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Koszul gradings on Brauer algebras

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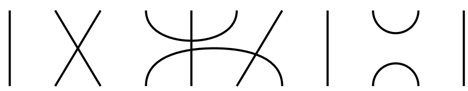
We show that the Brauer algebra $\mathrm{Br}_d(\delta)$ over the complex numbers for an integral parameter δ can be equipped with a grading. In case $\delta \neq 0$ it becomes a graded quasi-hereditary algebra which is moreover Morita equivalent to a Koszul algebra. These results are obtained by realizing the Brauer algebra as an idempotent truncation of a certain level two VW-algebra $\mathbb{W}_d^{\mathrm{cycl}}(N)$ for some large positive integral parameter N. The parameter δ appears here in the choice of a cyclotomic quotient. This cyclotomic VW-algebra arises naturally as an endomorphism algebra of a certain projective module in parabolic category $\mathcal{O}$ of type D. In particular, the graded decomposition numbers are given by the associated parabolic Kazhdan-Lusztig polynomials.

1 Introduction

We fix as ground ring the complex numbers $\mathbb{C}$. Given an integer $d \geq 1$ and $\delta \in \mathbb{C}$, the associated *Brauer algebra* $\mathrm{Br}_d(\delta)$ is a diagrammatically defined algebra with basis all Brauer diagrams on $2d$ points, that is all possible matchings of $2d$ points with d strands, such that each point is the endpoint of exactly one strand. In other words, the basis elements correspond precisely to subdivisions of the set of $2d$ points into subsets of order 2. Here is an example of such a Brauer diagram (with $d = 11$):



(1.1)

The multiplication is given on these basis vectors by a simple diagrammatic rule: we fix the positions of the $2d$ points as in the diagram (1.1) with d points at the bottom and d points at the top. Then the product bb' is equal to $\delta^k(b' \circ b)$, where $b' \circ b$ is the Brauer diagram obtained from the two basis elements b and b' by stacking b' on top of b (identifying the bottom points of b' with the top points of b in the obvious way) and then turning the result into a Brauer diagram by removing all internal circles; hereby, k is the number of such circles removed.

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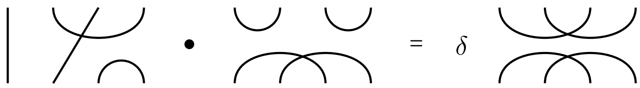
For instance:



(1.2)

Brauer algebras form important examples of cellular diagram algebras in the sense of [18]. In particular we have *cell modules* (or *Specht modules*) $\Delta(\lambda)$ indexed by $\lambda \in \Lambda_d$. Here

$$\Lambda_d = \bigcup_{m \in \mathbb{Z}_{\geq 0} \cap (d - 2\mathbb{Z}_{\geq 0})} \mathrm{Par}(m),$$

with $\mathrm{Par}(m)$ denoting the partitions of the integer m. We have simple modules $L(\lambda)$ for $\lambda \in \Lambda_d^\delta$, where $\Lambda_d^\delta = \Lambda_d$ in case of $\delta \neq 0$ and $\Lambda_d^\delta = \Lambda_d \setminus \mathrm{Par}(0)$ in case of $\delta = 0$, see [9].

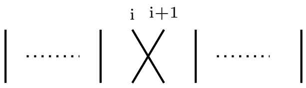
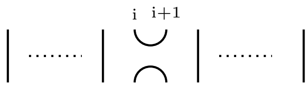
Although the Brauer algebra can be defined for arbitrary $\delta \in \mathbb{C}$ it turns out that it is always semi-simple for $\delta \notin \mathbb{Z}$, see [31] or [27]. For our purposes these cases are trivial, hence *we will always assume* $\delta \in \mathbb{Z}$.

Brauer algebras were originally introduced by Brauer [6] in the context of classical invariant theory as centralizer algebras of tensor products of the natural representation of orthogonal and symplectic Lie algebras. More precisely, assuming $d < n$ there is a canonical isomorphism of algebras

$$\mathrm{End}_{\mathfrak{g}}(V^{\otimes d}) \quad \cong \quad \mathrm{Br}_d(N), \tag{1.3}$$

where $\mathfrak{g}$ is an orthogonal or symplectic Lie algebra of rank n with vector representation V of dimension N in the orthogonal case and dimension $-N$ in the symplectic case, see e.g. [11] or [17] for details.

As an algebra, the Brauer algebra is generated by the following elements t_i, g_i, with $1 \leq i \leq d-1$,

t_i  g_i 

(1.4)

and t_i acts on $V^{\otimes d}$ in (1.3) by permuting the ith and $(i+1)$st tensor factor, and the element g_i acts by applying to the ith and $(i+1)$st factor the evaluation morphism $V \otimes V = V^* \otimes V \to \mathbb{C}$ followed by its adjoint.

The realization (1.3) as centralizers includes the cases $\mathrm{Br}_d(\delta)$ for $\delta \in \mathbb{Z}$ integral and δ large enough in comparison to d. Hence the Brauer algebra is semisimple in these cases. In fact it was shown by Rui, see [27], that $\mathrm{Br}_d(\delta)$ is semisimple except for δ integral of small absolute value, see also [1, Theorem 7.1] for an explicit statement and more detailed proofs. For arbitrary $\delta \in \mathbb{Z}$ and $d \geq 1$ the Brauer algebras still appear as centralizers of the form (1.3) if we replace $\mathfrak{g}$ by an orthosymplectic Lie *superalgebra* such that its vector representation has super dimension $k|2n$ with $\delta = k - 2n$, see [16]. This also gives an explanation why Brauer algebras might not be semisimple in general.

Whereas the semisimple cases were studied in detail in many papers, including for instance the semiorthogonal form in [26], the non-semisimple cases are still not well understood. In the pioneering work of Cox, De Visscher and Martin, [9], it was observed that the multiplicity $[\Delta(\lambda) : L(\mu)]$ how often a simple module $L(\mu)$ indexed by μ (that is the simple quotient of $\Delta(\mu)$) appears in a Jordan-Hölder series of the cell module $\Delta(\lambda)$, is given by a certain parabolic Kazhdan-Lusztig polynomial $n_{\lambda,\mu}$ of type D with maximal parabolic of type A, [5], [23], i.e.

$$[\Delta(\lambda) : L(\mu)] = n_{\lambda,\mu}(1). \tag{1.5}$$

This result connects the combinatorics of Brauer algebras with Kazhdan-Lusztig combinatorics of type D Lie algebras, i.e. with multiplicities of simple (usually infinite dimensional) highest weight modules appearing in a parabolic Verma module. Here two obvious questions arise: Is there an interpretation of the variable q in the Kazhdan-Lusztig polynomial $n_{\lambda,\mu}(q) \in \mathbb{Z}[q]$? Is there an equivalence of categories between modules over the Brauer algebra $\mathrm{Br}_d(\delta)$ for integral δ and some subcategory of the Bernstein-Gelfand-Gelfand (parabolic) category $\mathcal{O}$ for type D explaining the mysterious match in the combinatorics? In this paper we will give an answer to both questions.

Let us explain the results in more detail. Given a finite dimensional algebra A we denote by $A-\mathrm{mod}$ its category of finite dimensional modules. If the algebra A is $\mathbb{Z}$-graded we denote by $A-\mathrm{gmod}$ its category of finite dimensional *graded* modules with degree preserving morphisms, and by $F : A-\mathrm{gmod} \to A-\mathrm{mod}$ the grading forgetting functor. For $i \in \mathbb{Z}$ let $\langle i\rangle : M \mapsto M\langle i\rangle$ denote the autoequivalence of $A-\mathrm{gmod}$ which shifts the grading by i, i.e. $F(M\langle i\rangle) = FM$ and $(M\langle i\rangle)_j = M_{j-i}$ for any $M \in A-\mathrm{gmod}$. As an application of our main theorem below we obtain the following refinement of (1.5), summing up the results obtained in Section 5:

Theorem A. Let $\delta \in \mathbb{Z}$. The Brauer algebra $\mathrm{Br}_d(\delta)$ can be equipped with a $\mathbb{Z}$-grading turning it into a $\mathbb{Z}$-graded algebra $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$. Moreover, it satisfies the following:

1.) $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is Morita equivalent to a Koszul algebra if and only if $\delta \neq 0$ or if $\delta = 0$ and d odd.

2.) $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is graded cellular.

3.) $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is graded quasi-hereditary if and only if $\delta \neq 0$ or if $\delta = 0$ and d odd. Moreover, in this case

   a.) There are distinguished graded lifts of standard modules and simple modules in the following sense: For the cell module $\Delta(\lambda)$ of $\mathrm{Br}_d(\delta)$, $\lambda \in \Lambda_d$, there exists a module $\widehat{\Delta}(\lambda)$ for $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ such that $F\widehat{\Delta}(\lambda) = \Delta(\lambda)$. For a simple module $L(\mu)$ of $\mathrm{Br}_d(\delta)$, for $\mu \in \Lambda_d^{\delta}$, there exists a module $\widehat{L}(\mu)$ for $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ such that $F\widehat{L}(\mu) = L(\mu)$. Furthermore $\widehat{L}(\mu)$ is the simple quotient of $\widehat{\Delta}(\mu)$ concentrated in degree 0, making the choice of these lifts unique (up to isomorphism).

   b.) The $\widehat{\Delta}(\lambda)$ form the graded standard modules.

4.) The modules $\widehat{\Delta}(\lambda)$ have a Jordan-Hölder series in $\mathrm{Br}_d^{\mathrm{gr}}(\delta)-\mathrm{gmod}$ with multiplicities given by

$$[\widehat{\Delta}(\lambda) : \widehat{L}(\mu) < i >] = n_{\lambda,\mu,i}, \qquad \text{where} \qquad n_{\lambda,\mu}(q) = \sum_{i\geq 0} n_{\lambda,\mu,i} q^i.$$

□

For instance, $\mathrm{Br}_2^{\mathrm{gr}}(\delta)$ is isomorphic to the algebra $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$ in case $\delta\neq 0$ whereas it is isomorphic to $\mathbb{C}\oplus\mathbb{C}[x]/(x^2)$ with x in degree 2 in case $\delta=0$, see Section 7.

The above theorem is based on our main theorem which realizes $\mathrm{Br}_d(\delta)$ for integral δ as an *idempotent truncation* of a level 2 cyclotomic quotient of a VW-algebra $\mathbb{W}_d(\Xi)$, see Definition 3.1 for the exact parameter set Ξ. Here, the *VW-algebra* $\mathbb{W}_d(\Xi)$ is as a vector space isomorphic to $\mathrm{Br}_d(N)\otimes\mathbb{C}[y_1,\dots y_d]$ with both factors being in fact subalgebras. The defining relations imply that there is a surjective homomorphism of algebras

$$\mathbb{W}_d(\Xi)\quad\longrightarrow\quad\mathrm{Br}_d(N),\tag{1.6}$$

which extends the identity on $\mathrm{Br}_d(N)$ and sends y_1 to 0. The polynomial generators y_k are then sent to the famous *Jucys-Murpy elements* ξ_k in the Brauer algebra, see Proposition 4.4 for a definition. These elements form a commutative subalgebra which plays an important role in the theory of semiorthogonal forms for the Brauer algebras. In this way, the Brauer algebra $\mathrm{Br}_d(N)$ can be realized as a level 1 cyclotomic quotient of $\mathbb{W}_d(\Xi)$.

The connection to Lie theory is based however on a more subtle realization of the Brauer algebra as follows: Let $n\in\mathbb{Z}$ be large (say $N=2n\geq 2d$) and consider the type D_n Lie algebra $\mathfrak{so}(N)$ of rank n with its vector representation V. Let ϖ_0 be the fundamental weight corresponding to a spin representation (that is to one of the fork ends in the Dynkin diagram) and let $\mathfrak{p}\subset\mathfrak{so}(N)$ be the (type A) maximal parabolic corresponding to the simple roots orthogonal to ϖ_0. For any fixed $\delta\in\mathbb{Z}$ let $M^{\mathfrak{p}}(\underline{\delta})$ be the associated parabolic Verma module with highest weight $\delta\omega_0$, see [20]. Then [15, Theorem 3.1] gives a natural isomorphism of algebras

$$\mathrm{End}_{\mathfrak{so}(N)}(M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes d})^{\mathrm{opp}}\cong\mathbb{W}_d^{\mathrm{cycl}}\tag{1.7}$$

where $\mathbb{W}_d^{\mathrm{cycl}}=\mathbb{W}_d(\Xi)/(y_1-\alpha)(y_1-\beta)$ with $\alpha=\frac{1}{2}(1-\delta)$ and $\beta=\frac{1}{2}(\delta+N-1)$. The finite dimensional algebra $\mathbb{W}_d^{\mathrm{cycl}}$ decomposes into simultaneous generalized eigenspaces with respect to $\mathbb{C}[y_1,\dots y_d]$. Let $\mathbf{f}$ be the idempotent of $\mathbb{W}_d^{\mathrm{cycl}}$ which projects onto all common generalized eigenspaces with *small* eigenvalues γ_j with respect to y_j, i.e. where γ_j satisfy $|\gamma_j|<\beta$ for $1\leq j\leq d$.

Our main result, Theorem 4.3, is then the following:

Theorem B. For any fixed $\delta\in\mathbb{Z}$, there is an isomorphism of algebras

$$\Phi_\delta:\quad\mathrm{Br}_d(\delta)\quad\longrightarrow\quad\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$$

given on the standard generators of the Brauer algebra by

$$t_k\quad\longmapsto\quad -Q_ks_kQ_k+\frac{1}{b_k}\mathbf{f},\quad\text{and}\quad g_k\quad\longmapsto\quad Q_ke_kQ_k,\tag{1.8}$$

for certain elements Q_k and b_k, defined in (3.9) and (4.2), which can be expressed in term of the polynomial generators y_j, $1 \leq j \leq d$ and β. $\square$

Note that the the Brauer algebra on the left hand side is independent of N, but the right hand side as well as the map Φ_δ do in fact depend on N. In particular, the parameter N on the right hand side changes into the parameter δ on the left hand side. Under the isomorphism, the Jucys-Murphy elements ξ_k of the Brauer algebra are sent to $-y_k\mathbf{f}$ inside $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$, see Definition 4.4. In particular, generalized eigenspaces for Jucys-Murphy elements coincide with generalized eigenspaces of the polynomial generators y_k.

By general theory on category $\mathcal{O}$, [4], [3], the algebra $\mathbb{W}_d^{\mathrm{cycl}}$ can be equipped with a positive $\mathbb{Z}$-grading, see [15, Theorem 3.1]. Since the idempotent truncation $\mathbf{f}$ corresponds to successive projections onto blocks, see [15, Section 4.1], $B_d(\delta)$ inherits a grading which is then the grading in Theorem A. In contrast to a general block in category $\mathcal{O}$, the grading can be made totally explicit in our case using the theory of generalized Khovanov algebras of type D, [14], [13]. Note that the theorem implies that all of the combinatorics developed in [15] for $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$ are now applicable to the Brauer algebra.

As an application of our result note that understanding the degree of non-semisimplicity for Brauer algebras and decomposition numbers in the non-semisimple cases gives some first insight into the structure of the tensor product of the natural module for the orthosymplectic Lie superalgebra via the result from [16], or [22].

The idea and difficulty behind the formulas (1.8) stems from the fact that we connect directly the semiorthogonal form for $\mathbb{W}_d^{\mathrm{cycl}}$ from [2] and for $\mathrm{Br}_d(\delta)$ from [26] by realizing the latter as obtained from the first via a naive idempotent truncation to small eigenvalues corrected with some extra terms encoded in the rather complicated elements Q_k and b_k. The correction terms seriously depend on the generalized weight spaces. In the semisimple case, i.e if all generalized eigenvectors are actually proper eigenvectors, the formulas simplify drastically, since the operators act on an eigenspace just by a certain number which can be expressed combinatorially in terms of contents of tableaux. The point here is that our formulas also work in the non-semisimple cases. The prize to pay is that the definitions of the correction terms involves (inverses) of square roots. It is a non-trivial result, that the operators are well-defined and that the images of the generators of the Brauer algebra satisfy the Brauer algebra relations.

We should mention that a similar result for walled Brauer algebras was obtained in [28]. The two results are independent and, as far as we can see, neither of the two implies the other. In fact, the result [28, Lemma 8.1] seriously simplifies the set-up treated there, but doesn't hold in the Brauer algebra setting. As a result, the Brauer algebra requires a very different treatment.

Another approach defining gradings on Brauer algebras arising from semiorthogonal forms was taken independently in [24] and resulted in a KLR-type presentation of Brauer algebras. One can show that the two algebras are actually isomorphic as graded algebras, see [25]. More general choices of parabolic category $\mathcal{O}$ of type D should provide a general framework for higher level cyclotomic quotients of VW-algebras following the ideas of [15]. Moreover, they should give rise to a KLR-type presentation of these algebras similar to the beautiful results in type A, see e.g. [7], [8], [19].

2 Brauer algebra and VW algebras

We start with the definition of the Brauer algebra in terms of generators and relations. Then we recall the definition of its degenerate affine analogue, the so-called VW-algebra with its cyclotomic quotients. By an algebra we always mean an associative unitary algebra with unit 1.

