Double Affine Hecke Algebras and Their Representations

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1 Introduction

Double affine Hecke algebras were introduced by Ivan Cherednik in order to study Knizhnik-Zamolodchikov equations as described in [Che05]. Since then a prolific branch of research has been developed around double affine Hecke algebras and their applications range from special functions, over Verlinde algebras to various topics in representation theory, such as Schur algebras and quantum groups. The most famous example of such applications is presumably the proof of the Macdonald's conjectures by Cherednik [Che05, Chapter 0.2.4]. See also [Che05, Chapter 0] for an in-depth overview.

One can associate to any root system and any lattice $Q \subseteq L \subseteq P$ a double affine Hecke algebra as done in [Che05, Chapter 3.2]. In Chapter 4 we will consider an example of this construction for the root system A_1 associated to SL_2 and the lattice $L := P$: the so-called *one-dimensional double affine Hecke algebra* $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. But for the main part of this thesis we will work with a slightly different version, namely the *double affine Hecke algebra associated to* GL_n for $n \geq 2$, see Definition 3.1. Our definition is equivalent the construction in [SV05, Definition 4.1], whereas compared to the construction from [Che05, Chapter 3.7] one has to additionally demand that the generator π is invertible to match our definition. We will denote the double affine Hecke algebra associated to GL_n by $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ to emphasize the dependence of the algebra on two parameters q, t. This algebra is closely related to the double affine Hecke algebra associated to the root system A_{n-1}. In fact, the latter is a subquotient of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, see [Che05, Chapter 3.7].

1.1 Motivation: a topological construction

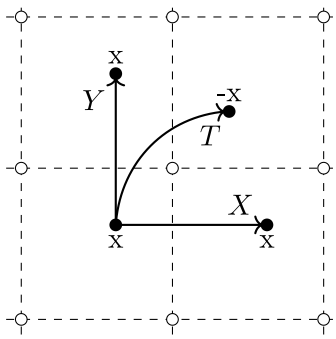
Let us give a topological motivation for the definition of the one-dimensional double affine Hecke algebra $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. We are following the outlines given in [Che05, Chapter 2.7.3] and [Sim17, Lecture 2, Chapter 2.3]. The point of this discussion is to recover via a topological construction the definition of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ as a $\mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-algebra with generators $X^{\pm 1}, Y^{\pm 1}, T$ subject to the relations

$$\textbf{(T):}\quad (T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) = 0, \qquad \textbf{(YT):}\quad TY^{-1}T = Y,$$
$$\textbf{(XT):}\ TXT = X^{-1}, \qquad \textbf{(XYT):}\ q^{\frac{1}{2}} Y^{-1} X^{-1} Y X T^2 = 1,$$

given in Lemma 4.2.

Consider the lattice $\Lambda := \mathbb{Z} \oplus \mathbb{Z}i \subseteq \mathbb{C}$ and let $E := \mathbb{C}/\Lambda$ be the corresponding elliptic curve. Let $0 \in E$ be the zero point and let $x \in E$ be a point with $-x \neq x \in E$. Furthermore, let $G := \mathbb{Z}/2\mathbb{Z}$ act on $E \setminus \{0\}$ via $e \mapsto -e$ for $e \in E$. We denote the *orbifold fundamental group* of $(E \setminus \{0\})/G$

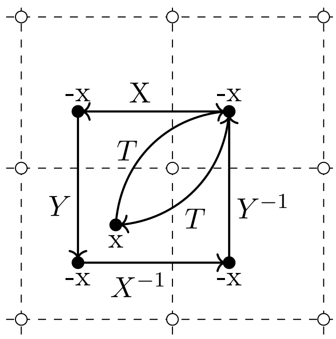
based at x by $\pi_1^{orb}((E \setminus \{0\})/G, x)$. It is defined to be the group of homotopy classes of paths from x to $\pm x$ in $E \setminus \{0\}$, where the product $\gamma_1\gamma_2$ of $\gamma_1, \gamma_2 \in \pi_1^{orb}((E \setminus \{0\})/G, x)$ is defined to be γ_2 followed by γ_1 if γ_2 ends in x and γ_2 followed by $-\gamma_1$ if γ_2 ends in $-x$. Since $E \setminus \{0\}$ is homotopy equivalent to the wedge of two circles, its (usual) fundamental group is generated by the homotopy classes of those two circles, which we denote by X and Y. We obtain from the definition of $\pi_1^{orb}((E \setminus \{0\})/G, x)$ that it is generated as a group by X, Y and a path from x to $-x$, which we denote by T. As in [Sim17] we can depict these elements graphically, where the next and all upcoming pictures represent lifts of elements in $\pi_1^{orb}((E \setminus \{0\})/G, x)$ to paths in $\mathbb{C} \setminus \Lambda$ connecting a fixed lift of x to some (possibly different) lift of $\pm x$. For simplicity we will denote lifts of $\pm x$ by $\pm x$ as well.



Since $\mathbb{C} \setminus \Lambda \to E \setminus \{0\}$ is a covering map, an element γ in the orbifold fundamental group $\pi_1^{orb}((E \setminus \{0\})/G, x)$ equals 0 if and only if it lifts to a null-homotopic loop in $\mathbb{C} \setminus \Lambda$. It is easy to see that the elements $X^{\pm 1}, Y^{\pm 1}, T \in \pi_1^{orb}((E \setminus \{0\})/G, x)$ satisfy the following relations:

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad Y^{-1}X^{-1}YXT^2 = 1. \tag{1}$$

For example we can verify the last relation graphically, since the following loop starting at x representing $TY^{-1}X^{-1}YXT$ is null-homotopic in $\mathbb{C} \setminus \Lambda$.

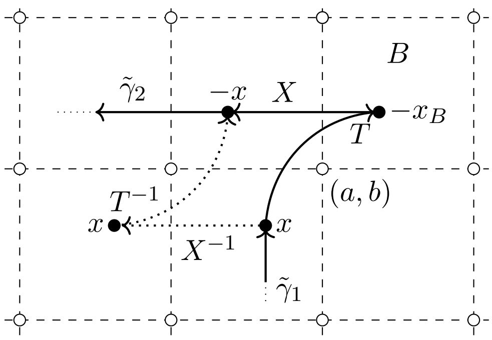


In fact, $\pi_1^{orb}((E \setminus \{0\})/G, x)$ is already isomorphic to the group generated by $X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}$ subject to the relations in (1). We sketch the idea. Let

γ be a path equal to 0 in $\pi_1^{orb}((E \setminus \{0\})/G, x)$ represented by a word in $X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}$. This means γ is a null-homotopic loop at $x \in E$. We have to show that the word representing γ can be reduced to the trivial word using the relations in (1). We can suppose that no generator appears next to its inverse and we will assume by induction that the claim holds for all paths with shorter word-length in the generators $X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}$. Lift γ to a path $\tilde{\gamma}$ in $\mathbb{C} \setminus \Lambda$ which consists of path segments from above corresponding to $X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}$. Since γ is a null-homotopic, the lift $\tilde{\gamma}$ must be a null-homotopic loop based at (a lift of) x in $\mathbb{C} \setminus \Lambda$. We will now use the relations from (1) to reduce $\tilde{\gamma}$ to the constant path in x. For this look at a top-right-extremal 1-by-1-box $B \subseteq \mathbb{C}$ for $\tilde{\gamma}$, by which we mean that $\tilde{\gamma}$ passes through this box, but not through any box to the top-right of it. Let $(a, b) \in \Lambda$ be the bottom-left corner of this box and let $\pm x_B$ be the lifts of $\pm x$ in B. Our aim is to use the relations from (1) to either reduce the word-length of $\tilde{\gamma}$ or to reduce the number of times that $\tilde{\gamma}$ passes through B, so that by induction we can assume that $\tilde{\gamma}$ does not pass through B at all. Since we assumed B to be extremal only the following path-segments (or their inverses) are possible whenever $\tilde{\gamma}$ passes through B:

$$\begin{aligned} &(1) : XT \text{ or } TY^{-1} && \text{passing through } -x_B, \\ &(2) : XT^{-1} \text{ or } T^{-1}Y^{-1} && \text{passing through } -x_B, \\ &(3) : X^{-1}Y^{-1} \text{ or } XY && \text{passing through } -x_B \text{ or } x_B. \end{aligned} \tag{2}$$

In the first case we can use the first two relations in (1) to reduce the number of times $\tilde{\gamma}$ passes through B. For example if we have $\tilde{\gamma} = \tilde{\gamma}_2 XT \tilde{\gamma}_1$, we can replace $\tilde{\gamma}$ with $\tilde{\gamma}' = \tilde{\gamma}_2 T^{-1} X^{-1} \tilde{\gamma}_1$ as follows.



In the other two cases the path segment seems to wind around $(a, b) \in \Lambda$. But since $\tilde{\gamma}$ is null-homotopic this winding must be resolved by a homotopy. Since B is extremal this means we can find a path-segment $\tilde{\gamma}_1$ of $\tilde{\gamma}$, which starts and ends in B and is homotopic to a path inside B from $\pm x_B$ to $\pm x_B$. If $\tilde{\gamma}_1$ starts and ends in the same point, we can use induction on the word-length to reduce $\tilde{\gamma}_1$ to the trivial path at $\pm x_B$ using the relations from (1) and we are done by induction, since we also reduced the length of $\tilde{\gamma}$.

Otherwise, we have that $\tilde{\gamma}_1(YXT)^{\pm 1}$ is a null-homotopic loop at $\pm x_B$, where the exponent depends on the start- and end-point of $\tilde{\gamma}_1$. After inspecting some cases for short word-lengths of $\tilde{\gamma}$ by hand we can assume that the word-length of $\tilde{\gamma}$ is larger than the word-length of $\tilde{\gamma}_1(TY^{-1}X^{-1})^{\pm 1}$. Hence, by induction on the word-length, we can replace $\tilde{\gamma}_1$ by $(TY^{-1}X^{-1})^{\mp 1}$ in $\tilde{\gamma}$. Now excluding some cases for short word-lengths of $\tilde{\gamma}_1$ by hand lets us assume that this reduces the word-length of $\tilde{\gamma}$ and we are done in this case by induction. The other extremal cases (top-left, bottom-right, bottom-left) can be handled similarly. If we do not reach a step where we can reduce the word-length as above, we can cancel extremal boxes successively to shrink our path until it lies in a 2-by-2 box. One can see now by inspection that this implies $\gamma = (Y^{-1}X^{-1}YXT^2)^k$ for some $k \in \mathbb{Z}$, which shows the claim.

Now look at the group algebra $\tilde{H} := \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})[\pi_1^{orb}((E \setminus \{0\})/G, x)]$. Set $\tilde{T} := q^{-\frac{1}{4}}T$, $\tilde{X} := q^{\frac{1}{4}}X$ and $\tilde{Y} := q^{-\frac{1}{4}}Y$. If necessary we add $q^{\frac{1}{4}}$ and $t^{\frac{1}{4}}$ to the base field. Then $\tilde{H}$ is isomorphic to the unital and associative $\mathbb{C}(q^{\frac{1}{4}}, t^{\frac{1}{4}})$-algebra generated by the elements $\tilde{X}^{\pm 1}, \tilde{Y}^{\pm 1}, \tilde{T}^{\pm 1}$ modulo the relations $(YT), (XT), (XYT)$ from the one-dimensional double affine Hecke algebra given above. Therefore the one-dimensional double affine Hecke algebra $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is a quotient of the group algebra of $\pi_1^{orb}((E \setminus \{0\})/G, x)$ by the quadratic (T)-relation, which concludes our topological motivation. We refer to [Sim17, Section 2.4.3] for a similar construction for the double affine Hecke algebra associated to GL_n.

1.2 Content of the thesis

In this thesis we will present some aspects of the representation theory of double affine Hecke algebras, short DAHA. More precisely, we study the DAHA associated to GL_n for $n \geq 2$, denoted by $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, in Chapter 2 and Chapter 3 and its spherical version in Chapter 5. The construction of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is described in Definition 3.1. Furthermore, we will study the so-called *one-dimensional DAHA* associated to SL_2 from Definition 4.1 in Chapter 4.

Classification of irreducible $\mathcal{X}$-semisimple modules in the generic case. The second and third chapters are based on results in [SV05] and aim to classify irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules for *generic* parameter q and $t = q^{\frac{1}{\kappa}}$ for some $\kappa \in \mathbb{Z} \setminus \{0\}$ by combinatorial means. We start with recalling basic definitions and facts about the affine root system $(\hat{\mathfrak{h}}^*, (\ |\ ), \dot{R})$ and the extended affine Weyl group $\dot{W}$ of type $\tilde{A}_{n-1}$ associated to $\widehat{\mathfrak{gl}}_n$ in Section 2.1. This includes various statements about the length function in the extended affine Weyl group $\dot{W}$ (Theorem 2.6), an action of $\dot{W}$ on $\mathbb{Z}$ (Proposition 2.12) and a discussion of parabolic subgroups $\dot{W}_I$ (Definition 2.17). In Section 2.2 we describe the combinatorial theory of so-called *periodic skew diagrams* and *tableaux* (Definitions 2.21 and 2.25). These can be seen as a generalization of (skew) Young diagrams and tableaux, which play a central role in the classification of irreducible modules for the symmetric group S_n, see for example [TCST10] or [Ful97, Notation and Chapter 7]. The ideas from the S_n-theory generalize quite nicely: to each periodic skew diagram we associate an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module (Theorem 3.15), whose basis is indexed by the standard tableaux on the diagram (Definition 2.26), which are defined to be the strictly row- and column-increasing tableaux. These modules are irreducible and $\mathcal{X}$-semisimple (Theorem 3.16), which means that they have a basis of weight vectors for the subalgebra $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ generated by X_i for $1 \leq i \leq n$. More precisely we construct for each skew diagram $\widehat{\lambda/\mu}$ an irreducible $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $V(\lambda, \mu)$ such that

$$V(\lambda, \mu) = \bigoplus_{T \in \mathrm{Tab}_{\gamma}^{RC}(\widehat{\lambda/\mu})} \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}}) v_T, \tag{3}$$

where the sum is taken over all standard tableaux on (λ, μ). The action of X_i is then given by $X_i v_T = t^{C_T(i)} v_T$, where C_T is the content function associated to the tableau T, see Definition 2.32. We conclude this chapter with the statement that these modules form a complete class of representatives of irreducible $\mathcal{X}$-semisimple modules for $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ (Theorems 3.18 and 3.19).

Finite-dimensional irreducible modules for the one-dimensional DAHA. In the fourth chapter we describe the *one-dimensional DAHA* (Definition 4.1) associated to SL_2, which we denote by $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. The goal is to classify its finite-dimensional irreducible modules following [Che05, Chapters 2.8 and 2.9]. For this goal the *polynomial representation* (Proposition 4.3) on the ring of Laurent polynomials $\mathcal{P} := \mathbb{K}[X^{\pm 1}]$ is of fundamental importance. It enables us to deduce a PBW-basis theorem for the one-dimensional DAHA (Corollary 4.5). We will show that many finite-dimensional irreducible modules are quotients of $\mathcal{P}$ up to some twists described in Lemma 4.22. Furthermore, it gives rise to the *non-symmetric polynomials* (Definition 4.12), which can be seen as a non-symmetric version of Macdonald's polynomials appearing in the Macdonald's conjectures. Another important feature of $\mathcal{P}$ is the existence of a symmetric bilinear form (Definition 4.17), whose radical Rad lets us construct finite-dimensional irreducible modules via $\mathcal{P}/\operatorname{Rad}$. Using that the non-symmetric polynomials are Y-eigenvectors (Corollary 4.15) and the evaluation formula (Lemma 4.21) will allow us to find an explicit description of Rad as an ideal with one generator, which is helpful to understand the structure of finite-dimensional irreducible modules. The classification is highly dependent on the parameters q, t of the DAHA. In particular, the *generic* case, where $q \in \mathbb{C}$ is not a root of unity, and the complementary *special* case will be treated separately in Sections 4.3 and 4.4. Albeit the simple construction of the one-dimensional DAHA its representation theory is quite non-trivial, as the main classification results of these sections in Theorem 4.28, Proposition 4.32 and the following propositions, Corollary 4.40 and Proposition 4.42 show.

