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# Categorification of (induced) cell modules and the rough structure of generalised Verma modules

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**Abstract**

This paper presents categorifications of (right) cell modules and induced cell modules for Hecke algebras of finite Weyl groups. In type $A$ we show that these categorifications depend only on the isomorphism class of the cell module, not on the cell itself. Our main application is multiplicity formulas for parabolically induced modules over a reductive Lie algebra of type $A$, which finally determines the so-called rough structure of generalised Verma modules. On the way we present several categorification results and give a positive answer to Kostant's problem from [A. Joseph, Kostant's problem, Goldie rank and the Gelfand–Kirillov conjecture, Invent. Math. 56 (3) (1980) 191–213] in many cases. We also present a general setup of decategorification, precategorification and categorification.


*Keywords:* Generalised Verma modules; Arbitrary irreducible module over a semisimple Lie algebra; Kazhdan–Lusztig; Cells; Categorification; Gelfand–Zetlin modules; Hecke algebra; Kostant's problem

**Contents**



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## 1. Introduction

The Weyl group acts via (exact) translation functors on the principal block of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated with a semisimple complex finite-dimensional Lie algebra, see [7]. On the level of the Grothendieck group, this becomes the regular representation of the Weyl group. The nature of translation functors is such that they obviously preserve several classes of modules—for example projective, injective or tilting modules.

This naturally leads to the question whether the isomorphism classes of such modules, considered as elements of the Grothendieck group, can be interpreted in terms of the representation theory of the Weyl group, in particular in terms of the regular representation of the Weyl group.

One of the most remarkable breakthrough results in the theory of semisimple complex Lie algebras is that such an interpretation actually exists. The connection is given by the so-called *Kazhdan–Lusztig Theory*, which first 'upgrades' the Weyl group to the corresponding Hecke algebra, and also the corresponding category $\mathcal{O}$ to its graded version, and then says that the isomorphism classes of the graded indecomposable projective modules in the regular block of the category $\mathcal{O}$ descend (on the level of the Grothendieck group) to what is now known as the *Kazhdan–Lusztig basis* of the Hecke algebra. The introduction of this Kazhdan–Lusztig basis together with the Kazhdan–Lusztig conjecture [38, Conjecture 1.5] was a milestone in combinatorial representation theory which finally turned the computation of the character of any simple highest weight module for a complex semisimple Lie algebra into a purely combinatorial task.

One main idea in this combinatorial representation theory showed up already before [38], namely, the idea of (left or right) *cells* for finite Weyl group, in particular for the symmetric group. The latter was first studied by combinatorialists (see e.g. [46]) and afterwards introduced into representation theory [34,84].

A natural consequence of the theory of cells is the definition of a special class of modules for the Hecke algebra, namely, the *cell modules*. In type $A$ these modules contribute an exhaustive list of all irreducible modules. For other types, however, they are not irreducible in general.

The first objective of the present paper is to give a categorical version of (right) cell modules. To each cell in the Weyl group $W$ we associate a certain quotient category of some subcategory of the category $\mathcal{O}$ (of the corresponding semisimple Lie algebra $\mathfrak{g}$) which is stable under the action of translation functors. The categories used for this categorification are indecomposable. When passing to the Grothendieck group we obtain the cell module corresponding to our chosen cell. In other words: we categorify cell modules for the Hecke algebra. Note that two different cells might have isomorphic cell modules. In type $A$ the isomorphism classes of cell modules are exactly the isomorphism classes of irreducible modules. We show that the categorical picture is the same:

**Theorem I** *(Uniqueness Theorem). Assume that $W$ is of type $A$. If two cell modules are isomorphic, then their categorifications are equivalent.*

We will make this equivalence concrete by giving an explicit functor which naturally commutes with the functorial action of the Hecke algebra. This is what we call the 'uniqueness' of categorifications (Theorem 18). As a result, we therefore have to each right cell $\mathbf{R}$ a categorification $\mathscr{C}_{\mathbf{R}}$ together with an equivalence $\Phi : \mathscr{C}_{\mathbf{R}_1} \to \mathscr{C}_{\mathbf{R}_2}$ whenever the cell modules corresponding to $\mathbf{R}_1$ and $\mathbf{R}_2$ are isomorphic (i.e., $\mathbf{R}_1$ and $\mathbf{R}_2$ are in the same double cell). The Kazhdan–Lusztig cell theory equips the cell modules with a distinct basis which corresponds in the categorification to the isomorphism classes of indecomposable projective modules.

Given a parabolic subgroup $W'$ of $W = S_n$, a right cell $\mathbf{R}'$ of $W'$ and the corresponding cell module $S(\mathbf{R}')$ of its Hecke algebra $\mathbb{H}(W')$ there is the induced cell module $S(\mathbf{R}') \otimes_{\mathbb{H}(W')} \mathbb{H}(W)$. To these data we associate a certain category $\mathscr{X} = \mathscr{X}(W, W', \mathbf{R}')$ of $\mathfrak{g}$-modules (in fact a subcategory of the category $\mathcal{O}$) such that the following holds (for details see Theorem 34, Proposition 35, Theorem 37):

**Theorem II.**

(i) *The category $\mathscr{X}$ is a categorification of $S(\mathbf{R}') \otimes_{\mathbb{H}(W')} \mathbb{H}(W)$, with the $\mathbb{H}$-action given by translation functors.*
(ii) *Up to equivalence $\mathscr{X}$ only depends on the isomorphism class of $S(\mathbf{R}')$, not on $\mathbf{R}'$ itself.*
(iii) *There is a combinatorial description of $\mathscr{X}$ in terms of Kazhdan–Lusztig polynomials in the following sense: the module $S(\mathbf{R}') \otimes_{\mathbb{H}(W')} \mathbb{H}(W)$ is equipped with four natural bases corresponding to four natural classes of modules in $\mathscr{X}$.*

A consequence of the (now proved) [38, Conjecture 1.5] is that the Kazhdan–Lusztig basis of the Hecke algebra turns the problem of finding multiplicities of composition factors of Verma modules into a purely combinatorial statement: the multiplicities are given by evaluating the corresponding Kazhdan–Lusztig polynomials. Verma modules are a special sort of induced modules obtained by inducing one-dimensional (irreducible) modules over a Borel subalgebra. In general, one would like to understand the structure of modules obtained by inducing from an arbitrary irreducible module over a parabolic subalgebra, ideally with a combinatorial description similar to the case of Verma modules. This is however a very difficult task for at least two reasons: Firstly, there is no classification or reasonable understanding of simple modules for finite-dimensional complex Lie algebras available (except for the Lie algebra $\mathfrak{sl}_2$, see [14]), hence the starting point for the induction process is not understood at all. Secondly, it might happen that the induced modules are of infinite length (due to a result of Stafford on existence of non-holonomic simple modules over the Weyl algebra and $U(\mathfrak{sl}_2 \times \mathfrak{sl}_2)$, see [78]).

Nevertheless, our paper goes a big step further in solving these problems. The principal idea is that we realise the induced module we are interested in, as a (proper) standard object in some category which is equivalent to some $\mathscr{X}$ as above. Then the Kazhdan–Lusztig Theory together with Theorem II(iii) provides the necessary combinatorics and as a result we can describe the so-called *rough structure* of parabolically induced arbitrary simple modules.

One of the difficulties is actually to give a precise definition of what is meant by *rough structure* (this is the topic of the last section of the article). In this introduction we just try to give the main idea. To do so let $\mathfrak{g}$ be a Lie algebra with triangular decomposition. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, and $V$ a simple module over the reductive part of $\mathfrak{p}$. Then $V$ trivially extends to a simple $\mathfrak{p}$-module, and the corresponding induced module

$$\Delta(\mathfrak{p}, V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$$

is called a *generalised Verma module*. We want to describe the composition factors of generalised Verma modules. Our main result is the following statement (for details and notation see Section 11, in particular Theorem 73):

**Theorem III.** *Assume that the reductive part of* $\mathfrak{p}$ *is of type* $A$. *For* $X, Y \in \mathrm{Irr}^{\mathfrak{g}}(\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(\overline{N} \otimes E)\}_{\mathrm{int}})$ *we have the following multiplicity formula in the category of* $\mathfrak{g}$*-modules:*

$$\big[\Delta(\mathfrak{p}, V_X) : L(\mathfrak{p}, V_Y)\big] = \big[\Delta(\mathfrak{p}, V_{\hat{\xi}(X)}) : L(\mathfrak{p}, V_{\hat{\xi}(Y)})\big]. \tag{1.1}$$

Here, the generalised Verma module $\Delta(\mathfrak{p}, V_X)$ is the one we are interested in, that means we want to describe the multiplicities of the left-hand side of Eq. (1.1). On the other hand, $\Delta(\mathfrak{p}, V_{\hat{\xi}(X)})$ is a generalised Verma module induced from a simple *highest weight* module, and hence is easier to understand. We will prove that the multiplicity on the right-hand side is given by Kazhdan–Lusztig combinatorics of a certain category $\mathscr{X}$ as in Theorem II above. In fact, the module $\Delta(\mathfrak{p}, V_{\hat{\xi}(X)})$ belongs to one of the four classes from Theorem II(iii). Therefore, it becomes in principle possible to compute the multiplicities completely. The only problem here is that simple subquotients of the form $L(\mathfrak{p}, V_Y)$, $Y \in \mathrm{Irr}^{\mathfrak{g}}(\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(\overline{N} \otimes E)\}_{\mathrm{int}})$, do not exhaust all simple subquotients of $\Delta(\mathfrak{p}, V_X)$. Roughly speaking, Theorem III gives information only about simple subquotients having small enough annihilator. It turns out that the number and multiplicities of such subquotients are always finite. *Knowing all the multiplicities for these 'allowed' simple subquotients is what we call 'knowing the rough structure of a generalised Verma module.'* To prove Theorem III we use the approach of [62] together with [42], which associates to simple modules of the form $L(\mathfrak{p}, V_Y)$ certain simple objects of some Coker-category. Without any restriction on the simple module $L$ to start with, our result seems to be the best possible, since all we know in general about $L$ is its annihilator. The complete (i.e., fine) structure of $\Delta(\mathfrak{p}, V_X)$ depends heavily on $V_X$, not just on its annihilator. This becomes transparent by comparing for instance the structure of generalised Verma modules induced from Gelfand–Zetlin modules on the one hand with generalised Verma modules induced from simple Verma modules on the other hand. In the first case the rough structure always coincides with the fine structure (see for example [55]), whereas in the second case the fine structure is different from the rough structure already in the case of the algebra $\mathfrak{sl}_3$ (this follows for example from [18, Theorem 7.6.23]).

Let now $\mathfrak{a}$ be the semisimple part of $\mathfrak{p}$ and $W'$ the corresponding Weyl group. As a consequence of Theorem III, we are able to deduce a criterion for the irreducibility of the generalised Verma module $\Delta(\mathfrak{p}, L)$, where $L$ is an *arbitrary* simple $\mathfrak{a}$-module (we formulate the statement in the case when $L$ has trivial central character, however, standard arguments extend this to the arbitrary type $A$ case, see Remark 76): We first associate in a combinatorial way to $L$ a pair $(x, w) \in W' \times W$ (see Section 11) and then deduce the following result:

**Theorem IV.** *The module* $\Delta(\mathfrak{p}, L)$ *is irreducible if and only if* $w$ *belongs to the same coset in* $W' \setminus W$ *as the longest element* $w_0$ *of* $W$.

An essential part of the approach from [62] is the study (including an answer in special cases) of the so-called *Kostant's problem* from [35]. If $M$ is a $\mathfrak{g}$-module and $\mathrm{Ann}(M)$ is the annihilator of $M$ in $U(\mathfrak{g})$, then the vector space $U(\mathfrak{g})/\mathrm{Ann}(M)$ canonically embeds into the vector space of all $\mathbb{C}$-linear automorphisms of $M$, which are locally finite with respect to the adjoint action of $\mathfrak{g}$. The question, which was called *Kostant's problem for $M$* in [35], is to determine for which modules $M$ the canonical injection above is in fact an isomorphism. We answer this question for several modules $M$. In particular, we prove the following statement:

**Theorem V.** *Let $\mathfrak{g}$ be of type $A$, and $x, y \in W = S_n$ be elements in the same left cell. Then Kostant's problem has a positive answer for the simple highest weight module $L(x \cdot 0)$ if and only if it has a positive answer for the simple highest weight module $L(y \cdot 0)$.*

The above result was for us a very strong evidence supporting the general belief that Kostant's problem should always have a positive answer in type $A$. For other types our approach fails, but the answer to Kostant's problem might be negative as well (as was shown in [35]). However, shortly after we completed this paper we found a counter-example to the general belief in the case of $\mathfrak{sl}_4$, see [61] (and also [37] for further results). Still lacking a general answer to Kostant's problem, Theorem V shows that the answer is an invariant of the left cell. A more detailed analysis of the problem, our partial solutions, and the obstacles for other types can be found in Section 11.2.

On the way to our main results we also obtain several categorification results which we think are of interest on their own. We also obtain some unexpected applications of the categorification procedure, in particular we define a canonical filtration on integral permutation and induced cell modules for the symmetric group $S_n$. This filtration is based on the Gelfand–Kirillov dimension which is closely related to Lusztig's $a$-function (see [50]) and should be compared with the appearance of the $a$-function in [12] and [28].

### *1.1. A structural overview*

The paper starts with a general discussion on the notion of *precategorification* and *categorification* in Section 2. In Section 3 we describe some categorifications of the regular representations of the Hecke algebra. These sections are preliminaries for the following main topics:

#### *1.1.1. The categorification of cell modules and the Uniqueness Theorem*

This is done in Section 4. Our approach here differs from other categorifications of cell structures, initiated by Lusztig [49] and subsequently studied in detail by Bezrukavnikov and Ostrik (see e.g. [10,12]). Their work crucially involves constructing monoidal categories and is mostly concerned with two-sided cells, whereas our interest is in right cells and the functorial action of the Hecke algebra. It is the additional categorification of different natural bases in the Grothendieck group (using projective, (proper) standard and simple objects) which are important for the application to the rough structure.

The categories appearing in our categorification are not very well understood. They are defined as quotients of certain subcategories of the category $\mathcal{O}$. In general they are not highest weight categories and might have infinite homological dimension (see Section 5.3).

From our Uniqueness Theorem (Theorem I), which is presented in Section 5, it follows that the categorifications of the cell modules are certain module categories over (in general non-commutative) symmetric algebras including as a special case Khovanov's algebra $\mathcal{H}^n$ (from

[40,83]). Together with [15] the uniqueness result also shows that the centres of these categories are isomorphic to the cohomology ring of a certain Springer fibre, that means the fixed point variety of the flag variety $GL(n, \mathbb{C})/B$ under a nilpotent matrix $N$.

*1.1.2. The categorification of induced cell modules*

The categories appearing here are the main players and most of the paper will be on constructing these, comparing them and describing their combinatorics.

For the standard examples of induced modules, namely, the induced sign or induced trivial (permutation) module, categorifications in terms of parabolic category $\mathcal{O}$ (see e.g. [74,82]) and Harish-Chandra bimodules (see e.g. [58]) are well known and well studied. Our categories corresponding to induced modules are generalisations of both, parabolic category $\mathcal{O}$ and certain categories of Harish-Chandra bimodules. A short summary can be found in Section 6.

In Section 8 we propose an alternative categorification for the permutation module, using the action of Arkhipov's twisting functors on singular blocks of $\mathcal{O}$. This categorification is Koszul dual to the categorification, described in Section 6. In Section 9 we present a categorical version of Schur–Weyl dualities. These results play important roles in the proof of Theorems III and V, given in Section 11.

In Section 6 we deal with Theorem II: we show that parabolic generalisations of $\mathcal{O}$ categorify induced cell modules for the Iwahori–Hecke algebra. In each induced cell module we have four special bases which will have a very natural categorical interpretation (Theorem 37 in Section 7) in terms of isomorphism classes of projective modules, simple modules, standard modules (which are induced projective modules from the categorification of the cell module) and proper standard modules (which are induced simple modules). The categorifications for induced cell modules are stratified in the sense of [16], and even weakly properly stratified in the sense of [22]. The latter structure plays an important role in several parts of the paper, especially for the proof of Theorem III.

In Section 10 we study properties of the categories used to categorify induced cell modules (in type $A$). Generalising Irving's results from [30], we classify all projective modules which are also injective (Theorem 48) and then deduce a double centraliser property (Theorem 51) which generalises Soergel's original Struktursatz from [71] highly non-trivially. Maybe the most surprising result here is the description of the centre of these induced categories (Theorem 55): the centre is isomorphic to the centre of a certain parabolic category $\mathcal{O}$. Therefore, we again have the explicit description of the centre as given in [15]. Moreover, the categories categorifying induced cell modules are all Ringel self-dual (Theorem 58), which means that there is an equivalence between the additive subcategory of all projective modules and the additive subcategory of all tilting modules. Understanding these categories is crucial for the proof of the Theorems II and III.

*1.1.3. Application: the rough structure of generalised Verma modules*

The categorifications of induced cell modules will finally be used to describe the best possible general result about generalised Verma modules, that means parabolically induced *arbitrary* simple modules. The generalised Verma modules as briefly explained above appear as the so-called (proper) standard objects in our categorifications. Our combinatorial description can then be used to deduce at least the multiplicity of certain composition factors (namely, the one which can be seen in our categories), and leads to what is called the 'rough structure' of generalised Verma modules. In this rough structure all the multiplicities become finite. A very special case of our setup was already considered in [62] and [42]. Our main results here are Theorems IV and V from the introduction, which are proved at the end of the paper, building on all the previous results.

**General terminology.** A *ring* always means an associative unitary ring. *Graded* always means $\mathbb{Z}$-graded. For a ring $R$ we denote by $R$-mod and mod-$R$ the categories of finitely generated left and right $R$-modules, respectively. If $R$ is graded, we denote by $R$-gmod and gmod-$R$ the categories of finitely generated graded left and right $R$-modules, respectively. Inclusions are denoted by $\subset$. If it is necessary to point out that some inclusion is proper, we use the symbol $\subsetneq$.

Let $\mathbb{F}$ be a commutative ring. We denote by $\mathbb{F}[v, v^{-1}]$ and $\mathbb{F}((v))$ the rings of Laurent polynomials and formal Laurent series in the variable $v$ with coefficients in $\mathbb{F}$, respectively. In the paper we usually work over $\mathbb{Z}$ or over $\mathbb{C}$. We abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$.

## 2. Decategorification, precategorification and categorification

In this section we define a general algebraic notion of categorification. The definition is based on and further develops the ideas of [44,45,58].

### 2.1. Ordinary setup

Let $\mathscr{C}$ be a category. If $\mathscr{C}$ is abelian or triangulated, we denote by $\mathrm{Gr}(\mathscr{C})$ the *Grothendieck group* of $\mathscr{C}$. The latter one is by definition the free abelian group generated by the isomorphism classes $[M]$ of objects $M$ of $\mathscr{C}$ modulo the relation $[C] = [A] + [B]$ whenever there is a short exact sequence $A \hookrightarrow C \twoheadrightarrow B$ if $\mathscr{C}$ is abelian; and whenever there is a triangle $(A, C, B, f, g, h)$ if $\mathscr{C}$ is triangulated. If $\mathscr{C}$ is additive, we denote by $\mathrm{Gr}(\mathscr{C})_{\oplus}$ the *split Grothendieck group* of $\mathscr{C}$, which is by definition the free abelian group generated by the isomorphism classes $[M]$ of objects $[M]$ of $\mathscr{C}$ modulo the relation $[C] = [A] + [B]$ whenever $C \cong A \oplus B$. For $M \in \mathscr{C}$ we denote by $[M]$ the image of $M$ in the (split) Grothendieck group. Let $\mathbb{F}$ be a commutative ring with 1.

**Definition 1.** Let $\mathscr{C}$ be an abelian or triangulated, respectively additive, category. Then the $\mathbb{F}$-*decategorification* of $\mathscr{C}$ is the $\mathbb{F}$-module $[\mathscr{C}]^{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathscr{C})$ (resp. $[\mathscr{C}]^{\mathbb{F}}_{\oplus} := \mathbb{F} \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathscr{C})_{\oplus}$).

The element $1 \otimes [M]$ of the $\mathbb{F}$-decategorification is abbreviated as $[M]$ as well. We set $[\mathscr{C}] := [\mathscr{C}]^{\mathbb{Z}} = \mathrm{Gr}(\mathscr{C})$ and $[\mathscr{C}]_{\oplus} := [\mathscr{C}]^{\mathbb{Z}}_{\oplus} = \mathrm{Gr}(\mathscr{C})_{\oplus}$.

**Definition 2.** Let $V$ be an $\mathbb{F}$-module. An $\mathbb{F}$-*precategorification* $(\mathscr{C}, \varphi)$ of $V$ is an abelian (resp. triangulated or additive) category $\mathscr{C}$ with a fixed monomorphism $\varphi$ from $V$ to the $\mathbb{F}$-decategorification of $\mathscr{C}$. If $\varphi$ is an isomorphism, then $(\mathscr{C}, \varphi)$ is called an $\mathbb{F}$-*categorification* of $V$.

Hence categorification is in some sense the 'inverse' of decategorification. Whereas the latter is uniquely defined, there are usually several different categorifications. In case $\mathbb{F} = \mathbb{Z}$ and $V$ is torsion-free there is always the (trivial) categorification given by a semisimple category of the appropriate size.

**Definition 3.** Let $V$ be an $\mathbb{F}$-module and $f : V \to V$ be an $\mathbb{F}$-endomorphism. Given an $\mathbb{F}$-precategorification $(\mathscr{C}, \varphi)$ of $V$, an $\mathbb{F}$-*categorification* of $f$ is an exact (resp. triangulated or additive) functor $F : \mathscr{C} \to \mathscr{C}$ such that $[F] \circ \varphi = \varphi \circ f$, where $[F]$ denotes the endomorphism of $[\mathscr{C}]^{\mathbb{F}}$ (or $[\mathscr{C}]^{\mathbb{F}}_{\oplus}$ if $\mathscr{C}$ is abelian) induced by $F$. In other words, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ [\mathscr{C}]^{\mathbb{F}}_{(\oplus)} & \xrightarrow{\quad [F] \quad} & [\mathscr{C}]^{\mathbb{F}}_{(\oplus)}. \end{array}$$

**Definition 4.** Assume $A$ is some $\mathbb{F}$-algebra defined by generators $a_1, \dots, a_k$ and relations $R_j$, $j \in J$. Given an $A$-module $M$, each generator $a_i$ of $A$ defines a linear endomorphism, $f_i$, of $M$. A *weak* $\mathbb{F}$*-(pre)categorification* of $M$ is a (pre)categorification $(\mathscr{C}, \varphi)$ of the vector space $M$ together with a categorification $F_i$, $i = 1, \dots, k$, of each $f_i$.

If there is an 'interpretation' of the relations $R_j$ between the generators of $A$ in terms of isomorphisms of functors, we will call $(\mathscr{C}, \varphi, F_1, \dots, F_k)$ a *(pre)categorification* of the $A$-module $M$. The interpretation of the relations will depend on the example.

**Example 5.** Let $R = \mathbb{C}[x]/(x^2)$ and $\mathscr{C} = R$-mod. Then $\mathrm{Gr}(\mathscr{C}) \cong \mathbb{Z}$, generated by the isomorphism class $[\mathbb{C}]$ of the unique simple $R$-module, and $[\mathscr{C}]^{\mathbb{C}} \cong \mathbb{C}$. Thus $\mathscr{C}$ is a $\mathbb{Z}$-categorification of $\mathbb{Z}$.

### 2.2. *Graded setup*

If $\mathscr{C}$ is equivalent to a category of modules over a graded ring, then $\mathrm{Gr}(\mathscr{C})$ (or $\mathrm{Gr}(\mathscr{C})_{\oplus}$) becomes a $\mathbb{Z}[v, v^{-1}]$-module via $v^i[M] = [M\langle i\rangle]$ for any $M \in \mathscr{C}$, $i \in \mathbb{Z}$, where $M\langle i\rangle$ is the module $M$, but in the grading shifted by $i$ such that $(M\langle i\rangle)_j = M_{j-i}$.

To define the notion of a decategorification for a category of graded modules (or complexes of graded modules) let $\mathbb{F}$ be a commutative ring with 1 and $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ be a fixed homomorphism of unitary rings. Then $\iota$ defines on $\mathbb{F}$ the structure of a (right) $\mathbb{Z}[v, v^{-1}]$-module.

**Definition 6.** The $(\mathbb{F}, \iota)$*-decategorification* of $\mathscr{C}$ is the $\mathbb{F}$-module

$$[\mathscr{C}]^{(\mathbb{F},\iota)} := \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{Gr}(\mathscr{C}) \qquad \bigl(\text{resp. } [\mathscr{C}]^{(\mathbb{F},\iota)}_{\oplus} := \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{Gr}(\mathscr{C})_{\oplus}\bigr).$$

In most of our examples the homomorphism $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ will be the obvious canonical inclusion. In such cases we will omit $\iota$ in the notation. We set

$$[\mathscr{C}] := [\mathscr{C}]^{(\mathbb{Z}[v,v^{-1}],\mathrm{id})}, \qquad [\mathscr{C}]_{\oplus} := [\mathscr{C}]^{(\mathbb{Z}[v,v^{-1}],\mathrm{id})}_{\oplus}.$$

**Definition 7.** Let $V$ be an $\mathbb{F}$-module. A $\iota$*-precategorification* $(\mathscr{C}, \varphi)$ of $V$ is an abelian or triangulated, respectively additive, category $\mathscr{C}$ with a fixed free action of $\mathbb{Z}$ and a fixed monomorphism $\varphi$ from $V$ to the $(\mathbb{F}, \iota)$-decategorification of $\mathscr{C}$. If $\varphi$ is an isomorphism, $(\mathscr{C}, \varphi)$ is called a $\iota$*-categorification* of $V$.

The definitions of a *ι-categorification* of an endomorphism, $f: V \to V$, and of a *ι-(pre)categorification* of a module over some $\mathbb{F}$-algebra are completely analogous to the corresponding definitions from the previous subsection.

**Example 8.** Let $R = \mathbb{C}[x]/(x^2)$. Consider $R$ as a graded ring (motivated by its realisation as a cohomology ring we put $x$ in degree two), and take $\mathscr{C} = R$-gmod. Then $[\mathscr{C}] \cong \mathbb{Z}[v, v^{-1}]$ as a $\mathbb{Z}[v, v^{-1}]$-module, hence the graded category $\mathscr{C}$ is a $(\mathbb{Z}[v, v^{-1}], \mathrm{id})$-categorification of $\mathbb{Z}[v, v^{-1}]$. Note that $\mathscr{C}$, considered just as an abelian category, is also a $\mathbb{Z}[v, v^{-1}]$-categorification of $\mathbb{Z}[v, v^{-1}]$.

## 3. The Hecke algebra as a bimodule over itself and its categorifications

In this section we recall the definition of Hecke algebras and give several examples of categorifications of regular (bi)modules over these algebras. We refer the reader to [45] for more examples of categorifications. The first written account of a categorification of the Hecke algebra seems to be Springer's realisation [77] in terms of perverse sheaves (although the terminology "categorification" was not used). Geometric categorifications of the Hecke algebra or its relatives like the affine Hecke algebra, degenerate affine Hecke algebra, etc., are various (see for example [11] and references therein). We recall now the algebraic versions which are directly related to the purposes of this paper.

From now on we fix a finite Weyl group $W$ with identity element $e$, set of simple reflections $S$, and length function $l$. Denote by $w_0$ the longest element of $W$. Let further $\leqslant$ be the Bruhat order on $W$. With respect to this order the element $e$ is the minimal and $w_0$ is the maximal element. Our main example will be $W = S_n$, the symmetric group on $n$ elements, and $S = \{(i, i+1), i = 1, \dots, n-1\}$, the set of elementary transpositions.

### *3.1. The Hecke algebra*

Denote by $\mathbb{H} = \mathbb{H}(W, S)$ the *Hecke algebra* associated with $W$ and $S$; that is, the $\mathbb{Z}$-algebra which is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_x \mid x \in W\}$ and multiplication given by

$$H_x H_y = H_{xy} \quad \text{if } l(x) + l(y) = l(xy), \quad \text{and} \quad H_s^2 = H_e + (v^{-1} - v) H_s \quad \text{for } s \in S. \tag{3.1}$$

The algebra $\mathbb{H}$ is a deformation of the group algebra $\mathbb{Z}[W]$. As a $\mathbb{Z}[v, v^{-1}]$-algebra it is generated by $\{H_s \mid s \in S\}$, or (which will turn out to be more convenient) by the set $\{\underline{H}_s = H_s + vH_e \mid s \in S\}$. Note that $\underline{H}_s$ is fixed under the involution $^-$, which maps $v \mapsto v^{-1}$ and $H_s \mapsto (H_s)^{-1}$. Moreover, $\underline{H}_s$ is a Kazhdan–Lusztig basis element. More generally, for $w \in W$ we denote by $\underline{H}_w$ the corresponding element from the Kazhdan–Lusztig bases for $\mathbb{H}$ in the normalisation of [74]. The Kazhdan–Lusztig polynomials $h_{x,y} \in \mathbb{Z}[v]$ are defined via $\underline{H}_x = \sum_{y \in W} h_{y,x} H_x$. With respect to the generators $\underline{H}_s$, $s \in S$, we have the following set of defining relations (in the case $W = S_n$):

$$\begin{aligned} \underline{H}_s^2 &= (v + v^{-1}) \underline{H}_s; \\ \underline{H}_s \underline{H}_t &= \underline{H}_t \underline{H}_s, \quad \text{if } ts = st; \\ \underline{H}_s \underline{H}_t \underline{H}_s + \underline{H}_t &= \underline{H}_t \underline{H}_s \underline{H}_t + \underline{H}_s, \quad \text{if } ts \neq st. \end{aligned} \tag{3.2}$$

Let $\mathbb{F}$ be any commutative ring and $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ be a homomorphism of unitary rings. Then we have the *specialised* Hecke algebra $\mathbb{H}^{(\mathbb{F},\iota)} = \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{H}$. Again if $\iota$ is clear from the context (for instance if $\iota$ is the natural inclusion), we will omit it in the notation.

**Example 9.** Let again $R = \mathbb{C}[x]/(x^2)$. Putting $x$ in degree two induces a grading on $R$ and $\hat{B}_s = (R \otimes_{\mathbb{C}} R)\langle -1\rangle$ becomes a graded $R$-bimodule. Let $\mathscr{S}$ be the additive category generated by the graded left $R$-modules $\mathbb{C}\langle j\rangle$ and $R\langle j\rangle$, $j \in \mathbb{Z}$. Then $\mathrm{Gr}(\mathscr{S})_{\oplus}$ is a free $\mathbb{Z}[v, v^{-1}]$-module of rank two, and is isomorphic to $\mathbb{H}(S_2, \{s\})$ via $[\mathbb{C}\langle j\rangle] \mapsto v^{-j} H_e$, $[R\langle j\rangle] \mapsto v^{-j} \underline{H}_s$. The functor $F_s^l = \hat{B}_s \otimes_R \_$ satisfies the condition $F_s^l \circ F_s^l \cong F_s^l\langle 1\rangle \oplus F_s^l\langle -1\rangle$ which is an interpretation of the first relation in (3.2). Hence we get a categorification of the left regular $\mathbb{H}(S_2, \{s\})$-module. This example generalises in several ways to arbitrary finite Weyl groups as we will describe in the next subsections.