Definition 2.1. Let $d \in \mathbb{N}$ and $\delta \in \mathbb{C}$. The *Brauer algebra* $\mathrm{Br}_d(\delta)$ is the associative $\mathbb{C}$-algebra generated by elements t_i, g_i, $1 \leq i \leq d-1$ subject to the relations

$$\begin{array}{llll} t_i^2 = 1, & t_i t_j = t_j t_i, & t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, & t_i g_{i+1} g_i = t_{i+1} g_i, \\ g_i^2 = \delta g_i, & g_i g_j = g_j g_i, & g_i g_{i+1} g_i = g_i, & t_{i+1} g_i g_{i+1} = t_i g_{i+1}, \\ t_i g_i = g_i = g_i t_i, & g_i t_j = t_j g_i, & g_{i+1} g_i g_{i+1} = g_{i+1}, & \end{array}$$

whenever the terms in the expressions are defined and $|i-j| > 1$ holds. □

Remark 2.2. Since $g_i g_j g_i = g_i t_i g_j g_i = g_i t_j g_i$ if $|i-j| = 1$, one can replace the relation $g_i g_j g_i = g_i$ by the more commonly used relation $g_i t_j g_i = g_i$. □

Definition 2.3. Let $d \in \mathbb{N}$ and $\Xi = (\omega_i)_{i \in \mathbb{N}_0}$ with $\omega_i \in \mathbb{C}$ for all i. Then the associated *VW-algebra* $\mathbb{W}_d(\Xi)$ is the algebra generated by

$$s_i, e_i, y_j \qquad \text{for } 1 \leq i \leq d-1 \text{ and } 1 \leq j \leq d, k \in \mathbb{N}_0, \tag{2.1}$$

subject to the following relations (for $1 \leq a, b \leq d-1$, $1 \leq c < d-1$, and $1 \leq i, j \leq d$)

(VW.1) $s_a^2 = 1$,

(VW.2) (a) $s_a s_b = s_b s_a$ for $\mid a - b \mid > 1$,

(b) $s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}$,

(c) $s_a y_i = y_i s_a$ for $i \notin \{a, a+1\}$,

(VW.3) $e_a^2 = \omega_0 e_a$,

(VW.4) $e_1 y_1^k e_1 = \omega_k e_1$ for $k \in \mathbb{N}_0$,

(VW.5) (a) $s_a e_b = e_b s_a$ and $e_a e_b = e_b e_a$ for $\mid a - b \mid > 1$,

(b) $e_a y_i = y_i e_a$ for $i \notin \{a, a+1\}$,

(c) $y_i y_j = y_j y_i$,

(VW.6) (a) $e_a s_a = e_a = s_a e_a$,

(b) $s_c e_{c+1} e_c = s_{c+1} e_c$ and $e_c e_{c+1} s_c = e_c s_{c+1}$,

(c) $e_{c+1} e_c s_{c+1} = e_{c+1} s_c$ and $s_{c+1} e_c e_{c+1} = s_c e_{c+1}$,

(d) $e_{c+1} e_c e_{c+1} = e_{c+1}$ and $e_c e_{c+1} e_c = e_c$,

(VW.7) $s_a y_a - y_{a+1} s_a = e_a - 1$ and $y_a s_a - s_a y_{a+1} = e_a - 1$,

(VW.8) (a) $e_a(y_a + y_{a+1}) = 0$,

(b) $(y_a + y_{a+1})e_a = 0$. □

Remark 2.4. The VW-algebra $\mathbb{W}_d$ is a degeneration of the affine BMW-algebra*, [10], hence plays the analogue role for the Brauer algebra as the degenerate affine Hecke algebra plays for the symmetric group. It was introduced originally by Nazarov in [26] under the name generalized Wenzl-algebra†. □

Finally we introduce, following [2], the cyclotomic quotients of $\mathbb{W}_d$ of level ℓ:

Definition 2.5. Given $\mathbf{u} = (u_1, u_2, \dots, u_\ell) \in \mathbb{C}^\ell$ we denote by $\mathbb{W}_d(\Xi; \mathbf{u})$ the quotient

$$\mathbb{W}_d(\Xi, \mathbf{u}) = \mathbb{W}_d(\Xi) / \prod_{i=1}^{\ell} (y_1 - u_i) \tag{2.2}$$

and call it the *cyclotomic VW-algebra of level ℓ* with parameters $\mathbf{u}$. □

Remark 2.6. As explained in [2], the tuple Ξ must satisfy some admissibility condition for the algebra $\mathbb{W}_d(\Xi)$ to have a nice basis. Furthermore, the Ξ must satisfy some $\mathbf{u}$-admissibility condition for this basis to be compatible with the quotient, see [2, Theorem A, Prop. 2.15]. □

Inside the VW-algebra $\mathbb{W}_d(\Xi)$, the elements $\{y_k \mid 1 \le k \le d\}$ generate a free commutative subalgebra, hence we can consider the simultaneous generalized eigenspace decompositions for these elements. Any finite dimensional $\mathbb{W}_d(\Xi)$-module M has a decomposition

$$M = \bigoplus_{\mathbf{i} \in \mathbb{C}^d} M_{\mathbf{i}}, \tag{2.3}$$

where $M_{\mathbf{i}}$ is the generalized eigenspace with eigenvalue $\mathbf{i}$, i.e., $(y_k - \mathbf{i}_k)^r M_{\mathbf{i}} = 0$ for $r \gg 0$ sufficiently large. We first describe how the generators e_k and s_k interact with this eigenspace decomposition.

Lemma 2.7. For all $1 \le k < d$ the following holds

$$e_k M_{\mathbf{i}} \subset \begin{cases} \{0\} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} \neq 0, \\ \bigoplus_{\mathbf{i}' \in \mathrm{I}} M_{\mathbf{i}'} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} = 0, \end{cases}$$

*as every car lover can probably imagine easily ...

†which translated to German is *Verallgemeinerte Wenzl Algebra*, abbreviated as $\mathbb{W}$. It is also sometimes called *Nazarov-Wenzl algebra* in the literature. Hence $\mathbb{W}$ can be viewed as composed of the letters N, W and V as well.

where $\mathrm{I} = \{\mathbf{i}' \in \mathbb{C}^d \mid \mathbf{i}'_j = \mathbf{i}_j \text{ for } j \neq k, k+1 \text{ and } \mathbf{i}'_k + \mathbf{i}'_{k+1} = 0\}$. □

Proof. Assume first that $a := \mathbf{i}_k + \mathbf{i}_{k+1} \neq 0$. Then the endomorphism induced by $(y_k + y_{k+1} - a)$ is nilpotent on $M_\mathbf{i}$ and hence $y_k + y_{k+1}$ induces an automorphism of $M_\mathbf{i}$. Hence $e_k M_\mathbf{i} = e_k(y_k + y_{k+1})M_\mathbf{i} = \{0\}$, where for the last equality (VW.8a) was used.

Now assume $\mathbf{i}_k + \mathbf{i}_{k+1} = 0$. Since $(y_k + y_{k+1})e_k = 0$ we know by (VW.8b) that on the image of e_k the endomorphism induced by $y_k + y_{k+1}$ has eigenvalue 0, hence y_k and y_{k+1} have eigenvalues that add up to 0. ■

The situation for s_k is more complicated and we need some preparations. Let k, $1 \leq k \leq d-1$ be fixed for the rest of this section.

Lemma 2.8. Set $\psi_k = s_k(y_k - y_{k+1}) + 1$. Then it holds

$$\psi_k M_\mathbf{i} \subset \begin{cases} M_{s_k \mathbf{i}} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} \neq 0, \\ \bigoplus_{\mathbf{i}' \in \mathrm{I}} M_{\mathbf{i}'} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} = 0. \end{cases}$$

In case $\mathbf{i}_k + \mathbf{i}_{k+1} \neq 0$ and additionally $|\mathbf{i}_k - \mathbf{i}_{k+1}| \neq 1$, the map ψ_k defines an isomorphism of vector spaces $M_\mathbf{i} \cong M_{s_k(\mathbf{i})}$. □

Proof. From (VW.7) we obtain $y_k \psi_k = \psi_k y_{k+1} + e_k(y_k - y_{k+1})$ and by (VW.2c) then $y_j \psi_k = \psi_k y_j$ for $|j-k| > 1$.

Assume $\mathbf{i}_k + \mathbf{i}_{k+1} \neq 0$ and let $m \in M_\mathbf{i}$. Then by Lemma 2.7 we have $y_k \psi_k m = \psi_k y_{k+1} m$ and $y_{k+1}\psi_k = \psi_k y_k m$, hence $(y_i - \mathbf{i}_{s_k(i)})^r \psi_k m = \psi_k (y_{s_k(i)} - \mathbf{i}_{s_k(i)})^r m$ for all $r \geq 1$ and all i. Thus $\psi_k M_\mathbf{i} \subset M_{s_k(\mathbf{i})}$.

Relation (VW.7) implies $(y_k + y_{k+1})\psi_k = \psi_k(y_k + y_{k+1})$. Assuming $\mathbf{i}_k + \mathbf{i}_{k+1} = 0$ we have $(y_k + y_{k+1})^r M_\mathbf{i} = \{0\}$ for some $r \geq 1$ and hence $(y_k + y_{k+1})^r \psi_k M_\mathbf{i} = \psi_k (y_k + y_{k+1})^r M_\mathbf{i} = \{0\}$. Thus, on the image of ψ_k, the endomorphism induced by $y_k + y_{k+1}$ has eigenvalue 0 and so y_k and y_{k+1} have eigenvalues adding up to 0.

Assuming now $\mathbf{i}_k + \mathbf{i}_{k+1} \neq 0$ and furthermore $|\mathbf{i}_k - \mathbf{i}_{k+1}| \neq 1$, then thanks to (VW.7) and Lemma 2.7 we have $\psi_k^2 = -(y_k - y_{k+1})^2 + 1$ as endomorphisms of $M_\mathbf{i}$. Setting $c = \mathbf{i}_k - \mathbf{i}_{k+1}$ it follows $((y_k - y_{k+1}) - c)^r M_\mathbf{i} = \{0\}$ for some $r \geq 1$ by definition. In particular, as endomorphisms of $M_\mathbf{i}$, this implies

$$\psi_k^2 \quad = \quad 1 - ((y_k - y_{k+1}) - c + c)^2 \quad = \quad 1 - c^2 - ((y_k - y_{k+1}) - c)^2 - c((y_k - y_{k+1}) - c) \quad = \quad 1 - c^2 - z$$

for some nilpotent endomorphism z. Since $c^2 \neq 1$ by assumption, ψ_k^2 is invertible and therefore also ψ_k. Note that the concrete form of the inverse depends on $\mathbf{i}$. ■

Corollary 2.9. Assume that $\mathbf{i}_k \neq \mathbf{i}_{k+1}$, then

$$s_k M_\mathbf{i} \quad \subset \quad \begin{cases} M_{s_k \mathbf{i}} \oplus M_\mathbf{i} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} \neq 0, \\ \bigoplus_{\mathbf{i}' \in \mathrm{I}} M_{\mathbf{i}'} & \text{if } \mathbf{i}_k + \mathbf{i}_{k+1} = 0. \end{cases}$$

□

Proof. Under the assumption that $\mathbf{i}_k \neq \mathbf{i}_{k+1}$ we have that $y_k - y_{k+1}$ is an automorphism of $M_{\mathbf{i}}$ and the statement follows directly from Lemma 2.8. ■

Corollary 2.10. Assume $M_{\mathbf{i}} \neq \{0\}$ for some $\mathbf{i} = (\mathbf{i}_1, \dots, \mathbf{i}_{d-1}, \mathbf{i}_d)$ such that $\mathbf{i}_k + \mathbf{i}_d \neq 0$ and $|\mathbf{i}_k - \mathbf{i}_d| \neq 1$ for all $k < d$. Let $\mathbf{i}' = (\mathbf{i}_d, \mathbf{i}_1, \dots, \mathbf{i}_{d-1})$, then

$$(y_d - \mathbf{i}_d)^r M_{\mathbf{i}} = \{0\} \Longleftrightarrow (y_1 - \mathbf{i}_d)^r M_{\mathbf{i}'} = \{0\}$$

for all positive integers r. □

Proof. By Lemma 2.8 the element $\psi := \psi_1 \cdots \psi_{d-1}$ acts as an isomorphism between $M_{\mathbf{i}}$ and $M_{\mathbf{i}'}$ intertwining the actions of y_1 and y_d, i.e. $\psi(y_1 m) = y_d \psi(m)$ for any $m \in M_{\mathbf{i}}$. ■

3 Cyclotomic quotients and category $\mathcal{O}$

Fix $\delta \in \mathbb{Z}$. Let $\mathfrak{g} = \mathfrak{so}(2n)$ be the complex special orthogonal Lie algebra corresponding to the Dynkin diagram Γ of type D_n and fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Denote by $\varepsilon_n, \dots, \varepsilon_1$ the standard basis of $\mathfrak{h}^*$. The ordering on these basis vectors and on the simple roots is chosen such that the labels 1 and 2 correspond to the two special nodes in the Dynkin diagram (at the fork end), whereas ϵ_n corresponds to the highest weight of the vector representation.[‡]

Fix $\mathfrak{l}$, a Levi subalgebra obtained from an embedding of the type A_{n-1} Dynkin diagram into Γ and denote by $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}^+$ the corresponding parabolic subalgebra.

By $\mathcal{O}^{\mathfrak{p}}(n) = \mathcal{O}^{\mathfrak{p}}_{\mathrm{int}}(\mathfrak{so}(2n))$ we denote the integral parabolic BGG category $\mathcal{O}$, i.e., the full subcategory of $\mathcal{U}(\mathfrak{g})$-modules consisting of finitely generated $\mathcal{U}(\mathfrak{g})$-modules, semisimple over $\mathfrak{h}$ with integral weights, and locally finite for $\mathfrak{p}$, see [20, Chapter 9]. Let

$$X^{\mathfrak{p}}_n = \{\lambda \in \mathfrak{h}^* \text{ integral} \mid \lambda + \rho = \sum_{i=1}^{n} \lambda_i \epsilon_i \text{ where } \lambda_1 < \lambda_2 < \cdots < \lambda_n\}, \tag{3.1}$$

where ρ denotes the half-sum of the positive roots, $\rho = \sum_{i=1}^{n} (i-1)\epsilon_i$. Then $X^{\mathfrak{p}}_n$ is precisely the set of highest weights of simple objects in $\mathcal{O}^{\mathfrak{p}}(n)$, see [20]. For an element $\lambda \in X^{\mathfrak{p}}_n$ we denote by $M^{\mathfrak{p}}(\lambda)$ the parabolic Verma module with highest weight λ. Note that a weight $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ written in the ϵ-basis is integral, if either $\lambda_i \in \mathbb{Z}$, i.e. $2\lambda_i$ is even for all i, or $\lambda_i \in \frac{1}{2} + \mathbb{Z}$, i.e. $2\lambda_i$ is odd for all i.