Spherical DAHA and an action on quantum cohomology rings. In the fifth and final chapter we will study a certain idempotent truncation $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$ of the double affine Hecke algebra of GL_n for $n \geq 2$, which is called the *spherical double affine Hecke algebra* (Definition 5.13). The goal of this chapter is to endow the quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{n,N})_{q=1}$ specialized at $q = 1$ with a module structure for the spherical DAHA. Here $\mathrm{Gr}_{n,N}$ denotes the Grassmannian of n-planes in $\mathbb{C}^N$. This construction has not been previously established in the literature. We begin the chapter by transporting some results and constructions for the one-dimensional DAHA from Chapter 4 to the DAHA of GL_n for $n \geq 2$. More precisely, we will define the polynomial representation of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ on the ring of Laurent polynomials $\mathcal{P} := \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ and look at the radical Rad of a certain bilinear form $\langle\ ,\ \rangle$ on $\mathcal{P}$ (Proposition 5.1 and Definition 5.7). As in the one-dimensional case $\mathcal{P}$ can be used to construct a PBW-type basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ (Theorem 5.4). After that we will define the idempotent $e \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ via

$$e := \frac{1}{\sum_{w \in W} t^{l(w)}} \sum_{w \in W} t^{\frac{l(w)}{2}} T_w, \tag{4}$$

which can be seen as an analogue of the symmetrizing element $\frac{1}{|W|}\sum_{w\in W} w$ in $\mathbb{C}[S_n]$. Unlike in the case of the symmetric group, e is not a priori well-defined. In fact, it is only well-defined for certain choices of the parameters q and t. One of these choices is to set $q = t$ to be a primitive N-th root of unity for some $N > n$. This choice allows us to define the *spherical DAHA* as $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$ and construct a certain module $\mathcal{M} := e\mathcal{P}/e\,\mathrm{Rad}$, which we want to identify with the quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{n,N})_{q=1}$. Setting $q = t$ will have one more nice consequence: the Macdonald's polynomials $P_\lambda \in \mathcal{P}$ specialize to the (rational) Schur polynomials s_λ (Remark 5.24). Because the quantum cohomology ring (Equation (178)) and the $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$-module $\mathcal{M}$ (Proposition 5.16) can both be described in terms of symmetric functions, this is an important result towards identifying them. Using one central statement from the theory of Macdonald's polynomials, namely that the Macdonald's polynomials are weight vectors for a certain subalgebra $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W \subseteq H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ (Theorem 5.25), together with some well-known facts from the theory of Schur polynomials will allow us to deduce important results about the structure of $\mathcal{M}$ in Section 5.5. More precisely, we will describe two bases of $\mathcal{M}$ consisting of weight vectors, once for the subalgebra $e\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W e$ and once for the subalgebra $e\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W e$ (Theorem 5.34 and Theorem 5.38), where $W = S_n$ is the Weyl group. In particular we will deduce that the dimension of $\mathcal{M}$ is $\binom{N}{n}$. Finally, we will identify $\mathcal{M}$ with the quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{n,N})_{q=1}$ of the Grassmannian $\mathrm{Gr}_{n,N}$ specialized at $q = 1$ in Theorem 5.40, which is the main result of the last chapter. The quantum cohomology ring is a certain deformation of the ordinary cohomology ring of the Grassmannian. It is studied in detail in [ST97] from an algebro-geometric point of view. In [KS10] the quantum cohomology ring was studied from the perspective of integrable systems. In this work the Bethe vectors, which from an eigenbasis for a certain family of commutative operators, were determined and used to describe the ring structure. Under the identification with the eH_ne-module $\mathcal{M}$ these Bethe vectors correspond to the above mentioned basis of $e\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W e$-eigenvectors. The main result of the last chapter is the construction of γ in the following theorem.

Theorem 1.1. *The following diagram of $\mathbb{C}$-algebras commutes and the morphism γ is an isomorphism. In particular, we obtain an eH_ne-action on $qH^\bullet(Gr_{n,N})_{q=1}$.*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\iota|_I"] & \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}] \arrow[r] \arrow[d, "\iota"] & qH^\bullet(Gr_{n,N})_{q=1} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & e\,\mathrm{Rad} \arrow[r] & e\mathcal{P} \arrow[r] & \mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

Here $\boldsymbol{e_1}, ..., \boldsymbol{e_n}$ are the elementary symmetric polynomials in n variables and $I := (h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n) \subseteq \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]$, where h_k is the k-th complete symmetric polynomial. The morphism ι is the inclusion of $\mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]$ into $e\mathcal{P} = \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$, where $W = S_n$. Furthermore, γ identifies the $e\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W e$-weight basis we constructed with the basis of Bethe vectors constructed in [KS10].

This result fits nicely into the philosophy of Cherednik from [Che05, Chapter 0.4], where he proposes a connection between double affine Hecke algebras and so-called *abstract Verlinde algebras*. By the results in [KS10] the quantum cohomology ring of the Grassmannian is an example of a Verlinde algebra. Thus, our construction of an action of the spherical DAHA eH_ne on $qH^\bullet(\mathrm{Gr}_{n,N})_{q=1}$ can be seen as an example of an explicit realization of Cherednik's philosophy.

1.3 Acknowledgement

Without the helpful counselling by Prof. Dr. Catharina Stroppel, writing this thesis would not have been possible for me. She gave me many insights into the subject, provided references when I needed them and helped me immensely to understand the concepts presented in this thesis. For this I wholeheartedly thank her. Moreover, I want to thank Prof. Dr. Jan Schröer for being the second advisor of this thesis.

2 Affine Weyl group and skew diagrams

2.1 Affine root system and affine Weyl group

In this section we recollect some facts about the affine root system and the (extended) affine Weyl group of type $\tilde{A}_{n-1}$ assigned to the affine Kac-Moody algebra $\widehat{\mathfrak{gl}_\mathfrak{n}}$. For a reference see [Car05], [Kac90] or [SV05, Chapter 2]. In particular, this section follows the outline given in the last reference. Some familiarity with root systems and Coxeter groups will be assumed.

Let $n \geq 2$. We will describe the root system of $\widehat{\mathfrak{gl}}_n$ now, where $\widehat{\mathfrak{gl}}_n$ is an affine Kac-Moody algebra defined in [KR87, Lecture 9]. For this let $\tilde{\mathfrak{h}}$ be a $(n+2)$-dimensional $\mathbb{Q}$-vector space with basis $\{e_1^\vee, ..., e_n^\vee, c, d\}$. Define a symmetric bilinear form on $\tilde{\mathfrak{h}}$ via

$$\begin{aligned} (e_i^\vee|e_j^\vee) = \delta_{ij}, \quad (e_i^\vee|c) = (e_i^\vee|d) = 0 \text{ for } 1 \leq i,j \leq n, \\ (c|d) = 1, \qquad (c|c) = (d|d) = 0. \end{aligned} \tag{5}$$

Note that this bilinear form is non-degenerate.

Let $\tilde{\mathfrak{h}}^*$ be the dual space of $\tilde{\mathfrak{h}}$ and let e_i for $1 \leq i \leq n$, c^* and $\delta \in \tilde{\mathfrak{h}}^*$ denote the dual vectors of $e_i^\vee$ for $1 \leq i \leq n$, c and d respectively. We define $\mathfrak{h}^*$ to be the $\mathbb{Q}$-span of $e_1, ..., e_n$. Denote by $\langle\ |\ \rangle : \tilde{\mathfrak{h}}^* \times \tilde{\mathfrak{h}} \to \mathbb{Q}$ the natural evaluation pairing. The assignment $c^* \mapsto d, \delta \mapsto c$ and $e_i \mapsto e_i^\vee$ for $1 \leq i \leq n$ defines a $\mathbb{Q}$-linear isomorphism $(\ \ )^\vee : \tilde{\mathfrak{h}}^* \to \tilde{\mathfrak{h}}$. Using this isomorphism we transport the bilinear form $(\ |\ )$ to $\tilde{\mathfrak{h}}^*$ and denote the resulting bilinear form on $\tilde{\mathfrak{h}}^*$ by $(\ |\ )$ as well. For $\zeta, \eta \in \tilde{\mathfrak{h}}^*$ we have $(\zeta|\eta) = \langle\zeta|\eta^\vee\rangle = (\zeta^\vee|\eta^\vee)$ by definition of $(\ \ )^\vee$.

We extend the definition of e_i for $1 \leq i \leq n$ to arbitrary $i \in \mathbb{Z}$ by setting $e_i := e_{\underline{i}} - k\delta$ for $i = \underline{i} + kn$ with $1 \leq \underline{i} \leq n$ and $k \in \mathbb{Z}$. Set $\alpha_{i,j} := e_i - e_j$ for $i, j \in \mathbb{Z}$ and abbreviate $\alpha_i := \alpha_{i,i+1}$ for $i \in \mathbb{Z}$. We call $\alpha_{i,j}$ for $i, j \in \mathbb{Z}$ a *root* and α_i for $i \in \mathbb{Z}$ a *simple root*. Note that this fits the well-known definition $\alpha_i = e_i - e_{i+1}$ from the finite case and furthermore $\alpha_0 = -\alpha_{1,n} + \delta$.

Definition 2.1. We define the set of *finite simple roots* Π, the set of *finite roots* R and the set of *finite positive roots* R^+ to be

$$\begin{aligned} \Pi &:= \{\alpha_1, ..., \alpha_{n-1}\}, \\ R &:= \{\alpha_{i,j} \mid 1 \leq i \neq j \leq n\}, \\ R^+ &:= \{\alpha_{i,j} \mid 1 \leq i < j \leq n\}. \end{aligned} \tag{6}$$

We also define the *finite root lattice* $Q := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$, the *finite weight lattice* $P := \bigoplus_{i=1}^n \mathbb{Z}e_i$ and furthermore the *weight lattice* $\dot{P} := P \oplus \mathbb{Z}c^*$. Finally, we define the set of *simple roots* $\dot{\Pi}$, the set of *(real) roots* $\dot{R}$ and the set of *positive roots* $\dot{R}^+$ by

$$\begin{aligned} \dot{\Pi} &:= \{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}, \\ \dot{R} &:= \{\alpha_{i,j} \mid i, j \in \mathbb{Z},\ i \neq j \bmod n\}, \\ \dot{R}^+ &:= \{\alpha_{i,j} \mid i, j \in \mathbb{Z},\ i \neq j \bmod n \text{ and } i < j\}. \end{aligned} \tag{7}$$

Note that the tuple $(\tilde{\mathfrak{h}}^*, (\ \mid\ ), \dot{R})$ matches the description of the affine root system of type $\tilde{A}_{n-1}$ for GL_n attached to the generalized Cartan matrix $A := ((\alpha_i \mid \alpha_j))_{0 \le i,j \le n}$. The tuple $(\mathfrak{h}^*, (\ \mid\ )|_{\mathfrak{h}^*}, R)$ matches the (finite) root system A_{n-1} for GL_n attached to the Cartan matrix $A' := ((\alpha_i \mid \alpha_j))_{1 \le i,j \le n}$. The difference to the (affine) root system associated to SL_n is that for GL_n we use a larger ambient space, which contains the (finite) weight lattice of GL_n. In particular, we do not assume that the roots span the ambient space in a root system. The finite and affine root systems of SL_n are described in detail in [Car05, Appendix] and this description can be used analogously for the GL_n-case.

Now we are prepared to define the *affine Weyl group of* $\tilde{A}_{n-1}$ and give some well-known properties.

Definition 2.2. For $\alpha \in \dot{R}$ let the *associated reflection* $s_\alpha : \tilde{\mathfrak{h}}^* \to \tilde{\mathfrak{h}}^*$ be the $\mathbb{Q}$-linear map defined by

$$s_\alpha : x \mapsto x - (\alpha \mid x)\alpha \text{ for } x \in \tilde{\mathfrak{h}}^*. \tag{8}$$

We call the group $\dot{W}_a := \langle s_0, ..., s_{n-1} \rangle$ the *affine Weyl group*, where $s_i := s_{\alpha_i}$ for $0 \le i \le n-1$.

Remark 2.3. We have $s_\alpha \in \dot{W}_a$ for all $\alpha \in \dot{R}$. For this observe that the definition of real roots in [Car05, Chapter 16.3] matches our definition of roots by [Car05, Theorem 17.17] up to changing the ambient space. By the definition in [Car05, Chapter 16.3] we can find for any $\alpha \in \dot{R}$ some $w \in \dot{W}_a$ such that $w(\alpha_i) = \alpha$ for some $0 \le i \le n-1$. Then we have $s_\alpha = w s_{\alpha_i} w^{-1}$ by the calculation in [Hum90, Chapter 5.7] and hence $s_\alpha \in \dot{W}_a$.

Theorem 2.4. *(a) The group $\dot{W}_a$ is isomorphic to the Coxeter group admitting the following presentation via generators and relations:*

$$\begin{aligned} \dot{W}_a = \langle s_0, ..., s_{n-1} \mid & s_i^2 = 1 \textit{ for } 0 \le i \le n-1, \\ & s_i s_j = s_j s_i \textit{ for } i \ne j \pm 1 \bmod n, \\ & s_i s_j s_i = s_j s_i s_j \textit{ for } j = i \pm 1 \bmod n \rangle. \end{aligned} \tag{9}$$

Also, $W := \langle s_1, ..., s_{n-1} \rangle$ is the Weyl group of A_{n-1} and hence isomorphic to the symmetric group S_n.

(b) The subgroup $\dot{W}_a \subseteq \mathrm{GL}(\tilde{\mathfrak{h}}^*)$ is a semi-direct product $\dot{W}_a = W \ltimes \tau(Q)$, where $\tau : P \to \mathrm{GL}(\tilde{\mathfrak{h}}^*)$ is a group monomorphism sending $x \in P$ to

$$\tau_x : y \mapsto y + (\delta \mid y)x - \left((x \mid y) + \frac{1}{2}(x \mid x)(\delta \mid y) \right) \delta. \tag{10}$$

For $w \in W$ and $x \in P$ we have $w \tau_x w^{-1} = \tau_{w(x)}$. Moreover, the equality $s_0 = \tau_\theta s_\theta$ where $\theta = e_1 - e_n \in R$ holds.

Proof. Part (a) follows from [Car05, Theorem 16.17], where we assume that the presentation of S_n as a Coxeter group is known. Part (b) is mostly shown in [Car05, Chapter 17.3]. The only thing left is the injectivity of τ, which follows easily from the description of τ_x for $x \in P$. Note that the author works over $\mathbb{C}$, but the proofs also work over $\mathbb{Q}$. $\square$

Definition 2.5. For $w \in \dot{W}_a$ let $l(w)$ denote the minimal length of a word in the Coxeter generators $\{s_0, ..., s_{n-1}\}$ representing w. We call an expression for w in the Coxeter generators of length $l(w)$ a *reduced expression.* Set

$$\dot{R}(w) := \dot{R}^+ \cap w^{-1}(\dot{R}^-), \tag{11}$$

where $\dot{R}^- := -\dot{R}^+ = \dot{R} \setminus \dot{R}^+$.

Theorem 2.6. *(a) For any $w \in \dot{W}_a$ we have $l(w) = |\dot{R}(w)|$.*

(b) For any $\alpha \in \dot{R}^+$ we have $l(ws_\alpha) > l(w)$ if and only if $w(\alpha) \in \dot{R}^+$.

(c) For $w = s_{i_1}...s_{i_m}$ a reduced expression we have

$$\dot{R}(w) = \{\alpha_{i_m}, s_{i_m}(\alpha_{i_{m-1}}), ..., s_{i_m}...s_{i_2}(\alpha_{i_1})\}. \tag{12}$$

(d) The following so-called strong exchange condition *holds. Let $w = s_{i_1}...s_{i_m}$ be a not necessarily reduced expression and let $s_\alpha \in \dot{W}_a$ be a reflection. If $l(ws_\alpha) < l(w)$ then there exists j such that $ws_\alpha = s_{i_1}...\widehat{s}_{i_j}...s_{i_m}$. If the expression is reduced then j is unique and we have $\alpha = s_{i_m}...s_{i_{j+1}}(\alpha_{i_j})$.*

Proof. Part (a) is proven in [Hum90, Chapter 5.6], part (b) is proven in [Hum90, Chapter 5.7] and part (c) can be easily deduced from part (a) and (b) via induction on $l(w)$. Part (d) is proven in [Hum90, Chapter 5.8]. Note that $\alpha = s_{i_m}...s_{i_{j+1}}(\alpha_{i_j})$ is not explicitly shown in the reference, but follows from $s_\alpha = s_{i_m}...s_{i_{j+1}}s_{i_j}s_{i_{j+1}}...s_{i_m}$ and the computation in the beginning of [Hum90, Chapter 5.7]. We have seen in Theorem 2.4 that $\dot{W}_a$ is a Coxeter group and furthermore our definition of (positive) roots coincides with the one given in [Hum90, Chapter 5.4] if we set V to be the $\mathbb{R}$-span of $\alpha_0, ..., \alpha_{n-1}$. This makes the proofs in [Hum90] applicable. $\square$

We will be mostly interested in the so-called *extended affine Weyl group*, which we obtain by replacing the root lattice inside $\dot{W}_a$ by the weight lattice.

