Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Journal of Algebra 268 (2003) 301–326
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Category $\mathcal{O}$: gradings and translation functors

Catharina Stroppel [1]

University of Leicester, England, UK

Received 22 February 2002

Communicated by Peter Littelmann

Abstract

In this article we consider a graded version of category $\mathcal{O}$. We reprove some results of [Beilinson et al., J. Amer. Math. Soc. 9 (1996) 473–527] using a different approach. Furthermore, we define a graded version of translation functors and duality. This provides the construction of various graded modules. On the other hand, we describe how to get modules which are not 'gradable.'

Introduction

For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ with Borel and Cartan subalgebras $\mathfrak{b}$ and $\mathfrak{h}$, respectively, we consider the so-called category $\mathcal{O}$ (originally defined in [BGG]). This category decomposes into blocks indexed by dominant weights, where each block $\mathcal{O}_\lambda$ has as objects certain $\mathfrak{g}$-modules with a fixed generalized central character.

For any weight λ there is a universal object, the so-called *Verma module* with highest weight λ. We denote it by $\Delta(\lambda)$. Each Verma module $\Delta(\lambda)$ has a simple head, denoted by $L(\lambda)$. All simple objects in $\mathcal{O}$ arise in this way.

A famous problem was to determine the multiplicities $[\Delta(\lambda) : L(\mu)]$, how often a simple module $L(\mu)$ occurs in a composition series of $\Delta(\lambda)$. This problem was turned into a combinatorial problem by a conjecture [KL, Conjecture 1.5] of Kazhdan and Lusztig. The conjecture states that the multiplicity is given by certain inductively defined polynomials evaluated at 1 and was subsequently proved in [BB1,BK].

The question now was, if there is also an interpretation of the exponents occurring in these Kazhdan–Lusztig polynomials. One answer to this problem is given in the article

E-mail address: stroppel@imf.au.dk.

[1] Partially supported by the EEC program ERB FMRX-CT97-0100.

0021-8693/$ – see front matter
doi:10.1016/S0021-8693(03)00308-9

[BGS] of Beilinson, Ginzburg, and Soergel. They consider each block of category $\mathcal{O}$ as a category of right modules over a finite-dimensional algebra A, namely the endomorphism ring of a minimal projective generator. This algebra is (see [So1]) isomorphic to an algebra of self-extensions of some module; so it inherits a natural $\mathbb{Z}$-grading. In [BGS, 3.11] it is shown that a Verma module $\Delta(\lambda)$ has a graded lift, i.e., that there is a graded A-module M such that $M \cong \widetilde{\Delta(\lambda)}$ after forgetting the grading. Here $\widetilde{\Delta(\lambda)}$ denotes the right A-module corresponding to $\Delta(\lambda)$. On the other hand, all simple modules have graded lifts concentrated in one single degree.

Since by definition the ring A is positively graded, the filtration associated to such a graded module is in fact a filtration by submodules. Moreover, the exponents of the Kazhdan–Lusztig polynomials indicate in which layer of this filtration a certain simple module occurs [BGS, Theorem 3.11.4]. By the results of [BB2], this filtration coincides for Verma modules with the Jantzen filtration (see [Ja1]). This observation was the motivation for us to ask whether there are other interesting objects in $\mathcal{O}$ which are 'gradable' in the sense described above. And if so, whether there is a combinatorial description of their filtrations induced by the grading in terms of Kazhdan–Lusztig polynomials.

For this reason we describe another approach to the graded version of category $\mathcal{O}$ using Soergel's functor $\mathbb{V}$ (defined in [So1]). In this approach, it is straightforward to define graded versions of translation functors. Inductively, this yields graded versions of 'important' modules, such as Verma modules, dual Verma modules and principal series in general. For Verma modules, it is just the same lift as described in [BGS]. For principal series, the combinatorics of these graded modules coincide with the ones described (in a geometric setup) in [CC].

We prove an Adjointness Theorem (8.4) for these graded translation functors. It turns out that the graded versions of translation *on* and *out* of the wall are adjoint to each other up to a shift. Therefore these graded functors carry more information than the 'usual' translation functors. An easy implication of these adjointness properties is the fact that, for type A_1, these graded versions of Verma modules are Koszul modules in the sense of [BGS]. We also explain what happens to the graded versions of Verma modules and simple modules when translated *through*, *onto* or *out of* the wall. We show the existence of a graded duality, which is a lift of the usual duality on $\mathcal{O}$. This gives the graded version of the Bernstein–Gelfand–Gelfand reciprocity (see also [BGS, Theorem 3.11.4]). Our main result is a combinatorial description of our graded versions of translation functors in terms of elements in the corresponding Hecke algebra (Theorem 7.1).

The advantage of our approach is that it provides a way to construct lifts of principal series. The motivation to define such lifts was given by A. Joseph's article [Jo] where he describes the connection between filtered versions of principal series and primitive ideals in the universal enveloping algebra of $\mathfrak{g}$. Details of how to construct graded versions of principal series from the graded versions of translation functors and why this is useful to determine composition factors of quotients of the universal enveloping algebra of $\mathfrak{g}$ by the annihilator of some simple $\mathfrak{g}$-module can be found with some explicit examples in [St2]. Another advantage of our approach is that it can easily be generalized to the categories of Harish-Chandra bimodules with generalized trivial central character from both sides, whose power is fixed from one side. We do not know, however, how to deduce a graded duality.

We briefly summarize the content of this paper. In the first section we recall some facts about category $\mathcal{O}$, translation functors and graded modules. In Section 2 we describe how to consider an integral block of $\mathcal{O}$ as a category of modules over a ring which is graded. This is done without using the main result of [BGS]. In Section 3 we introduce the notation of gradable modules and functors and give the main examples (Verma modules and translation trough the wall). How to find objects for which there exists no graded lift is described in Section 4. A graded version of duality can be found in Section 6. Our main results are the descriptions of graded translation functors in Section 3.2 and Section 8 which implies the short exact sequences given in Section 5. These results are summarized in the combinatorial description of Theorem 7.1. An Adjointness Theorem is proved in Section 8.

1. The category $\mathcal{O}$ and its main properties

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with a chosen Borel and a fixed Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. The corresponding universal enveloping algebras are denoted by $\mathcal{U} = \mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{b})$, etc.

We consider the category $\mathcal{O}$ which is a full subcategory of the category of all $\mathcal{U}(\mathfrak{g})$-modules and defined by the following set of objects:

$$\mathrm{Ob}(\mathcal{O}) := \left\{ M \in \mathcal{U}(\mathfrak{g})\text{-mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated as a } \mathcal{U}(\mathfrak{g})\text{-module,} \\ M \text{ is locally finite for } \mathfrak{n}, \\ \mathfrak{h} \text{ acts diagonally on } M \end{array} \right\},$$

where the second condition means that $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{n}) \cdot m < \infty$ for all $m \in M$ and the last says that $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$ denotes the μ-weight space of M.

Many results about this category can be found, for example, in [BGG,Ja1,Ja2]. The category $\mathcal{O}$ decomposes into a direct sum of full subcategories $\mathcal{O}_\chi$ indexed by central characters χ of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$. Let $S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$ be the symmetric algebra over $\mathfrak{h}$ considered as regular functions on $\mathfrak{h}^*$, together with the dot-action of the Weyl group W, defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$, where ρ is the half-sum of positive roots. Let $\mathcal{Z} = \mathcal{Z}(\mathcal{U})$ be the center of $\mathcal{U}$. Using the so-called Harish-Chandra isomorphism (see, e.g., [Ja1, Satz 1.5], [Di, Theorem 7.4.5]) $\mathcal{Z} \to S^{W\cdot}$ and the fact that S is integral over $S^{W\cdot}$ [Di, Theorem 7.4.8] we get an isomorphism $\xi : \mathfrak{h}^*/(W\cdot) \to \operatorname{Max} \mathcal{Z}$. Here $\operatorname{Max} \mathcal{Z}$ denotes the set of maximal ideals in $\mathcal{Z}$. This yields the following decomposition:

$$\mathcal{O} = \bigoplus_{\chi \in \operatorname{Max} \mathcal{Z}} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\cdot)} \mathcal{O}_\lambda, \tag{1.1}$$

where $\mathcal{O}_\chi$ denotes the subcategory of $\mathcal{O}$ consisting of all objects killed by some power of χ. It denotes the same block as $\mathcal{O}_\lambda$ if $\xi(\lambda) = \chi$.

$\mathcal{O}_\lambda$ is called a *regular block* of the category $\mathcal{O}$ if λ is regular, that is, if $\lambda+\rho$ is not zero at any coroot $\check{\alpha}$ belonging to $\mathfrak{b}$. We denote by $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$ the stabilizer of λ in W.

For all $\lambda \in \mathfrak{h}^*$ we have a standard module, the Verma module $\Delta(\lambda) = \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ denotes the irreducible $\mathfrak{h}$-module with weight λ enlarged by the trivial action to a module over the Borel subalgebra. This Verma module is a highest weight module of highest weight λ and has central character $\xi(\lambda)$. We denote by $L(\lambda)$ the unique irreducible quotient of $\Delta(\lambda)$. Let $\star$ denote the duality of $\mathcal{O}$, i.e., $M^\star$ is the maximal $\mathfrak{h}$-semisimple submodule of the contragradient representation M^* with the $\mathfrak{g}$-action twisted by the Chevalley antiautomorphism. We denote by $\nabla(\lambda)$ the dual Verma module $\Delta(\lambda)^\star$.

The category $\mathcal{O}$ has enough projectives. We denote the projective cover of $L(x\cdot\lambda)$ by $P(x\cdot\lambda)$ and its injective hull by $I(x\cdot\lambda)$. Thus the indecomposable projective objects in $\mathcal{O}_\lambda$ are in bijection with W/W_λ. By convention we choose for x a representative of minimal length.

1.1. *Translation functors and their combinatorial description*

Let $\lambda, \mu \in \mathfrak{h}^*$ be such that $\lambda - \mu$ is integral. The translation functor from the block $\mathcal{O}_\lambda$ to $\mathcal{O}_\mu$ is the functor

$$\theta_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu,$$
$$M \mapsto \mathrm{pr}_\mu\big(M \otimes E(\mu-\lambda)\big),$$

where pr_μ is the projection onto $\mathcal{O}_\mu$ and $E(\mu-\lambda)$ is the finite-dimensional simple $\mathfrak{g}$-module with extremal weight $\lambda-\mu$. Let s be a simple reflection and $W_\mu = \{1, s\}$; then translation *through* the s-wall is the composition of functors $\theta_s = \theta_\mu^\lambda \circ \theta_\lambda^\mu$. For more details concerning these functors see [Ja1,Ja2].