As in [15], a crucial player in the following will be the parabolic Verma module $M^{\mathfrak{p}}(\underline{\delta})$ of highest weight

$$\underline{\delta} \quad = \quad \frac{\delta}{2}(\varepsilon_1 + \ldots + \varepsilon_n), \tag{3.2}$$

[‡]Note that this is not the usual Bourbakian choice, but it is a choice which is better adapted to the idea of taking a limit for $n \mapsto \infty$. As a consequence dominant integral weights correspond to increasing sequences of (half)-integers.

i.e., a multiple of the fundamental weight $\frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. With an appropriate choice of parameters Ξ_δ, there is, see [15], a natural (right) action of $\mathbb{W}_d(\Xi_\delta)$ on $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ by $\mathfrak{g}$-endomorphisms. Hence we have an algebra homomorphism $\mathbb{W}_d(\Xi_\delta) \to \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}}$. The parameter set $\Xi_\delta = (\omega_a)_{a \in \mathbb{N}_0}$ appear as part of the following definition:

Definition 3.1. For $N = 2n$, we define the *cyclotomic parameters* ω_a, $a \in \mathbb{N}_0$ as follows

$$\omega_0 = N, \quad \omega_1 = N\frac{N-1}{2}, \quad \omega_a = (\alpha + \beta)\omega_{a-1} - \alpha\beta\omega_{a-2} \quad \text{for } a \geq 2, \tag{3.3}$$

where we set

$$\alpha = \tfrac{1}{2}(1 - \delta), \qquad \beta = \tfrac{1}{2}(\delta + N - 1). \tag{3.4}$$

(Observe that α is independent of N, whereas β depends linearly on N.) □

With these definitions the following important result holds:

Theorem 3.2 ([15])**.** If $n \geq 2d$, then the $\mathbb{W}_d(\Xi_\delta)$-action from above induces an isomorphism of algebras

$$\Psi(\underline{\delta}): \quad \mathbb{W}_d(\Xi_\delta; \alpha, \beta) \longrightarrow \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}}. \tag{3.5}$$

In the following we abbreviate $\mathbb{W}_d^{\mathrm{cycl}} = \mathbb{W}_d(\Xi_\delta; \alpha, \beta)$. □

Note that via (3.5), the space $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ becomes a (right) module for $\mathbb{W}_d^{\mathrm{cycl}}$ with the action preserving the finite dimensional $\mathfrak{g}$-weight spaces, hence we have a simultaneous generalized eigenspace decomposition (2.3). In the following we will always work with left modules and identify $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}) \cong (\mathbb{W}_d^{\mathrm{cycl}})^{\mathrm{opp}} = \mathbb{W}_d^{\mathrm{cycl}}$. We describe now this decomposition Lie theoretically and then combinatorially using the notion of up-down bitableaux and bipartitions. We start with the following well-known fact:

Lemma 3.3. Let $\mu \in X_n^{\mathfrak{p}}$. Then $M^{\mathfrak{p}}(\mu) \otimes V$ has a filtration (called *Verma flag*) with sections isomorphic to precisely the $M^{\mathfrak{p}}(\mu \pm \epsilon_j)$ for all $j = 1, \ldots, n$ such that $\mu \pm \epsilon_j \in X_n^{\mathfrak{p}}$. The sections are pairwise not isomorphic. □

Proof. This is a standard consequence of the tensor identity; see e.g. [20, Theorem 3.6], noting that V has precisely the weights $\pm\epsilon_j$ for $1 \leq j \leq n$. ■

In particular, $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ has a Verma flag with sections isomorphic to precisely the $M^{\mathfrak{p}}(\lambda)$ where $\lambda - \underline{\delta} = \sum_{j=i}^{d} a_i$ and the $(a_1, a_2, \ldots, a_d)$ run through all possible d-tuples with $a_i \in \{\pm\epsilon_j \mid 1 \leq j \leq n\}$. Given such a weight λ we can write

$$\lambda - \underline{\delta} \quad = \quad \sum_{i=1}^{n} m_i \epsilon_i \quad = \quad \sum_{i=1}^{s} m_i \epsilon_i + \sum_{i=s+1}^{n} m_i \epsilon_i, \qquad m_1 \leq m_2 \leq \ldots \leq m_n$$

with $m_i \le 0$ for $1 \le i \le s$ and $m_i > 0$ for $i > s$ for some (uniquely defined) s. Then we assign to λ the bipartition

$$\varphi(\lambda) = (\lambda^{(1)}, \lambda^{(2)}) \quad \text{with} \quad \begin{cases} \lambda_i^{(1)} = |m_i|, & \text{for} \quad 1 \le i \le s, \\ \lambda_i^{(2)} = m_{n-i+1} & \text{for} \quad 1 \le i \le n-s. \end{cases} \tag{3.6}$$

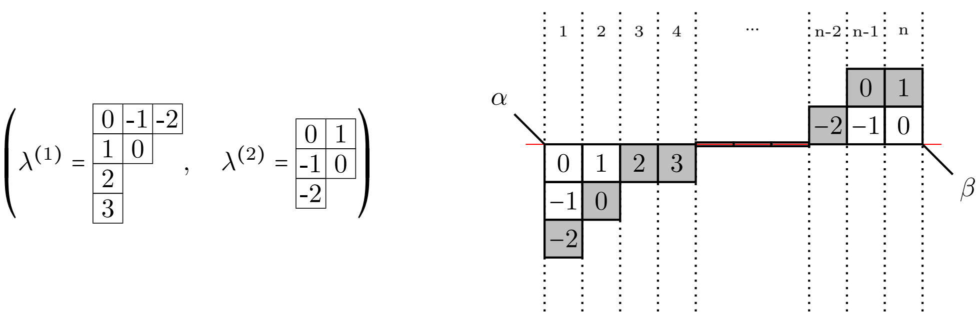
This will be seen as a pair of *Young diagrams*, which are both consisting of arrangements of boxes with left-justified rows and the number of boxes per row weakly decreasing from top to bottom. Each box b in such a pair of Young diagrams has a *content* $\mathrm{c}(b)$ defined as follows: let b be in the r-th row of its diagram and in the c-th column (counting from top to bottom and from left to right, starting with 1 in both cases), then $\mathrm{c}(b) = r - c$ if b is in $\lambda^{(1)}$ and $\mathrm{c}(b) = c - r$ if b is in $\lambda^{(2)}$. Here we display the pair of Young diagrams attached to the bipartition $(3,2,1,1),(2,2,1)$ with the contents of each box written in the box

$$\left(\lambda^{(1)} = \begin{array}{|c|c|c|} \hline 0 & \text{-}1 & \text{-}2 \\ \hline 1 & 0 \\ \cline{1-2} 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \; , \quad \lambda^{(2)} = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline \text{-}1 & 0 \\ \hline \text{-}2 \\ \cline{1-1} \end{array} \right) \tag{3.7}$$

To get back the weight λ we just add to $\underline{\delta}$ the vector $(-\lambda_1^{(1)}, -\lambda_2^{(1)}, \dots, -\lambda_s^{(1)}, 0, \dots, 0, \dots, \lambda_2^{(2)}, \lambda_1^{(2)})$. For instance (3.7) corresponds to the weight $\underline{\delta} + (-3,-2,-1,-1,0,0,\dots,0,1,2,2)$. Note that in this case $s = n-4$ the decomposition from (3.6) and $\lambda^{(1)} = (|-3|,|-2|,|-1|,|-1|,0,\dots,0)$ and $\lambda^{(2)} = (2,2,1)$.

Remark 3.4. To keep track of the contents it might be convenient to draw such a pair of Young diagrams $\varphi(\lambda)$ as a *double Young diagram* with a total of d boxes. For this consider a vertical infinite strip with n columns crossing a horizontal line o. This horizontal line will split the strip into two regions, an upper and a lower part. A double Young diagram consists of two Young diagrams, one placed in the upper half with center of gravity on the lower right point of that region and a second one placed in the lower part with center of gravity on the upper left such that no column contains boxes above and below the line o.

The double Young diagram attached to the weight λ is constructed as follows: the Young diagram at the bottom is just the Young diagram for $\lambda^{(1)}$ transposed; the Young diagram at the top is obtained from the Young diagram for $\lambda^{(2)}$ by transposing the diagram and then rotating it by 180 degrees. We denote the result $((\lambda^{(1)})^t, {}^t(\lambda^{(2)}))$. The contents for the boxes are transposed respectively rotated accordingly. Below, the double Young diagram attached to the bipartition $(3,2,1,1),(2,2,1)$ from (3.7) is displayed.



(Note that the contents just encode the corresponding diagonals; they increase by one when moving one box to the right or up and decrease by one when moving one box to the left or down. Note moreover that the condition $n \geq 2d$ ensures that the two diagrams do not overlap.) The associated weight λ is obtained by adding to $\underline{\delta}$ the n-tuple with entries the lengths of the columns read from left to right with a negative sign if the partition is below o. □

Definition 3.5. To incorporate some dependence on δ or better on the cyclotomic parameters (3.1), we *additionally shift the contents* by α respectively β as indicated in the diagram. That is if a box b is in the first component of a bipartition (i.e. inside the lower part of its double Young diagram) then define $\mathrm{c}_\delta(b) = \mathrm{c}(b) + \alpha$, while the shifted content of a box b in the second component (or the upper part of its double Young diagram) is defined to be $\mathrm{c}_\delta(b) = \mathrm{c}(b) + \beta$. □

Definition 3.6. An *up-down bitableaux* of length d is a sequence $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_d)$ of bipartitions such that $\mathbf{Y}_1$ is the trivial bipartition $(\varnothing, \varnothing)$ and two consecutive bipartitions differ just by one box (added or removed). Let $\mathrm{wt}(\mathbf{Y}) = (\mathrm{wt}(\mathbf{Y}_i))_{1 \leq i \leq d}$ be the weight sequence attached to $\mathbf{Y}$. The set of all corresponding up-down bitableaux of length d is denoted by $\mathcal{T}_d$. □

Example 3.7. Let us consider the case $d = 1$ (and $n \geq 4$). Then we have $\underline{\delta} + \rho = (\frac{\delta}{2}, \frac{\delta}{2} + 1, \dots, \frac{\delta}{2} + (n-1))$ for any $\delta \in \mathbb{Z}$, hence $\underline{\delta} \in X_n^{\mathfrak{p}}$. Now possible values for a_1 are $a_1 = -\epsilon_1$ and $a_1 = \epsilon_n$. In the first case the number s in (3.6) equals n and in the second case $s = n-1$. The bipartitions for $\lambda = \underline{\delta} + a_1$ are $(\square, \varnothing)$, respectively $(\varnothing, \square)$. Their double diagrams have just one box below (with content α) respectively above the line o (with content β). The corresponding up-down bitableaux are $((\varnothing, \varnothing), (\square, \varnothing))$ respectively $((\varnothing, \varnothing), (\varnothing, \square))$ and form the set $\mathcal{T}_1$. In case $d = 2$ we additionally have a_2, namely $a_2 \in \{-\epsilon_1, -\epsilon_2, \epsilon_1, \epsilon_n\}$ if $a_1 = -\epsilon_1$, and $a_2 \in \{-\epsilon_1, \epsilon_n, \epsilon_{n-1}, -\epsilon_n\}$ if $a_1 = -\epsilon_n$. They correspond to the bipartitions $(\square\square, \varnothing), (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing), (\varnothing, \varnothing), (\square, \square)$ respectively $(\square, \square), (\varnothing, \square\square), (\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}), (\varnothing, \varnothing)$. The contents of the added/removed boxes are $\alpha - 1, \alpha + 1, -\alpha, \beta$ respectively $\alpha, \beta + 1, \beta - 1, -\beta$. The eight up-down bitableaux in $\mathcal{T}_2$ are just obtained from the $d = 1$ ones by adding the additional bipartition given by the choice of a_2. □

The module $M := M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ decomposes into a direct sum of submodules

$$M = \bigoplus_{\mathbf{a} \in \mathbb{C}^d / S_d} M_{S_d \mathbf{a}}, \tag{3.8}$$

where a runs through a fixed set of representatives for the S_d-orbits on $\mathbb{C}^d$ and $M_{S_d \mathbf{a}} = \oplus_{w \in S_d} M_{w(\mathbf{a})}$, with the summands defined as in (2.3). These are then the subspaces of M where the multiset of occurring generalized eigenvalues of all the individual y_i's is fixed.

Proposition 3.8. Assume $n \geq 2d$. Then there is a canonical bijection between $\mathcal{T}_d$ and the parabolic Verma modules appearing as sections in a Verma filtration of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ counted with multiplicities such that the following holds

1.) The bijection is given by assigning to a up-down bitableau $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_d)$ the parabolic Verma module $M(\underline{\delta} + \mathrm{wt}(\mathbf{Y}_d))$.

2.) For $1 \le j \le d$, let $\sigma_j = 1$ if $\mathbf{Y}_j$ was obtained from $\mathbf{Y}_{j-1}$ by adding a box B_j and $\sigma_j = -1$ if $\mathbf{Y}_j$ was obtained from $\mathbf{Y}_{j-1}$ by removing a box B_j. Then the parabolic Verma module $M^{\mathfrak{p}}(\underline{\delta} + \mathrm{wt}(\mathbf{Y}_d))$ associated to $\mathbf{Y}$ appears as a subquotient of the summand in (3.8) containing the generalized eigenspace of the operator y_k for the eigenvalue $\sigma_k \, \mathrm{c}_\delta(B_k)$. □

Proof. In case $\delta \ge 0$ this follows directly from [15] and the bijection between up-down bitableaux and Verma paths describing the Verma modules appearing in a Verma filtration and their eigenvalues. For $\delta < 0$ the arguments are totally analogous. ■

Definition 3.9. We call an eigenvalue γ of y_k *small* if $|\gamma| < \beta$ and we call it *large* if $|\gamma| \ge \beta$. For the eigenvalues appearing in Proposition 3.8 the condition to be small is equivalent to the condition that the corresponding boxes in the bipartition are all in the first component. □

The idempotent we define now projects onto the generalized eigenspaces corresponding to small eigenvalues.

Definition 3.10. For $1 \le k \le d$ we denote by $\eta_k \in \mathbb{W}_d^{\mathrm{cycl}}$ the idempotent projecting onto the generalized eigenspace of y_k with eigenvalue different from β. Furthermore let $\mathbf{f}_k = \eta_1 \cdots \eta_k$ and $\mathbf{f} = \mathbf{f}_d$. □

Remark 3.11. Note that $M\mathbf{f} = (M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})\mathbf{f}$ is a direct summand of M and has therefore an induced Verma filtration. The occurring Verma modules are precisely those from M with small eigenvalues only. In case $d = 2$, see Example 3.7, they correspond to the up-down bitableaux ending on $(\square\square, \varnothing), (\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \varnothing), (\varnothing, \varnothing)$, whereas all eight bipartitions correspond to the parabolic Verma modules appearing in $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes 2}$ (note that one of them appears with multiplicity two). □

It is clear from the definitions that if y_k has a large eigenvalue on some Verma module in $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$, then some earlier polynomial generator already has eigenvalue β on it, that is there exists a $j \le k$ such that y_j has eigenvalue β on this module. This means combinatorially that in an up-down bitableau we have to place the first box into the second component before we place any further boxes into the second component. In particular, the element $\mathbf{f}$ *projects onto the common generalized eigenspaces of small eigenvalues for the commutative subalgebra generated by the* y_j*'s*. The idempotent $\mathbf{f}$ is not central, so we first deduce some commutation relations.

For $1 \le k \le d$ define the following important elements of $\mathbb{W}_d^{\mathrm{cycl}}$:

$$b_k = \beta + y_k \qquad \text{and} \qquad c_k = \beta - y_k. \tag{3.9}$$

Lemma 3.12. In $\mathbb{W}_d^{\mathrm{cycl}}$ the following equalities hold for $1 \le k \le d-1$:

1.) We have $b_{k+1} s_k = s_k b_k - e_k + 1$, and $\mathbf{f} s_k \frac{1}{b_k} \mathbf{f} = \frac{1}{b_{k+1}} \mathbf{f} s_k \mathbf{f} - \frac{1}{b_{k+1}} \mathbf{f} e_k \frac{1}{b_k} \mathbf{f} + \frac{1}{b_k b_{k+1}} \mathbf{f}$, 2.) We have $s_k b_{k+1} = b_k s_k - e_k + 1$, and $\frac{1}{b_k}\mathbf{f}s_k\mathbf{f} = \mathbf{f}s_k\frac{1}{b_{k+1}}\mathbf{f} - \frac{1}{b_k}\mathbf{f}e_k\frac{1}{b_{k+1}}\mathbf{f} + \frac{1}{b_k b_{k+1}}\mathbf{f}$,

3.) We have $c_{k+1}s_k = s_k c_k + e_k - 1$, and $\mathbf{f}s_k\frac{1}{c_k}\mathbf{f} = \frac{1}{c_{k+1}}\mathbf{f}s_k\mathbf{f} + \frac{1}{c_{k+1}}\mathbf{f}e_k\frac{1}{c_k}\mathbf{f} - \frac{1}{c_k c_{k+1}}\mathbf{f}$,

4.) We have $s_k c_{k+1} = c_k s_k + e_k - 1$, and $\frac{1}{c_k}\mathbf{f}s_k\mathbf{f} = \mathbf{f}s_k\frac{1}{c_{k+1}}\mathbf{f} + \frac{1}{c_k}\mathbf{f}e_k\frac{1}{c_{k+1}}\mathbf{f} - \frac{1}{c_k c_{k+1}}\mathbf{f}$. □

Proof. This follows from Relation (VW.7) on the nose or multiplied with the idempotent $\mathbf{f}$ from both sides, assuming however that the occurring fractions are defined. By Remark 3.14 below it is enough to make these expressions well-defined on each generalized eigenspaces $\mathbf{f}M_{\mathbf{i}}$ for the regular module $M = \mathbb{W}_d^{\mathrm{cycl}}$. If $\mathbf{i}_k + \mathbf{i}_{k+1} \neq 0$ then $\frac{1}{b_k}e_k\mathbf{f}$ acts by zero thanks to 2.7, and $s_k\mathbf{f}$ is contained by Corollary 2.9 in the direct sum of all generalized eigenspaces where b_k acts non-zero. If $i_k + i_{k+1} = 0$ then $\mathbf{f}M_{\mathbf{i}} \neq 0$ implies that $|i_r| < |\beta|$ for all $1 \leq r \leq d-1$ and hence $e_k\mathbf{f}M_{\mathbf{i}}$ is, again by Lemma 2.7, contained in the sum of generalized eigenspaces for $\mathbf{i}$ such that $|i_r| < \beta$ for $r < k$ and $|\mathbf{i}_k| < \beta$ or $\mathbf{i}_k = \beta$. But in any case $\mathbf{i}_k \neq -\beta$, hence $\frac{1}{b_k}e_k\mathbf{f}$ is well-defined. Similarly for $s_k\mathbf{f}M_{\mathbf{i}}$ using Corollary 2.9. For the other expressions the arguments are analogous and omitted. (Note however that the occurrence of $\mathbf{f}$ on the right is crucial.) ■

Proposition 3.13. In $\mathbb{W}_d^{\mathrm{cycl}}$ the following equalities hold:

1.) For $1 \leq k \leq d-1$

$$\begin{array}{llll} i) & c_k\mathbf{f}s_k\mathbf{f} = c_k s_k\mathbf{f}, & iii) & c_k\mathbf{f}e_k\mathbf{f} = c_k e_k\mathbf{f}, \\ ii) & b_{k+1}\mathbf{f}s_k\mathbf{f} = b_{k+1}s_k\mathbf{f}, & iv) & b_{k+1}\mathbf{f}e_k\mathbf{f} = b_{k+1}e_k\mathbf{f}. \end{array}$$