### *3.2. Special bimodules*

Associated with $W$ we have the additive category given by the so-called *special bimodules* $B_w$, $w \in W$, introduced by Soergel in [72], see also [75]. To define these bimodules we consider the geometric representation $(V_{\mathbb{R}}, \varphi)$ of $W$ and its complexification $(V, \varphi)$, see [13, 4.2]. Let $R$ be the ring of regular functions on $V$ with its natural $W$-action. This ring becomes graded by putting $V^*$ in degree 2. For any $s \in S$ let $R^s$ be the subring of $s$-invariants in $R$. Note that this is in fact a graded subring of $R$. Given $w \in W$ with a fixed reduced expression $[w] = s_1 s_2 \cdot \cdots \cdot s_k$ define the graded $R$-bimodule $R_{[w]}$ as follows:

$$R_{[w]} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R\bigl\langle -l(w)\bigr\rangle.$$

Following [75] we define $B_w$ as the unique indecomposable direct summand of $R_{[w]}$, which is not isomorphic to a direct summand of any $R_{[x]}$ with $l(x) < l(w)$. Let $\mathscr{S}$ be the smallest additive category which contains all special bimodules, and is closed under taking direct sums and graded shifts. There is a unique isomorphism $\mathcal{E}$ of $\mathbb{Z}[v, v^{-1}]$-modules, which satisfies

$$\begin{aligned} \mathcal{E}:\quad \mathbb{H} &\xrightarrow{\sim} [\mathscr{S}]_{\oplus}, \\ \underline{H}_w &\mapsto [B_w]. \end{aligned}$$

For any $s \in S$ we have the additive endofunctors $\mathrm{F}_s^l = B_s \otimes_R \_$ and $\mathrm{F}_s^r = \_ \otimes_R B_s$ of $\mathscr{S}$. Altogether we get a categorification of the regular Hecke module as follows (see [75, Theorem 1.10], [29, Satz 7.9] and [72, Theorem 1]):

**Proposition 10.**

(i) $(\mathscr{S}, \mathcal{E}, \{\mathrm{F}_s^r\}_{s\in S})$ *is a categorification of the right regular representation of* $\mathbb{H}$ *with respect to the generators* $\underline{H}_s$, $s \in S$.
(ii) $(\mathscr{S}, \mathcal{E}, \{\mathrm{F}_t^l\}_{t\in S})$ *is a categorification of the left regular representation of* $\mathbb{H}$ *with respect to the generators* $\underline{H}_t$, $t \in S$.

The interpretation of the relations (3.2) is given by the existence (see [72, Theorem 1]) of isomorphisms of functors as follows (in case $W = S_n$):

$$\left(F_s^\sharp\right)^2 \cong F_s^\sharp\langle 1\rangle \oplus F_s^\sharp\langle -1\rangle;$$
$$F_s^\sharp F_t^\sharp \cong F_t^\sharp F_s^\sharp, \quad \text{if } ts = st;$$
$$F_s^\sharp F_t^\sharp F_s^\sharp \oplus F_t^\sharp \cong F_t^\sharp F_s^\sharp F_t^\sharp \oplus F_s^\sharp, \quad \text{if } ts \neq st,$$

where $\sharp$ is either $l$ or $r$. For other types the interpretation is similar.

**Remark 11.**

(1) The functors $F_s^r$ and $F_t^l$ naturally commute (with each other), hence the parts (i) and (ii) of Proposition 10 together give a categorification of the regular Hecke **bimodule**.
(2) The above categorification is not completely satisfactory, mostly because it is given by an additive category which is not abelian. As a consequence, we cannot see the standard basis of the Hecke module in this categorification, hence we will present a categorification given by an abelian category. This will be done in the next subsection.
(3) The proof of Proposition 10 given in [72, Theorem 1] is quite involved and uses the full power of the Kazhdan–Lusztig Theory (or the decomposition theorem [5, Theorem 6.2.5]).
(4) If one prefers to work with finite-dimensional algebras and modules, one could replace the polynomial ring $R$ with the coinvariant ring $C$, which is the quotient of $R$ modulo the ideal generated by homogeneous $W$-invariant polynomials of positive degree. One can define the *special $C$-bimodules* $\hat{B}_w = B_w \otimes_R C$ and obtains a completely analogous result to Proposition 10, and Remark (1), see [72, Theorem 2].
(5) We could also define the *special right $C$-modules* $\overline{B}_w = \mathbb{C} \otimes_R B_w \otimes_R C$. Since they are preserved by the functors $F_s^r$, $s \in S$, Proposition 10(i) provides another categorification of the regular right $\mathbb{H}$-module [71, Zerlegungssatz 1 and Section 2.6].

## *3.3. Harish-Chandra bimodules*

In this section we would like to improve Proposition 10 and work with abelian categories. We start with introducing the setup, which then will also be used in the next subsection.

Let $\mathfrak{g}$ be a reductive finite-dimensional complex Lie algebra associated with the Weyl group $W$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ with its centre $Z(\mathfrak{g})$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$ contained in the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. For $\lambda \in \mathfrak{h}^*$ we denote by $M(\lambda)$ the Verma module with highest weight $\lambda$. Let $\rho$ be the half-sum of all positive roots. Define $\mathfrak{h}^*_{\mathrm{dom}} := \{\lambda \in \mathfrak{h}^*\colon \lambda + \rho \text{ is dominant}\}$, which is the dominant Weyl chamber with respect to the *dot-action* of $W$ on $\mathfrak{h}^*$ given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Denote by $\mathcal{H}$ the category of Harish-Chandra bimodules for $\mathfrak{g}$, that is, the category of finitely generated $U(\mathfrak{g})$-bimodules of finite length, which are locally finite with respect to the adjoint action of $\mathfrak{g}$ (which is defined for a bimodule $M$ as $x.m = xm - mx$ for any $x \in \mathfrak{g}$ and $m \in M$). The action of the centre defines the following block decomposition of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{\mathbf{m},\mathbf{n} \in \operatorname{Max} Z(\mathfrak{g})} {}_{\mathbf{m}}\mathcal{H}_{\mathbf{n}}, \quad \text{where } {}_{\mathbf{m}}\mathcal{H}_{\mathbf{n}} = \left\{M \in \mathcal{H} \,\middle|\, \exists k \in \mathbb{N}\colon \mathbf{m}^k M = 0 = M\mathbf{n}^k\right\}.$$

Note that $Z(\mathfrak{g}) \cong R$ (via the Harish-Chandra isomorphism and [36, 18-1]) hence it is positively graded (here $R$ is as in Section 3.2). Let $\mathbf{0} \in \operatorname{Max} Z(\mathfrak{g})$ denote the annihilator (in $Z(\mathfrak{g})$) of the trivial $U(\mathfrak{g})$-module. Consider the block ${}_{\mathbf{0}}\mathcal{H}_{\mathbf{0}}$. Tensoring with finite-dimensional left and right $U(\mathfrak{g})$-modules are endofunctors on $\mathcal{H}$ and their direct sums and summands are called *projective functors*. Indecomposable projective functors were classified in [7, Theorem 3.3]. It turns out that these summands are naturally labelled by the elements of $W$. For $w \in W$ we denote by $\theta_w^l$ the indecomposable projective endofunctor of ${}_{\mathbf{0}}\mathcal{H}_{\mathbf{0}}$ corresponding to $w$ and induced by tensoring with a finite-dimensional *left* $\mathfrak{g}$-module (as the supindex $l$ indicates). Similarly, we can consider projective functors given by tensoring with finite-dimensional *right* $U(\mathfrak{g})$-modules and obtain the corresponding functors $\theta_w^r$. For two $\mathfrak{g}$-modules $M$ and $N$ we denote by $\mathscr{L}(M,N)$ the largest ad($\mathfrak{g}$)-finite submodule of $\mathrm{Hom}_{\mathbb{C}}(M,N)$, see [18, 1.7.9]. The classes $[\mathscr{L}(M(0), M(w\cdot 0))]$, $w \in W$, form a basis of $[{}_{\mathbf{0}}\mathcal{H}_{\mathbf{0}}]$, see [7], [33, 6.15].

Following [72, Theorem 2] we form the positively graded algebra

$$A^\infty = \mathrm{End}_{R\text{-}R}\left(\bigoplus_{w\in W} B_w\right)$$

(here $B_w$ is as in Section 3.2) and we have an equivalence (see [72, Theorem 3]) of categories

$${}_{\mathbf{0}}\mathcal{H}_{\mathbf{0}} \cong \text{nil-}A^\infty,$$

where nil-$A^\infty$ is the category of all finite-dimensional right $A^\infty$-modules $M$ satisfying $MA_i^\infty = 0$ for all $i \gg 0$ (for example, this is obviously satisfied for any finite-dimensional *gradable* $A^\infty$-module). We consider the category gmod-$A^\infty$ of all finite-dimensional graded right $A^\infty$-modules. The functors $\theta_w^l$ and $\theta_w^r$ lift to endofunctors of gmod-$A^\infty$, see [56, Appendix]. The modules $\mathscr{L}(M(0), M(w\cdot 0))$ admit graded lifts as well and we fix standard lifts $\tilde{M}_w$ such that their heads are concentrated in degree 0. Let $\tilde{\mathcal{E}}$ be the unique isomorphism of the $\mathbb{Z}[v, v^{-1}]$-modules such that

$$\begin{aligned} \tilde{\mathcal{E}}:\ \ \mathbb{H} &\xrightarrow{\sim} [\text{gmod-}A^\infty], \\ H_w &\mapsto\ [\tilde{M}_w]. \end{aligned}$$

**Proposition 12.**

(i) (gmod-$A^\infty, \tilde{\mathcal{E}}, \{\theta_s^l\}_{s\in S}$) *is a categorification of the right regular representation of* $\mathbb{H}$ *with respect to the generators* $\underline{H}_s$, $s \in S$.
(ii) (gmod-$A^\infty, \tilde{\mathcal{E}}, \{\theta_t^r\}_{t\in S}$) *is a categorification of the left regular representation of* $\mathbb{H}$ *with respect to the generators* $\underline{H}_t$, $t \in S$.

This statement can be found for example in [79] and [41]. Basically, it follows from [72]. We would like to emphasise the difference between Propositions 12 and 10: In Proposition 10 the *right* regular representation of $\mathbb{H}$ was categorified using functors $\mathrm{F}_s^r$ of tensor product from the *right*, while in Proposition 12 the *right* regular representation of $\mathbb{H}$ was categorified using the *left* translation functors $\theta_s^l$. The interpretation of the relations (3.2) is similar to the one given after Proposition 10. Again, the functors $\theta_s^l$ and $\theta_t^r$ naturally commute with each other and hence the parts (i) and (ii) of Proposition 12 together give a categorification of the regular Hecke bimodule. The connection to Section 3.2 is given by [72, Section 3].

*3.4. Category $\mathcal{O}$*

We stick to the setup at the beginning of the previous subsection. Consider the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ [9] with its block decomposition

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathrm{dom}}} \mathcal{O}_\lambda, \quad \text{where } \mathcal{O}_\lambda = \{M \in \mathcal{O} \mid \exists k \in \mathbb{N}\colon \left(\mathrm{Ann}_{Z(\mathfrak{g})}\left(M(\lambda)\right)\right)^k M = 0\}.$$

For $\lambda \in \mathfrak{h}^*$ let $P(\lambda)$ be the projective cover of $M(\lambda)$ and $L(\lambda)$ be the simple quotient of $P(\lambda)$.

Following [71] we form the graded algebra $A = \mathrm{End}_C(\bigoplus_{w\in W} \overline{B}_w)$ (see Remark 11) and obtain an equivalence of categories between $\mathcal{O}_0$ and mod-$A$, the category of finite-dimensional right $A$-modules. We denote by $\mathcal{O}_0^{\mathbb{Z}}$ the category of finite-dimensional *graded* right $A$-modules. To connect this with the previous subsection let ${}_{\mathbf{0}}\mathcal{H}^1_{\mathbf{0}}$ denote the full subcategory of ${}_{\mathbf{0}}\mathcal{H}_{\mathbf{0}}$ consisting of all bimodules which are annihilated by $\mathbf{0}$ from the right-hand side. Then there is an equivalence of categories ${}_{\mathbf{0}}\mathcal{H}^1_{\mathbf{0}} \cong \mathcal{O}_0$, see [7, Theorem 5.9]. Via this equivalence the functors $\theta^l_w$, $w \in W$, restrict to exact endofunctors of $\mathcal{O}_0$, which admit graded lifts. Unfortunately, the functors $\theta^r_w$ do not preserve $\mathcal{O}_0$. However, for $s \in S$ there is a unique up to scalar natural transformation $\mathrm{Id}\langle 1\rangle \to \theta^r_s$, whose cokernel we denote by $\mathrm{T}_s$. These are the so-called *twisting functors* on $\mathcal{O}_0$, see [2] and [43]. Each $\mathrm{T}_s$ preserves $\mathcal{O}_0$ and has a graded lift by definition, but it is only right exact. Therefore we consider $\mathcal{D}^b(\mathcal{O}_0^{\mathbb{Z}})$, the bounded derived category of the category of finite-dimensional graded right $A$-modules with shift functor $[\![\cdot]\!]$. Let $\mathcal{L}\mathrm{T}_s$ be the left derived functor of $\mathrm{T}_s$.

For $w \in W$ we abbreviate $\Delta(w) = M(w \cdot 0)$, $L(w) = L(w \cdot 0)$ and $P(w) = P(w \cdot 0)$. All simple, standard and projective modules in $\mathcal{O}_0$ have standard graded lifts (i.e., their heads are concentrated in degree zero), which we will denote by the same symbols. We fix the unique isomorphism of the $\mathbb{Z}[v, v^{-1}]$-modules such that

$$\begin{aligned} \hat{\mathcal{E}}\colon \quad \mathbb{H} &\xrightarrow{\sim} [\mathcal{D}^b(\mathcal{O}_0^{\mathbb{Z}})], \\ H_w &\mapsto [\Delta(w)] \end{aligned}$$

and obtain the following well-known result:

**Proposition 13.**

(i) $(\mathcal{D}^b(\mathcal{O}_0^{\mathbb{Z}}), \hat{\mathcal{E}}, \{\theta^l_s\}_{s\in S})$ *is a categorification of the right regular representation of* $\mathbb{H}$ *with respect to the generators* $\underline{H}_s$, $s \in S$.
(ii) $(\mathcal{D}^b(\mathcal{O}_0^{\mathbb{Z}}), \hat{\mathcal{E}}, \{\mathcal{L}T_t\}_{t\in S})$ *is a categorification of the left regular representation of* $\mathbb{H}$ *with respect to the generators* $H_t$, $t \in S$.
(iii) $\hat{\mathcal{E}}(\underline{H}_w) = [P(w)]$ *for all* $w \in W$.

**Proof.** The ungraded resp. graded cases of (i) are treated in [7, Theorem 3.4(iv)] and [80, Theorem 7.1]. The ungraded resp. graded cases of (ii) follow from [2, (2.3) and Theorem 3.2] and [56, Appendix]. The claim (iii) follows from [6, Theorem 3.11.4(i) and (iv)]. □

For Proposition 13(i), the interpretation of the relations (3.2) is similar to the one given after Proposition 10. We note that the statements (i) and (iii) of Proposition 13 can be formulated entirely using the underlying abelian category, whereas the statement (ii) can not. The functors $\mathcal{L}T_t$, $t \in S$, satisfy braid relations (this can be proved analogously to [59, Proposition 11.1] using [2, Theorem 2.2] and [43, Section 6]), but we do not know any functorial interpretation for the relation $H_s^2 = H_e + (v^{-1} - v)H_s$. Hence (at least for the moment) Proposition 13(ii) gives only a *weak* categorification of the left regular representation of $\mathbb{H}$, but a categorification (in the stronger sense) of the underlying representation of the braid group, see [67].

The functors $\theta_s^l$ and $\mathcal{L}T_t$ naturally commute with each other and hence the parts (i) and (ii) of Proposition 13 together give a (weak) categorification of the regular Hecke bimodule. The connection to Remark 11(5) is given by Soergel's functor $\mathbb{V}$, see [71].

*3.4.1. $\mathfrak{gl}_2$-example, the basis given by standard modules*

Consider the case $W = S_2 = \{e, s\}$. In this case the category $\mathcal{O}_0$ is equivalent to the category of finite-dimensional right $A$-modules, where $A$ is the path algebra of the following quiver with relations:

$$s \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e\ , \quad \alpha\beta = 0.$$

The algebra $A$ is graded with respect to the length of paths. The algebra $A$ has a simple preserving duality and hence the categories of finite-dimensional right and left $A$-modules are equivalent. Working with left $A$-modules reflects better the natural $\mathfrak{gl}_2$-weight picture, so we will use it. The category $A$-mod has 5 indecomposable objects, namely,

$$\Delta(s) = L(s)\colon \quad \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0\ , \qquad L(e)\colon \quad 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C},$$

$$P(s)\colon \quad \mathbb{C} \oplus \mathbb{C} \underset{\binom{0}{1}}{\overset{(1\ 0)}{\rightleftarrows}} \mathbb{C}\ , \qquad \Delta(e) = P(e)\colon \quad \mathbb{C} \underset{\mathrm{id}}{\overset{0}{\rightleftarrows}} \mathbb{C},$$

$$I(e)\colon \quad \mathbb{C} \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} \mathbb{C}\ .$$

Let $f_s$ and $f_e$ denote the primitive idempotents of $A$ corresponding to the vertices $s$ and $e$. Then the functor $\theta_s^l$ is given by tensoring with the bimodule $Af_s \otimes_{\mathbb{C}} f_s A$, and the functor $\mathrm{T}_s$ is given by tensoring with the bimodule $Af_s A$. We have $\mathcal{L}_i \mathrm{T}_s = 0$, $i > 1$. The values of $\theta_s^l$, $\mathrm{T}_s$ and $\mathcal{L}_1 \mathrm{T}_s$ on the indecomposable objects from $\mathcal{O}_0^{\mathbb{Z}}$ are:

| $M$ | $L(s)$ | $L(e)$ | $P(s)$ | $P(e)$ | $I(e)$ |
|---|---|---|---|---|---|
| $\theta_s^l M$ | $P(s)\langle -1\rangle$ | $0$ | $P(s)\langle -1\rangle \oplus P(s)\langle 1\rangle$ | $P(s)$ | $P(s)\langle -2\rangle$ |
| $\mathrm{T}_s M$ | $I(e)$ | $0$ | $P(s)\langle -1\rangle$ | $L(s)$ | $I(e)\langle -1\rangle$ |
| $\mathcal{L}_1 \mathrm{T}_s M$ | $0$ | $L(e)\langle 1\rangle$ | $0$ | $0$ | $L(e)\langle 1\rangle$ |

There are several bases for the Grothendieck group, the *standard* choice is given by the isomorphism classes of the standard modules $\Delta(w)$, $w = e, s$. In this basis, the action of our functors is as follows:

$$\begin{aligned}
[\theta_s^l \Delta(e)] &= v\,[\Delta(e)] + [\Delta(s)];\\
[\theta_s^l \Delta(s)] &= [\Delta(e)] + v^{-1}\,[\Delta(s)];\\
[\mathcal{L}\mathrm{T}_s \Delta(e)] &= [\Delta(s)];\\
[\mathcal{L}\mathrm{T}_s \Delta(s)] &= [\Delta(e)] + (v^{-1} - v)\,[\Delta(s)].
\end{aligned}$$

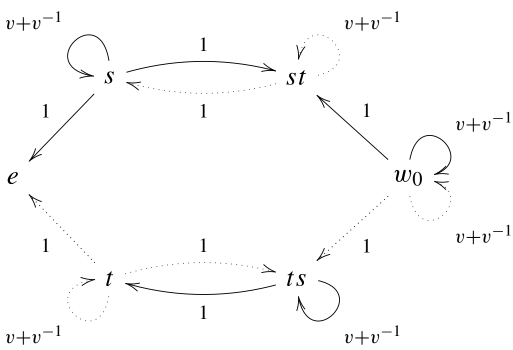
This is a weak categorification of the regular Hecke bimodule in the standard basis.

*3.4.2.* $\mathfrak{gl}_3$*-example, the basis given by simple modules*

Consider the case $W = S_3 = \{e, s, t, st, ts, sts = tst = w_0\}$. In this case the category $\mathcal{O}_0$ has infinitely many indecomposable objects (and is in fact wild, see [25]). However one can still compute the actions of $\theta_s^l, \theta_t^l, \mathrm{T}_s$ and $\mathrm{T}_t$ in various bases using known properties of these functors. The easiest basis is given by standard modules; here, however, we present the answer for $\theta_s^l, \theta_t^l$ in the most natural basis, namely, the one given by simple modules. To shorten the notation we will denote our simple modules just by the corresponding elements of the Weyl group. Here are the graded filtrations of the values of the translation functors $\theta_s^l$ and $\theta_t^l$ applied to simple modules:

| $M$ | $e$ | $s$ | $t$ | $st$ | $ts$ | $w_0$ |
|---|---|---|---|---|---|---|
| $\theta_s^l M$ | 0 | $s$<br>$st \quad e$<br>$s$ | 0 | 0 | $ts$<br>$t$<br>$ts$ | $w_0$<br>$st$<br>$w_0$ |
| $\theta_t^l M$ | 0 | 0 | $t$<br>$ts \quad e$<br>$t$ | $st$<br>$s$<br>$st$ | 0 | $w_0$<br>$ts$<br>$w_0$ |

From this we can draw the following graph which shows all the non-zero coefficients of the action of $\theta_s^l$ (indicated by solid arrows) and $\theta_t^l$ (indicated by dotted arrows) in the bases of simple modules:



. (3.3)

The graph (3.3) should be compared for example with [13, Fig. 6.2] (in order to get [13, Fig. 6.2] one should formally evaluate $v = 1$ and subtract the identity from $\theta_s^l$ and $\theta_t^l$). From (3.3) one can deduce immediately the existence of the following flag of $\mathbb{H}$-submodules inside our regular $\mathbb{H}$-module:

$$\langle [e] \rangle \subset \langle [e], [s], [st] \rangle \subset \langle [e], [s], [st], [t], [ts] \rangle \subset \langle [e], [s], [st], [t], [ts], [w_0] \rangle.$$

The subquotients of this flag are the Kazhdan–Lusztig cell modules for $\mathbb{H}$. As we will show later on, this can be extended to an explicit categorification of these cell modules via some subcategories of $\mathcal{O}_0$. The definition of left and right cells and the categorification of cell modules is the topic of the next section.

## 4. Categorifications of cell and Specht modules

In this section we will introduce two categorifications of cell modules—one which we believe is 'the correct one' and one which seems to be more canonical, easier, and straightforward on the first sight, but turns out to be less natural at the end. We do not know if the associated categories are in fact derived equivalent.

There also exists a different approach to categorification of cell structures, which was instigated by Lusztig in [49] and further extensively studied by Bezrukavnikov and Ostrik in [10,12].

### *4.1. Kazhdan–Lusztig's cell theory*

In this subsection we recall some facts from the Kazhdan–Lusztig cell theory. Our main references here are [38] and [13] and we refer the reader to these papers for details. We will use the notation from [74].

If $x \leqslant y$, then denote by $\mu(x, y)$ the coefficient of $v$ in the Kazhdan–Lusztig polynomial $h_{x,y}$ and extend it to a symmetric function $\mu : W \times W \to \mathbb{Z}$. In our normalisation the formula [38, (1.0.a)] reads then as follows:

$$\underline{H}_x \underline{H}_s = \begin{cases} \underline{H}_{xs} + \sum_{y<x, ys<y} \mu(y, x)\underline{H}_y, & xs > x; \\ (v + v^{-1})\underline{H}_x, & xs < x. \end{cases} \tag{4.1}$$

In particular, $\mu(x, xs) = \mu(xs, x) = 1$ for any $x \in W$ and $s \in S$. For $w \in W$ define the *left* and the *right descent* sets of $w$ as follows:

$$D_{\mathsf{L}}(w) := \{s \in S\colon sw < w\}, \qquad D_{\mathsf{R}}(w) := \{s \in S\colon ws < w\}.$$

Now for $x, y \in W$ we write $x \to_{\mathsf{L}} y$ provided that $\mu(x, y) \neq 0$ and there is some $s \in S$ such that $s \in D_{\mathsf{L}}(x)$ and $s \notin D_{\mathsf{L}}(y)$. Denote by $\geqslant_{\mathsf{L}}$ the transitive closure of the relation $\to_{\mathsf{L}}$. The relation $\geqslant_{\mathsf{L}}$ is called the *left* pre-order on $W$. The equivalence classes with respect to $\geqslant_{\mathsf{L}}$ are called the *left cells*. The fact that $x, y \in W$ belong to the same left cell will be denoted $x \sim_{\mathsf{L}} y$. The *right* versions $\geqslant_{\mathsf{R}}$ and $\sim_{\mathsf{R}}$ of the above are obtained by applying the involution $x \mapsto x^{-1}$, which yields the notion of *right cells*.

Given a right cell $\mathbf{R} \subset W$, the $\mathbb{C}[v, v^{-1}]$-span $X$ of $\underline{H}_x$, $x \geqslant_{\mathsf{R}} \mathbf{R}$, carries a natural structure of a right $\mathbb{H}$-module via (4.1). The $\mathbb{C}[v, v^{-1}]$-span $Y$ of $\underline{H}_x$, $x >_{\mathsf{R}} \mathbf{R}$, is a submodule of $X$. The $\mathbb{H}$-module $X/Y$ is called the *(right) cell module* associated with $\mathbf{R}$ and will be denoted by $S(\mathbf{R})$. We leave it as an exercise to the reader to verify that our definition of a cell module in fact agrees with the one from [38].

### 4.2. Presentable modules

Here we would like to recall the construction of the category of presentable modules from [3], a basic construction which will be crucial in the sequel. Let $\mathscr{A}$ be an abelian category and $\mathscr{B}$ be a full additive subcategory of $\mathscr{A}$. Denote by $\overline{\mathscr{B}}$ the full subcategory of $\mathscr{A}$, which consists of all $M \in \mathscr{A}$ for which there is an exact sequence $N_1 \to N_0 \to M \to 0$ with $N_1, N_0 \in \mathscr{B}$. This exact sequence is called a *$\mathscr{B}$-presentation* of $M$. In the special case when $\mathscr{B} = \mathrm{add}(P)$ for some projective object $P \in \mathscr{A}$ we have that $\overline{\mathscr{B}}$ is equivalent to the category of right $\mathrm{End}_{\mathscr{A}}(P)$-modules, see [3, Section 5]. In particular, $\overline{\mathrm{add}(P)}$ is abelian.

### 4.3. Categorification of cell modules

Let $\mathbf{R}$ be a right cell of $W$. Set

$$\hat{\mathbf{R}} = \{w \in W\colon\ w \leqslant_{\mathsf{R}} x \text{ for some } x \in \mathbf{R}\}.$$

Let $\mathcal{O}_0^{\hat{\mathbf{R}}}$ denote the full subcategory of $\mathcal{O}_0$, whose objects are all $M \in \mathcal{O}_0$ such that each composition subquotient of $M$ has the form $L(w)$, $w \in \hat{\mathbf{R}}$. For example if $\mathbf{R} = \{e\}$, the category $\mathcal{O}_0^{\hat{\mathbf{R}}}$ contains only finite direct sums of copies of the trivial $\mathfrak{g}$-module. In any case, the inclusion functor $\mathrm{i}^{\hat{\mathbf{R}}} : \mathcal{O}_0^{\hat{\mathbf{R}}} \hookrightarrow \mathcal{O}_0$ is exact and has as left adjoint the functor $\mathrm{Z}^{\hat{\mathbf{R}}}$ which picks out the maximal quotient contained in $\mathcal{O}_0^{\hat{\mathbf{R}}}$. In particular, the indecomposable projective modules in $\mathcal{O}_0^{\hat{\mathbf{R}}}$ are the $P^{\hat{\mathbf{R}}}(w) = \mathrm{Z}^{\hat{\mathbf{R}}} P(w)$, $w \in \hat{\mathbf{R}}$.

**Remark 14.** If $\mathbf{R}$ contains $w_0^{\mathfrak{p}} w_0$ where $w_0^{\mathfrak{p}}$ is the longest element in the parabolic subgroup of $W$ corresponding to a parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$ of $\mathfrak{g}$, then $\mathcal{O}_0^{\hat{\mathbf{R}}} = \mathcal{O}_0^{\mathfrak{p}}$, the principal block of the parabolic category $\mathcal{O}$ in the sense of [66]. This follows from [13, Proposition 6.2.7] and the fact that all simple modules in $\mathcal{O}_0^{\mathfrak{p}}$ can be obtained as subquotients of translations of the simple tilting module in $\mathcal{O}_0^{\mathfrak{p}}$ (as shown in [17]).

**Proposition 15.** *Let* $\mathbf{R}$ *be a right cell of* $W$.

(i) *The category* $\mathcal{O}_0^{\hat{\mathbf{R}}}$ *is stable under* $\theta_s^l$, $s \in S$.
(ii) *The additive category generated by* $P^{\hat{\mathbf{R}}}(w)$, $w \in \mathbf{R}$, *is stable under* $\theta_s^l$, $s \in S$.

**Proof.** To prove (i) it is enough to show that $\theta_s^l L(w) \in \mathcal{O}_0^{\hat{\mathbf{R}}}$ for all $w \in \hat{\mathbf{R}}$. For $z \in W$ using the self-adjointness of $\theta_s^l$, Eq. (4.1) and Proposition 13(iii) we have:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}\big(P(z), \theta_s^l L(w)\big) &= \mathrm{Hom}_{\mathcal{O}}\big(\theta_s^l P(z), L(w)\big) \\ &= \begin{cases} \mathrm{Hom}_{\mathcal{O}}(P(z) \oplus P(z), L(w)), & zs < z; \\ \mathrm{Hom}_{\mathcal{O}}(P(zs) \oplus \bigoplus_{y<z, ys<y} P(y)^{\mu(y,z)}, L(w)), & zs > z. \end{cases}\end{aligned}$$

The latter space can be non-zero only in the following cases: $z = w$ or $zs = w > z$, or, finally, $w < z$ where $ws < w$ and $\mu(w, z) \neq 0$. In all these cases $w \in \hat{\mathbf{R}}$ implies $z \in \hat{\mathbf{R}}$ and (i) follows.