For an abelian category $\mathcal{A}$ we denote by $[\mathcal{A}]$ the Grothendieck group of $\mathcal{A}$, i.e., it is the free abelian group generated by the isomorphism classes $[A]$ of objects A in $\mathcal{A}$ modulo relations $[C] = [A] + [B]$ whenever there is a short exact sequence of the form $A \hookrightarrow C \twoheadrightarrow B$. Consider the case where $\mathcal{A}$ is a block $\mathcal{O}_\lambda$ of $\mathcal{O}$. Each of the three sets $\{[L(x\cdot\lambda)] \mid x \in W/W_\lambda\}$, $\{[\Delta(x\cdot\lambda)] \mid x \in W/W_\lambda\}$, $\{[P(x\cdot\lambda)] \mid x \in W/W_\lambda\}$ forms a basis of the Grothendieck group of $\mathcal{O}_\lambda$.

Let us look at the situation where λ is a regular and integral dominant weight. The translation functors θ_s are exact functors and so they induce a group homomorphism θ_s on the Grothendieck group of the trivial block $\mathcal{O}_0$, giving rise to actions of the Weyl group on each side. For $[M] \in \mathcal{O}_0$ and s a simple reflection, the two actions are defined by

$$s.[M] = [\theta_s M] - [M] \quad \text{and} \quad [M].s = \theta_s^r[M] - [M],$$

where $\theta_s^r[\Delta(x\cdot 0)] = [\Delta(x\cdot 0)] + [\Delta(sx\cdot 0)]$.

Remarks 1.1. The notation θ_s^r should indicate that in fact this group homomorphism is induced via the Bernstein–Gelfand equivalence [BG,Ja2] by some translation functor acting from the right-hand side on Harish-Chandra bimodules. The Grothendieck group of the trivial block is isomorphic to the group ring $\mathbb{Z}[W]$ of W over $\mathbb{Z}$ via the map $[\Delta(x \cdot 0)] \mapsto x^{-1}$. Using this isomorphism the left and right action of s are just given respectively by the left and right multiplication of s on W.

We denote by H the Hecke algebra of W [Bo2, IV, 2, Example 22]. This is by definition the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_x \mid x \in W\}$ together with the relations

$$\begin{aligned} H_s^2 &= H_e + \left(v^{-1} - v\right) H_s \quad \text{for a simple reflection } s \quad \text{and} \\ H_x H_y &= H_{xy}, \quad \text{if } l(x) + l(y) = l(xy). \end{aligned}$$

With $v = 1$, the additive group of H is isomorphic to the Grothendieck group and the translation functors fit in the following commuting diagrams:

$$\begin{array}{ccc} \mathrm{H} & \xrightarrow[v=1]{H_x \mapsto [\Delta(x\cdot 0)]} & [\mathcal{O}_0] \\ {\scriptstyle \cdot(H_s+v)}\downarrow & & \downarrow{\scriptstyle [\theta_s]} \\ \mathrm{H} & \xrightarrow[v=1]{H_x \mapsto [\Delta(x\cdot 0)]} & [\mathcal{O}_0] \end{array}, \qquad \begin{array}{ccc} \mathrm{H} & \xrightarrow[v=1]{H_x \mapsto [\Delta(x\cdot 0)]} & [\mathcal{O}_0] \\ {\scriptstyle (H_s+v)\cdot}\downarrow & & \downarrow{\scriptstyle [\theta_s^r]} \\ \mathrm{H} & \xrightarrow[v=1]{H_x \mapsto [\Delta(x\cdot 0)]} & [\mathcal{O}_0] \end{array}. \tag{1.2}$$

1.2. Gradings

In this section we introduce first of all some notation and also recall some general results about graded modules which are important for the subsequent sections. In the following the word 'graded' always means $\mathbb{Z}$-graded. So let A be a graded ring and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded A-module. Let f denote the grading forgetting functor.

For $m \in \mathbb{Z}$ let $M\langle m\rangle$ be the graded module defined by $M\langle m\rangle_n := M_{n-m}$ with the same module structure as M, i.e., $\mathrm{f}(M\langle m\rangle) = \mathrm{f}(M)$. Given two graded A-modules M and N we denote by

$$\mathrm{Hom}_A(M, N) = \{A\text{-linear maps from } M \text{ to } N\}$$

the set of non-graded morphisms. This contains the set

$$\mathrm{Hom}_A(M, N)_i = \left\{\phi \in \mathrm{Hom}_A(M, N) \mid \phi(M_j) \subseteq N_{j+i},\ \forall j \in \mathbb{Z}\right\}$$

of all morphisms which are homogeneous of degree i. From the definitions we get

$$\mathrm{Hom}_A\big(M\langle i\rangle, N\big)_0 = \mathrm{Hom}_A(M, N)_i = \mathrm{Hom}_A\big(M, N\langle -i\rangle\big)_0. \tag{1.3}$$

Therefore, the ring $\mathrm{End}_A(M)$ inherits a natural grading. We denote by gmof-A the category of all finitely generated graded right A-modules with homogeneous morphisms of degree zero. For any ring R we denote by mof-R the category of finitely generated right R-modules.

The following fact about tensor products is needed later:

Lemma 1.2. *Let R, S be graded rings and let M be a graded R-S-bimodule and N a graded S-module. Then the module $M \otimes_S N$ is a graded R-module.*

Proof. The R-module $M \otimes_{\mathbb{Z}} N$ can be equipped with a grading, by setting

$$(M \otimes_{\mathbb{Z}} N)_i := \sum_k M_k \otimes_{\mathbb{Z}} N_{i-k} \subseteq M \otimes_{\mathbb{Z}} N.$$

Here, $M_k \otimes_{\mathbb{Z}} N_{i-k}$ denotes the subspace of (the tensor product of abelian groups) $M \otimes_{\mathbb{Z}} N$ generated by all elements of the form $m \otimes n$ with $m \in M_k$ and $n \in N_{i-k}$. Obviously, this is compatible with the R-module structure. We consider the canonical surjection of R-modules $M \otimes_{\mathbb{Z}} N \twoheadrightarrow M \otimes_S N$. The kernel is generated by elements of the form $ms \otimes n - m \otimes sn$, so it is generated by homogeneous elements. □

If we consider a category of graded modules over a graded algebra as a 'graded category' in the sense of [AJS], we have also the notion of 'functors of graded categories' or '$\mathbb{Z}$-functors.' The exact definition can be found in [AJS, E.3]. The previous lemma gives a standard example of such a functor.

Example 1.3. Let R and S be graded rings and let X be a finitely generated graded R-S-bimodule. The functor

$$\bullet \otimes_R X : \text{gmof-}R \to \text{gmof-}S$$

is a functor of graded categories, where the natural transformations are given by the natural isomorphisms

$$M\langle n\rangle \otimes_R X \cong (M \otimes_R X)\langle n\rangle.$$

To construct a graded version of the category $\mathcal{O}$, the following lemma is crucial.

Lemma 1.4. *Let M and N be finite-dimensional modules over a ring S which is graded. Let N be indecomposable and let N and $M \oplus N$ be graded S-modules. Then the module M inherits a grading.*

Proof. Let $i : N \hookrightarrow N \oplus M$ and $p : N \oplus M \to N$ be the canonical inclusion and projection, respectively. Let $i = \sum_j i_j$ and $p = \sum_j p_j$ be the grading decompositions. The composition $p \circ i = \sum_n p_n \circ i_{-n}$ is the identity on N. By assumption $\text{End}_A(N)$ is a local ring. So at least one of the summands, say $p_{n_0} \circ i_{-n_0}$, should be invertible. Denote by ϕ its inverse. This yields a splitting of $\phi \circ p_{n_0} : M \oplus N \to N\langle -n_0\rangle$, namely i_{-n_0}. The kernel of this morphism is a lift of M we are looking for. Moreover, it shows that the direct sum of the constructed lift of M and N is just the given lift of $M \oplus N$. □

For indecomposable modules these graded lifts are unique up to isomorphism and shift:

Lemma 1.5 (Uniqueness of the grading). *Let B be a graded ring and let $N \in$ mof-B be indecomposable. Furthermore we assume there exists a lift $M \in$ gmof-B of N. Then for each $M' \in$ gmof-B such that $\mathrm{f}(M) \cong \mathrm{f}(M') \cong N$ there is an isomorphism $M \cong M'\langle n\rangle$ for some $n \in \mathbb{Z}$.*

In the situation of the lemma, we will say, a bit sloppy, 'the grading is unique up to a shift.'

Proof. See [BGS, Lemma 2.5.3]. □

1.3. The combinatorial functor $\mathbb{V}$

Fix $\lambda \in \mathfrak{h}^*$ a dominant and integral weight. The center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}$ yields by multiplication a map $\mathcal{Z} \to \mathrm{End}_{\mathfrak{g}}(P(w_0^\lambda \cdot \lambda))$, where w_0^λ denotes the longest element of W/W_λ. On the other hand, we have a map $\mathcal{Z} \to S^{W\cdot} \to S/(S_+^W)$ by composing the Harish-Chandra isomorphism and the natural projection. Here S_+ denotes the maximal ideal of S consisting of all regular functions vanishing at zero and (S_+^W) is the ideal generated by polynomials without a constant term and invariant under the (usual!) action of the Weyl group. For $\lambda = 0$, both of these maps are surjective and have the same kernel. This gives the following key result.

Theorem 1.6 ([So1, Endomorphismensatz] and [Be] for regular λ). *Let $\lambda \in \mathfrak{h}^*$ be an integral and dominant weight and let W_λ be its stabilizer under the dot-action of the Weyl group. Let w_0^λ be the longest element of W/W_λ. Then there is an isomorphism of algebras*

$$\mathrm{End}_{\mathfrak{g}}\big(P\big(w_0^\lambda \cdot \lambda\big)\big) \cong \big(S/\big(S_+^W\big)\big)^{W_\lambda}.$$

Remark. The algebra $(S/(S_+^W))$ is the so-called 'algebra of coinvariants' and its dimension (as a complex vector space) is just the order of the Weyl group (see [Bo1]). In the following, we denote it by C and its invariants C^{W_λ} by C^λ. This algebra is commutative, so we can consider right C-modules also as left C-modules.

Convention 1.7. In order to be consistent with the literature, we consider $S = S(\mathfrak{h})$ as an evenly graded algebra, so $S = \bigoplus_{i\in\mathbb{N}} S^{2i}$. We also assume that $S^2 = \mathfrak{h}$ holds. The algebra of coinvariants C inherits a grading.

