $p = \{e, s_1, s_2, s_1s_2\}$. The remaining elements of S_3 form the other double coset $q = \{s_2s_1, s_1s_2s_1\}$. We have that

$${}_ID_J = \{w \in S_3 \mid s_1w > w, ws_2 > w\} = \{e, s_2s_1\}$$
$${}^ID^J = \{w \in S_3 \mid s_1w < w, ws_2 < w\} = \{s_1s_2, s_1s_2s_1\}.$$

Now one can observe that p and q both contain exactly one element out of each of these sets. Namely, p contains e and s_1s_2 which are the unique shortest and longest elements of p respectively.

Remark 2.23. Given a double coset $p \in W_I\backslash W/W_J$ we denote by p_- the unique element of minimal length in p. If I and J are finitary we denote by p_+ the unique element of maximal length in p. We call p_- and p_+ the *minimal* and *maximal double coset representatives.*
We call the polynomial

$$\pi(p) = v^{\ell(p_+)+\ell(p_-)} \cdot \sum_{x\in p} v^{-2\ell(x)}$$

Poincaré polynomial of p. ◊

The following result is due to Howlett, see e.g. [Wil11, Theorem 2.1.3].

Theorem 2.24. *Let $I, J \subset S$ and $p \in W_I\backslash W/W_J$. Define $K = I \cap p_-Jp_-^{-1}$. Then the map*

$$\begin{aligned}(D_K \cap W_I) \times W_J &\longrightarrow p\\ (u, v) &\longmapsto up_-v\end{aligned}$$

is a bijection satisfying $\ell(up_-v) = \ell(u) + \ell(p_-) + \ell(v)$.

Definition 2.25. We extend the Bruhat order to double cosets. For $p, q \in W_I\backslash W/W_J$ we define $p \leqslant q$ if and only if $p_- \leqslant q_-$. ◊

2.2 The Hecke algebra

Definition 2.26. Let (W, S) be a Coxeter system. The *Hecke algebra* $\mathcal{H} = \mathcal{H}(W, S)$ is the free $\mathbb{Z}[v, v^{-1}]$-algebra generated by symbols H_s for $s \in S$, modulo the following relations:

$$H_s^2 = 1 + (v^{-1} - v)H_s \qquad \text{for all } s \in S, \tag{2.1}$$

$$\underbrace{H_sH_tH_s\cdots}_{m_{st}\text{ factors}} = \underbrace{H_tH_sH_t\cdots}_{m_{st}\text{ factors}} \qquad \text{for all } s \neq t \in S. \tag{2.2}$$

If $m_{st} = \infty$ we have no relation of the form (2.2). ◊

For $w \in W$ we define $H_w = H_{s_{i_1}} \cdots H_{s_{i_d}}$ where $w = s_{i_1} \cdots s_{i_d}$ is a reduced expression for w. By convention this definition includes $H_e = 1$. Note that this definition is independent of the choice of reduced expression by Lemma 2.6 and (2.2).

Lemma 2.27. *$\mathcal{H}$ is a free $\mathbb{Z}\big[v, v^{-1}\big]$-module with basis $\{H_w \mid w \in W\}$. This basis is called standard basis.*

Remark 2.28. The following multiplication formula holds.

$$H_s \cdot H_w = \begin{cases} H_{sw} & \text{if } sw > w\\ (v^{-1} - v)H_w + H_{sw} & \text{if } sw < w.\end{cases} \tag{2.3}$$

One can alternatively define the Hecke algebra as the free $\mathbb{Z}\big[v, v^{-1}\big]$-algebra with basis given by the standard basis and the multiplication given by (2.3). ◊

Remark 2.29. With the multiplication formula (2.3) it is easy to check that H_s is invertible with inverse $H_s^{-1} = H_s + v + v^{-1}$. Thus, H_w is also invertible. ♢

Definition 2.30. We define the $\mathbb{Z}$-linear *bar involution* $\mathcal{H} \longrightarrow \mathcal{H}, h \longmapsto \overline{h}$ to be the unique algebra homomorphism specified by $v \longmapsto v^{-1}$ and $H_s \longmapsto H_s^{-1}$.
We call an element $h \in H$ *self-dual* if $h = \overline{h}$. ♢

Remark 2.31. The bar involution is well-defined, i.e. it respects relations (2.1) and (2.2). For (2.2) this is obvious and for (2.1) this is an easy calculation.
It is easy to check that the elements $C_s = H_s + v$ are self-dual. ♢

Theorem 2.32. *There exists a unique self-dual basis* $\{\underline{H}_w \mid w \in W\}$ *of* $\mathcal{H}$ *as a* $\mathbb{Z}[v, v^{-1}]$*-module which satisfies*

$$\underline{H}_w = H_w + \sum_{x \neq w} h_{x,w} H_x$$

where $h_{x,w} \in v\mathbb{Z}[v]$*. This basis is called Kazhdan–Lusztig basis and the polynomials* $h_{x,w}$ *are called Kazhdan–Lusztig polynomials.*

Remark 2.33. For $s \in S$ the Kazhdan–Lusztig basis element is given by $\underline{H}_s = C_s$. One can prove that $h_{x,w} = 0$ if $x \nleq w$.
For an expression $\underline{w} = (s_{i_1}, \dots, s_{i_d})$ we define

$$\underline{H}_{\underline{w}} = \underline{H}_{s_{i_1}} \cdots \underline{H}_{s_{i_d}}.$$

Warning! In general we have $\underline{H}_w \neq \underline{H}_{\underline{w}}$ for most $w \in W$. ♢

Example 2.34. The Kazhdan–Lusztig basis for S_3 with generators s_1 and s_2 is given by

$$\begin{aligned}
\underline{H}_e &= 1 \\
\underline{H}_{s_1} &= H_{s_1} + v \\
\underline{H}_{s_2} &= H_{s_2} + v \\
\underline{H}_{s_1 s_2} &= H_{s_1 s_2} + vH_{s_1} + vH_{s_2} + v^2 = \underline{H}_{s_1} \cdot \underline{H}_{s_2} \\
\underline{H}_{s_2 s_1} &= H_{s_2 s_1} + vH_{s_1} + vH_{s_2} + v^2 = \underline{H}_{s_2} \cdot \underline{H}_{s_1} \\
\underline{H}_{w_0} &= H_{s_1 s_2 s_1} + vH_{s_1 s_2} + vH_{s_2 s_1} + v^2 H_{s_1} + v^2 H_{s_2} + v^3.
\end{aligned}$$

For the expression $\underline{w_0} = (s_1, s_2, s_1)$ we see an example of the warning in the last remark.

$$\underline{H}_{\underline{w_0}} = H_{s_1 s_2 s_1} + vH_{s_1 s_2} + vH_{s_2 s_1} + (v^2 + 1)H_{s_1} + v^2 H_{s_2} + v^3 + v \neq \underline{H}_{w_0}.$$

Lemma 2.35. *If* (W, S) *is a finite Coxeter system and* w_0 *its longest element, we have*

$$\begin{aligned}
\underline{H}_{w_0} &= \sum_{x \in W} v^{\ell(w_0) - \ell(x)} H_x \\
H_s \underline{H}_{w_0} &= v^{-1} \underline{H}_{w_0}.
\end{aligned}$$

Remark 2.36. If $I \subset S$ is finitary we get from this and Lemma 2.16 that

$$\underline{H}_{w_I} = \sum_{x \in W_I} v^{\ell(w_I) - \ell(x)} H_x. \tag{2.4}$$

If $x \in W_I$ we can check inductively that

$$H_x \cdot \underline{H}_{w_I} = v^{-\ell(x)} \underline{H}_{w_I}. \tag{2.5}$$

It follows that

$$\underline{H}_{w_K} \cdot \underline{H}_{w_I} = \pi(K) \cdot \underline{H}_{w_I} \tag{2.6}$$

for $K \subset I$. ♢

Remark 2.37. As a $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$ is also generated by the elements $\underline{H}_s$ $(s \in S)$. However, the relations are less intuitive. One relation is

$$\underline{H}_s^2 = (v + v^{-1}) \underline{H}_s. \tag{2.7}$$

The other relations connect expressions of the form $\underline{H}_s \underline{H}_t \underline{H}_s \cdots$ for $s, t \in S$. For instance

$$m_{st} = 2: \qquad \underline{H}_s \underline{H}_t = \underline{H}_t \underline{H}_s \tag{2.8}$$

$$m_{st} = 3: \qquad \underline{H}_s \underline{H}_t \underline{H}_s + \underline{H}_t = \underline{H}_t \underline{H}_s \underline{H}_t + \underline{H}_s \tag{2.9}$$

are the first examples of these relations. ♢

Definition 2.38. We define a trace ϵ on $\mathcal{H}$ by $\epsilon\left(\sum_{w \in W} c_w H_w\right) = c_e$. We call ϵ *standard trace.*
We also define ω to be the $\mathbb{Z}[v, v^{-1}]$-antilinear (i.e. $\omega(v) = v^{-1}$) antiinvolution for which $\omega(\underline{H}_s) = \underline{H}_s$ holds. ♢

Remark 2.39. A trace on $\mathcal{H}$ is a $\mathbb{Z}[v, v^{-1}]$-linear map $\mathrm{tr} : \mathcal{H} \longrightarrow \mathbb{Z}[v, v^{-1}]$ satisfying $\mathrm{tr}(hh') = \mathrm{tr}(h'h)$ for all $h, h' \in \mathcal{H}$. The standard trace is a trace on $\mathcal{H}$.
Note that ω is not the same as the bar involution, since ω is an antiinvolution. That means $\omega(hh') = \omega(h') \cdot \omega(h)$ while $\overline{hh'} = \overline{h} \cdot \overline{h'}$ for all $h, h' \in \mathcal{H}$. ♢

Definition 2.40. We define a pairing $(-,-) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{Z}[v, v^{-1}]$ by $(h, h') = \epsilon\left(h' \cdot \omega(h)\right)$. This pairing will be called *standard pairing.* ♢

Remark 2.41. The standard pairing is $\mathbb{Z}[v, v^{-1}]$-antilinear in the first component and $\mathbb{Z}[v, v^{-1}]$-linear in the second component; that is $(v^{-1}h, h') = (h, vh') = v \cdot (h, h')$ for all $h, h' \in \mathcal{H}$. The element $\underline{H}_s$ is self-biadjoint under this pairing, i.e.

$$(\underline{H}_s x, y) = (x, \underline{H}_s y), \qquad (x \underline{H}_s, y) = (x, y \underline{H}_s).$$

One can define the standard pairing alternatively via

$$(H_x, H_y) = \delta_{x,y} \tag{2.10}$$

for all $x, y \in W$. ♢

2.3 The Hecke algebroid

In this section we will recall the definition of the Hecke algebroid and collect some basic facts. A reference for this is [Mat99].

Definition 2.42. Let $I, J \subset S$ be finitary subsets. We define

$$
\begin{aligned}
{}_I\mathcal{H} &= \underline{H}_{w_I}\mathcal{H}\\
\mathcal{H}_J &= \mathcal{H}\underline{H}_{w_J}\\
{}_I\mathcal{H}_J &= {}_I\mathcal{H} \cap \mathcal{H}_J.
\end{aligned}
$$

Given a third finitary subset $K \subset S$ we define a multiplication as follows

$$
\begin{aligned}
{}_I\mathcal{H}_J \times {}_J\mathcal{H}_K &\longrightarrow {}_I\mathcal{H}_K\\
(h_1, h_2) &\longmapsto h_1 *_J h_2 = \frac{1}{\pi(J)} h_1 h_2.
\end{aligned}
$$

This is well-defined by (2.6). If $J = \emptyset$ we write the normal multiplication $\cdot$ instead of $*_\emptyset$, since they agree. $\Diamond$

Definition 2.43. The *Hecke algebroid* is the $\mathbb{Z}[v, v^{-1}]$-linear category defined as follows. The objects are finitary subsets $I \subset S$. The morphisms between I and J are given by ${}_I\mathcal{H}_J$. Composition between morphisms ${}_I\mathcal{H}_J \times {}_J\mathcal{H}_K \longrightarrow {}_I\mathcal{H}_K$ is given by $*_J$. This defines a $\mathbb{Z}[v, v^{-1}]$-linear category with the identity endomorphism for $I \subset S$ given by $\underline{H}_{w_I}$. $\Diamond$

Remark 2.44. We can check that $h = \sum_{w\in W} a_w H_w \in {}_I\mathcal{H}_J$ if and only if, $a_{sw} = va_w$ and $a_{wt} = va_w$ for all $w \in W, s \in I$ and $t \in J$ such that $sw < w$ and $wt < w$. We define for all $p \in W_I\backslash W/W_J$

$$
{}^I H_p^J = \sum_{x\in p} v^{\ell(p_+)-\ell(x)} H_x.
$$

It follows that if $h = \sum_{w\in W} a_w H_w \in {}_I\mathcal{H}_J$, then

$$
h = \sum_{p\in W_I\backslash W/W_J} a_{p_+}\, {}^I H_p^J.
$$

The set $\{{}^I H_p^J \mid p \in W_I\backslash W/W_J\}$ is obviously linear independent over $\mathbb{Z}[v, v^{-1}]$, and thus it forms a basis for ${}_I\mathcal{H}_J$ over $\mathbb{Z}[v, v^{-1}]$. We will call it *standard basis.*
For a Kazhdan–Lusztig basis element we have $\underline{H}_w \in {}_I\mathcal{H}_J$ if and only if w is maximal in its (W_I, W_J)-double coset. That is why we define for $p \in W_I\backslash W/W_J$

$$
{}^I \underline{H}_p^J = \underline{H}_{p_+}.
$$

We have

$$
{}^I \underline{H}_p^J = {}^I H_p^J + \sum_{q<p} h_{q_+,p_+}\, {}^I H_q^J.
$$

It follows that $\{{}^I \underline{H}_p^J \mid p \in W_I\backslash W/W_J\}$ also forms a basis for ${}_I\mathcal{H}_J$ over $\mathbb{Z}[v, v^{-1}]$. We call this basis *Kazhdan–Lusztig basis.* $\Diamond$

Remark 2.45. For all finitary subsets $I, J \subset S$ satisfying $I \subset J$ or $J \subset I$ we define

$$^{I}H^{J} = {}^{I}H_{p}^{J} \text{ where } p = W_I W_J.$$

We call elements of the form ${}^{I}H^{J} \in {}_{I}\mathcal{H}_{J}$ *standard generators.* The standard generators have the following property:
Let $\{{}_{I}Z_{J} \subset {}_{I}\mathcal{H}_{J}\}$ be the smallest collection of subsets such that

1. If $I \subset J$ or $J \subset I$ we have ${}^{I}H^{J} \in {}_{I}Z_{J}$;
2. ${}_{I}Z_{J}$ is a $\mathbb{Z}[v, v^{-1}]$-submodule of ${}_{I}\mathcal{H}_{J}$;
3. The collection $\{{}_{I}Z_{J}\}$ is closed under composition in the Hecke algebroid.

Then ${}_{I}Z_{J} = {}_{I}\mathcal{H}_{J}$ for all finitary subsets $I, J \subset S$. We say that the standard generators generate the Hecke algebroid. $\diamondsuit$

Remark 2.46. Recall the antiinvolution ω we defined previously. One can check that $\omega\left(\underline{H}_{w_I}\right) = \underline{H}_{w_I}$ for all finitary $I \subset S$. Hence, ω restricts to an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules

$$\omega : {}_{I}\mathcal{H}_{J} \longrightarrow {}_{J}\mathcal{H}_{I}.$$

Now we can extend our standard pairing to

$$\begin{aligned} (-,-) : {}_{I}\mathcal{H}_{J} \times {}_{I}\mathcal{H}_{J} &\longrightarrow \mathbb{Z}[v, v^{-1}] \\ (h_1, h_2) &\longmapsto \langle h_1, h_2 \rangle = \epsilon\left(h_1 *_J \omega(h_2)\right). \end{aligned}$$

Note that one has for $h_1, h_2 \in {}_{I}\mathcal{H}_{J}$ the connection to the standard pairing given by $\pi(J) \cdot \langle h_1, h_2 \rangle = (h_1, h_2)$ where we regard h_1 and h_2 as elements in $\mathcal{H}$ in the second expression. One can check that for $I, J, K \subset S$ finitary and $h_1 \in {}_{I}\mathcal{H}_{J}, h_2 \in {}_{J}\mathcal{H}_{K}, h_3 \in {}_{I}\mathcal{H}_{K}$ we have

$$\langle h_1 *_J h_2, h_3 \rangle = \langle h_1, h_3 *_K \omega(h_2) \rangle.$$

We can also describe the standard pairing on the standard basis of ${}_{I}\mathcal{H}_{J}$. We have

$$\langle {}^{I}H_{p}^{J}, {}^{I}H_{q}^{J} \rangle = v^{\ell(p_+) - \ell(p_-)} \cdot \delta_{p,q}$$

for $p, q \in W_I \backslash W / W_J$. $\diamondsuit$

2.3.1 Some S_3-type relations

In this section we want to understand the Kazhdan–Lusztig bases in the Hecke algebroid for S_3 better. Let (W, S) be a Coxeter system with $s_i, s_j \in S$ such that $m_{ij} = 3$. Then the parabolic subgroup U generated by s_i and s_j is isomorphic to S_3 and the Hecke algebra $\mathcal{H}(W, S)$ has $\mathcal{H}(S_3, \{s_i, s_j\})$ as a subalgebra. Let now I and J be parabolic subsets of U (and thus also of W), then we want to understand the U-part of the Kazhdan–Lusztig basis in ${}_I\mathcal{H}_J$. More precisely, if we consider a Kazhdan–Lusztig basis element $\underline{H}_x$, $x \in U$, we can force it into ${}_I\mathcal{H}_J$ via

$$\underline{H}_{w_I}\underline{H}_x\underline{H}_{w_J} \in {}_I\mathcal{H}_J$$

Now we can decompose such an element in to our Kazhdan–Lusztig basis given by double cosets ${}^I\underline{H}^J_p$. For this we only need double cosets $p \subset U$. Hence, we may assume $W = U = S_3$ and $s_i = s_1, s_j = s_2$ and the calculations will also hold in the general case described above.
We will do these calculations for four choices of I and J. When we write ${}_I\mathcal{H}_J$ or ${}^I\underline{H}^J_p$ we will write 1 instead of $\{s_1\}$ and 2 instead or $\{s_2\}$, for example we will write ${}_1\mathcal{H}_2$.

Proposition 2.47. *Consider ${}_1\mathcal{H}_2$ and label the double cosets $W_1\backslash W/W_2$ by*

$$p = \{e, s_1, s_2, s_1s_2\}, \qquad q = \{s_2s_1, w_0\}.$$

Then we get the following decompositions.

1. $\underline{H}_{s_1}\underline{H}_e\underline{H}_{s_2} = {}^1\underline{H}^2_p$.
2. $\underline{H}_{s_1}\underline{H}_{s_1}\underline{H}_{s_2} = (v + v^{-1}) \cdot {}^1\underline{H}^2_p$.
3. $\underline{H}_{s_1}\underline{H}_{s_2}\underline{H}_{s_2} = (v + v^{-1}) \cdot {}^1\underline{H}^2_p$.
4. $\underline{H}_{s_1}\underline{H}_{s_1s_2}\underline{H}_{s_2} = (v^2 + 2 + v^{-2}) \cdot {}^1\underline{H}^2_p$.
5. $\underline{H}_{s_1}\underline{H}_{s_2s_1}\underline{H}_{s_2} = {}^1\underline{H}^2_p + (v + v^{-1}) \cdot {}^1\underline{H}^2_q$.
6. $\underline{H}_{s_1}\underline{H}_{w_0}\underline{H}_{s_2} = (v^2 + 2 + v^{-2}) \cdot {}^1\underline{H}^2_q$.

Proof. Recall that ${}^1\underline{H}^2_p = \underline{H}_{s_1s_2}$ and ${}^1\underline{H}^2_q = \underline{H}_{w_0}$. We will use the resolution of the Kazhdan–Lusztig basis into the standard basis from Example 2.34.

1. We compute

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_e\underline{H}_{s_2} &= \underline{H}_{s_1} \cdot \underline{H}_{s_2} = (H_{s_1} + v) \cdot (H_{s_2} + v) \\ &= H_{s_{12}} + vH_{s_1} + vH_{s_2} + v^2 = {}^1\underline{H}^2_p.\end{aligned}$$

2. Using the first part we get

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_{s_1}\underline{H}_{s_2} &= (H_{s_1} + v) \cdot (H_{s_1}) \cdot \underline{H}_{s_2} = \big((v^{-1} - v)H_{s_1} + 1 + 2vH_{s_1} + v^2\big) \cdot \underline{H}_{s_2} \\ &= (v + v^{-1}) \cdot \underline{H}_{s_1} \cdot \underline{H}_{s_2} = (v + v^{-1}) \cdot {}^1\underline{H}^2_p.\end{aligned}$$

3. Again using the first part we get

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_{s_2}\underline{H}_{s_2} &= \underline{H}_{s_1} \cdot (H_{s_2} + v) \cdot (H_{s_2}) = \underline{H}_{s_1} \cdot \big((v^{-1} - v)H_{s_2} + 1 + 2vH_{s_2} + v^2\big) \\ &= \underline{H}_{s_1} \cdot (v + v^{-1}) \cdot \underline{H}_{s_2} = (v + v^{-1}) \cdot {}^1\underline{H}^2_p.\end{aligned}$$

4. Using the calculations from the last three parts we compute

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_{s_1s_2}\underline{H}_{s_2} &= \underline{H}_{s_1}\cdot\underline{H}_{s_1}\cdot\underline{H}_{s_2}\cdot\underline{H}_{s_2} = (v+v^{-1})\cdot\underline{H}_{s_1}\cdot(v+v^{-1})\cdot\underline{H}_{s_2}\\ &= (v^2+2+v^{-2})\cdot\underline{H}_{s_1}\cdot\underline{H}_{s_2} = (v+v^{-1})\cdot{}^1\underline{H}_p^2.\end{aligned}$$

5. Here we compute that

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_{s_2s_1}\underline{H}_{s_2} &= (H_{s_1}+v)\cdot\big(H_{s_2s_1}+vH_{s_1}+vH_{s_2}+v^2\big)\cdot\underline{H}_{s_2}\\ &= \begin{pmatrix} H_{w_0}+v(v^{-1}-v)H_{s_1}+v+vH_{s_1s_2}+v^2H_{s_1}\\ +vH_{s_2s_1}+v^2H_{s_1}+v^2H_{s_2}+v^3 \end{pmatrix}\cdot\underline{H}_{s_2}\\ &= \big(\underline{H}_{w_0}+\underline{H}_{s_1}\big)\cdot\underline{H}_{s_2}\\ &= \big(H_{w_0}+vH_{s_1s_2}+vH_{s_2s_1}+v^2H_{s_1}+v^2H_{s_2}+v^3\big)\cdot(H_{s_2}+v)\\ &\quad+\underline{H}_{s_1s_2}\\ &= (v^{-1}-v)H_{w_0}+H_{s_2s_1}+v(v^{-1}-v)H_{s_1s_2}+vH_{s_1}+vH_{w_0}\\ &\quad+v^2H_{s_1s_2}+v^2(v^{-1}-v)H_{s_2}+v^2+v^3H_{s_2}+vH_{w_0}+v^2H_{s_1s_2}\\ &\quad+v^2H_{s_2s_1}+v^3H_{s_1}+v^3H_{s_2}+v^4+{}^1\underline{H}_p^2\\ &= (v+v^{-1})\underline{H}_{w_0}+{}^1\underline{H}_p^2 = {}^1\underline{H}_p^2+(v+v^{-1})\cdot{}^1\underline{H}_q^2.\end{aligned}$$

6. Using our last calculations we compute

$$\begin{aligned}\underline{H}_{s_1}\underline{H}_{w_0}\underline{H}_{s_2} &= \underline{H}_{s_1}\cdot\big(H_{w_0}+vH_{s_2s_1}+v\underline{H}_{s_1s_2}\big)\cdot\underline{H}_{s_2}\\ &= \underline{H}_{s_1}\cdot(H_{w_0}+vH_{s_2s_1})\cdot(H_{s_2}+v)+v(v^2+2+v^{-1})\cdot\underline{H}_{s_1s_1}\\ &= \underline{H}_{s_1}\cdot\big((v^{-1}-v)H_{w_0}+H_{s_2s_1}+vH_{w_0}+vH_{w_0}+v^2H_{s_2s_1}\big)\\ &\quad+v(v^2+2+v^{-1})\cdot\underline{H}_{s_1s_1}\\ &= \underline{H}_{s_1}\cdot(v+v^{-1})\cdot(H_{w_0}+vH_{s_2s_1})+v(v^2+2+v^{-1})\cdot\underline{H}_{s_1s_1}\\ &= (v+v^{-1})\cdot\big((v^{-1}-v)H_{w_0}+H_{s_2s_1}+vH_{w_0}+vH_{w_0}+v^2H_{s_2s_1}\big)\\ &\quad+v(v^2+2+v^{-1})\cdot\underline{H}_{s_1s_1}\\ &= (v^2+2+v^{-2})\cdot\big(H_{w_0}+vH_{s_2s_1}+v\underline{H}_{s_1s_1}\big)\\ &= (v^2+2+v^{-2})\cdot\underline{H}_{w_0} = (v^2+2+v^{-2})\cdot{}^1\underline{H}_q^2.\end{aligned}$$

□

Analogously one can for instance also verify the following equalities. We omit the details.

Proposition 2.48. *Consider ${}_1\mathcal{H}_1$ and label the double cosets $W_1\backslash W/W_1$ by*

$$p=\{e,s_1\},\qquad q=\{s_2,s_1s_2,s_2s_1,w_0\}.$$

Then we get the following decompositions.

1. $\underline{H}_{s_1}\underline{H}_e\underline{H}_{s_1} = (v+v^{-1})\cdot{}^1\underline{H}_p^1.$

2. $\underline{H}_{s_1}\underline{H}_{s_1}\underline{H}_{s_1} = (v^2+2+v^{-2})\cdot{}^1\underline{H}_p^1.$

3. $\underline{H}_{s_1}\underline{H}_{s_2}\underline{H}_{s_1} = {}^1\underline{H}_p^1 + {}^1\underline{H}_q^1$.

4. $\underline{H}_{s_1}\underline{H}_{s_1s_2}\underline{H}_{s_1} = (v+v^{-1})\cdot{}^1\underline{H}_p^1 + (v+v^{-1})\cdot{}^1\underline{H}_q^1$.

5. $\underline{H}_{s_1}\underline{H}_{s_2s_1}\underline{H}_{s_1} = (v+v^{-1})\cdot{}^1\underline{H}_p^1 + (v+v^{-1})\cdot{}^1\underline{H}_q^1$.

6. $\underline{H}_{s_1}\underline{H}_{w_0}\underline{H}_{s_1} = (v^2+2+v^{-2})\cdot{}^1\underline{H}_q^1$.

Proposition 2.49. *Consider ${}_1\mathcal{H}$ and label the double cosets $W_1\backslash W$ by*

$$p = \{e, s_1\}, \qquad q = \{s_2, s_1s_2\}, \qquad r = \{s_2s_1, w_0\}.$$

Then we get the following decompositions.

1. $\underline{H}_{s_1}\underline{H}_e = {}^1\underline{H}_p$.

2. $\underline{H}_{s_1}\underline{H}_{s_1} = (v+v^{-1})\cdot{}^1\underline{H}_p$.

3. $\underline{H}_{s_1}\underline{H}_{s_2} = {}^1\underline{H}_q$.

4. $\underline{H}_{s_1}\underline{H}_{s_1s_2} = (v+v^{-1})\cdot{}^1\underline{H}_q$.

5. $\underline{H}_{s_1}\underline{H}_{s_2s_1} = {}^1\underline{H}_p + {}^1\underline{H}_r$.

6. $\underline{H}_{s_1}\underline{H}_{w_0} = (v+v^{-1})\cdot{}^1\underline{H}_r$.

Proposition 2.50. *Consider $\mathcal{H}_1$ and label the double cosets W/W_1 by*

$$p = \{e, s_1\}, \qquad q = \{s_2, s_2s_1\}, \qquad r = \{s_1s_2, w_0\}.$$

Then we get the following decompositions.

1. $\underline{H}_e\underline{H}_{s_1} = \underline{H}_p^1$.

2. $\underline{H}_{s_1}\underline{H}_{s_1} = (v+v^{-1})\cdot\underline{H}_p^1$.

3. $\underline{H}_{s_2}\underline{H}_{s_1} = \underline{H}_q^1$.

4. $\underline{H}_{s_1s_2}\underline{H}_{s_1} = (v+v^{-1})\cdot\underline{H}_q^1$.

5. $\underline{H}_{s_2s_1}\underline{H}_{s_1} = \underline{H}_p^1 + \underline{H}_r^1$.

6. $\underline{H}_{w_0}\underline{H}_{s_1} = (v+v^{-1})\cdot\underline{H}_r^1$.

2.4 Graded bimodules

In the upcoming chapters we will work with graded bimodules. In this section we will fix some general terminology and observe some basic facts. We will always consider rings R satisfying

$$R = \bigoplus_{k=0} R_k \text{ is a finitely generated, positively graded commutative } \Bbbk\text{-algebra with } R_0 = \Bbbk$$

where $\Bbbk$ is some fixed commutative ring (in most cases $\Bbbk$ will be a field of characteristic zero). We denote by $(R, S) - \mathrm{Bim}$ the category of graded (R, S)-bimodules:

objects: (R, S)-bimodules M with a decomposition $M = \bigoplus_{k \in \mathbb{Z}} M_k$ where

a) The left and right action of $\Bbbk$ agrees.

b) The M_k is a free $\Bbbk$-module for all $k \in \mathbb{Z}$.

c) $R_l \cdot M_k \subseteq M_{l+k} \supseteq M_k \cdot S_l$ for all $k, l \in \mathbb{Z}$.

morphisms: homomorphisms $f : M \longrightarrow N$ of (R, S)-bimodules preserving degrees, i. e. $f(M_k) \subseteq N_k$ for all $k \in \mathbb{Z}$.

Remark 2.51. As all our rings are commutative we have an equivalence of categories between $(R, S) - \mathrm{Bim}$ and $R \otimes_\Bbbk S - \mathrm{Mod}$, the category of graded $R \otimes_\Bbbk S$-modules. This can be helpful sometimes to transfer known results for modules to bimodules. ◊

Definition 2.52. A category $\mathcal{C}$ is called a *graded category* if it is a $\Bbbk$-linear category enriched in $\Bbbk - \mathrm{Mod}$, the category of graded $\Bbbk$-modules. ◊

Remark 2.53. This basically means that the morphism spaces are graded $\Bbbk$-modules and the composition of morphisms is compatible with the grading. ◊

Lemma 2.54. *The category* $(R, S) - \mathrm{Bim}$ *is a graded category.*

Proof. We say that a morphism $f \in \mathrm{Hom}_{(R,S)}(M, N)$ is homogeneous of degree d if $f(M_k) \subseteq N_{k+d}$ for all $k \in \mathbb{Z}$. This defines a grading on morphism spaces which is compatible with compositions. □

Definition 2.55. In $(R, S) - \mathrm{Bim}$ we have *grading shifting functors* $(l \in \mathbb{Z})$

$$\begin{aligned} \langle l \rangle : (R, S) - \mathrm{Bim} &\longrightarrow (R, S) - \mathrm{Bim} \\ M &\longmapsto M\langle l \rangle \\ f &\longmapsto f \end{aligned}$$

where $M\langle l \rangle = M$ as an (R, S)-bimodule, but $(M\langle l \rangle)_k = M_{l+k}$. These functors define a free (this means that the stabilizer of objects is trivial) $\mathbb{Z}$-action on $(R, S) - \mathrm{Bim}$. ◊

A discussion on the following theorem can be found in e.g. [Str20b].

Theorem 2.56. *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \Bbbk\text{-linear categories with} \\ \text{free } \mathbb{Z}\text{-action} \end{array} \right\} \longleftrightarrow \{\text{graded categories}\}.$$

Proof. We define two functors

$$\mathcal{D} \longmapsto \mathcal{D}/\mathbb{Z}$$
$$\mathcal{C}^{\mathbb{Z}} \longmapsto \mathcal{C}.$$

The graded category $\mathcal{D}/\mathbb{Z}$ is defined as follows. The objects are $\mathbb{Z}$-orbits $\overline{M}$ of objects M in $\mathcal{D}$. The morphisms are given by

$$\mathrm{Hom}_{\mathcal{D}/\mathbb{Z}}(\overline{M}, \overline{N}) = \left(\bigoplus_{X \in \overline{M}, Y \in \overline{N}} \mathrm{Hom}_{\mathcal{D}}(X, Y) \right) \Big/ U$$

where U is generated by $f - l.f$ for all $l \in \mathbb{Z}$. Note that this implies

$$\mathrm{Hom}_{\mathcal{D}/\mathbb{Z}}(\overline{M}, \overline{N}) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, l.N).$$

The category $\mathcal{C}^{\mathbb{Z}}$ is defined as follows. The objects are pairs $(M, l) \in \mathrm{ob}(\mathcal{C}) \times \mathbb{Z}$. The morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}((M, l), (N, k)) = \mathrm{Hom}_{\mathcal{C}}(M, N)_{l-k}.$$

One then can check that these functors are inverse and give us the desired equivalence of categories. □

Remark 2.57. We will quickly write down the key differences between $(R, S) - \mathrm{Bim}$ and $(R, S) - \mathrm{Bim}/\mathbb{Z}$:

$(R, S) - \mathrm{Bim}$	$(R, S) - \mathrm{Bim}/\mathbb{Z}$
graded (R, S)-bimodules with grading shifting functors and morphisms of degree zero	graded (R, S)-bimodules (pick one up to grading shift) and morphisms of all degrees.

We will treat $\mathcal{C}$ and $\mathcal{C}/\mathbb{Z}$ as "the same" from now on. This means that we will sometimes talk about degrees of morphisms and other times we will talk about different shifts of objects while talking about the same category. This is justified by the previous theorem. ◊

Definition 2.58. We call an (R, S)-bimodule *indecomposable* if there are no non-trivial (R, S)-bimodules M_1 and M_2 such that $M \cong M_1 \oplus M_2$ as (R, S)-bimodules. ◊

Lemma 2.59. *Let M be a graded (R, S)-bimodule. Let $k \in \mathbb{Z}$ be the smallest number such that $M_k \neq 0$. Suppose that M_k has rank 1 and suppose that M is generated by some $m \in M_k$ as a bimodule. Then M is indecomposable.*

Proof. Suppose that there are (R, S)-bimodules N, L such that $\varphi : M \cong N \oplus L$. Then also $M_k \cong N_k \oplus L_k$. As M_k has rank one and N_k and L_k are free $\Bbbk$-modules we conclude that either $N_k \cong M_k$ or $L_k \cong M_k$. W.l.o.g assume that $N_k \cong M_k$. Then let $x = \varphi(m) \in N_k$. Now let $y \in M$. Since M is generated by m as a bimodule we find some $r_l \in R, s_l \in S$ such that

$$y = \sum_{l=1}^{N} r_l \cdot m \cdot s_l.$$

This implies that

$$\varphi^{-1}\big|_N \left(\sum_{l=1}^{N} r_l \cdot x \cdot s_l \right) = \sum_{l=1}^{N} r_l \cdot \varphi^{-1}\big|_N(x) \cdot s_l = \sum_{l=1}^{N} r_l \cdot m \cdot s_l = y.$$

Hence, $\varphi^{-1}\big|_N : N \longrightarrow M$ is surjective and obviously also injective. Thus, $M \cong N$ and the decomposition was trivial. Hence, M is indecomposable. $\square$

3 Soergel bimodules

In this chapter we will recall the definition and properties of Soergel bimodules. We will explain the connection between Soergel bimodules and the Hecke algebra and look at a few examples. This originally goes back to Soergel [Soe07, Soe92]. We will follow here the later treatments [EW16, Section 3]. We start with the definition of a realization of a Coxeter system.

3.1 Realizations

Definition 3.1. Let $\Bbbk$ be a commutative ring. A *realization* of a Coxeter system (W, S) over $\Bbbk$ is a free finite rank $\Bbbk$-module $\mathfrak{h}$ to together with subsets $\{\alpha_s^\vee \mid s \in S\} \subset \mathfrak{h}$ and $\{\alpha_s \mid s \in S\} \subset \mathfrak{h}^* = \mathrm{Hom}_\Bbbk(\mathfrak{h}, \Bbbk)$, satisfying:

1. $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ for all $s \in S$;
2. the assignment $s(v) = v - \langle v, \alpha_s \rangle \alpha_s^\vee$ for all $v \in \mathfrak{h}$ yields a representation of W;
3. $[m_{st}]_{a_{st}} = [m_{st}]_{a_{ts}} = 0$ for all $s, t \in S$.

The brackets in the third point stand for the 2-coloured quantum number and $a_{st} = \langle \alpha_s^\vee, \alpha_t \rangle$. For more details on this, see [EW16, Section 3.1]. $\Diamond$

Remark 3.2. In order for Soergel bimodules to behave well or for the theorems we will state to hold, one needs to put some assumptions on $\Bbbk$ and the realization. However, since we are only interested in the case S_n we will not discuss this in detail. We will soon come across a realization for S_n that is good in that sense and will mainly work with this. We just wanted to show the general definition to make the whole picture more clear. The details for the general case can be found in [EW16, Chapter 3]. $\Diamond$

Example 3.3. Suppose that W is finite. Let $\Bbbk = \mathbb{R}$ and $\mathfrak{h} = \bigoplus_{s \in S} \mathbb{R}\alpha_s^\vee$. Define elements $\{\alpha_s\} \subset \mathfrak{h}^*$ by

$$\langle \alpha_t^\vee, \alpha_s \rangle = -2\cos\left(\frac{\pi}{m_{st}}\right)$$

(by convention $m_{ss} = 1$). Then $\mathfrak{h}$ is a realization of (W, S), called the *geometric representation*. Note that the subset $\{\alpha_s\} \subset \mathfrak{h}^*$ is linearly independent and W acts faithfully on $\mathfrak{h}$ and hence also on $\mathfrak{h}^*$. This will be the main realization we will use for $W = S_n$.
We can extend this realization to a realization $\mathbb{C} \otimes_\mathbb{R} \mathfrak{h}$ over $\mathbb{C}$ by base change. So we may choose $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$.
Note that we have another realization of S_n, namely the *natural* n-dimensional *representation* $\mathfrak{h}'$. This is just an $\mathbb{R}^n$ where S_n acts by permuting the basis vectors. We can

pick $\alpha_i^\vee = v_i - v_{i+1}$ where v_i are the standard basis vectors. Then we pick $\alpha_i = e_i - e_{i+1}$ where the e_i are defined by $e_k(v_l) = \delta_{k,l}$ for $1 \leqslant k, l \leqslant n-1$. This gives us the desired realization.

This realization is connected to the geometric representation via $\mathfrak{h}' = \mathfrak{h} \oplus \mathbb{R}$ where S_n acts trivially on the extra summand $\mathbb{R}$. This comes simply from the fact that S_n is the Weyl group of $\mathfrak{sl}_n$ as well as $\mathfrak{gl}_n$. The geometric representation comes from $\mathfrak{sl}_n$ and the natural representation comes from $\mathfrak{gl}_n$ which immediately gives us the connection above. While we will mostly work with the geometric representation for the general theorems and definitions, we will use the natural representation for some examples as it is a bit nicer for explicit calculations. We will always state when we switch to the natural representation, so if nothing else is said the geometric representation is the one that is used.

Definition 3.4. For a fixed realization $(\mathfrak{h}, \{\alpha_s^\vee\}, \{\alpha_s\})$ of (W, S) denote by

$$R = S(\mathfrak{h}^*) = \bigoplus_{m \geqslant 0} S^m(\mathfrak{h}^*)$$

the symmetric algebra on $\mathfrak{h}^*$, which we view as a graded $\Bbbk$-algebra with $\deg(\mathfrak{h}^*) = 2$. Then W acts on $\mathfrak{h}^*$ via $s(\gamma) = \gamma - \langle \alpha_s^\vee, \gamma \rangle \alpha_s$ for all $\gamma \in \mathfrak{h}$ and this extends to an action of W on R by graded automorphisms. We think of R as polynomial functions on $\mathfrak{h}$. $\Diamond$

Example 3.5.

1. For a finite Coxeter system (W, S) with the geometric representation from Example 3.3 we have $R \cong \Bbbk[z_1, \ldots, z_{|S|}]$ where the z_i correspond to the $\alpha_s^\vee$. This gives us for $W = S_n$ that $R \cong \Bbbk[z_1, \ldots z_{n-1}]$.

2. For $W = S_n$ with the natural representation from Example 3.3 we have $R_1 = R \cong \Bbbk[x_1, \ldots, x_n]$ where S_n acts by permuting variables. Note that for the geometric representation we would have $R_0 \cong \Bbbk[z_1, \ldots, z_{n-1}]$ and we have an inclusion $R_0 \hookrightarrow R_1$ where z_i is sent to $x_i - x_{i+1}$. In this way these two realizations are connected and it does not really matter which one is used.

3.2 Demazure operators and rings of invariants

In this section we will recall the definition and basic properties of Demazure operators and investigate the rings of invariants of R under the action of W. These operators go back to Demazure [Dem73]. We will use them to understand how these rings are structured as modules over each other. A reference for this section is e.g. [Man98] and some topics are also covered nicely in [Str19]. Throughout the section we assume that $\Bbbk$ is a field of characteristic 0.

Definition 3.6. Let $I \subset S$ be a parabolic subset and W_I the corresponding parabolic subgroup. We define

$$\begin{aligned} R^I = R^{W_I} &= \{r \in R \mid w(r) = r \text{ for all } w \in W_I\} \\ &= \{r \in R \mid s(r) = r \text{ for all } s \in I\} \end{aligned}$$

to be the ring of W_I-invariant elements of R. If $I = \{i\}$ is a singleton we will write R^i instead of R^I. ◊

Remark 3.7. Note that R^I is actually a ring, since W acts by ring automorphisms on R. Moreover, we have $R^J \subset R^I$ if $I \subset J$, and thus R^I is even an R^J-algebra in this case. Also recall that the action of W on R preserves the grading. Thus, R^I is graded and the inclusion $R^I \hookrightarrow R$ preserves the grading. ◊

Example 3.8. We consider $(W, S) = (S_3, \{s_1, s_2\})$ with the natural representation from Example 3.3. Then we have $R_1 \cong \Bbbk[x_1, x_2, x_3]$. If we now consider rings of invariants the fundamental theorem of symmetric polynomials tells us that

$$\begin{aligned} R_1^{\{s_1\}} &= \Bbbk[x_1 + x_2, x_1x_2, x_3] \\ R_1^{\{s_2\}} &= \Bbbk[x_1, x_2 + x_3, x_2x_3] \\ R_1^{\{s_1,s_2\}} &= \Bbbk[x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3]. \end{aligned}$$

Thus, we can observe that the inclusions of rings $R_1^{\{s_1,s_2\}} \subset R_1^{\{s_1\}}, R_1^{\{s_2\}} \subset R_1$ hold true. This presentation can be generalized. Consider $(W, S) = (S_n, \{s_1, \dots, s_{n-1}\})$ with the natural representation from Example 3.3. Then we have for instance

$$R_1^{\{s_i\}} = \Bbbk[x_1, \dots x_{i-1}, x_i + x_{i+1}, x_ix_{i+1}, x_{i+2}, \dots, x_n].$$

Remark 3.9. Consider $W = S_n$ and let $J \subset S = \{\text{simple transpositions}\}$ be a parabolic subset. Recall that $W_J = S_{e_1} \times S_{e_2} \times \cdots \times S_{e_m}$ as in Remark 2.17. Let $R_1 \cong \Bbbk[x_1, \dots, x_n]$ be the ring corresponding to the natural representation of S_n. Now we can write R_1 a bit differently via

$$R_1 \cong \Bbbk[x_1, \dots, x_{e_1}] \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk[x_{n-e_m+1}, \dots, x_n].$$

Since $W_J = S_{e_1} \times \cdots \times S_{e_m}$ we get that

$$R_1^{W_J} \cong \Bbbk[x_1, \dots, x_{e_1}]^{S_{e_1}} \otimes_\Bbbk \cdots \otimes_\Bbbk \Bbbk[x_{n-e_m+1}, \dots, x_n]^{S_{e_m}}. \tag{3.1}$$

Hence, if we let $R_{e_k} = \Bbbk[x_1, \dots, x_{e_k}]$ be the polynomial ring in e_k variables viewed as a module over $\Bbbk[x_1, \dots, x_{e_k}]^{S_{e_k}}$, then we have as R_1^J-modules

$$R_1 \cong R_{e_1} \otimes_\Bbbk \cdots \otimes_\Bbbk R_{e_m}$$

where R_1^J acts on the right hand side via (3.1). ◊

Definition 3.10. For $i \in S$ the *Demazure operator* $\partial_i : R \longrightarrow R^i$ is defined by

$$\partial_i(r) = \frac{r - s_i(r)}{\alpha_i}$$

for all $r \in R$. ◊

Proposition 3.11. *The following hold:*

1. ∂_i *is well-defined for all* $i \in S$*, i.e.* $r - s_i(r) \in \alpha_i R$ *for all* $r \in R$ *and* $\text{im}(\partial_i) \subseteq R^i$*;*
2. $\ker(\partial_i) = R^i$*;*
3. $\partial_i(r_1 r_2) = \partial_i(r_1)r_2 + s_i(r_1)\partial_i(r_2)$ *for all* $r_1, r_2 \in R$*;*
4. ∂_i *is* R^i*-linear.*

Proof. **1.** First we check that $\text{im}(\partial_i) \in R^i$. Let $r \in R$, then we need to check that $s_i(\partial_i(r)) = \partial_i(r)$. We compute that

$$s_i(\partial_i(r)) = s_i\left(\frac{r - s_i(r)}{\alpha_i}\right) = \frac{s_i(r) - s_i(s_i(r))}{s_i(\alpha_i)} = \frac{s_i(r) - r}{-\alpha_i} = \partial_i(r).$$

Now we check that $r - s_i(r) \in \alpha_i R$. Recall that R is defined to be the symmetric algebra $S(\mathfrak{h}^*)$ of $\mathfrak{h}^*$. So by linearity of s_i it is enough to consider $r = x_1 \otimes x_2 \otimes \cdots \otimes x_N$. We know that $s_i(x_l) = x_l - \lambda_l \alpha_i$ for some $\lambda_l \in \mathbb{k}$. Thus, we compute that

$$\begin{aligned} r - s_i(r) &= r - s_i(x_1) \otimes s_i(x_2) \otimes \cdots \otimes s_i(x_N) \\ &= r - x_1 \otimes \cdots \otimes x_N + \sum_{l=1}^{N} x_1 \otimes \cdots \otimes x_{l-1} \otimes \lambda_l \alpha_i \otimes s_i(x_{l+1}) \otimes \cdots \otimes s_i(x_N) \\ &= \sum_{l=1}^{N} x_1 \otimes \cdots \otimes x_{l-1} \otimes \lambda_l \alpha_i \otimes s_i(x_{l+1}) \otimes \cdots \otimes s_i(x_N) \in \alpha_i R. \end{aligned}$$

2. Since $r - s_i(r) = r - r = 0$ for $r \in R^i$ we have $R^i \subseteq \ker(\partial_i)$. Now let $r \in \ker(\partial_i)$, then $0 = \partial_i(r) = \frac{r - s_i(r)}{\alpha_i}$ which implies $r - s_i(r) = 0$. This says that $r = s_i(r)$, and thus $r \in R^i$. Hence, we have $R^i = \ker(\partial_i)$.
3. Let $r_1, r_2 \in R$, then we compute that

$$\begin{aligned} \partial_i(r_1 r_2) &= \frac{r_1 r_2 - s_i(r_1 r_2)}{\alpha_i} = \frac{r_1 r_2 - s_i(r_1) r_2 + s_i(r_1) r_2 - s_i(r_1) s_i(r_2)}{\alpha_i} \\ &= \frac{r_1 - s_i(r_1)}{\alpha_i} \cdot r_2 + s_i(r_1) \cdot \frac{r_2 - s_i(r_2)}{\alpha_i} = \partial_i(r_1) r_2 + s_i(r_1) \partial_i(r_2). \end{aligned}$$

4. Let $r \in R$ and $r_i \in R^i$, then we compute that

$$\partial_i(r_i r) = \partial_i(r_i) r + s_i(r_i) \partial_i(r) = r_i \partial_i(r).$$

Thus, ∂_i is R^i-linear. □

Definition 3.12. Let $W = S_n$ with the usual realization. For $w \in W$ pick a reduced expression $w = s_{i_1} \cdots s_{i_d}$. We define the *Demazure operator* $\partial_w : R \longrightarrow R$ by

$$\partial_w = \partial_{i_1} \circ \partial_{i_2} \circ \cdots \circ \partial_{i_{d_j}}.$$

For $J \subset S$ a parabolic subset, let $w_J \in W_J$ be the unique longest element. Then we write ∂_J for ∂_{w_J}. ◊

Proposition 3.13. *Let $W = S_n$ and let $w \in W$. Suppose $J \subset S$ is a parabolic subset, then the following hold:*

1. *∂_w is well-defined for all $w \in W$, i.e. it is independent of the choice of reduced expression;*
2. *$\mathrm{im}(\partial_J) \subseteq R^J$;*
3. *$\ker(\partial_J) \supseteq R^J$;*
4. *∂_J is R^J-linear.*

Proof. **1.** If we can prove that

$$\underbrace{\partial_i \circ \partial_j \circ \partial_i \circ \cdots}_{m_{ij} \text{ terms}} = \underbrace{\partial_j \circ \partial_i \circ \partial_j \circ \cdots}_{m_{ij} \text{ terms}} \tag{3.2}$$

for all $i, j \in S$, then we be done by Lemma 2.6, since this would mean that the Demazure operators respect braid moves and every two reduced expressions for $w \in W$ can be transformed into one another via braid moves. We only have to check (3.2) for $m_{ij} = 2$ and $m_{ij} = 3$, since we are in the case $W = S_n$.
Let first $m_{ij} = 2$, then we compute for $r \in R$ that

$$\begin{aligned}
\partial_i(\partial_j(r)) &= \partial_i\left(\frac{r - s_j(r)}{\alpha_j}\right) \\
&= \frac{\frac{r-s_j(r)}{\alpha_j} - s_i\left(\frac{r-s_j(r)}{\alpha_j}\right)}{\alpha_i} \\
&= \frac{r - s_j(r)}{\alpha_j\alpha_i} - \frac{s_i(r) - (s_i \circ s_j)(r)}{s_i(\alpha_j)\alpha_i} \\
&= \frac{r - s_j(r) - s_i(r) + (s_i \circ s_j)(r)}{\alpha_j\alpha_i}.
\end{aligned}$$

Here we used that $s_i(\alpha_j) = \alpha_j$. The last expression if symmetric in i and j, since $s_i \circ s_j = s_j \circ s_i$ for $m_{ij} = 2$, and thus it follows that $\partial_i(\partial_j(r)) = \partial_j(\partial_i(r))$ for all $r \in R$.
Now let $m_{ij} = 3$. We can compute that $s_i(\alpha_j) = \alpha_j + \alpha_i$ and $s_j(\alpha_i) = \alpha_i + \alpha_j$. We can compute the following for all $r \in R$.

$$\begin{aligned}
&(\partial_j \circ \partial_i \circ \partial_j)(r) \\
&= \frac{1}{\alpha_j} \cdot \left(\frac{\frac{r-s_j(r)}{\alpha_j} - s_i\left(\frac{r-s_j(r)}{\alpha_j}\right)}{\alpha_i} - s_j\left(\frac{\frac{r-s_j(r)}{\alpha_j} - s_i\left(\frac{r-s_j(r)}{\alpha_j}\right)}{\alpha_i}\right)\right) \\
&= \frac{1}{\alpha_j} \cdot \left(\frac{r - s_j(r)}{\alpha_i\alpha_j} - \frac{s_i(r) - (s_is_j)(r)}{\alpha_i s_i(\alpha_j)} - \frac{s_j(r) - r}{s_j(\alpha_i)s_j(\alpha_j)} + \frac{(s_js_i)(r) - (s_js_is_j)(r)}{s_j(\alpha_i)(s_js_i)(\alpha_j)}\right) \\
&= \frac{1}{\alpha_j} \cdot \left(\frac{r - s_j(r)}{\alpha_i\alpha_j} - \frac{s_i(r) - (s_is_j)(r)}{\alpha_i(\alpha_j + \alpha_i)} - \frac{s_j(r) - r}{-(\alpha_i + \alpha_j)\alpha_j} + \frac{(s_js_i)(r) - (s_js_is_j)(r)}{(\alpha_i + \alpha_j)\alpha_i}\right)
\end{aligned}$$

$$= \frac{1}{\alpha_j} \cdot \left(\frac{r - s_j(r)}{\alpha_j} \cdot \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_i + \alpha_j} \right) - \frac{s_i(r) - (s_i s_j)(r)}{\alpha_i(\alpha_j + \alpha_i)} + \frac{(s_j s_i)(r) - (s_j s_i s_j)(r)}{\alpha_i(\alpha_j + \alpha_i)} \right)$$
$$= \frac{1}{\alpha_j} \cdot \left(\frac{r - s_j(r)}{\alpha_i(\alpha_j + \alpha_i)} - \frac{s_i(r) - (s_i s_j)(r)}{\alpha_i(\alpha_j + \alpha_i)} + \frac{(s_j s_i)(r) - (s_j s_i s_j)(r)}{\alpha_i(\alpha_j + \alpha_i)} \right)$$
$$= \frac{r - s_j(r) - s_i(r) + (s_i s_j)(r) + (s_j s_i)(r) - (s_j s_i s_j)(r)}{\alpha_j \alpha_i(\alpha_j + \alpha_i)}$$

The last expression is again symmetric in i and j, since $s_i s_j s_i = s_j s_i s_j$ for $m_{ij} = 3$, and thus we are done.
2. Let $w_J = s_{i_1} \cdots s_{i_d}$ be a reduced expression. Let $j \in J$, then we may assume that $s_{i_1} = s_j$ by Corollary 2.14. Hence, by Proposition 3.11 $\partial_J(r) = \partial_j(r') \in R^j$ for all $r \in R$ where $r' = (\partial_{i_2} \circ \cdots \circ \partial_{i_d})(r)$. Since $j \in J$ was arbitrary, we get that $\partial_J(r) \in \bigcap_{j \in J} R^j = R^J$ for all $r \in R$.
3. Let $r \in R^J$, then $r \in R^j$ for all $j \in J$. By definition we get $\partial_J(r) = (\partial_{i_1} \circ \cdots \circ \partial_{i_d})(r) = 0$ which follows from Proposition 3.11. Thus, $R^J \subseteq \ker(\partial_J)$.
4. Let $r_J \in R^J, r \in R$. Then we compute by Proposition 3.11, since $r_J \in R^j$ for all $j \in J$, that

$$\partial_J(r_J r) = (\partial_{i_1} \circ \cdots \circ \partial_{i_d})(r_J r) = r_J \cdot (\partial_{i_1} \circ \cdots \circ \partial_{i_d})(r) = r_J \cdot \partial_J(r).$$

Hence, ∂_J is R^J-linear. □

Lemma 3.14. *Let $I \subset S$ be finitary. Then R is a finitely generated R^I-module.*

Proof. Recall that R is defined to be the symmetric algebra of some vector space $\mathfrak{h}^*$. Hence, R is isomorphic to the polynomial ring $\Bbbk[x_1, \dots x_N]$ where $N = \dim_{\Bbbk}(\mathfrak{h}^*)$ is finite. Thus, we have $R = \Bbbk[a_1, \dots a_N]$ for some $a_i \in R$. Then obviously R is also generated by $a_1, \dots, a_N$ as R^I-algebra, since $\Bbbk = R^I_0$. Hence, R is of finite type over R^I.
Now we will prove that R is integral over R^I. Let $r \in R$ and consider the polynomial $p_r(t) = \prod_{w \in W_I} t - w(r)$. This polynomial is monic and has r as a zero. The coefficients of p_r are symmetric polynomials in $w(r)$ for $w \in W_I$. Thus, they are invariant under the action of W_I which implies $p_r \in R^I[t]$. Hence, r is integral over R^I and so, R is also integral over R^I.
Since R is integral and of finite type over R^I we get that R is a finitely generated R^I-module. □

Lemma 3.15. *Let $J \subset S$ be finitary and $I \subset J$. Assume $\sum_{i=1}^m g_i b_i = 0$ for some $g_i \in R^J$ and homogeneous $b_i \in R^I$. If $b_1 \notin R(I,J)_+$, where $R(I,J)_+$ is the ideal in R^I generated by $\bigoplus_{k>0} R^J_k$, then*

$$g_1 = \sum_{i=2}^m h_i g_i$$

for some homogeneous $h_i \in R^J$ where $\deg(h_i) = \deg(b_i) - \deg(b_1)$.