2.) For $1 \leq k < d-1$

$$\begin{array}{llll} i) & e_k\mathbf{f}s_{k+1}\mathbf{f} = e_k s_{k+1}\mathbf{f}, & iv) & \mathbf{f}s_k\mathbf{f}s_{k+1}\mathbf{f} = \mathbf{f}s_k s_{k+1}\mathbf{f}, \\ ii) & e_k\mathbf{f}e_{k+1}\mathbf{f} = e_k e_{k+1}\mathbf{f}, & v) & \mathbf{f}s_k\mathbf{f}e_{k+1}\mathbf{f} = \mathbf{f}s_k e_{k+1}\mathbf{f}, \\ iii) & \mathbf{f}e_k\mathbf{f}s_{k+1}\mathbf{f}e_k\mathbf{f} = \mathbf{f}e_k s_{k+1}e_k\mathbf{f}. & & \end{array}$$

as well as all of these equalities with k and $k+1$ swapped. □

Proof. We start with part *1.)*. For case *i)* we claim that

$$c_k\mathbf{f}s_k\mathbf{f} = c_k s_k\mathbf{f} - c_k(1-\mathbf{f})s_k\mathbf{f} = c_k s_k\mathbf{f}.$$

We only have to justify the last equality. But this holds due to Lemma 2.8; since the image of $(1-\mathbf{f})s_k\mathbf{f}$ is either 0 or consists of generalized eigenvectors for y_k with eigenvalue β. By Corollary 2.9 and the fact that y_1 always acts by a scalar by the assumptions on n, these are honest eigenvectors and thus c_k acts by zero. For *ii)* we calculate

$$
\begin{aligned}
b_{k+1}\mathbf{f}s_k\mathbf{f} &= b_{k+1}\mathbf{f}s_k\frac{c_{k+1}}{c_{k+1}}\mathbf{f}\\
&\overset{(a)}{=} b_{k+1}\mathbf{f}(c_ks_k+e_k-1)\frac{1}{c_{k+1}}\mathbf{f}\\
&\overset{(b)}{=} b_{k+1}c_ks_k\frac{1}{c_{k+1}}\mathbf{f}+b_{k+1}e_k\frac{1}{c_{k+1}}\mathbf{f}-\frac{b_{k+1}}{c_{k+1}}\mathbf{f})\\
&\overset{(c)}{=} b_{k+1}(s_kc_{k+1}-e_k+1)\frac{1}{c_{k+1}}\mathbf{f}+b_{k+1}e_k\frac{1}{c_{k+1}}\mathbf{f}-\frac{b_{k+1}}{c_{k+1}}\mathbf{f} \quad = \quad b_{k+1}s_k\mathbf{f}
\end{aligned}
$$

Here equalities (a) and (c) hold by Lemma 3.12, while equality (b) is due to the other cases of this lemma. The expressions are well-defined by Remark 3.14 below. For case *iii)* we have

$$c_ke_k\mathbf{f} = c_k\mathbf{f}_{k-1}e_k\mathbf{f} = c_k\mathbf{f}_k\mathbf{f}_{k-1}e_k\mathbf{f} + c_k(1-\mathbf{f}_k)\mathbf{f}_{k-1}e_k\mathbf{f} = c_k\mathbf{f}_ke_k\mathbf{f}.$$

The final equality holds because the image of $(1-\mathbf{f}_k)\mathbf{f}_{k-1}$ consists of eigenvectors for y_k with eigenvalue β, hence are annihilated by c_k. That the image consists of eigenvectors follows as in case *i)*. Furthermore, due to Lemma 2.7, $\mathbf{f}_ke_k = \mathbf{f}_{k+1}e_k$ and the statement follows. Case *iv)* is the same since $b_{k+1}e_k = c_ke_k$ by (VW.8b).

For part *2.)*, we note that all of these are more or less proven in the same way using Lemma 2.7 and Corollary 2.9. We will argue for $e_k\mathbf{f}s_{k+1}\mathbf{f} = e_ks_{k+1}\mathbf{f}$ and leave the others to the reader. It holds

$$e_ks_{k+1}\mathbf{f} = e_k\mathbf{f}s_{k+1}\mathbf{f} + e_k(1-\mathbf{f})s_{k+1}\mathbf{f}.$$

If we now look at a generalized eigenspace $M_{\mathbf{i}}$ in the image of $(1-\mathbf{f})s_{k+1}\mathbf{f}$, we see that, due to the diagram combinatorics, this can only be non-zero if $\mathbf{i}_{k+1} = \beta$ and $\mathbf{i}_{k+2} = -\beta$ while all other eigenvalues are small. Applying e_k to this eigenspace is zero, due to Lemma 2.7 since $\mathbf{i}_k$ is small and thus cannot be $-\beta$. Hence only the first summand survives which proofs the claim. Similar arguments have to be applied to the other cases. ■

The following remark deals with the well-definedness of expressions involving fractions of the form $\frac{1}{b_k}\mathbf{f}$, $\frac{1}{c_k}\mathbf{f}$, $\frac{1}{b_k}s_k\mathbf{f}$, $\frac{1}{b_k}e_k\mathbf{f}$ etc. This observation is very important for the whole paper.

Remark 3.14. To make sense of an expression like $\frac{1}{b_k}$ we first consider b_k as a formal polynomial in y_k with non-zero constant term β. Hence this polynomial has an inverse in the ring of formal powers series in y_k which is however not a well-defined element of $\mathbb{W}_d^{\mathrm{cycl}}$. Using Theorem 3.2 we realize b_k as an endomorphism of $M = M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$. By definition it preserves the decomposition (2.3). Hence it suffices to invert b_k restricted to $M_{\mathbf{i}}$ for each $\mathbf{i}$. This is possible in case $\mathbf{i}_k \neq -\beta$ by formally expanding $\frac{1}{b_k}$ around $\mathbf{i}_k$ noting that $(y_k-\mathbf{i}_k)^r$ acts by zero for large enough r. In particular, $\frac{1}{b_k}g$ makes sense for any endomorphism g which has image in these allowed generalized eigenspaces. Similarly for $\frac{1}{c_k}$ where $\mathbf{i}_k \neq \beta$ is allowed. In particular $\frac{1}{b_k}$ and $\frac{1}{c_k}$ are well-defined on eigenspaces corresponding to small eigenvalues. □

4 The isomorphism theorem

In the isomorphism of the main theorem will appear some square roots. We start by recalling their definition and the required setup from [28]. The crucial result is the following fact, see [28, Proposition 4.2]:

Lemma 4.1. Let $f(x) \in \mathbb{C}[x]$. Assume B is a finite-dimensional algebra, and let $x_0 \in B$. Suppose that $(x-a) \nmid f(x)$ for all $a \in \mathbb{C}$ which are generalized eigenvalues for the action of x_0 on the regular representation. Then $f(x_0) \in B$ has a (unique) inverse and a (non-unique) square root. □

Let as above $x_0 \in B$ be an element of a finite-dimensional algebra, and let $a \in \mathbb{C}$. If a is not a generalized eigenvalue of x_0 then by Proposition 4.1 we can write expressions like

$$\frac{1}{x_0 - a}, \qquad \sqrt{x_0 - a}, \qquad \sqrt{\frac{1}{x_0 - a}}. \tag{4.1}$$

The square root is not unique, but we make one choice once and for all, so that for example $\sqrt{x_0 - a}\sqrt{\frac{1}{x_0-a}} = 1$. For more details we refer to [28]. Recalling the elements b_k and c_k from (3.9) set (for our choice of square root)

$$Q_k \quad = \quad \sqrt{\frac{b_{k+1}}{b_k}\mathbf{f}}. \tag{4.2}$$

Note that $\frac{b_{k+1}}{b_k}\mathbf{f}$ is well-defined (in contrast to for instance $\frac{b_{k+1}}{b_k}$) by definition of b_k and $\mathbf{f}$ via Remark 3.14. By the above arguments, also the square root (4.2) makes sense.

Definition 4.2. For $1 \leq k \leq d-1$ define

$$\tilde{s}_k = -Q_k s_k Q_k + \frac{1}{b_k}\mathbf{f} \quad \text{and} \quad \tilde{e}_k = Q_k e_k Q_k.$$

Note that these are well-defined elements in $\mathbb{W}_d^{\mathrm{cycl}}$ by Remark 3.14 and in fact contained in $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$. □

Now we can finally state our main result:

Theorem 4.3 (Isomorphism theorem)**.** The map

$$\Phi_\delta : \quad \mathrm{Br}_d(\delta) \quad \longrightarrow \quad \mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}.$$

given on the standard generators by

$$t_k \quad \longmapsto \quad \tilde{s}_k = -Q_k s_k Q_k + \frac{1}{b_k}\mathbf{f}, \quad \text{and} \quad g_k \quad \longmapsto \quad \tilde{e}_k = Q_k e_k Q_k. \tag{4.3}$$

for $1 \leq k \leq d-1$ defines an isomorphism of algebras. □

The proof will be given in the next section except of the well-definedness of the map Φ_δ which is proved (independently of the other statements in the theorem) in Section 6.

4.1 The isomorphism Φ_δ and Jucys-Murphy elements

We first assume, see Section 6, that the map Φ_δ is a well-defined algebra homomorphism. Before the proof of the theorem we describe the preimages of the polynomial generators y_k.

Definition 4.4. The *Jucys-Murphy elements* ξ_k, for $1 \le k \le d$ in the Brauer algebra $\mathrm{Br}_d(\delta)$ are defined as follows:

$$\xi_1 = 0 \quad \text{and} \quad \xi_{k+1} = t_k \xi_k t_k + t_k - g_k \text{ for all } 1 < k < d.$$

□

Proposition 4.5. Let $1 \le k \le d$. The map Φ_δ from Theorem 4.3 maps $\xi_k - \alpha$ to $-y_k \mathbf{f}$. □

Proof. We prove this by induction on k; starting with $\Phi_\delta(\xi_1 - \alpha) = \Phi_\delta(-\alpha) = -\alpha\mathbf{f} = -y_1\mathbf{f}$. For $k+1 > 1$ we have

$$\begin{aligned}
\Phi_\delta(\xi_{k+1} - \alpha) &= \Phi_\delta(t_k\xi_k t_k + t_k - g_k - \alpha) \quad = \quad \Phi_\delta(t_k(\xi_k - \alpha)t_k + t_k - g_k) \\
&= -\tilde{s}_k y_k \tilde{s}_k + \tilde{s}_k - \tilde{e}_k \quad = \quad Q_k s_k Q_k y_k \tilde{s}_k + \tilde{s}_k - \tilde{e}_k - \frac{1}{b_k}\mathbf{f} y_k \tilde{s}_k \\
&= Q_k(y_{k+1}s_k + e_k - 1)Q_k\tilde{s}_k + \tilde{s}_k - \tilde{e}_k - \frac{1}{b_k}\mathbf{f}y_k\tilde{s}_k \\
&= y_{k+1}Q_k s_k Q_k \tilde{s}_k + \tilde{e}_k\tilde{s}_k - \frac{b_{k+1}}{b_k}\mathbf{f}\tilde{s}_k + \tilde{s}_k - \tilde{e}_k - \frac{1}{b_k}\mathbf{f}y_k\tilde{s}_k \\
&= -y_{k+1}\tilde{s}_k\tilde{s}_k + \frac{y_{k+1}}{b_k}\mathbf{f}\tilde{s}_k - \frac{b_{k+1}}{b_k}\mathbf{f}\tilde{s}_k + \tilde{s}_k - \frac{1}{b_k}\mathbf{f}y_k\tilde{s}_k \\
&= -y_{k+1}\mathbf{f}.
\end{aligned}$$

The proposition is proved. ■

We finish this section with the proof of Theorem 4.3.

Proof. That Φ_δ is well-defined follows from a series of statements in Section 6. Namely altogether the Lemmas 6.1 to 6.11 prove that the elements $\tilde{s}_k$ and $\tilde{e}_k$ for $1 \le k < d$ satisfy all the defining relations of the Brauer algebra $\mathrm{Br}_d(\delta)$.

It suffices now to prove surjectivity of Φ_δ, since the algebras have the same dimension, namely $(2d-1)!!$, by [15, Proposition 4.4]. To prove surjectivity we use the description of a basis of $\mathbb{W}_d^{\mathrm{cycl}}$ from [2, Theorem 5.5], see [15, Corollary 2.25] for our special case, which says in particular that any element in $\mathbb{W}_d^{\mathrm{cycl}}$ is a linear combination of elements of the form $p_1 w p_2$, where $p_1, p_2 \in \mathbb{C}[y_1, \dots, y_d]$ with degree ≤ 1 in each variable and $w = x_1 \cdots x_r$ where $x_j \in \{s_i, e_i \mid 1 \le i \le d-1\}$ for $1 \le j \le r$. We will call such a presentation $x_1 \cdots x_r$ for w a *reduced word* if r is chosen minimally to present w in such a form.

Since by Lemma 4.5 all the elements $y_k\mathbf{f}$ are in the image I of Φ_δ, it suffices to show that $\mathbf{f}x_1 \cdots x_r \mathbf{f} \in I$ for any reduced word $x_1 \cdots x_r$.

We show this by two inductions on the sum of the length of the word and the number of s's occurring in the expression. For $r = 0,1$ the claim is clear, since $y_k\mathbf{f} \in I$ for all k and so are all polynomial expressions in the y's, e.g. $Q_k^{-1}\mathbf{f}$ and $\frac{1}{b_k}\mathbf{f}$, and thus also $\mathbf{f}s_k\mathbf{f} \in I$ and $\mathbf{f}e_k\mathbf{f}$ for all k. Hence the claim is true for $r \leq 1$.

For $r > 1$, we first assume that the expression $\mathbf{f}x_1 \cdots x_r\mathbf{f}$ contains no s's. In this case assume that $x_r = e_l$ for some l, then by induction we know that

$$\mathbf{f}x_1 \cdots x_{r-1}\mathbf{f}c_l\mathbf{f}x_r\mathbf{f} \quad = \quad (\mathbf{f}x_1 \cdots x_{r-1}\mathbf{f})(\mathbf{f}c_l\mathbf{f})(\mathbf{f}x_r\mathbf{f})$$

is in the image of Φ_δ, since all three factors are in the image by induction.

Moreover by Proposition 3.13 we have

$$\begin{aligned}
\mathbf{f}x_1 \cdots x_{r-1}\mathbf{f}c_l\mathbf{f}x_r\mathbf{f} &= \mathbf{f}x_1 \cdots x_{r-1}c_lx_r\mathbf{f} \\
&= \beta\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} + \mathbf{f}x_1 \cdots x_{r-1}y_{l+1}x_r\mathbf{f} \\
&= \beta\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} \pm \begin{cases} y_j\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} & \text{for some } j \text{ or} \\ \mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f}y_j & \text{for some } j. \end{cases}
\end{aligned} \tag{4.4}$$

The last equality is possible because the word was assumed to be reduced, thus the element y_{l+1} can be moved to the outside by using repeatedly (VW.8a) and (VW.8b) - only creating a possible sign change. Since $\frac{1}{\beta \pm y_j}\mathbf{f} \in I$, it follows that $\mathbf{f}x_1 \cdots x_r\mathbf{f} \in I$. Assume now that $\mathbf{f}x_1 \cdots x_r\mathbf{f}$ for a reduced word $\mathbf{f}x_1 \cdots x_r\mathbf{f}$ contains a positive number, say m, of s's. Let l be such that $x_r \in \{e_l, s_l\}$. Then again by induction we know that $\mathbf{f}x_1 \cdots x_{r-1}\mathbf{f}c_l\mathbf{f}x_r\mathbf{f} \in I$.