The previous Endomorphism Theorem makes it possible to define the following combinatorial functor $\mathbb{V}$.

Theorem 1.8 [So1, Struktursatz 9]. *Let $\lambda \in \mathfrak{h}^*$ be an integral and dominant weight. The exact functor*

$$\mathbb{V} = \mathbb{V}_\lambda : \mathcal{O}_\lambda \to C^\lambda\text{-mof},$$
$$M \mapsto \mathrm{Hom}_{\mathfrak{g}}\big(P\big(w_0^\lambda \cdot \lambda\big), M\big)$$

is fully faithful on projective objects. In other words, for $x, y \in W/W_\lambda$, there is an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathfrak{g}}\big(P(x\cdot\lambda), P(y\cdot\lambda)\big) \cong \mathrm{Hom}_{C^\lambda}\big(\mathbb{V}P(x\cdot\lambda), \mathbb{V}P(y\cdot\lambda)\big).$$

For $x \in W$ with $x = s_r \cdots s_3 s_2 s_1$ a reduced expression and λ dominant, the module $P(x\cdot\lambda)$ is isomorphic to a direct summand of $\theta_{s_1}\cdots\theta_{s_r}M(\lambda)$. Moreover, it is the unique indecomposable direct summand of $M(\lambda)$ not isomorphic to some $P(y\cdot\lambda)$ with $y < x$ (more details can be found, e.g., in [Ja2,BG,So1]).

Translation through the wall and the functor $\mathbb{V}$ are related by the algebra of coinvariants in the following way.

Theorem 1.9 [So1, Theorem 10]. *Let $\lambda \in \mathfrak{h}^*$ be regular and let s be a simple reflection. Denote by C^s the invariants of C under the action of s. There is a natural equivalence of functors $\mathcal{O}_\lambda \to C$*-mof

$$\mathbb{V}\theta_s \cong C \otimes_{C^s} \mathbb{V}.$$

Corollary 1.10. *Let $x = s_r \ldots s_3 s_2 s_1$ be a reduced expression of $x \in W$. Then the module $\mathbb{V}P(x\cdot\lambda)$ is isomorphic to the unique direct summand of $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_3}} \cdots \otimes_{C^{s_r}} \mathbb{C}$ which is not isomorphic to some $\mathbb{V}P(y\cdot\lambda)$ with $y < x$.*

Remark. The theorem is also true for singular λ if we replace C by C^λ.

2. The category $\mathcal{O}$ as a category of modules over a graded ring

Let $\lambda \in \mathfrak{h}^*$ be an integral weight. The object $P_\lambda := \bigoplus_{x\in W/W_\lambda} P(x\cdot\lambda)$ is a (minimal) projective generator of $\mathcal{O}_\lambda$. So there is (see [Ba]) an equivalence of categories

$$\begin{aligned} \mathcal{O}_\lambda &\xrightarrow{\sim} \text{mof-}\mathrm{End}_{\mathfrak{g}}(P_\lambda), \\ M &\mapsto \mathrm{Hom}_{\mathfrak{g}}(P_\lambda, M). \end{aligned} \tag{2.1}$$

Now we are ready to explain how $A_\lambda := \mathrm{End}_{\mathfrak{g}}(P_\lambda)$ can be considered as a graded ring.

Theorem 2.1. *Let λ be an integral dominant weight and let $Q, Q' \in \mathcal{O}_\lambda$ be projective objects. The functor $\mathbb{V}$ induces a grading on $\mathrm{Hom}_{\mathfrak{g}}(Q, Q')$. In particular, $\mathrm{End}_{\mathfrak{g}}(P_\lambda)$ can be considered as a graded ring.*

Proof. A proof can be found in [BGS]. Since we do not need the stronger results of [BGS], we also give a proof for the assertion.

By Convention 1.7 the endomorphism ring of the antidominant projective module in $\mathcal{O}_\lambda$ can be considered as a graded ring. We assume for the moment that λ is regular. Given a simple reflection s, the subset $C^s \subset C$ of C is a graded subring, so C is a graded

C^s-module. On the other hand, the trivial module $\mathbb{C}$ is also a graded C^s-module. By Lemma 1.2, the module $C \otimes_{C^s} \mathbb{C}$ is a graded C-module. All projective objects in a block are given by direct summands of direct sums of such successive tensor products. Therefore, by Lemma 1.4 all $\mathbb{V}Q$ with Q projective are graded. Using (1.3) gives a natural grading on $\mathrm{Hom}_C(\mathbb{V}Q, \mathbb{V}Q')$ for projective modules Q and Q'. The property of faithfulness (see Proposition 1.8) induces therefore a grading on $\mathrm{Hom}_{\mathfrak{g}}(Q, Q')$. In particular, $\mathrm{End}_{\mathfrak{g}}(P_\lambda)$ becomes a graded ring. For singular λ the algebra C has to be replaced by the invariants C^λ. □

Convention 2.2. By the proof of Theorem 2.1, the module $\mathbb{V}P(x \cdot \lambda)$ becomes a graded C-module. In the following, we consider $\mathbb{V}P(x \cdot \lambda)$ as a graded C-module with the convention that its highest degree is $l(x)$. For λ singular, x should be chosen of minimal possible length. In the following A_λ has then the grading given by Theorem 2.1. For $\lambda = 0$ we omit the subindex λ of A and P.

By this convention the endomorphism ring A_λ is in fact a positively graded ring. Details for this can be found in [So4] and [BGS, Theorem 1.1.3].

3. Gradable modules and functors

In the following section we introduce lifts of Verma modules and their duals and also define a graded version of translation functors.

3.1. Lifts of objects in $\mathcal{O}_\lambda$

Definition 3.1 (Gradable modules). Let B be a graded ring. We call a module $M \in$ mof-B *gradable* if there exists a graded module $\widetilde{M} \in$ mof-B such that $\mathrm{f}(\widetilde{M}) \cong M$. In this case, the module $\widetilde{M}$ is a *lift* of M.

An object $M \in \mathcal{O}_\lambda$ is *gradable* if $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, M)$ is a gradable A_λ-module, where $A_\lambda = \mathrm{End}_{\mathfrak{g}}(P_\lambda)$ is graded via Conventions 2.2 and 1.7. By abuse of language, a lift of $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, M)$ is often called a *lift of* M.

We proved in Theorem 2.1 that all projective objects in any integral block are gradable. Moreover (see Lemma 1.5), the grading is 'unique up to a shift.' By Convention 2.2 the grading is unique up to isomorphism.

Another example for gradable modules are the simple objects.

Lemma 3.2. *Let λ be an integral dominant weight. The simple objects of $\mathcal{O}_\lambda$ are gradable. Their lifts are pure, i.e., they are concentrated in one degree.*

Proof. Consider in $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, P(x \cdot \lambda)) = \mathrm{Hom}_{C^\lambda}(\mathbb{V}(P_\lambda), \mathbb{V}P(x \cdot \lambda))$ the one-dimensional subspace L generated by the canonical projection, i.e., the projection onto its simple head as a right A_λ-module. Any complement as vector spaces is also a (right) A_λ-submodule since the image of $g \circ f$ is in the radical of $P(x \cdot \lambda)$ for any $g \in \mathrm{Hom}_{\mathfrak{g}}(P_\lambda, P(x \cdot \lambda))$ and

$f \in A_\lambda$. So the projection from $\mathrm{Hom}_C(\mathbb{V}(P_\lambda), \mathbb{V}P(x\cdot\lambda))$ onto L is a homogeneous map of degree zero. The quotient is concentrated in degree zero. □

Note that the previous lemma implies that the inductively defined lift of $P(x\cdot\lambda)$ is up to isomorphism independent of the choice of the reduced expression of x. Concerning the notation, we do not distinguish between the projective or simple objects and their graded lifts in the following.

3.2. Lifts of translation functors

Definition 3.3 (Lift of a gradable functor). Let B and D be graded rings. We call a functor $F:\text{mof-}B \to \text{mof-}D$ *gradable,* if there exists a functor of graded categories $\widetilde{F}:\text{gmof-}B \to \text{gmof-}D$ (in the sense of [AJS]) which induces F. If there is such a functor $\widetilde{F}$, we call it a *lift* of F. In other words, $\widetilde{F}$ is a lift of F, if it is a functor of graded categories such that the following diagram commutes:

$$\begin{array}{ccc} \text{gmof-}B & \xrightarrow{\widetilde{F}} & \text{gmof-}D \\ \Big\downarrow{\scriptstyle \mathrm{f}} & & \Big\downarrow{\scriptstyle \mathrm{f}} \\ \text{mof-}B & \xrightarrow{F} & \text{mof-}D \end{array}$$

For integral and dominant weights λ and μ, a functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_\mu$ is *gradable* if it induces a gradable functor from mof-A_λ to mof-A_μ.

The following lemma provides some gradable functors including the translation functors.

Lemma 3.4. *Let R and S be any rings. There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{right exact, compatible with direct sums}\\ \textit{functors}:\text{(g)mof-}R \to \text{(g)mof-}S\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} R\text{-(g)mof-}S,$$
$$F \mapsto F(R),$$
$$\bullet \otimes_R X \leftarrow X.$$

By definition, $F(R)$ is a right S-module. The left multiplication of R defines the left module structure on $F(R)$.

Proof. See [Ba, 2.2]. □

Consider $\theta_s:\mathcal{O}_0 \to \mathcal{O}_0$, the translation through the s-wall. The functor induced by Lemma 3.4 is the functor $\bullet \otimes_A \mathrm{Hom}(P, \theta_s P)$ on mof-A. This functor is (see Example 1.3) gradable by Theorem 2.1 and Lemma 1.2. We choose the following lift:

$$\bullet \otimes_A \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big)$$

and denote it also by θ_s.