Proof. We will do induction on $d = \deg(b_1)$. If $d = 0$ then $b_1 \in \Bbbk$ and $g_1 = \sum_{i=2}^{m} \frac{b_i}{b_1} g_i$. Then we can use that $g_1 \in R^J$ to get

$$g_1 = \frac{1}{|W_J|} \sum_{w \in W_J} w(g_1) = \frac{1}{|W_J|} \sum_{w \in W_J} w \left(\sum_{i=2}^{m} \frac{b_i}{b_1} g_i \right) = \sum_{i=2}^{m} \frac{\frac{1}{|W_J|} \sum_{w \in W_J} w(b_i)}{b_1} g_i.$$

Thus, $h_i = \frac{\frac{1}{|W_J|} \sum_{w \in W_J} w(b_i)}{b_1} \in R^J$ satisfy all conditions from the lemma, since they are homogeneous and have the right degree.
Now let $d > 0$ and pick $j \in J$ such that $\partial_j(b_1) \notin R(I,J)_+$. Then we get

$$0 = \partial_j(0) = \partial_j \left(\sum_{i=1}^{m} g_i b_i \right) = \sum_{i=1}^{m} g_i \partial_j(b_i),$$

where the last equality follows from Proposition 3.11. Now we are done by induction, since $\partial_j(b_i)$ is homogeneous and $\deg(\partial_j(b_1)) = \deg(b_1) - 2$.
So the only thing left to prove is that there exists a $j \in J$ such that $\partial_j(b_1) \notin R(I,J)_+$. Suppose that $\partial_j(b_1) \in R(I,J)_+$ for all $j \in J$, then

$$b_1 - s_j(b_1) = \alpha_j \partial_j(b_1) \in R(I,J)_+$$

or in other words $b_1 \equiv s_j(b_1) (\bmod\ R(I,J)_+)$ for all $j \in J$. Hence,

$$b_1 \equiv w(b_1) \pmod{R(I,J)_+}$$

for all $w \in W_J$. It follows that $|W_J| \cdot b_1 \equiv \sum_{w \in W_J} w(b_1) \equiv 0 \pmod{R(I,J)_+}$, since $\deg(b_1) > 0$ implies $\sum_{w \in W_J} w(b_1) \in R(I,J)_+$. Thus, $b_1 \in R(I,J)_+$ which is a contradiction. □

Theorem 3.16. *Let $J \subset S$ be a finitary subset and let $I \subset J$. Then R^I is a free R^J-module of rank $\frac{|W_J|}{|W_I|}$.*

Proof. Let $R(I,J)_+$ again be the ideal in R^I generated by $\bigoplus_{k>0} R^J_k$. Fix a homogeneous $\Bbbk$-basis $\overline{B}$ of ${R^I}/{R(I,J)_+}$ and let $B \subset R^I$ be a homogeneous lift of $\overline{B}$. We will prove that B is an R^J-basis for R^I.
Generating: Let $M \subset R^I$ be the R^J-submodule generated by B. We will prove inductively that $R^I_k = M_k$ for all $k \in \mathbb{N}_0$. For $k = 0$ we have $R^I_0 = \Bbbk$ and $M_0 = \Bbbk$, since $\left({R^I}/{R(I,J)_+}\right)_0 = \Bbbk = \mathbb{R}^J_0$, and thus $M_0 \neq 0$. Now let $r \in R^I_k$ for some $k \in \mathbb{N}$ and assume $M_l = R^I_l$ for all $l < k$. Now we can write

$$r = \sum_{b \in B} \lambda_b b + a$$

for some $\lambda_b \in \Bbbk$ and $a \in R(I,J)_+$, because B is a homogeneous lift of a $\Bbbk$-basis of ${R^I}/{R(I,J)_+}$. Since $a \in R(I,J)_+$ we can write

$$a = \sum a_i p_i$$

for some homogeneous $a_i \in R^I$ and $p_i \in \bigoplus_{k>0} R^J_k$ with $\deg(a_i p_i) = k$. However, since $\deg(p_i) > 0$ we get that $\deg(a_i) < k$, and thus $a_i \in M$. This implies $r \in M$, and thus $R^I_k = M_k$.

Linear independence: Consider all possible choices of bases $(B, \overline{B})$ and take a relation

$$\sum_{i=1}^{m} g_i b_i = 0, \qquad g_i \in R^J, b_i \in B$$

such that $m > 0$ is minimal among choices of such relations and $B, \overline{B}$. By Lemma 3.15 we have

$$g_1 = \sum_{i=2} h_i g_i$$

for some homogeneous $h_i \in R^J$ with $\deg(h_i) = \deg(b_i) - \deg(b_1)$. Hence, we get the smaller relation

$$\sum_{i=2}^{m} g_i(b_i - h_i b_1) = 0. \tag{3.3}$$

Note that, since $h_i \in R^J$ we have either $h_i \in \Bbbk = R^J_0$ or $h_i \in R(I,J)_+$. In the first case this implies

$$b_i - h_i b_1 \equiv \overline{b_i} - h_i \overline{b_1} \pmod{R(I,J)_+}, \qquad h_i \in \Bbbk$$

and in the second case we have

$$b_i - h_i b_1 \equiv \overline{b_i} \pmod{R(I,J)_+}.$$

Thus, the set

$$\overline{B_1} = \left(\overline{B} \setminus \{\overline{b_2}, \ldots, \overline{b_m}\}\right) \cup \{\overline{b_2 - h_2 b_1}, \ldots, \overline{b_m - h_m b_1}\}$$

is a basis for ${R^I}/{R(I,J)_+}$. Moreover, the set

$$B_1 = (B \setminus \{b_2, \ldots, b_m\}) \cup \{b_2 - h_2 b_1, \ldots, b_m - h_m b_1\}$$

is a homogeneous lift of this basis, since $\deg(h_i) = \deg(b_i) - \deg(b_1)$. Hence, (3.3) is a possible relation for the choice $(B_1, \overline{B_1})$ and has only $m-1$ summands which is a contradiction to the minimality of m. This implies that the elements of B are linear independent over R^J.

Rank: We know by Lemma 3.14 and the previous that $R \cong \left(R^I\right)^{N_1}$ as R^I-modules and thus also as R^J-modules for some $N_1 \in \mathbb{N}$. By the same reasoning we get that $R \cong \left(R^J\right)^{N_2}$ and $R^I \cong \bigoplus_{b\in B} R^J$ as R^J-modules for $N_2 \in \mathbb{N}$ and some set B. Together this gives $\left(R^J\right)^{N_2} \cong R \cong \left(\bigoplus_{b\in B} R^J\right)^{N_1}$. Hence, B is finite, and thus R^I is of finite rank over R^J.
The proof about the exact value will be omitted. However, we will see later that the rank is $\frac{|W_J|}{|W_I|}$ for $W = S_n$. $\square$

For the rest of this section we will consider the case $W = S_n$. The goal will be to find a basis for R as an R^J-module which has some nice properties. In order to do this we will consider the natural representation of S_n introduced in Example 3.3. This means we consider two rings simultaneously, our normal ring $R \cong \Bbbk[z_1, \dots, z_{n-1}]$ where z_i corresponds to α_i and the ring R_1 corresponding to the natural representation of S_n. Recall that $R_1 \cong \Bbbk[x_1, \dots, x_n]$ where S_n acts by permuting the x_i.
First we will discuss how exactly these realizations are connected which will be handy for understanding why we can switch between them. We can consider the inclusion map $\phi_n : R \hookrightarrow R_1, z_i \longmapsto x_i - x_{i+1}$. Note that this map is obviously injective.

Lemma 3.17. *The map ϕ_n preserves the action of W. Moreover, $R_1 \cong R[t]$ where R is included into R_1 via ϕ_n and t is invariant under the action of W.*

Proof. Since W acts on R by ring automorphisms it is enough to check that $\phi_n(s_i(z_j)) = s_i(\phi_n(z_j))$ for all $1 \leqslant i, j \leqslant n-1$. If $|i-j| > 1$ then, $s_i(z_j) = z_j$ and

$$s_j(\phi_n(z_j)) = s_i(x_j - x_{j+1}) = x_j - x_{j+1} = \phi_n(z_j).$$

Thus, we have three cases left. Let $j = i-1$, then $s_i(z_{i-1}) = z_{i-1} + z_i$. We have $s_i(x_{i-1} - x_i) = x_{i-1} - x_{i+1} = \phi_n(z_{i-1} + z_i)$.
Now let $j = i$. Then $s_i(z_i) = -z_i$ and $s_i(x_i - x_{i+1}) = x_{i+1} - x_i = \phi_n(-z_i)$.
At last let $j = i+1$. Then $s_i(z_{i+1}) = z_{i+1} + z_i$ and

$$s_i(x_{i+1} - x_{i+2}) = x_i - x_{i+2} = \phi_n(z_{i+1} + z_i).$$

Hence, ϕ_n respects the action of W. In order to prove the second claim note that

$$R_1 \cong \Bbbk[x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, x_1 + x_2 + \cdots + x_n].$$

This follows from the observation that we can write x_i as a $\Bbbk$-linear combination of $x_1 - x_2, \dots, x_{n-1} - x_n, x_1 + \cdots + x_n$. Indeed, we have

$$x_i = \frac{1}{n}\left(\sum_{j<i} -j(x_j - x_{j+1}) + \sum_{j\geqslant i}(n-j)(x_j - x_{j+1}) + (x_1 + \cdots + x_n)\right).$$

This implies that $R_1 \cong \phi_n(R)[x_1 + \cdots + x_n]$ which proves the claim since $x_1 + \cdots + x_n$ is invariant under W. $\square$

Remark 3.18. This lemma implies that the definition of Demazure operators for R_1 coincides with the one for R via ϕ. Thus, we have the Demazure operator

$$\partial_i : R_1 \longrightarrow R_1^i, \quad f \longmapsto \frac{f - s_i(f)}{x_i - x_{i+1}}$$

for R_1 as an extensions of $\partial_i : R \longrightarrow R$. Obviously the same holds true for the Demazure operators ∂_w for $w \in W$. We will abuse notation and write $\alpha_i = x_i - x_{i+1}$. ◊

Corollary 3.19. *We have* $R_1^W = R^W[t]$.

Proof. Obviously $R^W[t] \subseteq R_1^W$ as t is invariant under W. Now let $f \in R_1^W$, then $f \in R_1 \cong R[t]$. Thus, we can write

$$f = \sum_{k=0}^{N} r_k t^k, \qquad r_k \in R.$$

As the action of W preserves the grading, we get $r_k t^k \in R_1^W$, and thus $r_k \in R_1^W = R^W$. This implies $f \in R^W[t]$ and hence proves the corollary. □

Lemma 3.20. *If $B \subset R_1$ is a homogeneous R_1^W-basis of R_1, then $B \subset \mathrm{im}(\phi)$ and $\phi_n^{-1}(B)$ is an R^W-basis of R.*

Proof. Let $B \subset R_1$ be an R_1^W-basis of R_1. Consider $B' = \phi_n^{-1}(B) \subset R$. We claim that this is an R^W basis for R. Let $r \in R$, then we can write

$$\phi_n(r) = \sum_{b \in B} f_b b, \qquad f_b \in R_1^W, \tag{3.4}$$

since B is a basis. By Corollary 3.19 we can write $f_b = \sum_{k=0}^{N_b} f_{b,k} t^k$ with $f_{b,k} \in \phi_n(R)^W$. Now by looking at the degree of t in (3.4) we get that

$$\phi_n(r) = \sum_{b \in B, \deg(b)=0} f_{b,0} \cdot b = \sum_{b' \in B'} f_{b,0} \cdot \phi_n(b').$$

By the injectivity of ϕ_n we get that $r = \sum_{b' \in B'} \phi_n^{-1}(f_{b,0})\, b'$. Hence, B' generates R as an R^W-module, as $f_{b,0} \in \phi_n(R^W)$. It remains to check that the elements of B' are linearly independent. Suppose that

$$0 = \sum_{b' \in B'} r_{b'} b', \qquad r_{b'} \in R^W. \tag{3.5}$$

By applying ϕ_n to (3.5) we get

$$0 = \sum_{b' \in B'} \phi_n(r_{b'}) \cdot \phi_n(b').$$

Since $\phi_n(b') \in B$ and $\phi_n(r_{b'}) \in R_1^W$ we get $\phi_n(r_{b'}) = 0$, because B is a basis. Since ϕ_n is injective, this implies $r_{b'} = 0$, and thus the elements of B' are linearly independent which makes B' a basis.
Finally, we need to check that $B \subset \mathrm{im}(\phi_n)$. Note that $\phi_n(B') \subseteq B$. Let $f \in R_1 \cong \phi_n(R)[t]$, then we can write $f = \sum_{k=0}^{N} r_k t^k$ and since $r_k \in \phi_n(R)$ we can write

$$r_k = \sum_{b' \in B'} a_{b',k} \cdot \phi_n(b'), \qquad a_{b',k} \in \phi_n(R)^W.$$

Altogether we get that

$$f = \sum_{k=0}^{N} \sum_{b' \in B'} a_{b',k} \cdot \phi_n(b') \cdot t^k = \sum_{b' \in B} \left(\sum_{k=0}^{N} a_{b',k} \cdot t^k \right) \phi_n(b').$$

However, $\sum_{k=0}^{N} a_{b',k} \cdot t^k \in \phi_n(R)^W[t] \cong R_1^W$, and thus $\phi(B')$ is a basis of R_1 as an R_1^W-module. This implies that $B = \phi_n(B')$. □

Remark 3.21. In the last proof we also saw that if $B' \subset R$ is an R^W-basis of R, then $\phi_n(B')$ generates R_1 as an R_1^W-module. It is also easy to observe that $\phi_n(B')$ is a basis of R_1 as an R_1^W-module. For this suppose that

$$0 = \sum_{b \in B'} f_b \phi_n(b)$$

where $f_b \in R_1^W$. Then we can write $f_b = \sum_{k=0}^{N_b} \phi_n(a_{b,k}) t^k$ with $a_{b,k} \in R^W$. This gives

$$\begin{aligned} 0 = \sum_{b \in B'} f_b \phi_n(b) &= \sum_{b \in B'} \sum_{k=0}^{\max(N_b)} \phi_n(a_{b,k}) t^k \phi_n(b) \\ &= \sum_{k=0}^{\max(N_b)} \left(\sum_{b \in B'} \phi_n(a_{b,k} b) \right) t^k, \end{aligned}$$

where $a_{b,k} = 0$ if $k > N_b$. Then by comparing coefficients of t^k we get

$$0 = \sum_{b \in B'} \phi_n(a_{b,k} b) = \phi_n \left(\sum_{b \in B'} a_{b,k} b \right)$$

which implies $a_{b,k} = 0$ for all $b \in B'$ and $k = 0, \dots, N_b$. This implies $f_b = 0$ for all $b \in B'$, and thus the elements of $\phi_n(B')$ are linearly independent over R_1^W. Hence, $\phi_n(B')$ is a basis of R_1 over R_1^W. ♢

Remark 3.22. Note that the proofs of Corollary 3.19 and Lemma 3.20 still work if we replace W by a parabolic subgroup W_J, since W_J is again a Coxeter group (Lemma 2.16). ♢

Now we can begin to construct our basis for R. We will do this simultaneously for R and R_1 and will see soon that we can actually identify both bases via ϕ_n.

Definition 3.23. There are elements $\{\sigma_w\}_{w\in W} \subset R_1$, called *Schubert polynomials*, which are given by

$$\sigma_w = \partial_{w^{-1}w_0}\left(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\right).$$

By Corollary 3.19 we can write $\sigma_w = \sum_{k=0}^{N}\phi_n(a_{w,k})t^k$ with $a_{w,k} \in R$. We define $g_w = a_{w,0}$. ◊

Example 3.24. Consider $W = S_3$. We can now construct the Schubert polynomials step by step.

$$\begin{aligned}
\sigma_{w_0} &= \partial_e(x_1^2x_2) = x_1^2x_2\\
\sigma_{s_1s_2} &= \partial_{s_1}(x_1^2x_2) = x_1x_2\\
\sigma_{s_2s_1} &= \partial_{s_2}(x_1^2x_2) = x_1^2\\
\sigma_{s_1} &= \partial_{s_2s_1}(x_1^2x_2) = \partial_{s_2}(x_1x_2) = x_1\\
\sigma_{s_2} &= \partial_{s_1s_2}(x_1^2x_2) = \partial_{s_1}(x_1^2) = x_1 + x_2\\
\sigma_e &= \partial_{s_1s_2s_1}(x_1^2x_2) = \partial_{s_1}(x_1) = 1.
\end{aligned}$$

Lemma 3.25. *$\sigma_e = 1$.*

Proof. For $n = 1$ there is nothing to prove, as $\sigma_e = \sigma_{w_0} = 1$. Now let $n > 1$, then consider $w^k = s_{k-1}s_{k-2}\cdots s_1$. Note that this is a reduced expression of w^k. Moreover, we have $w_0 = w^2\cdots w^n$ and $\ell(w_0) = \ell(w^2) + \cdots + \ell(w_n)$. Thus, $\partial_{w_0} = \partial_{w^2}\circ\cdots\circ\partial_{w^n}$. At last we note that

$$\begin{aligned}
\partial_{w^k}(x_1^{k-1}x_2^{k-2}\cdots x_{k-1}) &= \partial_{w^n s_1}\left(x_1^{k-2}x_2^{k-2}x_3^{k-3}\cdots x_{k-1}\right)\\
&= \partial_{w^k s_1 s_2}\left(x_1^{k-2}x_2^{k-3}x_3^{k-3}x_4^{k-4}\cdots x_{k-1}\right)\\
&= \cdots = x_1^{k-2}x_2^{k-3}\cdots x_{k-2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sigma_e = \partial_{w_0}(x_1^{n-1}\cdots x_{n-1}) &= \left(\partial_{w^2}\circ\cdots\circ\partial_{w^n}\right)\left(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\right)\\
&= \left(\partial_{w^2}\circ\cdots\circ\partial_{w^{n-1}}\right)\left(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}\right)\\
&= \cdots = \partial_{w^2}(x_1^1) = 1.
\end{aligned}$$

□

Definition 3.26. Let $\{\tau_w\}_{w\in W_J} \subset R$ be a set of homogeneous elements. Then we say that $\{\tau_w\}_{w\in W_J}$ is *Demazure generated* if

$$\partial_u(\tau_w) = \begin{cases} \tau_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u)\\ 0 & \text{otherwise}\end{cases} \tag{3.6}$$

for all $u \in W_J$ and $\tau_e = 1$. ◊

Remark 3.27. Let $\{\tau_w\}_{w\in W_J}$ be Demazure generated. Since $\partial_s : R \longrightarrow R\langle -2\rangle$ reduces the degree by 2 we have that ∂_u reduces the degree by $2\ell(u)$. This implies, since

$$\partial_u(\tau_u) = \tau_e = 1,$$

that τ_u has degree $2\ell(u)$ for all $u \in W_J$. $\Diamond$

Theorem 3.28. *The elements $\{g_w\}_{w\in W_J}$ form a basis for R as R^J-module. Moreover, $\{g_w\}_{w\in W_J}$ is Demazure generated.*

Proof. **Demazure generated:** We will start by proving property (3.6) for the Schubert polynomials σ_w. By Lemma 3.25 we have $\sigma_e = 1$. Now let $w, u \in W_J$, then

$$\partial_u(\sigma_w) = \partial_u\left(\partial_{w^{-1}w_0}(\sigma_{w_0})\right) = \left(\partial_u \circ \partial_{w^{-1}w_0}\right)(\sigma_{w_0}).$$

Now note that we have $\ell(w) \leqslant \ell(wu^{-1}) + \ell(u)$ as $w = wu^{-1}u$. Suppose first that $\ell(w) < \ell(wu^{-1}) + \ell(u)$ and let $u = s_{i_1}\cdots s_{i_d}, w^{-1}w_0 = s_{j_1}\cdots s_{j_c}$ be reduced expressions. Now note that

$$\begin{aligned}\ell(uw^{-1}w_0) &= \ell(w_0) - \ell(uw^{-1}) = \ell(w_0) - \ell(wu^{-1})\\ &< \ell(w_0) - (\ell(w) - \ell(u)) = \ell(w_0) - \ell(w) + \ell(u) = \ell(w^{-1}w_0) + \ell(u).\end{aligned}$$

This implies that the expression $uw^{-1}w_0 = s_{i_1}\cdots s_{i_d}s_{j_1}\cdots s_{j_c}$ is not reduced. For simpler notation set $i_{d+k} = j_k$ for $k = 1, \ldots, c$. Then there must be some number k such that $\ell(s_{i_1}\cdots s_{i_{k-1}}) > \ell(s_{i_1}\cdots s_{i_k})$. Pick the smallest such k, then $v = s_{i_1}\cdots s_{i_{k-1}}$ is a reduced expression. By Theorem 2.11 there is an index l such that $vs_{i_k} = s_{i_1}\cdots\widehat{s_{i_l}}\cdots s_{i_{k-1}}$. Thus, $v = s_{i_1}\cdots\widehat{s_{i_l}}\cdots s_{i_{k-1}}s_{i_k}$ is a reduced expression. This implies that

$$\begin{aligned}\partial_u \circ \partial_{w^{-1}w_0} &= \partial_{i_1} \circ \cdots \circ \partial_{i_d} \circ \partial_{j_1} \circ \cdots \circ \partial_{j_c}\\ &= \partial_{i_1} \circ \cdots \partial_{i_{c+d}}\\ &= \partial_v \circ \partial_{i_k} \circ \cdots \circ \partial_{i_{c+d}}\\ &= \partial_{i_1} \circ \cdots \widehat{\partial_{i_l}} \circ \cdots \partial_{i_k} \circ \partial_{i_k} \circ \cdots \circ \partial_{i_{c+d}} = 0,\end{aligned}$$

since $\partial_{i_k} \circ \partial_{i_k} = 0$ by Proposition 3.11. Hence, $\partial_u(\sigma_w) = 0$. Now suppose that $\ell(w) = \ell(wu^{-1}) + \ell(u)$. Then

$$\begin{aligned}\ell(uw^{-1}w_0) &= \ell(w_0) - \ell(uw^{-1}) = \ell(w_0) - \ell(wu^{-1}) = \ell(w_0) - (\ell(w) - \ell(u))\\ &= \ell(w_0) - \ell(w) + \ell(u) = \ell(w^{-1}w_0) + \ell(u).\end{aligned}$$

Hence, $uw^{-1}w_0 = s_{i_1}\cdots s_{i_d}s_{j_1}\cdots s_{j_c}$ is a reduced expression. Thus,

$$\partial_u \circ \partial_{w^{-1}w_0} = \partial_{i_1} \circ \cdots \circ \partial_{i_d} \circ \partial_{j_1} \circ \cdots \circ \partial_{j_c} = \partial_{uw^{-1}w_0}.$$

This implies

$$\partial_u(\sigma_{w_0}) = \left(\partial_u \circ \partial_{w^{-1}w_0}\right)(\sigma_{w_0}) = \partial_{uw^{-1}w_0}(\sigma_{w_0}) = \partial_{(wu^{-1})^{-1}w_0}(\sigma_{w_0}) = \sigma_{wu^{-1}}.$$

Now we can check property (3.6) for the g_w. Let again $u, w \in W_J$, then by the previous

$$\partial_u\left(\sum_{k=0}^{N_w}\phi_n(a_{w,k})t^k\right) = \partial_u(\sigma_w) = \begin{cases} \sigma_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \sum_{k=0}^{N_{wu^{-1}}}\phi_n(a_{wu^{-1},k})t^k & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise.} \end{cases}$$

This implies by comparing coefficients of t^0, since $g_w = a_{w,0}$,

$$\partial_u(g_w) = \begin{cases} g_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise.} \end{cases}$$

By definition we also have $g_e = \sigma_e = 1$. Hence, $\{g_w\}_{w\in W_J}$ is Demazure generated. Next we prove that $\{g_w\}_{w\in W_J}$ is a basis.

Linear independence: First we check linear independence. Suppose that

$$0 = \sum_{w\in W_J} r_w g_w$$

for $r_w \in R^J$. Choose $u \in W_J$ of maximal length such that $r_u \neq 0$. Note that ∂_u is R^J-invariant as it is the composition of R^J-invariant morphisms ∂_j for some $j \in J$. Now for all $w \in W_J$ with $\ell(w) \leqslant \ell(u)$ we have $\ell(w) - \ell(u) \leqslant 0$. Hence, $\ell(wu^{-1}) = \ell(w) - \ell(u)$ is only possible if $\ell(wu^{-1}) = 0$ which means $w = u$. This implies

$$\partial_u(0) = \partial_u\left(\sum_{w\in W_J} r_w g_w\right) = \partial_u\left(\sum_{w\in W_J, \ell(w)\leqslant\ell(u)} r_w g_w\right)$$
$$= \sum_{w\in W_J, \ell(w)\leqslant\ell(u)} r_w \partial_u(g_w) = r_u g_{uu^{-1}} = r_u.$$

This is a contradiction and proves linear independence.

Generating: Now we prove that the g_w generate. Let $r \in R$. We define elements $b_w \in R$ for $w \in W_J$. Let $\ell = \ell(w_J) - \ell(w)$, we will define these elements by induction on ℓ:

$$b_{w_J} = \partial_{w_J}(r)$$
$$b_w = \partial_w\left(r - \sum_{\ell(u)>\ell(w)} b_u g_u\right).$$

Now we will prove by induction on ℓ that $b_w \in R^J$. For $\ell = 0$ we have $b_{w_J} = \partial_{w_J}(r) \in \mathrm{im}(\partial_{w_J}) \subseteq R^J$. Suppose now $\ell > 0$. It is enough to prove that $\partial_j(b_w) = 0$ for all $j \in J$, since this would imply $b_w \in \bigcap_{j\in J} R^j = R^J$ by Proposition 3.11. So let $j \in J$, then we have either $\ell(s_j w) < \ell(w)$ or $\ell(s_j jw) > \ell(w)$.

First suppose $\ell(s_j w) < \ell(w)$. This implies $\ell(w^{-1}s_j) = \ell(s_j w) < \ell(w) = \ell(w^{-1})$. Let $w = s_{i_1} \cdots s_{i_d}$ be a reduced expression, then by Theorem 2.11 we get that $w^{-1}s_j = s_{i_d} \cdots \widehat{s_{i_k}} \cdots s_{i_1}$ for some k. This implies that $w = s_j s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_d}$ is a reduced expression. Thus,

$$\begin{aligned}\partial_j(b_w) &= \partial_j \left(\partial_w \left(r - \sum_{\ell(u)>\ell(w)} b_u g_u \right)\right) \\ &= \left(\partial_j \circ \partial_j \circ \partial_{i_1} \circ \cdots \widehat{\partial_{i_k}} \circ \cdots \circ \partial_{i_d}\right) \left(r - \sum_{\ell(u)>\ell(w)} b_u g_u \right) = 0,\end{aligned}$$

as $\partial_j \circ \partial_j = 0$.

Now suppose $\ell(s_j w) > \ell(w)$ and write $w_1 = s_j w$. Note that $w_1 = s_j s_{i_1} \cdots s_{i_d}$ is a reduced expression for w_1 which implies $\partial_j \circ \partial_w = \partial_{w_1}$. If $\ell(u) = \ell(w_1)$ for $u \in W_J$, then $\ell(u) - \ell(w_1) = 0$. Thus, $\ell(uw_1^{-1}) = \ell(u) - \ell(w_1)$ only if $u = w_1$ which implies $\partial_{w_1}(g_u) = \delta_{u,w_1}$. Now we can compute that

$$\begin{aligned}\partial_j(b_w) &= \partial_j \left(\partial_w \left(r - \sum_{\ell(u)>\ell(w)} b_u g_u \right)\right) \\ &= \partial_{w_1} \left(r - \sum_{\ell(u)>\ell(w)} b_u g_u \right) \\ &= \partial_{w_1}(r) - \sum_{\ell(u)=\ell(w_1)} b_u \partial_{w_1}(g_u) - \partial_{w_1} \left(\sum_{\ell(u)>\ell(w_1)} b_u g_u \right) \\ &= \partial_{w_1}(r) - \partial_{w_1} \left(\sum_{\ell(u)>\ell(w_1)} b_u g_u \right) - \sum_{\ell(u)=\ell(w_1)} b_u \cdot \delta_{u,w_1} \\ &= \partial_{w_1} \left(r - \sum_{\ell(u)>\ell(w_1)} b_u g_u \right) - b_{w_1} = b_{w_1} - b_{w_1} = 0.\end{aligned}$$

This concludes our induction, and hence $b_w \in R^J$ for all $w \in W_J$. Moreover, we have that

$$R^J \ni b_e = \partial_e \left(r - \sum_{\ell(u)>\ell(e)} b_u g_u \right) = r - \sum_{\ell(u)>\ell(e)} b_u g_u$$

which implies that

$$r = b_e + \sum_{u \neq e} b_u g_u = \sum_{u \in WJ} b_u g_u.$$

Since $r \in R$ was arbitrary and $b_u \in R^J$ this proves that the g_w generate. Hence, $\{g_w\}_{w \in W_J}$ is a basis of R over R^J. □

Remark 3.29. Note that we could do the same proof if we replaced g_w by σ_w. This implies that $\{\sigma_w\}_{w\in W_J}$ is an R_1^J-basis of R_1. Hence, by Lemma 3.20 and Remark 3.22 we get that $\sigma_w = \phi_n(g_w)$. This now gives us the right to switch between R and R_1 whenever we want to prove something about this basis. Usually we want to state our theorems using R and $\{g_w\}_{w\in W_J}$, since this is the realization we will use in future chapters. However, in the proofs we will often need to work with R_1 and $\{\sigma_w\}_{w\in W_J}$, since we can calculate explicitly there. ♢

Remark 3.30. Note that this finishes the proof of Theorem 3.16 for $W = S_n$. We have proven that $\mathrm{rk}_{R^J}(R) = |W_J|$. Thus, we get that $\mathrm{rk}_{R^J}(R) = \mathrm{rk}_{R^I}(R)\cdot \mathrm{rk}_{R^J}(R^I)$ which implies

$$\mathrm{rk}_{R^J}\left(R^I\right) = \frac{\mathrm{rk}_{R^J}(R)}{\mathrm{rk}_{R^I}(R)} = \frac{|W_J|}{|W_I|}$$

for $I \subseteq J$. ♢

We will now construct a dual basis to our basis $\{g_w\}_{w\in W}$ and prove the duality. The definitions will only be done for R and the g_w, but work in the same way also for R_1 and σ_w which we will use in the proofs.

Definition 3.31. We define $\{g_w^*\}_{w\in W} \subset R$ by

$$g_w^* = (-1)^{\ell(ww_0)} w_0(g_{ww_0})$$

and call them *dual Schubert polynomials.* ♢

Lemma 3.32.

1. *For $u \in W$ we have that $w_0 \circ \partial_u \circ w_0 = (-1)^{\ell(u)} \partial_{w_0uw_0}$.*
2. *For $u \in W$ we have that*

$$\partial_u(g_w^*) = \begin{cases} g_{wu^{-1}}^* & \text{if } \ell(wu^{-1}) = \ell(w) + \ell(u) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. **1.** We begin with the case $u = s \in S$. Note that

$$\ell(w_0sw_0) = \ell(w_0) - \ell(w_0s) = \ell(w_0) - (\ell(w_0) - \ell(s)) = \ell(s) = 1.$$

Thus, $w_0sw_0 = \widetilde{s} \in S$. Moreover, $w_0(\alpha_s) = -\alpha_{\widetilde{s}}$. One can check this for example in R_1: Let $s = (i, i+1)$. As w_0 reverses the order of $1,\dots,n$ we get that $\widetilde{s} = (n-i, n-i+1)$. We compute that

$$w_0(\alpha_s) = w_0(x_i - x_{i+1}) = x_{n-i+1} - x_{n-i} = -\alpha_{\widetilde{s}}.$$

Bringing everything together we have for $r \in R$

$$\begin{aligned}(w_0 \circ \partial_s \circ w_0)(r) &= w_0\left(\frac{w_0(r) - sw_0(r)}{\alpha_s}\right) = \frac{r - w_0sw_0(r)}{w_0(\alpha_s)} \\ &= \frac{r - \widetilde{s}(r)}{-\alpha_{\widetilde{s}}} = -\partial_{\widetilde{s}}(r) = -\partial_{w_0sw_0}(r).\end{aligned}$$

Now let $u = s_{i_1} \cdots s_{i_d}$ be a reduced expression. Note that $\ell(w_0 u w_0) = \ell(u) = d$ as before and $\ell(w_0 s_{i_k} w_0) = 1$. Hence,

$$w_0 u w_0 = w_0 s_{i_1} \cdots s_{i_d} w_0 = (w_0 s_{i_1} w_0)(w_0 s_{i_2} w_0) \cdots (w_0 s_{i_d} w_0)$$

is a reduced expression. We compute that

$$\begin{aligned} w_0 \circ \partial_u \circ w_0 &= w_0 \circ (\partial_{i_1} \circ \cdots \circ \partial_{i_d}) \circ w_0 \\ &= (w_0 \circ \partial_{i_1} w_0) \circ (w_0 \circ \partial_{i_2} \circ w_0) \circ \cdots \circ (w_0 \circ \partial_{i_d} w_0) \\ &= \left(-\partial_{w_0 s_{i_1} w_0}\right) \circ \left(-\partial_{w_0 s_{i_2} w_0}\right) \circ \cdots \circ \left(-\partial_{w_0 s_{i_d} w_0}\right) \\ &= (-1)^d \partial_{w_0 u w_0}. \end{aligned}$$

This finishes part 1, since $\ell(u) = d$.
2. Let $w, u \in W$, then we just compute that

$$\begin{aligned} \partial_u (g_w^*) &= \partial_u \left((-1)^{\ell(ww_0)} w_0(g_{ww_0})\right) = (-1)^{\ell(ww_0)} \cdot (-1)^{\ell(u)} \cdot w_0 \left(\partial_{w_0 u w_0}(g_{ww_0})\right) \\ &= \begin{cases} (-1)^{\ell(ww_0)+\ell(u)} \cdot w_0 \left(g_{ww_0 w_0 u^{-1} w_0}\right) & \text{if } \begin{array}{l} \ell(ww_0 w_0 u^{-1} w_0) \\ = \ell(ww_0) - \ell(w_0 u w_0) \end{array} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (-1)^{\ell(w_0)-\ell(w)-\ell(u)} \cdot w_0 \left(g_{wu^{-1}w_0}\right) & \text{if } \ell(wu^{-1}w_0) = \ell(ww_0) - \ell(u) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (-1)^{\ell(w_0)-\ell(wu^{-1})} \cdot w_0 \left(g_{wu^{-1}w_0}\right) & \text{if } \ell(wu^{-1}) = \ell(w) + \ell(u) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} g^*_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) + \ell(u) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we used that $\ell(wu^{-1}w_0) = \ell(w_0) - \ell(wu^{-1})$ in the fourth line and Definition 3.26 in the second line. $\square$

Lemma 3.33. *Let $w, v \in W$ and let us expand $g_w g_v^*$ in the basis $\{g_u\}_{u \in W}$:*

$$g_w g_v^* = \sum_{u \in W} a_u g_u, \qquad a_u \in R^W.$$

Suppose $\ell(w) \neq \ell(v)$, then $a_{w_0} = 0$.

Proof. By Remark 3.29 we may consider σ_w instead of g_w and R_1 instead of R. Thus, we have $\sigma_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$ and $\sigma_v^* = (-1)^{\ell(vw_0)} w_0(\sigma_{vw_0})$. For $v = w_0$ the result is clear, since $\sigma_{w_0}^* = 1$. Hence, it is enough to check that the coefficient of σ_{w_0} in $\sigma_w w_0(\sigma_v)$ is zero for all $w \in W, v \in W \setminus \{e\}$.
Now let $v \neq e$. We will prove that σ_w is a $\Bbbk$-linear combination of monomials of the form $x_1^{b_1} \cdots x_n^{b_n}$ with $b_i \leqslant n - i$. We will prove this by induction on $\ell = \ell(w_0) - \ell(w)$. For $\ell = 0$ we have $w = w_0$ and we know that $\sigma_{w_0} = x_1^{n-1} \cdots x_{n-1}^1$.

Now let $\ell > 0$, then there is $s \in S$ such that $\ell(ws) > \ell(w)$. Thus, $\ell(w_0) - \ell(ws) < \ell$. So by inductions σ_{ws} is a $\mathbb{k}$-linear combination of such monomials. Since $\sigma_w = \partial_s(\sigma_{ws})$ it is enough to check that polynomials built out of monomials of the form $x_1^{b_1} \cdots x_n^{b_n}$ with $b_i \leqslant n - i$ are closed under applying Demazure operators. Let $s = (i, i+1)$, then we compute that

$$\begin{aligned}
\partial_s \left(x_1^{b_1} \cdots x_n^{b_n} \right) &= x_1^{b_1} \cdots x_{i-1}^{b_{i-1}} \cdot \partial_s \left(x_i^{b_i} x_{i+1}^{b_{i+1}} \right) \cdot x_{i+2}^{b_{i+2}} \cdots x_n^{b_n} \\
&= x_1^{b_1} \cdots x_{i-1}^{b_{i-1}} \cdot x_{i+2}^{b_{i+2}} \cdots x_n^{b_n} \cdot (x_i x_{i+1})^{b_{min}} \cdot \left(\pm \sum_{k=0}^{b-1} x_i^k \cdot x_{i+1}^{b-1-k} \right) \\
&= \pm \sum_{k=0}^{b-1} x_1^{b_1} \cdots x_{i-1}^{b_{i-1}} \cdot x_i^{b_{min}+k} \cdot x_{i+1}^{b_{max}-1-k} \cdot x_{i+2}^{b_{i+2}} \cdots x_n^{b_n}
\end{aligned}$$

where $b_{min} = \min(b_i, b_{i+1}), b_{max} = \max(b_i, b_{i+1})$ and $b = b_{max} - b_{min}$. Now we obviously still have $b_j \leqslant n - j$ for $j \notin \{i, i+1\}$. We also have $b_{min} + k \leqslant b_{min} + b - 1 = b_{max} - 1 \leqslant n - i - 1$ and $b_{max} - 1 - k \leqslant n - i - 1$ which concludes our induction argument.
Recall that w_0 is the permutation that reverses the order of $1, \ldots, n$. Then by the previous observation we get that $w_0(\sigma_v)$ is a $\mathbb{k}$-linear combination of monomials $x_1^{c_1} \cdots x_n^{c_n}$ where $c_i < i$. Hence, we have that $\sigma_w w_0(\sigma_v)$ is a $\mathbb{k}$-linear combination of monomials of the form

$$x_1^{b_1+c_1} x_2^{b_2+c_2} \cdots x_n^{b_n+c_n}.$$

Note that $0 \leqslant b_i + c_i < n - i + i = n$. Thus, we only have n possible exponents for n variables. This implies that two variables must have the same exponent, since the only other possibility is that each exponent $0, \ldots, n-1$ appears exactly once. Then $\sum_{i=1}^n b_i + c_i = \frac{n(n-1)}{2} = \deg(\sigma_{w_0})$, but then $\ell(w) + \ell(vw_0) = \ell(w_0)$. Thus, since $\ell(vw_0) = \ell(w_0) - \ell(v)$, we get $\ell(w) = \ell(v)$ which is not possible.
Hence, it is enough to prove that the coefficient of σ_{w_0} in such monomials (with the same exponent for some x_i, x_j) is 0. Note that such a monomial is fixed by a reflection $(i, j) \in S_n$ (where x_i and x_j have the same exponent). So, it is enough to prove that polynomials which are fixed by a reflection (i, j) have coefficient zero for σ_{w_0} when we expand them in the basis $\{\sigma_u\}_{u \in W}$.
Suppose $i < j$, then we can write

$$t = (i, j) = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_i = (s_i \cdots s_{j-2}) s_{j-1} (s_i \cdots s_{j-2})^{-1},$$

and thus t is a reflection in the Coxeter group, i.e. of the form $\widetilde{w} s \widetilde{w}^{-1}$. Let $r \in R_1$ be a polynomial with $t(r) = r$. We write

$$r = \sum_{u \in W} a_u \sigma_u, \qquad a_u \in R_1^W.$$

Then let us write

$$t(\sigma_x) = \sum_{u \in W} z_{u,x} \sigma_u, \qquad z_{u,x} \in R_1^W.$$

Note that by degree reasons $z_{w_0,x} = 0$ if $x \neq w_0$. Moreover, we get

$$\begin{aligned} t(r) &= \sum_{x\in W} a_x t(\sigma_x) \\ &= \sum_{x\in W}\sum_{u\in W} a_x z_{u,x}\sigma_u \\ &= \sum_{u\in W}\left(\sum_{x\in W} a_x z_{u,x}\right)\sigma_u. \end{aligned}$$

This implies that $a_{w_0} = a_{w_0} z_{w_0,w_0}$. If $z_{w_0,w_0} \neq 1$, then $a_{w_0} = 0$ and we would be done. So this is all we need to prove. By Theorem 3.28 we have that $\partial_{w_0}(\sigma_{w_0} - t(\sigma_{w_0})) = 1 - z_{w_0,w_0}$. Assume first $t = s \in S$, then

$$\begin{aligned} 1 - z_{w_0,w_0} &= \partial_{w_0}(\sigma_{w_0} - s(\sigma_{w_0})) = \partial_{w_0}(\alpha_s \cdot \partial_s(\sigma_{w_0})) \\ &= \partial_{w_0 s}\left(\partial_s(\alpha_s \cdot \partial_s(\sigma_{w_0}))\right) = \partial_{w_0 s}\left(2 \cdot \partial_s(\sigma_{w_0})\right) \\ &= 2 \cdot \partial_{w_0}(\sigma_{w_0}) = 2. \end{aligned}$$

This gives $z_{w_0,w_0} = -1$. Here we used that by Corollary 2.14 w_0 has a reduced expression which ends in s. Now let $t = s_{i_1}\cdots s_{i_d}$ be a reduced expression. Then we compute that

$$\begin{aligned} t(\sigma_{w_0}) &= (s_{i_1}\cdots s_{i_d})(\sigma_{w_0}) = (s_{i_1}\cdots s_{i_{d-1}})\left(-\sigma_{w_0} + \text{lower terms}\right) \\ &= (s_{i_1}\cdots s_{i_{d-2}})\left(\sigma_{w_0} + \text{lower terms}\right) = \cdots = (-1)^d\sigma_{w_0} + \text{lower terms}, \end{aligned}$$

where lower terms means polynomials of degree less than $\deg(\sigma_{w_0})$ (which are irrelevant for the coefficient of σ_{w_0}). Hence, $z_{w_0,w_0} = (-1)^{\ell(t)}$.
We have $t = \widetilde{w}s\widetilde{w}^{-1}$. We will prove by induction on $\ell(\widetilde{w})$ that $\ell(t)$ is odd. For $\ell(\widetilde{w}) = 0$ this is clear. Now write $\widetilde{w} = \widetilde{s}\widetilde{w}_1$. Then by induction $\ell(\widetilde{w}_1 s\widetilde{w}_1^{-1})$ is odd. Then $\ell(\widetilde{s}\widetilde{w}_1 s\widetilde{w}_1^{-1})$ is even. Thus, $\ell(\widetilde{w}s\widetilde{w}^{-1}) = \ell(\widetilde{s}\widetilde{w}_1 s\widetilde{w}_1^{-1}s)$ is odd. This finishes the induction.
Hence, $\ell(t)$ is odd and it follows that $z_{w_0,w_0} = -1$ which finishes the proof by the arguments above. □

Corollary 3.34. $\partial_{w_0}(g_w g_u^*) = \delta_{w,u}$ *for all* $w, u \in W$.

Proof. Note that $\partial_{w_0}(g_v) = 0$ if $v \neq w_0$, since the g_w are Demazure generated. If $\ell(w) \neq \ell(u)$, then by Lemma 3.33

$$g_w g_u^* = \sum_{v\neq w_0} a_v g_v, \qquad a_v \in R^W.$$

Hence,

$$\partial_{w_0}\left(g_w g_u^*\right) = \partial_{w_0}\left(\sum_{v\neq w_0} a_v g_v\right) = \sum_{v\neq w_0} a_v \partial_{w_0}\left(g_v\right) = 0.$$

Let's suppose $\ell(w) = \ell(u)$ now. Let $r_1, r_2 \in R, i \in S$, then have

$$\partial_{w_0}\left(\partial_i(r_1) \cdot r_2\right) = \partial_{w_0}\left(r_1 \cdot \partial_i(r_2)\right). \tag{3.7}$$

To see this let $w_0 = s_{i_1} \cdots s_{i_d}$ by a reduced expression. By Corollary 2.14 we may assume $i_d = i$. Thus, $\partial_{w_0} = \partial_{\widetilde{w}} \circ \partial_i$. Hence, we compute that

$$\begin{aligned}\partial_{w_0}\left(\partial_i(r_1) \cdot r_2\right) &= \partial_{\widetilde{w}}\left(\partial_i\left(\partial_i(r_1) \cdot r_2\right)\right) = \partial_{\widetilde{w}}\left(\partial_i(r_1) \cdot \partial_i(r_2)\right) \\ &= \partial_{\widetilde{w}}\left(\partial_i\left(r_1 \cdot \partial_i(r_2)\right)\right) = \partial_{w_0}\left(r_1 \cdot \partial_i(r_2)\right)\end{aligned}$$

which proves (3.7). Note that we can generalize (3.7). Let $v = s_{j_1} \cdots s_{j_c}$ be a reduced expression. Then

$$\begin{aligned}\partial_{w_0}\left(\partial_v(r_1) \cdot r_2\right) &= \partial_{w_0}\left(\left(\partial_{j_1} \circ \cdots \circ \partial_{j_c}\right)(r_1) \cdot r_2\right) \\ &= \partial_{w_0}\left(\left(\partial_{j_2} \circ \cdots \circ \partial_{j_c}\right)(r_1) \cdot \partial_{j_1}(r_2)\right) \\ &= \partial_{w_0}\left(\left(\partial_{j_3} \circ \cdots \circ \partial_{j_c}\right)(r_1) \cdot \left(\partial_{j_2} \circ \partial_{j_1}\right)(r_2)\right) \\ &= \cdots = \partial_{w_0}\left(r_1 \cdot \left(\partial_{j_c} \circ \cdots \circ \partial_{j_1}\right)(r_2)\right) \\ &= \partial_{w_0}\left(r_1 \cdot \partial_{v^{-1}}(r_2)\right).\end{aligned} \tag{3.8}$$

Now we write $g_w = \partial_{w^{-1}w_0}(g_{w_0})$ by Definition 3.26. Then we have $\ell(w_0 w) + \ell(u) = \ell(w_0) - \ell(w) + \ell(u) = \ell(w_0)$. Thus, $\ell(uw^{-1}w_0) = \ell(w_0 w) + \ell(u)$ if and only if $uw^{-1}w_0 = w_0$ which implies $u = w$. Hence, by Lemma 3.32

$$\begin{aligned}\partial_{w_0 w}(g_u^*) &= \begin{cases} g^*_{uw^{-1}w_0} & \text{if } \ell(uw^{-1}w_0) = \ell(u) + \ell(w_0 w) \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases} g^*_{w_0} & \text{if } u = w \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Now we can compute that

$$\begin{aligned}\partial_{w_0}\left(g_w g_u^*\right) &= \partial_{w_0}\left(\partial_{w^{-1}w_0}(g_{w_0}) \cdot g_u^*\right) \\ &= \partial_{w_0}\left(g_{w_0} \cdot \partial_{w_0 w}\left(g_u^*\right)\right) \\ &= \partial_{w_0}\left(g_{w_0} \cdot \delta_{w,u} g^*_{w_0}\right) \\ &= \delta_{w,u} \cdot \partial_{w_0}\left(g_{w_0}\right) \\ &= \delta_{w,u}.\end{aligned}$$

Here we used (3.8) in the second line and Definition 3.26 in the last line. This finishes the proof. ☐

Now we have proven that we have a dual basis for the basis $\{g_w\}_{w \in W}$. However, we would like to have a dual basis for $\{g_w\}_{w \in W_J}$ which is our basis for R as an R_J-module. In order to get this we will forget our basis $\{g_w\}_{w \in W_J}$ and instead look at a slightly different basis. The advantage is that we can then generalize the previous result to the case where R is viewed as an R^J-module.

Theorem 3.35. *There is an R^J-basis $\{\tau_w\}_{w\in W_J}$ of R which is Demazure generated with the following property. The set $\{\tau_w^*\}_{w\in W_J}$ where $\tau_w^* = (-1)^{\ell(ww_J)} w_J(\tau_{ww_J})$ is also an R^J-basis of R and we have*

$$\partial_{w_J}(\tau_w \tau_u^*) = \delta_{w,u}$$

for all $w, u \in W_J$.

Proof. By Remark 3.29 we may consider R_1 instead of R. Recall the notation from Remark 3.9 where we had

$$R_1 \cong R_{e_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} R_{e_m}$$

as R_1^J-modules. Each R_{e_k} has a basis $\{\sigma_{u,e_k}\}_{u\in S_{e_k}}$ given by Schubert polynomials. Thus, R_1 has an R_1^J-basis given by

$$\{\sigma_{u_1,e_1} \otimes \cdots \otimes \sigma_{u_m,e_m}\}.$$

We define $\tau_u = \tau_{(u_1,\ldots,u_m)} = \sigma_{u_1,e_1} \otimes \cdots \otimes \sigma_{u_m,e_m}$ for $u \in W_J$ where we identified u with the tuple $(u_1,\ldots,u_m) \in S_{e_1} \times \cdots \times S_{e_m}$. Then $\{\tau_w\}_{w\in W_J}$ is an R_1^J-basis for R_1. Moreover, $\{\tau_w^*\}_{w\in W_J}$ is also an R_1^J-basis for R_1, since w_J is an R_1^J-linear isomorphism. Since for $1 \leqslant k \neq l \leqslant m$ the elements of S_{e_k} and S_{e_l} (viewed as elements of S_n) are distant from one another, we get that the simple reflections $s_j \in J$ act only on one factor R_{e_k}. Hence, if we consider a Demazure operator ∂_j for j corresponding to (w.l.o.g.) S_{e_1}, we compute

$$\begin{aligned}
\partial_j (r_1 \otimes r_2 \otimes \cdots \otimes r_m) &= \frac{r_1 \otimes r_2 \otimes \cdots \otimes r_m - s_j (r_1 \otimes r_2 \otimes \cdots \otimes r_m)}{\alpha_j} \\
&= \frac{r_1 \otimes r_2 \otimes \cdots \otimes r_m - s_j(r_1) \otimes r_2 \otimes \cdots \otimes r_m}{\alpha_j} \\
&= \frac{r_1 - s_j(r_1)}{\alpha_j} \otimes r_2 \otimes \cdots \otimes r_m \\
&= \partial_j(r_1) \otimes r_2 \otimes \cdots \otimes r_m.
\end{aligned}$$

Hence, Demazure operators ∂_{u_k} for $u_k \in S_{e_k}$ acting on R_1 can be viewed as just acting on R_{e_k}. Moreover, since $u_k u_l = u_l u_k$ for $u_k \in S_{e_k}$ and $u_l \in S_{e_l}$ viewed as elements of S_n we have that ∂_{u_k} and ∂_{u_l} commute with each other if we let them act on R_1. Thus, if we write $u = (u_1,\ldots,u_m) \in W_J = S_{e_1} \times \cdots \times S_{e_m}$ we get that

$$\partial_u = \partial_{u_1} \otimes \cdots \otimes \partial_{u_m}.$$

From this we get that $\{\tau_w\}_{w\in W_J}$ is Demazure generated, as for $w = (w_1,\ldots,w_m), u = (u_1,\ldots,u_m) \in W_J = S_{e_1} \times \cdots \times S_{e_m}$ we have

$$
\begin{aligned}
\partial_u(\tau_w) &= (\partial_{u_1} \otimes \cdots \otimes \partial_{u_m})\,(\sigma_{w_1} \otimes \cdots \otimes \sigma_{w_m}) \\
&= \partial_{u_1}(\sigma_{w_1}) \otimes \cdots \otimes \partial_{u_m}(\sigma_{w_m}) \\
&= \begin{cases} \sigma_{w_1 u_1^{-1}} \otimes \cdots \otimes \sigma_{w_m u_m^{-1}} & \text{if } \ell(w_k u_k^{-1}) = \ell(w_k) - \ell(u_k) \text{ for all } 1 \leqslant k \leqslant m \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \tau_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Moreover, since we have $w_J = (w_{e_1}, \ldots, w_{e_m})$ by Remark 2.17 we get that (for $u = (u_1, \ldots, u_m)$)

$$
\begin{aligned}
\tau_u^* &= (-1)^{\ell(uw_J)} w_J(\tau_{uw_J}) \\
&= (-1)^{\ell\left(u_1 w_{e_1}, \ldots, u_m w_{e_m}\right)} \cdot (w_{e_1}, \ldots, w_{e_m}) \left(\sigma_{u_1 w_{e_1}} \otimes \cdots \otimes \sigma_{u_m w_{e_m}}\right) \\
&= (-1)^{\ell(u_1 w_{e_1}) + \cdots + \ell(u_m w_{e_m})} \cdot w_{e_1}(\sigma_{u_1 w_{e_1}}) \otimes \cdots \otimes w_{e_m}(\sigma_{u_m w_{e_m}}) \\
&= \sigma_{u_1}^* \otimes \cdots \otimes \sigma_{u_m}^*.
\end{aligned}
$$

Hence, we compute that

$$
\begin{aligned}
\partial_{w_J}\left(\tau_w \cdot \tau_u^*\right) &= \left(\partial_{w_{e_1}} \otimes \cdots \otimes \partial_{w_{e_m}}\right) \left((\sigma_{w_1} \otimes \cdots \otimes \sigma_{w_m}) \cdot \left(\sigma_{u_1}^* \otimes \cdots \otimes \sigma_{u_m}^*\right)\right) \\
&= \left(\partial_{w_{e_1}} \otimes \cdots \otimes \partial_{w_{e_m}}\right) \left(\sigma_{w_1} \cdot \sigma_{u_1}^* \otimes \cdots \otimes \sigma_{w_m} \cdot \sigma_{u_m}^*\right) \\
&= \partial_{w_{e_1}}\left(\sigma_{w_1} \cdot \sigma_{u_1}^*\right) \otimes \cdots \otimes \partial_{w_{e_m}}\left(\sigma_{w_m} \cdot \sigma_{u_m}^*\right) \\
&= \delta_{w_1,u_1} \otimes \cdots \otimes \delta_{w_m,u_m} = \delta_{w,u}
\end{aligned}
$$

for $w = (w_1, \ldots, w_m), u = (u_1, \ldots, u_m) \in W_J = S_{e_1} \times \cdots \times S_{e_m}$. This finishes the proof. □

3.3 (Regular) Soergel bimodules

Now we are set up to define Soergel bimodules. For $i \in S$, let $B_i = R \otimes_{R^i} R\langle 1\rangle$. From now on we will denote by $\otimes$ with no index the tensor product over R. If we consider tensor products of B_i's we will often omit the $\otimes$. Given a sequence $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_d}$ the corresponding *Bott–Samelson bimodule* is the tensor product

$$
B_{\underline{w}} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_d} = B_{i_1} B_{i_2} \cdots B_{i_d}
$$

viewed as an (R, R)-bimodule via left and right multiplication. Note that $B_{\underline{w}}$ is isomorphic to $R \otimes_{R^{i_1}} R \otimes_{R^{i_2}} R \otimes_{R^{i_3}} \cdots \otimes_{R^{i_d}} R\langle d\rangle$.