To prove (ii) we use (i) and note that $\theta_s^l$ maps projectives from $\mathcal{O}_0^{\hat{\mathbf{R}}}$ to projectives from $\mathcal{O}_0^{\hat{\mathbf{R}}}$ since it is self-adjoint. Now take $x \in \mathbf{R}$. Then $\theta_s^l P^{\hat{\mathbf{R}}}(x)$ is a direct sum of some $P^{\hat{\mathbf{R}}}(y)$'s. The possible $y$'s to occur are given by (4.1), hence either $y = x$ or $y \in \mathbf{R}$, or $y = xs > x$. In the last case we have either $y \in \mathbf{R}$ or $y \notin \hat{\mathbf{R}}$, which is not possible since $\theta_s^l$ preserves $\mathcal{O}_0^{\hat{\mathbf{R}}}$ by (i). This completes the proof. □

We know already that the indecomposable projective module $P(x) \in \mathcal{O}_0$ has a *standard* graded lift $\mathtt{P}(x)$ for all $x \in W$ (for the definition of graded lift we refer to [80, Section 3]; here and further a *standard* graded lift of a projective or simple or standard module is the lift in which the top of the module is concentrated in degree zero). Now for $x \in \hat{\mathbf{R}}$ the module $P^{\hat{\mathbf{R}}}(x) = \mathrm{Z}^{\hat{\mathbf{R}}} P(x)$ is the quotient of $P(x)$ modulo the trace of all $P(y)$ such that $y \not\leqslant_{\mathsf{R}} x$. The corresponding quotient $\mathtt{P}^{\hat{\mathbf{R}}}(x)$ of $\mathtt{P}(x)$ is then a standard graded lift of $P^{\hat{\mathbf{R}}}(x)$. Let $\mathcal{P}^{\mathbf{R}}$ be the additive category, closed under grading shifts, and generated by $\mathtt{P}^{\hat{\mathbf{R}}}(w)$, $w \in \mathbf{R}$. This category is the graded version of the additive category from Proposition 15(ii). Set $\mathscr{C}^{\mathbf{R}} = \overline{\mathcal{P}^{\mathbf{R}}}$ (see Section 4.2), which is equivalent to the category of graded finite-dimensional right modules over the algebra $B^{\mathbf{R}} := \mathrm{End}_{\mathcal{O}_0}(\bigoplus_{w \in \mathbf{R}} P^{\hat{\mathbf{R}}}(w))$, which inherits a grading from the algebra $A$ (Section 3.4). From Proposition 15(ii) it follows that $\mathscr{C}^{\mathbf{R}}$ is closed under $\theta_s^l$, $s \in S$. Our first result is the following statement (we recall that $\mathbb{Z}((v))$ denotes the ring of formal Laurent series in $v$ with integer coefficients):

**Theorem 16** *(Categorification of cell modules).*

(i) *There is a unique monomorphism of* $\mathbb{H}$*-modules such that*

$$\mathcal{E}^{\mathbf{R}}:\quad S(\mathbf{R}) \longrightarrow [\mathscr{C}^{\mathbf{R}}],$$
$$\underline{H}_w \mapsto \quad [\mathtt{P}^{\hat{\mathbf{R}}}(w)].$$

(ii) *The monomorphism* $\mathcal{E}^{\mathbf{R}}$ *defines a precategorification* $(\mathscr{C}^{\mathbf{R}}, \mathcal{E}^{\mathbf{R}}, \{\theta_s^l\}_{s\in S})$ *and induces a categorification* $(\mathcal{P}^{\mathbf{R}}, \mathcal{E}^{\mathbf{R}}, \{\theta_s^l\}_{s\in S})$ *of the right cell* $\mathbb{H}$*-module* $S(\mathbf{R})$ *with respect to the generators* $H_s$*,* $s \in S$*.*

(iii) *The monomorphism* $\mathcal{E}^{\mathbf{R}}$ *from* (i) *extends uniquely to a* $\mathbb{Z}((v))$*-categorification* $(\mathscr{C}^{\mathbf{R}}, \mathcal{E}^{\mathbf{R}}, \{\theta_s^l\}_{s\in S})$ *of the right cell* $\mathbb{H}^{\mathbb{Z}((v))}$*-module* $S(\mathbf{R})^{\mathbb{Z}((v))}$ *with respect to the generators* $H_s$*,* $s \in S$*.*

**Proof.** The statement (i) follows from Proposition 13(i), Proposition 15(ii) and the definitions. The statement (ii) follows from (i). Note that $B^{\mathbf{R}}$ has infinite homological dimension in general. Hence the statement (iii) follows from (ii) as the extension of scalars from $\mathbb{Z}[v, v^{-1}]$ to $\mathbb{Z}((v))$ allows one to work with infinite projective resolutions. □

*4.4. Remarks on another categorification of cell modules*

Formula (4.1) suggests another way to categorify cell modules. For a right cell $\mathbf{R}$ of $W$ set

$$\check{\mathbf{R}} = \{w \in W\colon\ x \leqslant_{\mathsf{R}} w \text{ for some } x \in \mathbf{R}\}$$

(note the difference to $\hat{\mathbf{R}}$). Let $\mathscr{A}$ denote the additive category, generated by $P(w)$, $w \in \check{\mathbf{R}}$. Denote also by $\mathscr{A}'$ the additive category, generated by $P(w)$, $w \in \check{\mathbf{R}} \setminus \mathbf{R}$. Consider the categories $\mathcal{O}_0^{\check{\mathbf{R}}} = \overline{\mathscr{A}}$ and $\tilde{\mathcal{O}}_0^{\check{\mathbf{R}}} = \overline{\mathscr{A}'}$.

Note that if $\mathbf{R}$ contains $w_0^{\mathfrak{p}}$ where $w_0^{\mathfrak{p}}$ is the longest element in the parabolic subgroup $W_{\mathfrak{p}}$ of $W$ corresponding to a parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$ of $\mathfrak{g}$, then $\mathcal{O}_0^{\check{\mathbf{R}}}$ coincides with the category of $\mathfrak{p}$-presentable modules in $\mathcal{O}_0$ [58, Section 2] and is equivalent to ${}_{\mathbf{0}}\mathcal{H}_\lambda^1$, where $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and has stabiliser $W_{\mathfrak{p}}$ [7, Theorem 5.9(ii)].

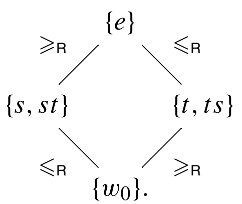
Formula (4.1) and Proposition 13(iii) immediately imply that both, the category $\mathcal{O}_0^{\check{\mathbf{R}}}$ and the category $\tilde{\mathcal{O}}_0^{\check{\mathbf{R}}}$, are stable under $\theta_s^l$, $s \in S$. And the 'quotient' should be exactly the cell module. To define this 'quotient' we let $\mathcal{Q}^{\mathbf{R}}$ denote the additive category, closed under grading shifts, and generated by $\mathtt{P}(w)$, $w \in \mathbf{R}$. Set $\mathscr{D}^{\mathbf{R}} = \overline{\mathcal{Q}^{\mathbf{R}}}$. The functors $\theta_s^l$, $s \in S$, do not preserve $\mathcal{Q}^{\mathbf{R}}$ unless $\mathbf{R} = \{w_0\}$. However, one can use them to define right exact functors $\tilde{\theta}_s^l$ on $\mathscr{D}^{\mathbf{R}}$ as follows: First we define the functor $\tilde{\theta}_s^l$ on the indecomposable projective module $\mathtt{P}(x)$. Let $s \in S$ and $x \in \mathbf{R}$. If $\theta_s^l \mathtt{P}(x) \in \mathcal{Q}^{\mathbf{R}}$, we set $\tilde{\theta}_s^l \mathtt{P}(x) = \theta_s^l \mathtt{P}(x)$, otherwise (4.1) gives

$$\theta_s^l \mathtt{P}(x) = \mathtt{P}(xs) \oplus \bigoplus_{y<x,\, ys<y} \mathtt{P}(y)^{\mu(y,x)}.$$

This decomposition into two summands is unique since the first summand coincides with the trace of the module $\mathtt{P}(xs)$ in $\theta_s^l \mathtt{P}(x)$ and the second summand coincides with the trace of the module $\bigoplus_{w \in \mathbf{R}} \mathtt{P}(w)$ in $\theta_s^l \mathtt{P}(x)$. Hence we can define $\tilde{\theta}_s^l \mathtt{P}(x) = \bigoplus_{y<x,\, ys<y} \mathtt{P}(y)^{\mu(y,x)}$ and define $\tilde{\theta}_s^l$ on morphisms via restriction. In the standard way $\tilde{\theta}_s^l$ extends uniquely to a right exact endofunctor on $\mathscr{D}^{\mathbf{R}}$. We do not know if $\tilde{\theta}_s^l$ is exact. By (4.1), the action of $\tilde{\theta}_s^l$, $s \in S$, on the Grothendieck group of $\mathcal{D}^b(\mathscr{D}^{\mathbf{R}})$ coincides with the action of $\underline{H}_s$ on $S(\mathbf{R})$ and hence we obtain a weak categorification of the cell module $S(\mathbf{R})$. We do not know whether this categorification is (derived) equivalent to the one constructed in Theorem 16 or not. The principal disadvantage with this categorification is that we do not know to which extend our uniqueness result from Section 5.1 holds in this setup.

### 4.5. $\mathfrak{gl}_3$-*example*

Let $W = \langle s, t\rangle \cong S_3$. Then there are four right cells and the Hasse diagram of the right order is as follows:




Consider first the case $\mathbf{R} = \{w_0\}$, where we have $\mathcal{O}_0^{\widehat{\{w_0\}}} = \mathcal{O}_0$. It contains all simple modules $L(w)$, $w \in S_3$. The presentation of this category as a module category over a finite-dimensional

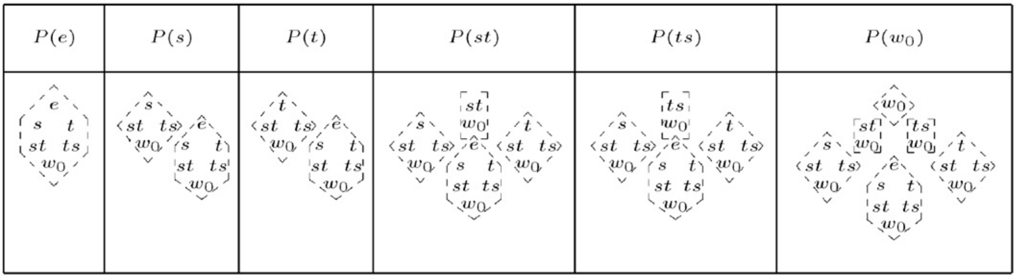



Fig. 1. Indecomposable projectives in $\mathcal{O}_0$.

algebra can be found in [81, 5.1.2]. The graded filtrations of the indecomposable projective modules (with indicated Verma subquotients) in this case are shown on Fig. 1. The category $\mathcal{P}^{\{w_0\}}$ contains (up to grading shift) a unique indecomposable module, namely, $P^{\{w_0\}}(w_0)$. The algebra $B^{\{w_0\}} = \mathrm{End}_{\mathfrak{g}}(P^{\{w_0\}}(w_0))$ is the coinvariant algebra of $W$, see [71, Endomorphismensatz].

Below we collect the analogous information for the three other choices for the right cells, in particular, we present all the algebras which appear there in terms of quivers and relations.

| **R** | $\{e\}$ | $\{s, st\}$ | $\{t, ts\}$ |
|---|---|---|---|
| Simple modules: | $e$ | $e, s, st$ | $e, t, ts$ |
| Projective modules: | $e$ | $P(e)$: $e$, $s$ <br> $P(s)$: $s$; $st$ $e$; $s$ <br> $P(st)$: $st$, $s$, $st$ | $P(e)$: $e$, $t$ <br> $P(t)$: $t$; $ts$ $e$; $t$ <br> $P(ts)$: $ts$, $t$, $ts$ |
| Quiver of $\mathcal{O}_0^{\hat{\mathbf{R}}}$: | $e$ | $st \underset{\beta}{\overset{\alpha}{\rightleftarrows}} s \underset{\delta}{\overset{\gamma}{\rightleftarrows}} e$ <br> $\beta\delta = \gamma\alpha = \gamma\delta = 0$ <br> $\alpha\beta = \delta\gamma$ | $ts \underset{\beta}{\overset{\alpha}{\rightleftarrows}} t \underset{\delta}{\overset{\gamma}{\rightleftarrows}} e$ <br> $\beta\delta = \gamma\alpha = \gamma\delta = 0$ <br> $\alpha\beta = \delta\gamma$ |
| Quiver of $\mathscr{C}^{\mathbf{R}}$: | $e$ | $st \underset{\beta}{\overset{\alpha}{\rightleftarrows}} s$ $\alpha\beta\alpha = \beta\alpha\beta = 0$ | $ts \underset{\beta}{\overset{\alpha}{\rightleftarrows}} t$ $\alpha\beta\alpha = \beta\alpha\beta = 0$ |

In the above example the category $\mathcal{O}_0^{\hat{\mathbf{R}}}$ always coincides with some parabolic category $\mathcal{O}_0^{\mathfrak{p}}$. This is not the case in general. The smallest such example is the right cell $\{s_1s_3, s_1s_3s_2\}$ of $S_4$.

## 4.6. Specht modules

In the special case $W = S_n$ we denote $\mathbb{H} = \mathbb{H}_n$. The (right) cell modules are exactly the irreducible $\mathbb{H}_n$-modules [38, Theorem 1.4]. However, cell modules for different right cells (namely, if they are in the same double cell) might be isomorphic. Theorem 16 gives therefore (several) categorifications for each irreducible $\mathbb{H}$-module. If we specialise $v = 1$ (i.e., we forget the grading) and work over a field of characteristic zero, the irreducible modules for the Hecke algebra specialise to irreducible modules for the symmetric group (for an explicit description see for example [64]), hence we get categorifications of Specht modules. In the special situation of Remark 14 we obtain the categorification of Specht modules constructed in [44].

Every cell module has a symmetric, non-degenerate, $\mathbb{H}_n$-invariant bilinear form $\langle\cdot,\cdot\rangle$ with values in $\mathbb{Z}[v, v^{-1}]$, which is unique up to a scalar, see [63, page 114]. There is a categorical interpretation of this form as follows: For any $\mathbb{Z}$-graded complex vector space $M = \bigoplus_{j\in\mathbb{Z}} M^j$ let $h(M) = \sum_{j\in\mathbb{Z}}(\dim_{\mathbb{C}} M^j)v^j \in \mathbb{Z}[v, v^{-1}]$ be the corresponding Hilbert polynomial. For all $M$, $N \in \mathscr{C}^{\mathbf{R}}$ and all $i \in \mathbb{Z}$ the vector space $E^i(M,N) := \mathrm{Ext}^i_{\mathscr{C}^{\mathbf{R}}}(M,N)$ is $\mathbb{Z}$-graded in the natural way. Set $h(E(M,N)) = \sum_{i\in\mathbb{Z}}(-1)^i h(E^i(M,N))$. Let $\mathrm{d}$ denote the graded lift of the standard duality on $\mathcal{O}_0$, restricted to the category $\mathscr{C}^{\mathbf{R}}$.

**Proposition 17.** *The form*

$$\beta(\cdot,\cdot) := h\big(E\big(\cdot, \mathrm{d}(\cdot)\big)\big) : \mathscr{C}^{\mathbf{R}} \times \mathscr{C}^{\mathbf{R}} \to \mathbb{Z}((v))$$

*descends to a symmetric, non-degenerate, $\mathcal{H}_n^{\mathbb{Z}((v))}$-invariant bilinear form $\langle\cdot,\cdot\rangle$ on the $\mathcal{H}_n^{\mathbb{Z}((v))}$-module $[\mathscr{C}^{\mathbf{R}}]^{\mathbb{Z}((v))}$. The restriction of this form to $[\mathcal{P}^{\mathbf{R}}]_{\oplus}$ has values in $\mathbb{Z}[v, v^{-1}]$.*

**Proof.** The same as the proof of [44, Proposition 4]. □

## 5. Uniqueness of the categorification for type $A$

In this section we stick to the case where $W = S_n$. In the previous section we constructed various categorifications for each single Specht module via cell modules. In this section we will show that all these categorifications are in fact equivalent. In particular, one can consider the categorification from [44] as a kind of 'universal one.'

### *5.1. Equivalence of categories*

**Theorem 18** *(Uniqueness Theorem). Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two right cells of $W = S_n$, which belong to the same double cell. Then there is an equivalence of categories*

$$\Phi = \Phi_{\mathbf{R}_1}^{\mathbf{R}_2}: \quad \mathscr{C}^{\mathbf{R}_1} \xrightarrow{\sim} \mathscr{C}^{\mathbf{R}_2},$$

*which (naturally) commutes with projective functors and induces an isomorphism of $\mathbb{H}$-modules $[\mathscr{C}^{\mathbf{R}_1}] \cong [\mathscr{C}^{\mathbf{R}_2}]$.*

We will only prove the ungraded version of this theorem. The graded version follows by standard arguments. For our proof we will need several new definitions and more notation. For any right cell $\mathbf{R}$ let $\mathscr{P}(\mathbf{R})$ denote the full additive subcategories of $\mathcal{O}$, generated by all indecomposable direct summands of the modules $E \otimes P^{\hat{\mathbf{R}}}(w)$, $w \in \mathbf{R}$, where $E$ runs through all finite-dimensional $\mathfrak{g}$-modules. Analogously we define $\mathscr{P}(\hat{\mathbf{R}})$ using the condition $w \in \hat{\mathbf{R}}$. Set $\mathcal{O}^{\mathbf{R}} = \overline{\mathscr{P}(\mathbf{R})}$ and $\mathcal{O}^{\hat{\mathbf{R}}} = \overline{\mathscr{P}(\hat{\mathbf{R}})}$.

Denote by $\mathcal{O}_{\mathrm{int}}$ the full subcategory of $\mathcal{O}$, which consists of all modules with integral support (i.e., those modules $M$ such that each weight of $M$ is also a weight of some finite-dimensional module). Further, for $s \in S$ we denote by $\mathcal{O}^s_{\mathrm{int}}$ the integral part of the $s$-parabolic category, that is, the full subcategory of $\mathcal{O}_{\mathrm{int}}$, which consists of all modules which have only composition factors of the form $L(w \cdot \lambda)$, where $\lambda$ is an integral weight in $\mathfrak{h}^*_{\mathrm{dom}}$, $sw \cdot \lambda \neq w \cdot \lambda$, and $sw > w$. For these categories we have the natural inclusion $\mathrm{i}_s : \mathcal{O}^s_{\mathrm{int}} \hookrightarrow \mathcal{O}_{\mathrm{int}}$ and we denote by $\mathrm{Z}_s$ and $\hat{\mathrm{Z}}_s$ the left and the right adjoint to this inclusion, respectively. These are the classical *Zuckerman functors*.

If $\mathbf{R}$ is a right cell such that $\mathbf{R} \leqslant_{\mathsf{R}} sw_0$, then we have the natural inclusion $\mathrm{i}_s^{\hat{\mathbf{R}}} : \mathcal{O}^{\hat{\mathbf{R}}} \hookrightarrow \mathcal{O}^s_{\mathrm{int}}$ and we denote by $\mathrm{Z}_s^{\hat{\mathbf{R}}}$ and $\hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}}$ the left and the right adjoint to this inclusion, respectively.

Let now $\mathbf{R}_1$ and $\mathbf{R}_2$ be two right cells. Assume that (see [38, Proof of Theorem 1.4])

$$\exists s, t \in S \quad \text{and} \quad w \in \mathbf{R}_1 \quad \text{such that } (st)^3 = e,\ sw \geqslant w,\ tw \leqslant w,\ tw \in \mathbf{R}_2. \tag{5.1}$$

In this case we have the following picture:

$$\mathcal{D}^b(\mathcal{O}^s_{\mathrm{int}}) \underset{\mathcal{R}\hat{\mathrm{Z}}_s}{\overset{\mathrm{i}_s}{\rightleftarrows}} \mathcal{D}^b(\mathcal{O}_{\mathrm{int}}) \underset{\mathrm{i}_t[\![1]\!]}{\overset{\mathcal{L}\mathrm{Z}_t[\![-1]\!]}{\rightleftarrows}} \mathcal{D}^b(\mathcal{O}^t_{\mathrm{int}}) .$$

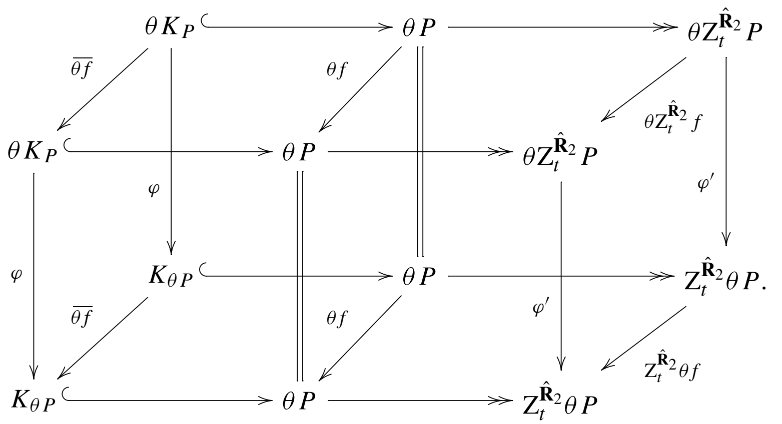
For this diagram we denote by F the composition from the left to the right and by G the composition from the right to the left. Directly from the definitions we have that (F, G) is an adjoint pair of functors. Furthermore, there are adjoint pairs $(\mathrm{i}_s^{\hat{\mathbf{R}}_1}, \hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1})$ and $(\mathrm{Z}_t^{\hat{\mathbf{R}}_2}, \mathrm{i}_t^{\hat{\mathbf{R}}_2})$ as follows:

$$\mathcal{O}^{\hat{\mathbf{R}}_1} \underset{\hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1}}{\overset{\mathrm{i}_s^{\hat{\mathbf{R}}_1}}{\rightleftarrows}} \mathcal{O}^s_{\mathrm{int}} \qquad \mathcal{O}^t_{\mathrm{int}} \underset{\mathrm{i}_t^{\hat{\mathbf{R}}_2}}{\overset{\mathrm{Z}_t^{\hat{\mathbf{R}}_2}}{\rightleftarrows}} \mathcal{O}^{\hat{\mathbf{R}}_2} .$$

**Lemma 19.** *The functors F, G, $\mathrm{i}_s^{\hat{\mathbf{R}}_1}$, $\hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1}$, $\mathrm{i}_t^{\hat{\mathbf{R}}_2}$ and $\mathrm{Z}_t^{\hat{\mathbf{R}}_2}$ commute with functors of tensoring with finite-dimensional $\mathfrak{g}$-modules, in particular with projective functors.*

**Proof.** Since all involved categories are stable under tensoring with finite-dimensional $\mathfrak{g}$-modules by definition, all involved inclusions commute with these functors. We will show how one derives from here that $\mathrm{Z}_t^{\hat{\mathbf{R}}_2}$ commutes with tensoring with finite-dimensional $\mathfrak{g}$-modules. For all other functors the arguments are similar and therefore omitted.

Let $E$ be a finite-dimensional $\mathfrak{g}$-module. For each $M \in \mathcal{O}^t_{\mathrm{int}}$ from the definition of $\mathrm{Z}_t^{\hat{\mathbf{R}}_2}$ we have the canonical projection $M \twoheadrightarrow \mathrm{Z}_t^{\hat{\mathbf{R}}_2} M$ with kernel $K_M$. Denote $\theta := E \otimes {}_-$, and let $P$ be a projective module in $\mathcal{O}^t_{\mathrm{int}}$ and $f \in \mathrm{End}_{\mathfrak{g}}(P)$. Consider the following diagram:




(5.2)

Both modules, $\theta \mathrm{Z}^{\hat{\mathbf{R}}_2}_t P$ and $\mathrm{Z}^{\hat{\mathbf{R}}_2}_t \theta P$, are obviously projective in $\mathcal{O}^{\hat{\mathbf{R}}_2}$. Let $\theta'$ be the adjoint of $\theta$. Then for any simple module $L \in \mathcal{O}^{\hat{\mathbf{R}}_2}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}\big(\theta \mathrm{Z}^{\hat{\mathbf{R}}_2}_t P, L\big) &= \mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Z}^{\hat{\mathbf{R}}_2}_t P, \theta' L\big) \\ &= \mathrm{Hom}_{\mathfrak{g}}(P, \theta' L) \\ &= \mathrm{Hom}_{\mathfrak{g}}(\theta P, L) \\ &= \mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Z}^{\hat{\mathbf{R}}_2}_t \theta P, L\big). \end{aligned}$$

Hence $\theta \mathrm{Z}^{\hat{\mathbf{R}}_2}_t P \cong \mathrm{Z}^{\hat{\mathbf{R}}_2}_t \theta P$. In particular, by definition of $\mathrm{Z}^{\hat{\mathbf{R}}_2}_t$, we have that $\theta K_P$ coincides with the maximal submodule of $\theta P$, whose head consists only of simple modules not in $\mathcal{O}^{\hat{\mathbf{R}}_2}$. In particular, the identity map on $\theta P$ restricts to an isomorphism $\varphi : \theta K_P \xrightarrow{\sim} K_{\theta P}$, and induces the isomorphism $\varphi' : \theta \mathrm{Z}^{\hat{\mathbf{R}}_2}_t P \xrightarrow{\sim} \mathrm{Z}^{\hat{\mathbf{R}}_2}_t \theta P$. It follows that cube on the left and the front, back, top and bottom faces of (5.2) commute. Therefore the face pointing to the right commutes as well. This implies $\theta \mathrm{Z}^{\hat{\mathbf{R}}_2}_t \cong \mathrm{Z}^{\hat{\mathbf{R}}_2}_t \theta$ since both functors are right exact. □

**Proposition 20.** *Assume that $\mathscr{P}(\mathbf{R}_1)$ has a simple projective module $L$. Then $\mathscr{P}(\mathbf{R}_2)$ has a simple projective module $L'$ given by $\mathrm{Z}^{\hat{\mathbf{R}}_2}_t \mathrm{Fi}^{\hat{\mathbf{R}}_1}_s L$.*

To prove Proposition 20 we will need a series of auxiliary statements. We start with verifying that the expression $\mathrm{Z}^{\hat{\mathbf{R}}_2}_t \mathrm{Fi}^{\hat{\mathbf{R}}_1}_s L$ makes sense, i.e., that it gives a module:

**Lemma 21.** *Let $X = L$ or $X = L(x)$ for some $x \in \mathbf{R}_1$. Then $\mathrm{Z}^{\hat{\mathbf{R}}_2}_t \mathrm{Fi}^{\hat{\mathbf{R}}_1}_s X \in \mathcal{O}^{\hat{\mathbf{R}}_2}$.*

**Proof.** The module $X$ does not belong to $\mathcal{O}^t_{\mathrm{int}}$ because of the condition (5.1). Hence by [20, Proposition 4.2] we have $\mathcal{L}_i \mathrm{Z}_t X = 0$ for $i = 0, 2$ and $\mathcal{L}_1 \mathrm{Z}_t X \in \mathcal{O}^t_{\mathrm{int}}$. Thus $\mathrm{F}X \in \mathcal{O}^t_{\mathrm{int}}$ and hence $\mathrm{Z}_t^{\hat{\mathbf{R}}_2} \mathrm{Fi}_s^{\hat{\mathbf{R}}_1} X \in \mathcal{O}^{\hat{\mathbf{R}}_2}$. □

**Lemma 22.**

(i) *$L' := \mathrm{Z}_t^{\hat{\mathbf{R}}_2} \mathrm{Fi}_s^{\hat{\mathbf{R}}_1} L$ is a simple module.*
(ii) *For each $L(x)$, $x \in \mathbf{R}_1$, the module $\mathrm{Z}_t^{\hat{\mathbf{R}}_2} \mathrm{Fi}_s^{\hat{\mathbf{R}}_1} L(x)$ is simple and has the form $L(y)$ for some $y \in \mathbf{R}_2$. Moreover, the map $\varphi : x \mapsto y$ is a bijection from $\mathbf{R}_1$ to $\mathbf{R}_2$.*

**Proof.** Let $L(z) \in \mathcal{O}_0$ be the (unique) simple module which translates to $L \in \mathscr{P}(\mathbf{R}_1)$ via translations to walls (see e.g. [33, 4.12(3)]). By [59, Theorem 2], [2, Theorem 6.3] and [2, Theorem 7.8] we have

$$\mathcal{L}_1 \mathrm{Z}_t L(z) \cong L(tz) \oplus \bigoplus_y L(y)^{a_y},$$

where $tz \in \mathbf{R}_2$, and $a_y \neq 0$ implies that $y \neq tz$ but both $y$ and $tz$ belong to the same left cell. Since the intersection of a left and a right cell inside a common two-sided cell consists of exactly one element (by the Robinson–Schensted correspondence, see e.g. [69, 3.1]), the later restrictions give that $a_y \neq 0$ implies $y \notin \mathbf{R}_2$. Hence $\mathrm{Z}_t^{\hat{\mathbf{R}}_2} \mathcal{L}_1 \mathrm{Z}_t L(z)$ is a simple module. Translating this onto the walls we obtain that the module $L'$ is simple. This proves (i) and also (ii) for the module $L(z)$. For other $x \in \mathbf{R}_1$ the proof is just the same as for $L(z)$. The fact that $\varphi : \mathbf{R}_1 \to \mathbf{R}_2$ is a bijection follows from [38, Section 4]. □

**Lemma 23.**

(i) $L = \hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1} \mathrm{Gi}_t^{\hat{\mathbf{R}}_2} L'$.
(ii) *For any $x \in \mathbf{R}_1$ we have $L(x) = \hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1} \mathrm{Gi}_t^{\hat{\mathbf{R}}_2} L(\varphi(x))$.*

**Proof.** Analogous to the proof of Lemma 22. □

As $L \in \mathscr{P}(\mathbf{R}_1)$, the category $\mathscr{P}(\mathbf{R}_1)$ is equivalent to the additive closure of the category with objects $L \otimes E$, where $E$ runs through all finite-dimensional $\mathfrak{g}$-modules. Set $\tilde{\mathrm{F}} = \mathrm{Z}_t^{\hat{\mathbf{R}}_2} \mathrm{Fi}_s^{\hat{\mathbf{R}}_1}$, $\tilde{\mathrm{G}} = \hat{\mathrm{Z}}_s^{\hat{\mathbf{R}}_1} \mathrm{Gi}_t^{\hat{\mathbf{R}}_2}$ and $\mathscr{Q} = \tilde{\mathrm{F}} \mathscr{P}(\mathbf{R}_1)$.