Remarks 3.5. The graded version θ_s is compatible with the Conventions 2.2 in the sense that for example $\theta_s P(0) \cong P(s \cdot 0)$ holds as graded modules. More generally, there is an isomorphism of graded modules

$$\begin{aligned} &\mathrm{Hom}_C\big(\mathbb{V}P, \mathbb{V}P(x\cdot 0)\big) \otimes_A \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big) \\ &\qquad \to \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P(x\cdot 0)\langle -1\rangle\big) \\ &g \otimes f \mapsto (\mathrm{id} \otimes g) \circ f. \end{aligned}$$

3.3. Lifts of Verma modules and their duals

The reason why we consider these graded versions of translation functors is the fact that these functors provide a construction of gradable modules. We first show, by a very general argument, that all Verma modules are gradable. Consider for $x \in W$ and s a simple reflection such that $xs > x$ the short exact sequence of A-modules

$$\mathrm{Hom}_{\mathfrak{g}}\big(P, \Delta(x\cdot 0)\big) \overset{j}{\hookrightarrow} \mathrm{Hom}_{\mathfrak{g}}\big(P, \theta_s\Delta(x\cdot 0)\big) \overset{k}{\twoheadrightarrow} \mathrm{Hom}_{\mathfrak{g}}\big(P, \Delta(xs\cdot 0)\big). \tag{3.1}$$

Consider inductively $\mathrm{Hom}_{\mathfrak{g}}(P, \Delta(x\cdot 0))$ and (by Lemma 3.4) $\mathrm{Hom}_{\mathfrak{g}}(P, \theta_s\Delta(x\cdot 0)) \cong \theta_s(\mathrm{Hom}_{\mathfrak{g}}(P, \Delta(x\cdot 0))$ as graded modules. Since $\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta(x\cdot 0), \theta_s\Delta(x\cdot 0)) = 1$ holds, the morphism j is homogeneous. Therefore, the cokernel is gradable. Starting with the projective Verma module, this method provides inductively lifts of all Verma modules. By Lemma 1.5, these graded lifts are unique up to isomorphism and grading shift. The lift of $\Delta(0) = P(0)$ is given by Convention 2.2. We choose the lift of $\Delta(xs\cdot 0)$ such that the surjection in (3.1) is homogeneous of degree zero. So the canonical surjection $P(x\cdot 0) \twoheadrightarrow \Delta(x\cdot 0)$ is homogeneous of degree zero and therefore the lifts do not depend on the reduced expression of x.

Concerning notation, we will not distinguish between Verma modules and their graded lifts. These graded Verma modules fit in a short exact sequence of the following form.

Theorem 3.6. *For $x \in W$ and s a simple reflection such that $xs > x$, there is an exact sequence of graded modules*

$$\Delta(x\cdot 0)\langle 1\rangle \hookrightarrow \theta_s\Delta(x\cdot 0) \twoheadrightarrow \Delta(xs\cdot 0).$$

Proof. We choose isomorphisms $\mathbb{V}P \cong \mathbb{V}P^* = \mathrm{Hom}_C(\mathbb{V}P, C)$ of graded right C-modules, in particular, a graded isomorphism $C \cong C^{\mathrm{opp}}$. The existence of such an isomorphism and also the description of a canonical isomorphism $(C \otimes_{C^s} M)^* \cong C \otimes_{C^s} M^*\langle -2\rangle$ can be found in [So4, Lemma 2.9.2]. So we get the following isomorphisms of graded vector spaces:

$$\begin{aligned} \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big) &\cong \mathrm{Hom}_C\big(\big(C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big)^*, \mathbb{V}P\big) \\ &\cong \mathrm{Hom}_C\big(\big(C \otimes_{C^s} \mathbb{V}P\big)^*\langle 1\rangle, \mathbb{V}P\big) \\ &\cong \mathrm{Hom}_C\big(\big(C \otimes_{C^s} \mathbb{V}P\big)\langle -1\rangle, \mathbb{V}P\big). \end{aligned} \tag{3.2}$$

Multiplication yields a homogeneous map of degree zero:

$$\begin{aligned} \operatorname{mult}: (C \otimes_{C^s} \mathbb{V}P) &\to \mathbb{V}P, \\ c \otimes m &\mapsto cm, \end{aligned} \tag{3.3}$$

so there is a C-linear map

$$f: \mathbb{V}P \to C \otimes_{C^s} \mathbb{V}P\langle -1\rangle \tag{3.4}$$

which is homogeneous of degree 1. This induces a non-trivial morphism

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}\big(P, \Delta(x \cdot 0)\big) &\to \operatorname{Hom}_{\mathfrak{g}}\big(P, \Delta(x \cdot 0)\big) \otimes_A \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big), \\ \phi &\mapsto \phi \otimes f. \end{aligned} \tag{3.5}$$

Up to a scalar, it has to be the map j of (3.1). By definition it is homogeneous of degree 1. So the theorem is proved. □

Corollary 3.7. *Let $x \in W$. Then the following multiplicity formulas hold*:

$$\big[\Delta(x \cdot 0) : L(x \cdot 0)\langle j\rangle\big] = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The (up to a scaler unique) surjection $P(y \cdot 0) \twoheadrightarrow \Delta(y \cdot 0)$ is homogeneous of degree zero for any $y \in W$. On the other hand, $P(y \cdot 0)$ is the projective cover of $L(y \cdot 0)$. □

Remarks 3.8.

(a) The graded versions of Verma modules can also be constructed by considering them as projective objects in some ‘truncated’ subcategory of $\mathcal{O}$. This is explained in detail in [BGS]. These graded versions coincide with our lifts. In our approach no further information such as ‘positively graded’ or the description of A as an algebra of self-extensions [BGS, Theorem 1.1.1] is needed.

(b) The homomorphism f as introduced in (3.4), the multiplication map mult of (3.3) and the identity (as a homogeneous map of degree 1) correspond under the following isomorphisms of graded vector spaces

$$\begin{aligned} \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1\rangle\big) &\cong \operatorname{Hom}_C\big((C \otimes_{C^s} \mathbb{V}P)\langle -1\rangle, \mathbb{V}P\big) \\ &\cong \operatorname{Hom}_{C^s}\big(\mathbb{V}P, \operatorname{Hom}_C(C, \mathbb{V}P)\big)\langle 1\rangle \\ &\cong \operatorname{Hom}_{C^s}(\mathbb{V}P, \mathbb{V}P)\langle 1\rangle. \end{aligned}$$

(c) The proof of Theorem 3.6 shows that the canonical map $M \to \theta_s M$ for any $M \in$ gmof-A is homogeneous of degree 1. The proof of Theorem 5.1 will show that the canonical map $\theta_s M \to M$ is also homogeneous of degree 1.

Lemma 3.9. *For all $x \in W$ the dual Verma module $\nabla(x \cdot 0)$ and the indecomposable injective module $I(x \cdot 0)$ are gradable.*

Proof. Consider the exact sequence

$$\nabla(xs \cdot 0) \overset{k}{\hookrightarrow} \theta_s \nabla(xs \cdot 0) \overset{j}{\twoheadrightarrow} \nabla(x \cdot 0) \tag{3.6}$$

with $x \in W$ and s a simple reflection such that $xs > x$ holds. The simple Verma module $\Delta(w_0 \cdot 0) = \nabla(w_0 \cdot 0)$ is gradable. By induction we may assume $\nabla(xs \cdot 0)$ is gradable. Hence so is $\theta_s \nabla(xs \cdot 0)$. Since

$$\begin{aligned} \dim \operatorname{Hom}_{\mathfrak{g}}\big(\nabla(xs \cdot 0), \theta_s \nabla(xs \cdot 0)\big) &= \dim \operatorname{Hom}_{\mathfrak{g}}\big(\nabla(xs \cdot 0), \theta_s \nabla(x \cdot 0)\big) \\ &= \dim \operatorname{Hom}_{\mathfrak{g}}\big(\theta_s \nabla(xs \cdot 0), \nabla(x \cdot 0)\big) \\ &= \big[\theta_s \nabla(xs \cdot 0) : L(x \cdot 0)\big] \\ &= 1 \end{aligned}$$

holds, k can be considered as a homogeneous map between the graded lifts. (The third equality holds, since $\nabla(x \cdot 0)$ is the injective hull of $L(x \cdot 0)$ in the full subcategory of $\mathcal{O}_0$ whose objects have only composition factors of the form $L(z \cdot 0)$ with $z \geqslant x$.) This induces a grading on the cokernel. This grading is determined up to isomorphism by the requiring that k should be homogeneous of degree zero. Inductively this gives the first statement.

The gradability of the injective modules then follows inductively by starting with the graded object $\nabla(0)$. That the lifts are independent of the chosen reduced expression of x can easily be seen using Proposition 5.1, which is proven independently. □

The dual statement to Theorem 3.6 is the following theorem.

Theorem 3.10. *Let $x \in W$ and let s be a simple reflection such that $xs > x$. The lifts of dual Verma modules defined in the proof of the previous lemma fit into the following graded exact sequence*:

$$\nabla(xs \cdot 0)\langle 1 \rangle \overset{k}{\hookrightarrow} \theta_s \nabla(xs \cdot 0) \overset{j}{\twoheadrightarrow} \nabla(x \cdot 0).$$

Proof. The map f as in (3.4) gives a non-trivial map

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}\big(P, \nabla(xs \cdot 0)\big) &\to \operatorname{Hom}_{\mathfrak{g}}\big(P, \nabla(xs \cdot 0)\big) \otimes_A \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1 \rangle\big), \\ \phi &\mapsto \phi \otimes f, \end{aligned}$$

which has to be, up to a scalar, the map k in (3.6). The surjection on the right is by definition homogeneous of degree zero. This shows the assertion. □

4. Non-gradable objects

Although, all the ‘important’ objects of $\mathcal{O}_0$ are gradable, there exist in general objects which are not. The following theorem shows that the existence of such non-gradable modules is guaranteed by the existence of an inhomogeneous ideal of C. Such an ideal does not exist for all rank-two cases, but it is not difficult to find one in the case A_3. (With the notation of [St1] we can choose the ideal generated by $x + yz$.)

Theorem 4.1 (Non-gradability). *Let $I \lhd \mathbb{V}P(w_0 \cdot 0) = C$ be an ideal and $Q \in \mathcal{O}_0$ gradable. We also assume $\mathbb{V}Q \cong C/I$. Then the ideal I is homogeneous.*

Proof. Assume the module $\mathrm{Hom}_{\mathfrak{g}}(P, Q)$ is a graded right $A = \mathrm{End}_C(\mathbb{V}P) = \mathrm{End}_{\mathfrak{g}}(P)$-module. Via restriction $\mathrm{Hom}_{\mathfrak{g}}(P(w_0 \cdot 0), Q)$ becomes a graded right $\mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot 0))$-module. Because of

$$\mathrm{Hom}_{\mathfrak{g}}\big(P(w_0 \cdot 0), Q\big) = \mathrm{Hom}_{\mathfrak{g}}\big(P(w_0 \cdot 0) \otimes_C \mathbb{V}P(w_0 \cdot 0), Q\big) = \mathrm{Hom}_C\big(\mathbb{V}P(w_0 \cdot 0), C/I\big)$$

(see, e.g., [CR, 2.19]), the module $X := \mathrm{Hom}_C(\mathbb{V}P(w_0 \cdot 0), \mathbb{V}P(w_0 \cdot 0)/I)$ becomes a graded right $\mathrm{End}_C(\mathbb{V}P(w_0 \cdot 0))$-module, where $\mathrm{End}_C(\mathbb{V}P(w_0 \cdot 0)) = \mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot 0)) \subset A$ inherits its grading from A.