Definition 3.36. We define $\mathbb{BS}$Bim to be the full monoidal subcategory of graded (R, R)-bimodules whose objects are Bott–Samelson bimodules and all their grading shifts.
Now we define $\mathbb{S}$Bim to be the Karoubi envelope of the additive closure of $\mathbb{BS}$Bim. $\mathbb{S}$Bim is called the *category of Soergel bimodules*. Note that $\mathbb{S}$Bim is additive but not abelian. ◊

For the next theorem to hold we need some assumptions on the realization. This is what was discussed in Remark 3.2. The theorem was proven by Soergel [Soe07, Satz 6.14].

Theorem 3.37. *There is a* 1*-to-*1 *correspondence*

$$\begin{aligned} W &\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{indecomposable Soergel bimodules} \\ \textit{up to isomorphism and grading shift} \end{array} \right\} \\ w &\longmapsto B_w. \end{aligned}$$

Here B_w is determined by being the only summand of $B_{\underline{w}}$, where $\underline{w} = s_{i_1} \dots s_{i_d}$ is a reduced expression for w, which is not a summand of (some shift of) $B_{\underline{y}}$ for any shorter sequence $\underline{y}$.

Remark 3.38. One could construct B_w by finding all summands of $B_{\underline{w}}$ which occur as shifts of summands of lower terms, removing them, and seeing what remains. The theorem implies that B_w is uniquely determined as being a direct summand for all $B_{\underline{w}}$ where $\underline{w} = s_{i_1} \dots s_{i_d}$ is a reduced expression for w. $\Diamond$

Theorem 3.39 (Categorification Theorem)**.** *Let $\mathfrak{h}$ a realization of W that behaves well, then there is a unique isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras:*

$$\begin{aligned} \varepsilon : \mathcal{H} &\longrightarrow [\mathbb{S}\mathrm{Bim}] \\ \underline{H}_i &\longmapsto [B_i], \end{aligned}$$

where $[\mathbb{S}\mathrm{Bim}]$ denotes the split Grothendieck group of $\mathbb{S}\mathrm{Bim}$. $[\mathbb{S}\mathrm{Bim}]$ becomes a $\mathbb{Z}\left[v, v^{-1}\right]$-algebra via $v \cdot [M] = [M\langle 1 \rangle]$.
Given two Soergel bimodules B and B' the graded rank of $\mathrm{Hom}_{(R,R)}(B, B')$ as a free left (or right) R-module is given by $\left(\varepsilon^{-1}\left([B]\right), \varepsilon^{-1}\left([B']\right)\right)$, where $(-,-)$ denotes the standard pairing in $\mathcal{H}$.

This Categorification Theorem goes back to Soergel [Soe07, Theorem 5.3]. He conjectured that if $\mathrm{char}(\Bbbk) = 0$, then $\varepsilon^{-1}\left([B_w]\right) = \underline{H}_w$. Soergel was able to prove this conjecture in particular for $W = S_n$ [Soe92]. The general case was established by Elias and Williamson [EW14].

3.4 Singular Soergel bimodules

We will present some results of Williamson [Wil11] in this section. This includes the definition of singular Soergel bimodules. The category of singular Soergel bimodules is a 2-category and we would like to view it in this context. In order to do that we will now first give the definition and for us most important example of 2-categories.

Definition 3.40. A (strict) 2-category $\mathcal{C}$ consists of the following data.

1. A set of objects $\mathrm{ob}(\mathcal{C})$.

2. For each pair of objects $x, y \in \text{ob}(\mathcal{C})$ a category $\text{Mor}_{\mathcal{C}}(x, y)$. The objects of $\text{Mor}_{\mathcal{C}}(x, y)$ are called 1-morphisms and will be denoted $M : x \longrightarrow y$. The morphisms between these 1-morphisms are called 2-morphisms and will be denoted $f : M \longrightarrow N$. The composition of 2-morphisms will be called *vertical composition* and will be denoted $f \circ g$ for $f : N \longrightarrow L, g : M \longrightarrow N$.

3. For each triple $x, y, z \in \text{ob}(\mathcal{C})$ a functor

$$\star : \text{Mor}_{\mathcal{C}}(y, z) \times \text{Mor}_{\mathcal{C}}(x, y) \longrightarrow \text{Mor}_{\mathcal{C}}(x, z).$$

The image of a pair of 1-morphisms (M, N) on the left hand side will be called the *composition* of M and N and denoted $M \star N$. The image of a pair of 2-morphisms (f, g) will be called *horizontal composition* and denoted $f \star g$.

These data are to satisfy the following conditions:

1. The set of objects together with the set of 1-morphisms endowed with the composition of 1-morphisms forms a category.

2. Horizontal composition of 2-morphisms is associative.

3. The identity 2-morphism id_{id_x} of the identity 1-morphism id_x is a unit for horizontal composition. ♢

Example 3.41. The most important example for us will be Bim.

objects:	rings R
1-morphisms:	bimodules
2-morphisms:	bimodule morphisms

This means that $\text{Mor}_{\text{Bim}}(R, S)$ is the category of (R, S)-bimodules. The horizontal composition is given by tensor products, i.e. ${}_S M_R \circ {}_P N_S = N \otimes_S M$ (here this notation means that $M \in \text{Mor}_{\text{Bim}}(R, S)$ for instance). The vertical composition is just the usual composition of bimodule morphisms.
Warning! This 2-category is not strict (i.e. identities only hold up to coherent isomorphisms). One calls such 2-categories weak 2-categories or bicategories.
All 2-categories that we will consider are subcategories of Bim. This means that the objects will be some set of rings and the categories $\text{Mor}(R, S)$ will be subcategories of $\text{Mor}_{\text{Bim}}(R, S)$.

For a more detailed introduction to 2-categories suitable for our purposes, see e.g. [Str20b].

Definition 3.42. Let $I, J \subset S$ be finitary parabolic subsets. We define the category ${}_I\mathbb{BS}\text{Bim}_J$ to be the full subcategory of (R^I, R^J)-bimodules that contains all Bott–Samelson bimodules (see Definition 3.36) viewed as (R^I, R^J)-bimodules by restricting the left and right action of R. ♢

Definition 3.43. We define the category ${}_I\text{s}\mathbb{BS}\text{Bim}_J$ to be the full subcategory of (R^I, R^J)-bimodules that contains all shifts of objects of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

where $I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset \cdots \subset J_{n-1} \supset I_n = J$ are finitary subsets of S. Objects of ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$ are called *singular Bott–Samelson bimodules.*
Finally, we define ${}_I\mathbb{S}\mathrm{Bim}_J$ to be Karoubi envelope of the additive closure of ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$ and call its objects *singular Soergel bimodules.* ♢

Remark 3.44. One can prove that every singular Bott–Samelson bimodule is a direct summand of some object in ${}_I\mathbb{BS}\mathrm{Bim}_J$. This mainly follows from the fact that R is free of finite rank over R^J for a finitary parabolic subset $J \subset S$ and another fact which we will come across in Section 4.3. This fact states that the objects $R \otimes_{R^J} R$ are direct summands of some Bott–Samelson bimodules.
Altogether we get that ${}_I\mathbb{S}\mathrm{Bim}_J$ is also the Karoubi envelope of the additive closure of ${}_I\mathbb{BS}\mathrm{Bim}_J$. ♢

Definition 3.45. In the following we define the 2-category of singular Bott–Samelson bimodules $\mathrm{s}\mathbb{BS}\mathrm{Bim}$. Objects are finitary parabolic subsets $I \subseteq S$. The categories $\mathrm{Mor}_{\mathrm{s}\mathbb{BS}\mathrm{Bim}}(I, J)$ are given by ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$.
The 2-category of singular Soergel bimodules $\mathbb{S}\mathrm{Bim}$ (note that we abused notation here) is defined similarly. Objects are finitary parabolic subsets $I \subseteq S$. The categories $\mathrm{Mor}_{\mathbb{S}\mathrm{Bim}}(I, J)$ are given by ${}_I\mathbb{S}\mathrm{Bim}_J$. The composition of 1-morphisms and the horizontal composition of 2-morphisms are induced from Bim. ♢

The following result is based on works of Soergel [Soe92] and Stroppel [Str04]. A proof as well as a detailed discussion can be found in [Wil11, Theorem 7.4.2].

Theorem 3.46. *There is a bijection*

$$W_I \backslash W / W_J \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{indecomposable bimodules in } {}_I\mathbb{S}\mathrm{Bim}_J \\ \textit{(up to grading shifts).} \end{array} \right\}$$

Remark 3.47. We want to give a small indication how to find these indecomposable bimodules. For a double coset $p \in W_I \backslash W / W_J$ choose an element $w \in p$ and fix a reduced expression $w = s_{i_1} \cdots s_{i_d}$. Then the indecomposable bimodule corresponding to p is a direct summand of $B_{i_1} \cdots B_{i_d}$. Note that this bimodule is actually an (R, R)-bimodule, but we can view it as an (R^I, R^J)-bimodule via restricting the actions.
The next lemma will give a little justification why the indecomposable bimodules are corresponding to double cosets and not just elements of W. ♢

Lemma 3.48. *Let $p \in W_I \backslash W / W_J$ and let $w \in p$. By Theorem 2.24 there are $u \in W_I$ and $v \in W_J$ such that $w = up_-v$ and $\ell(w) = \ell(u) + \ell(p_-) + \ell(v)$. Here p_- denotes the unique shortest element in p. Now by picking reduced expressions*

$$u = s_{j_1} \cdots s_{j_e}, p_- = s_{i_1} \cdots s_{i_d}, v = s_{l_1} \cdots s_{l_f}$$

we get a reduced expression $w = s_{j_1} \cdots s_{j_e} s_{i_1} \cdots s_{i_d} s_{l_1} \cdots s_{l_f}$. Write

$$\begin{aligned} B_{\underline{w}} &= B_{j_1} \cdots B_{j_e} B_{i_1} \cdots B_{i_d} B_{l_1} \cdots B_{l_f} \\ B_{\underline{p_-}} &= B_{i_1} \cdots B_{i_d} \end{aligned}$$

viewed as (R^I, R^J)-bimodules. Then

$$B_{\underline{w}} \cong \bigoplus_{k=0}^{e+f} \left(B_{\underline{p_-}}\langle e+f-2k\rangle\right)^{\oplus\binom{e+f}{k}}$$

as (R^I, R^J)-bimodules.

Proof. Claim:

$$B_{j_1}\cdots B_{j_e} \cong \bigoplus_{k=0}^{e} \left(R\langle e-2k\rangle\right)^{\oplus\binom{e}{k}} \qquad \text{as } (R^I, R)\text{-bimodule,}$$

$$B_{l_1}\cdots B_{l_f} \cong \bigoplus_{k=0}^{f} \left(R\langle f-2k\rangle\right)^{\oplus\binom{f}{k}} \qquad \text{as } (R, R^J)\text{-bimodule.}$$

Using this claim we can conclude the lemma, because $B_{\underline{w}}$ can be decomposed as a direct sum of copies with certain shifts of $R \otimes B_{p_-} \otimes R$. Explicitly using

$$\binom{e+f}{k} = \sum_{\substack{k_1, k_2 = 0 \\ k_1 + k_2 = k}}^{k} \binom{e}{k_1}\cdot\binom{f}{k_2}.$$

The proof then goes as follows.

$$\begin{aligned}
B_{\underline{w}} &= B_{j_1}\cdots B_{j_e} B_{i_1}\cdots B_{i_d} B_{l_1}\cdots B_{l_f} \\
&\cong \left(\bigoplus_{k=0}^{e} \left(R\langle e-2k\rangle\right)^{\oplus\binom{e}{k}}\right) \otimes B_{\underline{p_-}} \otimes \left(\bigoplus_{k=0}^{f} \left(R\langle f-2k\rangle\right)^{\oplus\binom{f}{k}}\right) \\
&\cong \bigoplus_{k_1=0}^{e}\bigoplus_{k_2=0}^{f} \left(B_{\underline{p_-}}\langle e+f-2k_1-2k_2\rangle\right)^{\oplus\binom{e}{k_1}\cdot\binom{f}{k_2}} \\
&\cong \bigoplus_{k=0}^{e+f} \left(B_{\underline{p_-}}\langle e+f-2k\rangle\right)^{\oplus\binom{e+f}{k}}.
\end{aligned}$$

We now prove the claim. It suffices to do this for the first isomorphism as the second proof is completely analogous. We do induction on e. For $e = 0$ there is nothing to do. For $e = 1$ we have by Remark 3.27 and Theorem 3.28

$$B_{j_1} = R \otimes_{R^{j_1}} R\langle 1\rangle \cong \left(R^{j_1} \oplus R^{j_1}\langle -2\rangle\right) \otimes_{R^{j_1}} R\langle 1\rangle \cong R\langle 1\rangle \oplus R\langle -1\rangle$$

as (R^I, R)-bimodules. Then we get by using first the case $e = 1$ and then applying induction the following isomorphism of (R^I, R)-bimodules (which is basically going from one row in Pascal's triangle to the next one)

$$
\begin{aligned}
B_{j_1}\cdots B_{j_e} &\cong (R\langle 1\rangle \oplus R\langle -1\rangle)\otimes B_{j_2}\cdots B_{j_e}\\
&\cong B_{j_2}\cdots B_{j_e}\langle 1\rangle \oplus B_{j_2}\cdots B_{j_e}\langle -1\rangle\\
&\cong \left(\bigoplus_{k=0}^{e-1}(R\langle e-1-2k\rangle)^{\oplus\binom{e-1}{k}}\right)\langle 1\rangle \oplus \left(\bigoplus_{k=0}^{e-1}(R\langle e-1-2k\rangle)^{\oplus\binom{e-1}{k}}\right)\langle -1\rangle\\
&\cong \left(\bigoplus_{k=0}^{e-1}(R\langle e-2k\rangle)^{\oplus\binom{e-1}{k}}\right) \oplus \left(\bigoplus_{k=0}^{e-1}(R\langle e-2-2k\rangle)^{\oplus\binom{e-1}{k}}\right)\\
&\cong R\langle e\rangle \oplus \left(\bigoplus_{k=1}^{e-1}(R\langle e-2k\rangle)^{\oplus\binom{e-1}{k}+\binom{e-1}{k+1}}\right)\oplus R\langle -e\rangle\\
&\cong \bigoplus_{k=0}^{e}(R\langle e-2k\rangle)^{\oplus\binom{e}{k}}\,.
\end{aligned}
$$

This finishes the proof. □

4 Soergel diagrammatics

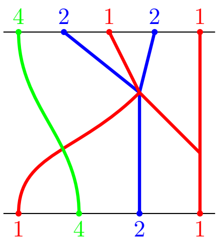
4.1 Soergel diagrammatics for S_n

In this section we consider the Coxeter system $(W, S) = (S_n, \{\text{simple transpositions}\})$. We label the elements of S with integers $1, \dots, n-1$ where i corresponds to the simple transposition $s_i = (i, i+1)$. Elias and Khovanov [EK10a] develop a diagrammatic presentation of a strictification of the monoidal category of Soergel bimodules $\mathbb{S}\mathrm{Bim}$ for S_n. We will revisit this presentation, since it is the foundation on which further diagrammatics in this thesis is based on. The main goal of this section is to define a diagrammatic category $\mathcal{D}$ and explain the following result from [EK10a] which says

Theorem. *There is a functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathbb{S}\mathrm{Bim}$ which is an equivalence of monoidal categories.*

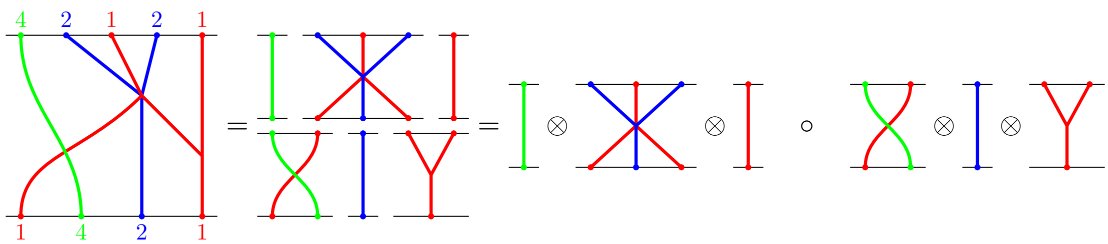
This will be done by defining an equivalence of monoidal categories $\mathcal{F}_1 : \mathcal{D}_1 \longrightarrow \mathbb{BS}\mathrm{Bim}$ and then extending it abstractly to the Karoubian closure $\mathcal{F} : \mathcal{D} = \mathrm{Kar}(\mathcal{D}_1) \longrightarrow \mathrm{Kar}(\mathbb{BS}\mathrm{Bim}) = \mathbb{S}\mathrm{Bim}$.

Before we go into the abstract definition of $\mathcal{D}_1$ we would like to give some insights on what the result will be. The objects in $\mathcal{D}_1$ will be sequences $\underline{i} = (i_1, \dots, i_d)$ for $i_j \in S$. They will later correspond to the bimodule $B_{\underline{i}} = B_{i_1} \otimes \cdots \otimes B_{i_d}$. A morphism could for example be given by the following picture.



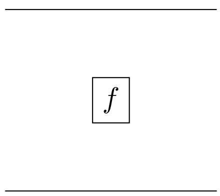
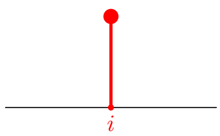
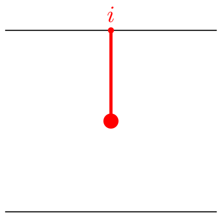
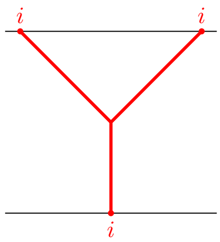
This would correspond to a morphism from $B_1 \otimes B_4 \otimes B_2 \otimes B_1$ to $B_4 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1$.

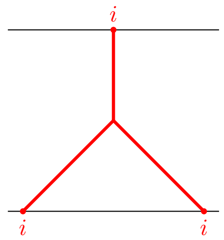
Glueing pictures vertically is interpreted as the composition of the corresponding morphisms. Glueing pictures horizontally is interpreted as the tensor product of the corresponding morphisms. This allows us to "build" each morphism out of small blocks. In the example this looks as following.



We will now give the definition for $\mathcal{D}_1$. Since our goal is the equivalence to $\mathbb{S}\mathrm{Bim}$ we will already write the corresponding morphisms in $\mathbb{S}\mathrm{Bim}$ to some of the morphisms we are about to define. This is technically not part of the definition, but it is nice to have everything at one place.

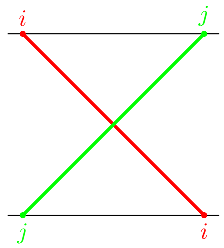
Definition 4.1. We construct a monoidal category $\mathcal{D}_1$ by generators and relations. It is generated on objects by S. This means that objects are sequences of indices $\underline{i} = (i_1, \dots, i_d)$ for $i_j \in S$. We visualize them as points on the real line $\mathbb{R}$, labelled or "coloured" by the indices from left to right.
On morphisms $\mathcal{D}_1$ is generated by the following generating morphisms modulo the relations (4.1) to (4.19).

	polynomial generator deg = deg(f) ($f \in R$ homogeneous)	$R \longrightarrow R$ $r \longmapsto f \cdot r$
	(end)dot deg = 1	$B_i \longrightarrow R$ $r_1 \otimes r_2 \longmapsto r_1 r_2$
	(start)dot deg = 1	$R \longrightarrow B_i$ $r \longmapsto \frac{r}{2} \cdot (\alpha_i \otimes 1 + 1 \otimes \alpha_i)$
	trivalent vertex (split) deg = −1	$B_i \longrightarrow B_i B_i$ $r_1 \otimes r_2 \longmapsto r_1 \otimes 1 \otimes r_2$



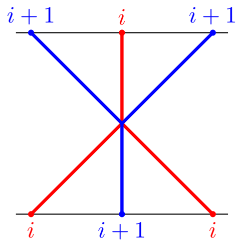
trivalent vertex
(merge)
deg = −1

$$B_iB_i \longrightarrow B_i$$
$$r_1 \otimes r_2 \otimes r_3 \longmapsto r_1\partial_i(r_2) \otimes r_3$$



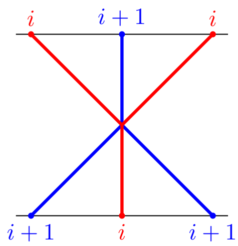
4-valent vertex
deg = 0
$(|\, i-j\, |> 1)$

$$B_jB_i \longrightarrow B_iB_j$$
$$r_1 \otimes 1 \otimes r_2 \longmapsto r_1 \otimes 1 \otimes r_2$$



6-valent vertex
deg = 0

$$B_iB_{i+1}B_i \longrightarrow B_{i+1}B_iB_{i+1}$$
$$1 \otimes 1 \otimes 1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1 \otimes 1$$
$$\begin{aligned} 1 \otimes x_i \otimes 1 \otimes 1 \longmapsto\ & x_i \otimes 1 \otimes 1 \otimes 1 \\ & + x_{i+1} \otimes 1 \otimes 1 \otimes 1 \\ & - 1 \otimes 1 \otimes 1 \otimes x_{i+2} \end{aligned}$$

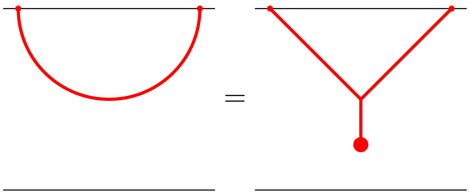


6-valent vertex
deg = 0

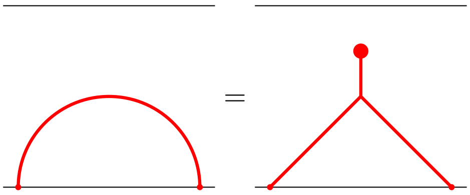
$$B_{i+1}B_iB_{i+1} \longrightarrow B_iB_{i+1}B_i$$
$$1 \otimes 1 \otimes 1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1 \otimes 1$$
$$\begin{aligned} 1 \otimes x_{i+2} \otimes 1 \otimes 1 \longmapsto\ & 1 \otimes 1 \otimes 1 \otimes x_{i+1} \\ & + 1 \otimes 1 \otimes 1 \otimes x_{i+2} \\ & - x_i \otimes 1 \otimes 1 \otimes 1 \end{aligned}$$

Thus, a morphism from $\underline{i}$ to $\underline{j}$ in $\mathcal{D}_1$ is given by a $\Bbbk$-linear sum of pictures embedded in the strip $\mathbb{R} \times [0, 1]$. The points in the line $\mathbb{R} \times \{0\}$ correspond to $\underline{i}$ and the points on the line $\mathbb{R} \times \{1\}$ correspond to $\underline{j}$. In-between are coloured graphs which are constructed by glueing the above generating morphisms (horizontally and vertically). $\Diamond$

Before we give the complete list of relations, we will discuss some abbreviations we will make. First, we will stop labelling the points on the boundary with explicit indices. Instead there will just be different colours that represent different indices. Often we will put some restrictions on the adjacency of colours. For example we could have introduced both 6-valent vertices together as just one of the pictures without explicit labels on the boundary by restricting the colours to being adjacent (however we wanted to state the corresponding bimodule morphism which slightly differs for the two types of 6-valent vertices).
Secondly, we need to define two abbreviating morphisms in order to state all relations.



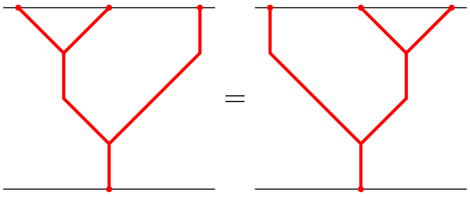
cup
deg = 0



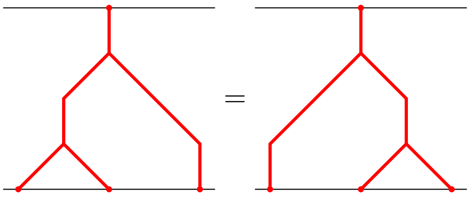
cap
deg = 0

Relations

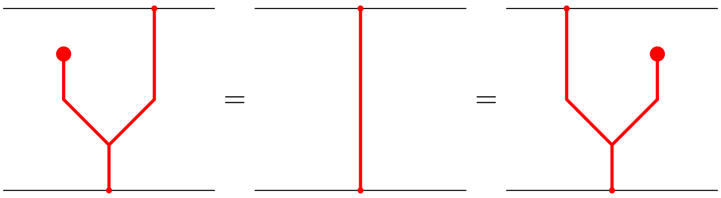
Now we give the complete list of relations. We will start with the *Frobenius relations*.



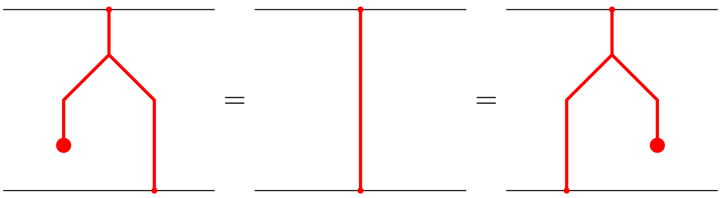
coassociativity of split (4.1)



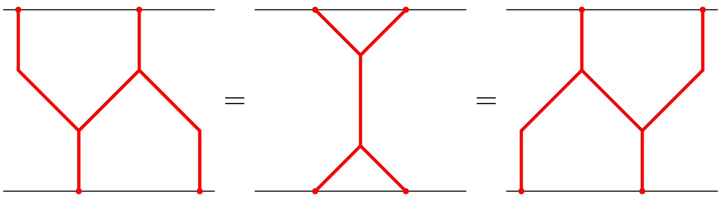
associativity of merge (4.2)



counit (4.3)

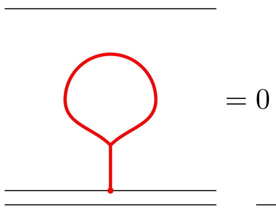


unit (4.4)

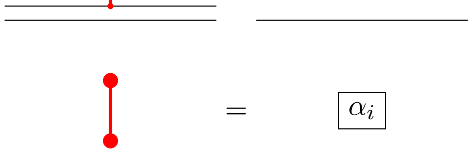


associativity (Frobenius condition) (4.5)

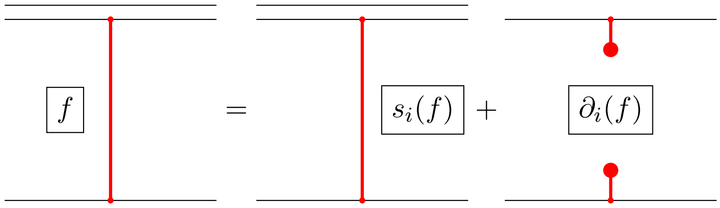
We will continue with the last *one-colour relations* that we need.



(4.6)



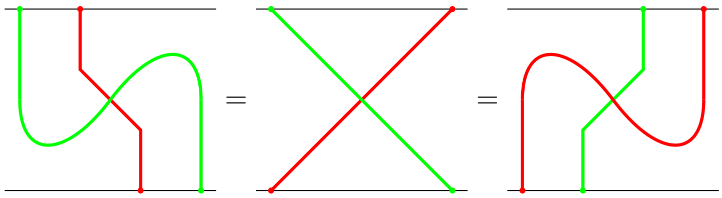
(4.7)



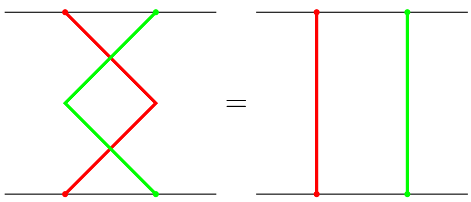
(4.8)

Remark 4.2. Relation (4.7) tells us that we can write every polynomial as $\Bbbk$-linear combination of many double dots (this is what we call the left side of (4.7)). Thus, the polynomial generator is actually not needed. We decided to include it anyway because it gives us a canonical way to give the morphism spaces the structure of an (R, R)-bimodule. In this way we can easily understand how the double dots are used for this which is an advantage. The disadvantage is that the pictures now contain these polynomials instead of just colourful graphs. ◊

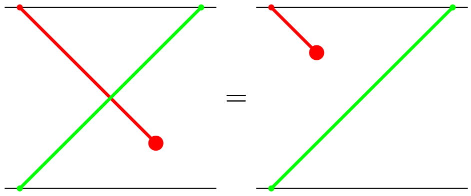
We continue with *multiple colour relations*. In the next relations red and green are distant, i.e. the corresponding simple transpositions $(i, i+1)$ and $(j, j+1)$ satisfy $|i-j| > 1$ (otherwise we call the colours adjacent).



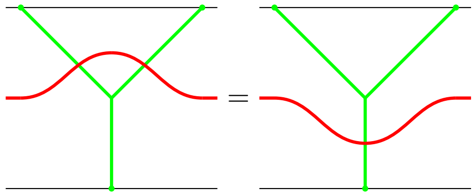
(4.9)



(4.10)

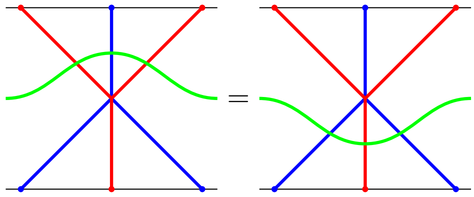


(4.11)



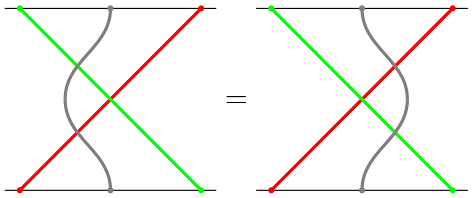
(4.12)

In the next relation red and blue are adjacent and green is distant from both of them.



(4.13)

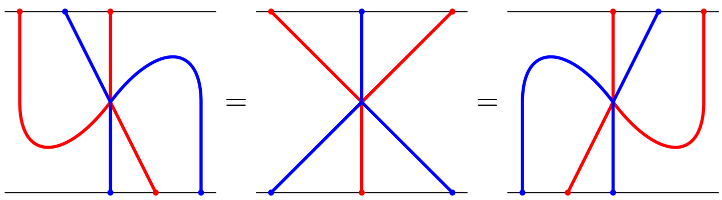
In the next relation all three colours are mutually distant.



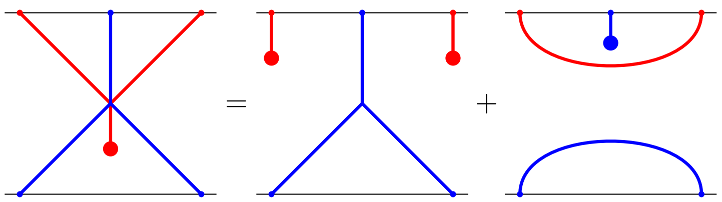
(4.14)

Remark 4.3. Relations (4.10) to (4.14) indicate that any part of the graph labelled i and any part labelled j for i and j distant do not interact with each other. This means we can slide the j-coloured part past the i-coloured part and it will not change the morphism. We call this *distant sliding property*. ♢

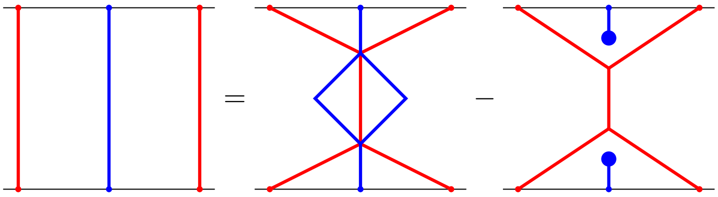
In the next relations red and blue are adjacent.



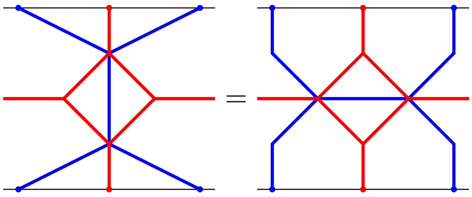
(4.15)



(4.16)

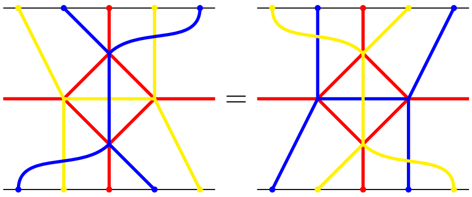


(4.17)



(4.18)

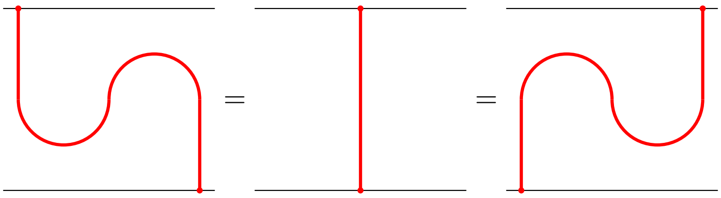
In the next relation the three colours have the same adjacency as $\{1, 2, 3\}$ (where red corresponds to 2).



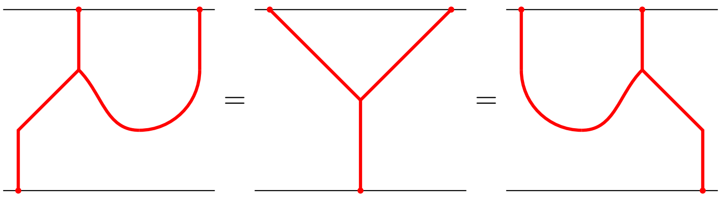
(4.19)

This concludes the list of relations for $\mathcal{D}_1$.

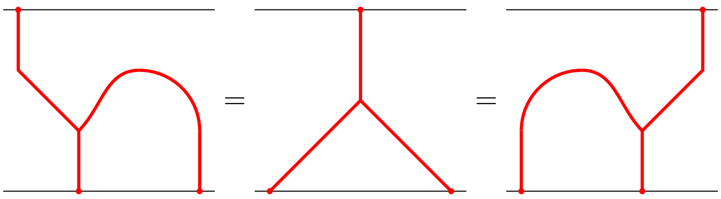
Remark 4.4. In some of the relations there are horizontal lines and lines which end neither in bottom or top. We will now explain how to interpret them.
Relations (4.1) to (4.5) turn the object i in $\mathcal{D}_1$ into a Frobenius object. They also imply some other relations which are quite useful and will help us to understand these horizontal lines. Therefore, we will also state them here.



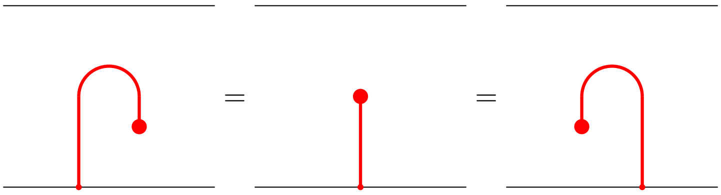
biadjunction (4.20)

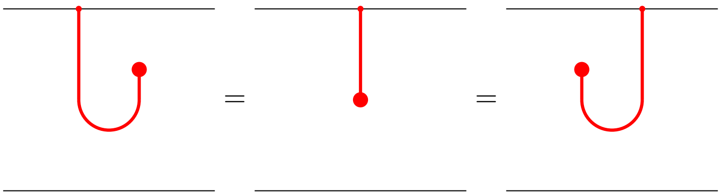


(4.21)

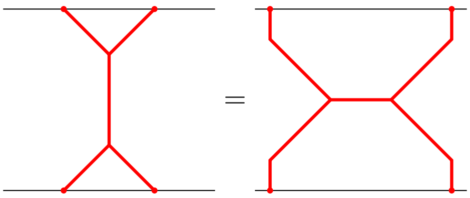


(4.22)

 (4.23)

 (4.24)

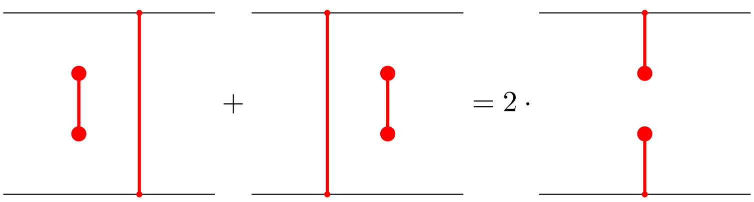
Now relations (4.20) to (4.24), (4.9) and (4.15) imply that the morphism specified by a particular graph embedding is independent of the isotopy class of the embedding. They are called *cyclicity relations*.
This is the reason for the usage of horizontal lines. They can be interpreted as either going up or going down (they just have to do the same on both sides of the equation). In this way one picture can encode many different morphisms. It is just a shortcut notation. For example we could rewrite (4.5) to

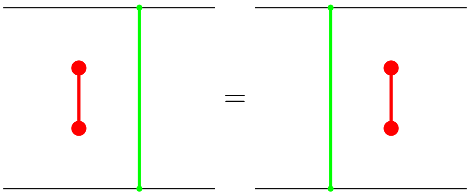
 (4.5)

which is a bit shorter and encodes even more information (think for example of the horizontal line as a cup or a cap). ♢

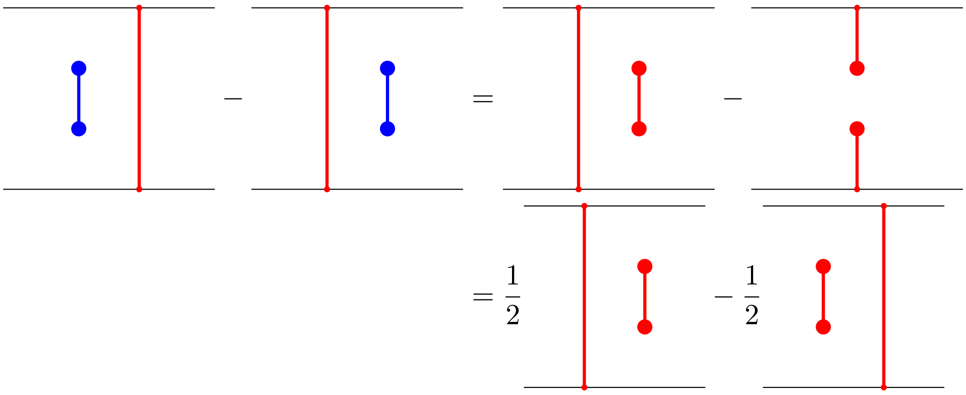
Remark 4.5 (Warning!)**.** The list of relations is not minimal. For instance (4.10) can be proven using the other relations. However, it is often to have a variety of relations to work with, since it makes it easier to simplify expressions and to prove things with these relations. That is why we included more relations than one actually needs. ♢

Remark 4.6. Since one can use double dots to write polynomials we can look at some consequences of (4.8) where we replace polynomials by double dots. In these relations red and green are distant while red and blue are adjacent.

 (4.25)



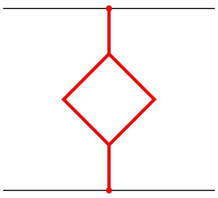
(4.26)



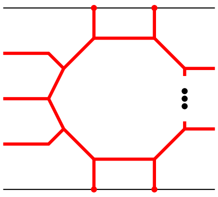
(4.27)

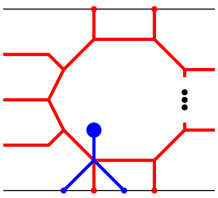
In the second equality of (4.27) one applies (4.25). ♢

Remark 4.7. There is a slight generalization of relation (4.6) which looks as follows.

 $= 0$ (4.28)

We can generalize this relation to get the following two relations (where red and blue are adjacent).

 $= 0$ (4.29)

 $= 0$ (4.30)

These relations tell us that if there is an empty area which is surrounded by lines of one colour (up to some dots) then the morphism is already zero. ♢

Definition 4.8. Note that $\mathcal{D}_1$ is a graded category (we have degrees for the morphisms). Let $\mathcal{D}_1'$ be the corresponding $\Bbbk$-linear category with free $\mathbb{Z}$-action (via Theorem 2.56). Then we define $\mathcal{D}_2$ to be the closure of $\mathcal{D}_1'$ under finite direct sums.
The category $\mathcal{D}$ then is the Karoubi envelope of $\mathcal{D}_2$. Thus, $\mathcal{D}$ is the closure of $\mathcal{D}_1'$ under finite direct sums and taking direct summands. $\Diamond$

Definition 4.9. We define the monoidal functor $\mathcal{F}_1 : \mathcal{D}_1 \longrightarrow \mathbb{BSB}\mathrm{im}$ on objects by sending i to B_i and on morphisms via the bimodule morphisms we associated to our generating morphisms in Definition 4.1.
The functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathbb{S}\mathrm{Bim}$ is the functor which is induced from $\mathcal{F}_1$ after taking the additive closure and the Karoubi envelope on both sides. $\Diamond$

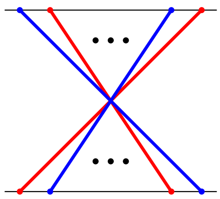
The following is one of the main results in [EK10a] and also the main theorem of this section.

Theorem 4.10. *The functors $\mathcal{F}_1$ and $\mathcal{F}$ are equivalences of monoidal categories.*

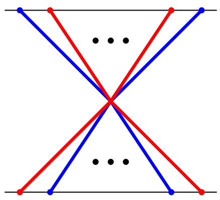
4.2 The general case

In this section we will see how to generalize this diagrammatic presentation to more general Coxeter systems (W, S). This was done by Elias and Williamson [EW16] and we will only present their results. We need to put some assumptions on (W, S) and $\Bbbk$ for this to work. First there needs to be a realization of (W, S) over $\Bbbk$ in order to define $\mathbb{S}\mathrm{Bim}$ and then we need to put a few assumptions on this realization in order for $\mathbb{S}\mathrm{Bim}$ to behave well. For details we refer the reader to [EW16, Section 3].
Now we can define a diagrammatic category $\mathcal{D}_1$ in the same way as in the last section and then the analogous of Definitions 4.8 and 4.9 and Theorem 4.10 hold. So we will just say what kind of generators and what kind of relations we need.

Definition 4.11. The generators will consist out of the one-colour generators that we already know: The two dots and the two trivalent vertices as well as the polynomial generator. The last generators are two-colour generators, namely for each ordered pair $(i, j) \in S^2$ we have the $(2m_{ij})$-valent vertex.



m_{ij} even

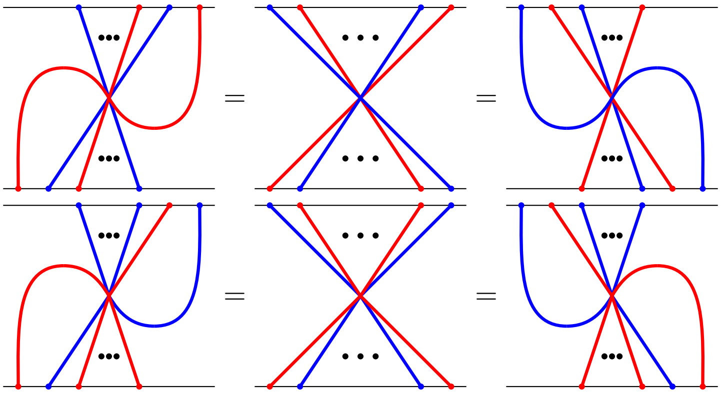


m_{ij} odd

Each of the bimodules $B_i \otimes B_j \otimes B_i \otimes \cdots \otimes B_j$ and $B_j \otimes B_i \otimes B_j \otimes \cdots \otimes B_i$ have the same indecomposable bimodule as a summand and this summand appears only once. The morphisms in $\mathbb{S}$Bim corresponding to these two generators are given by the projection to this summand composed with the inclusion of this summand into the other bimodule. $\Diamond$

The relations we require are all the one-colour relations that we have seen in the last section and then two more types of relations.

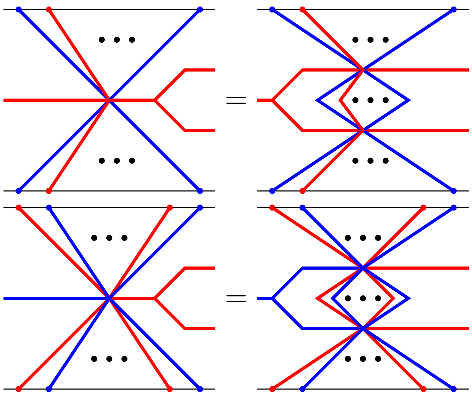
The first type of relations are the two-colour relations. We have three relations for each ordered pair $(i, j) \in S^2$. These relations depend again slightly on the parity of m_{ij}.



m_{ij} even

m_{ij} odd

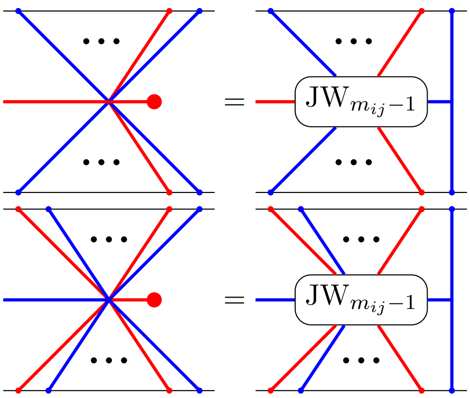
(4.31)



m_{ij} even

m_{ij} odd

(4.32)



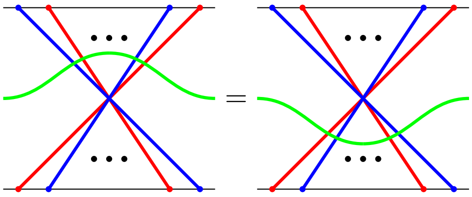
m_{ij} even

m_{ij} odd

(4.33)

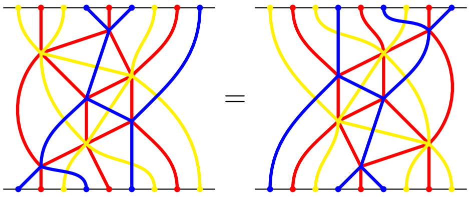
JW$_{m_{ij}-1}$ is the Jones–Wenzl morphism. It is a $\Bbbk$-linear combination of graphs constructed only out of dots and trivalent vertices. For more details we refer the reader to [EW16, Section 5.2].

The second type of relations are the three-colour relations. For a triplet forming a sub-Coxeter system of type $A_1 \times I_2(m)$, $m < \infty$, we have the following relation.



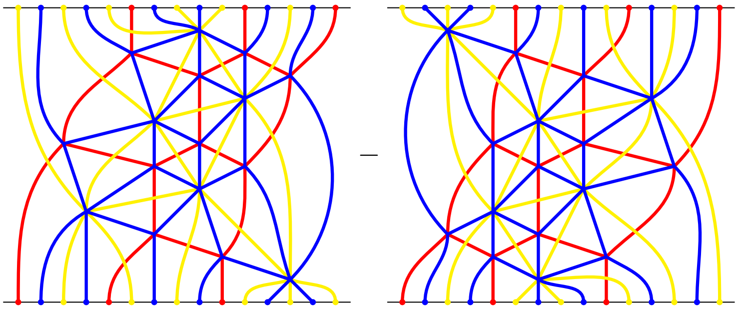
(4.34)

Then we have three relations corresponding to triplets forming sub-Coxeter systems of types A_3, B_3 and H_3. These relations are called *Zamolodzhikov relations*. For a motivation behind this name see [Str20b]. The relation for type A_3 is (4.19). The relation for type B_3 is the following.



(4.35)

The relation for type H_3 is quite complicated and was for a while not completely known. It looks as follows.



$=$ lower terms (4.36)

Here the "lower terms" on the right hand side are morphisms that vanish if we localize. These have been computed just recently. We will explain what localization means in the next remark.

Remark 4.12. Let Q be the quotient field of R. Let $\mathbb{BSB}\mathrm{im}_Q$ be the full monoidal subcategory of Q-bimodules generated by the bimodules $B_{i,Q} = Q \otimes_{Q^i} Q$. Let $\mathbb{S}\mathrm{Bim}_Q$ denote its Karoubi envelope. Then we have a faithful monoidal functor

$$\mathbb{S}\mathrm{Bim} \longrightarrow \mathbb{S}\mathrm{Bim}_Q$$

given by induction with Q on the right. This is called *localization*. For more details on this see [EW16, Section 3.6]. ♢

4.3 Thick lines

In this section we will give a diagrammatic presentation of the partial idempotent completion $g\mathbb{BSB}\mathrm{im}$ of $\mathbb{BSB}\mathrm{im}$ for $W = S_{n+1}$. This was done by Elias [Eli16, Chapter 4] and we will present his results here. First, we need to fix some terminology. For a more detailed presentation of the following have a look at [Eli16, Chapter 2].

Definition 4.13. Let J be a parabolic subset of S, the set of simple transpositions in $W = S_{n+1}$. We write d_J for length of the longest element w_J of W_J. We define $B_J = R \otimes_{R^J} R\langle d_J \rangle$.
For a sequence $\underline{\mathbf{J}} = J_1 J_2 \cdots J_r$ of parabolic subsets we let $B_{\underline{\mathbf{J}}} = B_{J_1} \cdots B_{J_r}$. These $B_{\underline{\mathbf{J}}}$ are called *generalized Bott–Samelson bimodules.* $\Diamond$

Lemma 4.14. *Let J be a parabolic subset of S. Let $w_J = s_{i_1} \cdots s_{i_r}$ be a reduced expression where $i_1, \dots, i_r \in J$. Then B_J is a direct summand of $B_{i_1} \otimes \cdots \otimes B_{i_r}$. Moreover, the inclusion $B_J \longrightarrow B_{i_1} \otimes \cdots \otimes B_{i_r}$ is given by $1 \otimes 1 \longmapsto 1 \otimes \cdots \otimes 1$.*

Proof. First consider the case $J = S$ and $W_J = W$. We know by Theorem 3.37 that there is a unique summand B_{w_0} of $B_{i_1} \otimes \cdots \otimes B_{i_r}$. One can prove that $B_{w_0} \cong B_S$ (see for instance [Str20b, Theorem II.3]). Thus, we are done in this case.
Now consider an arbitrary subset J of S. For (W, S) we used a realization $\mathfrak{h}$ to define Soergel bimodules. This vector space $\mathfrak{h}$ is also a realization for (W_J, J) with the induced action, because all the conditions of Definition 3.1 are still satisfied. Then, in the category of Soergel bimodules for (W_J, J), we get from the previous consideration that B_J is a direct summand of $B_{i_1} \otimes \cdots \otimes B_{i_r}$, because J is the maximal parabolic subset for (W_J, J). However, since the realization $\mathfrak{h}$ is the same for (W_J, J) and (W, S) we get the B_i and B_J in the category of Soergel bimodules for (W_J, J) are the same bimodules as they are in the category of Soergel bimodules for (W, S). This finishes the proof. $\square$

Definition 4.15. We define the category $g\mathbb{BS}\mathrm{Bim}$ to be the full subcategory of (R, R)-bimodules containing all grading shifts of the generalized Bott–Samelson bimodules $B_{\underline{\mathbf{J}}}$. This is a full monoidal graded subcategory of (R, R)-bimodules. By Lemma 4.14 this is also a full subcategory of $\mathbb{S}\mathrm{Bim}$. $\Diamond$

We will now define a category $g\mathcal{D}$ which is a partial idempotent completion of $\mathcal{D}_1$ and hence a full subcategory of $\mathcal{D}$. This means that we add some (not all) direct summands to $\mathcal{D}_1$. That is also what happens on the side of Soergel bimodules when transitioning from $\mathbb{BS}\mathrm{Bim}$ to $\mathrm{g}\mathbb{BS}\mathrm{Bim}$. We will then observe that the equivalence of categories $\mathcal{F} : \mathcal{D}_1 \longrightarrow \mathbb{BS}\mathrm{Bim}$ extends to an equivalence of categories $\mathcal{F} : g\mathcal{D} \longrightarrow g\mathbb{BS}\mathrm{Bim}$.