**Lemma 24.**

(i) *The functors $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ define mutually inverse equivalences between $\mathscr{P}(\mathbf{R}_1)$ and $\mathscr{Q}$.*
(ii) *$\mathscr{Q}$ is equivalent to the additive closure of the category with objects $L' \otimes E$, where $E$ runs through all finite-dimensional $\mathfrak{g}$-modules.*

**Proof.** We have already seen that $\tilde{\mathrm{F}} L = L'$ and $\tilde{\mathrm{G}} L' = L$. By Lemma 19 we thus have that

$$\tilde{\mathrm{G}}\tilde{\mathrm{F}}(E \otimes L) \cong E \otimes L \quad \text{and} \quad \tilde{\mathrm{F}}\tilde{\mathrm{G}}(E \otimes L') \cong E \otimes L' \tag{5.3}$$

for any finite-dimensional $\mathfrak{g}$-module $E$. By definition, we have the adjoint pair $(\tilde{\mathrm{F}}, \tilde{\mathrm{G}})$. Consider the adjunction morphisms $\mathrm{adj} : \tilde{\mathrm{F}}\tilde{\mathrm{G}} \to \mathrm{ID}$ and $\overline{\mathrm{adj}} : \mathrm{ID} \to \tilde{\mathrm{G}}\tilde{\mathrm{F}}$. Then the adjunction property says that $\mathrm{adj}_{\tilde{\mathrm{F}}(\bullet)} \circ \tilde{\mathrm{F}}(\overline{\mathrm{adj}}) = \mathrm{id}$. In particular $\mathrm{adj}_{E\otimes L'}$ must be surjective, hence an isomorphism by (5.3). Similarly $\overline{\mathrm{adj}}_{E\otimes L}$ is an isomorphism. This proves statement (i) and statement (ii) follows then from (i) and Lemma 19. □

Let now $\mathscr{Y}_1$ denote the full subcategory of $\mathcal{O}_0$, whose objects are the $P^{\hat{\mathbf{R}}_1}(x)$ and the $L(x)$, $x \in \mathbf{R}_1$. Denote further by $\mathscr{Y}_2$ the full subcategory of $\mathcal{O}_0$ whose objects are $\tilde{\mathrm{F}}P^{\hat{\mathbf{R}}_1}(x)$, $x \in \mathbf{R}_1$, and $L(y)$, $y \in \mathbf{R}_2$. Lemma 24 can be refined as follows:

**Lemma 25.** *The functors* $\tilde{\mathrm{F}}$ *and* $\tilde{\mathrm{G}}$ *induce mutually inverse equivalences of categories between* $\mathscr{Y}_1$ *and* $\mathscr{Y}_2$.

**Proof.** By definition and Lemma 24, $\tilde{\mathrm{F}}P^{\hat{\mathbf{R}}_1}(x) \in \mathscr{Y}_2$ for all $x \in \mathbf{R}_1$, and $\tilde{\mathrm{G}}\tilde{\mathrm{F}}P^{\hat{\mathbf{R}}_1}(x) \in \mathscr{Y}_1$ for all $x \in \mathbf{R}_1$. Analogously to the proof of Lemma 22 one shows that for each $x \in \mathbf{R}_1$ we have $\tilde{\mathrm{F}}L(x) \cong L(y)$ for some $y \in \mathbf{R}_2$, and that for each $y \in \mathbf{R}_2$ we have $\tilde{\mathrm{G}}L(y) \cong L(x)$ for some $x \in \mathbf{R}_1$. Hence $\tilde{\mathrm{F}} : \mathscr{Y}_1 \to \mathscr{Y}_2$ and $\tilde{\mathrm{G}} : \mathscr{Y}_2 \to \mathscr{Y}_1$. That these functors are mutually inverse equivalences is proved in the same way as in Lemma 24. □

For $x \in \mathbf{R}_1$ set $N_x = \tilde{\mathrm{F}}P^{\hat{\mathbf{R}}_1}(x)$.

**Corollary 26.** *For every* $x \in \mathbf{R}_1$ *we have* $P^{\hat{\mathbf{R}}_2}(\varphi(x)) \twoheadrightarrow N_x$.

**Proof.** Using Lemmas 21–25, for any $x \in \mathbf{R}_1$ and $y \in \mathbf{R}_2$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}\big(N_x, L(y)\big) &= \mathrm{Hom}_{\mathfrak{g}}\big(\tilde{\mathrm{F}}P^{\hat{\mathbf{R}}_1}(x), L(y)\big) \\
&= \mathrm{Hom}_{\mathfrak{g}}\big(P^{\hat{\mathbf{R}}_1}(x), \tilde{\mathrm{G}}L(y)\big) \\
&= \mathrm{Hom}_{\mathfrak{g}}\big(P^{\hat{\mathbf{R}}_1}(x), L\big(\varphi^{-1}(y)\big)\big) \\
&= \begin{cases} \mathbb{C}, & \varphi(x) = y; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

and the claim follows. □

**Lemma 27.**

(i) *For* $x, w \in W$ *we have* $\theta^l_w \theta^l_x \cong \bigoplus_{y \geqslant_{\mathsf{L}} w} (\theta^l_y)^{m_y}$.
(ii) *Let* $x, w \in W$ *be such that* $x <_{\mathsf{R}} w$. *Then* $\theta^l_w L(x) = 0$.
(iii) *For each* $w \in W$ *there exists* $x \in W$ *such that* $x \sim_{\mathsf{R}} w$ *and* $\theta^l_w L(x) \neq 0$.

**Proof.** To prove the first statement we use some ideas from the proof of [54, Theorem 11]. Denote by $\sigma$ the unique anti-automorphism of $\mathbb{H}$, which maps $H_w$ to $H_{w^{-1}}$ (and hence $\underline{H}_w$ to $\underline{H}_{w^{-1}}$) for each $w \in W$. Using (4.1) we have:

$$\underline{H}_w\underline{H}_x = \sigma\big(\sigma(\underline{H}_w\underline{H}_x)\big) = \sigma\big(\sigma(\underline{H}_x)\sigma(\underline{H}_w)\big) = \\ = \sigma(\underline{H}_{x^{-1}}\underline{H}_{w^{-1}}) = \sum_{y^{-1}\geqslant_{\mathsf{R}} w^{-1}} \sigma(a_y\underline{H}_{y^{-1}}) = \sum_{y\geqslant_{\mathsf{L}} w} a_y\underline{H}_y.$$

Now (i) follows from Proposition 13.

Let $x, w \in W$ be such that $x <_{\mathsf{R}} w$. We have $P(x) \cong \theta_x^l \Delta(e) \twoheadrightarrow L(x)$. Using (i) we have $\theta_w^l P(x) \cong \theta_w^l \theta_x^l \Delta(e) \cong \bigoplus_{y\geqslant_{\mathsf{L}} w} P(y)^{m_y}$. At the same time by Proposition 15(i), the head of $\theta_w^l L(x)$ can contain only $L(y)$ such that $y \leqslant_{\mathsf{R}} x$. Hence we have $y \leqslant_{\mathsf{R}} x <_{\mathsf{R}} w \leqslant_{\mathsf{L}} y$, which is not possible. Therefore $\theta_w^l L(x) = 0$ proving (ii).

Let $\mathbf{R}$ be the right cell of $w$. Using Lemma 19, we have

$$\theta_w^l \mathrm{Z}^{\hat{\mathbf{R}}} \Delta(e) \cong \mathrm{Z}^{\hat{\mathbf{R}}} \theta_w^l \Delta(e) \cong \mathrm{Z}^{\hat{\mathbf{R}}} P(w) \cong P^{\hat{\mathbf{R}}}(w) \neq 0.$$

Hence $\theta_w^l L(x) \neq 0$ for some simple subquotient $L(x)$ of $\mathrm{Z}^{\hat{\mathbf{R}}} \Delta(e)$. In particular, $x \leqslant_{\mathsf{R}} w$ (thanks to the definition of $\mathrm{Z}^{\hat{\mathbf{R}}}$) and then $x \sim_{\mathsf{R}} w$ follows from (ii). □

**Lemma 28.** *There exists $z \in \mathbf{R}_1$ such that $N_z \in \mathscr{P}(\mathbf{R}_2)$.*

**Proof.** We choose $w, y \in \mathbf{R}_2$ such that $\theta_w^l L(y) \neq 0$ (see Lemma 27(iii)). By Corollary 26, there exists some $x \in \mathbf{R}_1$ such that $P^{\hat{\mathbf{R}}_2}(y) \twoheadrightarrow N_x$. Let $K$ be the kernel of the latter map. Consider the short exact sequence $K' \hookrightarrow K \twoheadrightarrow K''$, where $K''$ is the maximal quotient of $K$, which contains only simple subquotients of the form $L(z)$, $z <_{\mathsf{R}} w$. By Lemma 27(ii) we have $\theta_w^l K' \cong \theta_w^l K$. Hence we have the short exact sequence of the form

$$\theta_w^l K' \hookrightarrow \theta_w^l P^{\hat{\mathbf{R}}_2}(y) \twoheadrightarrow \theta_w^l N_x. \tag{5.4}$$

Note that $\theta_w^l N_x \neq 0$ since $\theta$ is exact, $\theta_w^l L(y) \neq 0$ and $L(y)$ is the head of $N_x$. If $\theta_w^l K' = 0$, we immediately get that $0 \neq \theta_w^l N_x \in \mathscr{P}(\mathbf{R}_2)$. But the additive category, generated by indecomposable modules $N_z$, $z \in \mathbf{R}_1$, is stable with respect to projective functors by Lemma 19. This implies that $N_z \in \mathscr{P}(\mathbf{R}_2)$ for some $z \in \mathbf{R}_1$.

Assume hence that $\theta_w^l K' \neq 0$ and consider an arbitrary short exact sequence of the form $M' \hookrightarrow \theta_w^l K' \twoheadrightarrow M''$ such that $M''$ is simple. Then $M'' \cong L(v)$ for some $v \in \mathbf{R}_2$. If we factor $M'$ out in (5.4), we obtain the short exact sequence

$$L(v) \hookrightarrow X \twoheadrightarrow \theta_w^l N_x, \tag{5.5}$$

where $X = \theta_w^l P^{\hat{\mathbf{R}}_2}(y)/M'$. By Corollary 26, the heads of $X$ and $\theta_w^l N_x$ are isomorphic. Hence the sequence (5.5) is not split. Apply now the functor $\tilde{\mathrm{G}}$ to the sequence (5.5), which basically reduces to the application of the functor $\mathcal{L}_1\mathrm{Z}_t$ because of the definition of $\tilde{\mathrm{G}}$. As $\mathcal{L}_2\mathrm{Z}_t\theta_w^l N_x = 0$ and $\mathcal{L}_0\mathrm{Z}_t L(v) = 0$ (this follows for example from Lemma 25 and the definition of $\tilde{\mathrm{G}}$), we obtain a short exact sequence

$$\tilde{\mathrm{G}}L(v) \hookrightarrow \tilde{\mathrm{G}}X \twoheadrightarrow \tilde{\mathrm{G}}\theta_w^l N_x, \tag{5.6}$$

in particular, $\tilde{\mathrm{G}}X \in \mathcal{O}^{\hat{\mathbf{R}}_1}$. Analogously one shows that $\tilde{\mathrm{F}}\tilde{\mathrm{G}}X \in \mathcal{O}^{\hat{\mathbf{R}}_2}$, which, together with Lemma 25, implies that the adjunction morphism induces an isomorphism $\tilde{\mathrm{F}}\tilde{\mathrm{G}}X \cong X$, and thus the sequence (5.5) is obtained from the sequence (5.6) by applying $\tilde{\mathrm{F}}$. However, the sequence (5.6) splits as $\tilde{\mathrm{G}}\theta^l_w N_x$ is projective in $\mathcal{O}^{\hat{\mathbf{R}}_1}$. Therefore (5.5) must be split as well, a contradiction. Hence $\theta^l_w K' \neq 0$ is not possible. This completes the proof. □

**Proof of Proposition 20.** To prove Proposition 20 it is enough to show that $\mathscr{Q} = \mathscr{P}(\mathbf{R}_2)$. Let $\mathscr{Q}_0$ and $\mathscr{P}(\mathbf{R}_2)_0$ denote the intersections of $\mathcal{O}_0$ with $\mathscr{Q}$ and $\mathscr{P}(\mathbf{R}_2)$, respectively. The definition of $\mathscr{P}(\mathbf{R}_2)$ and Lemma 24 imply that it is even enough to show that $\mathscr{Q}_0 = \mathscr{P}(\mathbf{R}_2)_0$. From Lemma 28 we know that $\mathscr{Q}_0 \cap \mathscr{P}(\mathbf{R}_2)_0$ is not trivial. As $\mathscr{Q}_0$ is additively closed by Lemma 24(ii) we have that $\mathscr{Q}_0$ contains some indecomposable projective from $\mathscr{P}(\mathbf{R}_2)_0$. Applying projective functors and Theorem 16 we get that $\mathscr{Q}_0$ must contain all indecomposable projectives from $\mathscr{P}(\mathbf{R}_2)_0$. But by Lemma 24 the categories $\mathscr{Q}_0$ and $\mathscr{P}(\mathbf{R}_2)_0$ contain the same number of pairwise non-isomorphic indecomposable modules. Hence $\mathscr{Q}_0 = \mathscr{P}(\mathbf{R}_2)_0$. This completes the proof. □

Now we are prepared to prove Theorem 18.

**Proof of Theorem 18.** Assume first that $\mathbf{R}_1$ is of the form described in Remark 14. Then $\mathscr{P}(\mathbf{R}_1)$ has a simple projective module by [31, Section 3.1]. Let now $\mathbf{R}_2$ be any other right cell in the same two-sided cell as $\mathbf{R}_1$. By [38, Proof of Theorem 1.4] there is a sequence, $\mathbf{R}_1 = \mathbf{R}^{(0)}, \mathbf{R}^{(2)}, \dots, \mathbf{R}^{(k)} = \mathbf{R}_2$, such that $(\mathbf{R}^{(i)}, \mathbf{R}^{(i+1)})$ satisfies the condition (5.1) for each $i = 0, \dots, k-1$. Inductively applying Lemma 24 and Proposition 20 provides an equivalence between $\mathscr{P}(\mathbf{R}_1)$ and $\mathscr{P}(\mathbf{R}_2)$. This of course induces an equivalence of abelian categories. □

## 5.2. Consequences

Let $\mathbf{R}$ be a right cell of $S_n$. From Theorem 18 and Remark 14 one can deduce the following facts:

(I) The Koszul grading on the algebra $A$ [74] turns $\mathrm{End}_{\mathcal{O}_0}(\bigoplus_{w\in\mathbf{R}} P^{\hat{\mathbf{R}}}(w))$ into a positively graded self-injective symmetric algebra [60, Theorem 5.4].
(II) The centre of $\mathrm{End}_{\mathcal{O}_0}(\bigoplus_{w\in\mathbf{R}} P^{\hat{\mathbf{R}}}(w))$ is isomorphic to the cohomology algebra of the associated Springer fibre, see [15, Theorem 2] or [83, Theorem 4.1.1].
(III) For each $w \in \mathbf{R}$ there is a finite-dimensional $\mathfrak{g}$-module $E$ such that each $P^{\hat{\mathbf{R}}}(x)$, $x \in \mathbf{R}$, is a direct summand of $E \otimes L(w)$. This follows from [30, Proposition 4.3(ii)].
(IV) The projective modules in $\mathscr{P}(\mathbf{R})$ have all the same Loewy lengths [60, Theorem 5.2].

## 5.3. Counter-examples

Perhaps the most remarkable feature of Theorem 18 is that there is no way to extend this result to the categories $\mathcal{O}_0^{\hat{\mathbf{R}}}$. For two right cells satisfying the condition of Remark 14 this was already pointed out in [40, Proposition 6]. At the same time, in [40, Proposition 7], it was shown that the corresponding $\mathcal{O}_0^{\hat{\mathbf{R}}}$'s are derived equivalent. Even this weaker statement is not true in the general case. For example, take $W = S_4$, generated by the simple reflections $s, t, r$ such that $sr = rs$. Take the two right cells $\mathrm{R}_1 = \{sr, srt\}$ and $\mathrm{R}_2 = \{tsr, tsrt\}$. Then we have

$\hat{\mathbf{R}}_1 = \{e, s, r, ts, tr, sr, rts, str, srt\}$ whereas $\hat{\mathbf{R}}_2 = \{e, t, ts, tr, tsr, tsrt\}$. In particular, the categories $\mathcal{O}_0^{\hat{\mathbf{R}}_1}$ and $\mathcal{O}_0^{\hat{\mathbf{R}}_2}$ have different numbers of simple modules; hence they cannot be derived equivalent.

For right cells $\hat{\mathbf{R}}$ satisfying the condition of Remark 14, the categories $\mathcal{O}^{\hat{\mathbf{R}}}$ are special amongst the categories associated with right cells: they are equivalent to the principal block of some parabolic category $\mathcal{O}$, in particular are highest weight categories (i.e., described by quasi-hereditary algebras), see [66]. This is not true for arbitrary right cells. The smallest such example is again the case $W = S_4$ with $\mathrm{R} = \{t, ts, tr\}$. In this case $\hat{\mathbf{R}} = \{e, t, ts, tr\}$ and we have the following graded filtrations of projective and standard modules in $\mathcal{O}_0^{\hat{\mathbf{R}}}$:

| $w$ | $e$ | $t$ | $ts$ | $tr$ |
|---|---|---|---|---|
| $P(w)$ | $e$<br>$t$ | $t$<br>$ts\ e\ tr$<br>$t$ | $ts$<br>$t$<br>$ts$ | $tr$<br>$t$<br>$tr$ |
| $\Delta(w)$ | $e$<br>$t$ | $t$<br>$ts\quad tr$ | $ts$ | $tr$ |

We see that not all projective modules have standard filtrations and hence $\mathcal{O}_0^{\hat{\mathbf{R}}}$ is not a highest weight category.

## 6. Tensor products and parabolic induction

In this section we show how one can categorify some standard representation theoretical operations like tensor products and parabolic induction. As application we categorify induced cell modules. Up to equivalence, the resulting categories depend only on the isomorphism class of the cell module, not on the actual cell module itself.

### *6.1. Outer tensor products*

Let $W$ and $W'$ be arbitrary finite Weyl groups with sets of simple reflections $S$ and $S'$. Let $\mathbb{H}$, $\mathbb{H}'$ be the corresponding Hecke algebras. If $M$ is a right $\mathbb{H}$-module and $M'$ is a right $\mathbb{H}'$-module, then the *outer tensor product* $M \boxtimes M'$ is the right $\mathbb{H} \otimes \mathbb{H}'$-module whose underlying space is $M \otimes M'$ and the module structure is given by $m \otimes m'(h \otimes h') = mh \otimes m'h'$ for $m \in M$, $m' \in M'$, $h \in \mathbb{H}$ and $h' \in \mathbb{H}'$.

Given two categories $\mathscr{C}_1$ and $\mathscr{C}_2$ let $\mathscr{C}_1 \oplus \mathscr{C}_2$ be the category with objects being pairs $(C_1, C_2)$, where $C_i$ is an object in $\mathscr{C}_i$, and the morphisms from an object $(A_1, A_2)$ to an object $(B_1, B_2)$ being pairs of morphisms $(f_1, f_2)$, where $f_i : A_i \to B_i$ for $i = 1, 2$. We assume that each of these categories is either equivalent to a module categories over some finite-dimensional algebra $A$ or at least equivalent to its (bounded) derived category. Then $\mathrm{Gr}(\mathscr{C}_1 \oplus \mathscr{C}_2) \cong \mathrm{Gr}(\mathscr{C}_1) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathscr{C}_2)$ and hence also $[\mathscr{C}_1 \oplus \mathscr{C}_2] \cong [\mathscr{C}_1] \otimes_{\mathbb{Z}} [\mathscr{C}_2]$. Given two functors $F_i : \mathscr{C}_i \to \mathscr{C}_i$, $i = 1, 2$, we denote by $F_1 \boxtimes F_2$ the endofunctor of $\mathscr{C}_1 \oplus \mathscr{C}_2$ which maps $(A_1, A_2)$ to $(F_1(A_1), F_2(A_2))$ and $(f_1, f_2)$ to $(F_1(f_1), F_2(f_2))$. The following result gives a categorification of the outer tensor products:

**Proposition 29** *(Tensor products). Assume we are given a right $\mathbb{H}$-module $M$ and a right $\mathbb{H}'$-module $M'$ together with a categorification $(\mathscr{S}, \mathcal{E}, \{\mathrm{F}_s\}_{s \in S})$ of $M$ and a categorification*

*$(\mathscr{S}', \mathcal{E}', \{\mathrm{F}'_{s'}\}_{s'\in S'})$ of $M'$ with respect to the generators $\underline{H}_{s'}$, where $s' \in S'$, of $\mathbb{H}'$. Then the tuple*

$$\left(\mathscr{S} \oplus \mathscr{S}', \mathcal{E} \otimes \mathcal{E}', \left\{\mathrm{F}_s \boxtimes \mathrm{F}'_{s'}\right\}_{s\in S, s'\in S'}\right)$$

*is a categorification of $M \boxtimes M'$ with respect to the generators $\underline{H}_s \otimes \underline{H}_{s'}$, $s \in S$, $s' \in S'$.*

**Proof.** This follows directly from the definitions. □

*6.2. Examples of parabolic induction*

Let now $W'$ be a parabolic subgroup of $W$ which corresponds to a subset $S' \subset S$. Let $\mathbb{H}' = \mathbb{H}(W', S')$ be the corresponding subalgebra of $\mathbb{H}$, and let $M$ be a (right) $\mathbb{H}'$-module. The purpose of this section is to give a categorification of the induced module $\mathrm{Ind}_{\mathbb{H}'}^{\mathbb{H}} M = M \otimes_{\mathbb{H}'} \mathbb{H}$, where $M$ is a cell module over $\mathbb{H}'$. We start by recalling examples from the literature.

*6.2.1. Sign parabolic module*

The assignment $H_s \mapsto -v$ for all $s \in S'$ defines a surjection $\mathbb{H}' \twoheadrightarrow \mathbb{Z}[v, v^{-1}]$ and hence defines on $\mathbb{Z}[v, v^{-1}]$ the structure of an $\mathbb{H}'$-bimodule. Consider the *sign parabolic* $\mathbb{H}$-module $\mathcal{N} = \mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{H}'} \mathbb{H}$. The set $\{N_x = 1 \otimes H_x\}$, where $x$ runs through the set $(W'\backslash W)_{\mathrm{short}}$ of shortest coset representatives in $W'\backslash W$, forms a basis of $\mathcal{N}$. The action of $\underline{H}_s$, $s \in S$, in this basis is given by (see [74, Section 3]):

$$N_x \underline{H}_s = \begin{cases} N_{xs} + vN_x, & \text{if } xs \in (W'\backslash W)_{\mathrm{short}}, xs > x; \\ N_{xs} + v^{-1}N_x, & \text{if } xs \in (W'\backslash W)_{\mathrm{short}}, xs < x; \\ 0, & \text{if } xs \notin (W'\backslash W)_{\mathrm{short}}. \end{cases}$$

It is easy to see that the specialisation $v = 1$ gives the $W$-module $\mathrm{Ind}_{W'}^{W} M$, where $M$ is the *sign* $W'$-module, that is, $M = \mathbb{Z}$ with the alternating action $1s = -1$ for all $s \in S'$.

*Its categorification.* Let $\mathfrak{p} \supseteq \mathfrak{b}$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to $S'$. Let further $\mathcal{O}_0^{\mathfrak{p}}$ be the locally $\mathfrak{p}$-finite part of $\mathcal{O}_0$ (in the sense of [66]). This is the full extension closed subcategory of $\mathcal{O}_0$, generated by the simple modules $L(w)$, $w \in (W'\backslash W)_{\mathrm{short}}$. Finally, let $\mathcal{O}_0^{\mathfrak{p},\mathbb{Z}}$ be the graded version of $\mathcal{O}_0^{\mathfrak{p}}$ (as defined in [6]). Let $\Delta^{\mathfrak{p}}(w)$ denote the corresponding standard graded lift of the generalised Verma module, i.e., the corresponding standard module in $\mathcal{O}_0^{\mathfrak{p},\mathbb{Z}}$ with head concentrated in degree 0. The category $\mathcal{O}_0^{\mathfrak{p}}$ has finite homological dimension, and hence we have a unique isomorphism $\mathcal{E}^{\mathfrak{p}}$ of $\mathbb{Z}[v, v^{-1}]$-modules as follows:

$$\begin{aligned} \mathcal{E}^{\mathfrak{p}}\colon\ \mathcal{N} &\xrightarrow{\sim} \left[\mathcal{O}_0^{\mathfrak{p},\mathbb{Z}}\right], \\ N_w &\mapsto \left[\Delta^{\mathfrak{p}}(w)\right]. \end{aligned}$$

The following result is well known (see for example [82, Proposition 1.5]):

**Proposition 30.** *$(\mathcal{O}_0^{\mathfrak{p},\mathbb{Z}}, \mathcal{E}^{\mathfrak{p}}, \{\theta_s^l\}_{s\in S})$ is a categorification of $\mathcal{N}$ with respect to the generators $\underline{H}_s$, $s \in S$.*

*6.2.2. Permutation parabolic module*

The assignment $H_s \mapsto v^{-1}$ for all $s \in S'$ defines a surjection $\mathbb{H}' \twoheadrightarrow \mathbb{Z}[v, v^{-1}]$ and hence determines on $\mathbb{Z}[v, v^{-1}]$ the structure of an $\mathbb{H}'$-bimodule. The *permutation parabolic* $\mathbb{H}$-module is defined as follows: $\mathcal{M} = \mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{H}'} \mathbb{H}$. There is the standard basis of $\mathcal{M}$ given by $\{M_x = 1 \otimes H_x\}$, where $x$ runs through $(W'\backslash W)_{\text{short}}$. The action of $\underline{H}_s$, $s \in S$, in this basis is given as follows (see [74, Section 3]):

$$M_x \underline{H}_s = \begin{cases} M_{xs} + vM_x, & \text{if } xs \in (W'\backslash W)_{\text{short}}, xs > x; \\ M_{xs} + v^{-1}M_x, & \text{if } xs \in (W'\backslash W)_{\text{short}}, xs < x; \\ (v + v^{-1})M_x, & \text{if } xs \notin (W'\backslash W)_{\text{short}}. \end{cases}$$

It is easy to see that the specialisation $v = 1$ gives the $W$-module $\operatorname{Ind}_{W'}^{W} M$, where $M$ is the *trivial* $W'$-module, that is, $M = \mathbb{Z}$ with the trivial action $1s = 1$ for all $s \in S'$. The module $\operatorname{Ind}_{W'}^{W} M$ is usually called the *permutation module*, see [69, 2.1].

*Its categorification.* Let $\mathfrak{p}$ be as in the previous example. Let $\mathscr{P}(\mathfrak{p})$ be the additive category, closed with respect to the shift of grading, and generated by the indecomposable projective modules $\mathtt{P}(w) \in \mathcal{O}_0^{\mathbb{Z}}$, where $w$ runs through the set $(W'\backslash W)_{\text{long}}$ of longest coset representatives in $W'\backslash W$. The category $\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}} = \overline{\mathscr{P}(\mathfrak{p})}$ is the graded version of the category $\mathcal{O}_0^{\mathfrak{p}\text{-pres}}$ from [58] (see also Section 4.4). The simple objects of $\mathcal{O}_0^{\mathfrak{p}\text{-pres}}$ are in natural bijection with $w \in (W'\backslash W)_{\text{long}}$. For $w \in (W'\backslash W)_{\text{long}}$ denote by $\Delta^{\mathfrak{p}\text{-pres}}(w)$ the standard object of $\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}}$ corresponding to $w$ and with the head concentrated in degree 0 [58, Theorem 2.16, Lemma 7.2]. Let $w_0'$ be the longest element of $W'$. All this defines a unique homomorphism $\mathcal{E}^{\mathfrak{p}\text{-pres}}$ of $\mathbb{Z}[v, v^{-1}]$-modules as follows:

$$\begin{aligned} \mathcal{E}^{\mathfrak{p}\text{-pres}}: \quad \mathcal{M} &\xrightarrow{\sim} \big[\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}}\big], \\ M_{w_0'w} &\mapsto \big[\Delta^{\mathfrak{p}\text{-pres}}(w)\big]. \end{aligned}$$

The category $\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}}$ does not have finite homological dimension in general, although the projective dimension of all standard modules is finite. Hence we have $[\Delta^{\mathfrak{p}\text{-pres}}(w)] \in [\mathscr{P}(\mathfrak{p})]_{\oplus}$ for all $w \in (W'\backslash W)_{\text{long}}$. This can be extended to the following (see for example [58, Theorem 7.7]):

**Proposition 31.**

(i) $(\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}}, \mathcal{E}^{\mathfrak{p}\text{-pres}}, \{\theta_s^l\}_{s\in S})$ *is a precategorification whereas* $(\mathscr{P}(\mathfrak{p}), \mathcal{E}^{\mathfrak{p}\text{-pres}}, \{\theta_s^l\}_{s\in S})$ *is a categorification of* $\mathcal{M}$ *with respect to the generators* $\underline{H}_s$, $s \in S$.

(ii) *The homomorphism* $\mathcal{E}^{\mathfrak{p}\text{-pres}}$ *extends uniquely to the* $\mathbb{Z}((v))$*-categorification* $(\mathcal{O}_0^{\mathfrak{p}\text{-pres},\mathbb{Z}}, \mathcal{E}^{\mathfrak{p}\text{-pres}}, \{\theta_s^l\}_{s\in S})$ *of* $\mathcal{M}^{\mathbb{Z}((v))}$ *with respect to the generators* $\underline{H}_s$, $s \in S$.

*6.3. The 'unification': the category* $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$

In Section 6.2 we used certain parabolic categories to categorify the sign module, but also used categories of presentable modules to categorify the permutation parabolic modules. Both depend on a fixed parabolic $\mathfrak{p} \subset \mathfrak{g}$. In this section we actually want to put these two approaches under one roof using a series $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ of categories, depending on the (fixed) $\mathfrak{p}$, $\mathfrak{g}$ and a (varying) category $\mathscr{A}$. The categorifications from Section 6.2 will then emerge for a special choice of $\mathscr{A}$.

The categories $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ were first introduced in [24]—as certain parabolic generalisations of the category $\mathcal{O}$ which led to properly stratified algebras. The setup was afterwards extended in [52, 6.2] to include general stratified algebras (in the sense of [16]). Here we present a slight variation of the original definition. This variation seems to be more natural for us, and is better adapted to the examples we work with.

Let $\tilde{\mathfrak{a}}$ be a reductive complex finite-dimensional Lie algebra with semisimple part $\mathfrak{a}$ and centre $\mathfrak{z}(\tilde{\mathfrak{a}})$. Let $\mathscr{A}$ be a full subcategory of the category of finitely generated $\tilde{\mathfrak{a}}$-modules. Then $\mathscr{A}$ is called an *admissible* category (of $\tilde{\mathfrak{a}}$-modules) if the following holds:

(L1) $\mathscr{A}$ is stable under $E\otimes_-$ for each simple finite-dimensional $\tilde{\mathfrak{a}}$-module $E$;
(L2) the action of $Z(\tilde{\mathfrak{a}})$ gives a decomposition of $\mathscr{A}$ into a direct sum of full subcategories, each of which is equivalent to a module category over a finite-dimensional self-injective associative algebra;
(L3) the action of $\mathfrak{z}(\tilde{\mathfrak{a}})$ on any object $M$ from $\mathscr{A}$ is diagonalisable.