Using again Proposition 3.13 we obtain a similar expression as before, namely

$$\begin{aligned}
\mathbf{f}x_1 \cdots x_{r-1}\mathbf{f}c_l\mathbf{f}x_r\mathbf{f} &= \mathbf{f}x_1 \cdots x_{r-1}c_lx_r\mathbf{f} \\
&= \beta\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} + \mathbf{f}x_1 \cdots x_{r-1}y_{l+1}x_r\mathbf{f} \\
&= \beta\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} \pm \begin{cases} y_j\mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f} & \text{for some } j \text{ or} \\ \mathbf{f}x_1 \cdots x_{r-1}x_r\mathbf{f}y_j & \text{for some } j \end{cases} \quad + \text{ smaller summands.}
\end{aligned}$$

The smaller summands do not contain any y's in this case, see Relation (VW.7), and are moreover either of length smaller than r, or of length r but then with strictly less than m s's. Now by induction all these smaller terms are contained in I and the claim follows also in this case. Thus Φ_δ is surjective and the theorem follows. ■

Remark 4.6. The main feature of our isomorphism is the change of the parameter N for $\mathbb{W}_d^{\mathrm{cycl}}$ to the corresponding parameter δ of $\mathrm{Br}_d(\delta)$; the most important relation we have to verify is

$$(Q_ke_kQ_k)^2 = \delta\, Q_ke_kQ_k.$$

Essentially, this amounts to check that

$$Q_k e_k \frac{b_{k+1}}{b_k} \mathbf{f} e_k Q_k \quad = \quad (2\beta+1) Q_k e_k Q_k - N Q_k e_k Q_k = \delta\, Q_k e_k Q_k. \tag{4.5}$$

see Lemma 6.3. By [26] we can take a formal variable u and write

$$e_k \frac{1}{u - y_k} e_k \mathbf{f} \quad = \quad \frac{W_k(u)}{u} e_k \mathbf{f}, \tag{4.6}$$

where $W_k(u)$ is a formal power series in u^{-1} as in [26]. We may now be tempted to replace $u = -\beta$, compute $W_k(-\beta) = \beta_1$, and hence obtain (4.5) from (4.6). Now, while this can be formalized in the semisimple case (by using the eigenvalues of y_k, as done several times in [26]), it gets much more tricky in the non-semisimple case. Hence we need to take another way using the formalism from Section 3 and the beginning of this section. □

Corollary 4.7. The algebra $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$ is generated by the elements $\mathbf{f}s_i\mathbf{f}$, $\mathbf{f}e_i\mathbf{f}$, $y_k\mathbf{f}$ for $1 \leq i < d$ and $1 \leq k \leq d$. □

Proof. By Theorem 4.3 the algebra is generated by the elements $\tilde{s}_k, \tilde{e}_k$ for $1 \leq k \leq d-1$. Then the claim follows from the definitions. ■

5 Consequences: Koszulity and graded decomposition numbers

We deduce now some non-trivial consequences of the main theorem. For the whole section d is a positive integer and $\delta \in \mathbb{Z}$. First, one of the main results of [15] allows us to equip the Brauer algebra with a grading. To state the result we need some additional notation for graded modules. Let A be a $\mathbb{Z}$-graded algebra. For $M \in A-\mathrm{gmod}$ we denote its *graded endomorphism ring* by

$$\mathrm{end}_A(M) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{A-\mathrm{gmod}}(M, M\langle r\rangle),$$

which becomes a graded ring by putting $\mathrm{end}_A(M)_r = \mathrm{Hom}_{A-\mathrm{gmod}}(M, M\langle r\rangle)$. The composition of $f \in \mathrm{Hom}_{A-\mathrm{gmod}}(M, M\langle r\rangle)$ and $g \in \mathrm{Hom}_{A-\mathrm{gmod}}(M, M\langle s\rangle)$ is given by $g\langle r\rangle \circ f$ in the category of graded modules. Note that for a graded lift $\widehat{M} \in A-\mathrm{gmod}$ of $M \in A-\mathrm{mod}$ it holds

$$\mathrm{End}_{A-\mathrm{mod}}(M) \cong \mathrm{end}_A(\widehat{M})$$

as (ungraded) algebras.

Proposition 5.1. Let $\delta \in \mathbb{C}$. The Brauer algebra $\mathrm{Br}_d(\delta)$ can be equipped with a $\mathbb{Z}$-grading turning it into a $\mathbb{Z}$-graded algebra $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$. □

Proof. We again assume $\delta \in \mathbb{Z}$, since otherwise the statement is trivial. Recall from Section 3 the parabolic category $\mathcal{O}^{\mathfrak{p}}(n)$. Consider the endofunctor $\mathcal{F} = ? \otimes V$ of $\mathcal{O}^{\mathfrak{p}}(n)$. Following [15, Sections 4] we have the summand $\mathcal{G}$ of this functor that corresponds to projecting onto blocks with small weights (in [15] this functor was denoted by $\widetilde{\mathcal{F}}$) such that

$$\mathrm{Br}_d(\delta) \cong \mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f} \cong \mathbf{f}\mathrm{End}_{\mathfrak{g}}(\mathcal{F}^d M^{\mathfrak{p}}(\underline{\delta}))\mathbf{f} \cong \mathrm{End}_{\mathfrak{g}}(\mathcal{G}^d M^{\mathfrak{p}}(\underline{\delta})),$$

as algebras. By choosing a minimal projective generator P of $\mathcal{O}^{\mathfrak{p}}(n)$ we have an equivalence of categories

$$\mathcal{O}^{\mathfrak{p}}(n) \cong \mathrm{mod} - \mathrm{End}_{\mathfrak{g}}(P).$$

Following [14] we equip $A := \mathrm{End}_{\mathfrak{g}}(P)$ with a Koszul grading and denote by $\widehat{\mathcal{O}}^{\mathfrak{p}}(n) := A - \mathrm{gmod}$ its associated category of graded modules. In [15, Section 5] a graded lift $\widehat{\mathcal{F}}$ of $\mathcal{F}$ is constructed by choosing graded lifts for each summand obtained by projecting onto blocks, see [15, Lemma 5.3]. Thus it also yields a graded lift $\widehat{\mathcal{G}}$ of $\mathcal{G}$ and gives

$$\mathrm{Br}_d^{\mathrm{gr}}(\delta) := \mathrm{end}_A(\widehat{\mathcal{G}}^d \widehat{M^{\mathfrak{p}}(\underline{\delta})}),$$

where $\widehat{M^{\mathfrak{p}}(\underline{\delta})}$ is the standard graded lift of the parabolic Verma module. ■

With this grading one can establish Koszulity.

Theorem 5.2. The Brauer algebra $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is Morita equivalent to a Koszul algebra if and only if $\delta \neq 0$ or $\delta = 0$ and d odd. □

Proof. This follows directly from our main Theorem 4.3 together with [15, Theorem 5.1]. ■

Proposition 5.3. The Brauer algebra $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is graded cellular. □

Proof. It follows from [21] that the idempotent truncation of the quasi-hereditary algebra $\mathbb{W}_d^{\mathrm{cycl}}$ is always cellular. It is straightforward to see that this is compatible with the grading. ■

Theorem 5.4. The Brauer algebra $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is graded quasi-hereditary if and only if $\delta \neq 0$ or $\delta = 0$ and d odd. □

Proof. By [15, Theorem 4.13 and Remark 4.14] the highest weight structure of $\mathbb{W}_d^{\mathrm{cycl}}$ induces a highest weight structure on $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$ if and only if $\delta \neq 0$ or $\delta = 0$ and d odd. Since by [15, Definition 4.11] the labelling posets of standard modules for $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$ agrees with the one for the Brauer algebra from [9] the isomorphism from Theorem 4.3 is an isomorphism of quasi-hereditary algebras. The result then follows from the general theory of graded category $\mathcal{O}$ (see [30]) using [15, Theorem 4.9]. ■

Denote by $\Delta(\lambda)$ for $\lambda \in \Lambda_d$ the standard module for $\mathrm{Br}_d(\delta)$ and by $L(\lambda)$ for $\lambda \in \Lambda_d^{\delta}$ the corresponding simple quotient, see [9]. As in the introduction denote by F the grading forgetting functor from $\mathrm{Br}_d^{\mathrm{gr}}(\delta) - \mathrm{gmod}$ to $\mathrm{Br}_d(\delta) - \mathrm{mod}$.

Theorem 5.5. Assume that $\delta \neq 0$ or $\delta = 0$ and d is odd, i.e. the case where $\mathrm{Br}_d^{\mathrm{gr}}(\delta)$ is graded quasi-hereditary. For any $\lambda \in \Lambda_d$ there exists a unique modules $\widehat{\Delta}(\lambda) \in \mathrm{Br}_d^{\mathrm{gr}}(\delta) - \mathrm{gmod}$ such that $F\widehat{\Delta}(\lambda) \cong \Delta(\lambda)$ and for $\lambda \in \Lambda_d^\delta$ there exists a unique $\widehat{L}(\lambda) \in \mathrm{Br}_d^{\mathrm{gr}}(\delta) - \mathrm{gmod}$ such that $F\widehat{L}(\lambda) \cong L(\lambda)$, and both modules are concentrated in non-negative degrees with non-vanishing degree zero. □

Proof. From [30, Lemma 1.5] it follows that graded lifts, if they exist, are unique up to isomorphism and grading shifts. With the assumption on the degree of the modules the graded lifts will be unique up to isomorphism in our case. For $\mathbb{W}_d^{\mathrm{cycl}}$ the existence of graded lifts follows from [15, Theorem 4.9] and general theory of category $\mathcal{O}$, see [30]. The existence of the graded lifts for $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$ then follows from [15, Theorem 4.13] via a quotient functor construction. ■

We now match the multiplicities of simple modules $\widehat{L}(\mu)\langle i\rangle$ occurring in a standard module $\widehat{\Delta}(\lambda)$ with the coefficients of certain Kazhdan-Lusztig polynomials. Denote by W the Weyl group of $\mathfrak{g}$ and by $W_{\mathfrak{p}}$ the parabolic subgroup generated by all simple roots except α_0, i.e. the one corresponding to the parabolic $\mathfrak{p}$ from the introduction and Section 3. By $W^{\mathfrak{p}}$ we denote the shortest coset representatives in $W_{\mathfrak{p}}\backslash W$. For $x, y \in W^{\mathfrak{p}}$ let $n_{x,y}(q) \in \mathbb{Z}[q]$ be the parabolic Kazhdan-Lusztig polynomial of type $(\mathrm{D}_n, \mathrm{A}_{n-1})$ in the normalization from [29, Remark 3.2]; see [5] and in particular [23] for the special case needed here.

Given $\nu \in X_n^{\mathfrak{p}}$ there is a unique $\nu_{\mathrm{dom}} \in X_n^{\mathfrak{p}}$ and $x_\nu \in W^{\mathfrak{p}}$ such that $\nu_{\mathrm{dom}} + \rho$ is dominant and

$$x_\nu(\nu_{\mathrm{dom}} + \rho) = \nu + \rho.$$

We now give a dictionary how to translate between the labelling set of standard modules and Weyl group elements. To a partition λ we associate a double Young diagram $Y(\lambda)$ via the bipartition $(\lambda, \varnothing)$ and a weight $\mathrm{wt}(\lambda) = \underline{\delta} + \mathrm{wt}(Y(\lambda))$. For $\lambda, \mu \in \Lambda_d$ we put

$$n_{\lambda,\mu}(q) = \begin{cases} n_{x_{\mathrm{wt}(\lambda)}, x_{\mathrm{wt}(\mu)}}(q) & \text{if } \mathrm{wt}(\lambda) + \rho \in W \cdot (\mathrm{wt}(\mu) + \rho), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.6. For $\lambda \in \Lambda_d$, the module $\widehat{\Delta}(\lambda)$ has a Jordan-Hölder series in $\mathrm{Br}_d^{\mathrm{gr}}(\delta) - \mathrm{gmod}$ with multiplicities given by

$$\left[\widehat{\Delta}(\lambda) : \widehat{L}(\mu) < i >\right] = n_{\lambda,\mu,i},$$

where $n_{\lambda,\mu}(q) = \sum_{i \geq 0} n_{\lambda,\mu,i} q^i$ and $\mu \in \Lambda_d^\delta$. □

Proof. Denote by $\widetilde{\Delta}(\lambda)$ the standard module in $\mathbb{W}_d^{\mathrm{cycl}} - \mathrm{gmod}$ that is sent to $\widehat{\Delta}(\lambda)$ via the quotient functor $\mathcal{Q} : \mathbb{W}_d^{\mathrm{cycl}}\mathbf{f} - \mathrm{gmod} \to \mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f} - \mathrm{gmod}$ and analogously by $\widetilde{L}(\mu)$ the simple module in $\mathbb{W}_d^{\mathrm{cycl}} - \mathrm{gmod}$. Then the following equality holds

$$n_{\lambda,\mu,i} \overset{(a)}{=} \left[\widetilde{\Delta}(\lambda) : \widetilde{L}(\mu) < i >\right] \overset{(b)}{=} \left[\mathcal{Q}\widetilde{\Delta}(\lambda) : \mathcal{Q}\widetilde{L}(\mu) < i >\right] \overset{(c)}{=} \left[\widehat{\Delta}(\lambda) : \widehat{L}(\mu) < i >\right],$$

where (a) is due to [15, Theorem 4.9], (b) is due to [12, p.136 (4)(vii)], and (c) is due to [12, p.136 (4)(iv)] for the simple module and [21, Prop. 4.3] for the standard module. ■

6 Well-definedness of Φ_δ

In this section we establish the remaining part of Theorem 4.3, namely show that the map Φ_δ is well-defined. In other words, we have to verify that the $\tilde{s}_i$ and $\tilde{e}_i$ satisfy the Brauer relations. We will commence with a few lemmas that will help in the calculations, allowing us to simplify various expressions.

Lemma 6.1. It holds $e_k\eta_{k+1} = e_k\eta_k$ for $1 \le k \le d-1$, hence also $e_k\mathbf{f}_{k+1} = e_k\mathbf{f}_k$. □

Proof. Let $\mathbb{W}_d^{\text{cycl}}$ act on itself by the regular representation. By Lemma 2.7 it follows that if either side of the equation acts non-trivially on some generalized eigenspace, then the eigenvalues of y_k and y_{k+1} add up to zero. Hence if one is small so is the other. ■

We often need to simplify expressions involving fractions, such as in the definitions of $\tilde{s}_k$ and $\tilde{e}_k$. The following proposition collects a few useful formulas for this.

Proposition 6.2. In $\mathbb{W}_d^{\text{cycl}}$ the following equalities hold for $1 \le k \le d-1$:

$$i) \quad e_k\tfrac{1}{b_k}s_k\mathbf{f} = \tfrac{1}{2\beta}e_k\tfrac{1}{b_k}\mathbf{f}, \qquad ii) \quad \mathbf{f}s_k\tfrac{1}{b_k}e_k\mathbf{f} = \tfrac{1}{2\beta}\tfrac{1}{b_k}\mathbf{f}e_k\mathbf{f}, \qquad iii) \quad e_k\tfrac{1}{b_k}e_k\mathbf{f} = (1+\tfrac{1}{2\beta})e_k\mathbf{f}.$$

(For the well-definedness we again refer to Remark 3.14.) □

Proof. Let us first assume $iii)$ is proven already. To verify $i)$ we calculate

$$\begin{aligned}
e_k\frac{1}{b_k}s_k\mathbf{f} &\overset{(a)}{=} e_ks_k\frac{1}{b_{k+1}}\mathbf{f} - e_k\frac{1}{b_k}e_k\frac{1}{b_{k+1}}\mathbf{f} + e_k\frac{1}{b_kb_{k+1}}\mathbf{f}\\
&\overset{(b)}{=} e_ks_k\frac{1}{b_{k+1}}\mathbf{f} - (1+\frac{1}{2\beta})e_k\frac{1}{b_{k+1}}\mathbf{f} + e_k\frac{1}{b_kb_{k+1}}\mathbf{f}\\
&= e_k\left(\frac{2\beta - b_k}{2\beta b_kb_{k+2}}\right) = e_k\left(\frac{c_k}{2\beta b_kb_{k+1}}\right) = e_k\left(\frac{b_{k+1}}{2\beta b_kb_{k+1}}\right) = \frac{1}{2\beta}e_k\frac{1}{b_k}\mathbf{f}
\end{aligned}$$

where equality (a) holds by Lemma 3.12 and equality (b) is valid thanks to part $iii)$ of this proposition.

Formula $ii)$ is shown analogously, but note that since $\frac{1}{b_{k+1}}e_k$ is in general not defined we have to multiply the whole equation by $\mathbf{f}$ from the left to make it well-defined.