Definition 4.16. Let $\mathcal{C}$ be a full subcategory of some ambient module category. If $\mathcal{S}$ is a set of objects in the idempotent completion for $\mathcal{C}$ we define $\mathcal{C}(\mathcal{S})$ to be the full subcategory of the ambient module category whose objects are the objects of $\mathcal{C}$ as well as $\mathcal{S}$. We call this a *partial idempotent completion* of $\mathcal{C}$. If $\mathcal{S}$ consists of a single object M, we denote the partial idempotent completion by $\mathcal{C}(M)$. $\Diamond$

Definition 4.17. We call a collection of morphisms $\varphi_{\alpha,\beta} : X_\alpha \longrightarrow X_\beta$ in a category $\mathcal{C}$ satisfying $\varphi_{\alpha,\gamma} = \varphi_{\beta,\gamma}\varphi_{\alpha,\beta}$ a *consistent family of projectors.* $\Diamond$

Remark 4.18. Given a collection of morphisms $\{\varphi_{\alpha,\beta}\}$ we have that $\{\varphi_{\alpha,\beta}\}$ is a consistent family of projectors if and only if the corresponding objects X_α have a mutual summand M. The morphisms $\varphi_{\alpha,\beta} : X_\alpha \longrightarrow X_\beta$ are then given by the composition $X_\alpha \xrightarrow{\mathrm{p}_\alpha} M \xrightarrow{\mathrm{i}_\beta} X_\beta$ of projection and inclusion.

If we then assume that we have a presentation for $\mathcal{C}$ we can obtain a presentation for $\mathcal{C}(M)$ as follows. The generators will be the generators of $\mathcal{C}$ as well as the new morphisms $\mathrm{p}_\alpha : X_\alpha \longrightarrow M$ and $\mathrm{i}_\alpha : M \longrightarrow X_\alpha$. The relations will consist of those relations in $\mathcal{C}$ and the new relations $\mathrm{i}_\beta \, \mathrm{p}_\alpha = \varphi_{\alpha,\beta}$ and $\mathrm{p}_\alpha \, \mathrm{i}_\alpha = \mathrm{id}_M$. ♢

Definition 4.19. A parabolic subset J is *connected* if for every $i \notin J$ either $j \notin J$ for all $j < i$ or $j \notin J$ for all $j > i$. ♢

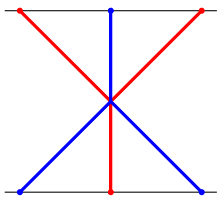
Proposition 4.20. *Let J be a connected parabolic subset. In $\mathcal{D}_1$ there is a family of morphisms $\phi_J = \{\phi_{\mathbf{x},\mathbf{y}}\}$ for each pair $(\mathbf{x}, \mathbf{y})$ of reduced expressions for w_J which satisfies the following three properties.*

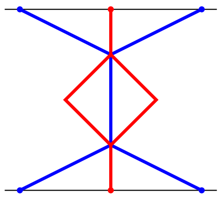
1. *The family ϕ_J is a consistent family of projectors, picking out a summand X.*
2. *The summand X satisfies $X \otimes i \cong X\langle 1\rangle \oplus X\langle -1\rangle$ for each $i \in J$.*
3. *The space $\mathrm{Hom}_{\mathcal{D}_1}(X, \emptyset)$ is a cyclic R-module, generated in degree d_J.*

Moreover, X is indecomposable, and is sent to the Soergel bimodule B_J by the functor $\mathcal{F}$.

Proof. [Eli16, Proposition 2.16, Theorem 3.18]. □

Example 4.21. Let $W = S_3$ and $S = \{s_1, s_2\}$. We consider the parabolic subset $S \subseteq S$. There are two reduced expression for the longest element of $W = W_S$, namely $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$. Let us write $\mathbf{x} = (s_1, s_2, s_1)$ and $\mathbf{y} = (s_2, s_1, s_2)$. Then ϕ_S is given by the following.

$\phi_{\mathbf{x},\mathbf{y}} =$ 

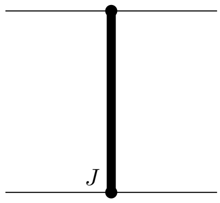
$\phi_{\mathbf{x},\mathbf{x}} =$ 

$\phi_{\mathbf{y},\mathbf{x}}$ and $\phi_{\mathbf{y},\mathbf{y}}$ are given by swapping colours above.

We will call the summand $X \in \mathcal{D}_1$ in Proposition 4.20 from now on J. Now we are ready to define the diagrammatic category $g\mathcal{D}$.

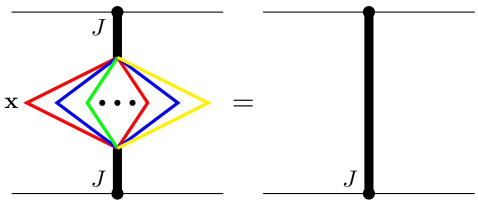
Remark 4.22. The elements of ϕ_J are constructed only out of 4-valent and 6-valent vertices [Eli16, Definition 3.9]. ♢

Definition 4.23. Let $g\mathcal{D}$ be the graded monoidal category presented diagrammatically as follows. The generating objects are connected subsets J of S (thus, general objects are sequences $\underline{\mathbf{J}} = J_1 J_2 \ldots J_r$ of connected subsets of S). When $J = \{j\}$ is a singleton, we write the element j instead of J and identify it with an object in $\mathcal{D}_1$. We draw the identity of J as follows.

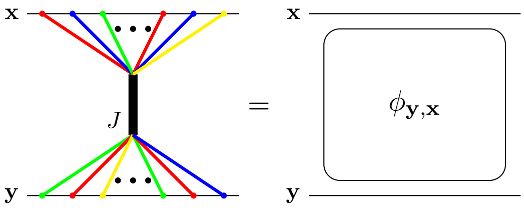


The generating morphisms are the usual generators of $\mathcal{D}_1$, in addition to J-inclusions and J-projections. The J-inclusion is a morphism from J to $\mathbf{x}$ where $\mathbf{x}$ is any reduced expression for w_J. The J-projection is a morphism in the other direction. Both have degree 0.

The defining relations consist of



(4.37)



(4.38)

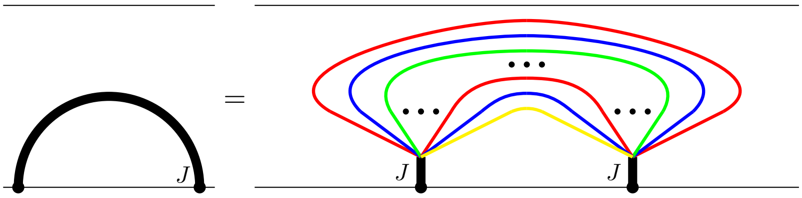
together with the defining relations of $\mathcal{D}_1$. ◊

Theorem 4.24. *This category $g\mathcal{D}$ is equivalent to the partial idempotent completion of $\mathcal{D}_1$ by the images of ϕ_J for $J \subset S$. The functor $\mathcal{F}$ from $\mathcal{D}_1$ to $\mathbb{BS}$Bim extends to a functor $g\mathcal{F}$ from $g\mathcal{D}$ to $g\mathbb{BS}$Bim which is an equivalence of categories if $\mathcal{F}$ is one.*

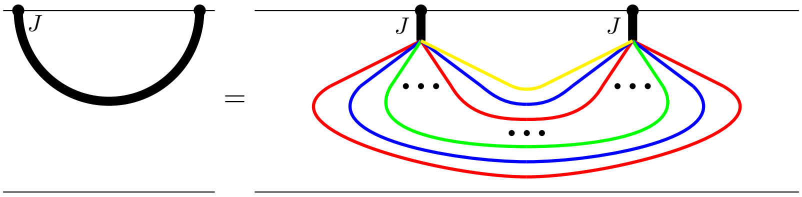
Proof. This follows from the discussion in Remark 4.18 and Proposition 4.20. □

We will now identify some morphisms with more special pictures (like we did for cup and cap) and give relations for them. This makes many statements more intuitive. We will only cover a part of what is done in [Eli16, Chapter 4], since we only need some of the morphisms for the next sections.

Definition 4.25. The first new morphisms are the *thick cap* and *thick cup*.



(4.39)



(4.40)

They are independent of the choice of reduced expression. ◊

Note that one can check that the thick cap corresponds to the bimodule morphism

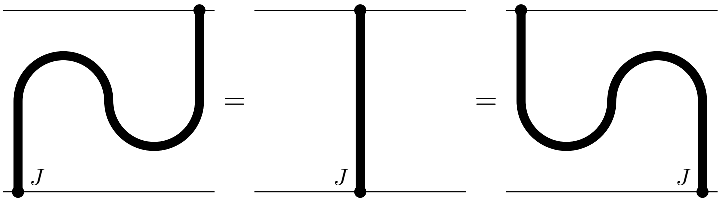
$$\begin{aligned} B_J B_J = R \otimes_{R^J} R \otimes_{R^J} R &\longrightarrow R \\ r_1 \otimes r_2 \otimes r_3 &\longmapsto r_1 \partial_J(r_2) r_3 \end{aligned}$$

and the thick cup corresponds to the bimodule morphism

$$\begin{aligned} R &\longrightarrow B_J B_J = R \otimes_{R^J} R \otimes_{R^J} R \\ 1 &\longmapsto 1 \otimes 1 \otimes 1. \end{aligned}$$

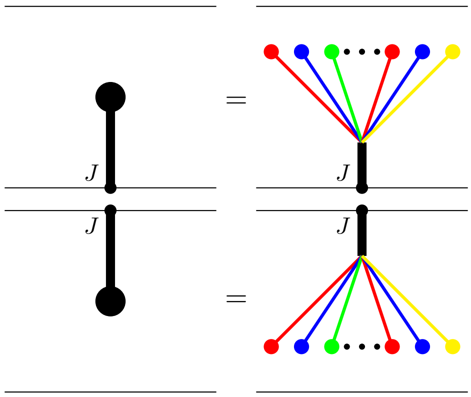
The following relation is the important one for cap and cup.

Lemma 4.26. *We have the following relation in $g\mathcal{D}$.*



(4.41)

Definition 4.27. The next morphisms are the *thick dots*. They are obtained by choosing a reduced expression $\mathbf{x}$ for w_J and composing $J \longrightarrow \mathbf{x} \longrightarrow \emptyset$, where the latter morphism consists of a dot on every strand.



(4.42)

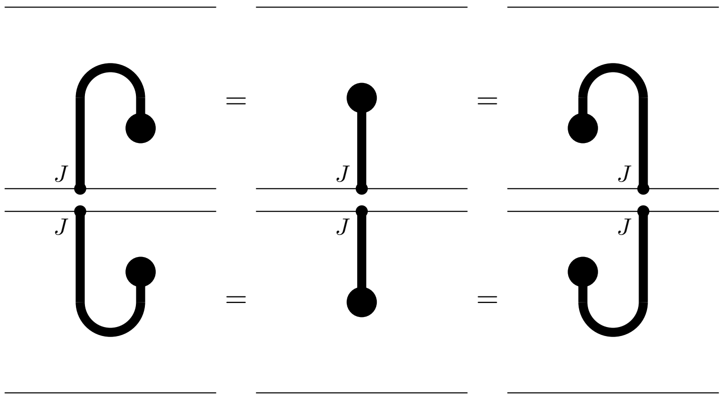
(4.43)

They are independent of the choice of reduced expression. ◊

Lemma 4.28. *The two morphims above are both non-zero and independent of the choice of* $\mathbf{x}$*, so they are well defined. It is the generator of* $\mathrm{Hom}_{g\mathcal{D}}(J, \emptyset)$ *as an* R*-bimodule.*

Proof. See [Eli16, Proposition 3.49 and Claim 4.5]. □

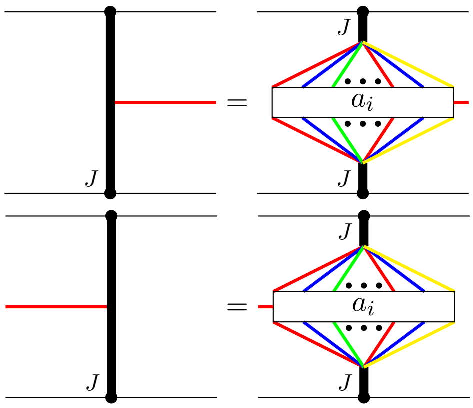
Lemma 4.29. *We have the following cyclicity relations for the thick dots.*



(4.44)

(4.45)

Definition 4.30. The *thick trivalent vertex* exists only if $i \in J$. There are two versions of the thick trivalent vertex, a right-facing one and a left-facing one.



(4.46)

(4.47)

For the definition of a_i see [Eli16, §3.4]. $\diamondsuit$

Note that we abused notation here by writing a_i in both boxes, but meaning two different morphisms (one with a right-facing strand i and one with a left-facing strand i). The colour of i in these pictures is red. Note that the reduced expression used for the J-projections and J-inclusions starts with red, but this could be totally different and the definition of a_i depends on the reduced expression we choose.

The easiest way to understand a_i is to choose a reduced expression that ends (respectively starts) in i. Then a_i is just the identity and we have the usual trivalent vertex on the right (respectively left).

In this way we can also observe what the thick trivalent vertex is on the bimodule side. As a morphism $B_J \otimes B_i \longrightarrow B_J$ it is given by

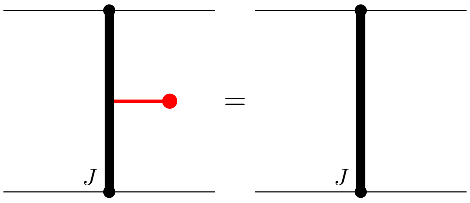
$$r_1 \otimes r_2 \otimes r_3 \longmapsto r_1 \otimes \partial_i(r_2)r_3.$$

As a morphism $B_J \longrightarrow B_J \otimes B_i$ it is given by

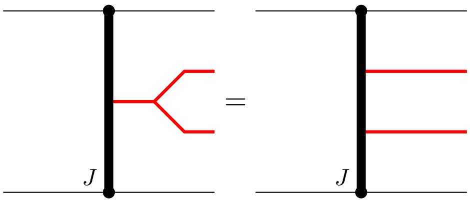
$$r_1 \otimes r_2 \longmapsto r_1 \otimes 1 \otimes r_2.$$

The analogous morphisms correspond to the left-facing thick trivalent vertex. Now we can give some relations for the thick trivalent vertices. We will only draw the right-facing versions of the relations. The left-facing versions are also true.

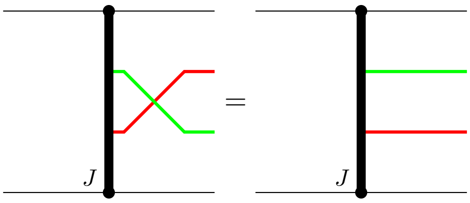
Lemma 4.31. *We have the following relations in $g\mathcal{D}$ where red and green are distant while red and blue are adjacent.*



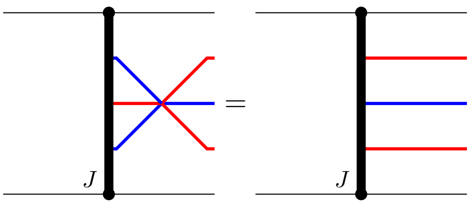
(4.48)



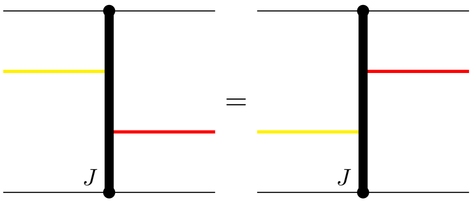
(4.49)



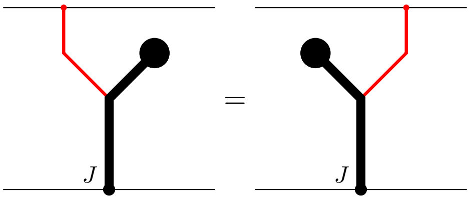
(4.50)



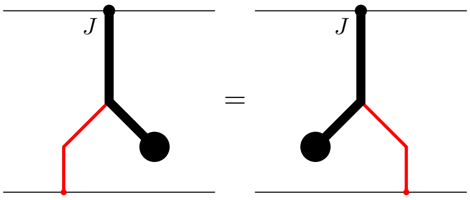
(4.51)



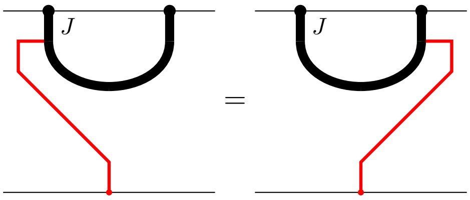
(4.52)



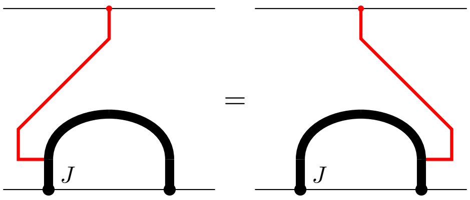
(4.53)



(4.54)

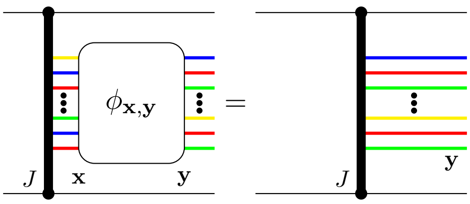


(4.55)



(4.56)

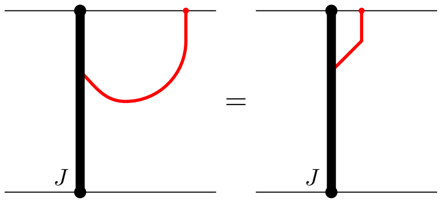
Remark 4.32. Recall Remark 4.22 says that $\phi_{\mathbf{x},\mathbf{y}}$ is constructed only out of 4-valent and 6-valent vertices. Thus, (4.50) and (4.51) imply that if we write $\phi_{\mathbf{x},\mathbf{y}}$ rotated by 90 degrees next to a thick line labelled J it will get sucked in completely and just changes the ordering of the strings:



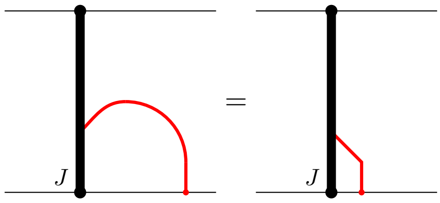
(4.57)

The same relation holds on the left side. ♢

Corollary 4.33. *We have the following isotopy relations for the thick trivalent vertex. We will again only show the right version, but the left version works completely analogous.*

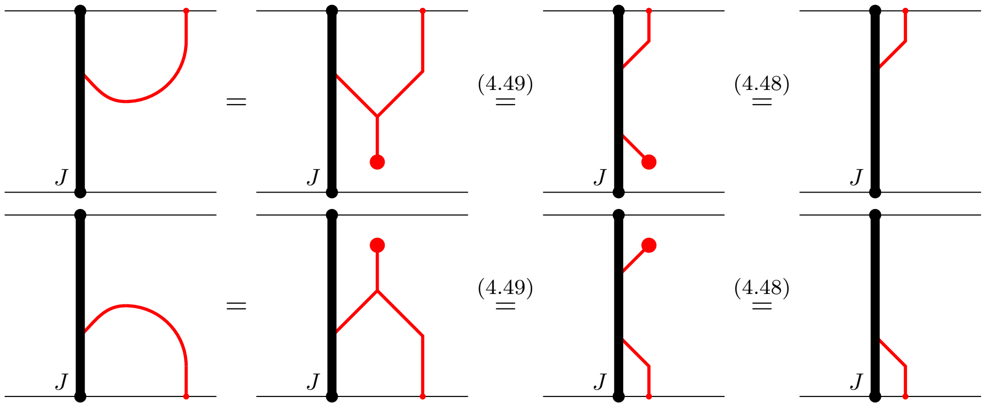


(4.58)



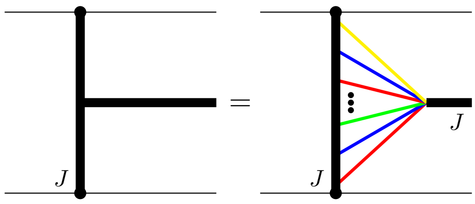
(4.59)

Proof. We use (4.48) and (4.49) to get the following chain of equalities.



This finishes the proof. □

Definition 4.34. The *very thick trivalent vertex* is constructed as follows. Rotate the J-inclusion by 90 degrees, and then connect the output sequence $\mathbf{x}$ to another J-coloured strand by a sequence of thick trivalent vertices. There are d_J thick trivalent vertices, so this morphism has degree $-d_J$.



(4.60)

Again this morphism is independent of the choice of reduced expression. ◊

We can again analyse what the very thick trivalent vertex corresponds to on the bimodule side. First consider it as a morphism $B_J \otimes B_J \longrightarrow B_J$. If we look at an element $r_1 \otimes r_2 \otimes r_3 \in B_J \otimes B_J$, then this gets sent by the J-inclusion to $r_1 \otimes r_2 \otimes 1 \otimes \cdots \otimes 1 \otimes r_3$.

Now we apply the d_J thick trivalent vertices. Each of them will just apply a Demazure operator ∂_i to r_2 and cancel on of the middle tensor signs. Thus, in the end we are left with the following expression:

$$r_1 \otimes \partial_{i_1}(\partial_{i_2}(\cdots(\partial_{i_{d_J}}(r_2))\cdots))r_3 = r_1 \otimes \partial_J(r_2)r_3,$$

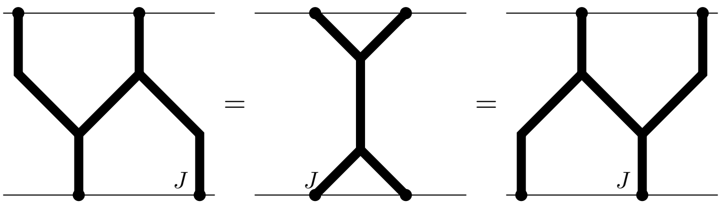
where the last equality comes from the fact that $s_{i_1} \cdots s_{i_{d_J}}$ is a reduced expression for w_J, since the i_j are coming from the J-inclusion. Hence, the very thick trivalent vertex as a morphism $B_J \otimes B_J \longrightarrow B_J$ is given by

$$r_1 \otimes r_2 \otimes r_3 \longmapsto r_1 \otimes \partial_J(r_2)r_3.$$

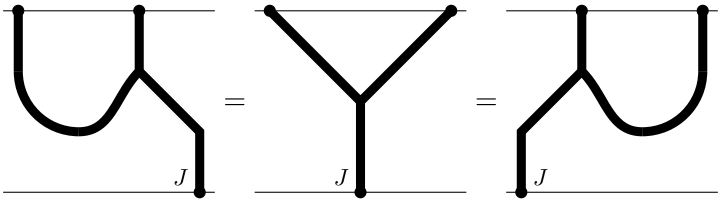
If we consider the very thick trivalent vertex as a morphism $B_J \longrightarrow B_J \otimes B_J$ we can do a similar analysis and get that it is given by

$$r_1 \otimes r_2 \longmapsto r_1 \otimes 1 \otimes r_2.$$

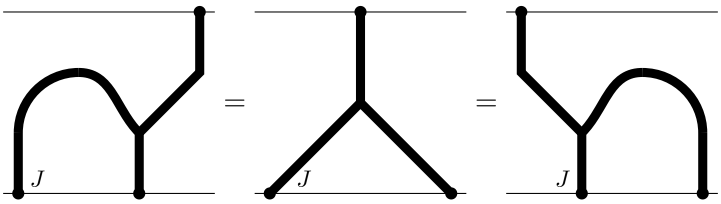
Lemma 4.35. *We have the following relations for the very thick trivalent vertex.*



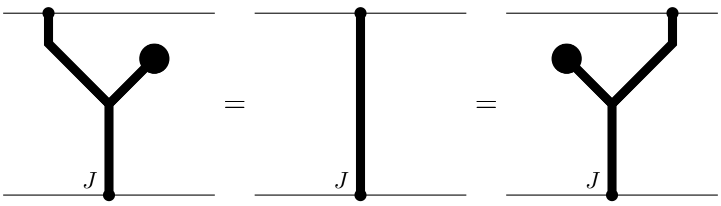
(4.61)



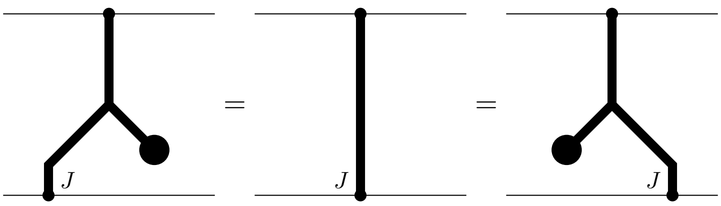
(4.62)



(4.63)

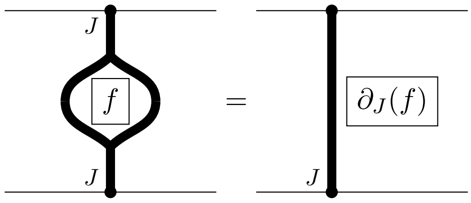


(4.64)

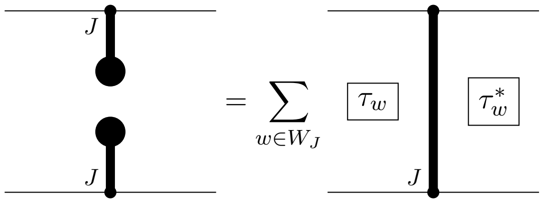


(4.65)

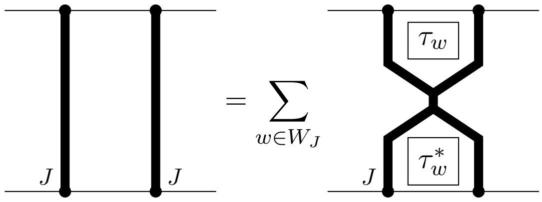
Lemma 4.36. *There are three more relations which we will state. For some of these we need the bases* $\{\tau_w\}_{w \in W_J}$ *and* $\{\tau_w^*\}_{w \in W_J}$ *from Theorem 3.35. In the first of the three relations we have* $f \in R$.



(4.66)



(4.67)



(4.68)

Remark 4.37. This diagrammatic presentation of $g\mathbb{BSB}\mathrm{im}$ (see Definition 4.23) only contains thick strands for connected parabolic subsets J. Suppose that J is disconnected. Then $J = J_1 \sqcup \cdots \sqcup J_r$ for connected, mutually distant parabolic subsets J_i. Thus, $W_J = W_{J_1} \times \cdots \times W_{J_r}$, w_J is the product of various w_{J_i}, and B_J is the tensor product of the B_{J_i} in $\mathbb{S}\mathrm{Bim}$. So the object B_J is already isomorphic to an object in $g\mathcal{D}$. $\diamondsuit$

5 The case S_3

In this chapter we are going to describe the 2-category of singular Soergel bimodules (Definition 3.43) for $W = S_3$ with $S = \{s_1, s_2\}$ where $s_1 = (12)$ and $s_2 = (23)$. In order to do so we need to understand the categories ${}_I\mathbb{S}\mathrm{Bim}_J$ for all parabolic subsets $I, J \subseteq S$. There are four parabolic subsets, namely $\emptyset, \{s_1\}, \{s_2\}, S$. Thus, there are sixteen categories which we need to consider.
For each such category we will go by the same procedure. We only need to understand the category ${}_I\mathbb{BS}\mathrm{Bim}_J$ or the category ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$, since ${}_I\mathbb{S}\mathrm{Bim}_J$ is their Karoubi envelope. We will find some indecomposable bimodules and show how each object in ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$ decomposes into these indecomposable bimodules. By doing so we also prove that these then are all indecomposable bimodules and prove Theorem 3.46 for S_3.
All that is left then is to understand the morphisms. We will compute bases for the homomorphism spaces between two indecomposable bimodules. Together with the first part we can then express every morphism between two arbitrary objects in ${}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$ by decomposing them into indecomposables and considering the morphisms on summands.
We put the sixteen categories in some classes depending on how many indecomposable they have which roughly measures how hard it is to understand them.

(1) $\mathbb{S}\mathrm{Bim}$
(2) ${}_S\mathbb{S}\mathrm{Bim}$, ${}_S\mathbb{S}\mathrm{Bim}_1$, ${}_S\mathbb{S}\mathrm{Bim}_2$, ${}_S\mathbb{S}\mathrm{Bim}_S, \mathbb{S}\mathrm{Bim}_S$, ${}_1\mathbb{S}\mathrm{Bim}_S$, ${}_2\mathbb{S}\mathrm{Bim}_S$
(3) ${}_1\mathbb{S}\mathrm{Bim}$, ${}_2\mathbb{S}\mathrm{Bim}, \mathbb{S}\mathrm{Bim}_1, \mathbb{S}\mathrm{Bim}_2$
(4) ${}_1\mathbb{S}\mathrm{Bim}_1$, ${}_1\mathbb{S}\mathrm{Bim}_2$, ${}_2\mathbb{S}\mathrm{Bim}_1$, ${}_2\mathbb{S}\mathrm{Bim}_2$

Here we wrote 1 instead of $\{s_1\}$, 2 instead of $\{s_2\}$ and nothing instead of $\emptyset$. The first class just contains the category of (regular) Soergel bimodules. This is already quite well understood and we will only cite results of Libedinsky [Lib19]. The second class is quite simple as there will only be one indecomposable bimodule. The third and the fourth case will be the harder ones. We will do one category in detail and only give the results for the other categories as the procedure is always the same.

5.1 (Regular) Soergel bimodules for S_3

In this section we will describe the category of (regular) Soergel bimodules for S_3. This will be the starting point for all our calculation in this chapter. For explicit calculations in the regular case we refer the reader to [Lib19] and focus on the singular case instead. We start by recalling some results from [Lib19].

Definition 5.1. We have the following objects in $\mathbb{S}$Bim:

$$B_{w_0} = R \otimes_{R^S} R\langle 3\rangle$$
$$B_{12} = R \otimes_{R^1} R \otimes_{R^2} R\langle 2\rangle \qquad B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle$$
$$B_1 = R \otimes_{R^1} R\langle 1\rangle \qquad B_2 = R \otimes_{R^2} R\langle 1\rangle$$
$$B_e = R.$$

For B_{w_0} this follows from Lemma 4.14. ♢

Remark 5.2. These six bimodules are generated by the 1-tensor (the element $1\otimes\cdots\otimes 1$) as bimodules. For B_{12} and B_{21} this follows from the fact that R is generated by 1 as an (R^1, R^2)-bimodule or (R^2, R^1)-bimodule respectively. For each of these bimodules the graded component of minimal degree which is not zero is one-dimensional. Thus, they are indecomposable by Lemma 2.59.
Moreover, note that $B_{12} = B_1 \otimes B_2$ and $B_{21} = B_2 \otimes B_1$. ♢

Remark 5.3. Since by Theorem 3.16 R is a free R^I-module of finite rank for $I = \emptyset, \{s_1\}, \{s_2\}, S$ we get the following. Let

$$M = R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_N}} R^{I_{N+1}}$$

be an object of s$\mathbb{BS}$Bim. Then we have

$$R \otimes_{R^{J_1}} R \otimes_{R^{J_2}} R \otimes_{R^{J_3}} \cdots \otimes_{R^{J_N}} R \cong M^{\oplus L}$$

for some $L \in \mathbb{N}$. Hence, if we can decompose all objects of the form

$$R \otimes_{R^{J_1}} R \otimes_{R^{J_2}} R \otimes_{R^{J_3}} \cdots \otimes_{R^{J_N}} R$$

into direct sums of $B_e, B_1, B_2, B_{12}, B_{21}, B_{w_0}$, then we can do this for all objects in s$\mathbb{BS}$Bim. Note that $J_i \in \{\emptyset, \{s_1\}, \{s_2\}, S\}$. Thus, we can write

$$R \otimes_{R^{J_1}} R \otimes_{R^{J_2}} \cdots \otimes_{R^{J_N}} R = B_{J_1} \otimes B_{J_2} \otimes \cdots \otimes B_{J_N}$$

where B_{J_i} is one of the following for all $i = 1, \dots, N$: B_e, B_1, B_2, B_{w_0}. Hence, it would be sufficient if we were able to decompose all bimodules of the form

$$M_1 \otimes M_2$$

for $M_1, M_2 \in \mathcal{I} = \{B_e, B_1, B_2, B_{12}, B_{21}, B_{w_0}\}$ into sums of elements of $\mathcal{I}$. This is what we will do. ♢

We have by Remark 3.27 and Theorem 3.28 the following isomorphisms

$$\begin{aligned}
R &\cong R^1 \oplus R^1\langle -2\rangle && \text{as } (R^1, R^1)\text{-bimodules} \\
R &\cong R^2 \oplus R^2\langle -2\rangle && \text{as } (R^2, R^2)\text{-bimodules} \\
R &\cong \begin{matrix} R^S \oplus R^S\langle -2\rangle \oplus R^S\langle -2\rangle \oplus R^S\langle -4\rangle \\ \oplus R^S\langle -4\rangle \oplus R^S\langle -6\rangle \end{matrix} && \text{as } (R^S, R^S)\text{-bimodules.}
\end{aligned} \tag{5.1}$$

We will now go through all the choices for $M_1, M_2 \in \mathcal{I}$. If $M_1 = B_e = R$, then $M_1 \otimes M_2 = M_2$ and we are done. If we have $M_1 = B_{12}$ or $M_1 = B_{21}$ we can use $B_{12} = B_1 \otimes B_2$ and $B_{21} = B_2 \otimes B_1$ respectively to reduce it to the case $M_1 = B_1, B_2$. Let us start with $M_1 = B_{w_0}$. We have $B_{w_0} \otimes B_e = B_{w_0}$.

Lemma 5.4. *We have the following isomorphisms in* $\mathbb{S}\mathrm{Bim}$*.*

1. $B_{w_0} \otimes B_1 \cong B_{w_0}\langle 1\rangle \oplus B_{w_0}\langle -1\rangle$.
2. $B_{w_0} \otimes B_2 \cong B_{w_0}\langle 1\rangle \oplus B_{w_0}\langle -1\rangle$.
3. $B_{w_0} \otimes B_{w_0} \cong B_{w_0}\langle 3\rangle \oplus (B_{w_0}\langle 1\rangle)^{\oplus 2} \oplus (B_{w_0}\langle -1\rangle)^{\oplus 2} \oplus B_{w_0}\langle -3\rangle$.

Proof. **1.** We can compute that

$$\begin{aligned} B_{w_0} \otimes B_1 &= R \otimes_{R^S} R \otimes_{R^1} R\langle 4\rangle \\ &\cong R \otimes_{R^S} \big(R^1 \oplus R^1\langle -2\rangle\big) \otimes_{R^1} R\langle 4\rangle \\ &\cong R \otimes_{R^S} R\langle 4\rangle \oplus R \otimes_{R^S} R\langle 2\rangle \\ &= B_{w_0}\langle 1\rangle \oplus B_{w_0}\langle -1\rangle. \end{aligned}$$

2. This is completely analogous to 1.
3. Here we compute that

$$\begin{aligned} B_{w_0} \otimes B_{w_0} &= R \otimes_{R^S} R \otimes_{R^S} R\langle 6\rangle \\ &\cong R \otimes_{R^S} \begin{pmatrix} R^S \oplus R^S\langle -2\rangle \oplus R^S\langle -2\rangle \oplus R^S\langle -4\rangle \\ \oplus R^S\langle -4\rangle \oplus R^S\langle -6\rangle \end{pmatrix} \otimes_{R^S} R\langle 6\rangle \\ &\cong R \otimes_{R^S} R\langle 6\rangle \oplus R \otimes_{R^S} R\langle 4\rangle \oplus R \otimes_{R^S} R\langle 4\rangle \oplus R \otimes_{R^S} R\langle 2\rangle \\ &\quad \oplus R \otimes_{R^S} R\langle 2\rangle \oplus R \otimes_{R^S} R \\ &= B_{w_0}\langle 3\rangle \oplus (B_{w_0}\langle 1\rangle)^{\oplus 2} \oplus (B_{w_0}\langle -1\rangle)^{\oplus 2} \oplus B_{w_0}\langle -3\rangle. \end{aligned}$$ □

Again we do not need to consider $M_2 = B_{12}, B_{21}$, since we can reduce to the case $M_2 = B_1, B_2$. At last we consider $M_1 = B_1$. This is enough, since $M_1 = B_2$ works completely analogous.

Lemma 5.5. *We have the following isomorphisms in* $\mathbb{S}\mathrm{Bim}$*.*

1. $B_1 \otimes B_e \cong B_1$.
2. $B_1 \otimes B_2 \cong B_{12}$.
3. $B_1 \otimes B_1 \cong B_1\langle 1\rangle \oplus B_1\langle -1\rangle$.
4. $B_1 \otimes B_{12} \cong B_{12}\langle 1\rangle \oplus B_{12}\langle -1\rangle$.
5. $B_1 \otimes B_{w_0} \cong B_{w_0}\langle 1\rangle \oplus B_{w_0}\langle -1\rangle$.
6. $B_1 \otimes B_{21} \cong B_1 \oplus B_{w_0}$.

Proof. **1.** There is nothing to do here.
2. We already know this isomorphism.

3. We compute that

$$\begin{aligned} B_1 \otimes B_1 &= R \otimes_{R^1} R \otimes_{R^1} R\langle 2\rangle \\ &\cong R \otimes_{R^1} \left(R^1 \oplus R^1\langle -2\rangle\right) \otimes_{R^1} R\langle 2\rangle \\ &\cong R \otimes_{R^1} R\langle 2\rangle \oplus R \otimes_{R^1} R \\ &= B_1\langle 1\rangle \oplus B_1\langle -1\rangle. \end{aligned}$$

4. Here we can use part 3 to get

$$\begin{aligned} B_1 \otimes B_{12} &\cong B_1 \otimes B_1 \otimes B_2 \cong (B_1\langle 1\rangle \oplus B_1\langle -1\rangle) \otimes B_2 \\ &\cong B_{12}\langle 1\rangle \oplus B_{12}\langle -1\rangle. \end{aligned}$$

5. We compute similar to the previous Lemma that

$$\begin{aligned} B_1 \otimes B_{w_0} &= R \otimes_{R^1} R \otimes_{R^S} R\langle 4\rangle \\ &\cong R \otimes_{R^1} \left(R^1 \oplus R^1\langle -2\rangle\right) \otimes_{R^S} R\langle 4\rangle \\ &\cong R \otimes_{R^S} R\langle 4\rangle \oplus R \otimes_{R^S} R\langle 2\rangle \\ &= B_{w_0}\langle 1\rangle \oplus B_{w_0}\langle -1\rangle. \end{aligned}$$

6. This is proven in [Lib19, 4.3]. The idempotent which picks out the summand B_1 is given by

$$\begin{aligned} R \otimes_{R^1} R \otimes_{R^2} R \otimes_{R^1} R &\longrightarrow R \otimes_{R^1} R \otimes_{R^2} R \otimes_{R^1} R \\ r_1 \otimes r_2 \otimes r_3 \otimes r_4 &\longmapsto -r_1\partial_1(r_2 r_3) \otimes \alpha_2 \otimes 1 \otimes r_4 - r_1\partial_1(r_2 r_3) \otimes 1 \otimes \alpha_2 \otimes r_4. \end{aligned}$$

Note that the other idempotent for B_{w_0} is then given by $1 - e$ where e is the above idempotent. □

Now we have decomposed all products $M_1 \otimes M_2$ for $M_1, M_2 \in \mathcal{I}$ into sums of objects in $\mathcal{I}$ which tells us how to decompose any object in s𝔹𝕊Bim. Next we want to construct bases for the morphism spaces. The construction is motivated from highest weight theory. The outcome will be a so-called light leaves basis which encodes certain standard and costandard filtrations of Soergel bimodules. The following combinatorics can be understood without any knowledge of this theory, but may become more intuitive when put into this context.

Definition 5.6. Let $\underline{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_N}) \in S^N$ be a fixed sequence of simple reflections. We will construct a perfect binary tree $\mathbb{T}_{\underline{w}}$ (this is a tree in which all interior nodes have exactly two children and all leaves have the same depth). The node at the top is labelled

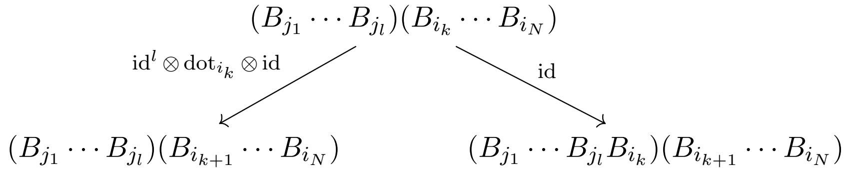
$$()(B_{i_1} B_{i_2} \cdots B_{i_N}).$$

Note that we wrote $B_{i_1} B_{i_2}$ instead of $B_{i_1} \otimes B_{i_2}$. We will use this abbreviation from now on. We will now construct this tree inductively. Let $k \in \mathbb{N}$, then a node of depth $k-1$ will be labelled

$$(B_{j_1} B_{j_2} \cdots B_{j_l})(B_{i_k} \cdots B_{i_N}),$$

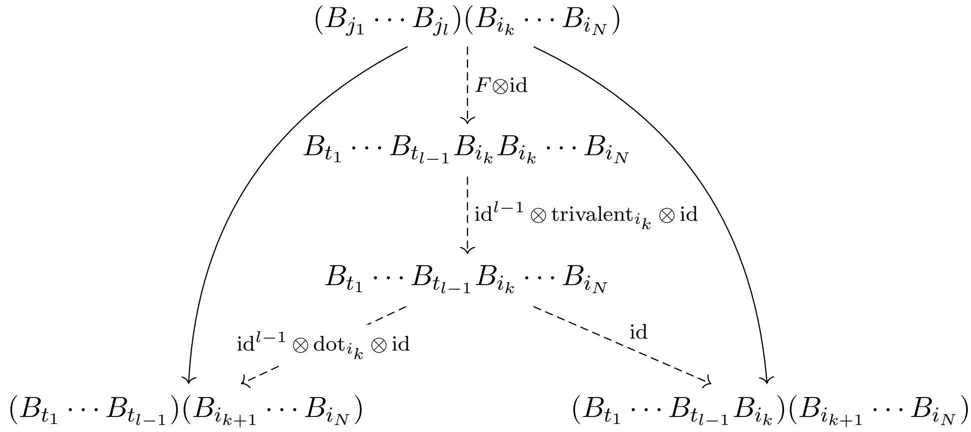
where $l \in \mathbb{N}$ is some number. Let us call this node $\mathcal{N}$. Now we have two cases.

1. If $\ell(s_{j_1} \cdots s_{j_l} s_{i_k}) > \ell(s_{j_1} \cdots s_{j_l})$, then the child nodes and child edges of $\mathcal{N}$ are labelled in the following way.



Here dot_{i_k} stands for the morphism $B_{i_k} \longrightarrow R$ given by the enddot (see Definition 4.1). Note that we are in this case for the top node.

2. If $\ell(s_{j_1} \cdots s_{j_l} s_{i_k}) < \ell(s_{j_1} \cdots s_{j_l})$, then the child nodes and child edges of $\mathcal{N}$ are labelled in the following way (the arrows are the composition of the corresponding dashed arrows).



Here $\mathrm{trivalent}_{i_k}$ is the morphism $B_{i_k} B_{i_k} \longrightarrow B_{i_k}$ given by the Merge (see Definition 4.1). In order to explain the morphism F we need some observations. First note that the expression $u = s_{j_1} \cdots s_{j_l}$ is always reduced which we can check inductively. Now by Theorem 2.11 and the condition $\ell(s_{j_1} \cdots s_{j_l} s_{i_k}) < \ell(s_{j_1} \cdots s_{j_l})$ we have that $us_{i_k} = s_{j_1} \cdots \widehat{s_{j_a}} \cdots s_{j_l}$, and thus $u = s_{j_1} \cdots \widehat{s_{j_a}} \cdots s_{j_l} s_{i_k}$ is a reduced expression. We write $s_{t_1} \cdots s_{t_{l-1}}$ for $s_{j_1} \cdots \widehat{s_{j_a}} \cdots s_{j_l}$. Now we have two reduced expressions for u and by Lemma 2.6 we can get from one to the other by braid moves $s_1 s_2 s_1 \longleftrightarrow s_2 s_1 s_2$. For each such braid move we have a morphism $B_1 B_2 B_1 \longrightarrow B_2 B_1 B_2$ (or the other way around) given by the 6-valent vertex (see Definition 4.1). Applying a braid move to a reduced expression of $u = s_{j_1} \cdots s_{j_l}$ stands for applying the corresponding 6-valent vertex tensored with identities to $B_{j_1} \cdots B_{j_l}$. If we now compose all the morphisms corresponding to the braid moves we get a morphism

$$F : B_{j_1} \cdots B_{j_l} \longrightarrow B_{t_1} \cdots B_{t_{l-1}} B_{i_k}.$$

This finishes the definition of $\mathbb{T}_{\underline{w}}$. $\Diamond$

Remark 5.7. Note that the sequence of braid moves we apply to get from one reduced expression to another is not unique. Thus, we could have multiple choices for the morphism F. It doesn't matter which one we choose, but we need to choose one once and for all. However, since we are in S_3 there is only one element with more than one reduced expression, namely w_0. For this we just choose F to be the 6-valent vertex.
Note that at the leaves of $\mathbb{T}_{\underline{w}}$ we have expressions of the form $(B_{j_1} \cdots B_{j_L})()$. So each leave corresponds to a Bott–Samelson bimodule $B_{\underline{x}}$ where $\underline{x} = (s_{j_1}, \cdots, s_{j_L})$ is a tuple of simple reflections. Moreover, we already noticed that expressions in the first bracket are reduced. Hence, $x = s_{j_1} \cdots s_{j_L}$ is a reduced expression.
Each edge in $\mathbb{T}_{\underline{w}}$ is labelled by a morphism between the two Bott–Samelson bimodules adjacent to this edge. For each leave there is a unique path from the top node to this leaf, and hence by composing the morphisms on the edges of this path we get a unique morphism $f_{\underline{x}} : B_{\underline{w}} \longrightarrow B_{\underline{x}}$. Thus, each leaf encodes a pair $(B_{\underline{x}}, f_{\underline{x}})$. ◊

Definition 5.8. We denote by $\mathbb{L}_{\underline{w}}$ the set of all morphisms $f_{\underline{x}}$ corresponding to a leaf in $\mathbb{T}_{\underline{w}}$.
For each morphism $f_{\underline{x}} \in \mathbb{L}_{\underline{w}}$ we have a morphism $f^a_{\underline{x}} : B_{\underline{x}} \longrightarrow B_{\underline{w}}$. If we write $f_{\underline{x}}$ in diagrammatic language, then $f^a_{\underline{x}}$ is just the picture of $f_{\underline{x}}$ flipped upside down (or equivalently read from top to bottom). We denote by $\mathbb{L}^a_{\underline{w}}$ the set of all the morphisms $f^a_{\underline{x}}$.
For $\underline{x} = (s_{j_1}, \cdots, s_{j_L})$ we write $x = s_{j_1} \cdots s_{j_L}$. Let $f_{\underline{x}} \in \mathbb{L}_{\underline{w}}$ and $f^a_{\underline{y}} \in \mathbb{L}^a_{\underline{u}}$. Then we define

$$f^a_{\underline{y}} \cdot f_{\underline{x}} = \begin{cases} f^a_{\underline{y}} \circ F \circ f_{\underline{x}} & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$

where $F : B_{\underline{x}} \longrightarrow B_{\underline{y}}$ is again the fixed morphism corresponding to a sequence of braid moves from $\underline{x}$ to $\underline{y}$. We call the set

$$\mathbb{L}^a_{\underline{u}} \cdot \mathbb{L}_{\underline{w}} = \left\{ f^a_{\underline{y}} \cdot f_{\underline{x}} \mid f^a_{\underline{y}} \in \mathbb{L}^a_{\underline{u}}, f_{\underline{x}} \in \mathbb{L}_{\underline{w}} \right\} \subset \mathrm{Hom}_{(R,R)}(B_{\underline{w}}, B_{\underline{u}})$$

the *double leaves basis* of $\mathrm{Hom}_{(R,R)}(B_{\underline{w}}, B_{\underline{u}})$. ◊

The following is a theorem of Libedinsky [Lib19, Theorem 6.4].

Theorem 5.9. *The double leaves basis $\mathbb{L}^a_{\underline{u}} \cdot \mathbb{L}_{\underline{w}}$ of $\mathrm{Hom}_{(R,R)}(B_{\underline{w}}, B_{\underline{u}})$ is a basis of $\mathrm{Hom}_{(R,R)}(B_{\underline{w}}, B_{\underline{u}})$ as a left (or right) R-module.*

Sketch. We will give the general idea of the proof. The rank of $\mathrm{Hom}_{(R,R)}(B_{\underline{w}}, B_{\underline{u}})$ can be computed using Theorem 3.39. One can also count the elements of $\mathbb{L}^a_{\underline{u}} \cdot \mathbb{L}_{\underline{w}}$ and observe that the two numbers are the same. Thus, it suffices to prove that the elements of $\mathbb{L}^a_{\underline{u}} \cdot \mathbb{L}_{\underline{w}}$ are linearly independent. This can be done, but is not easy. □

Remark 5.10. This theorem gives us bases for all the homomorphism spaces of our indecomposable bimodules $\mathcal{I}$ except for B_{w_0}, since all other elements of $\mathcal{I}$ are Bott-Samelson bimodules. However, since we know the idempotent for the decomposition

$B_1B_2B_1 \cong B_1 \oplus B_{w_0}$ explicitly, we can use

$$\mathrm{Hom}_{(R,R)}(B_1B_2B_1, M) \cong \mathrm{Hom}_{(R,R)}(B_1, M) \oplus \mathrm{Hom}_{(R,R)}(B_{w_0}, M)$$

to get bases for the remaining homomorphism spaces. $\diamondsuit$

5.2 Bases of homomorphism spaces

In this section we will observe some general results which let us understand the morphisms in ${}_I\mathbb{S}\mathrm{Bim}_J$ by tracing them back to the morphisms of $\mathbb{S}\mathrm{Bim}$. The results and proofs we will do work for $W = S_n$, but we only need them for S_3 in this chapter.

Definition 5.11. Let ${}_I\mathrm{Bim}_J$ be the category of (R^I, R^J)-bimodules. We define three functors that will help us to switch between categories:

- The *restriction* functor ${}_I\mathrm{res}_J : \mathrm{Bim} \longrightarrow {}_I\mathrm{Bim}_J$ is defined by $M \mapsto {}_IM_J$ where ${}_IM_J$ is M viewed as an (R^I, R^J)-bimodule with actions coming from the inclusions $R^I, R^J \subseteq R$.
- The *induction* functor ${}_I\mathrm{ind}_J : {}_I\mathrm{Bim}_J \longrightarrow \mathrm{Bim}$ is defined by $M \mapsto R \otimes_{R^I} M \otimes_{R^J} R$ with R acting on the left and right by multiplication.
- The *coinduction* functor ${}_I\mathrm{coind}_J : {}_I\mathrm{Bim}_J \longrightarrow \mathrm{Bim}$ is defined by

$$M \mapsto \mathrm{Hom}_{(R^I,R^J)}(R \otimes_{\mathbb{Z}} R, M)$$

  where the actions are given by $r_i \cdot f \cdot r_j = (r \otimes r' \mapsto f(r_ir \otimes r'r_j))$ for $r_i \in R^I, r_j \in R^J, f \in \mathrm{Hom}_{(R^I,R^J)}(R \otimes_{\mathbb{Z}} R, M)$. $\diamondsuit$

There are some well-known adjunctions which we will use.

Lemma 5.12.