Definition 2.7. Using $\tau : P \to \mathrm{GL}(\tilde{\mathfrak{h}}^*)$ from Theorem 2.4 we define the *extended affine Weyl group* to be the group generated by W and $\tau(P)$.

Proposition 2.8. *We have $\dot{W} = W \ltimes \tau(P)$ as subgroups of* $\mathrm{GL}(\tilde{\mathfrak{h}}^*)$.

Proof. The subgroup $W \subseteq \dot{W}$ normalizes $\tau(P)$ by Theorem 2.4 part (b). The group W is finite, whereas $\tau(P)$ is free abelian by the injectivity of τ and because P is free abelian. Therefore we have $W \cap \tau(P) = 1$ and the claim follows. $\square$

We want to give another description of $\dot{W}$ now. For this observe that $P/Q \cong \mathbb{Z}$ by sending $\hat{\pi} := [e_1] \in P/Q$ to $1 \in \mathbb{Z}$.

Proposition 2.9. *We have $\dot{W} \cong P/Q \ltimes \dot{W}_a$ and an explicit split is given by $s : P/Q \hookrightarrow \dot{W}$ sending $\hat{\pi} \mapsto \pi$ with $\pi := \tau_{e_1} s_1 ... s_{n-1}$.*

Proof. The inclusion $Q \hookrightarrow P$ induces a monomorphism $\dot{W}_a \hookrightarrow \dot{W}$. Its image is normal, because we have $\tau_x w \tau_{-x} = \tau_{x-w(x)} w \in \dot{W}_a$ for $x \in P$ and $w \in \dot{W}_a$ by Theorem 2.4 (b). By identifying P with $\tau(P)$ we get a surjective morphism $p : \dot{W} \twoheadrightarrow \dot{W}/\dot{W}_a \cong P/Q$. Since $P/Q \cong \mathbb{Z}$ and π lies in the preimage of $\hat{\pi}$ under p, the above described map is actually a split, which shows the claim.

□

Proposition 2.10. *The group $\dot{W}$ is isomorphic to the group generated by the elements $\pi, s_0, ..., s_{n-1}$ subject to the relations*

$$\begin{aligned} s_i^2 &= 1 && \text{for } 0 \le i \le n-1, \\ s_i s_j &= s_j s_i && \text{for } 0 \le i, j \le n-1 \text{ and } i - j \ne \pm 1 \bmod n, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } 0 \le i \le n-1 \text{ and } i - j = \pm 1 \bmod n, \\ \pi s_i &= s_j \pi && \text{for } 0 \le i \le n-1 \text{ and } j = i+1 \bmod n. \end{aligned} \tag{13}$$

Proof. From Proposition Theorem 2.4 (a) and 2.9 we can deduce all relations except $\pi s_i = s_j \pi$ for $0 \le i \le n-1$ and $j = i+1 \bmod n$. For this relation we use the explicit description $\pi = \tau_{e_1} s_1 ... s_{n-1}$ and calculate for $1 \le i \le n-2$:

$$\pi s_i = \tau_{e_1} s_1 ... s_{n-1} s_i = \tau_{e_1} s_{i+1} s_1 ... s_{n-1} = s_{i+1} \tau_{e_1} s_1 ... s_{n-1} = s_{i+1} \pi. \tag{14}$$

For $i = 0$ we have

$$\begin{aligned} \pi s_0 &= \tau_{e_1} s_1 ... s_{n-1} \tau_{e_1} \tau_{-e_n} s_\theta = s_1 ... s_{n-1} \tau_{e_n} \tau_{e_1} \tau_{-e_n} s_\theta = s_1 \tau_{e_1} s_2 ... s_{n-1} s_\theta \\ &= s_1 \pi, \end{aligned} \tag{15}$$

where we used $s_0 = \tau_\theta s_\theta$ from Theorem 2.4 and $s_\theta = s_{n-1} ... s_2 s_1 s_2 ... s_{n-2}$. Lastly, for $i = n-1$ we have

$$\begin{aligned} s_0 \pi &= \tau_{e_1} \tau_{-e_n} s_\theta \tau_{e_1} s_1 ... s_{n-1} = \tau_{e_1} s_\theta s_1 ... s_{n-1} = \tau_{e_1} s_1 ... s_{n-2} \\ &= \pi s_{n-1}, \end{aligned} \tag{16}$$

where we used the equality $s_\theta = s_1 ... s_{n-2} s_{n-1} s_{n-2} ... s_1$. This proves the existence of a morphism from the group above to $\hat{W}$. The inverse can be constructed using the universal property of the semidirect product. □

The main use of this description of $\dot{W}$ for us is the construction of the following action of $\dot{W}$ on $\mathbb{Z}$.

Definition 2.11. Let $\mathrm{Per}_n(\mathbb{Z})$ denote the group of n-periodic permutations of $\mathbb{Z}$, in other words the group of all bijections $\phi : \mathbb{Z} \to \mathbb{Z}$ for which $\phi(z+n) = \phi(z) + n$ for all $z \in \mathbb{Z}$.

Proposition 2.12. *Setting for* $0 \leq i \leq n-1$ *and* $j \in \mathbb{Z}$

$$\begin{aligned} s_i(j) &= j+1 && \text{if } j = i \bmod n, \\ s_i(j) &= j-1 && \text{if } j = i+1 \bmod n, \\ s_i(j) &= j && \text{if } j \neq i, i+1 \bmod n, \\ \pi(j) &= j+1, \end{aligned} \tag{17}$$

defines a group isomorphism $\dot{W} \to \mathrm{Per}_n(\mathbb{Z})$.

Proof. Verifying the relations is easily done and hence we only prove the bijectivity. Recall from Proposition 2.9 that $\tau_{e_1} = \pi s_{n-1} ... s_1$ and hence τ_{e_1} acts on $j \in \mathbb{Z}$ via

$$\tau_{e_1}(j) = j+n \text{ if } j = 1 \bmod n, \quad \tau_{e_1}(j) = j \text{ otherwise.} \tag{18}$$

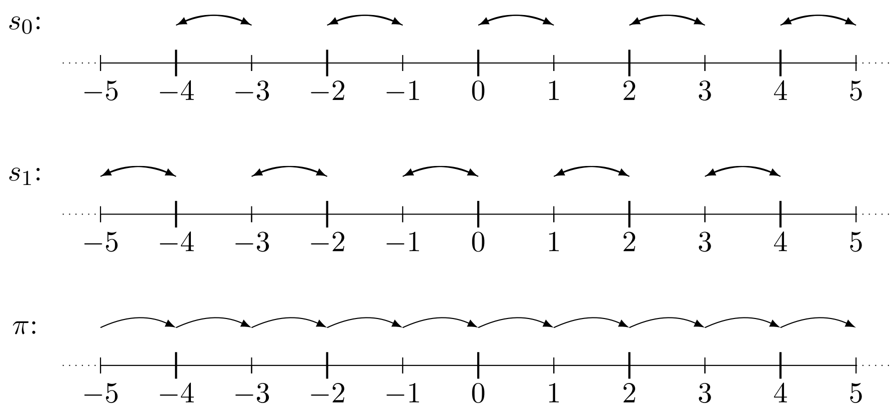
Using $\pi^{i-1}\tau_{e_1}\pi^{-i+1} = \tau_{e_i}$ for $1 \leq i \leq n$, which is obtainable via the explicit description of π in Proposition 2.9, we get for $1 \leq i \leq n$ and $j \in \mathbb{Z}$

$$\tau_{e_i}(j) = j+n \text{ if } j = i \bmod n, \quad \tau_{e_i}(j) = j \text{ otherwise.} \tag{19}$$

To prove injectivity assume that an element $w \in \dot{W}$ acts trivially. Write $w = \tau_x w'$ with $x \in P$ and $w' \in W$. Note that w' preserves the interval $\{1, ..., n\}$ and acts on it via the corresponding permutation in $S_n \cong W$. By the description of τ_{e_i} above we see that τ_x only preserves this interval if $x = 0$. But for w to act trivially it must preserve this interval and we can deduce that $x = 0$. Hence w' acts trivially on $\{1, ..., n\}$, which means $w = w' = 1$.

For the surjectivity let $\phi \in \mathrm{Per}_n(\mathbb{Z})$ be an arbitrary element. Define k_i, r_i for $1 \leq i \leq n$ with $r_i \in \{1, ..., n\}$ and $k_i \in \mathbb{Z}$ via $\phi(i) = r_i + k_i n$. If $r_i = r_j$ for some $i \neq j$ then $\phi(j + (k_i - k_j)n) = \phi(i)$ by the n-periodicity, which contradicts the bijectivity of ϕ. Let $w' \in W \subseteq \dot{W}$ correspond to the permutation $r_i \mapsto i$ in S_n. We can replace ϕ by $w'\phi$ and can now assume that $r_i = i$ for all $1 \leq i \leq n$. Setting $x = k_1 e_1 + ... + k_n e_n \in P$ gives $\phi = \tau_x$, which shows surjectivity. $\square$

Remark 2.13. Let us visualize the action of $\dot{W}$ on $\mathbb{Z}$ in the case $n = 2$. Here we view $\mathbb{Z}$ as a subset of the real number line. The restriction of s_1 to $\{1, 2\}$ is the transposition of 1 and 2. In fact, for arbitrary n we see from the definition of the action that $S_n \cong W \subseteq \dot{W}$ acts via the corresponding permutations on $\{1, ..., n\} \subseteq \mathbb{Z}$.



We obtain an interplay between the actions of $\dot{W}$ on $\mathbb{Z}$ and on $\tilde{\mathfrak{h}}^*$.

Lemma 2.14. *For $w \in \dot{W}$ we have*

$$\begin{aligned} &(a)\ w(e_i) = e_{w(i)} && \textit{for } i \in \mathbb{Z}, \\ &(b)\ w(\alpha_{i,j}) = \alpha_{w(i),w(j)} && \textit{for } i \neq j \in \mathbb{Z}. \end{aligned} \tag{20}$$

Proof. Claim (b) follows directly from (a). We prove (a) for the generators s_i for $1 \le i \le n-1$ and τ_{e_i} for $1 \le i \le n$. Take an arbitrary $j \in \mathbb{Z}$ and write $j = \underline{j} + kn$ for some uniquely determined $1 \le \underline{j} \le n$ and $k \in \mathbb{Z}$. We have

$$s_i(e_j) = s_i(e_{\underline{j}}) - ks_i(\delta) = e_{s_i(\underline{j})} - k\delta = e_{s_i(j)} \text{ for } 1 \le i \le n-1. \tag{21}$$

With the Kronecker delta δ_{ij} we obtain

$$\tau_{e_i}(e_j) = e_j - (e_i \mid e_j)\delta = e_j - \delta_{ij}\delta = e_{j+\delta_{ij}n} = e_{\tau_i(j)} \text{ for } 1 \le i \le n. \tag{22}$$

This proves (a) and hence the lemma. □

We deduce now one of the most useful properties of the $\dot{W}$-action on $\tilde{\mathfrak{h}}^*$.

Lemma 2.15. *The bilinear form $(\ \mid\ )$ is $\dot{W}$-invariant.*

Proof. Let $x, y \in \tilde{\mathfrak{h}}^*$. For $1 \le i \le n$ we have

$$\begin{aligned} (\tau_{e_i}(x) \mid \tau_{e_i}(y)) = (x \mid y) &+ (\delta \mid y)(e_i \mid y) + (\delta \mid x)(e_i \mid x) \\ &+ (\delta \mid x)(\delta \mid y)(e_i \mid e_i) - \left((e_i \mid y) + \frac{1}{2}(\delta \mid y)\right)(\delta \mid x) \\ &- \left((e_i \mid x) + \frac{1}{2}(\delta \mid x)\right)(\delta \mid y) = (x \mid y). \end{aligned} \tag{23}$$

For $1 \le i \le n-1$ we have by using $(\alpha_i \mid \alpha_i) = 2$

$$\begin{aligned} (s_i(x) \mid s_i(y)) &= (x \mid y) - 2(\alpha_i \mid x)(\alpha_i \mid y) + (\alpha_i \mid y)(\alpha_i \mid x)(\alpha_i \mid \alpha_i) \\ &= (x \mid y). \end{aligned} \tag{24}$$

This proves the invariance for the generators of $\dot{W}$ and therefore for $\dot{W}$ itself. □

Remark 2.16. Recall l and $\dot{R}$ from Definition 2.5. We extend the length function l from $\dot{W}_a$ to $\dot{W}$ by setting $l(\pi^k w) = l(w)$ for arbitrary $\pi^k w \in \dot{W}$ with $w \in \dot{W}_a$ and $k \in \mathbb{Z}$. This is well-defined by the properties of the semidirect product $\dot{W} = P/Q \ltimes \dot{W}_a$. We also extend the definition of $\dot{R}$ to arbitrary $w = \pi^k w' \in \dot{W}$. By Lemma 2.14 we have $\dot{R}(\pi^k w') = \dot{R}(w')$, since π preserves $\dot{R}^+$. In particular, Theorem 2.6 holds analogously for $\dot{W}$.

Let us give a short discussion of parabolic subgroups of $\dot{W}$.

Definition 2.17. For $I \subseteq \{0, ..., n\}$ define

$$\begin{aligned} \dot{W}_I &:= \langle s_i \mid i \in I \rangle \subseteq \dot{W}, \\ \dot{\Pi}_I &:= \{\alpha_i \mid i \in I\} \subseteq \dot{\Pi}, \\ \dot{R}_I^+ &:= \{\alpha \in \dot{R}^+ \mid s_\alpha \in \dot{W}_I\} \subseteq \dot{R}^+. \end{aligned} \tag{25}$$

We call $\dot{W}_I$ the *parabolic subgroup corresponding to I*. We also define the following subset of $\dot{W}$

$$\dot{W}^I := \{w \in \dot{W} \mid \dot{R}(w) \cap \dot{R}_I^+ = \emptyset\}. \tag{26}$$

Lemma 2.18. *For any $I \subseteq \{0, ..., n\}$ we have*

$$\dot{W}^I = \{w \in \dot{W} \mid l(ws_\alpha) > l(w) \text{ for all } \alpha \in \dot{R}_I^+\}. \tag{27}$$

Proof. By Theorem 2.6 part (b) we have $l(ws_\alpha) > l(w)$ for all $\alpha \in \dot{R}_I^+$ if and only if $w(\alpha) \in \dot{R}^+$ for all $\alpha \in \dot{R}_I^+$. But this is then equivalent to $\dot{R}^- \cap w(R_I^+) = \emptyset$, which is equivalent to $\dot{R}(w) \cap \dot{R}_I^+ = w^{-1}(\dot{R}^-) \cap \dot{R}_I^+ = \emptyset$. $\square$

We conclude this section with a description of the *affine action of $\dot{W}$* and its stabilizers.

Remark 2.19. Note that the action of $\dot{W}$ on $\tilde{\mathfrak{h}}^*$ fixes $\mathbb{Q}\delta$ and hence we can define the *affine action of $\dot{W}$* on $\mathfrak{h}^* \oplus \mathbb{Q}c^*$ as the induced action action on the quotient $\tilde{\mathfrak{h}}^*/\mathbb{Q}\delta \cong \mathfrak{h}^* \oplus \mathbb{Q}c^*$. We will denote the affine action of $w \in \dot{W}$ on $h \in \mathfrak{h}^* \oplus \mathbb{Q}c^*$ by $\bar{w}(h)$. Let $\dot{W}[\zeta]$ denote the stabilizer of a weight $\zeta = \zeta_1 e_1 + ... + \zeta_n e_n + \zeta_c c^* \in \dot{P}$ with respect to the affine action. We also define $\dot{R}[\zeta] := \{\alpha \in \dot{R} \mid (\zeta \mid \alpha) = 0\}$.