Finally let us consider the formula $iii)$ which we will prove by induction on k. If $k = 1$ then note that by Definition 3.1 the element y_1 has exactly two eigenvalues, namely α and β as in (3.4), with the projections $\frac{y_1-\beta}{\alpha-\beta}$ respectively $\frac{y_1-\alpha}{\beta-\alpha}$ onto the eigenspaces. Then we obtain

$$\begin{aligned} e_1\frac{1}{b_1}e_1\mathbf{f} &= e_1\left(\frac{1}{\alpha+\beta}\frac{y_1-\beta}{\alpha-\beta}+\frac{1}{2\beta}\frac{y_1-\alpha}{\beta-\alpha}\right)e_1\mathbf{f} &&= \frac{1}{\alpha-\beta}e_1\left(\frac{y_1-\beta}{\alpha+\beta}+\frac{\alpha-y_1}{2\beta}\right)e_1\mathbf{f} \\ &= e_1\frac{1}{(\alpha+\beta)2\beta}\left(-y_1+\alpha+2\beta\right)e_1\mathbf{f} &&\overset{(3.4)}{=} \frac{1}{N\beta}\left(-e_1y_1e_1+(\alpha+2\beta)e_1^2\right)\mathbf{f} \\ &= \frac{1}{2N\beta}\left(N(1-N)+N^2+2\beta\right)\mathbf{f} &&= \left(1+\frac{1}{2\beta}\right)e_1\mathbf{f}. \end{aligned}$$

Now assume the formula holds for k and, by applying Lemma 3.12 repeatedly, we obtain

$$\begin{aligned} e_{k+1}\frac{1}{b_{k+1}}e_{k+1}\mathbf{f} &= e_{k+1}s_k(s_k\frac{1}{b_{k+1}})e_{k+1}\mathbf{f} \\ &= e_{k+1}s_k\left(\frac{1}{b_k}\right)s_ke_{k+1}\mathbf{f} + e_{k+1}s_k\left(\frac{1}{b_k}e_k\frac{1}{b_{k+1}}\right)e_{k+1}\mathbf{f} - e_{k+1}s_k\left(\frac{1}{b_k}\frac{1}{b_{k+1}}\right)e_{k+1}\mathbf{f} \end{aligned} \tag{6.1}$$

by Lemma 3.12. Now the first summand in (6.1) equals, by (VW.6c),

$$e_{k+1}e_ks_{k+1}\left(\frac{1}{b_k}\right)s_{k+1}e_ke_{k+1}\mathbf{f} = e_{k+1}e_k\frac{1}{b_k}e_ke_{k+1}\mathbf{f} = \left(1+\frac{1}{2\beta}\right)e_{k+1}e_ke_{k+1}\mathbf{f} = \left(1+\frac{1}{2\beta}\right)e_{k+1}\mathbf{f}$$

by induction, whereas the second summand equals

$$\begin{aligned} & e_{k+1}s_k\frac{1}{b_k}e_k\frac{1}{b_{k+1}}e_{k+1}\mathbf{f} \\ =\ & e_{k+1}\left(s_k\frac{1}{b_k}\right)e_ke_{k+1}\frac{1}{c_k}\mathbf{f} \\ =\ & e_{k+1}\left(\frac{1}{b_{k+1}}s_k\right)e_ke_{k+1}\frac{1}{c_k}\mathbf{f} - e_{k+1}\left(\frac{1}{b_{k+1}}e_k\frac{1}{b_k}\right)e_ke_{k+1}\frac{1}{c_k}\mathbf{f} + e_{k+1}\left(\frac{1}{b_{k+1}}\frac{1}{b_k}\right)e_ke_{k+1}\frac{1}{c_k}\mathbf{f} \\ =\ & \frac{1}{c_k}e_{k+1}e_ke_{k+1}\frac{1}{c_k}\mathbf{f} - \left(1+\frac{1}{2\beta}\right)\frac{1}{c_k}e_{k+1}e_ke_{k+1}\frac{1}{c_k}\mathbf{f} + \frac{1}{b_kc_k}e_{k+1}e_ke_{k+1}\frac{1}{c_k}\mathbf{f} \\ =\ & -\frac{1}{2\beta}\frac{1}{c_k^2}e_{k+1}\mathbf{f} + \frac{1}{b_kc_k^2}e_{k+1}\mathbf{f} \end{aligned}$$

by (VW.8a) and (VW.5a), Lemma 3.12 and induction hypothesis. Finally the third summand in (6.1) equals

$$\begin{aligned} & -e_{k+1}\left(s_k\frac{1}{b_{k+1}}\right)e_{k+1}\frac{1}{b_k}\mathbf{f} \\ =\ & -e_{k+1}\left(\frac{1}{b_ks_k}\right)e_{k+1}\frac{1}{b_k}\mathbf{f} - e_{k+1}\left(\frac{1}{b_k}e_k\frac{1}{b_{k+1}}\right)e_{k+1}\frac{1}{b_k}\mathbf{f} + e_{k+1}\left(\frac{1}{b_kb_{k+1}}\right)e_{k+1}\frac{1}{b_k}\mathbf{f} \\ =\ & -\frac{1}{b_k^2}e_{k+1}\mathbf{f} - \frac{1}{b_k^2c_k}e_{k+1}\mathbf{f} + \frac{1}{b_k^2}e_{k+1}\frac{1}{b_{k+1}}e_{k+1}\mathbf{f}. \end{aligned}$$

Hence altogether we obtain

$$\begin{aligned}\left(1-\frac{1}{b_k^2}\right)e_{k+1}\frac{1}{b_{k+1}}e_{k+1}\mathbf{f} &= \left(1+\frac{1}{2\beta}\right)e_{k+1}\mathbf{f}+\left(-\frac{1}{2\beta}\frac{1}{c_k^2}+\frac{1}{b_kc_k^2}-\frac{1}{b_k^2}-\frac{1}{b_k^2c_k}\right)e_{k+1}\mathbf{f}\\ &= \left(1-\frac{1}{b_k^2}\right)\left(1+\frac{1}{2\beta}\right)e_{k+1}\mathbf{f}+\left(\frac{1}{2\beta}\frac{1}{b_k^2}-\frac{1}{2\beta}\frac{1}{c_k^2}+\frac{1}{b_kc_k^2}-\frac{1}{b_k^2c_k}\right)e_{k+1}\mathbf{f}.\end{aligned}$$

Now one easily checks that the last coefficient in front of the final $e_{k+1}\mathbf{f}$ is zero and since $\left(1-\frac{1}{b_k^2}\right)$ is invertible on the image of $e_{k+1}\mathbf{f}$ we obtain

$$e_{k+1}\frac{1}{b_{k+1}}e_{k+1}\mathbf{f} = \left(1+\frac{1}{2\beta}\right)e_{k+1}\mathbf{f}$$

which finishes the proof. ■

6.1 The key relation $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$

The following is the most crucial point of the proof (see also Remark 4.6):

Lemma 6.3. We have $\tilde{e}_k^2 = \delta\tilde{e}_k$ for $1 \le k \le d-1$. □

Proof. We compute

$$\begin{aligned}\tilde{e}_k^2 &= Q_ke_k\tfrac{b_{k+1}}{b_k}\mathbf{f}e_kQ_k \overset{(a)}{=} Q_ke_k\tfrac{c_k}{b_k}\mathbf{f}e_kQ_k \overset{(b)}{=} Q_ke_k\tfrac{c_k}{b_k}e_kQ_k\\ &\overset{(c)}{=} 2\beta Q_ke_k\tfrac{1}{b_k}e_kQ_k - Q_ke_k^2Q_k \overset{(d)}{=} (2\beta+1)Q_ke_kQ_k - NQ_ke_kQ_k = \delta\tilde{e}_k.\end{aligned}$$

Where equality (a) follows from (VW.8b), equality (b) holds by Proposition 3.13, equality (c) just expands c_k as $2\beta - b_k$ and equality (d) is valid thanks to Proposition 6.2. ■

6.2 Symmetric group relations in $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$

In this part we show that the $\tilde{s}_j$'s satisfy the defining relation of the symmetric group. One of the difficulties in the following calculations is to use only manipulations that result in well-defined terms. In some situations this will make it necessary to expand terms instead of directly simplifying them. However, the well-definedness of all occurring terms follows easily from Remark 3.14.

Lemma 6.4. We have $\tilde{s}_k^2 = \mathbf{f}$ for $1 \le k \le d-1$. □

Proof. We compute

$$\begin{aligned}
\tilde{s}_k^2 &= Q_k\left(s_k\frac{b_{k+1}}{b_k}\mathbf{f}s_k - s_k\frac{1}{b_k} - \frac{1}{b_k}\mathbf{f}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(a)}{=} Q_k\left(s_k\frac{b_{k+1}}{b_k}s_k - s_k\frac{1}{b_k} - \frac{1}{b_k}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(b)}{=} Q_k\left(s_kb_{k+1}s_k\frac{1}{b_{k+1}} - s_k\frac{b_{k+1}}{b_k}e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(c)}{=} Q_k\left(\frac{b_k}{b_{k+1}} + (s_k - e_k)\frac{1}{b_{k+1}} - s_k\frac{b_{k+1}}{b_k}e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(d)}{=} Q_k\left(\frac{b_k}{b_{k+1}} + (s_k - e_k)\frac{1}{b_{k+1}} - b_ks_k\frac{1}{b_k}e_k\frac{1}{b_{k+1}} + e_k\frac{1}{b_k}e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(e)}{=} Q_k\left(\frac{b_k}{b_{k+1}} + (s_k - e_k)\frac{1}{b_{k+1}} - \frac{1}{2\beta}e_k\frac{1}{b_{k+1}} + (1+\frac{1}{2\beta})e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}e_k\frac{1}{b_{k+1}} - \frac{1}{b_k}s_k\right)Q_k + \frac{1}{b_k^2}\mathbf{f}\\
&\overset{(f)}{=} Q_k\left(\frac{b_k}{b_{k+1}} - \frac{1}{b_kb_{k+1}}\right)\mathbf{f}Q_k + \frac{1}{b_k^2}\mathbf{f} = \mathbf{f}
\end{aligned}$$

where (a) follows from Proposition 3.13 and the fact that $\mathbf{f}$ commutes with $\frac{1}{b_k}$ on the image of s_k, (b) is due to Lemma 3.12, (c) and (d) are applications of Relation (VW.7), (e) uses Proposition 6.2, and finally (f) uses again Lemma 3.12. ■

Since $\mathbf{f}s_i\mathbf{f}$ commutes with Q_j and $\frac{1}{b_j}\mathbf{f}$ in case $|i-j|>1$, the following lemma holds.

Lemma 6.5. We have $\tilde{s}_i\tilde{s}_j = \tilde{s}_j\tilde{s}_i$ for $1 \le i, j \le d-1$ with $|i-j|>1$. □

We verify now the braid relations, which is a surprisingly non-trivial task.

Proposition 6.6. The braid relation $\tilde{s}_i\tilde{s}_{i+1}\tilde{s}_i = \tilde{s}_{i+1}\tilde{s}_i\tilde{s}_{i+1}$ holds for $1 \le i < d-1$. □

Proof. We expand both sides of the asserted equality and show that their difference is zero. The left hand side is equal to the following sum

$$\tilde{s}_k\tilde{s}_{k+1}\tilde{s}_k =$$

$$-Q_ks_kQ_kQ_{k+1}s_{k+1}Q_{k+1}Q_ks_kQ_k \tag{6.2}$$

$$+\frac{1}{b_k}\mathbf{f}Q_{k+1}s_{k+1}Q_{k+1}Q_ks_kQ_k \tag{6.3}$$

$$+Q_ks_kQ_k\frac{1}{b_{k+1}}\mathbf{f}Q_ks_kQ_k \tag{6.4}$$

$$+Q_ks_kQ_kQ_{k+1}s_{k+1}Q_{k+1}\frac{1}{b_k}\mathbf{f} \tag{6.5}$$

$$-\frac{1}{b_kb_{k+1}}\mathbf{f}Q_ks_kQ_k \tag{6.6}$$

$$-\frac{1}{b_k}\mathbf{f}Q_{k+1}s_{k+1}Q_{k+1}\frac{1}{b_k}\mathbf{f} \tag{6.7}$$

$$-Q_ks_kQ_k\frac{1}{b_kb_{k+1}}\mathbf{f} \tag{6.8}$$

$$+\frac{1}{b_k^2b_{k+1}}\mathbf{f}. \tag{6.9}$$

and the right hand side is equal to the sum of the following elements

$$\tilde{s}_{k+1}\tilde{s}_k\tilde{s}_{k+1} =$$

$$-Q_{k+1}s_{k+1}Q_{k+1}Q_k s_k Q_k Q_{k+1}s_{k+1}Q_{k+1} \tag{6.10}$$

$$+\frac{1}{b_{k+1}}\mathbf{f}Q_k s_k Q_k Q_{k+1}s_{k+1}Q_{k+1} \tag{6.11}$$

$$+Q_{k+1}s_{k+1}Q_{k+1}\frac{1}{b_k}\mathbf{f}Q_{k+1}s_{k+1}Q_{k+1} \tag{6.12}$$

$$+Q_{k+1}s_{k+1}Q_{k+1}Q_k s_k Q_k\frac{1}{b_{k+1}}\mathbf{f} \tag{6.13}$$

$$-\frac{1}{b_k b_{k+1}}\mathbf{f}Q_{k+1}s_{k+1}Q_{k+1} \tag{6.14}$$

$$-\frac{1}{b_{k+1}}\mathbf{f}Q_k s_k Q_k\frac{1}{b_{k+1}}\mathbf{f} \tag{6.15}$$

$$-Q_{k+1}s_{k+1}Q_{k+1}\frac{1}{b_k b_{k+1}}\mathbf{f} \tag{6.16}$$

$$+\frac{1}{b_k b_{k+1}^2}\mathbf{f}. \tag{6.17}$$

We first simplify some of these expressions using the abbreviation $A := Q_{k+1}\frac{1}{\sqrt{b_k}}\mathbf{f}$. To improve readability we highlight those terms that are modified in each step, using either Lemma 3.12 to move terms past s_j's, Proposition 3.13 to eliminate $\mathbf{f}$'s or Proposition 6.2 to modify terms involving fractions. First we obtain from

$$\begin{aligned}(6.16)+(6.17) &= -As_{k+1}\frac{1}{b_{k+1}}\mathbf{f}A+\frac{1}{b_k b_{k+1}^2}\mathbf{f}\\ &= -A\left(\frac{1}{b_{k+2}}\mathbf{f}s_{k+1}-\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}\frac{1}{b_{k+1}}+\frac{1}{b_{k+1}b_{k+2}}\right)A+\frac{1}{b_k b_{k+1}^2}\mathbf{f}\\ &= A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}\frac{1}{b_{k+1}}A-A\frac{1}{b_{k+2}}\mathbf{f}s_{k+1}A\end{aligned}$$

and the calculation

$$\begin{aligned}(6.12)+(6.14) &= A\left(s_{k+1}\frac{b_{k+2}}{b_{k+1}}\mathbf{f}s_{k+1}-\frac{1}{b_{k+1}}\mathbf{f}s_{k+1}\right)A \quad +A\left(s_{k+1}b_{k+2}\frac{1}{b_{k+1}}s_{k+1}-\frac{1}{b_{k+1}}s_{k+1}\right)A\\ &= A\left((b_{k+1}s_{k+1}-e_{k+1}+1)\frac{1}{b_{k+1}}s_{k+1}-\frac{1}{b_{k+1}}s_{k+1}\right)A\\ &= A\left(b_{k+1}s_{k+1}\frac{1}{b_{k+1}}s_{k+1}-e_{k+1}\frac{1}{b_{k+1}}s_{k+1}\right)A\\ &= A\left(b_{k+1}s_{k+1}\left(s_{k+1}\frac{1}{b_{k+2}}-\frac{1}{b_{k+1}}e_{k+1}\frac{1}{b_{k+2}}+\frac{1}{b_{k+1}b_{k+2}}\right)-\frac{1}{2\beta}e_{k+1}\frac{1}{b_{k+1}}\right)A\\ &= A\left(\frac{b_{k+1}}{b_{k+2}}-b_{k+1}s_{k+1}\frac{1}{b_{k+1}}e_{k+1}\frac{1}{b_{k+2}}+b_{k+1}s_{k+1}\frac{1}{b_{k+1}b_{k+2}}-\frac{1}{2\beta}e_{k+1}\frac{1}{b_{k+1}}\right)A\\ &= A\left(\frac{b_{k+1}}{b_{k+2}}-\frac{1}{2\beta}e_{k+1}\frac{1}{b_{k+2}}+b_{k+1}s_{k+1}\frac{1}{b_{k+1}b_{k+2}}-\frac{1}{2\beta}e_{k+1}\frac{1}{b_{k+1}}\right)A\\ &= A\left(\frac{b_{k+1}}{b_{k+2}}-e_{k+1}\frac{1}{b_{k+1}b_{k+2}}+b_{k+1}s_{k+1}\frac{1}{b_{k+1}b_{k+2}}\right)A\\ &= A\left(\frac{b_{k+1}}{b_{k+2}}-e_{k+1}\frac{1}{b_{k+1}b_{k+2}}+(s_{k+1}b_{k+2}+e_{k+1}-1)\frac{1}{b_{k+1}b_{k+2}}\right)A\\ &= A\left(\frac{b_{k+1}}{b_{k+2}}+s_{k+1}\frac{1}{b_{k+1}}-\frac{1}{b_{k+1}b_{k+2}}\right)A \quad = \quad \frac{1}{b_k}\mathbf{f}+As_{k+1}\frac{1}{b_{k+1}}A-\frac{1}{b_k b_{k+1}^2}\mathbf{f}\\ &= \frac{1}{b_k}\mathbf{f}+A\left(\frac{1}{b_{k+2}}\mathbf{f}s_{k+1}\right)A-A\left(\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}\frac{1}{b_{k+1}}\right)A\end{aligned}$$

the equality

$$(6.12)+(6.14)+(6.16)+(6.17)=\frac{1}{b_k}\mathbf{f}.$$

For the left hand side of our equation we simplify

$$
\begin{aligned}
-(6.4)-(6.6) &= A\left(-\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}\mathbf{f}s_kb_{k+1}+\frac{1}{b_kb_{k+2}}\mathbf{f}s_kb_{k+1}\right)A\\
&= A\left(-b_{k+1}s_k\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}+\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}\right)A\\
&= A\left(-(s_kb_k-e_k+1)\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}+\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}\right)A\\
&= A\left(-s_k\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}+e_k\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}\right)A\\
&= A\left(-\frac{b_{k+1}}{b_{k+2}}+s_k(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}+e_k\frac{1}{b_k}s_k\frac{b_{k+1}}{b_{k+2}}-e_k\frac{1}{b_k}(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}\right)A\\
&= -\frac{1}{b_k}\mathbf{f}+As_k(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A+A\frac{1}{2\beta}e_k\frac{b_{k+1}}{b_kb_{k+2}}A-Ae_k\frac{1}{b_k}(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A.
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
(6.15)-(6.8)-(6.9) &= -\frac{1}{b_{k+1}}\mathbf{f}Q_ks_k\frac{Q_k}{b_{k+1}}\mathbf{f}+Q_ks_k\frac{1}{b_k}\frac{Q_k}{b_{k+1}}\mathbf{f}-\frac{1}{b_k^2b_{k+1}}\mathbf{f}\\
&= -\frac{1}{b_{k+1}}\mathbf{f}Q_ks_k\frac{Q_k}{b_{k+1}}\mathbf{f}+Q_k\left(\frac{1}{b_{k+1}}\mathbf{f}s_k-\frac{1}{b_{k+1}}\mathbf{f}e_k\frac{1}{b_k}+\frac{1}{b_kb_{k+1}}\right)\frac{Q_k}{b_{k+1}}\mathbf{f}-\frac{1}{b_k^2b_{k+1}}\mathbf{f}\\
&= Q_k\left(-\frac{1}{b_{k+1}}\mathbf{f}e_k\frac{1}{b_k}+\frac{1}{b_kb_{k+1}}\right)\frac{Q_k}{b_{k+1}}\mathbf{f}-\frac{1}{b_k^2b_{k+1}}\mathbf{f}=-Q_k\frac{1}{b_{k+1}}\mathbf{f}e_k\frac{Q_k}{b_kb_{k+1}}\mathbf{f}\\
&= -Ae_k\frac{1}{b_kb_{k+2}}A.
\end{aligned}
$$