We have $I = \mathrm{Ann}_{\mathrm{End}(\mathbb{V}P(w_0 \cdot 0))} X$, hence I is homogeneous (see [Bo1, II, 11.3, Proposition 4]). $\square$

5. Some short exact sequences

In Section 3 we described the behavior of Verma modules $\Delta(x \cdot 0)$ and dual Verma modules $\nabla(xs \cdot 0)$ for $xs > x$ under translations through the s-wall. After forgetting the gradings, there are isomorphisms $\theta_s \Delta(x \cdot 0) \cong \theta_s \Delta(xs \cdot 0)$ and also $\theta_s \nabla(x \cdot 0) \cong \theta_s \nabla(xs \cdot 0)$. So there is no difference between the translation through the s-wall from ‘above’ or from ‘below’ the wall. In the graded case, with which we deal with in this section, the situation is different.

Simple objects lying ‘above’ the s-wall are sent to zero by applying translation through this wall (see [Ja2]). For simple objects lying ‘below the wall,’ the situation is much more complicated, but we can determine some multiplicities (see also Corollary 5.4).

Theorem 5.1. *Let $x \in W$ and let s be a simple reflection satisfying $xs > x$. Then the formula*

$$\big[\theta_s L(xs \cdot 0) : L(xs \cdot 0)\langle j \rangle\big] = \begin{cases} 1 & \text{if } j = \pm 1, \\ 0 & \text{otherwise}, \end{cases}$$

holds. Here, the simple module $L(xs \cdot 0)\langle 1 \rangle$ is a submodule and $L(xs \cdot 0)\langle -1 \rangle$ is a quotient of the translated module.

Proof. It is well known (see, e.g., [Ja2, 4.13(3')]) that after forgetting the grading $[\theta_s L(xs\cdot 0):L(xs\cdot 0)]=2$ holds. So it is sufficient to find two shifts such that the multiplicities in question are not zero.

The map $\bullet\otimes f$, as defined in (3.4), gives an inclusion $L(xs\cdot 0)\hookrightarrow \theta_s L(xs\cdot 0)$ which is homogeneous of degree 1. On the other hand, for a C-module M the multiplication $\mathrm{mult}: C\otimes_{C^s} M\to M$ induces a morphism

$$\mathrm{m}:\mathrm{Hom}_C\big(\mathbb{V}P, C\otimes_{C^s}\mathbb{V}P\langle -1\rangle\big)\xrightarrow{\mathrm{mult}\circ}\mathrm{Hom}_C(\mathbb{V}P,\mathbb{V}P)=\mathrm{End}_{\mathfrak{g}}(P) \tag{5.1}$$

of degree -1. So the map

$$h:\mathrm{Hom}_{\mathfrak{g}}\big(P, L(xs\cdot 0)\big)\otimes_A \mathrm{Hom}_C(\mathbb{V}P, C\otimes_{C^s}\mathbb{V}P)\to \mathrm{Hom}_{\mathfrak{g}}\big(P, L(xs\cdot 0)\big),$$
$$f\otimes g\mapsto f\circ \mathrm{m}(g),$$

gives a surjection being homogeneous of degree 1 in the other direction as $\bullet\otimes f$. Both maps together yield the two required shifts. □

The previous theorem shows in particular that the inductively defined lifts of the Verma modules and their duals do not depend (up to isomorphism) on x. The graded versions of the dual Verma modules have the following multiplicity formulas

Corollary 5.2. *For all $x\in W$ we have*

$$\big[\nabla(x\cdot 0):L(x\cdot 0)\langle i\rangle\big]=\begin{cases}0 & \text{if } i\neq 0,\\ 1 & \text{if } i=0.\end{cases}$$

Proof. The statement follows inductively, starting with $x=w_0$, from Theorem 5.1 using the results of Theorem 3.10. □

We are now ready to state some more graded short exact sequences.

Theorem 5.3. *Let $x\in W$ and let s be a simple reflection such that $xs>x$ holds. There exist the following exact sequences of graded modules*:

$$\Delta(x\cdot 0)\overset{j}{\hookrightarrow}\theta_s\Delta(xs\cdot 0)\overset{k}{\twoheadrightarrow}\Delta(xs\cdot 0)\langle -1\rangle, \tag{5.2}$$

$$\nabla(xs\cdot 0)\overset{k'}{\hookrightarrow}\theta_s\nabla(x\cdot 0)\overset{j'}{\twoheadrightarrow}\nabla(x\cdot 0)\langle -1\rangle. \tag{5.3}$$

Proof. All the maps in question are homogeneous, because of $\dim\mathrm{Hom}_{\mathfrak{g}}(\theta_s\Delta(xs\cdot 0),\Delta(xs\cdot 0))=1$. Consider the following map:

$$h:\mathrm{Hom}_{\mathfrak{g}}\big(P,\Delta(xs\cdot 0)\big)\otimes_A \mathrm{Hom}_C\big(\mathbb{V}P, C\otimes_{C^s}\mathbb{V}P\langle -1\rangle\big)\to \mathrm{Hom}_{\mathfrak{g}}\big(P,\Delta(xs\cdot 0)\big),$$
$$f\otimes g\mapsto f\circ\mathrm{m}(g),$$

where m is defined as in (5.1). This is a morphism of right A-modules, homogeneous of degree 1 and non-trivial. So it is, up to a scalar, the map k. Therefore, there is an exact sequence of graded modules of the form

$$\Delta(x\cdot 0)\langle n\rangle \overset{j}{\hookrightarrow} \theta_s\Delta(xs\cdot 0) \overset{k}{\twoheadrightarrow} \Delta(xs\cdot 0)\langle -1\rangle,$$

for some $n\in\mathbb{Z}$. On the other hand, we know that $\mathrm{f}(\theta_s\Delta(xs\cdot 0))\cong \mathrm{f}(\theta_s\Delta(x\cdot 0))$ is an indecomposable module. From Theorem 3.6 it follows that $n=0$ and so j has to be homogeneous of degree 0. The proof of the second statement is analogous. □

Without using results of Kazhdan–Lusztig theory, we can prove the following result.

Corollary 5.4. *With the assumptions of the previous theorem, the following equalities hold*:

$$\left[\theta_s L(xs\cdot 0):L(x\cdot 0)\langle j\rangle\right]=\begin{cases}1 & \textit{if } j=0,\\ 0 & \textit{otherwise}.\end{cases} \tag{5.4}$$

Proof. It follows directly from the sequence (5.2) and Corollary 3.7. □

Corollary 5.5. *With the assumptions from previous Theorem* 5.3, *there are isomorphisms of graded modules*

$$\theta_s\Delta(x\cdot 0)\cong\theta_s\Delta(xs\cdot 0)\langle 1\rangle, \tag{5.5}$$

$$\theta_s\nabla(x\cdot 0)\cong\theta_s\nabla(xs\cdot 0)\langle -1\rangle. \tag{5.6}$$

Proof. The assertion follows directly from the Theorems 3.6 and 3.10. □

Corollary 5.6. *Let $x\in W$. Then $I(x\cdot 0)$ is the injective hull of $L(x\cdot 0)$ in* gmof-A.

Proof. By definition $I(x\cdot 0)$ is indecomposable and injective. The inductive construction of these graded modules (see (3.6)) together with the previous corollary and Theorem 3.10 provides an injection $\nabla(x\cdot 0)\hookrightarrow I(x\cdot 0)$ which is homogeneous of degree 0. So the statement follows from Corollary 5.2. □

Corollary 5.7. *With the assumption of Theorem* 5.3, *the following formulas hold*:

$$\left[\Delta(x\cdot 0):L(xs\cdot 0)\langle j\rangle\right]=\begin{cases}1 & \textit{if } j=1,\\ 0 & \textit{otherwise}.\end{cases}$$

Proof. Assume $[\Delta(x\cdot 0):L(xs\cdot 0)\langle j\rangle]\neq 0$ for some $j\in\mathbb{Z}$. By Theorem 5.1 we have

$$\left[\theta_s\Delta(x\cdot 0):L(xs\cdot 0)\langle k\rangle\right]=\begin{cases}1 & \text{if } k=j\pm 1,\\ 0 & \text{otherwise}.\end{cases}$$

Hence, by Corollary 5.5,

$$\left[\theta_s\Delta(xs\cdot 0)\langle 1\rangle:L(xs\cdot 0)\langle k\rangle\right]=\begin{cases}1 & \text{if } k=j\pm 1,\\ 0 & \text{otherwise}.\end{cases}$$

By functoriality we get

$$\left[\Delta(x\cdot 0)\langle 1\rangle : L(xs\cdot 0)\langle j+1\rangle\right] = 1 \quad \text{and} \quad \left[\Delta(xs\cdot 0) : L(xs\cdot 0)\langle j-1\rangle\right] = 1.$$

On the other hand, the simple module $L(xs\cdot 0)$ is the head of $\Delta(xs\cdot 0)$ by Corollary 3.7, hence $j=1$ and the statement follows. □

Corollary 5.8.

$$\left[\nabla(x\cdot 0) : L(xs\cdot 0)\langle j\rangle\right] = \begin{cases} 1 & \text{if } j=-1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. With analogous arguments as in Corollary 5.7 the statement follows. □

6. Gradings and duality

In the following section we define a duality functor d on gmof-A corresponding to a regular integral block. Without any additional effort (but with more indices) this can also be done for singular integral blocks.

6.1. A graded version of duality

The question is, whether there is a ‘graded duality,’ i.e., a functor d on gmof-A which induces the duality $\star$ on $\mathcal{O}_0$ after forgetting the grading. It should also fulfill the following conditions: *We require that* d *fixes the lift of the simple Verma module and commutes with translation functors.*

6.1.1. Properties of a ‘graded duality’

A ‘graded duality’ d (if it exists) is uniquely defined by the image of the dominant Verma module. The whole information is given by the following lemma.