Lemma 2.20. *Let $\zeta = \zeta_1 e_1 + ... + \zeta_n e_n + \zeta_c c^* \in \dot{P}$ with $\zeta_c \neq 0$ and $w \in \dot{W}[\zeta] \setminus \{1\}$. Then $\dot{R}(w) \cap \dot{R}[\zeta] \neq \emptyset$.*

Proof. Let $1 \neq w = \tau_x w' \in \dot{W}[\zeta]$ with $w' \in W$ and $x = x_1 e_1 + ... + x_n e_n$. Then $w \in \dot{W}[\zeta]$ implies

$$\zeta_i = \zeta_{w'^{-1}(i)} + x_i \zeta_c \text{ for } 1 \leq i \leq n. \tag{28}$$

For $1 \leq i \leq n$ set $O_i := \{i, w'(i), ..., w'^{k_i}(i)\}$ for appropriate $k_i \in \mathbb{Z}_{>0}$ to be the w'-orbit of i. If all $x_i = 0$ for $1 \leq i \leq n$, then $w' = w \neq 1$ and we can find $1 \leq i \leq n$ and $j \in O_i$ such that $w'(j) > j$ and $w'^{-1}(j) > j$. But then $\alpha_{j,w'^{-1}(j)} \in \dot{R}^+ \cap \dot{R}[\zeta]$ and $w'(\alpha_{j,w'^{-1}(j)}) = \alpha_{w'(j),j} \in \dot{R}^- \cap \dot{R}[\zeta]$, which shows our claim. Now assume that some $x_i \neq 0$. We have $x_i + x_{w'(i)} + ... + x_{w'^{k_i}(i)} = 0$ by Equation (28) and since $\zeta_c \neq 0$. Thus we can find some $1 \leq j \leq k_i$ and some $1 \leq b \leq k_i$ such that

$$x_{w'^j(i)} < 0, \quad x_{w'^{j+1}(i)} = ... = x_{w'^{j+b-1}(i)} = 0, \quad x_{w'^{j+b}(i)} > 0. \tag{29}$$

By Equation (28) we have $\alpha_{w'^{j+b-1}(i),w'^{j-1}(i)} - x_{w'^j(i)}\delta \in \dot{R}[\zeta]$. By $x_{w'^j(i)} < 0$ we even have

$$\alpha_{w'^{b-1+j}(i),w'^{j-1}(i)} - x_{w'^j(i)}\delta \in \dot{R}[\zeta] \cap \dot{R}^+. \tag{30}$$

Applying $w = \tau_x w'$ we obtain

$$\begin{aligned} w(\alpha_{w'^{b-1+j}(i),w'^{j-1}(i)} - x_{w'^j(i)}\delta) &= \alpha_{w'^{b+j}(i),w'^j(i)} \\ &\quad - (x_{w'^j(i)} + x_{w'^{b+j}(i)} - x_{w'^j(i)})\delta \\ &= \alpha_{w'^{b+j}(i),w'^j(i)} - x_{w'^{b+j}(i)}\delta \end{aligned} \tag{31}$$

and since $x_{w'^{b+j}(i)} > 0$ this element lies in $\dot{R}^-$, which shows the claim.

□

2.2 Periodic skew diagrams and tableaux

In the representation theory of S_n Young diagrams and tableaux play a very important role. To each Young diagram a certain irreducible S_n-module, the so-called *Specht module*, is associated, whose basis is given by the standard tableaux on the diagram. Furthermore, the action of S_n with respect to this basis can be described using an action of S_n on the Young tableaux. For an overview of this theory see [Ful97], especially Chapter 7. The goal of this and the following sections is to mimic these ideas in a 'double affine context'. In this section we will generalize the notion of Young diagrams and (standard) tableaux and define an action of the extended affine Weyl group $\dot{W}$ on the tableaux. We will again fix $n \geq 2$. The discussion presented in this and the following sections is based on [SV05].

Recall that a skew Young diagram λ is a subset of $\mathbb{Z}^2$ such that if $(a,b) \in \lambda$ and $(a+i, b+j) \in \lambda$ for some $i,j \in \mathbb{Z}_{\geq 1}$ then also the full rectangle $[(a,b),(a+i,b+j)]$ lies in λ. See Definition 5.19 for a description of the non-skew version. The following definition can be seen as a double affine analogue.

Definition 2.21. Let $m \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 0}$ and $\gamma = (m, -l)$. A *γ-periodic skew diagram of degree n* is a subset of $\Lambda \subseteq \mathbb{Z}^2$ with the following properties.

(D1) We have $\Lambda = \Lambda + \gamma$.

(D2) The set $\{(a,b) \in \Lambda \mid 1 \leq a \leq m\}$ has size n.

(D3) For (a,b) and $(a+i, b+j) \in \Lambda$ with $i, j \in \mathbb{Z}_{\geq 1}$ we have $(a+i', b+j') \in \Lambda$ for all $1 \leq i' \leq i$ and $1 \leq j' \leq j$.

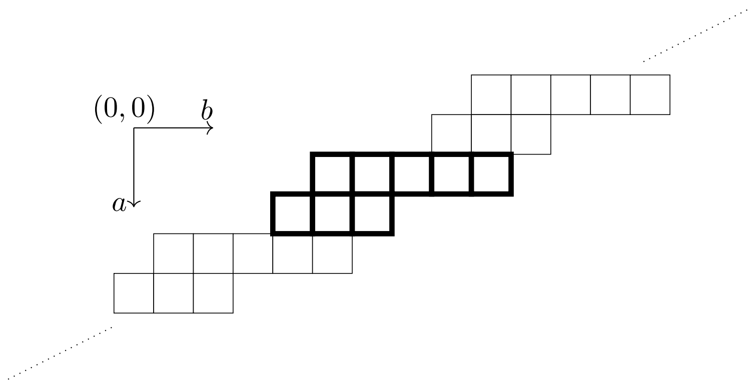
Let $\mathcal{D}^n_{m,-l}$ denote the set of all γ-periodic skew diagrams of degree n for $\gamma = (m, -l)$ and let

$$\mathcal{D}^{*n}_{m,-l} := \{\Lambda \in \mathcal{D}_{m,-l} \mid \forall a \in \mathbb{Z} : \exists b \text{ such that } (a,b) \in \Lambda\} \tag{32}$$

denote the subset of γ-periodic skew diagrams with no empty rows.

From now on we will assume $m \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 0}$ and $\gamma = (m, -l) \in \mathbb{Z}^2$.

Example 2.22. The following picture shows the γ-periodic skew diagram for $\gamma = (m, -l) = (2, -4)$ and the boldly framed fundamental domain $\{(1,5),(1,6),(1,7),(1,8),(1,9),(2,4),(2,5),(2,6)\}$ for the translation by γ.



For two nested partitions $\lambda = (\lambda_1 \geq ... \geq \lambda_m)$ and $\mu = (\mu_1 \geq ... \geq \mu_m)$ with $\lambda_i \geq \mu_i$ for all i one can construct a skew Young diagram by taking the complement of μ in λ as described in [Ful97, Notation]. In a similar fashion we want to use a generalized form of partitions to construct periodic skew diagrams.

Definition 2.23. Define the *set of m-partitions of width l* to be

$$\mathcal{P}^+_{m,l} := \{\mu = (\mu_1, ..., \mu_m) \in \mathbb{Z}^m \mid \mu_1 \geq ... \geq \mu_m,\ l \geq \mu_1 - \mu_m\}. \tag{33}$$

We define the set of *(strictly) nested m-partitions of width l* to be

$$\mathcal{J}^n_{m,l} := \{(\lambda, \mu) \in \mathcal{P}^+_{m,l} \times \mathcal{P}^+_{m,l} \mid \lambda_i \geq \mu_i \text{ for all } i,\ \sum_{i=1}^m (\lambda_i - \mu_i) = n\}, \tag{34}$$

$$\mathcal{J}^{*n}_{m,l} := \{(\lambda, \mu) \in \mathcal{P}^+_{m,l} \times \mathcal{P}^+_{m,l} \mid \lambda_i > \mu_i \text{ for all } i,\ \sum_{i=1}^m (\lambda_i - \mu_i) = n\}. \tag{35}$$

To each $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ we want to associate a γ-periodic skew diagram for $\gamma = (m, -l)$. For this we define the following subsets of $\mathbb{Z}^2$.

$$\lambda/\mu := \{(a, b) \in \mathbb{Z}^2 \mid 1 \leq a \leq m,\ \mu_a + 1 \leq b \leq \lambda_a\}, \tag{36}$$

$$\lambda/\mu[k] := \lambda/\mu + k\gamma \text{ for } k \in \mathbb{Z}, \tag{37}$$

$$\widehat{\lambda/\mu} := \bigcup_{k\in\mathbb{Z}} \lambda/\mu[k]. \tag{38}$$

We also extend the definition of λ_a and μ_a to all $a \in \mathbb{Z}$ by writing $a \in \mathbb{Z}$ as $a = \underline{a} + km$ with $\underline{a} \in \{1, ..., m\}$, $k \in \mathbb{Z}$ and setting $\lambda_a = \lambda_{\underline{a}} - kl$. Similarly we set $\mu_a = \mu_{\underline{a}} - kl$. We obtain that the a-th row of $\widehat{\lambda/\mu}$ equals $\{(a, \mu_a + 1), ..., (a, \lambda_a)\}$.

Proposition 2.24. *Let $m \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 0}$ and $\gamma = (m, -l)$.*
(a) Sending (λ, μ) to $\widehat{\lambda/\mu}$ defines a surjective map $\Phi : \mathcal{J}^n_{m,l} \to \mathcal{D}^n_{m,-l}$.
(b) Φ restricts to a bijective map $\Phi^* : \mathcal{J}^{*n}_{m,l} \to \mathcal{D}^{*n}_{m,-l}$.
Pictured as a diagram the proposition reads as:

$$\begin{tikzcd}
\left\{ \begin{array}{c} \textit{Strictly nested m-partitions} \\ \textit{of width } l \end{array} \right\} \arrow[r, leftrightarrow, "\Phi^*"] \arrow[d, phantom, "\bigcap"] & \left\{ \begin{array}{c} \gamma\textit{-periodic skew diagrams} \\ \textit{without empty rows} \end{array} \right\} \arrow[d, phantom, "\bigcap"] \\
\left\{ \begin{array}{c} \textit{Nested m-partitions} \\ \textit{of width } l \end{array} \right\} \arrow[r, twoheadrightarrow, "\Phi"] & \left\{ \gamma\textit{-periodic skew diagrams} \right\}
\end{tikzcd}$$

Proof. First we have to show that for any $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ we have $\widehat{\lambda/\mu} \in \mathcal{D}^n_{m,-l}$. We see that property (D1) follows directly from the definition of $\widehat{\lambda/\mu}$ as the union of $\lambda/\mu[k]$ for $k \in \mathbb{Z}$. Since $\gamma = (m, -l)$ we obtain that $\widehat{\lambda/\mu} \cap (\{1, ..., m\} \times \mathbb{Z}) = \lambda/\mu[0] = \lambda/\mu$. This set has size n as $\lambda/\mu \in \mathcal{J}^n_{m,l}$ and (D2) follows. To prove (D3) let $(a, b), (a + i, b + j) \in \widehat{\lambda/\mu}$ with $i, j \in \mathbb{Z}_{\geq 0}$. Note that for $(a_1, b_1), (a_2, b_2) \in \widehat{\lambda/\mu}$ with $a_1 \geq a_2$ we have $\lambda_{a_1} \leq \lambda_{a_2}$. Indeed, if $(a_1, b_1), (a_2, b_2) \in \lambda/\mu[k]$ then this holds by definition of $\lambda/\mu[k] = \lambda/\mu + k\gamma$. Otherwise we can use that λ/μ and hence any $\lambda/\mu[k]$ is not empty and by transitivity we only need to prove the claim for some fixed k and for some $(a_1, b_1) \in \lambda/\mu[k+1]$ and $(a_2, b_2) \in \lambda/\mu[k]$. We use $l \geq \lambda_1 - \lambda_m = \lambda_{1+(k+1)m} - \lambda_{m+(k+1)m}$ and obtain $\lambda_{a_1} \leq \lambda_{1+(k+1)m} \leq \lambda_{m+(k+1)m} + l = \lambda_{m+km} \leq \lambda_{a_2}$. Analogously we obtain $\mu_{a_1} \leq \mu_{a_2}$ for $a_1 \geq a_2$. In our setting this implies $\mu_{\tilde{a}} + 1 \leq \mu_a + 1 \leq b$ and $b + j \leq \lambda_{a+i} \leq \lambda_{\tilde{a}}$ for all $\tilde{a} \in \{a, ..., a + i\}$. Hence $(\tilde{a}, \tilde{b}) \in \widehat{\lambda/\mu}$ for all $a \leq \tilde{a} \leq a + i$ and $b \leq \tilde{b} \leq b + j$, which shows (D3).

Now we prove the surjectivity of Φ. Take any $\Lambda \in \mathcal{D}^n_{m,-l}$. Pick an

$i_0 \in \mathbb{Z}_{\leq 0}$ such that the i_0-th row of Λ is not empty. For $i \geq i_0$ define

$$\lambda_i \quad := \begin{cases} \max\{b \mid (i,b) \in \Lambda\} \text{ if the i-th row is not empty,} \\ \lambda_{i-1} \text{ otherwise,} \end{cases} \tag{39}$$

$$\mu_i \quad := \begin{cases} \min\{b \mid (i,b) \in \Lambda\} - 1 \text{ if the i-th row is not empty,} \\ \lambda_{i-1} \text{ otherwise.} \end{cases} \tag{40}$$

Set $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mu = (\mu_1, ..., \mu_m)$. We want to show $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. Using condition (D3) we obtain that the a-th row of Λ is $\{(a, \mu_a + 1), ..., (a, \lambda_a)\}$, where emptiness of this set is possible. Also, we immediately obtain $\mu_a \leq \lambda_a$ for $a \in \mathbb{Z}$. Let $a, a' \in \{1, ..., m\}$ with $a < a'$. We claim that $\lambda_a \geq \lambda_{a'}$. If the row of a or a' is empty we can replace a respectively a' by the greatest $\tilde{a} \leq a$ respectively $\tilde{a}' \leq a'$ such that the corresponding row is not empty. Note that $\tilde{a} \leq \tilde{a}'$. But then by condition (D3) we have $(\tilde{a}, \lambda_{\tilde{a}'}) \in \Lambda$ and since $\lambda_{\tilde{a}}$ is the maximal b for which $(\tilde{a}, b) \in \Lambda$ we have $\lambda_a \geq \lambda_{a'}$ for all $a < a'$ in $\{1, ..., m\}$. Similarly we obtain $\mu_a \geq \mu_{a'}$. Using (D1) we obtain $\lambda_1 \leq \lambda_0 = \lambda_m + l$ and using (D2) we obtain that the union of the rows 1 up to m has size n, which altogether shows $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. It now follows that $\Phi((\lambda, \mu))$ agrees with Λ on the rows 1 up to m and hence by periodicity on all rows, which proves the surjectivity. For the injectivity on $\mathcal{J}^{*n}_{m,l}$ note that if the i-th row of $\Phi((\lambda, \mu))$ for $1 \leq i \leq m$ is not empty then λ_i and μ_i are uniquely determined by the maximal and minimal element in the i-th row. □

We want to generalize the notion of a Young tableau on a (skew) Young diagram from [Ful97, Notation] to our context.

Definition 2.25. Let $(\lambda, \mu) \in \mathcal{J}^n_{m,-l}$. A bijective map $T : \widehat{\lambda/\mu} \to \mathbb{Z}$ with $T(u + \gamma) = T(u) + n$ is called a *γ-tableau*. The set of all γ-tableaux on $\widehat{\lambda/\mu}$ is denoted by $\mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$. We call $T(a, b)$ for $(a, b) \in \widehat{\lambda/\mu}$ the *label* of (a, b).

By the periodicity constraint a tableau on $\widehat{\lambda/\mu}$ is uniquely determined by its values on λ/μ. As in the finite case described in [Ful97, Notation and Chapter 7] we are mostly interested in a certain subset of the periodic tableaux, the so-called *standard tableaux*, which will index bases of our irreducible representations by Theorem 3.15.

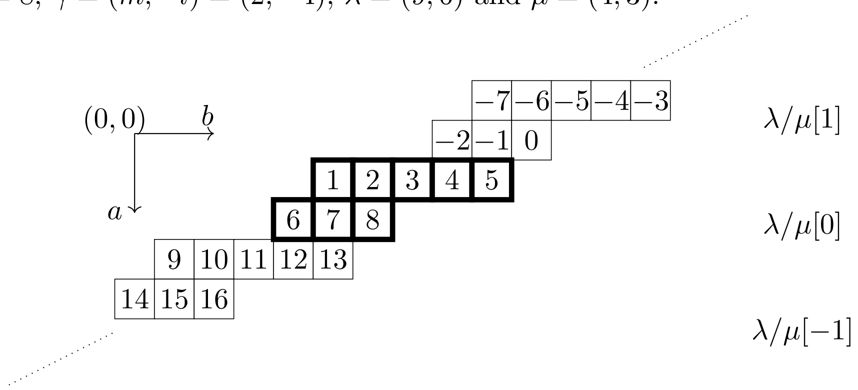
Definition 2.26. Let $(\lambda, \mu) \in \mathcal{J}^n_{m,-l}$. A γ-tableau T is called *row increasing* (respectively *column increasing*) if $(a, b), (a, b + 1) \in \widehat{\lambda/\mu}$ implies $T(a, b) < T(a, b+1)$ (respectively $(a, b), (a+1, b) \in \widehat{\lambda/\mu}$ implies $T(a, b) < T(a+1, b)$). A γ-tableau which is column increasing and row increasing is called a *standard γ-tableau*. The set of all row increasing γ-tableaux on $\widehat{\lambda/\mu}$ is denoted by $\mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$ and the set of all standard γ-tableaux on $\widehat{\lambda/\mu}$ is denoted by $\mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})$.