To summarize, we showed that

$$
\begin{aligned}
&(6.12)+(6.14)+(6.16)+(6.17)-(6.4)-(6.6)+(6.15)-(6.8)-(6.9)\\
=\;& As_k(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A+A\frac{1}{2\beta}e_k\frac{b_{k+1}}{b_kb_{k+2}}A-Ae_k\frac{1}{b_k}(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A-Ae_k\frac{1}{b_kb_{k+2}}A. \qquad (6.18)
\end{aligned}
$$

Now consider the following expressions

$$
\begin{aligned}
(6.11)-(6.5)-(6.7) &= A\left(s_k\mathbf{f}s_{k+1}-b_{k+1}s_k\mathbf{f}s_{k+1}\frac{1}{b_k}\mathbf{f}+\frac{1}{b_k}\mathbf{f}s_{k+1}\right)A\\
&= A\left(s_ks_{k+1}-b_{k+1}s_ks_{k+1}\frac{1}{b_k}+\frac{1}{b_k}s_{k+1}\right)A\\
&= A\left(s_ks_{k+1}-(s_kb_k-e_k+1)s_{k+1}\frac{1}{b_k}+\frac{1}{b_k}s_{k+1}\right)A \quad = \quad Ae_ks_{k+1}\frac{1}{b_k}A, \qquad (6.19)
\end{aligned}
$$

and

$$\begin{aligned}
(6.13) - (6.3) &= Q_{k+1}s_{k+1}Q_{k+1}Q_k s_k Q_k \frac{1}{b_{k+1}}\mathbf{f} - \frac{1}{b_k}\mathbf{f}Q_{k+1}s_{k+1}Q_{k+1}Q_k s_k Q_k \\
&= A\left(s_{k+1}\mathbf{f}s_k - s_{k+1}\frac{1}{b_k}\mathbf{f}s_k b_{k+1}\right)A \\
&= A\left(s_{k+1}s_k - s_{k+1}\frac{1}{b_k}s_k b_{k+1}\right)A \\
&= A\left(s_{k+1}s_k - s_{k+1}\left(s_k\frac{1}{b_{k+1}} - \frac{1}{b_k}\mathbf{f}e_k\frac{1}{b_{k+1}} + \frac{1}{b_k b_{k+1}}\right)b_{k+1}\right)A \\
&= A\left(s_{k+1}\frac{1}{b_k}\mathbf{f}e_k - s_{k+1}\frac{1}{b_k}\right)A \\
&= As_{k+1}\frac{1}{b_k}e_k A - As_{k+1}\frac{1}{b_k}A. \qquad (6.20)
\end{aligned}$$

The only terms not dealt with so far are (6.2) and (6.10). Instead of simplifying them we will need to expand them into a large number of terms which then cancel with the remaining terms from the previous calculations. For the first term we obtain

$$\begin{aligned}
-(6.2) &= A\left(b_{k+1}s_k\mathbf{f}s_{k+1}\frac{1}{b_k}\mathbf{f}s_k b_{k+1}\right)A \\
&= A\left(\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k b_{k+2}s_{k+1}\frac{1}{b_k}\mathbf{f}s_k b_{k+1}\right)A \\
&= A\frac{b_{k+1}}{b_{k+2}}\mathbf{f}\left(s_k\frac{1}{b_k}s_{k+1}b_{k+1}\mathbf{f}s_k b_{k+1} - s_k\frac{1}{b_k}e_{k+1}\mathbf{f}s_k b_{k+1} + s_k\frac{1}{b_k}\mathbf{f}s_k b_{k+1}\right)A \\
&= \underbrace{A\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}s_{k+1}b_{k+1}s_k b_{k+1}A}_{(a)} \underbrace{-A\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}e_{k+1}s_k b_{k+1}A}_{(b)} \underbrace{+A\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}\mathbf{f}s_k b_{k+1}A}_{(c)}. \qquad (6.21)
\end{aligned}$$

Similarly, for the other term we obtain

$$\begin{aligned}
(6.10) &= -A\left(s_{k+1}\mathbf{f}s_k b_{k+2}\mathbf{f}s_{k+1}\right)A \quad = \quad -A\left(\frac{1}{b_{k+2}}\mathbf{f}b_{k+2}s_{k+1}\mathbf{f}s_k b_{k+2}s_{k+1}\right)A \\
&= -A\frac{1}{b_{k+2}}\mathbf{f}s_{k+1}b_{k+1}\mathbf{f}s_k b_{k+2}s_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}\mathbf{f}s_k b_{k+2}s_{k+1}A - As_k s_{k+1}A \\
&= -A\frac{1}{b_{k+2}}\mathbf{f}s_{k+1}b_{k+1}s_k b_{k+2}s_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k b_{k+2}s_{k+1}A - As_k s_{k+1}A \\
&= -As_{k+1}s_k b_{k+2}s_{k+1}A - A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k b_{k+2}s_{k+1}A + As_k s_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k s_{k+1}b_{k+1}A \\
&\quad -A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k e_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k A - As_k s_{k+1}A \\
&= -As_{k+1}s_k s_{k+1}b_{k+1}A + As_{k+1}s_k e_{k+1}A - As_{k+1}s_k A - A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k s_{k+1}b_{k+1}A \\
&\quad +A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k e_{k+1}A - A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k s_{k+1}b_{k+1}A \\
&\quad -A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k e_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_{k+1}s_k A \\
&= \underbrace{-As_{k+1}s_k s_{k+1}b_{k+1}A}_{(a)} \quad \underbrace{+As_{k+1}s_k e_{k+1}A}_{(b)} \quad \underbrace{-As_{k+1}s_k A.}_{(c)} \qquad (6.22)
\end{aligned}$$

We examine the three terms in (6.21)) separately. For the first one we obtain

$$
\begin{aligned}
(6.21.a) &= A\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}s_{k+1}b_{k+1}s_kb_{k+1}A \\
&= A\left(\frac{1}{b_{k+2}}\mathbf{f}s_ks_{k+1}b_{k+1}s_k - \frac{1}{b_{k+2}}\mathbf{f}e_k\frac{1}{b_k}s_{k+1}b_{k+1}s_k + \frac{1}{b_{k+2}b_k}\mathbf{f}s_{k+1}b_{k+1}s_k\right)b_{k+1}A \\
&= As_ks_{k+1}s_kb_{k+1}A + A\frac{1}{b_k}\mathbf{f}s_ke_{k+1}s_kb_{k+1}A - A\frac{b_{k+1}}{b_{k+2}}A \qquad (6.23)\\
&\quad -Ae_k\frac{1}{b_k}s_{k+1}s_kb_{k+1}A - A\frac{1}{b_{k+2}}e_k\frac{1}{b_k}e_{k+1}s_kb_{k+1}A + A\frac{1}{2\beta}\frac{1}{b_{k+2}}e_k\frac{b_{k+1}}{b_k}A \qquad (6.24)\\
&\quad +A\frac{1}{b_k}\mathbf{f}s_{k+1}s_kb_{k+1}A + A\frac{1}{b_{k+2}b_k}\mathbf{f}e_{k+1}s_kb_{k+1}A - A\frac{1}{b_{k+2}b_k}\mathbf{f}s_kb_{k+1}A, \qquad (6.25)
\end{aligned}
$$

whereas the second equals

$$
\begin{aligned}
(6.21.b) &= -A\left(\frac{b_{k+1}}{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}e_{k+1}s_kb_{k+1}\right)A \\
&= -A\frac{1}{b_{k+2}}\mathbf{f}s_ke_{k+1}s_kb_{k+1}A + A\frac{1}{b_{k+2}}\mathbf{f}e_k\frac{1}{b_k}e_{k+1}s_kb_{k+1}A - A\frac{1}{b_kb_{k+2}}\mathbf{f}e_{k+1}s_kb_{k+1}A. \qquad (6.26)
\end{aligned}
$$

The sum of the third term, denoted (c), in (6.21) and (6.18) simplifies to

$$
\begin{aligned}
(6.21.c) + (6.18) &= A\left(\frac{1}{b_{k+2}}\mathbf{f}b_{k+1}s_k\frac{1}{b_k}\mathbf{f}s_kb_{k+1} + s_k(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A + A\frac{1}{2\beta}e_k\frac{b_{k+1}}{b_kb_{k+2}}\right.\\
&\quad \left. - e_k\frac{1}{b_k}(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}}A - Ae_k\frac{1}{b_kb_{k+2}}\right)A \\
&= A\left(\frac{1}{b_{k+2}}\mathbf{f}s_k\mathbf{f}s_kb_{k+1} - \frac{1}{b_{k+2}}\mathbf{f}e_k\frac{1}{b_k}\mathbf{f}s_kb_{k+1} + \frac{1}{b_{k+2}b_k}\mathbf{f}s_kb_{k+1}\right.\\
&\quad \left. +s_k(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}} + \frac{1}{2\beta}e_k\frac{b_{k+1}}{b_kb_{k+2}} - e_k\frac{1}{b_k}(1-\mathbf{f})s_k\frac{b_{k+1}}{b_{k+2}} - e_k\frac{1}{b_kb_{k+2}}\right)A \\
&= A\left(\frac{b_{k+1}}{b_{k+2}} - e_k\frac{1}{b_k}s_k\frac{b_{k+1}}{b_{k+2}} + \frac{1}{b_{k+2}b_k}\mathbf{f}s_kb_{k+1} + \frac{1}{2\beta}e_k\frac{b_{k+1}}{b_kb_{k+2}} - e_k\frac{1}{b_kb_{k+2}}\right)A \\
&= \underbrace{\frac{1}{b_k}\mathbf{f}}_{(a)} \quad \underbrace{+A\frac{1}{b_k}\mathbf{f}s_k\frac{b_{k+1}}{b_{k+2}}A}_{(b)} \quad \underbrace{-Ae_k\frac{1}{b_kb_{k+2}}A}_{(c)}. \qquad (6.27)
\end{aligned}
$$

We now compare all the results and see that the following summands cancel each other: The three summands from (6.26) cancel with the second summand in (6.23), (6.24); and (6.25) respectively. Moreover, the first from (6.27) cancels with the last from (6.23); the second from (6.27) with the last in (6.25); and (6.22.a) with the first in (6.23).

If we take from the remaining summands the first, call it (6.25-1), in (6.25), (6.22.c) and (6.20) we get one more cancelation:

$$
\begin{aligned}
(6.25\text{-}1) + (6.22) + (6.20) &= A\left(s_{k+1}\frac{1}{b_k}s_kb_{k+1} - s_{k+1}s_k + s_{k+1}\frac{1}{b_k}e_k - s_{k+1}\frac{1}{b_k}\right)A \\
&= A\left(s_{k+1}s_k - s_{k+1}\frac{1}{b_k}e_k + s_{k+1}\frac{1}{b_k} - s_{k+1}s_k + s_{k+1}\frac{1}{b_k}e_k - s_{k+1}\frac{1}{b_k}\right)A \quad = \quad 0.
\end{aligned}
$$

Then we consider the first summands in (6.24), the second in (6.22), and (6.19)) and obtain

$$\begin{aligned}(6.24\text{-}1)+(6.22.\text{b})+(6.19) &= A\left(-e_k s_{k+1}\frac{1}{b_k}s_k b_{k+1}+s_{k+1}s_k e_{k+1}+e_k s_{k+1}\frac{1}{b_k}\right)A\\ &= A\left(-e_k s_{k+1}s_k+e_k s_{k+1}\frac{1}{b_k}e_k-e_k s_{k+1}\frac{1}{b_k}+e_k e_{k+1}+e_k s_{k+1}\frac{1}{b_k}\right)A\\ &= A\left(-e_k e_{k+1}+e_k s_{k+1}\frac{1}{b_k}e_k+e_k e_{k+1}\right)A\\ &= Ae_k s_{k+1}\frac{1}{b_k}e_k A \end{aligned} \tag{6.28}$$

Collecting all the remaining summands, i.e. the thirds in (6.24) and in (6.27) and the term (6.28), we get the following expression:

$$\begin{aligned}(6.24\text{-}3)+(6.27.\text{c})+(6.28) &= A\left(\frac{1}{2\beta}\frac{1}{b_{k+2}}e_k\frac{b_{k+1}}{b_k}-e_k\frac{1}{b_kb_{k+2}}+e_k s_{k+1}\frac{1}{b_k}e_k\right)A\\ &= A\left(\frac{1}{2\beta}\frac{1}{b_{k+2}}e_k\frac{b_{k+1}}{b_k}-e_k\frac{1}{b_kb_{k+2}}+e_k e_{k+1}\mathbf{f}s_k\frac{1}{b_k}e_k\right)A\\ &= A\left(\frac{1}{2\beta}\frac{1}{b_{k+2}}e_k\frac{b_{k+1}}{b_k}-e_k\frac{1}{b_kb_{k+2}}+\frac{1}{2\beta}\frac{1}{b_{k+2}}e_k e_{k+1}\mathbf{f}e_k\right)A\\ &= A\left(e_k\frac{b_{k+1}-2\beta+b_k}{2\beta b_{k+2}b_k}\right)A=A\left(e_k\frac{y_{k+1}+y_k}{2\beta b_{k+2}b_k}\right)A=0.\end{aligned}$$

This proves the claim of the proposition. ■

6.3 Relations involving only $\tilde{e}_k$'s in $\mathbf{f}\mathbb{W}_d^{\text{cycl}}\mathbf{f}$

We continue with the defining relations that will only involve the $\tilde{e}_k$'s. The key relation that $\tilde{e}_k$ squares to $\delta\tilde{e}_k$ was already proven in Lemma 6.3. We continue with the remaining relations:

Lemma 6.7. We have $\tilde{e}_i\tilde{e}_j=\tilde{e}_j\tilde{e}_i$ for $1\le i,j<d$ with $|i-j|>1$. □

Proof. Since $\mathbf{f}e_i\mathbf{f}$ commutes with Q_j when $|i-j|>1$ the statement follows. ■

Lemma 6.8. We have $\tilde{e}_k\tilde{e}_{k+1}\tilde{e}_k=\tilde{e}_k$ and $\tilde{e}_{k+1}\tilde{e}_k\tilde{e}_{k+1}=\tilde{e}_{k+1}$ for $1\le k<d-1$. □

Proof. We only prove the first equality, the second is done in an analogous way. We compute

$$\begin{aligned}\tilde{e}_k\tilde{e}_{k+1}\tilde{e}_k &= Q_ke_kQ_kQ_{k+1}e_{k+1}Q_{k+1}Q_ke_kQ_k\\ &= Q_k\sqrt{b_{k+2}}\mathbf{f}e_k\mathbf{f}e_{k+1}\frac{1}{b_k}\mathbf{f}e_k\sqrt{b_{k+2}}\mathbf{f}Q_k\\ &\overset{(a)}{=} Q_k\sqrt{b_{k+2}}\mathbf{f}e_k\mathbf{f}e_{k+1}\frac{1}{c_{k+1}}\mathbf{f}e_k\sqrt{b_{k+2}}\mathbf{f}Q_k\\ &\overset{(b)}{=} Q_k\sqrt{b_{k+2}}\mathbf{f}e_k\mathbf{f}e_{k+1}\frac{1}{b_{k+2}}\mathbf{f}e_k\sqrt{b_{k+2}}\mathbf{f}Q_k\\ &= Q_k\sqrt{b_{k+2}}\mathbf{f}e_k\mathbf{f}e_{k+1}\mathbf{f}e_k\frac{\sqrt{b_{k+2}}}{b_{k+2}}\mathbf{f}Q_k \overset{(c)}{=} \tilde{e}_k\end{aligned}$$

Where (a) is due to (VW.8b), since $\frac{1}{b_k}\mathbf{f}e_k = \frac{1}{c_{k+1}}\mathbf{f}e_k$, and (b) follows from $e_{k+1}\frac{1}{c_{k+1}}\mathbf{f} = e_{k+1}\frac{1}{b_{k+2}}\mathbf{f}$ due to (VW.8a). Finally (c) is again a consequence of Lemma 3.12 and Relation (VW.6d). ■

6.4 Mixed relations in $\mathbf{f}\mathbb{W}_d^{\mathrm{cycl}}\mathbf{f}$

We are now left with proving the relations involving both $\tilde{s}_j$'s and $\tilde{e}_j$'s. Again we refer to Remark 3.14 for the well-definedness of all occurring terms.