1. $({}_I\mathrm{ind}_J, {}_I\mathrm{res}_J)$ *is an adjoint pair.*
2. $({}_I\mathrm{res}_J, {}_I\mathrm{coind}_J)$ *is an adjoint pair.*
3. ${}_I\mathrm{ind}_J$ *and* ${}_I\mathrm{coind}_J$ *are isomorphic.*
4. $({}_I\mathrm{res}_J, {}_I\mathrm{ind}_J)$ *is an adjoint pair.*

Proof. The first two points are known adjunctions (the standard tensor-hom adjunction). The fourth point follows immediately from the second and third. Thus, we will just prove the third point.
To prove that induction and coinduction are isomorphic we need to find an isomorphism

$$R \otimes_{R^I} M \otimes_{R^J} R \cong \mathrm{Hom}_{(R^I,R^J)}(R \otimes_{\mathbb{Z}} R, M)$$

for all $M \in {}_I\mathrm{Bim}_J$ which is natural in M. We will do this in two steps. First consider the following map

$$\begin{aligned} \mathrm{Hom}_{R^I}(R, R^I) \otimes_{R^I} M \otimes_{R^J} \mathrm{Hom}_{R^J}(R, R^J) &\longrightarrow \mathrm{Hom}_{(R^I,R^J)}(R \otimes_{\mathbb{Z}} R, M) \\ \varphi \otimes m \otimes \psi &\longmapsto (r \otimes r' \mapsto \varphi(r) \cdot m \cdot \psi(r')) . \end{aligned} \tag{5.2}$$

Note that this is a morphism of (R, R)-bimodules and it is natural in M. We have the following chain of isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{R^I}(R^I, R^I) \otimes_{R^I} M \otimes_{R^J} \mathrm{Hom}_{R^J}(R^J, R^J) & & \varphi \otimes m \otimes \psi \\ \downarrow \cong & & \downarrow \\ R^I \otimes_{R^I} M \otimes_{R^J} R^J & & \varphi(1) \otimes m \otimes \psi(1) \\ \downarrow \cong & & \downarrow \\ M & & \varphi(1) \cdot m \cdot \psi(1) \\ \downarrow \cong & & \downarrow \\ \mathrm{Hom}_{(R^I,R^J)}(R^I \otimes_{\mathbb{Z}} R^J, M) & & (r \otimes r' \longmapsto r\varphi(1) \cdot m \cdot \psi(1)r') . \end{array}$$

Since $r\varphi(1) = \varphi(r)$ and $\psi(1)r' = \psi(r')$ this is the same morphisms as (5.2) just with R^I and R^J instead of R. As R is free over R^I and R^J by Theorem 3.16 we get from this that (5.2) is also bijective and hence an isomorphism of (R, R)-bimodules.
Now we just need to find an isomorphism $R \cong \mathrm{Hom}_{R^I}(R, R^I)$ to finish the proof. For this we use the map

$$\Phi : R \longrightarrow \mathrm{Hom}_{R^I}(R, R^I), \qquad r \longmapsto \big(r' \mapsto \partial_I(rr')\big) .$$

This map is R-linear and well-defined by Proposition 3.13. Suppose that $\Phi(r) = 0$. Write $r = \sum_{w\in W_I} \beta_w \tau_w$ where $\{\tau_w\}_{w\in W_I}$ is the R^I-basis of R from Theorem 3.35. Then $0 = \Phi(r)(\tau_u^*) = \sum_{w\in W_I} \beta_w \partial_I(\tau_w \tau_u^*) = \beta_u$. Thus, $r = 0$ and Φ is injective.
Let $\varphi \in \mathrm{Hom}_{R^I}(R, R^I)$. Then φ is determined by $\beta_w = \varphi(\tau_w^*)$. Now choose $r = \sum_{w\in W_I} \beta_w \tau_w$, then $\Phi(r)(\tau_u^*) = \beta_u$ as before, and hence $\Phi(r) = \varphi$ and Φ is surjective. □

Now let $M, N \in {}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J$. We would like to understand $\mathrm{Hom}_{(R^I,R^J)}(M, N)$. If we write

$$\begin{aligned} M &= R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_k}} R^{I_{k+1}} \\ N &= R^{I'_1} \otimes_{R^{J'_1}} R^{I'_2} \otimes_{R^{J'_2}} \cdots \otimes_{R^{J'_l}} R^{I'_{l+1}} \end{aligned}$$

we can consider the bimodules

$$\begin{aligned} M_1 &= R \otimes_{R^{J_1}} R \otimes_{R^{J_2}} \cdots \otimes_{R^{J_k}} R \in \mathrm{Bim} \\ N_1 &= R \otimes_{R^{J'_1}} R \otimes_{R^{J'_2}} \cdots \otimes_{R^{J'_l}} R \in \mathrm{Bim} \\ \widetilde{M} &= {}_I\mathrm{res}_J (M_1) \\ \widetilde{N} &= {}_I\mathrm{res}_J (N_1) . \end{aligned}$$

By Theorem 3.16 we have $\widetilde{M} \cong M^{\oplus K}$ and $\widetilde{N} \cong N^{\oplus K'}$ for some $K, K' \in \mathbb{N}$, and thus it suffices to understand $\mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right)$.

Lemma 5.13. *There is an isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right) & \cong & \mathrm{Hom}_{(R,R)}\left(R \otimes_{R^I} \widetilde{M} \otimes_{R^J} R, N_1\right) \\ (m \longmapsto \varphi(1 \otimes m \otimes 1)) & \longleftarrow & \varphi. \end{array}$$

Proof. We have the following chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right) &\cong \mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, {}_I\mathrm{res}_J\left(N_1\right)\right) \\ &\cong \mathrm{Hom}_{(R,R)}\left({}_I\mathrm{ind}_J\left(\widetilde{M}\right), N_1\right). \end{aligned}$$

Note that ${}_I\mathrm{ind}_J\left(\widetilde{M}\right) = R \otimes_{R^I} \widetilde{M} \otimes_{R^J} R$. This finishes the proof. □

This is a useful statement, because we understand the morphisms on the right already by Section 4.1 and want to understand the morphims on the left.

Lemma 5.14. *Suppose* $\{\varphi_1, \ldots, \varphi_k\} \subset \mathrm{Hom}_{(R,R)}\left(R \otimes_{R^I} \widetilde{M} \otimes_{R^J} R, N_1\right)$ *is basis as left* R*-module. Define* $\psi_{l,w} \in \mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right)$ *for* $l = 1, \ldots, k$ *and* $w \in W_I$ *by*

$$\psi_{l,w}(m) = \varphi_l(\tau_w \otimes m \otimes 1)$$

where $\{\tau_w\}_{w \in W_I}$ *is the basis from Theorem 3.35. Then* $\{\psi_{l,w} \mid l = 1, \ldots, k, w \in W_I\}$ *is a basis for* $\mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right)$ *as left* R^I*-module.*

Proof. We start by proving that this set is a generating set. Let $\psi \in \mathrm{Hom}_{(R^I,R^J)}\left(\widetilde{M}, \widetilde{N}\right)$. Then by Lemma 5.13 there is $\varphi \in \mathrm{Hom}_{(R,R)}\left(R \otimes_{R^I} \widetilde{M} \otimes_{R^J} R, N_1\right)$ such that $\psi(m) = \varphi(1 \otimes m \otimes 1)$. We can write

$$\varphi = \sum_{l=1}^{k} r_l \varphi_l$$

for some $r_l \in R$. We have $r_l = \sum_{w \in W_I} r_{l,w} \tau_w$ where $r_{l,w} \in R^I$ by Theorem 3.35. This gives

$$\begin{aligned} \psi(m) = \varphi(1 \otimes m \otimes 1) = \sum_{l=1}^{k} r_l \cdot \varphi_l(1 \otimes m \otimes 1) = \sum_{l=1}^{k} \sum_{w \in W_I} r_{w,l} \cdot \tau_w \cdot \varphi_l(1 \otimes m \otimes 1) \\ = \sum_{l_1}^{k} \sum_{w \in W_I} r_{w,l} \cdot \varphi_l(\tau_w \otimes m \otimes 1) = \sum_{l_1}^{k} \sum_{w \in W_I} r_{w,l} \cdot \psi_{l,w}(m). \end{aligned}$$

Now we prove linear independence. Suppose

$$0 = \sum_{l=1}^{k} \sum_{w \in W_I} r_{l,w} \cdot \psi_{l,w}$$

for some $r_{l,w} \in R^I$. This implies for all $m \in M$

$$\begin{aligned} 0 &= \sum_{l=1}^{k} \sum_{w \in W_I} r_{l,w} \cdot \psi_{l,w}(m) = \sum_{l=1}^{k} \sum_{w \in W_I} r_{l,w} \cdot \varphi_l(\tau_w \otimes m \otimes 1) \\ &= \sum_{l=1}^{k} \left(\sum_{w \in W_I} r_{l,w} \tau_w \right) \varphi_l(1 \otimes m \otimes 1). \end{aligned}$$

By multiplying with r from the left and r' from the right, this gives

$$0 = \sum_{l=1}^{k} \left(\sum_{w \in W_I} r_{l,w} \tau_w \right) \varphi_l(r \otimes m \otimes r')$$

for all $r, r' \in R, m \in M$, and thus

$$0 = \sum_{l=1}^{k} \left(\sum_{w \in W_I} r_{l,w} \tau_w \right) \varphi_l.$$

As $\{\varphi_1, \ldots, \varphi_k\}$ is a basis this implies

$$0 = \sum_{w \in W_I} r_{l,w} \tau_w$$

for $l = 1, \ldots, k$ and since $\{\tau_w\}_{w \in W_I}$ is a basis we get $r_{l,w} = 0$. This gives us linear independence and finishes the proof. □

5.3 The category ${}_1\mathbb{S}\mathrm{Bim}_2$

We consider now ${}_1\mathbb{S}\mathrm{Bim}_2$ whose elements are (R^{s_1}, R^{s_2})-bimodules. In ${}_1\mathbb{S}\mathrm{Bim}_2$ we have the following bimodules

$$\mathbb{I}_1 = R, \mathbb{I}_2 = R^1 \otimes_{R^S} R^2\langle 1 \rangle.$$

As they are generated by 1 and $1 \otimes 1$ as bimodules Lemma 2.59 implies that these are indecomposable.

Remark 5.15. Let $M \in {}_1\text{s}\mathbb{B}\mathbb{S}\text{Bim}_2$. Then as in Remark 5.3 we get that

$$M^{\oplus L} \cong R \otimes_{R^{J_1}} \cdots \otimes_{R^{J_N}} R.$$

The right hand side can be decomposed into the six indecomposable bimodules for $\mathbb{S}$Bim. Since this decomposition is an isomorphism of (R, R)-bimodules it is also an isomorphism of (R^1, R^2)-bimodules. Hence, it is enough to decompose the six indecomposables of $\mathbb{S}$Bim into $\mathbb{I}_1$ and $\mathbb{I}_2$.
Note that this reduction works for all the categories ${}_I\mathbb{S}\text{Bim}_J$. So, for all the other cases we will just decompose the six indecomposables for $\mathbb{S}$Bim and not repeat this argument. ♢

Lemma 5.16. *We have isomorphisms in* ${}_1\mathbb{S}\text{Bim}_2$*:*

1. $B_e \cong \mathbb{I}_1$.
2. $B_1 \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle$.
3. $B_2 \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle$.
4. $B_{12} \cong \mathbb{I}_1\langle 2\rangle \oplus \mathbb{I}_1^{\oplus 2} \oplus \mathbb{I}_1\langle -2\rangle$.
5. $B_{21} \cong \mathbb{I}_1 \oplus \mathbb{I}_2\langle 1\rangle \oplus \mathbb{I}_2\langle -1\rangle$.
6. $B_{w_0} \cong \mathbb{I}_2\langle -2\rangle \oplus \mathbb{I}_2 \oplus \mathbb{I}_2\langle 2\rangle$.

Proof.

1. This is actually an equality.
2. We can use (5.1) to get

$$\begin{aligned} B_1 &= R \otimes_{R^1} R\langle 1\rangle \cong R^1 \otimes_{R^1} R\langle 1\rangle \oplus R^1\langle -2\rangle \otimes_{R^1} R\langle 1\rangle \\ &= \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle. \end{aligned}$$

3. We again use (5.1) to get

$$\begin{aligned} B_2 &= R \otimes_{R^2} R\langle 1\rangle \cong R \otimes_{R^2} R^2\langle 1\rangle \oplus R \otimes_{R^2} R^2\langle -1\rangle \\ &= \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle. \end{aligned}$$

4. Similar to the previous points (5.1) implies

$$\begin{aligned} B_{12} &= R \otimes_{R^1} R \otimes_{R^2} R\langle 2\rangle \\ &\cong \left(R^1 \oplus R^1\langle -2\rangle\right) \otimes_{R^1} R \otimes_{R^2} \left(R^2 \oplus R^2\langle -2\rangle\right)\langle 2\rangle \\ &\cong \mathbb{I}_1\langle 2\rangle \oplus \mathbb{I}_1^{\oplus 2} \oplus \mathbb{I}_1\langle -2\rangle. \end{aligned}$$

5. In this case we need write out the projections and inclusions explicitly. For this we will use the presentation $R \cong \Bbbk[x,y,z]$.

$$\begin{array}{rcl} B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle & \longrightarrow & \mathbb{I}_1 = R \\ r_1 \otimes r_2 \otimes r_3 & \longmapsto & -\partial_1(r_1 r_2) r_3 \end{array}$$

$$\begin{array}{rcl} \mathbb{I}_1 = R & \longrightarrow & B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle \\ r & \longmapsto & (\alpha_2 \otimes 1 \otimes 1 + 1 \otimes \alpha_2 \otimes 1) \cdot \frac{r}{2}. \end{array}$$

This gives us the first summand. For the next morphisms let $P_i : R \longrightarrow R^i, r \longmapsto \frac{r+s_i(r)}{2}$ for $i \in S$.

$$\begin{array}{rcl} B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle & \longrightarrow & \mathbb{I}_2\langle -1\rangle = R^1 \otimes_{R^S} R^2 \\ r_1 \otimes r_2 \otimes r_3 & \longmapsto & A \end{array}$$

$$\begin{array}{rcl} \mathbb{I}_2\langle -1\rangle = R^1 \otimes_{R^S} R^2 & \longrightarrow & B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle \\ r_1 \otimes r_2 & \longmapsto & r_1 \otimes 1 \otimes r_2\alpha_2 \end{array}$$

where A is defined as follows.

$$\begin{aligned} A = & \frac{1}{8}\partial_1(r_1) \cdot \left(\begin{array}{c} \left(x + y \otimes \frac{2x+y+z}{2}\right) - (2xy \otimes 1) \\ -(1 \otimes xy + xz) \end{array} \right) \cdot \partial_2\left(\partial_1(r_2) r_3\right) \\ & + \frac{1}{4}\partial_1(r_1) \cdot \left(\left(x + y \otimes \frac{1}{2}\right) - (1 \otimes x) \right) \cdot P_2\left(\partial_1(r_2) r_3\right) \\ & + \frac{1}{4}\partial_1(r_1) \cdot \left((1 \otimes x) - (z \otimes 1)\right) \cdot \partial_2\left(P_1(r_2) r_3\right) \\ & + \frac{1}{2}\partial_1(r_1) \cdot (1 \otimes 1) \cdot P_2\left(P_1(r_2) r_3\right) \\ & + \frac{1}{4}P_1(r_1) \cdot \left((x + y - z \otimes 1) - \left(1 \otimes \frac{y+z}{2}\right) \right) \cdot \partial_2\left(\partial_1(r_2) r_3\right) \\ & + \frac{1}{2}P_1(r_1) \cdot \left(1 \otimes \frac{1}{2}\right) \cdot P_2\left(\partial_1(r_2) r_3\right) \\ & + \frac{1}{2}P_1(r_1) \cdot (1 \otimes 1) \cdot \partial_2\left(P_1(r_2) r_3\right) \end{aligned}$$

This gives us the second summand. The last summand will be given by the following morphisms.

$$\begin{array}{rcl} B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle & \longrightarrow & \mathbb{I}_2\langle 1\rangle = R^1 \otimes_{R^S} R^2\langle 2\rangle \\ r_1 \otimes r_2 \otimes r_3 & \longmapsto & A' \end{array}$$

$$\begin{array}{rcl} \mathbb{I}_2\langle 1\rangle = R^1 \otimes_{R^S} R^2\langle 2\rangle & \longrightarrow & B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle \\ r_1 \otimes r_2 & \longmapsto & r_1 \otimes 1 \otimes r_2 \end{array}$$

where

$$
\begin{aligned}
A' = \frac{1}{8}\partial_1(r_1)\cdot &\left(\left(x+y\otimes\frac{(y-z)^2}{2}\right)-\left(1\otimes x(y-z)^2\right)\right)\cdot\partial_2\left(\partial_1(r_2)r_3\right)\\
&+\frac{1}{4}\partial_1(r_1)\cdot\left(\begin{array}{c}\left(x+y\otimes\frac{2x+y+z}{2}\right)-(2xy\otimes 1)\\ -(1\otimes xy+xz)\end{array}\right)\cdot P_2\left(\partial_1(r_2)r_3\right)\\
&+\frac{1}{4}\partial_1(r_1)\cdot\left(1\otimes(y-z)^2\right)\cdot\partial_2\left(P_1(r_2)r_3\right)\\
&+\frac{1}{2}\partial_1(r_1)\cdot((1\otimes x)-(z\otimes 1))\cdot P_2\left(P_1(r_2)r_3\right)\\
&+\frac{1}{4}P_1(r_1)\cdot\left(1\otimes\frac{(y-z)^2}{2}\right)\cdot\partial_2\left(\partial_1(r_2)r_3\right)\\
&+\frac{1}{2}P_1(r_1)\cdot\left((x+y-z\otimes 1)-\left(1\otimes\frac{y+z}{2}\right)\right)\cdot P_2\left(\partial_1(r_2)r_3\right)\\
&+P_1(r_1)\cdot(1\otimes 1)\cdot P_2\left(P_1(r_2)r_3\right).
\end{aligned}
$$

Now we can compose the projections and inclusions to get three idempotents. Then one can check that these idempotents are orthogonal and their sum is the identity. Thus,

$$B_{21}\cong\mathbb{I}_1\oplus\mathbb{I}_2\langle 1\rangle\oplus\mathbb{I}_2\langle -1\rangle.$$

6. From (5.1) we get that

$$
\begin{aligned}
B_{w_0} &= R\otimes_{R^S}R\langle 3\rangle\cong\left(R^1\oplus R^1\langle -2\rangle\right)\otimes_{R^S}\left(R^2\oplus R^2\langle -2\rangle\right)\langle 3\rangle\\
&=\mathbb{I}_2\langle -2\rangle\oplus\mathbb{I}_2\oplus\mathbb{I}_2\langle 2\rangle.
\end{aligned}
$$ □

Remark 5.17. Note that via the identification

$$
\begin{aligned}
B_w &\longleftrightarrow \underline{H}_w\\
\mathbb{I}_1 &\longleftrightarrow {}^1\underline{H}_p^2\\
\mathbb{I}_2 &\longleftrightarrow {}^1\underline{H}_q^2
\end{aligned}
$$

this lemma categorifies Proposition 2.47. ◊

Now all that is left is to find bases for the homomorphism spaces between $\mathbb{I}_1$ and $\mathbb{I}_2$. Let $k,l\in\{1,2\}$. Since $\mathbb{I}_k$ is generated by the 1-tensor we get that every element of $\mathrm{Hom}_{(R^1,R^2)}(\mathbb{I}_k,\mathbb{I}_l)$ is the determined by its image of the 1-tensor. This gives us the following.

Theorem 5.18. *We have isomorphisms in* ${}_1\mathbb{S}\mathrm{Bim}_2$*:*

1. $\mathrm{Hom}_{(R^1,R^2)}(\mathbb{I}_1,\mathbb{I}_1)\cong\mathbb{I}_1,\varphi\longmapsto\varphi(1)$.
2. $\mathrm{Hom}_{(R^1,R^2)}(\mathbb{I}_2,\mathbb{I}_1)\cong\mathbb{I}_1,\varphi\longmapsto\varphi(1\otimes 1)$.

3. $\mathrm{Hom}_{(R^1,R^2)}(\mathbb{I}_2,\mathbb{I}_2) \cong \mathbb{I}_2, \varphi \longmapsto \varphi(1\otimes 1)$.

This does not work for $\mathrm{Hom}_{(R^1,R^2)}(\mathbb{I}_1,\mathbb{I}_2)$, as for example the map $1 \longmapsto 1\otimes 1$ is not well-defined (it is not a morphism of (R^1,R^2)-bimodules). So, we need to do some work to understand this homomorphism space.
By Lemma 5.13 and the discussion leading to this lemma we need to find a basis of $\mathrm{Hom}_{(R,R)}(R\otimes_{R^1} R\otimes_{R^2} R, R\otimes_{R^S} R)$. We will use the fact that $B_1B_2B_1 \cong B_1 \oplus R\otimes_{R^S} R$. Thus, we want a basis of $\mathrm{Hom}_{(R,R)}(B_1B_2, B_1B_2B_1)$. For this we can use Theorem 5.9 and get the following basis.

$$\varphi_1 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{8} r_1r_2r_3 \cdot \begin{pmatrix} \alpha_1\otimes\alpha_2\otimes\alpha_1\otimes 1 + \alpha_1\otimes\alpha_2\otimes 1\otimes\alpha_1 \\ +\alpha_1\otimes 1\otimes\alpha_2\alpha_1\otimes 1 + \alpha_1\otimes 1\otimes\alpha_2\otimes\alpha_1 \\ +1\otimes\alpha_1\alpha_2\otimes\alpha_1\otimes 1 + 1\otimes\alpha_1\alpha_2\otimes 1\otimes\alpha_1 \\ +1\otimes\alpha_1\otimes\alpha_2\alpha_1\otimes 1 + 1\otimes\alpha_1\otimes\alpha_2\otimes\alpha_1 \end{pmatrix}$$

$$\varphi_2 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{4} r_1r_2 \cdot \begin{pmatrix} \alpha_1\otimes 1\otimes\alpha_1\otimes 1 + \alpha_1\otimes 1\otimes 1\otimes\alpha_1 \\ +1\otimes\alpha_1\otimes\alpha_1\otimes 1 + 1\otimes\alpha_1\otimes 1\otimes\alpha_1 \end{pmatrix} \cdot r_3$$

$$\varphi_3 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{4} r_1 \cdot \begin{pmatrix} \alpha_1\otimes\alpha_2\otimes 1\otimes 1 + \alpha_1\otimes 1\otimes\alpha_2\otimes 1 \\ +1\otimes\alpha_1\alpha_2\otimes 1\otimes 1 + 1\otimes\alpha_1\otimes\alpha_2\otimes 1 \end{pmatrix} \cdot r_2r_3$$

$$\varphi_4 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{2}\left(r_1\otimes r_2\otimes\alpha_1\otimes r_3 + r_1\otimes r_2\otimes 1\otimes\alpha_1 r_3\right)$$

$$\varphi_5 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{4} r_1r_2r_3 \cdot \begin{pmatrix} \alpha_1\otimes\alpha_2\otimes 1\otimes 1 + \alpha_1\otimes 1\otimes\alpha_2\otimes 1 \\ +1\otimes\alpha_2\otimes 1\otimes\alpha_1 + 1\otimes 1\otimes\alpha_2\otimes\alpha_1 \end{pmatrix}$$

$$\varphi_6 : r_1\otimes r_2\otimes r_3 \longmapsto \frac{1}{2} r_1 \cdot (1\otimes\alpha_2\otimes 1\otimes 1 + 1\otimes 1\otimes\alpha_2\otimes 1)\cdot r_2r_3$$

Let $\mathrm{pr} : B_1B_2B_1 \longrightarrow R\otimes_{R^S} R$ be the projection. Then we can check that $\mathrm{pr}\circ\varphi_5 = \mathrm{pr}\circ\varphi_6 = 0$. Hence, $\{\mathrm{pr}\circ\varphi_1, \mathrm{pr}\circ\varphi_2, \mathrm{pr}\circ\varphi_3, \mathrm{pr}\circ\varphi_4\}$ is a basis of $\mathrm{Hom}_{(R,R)}(R\otimes_{R^1} R\otimes_{R^2} R, R\otimes_{R^S} R)$. By Lemma 5.14 we now get a basis as left R^1-module

$$\{\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8\} \subset \mathrm{Hom}_{(R^1,R^2)}(R, R\otimes_{R^S} R).$$

We can then use that $R\otimes_{R^S} R \cong \left(R^1\otimes_{R^S} R^2\right)^{\oplus 4}$ via the projections $P_1\otimes P_2, P_1\otimes\partial_2, \partial_1\otimes P_2, \partial_1\otimes\partial_2$. From this we get that $\mathrm{Hom}_{(R^1,R^2)}(R, R^1\otimes_{R^S} R^2)$ has the following basis as left R^1-module

$$\begin{aligned} \phi_1 : r \longmapsto\ & P_1(r)\cdot((1\otimes x) - (z\otimes 1)) \\ & +\tfrac{1}{2}\partial_1(r)\cdot\left(\left(1\otimes x^2 + yz\right) - (xy\otimes 1) - (z\otimes x)\right) \\ \phi_2 : r \longmapsto\ & P_1(r)\cdot\left(\left(1\otimes x^2 + yz\right) - (xy\otimes 1) - (z\otimes x)\right) \\ & +\tfrac{1}{2}\partial_1(r)(x-y)^2\cdot((1\otimes x) - (z\otimes 1)). \end{aligned} \tag{5.3}$$

5.4 The other categories

In this section we will only state the results. All the proofs work similar to the proofs in the last section.

5.4.1 ${}_2\mathbb{S}\mathrm{Bim}_1$

We can swap the roles of s_1 and s_2 in S_3 and get S_3 again. Via this symmetry the category ${}_2\mathbb{S}\mathrm{Bim}_1$ is completely symmetric to ${}_1\mathbb{S}\mathrm{Bim}_2$.

5.4.2 ${}_1\mathbb{S}\mathrm{Bim}_1$ and ${}_2\mathbb{S}\mathrm{Bim}_2$

Again via the symmetry of s_1 and s_2 it is enough to state results for ${}_1\mathbb{S}\mathrm{Bim}_1$. We have the following indecomposable bimodules in ${}_1\mathbb{S}\mathrm{Bim}_1$

$$\mathbb{I}_1 = R^1\langle -1\rangle, \mathbb{I}_2 = R^1 \otimes_{R^S} R^1\langle 1\rangle.$$

Theorem 5.19. *We have isomorphisms in* ${}_1\mathbb{S}\mathrm{Bim}_1$*:*

1. $B_e \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle$.
2. $B_1 \cong \mathbb{I}_1\langle 2\rangle \oplus \mathbb{I}_1^{\oplus 2} \oplus \mathbb{I}_1\langle -2\rangle$.
3. $B_2 \cong \mathbb{I}_1 \oplus \mathbb{I}_2$.
4. $B_{12} \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle \oplus \mathbb{I}_2\langle 1\rangle \oplus \mathbb{I}_2\langle -1\rangle$.
5. $B_{21} \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle \oplus \mathbb{I}_2\langle 1\rangle \oplus \mathbb{I}_2\langle -1\rangle$.
6. $B_{w_0} \cong \mathbb{I}_2\langle 2\rangle \oplus \mathbb{I}_2^{\oplus 2} \oplus \mathbb{I}_2\langle -2\rangle$.

Remark 5.20. Note that via the identification

$$\begin{aligned} B_w &\longleftrightarrow \underline{H}_w \\ \mathbb{I}_1 &\longleftrightarrow {}^1\underline{H}_p^1 \\ \mathbb{I}_2 &\longleftrightarrow {}^1\underline{H}_q^1 \end{aligned}$$

this theorem categorifies Proposition 2.48. ◊

Theorem 5.21.

1. *The space* $\mathrm{Hom}_{(R^1,R^1)}(\mathbb{I}_1, \mathbb{I}_2)$ *has rank* 1 *as left* R^1*-module with basis given by*

$$\phi : r_i \longmapsto r_i \cdot ((xy - xz - yz \otimes 1) + (1 \otimes xy - xz - yz) + (2z \otimes z)).$$

2. *The remaining spaces of the form* $\mathrm{Hom}_{(R^1,R^1)}(\mathbb{I}_k, \mathbb{I}_l)$ *for* $k, l \in \{1, 2\}$ *are isomorphic to* $\mathbb{I}_l$ *via the mapping* $\varphi \longmapsto \varphi(1^{\otimes})$.

5.4.3 ${}_1\mathbb{S}\mathrm{Bim}$ and ${}_2\mathbb{S}\mathrm{Bim}$

Again via the symmetry of s_1 and s_2 it is enough to state results for ${}_1\mathbb{S}\mathrm{Bim}$. We have the following indecomposable bimodules in ${}_1\mathbb{S}\mathrm{Bim}$

$$\mathbb{I}_1 = R, \mathbb{I}_2 = R \otimes_{R^2} R\langle 1\rangle, \mathbb{I}_3 = R^1 \otimes_{R^S} R\langle 2\rangle.$$

Theorem 5.22. *We have isomorphisms in* ${}_1\mathbb{S}\mathrm{Bim}$*:*

1. $B_e \cong \mathbb{I}_1$.
2. $B_1 \cong \mathbb{I}_1\langle 1\rangle \oplus \mathbb{I}_1\langle -1\rangle$.
3. $B_2 \cong \mathbb{I}_2$.
4. $B_{12} \cong \mathbb{I}_2\langle 1\rangle \oplus \mathbb{I}_2\langle -1\rangle$.
5. $B_{21} \cong \mathbb{I}_1 \oplus \mathbb{I}_3$.
6. $B_{w_0} \cong \mathbb{I}_3\langle 1\rangle \oplus \mathbb{I}_3\langle -1\rangle$.

Remark 5.23. Note that via the identification

$$\begin{aligned} B_w &\longleftrightarrow \underline{H}_w \\ \mathbb{I}_1 &\longleftrightarrow {}^1\underline{H}_p \\ \mathbb{I}_2 &\longleftrightarrow {}^1\underline{H}_q \\ \mathbb{I}_3 &\longleftrightarrow {}^1\underline{H}_r \end{aligned}$$

this theorem categorifies Proposition 2.49. $\Diamond$

Theorem 5.24.

1. *The space* $\mathrm{Hom}_{(R^1,R)}(\mathbb{I}_1, \mathbb{I}_2)$ *has rank* 2 *as left* R^1*-module with basis given by*

$$\begin{aligned} \phi_1 &: r \longmapsto r \cdot (\alpha_2 \otimes 1 + 1 \otimes \alpha_2) \\ \phi_2 &: r \longmapsto r\alpha_1 \cdot (\alpha_2 \otimes 1 + 1 \otimes \alpha_2) . \end{aligned}$$

2. *The space* $\mathrm{Hom}_{(R^1,R)}(\mathbb{I}_1, \mathbb{I}_3)$ *has rank* 2 *as left* R^1*-module with basis given by*

$$\begin{aligned} \phi_1 : r \longmapsto & \; P_1(r) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \\ & + \tfrac{1}{2}\partial_1(r) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot \alpha_1 \\ \phi_2 : r \longmapsto & \; P_1(r) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot \alpha_1 \\ & + \tfrac{1}{2}\partial_1(r)\alpha_1^2 \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) . \end{aligned}$$

3. The space $\mathrm{Hom}_{(R^1,R)}(\mathbb{I}_2, \mathbb{I}_3)$ *has rank* 4 *as left* R^1*-module with basis given by*

$$\begin{aligned}
\phi_1 : r_1 \otimes r_2 \longmapsto & \; P_1(r_1) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot r_2 \\
& + \tfrac{1}{2}\partial_1(r_1) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot \alpha_1 r_2 \\
\phi_2 : r_1 \otimes r_2 \longmapsto & \; P_1(r_1) \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot \alpha_1 r_2 \\
& + \tfrac{1}{2}\partial_1(r_1)\alpha_1^2 \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz)) \cdot r_2 \\
\phi_3 : r_1 \otimes r_2 \longmapsto & \; P_1(r_1) \cdot ((1 \otimes x) - (z \otimes 1)) \cdot r_2 \\
& + \tfrac{1}{2}\partial_1(r_1) \cdot \big((1 \otimes x^2 + yz) - (xy \otimes 1) - (z \otimes x)\big) \cdot r_2 \\
\phi_4 : r_1 \otimes r_2 \longmapsto & \; P_1(r_1) \cdot \big((1 \otimes x^2 + yz) - (xy \otimes 1) - (z \otimes x)\big) \cdot r_2 \\
& + \tfrac{1}{2}\partial_1(r_1)\alpha_1^2 \cdot ((1 \otimes x) - (z \otimes 1)) \cdot r_2.
\end{aligned}$$

4. The remaining spaces of the form $\mathrm{Hom}_{(R^1,R)}(\mathbb{I}_k, \mathbb{I}_l)$ *for* $k, l \in \{1, 2, 3\}$ *are isomorphic to* $\mathbb{I}_l$ *via the mapping* $\varphi \longmapsto \varphi(1^{\otimes})$.

5.4.4 $\mathbb{S}\mathrm{Bim}_1$ and $\mathbb{S}\mathrm{Bim}_2$

Again via the symmetry of s_1 and s_2 it is enough to state results for $\mathbb{S}\mathrm{Bim}_1$. We have the following indecomposable bimodules in $\mathbb{S}\mathrm{Bim}_1$

$$\mathbb{I}_1 = R, \mathbb{I}_2 = R \otimes_{R^2} R\langle 1 \rangle, \mathbb{I}_3 = R \otimes_{R^S} R^1\langle 2 \rangle.$$

Theorem 5.25. *We have isomorphisms in* $\mathbb{S}\mathrm{Bim}_1$*:*

1. $B_e \cong \mathbb{I}_1$.
2. $B_1 \cong \mathbb{I}_1\langle 1 \rangle \oplus \mathbb{I}_1\langle -1 \rangle$.
3. $B_2 \cong \mathbb{I}_2$.
4. $B_{12} \cong \mathbb{I}_2\langle 1 \rangle \oplus \mathbb{I}_2\langle -1 \rangle$.
5. $B_{21} \cong \mathbb{I}_1 \oplus \mathbb{I}_3$.
6. $B_{w_0} \cong \mathbb{I}_3\langle 1 \rangle \oplus \mathbb{I}_3\langle -1 \rangle$.

Remark 5.26. Note that via the identification

$$\begin{aligned}
B_w &\longleftrightarrow \underline{H}_w \\
\mathbb{I}_1 &\longleftrightarrow \underline{H}_p^1 \\
\mathbb{I}_2 &\longleftrightarrow \underline{H}_q^1 \\
\mathbb{I}_3 &\longleftrightarrow \underline{H}_r^1
\end{aligned}$$

this theorem categorifies Proposition 2.50. ◊

Theorem 5.27.

1. *The space* $\mathrm{Hom}_{(R,R^1)}(\mathbb{I}_1,\mathbb{I}_2)$ *has rank* 1 *as left* R*-module with basis given by*

$$\phi : r \longmapsto r \cdot (\alpha_2 \otimes 1 + 1 \otimes \alpha_2)\,.$$

2. *The space* $\mathrm{Hom}_{(R,R^1)}(\mathbb{I}_1,\mathbb{I}_3)$ *has rank* 1 *as left* R*-module with basis given by*

$$\phi : r \longmapsto r \cdot ((z \otimes z) + (xy \otimes 1) - (1 \otimes xz + yz))\,.$$

3. *The space* $\mathrm{Hom}_{(R,R^1)}(\mathbb{I}_2,\mathbb{I}_3)$ *has rank* 2 *as left* R*-module with basis given by*

$$\begin{aligned}\phi_1 : r_1 \otimes r_2 &\longmapsto r_1 r_2 \cdot ((xy \otimes 1) + (z \otimes z) - (1 \otimes xz + yz))\\ \phi_2 : r_1 \otimes r_2 &\longmapsto \begin{array}{l} r_1 \cdot ((x \otimes 1) - (1 \otimes z)) \cdot P_1(r_2) \\ + r_1 \cdot \left(\left(x^2 + yz \otimes 1\right) - (x \otimes z) - (1 \otimes xy)\right) \cdot \frac{1}{2}\partial_1(r_2). \end{array}\end{aligned}$$

4. *The remaining spaces of the form* $\mathrm{Hom}_{(R,R^1)}(\mathbb{I}_k,\mathbb{I}_l)$ *for* $k,l \in \{1,2,3\}$ *are isomorphic to* $\mathbb{I}_l$ *via the mapping* $\varphi \longmapsto \varphi(1^{\otimes})$.

5.4.5 All remaining categories

We can consider all the remaining categories ${}_I\mathbb{S}\mathrm{Bim}_J$ together. They have one thing in common, namely that $I = S$ or $J = S$. There will only be one indecomposable bimodule

$$\mathbb{I} = R^{I \cap J}\langle -|I \cap J|\rangle.$$

Then the space $\mathrm{Hom}_{(R^I,R^J)}(\mathbb{I},\mathbb{I})$ is isomorphic to $\mathbb{I}$ via $\varphi \longmapsto \varphi(1)$. We have the following decomposition lemma.

Lemma 5.28. *If* $I = S$ *or* $J = S$*, then all objects in* ${}_I\mathbb{S}\mathrm{Bim}_J$ *decompose into sums of shifts of* $\mathbb{I}$.

One can observe this by decomposing the indecomposable bimodules for $\mathbb{S}\mathrm{Bim}$ into $\mathbb{I}$ by using the R^S-module structure from one side to erase all the tensor products. We will do one example which makes clear what is meant by that. Consider ${}_S\mathbb{S}\mathrm{Bim}_1$ where $\mathbb{I} = R^1\langle -1\rangle$. We will decompose B_{21}.

$$\begin{aligned} B_{21} = R \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle &\cong \left(R^2 \oplus R^2\langle -2\rangle\right) \otimes_{R^2} R \otimes_{R^1} R\langle 2\rangle \\ &= R \otimes_{R^1} R\langle 2\rangle \oplus R \otimes_{R^1} R \\ &\cong \left(R^1 \oplus R^1\langle -2\rangle\right) \otimes_{R^1} R\langle 2\rangle \oplus \left(R^1 \oplus R^1\langle -2\rangle\right) \otimes_{R^1} R \\ &= R\langle 2\rangle \oplus R^{\oplus 2} \oplus R\langle -2\rangle \\ &\cong R^1\langle 2\rangle \oplus \left(R^1\right)^{\oplus 3} \oplus \left(R^1\langle -2\rangle\right)^{\oplus 3} \oplus R^1\langle -4\rangle \\ &= \mathbb{I}\langle 3\rangle \oplus \mathbb{I}\langle 1\rangle^{\oplus 3} \oplus \mathbb{I}\langle -1\rangle^{\oplus 3} \oplus \mathbb{I}\langle -3\rangle. \end{aligned}$$

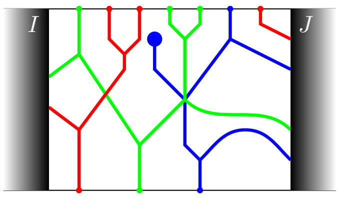
6 Diagrammatics in the singular case

6.1 Diagrammatics for (R^I, R^J)-bimodules

In this section we want to develop a diagrammatic presentation for the category ${}_I\mathbb{BS}\mathrm{Bim}_J$ of Bott–Samelson bimodules viewed as (R^I, R^J)-bimodules via restriction for some parabolic subsets I and J. This is a good first step to finding a diagrammatic presentation for singular Soergel bimodules as they are the Karoubi envelope of ${}_I\mathbb{BS}\mathrm{Bim}_J$. The results in this section are a generalization of the results of Elias [Eli16, Section 5] and the proofs are very similar to his work.

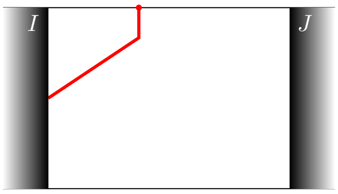
Definition 6.1. We define the category ${}_I\mathcal{T}_J$ as follows. Objects are sequences $\underline{i}$ of indices in S, just as for $\mathcal{D}_1$. Morphisms between $\underline{i}$ and $\underline{j}$ are again given by ($\Bbbk$-linear combinations of) coloured graphs in the strand $\mathbb{R} \times [0, 1]$ with appropriate top and bottom boundary. This time these pictures include a membrane on the left, labelled I, and a membrane on the right, labelled J. The pictures are constructed out of the generators of $\mathcal{D}_1$ and the thick trivalent vertex (see Definition 4.30) which is the only interaction with the membranes. Strands running into a membrane (via the thick trivalent vertex) must be labelled $i \in I$ on the left and $j \in J$ on the right.
The relations for the morphisms are given by those of $\mathcal{D}_1$ and the relations (4.48) to (4.51) (where the thick lines are substituted by the membranes). ♢

For example a morphism in ${}_I\mathcal{T}_J$ could look like this.

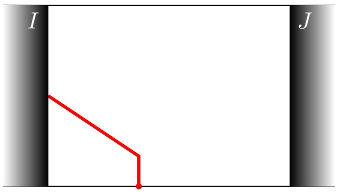


We view the morphisms as being equipped with a left-R^I-module structure and a right-R^J-module structure by placing symmetric polynomials directly on the right of the left membrane respectively directly on the left of the right membrane. This is well-defined, i.e. it does not matter in which region directly next to a membrane we place the polynomials. That is because the polynomials can slide (via (4.8)) through every strand that is connected to the membrane, since such a strand is labelled with $i \in I$ or $j \in J$ and the polynomials live in R^I and R^J respectively.

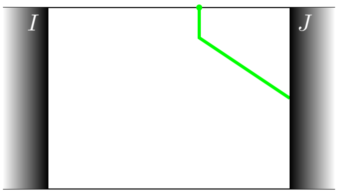
Definition 6.2. There is a functor ${}_I\mathcal{F}_J : {}_I\mathcal{T}_J \longrightarrow {}_I\mathbb{BS}\mathrm{Bim}_J$ defined as follows. The object $\underline{i}$ is sent to $B_{\underline{i}}$ restricted on the left to R^I and on the right side to R^J. Morphisms in ${}_I\mathcal{T}_J$ which do not interact with the membranes are sent to (R, R)-bimodule morphisms, which are also (R^I, R^J)-bimodule morphisms, via $\mathcal{F}_1$ (see Definition 4.9). The images of the thick trivalent vertices are the following.



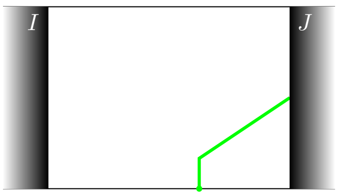
$$\begin{aligned} R &\longrightarrow R \otimes_{R^i} R \\ r &\longmapsto 1 \otimes r \end{aligned}$$



$$\begin{aligned} R \otimes_{R^i} R &\longrightarrow R \\ r_1 \otimes r_2 &\longmapsto \partial_i(r_1) r_2 \end{aligned}$$



$$\begin{aligned} R &\longrightarrow R \otimes_{R^j} R \\ r &\longmapsto r \otimes 1 \end{aligned}$$



$$\begin{aligned} R \otimes_{R^j} R &\longrightarrow R \\ r_1 \otimes r_2 &\longmapsto r_1 \partial_j(r_2) \end{aligned}$$

${}_I\mathcal{F}_J$ is required to respect compositions and tensor products and is thus defined for all morphisms of ${}_I\mathcal{T}_J$. $\Diamond$

Definition 6.3. There is a functor ${}_I\mathcal{G}_J : {}_I\mathcal{T}_J \longrightarrow g\mathcal{D}$ defined as follows. The object $\underline{i}$ in ${}_I\mathcal{T}_J$ is sent to $I \otimes \underline{i} \otimes J$ in $g\mathcal{D}$. The functor is given on morphisms by interpreting the two membranes as thick lines labelled I and J respectively. $\Diamond$

Proposition 6.4.

1. *The functors ${}_I\mathcal{F}_J$ and ${}_I\mathcal{G}_J$ are well-defined and preserve the (R^I, R^J)-bimodule structure on* Hom *spaces.*

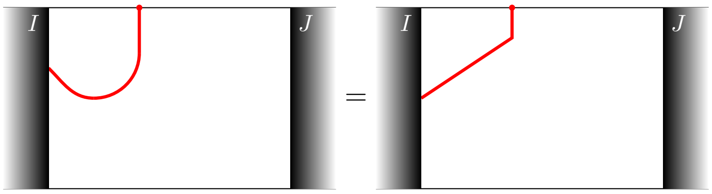
2. *The composition of functors $g\mathcal{F} \circ {}_I\mathcal{G}_J : {}_I\mathcal{T}_J \longrightarrow g\mathcal{D} \longrightarrow g\mathbb{BS}\mathrm{Bim}$ is equal to the composition of functors ${}_I\mathrm{ind}_J \circ {}_I\mathcal{F}_J : {}_I\mathcal{T}_J \longrightarrow {}_I\mathbb{BS}\mathrm{Bim}_J \longrightarrow g\mathbb{BS}\mathrm{Bim}$ where ${}_I\mathrm{ind}_J$ is induction from R^I to R on the left and from R^J to R on the right.*

Proof. The functor $g\mathcal{F}$ is well-defined as we know and the same is true for the induction functor. For the functors ${}_I\mathcal{F}_J$ and ${}_I\mathcal{G}_J$ all we need to check is that the relations in ${}_I\mathcal{T}_J$ hold true when sent to ${}_I\mathbb{BS}\mathrm{Bim}_J$ and $g\mathcal{D}$ via ${}_I\mathcal{F}_J$ and ${}_I\mathcal{G}_J$ respectively. For ${}_I\mathcal{G}_J$ this is

clear, since all relations in ${}_I\mathcal{T}_J$ come from relations we derived for $g\mathcal{D}$.
For ${}_I\mathcal{F}_J$ we already know that the relations coming from $\mathcal{D}$ are satisfied in $\mathbb{BS}$Bim, since we know that $\mathcal{F}_1$ is well-defined. Thus, these relations also hold in ${}_I\mathbb{BS}\mathrm{Bim}_J$, because restricting the module action on the sides does not influence them. Hence, all we need to check are the relations (4.48) to (4.51). This is done in Lemma 6.5.
For the two compositions to be equal we easily check that both of them sent the object $\underline{i}$ to $B_I B_{\underline{i}} B_J = R \otimes_{R^I} B_{\underline{i}} \otimes_{R^J} R$, and thus they are equal on objects. Hence, all there is to do is to check that the generating morphisms are sent to the same. This is done in Lemma 6.6.
The (R^I, R^J)-bimodule structure gets preserved by all four functors. For ${}_I\mathcal{F}_J$ and $g\mathcal{F}$ this is true by definition. For ${}_I\mathcal{G}_J$ and the induction functor this follows from the fact that symmetric polynomials slide through a thick line or a tensor product respectively. □

Lemma 6.5. *The relations (4.48) to (4.51) are preserved when passing to* ${}_I\mathbb{BS}\mathrm{Bim}_J$ *via* ${}_I\mathcal{F}_J$.

Proof. We will only check these relations for the left membrane as the calculations are completely analogous for the right membrane. Also note that all four relations are stated in a way that uses isotopy invariance, so we would need to check multiple iterations of them (for example (4.50) could be a morphism $B_i B_j \longrightarrow R$, but could also be a morphism $R \longrightarrow B_j B_i$). Instead we will check the relations (4.58) and (4.59) which give us the isotopy invariance we need and then we just check one iteration of each of the relations (4.48) to (4.51).
We begin with relation (4.58).



(4.58)

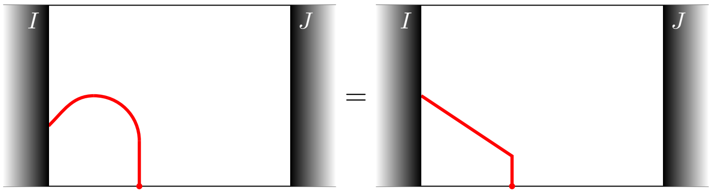
The right hand side is sent to

$$\begin{aligned} R &\longrightarrow R \otimes_{R^i} R \\ r &\longmapsto 1 \otimes r \end{aligned}$$

under ${}_I\mathcal{F}_J$. Now we just compute what the left hand side becomes under ${}_I\mathcal{F}_J$.

$$\begin{array}{ccccc} R & \xrightarrow{cup} & R \otimes_{R^i} R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \\ r & \longmapsto & (\alpha_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha_i) \cdot \frac{r}{2} & \longmapsto & (2 \otimes 1 + 0 \otimes \alpha_i) \cdot \frac{r}{2} = 1 \otimes r \end{array}$$

Here the second arrow was the image of the very thick trivalent vertex. Now we can check relation (4.59).



(4.59)

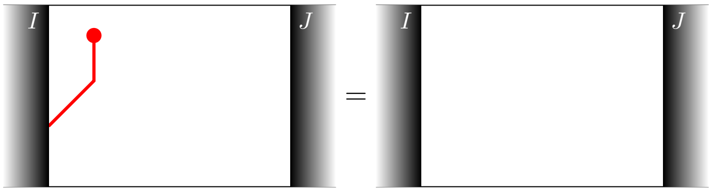
The right hand side is sent to

$$\begin{aligned} R \otimes_{R^i} R &\longrightarrow R \\ r_1 \otimes r_2 &\longmapsto \partial_i(r_1) r_2 \end{aligned}$$

under ${}_I\mathcal{F}_J$. Now we compute what the left hand side becomes under ${}_I\mathcal{F}_J$.

$$\begin{array}{ccccc} R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^i} R & \xrightarrow{cap} & R \\ r_1 \otimes r_2 & \longmapsto & 1 \otimes r_1 \otimes r_2 & \longmapsto & \partial_i(r_1) r_2 \end{array}$$

Here the first arrow was the image of the very thick trivalent vertex. Next we check one iteration of (4.48).

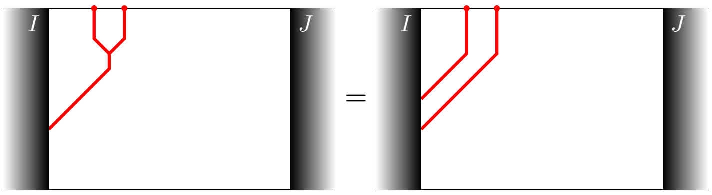


(4.48)

The right hand side is sent to the identity on R under ${}_I\mathcal{F}_J$. The left hand side is sent to the following composition under ${}_I\mathcal{F}_J$.

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^i} R & \xrightarrow{dot} & R \\ r & \longmapsto & 1 \otimes r & \longmapsto & r \end{array}$$

We will continue with checking one iteration of (4.49).



(4.49)

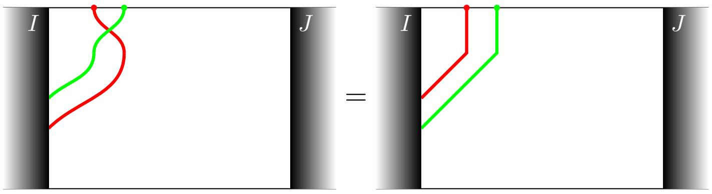
The right hand side is just two thick trivalent vertices, and thus is sent to

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^i} R \\ r & \longmapsto & 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes r \end{array}$$

under ${}_I\mathcal{F}_J$. So we just have to observe what the left hand side becomes after applying ${}_I\mathcal{F}_J$.

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^i} R \\ r & \longmapsto & 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes r \end{array}$$

Here the first arrow is the image of the thick trivalent vertex and the second arrow is the image of the normal trivalent vertex. We will continue with checking one iteration of (4.50).



(4.50)

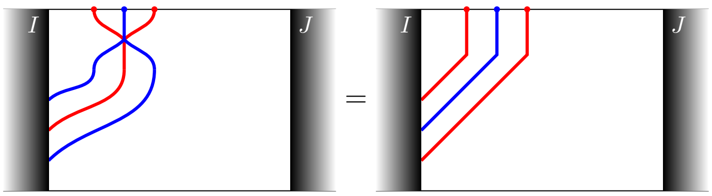
The right hand side is again just two thick trivalent vertices, and thus is sent to

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^j} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^j} R \\ r & \longmapsto & 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes r \end{array}$$

under ${}_I\mathcal{F}_J$. We compute that the left hand side is sent to

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^j} R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^j} R \\ r & \longmapsto & 1 \otimes 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes r \end{array}$$

under ${}_I\mathcal{F}_J$. Here the first arrow is the image of two thick trivalent vertices and the second arrow is the image of the 4-valent vertex. Now we are left with checking (4.51).



(4.51)

The right hand side is this time given by three thick trivalent vertices, and hence it is sent to

$$\begin{array}{ccccccc} R & \longrightarrow & R \otimes_{R^i} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^{i+1}} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^{i+1}} R \otimes_{R^i} R \\ r & \longmapsto & 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes 1 \otimes r \end{array}$$

under ${}_I\mathcal{F}_J$. So we compute what the left hand side is sent under ${}_I\mathcal{F}_J$.

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_{R^{i+1}} R \otimes_{R^i} R \otimes_{R^{i+1}} R & \longrightarrow & R \otimes_{R^i} R \otimes_{R^{i+1}} R \otimes_{R^i} R \\ r & \longmapsto & 1 \otimes 1 \otimes 1 \otimes r & \longmapsto & 1 \otimes 1 \otimes 1 \otimes r \end{array}$$

Here the first arrow is again the image of three thick trivalent vertices and the second arrow is the image of the 6-valent vertex. To be precise we would need to do the same calculation with i and $i+1$ swapped, but this is completely analogous, and thus we will omit it. This finishes the proof. □

Lemma 6.6. *The compositions $g\mathcal{F} \circ {}_I\mathcal{G}_J$ and ${}_I\mathrm{ind}_J \circ_I \mathcal{F}_J$, where ${}_I\mathrm{ind}_J$ is the induction functor, are the same on generating morphisms of ${}_I\mathcal{T}_J$.*

Proof. We will abbreviate the compositions as $\mathcal{G}_1 = g\mathcal{F} \circ {}_I\mathcal{G}_J$ and $\mathcal{G}_2 = {}_I\mathrm{ind}_J \circ_I \mathcal{F}_J$. Let first $\varphi \in \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j})$ be a generator from $\mathcal{D}$. Then we compute $\mathcal{G}_1(\varphi)$.

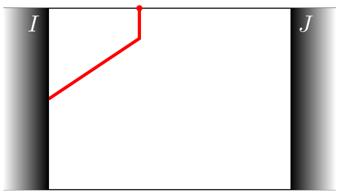
$$\begin{aligned}\mathcal{G}_1(\varphi) &= (g\mathcal{F} \circ {}_I\mathcal{G}_J)(\varphi) = g\mathcal{F}(\mathrm{id}_I \otimes \varphi \otimes \mathrm{id}_J) \\ &= \mathrm{id}_{B_I} \otimes \mathcal{F}(\varphi) \otimes \mathrm{id}_{B_J}\end{aligned}$$

Next we can compute $\mathcal{G}_2(\varphi)$ and observe that the two values are equal.

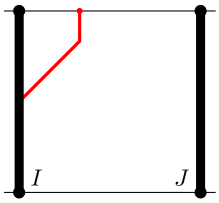
$$\begin{aligned}\mathcal{G}_2(\varphi) &= ({}_I\mathrm{ind}_J \circ_I \mathcal{F}_J)(\varphi) = {}_I\mathrm{ind}_J(\mathcal{F}(\varphi)) \\ &= \mathrm{id}_R \otimes_{R^I} \mathcal{F}(\varphi) \otimes_{R^J} \mathrm{id}_R = \mathrm{id}_{B_I} \otimes \mathcal{F}(\varphi) \otimes \mathrm{id}_{B_J} = \mathcal{G}_1(\varphi)\end{aligned}$$

So all the is left to do is to check it for the thick trivalent vertices. We will only do this for the left membrane, since the right side is completely analogous. Thus, we are left with two thick trivalent vertices and just need to send them through $\mathcal{G}_1$ and $\mathcal{G}_2$.