Definition 2.27. Let $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. We define a standard tableau T_0 on $\widehat{\lambda/\mu}$ by setting

$$T_0(a, \mu_i + j) := \left(\sum_{k=1}^{i-1} (\lambda_k - \mu_k) \right) + j \text{ for } i \in \{1, ..., m\} \text{ and } j \in \{1, ..., \lambda_i - \mu_i\}. \tag{41}$$

on the fundamental domain $\lambda/\mu \subseteq \widehat{\lambda/\mu}$ and then extending the assignment to $\widehat{\lambda/\mu}$ via $\gamma = (m, -l)$. Note that this is possible, because $T_0(\lambda/\mu) = \{1, ..., n\}$. We call T_0 the *ground state tableau* or *row reading tableau.*

Example 2.28. Here is an example of the ground state tableau on $\widehat{\lambda/\mu}$ for $n = 8$, $\gamma = (m, -l) = (2, -4)$, $\lambda = (9, 6)$ and $\mu = (4, 3)$.



We want to study an action of the extended affine Weyl group $\dot{W}$ from Definition 2.7 on the set $\mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$. It is induced by the action of $\dot{W}$ on $\mathbb{Z}$ described in Proposition 2.12.

Proposition 2.29. *Let* $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. *Then* $\dot{W}$ *acts on* $\mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$ *via*

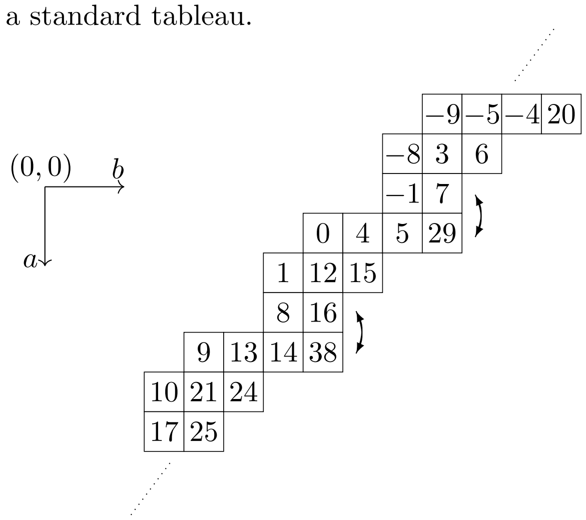
$$(wT)(u) := w(T(u)) \text{ for all } u \in \widehat{\lambda/\mu}, \tag{42}$$

Proof. The element $w \in \dot{W}$ acts via $\mathrm{Per}_n(\mathbb{Z})$ on $\mathbb{Z}$ and hence $wT : \widehat{\lambda/\mu} \to \mathbb{Z}$ is a bijection with $wT(u + \gamma) = w(T(u) + n) = wT(u) + n$ for all $u \in \widehat{\lambda/\mu}$, therefore it is again a γ-tableaux. $\square$

Proposition 2.30. *Let* $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. *The action of* $\dot{W}$ *on* $\mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$ *is simply transitive.*

Proof. Let T_0 denote the ground state tableau and S any γ-periodic tableau on $\widehat{\lambda/\mu}$. Then $S \circ T_0^{-1} : \mathbb{Z} \to \mathbb{Z}$ is a n-periodic permutation on $\mathbb{Z}$ and hence by Proposition 2.12 there exists a unique $w \in \dot{W}$ such that w acts on $\mathbb{Z}$ via $S \circ T_0^{-1}$. Precomposing with the bijective T_0 this shows that there exists a unique $w \in \dot{W}$ such that $wT_0 = S$, which implies the simple transitivity. $\square$

Remark 2.31. Observe that Proposition 2.30 implies that in general $\dot{W}$ does not preserve the set of standard tableaux on a given γ-periodic skew diagram $\widehat{\lambda/\mu}$. Take for example $n = 9$, $m = 3$, $l = 5$, $\mu = (9, 8, 8)$ and $\lambda = (13, 11, 10)$. The corresponding periodic skew diagram is depicted below with a standard tableau T on it. Take $w \in \dot{W}$ such that the element in $\mathrm{Per}_9(\mathbb{Z})$ corresponding to w via Proposition 2.12 permutes 7 and 29 periodically modulo 9 and is the identity otherwise. The action of w on the tableau T is depicted below and one sees that wT is not column increasing and hence not a standard tableau.



2.3 Standard tableaux and $Z_T^{(\lambda/\mu)}$

In order to understand the action of the DAHA on certain irreducible modules defined later in Theorem 3.15 we will need to know for which $w \in \dot{W}$ and T a given standard tableau wT is again a standard tableau. On the following pages we will deduce via some technical work that the set of such w coincides with $Z_T^{(\lambda,\mu)}$ defined below. For this task the content function $C_T^{\widehat{\lambda/\mu}}$ plays a crucial role. We still assume $n \geq 2$, $m \geq 1$, $l \geq 0$ and $\gamma = (m, -l)$.

Definition 2.32. Let $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. The *diagonal function* $C : \mathbb{Z}^2 \to \mathbb{Z}$ is defined via $C(a, b) = b - a$ for $(a, b) \in \mathbb{Z}^2$. For any γ-periodic tableau T on $\widehat{\lambda/\mu}$ define the *content of* T to be the function which associates to each element in $\mathbb{Z}$ the diagonal value of its position in T. In formulas:

$$\begin{aligned} C_T^{\widehat{\lambda/\mu}} : \mathbb{Z} &\longrightarrow \mathbb{Z}, \\ u &\longmapsto C(T^{-1}(u)). \end{aligned} \tag{43}$$

We also define the *associated weight* of $C_T^{\widehat{\lambda/\mu}}$ to be

$$\zeta_T := \sum_{i=1}^{n} C_T^{\widehat{\lambda/\mu}}(i) e_i + (l + m)c^* \in \dot{P}. \tag{44}$$

Furthermore we set

$$Z_T^{(\lambda,\mu)} := \{w \in \dot{W} \mid (\zeta_T \mid \alpha) \notin \{-1,1\} \text{ for all } \alpha \in \dot{R}(w)\}. \tag{45}$$

Lemma 2.33. *Let* $(\lambda,\mu) \in \mathcal{J}^n_{m,l}$ *and* $T \in \mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$. *We have*
(a) $C_T(i+n) = C_T(i) - (l+m)$ *for all* $i \in \mathbb{Z}$,
(b) $C_{wT}(i) = C_T(w^{-1}(i))$ *for all* $i \in \mathbb{Z}$ *and* $w \in \dot{W}$,
(c) $(\zeta_T \mid e_i) = C_T(i)$ *for all* $i \in \mathbb{Z}$,
(d) $\bar{w}(\zeta_T) = \zeta_{wT}$ *and in particular* $(w(\zeta_T) \mid \alpha) = (\zeta_{wT} \mid \alpha)$ *for all* $\alpha \in \dot{R}$.

Proof. For (a) we calculate for $i \in \mathbb{Z}$

$$C_T(i+n) = C(T^{-1}(i+n)) = C(T^{-1}(i) + (m,-l)) = C(T^{-1}(i)) - (l+m), \tag{46}$$

which proves the claim using $C(T^{-1}(i)) = C_T(i)$. For (b) we have

$$C_{wT}(i) = C((wT)^{-1}(i)) = C(T^{-1}w^{-1}(i)) = C_T(w^{-1}(i)). \tag{47}$$

Claim (c) follows directly from the definition of $(\ \mid\ )$ in Equation (5) and the definition of ζ_T. Finally, the first part of (d) can be easily deduced from (a), (b) and the definition of the affine action from Remark 2.19. The second part then follows since $(w(\zeta) \mid \alpha) = (\bar{w}(\zeta) \mid \alpha)$ for $\alpha \in \dot{R}$. $\square$

We will first deal with the row increasing case and find out for which $w \in \dot{W}$ we have $wT_0 \in \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$, where T_0 is the ground state tableau of $\widehat{\lambda/\mu}$ from Definition 2.27. For this we associate to each $(\lambda,\mu) \in \mathcal{J}^n_{m,l}$ a parabolic subgroup $\dot{W}_{(\lambda,\mu)} \subseteq \dot{W}$.

Definition 2.34. Let $(\lambda,\mu) \in \mathcal{J}^n_{m,l}$. We define $n_i := \sum_{j=1}^{i}(\lambda_j - \mu_j)$ for $i \in \{1,...,m\}$ and set

$$I_{(\lambda,\mu)} := \{1,...,n\} \setminus \{n_1,...,n_m\}. \tag{48}$$

Furthermore, we define $\dot{W}_{(\lambda,\mu)} := \dot{W}_{I_{\lambda,\mu}}$, $\dot{R}^+_{(\lambda,\mu)} := \dot{R}^+_{I_{(\lambda,\mu)}}$ and $\dot{W}^{(\lambda,\mu)} := \dot{W}^{I_{(\lambda,\mu)}}$, where we use the notation from Definition 2.17.

The group $\dot{W}_{(\lambda,\mu)}$ is nothing but the subgroup containing the $w \in \dot{W}$ that act on T_0 by permuting the labels inside each row. Indeed, any of the generators s_i for $i \in I_{(\lambda,\mu)}$ only permutes the labels inside each row, hence this holds for any $w \in W_{(\lambda,\mu)}$. Also, any element $w \in \dot{W}$ which permutes only labels inside the rows of T_0 lies in $\dot{W}_{(\lambda,\mu)}$, as one can deduce by going through the rows $1,...,m$ and writing the permutation in each row via the generators $s_i \in \dot{W}_{(\lambda,\mu)}$.

In the following we will frequently use Theorem 2.6 in the context of the *extended* affine Weyl group. This is justified by Remark 2.16.

Proposition 2.35. *Let $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ and let T_0 be the ground state tableau on $\widehat{\lambda/\mu}$. For $w \in \dot{W}$ we have $wT_0 \in \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$ if and only if $w \in \dot{W}^{(\lambda,\mu)}$. Moreover we have $\dot{W}^{(\lambda,\mu)}T_0 = \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$.*

Proof. First let $w \in \dot{W}^{(\lambda,\mu)}$ and let $(a, b), (a, b+1) \in \widehat{\lambda/\mu}$. We have to show $wT_0(a, b) < wT_0(a, b + 1)$. Let $i := T_0(a, b)$. By definition of T_0 we have $i + 1 = T_0(a, b + 1)$ and since $(a, b), (a, b + 1) \in \widehat{\lambda/\mu}$ we have $i \in I_{(\lambda,\mu)}$. If $wT_0(a, b) > wT_0(a, b + 1)$ then $w(i) > w(i + 1)$ and hence by Lemma 2.14 $w(\alpha_i) \in \dot{R}^-$. Then by Theorem 2.6 part (a) we have $l(ws_i) < l(w)$, which contradicts $w \in \dot{W}^{(\lambda,\mu)}$ by Lemma 2.18.

For the other direction assume $T := wT_0 \in \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$ and $w \notin \dot{W}^{(\lambda,\mu)}$. Lemma 2.18 implies that we can find $\alpha_{i,j} \in \dot{R}^+_{(\lambda,\mu)}$ for some $i, j \in \mathbb{Z}$ with $i \neq j \bmod n$ and $i < j$ such that $l(ws_{\alpha_{i,j}}) < l(w)$. Now $\alpha_{i,j} \in \dot{R}^+_{(\lambda,\mu)}$ implies $s_{\alpha_{i,j}} \in \dot{W}_{(\lambda,\mu)}$ and therefore the labels i and j appear in the same row of $\widehat{\lambda/\mu}$. By setting $(a, b) := T_0^{-1}(i)$ we obtain $T_0^{-1}(j) = (a, b + j - i)$. But now $l(ws_{\alpha_{i,j}}) < l(w)$ implies $w(i) > w(j)$ by Theorem 2.6 and Lemma 2.14. Therefore we have $T(a, b) = w(i) > w(j) = T(a, b + j - i)$ contradicting $T \in \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$.

The last statement follows immediately from the simple transitivity of the action of $\dot{W}$ on $\mathrm{Tab}_\gamma(\widehat{\lambda/\mu})$ from Proposition 2.30.

□

The first step in proving an analogue for standard tableaux instead of just row increasing tableaux is the following lemma.

Lemma 2.36. *Let $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ and T_0 the ground state tableau on $\widehat{\lambda/\mu}$. Let $0 \leq i \leq n - 1$ and $w \in \dot{W}$ such that $wT_0 \in \mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})$ and $l(s_iw) < l(w)$. Then also $s_iwT_0 \in \mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})$.*

Proof. Since wT_0 is row increasing we have $w \in \dot{W}^{(\lambda,\mu)}$ by Proposition 2.35. Set $x := s_iw$. Then from Theorem 2.6 part (c) and $l(x) < l(w)$ it follows that $R(x) \subseteq \dot{R}(w) = \dot{R}(x) \cup \{x^{-1}(\alpha_i)\}$ and hence $x \in \dot{W}^{(\lambda,\mu)}$, which implies $xT_0 \in \mathrm{Tab}^R_\gamma(\widehat{\lambda/\mu})$. Assume xT_0 is not be a standard tableau, which by the above implies that it is not column increasing. Hence we can find $(a, b), (a + 1, b) \in \widehat{\lambda/\mu}$ such that $xT_0(a, b) > xT_0(a + 1, b)$. By $wT_0 \in \mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})$ we have $wT_0(a, b) < wT_0(a + 1, b)$. Since $w = s_ix$ we therefore must have $xT_0(a, b) = i + 1 + kn$ and $xT_0(a + 1, b) = i + kn$ for some $k \in \mathbb{Z}$. But we also have $x^{-1}(\alpha_i) \in \dot{R}(w) \subseteq \dot{R}^+$, hence $x^{-1}(i) < x^{-1}(i + 1)$ and the periodicity of x^{-1} implies $x^{-1}(i + kn) < x^{-1}(i + 1 + kn)$. This now gives $T_0(a, b) > T_0(a + 1, b)$, which is impossible as T_0 is column increasing. □

As an intermediate step we prove that $Z_{T_0}^{(\lambda,\mu)}$ from Definition 2.32 sends the ground state tableau T_0 on $\widehat{\lambda/\mu}$ to row increasing tableaux.

Lemma 2.37. *Let $(\lambda,\mu) \in \mathcal{J}_{m,l}^n$. Then we have*

$$Z_{T_0}^{(\lambda,\mu)} \subseteq \dot{W}^{(\lambda,\mu)}. \tag{49}$$

In other words, $w \in Z_{T_0}^{(\lambda,\mu)}$ implies that wT_0 is row increasing.

Proof. Let $w \in \dot{W} \backslash \dot{W}^{(\lambda,\mu)}$. Then by Lemma 2.18 there exists $s_j \in \dot{W}_{(\lambda,\mu)}$ for some $1 \leq j \leq n-1$ such that $l(ws_j) < l(w)$. By Theorem 2.6 part (c) it follows that $\alpha_j \in \dot{R}(w)$, since we can take a reduced expression $ws_j = \pi^k s_{i_1} ... s_{i_l}$ and by $l(ws_j) < l(w)$ we have that $\pi^k s_{i_1} ... s_{i_l} s_j$ is a reduced expression for w. Furthermore, $s_j \in \dot{W}_{(\lambda,\mu)}$ implies that the labels j and $j+1$ appear in the same row of the tableau T_0 and hence by Lemma 2.33

$$(\zeta_{T_0} \mid \alpha_j) = C_{T_0}(j) - C_{T_0}(j+1) = -1, \tag{50}$$

which shows $w \notin Z_{T_0}^{(\lambda,\mu)}$ and thus $Z_{T_0}^{(\lambda,\mu)} \subseteq \dot{W}^{(\lambda,\mu)}$. The reformulation in the end of the statement follows now directly from Proposition 2.35. $\square$

We are now able to deduce the aforementioned goal.

Theorem 2.38. *Let $(\lambda,\mu) \in \mathcal{J}_{m,l}^n$. For $T \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ and $w \in \dot{W}$ we have*

$$wT \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu}) \text{ if and only if } w \in Z_T^{(\lambda,\mu)}. \tag{51}$$

In particular, for all $T' \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ there exists a unique $w \in Z_T^{(\lambda,\mu)}$ such that $wT = T'$.