Since the functors $E\otimes_-$ and $E^*\otimes_-$ are both left and right adjoint to each other on the category of all $\tilde{\mathfrak{a}}$-modules, (L1) implies that $E\otimes_-$ is in fact exact (as endofunctor of $\mathscr{A}$). (L2) guarantees that $\mathscr{A}$ is abelian, has enough projectives (which are also injective) and that each object of $\mathscr{A}$ has finite length (with respect to the abelian structure of $\mathscr{A}$, but not as an $\tilde{\mathfrak{a}}$-module in general).

Given an admissible $\mathscr{A}$, we can construct a series of categories $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ as follows: We take a semisimple (or reductive) Lie algebra $\mathfrak{g}$ with a chosen Borel subalgebra $\mathfrak{b}$, and require that $\mathfrak{p} \supset \mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p} = \tilde{\mathfrak{a}} \oplus \mathfrak{n}_{\mathfrak{p}}$ is the Levi decomposition of $\mathfrak{p}$. Given these data it makes sense to make the following definition:

**Definition 32.** The category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ is the full subcategory of the category of $\mathfrak{g}$-modules given by all objects which are

(PL1) finitely generated,
(PL2) locally $\mathfrak{n}_{\mathfrak{p}}$-finite,
(PL3) direct sums of objects from $\mathscr{A}$ when viewed as $\tilde{\mathfrak{a}}$-modules.

### 6.4. *Special cases of* $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$

#### 6.4.1. *Category* $\mathcal{O}$

If $\mathfrak{p} = \mathfrak{b}$, then $\tilde{\mathfrak{a}} = \mathfrak{h}$ is abelian. Let $\mathscr{A}$ be the category of all finite-dimensional semisimple $\mathfrak{h}$-modules. This category is obviously admissible. The category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ in this case is nothing else than the usual category $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b})$. Note that the property (PL3) in this case just means that the modules from $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ have a weight space decomposition. The category $\mathcal{O}$ is a highest weight category with standard modules given by the Verma modules.

#### 6.4.2. *The parabolic category* $\mathcal{O}^{\mathfrak{p}}$

If $\mathfrak{p}$ is any parabolic and $\mathscr{A}$ is the category of finite-dimensional semisimple $\tilde{\mathfrak{a}}$-modules, then $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ is the parabolic category $\mathcal{O}^{\mathfrak{p}}$. The category $\mathcal{O}^{\mathfrak{p}}$ is a highest weight category with standard modules given by the parabolic Verma modules.

*6.4.3. The category $\mathcal{O}_0^{\mathfrak{p}\text{-pres}}$*

Let $\mathfrak{p}$ be any parabolic subalgebra with Weyl group $W'$ and longest element $w_0'$. Consider the corresponding indecomposable projective module $P^{\mathfrak{a}}(w_0' \cdot 0)$ in the category $\mathcal{O}(\mathfrak{a}, \mathfrak{a} \cap \mathfrak{b})$ corresponding to $\mathfrak{a}$. Let $\mathscr{A}$ be the smallest abelian category which contains this $P^{\mathfrak{a}}(w_0' \cdot 0)$ and is closed under tensoring with finite-dimensional simple $\mathfrak{a}$-modules and taking quotients. Extend $\mathscr{A}$ to a category of $\tilde{\mathfrak{a}}$-modules by allowing diagonalisable action of $\mathfrak{z}(\tilde{\mathfrak{a}})$. Then the category $\mathscr{A}$ is admissible and $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ is the category of modules which are presentable by the $P(w \cdot \lambda) \in \mathcal{O}$, where $w$ runs through $(W' \backslash W)_{\text{long}}$ and $\lambda$ is an integral weight in $\mathfrak{h}^*_{\text{dom}}$ (for details see [58]). This category is also equivalent to the category of Harish-Chandra bimodules with generalised trivial integral central character from the left-hand side and the singular central character given by $W'$ from the right-hand side (for details see e.g. [7], [33, Kapitel 6]). This category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ is not a highest weight category in general, but still equivalent to a module category over a so-called properly stratified algebra, see [58].

*6.5. From highest weight categories to stratified algebras*

As usual, the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ decomposes into a direct sum of full subcategories, each of which is equivalent to a module category over a finite-dimensional associative algebra. Any block of the (parabolic) category $\mathcal{O}$ is a highest weight category, hence the associated algebra is quasi-hereditary. In general, this is not true for a block of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ (see for example [58]). The algebras which appear from blocks of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ are however always *weakly properly stratified* in the sense of [22, Section 2]. The proof of this fact is completely analogous to the properly stratified case, and we refer to [24, Section 3] for details.

A weakly properly stratified structure of an algebra means the following: the isomorphism classes of simple modules are indexed by a partially pre-ordered set $I$ and we have so-called *standard* and *proper standard* modules (both indexed by $I$ again) such that projective modules have standard filtrations, i.e., filtrations with subquotients isomorphic to standard modules, and standard modules have proper standard filtrations. Which subquotients are allowed to occur in the above filtrations and in the Jordan–Hölder filtrations of proper standard modules is given by the partial pre-order (for a precise definition we refer to [22]).

The modules defining the stratified structure are given in terms of parabolically induced modules as follows: If $V$ is any $\tilde{\mathfrak{a}}$-module, we consider $V$ as a $\mathfrak{p}$-module by letting $\mathfrak{n}_{\mathfrak{p}}$ act trivially and define the *parabolically induced module*

$$\Delta(\mathfrak{p}, V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

If $V$ is a simple object of $\mathscr{A}$, then $\Delta(\mathfrak{p}, V)$ is a *proper standard* module; if $V$ is projective, then $\Delta(\mathfrak{p}, V)$ is a *standard* module. The dual construction (using conduction) gives rise to *(proper) costandard module*. If $V$ is a simple $\tilde{\mathfrak{a}}$-module, then $\Delta(\mathfrak{p}, V)$ is usually called a *generalised Verma module*, or simply a *GVM*.

Let $\mathscr{F}(\Delta)$ denote the full subcategory of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$, given by all modules, which admit a standard filtration, that is, a filtration, whose subquotients are standard modules. Analogously one defines $\mathscr{F}(\overline{\Delta})$ for modules with proper standard filtration, $\mathscr{F}(\nabla)$ for modules with costandard filtration, and $\mathscr{F}(\overline{\nabla})$ for modules with proper costandard filtration. In this notation the property to be weakly properly stratified is equivalent to the claim that all projective modules in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$ belong to both $\mathscr{F}(\Delta)$ and $\mathscr{F}(\overline{\Delta})$.

We would like to point out that weakly properly stratified algebras form a strictly bigger class than properly stratified algebras as the classes of simple modules might be only partially pre-ordered, not partially ordered. As a consequence there could be non-isomorphic standard modules $\Delta_1$ and $\Delta_2$ such that $\mathrm{Hom}(\Delta_1, \Delta_2) \neq 0 \neq \mathrm{Hom}(\Delta_2, \Delta_1)$ (which will be in fact the case in almost all the examples occurring from now on in this paper).

### 6.6. *Parabolic induction via $\mathcal{O}\{\mathfrak{p}, \mathscr{A}\}$*

Let us return to the case $W = S_n$ with some fixed parabolic subgroup $W' = S_{i_1} \times S_{i_2} \times \cdots \times S_{i_r}$, where $i_1 + i_2 + \cdots + i_r = n$. Let $\mathbb{H}$ and $\mathbb{H}'$ be the corresponding Hecke algebras. Assume we are given a right cell $\mathbf{R}'$ of $W'$, then $\mathbf{R}' = \mathbf{R}'_{i_1} \times \mathbf{R}'_{i_2} \times \cdots \times \mathbf{R}'_{i_r}$ for some right cells $\mathbf{R}'_{i_j}$ in $S_{i_j}$.

Recall from Theorem 18 the categorification $\mathscr{C}^{\mathbf{R}'_{i_j}}$ of the right cell module $S(\mathbf{R}'_{i_j})$ associated with $\mathbf{R}'_{i_j}$. From Section 6.1 we deduce that the outer product, call it $\mathscr{C}^{\mathbf{R}'}$, of these categories categorifies the cell module corresponding to $\mathbf{R}'$. The objects of $\mathscr{C}^{\mathbf{R}'}$ are certain $\tilde{\mathfrak{a}} := \mathfrak{gl}_{i_1} \oplus \mathfrak{gl}_{i_2} \oplus \cdots \oplus \mathfrak{gl}_{i_r}$-modules. Let $\mathscr{P}$ denote the additive closure of the category of all modules, which have the form $E \otimes P$, where $P \in \mathscr{C}^{\mathbf{R}'}$ is projective and $E$ is a simple finite-dimensional $\tilde{\mathfrak{a}}$-module. Set $\mathscr{A}^{\mathbf{R}'} = \overline{\mathscr{P}}$.

**Lemma 33.** *$\mathscr{A}^{\mathbf{R}'}$ is admissible.*

**Proof.** As translations are exact, condition (L1) is satisfied by definition. Condition (L3) follows again from the definitions as the action of $\mathfrak{z}(\tilde{\mathfrak{a}})$ on any simple finite-dimensional $\tilde{\mathfrak{a}}$-module is diagonalisable. It is left to check (L2). By definition, $\mathscr{A}^{\mathbf{R}'}$ is a subcategory of $\mathcal{O}\{\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}} \cap \mathfrak{b}\}$. The block decomposition of the latter (with respect to the action of the centre of $Z(\tilde{\mathfrak{a}})$) induces a block decomposition of the former. Since translations are exact and send projectives to projectives, $\mathscr{A}^{\mathbf{R}'}$ has enough projectives. These projective modules are also injective by (I) from Section 5.2. Therefore the condition (L2) follows from the definitions and [3, Section 5]. □

By Lemma 33, the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ is defined. By construction, it is a subcategory of $\mathcal{O}$ and hence inherits a decomposition from the block decomposition of $\mathcal{O}$ which we call the block decomposition of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$. Denote by $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ the block of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ corresponding to the trivial central character. We note that one can show that $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ is indecomposable by invoking Theorem 18. We omit a proof, since the result will not be relevant for the following.

From the definition of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ we have that simple objects in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ are in natural bijection with the elements from $W$ of the form $xw$, where $w \in (W' \backslash W)_{\text{short}}$ and $x \in \mathbf{R}'$. Denote by $\Delta(\mathfrak{p}, xw)$ and $\overline{\Delta}(\mathfrak{p}, xw)$ the standard, respectively proper standard, module in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ corresponding to $xw$. Dually we also have the (proper) costandard module $\nabla(\mathfrak{p}, xw)$ ($\overline{\nabla}(\mathfrak{p}, xw)$). For details see [22, Section 2]. Finally, let $P(\mathfrak{p}, xw)$ be the projective cover of $\Delta(\mathfrak{p}, xw)$ in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$.

Set

$$\mathbb{I}(\mathbf{R}') = \{(x, w)\colon x \in \mathbf{R}',\ w \in (W' \backslash W)_{\text{short}}\},$$

$$\mathbb{J}(\mathbf{R}') = \{y \in W\colon y \geqslant_{\mathsf{R}} \mathbf{R}',\ y \neq xw \text{ for any } (x, w) \in \mathcal{I}(\mathbf{R}')\}.$$

In particular, by above the set $\mathbb{I}(\mathbf{R}')$ indexes bijectively the isomorphism classes of indecomposable projective modules in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$. From Section 4.3 and the definitions it follows that for any $(x, w) \in \mathbb{I}(\mathbf{R}')$ the module $P(\mathfrak{p}, xw)$ is the quotient of $P(xw)$ modulo the trace of all $P(y)$, $y \in \mathbb{J}(\mathbf{R}')$. In particular, all projectives in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ are gradable and hence the endomorphism ring $B$ of a minimal projective generator of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$ inherits a $\mathbb{Z}$-grading from the ring $A$ from Section 3.4. We denote by $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$ the category of finite-dimensional graded $B$-modules.

Let $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ be the induced cell module. By definition, it has a $\mathbb{Z}[v, v^{-1}]$-basis given by $\Delta_{x,w} := \underline{H}_x \otimes H_w$, where $(x, w) \in \mathbb{I}(\mathbf{R}')$. Hence we can define a homomorphism of $\mathbb{Z}[v, v^{-1}]$-modules as follows:

$$\begin{aligned} \Psi_{\mathbf{R}'}: \quad S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H} &\xrightarrow{\sim} \big[\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}\big], \\ \Delta_{x,w} &\mapsto \big[\Delta(\mathfrak{p}, xw)\big]. \end{aligned} \tag{6.1}$$

Let $\mathscr{P}(\mathfrak{p}, \mathscr{A}^{\mathbf{R}'})$ denote the additive category of all (graded) projective modules in $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0$. We obtain the following main result:

**Theorem 34** *(Categorification of induced cell modules).*

(i) *The map $\Psi_{\mathbf{R}'}$ is a homomorphism of $\mathbb{H}$-modules.*
(ii) *$(\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}, \Psi_{\mathbf{R}'}, \{\theta_s^l\}_{s\in S})$ is a precategorification of the induced (right) cell $\mathbb{H}$-module $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ whereas $(\mathscr{P}(\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}), \Psi_{\mathbf{R}'}, \{\theta_s^l\}_{s\in S})$ is a categorification of this module with respect to the generators $\underline{H}_s$, $s \in S$.*
(iii) *The map $\Psi_{\mathbf{R}'}$ defines a $\mathbb{Z}((v))$-categorification $(\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}, \Psi_{\mathbf{R}'}, \{\theta_s^l\}_{s\in S})$ of the induced cell $\mathbb{H}^{\mathbb{Z}((v))}$-module $S(\mathbf{R}')^{\mathbb{Z}((v))} \otimes_{(\mathbb{H}')^{\mathbb{Z}((v))}} \mathbb{H}^{\mathbb{Z}((v))}$ with respect to the generators $\underline{H}_s$, $s \in S$.*

**Proof.** In order to prove our theorem we only have to prove that $\Psi_{\mathbf{R}'}$ is a homomorphism of $\mathbb{H}$-modules. In other words, we have to compare the combinatorics of the action of $\mathbb{H}$ on the module $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ with the combinatorics of the action of $\{\theta_s^l\}_{s\in S}$ on $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$. Fix $(x, w) \in \mathbb{I}(\mathbf{R}')$ and $s \in S$.

If $ws \in (W'\backslash W)_{\mathrm{short}}$, then the definition of $\mathbb{H}$ (see Section 3.1) gives

$$\Delta_{x,w}\underline{H}_s = \begin{cases} \Delta_{x,ws} + v\Delta_{x,w}, & \text{if } ws > w; \\ \Delta_{x,ws} + v^{-1}\Delta_{x,w}, & \text{if } ws < w. \end{cases}$$

If $ws \notin (W'\backslash W)_{\mathrm{short}}$, we have that $ws = s'w$ for $s' \in S \cap W'$. In particular $ws > w$ and the definition of $\mathbb{H}$ gives $H_w\underline{H}_s = \underline{H}_{s'}H_w$. Therefore

$$\Delta_{x,w}\underline{H}_s = (\underline{H}_x \otimes H_w)\underline{H}_s = \underline{H}_x \otimes \underline{H}_{s'}H_w = \underline{H}_x\underline{H}_{s'} \otimes H_w, \tag{6.2}$$

and $\underline{H}_x\underline{H}_{s'}$ can be computed using the definition of $S(\mathbf{R}')$, i.e., (4.1).

Now let us compare this with the combinatorics of the translation functors. Assume first that $ws = s'w \notin (W'\backslash W)_{\mathrm{short}}$. If $M$ is an $\tilde{\mathfrak{a}}$-module and $E$ is a finite-dimensional $\mathfrak{g}$-module (which we can also view as a finite-dimensional $\tilde{\mathfrak{a}}$-module), then the Poincaré–Birkhoff–Witt Theorem implies the so-called tensor identity $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (E \otimes M) \cong E \otimes U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M$ as $\tilde{\mathfrak{a}}$-modules (in both cases the $\tilde{\mathfrak{a}}$-module structure is given by restriction). This implies that the computation of $[\theta_s^l \Delta(\mathfrak{p}, xw)]$ reduces to the computation of $[\theta_{s'}^l V]$, where $V$ is the indecomposable projective module in $\mathscr{A}^{\mathbf{R}'}$ such that $\Delta(\mathfrak{p}, xw) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$. From Theorem 16 it follows that the result is given by (4.1). Hence it perfectly fits with the computation of (6.2). Finally, let us assume that $ws \in (W'\backslash W)_{\text{short}}$. We have to show that in this case

$$\left[\theta_s^l \Delta(\mathfrak{p}, xw)\right] = \begin{cases} [\Delta(\mathfrak{p}, xws)] + v[\Delta(\mathfrak{p}, xw)], & ws > w, \\ [\Delta(\mathfrak{p}, xws)] + v^{-1}[\Delta(\mathfrak{p}, xw)], & ws < w. \end{cases} \tag{6.3}$$

Since all (proper) standard modules are parabolically induced, from our observation about the parabolic induction and projective functors above it follows that projective functors preserve both $\mathscr{F}(\Delta)$ and $\mathscr{F}(\overline{\Delta})$. By duality, projective functors also preserve both $\mathscr{F}(\nabla)$ and $\mathscr{F}(\overline{\nabla})$. Hence $\theta_s^l \Delta(\mathfrak{p}, xw) \in \mathscr{F}(\Delta)$ and we only have to compute which standard modules occur in the standard filtration of $\Delta(\mathfrak{p}, xw)$ and with which multiplicity. From [22, 4.1] it follows that the multiplicity of $\Delta(\mathfrak{p}, y)\langle k\rangle$ in the standard filtration of $\theta_s^l \Delta(\mathfrak{p}, xw)$ equals the dimension of

$$\mathrm{Hom}_{\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}}\big(\theta_s^l \Delta(\mathfrak{p}, xw), \overline{\nabla}(\mathfrak{p}, y)\langle k\rangle\big).$$

Write $\theta_s^l = \theta_s^{\text{out}}\theta_s^{\text{on}}$, where $\theta_s^{\text{on}}$ and $\theta_s^{\text{out}}$ are the graded translations onto and out of the $s$-wall (see [80, Corollary 8.3]). Adjunction properties [80, Theorem 8.4] give

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}}\big(\theta_s^{\text{out}}\theta_s^{\text{on}} \Delta^{\mathfrak{p}}(xw), \overline{\nabla}(\mathfrak{p}, y)\langle k\rangle\big) \\ &\quad = \mathrm{Hom}_{\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}}\big(\theta_s^{\text{on}} \Delta^{\mathfrak{p}}(xw), \theta_s^{\text{on}}\overline{\nabla}(\mathfrak{p}, y)\langle k+1\rangle\big). \end{aligned}$$

A character argument shows that $\theta_s^{\text{on}}\Delta(\mathfrak{p}, xw)$ is a graded lift of a standard module and $\theta_s^{\text{on}}\overline{\nabla}(\mathfrak{p}, y)$ is a graded lift of a proper standard module on the wall, and a direct calculation (using [80, Theorem 8.1]) gives

$$\mathrm{Hom}_{\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}}\big(\theta_s^{\text{on}} \Delta(\mathfrak{p}, xw), \theta_s^{\text{on}}\overline{\nabla}(\mathfrak{p}, y)\langle k+1\rangle\big) = \begin{cases} \mathbb{C}, & y = xws, k = 0; \\ \mathbb{C}, & y = xw, k = 1, ws > w; \\ \mathbb{C}, & y = xw, k = -1, ws < w; \\ 0, & \text{otherwise.} \end{cases}$$

Formula (6.3) follows and the proof is complete. □

## *6.7. Uniqueness of categorification*

Assume that we are still in the situation of Section 6.6.

**Proposition 35.** *Let $\mathbf{R}'_1$ and $\mathbf{R}'_2$ be two right cells of $W'$ inside the same two-sided cell. Then the categories $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'_1}\}$ and $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'_2}\}$ are equivalent.*

**Proof.** The equivalence between $\mathscr{A}^{\mathbf{R}'}$ and $\mathscr{A}^{\mathbf{R}'_2}$, constructed in Theorem 18 extends in a straightforward way to an equivalence between the categories $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ and $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'_2}\}$. □

## 7. Combinatorics and filtrations of induced cell modules

In this section we first introduce a non-degenerate bilinear form on the induced cell modules and establish a categorical version of it. As a result we get four different distinguished bases in any induced cell modules which we then will interpret via four distinguished classes of objects in the corresponding categorification. Afterwards we describe the resulting refined Kazhdan–Lusztig combinatorics and also introduce a natural filtration on induced cell modules which are induced from a natural counterpart on their categorifications.

### *7.1. Different bases and the combinatorics of induced cell modules*

Assume that we are still in the situation of Section 6.6. Any cell module $S(\mathbf{R}')$ has a unique up to a scalar non-degenerate symmetric bilinear $\mathbb{H}'$-invariant form $\langle\cdot,\cdot\rangle$. We normalise this form such that its categorification is given by Proposition 17. We first state the following easy lemma:

**Lemma 36.** *The induced module $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ has a non-degenerate symmetric $\mathbb{H}$-invariant bilinear form $(\cdot,\cdot)$ with values in $\mathbb{Z}[v, v^{-1}]$ given by*

$$(m \otimes H_x, n \otimes H_y) = \delta_{x,y}\langle m, n\rangle,$$

*for any $x$, $y \in (W'\backslash W)_{\text{short}}$ and $m$, $n \in S(\mathbf{R}')$.*

**Proof.** The form is obviously symmetric and non-degenerate, since so is $\langle\cdot,\cdot\rangle$. It is left to show the $\mathbb{H}$-invariance. Let $s \in S \subset W$ and $m, n, x, y$ as above. For the rest of the proof we set $X = (m \otimes H_x H_s, n \otimes H_y)$ and $Y = (m \otimes H_x, n \otimes H_y H_s)$.

Assume first that $xs, ys \in (W'\backslash W)_{\text{short}}$. If $xs > x$ and $ys > y$, then $xs \neq y$, $x \neq ys$ and hence $X = (m \otimes H_{xs}, n \otimes H_y) = \delta_{xs,y}\langle m, n\rangle = 0$, and $Y = (m \otimes H_x, n \otimes H_{ys}) = \delta_{x,ys}\langle m, n\rangle = 0$. If $xs > x$ and $ys < y$, then $X = (m \otimes H_{xs}, n \otimes H_y) = \delta_{xs,y}\langle m, n\rangle$, and $Y = (m \otimes H_x, n \otimes H_{ys} + (v^{-1} - v)H_y) = \delta_{x,ys}\langle m, n\rangle = \delta_{xs,y}\langle m, n\rangle$ (as $x \neq y$, and $xs = y$ if and only if $x = ys$). If $xs < x$ and $ys > y$, then the argument is analogous (by symmetry). If $xs < x$ and $ys < y$, then $X = (m \otimes H_{xs} + (v^{-1} - v)H_x, n \otimes H_y) = (v^{-1} - v)\delta_{x,y}\langle m, n\rangle$, and $Y = (m \otimes H_x, n \otimes H_{ys} + (v^{-1} - v)H_y) = (v^{-1} - v)\delta_{x,y}\langle m, n\rangle$.

Now let us assume $xs \notin (W'\backslash W)_{\text{short}}$, $ys \in (W'\backslash W)_{\text{short}}$. We write $xs = s'x$ and get $X = (mH_{s'} \otimes H_x, n \otimes H_y) = \delta_{x,y}\langle m, n\rangle = 0$. On the other hand, $Y = (m \otimes H_x, n \otimes H_y H_s) = 0$ as $x \notin y, ys$.

Finally let us assume $xs \notin (W'\backslash W)_{\text{short}}$, $ys \notin (W'\backslash W)_{\text{short}}$. We write $xs = s'x$ and $ys = ty$, where $s', t \in W'$ are simple reflections. Then $X = (mH_{s'} \otimes H_x, n \otimes H_y) \neq 0$ implies $x = y$, and then also $s' = t$. The same holds if $Y = (m \otimes H_x, nH_t \otimes H_y) \neq 0$; and both terms have the same value, namely, $\langle mH_t, n\rangle = \langle m, nH_t\rangle$, since $\langle\cdot,\cdot\rangle$ is $\mathbb{H}'$-invariant. The statement of the lemma follows. □

The involution $h' \mapsto \overline{h'}$ on $\mathbb{H}'$ restricts to an involution on any right cell module and is on the other hand itself the restriction of the involution $h \mapsto \bar{h}$ on $\mathbb{H}$. Therefore, we get an involution

$$S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H} \to S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H},$$
$$m \otimes h \mapsto \overline{m \otimes h} := \overline{m} \otimes \bar{h}.$$

For $(x, w) \in \mathbb{I}(\mathbf{R}')$ we define the *Kazhdan–Lusztig element* $\underline{H}_x \boxdot H_w$ as the unique self-dual element in $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$, satisfying

$$\underline{H}_x \boxdot H_w \in \underline{H}_x \otimes H_w + \sum_{(x',w')\in\mathbb{I}(\mathbf{R}')} v\mathbb{Z}[v]\underline{H}_{x'} \otimes H_{w'}.$$

The existence and uniqueness of such elements is obtained by standard arguments (see e.g. [74, Theorem 2.1]).

The induced module $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ has then four distinguished bases:

- The *Kazhdan–Lusztig basis* (or short *KL basis*) given by the Kazhdan–Lusztig elements $\underline{H}_x \boxdot H_w$, where $x \in \mathbf{R}'$ and $w \in (W'\backslash W)_{\text{short}}$.
- The *Kazhdan–Lusztig-standard basis* (or short *KL-s basis*) given by the elements $\Delta_{x,w} = \underline{H}_x \otimes H_w$, where $x \in \mathbf{R}'$ and $w \in (W'\backslash W)_{\text{short}}$.
- The *dual Kazhdan–Lusztig basis* (or short *dual KL basis*) which is the dual of the KL-basis with respect to the form $(\cdot, \cdot)$.
- The *dual Kazhdan–Lusztig-standard basis* (or short *dual KL-s basis*).

These bases have the following categorical interpretation:

**Theorem 37** *(Combinatorics). The isomorphism $\Psi_{\mathbf{R}'}$ from* (6.1) *defines the following correspondences:*

| | | |
|---|---|---|
| *KL-s basis* | $\leftrightarrow$ | *isoclasses of standard lifts of standard modules,* |
| *KL basis* | $\leftrightarrow$ | *isoclasses of standard lifts of indecomposable projectives,* |
| *dual KL-s basis* | $\leftrightarrow$ | *isoclasses of standard lifts of proper costandard modules,* |
| *dual KL basis* | $\leftrightarrow$ | *isoclasses of standard lifts of simple modules.* |

**Proof.** Let $(x, w) \in \mathbb{I}(\mathbf{R}')$. The isomorphism class $[\Delta(\mathfrak{p}, xw)]$ is mapped to $\Delta_{x,w}$ by definition, hence the first statement of the theorem is obvious. Note that for $w = e$, the module $\Delta(\mathfrak{p}, x)$ is always projective and $\Delta_{x,e} = \underline{H}_x \boxdot H_e = \underline{H}_x \otimes H_e$. This provides the starting point for an induction argument which proves the remaining part of the theorem.

Before we do the induction argument we have to recall a few facts. First recall that for $s \in S$ the functor $\theta_s^l$ sends projectives to projectives, since it is left adjoint to an exact functor. The usual weight argument also shows that if $ws \in (W'\backslash W)_{\text{short}}$ and $ws > w$, then

$$\theta_s^l P(\mathfrak{p}, xw) \cong P(\mathfrak{p}, xws) \bigoplus_{(y,z)\neq(x,w)} a_{y,z} P(\mathfrak{p}, yz), \tag{7.1}$$

at least if we forget the grading. Since the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ is by definition a subcategory of $\mathcal{O}$, we could take the projective cover $P(xw) \in \mathcal{O}_0^{\mathbb{Z}}$ of $P(\mathfrak{p}, xw) \in \mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$, and the decomposition (7.1) is controlled by that of $\theta_s^l P(xw)$. In particular, it is of the form as stated in (7.1) (even as graded modules).

Assume now that the statement of the theorem is true for some $(x, w) \in \mathbb{I}(\mathbf{R}')$ and let $s \in W$ such that $ws > w$ and $ws \in (W'\backslash W)_{\text{short}}$. From Theorem 34 we know that $\theta_s^l P(\mathfrak{p}, xw)$ corresponds to $H := (\underline{H}_x \boxdot H_w)\underline{H}_s$ under $\Psi_{\mathbf{R}'}$. In particular

$$H = \underline{H}_x \otimes H_{ws} + \sum_{(x',w') \neq (x,ws)} \beta_{(x',w')}(v) \underline{H}_{x'} \otimes H_{w'},$$

where $\beta_{x',w'}(v) \in \mathbb{Z}[v]$. From (7.1) and the explanation afterwards we get then that $P(\mathfrak{p}, xw)$ corresponds to

$$H' := (\underline{H}_x \boxdot H_w)\underline{H}_s - \beta_{x',w'}(0)\underline{H}_{x'} \otimes H_{w'}.$$

Note that $H'$ can be characterised as the unique self-dual element with the property that $H' \in \underline{H}_x \otimes H_{ws} + \sum_{(x',w')} v\mathbb{Z}[v]\underline{H}_{x'} \otimes H_{w'}$. The same characterisation holds for the element $\underline{H}_x \boxdot H_{ws}$. Hence $\underline{H}_x \boxdot H_{ws}$ is mapped to $[P(\mathfrak{p}, xws)]$ under the isomorphism $\Psi_{\mathbf{R}'}$ and the second part of the theorem follows.

It is not difficult to verify that the bilinear form $(\cdot,\cdot)$ has again a categorical version as in Proposition 17. In particular, the isomorphism classes of simple modules are dual to the ones of indecomposable projective modules. Finally, the proper costandard modules form a dual basis to the standard modules thanks to the duality on $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ and the usual Ext-orthogonality between standard and proper costandard modules [22, Theorem 3]. The theorem follows. □

**Example 38.** Let $W = S_3 = \langle s, t\rangle$ and $W' = \langle s\rangle \cong S_2 \times S_1$. Then $(W'\backslash W)_{\text{short}} = \{e, t, ts\}$. Choose the right cell $\mathbf{R}'$ of $W'$ corresponding to the (longest) element $s$. The categorification $\mathscr{C}^{\mathbf{R}'}$ is then equivalent to the category of graded $R = \mathbb{C}[x]/(x^2) = \operatorname{End}_{\mathfrak{gl}(2)}(\mathrm{P}(s \cdot 0))$ as in Example 9, and $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\} \cong \mathcal{O}_0^{\mathfrak{p}\text{-pres}}$ from Section 6.4. The module $\Delta(\mathfrak{p}, se)$ is projective, hence $\Delta(\mathfrak{p}, se) = P(\mathfrak{p}, se)$. A direct calculation shows that the projective module $P(\mathfrak{p}, st)$ has a standard-filtration of length two, with $\Delta(\mathfrak{p}, st)$ as a quotient, and $\Delta(\mathfrak{p}, se)$ as a submodule; whereas $P(\mathfrak{p}, sts)$ has a standard filtration with $\Delta(\mathfrak{p}, sts)$ occurring as a quotient, $\Delta(\mathfrak{p}, st)$ as a subquotient, and $\Delta(\mathfrak{p}, se)$ as a submodule (see the detailed example in [58, Section 9]). On the combinatorial side, the standard basis element $\underline{H}_s \otimes H_e$ is a self-dual KL-basis element. The element $\underline{H}_s \otimes H_t + v\underline{H}_s \otimes H_e$ is a KL-basis element. Now, $\underline{H}_s \otimes \underline{H}_{ts} = \underline{H}_s \otimes (H_{ts} + v(H_t + H_s) + v^2 H_e)$ is self-dual and equal to $\underline{H}_s \otimes H_{ts} + v\underline{H}_s \otimes H_t + \underline{H}_s \otimes H_e + v^2\underline{H}_s \otimes H_e$. Hence subtracting $\underline{H}_s \otimes H_e$ gives $\underline{H}_s \boxdot H_{ts} = \underline{H}_s \otimes H_{ts} + v\underline{H}_s \otimes H_t + v^2\underline{H}_s \otimes H_e$.