$\mathcal{G}_1$:

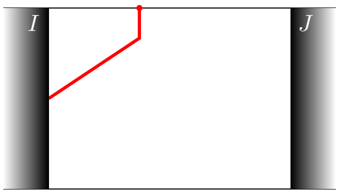


$\xmapsto{{}_I\mathcal{G}_J}$



$$\xmapsto{g\mathcal{F}} \begin{pmatrix} R \otimes_{R^I} \otimes R \otimes_{R^J} R & \longrightarrow & R \otimes_{R^I} R \otimes_{R^i} R \otimes_{R^J} R \\ r_1 \otimes r_2 \otimes r_3 & \longmapsto & r_1 \otimes 1 \otimes r_2 \otimes r_3 \end{pmatrix}$$

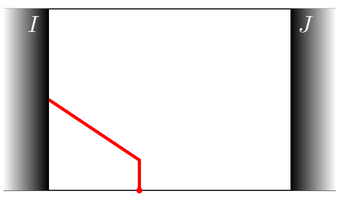
$\mathcal{G}_2$:



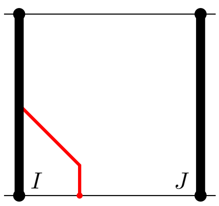
$$\xmapsto{{}_I\mathcal{F}_J} \begin{pmatrix} R & \longrightarrow & R \otimes_{R^i} R \\ r & \longmapsto & 1 \otimes r \end{pmatrix}$$

$$\xmapsto{{}_I\mathrm{ind}_J} \begin{pmatrix} R \otimes_{R^I} \otimes R \otimes_{R^J} R & \longrightarrow & R \otimes_{R^I} R \otimes_{R^i} R \otimes_{R^J} R \\ r_1 \otimes r_2 \otimes r_3 & \longmapsto & r_1 \otimes 1 \otimes r_2 \otimes r_3 \end{pmatrix}$$

$\mathcal{G}_1$:

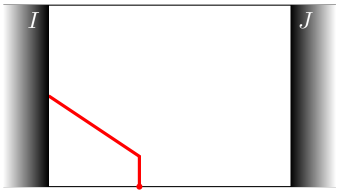


$\xmapsto{{}_I\mathcal{G}_J}$



$$\xmapsto{g\mathcal{F}} \begin{pmatrix} R \otimes_{R^I} R \otimes_{R^i} R \otimes_{R^J} R & \longrightarrow & R \otimes_{R^I} \otimes R \otimes_{R^J} R \\ r_1 \otimes r_2 \otimes r_3 \otimes r_4 & \longmapsto & r_1 \otimes \partial_i(r_2) r_3 \otimes r_4 \end{pmatrix}$$

$\mathcal{G}_2$:



$$\overset{{}_I\mathcal{F}_J}{\longmapsto} \begin{pmatrix} R \otimes_{R^i} R & \longrightarrow & R \\ r_1 \otimes r_2 & \longmapsto & \partial_i(r_1) r_2 \end{pmatrix}$$

$$\overset{{}_I\mathrm{ind}_J}{\longmapsto} \begin{pmatrix} R \otimes_{R^I} R \otimes_{R^i} R \otimes_{R^J} R & \longrightarrow & R \otimes_{R^I} \otimes R \otimes_{R^J} R \\ r_1 \otimes r_2 \otimes r_3 \otimes r_4 & \longmapsto & r_1 \otimes \partial_i(r_2) r_3 \otimes r_4 \end{pmatrix}$$

We observe that the thick trivalent vertices are sent to the same morphism under $\mathcal{G}_1$ and $\mathcal{G}_2$. This finishes the proof. □

Definition 6.7. We define the following $\Bbbk$-linear map for $\underline{i}, \underline{j}$ two sequences of indices in S.

$$\Phi : \begin{array}{rcl} R \otimes_{R^I} \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \otimes_{R^J} R & \longrightarrow & \mathrm{Hom}_{g\mathcal{D}}(I\underline{i}J, I\underline{j}J) \\ r_1 \otimes \varphi \otimes r_2 & \longmapsto & r_1 \cdot {}_I\mathcal{G}_J(\varphi) \cdot r_2 \end{array} \tag{6.1}$$

This map is well-defined, i.e. the symmetric polynomials which slide through the tensor products also slide through ${}_I\mathcal{G}_J$, because ${}_I\mathcal{G}_J$ respects the (R^I, R^J)-bimodule structure of Hom spaces. ◊

Lemma 6.8. *For $X, Y \in {}_I\mathbb{BSB}\mathrm{im}_J$ we have an (R,R)-bimodule isomorphism Ψ*

$$\begin{array}{rcl} \mathrm{Hom}_{(R,R)}(R \otimes_{R^I} X \otimes_{R^J} R, R \otimes_{R^I} Y \otimes_{R^J} R) & \cong & R \otimes_{R^I} \mathrm{Hom}_{(R^I,R^J)}(X,Y) \otimes_{R^J} R \\ (\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto r_1\widetilde{r_1} \otimes \varphi(x) \otimes \widetilde{r_2} r_2) & \longleftarrow & r_1 \otimes \varphi \otimes r_2. \end{array}$$

Proof. **Well-defined:** We first observe that Ψ is obviously a homomorphism of (R,R)-bimodules as r_1 and r_2 exactly act by the (R,R)-bimodule action on the left hand side. Now we need to check that Ψ is well-defined. We compute that

$$\begin{aligned} \Psi(r_1 r_i \otimes \varphi \otimes r_j r_2) &= (\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto r_1 r_i \widetilde{r_1} \otimes \varphi(x) \otimes \widetilde{r_2} r_j r_2) \\ &= (\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto r_1 \widetilde{r_1} \otimes r_i \varphi(x) r_j \otimes \widetilde{r_2} r_2) \\ &= \Psi(r_1 \otimes r_i \varphi r_j \otimes r_2) \end{aligned}$$

for $r_1, r_2 \in R, r_i \in R^I, r_j \in R^J$ and $\varphi \in \mathrm{Hom}_{(R^I,R^J)}(X,Y)$. So Ψ is well-defined if an image of Ψ is actually a well-defined morphism of (R,R)-bimodules. Let $r_1, r_2 \in R$ and $\varphi \in \mathrm{Hom}_{(R^I,R^J)}(X,Y)$, then we easily observe that the map

$$R \otimes_{R^I} X \otimes_{R^J} R \longrightarrow R \otimes_{R^I} Y \otimes_{R^J} R, \qquad \widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto r_1 \widetilde{r_1} \otimes \varphi(x) \otimes \widetilde{r_2} r_2$$

is a homomorphism of (R,R)-bimodules. Note that under this morphism we also have

$$\begin{aligned} \widetilde{r_1}\widetilde{r_i} \otimes x \otimes \widetilde{r_j}\widetilde{r_2} \longmapsto r_1 \widetilde{r_1}\widetilde{r_i} \otimes \varphi(x) \otimes \widetilde{r_j}\widetilde{r_2} r_2 &= r_1 \widetilde{r_1} \otimes \widetilde{r_i}\varphi(x)\widetilde{r_j} \otimes \widetilde{r_2} r_2 \\ &= r_1 \widetilde{r_1} \otimes \varphi(\widetilde{r_i} x \widetilde{r_j}) \otimes \widetilde{r_2} r_2 \\ \widetilde{r_1} \otimes \widetilde{r_i} x \widetilde{r_j} \otimes \widetilde{r_2} \longmapsto r_1 \widetilde{r_1} \otimes \varphi(\widetilde{r_i} x \widetilde{r_j}) \otimes \widetilde{r_2} r_2 & \end{aligned}$$

if $\widetilde{r_i} \in R^I$ and $\widetilde{r_j} \in R^J$, and thus the morphism is well-defined. Hence, Ψ is well-defined. It remains to prove that Ψ is bijective.

Injectivity: We start with injectivity. Let $A = \sum_{k=1}^N r_{1k} \otimes \varphi_k \otimes r_{2k} \in \ker(\Psi)$. By Theorem 3.35 we know that R has an R^I-basis given by $\{\tau_w\}_{w \in W_I}$ and an R^J-basis given by $\{\pi_r\}_{r \in W_J}$ together with dual bases $\{\tau_w^*\}_{w \in W_I}$ and $\{\pi_r^*\}_{r \in W_J}$ respectively which have the property that $\partial_I(\tau_w \tau_u^*) = \delta_{w,u}$ and $\partial_J(\pi_r \pi_t^*) = \delta_{r,t}$. This implies that we can write

$$A = \sum_{k=1}^N r_{1k} \otimes \varphi_k \otimes r_{2k} = \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r$$

for some $\varphi_{w,r} \in \mathrm{Hom}_{(R^I,R^J)}(X,Y)$. Now we can compute that

$$\begin{aligned} 0 &= \Psi(A) \\ &= \Psi\left(\sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r \right) \\ &= \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \widetilde{r_1} \left(\sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r \right) \widetilde{r_2} \right). \end{aligned}$$

If we choose $\widetilde{r_1} = \tau_u^*$ and $\widetilde{r_2} = \pi_t^*$ this implies

$$0 = \sum_{w \in W_I, r \in W_J} \tau_w \tau_u^* \otimes \varphi_{w,r}(x) \otimes \pi_r \pi_t^*$$

for all $x \in X$. If we now apply the $\Bbbk$-linear map

$$\partial_I \otimes \mathrm{id}_Y \otimes \partial_J : R \otimes_{R^I} Y \otimes_{R^J} R \longrightarrow R^I \otimes_{R^I} Y \otimes_{R^J} R^J \cong Y$$

to this, we get

$$\begin{aligned} 0 &= \sum_{w \in W_I, r \in W_J} \partial_I(\tau_w \tau_u^*) \otimes \varphi_{w,r}(x) \otimes \partial_J(\pi_r \pi_t^*) \\ &= 1 \otimes \varphi_{u,t}(x) \otimes 1 \cong \varphi_{u,t}(x) \end{aligned}$$

for all $x \in X$ and all $u \in W_I, t \in W_J$, where the last equality corresponds to the isomorphism $R^I \otimes_{R^I} Y \otimes_{R^J} R^J \cong Y$. So we have $\varphi_{u,t} = 0$ for all $u \in W_I, t \in W_J$ which implies

$$A = \sum_{k=1}^N r_{1k} \otimes \varphi_k \otimes r_{2k} = \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r = 0.$$

Thus, $\ker(\Psi) = 0$ and Ψ is injective.

Surjectivity: Now we need to prove surjectivity. Let $\psi \in \mathrm{Hom}_{(R,R)}(R \otimes_{R^I} X \otimes_{R^J}$

$R, R \otimes_{R^I} Y \otimes_{R^J} R$). We still have the R^I-basis $\{\tau_w\}_{w \in W_I}$ of R and the R^J-basis $\{\pi_r\}_{r \in W_J}$ of R. Hence, we can write $\psi(1 \otimes x \otimes 1)$ uniquely as

$$\psi(1 \otimes x \otimes 1) = \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r$$

for all $x \in X$, where $\varphi_{w,r}(x) \in Y$ are some elements of Y that only depend on x. In this way we have defined some maps $\varphi_{w,r} : X \longrightarrow Y$. Now we want to check that these maps are homomorphisms of (R^I, R^J)-bimodules. So let $x, x' \in X$, then

$$\begin{aligned}\sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x + x') \otimes \pi_r &= \psi(1 \otimes x + x' \otimes 1) = \psi(1 \otimes x \otimes 1) + \psi(1 \otimes x' \otimes 1) \\ &= \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r \\ &\quad + \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x') \otimes \pi_r \\ &= \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x) + \varphi_{r,w}(x') \otimes \pi_r\end{aligned}$$

and from this we get by the uniqueness of this description that $\varphi_{w,r}(x + x') = \varphi_{w,r}(x) + \varphi_{w,r}(x')$ for all $w \in W_I, r \in W_J$. Now let $r_i \in R^I, r_j \in R^J$ and $x \in X$, then we compute

$$\begin{aligned}\sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(r_i x r_j) \otimes \pi_r &= \psi(1 \otimes r_i x r_j \otimes 1) = \psi(r_i \otimes x \otimes r_j) \\ &= r_i \cdot \psi(1 \otimes x \otimes 1) \cdot r_j \\ &= r_i \cdot \left(\sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r \right) \cdot r_j \\ &= \sum_{w \in W_I, r \in W_J} r_i \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r r_j \\ &= \sum_{w \in W_I, r \in W_J} \tau_w \otimes r_i \varphi_{w,r}(x) r_j \otimes \pi_r\end{aligned}$$

and from this we get that $\varphi_{w,r}(r_i x r_j) = r_i \varphi_{w,r}(x) r_j$ for all $w \in W_I, r \in W_J$ again by uniqueness of this description. Hence, $\varphi_{w,r} \in \mathrm{Hom}_{(R^I, R^J)}(X, Y)$. Now we can define

$$A = \sum_{w \in W_I, r \in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r \in R \otimes_{R^I} \mathrm{Hom}_{(R^I, R^J)}(X, Y) \otimes_{R^J} R$$

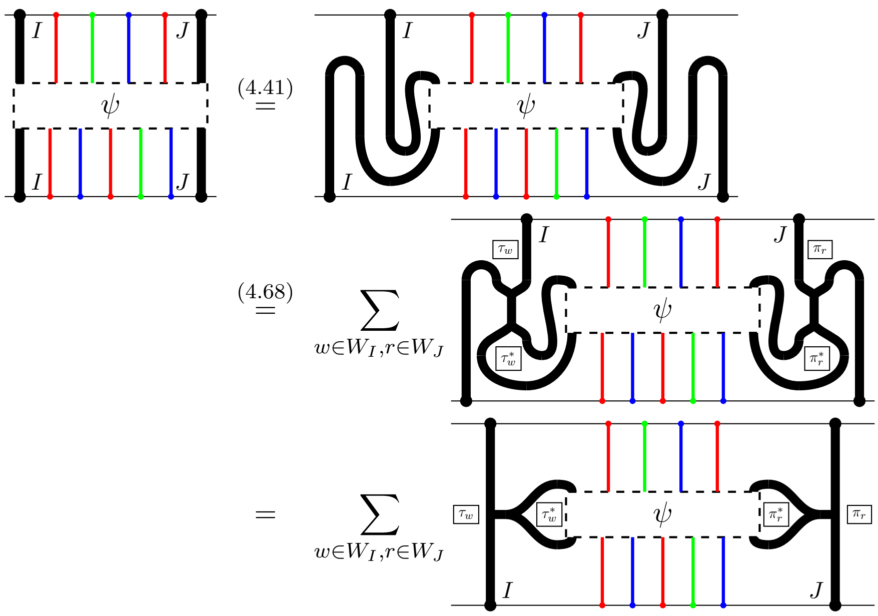
and apply Ψ to it:

$$\begin{aligned}
\Psi(A) &= \Psi\left(\sum_{w\in W_I, r\in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r\right) = \sum_{w\in W_I, r\in W_J} \Psi\left(\tau_w \otimes \varphi_{w,r} \otimes \pi_r\right) \\
&= \sum_{w\in W_I, r\in W_J} \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \tau_w \widetilde{r_1} \otimes \varphi_{w,r}(x) \otimes \widetilde{r_2}\pi_r\right) \\
&= \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \sum_{w\in W_I, r\in W_J} \tau_w \widetilde{r_1} \otimes \varphi_{w,r}(x) \otimes \widetilde{r_2}\pi_r\right) \\
&= \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \widetilde{r_1} \cdot \left(\sum_{w\in W_I, r\in W_J} \tau_w \otimes \varphi_{w,r}(x) \otimes \pi_r\right) \cdot \widetilde{r_2}\right) \\
&= \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \widetilde{r_1} \cdot (\psi(1 \otimes x \otimes 1)) \cdot \widetilde{r_2}\right) \\
&= \left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2} \longmapsto \psi\left(\widetilde{r_1} \otimes x \otimes \widetilde{r_2}\right)\right) = \psi.
\end{aligned}$$

Thus, $\psi \in \mathrm{im}(\Psi)$ and Ψ is surjective and hence bijective. This finishes the proof. □

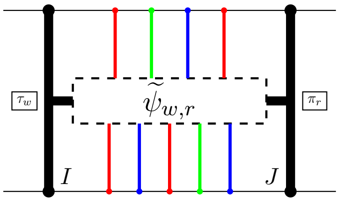
Proposition 6.9. *The map Φ from (6.1) is an isomorphism of (R, R)-bimodules.*

Proof. It is obvious that Φ is a homomorphism of (R, R)-bimodules. So it is enough to check that Φ is bijective. We begin to with looking at an arbitrary morphism $\psi \in \mathrm{Hom}_{g\mathcal{D}}(I\underline{i}J, I\underline{j}J)$.

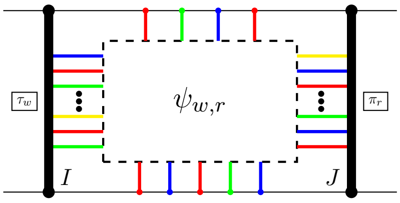


(6.2)

$$= \sum_{w\in W_I, r\in W_J}$$

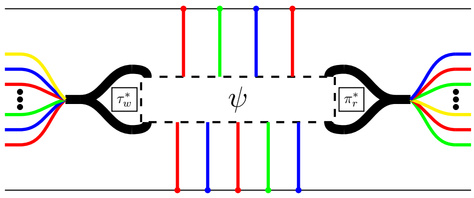


$$= \sum_{w\in W_I, r\in W_J}$$



Here we used the R^I-bases $\{\tau_w\}_{w\in W_I}$ and $\{\tau_w^*\}_{w\in W_I}$ of R and the R^J-bases $\{\pi_r\}_{r\in W_J}$ and $\{\pi_r^*\}_{r\in W_J}$ of R given by Theorem 3.35 again. The morphisms $\psi_{w,r}$ are defined as follows.

$$\psi_{w,r} = \qquad (6.3)$$



Note that $\psi_{w,r}$ is now a morphism between objects in $\mathcal{D}_1$ and thus can be written using thin lines only. Thus, we can view $\psi_{w,r}$ as a morphism in $\mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j})$ if we let the lines coming out from the sides run into the membranes. Note that this is possible, since they form reduced expressions for w_I and w_J respectively, and hence the indices lie in I and J respectively. In this way we can define a $\Bbbk$-linear map

$$\overline{\Phi} : \begin{array}{rcl} \mathrm{Hom}_{g\mathcal{D}}(I\underline{i}J, I\underline{j}J) & \longrightarrow & R \otimes_{R^I} \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \otimes_{R^J} R \\ \psi & \longmapsto & \sum_{w\in W_I, r\in W_J} \tau_w \otimes \psi_{w,r} \otimes \pi_r. \end{array}$$

The calculation (6.2) shows us that $\Phi \circ \overline{\Phi} = \mathrm{id}$. So all that is left do is to prove the other direction. For this let

$$A = \sum_{k=1}^{N} r_{1k} \otimes \varphi_k \otimes r_{2k} \in R \otimes_{R^I} \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \otimes_{R^J} R$$

be an arbitrary element. Again we can rewrite this element using the bases $\{\tau_w\}_{w\in W_I}$ and $\{\pi_r\}_{r\in W_J}$ for R as an R^I-module or R^J-module respectively.

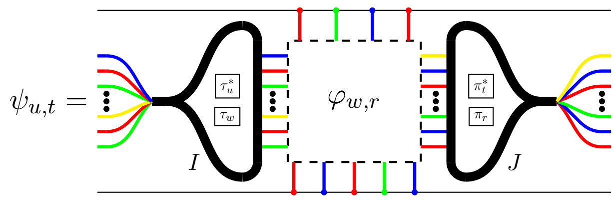
$$A = \sum_{w\in W_I, r\in W_J} \tau_w \otimes \varphi_{w,r} \otimes \pi_r$$

Before we start to apply Φ and $\overline{\Phi}$ to A we need to make some observations about the $\varphi_{w,r}$. We are interested in the lines that run into the membranes on the sides. We will

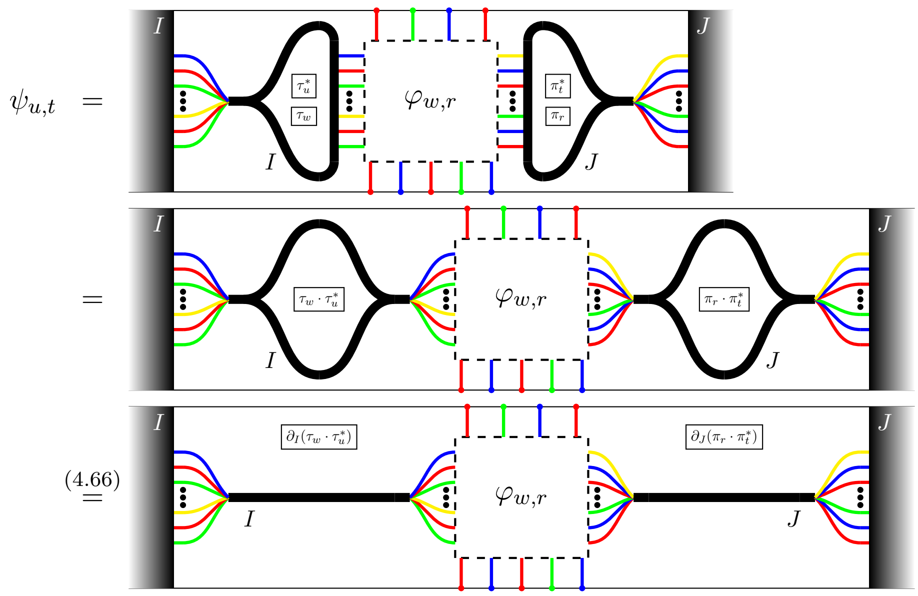
only talk about the left membrane, the right one is completely analogous. The lines running into the left membrane write a word with the elements of I (read from bottom to top). This word corresponds to an element in W_I.
We can then use relations (4.49) to (4.51) to change the order in which the lines hit the membrane or reduce the number of lines. This corresponds to using all relations in the group W_I. Thus, we can reduce the word to a reduced expression for the corresponding element. Now we can use relation (4.48) to increase the reduced expression to a reduced expression of w_I.
The upshot is now that we can w.l.o.g. assume that the lines running into the left membrane form a reduced expression for w_I and we will do this. The same goes for the right membrane and w_J.
To observe that $\left(\overline{\Phi} \circ \Phi\right)(A) = A$ it is enough to check that

$$\left(\overline{\Phi} \circ \Phi\right)(\tau_w \otimes \varphi_{w,r} \otimes \pi_r) = \tau_w \otimes \varphi_{w,r} \otimes \pi_r,$$

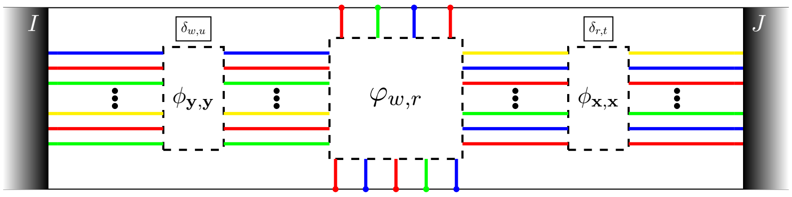
because $\overline{\Phi}$ respects sums (since we have to extend it $\Bbbk$-linearly anyway for a complete definition). We have $\Phi(\tau_w \otimes \varphi_{w,r} \otimes \pi_r) = \tau_w \cdot {}_I\mathcal{G}_J(\varphi_{w,r}) \cdot \pi_r$. In order to apply $\overline{\Phi}$ to this we need to calculate $\psi_{u,t} = (\tau_w \cdot {}_I\mathcal{G}_J(\varphi_{w,r}) \cdot \pi_r)_{u,t}$ as in (6.3) for $u \in W_I, t \in W_J$. This would look like the following.



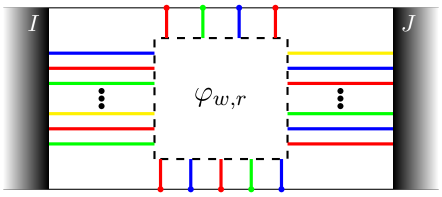
Now we know that $\overline{\Phi}(\tau_w \cdot {}_I\mathcal{G}_J(\varphi_{w,r}) \cdot \pi_r) = \sum_{u \in W_I, t \in W_J} \tau_u \otimes \psi_{u,t} \otimes \pi_t$. The next step is to rewrite $\psi_{u,t}$ in ${}_I\mathcal{T}_J$.



$\overset{(4.38)}{=}$



$\overset{(4.57)}{=} \delta_{w,u} \cdot \delta_{r,t} \cdot$



$= \delta_{w,u} \cdot \delta_{r,t} \cdot \varphi_{w,r}$

Note that we have used that the lines coming out of the sides of $\varphi_{w,r}$ form the same reduced expressions for w_I and w_J as the lines running into the membranes. This follows from our discussion above which explained that we can use (4.50) and (4.51) to let the lines running out of the sides of $\varphi_{w,r}$ form reduced expressions of w_I and w_J of our choice. Now we can finish our calculation.

$$\left(\overline{\Phi} \circ \Phi\right) (\tau_w \otimes \varphi_{w,r} \otimes \pi_r) = \overline{\Phi} \left(\tau_w \cdot {}_I\mathcal{G}_J(\varphi_{w,r}) \cdot \pi_r\right) = \sum_{u \in W_I, t \in W_J} \tau_u \otimes \psi_{u,t} \otimes \pi_t$$
$$= \sum_{u \in W_I, t \in W_J} \tau_u \otimes \delta_{w,u} \cdot \delta_{r,t} \cdot \varphi_{w,r} \otimes \pi_t = \tau_w \otimes \varphi_{w,r} \otimes \pi_r$$

This finally tells us that $\overline{\Phi} \circ \Phi = \mathrm{id}$, and thus $\overline{\Phi}$ is the inverse of Φ. Hence, Φ is an isomorphism and the proof is finished. $\square$

Finally, we can prove the main theorem of this section.

Theorem 6.10. *The functor ${}_I\mathcal{F}_J : {}_I\mathcal{T}_J \longrightarrow {}_I\mathbb{BS}\mathrm{Bim}_J$ is an equivalence of categories.*

Proof. ${}_I\mathcal{F}_J$ is obviously essentially surjective. If ${}_I\mathcal{F}_J$ would not be fully faithful, then there would be $\underline{i}, \underline{j} \in {}_I\mathcal{T}_J$ such that

$$\mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \xrightarrow{{}_I\mathcal{F}_J} \mathrm{Hom}_{(R^I, R^J)}(B_{\underline{i}}, B_{\underline{j}})$$

is not an isomorphism. So let us assume this and derive a contradiction. We consider the following diagram.

$$\begin{array}{ccc} R \otimes_{R^I} \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \otimes_{R^J} R & \xrightarrow{\mathrm{id} \otimes {}_I\mathcal{F}_J \otimes \mathrm{id}} & R \otimes_{R^I} \mathrm{Hom}_{(R^I, R^J)}(B_{\underline{i}}, B_{\underline{j}}) \otimes_{R^J} R \\ \Phi \big\downarrow & & \big\downarrow \Psi \\ \mathrm{Hom}_{g\mathcal{D}}(I\underline{i}J, I\underline{j}J) & \xrightarrow{g\mathcal{F}} & \mathrm{Hom}_{(R,R)}(R \otimes_{R^I} B_{\underline{i}} \otimes_{R^J} R, R \otimes_{R^I} B_{\underline{j}} \otimes_{R^J} R) \end{array}$$

Here we used the isomorphisms Φ from (6.1) and Ψ from Lemma 6.8. Note that $g\mathcal{F}$ is an equivalence of categories, and thus the bottom arrow is also an isomorphism. Also note that $\Psi(r_1 \otimes \psi \otimes r_2) = r_1 \cdot {}_I\mathrm{ind}_J(\psi) \cdot r_2$ where ${}_I\mathrm{ind}_J$ is the induction functor. We check now that the diagram commutes, for this we need

$$\begin{aligned} & (\Psi \circ (\mathrm{id} \otimes_I \mathcal{F}_J \otimes \mathrm{id}))\,(r_1 \otimes \varphi \otimes r_2) \overset{!}{=} (g\mathcal{F} \circ \Phi)\,(r_1 \otimes \varphi \otimes r_2) \\ \iff \quad & \Psi\,(r_1 \otimes {}_I\mathcal{F}_J(\varphi) \otimes r_2) \overset{!}{=} g\mathcal{F}\,(r_1 \cdot {}_I\mathcal{G}_J(\varphi) \cdot r_2) \\ \iff \quad & r_1 \cdot (\,{}_I\mathrm{ind}_J \circ_I \mathcal{F}_J)\,(\varphi) \cdot r_2 \overset{!}{=} r_1 \cdot (g\mathcal{F} \circ {}_I\mathcal{G}_J)\,(\varphi) \cdot r_2 \end{aligned}$$

to hold. However, the last equation is true due to Proposition 6.4. So we have a commutative diagram where three arrows are isomorphisms. Then the fourth arrow (the top arrow) also needs to be an isomorphism. Hence,

$$R \otimes_{R^I} \mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \otimes_{R^J} R \xrightarrow{\mathrm{id} \otimes_I \mathcal{F}_J \otimes \mathrm{id}} R \otimes_{R^I} \mathrm{Hom}_{(R^I, R^J)}(B_{\underline{i}}, B_{\underline{j}}) \otimes_{R^J} R$$

is an isomorphism. Since R is free over R^I and R^J we can write the left side as $\left(\mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j})\right)^N$ and the right side as $\left(\mathrm{Hom}_{(R^I, R^J)}(B_{\underline{i}}, B_{\underline{j}})\right)^N$, where $N = |W_I| \cdot |W_J|$, and the isomorphism above is then given by $({}_I\mathcal{F}_J)^N$. This implies that

$$\mathrm{Hom}_{{}_I\mathcal{T}_J}(\underline{i}, \underline{j}) \xrightarrow{{}_I\mathcal{F}_J} \mathrm{Hom}_{(R^I, R^J)}(B_{\underline{i}}, B_{\underline{j}})$$

is an isomorphism and we have a contradiction. Thus, ${}_I\mathcal{F}_J$ is fully faithful and the proof is finished. □

6.2 Diagrammatics for singular Soergel bimodules

In this section we will use the concept of idempotent completions to obtain a new diagrammatic category which is equivalent to the category of singular Soergel bimodules. This diagrammatic category will have the same problems as the category $g\mathcal{D}$: In order to understand these categories we need to understand how the complicated idempotents behave. This makes these diagrammatic categories hard to work with, but they are still a good starting point for calculations. In a later chapter we will give a diagrammatic presentation of singular Soergel bimodules for S_3 with generators and relations and use the work from this chapter to achieve this.
Before we can construct the idempotents in ${}_I\mathcal{T}_J$ we need some preparation.

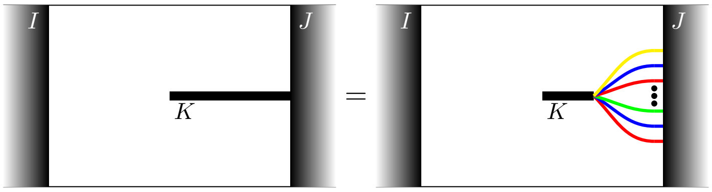
Definition 6.11. We define the diagrammatic category ${}_Ig\mathcal{T}_J$ for $I, J \subseteq S$ parabolic subsets. This category is derived from ${}_I\mathcal{T}_J$ in the same way as $g\mathcal{D}$ is derived from $\mathcal{D}_1$. Objects are sequences $\underline{\mathbf{J}} = J_1 J_2 \ldots J_r$ of connected subsets of S. The generating morphisms are the generators of ${}_I\mathcal{T}_J$ together with the J-inclusions and J-projections (with membranes on both sides). The defining relations are the ones from ${}_I\mathcal{T}_J$ as well as relations (4.37) and (4.38) (with membranes on both sides). ◊

Remark 6.12. Note that all morphisms in $g\mathcal{D}$ are (R,R)-bimodule morphisms and thus become (R^I, R^J)-bimodule morphisms via restriction. Hence, all the relations from $g\mathcal{D}$ also hold in ${}_Ig\mathcal{T}_J$. $\diamondsuit$

Theorem 6.13. *The category ${}_Ig\mathcal{T}_J$ is equivalent to the partial idempotent completion of ${}_I\mathcal{T}_J$ by the images of ϕ_J for $J \subset S$. The functor ${}_I\mathcal{F}_J$ from ${}_I\mathcal{T}_J$ to ${}_I\mathbb{BS}\mathrm{Bim}_J$ extends to a functor ${}_Ig\mathcal{F}_J$ from ${}_Ig\mathcal{T}_J$ to ${}_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J$ which is an equivalence of categories. Here ${}_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J$ is the full subcategory of (R^I, R^J)-bimodules containing all grading shifts of the generalized Bott–Samelson bimodules $B_{\underline{\mathbf{J}}}$.*

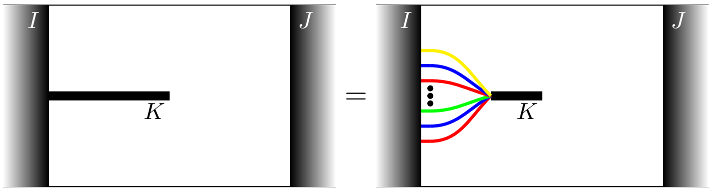
Proof. This follows from the discussion in Remark 4.18 and Proposition 4.20. $\square$

Definition 6.14. Let $K \subseteq S$. If $K \subseteq J$ we define the following morphism in ${}_Ig\mathcal{T}_J$.



(6.4)

If $K \subseteq I$ we have can define the same morphism on the other side.



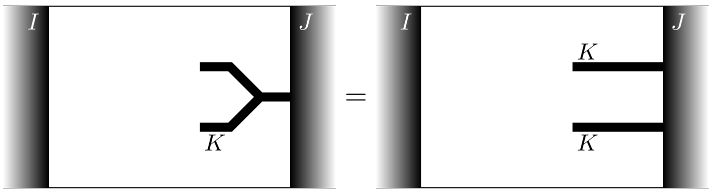
(6.5)

We call these morphisms *very thick trivalent vertices.* $\diamondsuit$

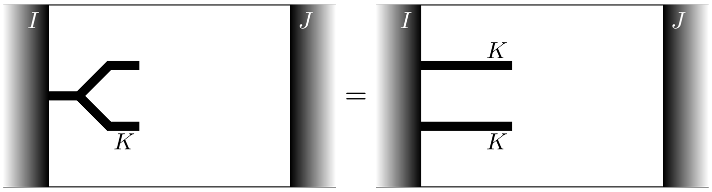
Remark 6.15. Note that these morphisms are well-defined, i.e. they do not depend on the reduced expression for w_K which is chosen on the right hand side. This follows from relation (4.57) (although one uses it with the membrane here which behaves like a thick line) and the fact that applying $\phi_{\mathbf{x},\mathbf{y}}$ after a J-inclusions just gives the J-inclusion to $\mathbf{y}$. $\diamondsuit$

Remark 6.16. One can compute what the images of (6.4) and (6.5) under ${}_Ig\mathcal{F}_J$ are. If (6.4) is going up it is sent to the morphism $R \longrightarrow B_K, r \longmapsto r \otimes 1$. If (6.4) is going down it is sent to $B_K \longrightarrow R, r_1 \otimes r_2 \longmapsto r_1 \partial_K(r_2)$. The morphisms corresponding to (6.5) are similar (just swapped left to right). $\diamondsuit$

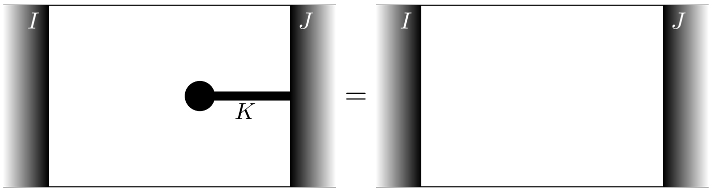
Lemma 6.17. *The following relations hold in ${}_Ig\mathcal{T}_J$ where $K \subseteq J$ in the first and third relation, $K \subseteq I$ in the second and fourth relation and $K \subseteq I, J$ in the fifth relation.*



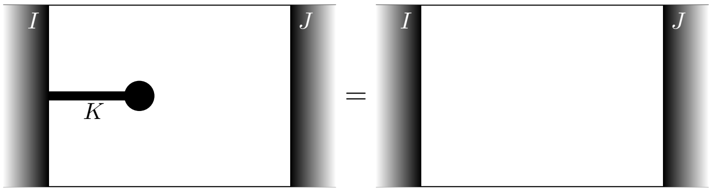
(6.6)



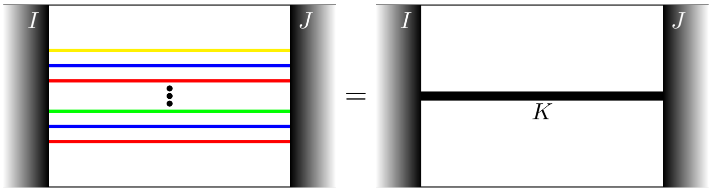
(6.7)



(6.8)



(6.9)



(6.10)

In the last relation the lines of the left hand side give a reduced expression for w_K.

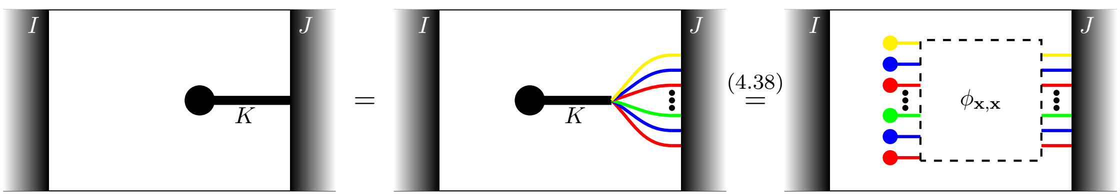
Proof. The first two relations can be checked in $_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J$ and then hold in $_Ig\mathcal{T}_J$ because of Theorem 6.13. For instance, if we consider the version of (6.6) were both strands end in the top, the left hand side would be given by the following composition.

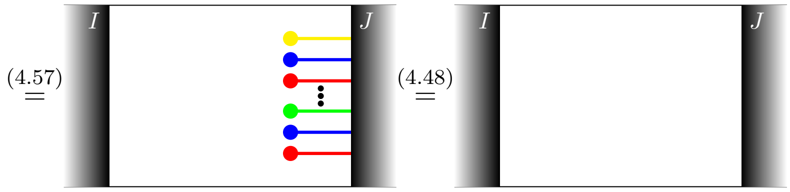
$$\begin{array}{ccccc} R & \longrightarrow & B_K & \longrightarrow & B_K B_K \\ r & \longmapsto & r \otimes 1 & \longmapsto & r \otimes 1 \otimes 1. \end{array}$$

The right hand side of (6.6) is given by the following composition.

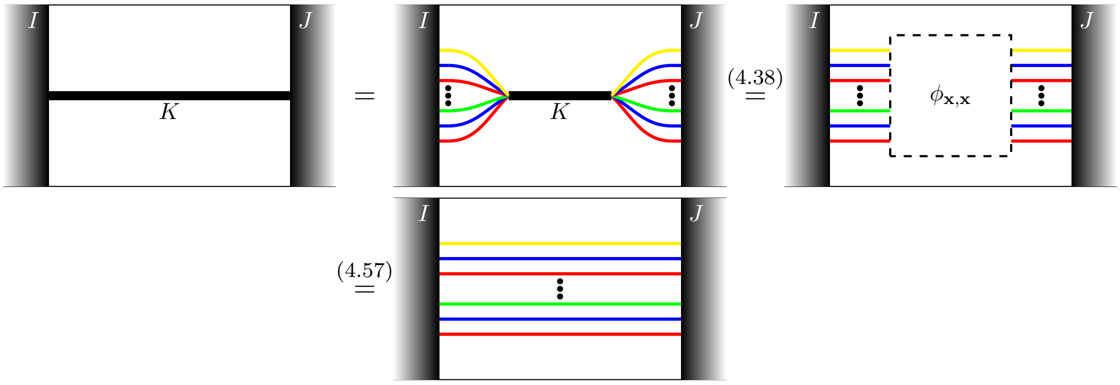
$$\begin{array}{ccccc} R & \longrightarrow & B_K & \longrightarrow & B_K B_K \\ r & \longmapsto & r \otimes 1 & \longmapsto & r \otimes 1 \otimes 1. \end{array}$$

This proves this version of (6.6). The others can be done similar. The proof of the third and fourth relation is basically the same. Thus, we will just prove the third relation.



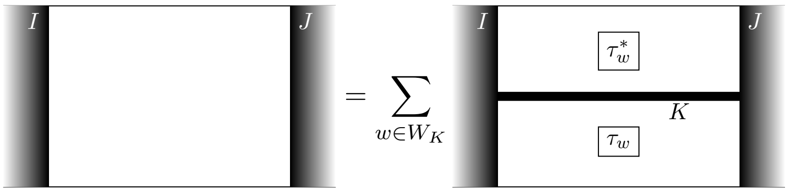


The last relation follows from the following calculation.



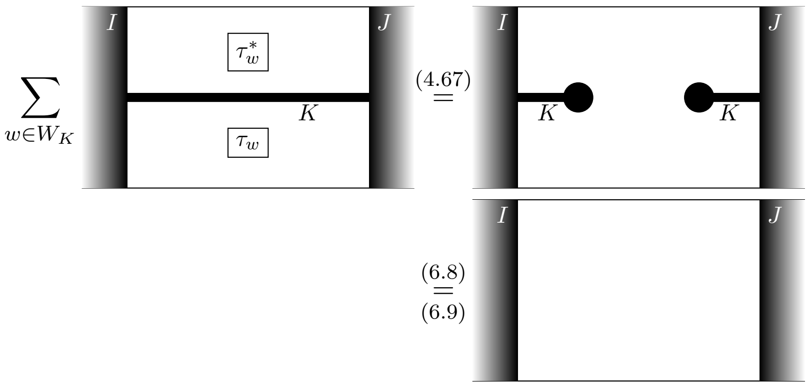
This finishes the proof. □

Lemma 6.18. *Let $K \subseteq S$ be such that $K \subseteq I, J$. Let $\{\tau_w\}_{w \in W_K}$ be the R^K-basis of R from Theorem 3.35 and let $\{\tau_w^*\}_{w \in W_K}$ be its dual basis. Then we have the following decomposition in pairwise orthogonal idempotents.*

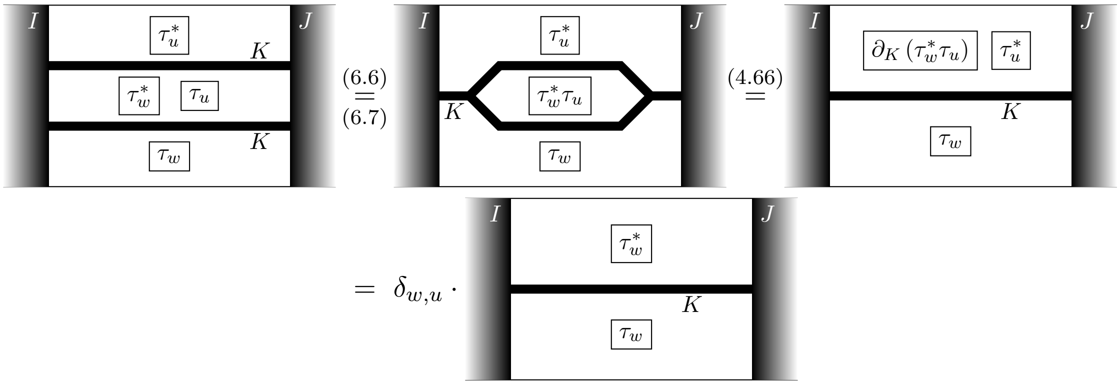


(6.11)

Proof. We will first prove that the relation (6.11) is true.



Now we will prove that the summands on the right side of (6.11) are pairwise orthogonal idempotents. For this let $w, u \in W_K$. We will compute the composition of the summand corresponding to w and the summand corresponding to u.



This finishes the proof. □

Remark 6.19. These are exactly the diagrammatic pictures for the decomposition $R \cong (R^K)^{|W_K|}$. We will now use these idempotents to extend our category as we did for g$\mathbb{BS}$Bim. ♢

The following is a crucial definition introducing an important category underlying all further categories.

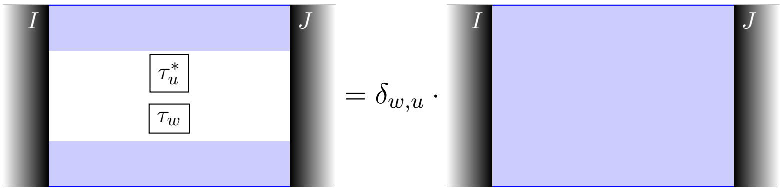
Definition 6.20. We construct a category $\widetilde{{}_I g\mathcal{T}_J}$.

Objects: Objects are the same as in ${}_I g\mathcal{T}_J$ and for each $K \subseteq I, J$ we add another object which is an empty sequence labelled K (we identify the original empty sequence with the empty sequence labelled $\emptyset$). We draw the identity on the empty sequence labelled K as follows.

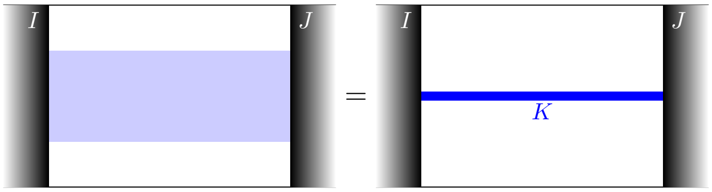
Morphisms: The generating morphisms are the same as in ${}_I g\mathcal{T}_J$ as well as two new morphisms for each $K \subseteq I, J$. These are morphisms between the empty sequence labelled K and the empty sequence labelled $\emptyset$ and look as follows.

(6.12)

Relations: The defining relations are



(6.13)



(6.14)

as well as the defining relations of ${}_Ig\mathcal{T}_J$. ♢

As described in the introduction we will now put all the individual categories $\widetilde{{}_Ig\mathcal{T}_J}$ for fixed $I, J \subseteq S$ together to obtain a 2-category in order to mirror the fact that singular Soergel bimodules are a 2-category.

Definition 6.21. We define the collection of categories $\{{}_Is\mathcal{T}_J\}_{I,J\subseteq S}$ to be the smallest (with respect to taking full subcategories) such collection with the following properties:

- For each $I, J \subseteq S$ the category $\widetilde{{}_Ig\mathcal{T}_J}$ is a full subcategory of ${}_Is\mathcal{T}_J$;
- The set of subsets of S together with the arrangement $\mathrm{Mor}(I,J) = {}_Is\mathcal{T}_J$ forms a 2-category.

We call the 2-category from the second property $s\mathcal{T}$. ♢

Remark 6.22. This 2-category is well-defined, i.e. there exists a unique such collection of categories. Existence of such a collection is given, since the 2-category Bim satisfies both properties if we restrict ourselves to the objects R^I for $I \subseteq S$. For uniqueness assume that we would have two such collections $\{{}_Is\mathcal{T}_J\}_{I,J\subseteq S}$ and $\{\widetilde{{}_Is\mathcal{T}_J}\}_{I,J\subseteq S}$. Then let $\widehat{{}_Is\mathcal{T}_J}$ be the full subcategory of ${}_Is\mathcal{T}_J$ which only contains objects that are also contained in $\widetilde{{}_Is\mathcal{T}_J}$. Then the collection $\{\widehat{{}_Is\mathcal{T}_J}\}_{I,J\subseteq S}$ has both properties and is smaller than both of our original collections. ♢

Definition 6.23. We define the category $\overline{{}_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J}$ to be the full subcategory of (R^I, R^J)-bimodules that contains all objects of ${}_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J$ as well as the bimodules R^K for $K \subseteq I, J$. ♢

Definition 6.24. We define the collection of categories $\{\overline{{}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J}\}_{I,J\subseteq S}$ to be the smallest (with respect to taking full subcategories) such collection with the following properties:

- For each $I, J \subseteq S$ the category $\overline{{}_I\mathrm{g}\mathbb{BS}\mathrm{Bim}_J}$ is a full subcategory of $\overline{{}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J}$;
- The set of subsets of S together with the arrangement $\mathrm{Mor}(I,J) = \overline{{}_I\mathrm{s}\mathbb{BS}\mathrm{Bim}_J}$ forms a 2-category.

We call the 2-category from the second property $\overline{\text{s}\mathbb{BS}\text{Bim}}$. ◊

Lemma 6.25. *The equivalences* ${}_I g\mathcal{F}_J$ *extends to an equivalence of* 2*-categories* $\overline{s\mathcal{F}}$: $s\mathcal{T} \longrightarrow \overline{\text{s}\mathbb{BS}\text{Bim}}$.

Proof. It follows from Remark 6.19 that the category $\widetilde{{}_I g\mathcal{T}_J}$ formally adds pictures for the inclusions and projections between R^K and R. Then Remark 4.18 implies that $\widetilde{{}_I g\mathcal{T}_J}$ is equivalent to the partial idempotent completion of ${}_I\text{g}\mathbb{BS}\text{Bim}_J$ by the decompositions $R \cong \left(R^K\right)^{|W_K|}$. However, this is exactly the category $\overline{{}_I\text{g}\mathbb{BS}\text{Bim}_J}$, and thus $\widetilde{{}_I g\mathcal{T}_J}$ and $\overline{{}_I\text{g}\mathbb{BS}\text{Bim}_J}$ are equivalent.
Since $s\mathcal{T}$ and $\overline{\text{s}\mathbb{BS}\text{Bim}}$ are built in the same way from $\widetilde{{}_I g\mathcal{T}_J}$ and $\overline{{}_I\text{g}\mathbb{BS}\text{Bim}_J}$ respectively it follows that these two 2-categories need to be equivalent as well. □

Theorem 6.26. *The* 2*-categories* $\overline{\text{s}\mathbb{BS}\text{Bim}}$ *and* $\text{s}\mathbb{BS}\text{Bim}$ *coincide.*

Proof. Obviously both categories have the same sets of objects. Moreover, all the compositions are induced from Bim and thus are the same. Hence, it is enough to prove that $\overline{{}_I\text{s}\mathbb{BS}\text{Bim}_J}$ and ${}_I\text{s}\mathbb{BS}\text{Bim}_J$ are the same. Note that the collection $\{\, {}_I\text{s}\mathbb{BS}\text{Bim}_J\}_{I,J\subseteq S}$ satisfies both conditions of Definition 6.24. Thus, $\overline{{}_I\text{s}\mathbb{BS}\text{Bim}_J}$ is a full subcategory of ${}_I\text{s}\mathbb{BS}\text{Bim}_J$. It is now enough to check that every object of ${}_I\text{s}\mathbb{BS}\text{Bim}_J$ lies in $\overline{{}_I\text{s}\mathbb{BS}\text{Bim}_J}$. Let

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

be an arbitrary object of ${}_I\text{s}\mathbb{BS}\text{Bim}_J$ where $I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset \cdots \subset J_{n-1} \supset I_n = J$ are subsets of S. Then we have $R^{I_l} \in \overline{{}_{J_{l-1}}\text{s}\mathbb{BS}\text{Bim}_{J_l}}$ where $J_0 = I, J_n = J$. Now we can use that $\overline{\text{s}\mathbb{BS}\text{Bim}}$ is closed under composition of 1-morphisms to successively get

$$\begin{aligned} R^{I_1} &\in \overline{{}_I\text{s}\mathbb{BS}\text{Bim}_{J_1}} \\ R^{I_1} \otimes_{R^{J_1}} R^{I_2} &\in \overline{{}_I\text{s}\mathbb{BS}\text{Bim}_{J_2}} \\ &\vdots \\ R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n} &\in \overline{{}_I\text{s}\mathbb{BS}\text{Bim}_{J_n}}. \end{aligned}$$

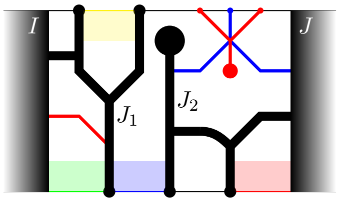
This finishes the proof. □

From the previous discussions we can easily deduce the following result which is also the main result of this chapter.

Corollary 6.27. *The equivalences* ${}_I g\mathcal{F}_J$ *extend to an equivalence of* 2*-categories*

$$s\mathcal{F} : s\mathcal{T} \longrightarrow \text{s}\mathbb{BS}\text{Bim}.$$

Remark 6.28. The definition of $s\mathcal{T}$ we gave is rather abstract and might seem hard to work with. This is in opposition to our goal to use these diagrammatic categories to better understand Soergel bimodules. That is why we will now give a different description of $s\mathcal{T}$ which is more concrete. We will first show an example of a morphism in $s\mathcal{T}$ and afterwards give the alternative description while relating it to the example.



The category ${}_I s\mathcal{T}_J$ can be described with generators and relations.

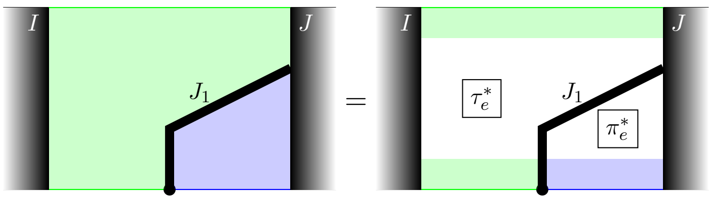
Objects: The objects of ${}_I s\mathcal{T}_J$ are sequences of parabolic subsets $J \subseteq S$ where the gaps between two subsets J_1, J_2 in such a sequence are labelled by some parabolic subset $K \subseteq J_1, J_2$. Such a sequence would look like this: ${}_{K_0}J_1{}_{K_1}J_2{}_{K_2} \cdots {}_{K_{l-1}}J_l{}_{K_l}$. Note that the beginning and the end also count as gaps (we require $K_0 \subseteq I, K_l \subseteq J$). Such a sequence will be viewed as dots (labelled $J_1, \cdots, J_l$) on a line in the plane and the gaps between the dots are labelled/coloured with the K_i's. In the example this can especially be seen with the very thin green, blue, red and yellow lines on the boundary.

Morphisms: The generating morphisms are the same as for ${}_I g\mathcal{T}_J$ together with the two generators (6.12). However, the two membranes in the pictures (6.12) can be replaced by two thick lines labelled J_1, J_2 as long as $K \subseteq J_1, J_2$. Basically we consider these generators locally between two thick lines. We have four of them in the example above, namely the green, blue, red and yellow areas.

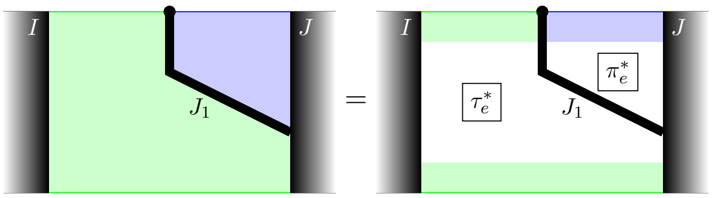
Relations: The relations are the ones for ${}_I g\mathcal{T}_J$ as well as the relations (6.13) and (6.14) where we again allow the two membranes to be replaced by two thick lines labelled J_1, J_2 as long as $K \subseteq J_1, J_2$ is satisfied. This concludes the description. $\diamondsuit$

As we did for $g\mathcal{D}$ we will now identify some morphisms with new pictures and show some extra relations that hold in $s\mathcal{T}$.

Definition 6.29. We define the *coloured trivalent vertices* by the following pictures.