Proof. First we will deal with $T = T_0$ and deduce the general case afterwards. Let $w \in Z_{T_0}^{(\lambda,\mu)}$. We want to show $wT_0 \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ by induction on the length of w. If $l(w) = 0$ then by definition of the length function we have $w = \pi^k$ and hence the claim is obvious, since wT_0 is just the row reading tableau with k added to each label. Now assume $w = \pi^k s_{i_1} ... s_{i_l}$ with $l(w) = l > 0$ and that the claim has been proven for elements with length less than l. Let $i_0 \in \{0, ..., n-1\}$ defined by $i_0 + k = i_1 \bmod n$. We have $s_{i_0} w = \pi^k s_{i_2} ... s_{i_l}$ and $l(s_{i_0} w) = l - 1$. Setting $x := s_{i_0} w$ we obtain from Theorem 2.6 part (c) that $\dot{R}(w) = \dot{R}(x) \cup \{x^{-1}(a_{i_0})\}$. Hence we have $\dot{R}(x) \subseteq \dot{R}(w)$ and therefore $x \in Z_{T_0}^{(\lambda,\mu)}$, which by the induction hypothesis gives us $xT_0 \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$. If $wT_0 \notin \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ then by Proposition 2.35 and Lemma 2.37 wT_0 is not column increasing. So we can find $(a,b), (a+1,b) \in \widehat{\lambda/\mu}$ such that $wT_0(a,b) > wT_0(a+1,b)$. But since xT_0 is column increasing by induction hypothesis we also have $xT_0(a,b) < xT_0(a+1,b)$. As $w = s_{i_0} x$ this is only possible if $xT_0(a,b) = i_0 + rn$ and

$xT_0(a+1,b) = i_0+1+rn$ for some $r \in \mathbb{Z}$. We can calculate the following using Lemma 2.14 and also using $\alpha_{i_0+rn} = \alpha_{i_0}$ by definition of the simple roots α_i.

$$\begin{aligned}(\zeta_{T_0} \mid x^{-1}(\alpha_{i_0})) &= (\zeta_{T_0} \mid \alpha_{x^{-1}(i_0+rn)}) \\ &= C_{T_0}(x^{-1}(i_0+rn)) - C_{T_0}(x^{-1}(i_0+1+rn)) \\ &= b-a-(b-a-1) = 1.\end{aligned} \tag{52}$$

Since $x^{-1}(\alpha_{i_0}) \in \dot{R}(w)$ this contradicts $w \in Z_{T_0}^{(\lambda,\mu)}$.

For the other direction let $w \in \dot{W}$ be such that $wT_0 \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$. We argue again by induction on the length of w to prove that $w \in Z_{T_0}^{(\lambda,\mu)}$. If $l(w) = 0$ then $\dot{R}(w)$ is empty and the claim follows. Therefore assume that $l(w) = l > 0$ and that the claim has been proven for all elements with length less than l. As before we can find s_{i_0} such that for $x := s_{i_0}w$ we have $l(x) = l(w)-1$ and $\dot{R}(w) = \dot{R}(x) \cup \{x^{-1}(\alpha_{i_0})\}$. Thus we can use Lemma 2.36 and deduce $xT_0 \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ and by the induction hypothesis we have $x \in Z_{T_0}^{(\lambda,\mu)}$. By the above description of $\dot{R}(w)$ we only have to prove

$$\sigma := (\zeta_{T_0} \mid x^{-1}(\alpha_{i_0})) \neq \pm 1, \tag{53}$$

in order to obtain $w \in Z_{T_0}^{(\lambda,\mu)}$. Assume $\sigma = 1$. The case $\sigma = -1$ works analogously and is omitted. Let $T := xT_0$ and $(a,b) := T^{-1}(i_0)$. We deduce from Lemma 2.33 (c) that $T^{-1}(i_0+1) = (a+j+1, b+j)$ for some $j \in \mathbb{Z}$, since $\sigma = 1$. If $j < 0$ then by property (D3) in Definition 2.21 we have $(a,b-1) \in \widehat{\lambda/\mu}$ and then $i_0+1 = T(a+j+1,b+j) \leq T(a,b-1) < T(a,b) = i_0$ gives a contradiction to T being column increasing. If $j > 0$ we obtain $(a+1,b) \in \widehat{\lambda/\mu}$ and therefore $i_0+1 = T(a+j+1,b+j) > T(a+1,b) > T(a,b) = i_0$, which again is a contradiction. Thus we must have $j = 0$ and we calculate

$$wT_0(a,b) = s_{i_0}T(a,b) = i_0+1 > i_0 = s_{i_0}T(a+1,b) = wT_0(a+1,b), \tag{54}$$

which contradicts that wT_0 is column increasing. This and the simple transitivity of the $\dot{W}$-action from Proposition 2.30 finishes the claim for $T = T_0$.

Now let $T \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ be arbitrary. By the proof so far we can find an element $w_T \in Z_{T_0}^{(\lambda,\mu)}$ such that $w_T T_0 = T$. We want to show that the map $z \mapsto zw_T$ gives a bijection between $Z_T^{(\lambda,\mu)}$ and $Z_{T_0}^{(\lambda,\mu)}$ with inverse $z' \mapsto z'w_T^{-1}$. Once we have this, we can argue that $zT = zw_T T_0 \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ if and only if $zw_T \in Z_{T_0}^{(\lambda,\mu)}$ if and only if $z \in Z_T^{(\lambda,\mu)}$, which together with the simple transitivity from Proposition 2.30 finishes the proof.

Let us show that $zw_T \in Z_{T_0}^{(\lambda,\mu)}$ for all $z \in Z_T^{(\lambda,\mu)}$. If there exists z such that this is not the case then there exists an $\alpha \in \dot{R}(zw_T)$ such that

$(\zeta_{T_0} \mid \alpha) = \pm 1$. By definition of $\dot{R}(zw_T)$ we can find $\alpha' \in \dot{R}^-$ such that $(zw_T)^{-1}(\alpha') = \alpha$. If $\alpha \in w_T^{-1}\dot{R}^+$ then set $\beta := w_T(\alpha) \in \dot{R}^+$. We have by definition $\beta = z^{-1}(\alpha')$ and hence $\beta \in \dot{R}(z)$. Now we see using Lemma 2.15 and Lemma 2.33 (d) $(\zeta_T \mid \beta) = (w_T^{-1}\zeta_T \mid \alpha) = (\bar{w}_T^{-1}\zeta_T \mid \alpha) = (\zeta_{w_T^{-1}T} \mid \alpha) = (\zeta_{T_0} \mid \alpha) = \pm 1$ by assumption. This contradicts $z \in Z_T^{(\lambda,\mu)}$. Otherwise we have $\alpha \in w_T^{-1}\dot{R}^-$ and hence $\alpha \in \dot{R}(w_T)$. Then $(\zeta_{T_0} \mid \alpha) = \pm 1$ contradicts $w_T \in Z_{T_0}^{(\lambda,\mu)}$. Therefore we obtain $zw_T \in Z_{T_0}^{(\lambda,\mu)}$. Similarly one can show that $zw_T^{-1} \in Z_T^{(\lambda,\mu)}$ for all $z \in Z_{T_0}^{(\lambda,\mu)}$, which shows that the map described above is a bijection and hence our claim follows. $\square$

2.4 Properties of the content function

We still assume $n \geq 2$, $m \geq$, $l \geq 0$ and $\gamma = (m, -l)$. Let $(\lambda, \mu) \in \mathcal{J}_{m,l}^n$. We have seen in Theorem 2.38 that the content functions C_T for $T \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ play an important role in understanding the interplay between the extended affine Weyl group $\dot{W}$ and the standard tableaux. The content functions will continue to be of importance in the upcoming sections, when we study certain irreducible representations associated to skew diagrams. Therefore let us exhibit two properties of content functions, which in fact characterize them as we will see in Theorem 2.39. Let $\kappa := m + l$. We say that a function $F : \mathbb{Z} \to \mathbb{Z}$ has property (C1) respectively (C2) if

(C1) We have $F(i + n) = F(i) - \kappa$ for all $i \in \mathbb{Z}$.

(C2) For any $p \in \mathbb{Z}$ and $i, j \in F^{-1}(p)$ with $i < j$ and $F^{-1}(p) \cap \{i, ..., j\} = \{i, j\}$ there exists a unique $k_- \in F^{-1}(p-1)$ and a unique $k_+ \in F^{-1}(p+1)$ such that $i < k_\pm < j$.

It follows directly from the periodicity of T that C_T satisfies (C1), since $C_T(i + n) = C(T^{-1}(i + n)) = C(T^{-1}(i) + (m, -l)) = C_T(i) - (l + m) = C_T(i) - \kappa$. To prove that C_T satisfies (C2) observe that the set-up just says that i and j are two consecutive labels in the same diagonal p, say on the elements (a, b) and $(a + 1, b + 1)$. Then by property (D3) from Definition 2.21 we have that $(a + 1, b)$ and $(a, b + 1)$ lie in $\widehat{\lambda/\mu}$ and because T is a standard tableau the labels $k_\pm$ on these elements lie between i and j. The uniqueness also follows by the row and column increasing property of T, since any other label in $C_T^{-1}(p-1)$ or $C_T^{-1}(p+1)$ lies to the top left of (a, b) or to the bottom right of $(a + 1, b + 1)$.

Theorem 2.39. *Let $n \in \mathbb{Z}_{\geq 2}$, $\kappa \in \mathbb{Z}_{\geq 1}$ and $F : \mathbb{Z} \mapsto \mathbb{Z}$ be a map with properties (C1) and (C2) from above. Then there exist $m \in \{1, ..., \kappa\}$, $l = \kappa - m$, $(\lambda, \mu) \in \mathcal{J}^{*n}_{(m,l)}$ and $T \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$ such that $F = C_T$.*

Proof. Let $F : \mathbb{Z} \to \mathbb{Z}$ have properties (C1) and (C2). For p in the image of F set $d_p := |F^{-1}(p)|$. Note that property (C1) implies that for each $i \in \{1, ..., n\}$ there exits at most one $j \in F^{-1}(p)$ with $j = i \bmod n$, hence

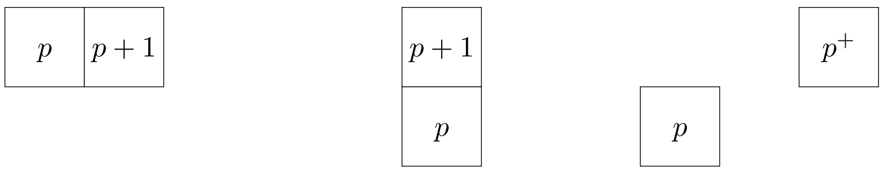
d_p is finite. For each p in the image of F we let $F^{-1}(p) = \{i_p^1, ..., i_p^{d_p}\}$ with $i_p^1 < ... < i_p^{d_p}$. Also from (C1) we get that $d_p = d_{p-\kappa}$ and $i_{p-\kappa}^j = i_p^j + n$ for $j = 1, ..., d_p = d_{p-\kappa}$.

We want to study the behaviour of $F^{-1}(p) \cup F^{-1}(p+1)$ now. For this let us order this set by the standard order $<$ on $\mathbb{Z}$. Directly from property (C2) it follows, that then the elements must alternate between $F^{-1}(p)$ and $F^{-1}(p+1)$. We get $d_{p+1} - d_p \in \{-1, 0, 1\}$ and the value -1 is only possible if $i_p^1 < i_{p+1}^1$ while the value 1 is only possible if $i_p^1 > i_{p+1}^1$. In particular, $F^{-1}(p+1) = \emptyset$ or $F^{-1}(p-1) = \emptyset$ implies that $d_p = 0$ or 1.

We want to associate for any $p_0 \in F(\mathbb{Z})$ and any $r \in \mathbb{Z}$ a skew diagram without empty rows Λ_{r,p_0} to F in order to prove $F = C_T$ for an appropriate standard tableau T on Λ_{r,p_0}. For any p in the image of F we will denote by p^+ the minimum of $F(\mathbb{Z}) \cap \mathbb{Z}_{>p}$, in other words p^+ is the next element in the image of F. Similarly let p^- denote the maximum of $F(\mathbb{Z}) \cap \mathbb{Z}_{<p}$. Note that $p^\pm$ is finite, as $F(\mathbb{Z}) \cap \mathbb{Z}_{<p}$ respectively $F(\mathbb{Z}) \cap \mathbb{Z}_{>p}$ are never empty because of (C1). We define a subset $\{(a_p^1, b_p^1) \mid p \in F(\mathbb{Z})\} \subseteq \mathbb{Z}^2$ as follows. First set $(a_{p_0}^1, b_{p_0}^1) = (r, p_0 + r)$ and define (a_p^1, b_p^1) recursively for $p \in F(\mathbb{Z}) \cap \mathbb{Z}_{>p_0}$ by setting

$$(a_{p^+}^1, b_{p^+}^1) := \begin{cases} (a_p^1, b_p^1 + 1) & \text{if } p^+ = p+1, i_p^1 < i_{p+1}^1, \\ (a_p^1 - 1, b_p^1) & \text{if } p^+ = p+1, i_p^1 > i_{p+1}^1, \\ (a_p^1 - 1, b_p^1 + p^+ - p - 1) & \text{if } p^+ > p+1. \end{cases} \tag{55}$$

One can visualize the three cases appearing in the recursion as follows.



Case 1 Case 2 Case 3

We extend this recursive relation to $F(\mathbb{Z}) \cap \mathbb{Z}_{<p_0}$ by setting

$$(a_{p^-}^1, b_{p^-}^1) := \begin{cases} (a_p^1, b_p^1 - 1) & \text{if } p^- = p-1, i_{p-1}^1 < i_p^1, \\ (a_p^1 + 1, b_p^1) & \text{if } p^- = p-1, i_{p-1}^1 > i_p^1, \\ (a_p^1 + 1, b_p^1 - p^- + p + 1) & \text{if } p^- < p-1. \end{cases} \tag{56}$$

We also set $(a_p^j, b_p^j) := (a_p^1 + j - 1, b_p^1 + j - 1)$ for all $p \in F(\mathbb{Z})$ and $2 \leq j \leq d_p$ and put

$$\Lambda_{r,p_0} := \{(a_p^j, b_p^j) \mid p \in F(\mathbb{Z}), 1 \leq j \leq d_p\} \subseteq \mathbb{Z}^2. \tag{57}$$

Note that the p-diagonal of Λ_{r,p_0} for $p \in F(\mathbb{Z})$ is $\{(a_p^1, b_p^1), ..., (a_p^{d_p}, b_p^{d_p})\}$ and empty for $p \notin F(\mathbb{Z})$. Set for $p \in F(\mathbb{Z})$

$$m_p := |\{s \in \{p, ..., p+\kappa-1\} \mid (s^+ = s+1, i_s^1 > i_{s+1}^1) \text{ or } s^+ > s+1\}| \tag{58}$$

and $l_p := \kappa - m_p$. The cases in the recursive construction of Λ_{r,p_0} appear κ-periodically: if $(a^1_{p^+}, b^1_{p^+})$ was constructed using case 1,2 or 3, then the same is true for $(a^1_{(p+\kappa)^+}, b^1_{(b+\kappa)^+})$. This follows from the κ-periodicity of the i^1_p discussed in the beginning of the proof. This periodicity also shows that $m := m_p$ and $l := l_p$ are independent of p. Using the definition of m we see that $m = a^1_p - a^1_{p+\kappa}$ for any $p \in F(\mathbb{Z})$. Thus we set $\gamma = (m, -l)$. We will now verify conditions (D1) - (D3) to see that Λ_{r,p_0} defines a γ-skew diagram.

(D1): We have to show that $(a^j_p, b^j_p) \in \Lambda_{r,p_0}$ implies $(a^j_p + m, b^j_p - l) \in \Lambda_{r,p_0}$. By construction and the periodicity of the d_p we can reduce to the claim for $j = 1$. We already know $m = a^1_p - a^1_{p+\kappa}$. We have $b^1_p - a^1_p = p$, since this holds for the initial element $(r, p_0 + r) \in \Lambda_{r,p_0}$ and is preserved under the three cases appearing in the recursive relations. Therefore $(b^1_{p+\kappa} - b^1_p) - (a^1_{p+\kappa} - a^1_p) = \kappa$ or equivalently $b^1_p - b^1_{p+\kappa} = -\kappa + m = -l$, which shows the claim.