Lemma 6.1. *If there is a ‘graded duality’* d *of the duality $\star$ on $\mathcal{O}_0$, the following statements have to be true for all $x\in W$ and $j\in\mathbb{Z}$:*

$$\begin{aligned} \mathrm{d}\Delta(x\cdot 0)\langle j\rangle &\cong \nabla(x\cdot 0)\langle -j\rangle, \\ \mathrm{d}L(x\cdot 0)\langle j\rangle &\cong L(x\cdot 0)\langle -j\rangle, \quad \textit{and} \\ \mathrm{d}P(x\cdot 0) &\cong I(x\cdot 0). \end{aligned}$$

Proof. Dualizing (5.2) for $xs=w_0$ yields

$$\mathrm{d}\big(\Delta(w_0\cdot 0)\langle -1\rangle\big) \hookrightarrow \theta_s\nabla(w_0\cdot 0) \twoheadrightarrow \nabla(w_0 s\cdot 0)\langle j\rangle$$

for some j. Theorem 5.3 and Corollary 5.5 implies $j=0$ and so we get $\mathrm{d}(\Delta(w_0 s\cdot 0))\cong \nabla(w_0 s\cdot 0)$ and $\mathrm{d}(\Delta(w_0\cdot 0)\langle -1\rangle)\cong \nabla(w_0\cdot 0)\langle 1\rangle$. Inductively the first statement follows.

The Lemmata 5.2 and 3.7 imply the second assertion.

For the last one, it is sufficient to show that $I(x \cdot 0)$ is the injective hull of $L(x \cdot 0)$, because $P(x \cdot 0)$ is its projective cover. This is just Corollary 5.6. □

6.1.2. Existence of a 'graded duality'

Recall the duality for graded C-modules. Given $M \in C$-gmof, the duality is defined as $M^* := \mathrm{Hom}_C(M, C) \cong \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ (note C is commutative). Due to [So4] all the images $\mathbb{V}P(x \cdot 0)$ of projectives modules under $\mathbb{V}$ are self-dual. We fix for each indecomposable projective such an isomorphism. This implies

$$\begin{aligned}\mathrm{End}_{\mathfrak{g}}(P) &= \mathrm{End}_{\mathfrak{g}}\Big(\bigoplus_{x \in W} P(x \cdot 0)\Big) = \mathrm{End}_C\Big(\bigoplus_{x \in W} \mathbb{V}P(x \cdot 0)\Big) = \mathrm{End}_C\Big(\bigoplus_{x \in W} \mathbb{V}P(x \cdot 0)^*\Big) \\ &= \mathrm{End}_{\mathfrak{g}}(P)^{\mathrm{opp}}. \end{aligned} \tag{6.1}$$

This provides a duality $*$ on mof-A defined as $M^* := \mathrm{Hom}_A(M, A)$ for any $M \in$ mof-A and even a duality $\circledast$ on gmof-A by setting $(M^{\circledast})_{-i} := M_i^*$.

This duality is contravariant and exact and maps a simple module $L\langle j \rangle$ concentrated in degree j to $L\langle -j \rangle$. Therefore, its projective cover is mapped to the injective hull of $L\langle -j \rangle$. Moreover, this duality commutes with translation through the walls. To see this take $M \in$ gmof-A and let $V := \mathrm{Hom}_C(\mathbb{V}P, C \otimes_{C^s} \mathbb{V}P\langle -1 \rangle)$ be the graded A-bimodule, which describes the translation θ_s. We get the following isomorphisms of right A-modules

$$\begin{aligned}(\theta_s M^{\circledast})^{\circledast} &\cong \mathrm{Hom}_A(M^{\circledast} \otimes_A V, A) \cong \mathrm{Hom}_A(M^{\circledast}, \mathrm{Hom}_A(V, A)) \\ &\cong \mathrm{Hom}_A(A, M \otimes_A V^{\circledast}) \cong M \otimes_A V^{\circledast} \cong M \otimes_A V \cong \theta_s M.\end{aligned}$$

We used the isomorphism $V \cong V^{\circledast}$, which follows from the choices of isomorphisms in (6.1).

7. Gradings, combinatorics, and Hecke algebra

In the following section we describe the combinatorics of graded translation functors. This is a generalization of the results in (1.2).

We denote by $[\mathcal{O}_0^{\mathbb{Z}}] = [\text{gmof-}A]$ the *'graded' Grothendieck group of* $\mathcal{O}_0$, which is in fact the Grothendieck group of the graded version gmof-A of mof-$A \cong \mathcal{O}_0$. Each of the sets $\{[\Delta(x \cdot 0)\langle n \rangle] \mid x \in W, n \in \mathbb{Z}\}$ and $\{[L(x \cdot 0)\langle n \rangle] \mid x \in W,\ n \in \mathbb{Z}\}$ forms a $\mathbb{Z}$-basis of $[\mathcal{O}_0^{\mathbb{Z}}]$. Moreover, $[\mathcal{O}_0^{\mathbb{Z}}]$ can be considered as a $\mathbb{Z}[v, v^{-1}]$-module defined by $v^n[M] := [M\langle n \rangle]$. We get isomorphisms of abelian groups

$$\begin{aligned} \mathrm{H} &\longrightarrow [\mathcal{O}_0^{\mathbb{Z}}] \longrightarrow (\mathbb{Z}[v, v^{-1}])[W], \\ v^n H_x &\mapsto [\Delta(x \cdot 0)\langle n \rangle] \mapsto v^n x^{-1}, \end{aligned} \tag{7.1}$$

which are in fact morphisms of $\mathbb{Z}[v, v^{-1}]$-modules.

So far, the main results of the paper can be summarized in the following commuting diagram.

Theorem 7.1 (Graded combinatorics). *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{H} & \xrightarrow{v^n H_x \mapsto [\Delta(x\cdot 0)\langle n\rangle]} & [\mathcal{O}_0^{\mathbb{Z}}] \\ {\scriptstyle \cdot(H_s+v)}\downarrow & & \downarrow{\scriptstyle [\theta_s]} \\ \mathrm{H} & \xrightarrow{v^n H_x \mapsto [\Delta(x\cdot 0)\langle n\rangle]} & [\mathcal{O}_0^{\mathbb{Z}}] \end{array} \; . \tag{7.2}$$

Proof. This follows from the isomorphism (7.1), Theorems 3.6 and 5.3. □

7.1. Projectives and the graded reciprocity formula

Our next aim is to compute multiplicities for the lifts of indecomposable projective modules. Let $M \in$ gmof-A. The 'multiplicities of graded Verma modules' $[M : \Delta(x \cdot 0)\langle i\rangle]$ are defined by the following equality:

$$[M] = \bigoplus_{x \in W} \big[M : \Delta(x \cdot 0)\langle i\rangle\big]\big[\Delta(x \cdot 0)\langle i\rangle\big]. \tag{7.3}$$

A module $M \in$ gmof-A *has a graded Verma flag* (or *has a graded dual-Verma-flag*) if there is a filtration of graded A-modules for M whose subquotients are all isomorphic to lifts of Verma modules (or dual Verma modules, respectively). The multiplicities (7.3) are in fact the number of times a Verma module occurs in a graded Verma flag:

Theorem 7.2 (see [Ja3]). *Let $\widetilde{M} \in \mathcal{O}_0$ be gradable with lift M. Assume that for all $x \in W$ and $j \in \mathbb{Z}$*

$$\mathrm{Ext}^1_{\text{gmof-}A}\big(M, \nabla(x \cdot 0)\langle j\rangle\big) = 0$$

holds. Then M has a graded Verma flag.

Proof. The proof is just a mimic of [Ja3, II, 4.16]. Details can be found in [St2]. □

Corollary 7.3. *All graded lifts of the projective objects have a graded Verma flag.*

Proof. Given two graded A-modules M and N there is an isomorphism of vector spaces

$$\bigoplus_j \mathrm{Ext}^1_{\text{gmof-}A}\big(M\langle j\rangle, N\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\text{mof-}A}(\mathrm{f}M, \mathrm{f}N).$$

So the projective objects fulfill the assertion of the theorem. □

Corollary 7.4. *All graded lifts of injective modules have a graded dual-Verma-flag. More precisely, for $x, y \in W$ and $j \in \mathbb{Z}$ the following multiplicities hold*:

$$\big[P(x\cdot 0):\Delta(y\cdot 0)\langle j\rangle\big] = \big[I(x\cdot 0):\nabla(y\cdot 0)\langle -j\rangle\big].$$

Proof. The proof is given by dualizing the arguments above. □

The following lemma is a graded version of a well-known result.

Lemma 7.5. *Let $x, y \in W$ and $i \in \mathbb{Z}$. Then the following formula is true*:

$$\dim \operatorname{Hom}_{\text{gmof-}A}\big(\Delta(x\cdot 0), \nabla(y\cdot 0)\langle i\rangle\big) = \begin{cases} 1 & \text{if } x = y \text{ and } i = 0,\\ 0 & \text{otherwise}.\end{cases}$$

Proof. Standard argumentation (see, e.g., [KK]) shows that for the existence of such a homomorphism, it is necessary that $x = y$ holds. Then the Lemmata 3.7 and 5.2 imply also $i = 0$. □

Now we are ready to prove the graded version of the so-called BGG-reciprocity (see [BGS, Theorem 3.11.4]).

Theorem 7.6 (Reciprocity). *For $x, y \in W$ and $i \in \mathbb{Z}$, the following reciprocity formulae hold*:

$$\big[P(x\cdot 0):\Delta(y\cdot 0)\langle i\rangle\big] \overset{\text{(a)}}{=} \big[\nabla(y\cdot 0):L(x\cdot 0)\langle -i\rangle\big] \overset{\text{(b)}}{=} \big[\Delta(y\cdot 0):L(x\cdot 0)\langle i\rangle\big].$$

Proof. If $y \nleqslant x$, the statement is trivial. So let $y \leqslant x$. The Lemma 7.5 implies

$$\begin{aligned} &\dim \operatorname{Hom}_{\text{gmof-}A}\big(P(x\cdot 0), \nabla(y\cdot 0)\langle i\rangle\big)\\ &\quad = \sum_{z,j}\big[P(x\cdot 0):\Delta(z\cdot 0)\langle j\rangle\big] \dim \operatorname{Hom}_{\text{gmof-}A}\big(\Delta(z\cdot 0)\langle j\rangle, \nabla(y\cdot 0)\langle i\rangle\big)\\ &\quad = \big[P(x\cdot 0):\Delta(y\cdot 0)\langle i\rangle\big]. \end{aligned}$$

So this leads to

$$\begin{aligned} \big[P(x\cdot 0):\Delta(y\cdot 0)\langle i\rangle\big] &= \dim \operatorname{Hom}_{\text{gmof-}A}\big(P(x\cdot 0), \nabla(y\cdot 0)\langle i\rangle\big)\\ &= \big[\nabla(y\cdot 0)\langle i\rangle : L(x\cdot 0)\big]\\ &= \big[\nabla(y\cdot 0) : L(x\cdot 0)\langle -i\rangle\big]. \end{aligned}$$

Thus statement (a) is proven. The second statement, by Lemma 6.1, is just the dual of the first one. □

So far, we have not used the results of the Kazhdan–Lusztig theory or the fact, that the endomorphism ring of a projective generator of a block in $\mathcal{O}$ is positively graded (see [BGS]). We can reprove the latter using Kazhdan–Lusztig theory.