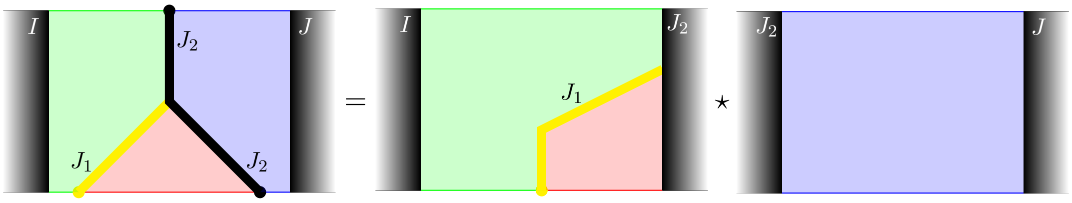
(6.15)



(6.16)

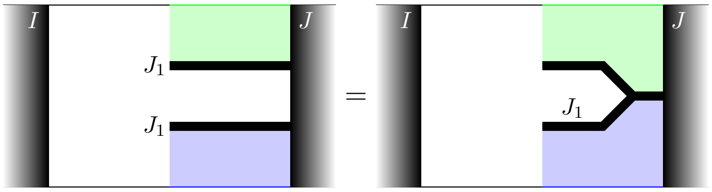
Here $\{\tau_w^*\}_{w \in W_{K_1}}$ and $\{\pi_w^*\}_{w \in W_{K_2}}$ are the dual bases for R over R^{K_1} and R^{K_2} respectively (from Theorem 3.35) where K_1 is the subset corresponding to the green coloured plane and K_2 is the subset corresponding to the blue coloured plane. We also define the same morphisms where the thick strand ends in the left membrane analogously. $\diamondsuit$

Remark 6.30. Note that with this definition we have already defined another morphism as follows.

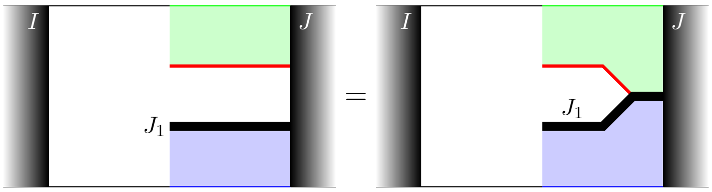


We will call this morphism coloured trivalent vertex as well. ◊

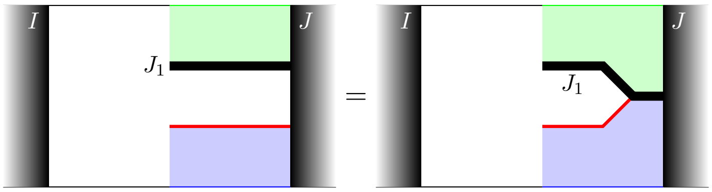
We have the following relations.



(6.17)

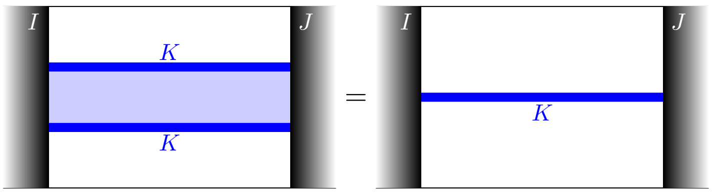


(6.18)

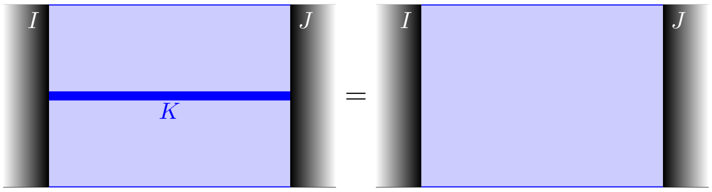


(6.19)

Here we used the shortcut notation that we got from the isotopy invariance relations again (note that we do not have proven isotopy invariance in this setting, but we can still use this notation as a shortcut). The two lines in each picture that end nowhere could end in either bottom, top or the other membrane as long as they do the same thing on both sides of the equation. Moreover, by using Remark 6.30 we could replace the membrane by an appropriately labelled thick line again. By symmetry we also have the same relations with everything at the left membrane. There are two more relations for the coloured trivalent vertex.



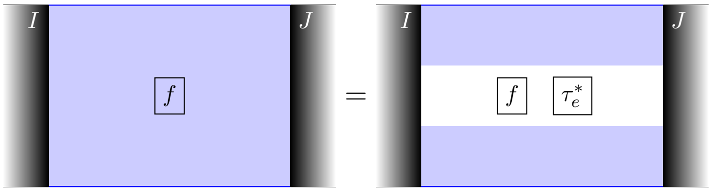
(6.20)



(6.21)

Note that we required that the blue thick line is labelled K which is also the label of the blue coloured area. We could again replace the membranes with appropriately labelled thick lines.

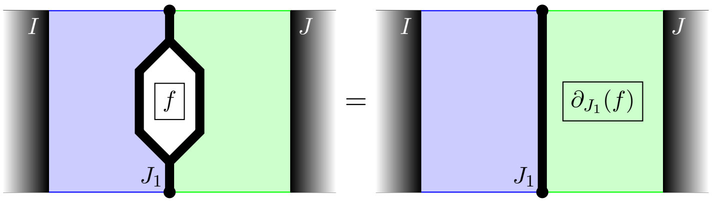
Definition 6.31. We define the *coloured polynomial morphism* for a polynomial $f \in R^K$ (where the blue area is labelled K) as follows.



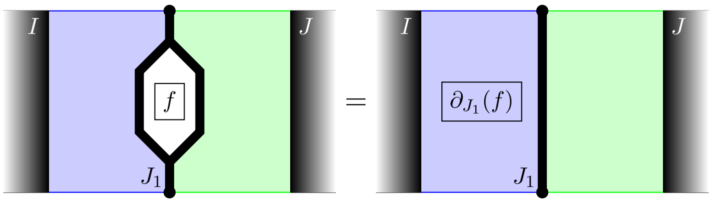
(6.22)

Here $\{\tau^*_w\}_{w \in W_K}$ is again the dual basis for R over R^K from Theorem 3.35. ♢

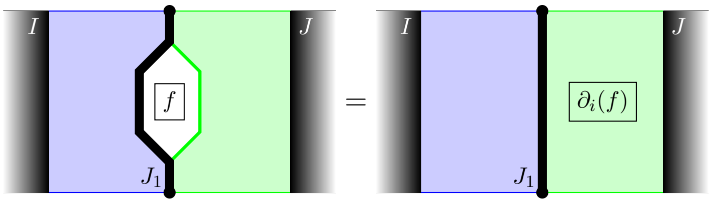
We have the following relations for $f \in R$.



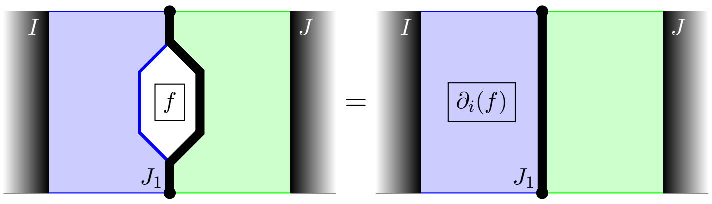
(6.23)



(6.24)



(6.25)



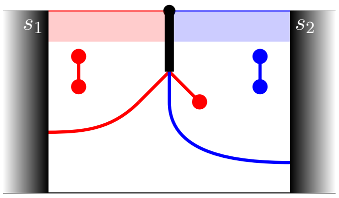
(6.26)

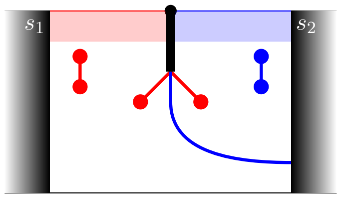
In the first relation we require that $K_1 \subseteq J_1$ where K_1 is the label of the green area. In the second relation we require $K_2 \subseteq J_1$ where K_2 is the label of the blue area. In the third relation we require that the green area is labelled with i, the colour of the green strand. In the last relation we require that the blue area is labelled with i, the label of the blue strand.

Examples of morphisms for S_3

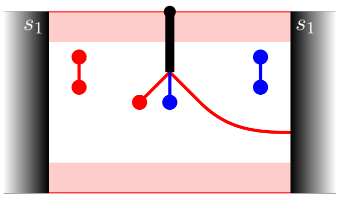
We will finish this chapter by applying this diagrammatic presentation of s$\mathbb{BS}$Bim to our calculation from Chapter 5. There we calculated various bases for homomorphisms spaces between indecomposable bimodules. We will now observe which morphisms in $s\mathcal{T}$ correspond to these morphisms in s$\mathbb{BS}$Bim.
We first fix some notation. We are now in the case $W = S_3$ and $S = \{s_1, s_2\}$. The strands and areas labelled s_1 will be coloured red. The strands labelled s_2 will be coloured blue. Thick black lines will always be labelled S.

Lemma 6.32. *Under the $s\mathcal{F}$ the two morphisms from (5.3) correspond (up to some scalars in $\Bbbk$) to the following morphisms in $s\mathcal{T}$.*

$\phi_1 \mathrel{\hat{=}}$ 

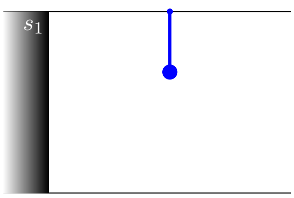
$\phi_2 \mathrel{\hat{=}}$ 

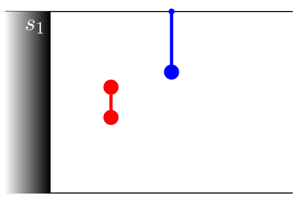
Lemma 6.33. *The morphism from the first point of Theorem 5.21 corresponds under $s\mathcal{F}$ (up to some scalar in $\Bbbk$) to the following morphism in $s\mathcal{T}$.*

$\phi \mathrel{\hat{=}}$ 

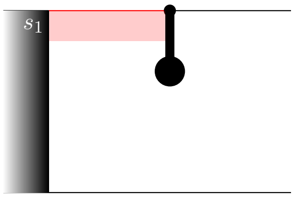
Lemma 6.34.

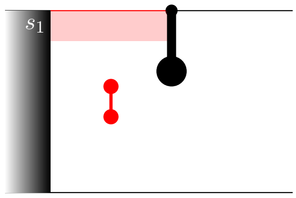
1. *The morphisms from the first point of Theorem 5.24 correspond under $s\mathcal{F}$ (up to some scalars in $\Bbbk$) to the following morphisms in $s\mathcal{T}$.*

$\phi_1 \mathrel{\hat{=}}$ 

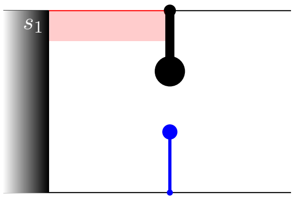
$\phi_2 \mathrel{\hat{=}}$ 

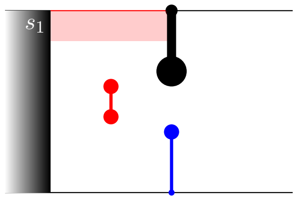
2. *The morphisms from the second point of Theorem 5.24 correspond under $s\mathcal{F}$ (up to some scalars in $\Bbbk$) to the following morphisms in $s\mathcal{T}$.*

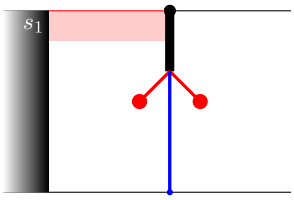
$\phi_1 \mathrel{\widehat{=}}$ 

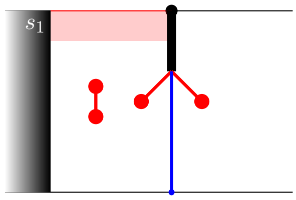
$\phi_2 \mathrel{\widehat{=}}$ 

3. *The morphisms from the third point of Theorem 5.24 correspond under $s\mathcal{F}$ (up to some scalars in $\Bbbk$) to the following morphisms in $s\mathcal{T}$.*

$\phi_1 \mathrel{\widehat{=}}$ 

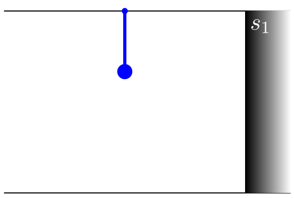
$\phi_2 \mathrel{\widehat{=}}$ 

$\phi_3 \mathrel{\widehat{=}}$ 

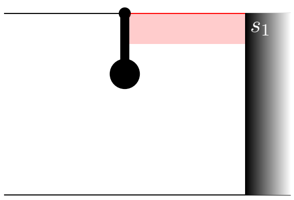
$\phi_4 \mathrel{\widehat{=}}$ 

Lemma 6.35.

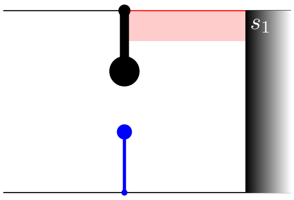
1. *The morphism from the first point of Theorem 5.27 corresponds under $s\mathcal{F}$ (up to some scalar in $\Bbbk$) to the following morphism in $s\mathcal{T}$.*

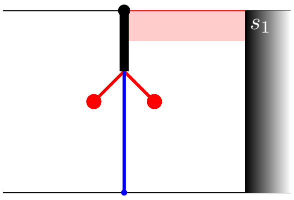
$\phi \mathrel{\widehat{=}}$ 

2. *The morphism from the second point of Theorem 5.27 corresponds under $s\mathcal{F}$ (up to some scalar in $\Bbbk$) to the following morphism in $s\mathcal{T}$.*

$\phi \mathrel{\widehat{=}}$ 

3. *The morphisms from the third point of Theorem 5.27 correspond under $s\mathcal{F}$ (up to some scalars in $\Bbbk$) to the following morphisms in $s\mathcal{T}$.*

$\phi_1 \widehat{=}$ 

$\phi_2 \widehat{=}$ 

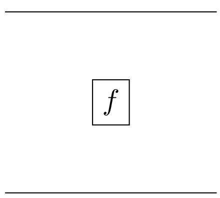
7 Diagrammatics for S_3

In this chapter we will consider the case $W = S_3$. Our goal is to give new descriptions for the categories $g\mathcal{D}$ and $s\mathcal{T}$. We would like to describe these categories by generators and relations without using rather abstract inclusion and projection morphisms and the complicated idempotent relations.
Before start to give such descriptions and prove that they are equivalent to the definition we know we will fix some notations. Note that for $W = S_3$ we have $S = \{s_1, s_2\}$ and only the four subsets $\emptyset, \{s_1\}, \{s_2\}, S$. We will use the colours red and blue for the strands labelled s_1 and s_2. We will not specify which colour corresponds to which simple transposition as everything is symmetric under swapping these two transpositions. We will use the colour violet if we mean a strand which is allowed to be blue or red. We will use the colour **black** for thick strands labelled S. If we colour certain areas in the description for $s\mathcal{T}$ we will use the colour white for $\emptyset$ and red, blue and **black** for the other subsets according to our colouring of the strands.

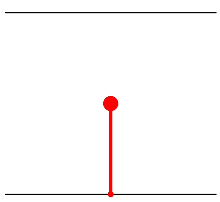
7.1 $g\mathcal{D}$ by generators and relations

We will now define a category $g\mathcal{D}_1$ by generators and relations and later prove that this category is equivalent to $g\mathcal{D}$.

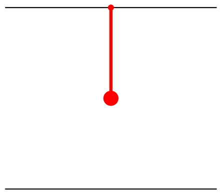
Definition 7.1. We define a monoidal category $g\mathcal{D}_1$ by generators and relations. It is generated on objects by s_1, s_2 and S viewed as coloured dots on a line. On morphisms it is generated by the following morphisms



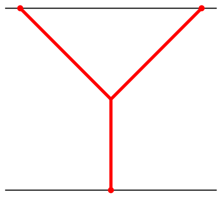
polynomial generator
$\deg = \deg(f)$
($f \in R$ homogeneous)



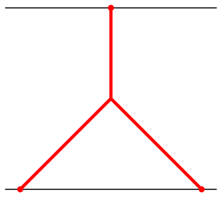
(end)dot
$\deg = 1$



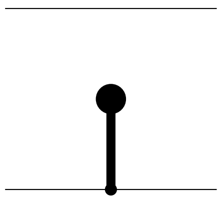
(start)dot
deg = 1



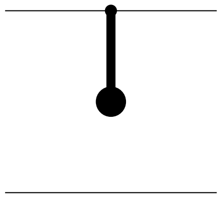
trivalent vertex (split)
deg = −1



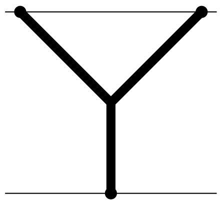
trivalent vertex (merge)
deg = −1



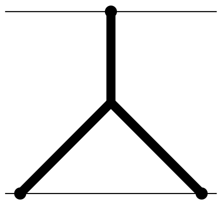
thick (end)dot
deg = 3



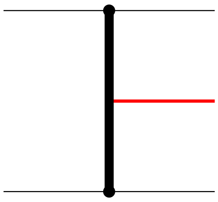
thick (start)dot
deg = 3



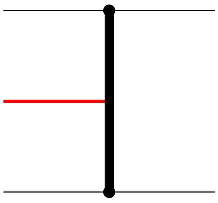
very thick trivalent vertex (split)
deg = −3



very thick trivalent vertex (merge)
deg = −3



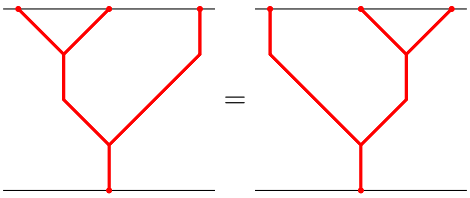
thick trivalent vertex (right-facing)
deg $= -1$



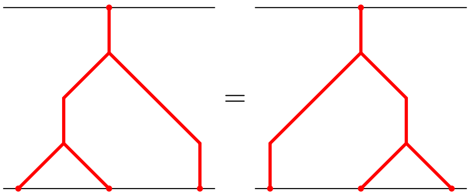
thick trivalent vertex (left-facing)
deg $= -1$

modulo the relations (7.1) to (7.27). ◊

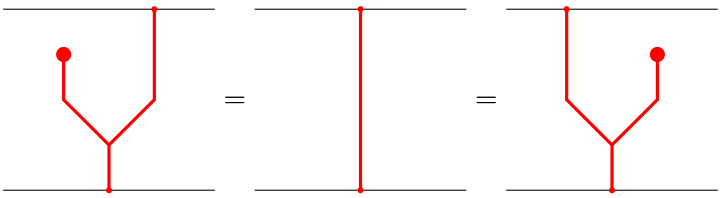
Relations (cup and cap are defined as usual)



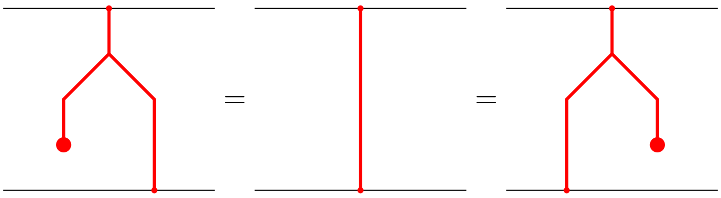
(7.1)



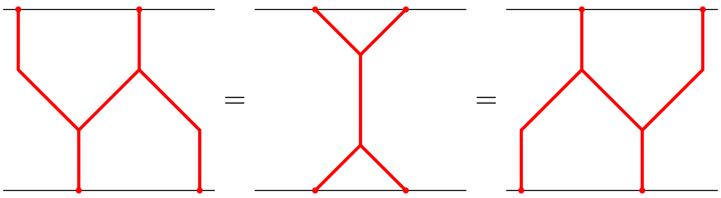
(7.2)



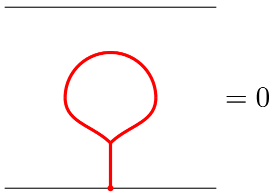
(7.3)



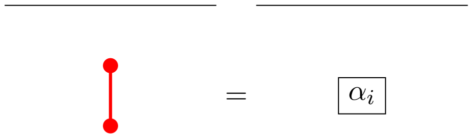
(7.4)



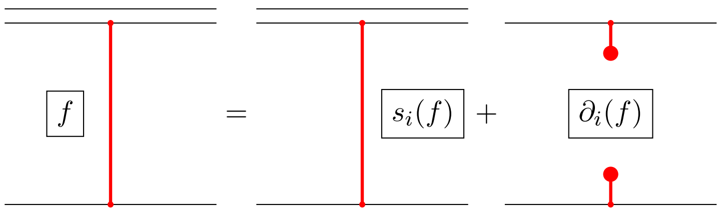
(7.5)



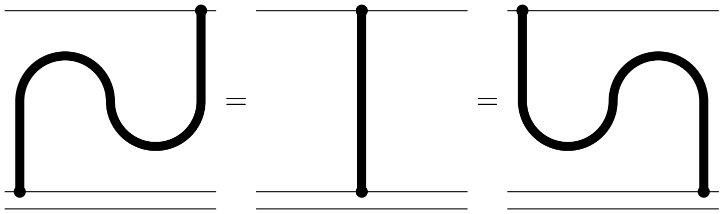
(7.6)



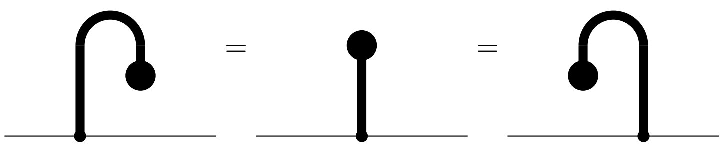
(7.7)



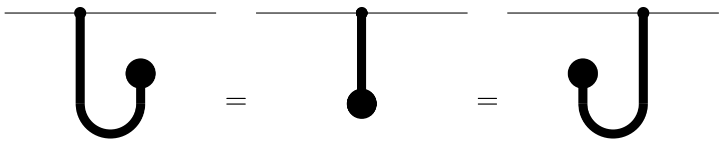
(7.8)



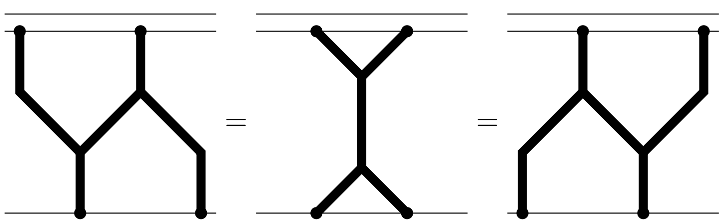
(7.9)



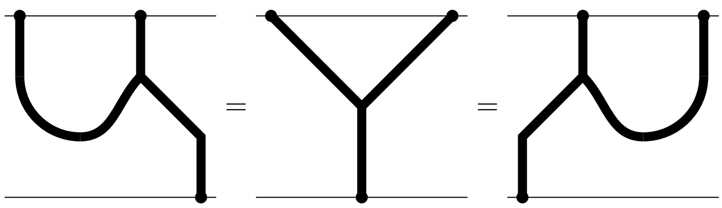
(7.10)



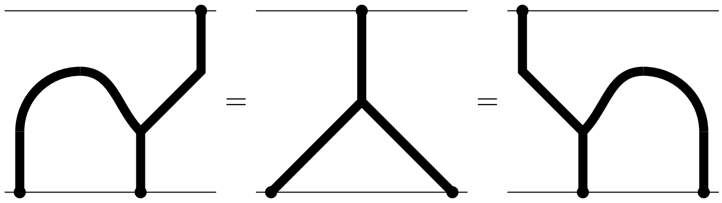
(7.11)



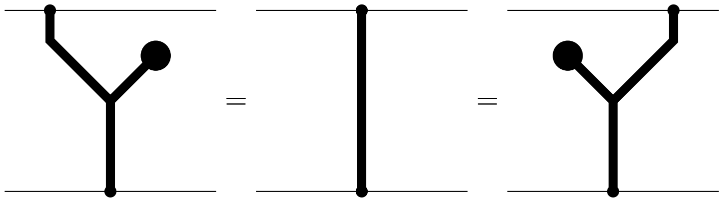
(7.12)



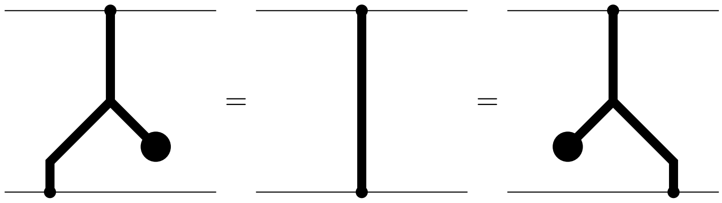
(7.13)



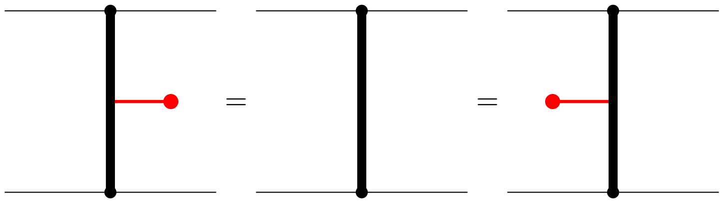
(7.14)



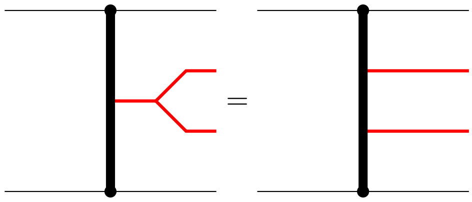
(7.15)



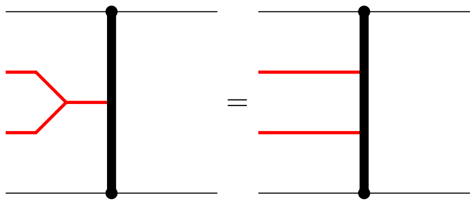
(7.16)



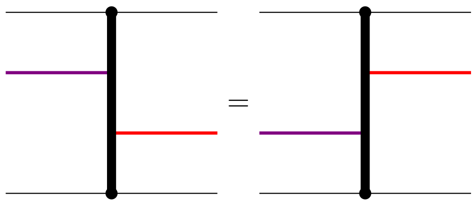
(7.17)



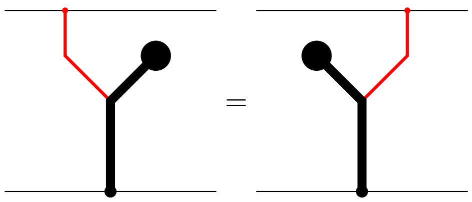
(7.18)



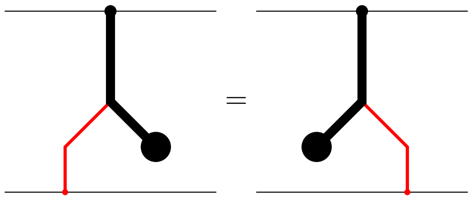
(7.19)



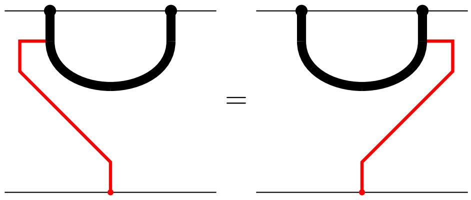
(7.20)



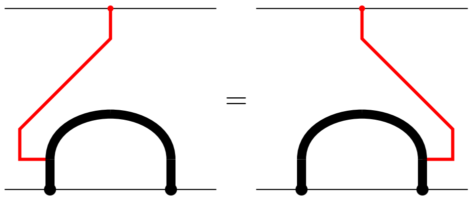
(7.21)



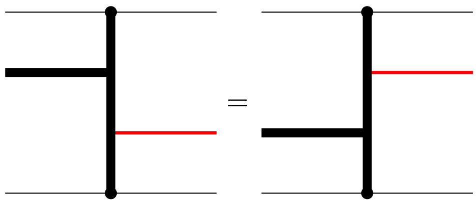
(7.22)



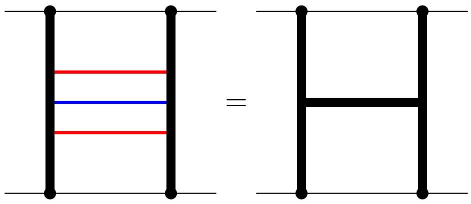
(7.23)



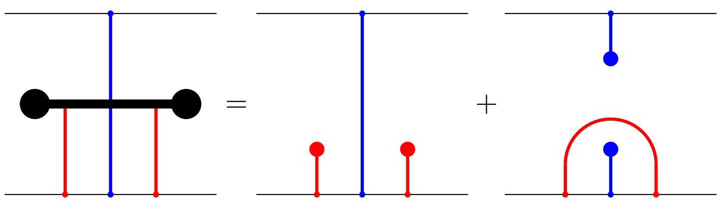
(7.24)



(7.25)



(7.26)

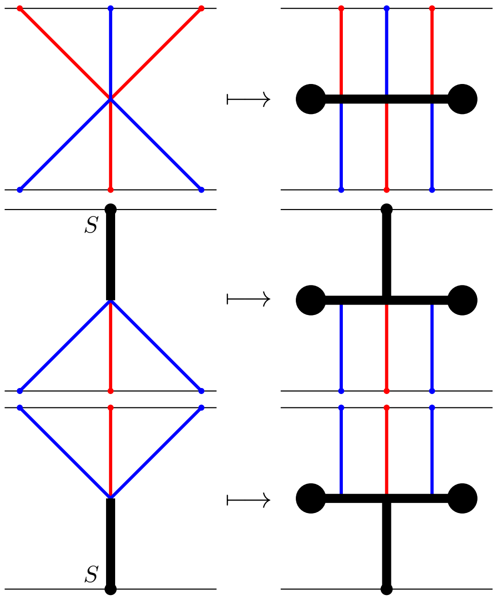


(7.27)

Remark 7.2. This list of relations is not minimal! For instance (7.23) and (7.24) are consequences of (7.21), (7.22) and (7.25). However, since we want to use all these relations and most of them are also very intuitive we put them in the definition. In this way we do not need to spend time on proving some of these relations as consequences of others and can instead concentrate on our main goal which is the equivalence (Theorem 7.5). ◊

Definition 7.3. We define a functor $\mathcal{G}_1 : g\mathcal{D}_1 \longrightarrow g\mathcal{D}$. On objects $\mathcal{G}_1$ is just the identity ($g\mathcal{D}$ and $g\mathcal{D}_1$ have the same objects). On morphisms each of the generators from Definition 7.1 is sent to the same picture in $g\mathcal{D}$. Note that this is possible as we defined the thick dots, thick trivalent vertices and very thick trivalent vertices in $g\mathcal{D}$. $\Diamond$

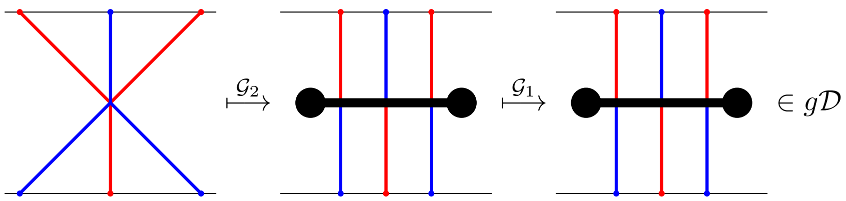
Definition 7.4. We define a functor $\mathcal{G}_2 : g\mathcal{D} \longrightarrow g\mathcal{D}_1$. On objects $\mathcal{G}_2$ is just the identity ($g\mathcal{D}$ and $g\mathcal{D}_1$ have the same objects). On morphisms we define the image for each of the generators of $g\mathcal{D}$.

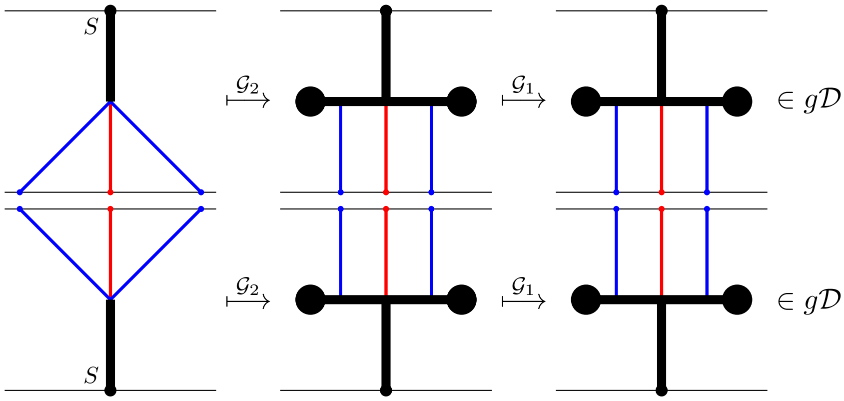


The remaining generators are the 1-colour generators from $\mathcal{D}_1$. They are just sent to there counterparts in $g\mathcal{D}_1$. $\Diamond$

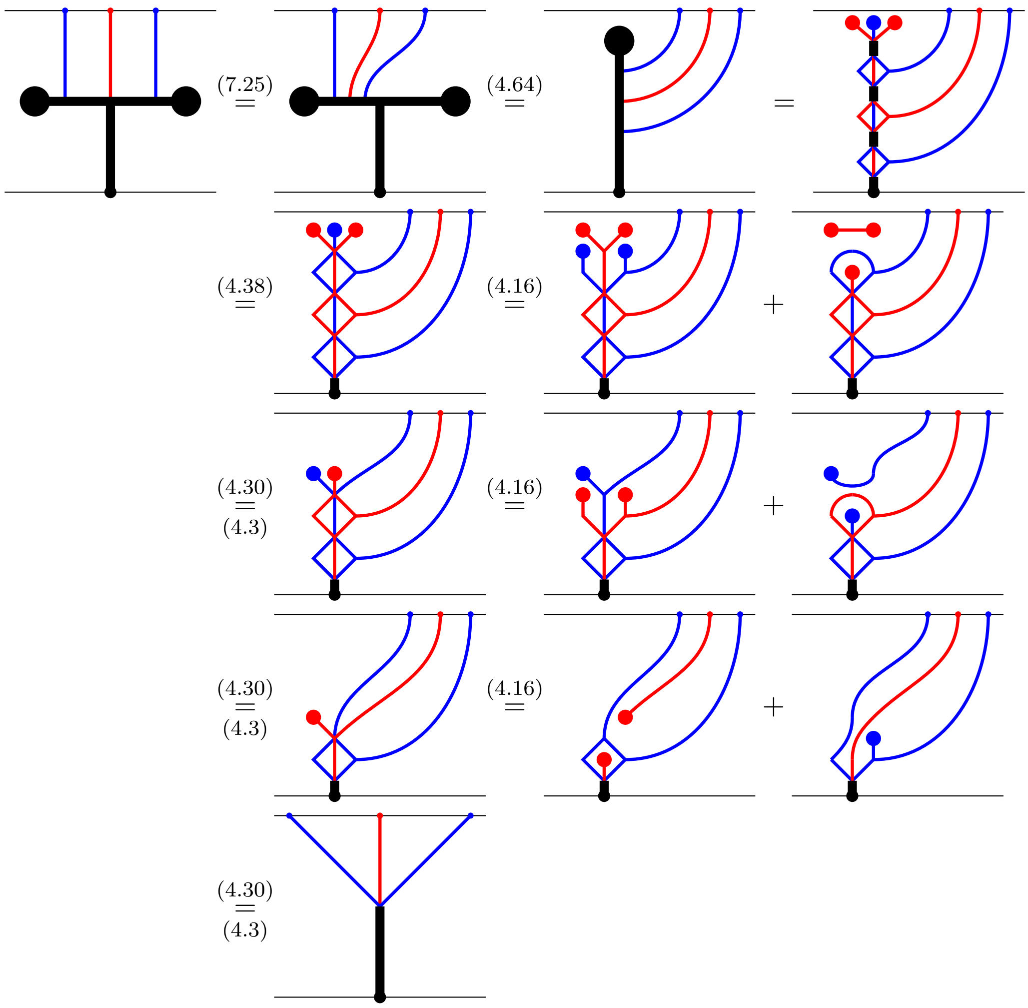
Theorem 7.5. *Assume that the functors $\mathcal{G}_1$ and $\mathcal{G}_2$ are well-defined. Then they are inverse to each other and yield an equivalence (even an isomorphism) of categories between $g\mathcal{D}_1$ and $g\mathcal{D}$.*

Proof. All we need to prove is that $\mathcal{G}_1 \circ \mathcal{G}_2$ and $\mathcal{G}_2 \circ \mathcal{G}_1$ are the identity functors on $g\mathcal{D}$ and $g\mathcal{D}_1$ respectively. On objects this is obvious. Hence, we only need to check it on generating morphisms. We start with $\mathcal{G}_1 \circ \mathcal{G}_2$. Both functors send the 1-colour morphisms to their respective version in the other category. Thus, $\mathcal{G}_1 \circ \mathcal{G}_2$ is obviously the identity on them. So we just need to check this for the other three generators. We have the following.

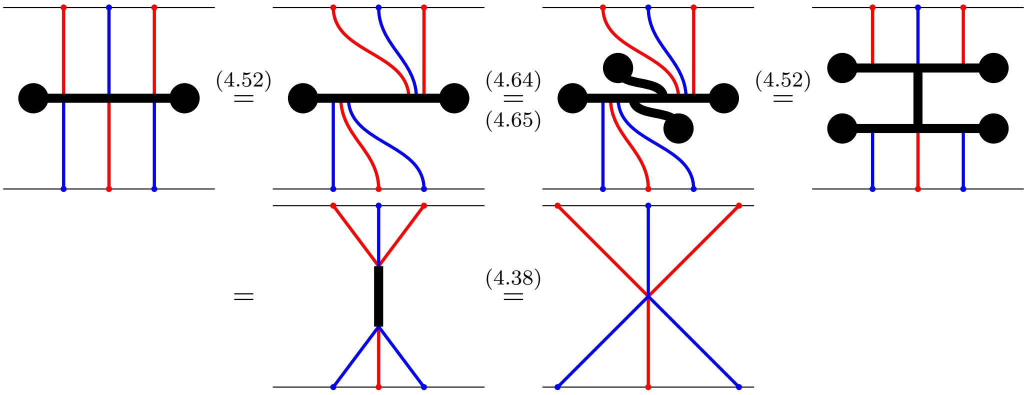




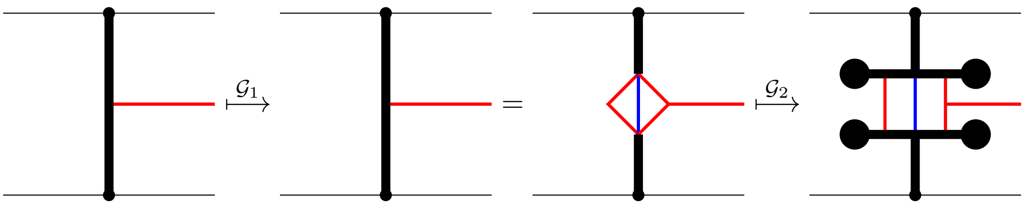
Thus, we have three equations to prove. We will start with the last one. Note that since we assumed that the functors are well-defined we know that all the relations from Definition 7.1 hold in $g\mathcal{D}$.



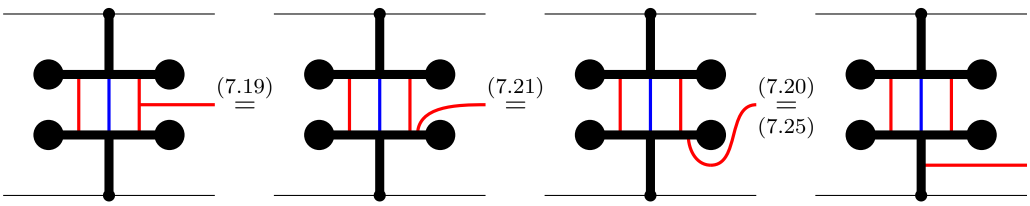
Note that the last application of (4.30) has an S-inclusion instead of a 6-valent vertex at the bottom. However, by the nature of the S-inclusion we can always replace it by an S-inclusion composed with a 6-valent vertex and then we could use (4.30). That is what happened there.
The proof for the second equation for the S-projection is exactly the same as for the S-inclusion just everything turned upside down. Thus, we are left with the first equation for the 6-valent vertex.

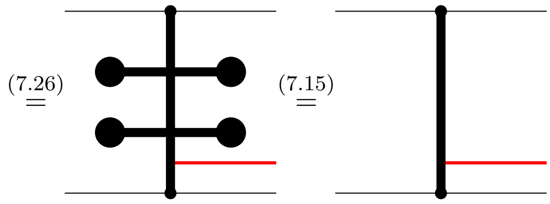


This finishes the proof that $\mathcal{G}_1 \circ \mathcal{G}_2$ is the identity functor. We proceed with proving the same for $\mathcal{G}_2 \circ \mathcal{G}_1$. We need to check that $\mathcal{G}_2 \circ \mathcal{G}_1$ is the identity functor on the generating morphisms of $g\mathcal{D}_1$. This is obvious for the (thin) 1-colour generators as they are sent to their respective versions by $\mathcal{G}_1$ as well as $\mathcal{G}_2$. We now need to check this for the thick dots, the very thick trivalent vertices and the thick trivalent vertices. We will only do this only for one of the two iterations of each of these because the other ones can be done in the exact same way. We will start with the thick trivalent vertex.

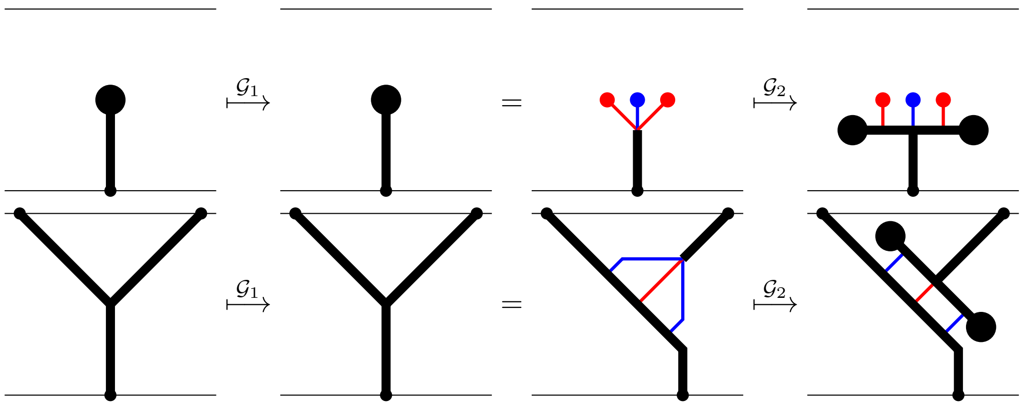


We have to prove that the first and the last picture above are the equal in $g\mathcal{D}_1$. This follows from the following chain of equalities.

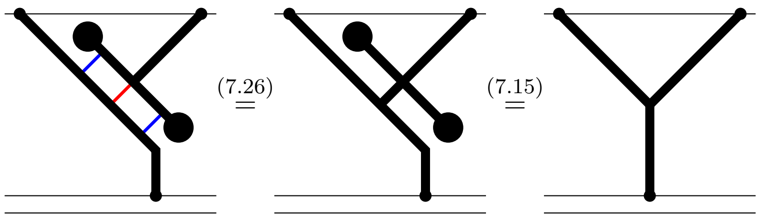


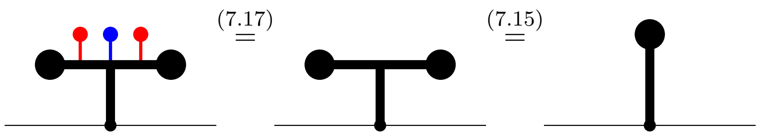


We will continue with the very thick dot and the very thick trivalent vertex. First we need to compute their images under $\mathcal{G}_2 \circ \mathcal{G}_1$.



Note that we used the fact that $\mathcal{G}_2$ sends the thick trivalent vertex to its counterpart in $g\mathcal{D}_1$ when we computed the image of the very thick trivalent vertex. This is however no problem as we checked exactly this in our last calculation. Now we just need to prove the remaining two equations which arise from the above computations.



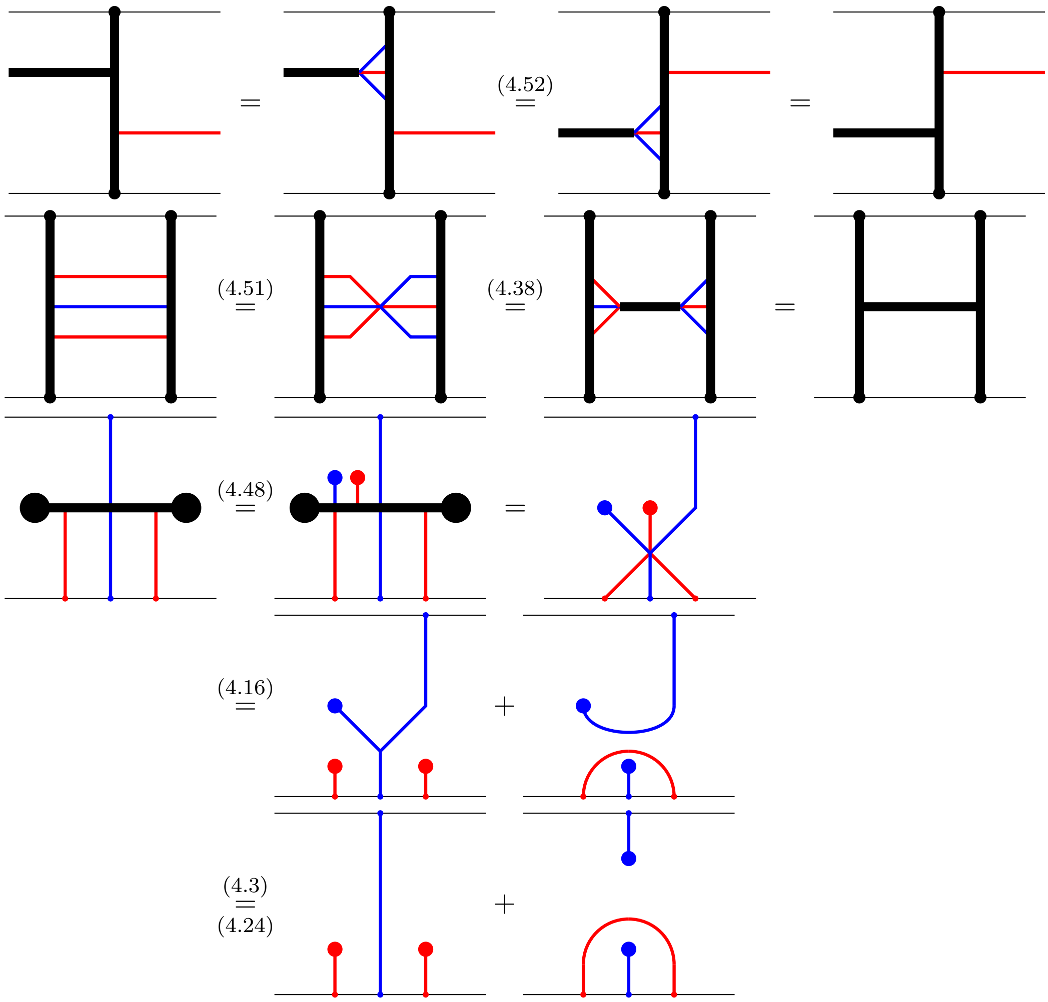


This finishes the proof. □

Lemma 7.6. *The functor $\mathcal{G}_1$ is well-defined.*

Proof. For $\mathcal{G}_1$ to be well-defined we need that all relations from $g\mathcal{D}_1$ hold in $g\mathcal{D}$ when sent there by $\mathcal{G}_1$. However, we have seen almost all relations from $g\mathcal{D}_1$ in $g\mathcal{D}$ already.

The only exceptions are the relations (7.25) to (7.27). Thus, we only have to prove that these hold in $g\mathcal{D}$. For this we compute in $g\mathcal{D}$.

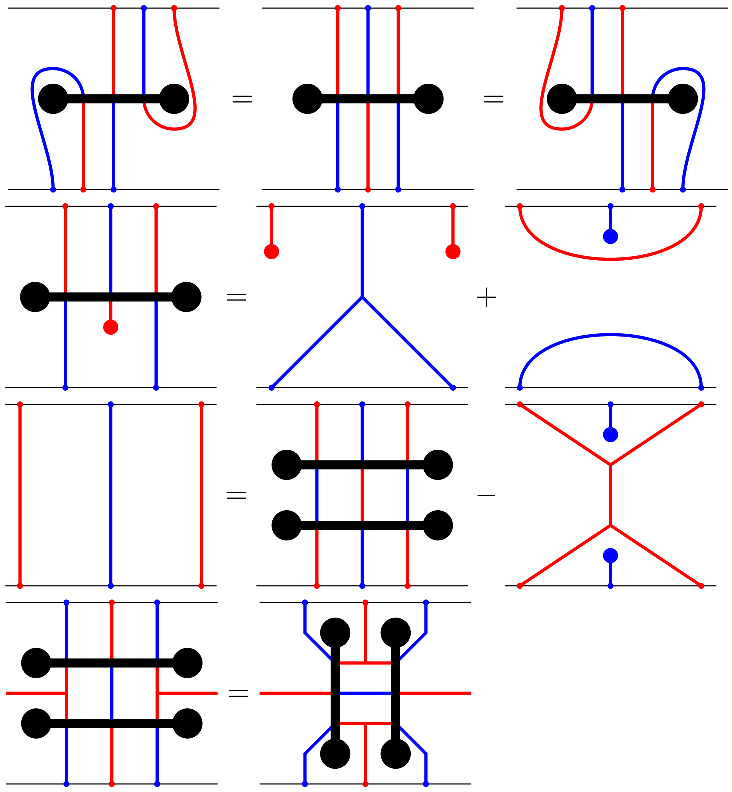


Note that we used the proof of Theorem 7.5, where we saw a way to rewrite the 6-valent vertex with thick lines, in the proof of the last relation (the second equality there). ☐

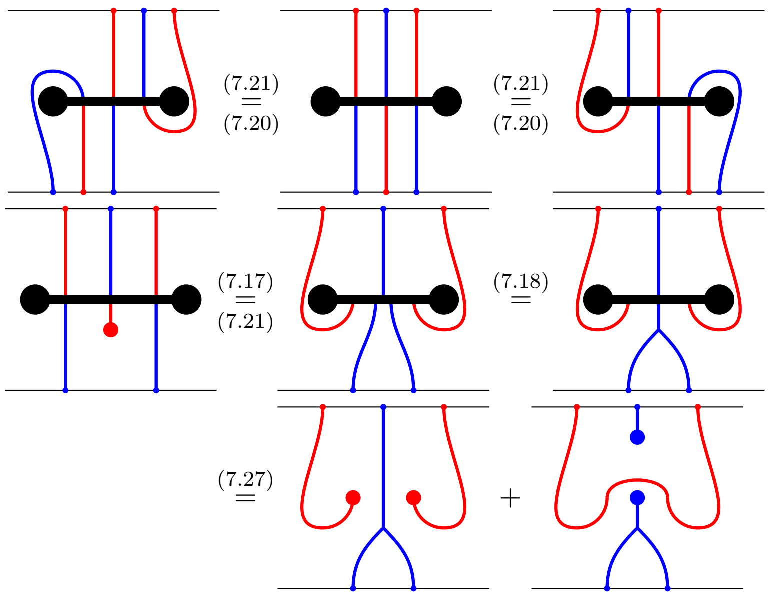
Lemma 7.7. *The functor $\mathcal{G}_2$ is well-defined.*

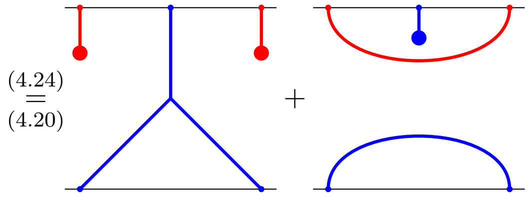
Proof. For $\mathcal{G}_2$ to be well-defined we need that all relations from $g\mathcal{D}$ hold in $g\mathcal{D}_1$ when sent there by $\mathcal{G}_2$. Recall that the relations for $g\mathcal{D}$ are the relations for $\mathcal{D}_1$ as well as the relations (4.37) and (4.38). We know that $\mathcal{G}_2$ sends the one-colour relations from $\mathcal{D}_1$ to their respective versions in $g\mathcal{D}_1$. Thus, we don't have to check anything for the one-colour relations. The remaining relations in $\mathcal{D}_1$ are the four relations (4.15) to (4.18).

They are sent to the following equations by $\mathcal{G}_2$.

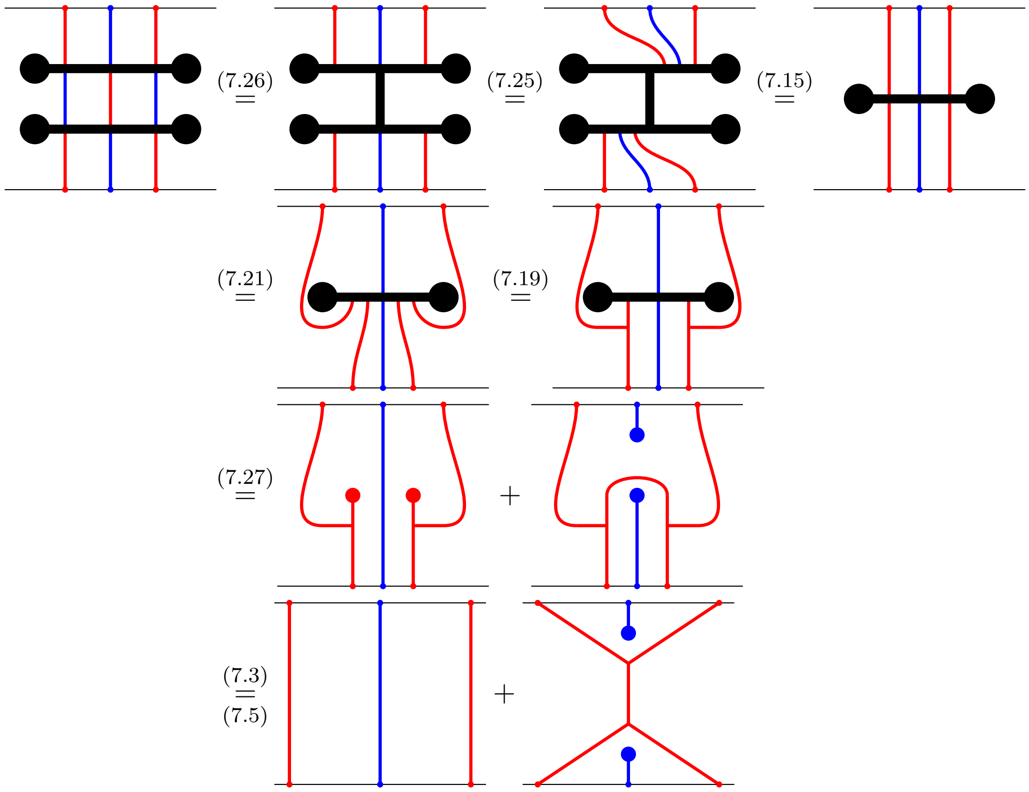


Now we will prove these relations in $g\mathcal{D}_1$.