## 7.2. *Stratifications of induced modules*

Let us come back to the examples in Section 6.2 and assume $W = S_n$ with parabolic subgroup $W'$. Let $\mu$ be the composition of $n$ which defines $W'$ and let $\lambda$ be the corresponding partition. Consider again the permutation module $\mathcal{M} = \mathcal{M}^\lambda$ and the irreducible cell module $S(\lambda)$, which specialises to the irreducible Specht module $S^\lambda$ corresponding to $\lambda$. This is naturally a submodule of $\mathcal{M}^\lambda$. Over the complex numbers, however, $\mathcal{M}^\lambda$ is completely reducible and contains $S(\lambda)^{\mathbb{C}}$ as a unique direct summand. Furthermore, over the complex numbers, any finite-dimensional (right) $\mathbb{H}^{\mathbb{C}}$-module $M$ has a decomposition into isotypic components. This special feature is however not independent of the ground field (as the Specht module is only indecomposable but not irreducible in general), in particular it is not an integral phenomenon. However, there is a natural filtration of $\mathcal{M}$ by Specht modules which always exists (see e.g. [51, 4.10 Corollary]). The purpose of this subsection is to give a very natural categorical construction of a somewhat coarser filtration on all induced cell modules. The idea is to use the notion of Gelfand–Kirillov-dimension.

Consider the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$. The objects of this category are certain $\mathfrak{gl}_n$-modules. Any such module $M$ has a well-defined *Gelfand–Kirillov-dimension* $\operatorname{GKdim}(M)$. Recall the following easy facts:

**Lemma 39.**

(1) *For any* $s \in S \subset W$ *we have* $\mathrm{GKdim}(\theta_s^l M) \leqslant \mathrm{GKdim}(M)$.
(2) $\mathrm{GKdim}(M) = \max\{\mathrm{GKdim}(L_j)\}$, *where* $L_j$ *runs through the composition factors of* $M$.

**Proof.** See for example [33, Lemmas 8.6, 8.8 and 8.7(1)]. □

For any positive integer $j$ we define $(\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}})_{\leqslant j}$ to be the full subcategory of $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$ consisting of all modules which have Gelfand–Kirillov dimension at most $j$. From the lemma above it follows that this subcategory is closed under taking submodules, quotients and extensions, and also stable under translations through walls. Therefore, we have a filtration of the $\mathbb{H}$-module $[\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}]$. For simplicity we relabel the filtration such that we have:

$$\{0\} \subsetneq \big[\big(\mathcal{O}\big\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\big\}_0^{\mathbb{Z}}\big)_1\big] \subsetneq \big[\big(\mathcal{O}\big\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\big\}_0^{\mathbb{Z}}\big)_2\big] \subsetneq \cdots \subsetneq \big[\big(\mathcal{O}\big\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\big\}_0^{\mathbb{Z}}\big)_r\big] = \big[\mathcal{O}\big\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\big\}_0^{\mathbb{Z}}\big].$$

The set of partitions of $n$ is ordered via the so-called dominance ordering which we denote by $\trianglerighteq$. Given two partitions $\nu = \nu_1 \geqslant \nu_2 \geqslant \cdots$ and $\mu = \mu_1 \geqslant \mu_2 \geqslant \cdots$ we have $\nu \trianglerighteq \mu$ if and only if $\sum_{j=1}^{i} \nu_j \geqslant \sum_{j=1}^{i} \mu_j$ for any $i \geqslant 1$. The simple composition factors of the module $\mathcal{M}^\lambda$ are all of the form $S(\mu)$, where $\lambda \trianglelefteq \mu$ (see e.g. [51, 4.10, Exercise 1] or [69, Corollary 2.4.7]).

The following result is the technical formulation of a fact which is quite easy to describe: For every induced cell module $\mathbb{H}$-module $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ we have a corresponding categorification, hence an attached category $\mathscr{C}$, of modules over some Lie algebra. The Gelfand–Kirillov dimension induces a filtration on $\mathscr{C}$ that corresponds to a filtration of $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ which is an analogue of the Specht filtration of the induced cell module given by the dominance ordering. More precisely we have the following:

**Theorem 40.** *Assume that we are in the setup of Section* 6.6. *For* $i \geqslant 0$ *set*

$$Q_i = \big\{v \in S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}\colon\ \Psi_{\mathbf{R}'}(v) \in \big[\big(\mathcal{O}\big\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\big\}_0^{\mathbb{Z}}\big)_i\big]\big\}.$$

*Then we have:*

(i) $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_r = S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$ *is a filtration of the induced cell module* $S(\mathbf{R}') \otimes_{\mathbb{H}'} \mathbb{H}$.
(ii) *Assume,* $S(\lambda)$ *occurs in the* $i$*th and* $S(\mu)$ *in the* $j$*th filtration in* (i)*, respectively. Then* $\lambda \lhd \mu$ *implies* $i < j$ *(in other words: if* $\lambda \lhd \mu$*, then* $S(\lambda)$ *occurs earlier than* $S(\mu)$ *as a subquotient of* (i)*).*
(iii) *All subquotients of the filtration* (i) *are direct sums of Specht modules.*
(iv) *In the permutation module* $\mathcal{M}^\lambda$ *the Specht submodule* $S(\lambda)$ *coincides with* $Q_1$ *(i.e., it is given by the subcategory of modules of the minimal possible Gelfand–Kirillov-dimension from* $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}_0^{\mathbb{Z}}$*).*

**Proof.** The statement (i) follows from Theorem 34 and the definitions. To prove (ii) recall that there is Joseph's explicit formula (see e.g. [33, 10.11(2)])

$$2\,\mathrm{GKdim}\big(L(w \cdot 0)\big) = n(n-1) - \sum_i \mu_i(\mu_i - 1), \tag{7.2}$$

where $\mu$ is the shape of the tableaux associated with $w \in S_n$ via the Robinson–Schensted correspondence (in particular, simple modules in the same right cell have the same Gelfand–Kirillov dimension). Hence the statement (ii) follows from Lemma 41 below, since for two partitions $\mu$ and $\nu$ of $n$ we have $\sum_i \mu_i(\mu_i - 1) < \sum_i \nu_i(\nu_i - 1)$ if and only if $\sum_i \mu_i^2 < \sum_i \nu_i^2$.

**Lemma 41.** *Let $\mu$ and $\nu$ be partitions of $n$ and $l$ be the maximum of the lengths of the partition. Then $\mu \lhd \nu$ implies $\sum_{i=1}^n \mu_i^2 < \sum_{i=1}^n \nu_i^2$.*

**Proof.** If $l = 2$ then $2(\mu_1^2 + \mu_2^2) = (\mu_1 + \mu_2)^2 + (\mu_1 - \mu_2)^2 < (\nu_1 + \nu_2)^2 + (\nu_1 - \nu_2) = 2(\nu_1^2 + \nu_2^2)$. We will do induction on $l$. Without loss of generality assume $\mu_i \neq \nu_i$ for $1 \leqslant i \leqslant l$. Choose now $i$ minimal such that $\mu_i < \nu_i$, but $\mu_{i+1} > \nu_{i+1}$ and set $m := \min\{\mu_i - \nu_i, \nu_{i+1} - \mu_{i+1}\}$. It is easy to check that we get a new partition $\sigma$, where $\sigma_i = \mu_i - m$, $\sigma_{i+1} = \mu_{i+i} + m$ and $\sigma_j = \mu_j$ for all other $j$. Note that $\sigma_k = \nu_k$ for some $k \in \{i, i+1\}$. So we may apply the induction hypothesis to the partitions $\sigma$ and $\nu$ with the common part removed. On the other hand, $(\mu_i, \mu_{i+1}) \lhd (\sigma_i, \sigma_{i+1})$ satisfies the assumption of the lemma, hence $\mu_i^2 + \mu_{i+1}^2 < \sigma_i^2 + \sigma_{i+1}^2$ and so $\sum_{j=1}^l \mu_j^2 < \sum_{j=1}^l \sigma_j^2 < \sum_{j=1}^l \nu_j^2$. □

From (ii) and [26, Theorem 5.1] it follows that the indexing partitions of the Specht modules occurring in a fixed subquotient of the filtration from (i) are not comparable in the right order. This implies (iii). The claim (iv) follows immediately from (ii) and [69, Corollary 2.4.7] (see the remark before the formulation of the theorem). Theorem 40 follows. □

**Remark 42.** The Gelfand–Kirillov-dimension of $L(w \cdot 0)$ is closely related to the value of Lusztig's **a**-function on $W$, see for example [50, Section 20] for the latter. In fact, $\mathbf{a}(w_0 w) + \mathrm{GKdim}(L(w \cdot 0)) = l(w_0)$, see for example [4, Theorem 2.6]. The **a**-function also agrees with the dimension of the corresponding Springer fibre [48, Theorem 24.8].

**Remark 43.** Using [69, Corollary 2.4.7] and [51, 4.10 Corollary] one can construct the following natural integral filtration of the permutation module $\mathcal{M}^\lambda$: For the first step of the filtration we take the submodule $S(\lambda)$ (note again that with respect to the dominance order the partition $\lambda$ is minimal amongst those partitions which index the subquotients of $\mathcal{M}^\lambda$). To construct the second step in the quotient we take the direct sum of all Specht modules, whose partitions are minimal elements in the dominance order among all other partitions which occur; and so on. For $n \leqslant 6$ the constructed *dominance order filtration* will coincide with the one given by Theorem 40(i). However, already for $n = 7$ one gets that the filtration given by Theorem 40(i) is a proper refinement of the dominance order filtration. For example if we take $n = 7$ and the permutation module $\mathcal{M}^{(1^n)}$ (this permutation module is isomorphic to the regular representation of the Hecke algebra), it turns out that the dominance order filtration contains a step in which the subquotients are Specht modules corresponding to the partitions $(5, 1, 1)$ and $(4, 3)$. However, $5^2 + 1^2 + 1^2 = 27 \neq 25 = 4^2 + 3^2$ and so by (7.2) these Specht modules occur in different layers of the filtration given by Theorem 40(i).

## 8. An alternative categorification of the permutation module

In this section we propose an alternative categorification of the permutation parabolic modules. The connection to the categorification from Section 6.2.2 is not completely obvious (but can be made precise using [57, 6.4–6.5]).

Let $W'$ be a subgroup of $W$ and let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be an integral weight with stabiliser $W'$ with respect to the dot-action. The isomorphism classes of the Verma modules in $\mathcal{O}_\lambda$ are exactly given by the $M(x \cdot \lambda)$, where $x \in (W/W')_{\mathrm{short}}$.

For any simple reflection $s \in S$, the *twisting functor* $T_s : \mathcal{O} \to \mathcal{O}$ (see Section 3.4) preserves blocks, in particular induces $T_s : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$. The most convenient description (for our purposes) of these functors is given in [43] in terms of partial coapproximation: Let $M \in \mathcal{O}_\lambda$ be projective. Let $M' \subset M$ be the smallest submodule such that $M/M'$ has only composition factors of the form $L(x \cdot \lambda)$, where $sx > x$. Then $M \mapsto M'$ defines a functor $T_s$ from the additive category of projective modules in $\mathcal{O}_\lambda$ to $\mathcal{O}_\lambda$. This functor extends in a unique way to a right exact functor $T_s : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ (for details see [43]). From this definition of $T_s$ it is immediately clear that this functor is gradable. More precisely, we have the following:

**Lemma 44.** *(See [21, Proposition 5.1].) For any simple reflection $s \in W$ and integral weight $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$, the twisting functor $T_s : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ is gradable. A graded lift is unique up to isomorphism and shift in the grading.*

**Proposition 45.** *Let $s \in S$.*

(1) *The twisting functor $T_s$ is right exact, and exact when restricted the subcategory $\mathcal{V}_\lambda$ of $\mathcal{O}_\lambda$ of modules having a filtration with subquotients isomorphic to Verma modules.*
(2) *One can choose a graded lift $\mathbf{T}_s$ satisfying the following properties:*

$$\begin{aligned}&\big[\mathbf{T}_s M(x\cdot\lambda)\big]\\ &\quad= \begin{cases} [(M(sx\cdot\lambda))] + (v^{-1}-v)[(M(x\cdot\lambda))] & \text{if } sx < x,\ sx \in W/W'_{\mathrm{short}},\\ [(M(sx\cdot\lambda))] & \text{if } sx > x,\ sx \in (W/W')_{\mathrm{short}},\\ v^{-1}[(M(x\cdot\lambda))] & \text{if } sx \notin (W/W')_{\mathrm{short}}.\end{cases}\end{aligned} \tag{8.1}$$

(3) *There is an isomorphism of (left) $\mathbb{Z}[W]$-modules*

$$\begin{aligned}\Psi_\lambda : \mathbb{Z}[W] \otimes_{\mathbb{Z}[W']} \mathbb{Z} &\longrightarrow \big[\mathcal{D}^b(\mathcal{O}_\lambda)\big],\\ x \otimes 1 &\longmapsto \big[M(x\cdot\lambda)\big],\end{aligned}$$

*where the $\mathbb{Z}[W']$-structure on $\mathbb{Z}$ is trivial, and the $\mathbb{Z}[W]$-structure on the right-hand side is induced by the action of the left derived twisting functors $\mathcal{L}T_s$.*

**Proof.** The first statement follows directly from [2, Lemma 2.1]. If we forget the grading (and put $v = 1$), then the second statement follows directly from [1, Theorem 6.2, Definition 5.1(ii)] and implies the last statement. For the graded setup we refer to the proof of [21, Proposition 5.2]. □

## 9. Remarks on Schur–Weyl dualities

For completeness we would like to formulate here a categorical version of the Schur–Weyl duality generalising the approach of [21]. Complete proofs and also a geometric interpretation in terms of generalised Steinberg varieties will appear in [76].

### 9.1. For permutation parabolic modules

We assume again the setup of Section 6.2. Let $\lambda, \mu \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral. If $F : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is a projective functor, then it induces a homomorphism $F^G : [\mathcal{O}_\lambda] \to [\mathcal{O}_\mu]$. Since finite direct sums of projective functors are again projective functors, they form a monoid. On the other hand, the composition of two projective functors (if defined) is again a projective functor. The same holds if we work in the graded setup with graded translation functors between the graded versions $\mathcal{O}^{\mathbb{Z}}_\lambda$ and $\mathcal{O}^{\mathbb{Z}}_\mu$ of $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ (see [6]). This means we have the additive category of (graded) projective functors from $\mathcal{O}^{\mathbb{Z}}_\lambda$ to $\mathcal{O}^{\mathbb{Z}}_\mu$ with its complexified split Grothendieck group $[\text{projective functors}: \mathcal{O}_\lambda \to \mathcal{O}_\mu]^{\mathbb{C}}_{\oplus}$.

**Theorem 46.** *With the notation from Section* 6.2 *we have the following: There is an isomorphism of* $\mathbb{C}[v, v^{-1}]$*-modules*

$$\Psi_{\lambda,\mu}\colon [\textit{projective functors}: \mathcal{O}_\lambda \to \mathcal{O}_\mu]^{\mathbb{C}}_{\oplus} \cong \mathrm{Hom}_{\mathbb{H}^{\mathbb{C}}}\big(\mathcal{M}^\lambda, \mathcal{M}^\mu\big),$$
$$F \mapsto \Psi_\mu^{-1} F^G \Psi_\lambda.$$

The latter result is true for any reductive complex Lie algebra $\mathfrak{g}$. In the following we assume however $\mathfrak{g} = \mathfrak{sl}_n$. For any Young subgroup $S_\lambda$ of $S_n$ we **pick** some integral weight $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ where $W_\lambda \cong S_\lambda$. Let $\Lambda$ be the set of all these $\lambda$'s. For any positive integer $d$ let $\Lambda(d)$ denote the subset of $\Lambda$ whose elements correspond to partitions with at most $d$ rows. The complexified Grothendieck group of all projective functors from $\bigoplus_{\lambda\in\Lambda(d)} \mathcal{O}^{\mathbb{Z}}_\lambda$ to $\bigoplus_{\lambda\in\Lambda(d)} \mathcal{O}^{\mathbb{Z}}_\lambda$ has also a multiplication induced from the composition of projective functors which induces a ring structure. Let $\mathrm{Func}(d)$ denote the complexification of this Grothendieck ring of all projective functors from $\bigoplus_{\lambda\in\Lambda(d)} \mathcal{O}^{\mathbb{Z}}_\lambda$ to $\bigoplus_{\lambda\in\Lambda(d)} \mathcal{O}^{\mathbb{Z}}_\lambda$. Finally let $\mathbf{S}^{\mathbb{C}}_{\mathbb{Z},v}(d,n) = \mathrm{End}_{\mathbb{H}}(\bigoplus_{\lambda\in\Lambda(d)} M^\lambda)$ be the (generic) Schur algebra attached to the numbers $d, n$. Then the following holds:

**Theorem 47.** *There is an isomorphism of* $\mathbb{C}[v, v^{-1}]$*-algebras*

$$\mathrm{Func}(d) \cong \mathbf{S}^{\mathbb{C}}_{\mathbb{Z},v}(d,n).$$

The double centraliser property (see [51, Theorem 4.19]) of the Hecke algebra $\mathbb{H}^{\mathbb{C}}$ for the symmetric group $S_n$ is an isomorphism

$$\mathbb{H}^{\mathbb{C}} \cong \mathrm{End}_{\mathbf{S}^{\mathbb{C}}_{\mathbb{Z},v}(d,n)}\Bigg(\bigoplus_{\lambda\in\Lambda} \mathcal{M}^\lambda\Bigg).$$

It is well known (see [2, Theorem 3.2]) that twisting functors commute naturally (in the sense of [39]) with translation functors. From Proposition 45 we know that the permutation parabolic modules can be categorified via certain singular blocks of category $\mathcal{O}$ together with the action of the twisting functors. Together with the remarks of this section one can deduce the following categorical version of the double centraliser property: *The left derived functors of the graded versions of twisting functors categorify the above action of the Schur algebra and commute naturally with projective functors.*

*9.2. For sign parabolic modules*

Here we get the analogous result using Koszul duality. Translation functors should be replaced by the so-called Zuckerman functors and twisting functors should be replaced by Irving's shuffling functors. For the Koszul duality of these functors see [57, Section 6] and [68].

## 10. Properties of $\mathscr{X} := \mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ in case of type $A$

This section describes in more detail the categories $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$, which were used to categorify induced cell modules, in the special case where $\mathfrak{g} = \mathfrak{sl}_n$. We will describe projective–injective modules, the associated Serre functor and show that the categories are always Ringel self-dual.

From now on we assume that we are in the situation of Section 6.6 and will use the notation introduced there. Additionally we assume that the Lie algebra $\mathfrak{g}$ is of type $A$.

We fix a right cell $\mathbf{R}'$ of $W'$ and for simplicity put

$$\mathscr{X} := \mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}''}\}_0.$$

For $(x, w) \in \mathbb{I}(\mathbf{R}')$ we denote by $L^{\mathscr{X}}(xw)$ the simple object of $\mathscr{X}$, which corresponds to $x$ and $w$. We also have the corresponding standard module $\Delta^{\mathscr{X}}(xw)$, proper standard module $\overline{\Delta}^{\mathscr{X}}(xw)$, indecomposable projective module $P^{\mathscr{X}}(xw)$, and indecomposable injective module $I^{\mathscr{X}}(xw)$. We further denote by $T^{\mathscr{X}}(xw)$ the indecomposable tilting module in $\mathscr{X}$ whose standard filtration starts with a submodule $\Delta^{\mathscr{X}}(xw)$ (see [22, 4.2] for its existence and properties). We denote by $w_0'$ the longest element in $W' \subset W$ and $\overline{w} = w_0' w_0$ the longest element in $(W' \backslash W)_{\mathrm{short}}$.

*10.1. Irving-type properties*

The following theorem is a generalisation of both [30, Main result, Proposition 4.3].

**Theorem 48.** *Let $(x, w) \in \mathbb{I}(\mathbf{R}')$. Then the following conditions are equivalent:*

(a) *$L^{\mathscr{X}}(x, w)$ occurs in the socle of some standard module from $\mathscr{X}$.*
(b) *$L^{\mathscr{X}}(x, w)$ occurs in the socle of some proper standard module from $\mathscr{X}$.*
(c) *$L^{\mathscr{X}}(x, w)$ occurs in the socle of some tilting module from $\mathscr{X}$.*
(d) *$P^{\mathscr{X}}(x, w)$ is injective.*
(e) *$P^{\mathscr{X}}(x, w)$ is tilting.*
(f) *$xw \in W$ belongs to the same right cell $\tilde{\mathbf{R}}$ of $W$ as $x\overline{w}$.*

**Remark 49.** As $\mathbf{R}'$ is a right cell of $W'$, we have that with our fixed $\overline{w}$ all the $y\overline{w}$, where $y$ runs through $\mathbf{R}'$, are in the same right cell of $W$. We denote this right cell by $\tilde{\mathbf{R}}$, see the condition (f) above.

**Proof of Theorem 48.** Since the parabolic induction is exact, consequence (I) from Section 5.2, and the definition of proper standard modules as induced simple modules, implies that $\overline{\Delta}^{\mathscr{X}}(xw)$ is a submodule of $\Delta^{\mathscr{X}}(xw)$, hence (b) ⇒ (a). Since any standard module has a proper standard filtration we also have (a) ⇒ (b).

Analogously, as $\Delta^{\mathscr{X}}(xw) \subset T^{\mathscr{X}}(xw)$ and tilting modules have standard filtrations, the equivalence (a) $\Leftrightarrow$ (c) is clear.

Consequence (I) from Section 5.2 implies that for each $x \in \mathbf{R}' \subset W' \subset W$ the module $\Delta^{\mathscr{X}}(x\overline{w})$ is both standard and costandard, hence tilting, and that the socle of $\Delta^{\mathscr{X}}(x\overline{w})$ is isomorphic to $L^{\mathscr{X}}(x\overline{w})$. Let $\theta$ be a projective functor and $\theta'$ be its adjoint. For any $(y, w) \in \mathbb{I}(\mathbf{R}')$ we have

$$\mathrm{Hom}_{\mathscr{X}}\big(L^{\mathscr{X}}(yw), \theta\Delta^{\mathscr{X}}(x\overline{w})\big) = \mathrm{Hom}_{\mathscr{X}}\big(\theta' L^{\mathscr{X}}(yw), \Delta^{\mathscr{X}}(x\overline{w})\big).$$

Since projective functors respect the right order (Proposition 15), the latter space can be nonzero only if $yw \geqslant_{\mathsf{R}} x\overline{w}$ in the right order. Since $\overline{w}$ is the longest element in $(W'\backslash W)_{\text{short}}$ and $\mathbf{R}'$ is a right cell, it follows that $yw$ is in the same right cell than $x\overline{w}$. From the proof of Theorem 34 (namely, from the formula (6.3)) it follows that, translating the tilting module $\Delta^{\mathscr{X}}(x\overline{w})$ inductively through the walls, we obtain, as direct summands, all indecomposable tilting modules in $\mathscr{X}$. The equivalence (c) $\Leftrightarrow$ (f) follows.

A module which is projective and injective, is in particular tilting (since it has a standard and a proper costandard filtration). On the other hand, a tilting module has by definition a standard filtration and a proper costandard filtration, but by the construction described above even a costandard filtration and a proper standard filtration. Hence the dual module of a tilting module is again tilting. By weight arguments, it is isomorphic to the original tilting module. Hence a projective tilting module is also injective and so (d) $\Leftrightarrow$ (e).

By Proposition 20, the category $\mathscr{A}^{\mathbf{R}'}$ has a simple projective module. Using this and [19, Theorem 1] one shows that $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ has a simple projective module (this statement also follows from consequence (III) and [31, 3.1]). Translating this module out of the wall one gets that there is at least one indecomposable projective module in $\mathscr{X}$ which is also injective. As we have seen already, this module must be then of the form $P^{\mathscr{X}}(xw)$ for some $(x, w) \in \mathbb{I}(\mathbf{R}')$ such that $xw \in \tilde{\mathbf{R}}$. Applying to $P^{\mathscr{X}}(xw)$ projective functors we get that $P^{\mathscr{X}}(yu)$ is both projective and injective for all $(y, u) \in \mathbb{I}(\mathbf{R}')$ such that $yu \in \tilde{\mathbf{R}}$. Hence, finally, (d) $\Leftrightarrow$ (f). □

**Remark 50.** The following statements from Theorem 48 do not require the additional assumption that $\mathfrak{g}$ is of type $A$: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (f), (d) $\Leftrightarrow$ (e) $\Rightarrow$ $(c)$. We use that $\mathfrak{g}$ is of type $A$ when we refer to [31, 3.1] in the last paragraph of the proof (in particular, using [31] the complete statement of Theorem 48 extends to some other special cases treated in [31], but not to the general case because of the counterexample from [31, 5.1]). We believe, however, that the whole Theorem 48 holds for arbitrary type, but do not have a complete argument. Basically, to complete the proof for arbitrary type one has to show that $\mathscr{X}$ always contains a projective–injective module.

### *10.2. Double centraliser and the centre*

Recall that an algebra $R$ has the *double centraliser property* with respect to an $R$-module $M$, if there is an algebra isomorphism

$$R \cong \mathrm{End}_{\mathrm{End}_R(M)}(M).$$

If now $R$ has a double centraliser property with respect to a module $M$ and $\mathscr{C} \cong \text{mod-}R$, then we also say that $\mathscr{C}$ has the double centraliser property (with respect to the image of $M$ under the equivalence). We call a module $M \in \text{mod-}R$ projective–injective, if it is both projective and injective. If it is a direct sum of non-isomorphic indecomposable projective–injective modules, exactly one from each isomorphism class, then we call the module a *full basic projective–injective module*.

The following statement is a generalisation of [60, Theorem 5.2(ii)]:

**Proposition 51.** *The category $\mathscr{X}$ satisfies the double centraliser property with respect to any full basic projective–injective module.*

To prove the statement we first need a generalisation of [60, Lemma 4.7]:

**Lemma 52.** *Let $\mathbf{R}'$ be a right cell in $W' \subset W$ and $x \in \mathbf{R}'$. Then the socle of $\Delta^{\mathscr{X}}(xe) \in \mathscr{X}$ is simple.*

**Proof.** Theorem 48 ensures the existence of projective–injective tilting modules in $\mathscr{X}$. Translation functors preserve the category of projective–injective tilting modules. Any translation of a module with standard filtration has a standard filtration. Further, from the combinatorics in the proof of Theorem 34 it follows that any standard module can be translated to a module, whose standard filtration contains $\Delta^{\mathscr{X}}(xe)$ as a subquotient. Therefore, any projective–injective tilting module can be translated to some projective–injective tilting module $T$, whose standard filtration contains $\Delta^{\mathscr{X}}(xe)$ as a subquotient. The module $T$ contains $\Delta^{\mathscr{X}}(xe)$ as a submodule since $\Delta^{\mathscr{X}}(xe)$ is projective, and hence $T^{\mathscr{X}}(xe)$ is a direct summand of $T$. Thus $T^{\mathscr{X}}(xe)$ is projective–injective, in particular, has simple socle. As $\Delta^{\mathscr{X}}(xe) \hookrightarrow T^{\mathscr{X}}(xe)$, the claim follows. □

**Proof of Proposition 51.** Let $x \in \mathbf{R}'$. Then the inclusion $\Delta^{\mathscr{X}}(xe) \hookrightarrow T^{\mathscr{X}}(xe)$ extends to a short exact sequence of the following form:

$$0 \to \Delta^{\mathscr{X}}(xe) \to T^{\mathscr{X}}(xe) \to K \to 0, \tag{10.1}$$

where $K \in \mathscr{F}(\Delta^{\mathscr{X}})$. The module $T^{\mathscr{X}}(xw)$ is projective–injective by Lemma 52. Projective functors are exact and preserve $\mathscr{F}(\Delta^{\mathscr{X}})$. Hence, applying to (10.1) appropriate projective functors and taking the direct sum over all $(y, w) \in \mathbb{I}(\mathbf{R}')$, we get an exact sequence

$$0 \to P^{\mathscr{X}} \to M_1 \to M_2 \to 0,$$

where $P^{\mathscr{X}}$ is a projective generator for $\mathscr{X}$, while $M_1$ is projective–injective and $M_2 \in \mathscr{F}(\Delta^{\mathscr{X}})$. By Theorem 48, the injective envelope of $M_2$ is projective. The statement now follows from [47, Theorem 2.8]. □

**Corollary 53.** *Let $Q^{\mathscr{X}}$ denote a full basic projective–injective module of $\mathscr{X}$. Then the centres of $\mathscr{X}$ and $\mathrm{End}_{\mathscr{X}}(Q^{\mathscr{X}})$ are isomorphic.*

**Proof.** The centre of $\mathscr{X}$ is isomorphic to the centre of $\mathrm{End}_{\mathscr{X}}(P^{\mathscr{X}})$, where $P^{\mathscr{X}}$ is a projective generator. Thanks to Proposition 51, the centres of $\mathrm{End}_{\mathscr{X}}(P^{\mathscr{X}})$ and $\mathrm{End}_{\mathscr{X}}(Q^{\mathscr{X}})$ are isomorphic (see [60, Theorem 5.2(ii)] for details). □

Because of Proposition 35 we can now assume that there is a parabolic subgroup, $W''$ of $W'$ such that $\mathbf{R}'$ contains the element $w_0''w_0'$, where $w_0'$ and $w_0''$ denote the longest elements in $W'$ and $W''$, respectively. Set $S'' = W'' \cap S'$. Let $\mathfrak{q}$ denote the parabolic subalgebra of $\mathfrak{g}$, which contains $\mathfrak{b}$ and such that the Weyl group of its Levi factor is $W''$. Both $\mathscr{X}$ and $\mathcal{O}_0^{\mathfrak{q}}$ are subcategories of the category $\mathcal{O}_0$ for $\mathfrak{g}$ and we have the following result:

**Lemma 54.**

(i) *$\mathscr{X}$ is a subcategory of $\mathcal{O}_0^{\mathfrak{q}}$.*
(ii) *The projective–injective modules in $\mathscr{X}$ and $\mathcal{O}_0^{\mathfrak{q}}$, considered as objects in $\mathcal{O}_0$, coincide.*

**Proof.** For $s \in S''$ we obviously have $sw_0''w_0' > w_0''w_0'$. As $w_0''w_0' \in \mathbf{R}'$ and $\mathbf{R}'$ is a right cell, it follows that $sxw > xw$ for all $s \in S''$ and $(x, w) \in \mathbb{I}(\mathbf{R}')$. In particular, $L^{\mathscr{X}}(xw) \in \mathcal{O}_0^{\mathfrak{q}}$ $(x, w) \in \mathbb{I}(\mathbf{R}')$, which implies (i).