Lemma 7.7. *All the inductively defined lifts of Verma modules and projective modules are positively graded. In particular, the projective generator is positively graded.*

Proof. One result of Kazhdan–Lusztig theory is that for a reduced expression $x = s_1 \cdot \cdots \cdot s_r$ for any $x \in W$ the coefficients of $(H_{s_1} + v) \cdot \cdots \cdot (H_{s_r} + v)$, expressed in the standard basis, are polynomials, i.e., elements of $\mathbb{Z}[v]$. By Theorem 7.1 and Lemma 7.6, all Verma modules and all projective modules are positively graded. □

Remarks 7.8.

(a) Using the results of Kazhdan–Lusztig theory it is also possible to prove that our graded versions of indecomposable projective modules correspond via Theorem 7.1 to the elements of the self-dual Kazhdan–Lusztig basis $\underline{H}_x$ (in the notation of [So3]).
(b) In [So2] a generalized functor $\mathbb{V}$ is defined for the category of Harish-Chandra bimodules with generalized trivial central character from both sides. The subcategory of all bimodules with fixed generalized trivial character from one side has enough projectives and can also be considered as a module category over some graded ring. But in general, the images of the projective modules are not necessarily self-dual. That means that graded versions of translation functors can be defined, but it is not obvious how to get a graded duality.

8. Translations onto and out of the wall

Let λ be a semi-regular integral weight, i.e., $|W_\lambda| = |\{1, s\}| = 2$ for some simple reflection s. We denote by $\text{res} = \text{res}^C_{C^\lambda}$ the restriction functor. The right A_λ-module $\text{Hom}_\mathfrak{g}(P_\lambda, \theta_0^\lambda P) = \text{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \text{res}\,\mathbb{V}P)$ becomes via composition a left A-module, i.e., $g.f := (\mathbb{V}f) \circ g$ for $g \in A = \text{End}_\mathfrak{g}(P)$ and $f \in \text{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \text{res}\,\mathbb{V}P)$.

Theorem 8.1 (Translation onto the wall).

(1) *Let λ be as above. Then the functor θ_0^λ is gradable with lift*

$$\bullet \otimes_A \text{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \text{res}\,\mathbb{V}P). \tag{8.1}$$

(2) *The lifts for the simple modules concentrated in degree zero are annihilated or are mapped to a simple module concentrated in degree -1 by the functor defined in* (8.1).
(3) *Let $xs > x$ for some $x \in W$ and a simple reflection s. Concerning Verma modules, there are isomorphisms of graded modules as follows*:

$$\theta_0^\lambda \Delta(x \cdot 0) \cong \Delta(x \cdot \lambda) \quad \textit{and}$$
$$\theta_0^\lambda \Delta(xs \cdot 0) \cong \Delta(xs \cdot \lambda)\langle -1\rangle \cong \Delta(x \cdot \lambda)\langle -1\rangle.$$

Proof. The first statement follows directly from Lemma 3.4.

To prove the second statement, we can assume that the given lift of a simple module is not annihilated by the functor. So we can assume that $xs > x$ holds. By construction, the module $P(xs \cdot 0)$ is a direct summand of $\theta_s P(x \cdot 0)$. More precisely, there is a decomposition as graded C-modules of the form

$$\mathbb{V}P(xs \cdot 0)\langle 1\rangle \oplus \mathbb{V}\mathrm{R} = C \otimes_{C^s} \mathbb{V}P(x \cdot 0)$$

for some graded C-module R. Restriction to C^λ-modules yields a decomposition

$$\begin{aligned}\mathbb{V}P(xs \cdot 0)\langle 1\rangle \oplus \mathbb{V}\mathrm{R} &= \mathbb{V}P(x \cdot 0) \oplus \mathbb{V}P(x \cdot 0)\langle 2\rangle \\ &= \mathbb{V}_\lambda\big(P(x \cdot \lambda)\big) \oplus \mathbb{V}_\lambda\big(P(x \cdot \lambda)\big)\langle 2\rangle \oplus N \end{aligned} \tag{8.2}$$

for some N. (The second equality comes from the fact that $P(x \cdot \lambda)$ is a direct summand of $\theta_0^\lambda P(x \cdot 0)$, hence $\mathbb{V}_\lambda(P(x \cdot \lambda))$ is a direct summand of $\mathbb{V}\theta_0^\lambda P(x \cdot 0) \cong \mathrm{res}_{C^\lambda}^C \mathbb{V}P(x \cdot 0)$, see [So1, Theorem 10].)

Using these decompositions, it is possible to define a morphism of C^λ-modules as follows:

$$\tilde{h} : \mathbb{V}_\lambda(P_\lambda) \twoheadrightarrow \mathbb{V}_\lambda\big(P(x \cdot \lambda)\big) \xrightarrow{\mathrm{id}} \mathbb{V}_\lambda\big(P(x \cdot \lambda)\big)\langle 2\rangle \xrightarrow{f} \mathbb{V}P(xs \cdot 0)\langle 1\rangle,$$

where the first map is the canonical projection and where f is given by the decomposition (8.2). The composition with the natural inclusion provides a homogeneous morphism $h : \mathbb{V}_\lambda(P_\lambda) \to \mathbb{V}P$ of degree -1.

We consider the map

$$\begin{aligned}&\mathrm{Hom}_C\big(\mathbb{V}P, \mathbb{V}P(xs \cdot 0)\big) \otimes_A \mathrm{Hom}_{C^\lambda}\big(\mathbb{V}_\lambda(P_\lambda), \mathrm{res}\,\mathbb{V}P\big)\\ &\quad \to \mathrm{Hom}_{C^\lambda}\big(\mathbb{V}_\lambda(P_\lambda), \mathbb{V}_\lambda\big(P(x \cdot \lambda)\big)\langle 1\rangle\big),\\ &f \otimes g \mapsto \mathrm{p}_2 \circ \big(\mathrm{res}(f) \circ g\big), \end{aligned} \tag{8.3}$$

where p_2 denotes the projection onto the second summand of (8.2). This map is well defined and homogeneous of degree 0. Taking for f the canonical projection onto the direct summand $\mathbb{V}P(xs \cdot 0)$, the image of $f \otimes h$ is non-trivial. More precisely, the canonical projection $P_\lambda \twoheadrightarrow P(x \cdot \lambda)$ is of degree -1. So the second assertion is proven.

The third statement is well-known after forgetting the grading. So we have to find the correct shifts. Using the result

$$\big[\Delta(x \cdot 0) : L(xs \cdot 0)\langle j\rangle\big] = \begin{cases} 1 & \text{if } j = 1,\\ 0 & \text{otherwise,} \end{cases} \tag{8.4}$$

of Corollary 5.7 and the second statement, we get the following multiplicities:

$$\big[\theta_0^\lambda \Delta(x \cdot 0) : L(x \cdot \lambda)\langle j\rangle\big] = \begin{cases} 1 & \text{if } j = 0,\\ 0 & \text{otherwise.} \end{cases} \tag{8.5}$$

So, the existence of the first of these isomorphisms is proven. For the existence of the second the arguments are similar. □

The following theorem describes translation out of the wall.

Theorem 8.2 (Translation out of the wall).

(1) *With λ as above, the functor θ_λ^0 is gradable with lift*

$$\bullet \otimes_{A_\lambda} \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle\big).$$

(2) *For $xs > x$, there are natural isomorphisms of graded modules*

$$\theta_\lambda^0 \Delta(x\cdot\lambda) \cong \theta_s \Delta(x\cdot 0) \quad \textit{and} \quad \theta_\lambda^0 \Delta(xs\cdot\lambda)\langle -1\rangle \cong \theta_s \Delta(xs\cdot 0).$$

(3) *Let $xs > x$ for some $x \in W$ and a simple reflection s. Then there is an isomorphism of graded modules $\theta_\lambda^0 P(x\cdot\lambda) \cong P(xs\cdot 0)$.*

(4) *Considering the simple modules for $xs > x$ the following statements hold*:

$$\big[\theta_\lambda^0 L(x\cdot\lambda) : L(x\cdot 0)\langle j\rangle\big] = \begin{cases} 1 & \textit{if } j = 1, \\ 0 & \textit{otherwise}, \end{cases} \quad \textit{and}$$

$$\big[\theta_\lambda^0 L(x\cdot\lambda) : L(xs\cdot 0)\langle j\rangle\big] = \begin{cases} 1 & \textit{if } j \in \{0, 2\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The first statement follows from Lemma 3.4.

To show the second one, we use induction on the length of x. If $x = e$, the natural isomorphisms are given by the composition

$$\begin{array}{ccc}
\operatorname{Hom}_{C^\lambda}\big(\mathbb{V}P_\lambda, \mathbb{V}\Delta(\lambda)\big) \otimes_{A_\lambda} \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle\big) & & f \otimes g \\
\downarrow\wr & & \downarrow \\
\operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}\Delta(\lambda)\langle -1\rangle\big) & & (\mathrm{id} \otimes f) \circ g \\
\|\wr & & \\
\operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}\Delta(0)\langle -1\rangle\big) & & (\mathrm{id} \otimes f) \circ g \\
\wr\uparrow & & \uparrow \\
\operatorname{Hom}_{C^\lambda}\big(\mathbb{V}P_\lambda, \mathbb{V}\Delta(0)\big) \otimes_{A_\lambda} \operatorname{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle\big) & & f \otimes g
\end{array}$$

Let $x \in W$ be such that $xs > x$ holds and let t be a simple reflection satisfying $xt > x$. There are two cases to consider:

$t \neq s$: In this case there are canonical inclusions

$$\Delta(xt\cdot 0)\langle 1\rangle \hookrightarrow \Delta(x\cdot 0) \quad \text{and} \quad \Delta(xt\cdot\lambda)\langle 1\rangle \hookrightarrow \Delta(x\cdot\lambda).$$

Hence, the induced isomorphism $\theta_s \Delta(xt \cdot 0) \cong \theta_\lambda^0 \Delta(xt \cdot \lambda)$ is homogeneous of degree 0.

$t = s$: In this case, there is an isomorphism of graded modules

$$\theta_s \Delta(xt \cdot 0)\langle 1\rangle \cong \theta_\lambda^0 \Delta(xt \cdot \lambda).$$

Inductively the second statement follows.