(D2): Set $E := \{(a^j_p, b^j_p) \mid p \in \{1, ..., \kappa\} \cap F(\mathbb{Z}), j \in \{1, ..., d_p\}\}$. Since we have $(a^j_p, b^j_p) = (a^j_{p+\kappa}, b^j_{p+\kappa}) + \gamma$ for all $p \in F(\mathbb{Z})$ and $j \in \{1, ..., d_p\}$, this defines a fundamental domain for γ. Sending $(a^j_p, b^j_p) \in E$ to i^j_p gives a bijection from E to $F^{-1}(\{1, ..., \kappa\})$. Define a map $\phi : F^{-1}(\{1, ..., \kappa\}) \to \{1, ..., n\}$ by sending $z \in F^{-1}(\{1, ..., k\})$ to $z \bmod n$. By (C1) this map is injective, because $i,\ i + jn \in F^{-1}(\{1, ..., \kappa\})$ would imply that $F(i)$ and $F(i) - j\kappa$ lie in $\{1, ..., \kappa\}$ and hence $j = 0$. If ϕ was not surjective, then $F^{-1}(\{1, ..., \kappa\})$ would not contain any element with n-modulus z for some $z \in \{1, ..., n\}$. By (C1) this would also imply that $F^{-1}(\mathbb{Z}) = \mathbb{Z}$ would not contain any such element, which is absurd. Hence the fundamental domain has size n from which (D2) follows.

(D3): The description of the diagonals above shows that if we have $(a^l_p, b^l_p), (a^l_p + 1, b^l_p + 1) \in \Lambda_{r,p_0}$ then $(a^l_p + 1, b^l_p + 1) = (a^{l+1}_p, b^{l+1}_p)$. We have in that case $d_p > 1$ and hence by the consideration in the beginning $p^- = p - 1$ and $p^+ = p + 1$. Furthermore, $i^1_p > i^1_{p+1}$ implies $d_{p+1} \geq d_p$ respectively $i^1_{p-1} < i^1_p$ implies $d_{p-1} \geq d_p$. By construction of Λ_{r,p_0} we therefore obtain $(a^l_p + 1, b^l_p), (a^l_p, b^l_p - 1) \in \Lambda_{r,p_0}$. Suppose now that (D3) does not hold. Then we can find $(a^l_p, b^l_p) = (a, b) \in \Lambda_{r,p_0}$ and $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with $(i, j) \neq (0, 0), (0, 1), (1, 0)$ such that for all $0 \leq i' \leq i$ and $0 \leq j' \leq j$ we have

$$(a + i', b + j') \in \Lambda_{r,p_0} \text{ if and only if } (i', j') = (0, 0) \text{ or } (i, j). \tag{59}$$

Assume that $j - i = 0$. Then $(a + i, b + j) = (a^{l+i}_p, b^{l+i}_p)$ and only $i = j = 1$ is possible. But then the above discussion gives a contradiction.

Now assume that $j - i > 0$ and let $(a^r_s, b^r_s) = (a + i, b + j)$. By construction we have that $s - p$ and $j - i$ both give the distance in the direction of the north-east diagonal between these two points and hence $s - p = j - i > 0$. Assume $r = 1$. By construction we have for $l \geq 1$ that $a^l_p \geq a^1_p \geq a^1_s$. Since $0 \leq i = (a + i) - a = a^1_s - a^l_p$, we need both of these inequalities to be equalities. This is only possible for $l = 1$, $p, p + 1, ..., s \in F(\mathbb{Z})$ and

$i_p^1 < ... < i_s^1$. In that case $i = 0$ and we can use the elements $(a_{p+j'}^1, b_{p+j'}^1) = (a, b + j') \in \Lambda_{r,p_0}$ for $0 \leq j' \leq j$ and our assumption in (59) to obtain $j = 1$ and hence a contradiction to our choice of $(i, j) \neq (0, 1)$. If $r > 1$ then $(a_s^{r-1}, b_s^{r-1}) = (a+i-1, b+j-1) \in \Lambda_{r,p_0}$ and hence also $(a+i, b+j-1) \in \Lambda_{r,p_0}$ by the discussion at the beginning of (D3). Because we have $j > 0$, this is a contradiction to assumption (59). The case $i - j > 0$ works analogously and hence (D3) holds as well.

Assume Λ_{r,p_0} contains an empty row. By the recursive construction of the points (a_p^1, b_p^1) this is only possible if there exists p_1 such that for all $p > p_1$ with $p \in F(\mathbb{Z})$ we have $p^+ = p+1$ and $i_p^1 < i_{p+1}^1$ or if there exits a p_2 such that for all $p < p_2$ with $p \in F(\mathbb{Z})$ we have $p^- = p-1$ and $i_{p-1}^1 < i_p^1$. In both cases we get a contradiction to the periodicity property $i_{p-\kappa}^1 = i_p^1 + n$. Hence $\Lambda_{r,p_0} \in \mathcal{D}^{*n}_{m,-l}$ and we can find $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$ such that $\Lambda_{r,p_0} = \widehat{\lambda/\mu}$ by Proposition 2.24.

Finally, we define $T : \Lambda_{r,p_0} \to \mathbb{Z}$ by setting $T(a_p^j, b_p^j) = i_p^j$. Then T is a bijection, since the i_p^j are pairwise different and any $i \in \mathbb{Z} = F^{-1}(\mathbb{Z})$ equals i_p^j for appropriate j and p. We have $T^{-1}(i_p^j) = (a_p^j, b_p^j)$ and $C(a_p^j, b_p^j) = b_p^j - a_p^j = b_p^1 - a_p^1 = p$. Hence $F = C_T$. The fact that T is a tableau follows from $i_{p-\kappa}^j = i_p^j + n$ deduced in the beginning of the proof and since $(a_{p-\kappa}^j, b_{p-\kappa}^j) = (a_p^j, b_p^j) + \gamma$. Also, the fact that T is standard follows from the description of $F^{-1}(p) \cup F^{-1}(p+1)$ from the beginning of the proof: if $(a, b) = (a_p^j, b_p^j)$ and $(a, b+1)$ lie in Λ_{r,p_0} then necessarily $(a, b+1) = (a_{p+1}^k, b_{p+1}^k)$ with either $k = j$ and $i_p^1 < i_{p+1}^1$ or $k = j+1$ and $i_p^1 > i_{p+1}^1$. Since the elements in $F^{-1}(p) \cup F^{-1}(p+1)$ alternate between $F^{-1}(p)$ and $F^{-1}(p+1)$ we get in both cases $T(a, b) < T(a, b+1)$. Analogously one shows that T is also column increasing, which finishes the proof. □

Let us deduce when two content functions are identical. We will need the automorphisms $\omega_{m,l} : \mathbb{Z}^m \to \mathbb{Z}^m$ for $m \geq 1$ and $l \geq 0$ defined on $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{Z}^m$ via

$$\omega_{m,l}\lambda := (\lambda_m + l + 1, \lambda_1 + 1, ..., \lambda_{m-1} + 1). \tag{60}$$

Note that under the diagonal action $\omega_{m,l}$ preserves $\mathcal{J}^n_{m,l}$ and $\mathcal{J}^{*n}_{m,l}$. One can visualize the action as cutting off the bottom 'row' of $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ and putting it on top shifted by l to the right, while simultaneously shifting the whole partition by 1 to the right. One can see that this corresponds to sending $\widehat{\lambda/\mu}$ to $\widehat{\lambda/\mu} + (1, 1)$ under Φ from Proposition 2.24.

Proposition 2.40. *Let $m, m' \in \{1, ..., n\}$ and $l, l' \geq 0$. For $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$ and $(\eta, \nu) \in \mathcal{J}^{*n}_{m',l'}$ the following are equivalent:*

(a) $C_T^{\widehat{\lambda/\mu}} = C_S^{\widehat{\eta/\nu}}$ for some $T \in \mathrm{Tab}^{RC}_{(m,-l)}(\widehat{\lambda/\mu})$ and $S \in \mathrm{Tab}^{RC}_{(m',-l')}(\widehat{\eta/\nu})$,

(b) $m = m', l = l'$ and $\widehat{\lambda/\mu} = \widehat{\eta/\nu} + (r, r)$ for some $r \in \mathbb{Z}$,

(c) $m = m', l = l'$ and $(\eta, \nu) = \omega_{m,l}^r(\lambda, \mu)$ for some $r \in \mathbb{Z}$.

Proof. Concluding (b) $\Rightarrow$ (a) is trivial. To prove (a) $\Rightarrow$ (b) choose any $p_0 \in C_T^{\widehat{\lambda/\mu}}(\mathbb{Z})$. We now note some facts about $\widehat{\lambda/\mu}$ and compare them with the construction of Λ_{r,p_0} from Theorem 2.39 to prove that these skew diagrams coincide up to translation on the (x,x)-diagonal. We put $d_p = |C_T^{-1}(p)|$ for all $p \in C_T(\mathbb{Z})$ and let $C_T^{-1}(p) = \{i_p^1, ..., i_p^{d_p}\}$ with $i_p^1 < ... < i_p^{d_p}$. We also set $(a_p^j, b_p^j) := T^{-1}(i_p^j)$. We then have $(a_p^j, b_p^j) = (a_p^1, b_p^1) + (j-1, j-1)$ for all $j \in \{1, ..., d_p\}$, since these elements have the same diagonal value and by definition of i_p^j. We check that the (a_p^1, b_p^1) satisfy the recursion relation (55): if $p^+ = p+1$ and $i_p^1 < i_{p+1}^1$ then the $(p+1)$-th diagonal of $\widehat{\lambda/\mu}$ is not empty and $T^{-1}(i_{p+1}^1)$ must be the top left entry of this diagonal. Since $i_p^1 < i_{p+1}^1$, the point $T^{-1}(i_{p+1}^1)$ must be lower and strictly right of $T^{-1}(i_p^1)$. Then by (D3) from Definition 2.21 we have $T^{-1}(i_p^1) + (0,1) \in \widehat{\lambda/\mu}$, which must therefore equal $T^{-1}(i_{p+1}^1)$. The second case works similar. For the third case let $T^{-1}(i_{p^+}^1) = (a,b)$. This is the top left entry of the p^+-th diagonal. We just have to show that $a = a_p^1 - 1$. If $a > a_p^1 - 1$ then we have $b \geq b_p^1$ and we can use (D3) for the elements $(a_p^1, b_p^1), (a,b)$ to obtain a contradiction to $p^+ > p+1$. If $a < a_p^1 - 1$ we can use that $\widehat{\lambda/\mu}$ has no empty rows and find an element $(a_p^1 - 1, b')$ in the (a_p^1-1)-th row. If $b' \leq b_p^1$ we can use (D3) for $(a_p^1 - 1, b'), (a_p^1, b_p^1)$ to obtain a contradiction to the minimal choice of $p^+ > p$. If $b_p^1 < b' < b$ we directly obtain a contradiction to the minimal choice of $p^+ > p$,. If $b_p^1 < b' \geq b$ we can use (D3) for $(a,b), (a_p^1 - 1, b')$ to again obtain a contradiction to the minimal choice of $p^+ > p$. This shows that the relations from (55) are satisfied. Furthermore, $T^{-1}(i_p^1) - T^{-1}(i_{p-\kappa}^1) = \gamma$, which matches the construction of γ for Λ_{r,p_0}. If we choose r now so that $(a_{p_0}^1, b_{p_0}^1) = (r, p_0 + r)$, which is possible because $(r, p_0 + r)$ and $(a_{p_0}^1, b_{p_0}^1)$ both have p_0 as the diagonal value, we obtain that $\widehat{\lambda/\mu} = \Lambda_{r,p_0}$. Doing the same for C_S shows that (a) implies (b).

For (b) $\Leftrightarrow$ (c) by the bijection from Proposition 2.24 we only need to show $(\omega_{m,l}\lambda/\omega_{m,l}\mu)^\wedge = \widehat{\lambda/\mu} + (1,1)$. This follows from putting in the definitions. $\square$

3 DAHA for GL_n

3.1 DAHA and $\mathcal{X}$-semsimiple irreducible modules

We will now give the definition of the double affine Hecke algebra of GL_n. The definition goes back to Ivan Cherednik, who introduced double affine Hecke algebras in order to study (quantum) Knizhnik-Zamolodchikov equations [Che05, Chapter 0.2]. For the rest of this chapter we will work over the field $\mathbb{K} := \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, where $t = q^k$ for some fixed $k \in \mathbb{C} \setminus \{0\}$ and $q \neq 0$. The element q can be transcendental over $\mathbb{C}$ or it can lie in $\mathbb{C}$ from which

$\mathbb{K} = \mathbb{C}$ follows. We fix again $n \geq 2$ and note that the discussion in this chapter is based on [SV05].

Definition 3.1. The *double affine Hecke algebra* of GL_n (or short DAHA) denoted by $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is the unital, associative $\mathbb{K}$-algebra generated by the elements $T_0, ..., T_{n-1}, \pi^{\pm 1}, X_1^{\pm 1}, ..., X_n^{\pm 1}$ subject to the relations:

- **(T1)** $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}})$ for $0 \leq i \leq n-1$,
- **(T2)** $T_iT_jT_i = T_jT_iT_j$ for $i = j \pm 1 \bmod n$,
- **(T3)** $T_iT_j = T_jT_i$ for $i \neq j \pm 1 \bmod n$,
- **(PT)** $\pi T_i \pi^{-1} = T_j$ for $j = i+1 \bmod n$,
- **(X)** $X_iX_j = X_jX_i$ for $1 \leq i, j \leq n$,
- **(XT1)** $T_iX_iT_i = X_{i+1}$ for $1 \leq i \leq n-1$ and $T_0X_nT_0 = q^{-1}X_1$,
- **(XT2)** $T_iX_j = X_jT_i$ for $j \neq i, i+1 \bmod n$,
- **(PX)** $\pi X_i \pi^{-1} = X_{i+1}$ for $1 \leq i \leq n-1$ and $\pi X_n \pi^{-1} = q^{-1}X_1$.

Remark 3.2. For any $v = \sum_{i=1}^n v_ie_i + v_d\delta \in \tilde{\mathfrak{h}}^*$ with $v_i, v_d \in \mathbb{Z}$ we define $X^v := X_1^{v_1}...X_n^{v_n}q^{v_d}$. Also, for any $w = \pi^k w'$ with $w' \in \dot{W}_a$ choose a reduced expression $w' = s_{i_1}...s_{i_m}$ and define $T_w := \pi^k T_{i_1}...T_{i_m}$. Note that the k in the decomposition of w does not depend on any choices. To show that T_w also does not depend on the choice of the reduced expression for $w' \in \dot{W}_a$ it is enough to see that any two reduced expressions for w' are related via braid relations, which are the relations in the affine Weyl group corresponding to (T2) and (T3) above. This fact is proven in [IM65, Proposition 1.15] for affine Weyl groups.

Proposition 3.3. *Any element $h \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ can be written as*

$$h = \sum_{w \in \dot{W}} T_w F_w, \tag{61}$$

for some $F_w \in \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$.

Proof. Using the relations it is easy to see that we can arrange the π to the left and the X_i to the right in any monomial in the generators. Hence the claim reduces to showing that any monomial in the T_i is a sum of T_w. Look at a monomial $T_{i_1}...T_{i_k}$ and set $w = s_{i_1}...s_{i_k}$. By induction on k we can assume that $w' = s_{i_1}...s_{i_{k-1}}$ is reduced. If $s_{i_1}...s_{i_k}$ is also reduced we are done. Otherwise, we have by the strong exchange condition in Theorem 2.6 that $w = w's_{i_k} = s_{i_1}...\hat{s}_{i_j}...s_{i_{k-1}}$ for some $1 \leq j \leq k-1$. Hence $w' = s_{i_1}...\hat{s}_{i_j}...s_{i_{k-1}}s_{i_k}$ is a reduced expression. By Remark 3.2 we know $T_{i_1}...T_{i_{k-1}} = T_{i_1}...\hat{T}_{i_j}...T_{i_k}$. Multiplying by T_{i_k} from the right finishes the proof by relation (T1) and the induction hypothesis for $k-1$ and $k-2$. □

By Remark 5.5 the elements $\{T_w X^v \mid w \in \dot{W}, v \in P\}$ in fact form a $\mathbb{K}$-basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$.

Definition 3.4. An $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module M is called *$\mathcal{X}$-semisimple* if it admits a $\mathbb{K}$-vector space decomposition $M = \bigoplus_{\zeta \in \dot{P}_\kappa} M_\zeta$, where $\kappa := \frac{1}{k}$ for k such that $t = q^k$,

$$\dot{P}_\kappa := \{\zeta_1 e_1 + ... + \zeta_n e_n + \kappa c* \mid \zeta_i \in \mathbb{Z} \text{ for } 1 \leq i \leq n\}, \tag{62}$$

and M_ζ is the *weight space for* ζ, in other words

$$M_\zeta := \{m \in M \mid (X_i - t^{(\zeta|e_i)})m = 0 \text{ for all } 1 \leq i \leq n\}. \tag{63}$$

If $M_\zeta \neq 0$ we say that $\zeta \in \dot{P}_\kappa$ is a weight of M. We also define the *generalized weight space of* $\zeta \in \dot{P}_\kappa$ to be

$$M_\zeta^\infty := \bigcup_{k \geq 0} \{m \in M \mid (X_i - t^{(\zeta|e_i)})^k m = 0 \text{ for all } 1 \leq i \leq n\}. \tag{64}$$

Note that $(c^* \mid e_i) = 0$ for $1 \leq i \leq n$ and hence the choice of $\dot{P}_\kappa$ instead of P seems to not make a difference now. But in the theory described below it will become necessary to index the weights over $\dot{P}_\kappa$ and not over P.