Lemma 6.9. We have $\tilde{e}_k\tilde{s}_k = \tilde{e}_k = \tilde{s}_k\tilde{e}_k$ for $1 \le k < d$. □

Proof. We compute

$$\begin{aligned}
\tilde{e}_k\tilde{s}_k &= -Q_ke_k\frac{b_{k+1}}{b_k}\mathbf{f}s_kQ_k + Q_ke_kQ_k\frac{1}{b_k}\mathbf{f} \\
&\overset{(a)}{=} -Q_ke_k\frac{b_{k+1}}{b_k}s_kQ_k + Q_ke_kQ_k\frac{1}{b_k}\mathbf{f} \\
&\overset{(b)}{=} -Q_ke_k\frac{1}{b_k}s_kb_kQ_k + Q_ke_k\frac{1}{b_k}e_kQ_k \\
&\overset{(c)}{=} -\frac{1}{2\beta}Q_ke_kQ_k + (1+\frac{1}{2\beta})Q_ke_kQ_k \quad = \quad \tilde{e}_k
\end{aligned}$$

where equality (a) is due to Proposition 3.13, (b) is due to (VW.7), and finally (c) is due to Proposition 6.2. The second equality in the claim follows analogously. ■

Lemma 6.10. We have $\tilde{s}_i\tilde{e}_j = \tilde{e}_j\tilde{s}_i$ for $1 \le i, j < d$ with $|i-j| > 1$. □

Proof. This follows by the same arguments as for Lemmas 6.5 and 6.7. ■

Lemma 6.11. We have $\tilde{s}_k\tilde{e}_{k+1}\tilde{e}_k = \tilde{s}_{k+1}\tilde{e}_k$ and $\tilde{s}_{k+1}\tilde{e}_k\tilde{e}_{k+1} = \tilde{s}_k\tilde{e}_{k+1}$ for $1 \le k < d-1$. □

Proof. We only prove the first equality, the second is done analogously,

$$\begin{aligned}
\tilde{s}_k\tilde{e}_{k+1}\tilde{e}_k &= -Q_ks_kQ_kQ_{k+1}e_{k+1}Q_{k+1}Q_ke_kQ_k + \frac{1}{b_k}\mathbf{f}Q_{k+1}e_{k+1}Q_{k+1}Q_ke_kQ_k \\
&\overset{(a)}{=} -Q_k\sqrt{b_{k+2}}\mathbf{f}s_k\frac{1}{b_k}e_{k+1}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k + \frac{\sqrt{b_{k+2}}}{b_k\sqrt{b_k}\sqrt{b_{k+1}}}\mathbf{f}e_{k+1}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k \\
&\overset{(b)}{=} -Q_{k+1}\frac{1}{\sqrt{b_k}}\mathbf{f}s_ke_{k+1}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k + Q_{k+1}\frac{1}{\sqrt{b_k}}\mathbf{f}e_k\frac{1}{b_k}e_{k+1}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k \\
&\overset{(c)}{=} -Q_{k+1}\frac{1}{\sqrt{b_k}}\mathbf{f}s_{k+1}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k + \frac{1}{b_{k+2}}Q_{k+1}\frac{1}{\sqrt{b_k}}\mathbf{f}e_k\mathbf{f}\sqrt{b_{k+2}}\mathbf{f}Q_k \\
&\overset{(d)}{=} -Q_{k+1}s_{k+1}Q_{k+1}Q_ke_kQ_k + \frac{1}{b_{k+1}}Q_ke_kQ_k \quad = \quad \tilde{s}_{k+1}\tilde{e}_k.
\end{aligned}$$

Where equality (a) follows from Proposition 3.13, (b) is a consequence of Lemma 3.12 and again Proposition 3.13. Equality (c) follows by using relations (8a) and (8b) to rewrite the second summand and then applying relations (6b) and (6d). Finally equality (d) is using Proposition 3.13 and reordering the factors afterwards. ■

7 Example: the graded Brauer algebras $\mathrm{Br}_2^{\mathrm{gr}}(\delta)$

In this section we will illustrate explicitly the construction of the isomorphism for the Brauer algebras $\mathrm{Br}_2(\delta)$ and describe their graded version $\mathrm{Br}_2^{\mathrm{gr}}(\delta)$.

7.1 Case $\delta \neq 0$

We first consider the case $\mathrm{Br}_2(\delta)$ for $\delta \neq 0$. By [27] this Brauer algebra is semisimple with basis 1, $t = t_1$, and $g = g_1$. The set of orthogonal idempotents is

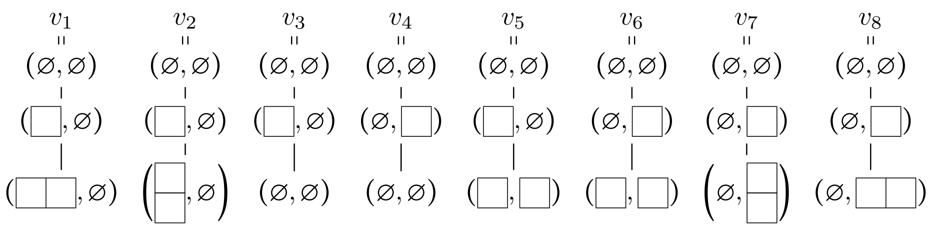
$$\left\{ \frac{1+t}{2} - \frac{1}{\delta} g,\ \frac{1-t}{2} + \frac{1}{\delta} g,\ \frac{1}{\delta} g \right\}$$

which gives rise to an isomorphism

$$\mathrm{Br}_2(\delta) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

Note that the grading on $\mathrm{Br}_2(\delta)$ needs to be trivial since all idempotents have to have degree 0. We now want to illustrate the idempotent truncation of the level 2 cyclotomic quotient $\mathbb{W}_2(\Xi)$ with parameters from Definition3.1.

We describe $\mathbb{W}_2^{\mathrm{cycl}}$ in terms of the seminormal representation of $\mathbb{W}_2^{\mathrm{cycl}}$ from [2, Theorem 4.13] by an action of $\mathbb{W}_2^{\mathrm{cycl}}$ on the vector space with basis given by all up-down bitableaux of length 2. Explicitly, this basis consists of



all of which are common eigenvectors for y_1 and y_2. The corresponding pairs of eigenvalues are the following:

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$(\alpha, \alpha - 1)$	$(\alpha, \alpha + 1)$	$(\alpha, -\alpha)$	$(\beta, -\beta)$	(α, β)	(β, α)	$(\beta, \beta - 1)$	$(\beta, \beta + 1)$

where the first entry denotes the eigenvalue for y_1 and the second for y_2. Using these eigenvalues one can calculate via [2, Theorem 4.13] the matrix of s_1

$$s_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha+\beta-1}{\alpha-\beta} & \frac{\sqrt{-(2\beta-1)(2\alpha-1)}}{\alpha-\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{-(2\beta-1)(2\alpha-1)}}{\alpha-\beta} & -\frac{\alpha+\beta-1}{\alpha-\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix}.$$

The $\star$'s indicate some further non-zero entries which are irrelevant for the construction. Similarly one obtains the matrix for e_1 as

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (2\alpha-1)\frac{\alpha+\beta}{\alpha-\beta} & \sqrt{-(2\beta-1)(2\alpha-1)}\frac{\alpha+\beta}{\alpha-\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-(2\beta-1)(2\alpha-1)}\frac{\alpha+\beta}{\alpha-\beta} & -(2\beta-1)\frac{\alpha+\beta}{\alpha-\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Although tedious, it is of course straightforward to check that these matrices together with the action of y_1, y_2 satisfy all the defining relations of $\mathbb{W}_2^{\mathrm{cycl}}$.

For the isomorphism in Theorem 4.3 we first need to apply the idempotent $\mathbf{f}$ from both sides. In the chosen basis this amounts to truncation with respect to the basis vectors where the second partition is always empty (i.e. v_1, v_2 and v_3) that means we look at the submatrices consisting of the first three columns in the first three rows. Clearly, the submatrices $\mathbf{f}s_1\mathbf{f}$ and $\mathbf{f}e_1\mathbf{f}$ do not even satisfy the most basic Brauer algebra relations, i.e. the relation for squares. This deficiency is overcome by the correction term $Q = Q_1 = \sqrt{\frac{b_2}{b_1}}\mathbf{f}$ and $\frac{1}{b_1}\mathbf{f}$ from the definition of the isomorphism in Theorem 4.3 as we show now explicitly. We have

$$Q = \begin{pmatrix} \sqrt{\frac{\alpha+\beta-1}{\alpha+\beta}} & 0 & 0 \\ 0 & \sqrt{\frac{\alpha+\beta+1}{\alpha+\beta}} & 0 \\ 0 & 0 & \sqrt{\frac{\beta-\alpha}{\alpha+\beta}} \end{pmatrix} \text{ and } \frac{1}{b_1}\mathbf{f} = \begin{pmatrix} \frac{1}{\alpha+\beta} & 0 & 0 \\ 0 & \frac{1}{\alpha+\beta} & 0 \\ 0 & 0 & \frac{1}{\alpha+\beta} \end{pmatrix}.$$

By multiplying $\mathbf{f}e_1\mathbf{f}$ from both sides with Q we obtain

$$Q\mathbf{f}e_1\mathbf{f}Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

which is obviously a matrix that squares to δ times itself. The analogous construction for $\mathbf{f}s_1\mathbf{f}$ needs an extra correction term (as given in Theorem 4.3):

$$Q\mathbf{f}s_1\mathbf{f}Q = \begin{pmatrix} -\frac{\alpha+\beta-1}{\alpha+\beta} & 0 & 0 \\ 0 & \frac{\alpha+\beta+1}{\alpha+\beta} & 0 \\ 0 & 0 & -\frac{\alpha+\beta-1}{\alpha+\beta} \end{pmatrix} \text{ and } -Q\mathbf{f}s_1\mathbf{f}Q + \frac{1}{b_1}\mathbf{f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

7.2 **Case** $\delta \neq 0$

If we try to do the same for the Brauer algebra $\mathrm{Br}_2(0)$ we immediately encounter a problem, since the algebra is not semisimple and so we cannot apply the formulas and constructions from [2]. The whole difficulty of our proof is to show that the formulas in Theorem 4.3 still make sense and give the required correction terms even in the non-semisimple case.

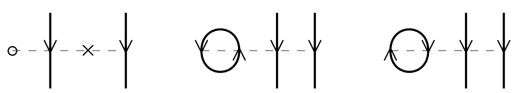
One can show that via the orthogonal idempotents $\{\frac{1+t}{2}, \frac{1-t}{2}\}$ one obtains

$$\mathrm{Br}_2(0) \quad \cong \quad \mathbb{C} \oplus \mathbb{C}[x]/(x^2), \tag{7.1}$$

with the element x corresponding to the element $g \in \mathrm{Br}_2(0)$ It becomes graded in the obvious way by putting the idempotents in degree 0 and x in degree 2.

There is in fact a generalization of the up-down tableaux basis in the graded setting via the diagram calculus developed in [15]. The explicit isomorphism between this description and the Brauer algebra itself is being worked out in [25].

Remark 7.1. To make the connection from (7.1) to [15] we note that $\mathrm{Br}_2^{\mathrm{gr}}(0)$ can be realized as the subalgebra of the generalized type D Khovanov algebra from [14] (using the notation from there) with the following diagrams as basis



Here the first diagram spans the copy of $\mathbb{C}$ under the identification with (7.1), while the other two span a copy of $\mathbb{C}[x]/(x^2)$, with the second diagram being the unit and the third one being the element of degree 2, i.e. it corresponds to x. □

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References

[1] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Semisimplicity of Hecke and (walled) Brauer algebra. arXiv:1507.07676, 2015.

[2] S. Ariki, A. Mathas, and H. Rui. Cyclotomic Nazarov-Wenzl algebras. *Nagoya Math. J.*, 182:47–134, 2006.

[3] E. Backelin. Koszul duality for parabolic and singular category $\mathcal{O}$. *Represent. Theory*, 3:139–152., 1999. ISSN 1088-4165. doi: 10.1090/S1088-4165-99-00055-2. URL `http://dx.doi.org/10.1090/S1088-4165-99-00055-2`.

[4] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.

[5] B. D. Boe. Kazhdan-Lusztig polynomials for Hermitian symmetric spaces. *Trans. Amer. Math. Soc.*, 309 (1):279–294, 1988.

[6] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937.

[7] J. Brundan and A. Kleshchev. Schur-Weyl duality for higher levels. *Selecta Math. (N.S.)*, 14(1): 1–57, 2008. ISSN 1022-1824. doi: 10.1007/s00029-008-0059-7. URL `http://dx.doi.org/10.1007/s00029-008-0059-7`.

[8] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$. *Represent. Theory*, 15:170–243, 2011. ISSN 1088-4165. doi: 10.1090/S1088-4165-2011-00389-7. URL `http://dx.doi.org/10.1090/S1088-4165-2011-00389-7`.

[9] A. Cox, M. De Visscher, and P. Martin. The blocks of the Brauer algebra in characteristic zero. *Represent. Theory*, 13:272–308, 2009. ISSN 1088-4165. doi: 10.1090/S1088-4165-09-00305-7. URL `http://dx.doi.org/10.1090/S1088-4165-09-00305-7`.

[10] Z. Daugherty, A. Ram, and R. Virk. Affine and degenerate affine BMW algebras: actions on tensor space. *Selecta Math. (N.S.)*, 19(2):611–653, 2013. ISSN 1022-1824. doi: 10.1007/s00029-012-0105-3. URL `http://dx.doi.org/10.1007/s00029-012-0105-3`.

[11] C. de Concini and C. Procesi. A characteristic free approach to invariant theory. *Advances in Math.*, 21 (3):330–354, 1976.

[12] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. ISBN 0-521-64558-1. doi: 10.1017/CBO9780511600708. URL `http://dx.doi.org/10.1017/CBO9780511600708`.

[13] M. Ehrig and C. Stroppel. 2-row Springer fibres and Khovanov diagram algebras for type D. *to appear in CJM*, 2012.

[14] M. Ehrig and C. Stroppel. Diagramatic description of the category of perverse sheaves on isotropic Grassmannians and the supergroup SOSP(m,2n). ArXiv:1306.4043, 2013.

[15] M. Ehrig and C. Stroppel. Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality. arXiv:1310.1972, 2013.

[16] M. Ehrig and C. Stroppel. Schur-Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra. arXiv:1412.7853, 2014.

[17] R. Goodman and N. R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009. ISBN 978-0-387-79851-6. doi: 10.1007/978-0-387-79852-3. URL `http://dx.doi.org/10.1007/978-0-387-79852-3`.

[18] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[19] J. Hu and A. Mathas. Quiver Schur algebras for linear quivers. *Proc. Lond. Math. Soc. (3)*, 110(6): 1315–1386, 2015.

[20] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. ISBN 978-0-8218-4678-0.

[21] S. König and C. Xi. On the structure of cellular algebras. In *Algebras and modules, II (Geiranger, 1996)*, volume 24 of *CMS Conf. Proc.*, pages 365–386. Amer. Math. Soc., Providence, RI, 1998.

[22] G. Lehrer and R. Zhang. The first fundamental theorem of invariant theory for the orthosymplectic supergroup, 2014. arXiv:1401.7395.

[23] T. Lejczyk and C. Stroppel. A graphical description of (D_n, A_{n-1}) Kazhdan-Lusztig polynomials. *GMJ*, 55:313–340, 2012.

[24] G. Li. A KLR Grading of the Brauer Algebras. arXiv:1409.1195, 2014.

[25] G. Li and C Stroppel. An isomorphism of graded Brauer algebras, 2015. in preparation.

[26] M. Nazarov. Young's orthogonal form for Brauer's centralizer algebra. *J. Algebra*, 182(3):664–693, 1996. ISSN 0021-8693. doi: 10.1006/jabr.1996.0195. URL `http://dx.doi.org/10.1006/jabr.1996.0195`.

[27] H. Rui. A criterion on the semisimple Brauer algebras. *J. Combin. Theory Ser. A*, 111(1):78–88, 2005.

[28] A. Sartori and C. Stroppel. Walled Brauer algebras as idempotent truncations of level 2 cyclotomic quotients. *J. Algebra*, 440:602–638, 2015.

[29] W. Soergel. Kazhdan-Lusztig polynomials and a combinatorics for tilting modules. *Represent. Theory*, 1: 83–114, 1997.

[30] C. Stroppel. Category $\mathcal{O}$: gradings and translation functors. *J. Algebra*, 268(1):301–326, 2003. ISSN 0021-8693. doi: 10.1016/S0021-8693(03)00308-9. URL `http://dx.doi.org/10.1016/S0021-8693(03)00308-9`.

[31] H. Wenzl. On the structure of Brauer's centralizer algebras. *Ann. of Math. (2)*, 128(1):173–193, 1988.