Consider now the element $w_0''w_0'\overline{w} = w_0''w_0'w_0'w_0 = w_0''w_0$. Then the module $P^{\mathfrak{q}}(w_0''w_0)$ is projective–injective in $\mathcal{O}_0^{\mathfrak{q}}$ (see [44] for details). As a $\mathfrak{g}$-module, the module $P^{\mathscr{X}}(w_0''w_0'\overline{w})$ has simple top $L(w_0''w_0 \cdot 0)$. Hence, as $P^{\mathscr{X}}(w_0''w_0'\overline{w}) \in \mathcal{O}_0^{\mathfrak{q}}$ by (i), we get that $P^{\mathscr{X}}(w_0''w_0'\overline{w})$ is a quotient of $P^{\mathfrak{q}}(w_0''w_0)$.

On the other hand, from the existence of a simple projective module in $\mathcal{O}^{\mathfrak{q}}$ (see [31, 3.1]) it follows that $P^{\mathfrak{q}}(w_0''w_0)$ is a direct summand of some translation of $L(w_0''w_0)$ (see consequence (III) in Section 5.2), which, in turn, is the simple quotient of $P^{\mathscr{X}}(w_0''w_0'\overline{w})$. Hence $P^{\mathfrak{q}}(w_0''w_0)$ is a quotient of some translation of $P^{\mathscr{X}}(w_0''w_0'\overline{w})$. As $P^{\mathfrak{q}}(w_0''w_0)$ has simple top, it follows that the only possibility is that $P^{\mathfrak{q}}(w_0''w_0)$ is a quotient of $P^{\mathscr{X}}(w_0''w_0'\overline{w})$.

The above implies that the $\mathfrak{g}$-modules $P^{\mathfrak{q}}(w_0''w_0)$ and $P^{\mathscr{X}}(w_0''w_0'\overline{w})$ are isomorphic, and the claim (ii) follows by applying projective functors. □

Lemma 54 implies the following result:

**Proposition 55.** *The algebra* $\mathrm{End}_{\mathscr{X}}(Q^{\mathscr{X}})$ *is symmetric. The centre of $\mathscr{X}$ is isomorphic to the centre of $\mathcal{O}_0^{\mathfrak{q}}$.*

**Proof.** By Lemma 54, the first statement is nothing else than [60, Theorem 4.6]. The second statement is given by Corollary 53. □

**Remark 56.** Recall our assumption that $\mathfrak{g}$ is of type $A$. In this case the centre of $\mathcal{O}_0^{\mathfrak{q}}$ has a nice geometric description: it is isomorphic to the cohomology algebra of a certain Springer fibre. This is described in [15] and [81].

### *10.3. The Serre functor for $\mathcal{D}^p(\mathscr{X})$*

Let $\mathcal{D}^p(\mathscr{X})$ denote the full subcategory of $\mathcal{D}^b(\mathscr{X})$ given by perfect complexes, that is, complexes which are quasi-isomorphic to finite complexes of projective objects from $\mathscr{X}$.

Recall that if $\mathscr{C}$ is a $\Bbbk$-linear additive category with finite-dimensional homomorphism spaces, then a *Serre functor* on $\mathscr{C}$ is an auto-equivalence $\mathrm{F}$ of $\mathscr{C}$ such that the bifunctors $(X, Y) \mapsto \mathscr{C}(X, \mathrm{F}Y)$ and $(X, Y) \mapsto \mathscr{C}(Y, X)^*$ are isomorphic (here, $*$ denotes the ordinary duality of vector spaces).

Denote by $\mathrm{Coapp}_{\mathbf{R}'} : \mathscr{X} \to \mathscr{X}$ the functor of partial coapproximation with respect to a fixed full basic projective–injective module $Q^{\mathscr{X}}$. It is constructed as follows (see [43, 2.5] for details): If $M \in \mathscr{X}$, then $\mathrm{Coapp}_{\mathbf{R}'}(M)$ is obtained from $M$ by first maximally extending $M$ using simple modules, which do not occur in the top of $Q^{\mathscr{X}}$, and afterwards deleting all occurrences of such modules in the top part.

**Proposition 57.** *The functor* $\mathcal{R}\mathrm{Coapp}^2_{\mathbf{R}'}$ *is a Serre functor for* $\mathcal{D}^p(\mathscr{X})$.

**Proof.** Thanks to Propositions 51 and 55, we are in the situation of [60, Theorem 3.7], except that the category $\mathscr{X}$ usually does not have finite global dimension. Using [27, Proposition 20.5.5(i)] (see [60, 4.3] for details), one can get rid of the assumption of finite global dimension by working with the category of perfect complexes instead of the bounded derived category. □

*10.4. Ringel self-duality of* $\mathscr{X}$

Consider the module

$$T^{\mathscr{X}} = \bigoplus_{(x,w)\in\mathbb{I}(\mathbf{R}')} T^{\mathscr{X}}(xw).$$

Based on [65], the algebra $\mathrm{End}_{\mathscr{X}}(T^{\mathscr{X}})$ is called the *Ringel dual* of the algebra $\mathrm{End}_{\mathscr{X}}(P^{\mathscr{X}})$, see [22]. If $\mathbf{R}' = \{e\}$, the category $\mathscr{X}$ is *Ringel self-dual* (that is, $\mathrm{End}_{\mathscr{X}}(P^{\mathscr{X}}) \cong \mathrm{End}_{\mathscr{X}}(T^{\mathscr{X}})$) by [73, Section 7]. If $\mathbf{R}' = \{w_0'\}$, the category $\mathscr{X}$ is Ringel self-dual by [23, Theorem 3], see also [60, Proposition 4.9]. The following theorem generalises both these results:

**Theorem 58.** *The category* $\mathscr{X}$ *is Ringel self-dual for each* $\mathbf{R}'$.

**Proof.** We retain all assumptions and notation from Section 10.2 (especially the ones before Lemma 54). To prove this statement we will construct an endofunctor $\mathrm{F} = \mathrm{F}_2\mathrm{F}_1$ on $\mathcal{O}$ which maps $P^{\mathscr{X}}$ to $T^{\mathscr{X}}$ preserving the endomorphism ring. The functor $\mathrm{F}_2$ is an auto-equivalence of $\mathcal{O}$ which is easy to describe: Since $\mathfrak{g}$ is assumed to be of type $A$, the Dynkin diagram has an involution which is on any $A_n$-component just the flip mapping the $i$th vertex to the $(n+1-i)$th vertex. This involution induces an automorphism $\phi$ of $\mathfrak{g}$, and $F_2$ maps a module $M$ to $M^{\phi}$, the same vector space with the $\mathfrak{g}$-action twisted by $\phi$. The functor $\mathrm{F}_1$ is more complicated. Let $w_0 = s_{i_1}s_{i_2}\cdots s_{i_{l(w_0)}}$ be a reduced expression. Consider the twisting functor

$$\mathrm{T} := \mathrm{T}_{i_1}\mathrm{T}_{i_2}\cdots\mathrm{T}_{i_{l(w_0)}} : \mathcal{O} \to \mathcal{O}$$

(see Section 3.4 and then [2,73] for details). This functor is right exact, commutes with projective functors, and $\mathcal{L}\mathrm{T}$ is a self-equivalence of $\mathcal{D}^b(\mathcal{O})$, see [2]. We define $\mathrm{F}_1 = \mathcal{L}_{l(w_0'')}\mathrm{T}$ and claim that $\mathrm{F} = \mathrm{F}_2\mathrm{F}_1$ does the required job. The arguments to deduce this are very much along the lines of [60, Proposition 4.4]. Here we just outline the arguments leaving to work out the details (following [60, Proposition 4.4]) to the reader.

Denote by $\mathfrak{k}$ the semisimple part of the Levi factor of $\mathfrak{q}$. Then each finite-dimensional simple $\mathfrak{k}$-module $M$ comes along with its so-called BGG-resolution (see [8]), that is, a resolution by (direct sums of) Verma modules. The (exact) parabolic induction functor from $\mathfrak{k}$ to $\mathfrak{a}$ can be applied to the BGG-resolution, and we obtain a resolution of $M' = \mathcal{U}(\mathfrak{a}) \otimes_{\mathcal{U}(\mathfrak{k}+(\mathfrak{b}\cap\mathfrak{a}))} M$ by (direct sums of) Verma $\mathfrak{a}$-modules.

From the uniqueness result, Remark 14 and [31, 3.1], there is a simple, projective object in $\mathscr{A}^{\mathbf{R}'}$ which is parabolically induced from a simple finite-dimensional $\mathfrak{k}$-module (see also Section 11.4 for details). This is the $M'$ we want to consider. Its resolution by (direct sums of) Verma $\mathfrak{a}$-modules gives rise to a resolution of the projective module $\Delta(\mathfrak{p}, M') := U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M'$ by (direct sums of) Verma $\mathfrak{g}$-modules. Then $\mathcal{L}\mathrm{T}\Delta(\mathfrak{p}, M') = \mathcal{L}_{l(w_0'')}\mathrm{T}\Delta(\mathfrak{p}, M')$ (following the arguments in [60]), and the latter becomes a dual parabolic Verma module.

From the construction of $\mathscr{X}$ we know that each projective in $\mathscr{X}$ can be obtained as a direct summand of some translation of $\Delta(\mathfrak{p}, M')$. The previous paragraph says that $\Delta(\mathfrak{p}, M')$ has an (explicitly given) resolution by (direct sums of) Verma $\mathfrak{g}$-modules.

From [1] and [2] (see also Proposition 45), we have explicit formulas for the action of the functor T on Verma modules. Using these formulas one shows by a direct computation that F maps $\Delta(\mathfrak{p}, M')$ to a tilting module from $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$. As F commutes (up to the automorphism defining $\mathrm{F}_2$) with projective functors, it follows that F sends projective modules from $\mathscr{X}$ to tilting modules from $\mathscr{X}$. Finally, as both $\mathrm{F}_2$ and T are equivalences, it also follows that F preserves the endomorphism ring. This completes the proof. □

## 11. The rough structure of generalised Verma modules

In this section we want to apply the results of the paper to determine the 'rough structure' of generalised Verma modules. We will start by giving some background information.

### 11.1. Basic questions

Let $\mathfrak{g}$ be a Lie algebra with the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be a parabolic subalgebra of $\mathfrak{g}$, and $V$ a simple $\mathfrak{p}$-module, annihilated by the nilpotent radical of $\mathfrak{p}$. The module

$$\Delta(\mathfrak{p}, V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$$

is usually called the *generalised Verma module* (or simply GVM) associated with $\mathfrak{p}$ and $V$.

If $\mathfrak{p} = \mathfrak{b}$ and $V$ is one-dimensional, then we get an ordinary Verma module. If $\mathfrak{p}$ is arbitrary, but $V$ still finite-dimensional, then the resulting module is a parabolic generalised Verma module as studied for example in [32].

The most basic questions about GVMs are:

- In which case is $\Delta(\mathfrak{p}, V)$ irreducible?
- If $\Delta(\mathfrak{p}, V)$ is not irreducible: which simple $\mathfrak{g}$-modules occur as subquotients of $\Delta(\mathfrak{p}, V)$, and what are their multiplicity (in case this makes sense at all)?

These questions were studied in special cases by many authors, we refer the reader to [42, Introduction] for a more detailed survey. The answer to the questions above is also of interest in theoretical physics, since the structure of generalised Verma modules determines the structure of Verma modules for (super)algebras appearing in conformal field theory (see for example [70] for an affine setup).

The most general known facts in the theory of generalised Verma modules are the main results of [42] (based on [62]) under the assumption that the module $V$ has minimal possible annihilator: [42, Theorem 22] gives an explicit criterion for the irreducibility of $\Delta(\mathfrak{p}, V)$; and [42, Theorem 23] describes what is called the *rough* structure of $\Delta(\mathfrak{p}, V)$, defined as follows: each $\Delta(\mathfrak{p}, V)$ has a unique simple quotient, denoted by $L(\mathfrak{p}, V)$. If $V'$ is another simple $\mathfrak{p}$-module with minimal annihilator, then [42, Theorem 23] says that the multiplicity $[\Delta(\mathfrak{p}, V) : L(\mathfrak{p}, V')]$ is well defined (in particular, it is always finite); an explicit formula for its computation in terms of Kazhdan–Lusztig polynomials is also provided. In general, this does not describe the structure of $\Delta(\mathfrak{p}, V)$ completely: $\Delta(\mathfrak{p}, V)$ may have many other subquotients, it even might be of infinite length (because of the example due to Stafford, see [78, Theorem 4.1]). No reasonable information about this so-called *fine structure* of $\Delta(\mathfrak{p}, V)$ is known so far.

In what follows we want to explain how one can drop the restriction on the minimality of the annihilator of $V$ by applying the techniques we have developed so far in this paper. Following the approach proposed in [62] and developed further in [42], an essential part of the argument is an improved answer to the so-called 'Kostant's problem' for certain simple and induced modules.

### *11.2. Kostant's problem*

Let $\mathfrak{g}$ be a complex reductive finite-dimensional Lie algebra. For every $\mathfrak{g}$-module $M$ we have the bimodule $\mathscr{L}(M, M)$ of all $\mathbb{C}$-linear endomorphisms of $M$, on which the adjoint action of $U(\mathfrak{g})$ is locally finite (that means any vector $f \in \mathscr{L}(M, M)$ lies inside a finite-dimensional subspace which is stable under the adjoint action defined as $x.f(m) = x(f(m)) - f(xm)$ for $x \in \mathfrak{g}$, $m \in M$). Initiated by [35], *Kostant's problem* became the standard terminology for the following question concerning an arbitrary $\mathfrak{g}$-module $M$:

*Is the natural injection $U(\mathfrak{g})/\mathrm{Ann}(M) \hookrightarrow \mathscr{L}(M, M)$ surjective?*

Although there are several classes of modules for which the answer is known to be positive (see [35,53] and references therein), a complete answer to this problem seems to be far away—not even for simple highest weight modules the problem is solved. There is even an instance of a simple highest weight module for which the answer is negative. The details of such an example (which was first mentioned in [35, 9.5]) will be discussed in Section 11.5.

In the following we will show that for certain simple and induced modules which appeared earlier in the present paper, the answer to Kostant's problem is positive. Moreover, Theorem 61 shows that for a simple highest weight module in type $A$ the answer only depends on the left cell associated with the indexing element of the Weyl group.

### *11.3. General assumptions*

For the rest of the paper let $\mathfrak{g}$ be an arbitrary complex reductive finite-dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{a}'$ and nilpotent radical $\mathfrak{n}$. Finally, denote by $\mathfrak{a}$ the semisimple part of $\mathfrak{a}'$. Then $\mathfrak{a}$ is a semisimple finite-dimensional Lie algebra with induced triangular decomposition. We **assume** that $\mathfrak{a}$ is of type $A$, that means:

*We assume that $\mathfrak{a} \cong \bigoplus_{i \in I} \mathfrak{sl}_{k_i}$, where $I$ is some finite set and $k_i \in \{2, 3, \dots\}$.*

*11.4. Kostant's problem for the IS-module*

We have to start with some technical statements which involve explicit definitions of certain weights. Assume $\mathfrak{a} \cong \mathfrak{sl}_k$ for some $k \geqslant 2$. We consider $\mathfrak{a}$ as a subalgebra of $\mathfrak{gl}_k$ in the canonical way. In particular, all simple highest weight $\mathfrak{gl}_k$-modules are simple highest weight $\mathfrak{a}$-modules via restriction. Let $\alpha_i$ ($i = 1, \dots, k-1$) be the list of simple roots of $\mathfrak{a}$ in the usual ordering. As before we denote the Weyl group of $\mathfrak{a}$ by $W'$ and note that $W' \cong S_k$. Let $r$ be a partition of $k$ of length $s$, that is, $r = (r_1, \dots, r_s) \in \mathbb{N}^s$, $r_1 + \cdots + r_s = k$ and $r_1 \geqslant r_2 \geqslant \cdots$. Set $r_0 = 0$. Depending on $r$, we define $\pi = \{\alpha_i \colon i \in I\}$, where

$$I = \{1, 2, \dots, r_1 - 1\} \cup \{r_1 + 1, \dots, r_1 + r_2 - 1\} \cup \cdots$$
$$\cup \{r_1 + \cdots + r_{s-1} + 1, \dots, r_1 + \cdots + r_s - 1\}$$

and then the $\mathfrak{gl}_k$-weight $\nu$ as

$$\nu = (b_1, \dots, b_k), \quad b_{r_{j-1}+m} = r_j - m, \text{ for } m \in \{1, \dots, r_j\}.$$

In [31, Proposition 3.1], it is shown that $\nu$ is the only $\pi$-dominant weight in $W'\nu$ and hence the corresponding simple highest weight module $L(\nu - \rho)$ is a projective simple module in the parabolic category $\mathcal{O}$ associated with $\pi$. This is what we call the *simple projective IS-module*.

Denote by $\mu$ the weight such that $\mu + \rho$ is the dominant weight in $W'\nu$. To proceed we have to construct Weyl group elements $x_\nu$, $x_\mu$ such that $L(\nu - \rho)$ is the translation of $L(x_\nu \cdot 0)$ to $\mathcal{O}_\mu$, and $L(\mu)$ is the translation of $L(x_\mu \cdot 0)$ to $\mathcal{O}_\mu$.

Let $\xi = (\xi_1, \dots, \xi_k)$ be a $k$-tuple of non-negative integers. We convert the coordinates of $\xi$ into the sequence $(\eta_1, \dots, \eta_k)$ without repetitions, which differs from $(k-1, k-2, \dots, 0)$ only by a permutation, and satisfies $\eta_j < \eta_k$ if $j < k$ or $\xi_j < \xi_k$ (in practice we first replace all occurring zeros from the left to the right by $0, 1, \dots, m_0$, where $m_0 + 1$ is the total number of zeros in $\xi$, then all occurring ones by $m_0 + 1$, $m_0 + 2$, etc.). Applying this procedure to $\nu$ and $\mu + \rho$ we obtain weights $\nu' + \rho$ and $\mu' + \rho$ from the orbit $W'(k-1, k-2, \dots, 0)$. Then $\nu' + \rho = x_\nu(k-1, k-2, \dots, 0)$ and $\mu' + \rho = x_\mu(k-1, k-2, \dots, 0)$ for some $x_\nu$, $x_\mu \in W'$. By construction, $L(x_\nu \cdot 0)$ and $L(x_\mu \cdot 0)$ are simple highest weight modules with the desired properties described above.

**Example 59.** Consider the case where $\mathfrak{a} = \mathfrak{sl}_4$ with the three simple reflections $s_1, s_2, s_3$, where $s_1$ and $s_3$ commute. The partition $r = (2, 2)$ gives $\pi = \{\alpha_1, \alpha_3\}$ and $\nu = (1, 0, 1, 0)$. Then $\mu + \rho = (1, 1, 0, 0)$, $\nu' + \rho = (2, 0, 3, 1)$ and $\mu' + \rho = (2, 3, 0, 1)$. Hence $x_\nu = s_2 s_1 s_3$, $x_\mu = s_1 s_3$.

Our crucial technical observation is the following

**Lemma 60.** *$x_\nu$ and $x_\mu$ belong to the same left cell.*

**Proof.** We prove this by induction on $k$. If $k = 2$, there is nothing to prove. If $r_1 > r_2$, then in both, $\nu' + \rho$ and $\mu' + \rho$, the element $k - 1$ stays at the leftmost place, and the induction hypothesis applies to the remaining parts of $\nu' + \rho$ and $\mu' + \rho$. The only tricky part is therefore the case $r_1 = r_2$, which may in fact easily be reduced inductively to the case $r_1 = r_2 = \cdots = r_s$. Consider first the case $s = 2$. Then $x_\nu$ is the following permutation on $\{0, \dots, k-1\}$, which we consider as an element of $W'$:

$$x_\nu = \begin{pmatrix} 0 & 1 & \dots & r_1 - 1 & r_1 & r_1 + 1 & r_1 + 2 & \dots & m \\ m-1 & m-3 & \dots & 2 & 0 & m & m-2 & \dots & 1 \end{pmatrix},$$

where $m = r_1 + r_2 - 1$. Since $0 < 2 < m$, we can apply Knuth transformation (see [69, Definition 3.6.8]) to interchange 0 and $m$ in the second row of the above permutation. This can be continued until $m$ appears at the second left position, where the procedure stops. Since the Knuth transformations preserve left cells [69, Lemma 3.6.9], the new permutation $\sigma$ will be in the same left cell as $x_\nu$. Now in $\sigma$ and $x_\mu$ the first two elements coincide. So, applying the induction hypothesis to the remaining parts, we get that $x_\mu$ and $x_\nu$ are in the same left cell. The case $s > 2$ follows now inductively. We omit the details. □

The following result is crucial an its proof is based on the categorification results from Section 8:

**Theorem 61.**

(i) *The modules $L(\nu - \rho)$ and $L(\mu)$ have the same annihilator.*

(ii) *For any projective functor $\theta$ we have*

$$\dim \operatorname{Hom}_{\mathfrak{a}}\big(L(\nu - \rho), \theta L(\nu - \rho)\big) = \dim \operatorname{Hom}_{\mathfrak{a}}\big(L(\mu), \theta L(\mu)\big).$$

(iii) *Kostant's problem has a positive answer for $L(\mu)$ and $L(\nu - \rho)$.*

(iv) *For any projective functor $\theta$ we have*

$$\dim \operatorname{Hom}_{\mathfrak{a}}\big(L(x \cdot 0), \theta L(x \cdot 0)\big) = \dim \operatorname{Hom}_{\mathfrak{a}}\big(L(y \cdot 0), \theta L(y \cdot 0)\big)$$

*whenever $x$ and $y$ are in the same left cell of $W'$. In particular, Kostant's problem has a positive answer for $L(x \cdot 0)$ if and only if it has a positive answer for $L(y \cdot 0)$.*

**Proof.** The annihilators of the modules $L(\nu')$ and $L(\mu')$ coincide since $x_\nu$ and $x_\mu$ belong to the same left cell by Lemma 60. The statement (i) is now obtained by translating to the wall and applying [33, 5.4(3)].

We will see later that the statement (ii) follows from (iv). To prove (iv) we have to work much harder. The principal idea is the following: Given two simple modules in the same block, and indexed by elements in the same left cell, then Proposition 45 tells us that they are connected via twisting functors. These twisting functors commute with projective functors and therefore they can be used to obtain estimates for the dimensions of homomorphism spaces, which would result in (iv). Let us make this idea precise. Assume $x \in W$ and $s$ is a simple reflection such that $sx < x$, and the elements $sx$ an $x$ belong to the same left cell. For the twisting functor $\mathrm{T}_s : \mathcal{O} \to \mathcal{O}$ we have:

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{a}}\big(\theta L(x \cdot 0), L(x \cdot 0)\big) &\cong \operatorname{Hom}_{\mathfrak{a}}\big(\mathcal{L}\mathrm{T}_s \theta L(x \cdot 0), \mathcal{L}\mathrm{T}_s L(x \cdot 0)\big) \\ &\cong \operatorname{Hom}_{\mathfrak{a}}\big(\theta \mathcal{L}\mathrm{T}_s L(x \cdot 0), \mathcal{L}\mathrm{T}_s L(x \cdot 0)\big) \end{aligned}$$

$$\cong \mathrm{Hom}_{\mathfrak{a}}\big(\theta \mathrm{T}_s L(x\cdot 0), \mathrm{T}_s L(x\cdot 0)\big)$$
$$\cong \mathrm{Hom}_{\mathfrak{a}}\big(\mathrm{T}_s \theta L(x\cdot 0), \mathrm{T}_s L(x\cdot 0)\big) \tag{11.1}$$

by [2, Corollary 4.2, Theorems 2.2, 6.1, 3.2]. Moreover, we also have $\mathrm{T}_s L(x\cdot 0) \neq 0$ and a short exact sequence

$$0 \to U \longrightarrow \mathrm{T}_s L(x\cdot 0) \xrightarrow{\mathrm{nat}} L(x\cdot 0) \to 0, \tag{11.2}$$

where nat is the evaluation at $L(x\cdot 0)$ of the natural transformation from $\mathrm{T}_s$ to the identity functor, given by [43, Theorem 4], and $U$ is the kernel of nat. Further, the module $U$ is semisimple, and has $L(sx\cdot 0)$ as a simple subquotient with multiplicity one by [2, Section 7]. As $L(x\cdot 0)$ is simple and $s$-infinite, the module $U$ coincides with the maximal $s$-finite submodule of $\mathrm{T}_s L(x\cdot 0)$, see [2, Proposition 5.4] and [43, 2.5 and Theorem 10].

Analogously we have a short exact sequence

$$0 \to U' \longrightarrow \mathrm{T}_s \theta L(x\cdot 0) \longrightarrow \mathrm{nat}\theta L(x\cdot 0) \to 0, \tag{11.3}$$

where $U'$ is just the kernel of nat. As all simple submodules of the socle of $\theta L(x\cdot 0)$ are $s$-infinite, the module $U'$ again coincides with the maximal $s$-finite submodule of $\mathrm{T}_s\theta L(x\cdot 0)$. This implies $U' \cong \theta U$. Now, any non-zero homomorphism $f \in \mathrm{Hom}_{\mathfrak{a}}(\theta L(x\cdot 0), L(x\cdot 0))$ is automatically surjective and gives rise to a diagram as follows (in which the square of solid arrows commutes):

$$\begin{array}{ccccc}
\theta U & \hookrightarrow & \mathrm{T}_s\theta L(x\cdot 0) & \xrightarrow{\mathrm{nat}} & \theta L(x\cdot 0) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \mathrm{T}_s f} & & \downarrow{\scriptstyle f} \\
U & \hookrightarrow & \mathrm{T}_s L(x\cdot 0) & \xrightarrow{\mathrm{nat}} & L(x\cdot 0),
\end{array}$$

inducing the map $f'$. We claim that the map $f'$ restricts to a non-zero map

$$\overline{f}' \in \mathrm{Hom}_{\mathfrak{a}}\big(\theta L(sx\cdot 0), L(sx\cdot 0)\big).$$

We first claim that the cokernel of $f'$ does not contain any simple module $L(z)$, where $z$ is in the same left cell as $x$. By the Snake Lemma, the cokernel of $f'$ embeds into $\theta L(x\cdot 0)$. From Proposition 15(i) it follows that $\theta L(x\cdot 0)$ only has composition factors indexed by $z$'s either in the same right cell as $x$ or in smaller right cells. From the Robinson–Schensted algorithm it is directly clear that smaller right cells intersect the left cell of $x$ trivially. Robinson–Schensted also implies that any given left and right cell inside the same two-sided cell intersect in exactly one point; so the only possible $z$ is $z = x$. Since $L(x\cdot 0)$ is not a composition factor of $U$ by [2, Theorem 6.3(ii)] the claim follows. In particular, $L(sx\cdot 0)$ occurs in the image of $f'$. Let now $L(z\cdot 0)$ be a simple subquotient of $U$. If $z$ belongs to a smaller two-sided cell than $sx$, then the arguments of Theorem 40 imply that the GK-dimension of $L(z\cdot 0)$ is strictly smaller than that of $L(sx\cdot 0)$. Hence $\mathrm{Hom}_{\mathfrak{a}}(\theta L(z\cdot 0), L(sx\cdot 0)) = 0$. If $z$ is in the same left cell as $sx$, then by Proposition 15(i) and Robinson–Schensted the inequality $\mathrm{Hom}_{\mathfrak{a}}(\theta L(z\cdot 0), L(sx\cdot 0)) \neq 0$ is possible only for $z = sx$. This implies $\overline{f}' \neq 0$ and the claim follows. Hence we get the inequality

$$\dim \mathrm{Hom}_{\mathfrak{a}}\big(\theta L(x\cdot 0), L(x\cdot 0)\big) \leqslant \dim \mathrm{Hom}_{\mathfrak{a}}\big(\theta L(sx\cdot 0), L(sx\cdot 0)\big). \tag{11.4}$$

Analogously one obtains the inequality

$$\dim \operatorname{Hom}_{\mathfrak{a}}\big(\theta L(x\cdot 0), L(x\cdot 0)\big) \leqslant \dim \operatorname{Hom}_{\mathfrak{a}}\big(\theta L(tx\cdot 0), L(tx\cdot 0)\big) \tag{11.5}$$

in the case when $sx$ belongs to the smaller left cell than $x$ and $t$ is a simple reflection such that $(st)^3 = e$ and the element $tx$ belongs to the same left cell as $x$.

Since left cell modules are irreducible, using (11.4) and (11.5) inductively one also obtains the opposite inequalities, which implies that

$$\operatorname{Hom}_{\mathfrak{a}}\big(\theta L(x\cdot 0), L(x\cdot 0)\big) = \operatorname{Hom}_{\mathfrak{a}}\big(\theta L(y\cdot 0), L(y\cdot 0)\big) \tag{11.6}$$

if $x$ and $y$ are in the same left cell. By [33, 6.8(3)] for any $\mathfrak{a}$-module $M$ and any simple finite-dimensional $\mathfrak{a}$-module $F$ we have

$$\big[\mathscr{L}(M, M) : F\big] = \operatorname{Hom}_{\mathfrak{a}}(F \otimes M, M). \tag{11.7}$$

Since the modules $L(x\cdot 0)$ and $L(y\cdot 0)$ have the same annihilator by [33, 5.25], the formulas (11.7) and (11.6) imply that Kostant's problem has a positive answer either for both $L(x\cdot 0)$ and $L(y\cdot 0)$ or for none of them. This proves (iv).

By Lemma 60, both $L(\nu-\rho)$ and $L(\mu)$ are obtained by translating two simple modules, indexed by elements from the same left cell, from $\mathcal{O}_0$ to a fixed singular block. Hence, using Proposition 45, the statement (ii) is proved just in the same way as the statement (iv) is proved above.