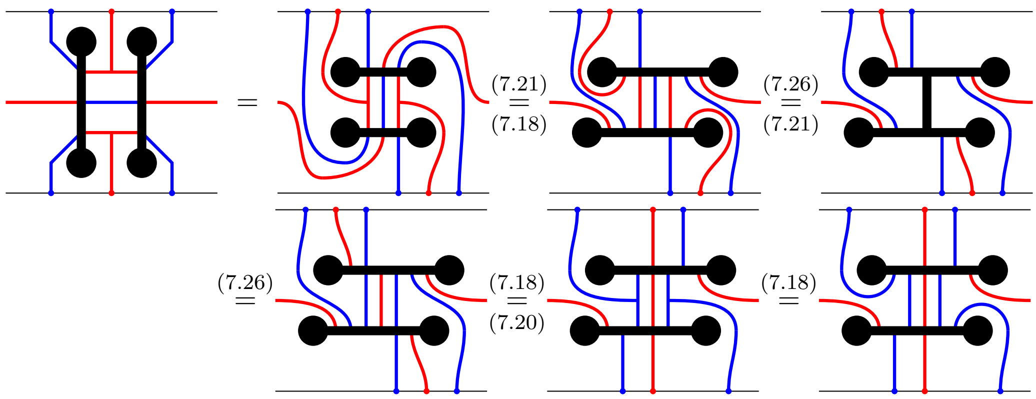


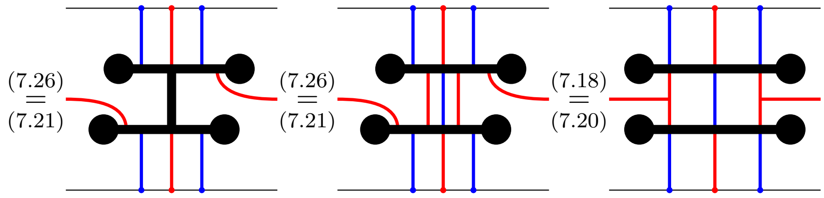


Note that we used in the last step that (4.20) and (4.24) are consequences of the other one-colour relations and thus also hold in $g\mathcal{D}_1$. We continue with the third equation.

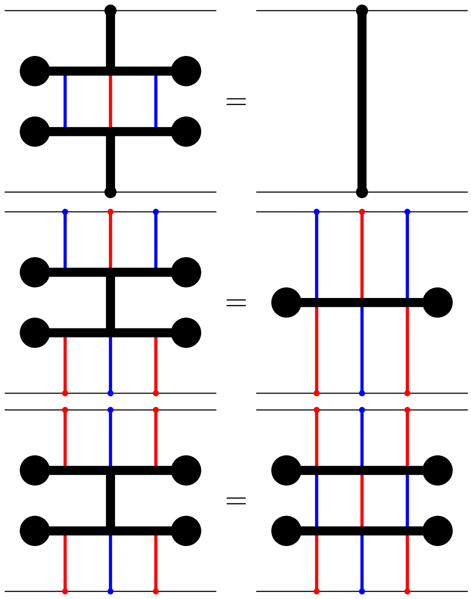


Now we just need to check the last equation.

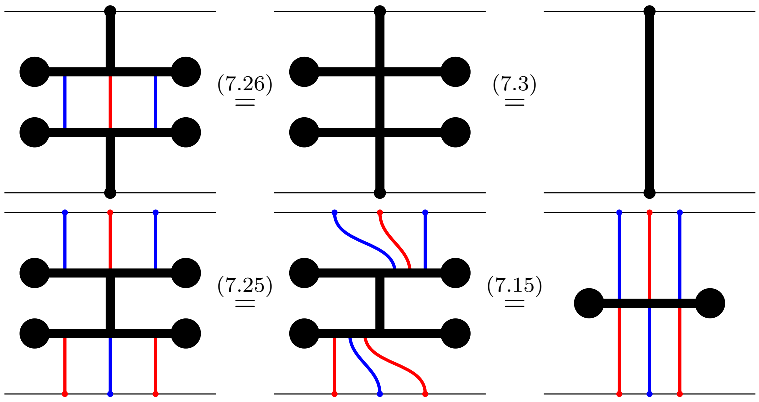


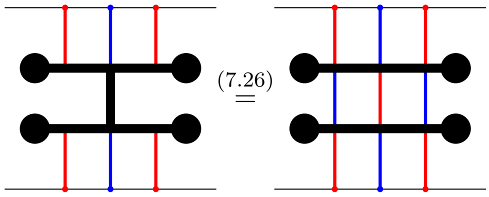


Hence, we have checked that all relations from $\mathcal{D}_1$ still hold when sent to $g\mathcal{D}_1$ by $\mathcal{G}_2$. All that is left to do is to prove that the same is true for (4.37) and (4.38) which are the last relations for $g\mathcal{D}$. They are sent to the following equations by $\mathcal{G}_2$.



Now we will prove these relations in $g\mathcal{D}_1$.





This finishes the proof. □

7.2 $s\mathcal{T}$ by generators and relations

Before we begin with the definitions we fix some notation. We keep our colouring of the strands as in the last section. In this section we will also colour areas. The colours red, blue and **black** represent the same subsets of S for areas as they do for strands. White represents the empty subset of S. We use the colours green and yellow to indicate that the area is allowed to be coloured with any subset of S (as long as all conditions that may be imposed are satisfied).

Definition 7.8. We define a 2-category $s\mathfrak{T}$. The objects are the sets $\emptyset, \{s_1\}, \{s_2\}, S$. The 1-morphisms will be generated by labelled empty sequences of dots. Namely the generating 1-morphisms in ${}_I s\mathfrak{T}_J = \mathrm{Mor}_{s\mathfrak{T}}(I, J)$ are $\emptyset_K$ where $K \subseteq I, J$ and the horizontal composition of 1-morphisms will be written as

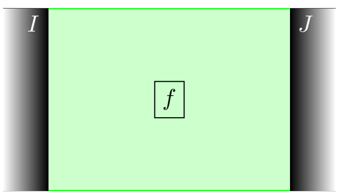
$$\emptyset_{K_1} \star \emptyset_{K_2} = {}_{K_1}{}^{J_1}{}_{K_2} \in {}_I s\mathfrak{T}_J$$

for $\emptyset_{K_1} \in {}_I s\mathfrak{T}_{J_1}, \emptyset_{K_2} \in {}_{J_1} s\mathfrak{T}_J$. So the resulting objects are sequences

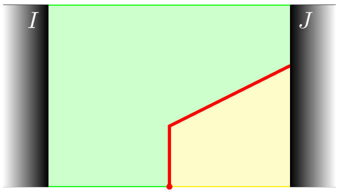
$${}_{K_0}{}^{J_1}{}_{K_1}{}^{J_2}{}_{K_2} \cdots {}_{K_{l-1}}{}^{J_l}{}_{K_l} \in {}_I s\mathfrak{T}_J$$

with $K_i \subseteq J_i, J_{i+1}$ for $i = 0, \ldots, l$ where $J_0 = I, J_{l+1} = J$.
The 2-morphisms will be generated by the following morphisms modulo the relations we list at the end.

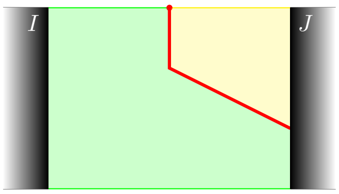
- All generators from $g\mathcal{D}$, but with membranes on the sides.



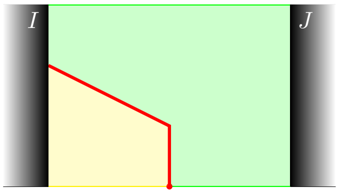
coloured polynomial generator
$\deg = \deg(f)$
($f \in R^K$ homogenous)



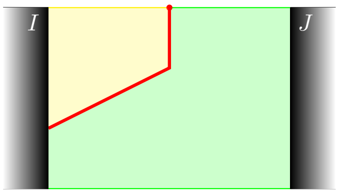
coloured trivalent vertex
$\deg = \ell(w_K) - 1$
(where K corresponds to the yellow area)
$i \in J$ is required



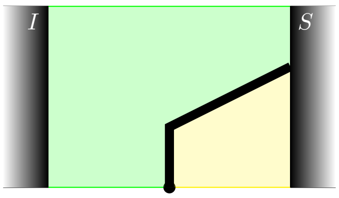
coloured trivalent vertex
$\deg = \ell(w_K) - 1$
(where K corresponds to the yellow area)
$i \in J$ is required



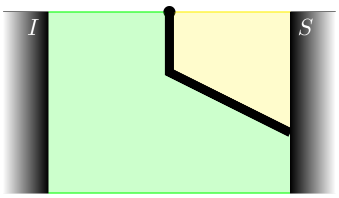
coloured trivalent vertex
$\deg = \ell(w_K) - 1$
(where K corresponds to the yellow area)
$i \in I$ is required



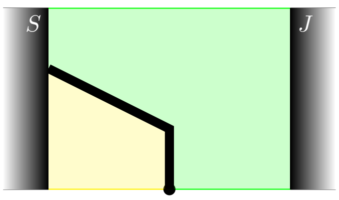
coloured trivalent vertex
$\deg = \ell(w_K) - 1$
(where K corresponds to the yellow area)
$i \in I$ is required



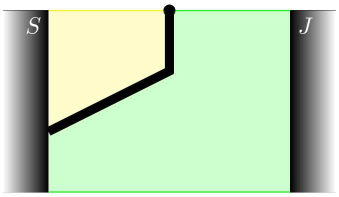
coloured thick trivalent vertex
$\deg = \ell(w_K) - 3$
(where K corresponds to the yellow area)



coloured thick trivalent vertex
$\deg = \ell(w_K) - 3$
(where K corresponds to the yellow area)

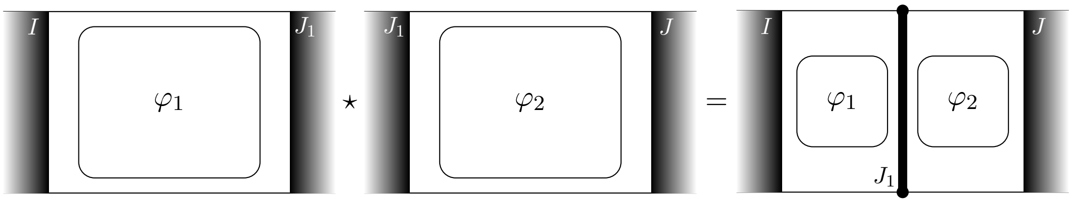


coloured thick trivalent vertex
$\deg = \ell(w_K) - 3$
(where K corresponds to the yellow area)



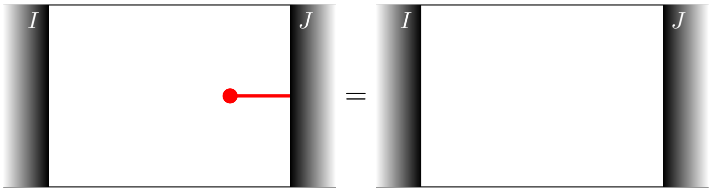
coloured thick trivalent vertex
deg $= \ell(w_K) - 3$
(where K corresponds to the yellow area)

The relations are all relations from $g\mathcal{D}$ with membranes on the sides together with the relations (7.28) to (7.50). The horizontal composition of such 2-morphisms will be given by the following relation.

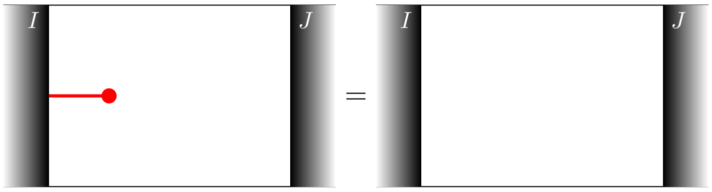


If $J_1 = \emptyset$, then a line labelled $\emptyset$ is just no line. ◊

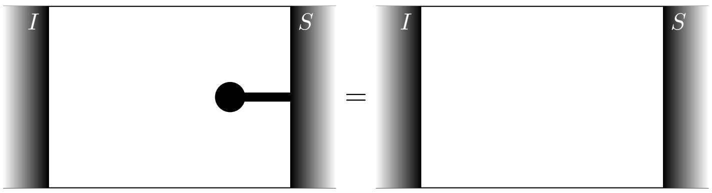
Relations



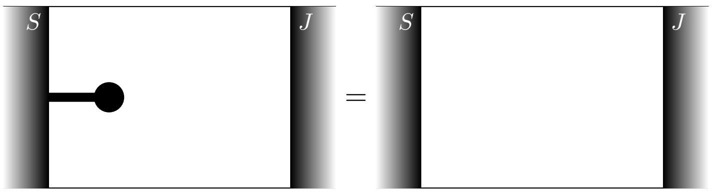
(7.28)



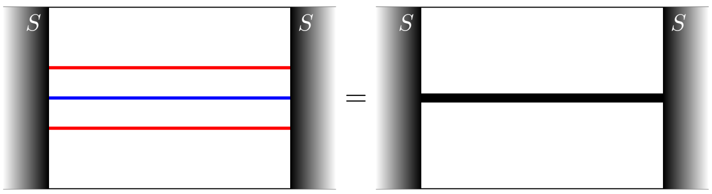
(7.29)



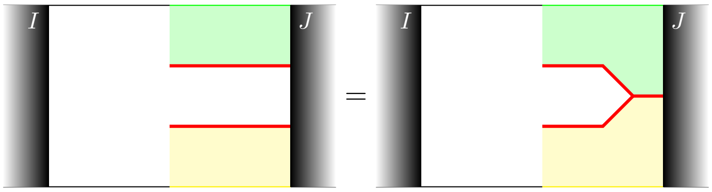
(7.30)



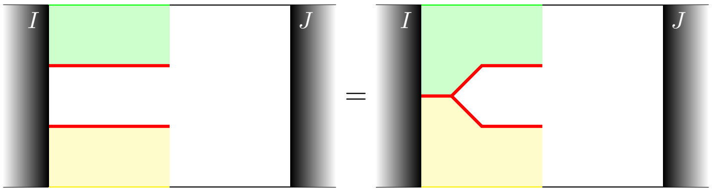
(7.31)



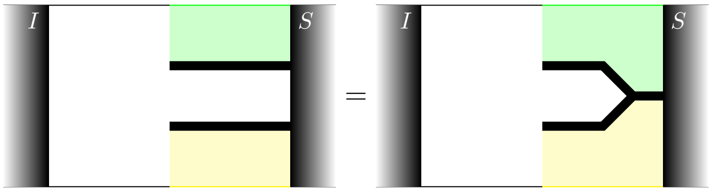
(7.32)



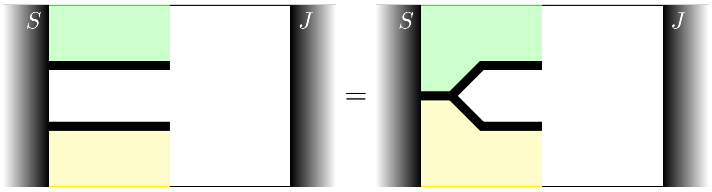
(7.33)



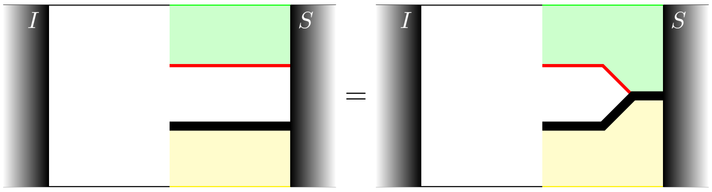
(7.34)



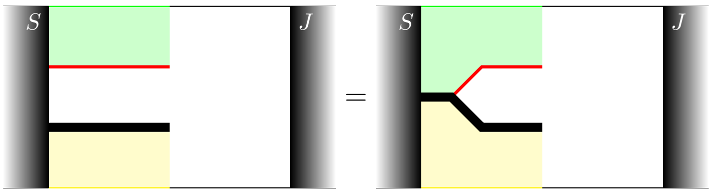
(7.35)



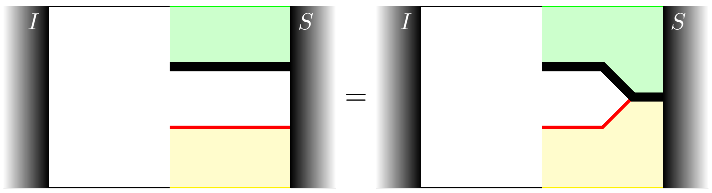
(7.36)



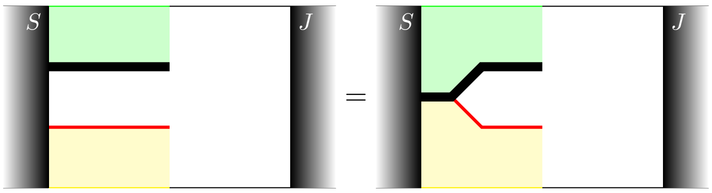
(7.37)



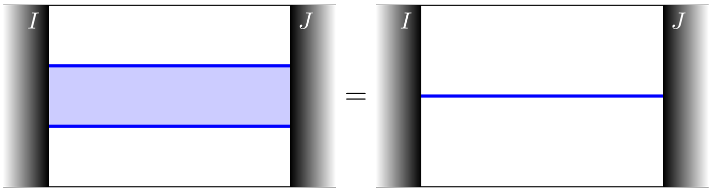
(7.38)



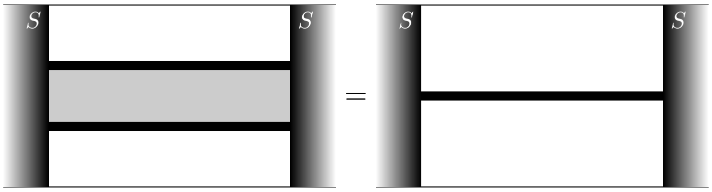
(7.39)



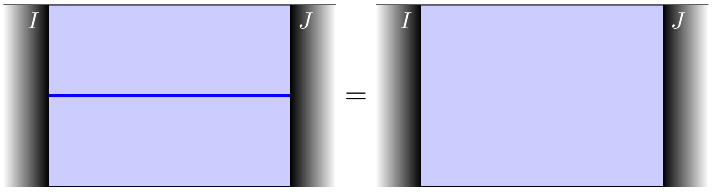
(7.40)



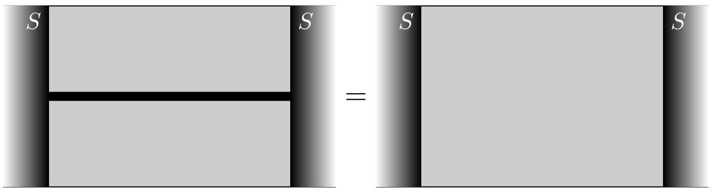
(7.41)



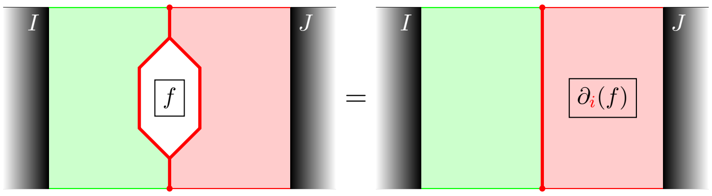
(7.42)



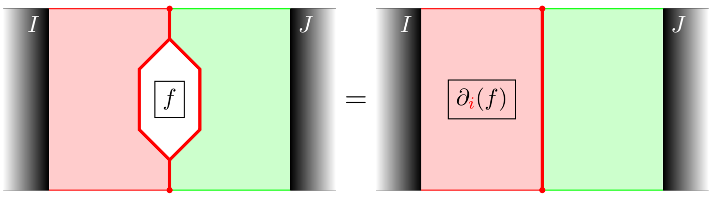
(7.43)



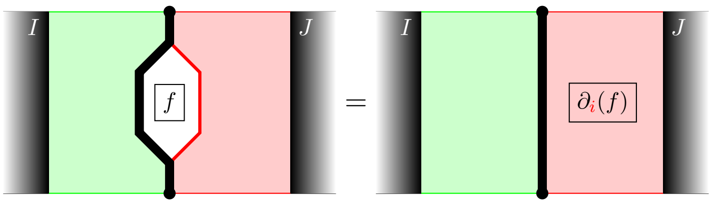
(7.44)



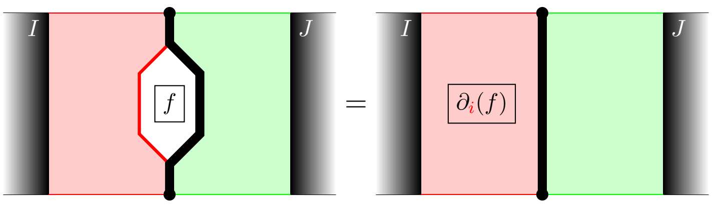
(7.45)



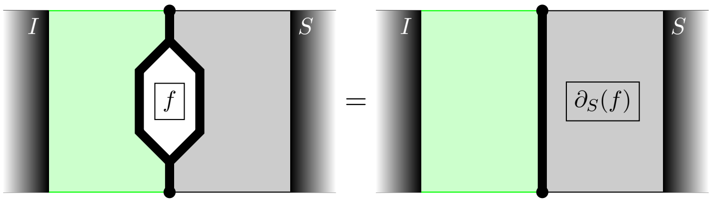
(7.46)



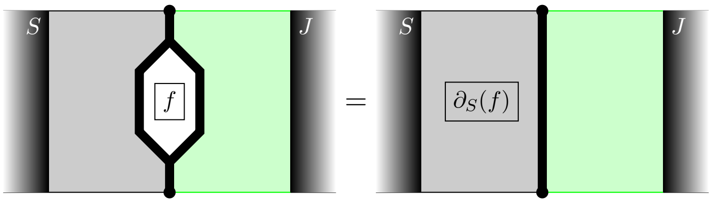
(7.47)



(7.48)

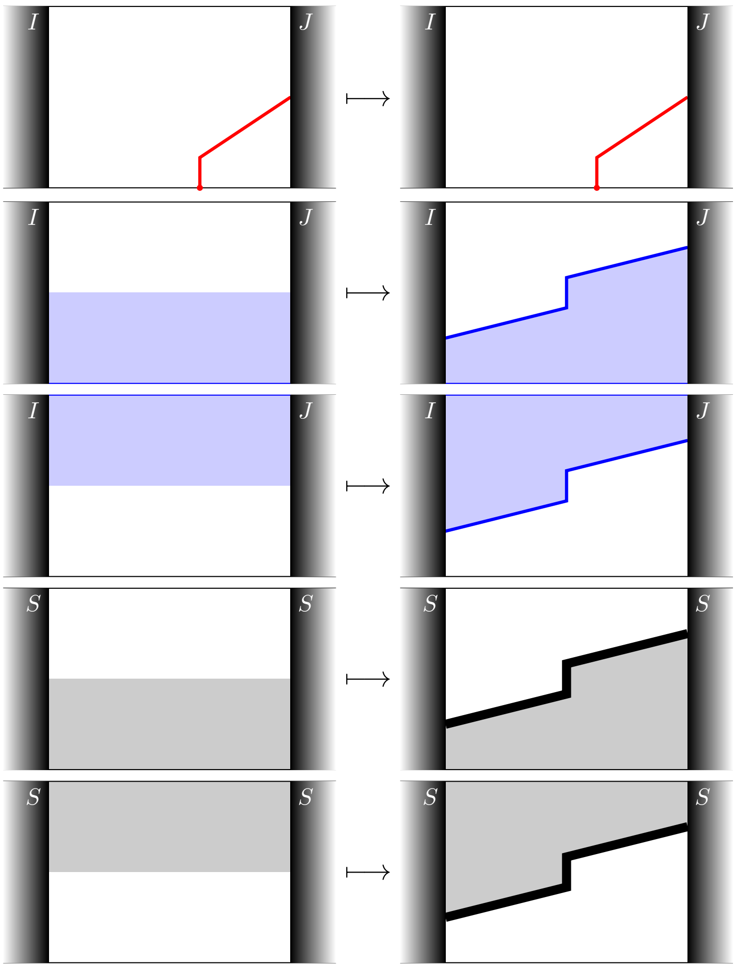


(7.49)



(7.50)

Definition 7.9. We define a functor $\mathcal{G}_3 : s\mathcal{T} \longrightarrow s\mathfrak{T}$ as follows. On objects $\mathcal{G}_3$ is just the identity ($s\mathcal{T}$ and $s\mathfrak{T}$ have the same objects). On morphisms each of the generators from $g\mathcal{D}$ in $s\mathcal{T}$ is sent to the corresponding generator from $g\mathcal{D}$ in $s\mathfrak{T}$. The images of the remaining generators are defined as follows.

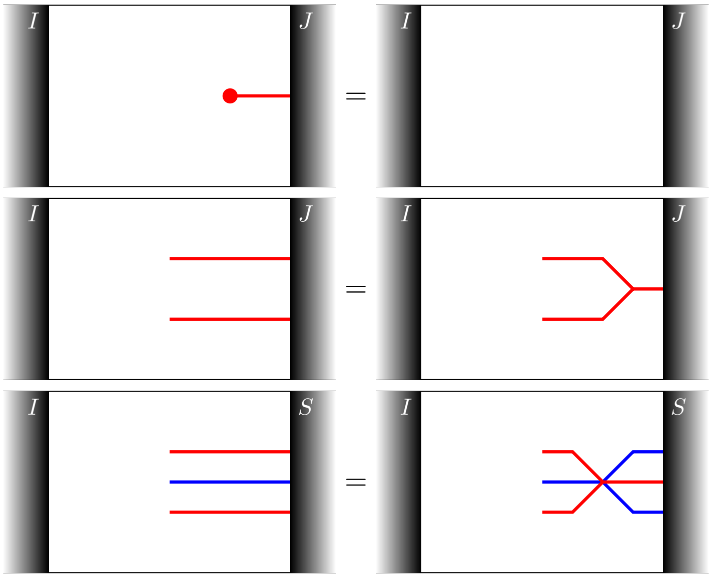


In the first picture the image is the coloured trivalent vertex with white colouring. We also use the same definition if the red strand would end in the left membrane or the top boundary. ◊

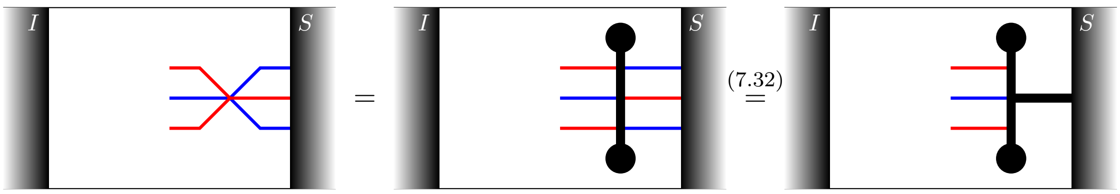
Definition 7.10. We define a functor $\mathcal{G}_4 : s\mathfrak{T} \longrightarrow s\mathcal{T}$ as follows. On objects $\mathcal{G}_4$ is just the identity ($s\mathcal{T}$ and $s\mathfrak{T}$ have the same objects). On morphisms each of the generators from $g\mathcal{D}$ in $s\mathfrak{T}$ is sent to the corresponding generator from $g\mathcal{D}$ in $s\mathcal{T}$. The coloured polynomial generator is sent to its corresponding version is $s\mathcal{T}$ (see Definition 6.31). The coloured (thick) trivalent vertices are sent to there respective versions in $s\mathcal{T}$ (see Definition 6.29). ◊

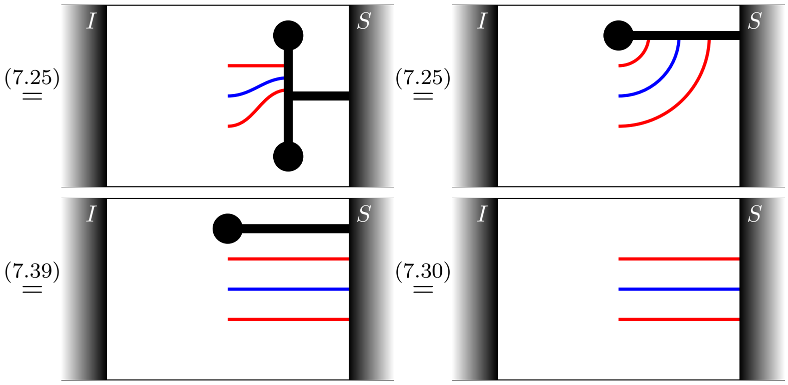
Lemma 7.11. *The functor $\mathcal{G}_3$ is well-defined.*

Proof. For $\mathcal{G}_3$ to be well-defined we need that any relation from $s\mathcal{T}$ holds in $s\mathfrak{T}$ when sent there by $\mathcal{G}_3$. We know the $\mathcal{G}_3$ sends any relation from $g\mathcal{D}$ in $s\mathcal{T}$ to their respective version in $s\mathfrak{T}$ and there it holds by definition of $s\mathfrak{T}$. Thus, there is nothing to check in this case. Then we have three relations coming from ${}_I\mathcal{T}_J$ in $s\mathcal{T}$, namely the following (and their left-side versions).

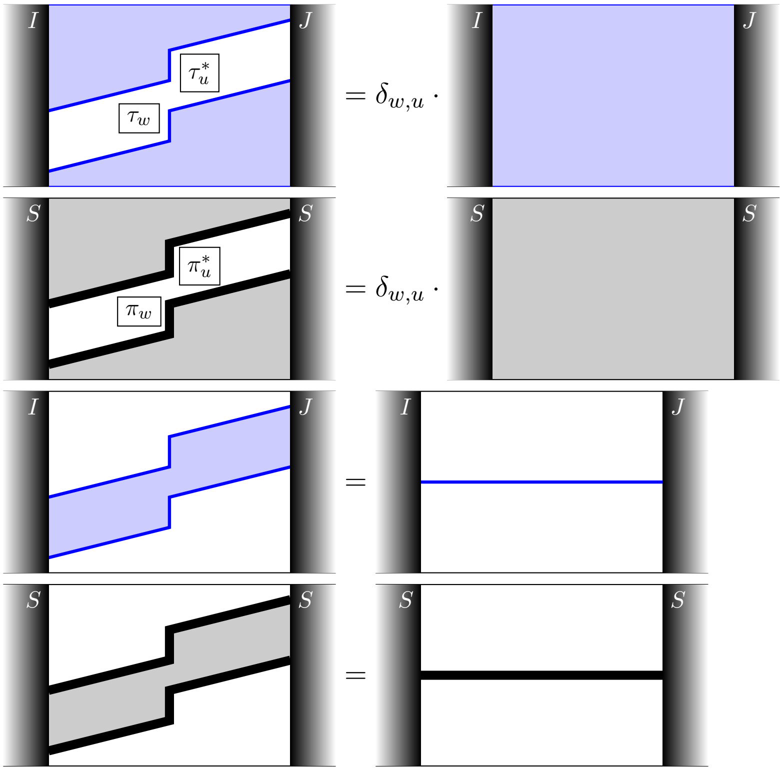


These relations get sent to the same equations in $s\mathfrak{T}$. The first two relations follow from (7.28) and (7.33). The third relations can be proven as follows.

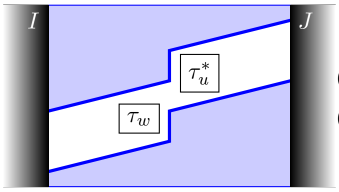




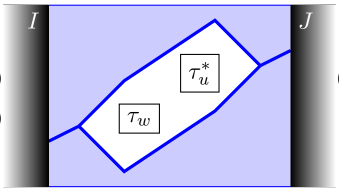
The last relations for $s\mathcal{T}$ are the relations (6.13) and (6.14). The will be sent to the following equations in $s\mathfrak{T}$.



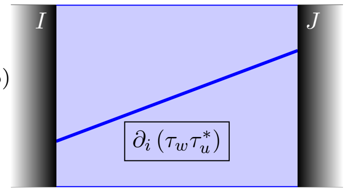
In the first equation $\{\tau_w\}_{w \in W_i}$ and $\{\tau_w^*\}_{w \in W_i}$ are the dual bases for R over R^i from Theorem 3.35. In the second equation $\{\pi_w\}_{w \in W}$ and $\{\pi_w^*\}_{w \in W}$ are the dual bases for R over R^S from Theorem 3.35. The third and the fourth equation follow immediately from (7.41) and (7.42). For the other two equations we have.



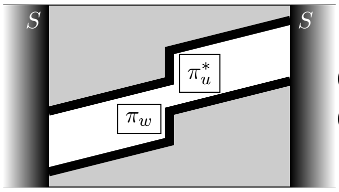
$\overset{(7.34)}{=}$ (7.33)



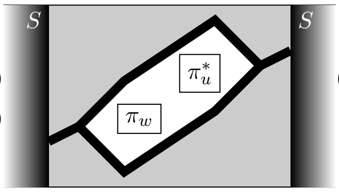
$\overset{(7.45)}{=}$



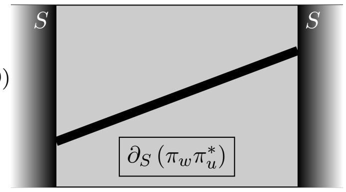
$\overset{(7.43)}{=}\ \delta_{w,u}\cdot$



$\overset{(7.35)}{=}$ (7.36)



$\overset{(7.49)}{=}$



$\overset{(7.44)}{=}\ \delta_{w,u}\cdot$

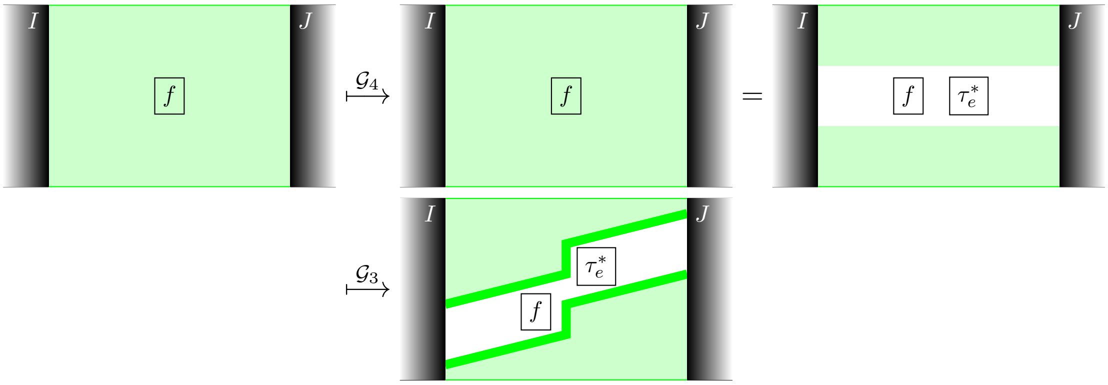
This finishes the proof. □

Lemma 7.12. *The functor $\mathcal{G}_4$ is well-defined.*

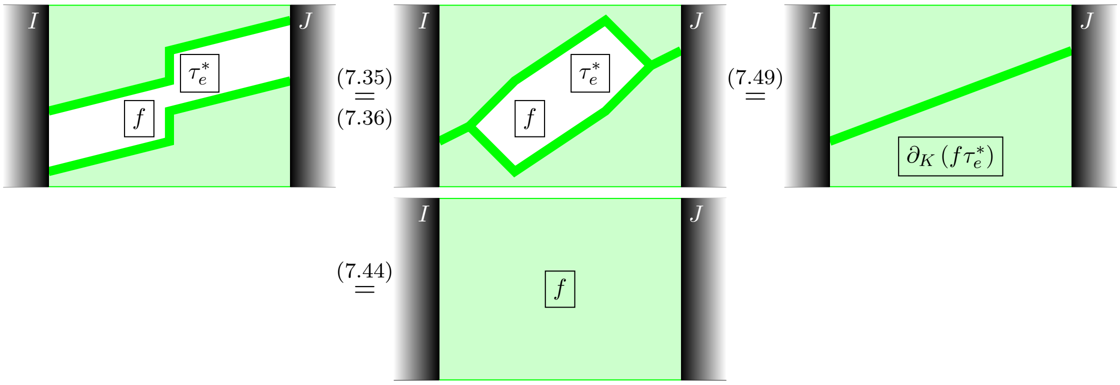
Proof. For $\mathcal{G}_4$ to be well-defined we need that any relation from $s\mathfrak{T}$ holds in $s\mathcal{T}$ when sent there by $\mathcal{G}_4$. We know that $\mathcal{G}_4$ sends any relation from $g\mathcal{D}$ in $s\mathfrak{T}$ to their respective version in $s\mathcal{T}$ and there it holds by definition of $s\mathcal{T}$. Thus, there is nothing to check in this case. The other relations of $s\mathfrak{T}$ are sent to respective equations with the same pictures in $s\mathcal{T}$. However, we have already seen that all these relations hold in $s\mathcal{T}$. Thus, there is nothing to do here. □

Theorem 7.13. *The functors $\mathcal{G}_3$ and $\mathcal{G}_4$ are inverse to each other and yield an equivalence (even an isomorphism) of 2-categories between $s\mathcal{T}$ and $s\mathfrak{T}$.*

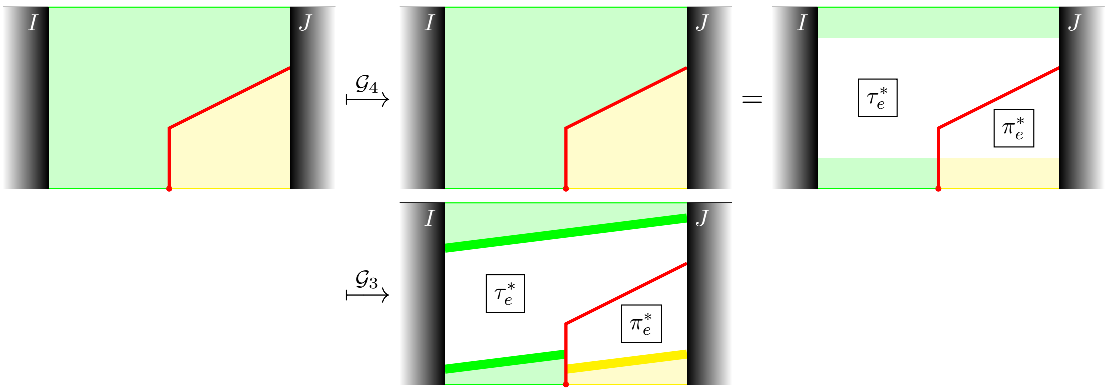
Proof. All we need to prove is that $\mathcal{G}_3 \circ \mathcal{G}_4$ and $\mathcal{G}_4 \circ \mathcal{G}_3$ are the identity functors on $s\mathfrak{T}$ and $s\mathcal{T}$ respectively. On objects this is obvious. Hence, we only need to check it on generating morphisms. We start with $\mathcal{G}_3 \circ \mathcal{G}_4$. Both functors send the morphisms from $g\mathcal{D}$ to their respective version in the other category. Thus, $\mathcal{G}_3 \circ \mathcal{G}_4$ is obviously the identity on them. So we just need to check this for the other generators. It will be enough to check this for one of the coloured trivalent vertices and one of the coloured thick trivalent vertices, since the proofs for the rest of them will be a symmetric version of the proof we are about to show. We have the following.



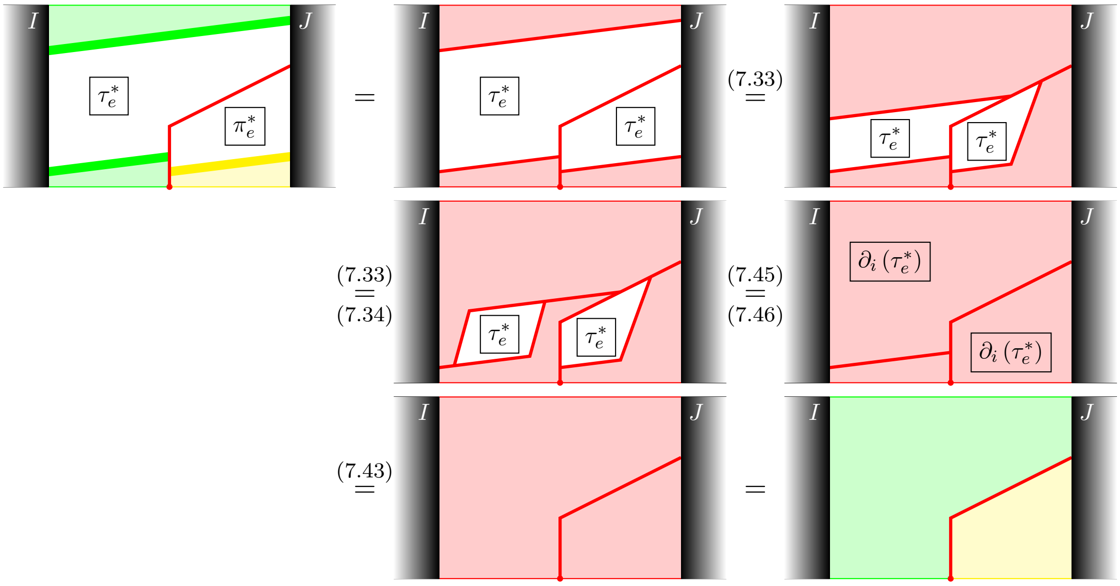
Here $\{\tau_w\}_{w\in W_K}$ and $\{\tau_w^*\}_{w\in W_K}$ are dual bases for R over R^K from Theorem 3.35 where K corresponds to the green area. Note that the thick green line should be interpreted as corresponding to s_1, s_2 or S (and the green area to the same). We will now do the prove where we think of this green line as corresponding to S. However, the proof works in the same way for s_1 and s_2 and we will state at the end which relations one needs to replace.



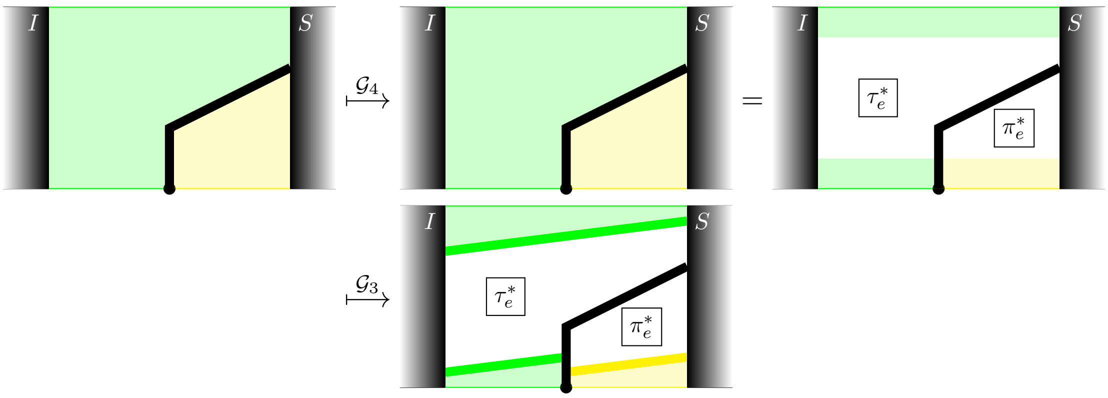
If green represents s_1 or s_2 one needs to exchange the relations (7.35), (7.36), (7.44) and (7.49) with the relations (7.33), (7.34), (7.43) and (7.45). We will continue with the coloured trivalent vertex.



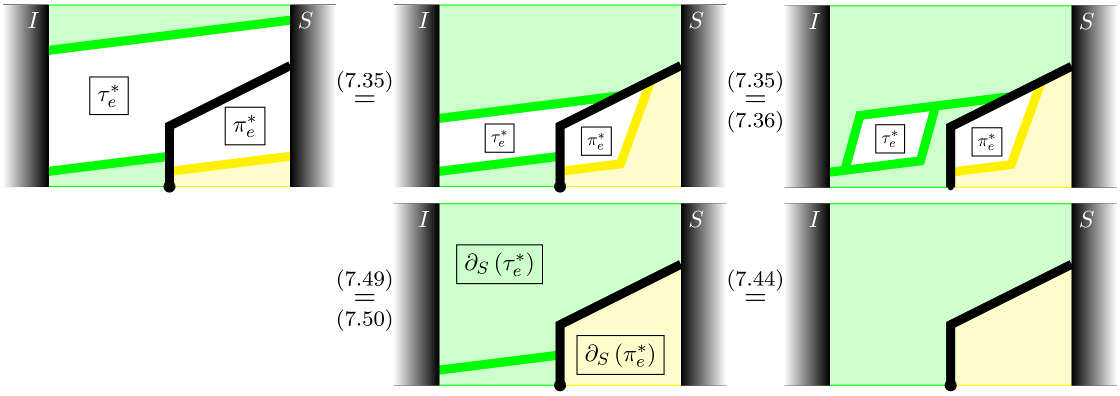
Here $\{\tau_w\}_{w \in K_1}$ and $\{\tau_w^*\}_{w \in K_1}$ are again dual bases for R over R^{K_1} from Theorem 3.35 where K_1 corresponds to the green area. Similarly, $\{\pi_w\}_{w \in K_2}$ and $\{\pi_w^*\}_{w \in K_2}$ are dual bases for R over R^{K_2} from Theorem 3.35 where K_2 corresponds to the yellow area. Note that the green and the yellow areas are next to a thin red strand. Thus, by definition of $s\mathfrak{T}$ they can only be white or red. If they are white the (local) calculation is trivial. Hence, we will now assume that the green and the yellow areas are red. Then we compute.



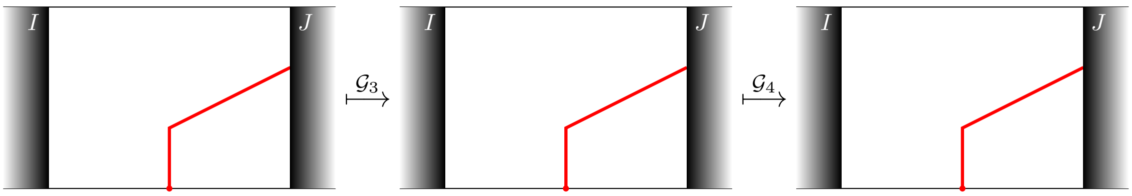
We continue with the coloured thick trivalent vertex.



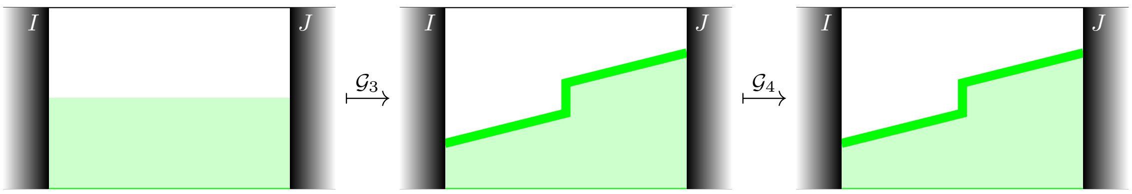
Here $\{\tau_w\}_{w \in K_1}$ and $\{\tau_w^*\}_{w \in K_1}$ are again dual bases for R over R^{K_1} from Theorem 3.35 where K_1 corresponds to the green area. Similarly, $\{\pi_w\}_{w \in K_2}$ and $\{\pi_w^*\}_{w \in K_2}$ are dual bases for R over R^{K_2} from Theorem 3.35 where K_2 corresponds to the yellow area. Note that the thick green lines and thick yellow line should again be interpreted as corresponding to s_1, s_2 or S (and the green and yellow areas to the same respectively). We will now do the prove where we think of the green and yellow lines as corresponding to S. However, the proof works in the same way for s_1 and s_2 and we will state at the end which relations one needs to replace.



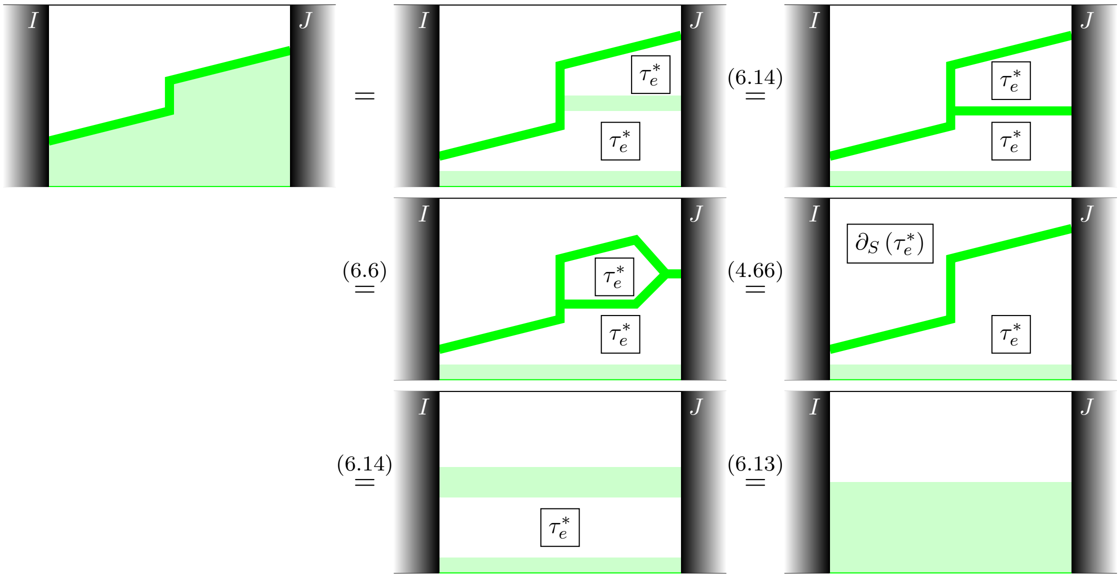
If green or yellow were corresponding to s_1 or s_2 one would need to replace the relations as follows: In the first equality we need to replace (7.35) with (7.37) for green and with (7.39) for yellow. In the second equality we need to replace (7.35) and (7.36) with (7.33) and (7.34) for green (nothing happens with yellow). In the third equality we need to replace (7.50) with (7.46) for green and (7.49) with (7.47) for yellow. In the last equality we need to replace (7.44) with (7.43) for green (again nothing happens with yellow). Hence, we are finished proving that $\mathcal{G}_3 \circ \mathcal{G}_4$ is the identity functor.
We proceed with proving the same for $\mathcal{G}_4 \circ \mathcal{G}_3$. We need to check that $\mathcal{G}_4 \circ \mathcal{G}_3$ is the identity functor on the generating morphisms of $s\mathcal{T}$. Again both functors send the morphisms from $g\mathcal{D}$ to their respective versions in the other category. Thus, there is nothing to check for them. Then there are the generators coming from ${}_I\mathcal{T}_J$. However, for these it is trivial that $\mathcal{G}_4 \circ \mathcal{G}_3$ is the identity on them. For instance, we have the following.



So all that is left to do is to check that $\mathcal{G}_4 \circ \mathcal{G}_3$ is the identity functor on the generators (6.12). It is enough to check it on one of the two generators, since the proof for the other one is symmetrical. We will once more use the colour green to represent s_1, s_2 or S. Then we have the following.



We will now do the proof thinking of the green lines and areas as corresponding to S. After that we state how to replace the relations if green corresponds to s_1 or s_2.



Here $\{\tau_w\}_{w\in W}$ and $\{\tau_w^*\}_{w\in W}$ are again dual bases from Theorem 3.35 (replace W with W_i if green corresponds to s_1 or s_2). If green corresponds to s_1 or s_2 one would just need to replace (6.6) with (4.49) and (4.66) with (4.3), (4.8) and (4.28). This finishes the proof that $\mathcal{G}_4 \circ \mathcal{G}_3$ is the identity functor, and hence the theorem is proven as well. $\square$

Bibliography

[AEHL18] Andrea Appel, Ilknur Egilmez, Matthew Hogancamp, and Aaron D. Lauda. A DG-extension of symmetric functions arising from higher representation theory. *J. Comb. Algebra*, 2(2):169–214, 2018.

[BB81] Alexandre Beĭlinson and Joseph Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BK81] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64(3):387–410, 1981.

[Bou81] Nicolas Bourbaki. *Éléments de mathématique*. Masson, Paris, 1981. Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6].

[Dem73] Michel Demazure. Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math.*, 21:287–301, 1973.

[EK10a] Ben Elias and Mikhail Khovanov. Diagrammatics for Soergel categories. *Int. J. Math. Math. Sci.*, pages Art. ID 978635, 58, 2010.

[EK10b] Ben Elias and Dan Krasner. Rouquier complexes are functorial over braid cobordisms. *Homology Homotopy Appl.*, 12(2):109–146, 2010.

[Eli16] Ben Elias. Thicker Soergel calculus in type A. *Proc. Lond. Math. Soc. (3)*, 112(5):924–978, 2016.

[EMTW20] Ben Elias, Shotaro Makisumi, Ulrich Thiel, and Geordie Williamson. *Introduction to Soergel bimodules*, volume 5 of *RSME Springer Series*. Springer, Cham, [2020] ©2020.

[EW14] Ben Elias and Geordie Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.

[EW16] Ben Elias and Geordie Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[Lib19] Nicolas Libedinsky. Gentle introduction to Soergel bimodules I: the basics. *São Paulo J. Math. Sci.*, 13(2):499–538, 2019.

[Man98] Laurent Manivel. *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, volume 3 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 1998.

[Mat64] Hideya Matsumoto. Générateurs et relations des groupes de Weyl généralisés. *C. R. Acad. Sci. Paris*, 258:3419–3422, 1964.

[Mat99] Andrew Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[Soe92] Wolfgang Soergel. The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math.*, 429:49–74, 1992.

[Soe07] Wolfgang Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.

[Str04] Catharina Stroppel. A structure theorem for Harish-Chandra bimodules via coinvariants and Golod rings. *J. Algebra*, 282(1):349–367, 2004.

[Str19] Catharina Stroppel. Representation theory II. Lecture course at the University of Bonn, 2019.

[Str20a] Catharina Stroppel. Category $\mathcal{O}$ and Categorification. Lecture course at the University of Bonn, 2020.

[Str20b] Catharina Stroppel. On some developements in representation theory in the last 20 years. Lecture course at the University of Bonn, 2020.

[Wil11] Geordie Williamson. Singular Soergel bimodules. *Int. Math. Res. Not. IMRN*, (20):4555–4632, 2011.