The assertion concerning the projective objects is well-known after forgetting the grading. The second statement gives also the graded version.

Using Theorems 8.1, 5.1, and Corollary 5.4, the following statements for simples are true:

$$\left[\theta_\lambda^0 L(x \cdot \lambda) : L(x \cdot 0)\langle j\rangle\right] = \left[\theta_s L(xs \cdot 0)\langle 1\rangle : L(x \cdot 0)\langle j\rangle\right] = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\left[\theta_\lambda^0 L(x \cdot \lambda) : L(xs \cdot 0)\langle j\rangle\right] = \left[\theta_s L(xs \cdot 0)\langle 1\rangle : L(xs \cdot 0)\langle j\rangle\right] = \begin{cases} 1 & \text{if } j \in \{0, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

So the theorem is proved. □

The translation functors are related to each other by the following

Corollary 8.3. *There is a natural equivalence of functors* gmof-$A \to$ gmof-A:

$$\theta_s \cong \theta_\lambda^0 \theta_0^\lambda.$$

Proof. The previous theorem shows that the natural isomorphisms are compatible with the grading. □

The following Adjointness Theorem is a very strong tool. On the one hand, it is a generalization of the non-graded case, but on the other hand, the adjointness property is satisfied only up to a grading shift.

Theorem 8.4 (Adjointness). *There are the following two adjoint pairs of graded versions of translation functors*:

$$\left(\theta_0^\lambda, \theta_\lambda^0\langle -1\rangle\right) \quad \textit{and} \quad \left(\theta_\lambda^0, \theta_0^\lambda\langle 1\rangle\right).$$

Proof. Let $M \in$ gmof-A and $N \in$ gmof-A_λ. There are isomorphisms of graded vector spaces

$$\mathrm{Hom}_{\text{gmof-}A_\lambda}\left(M \otimes_A \mathrm{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \operatorname{res}\mathbb{V}P), N\right) \cong \mathrm{Hom}_{\text{gmof-}A}(M, X)$$

with $X = \mathrm{Hom}_{\text{gmof-}A_\lambda}(\mathrm{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \operatorname{res}\mathbb{V}P), N)$. Setting $Y = \mathrm{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \operatorname{res}\mathbb{V}P)$, we get the following isomorphisms of graded vector spaces:

$$X \cong \mathrm{Hom}_{A_\lambda\text{-gmof}}\big(N^{\circledast}, Y^{\circledast}\big) \cong \mathrm{Hom}_{A_\lambda\text{-gmof}}\big(N^{\circledast}, \mathrm{Hom}_{\mathrm{gmof}\text{-}A_\lambda}(Y, A_\lambda)\big)$$
$$\cong \mathrm{Hom}_{A_\lambda\text{-gmof}}\big(Y \otimes_{A_\lambda} N^{\circledast}, A_\lambda\big) \cong \big(Y \otimes_{A_\lambda} N^{\circledast}\big)^{\circledast} \cong N \otimes_{A_\lambda} Y^{\circledast}. \tag{8.6}$$

With fixed isomorphisms $\mathbb{V}Q \cong \mathrm{d}\mathbb{V}Q$ for each indecomposable projective module Q this yields isomorphisms of graded vector spaces

$$Y^{\circledast} = \mathrm{Hom}_{\mathrm{gmof}\text{-}A_\lambda}(Y, A_\lambda) \cong \mathrm{Hom}_{C^\lambda}(\mathrm{res}\,\mathbb{V}P, \mathbb{V}P_\lambda) \cong \mathrm{Hom}_{C^\lambda}(\mathbb{V}P_\lambda, \mathrm{res}\,\mathbb{V}P)$$
$$\cong \mathrm{Hom}_C(C \otimes_{C^\lambda} \mathbb{V}P_\lambda, \mathbb{V}P) \cong \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -2\rangle\big)$$
$$\cong \mathrm{Hom}_C\big(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle\big)\langle 1\rangle.$$

For the second and the fourth isomorphism we used the self-duality of $\mathbb{V}Q$ for projective modules Q and the canonical isomorphism $(C \otimes_{C^\lambda} M)^* \cong C \otimes_{C^\lambda} (M^*)\langle -2\rangle$ (see Section 6.1.2 and [So4]). The third step is just the adjointness property of restriction and induction. Altogether, we get an isomorphism compatible with the grading

$$\mathrm{Hom}_{\mathrm{gmof}\text{-}A_\lambda}\big(\theta_0^\lambda M, N\big) \cong \mathrm{Hom}_{\mathrm{gmof}\text{-}A}\big(M, \theta_\lambda^0 N\langle -1\rangle\big).$$

To see the second adjunction let $M \in \mathrm{gmof}\text{-}A_\lambda$ and $N \in \mathrm{gmof}\text{-}A$. There is an isomorphism of graded vector spaces

$$\mathrm{Hom}_{\mathrm{gmof}\text{-}A}\big(M \otimes_A \mathrm{Hom}_C(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda)\langle -1\rangle, N\big) \cong \mathrm{Hom}_{\mathrm{gmof}\text{-}A}(M, X)$$

with $X = \mathrm{Hom}_{\mathrm{gmof}\text{-}A_\lambda}(\mathrm{Hom}_C(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle), N)$. Setting $W = \mathrm{Hom}_C(\mathbb{V}P, C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle)$, we get in a similar way to (8.6) an isomorphism

$$X \cong N \otimes_A W^{\circledast}.$$

On the other hand,

$$W^{\circledast} = \mathrm{Hom}_{\mathrm{gmof}\text{-}A}(W, A) \cong \mathrm{Hom}_C\big(C \otimes_{C^\lambda} \mathbb{V}P_\lambda\langle -1\rangle, \mathbb{V}P\big)$$
$$\cong \mathrm{Hom}_{C^\lambda}\big(\mathbb{V}P_\lambda\langle -1\rangle, \mathrm{res}\,\mathbb{V}P\big) = \mathrm{Hom}_{C^\lambda}\big(\mathbb{V}P_\lambda, \mathrm{res}\,\mathbb{V}P\big)\langle -1\rangle$$

holds. This gives the desired isomorphism of graded vector spaces:

$$\mathrm{Hom}_{\mathrm{gmof}\text{-}A}\big(\theta_\lambda^0 M, N\big) \cong \mathrm{Hom}_{\mathrm{gmof}\text{-}A_\lambda}\big(M, \theta_0^\lambda N\langle 1\rangle\big). \quad \square$$

Corollary 8.5. *The functor θ_s (as a functor on* gmof-*A) is self-adjoint.*

Proof. The previous theorem together with Theorem 8.3 implies natural isomorphisms

$$\mathrm{Hom}_{\mathrm{gmof}\text{-}A}(\theta_s M, N) \cong \mathrm{Hom}_{\mathrm{gmof}\text{-}A}\big(\theta_0^\lambda M, \theta_0^\lambda N\langle 1\rangle\big) \cong \mathrm{Hom}_{\mathrm{gmof}\text{-}A}(M, \theta_s N)$$

of graded vector spaces. $\square$

Acknowledgments

I thank Wolfgang Soergel for many helpful discussions and comments. My thanks are also to Martin Härterich and Steffen König for comments on a very first draft of this paper.

References

[AJS] H. Andersen, J. Jantzen, W. Soergel, Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p, Astérisque 220 (1994).

[Ba] H. Bass, Algebraic K-theory, Benjamin, 1968.

[BB1] A. Beilinson, J.N. Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris 292 (1981) 15–18.

[BB2] A. Beilinson, J.N. Bernstein, A proof of Jantzen conjectures, Adv. Soviet Math. 16 (1993) 1–50.

[Be] I.N. Bernstein, Trace in categories, in: A. Connes, et al. (Eds.), Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, in: Progr. Math., Vol. 92, 1990, pp. 417–423.

[BG] I. Bernstein, S.I. Gelfand, Tensor products of finite and infinite dimensional representations of semisimple Lie algebras, Compositio Math. 41 (1980) 245–285.

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, A category of $\mathfrak{g}$-modules, Funct. Anal. Appl. 10 (1976) 87–92.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996) 473–527.

[BK] J.-L. Brylinski, M. Kashiwara, Kazhdan–Lustzig conjecture and holonomic systems, Inventiones 64 (1981) 387–410.

[Bo1] N. Bourbaki, Algebra I, Springer, 1989.

[Bo2] N. Bourbaki, Groupes et algèbres de Lie, Masson, 1994.

[CC] L. Casian, D. Collingwood, Weight filtrations for induced representations of real reductive Lie groups, Adv. Math. 73 (1989) 79–146.

[CR] C. Curtis, I. Reiner, Methods of Representation Theory I, Wiley, 1981.

[Di] J. Dixmier, Enveloping Algebras, in: Grad. Stud. Math., Vol. 11, AMS, 1996.

[Ja1] J.C. Jantzen, Moduln mit einem höchsten Gewicht, Springer, 1979.

[Ja2] J.C. Jantzen, Einhüllende Algebren halbeinfacher Liealgebren, Springer, 1983.

[Ja3] J.C. Jantzen, Representations of Algebraic Groups, in: Pure Appl. Math., Vol. 131, Academic Press, 1987.

[Jo] A. Joseph, Problems old and new, in: J.-L. Brylinski (Ed.), In honor of Bertram Kostant, Birkhäuser, 1994, pp. 385–414.

[KK] M. Klucznik, S. König, Characteristic tilting modules over quasi-hereditary algebras, Preprint-Server of the SFB 343 E99-004, Universität Bielefeld.

[KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Inventiones 53 (1979) 191–213.

[So1] W. Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, J. Amer. Math. Soc. 3 (1990) 421–445.

[So2] W. Soergel, The combinatorics of Harish-Chandra bimodules, J. Reine Angew. Math. 429 (1992) 49–74.

[So3] W. Soergel, Kazhdan–Lusztig polynomials and a combinatoric for tilting modules, Represent. Theory 1 (1997) 83–114.

[So4] W. Soergel, Über die Beziehung von Schnittkohomologie und Darstellungstheorie in endlicher Charakteristik, preprint; English version: J. Pure Appl. Algebra 152 (2000) 311–335.

[St1] C. Stroppel, Category $\mathcal{O}$: quivers and endomorphism rings of projectives, preprint, 2001.

[St2] C. Stroppel, Der Kombinatorikfunktor $\mathbb{V}$: Graduierte Kategorie $\mathcal{O}$, Hauptserien und primitive Ideale, Dissertation, Universität Freiburg i. Br., 2001.