As in (iv), the statement (ii) implies that Kostant's problem has a positive answer either for both $L(\nu-\rho)$ and $L(\mu)$ or for none of them. Since $\mu$ is dominant, Kostant's problem has a positive answer for $L(\mu)$ by [33, 6.9]. Hence the answer to Kostant's problem for $L(\nu-\rho)$ is positive as well. This completes the proof of Theorem 61. □

### *11.5. A negative answer to Kostant's problem: type $B_2$*

The answer to Kostant's problem is not positive in general. The following negative example was constructed first in [35, 9.5]: Let

$$W' = \{e, s, t, st, ts, sts, tst, stst = tsts\}$$

be the Weyl group of type $B_2$ with the two simple reflections $s$ and $t$ as generators. Then the requirement of Kostant's problem fails for $L(ts)$ and $L(st)$. What goes wrong in our arguments? The right cells of $W'$ are $\{e\}$, $\{s, st, sts\}$, $\{t, ts, tst\}$, $\{stst\}$. In particular, the elements $s$ and $sts$ are both in the same left and in the same right cell. In our arguments we used several times that the intersection of a given left with a given right cell contains at most one element. An easy direct calculation also shows that $\operatorname{Hom}_{\mathcal{O}}(\theta L(s), L(s)) = \mathbb{C} = \operatorname{Hom}_{\mathcal{O}}(\theta L(sts), L(sts))$, whereas $\operatorname{Hom}_{\mathcal{O}}(\theta L(ts), L(ts)) = \mathbb{C}^2$ for $\theta = \theta_s\theta_t\theta_s$. Hence Theorem 61(iv) fails in this case. Even in type $A$, there are now examples with a negative answer, see [61] and further [37].

### *11.6. Coker-categories and their equivalence*

Recall the setup from Section 11.3, in particular that $\mathfrak{a}$ is assumed to be of type $A$. Let $V$ be an arbitrary simple $\mathfrak{a}'$-module. Let $L$ be the $\mathfrak{a}$-module obtained by restriction. For simplicity

$$\text{we assume for the moment that } L \text{ has integral and regular central character} \tag{11.8}$$

and refer to Remark 76 for the general case.

Now, $V$ is determined uniquely by the underlying simple $\mathfrak{a}$-module $L$ and some functional $\eta$ on the centre of $\mathfrak{a}'$. We first construct an admissible category attached to these data.

Let $L(x \cdot 0)$ be a simple highest weight module with the same annihilator as $L$. Without loss of generality we assume that $x$ is contained in a right cell associated with a parabolic subalgebra $\mathfrak{p}$ as in Remark 14 (the latter is possible as $x$ can be chosen arbitrarily in its left cell by [33, 5.25]). By [31, 3.1], there is a block $\mathcal{O}_\mu^{\mathfrak{p}}$ (for some integral weight $\mu \in \mathfrak{h}^*_{\mathrm{dom}}$) which contains exactly one simple (highest weight) module $L(y \cdot \mu)$, and this module is also projective. We assume that $ys < y$ for any simple reflection $s$ such that $s \cdot \mu = \mu$. The module $L(y \cdot \mu)$ is a tensor product of simple highest weight modules over all simple components of $\mathfrak{a}$. Each of the factors has the form $L(\nu)$, where $\nu$ is as in Section 11.4. Because of our assumptions we also have that $L(y \cdot \mu)$ is the translation of $L(y \cdot 0)$ to the $\mu$-wall, and that $x$ and $y$ belong to the same right cell (consequence (III) in Section 5.2).

**Proposition 62.** *There is some projective functor $F : \mathcal{O}(\mathfrak{a}, \mathfrak{a} \cap \mathfrak{b}) \to \mathcal{O}(\mathfrak{a}, \mathfrak{a} \cap \mathfrak{b})$ such that $FL(x \cdot 0) \cong \bigoplus_{i=1}^k L(y \cdot \mu)$ for some finite number $k > 0$.*

**Proof.** First we claim that there is a projective functor $\theta$ such that $L(y \cdot 0)$ occurs as a composition factor in $\theta L(x \cdot 0)$. Indeed, recall that the elements $x$ and $y$ are in the same right cell of $W'$. Consider the basis of simple modules for the categorification of the cell module (corresponding to $x$ and $y$) given by Theorem 16. As cell modules are irreducible in type $A$, there is a projective functor $\theta$ such that $[\theta L(x \cdot 0)]$ has a non-zero coefficient at $[L(y \cdot 0)]$, when expressed with respect to the basis of simple modules. This means exactly that $L(y \cdot 0)$ occurs as a composition factor in $\theta L(x \cdot 0)$.

Let $\theta'$ be the translation to the $\mu$-wall. Then the functor $F = \theta'\theta$ satisfies the requirement of the lemma as the module $L(y \cdot \mu)$ is simple projective and is a unique simple modules in its parabolic block (see the definition of $L(y \cdot \mu)$ and [31, 3.1]). □

We fix $F$ as in Proposition 62 and set $N := FL$.

**Lemma 63.**

(i) $N = FL \neq 0$.
(ii) $\mathrm{Ann}_{U(\mathfrak{a})} N = \mathrm{Ann}_{U(\mathfrak{a})} L(y \cdot \mu) = \mathrm{Ann}_{U(\mathfrak{a})} L(\mu)$.

**Proof.** Since $\mathrm{Ann}\, L = \mathrm{Ann}\, L(x \cdot 0)$, we have

$$\mathrm{Ann}_{U(\mathfrak{a})} N = \mathrm{Ann}_{U(\mathfrak{a})} FL = \mathrm{Ann}_{U(\mathfrak{a})} FL(x \cdot 0)$$

(see [33, 5.4]). The second statement follows then directly from Proposition 62, Theorem 61 and the definition of $L(y \cdot \mu)$. Since $FL(x \cdot 0) \neq 0$, we also have $FL \neq 0$. □

If $\mathfrak{g}$ is any complex Lie algebra and $Q$ a $\mathfrak{g}$-module, then we denote by $\mathrm{Coker}(Q \otimes E)$ the full subcategory of $\mathfrak{g}$-mod, which consists of all modules $M$ having a presentation $X \to Y \twoheadrightarrow M$, where both $X$ and $Y$ are direct summands of $Q \otimes E$ for some finite-dimensional module $E$. In particular, if we choose the Lie algebra to be $\mathfrak{a}$, then we have the two categories $\mathrm{Coker}(L(y \cdot \mu) \otimes E)$ and $\mathrm{Coker}(N \otimes E)$.

**Lemma 64.** *$L(y \cdot \mu)$ is projective in* $\mathrm{Coker}(L(y \cdot \mu) \otimes E)$.

**Proof.** Of course $L(y \cdot \mu)$ is contained in $\mathrm{Coker}(L(y \cdot \mu) \otimes E)$. On the other hand, $L(y \cdot \mu) \in \mathcal{O}_\mu^{\mathfrak{p}}$ is projective. It is even projective in $\mathcal{O}^{\mathfrak{p}}$. The latter is stable under tensoring with finite-dimensional modules, hence contains $\mathrm{Coker}(L(y \cdot \mu) \otimes E)$ as a full subcategory. The statement follows. □

Unfortunately, we do not know how to prove directly that the module $N$ is semisimple. To get around this problem we have to make sure that there is a 'nice' simple subquotient $\overline{N}$ of $N$. Let $G$ be the adjoint functor of $F$. Then the adjunction morphism $a : L \to GFL$ is injective, since $L$ is simple. Since $G$ is exact, there must therefore be a simple subquotient $\overline{N}$ of $N$ such that $G\overline{N}$ contains $L$ as a quotient.

Let $\chi_\mu = \mathrm{Ann}_{Z(\mathfrak{a})} M(\mu)$ be the central character of the Verma module with highest weight $\mu$. Then we denote by $\mathcal{M}(\mu)$ the category of all $\mathfrak{a}$-modules $M$ such that $(\chi_\mu)^n M = 0$ for some $n$ (depending on $M$), i.e., $M$ has generalised central character $\chi_\mu$. With this notation the following holds:

**Lemma 65.** *Let $\theta : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ be an indecomposable projective functor which is not the identity. Then $\theta L(\mu) = 0$, $\theta N = 0$ and $\theta \overline{N} = 0$.*

**Proof.** We first show the statement for $L(\mu)$: if $\theta : \mathcal{O}_\mu \to \mathcal{O}_\mu$ is an indecomposable projective functor, then $\theta L(\mu) \neq 0$ means that $\theta$ is the identity functor. To see this, take the projective cover $P(\mu)$ of $L(\mu)$. Then

$$\mathrm{Hom}_{\mathfrak{a}}\big(P(\mu), \theta L(\mu)\big) = \mathrm{Hom}_{\mathfrak{a}}\big(\theta' P(\mu), L(\mu)\big), \tag{11.9}$$

where $\theta'$ is the adjoint functor of $\theta$. Note that $\theta'$ is an indecomposable projective functor if so is $\theta$. The classification theorem of projective functors gives $\theta' M(\mu) = P(\zeta)$ for some $\zeta$. If we assume the space (11.9) to be non-trivial, then we have $\zeta = \mu$, which forces (by the classification theorem again) $\theta'$ to be the identity functor, and then $\theta$ is the identity functor as well.

Assume therefore (11.9) is trivial, but $\theta L(\mu) \neq 0$. Recall the categorification result of Proposition 45 and extend the scalars to $\mathbb{C}$. Together with Theorem 47 we get that $\theta$ induces an endomorphism of the complexified Grothendieck group of $\mathcal{O}_\mu$. The module $L(\mu)$ has minimal Gelfand–Kirillov dimension and is contained in the categorification of the irreducible (Specht) submodule of $\mathcal{M}^\mu$ corresponding to the partition given by $\mu$. The endomorphism of the parabolic permutation module given by $\theta$ is a homomorphism of the symmetric group which underlies the Hecke algebra $\mathbb{H}$ and restricts to an endomorphism of the irreducible submodule which has to be a multiple $c \in \mathbb{C}$ of the identity. But since $0 \neq \theta L(\mu)$ has at most the same Gelfand–Kirillov dimension as $L(\mu)$ (by Lemma 39), we deduce that $c \neq 0$. On the other hand the fact that both sides of the equality (11.9) are equal to 0 is equivalent to the statement that $L(\mu)$ does not occur as a composition factor in $\theta L(\mu)$, a contradiction. In particular, $c \neq 0$ forces $\theta$ to be the identity functor. Hence the claim is true for $L(\mu)$.

Assume again that $\theta$ is not the identity functor. To see that $\theta N = \theta F L = 0$ we consider the annihilator $\operatorname{Ann}\theta N = \operatorname{Ann}_{\mathcal{U}(\mathfrak{a})}(\theta N)$. By [33, 6.35(1)] we have

$$\mathcal{U}(\mathfrak{a})/\operatorname{Ann}\theta N = \theta^l\theta^r\big(\mathcal{U}(\mathfrak{a})/\operatorname{Ann} N\big), \tag{11.10}$$

where $\mathcal{U}(\mathfrak{a})/\operatorname{Ann} N$ is considered as a $\mathcal{U}(\mathfrak{a})$-bimodule, and $\theta^l$ is the projective functor $\theta$ when considering left $\mathcal{U}(\mathfrak{a})$-modules, whereas $\theta^r$ is the projective functor $\theta$ when considering right $\mathcal{U}(\mathfrak{a})$-modules (see also Section 3.3). On the other hand we have an equality of bimodules

$$\mathcal{U}(\mathfrak{a})/\operatorname{Ann} N = \mathcal{U}(\mathfrak{a})/\operatorname{Ann} L(\mu) = \mathscr{L}\big(L(\mu), L(\mu)\big). \tag{11.11}$$

Here the first equality is [33, 5.4]. The second equality is given by the natural map, since Kostant's problem has a positive answer in this case (Theorem 61(iii)). Putting everything together we get

$$\begin{aligned}\mathcal{U}(\mathfrak{a})/\operatorname{Ann}\theta N &= \theta^l\theta^r\big(\mathcal{U}(\mathfrak{a})/\operatorname{Ann} N\big) = \theta^l\theta^r\mathscr{L}\big(L(\mu), L(\mu)\big)\\ &\cong \mathscr{L}\big(\theta L(\mu), \theta L(\mu)\big) = 0.\end{aligned}$$

For the penultimate isomorphism we refer to [33, 6.33(6)]. It follows that $\theta N = 0$. As $\theta$ is exact and $\overline{N}$ is a subquotient of $N$ we also get that $\theta\overline{N} = 0$. □

**Proposition 66.** *The following holds:*

(i) $\operatorname{Ann}_{U(\mathfrak{a})}\overline{N} = \operatorname{Ann}_{U(\mathfrak{a})} L(y\cdot\mu) = \operatorname{Ann}_{U(\mathfrak{a})} L(\mu)$.
(ii) $\overline{N}$ *is projective in* $\operatorname{Coker}(\overline{N}\otimes E)$.
(iii) *Kostant's problem has a positive solution for the module* $\overline{N}$.

**Proof.** Of course, $\operatorname{Ann}_{U(\mathfrak{a})}\overline{N} \supseteq \operatorname{Ann}_{U(\mathfrak{a})} N$. Let us assume $\operatorname{Ann}_{U(\mathfrak{a})}\overline{N}$ is strictly bigger than $\operatorname{Ann}_{U(\mathfrak{a})} N$. Choose $z$, such that $\operatorname{Ann}_{U(\mathfrak{a})}\overline{N} = \operatorname{Ann}_{U(\mathfrak{a})} L(z\cdot\mu)$. Since $\operatorname{Ann}_{U(\mathfrak{a})} N = \operatorname{Ann}_{U(\mathfrak{a})} L(y\cdot\mu)$ (Lemma 63), it follows that $z$ is strictly smaller than $y$ in the left order. Hence, also strictly smaller than $x$ in the left order. On the other hand (by definition of the modules) $L(y\cdot 0)$ can be obtained as a subquotient in a translation of $L(z\cdot 0)$. Proposition 15(i) tells then that $y \leqslant_{\mathsf{R}} z$. In particular, $y$ is smaller than or equal to $x$ in the twosided order. This contradicts the fact that $z$ should be strictly smaller than $y$ in the left order. The first statement follows.

By definition $\overline{N} \in \operatorname{Coker}(\overline{N}\otimes E)$. Moreover, $\overline{N} \in \mathcal{M}(\mu)$ as $\overline{N}$ is simple [18, Proposition 2.6.8]. Hence we can apply Lemma 65 and obtain that the intersection of $\mathcal{M}(\mu)$ with the additive closure of $\overline{N}\otimes E$ consists just of direct sums of copies of $\overline{N}$. Since $\overline{N}$ is simple, the cokernel of any homomorphism between direct sums of copies of $\overline{N}$ is isomorphic to a direct sum of copies of $\overline{N}$ as well [18, Proposition 2.6.5(iii)]. Hence the intersection of $\mathcal{M}(\mu)$ with $\operatorname{Coker}(\overline{N}\otimes E)$ also consists just of direct sums of copies of $\overline{N}$. This implies that $\overline{N}$ is projective in $\operatorname{Coker}(\overline{N}\otimes E)$.

By Theorem 61 we know that Kostant's problem has a positive answer for the module $L(y\cdot\mu)$. So, by part (i), it would suffice to show that

$$\dim \mathrm{Hom}_{\mathfrak{a}}(\overline{N}, \theta\overline{N}) = \dim \mathrm{Hom}_{\mathfrak{a}}\big(L(y\cdot\mu), \theta L(y\cdot\mu)\big)$$

for all indecomposable projective functors $\theta$. This is true if $\theta$ is not the identity functor (Lemma 65), otherwise Schur's Lemma [18, 2.6.5] does the job. □

Finally we get the following result:

**Theorem 67.** *With the notation from above there is an equivalence of categories*

$$\mathrm{Coker}\big(L(y\cdot\mu)\otimes E\big) \cong \mathrm{Coker}(\overline{N}\otimes E).$$

**Proof.** By Lemma 64 and Proposition 66, $L(y\cdot\mu)$ is projective in $\mathrm{Coker}(L(y\cdot\mu)\otimes E)$ and $\overline{N}$ is projective in $\mathrm{Coker}(\overline{N}\otimes E)$. By Theorem 61(iii), Kostant's problem has a positive solution for $L(y\cdot\mu)$. By Proposition 66, Kostant's problem has a positive solution for $\overline{N}$. Hence the claim follows from Proposition 66 and [42, Theorem 5]. □

### *11.7. Categories of induced modules and their equivalence*

We extend the categories $\mathrm{Coker}(L(y\cdot\mu)\otimes E)$ and $\mathrm{Coker}(\overline{N}\otimes E)$ of $\mathfrak{a}$-modules to categories of $\mathfrak{a}'$-modules by allowing arbitrary scalar actions of the centre of $\mathfrak{a}'$. Abusing notation we denote the resulting categories by the same symbols.

**Lemma 68.** $\mathrm{Coker}(L(y\cdot\mu)\otimes E)$ *and* $\mathrm{Coker}(\overline{N}\otimes E)$ *are both admissible.*

**Proof.** The conditions (L1) and (L3) are clear by definition, so we have only to check the condition (L2). By Lemma 64, $L(y\cdot\mu)$ is projective in $\mathrm{Coker}(L(y\cdot\mu)\otimes E)$ and $\overline{N}$ is projective in $\mathrm{Coker}(\overline{N}\otimes E)$ by Proposition 66. In particular, all modules of the form $L(y\cdot\mu)\otimes E$ and $\overline{N}\otimes E$, where $E$ is finite-dimensional, are projective in the corresponding categories. It follows that both categories $\mathrm{Coker}(L(y\cdot\mu)\otimes E)$ and $\mathrm{Coker}(\overline{N}\otimes E)$ have enough projectives. Now the condition (L2) follows for instance from [3, Section 5]. □

Lemma 68 allows us to consider the category $\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(L(y\cdot\mu)\otimes E)\}$ and the category $\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(\overline{N}\otimes E)\}$. Both categories have a block decomposition with respect to central characters. By [52, Theorem 6.1], these blocks are equivalent to module categories over some standardly stratified algebras (it is easy to see that these algebras are even weakly properly stratified in the sense of [22]). Denote by $\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(L(y\cdot\mu)\otimes E)\}_{\mathrm{int}}$ and $\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(\overline{N}\otimes E)\}_{\mathrm{int}}$ the direct sums of all blocks corresponding to integral central characters. The main result of this section is the following statement:

**Theorem 69.** *There is a blockwise equivalence of categories*

$$\xi:\quad \mathcal{O}\big\{\mathfrak{p}, \mathrm{Coker}(\overline{N}\otimes E)\big\}_{\mathrm{int}} \cong \mathcal{O}\big\{\mathfrak{p}, \mathrm{Coker}\big(L(y\cdot\mu)\otimes E\big)\big\}_{\mathrm{int}},$$

*which sends proper standard modules to proper standard modules.*

**Proof.** To construct a blockwise equivalence it is enough to verify the assumptions of [42, Theorem 5]. The module $L(y \cdot \mu)$ is projective in $\mathscr{C} := \mathrm{Coker}(L(y \cdot \mu) \otimes E)$ (Lemma 64). Hence the induced module $\Delta(\mathfrak{p}, L(y \cdot \mu))$ is both standard and proper standard in $\mathcal{O}\{\mathfrak{p}, \mathscr{C}\}$ for any linear functional on the centre of $\mathfrak{a}'$ which extends the $\mathfrak{a}$-action on $L(y \cdot \mu)$. We pick the linear functional such that the module $\Delta(\mathfrak{p}, L(y \cdot \mu))$ is projective in some regular block of $\mathcal{O}\{\mathfrak{p}, \mathscr{C}\}_{\mathrm{int}}$. It is easy to see that all projective modules in $\mathcal{O}\{\mathfrak{p}, \mathscr{C}\}_{\mathrm{int}}$ can be obtained by translating $\Delta(\mathfrak{p}, L(y \cdot \mu))$. In particular, we have

$$\mathcal{O}\{\mathfrak{p}, \mathscr{C}\}_{\mathrm{int}} \cong \mathrm{Coker}\big(\Delta\big(\mathfrak{p}, L(y \cdot \mu)\big) \otimes E\big).$$

Analogously

$$\mathcal{O}\big(\mathfrak{p}, \mathrm{Coker}(\overline{N} \otimes E)\big)_{\mathrm{int}} \cong \mathrm{Coker}\big(\Delta(\mathfrak{p}, \overline{N}) \otimes E\big)$$

for the same linear functional. To be able to apply [42, Theorem 5] we just have to verify that Kostant's problem has a positive answer for the modules $\Delta(\mathfrak{p}, \overline{N})$ and $\Delta(\mathfrak{p}, L(y \cdot \mu))$. This will be done in the following Lemmas 71 and 72. Hence there is an equivalence of categories $\xi$, and it remains to show that such an equivalence preserves proper standard objects. The partial ordering on the simple modules in $\mathcal{O}\{\mathfrak{p}, \mathscr{C}\}_{\mathrm{int}}$ induces a partial ordering on the simple modules in $\mathcal{O}\{\mathfrak{p}, \mathrm{Coker}(\overline{N} \otimes E)\}_{\mathrm{int}}$, which defines a stratified structure. Since proper standard modules have a categorical definition, they will be sent to proper standard modules by any blockwise equivalence. □

For a finite-dimensional $\mathfrak{g}$-module $E$ we denote by $\tilde{E}$ its underlying $\mathfrak{a}$-module. Let $\tilde{E}_0$ be the direct sum of finite-dimensional $\mathfrak{a}$-submodules of $\tilde{E}$ where the centre of the reductive Lie algebra $\mathfrak{a}'$ acts trivially.

**Lemma 70.** *Kostant's problem has a positive answer for $\Delta(\mathfrak{p}, L(\mu))$.*

**Proof.** The module $\Delta(\mathfrak{p}, L(\mu))$ is a quotient of the dominant Verma module and therefore Kostant's problem is affirmative by [33, 6.9(10)]. □

**Lemma 71.** *Kostant's problem has a positive answer for $\Delta(\mathfrak{p}, L(y \cdot \mu))$.*

**Proof.** For any simple finite-dimensional $\mathfrak{g}$-module $E$ we have

$$\big[U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}\big(\Delta\big(\mathfrak{p}, L(\mu)\big)\big) : E\big] = \dim \mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(\mu)\big) \otimes E, \Delta\big(\mathfrak{p}, L(\mu)\big)\big) \tag{11.12}$$

by Lemma 70 and [33, 6.8(3)]. Since for $\zeta \in \{\mu, y \cdot \mu\}$, the module $\Delta(\mathfrak{p}, L(\zeta))$ is a projective standard module in its corresponding Coker-category, the standard adjointness gives

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(\zeta)\big) \otimes E, \Delta\big(\mathfrak{p}, L(\zeta)\big)\big) &= \mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(\zeta)\big), \Delta\big(\mathfrak{p}, L(\zeta)\big) \otimes E^*\big)\\ &= \mathrm{Hom}_{\mathfrak{a}}\big(L(\zeta), L(\zeta) \otimes E_0^*\big)\\ &= \mathrm{Hom}_{\mathfrak{a}}\big(L(\zeta) \otimes E_0, L(\zeta)\big).\end{aligned}$$

The latter is however independent of the choice of $\zeta$ by Theorem 61, and therefore

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(\mu)\big) \otimes E, \Delta\big(\mathfrak{p}, L(\mu)\big)\big)\\&\quad= \mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(y\cdot\mu)\big) \otimes E, \Delta\big(\mathfrak{p}, L(y\cdot\mu)\big)\big).\end{aligned} \tag{11.13}$$

The modules $L(y \cdot \mu)$ and $L(\mu)$ have the same annihilator (by Theorem 61 again), therefore the modules $\Delta(\mathfrak{p}, L(y\cdot\mu))$ and $\Delta(\mathfrak{p}, L(\mu))$ have the same annihilator by [18, Proposition 5.1.7]. Together with (11.13) and Lemma 70 we deduce that Kostant's problem has a positive answer for $\Delta(\mathfrak{p}, L(y\cdot\mu))$. □

**Lemma 72.** *Kostant's problem has a positive answer for* $\Delta(\mathfrak{p}, \overline{N})$.

**Proof.** Since $\Delta(\mathfrak{p}, \overline{N})$ is a projective standard module in the corresponding Coker-category, as in (11.13) we have

$$\mathrm{Hom}_{\mathfrak{g}}\big(\Delta(\mathfrak{p}, \overline{N}), \Delta(\mathfrak{p}, \overline{N}) \otimes E\big) = \mathrm{Hom}_{\mathfrak{a}}(\overline{N}, \overline{N} \otimes E_0).$$

Recall that $\overline{N}$ and $L(\mu)$ have the same annihilator (Proposition 66), and Kostant's map is surjective in both cases (Theorem 61 and Proposition 66). Together with (11.12) we have

$$\mathrm{Hom}_{\mathfrak{g}}\big(\Delta(\mathfrak{p}, \overline{N}), \Delta(\mathfrak{p}, \overline{N}) \otimes E\big) \cong \mathrm{Hom}_{\mathfrak{g}}\big(\Delta\big(\mathfrak{p}, L(y\cdot\mu)\big), \Delta\big(\mathfrak{p}, L(y\cdot\mu)\big) \otimes E\big).$$

Now, $\Delta(\mathfrak{p}, L(y\cdot\mu))$ and $\Delta(\mathfrak{p}, \overline{N})$ have the same annihilator (Proposition 66 and [18, Proposition 5.1.7]). So, the latter equality and the fact that Kostant's problem has a positive answer for $\Delta(\mathfrak{p}, L(y\cdot\mu))$ imply that Kostant's problem has a positive answer for $\Delta(\mathfrak{p}, \overline{N})$. This completes the proof. □

*11.8. The rough structure of generalised Verma modules: main results*

The equivalence $\xi$ from Theorem 69 induces a bijection between the sets of the isomorphism classes of indecomposable projective modules in the categories

$$\mathscr{Y}_{\overline{N}} = \mathcal{O}\big\{\mathfrak{p}, \mathrm{Coker}(\overline{N} \otimes E)\big\}_{\mathrm{int}} \quad \text{and} \quad \mathscr{Y}_{L(y\cdot\mu)} = \mathcal{O}\big\{\mathfrak{p}, \mathrm{Coker}\big(L(y\cdot\mu) \otimes E\big)\big\}_{\mathrm{int}}.$$

Therefore $\xi$ also induces a bijection

$$\bar{\xi} : \mathrm{Irr}(\mathscr{Y}_{L(y\cdot\mu)}) \to \mathrm{Irr}(\mathscr{Y}_{\overline{N}})$$

between the sets of isomorphism classes of simple objects in $\mathscr{Y}_{L(y\cdot\mu)}$ and $\mathscr{Y}_{\overline{N}}$, respectively. This induces moreover a bijection

$$\hat{\bar{\xi}} : \mathrm{Irr}^{\mathfrak{g}}(\mathscr{Y}_{\overline{N}}) \to \mathrm{Irr}^{\mathfrak{g}}(\mathscr{Y}_{L(y\cdot\mu)})$$

between the sets of isomorphism classes of the simple quotients, as $\mathfrak{g}$-modules, of the modules from $\mathrm{Irr}(\mathscr{Y}_{\overline{N}})$ and $\mathrm{Irr}(\mathscr{Y}_{L(y\cdot\mu)})$, respectively. Each module $X \in \mathrm{Irr}^{\mathfrak{g}}(\mathscr{Y}_{\overline{N}})$ or $\mathrm{Irr}^{\mathfrak{g}}(\mathscr{Y}_{L(y\cdot\mu)})$ has the form $L(\mathfrak{p}, V_X)$ for a uniquely defined simple $\mathfrak{a}'$-module $V_X$.

As a consequence of Theorem 69 we obtain the following result:

**Theorem 73.** *For $X, Y \in \mathrm{Irr}^{\mathfrak{g}}(\mathscr{Y}_{\overline{N}})$ we have the following multiplicity formula in the category of $\mathfrak{g}$-modules:*

$$\big[\Delta(\mathfrak{p}, V_X) : L(\mathfrak{p}, V_Y)\big] = \big[\Delta(\mathfrak{p}, V_{\hat{\xi}(X)}) : L(\mathfrak{p}, V_{\hat{\xi}(Y)})\big].$$

**Proof.** Let $P(X) \in \mathscr{Y}_{\overline{N}}$ be an indecomposable projective, whose head (as a $\mathfrak{g}$-module) is isomorphic to $X$. Then $[\Delta(\mathfrak{p}, V_X) : L(\mathfrak{p}, V_Y)]$ is just the dimension of the homomorphism space from $P(X)$ to the proper standard module in $\mathscr{Y}_{\overline{N}}$ corresponding to $X$ (see [42, Section 5]). Exactly the same holds if we replace $X$ by $\hat{\xi}(X)$ and work with the category $\mathscr{Y}_{L(y\cdot\mu)}$ instead of $\mathscr{Y}_{\overline{N}}$. Since $\xi$ is an equivalence of categories (Theorem 69) sending proper standard objects to proper standard objects, the claim follows. □

**Remark 74** *(Additional remarks to Theorem 73).* Theorem 73 describes only multiplicities of certain simple subquotients of $\Delta(\mathfrak{p}, V_X)$, namely, multiplicities of those simple subquotients, which occur as heads of indecomposable projectives in $\mathscr{Y}_{\overline{N}}$. Following [42] we call this the *rough structure* of $\Delta(\mathfrak{p}, V_X)$. The theorem reduces the question about the rough structure of the module $\Delta(\mathfrak{p}, V_X)$ to the analogous question for the module $\Delta(\mathfrak{p}, V_{\hat{\xi}(X)})$. The latter module is an object of $\mathcal{O}$ and hence the problem can be solved inductively using the Kazhdan–Lusztig combinatorics (see Theorem 37).

Let $L$ be as in Section 11.4. Then the module $\Delta(\mathfrak{p}, L)$ has generalised trivial integral central character, and $L(\mathfrak{p}, L)$ is the simple top of some indecomposable projective module, $P$ say, in $\mathscr{Y}_{\overline{N}}$. Let $\mathscr{X}$ be the block of $\mathscr{Y}_{\overline{N}}$ corresponding to the trivial central character. It contains $P$ by construction. By Theorem 69 and Section 6.6, simple modules in $\mathscr{X}$ are (bijectively) indexed by $(x, w) \in \mathbb{I}(\mathbf{R}')$. Therefore, there is a pair $(x, w)$ for each $\Delta(\mathfrak{p}, L)$ in $\mathscr{X}$. Theorem 73 allows us to formulate the following irreducibility criterion for generalised Verma modules:

**Theorem 75.** *Let $(x, w)$ be the pair associated with $\Delta(\mathfrak{p}, L)$. Then the module $\Delta(\mathfrak{p}, L)$ is irreducible if and only if $w = \overline{w}$.*

**Proof.** Theorem 73 reduces this to the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ from Section 6.6. For the category $\mathcal{O}\{\mathfrak{p}, \mathscr{A}^{\mathbf{R}'}\}$ the statement follows from the proof of Theorem 34. □

**Remark 76** *(Unnecessary restrictions).*

(i) The restriction of integrability for the central character is not really essential and can be taken away using methods proposed by Soergel in [71, Bemerkung 1] on the reduction of the Kazhdan–Lusztig conjecture to the integral case.
(ii) In this paper we only worked with the trivial central character to avoid even more notation. The singular case follows by translation to the regular case, using our results there and translating back (invoking the fact that the composition of these translation functors is just a multiple of the identity).

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