The goal of this chapter is to classify the irreducible $\mathcal{X}$-semisimple modules of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. For this the *intertwining operators* will play an essential role, because they allow us to move between the weight spaces, as described later in Propositions 3.8 and 3.9.

Definition 3.5. We define the *intertwining operators* $\phi_i \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ by

$$\phi_i := T_i(1 - X^{\alpha_i}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \text{ for } 0 \leq i \leq n-1 \tag{65}$$

Lemma 3.6. *The following equations hold in $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$:*
(a) $\phi_i^2 = t^{-1}(1 - tX^{\alpha_i})(1 - tX^{-\alpha_i})$ for $0 \leq i \leq n-1$,
(b) $\phi_i\phi_j = \phi_j\phi_i$ for $0 \leq i, j \leq n-1$, $i \neq j \pm 1 \bmod n$,
(c) $\phi_i\phi_j\phi_i = \phi_j\phi_i\phi_j$ for $0 \leq i \leq n-1$ and $j = i \pm 1 \bmod n$.

Proof. The following identities for $0 \leq i \leq n-1$ follow from (T1) and turn out to be useful during the proof:

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad T_i^{-1} = T_i + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}). \tag{66}$$

(a) and (b) can be easily verified using the relations of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. We omit these calculations. The computation for (c) is more involved. We will go through it in the case $i \neq 0, n-1$ and $j = i+1$. The other cases work in the same way, but pose a notational inconvenience, because of the additional q

coming from (XT1) and α_0. Multiplying $\phi_i\phi_j\phi_i$ respectively $\phi_j\phi_i\phi_j$ gives us 8 terms for each of them:

$$\begin{aligned}
\phi_i\phi_j\phi_i = {} & T_i(1-X^{\alpha_i})T_j(1-X^{\alpha_j})T_i(1-X^{\alpha_i}) && (A) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_i(1-X^{\alpha_i})T_j(1-X^{\alpha_j}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_j(1-X^{\alpha_j})T_i(1-X^{\alpha_i}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_i(1-X^{\alpha_i})T_i(1-X^{\alpha_i}) && (B) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_i(1-X^{\alpha_i}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_j(1-X^{\alpha_j}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_i(1-X^{\alpha_i}) && (C) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^3,
\end{aligned} \tag{67}$$

$$\begin{aligned}
\phi_j\phi_i\phi_j = {} & T_j(1-X^{\alpha_j})T_i(1-X^{\alpha_i})T_j(1-X^{\alpha_j}) && (A') \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_i(1-X^{\alpha_i})T_j(1-X^{\alpha_j}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_j(1-X^{\alpha_j})T_i(1-X^{\alpha_i}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})T_j(1-X^{\alpha_j})T_j(1-X^{\alpha_j}) && (B') \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_i(1-X^{\alpha_i}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_j(1-X^{\alpha_j}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_j(1-X^{\alpha_j}) && (C') \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^3.
\end{aligned} \tag{68}$$

Cancelling leaves us with showing that $(L) = (R)$, where we set $(L) := (A) + (B) + (C)$ and $(R) := (A') + (B') + (C')$. From the second part of Equation (66) we see

$$\begin{aligned}
(L) &= T_i(1-X^{\alpha_i})(T_j^{-1} - T_jX^{\alpha_j})T_i(1-X^{\alpha_i}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_i(1-X^{\alpha_i}), \\
(R) &= T_j(1-X^{\alpha_j})(T_i^{-1} - T_iX^{\alpha_i})T_j(1-X^{\alpha_j}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_j(1-X^{\alpha_j}).
\end{aligned}$$

By multiplication this leads to:

$$\begin{aligned}
(L) = {} & (T_iT_j^{-1}T_i - T_iT_jX^{\alpha_j}T_i - T_iX^{\alpha_i}T_j^{-1}T_i + T_iX^{\alpha_i}T_jX^{\alpha_j}T_i)(1-X^{\alpha_i}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_i(1-X^{\alpha_i}), \\
(R) = {} & (T_jT_i^{-1}T_j - T_jT_iX^{\alpha_i}T_j - T_jX^{\alpha_j}T_i^{-1}T_j + T_jX^{\alpha_j}T_iX^{\alpha_i}T_j)(1-X^{\alpha_j}) \\
& + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 T_j(1-X^{\alpha_j}).
\end{aligned}$$

To bring (L) into a form where the T_i and T_j appear only in the beginning we need the three equalities below. We omit all but the first calculation, because they are very similar in nature.

$$
\begin{aligned}
X^{\alpha_{i+1}}T_i = X_{i+1}X_{i+2}^{-1}T_i = X_{i+1}T_iX_{i+2}^{-1} = X_{i+1}(T_i^{-1} + (t^{\frac{1}{1}} - t^{-\frac{1}{2}}))X_{i+2}^{-1} \\
= T_iX_iX_{i+2}^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1}X_{i+2}^{-1},
\end{aligned}
\tag{69}
$$

$$
X^{\alpha_i}T_{i+1}^{-1}T_i = T_{i+1}T_iX_{i+1}X_{i+2}^{-1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{i+1}X_{i+1}X_{i+2}^{-1}, \tag{70}
$$

$$
\begin{aligned}
X^{\alpha_i}T_{i+1}X^{\alpha_{i+1}}T_i = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_iX_{i+1}X_{i+2}^{-1} \\
- (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2X_{i+1}X_{i+2}^{-1} + T_{i+1}T_iX_iX_{i+1}X_{i+2}^{-2}.
\end{aligned}
\tag{71}
$$

This gives us:

$$
\begin{aligned}
(L) =& \Big(T_iT_{i+1}T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\
& - T_iT_{i+1}T_iX_iX_{i+2}^{-1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_iT_{i+1}X_{i+1}X_{i+2}^{-1} \\
& - T_iT_{i+1}T_iX_{i+1}X_{i+2}^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_iT_{i+1}X_{i+1}X_{i+2}^{-1} \\
& + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i^2X_{i+1}X_{i+2}^{-1} \\
& - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2T_iX_{i+1}X_{i+2}^{-1} + T_iT_{i+1}T_iX_iX_{i+1}X_{i+2}^{-2}\Big)(1 - X_iX_{i+1}^{-1}) \\
=& \Big(T_iT_{i+1}T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\
& - T_iT_{i+1}T_iX_iX_{i+2}^{-1} \\
& - T_iT_{i+1}T_iX_{i+1}X_{i+2}^{-1} \\
& + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1}X_{i+2}^{-1} \\
& + T_iT_{i+1}T_iX_iX_{i+1}X_{i+2}^{-2}\Big)(1 - X_iX_{i+1}^{-1}).
\end{aligned}
\tag{72}
$$

Analogously we use for (R):

$$
X^{\alpha_i}T_{i+1} = T_{i+1}X_iX_{i+2}^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_iX_{i+1}^{-1}, \tag{73}
$$

$$
X^{\alpha_{i+1}}T_i^{-1}T_{i+1} = T_iT_{i+1}X_iX_{i+1}^{-1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_iX_iX_{i+1}^{-1}, \tag{74}
$$

$$
\begin{aligned}
X^{\alpha_{i+1}}T_iX^{\alpha_i}T_{i+1} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{i+1}X_iX_{i+1}^{-1} \\
- (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2X_iX_{i+1}^{-1} + T_iT_{i+1}X_i^2X_{i+1}^{-1}X_{i+2}^{-1}.
\end{aligned}
\tag{75}
$$

We deduce:

$$
\begin{aligned}
(R) =&\Big(T_{i+1}T_iT_{i+1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\
&- T_{i+1}T_iT_{i+1}X_iX_{i+2}^{-1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{i+1}T_iX_iX_{i+1}^{-1} \\
&- T_{i+1}T_iT_{i+1}X_iX_{i+1}^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{i+1}T_iX_iX_{i+1}^{-1} \\
&+ (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{i+1}^2X_iX_{i+1}^{-1} \\
&- (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2T_{i+1}X_iX_{i+1}^{-1} + T_{i+1}T_iT_{i+1}X_i^2X_{i+1}^{-1}X_{i+2}^{-1}\Big)(1 - X_{i+1}X_{i+2}^{-1}) \\
=&\Big(T_{i+1}T_iT_{i+1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \qquad (76) \\
&- T_{i+1}T_iT_{i+1}X_iX_{i+2}^{-1} \\
&- T_{i+1}T_iT_{i+1}X_iX_{i+1}^{-1} \\
&+ (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_iX_{i+1}^{-1} \\
&+ T_{i+1}T_iT_{i+1}X_i^2X_{i+1}^{-1}X_{i+2}^{-1}\Big)(1 - X_{i+1}X_{i+2}^{-1}).
\end{aligned}
$$

Comparing the expressions in (72) and (76) using $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ shows that $(L) = (R)$ and hence $\phi_i\phi_j\phi_i = \phi_j\phi_i\phi_j$. □

The braid relations in Proposition 3.6 imply by [IM65, Proposition 1.15], as before for T_w in Remark 3.2, that we can associate to any $w \in \dot{W}$ with reduced expression $w = \pi^k s_{i_1}...s_{i_k}$ a well-defined element $\phi_w := \pi^k\phi_{i_1}...\phi_{i_k} \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ independent of the choice of reduced expression. Also note that $\pi\phi_i\pi^{-1} = \phi_{\pi(i)}$. These operators have the following properties.

Lemma 3.7. *Let $w \in \dot{W}$ and $v = \sum_{i=1}^n v_ie_i + v_d\delta \in \tilde{\mathfrak{h}}^*$ with $v_i, v_d \in \mathbb{Z}$.*

(a) $\phi_wX^v = X^{w(v)}\phi_w$,

(b) $\phi_w = T_w\prod_{\alpha\in\dot{R}(w)}(1 - X^\alpha) + \sum_{y\in\dot{W},y\prec w} T_yF_y$, where the sum ranges over all reduced subexpressions of some reduced expression for w and we have $F_y \in \mathbb{K}[X_1^{\pm1}, ..., X_n^{\pm1}]$.

Proof. For (a) we can easily reduce by induction to the case $w = s_i$ for some $0 \le i \le n-1$ and $v = \pm e_j$ for some $1 \le j \le n$. Note that for $\phi_\pi = \pi$ the claim follows from (PX). For simplicity let us assume $j \notin \{1, n\}$ and $i \ne 0$. The calculations for the other cases only involve some additional q. We have

$$\phi_iX_j^{\pm1} = T_i(1 - X_iX_{i+1}^{-1})X_j^{\pm1} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})X_j^{\pm1} \qquad (77)$$

If $j \ne i, i+1$ we can just pull $X_j^{\pm1}$ to the left and we are done. If $j = i$ and the exponent of X_i is positive we have

$$\begin{aligned}\phi_i X_i &= X_{i+1}(T_i + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}))(1 - X_i X_{i+1}^{-1}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})X_i \\ &= X_{i+1}T_i(1 - X_i X_{i+1}^{-1}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(X_{i+1} - X_i + X_i) \\ &= X_{s_i(i)}\phi_i.\end{aligned} \tag{78}$$

The remaining cases for $j = i + 1$ or for X_i^{-1} work similar and are omitted.

Statement (b) is proven via induction on $l(w)$. We will not give an explicit proof, but only remark that it uses (a) and Theorem 2.6 (c) in the computation. □

3.2 Classification of $\mathcal{X}$-semisimple modules

In this section we will give a classification of $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules using the combinatorial tools developed in Section 2. Let again $n \geq 2$. We assume from now on that $t = q^k$ for $k = \frac{1}{\kappa}$ and $\kappa \in \mathbb{Z}_{>0}$. Furthermore, we assume that q, hence also t, is not a root of unity. In Remark 3.20 we will see how to extend the results to $\kappa \in \mathbb{Z}_{<0}$. Let $(\lambda, \mu) \in \mathcal{J}_{m,l}^n$ be a nested partition as in Definition 2.23 with $m > 0$ and $l \geq 0$ such that $\kappa = m + l$. We will associate to (λ, μ) an irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $V(\lambda, \mu)$ in Theorem 3.15, whose basis is given by the standard tableaux on $\widehat{\lambda/\mu}$ from Definition 2.26. We will show that any irreducible and $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module is isomorphic to $V(\lambda, \mu)$ for an appropriate choice of $(\lambda, \mu) \in {\mathcal{J}^*}_{m,l}^n$ in Theorem 3.18. Finally, we will describe the isomorphism classes of the modules $V(\lambda, \mu)$ in Theorem 3.19, which finishes the classification.

Let us start by looking at the intertwining operators ϕ_w from Definition 3.5 again to obtain a first hint of the connection to the combinatorics of skew diagrams. For this recall the definition of weights from Definition 3.4 and the definition of the affine action of $\dot{W}$ from Remark 2.19.

Proposition 3.8. *Let M be an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module, $v \in M_\zeta$ for some $\zeta \in \dot{P}_\kappa$ and $w \in \dot{W}$. We have*

(a) $\phi_w M_\zeta \subseteq M_{\bar{w}(\zeta)}$ and $\phi_w M_\zeta^\infty \subseteq M_{\bar{w}(\zeta)}^\infty$,

(b) $\phi_{w^{-1}}\phi_w v = \prod_{\alpha \in \dot{R}(w)} t^{-1}(1 - t^{1+(\zeta|\alpha)})(1 - t^{1-(\zeta|\alpha)})v$.

Proof. Statement (a) follows from Lemma 3.7 (a) and since $(\bar{w}(\zeta) \mid v) = (w(\zeta) \mid v)$ for any $v = \sum_{i=1}^n v_i e_i + v_d \delta$. To prove (b) one can argue by induction on $l(w)$. Using Lemma 3.6 (a) together with the fact that $v \in M_\zeta$ and hence $X^{\pm\alpha}(v) = t^{\pm(\zeta|\alpha)}v$ gives the claim immediately for $l(w) \in \{0, 1\}$. For $l(w) > 1$ we can write $\phi_w = \phi_{s_i}\phi_{w'}$ for some w' with $l(w') = l(w) - 1$ and $0 \leq i \leq n - 1$. The claim then follows by statement (a) and Lemma 3.7 part (a) together with the explicit description of $\dot{R}(w)$ from Theorem 2.6. □

From (b) we can deduce in the following proposition a sufficient condition for $\phi_w : M_\zeta \to M_{\bar{w}(\zeta)}$ to be an isomorphism. For $\zeta \in \dot{P}_\kappa$ set

$$Z_\zeta := \{w \in \dot{W} \mid (\zeta \mid \alpha) \neq \pm 1 \text{ for all } \alpha \in \dot{R}(w)\}. \tag{79}$$

For T a γ-tableau on some γ-skew diagram $\widehat{\lambda/\mu}$ we have $Z_{\zeta_T} = Z_T^{(\lambda,\mu)}$, where ζ_T and $Z_T^{(\lambda,\mu)}$ are described in Definition 2.32. This aligns very nicely with the theory developed in Section 2.3 and in fact can be seen as a central reason why the combinatorics of periodic skew diagrams are applicable to the representation theory of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$.

Proposition 3.9. *Let M be an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module, $\zeta \in \dot{P}_\kappa$ and $w \in Z_\zeta$. Then $\phi_w : M_\zeta \to M_{\bar{w}(\zeta)}$ is a linear isomorphism.*

Proof. The claim directly follows from Proposition 3.8, since $w \in Z_\zeta$ implies $\prod_{\alpha \in \dot{R}(w)} t^{-1}(1-t^{1+(\zeta|\alpha)})(1-t^{1-(\zeta|\alpha)}) \neq 0$, because t is not a root of unity. $\square$

Lemma 3.10. *Let M be an $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. If $(\zeta \mid \alpha_i) = 0$ for some $0 \leq i \leq n-1$, we have $M_\zeta = 0$.*

Proof. Let $v \in M_\zeta \backslash \{0\}$. We obtain via a simple computation $(X^{\alpha_i} - 1)T_i v = 2(t^{-\frac{1}{2}} - t^{\frac{1}{2}})v \neq 0$ as $t \neq 1$. From this we conclude $(X^{\alpha_i} - 1)^2 T_i v = 0$ and hence $T_i v$ is a generalized X^{α_i}-eigenvector, which is not a proper eigenvector. Write

$$T_i v = \sum_{j \in I} c_j v_j \text{ with } v_j \in M_{\zeta_j}, c_j \in \mathbb{K}, \tag{80}$$

where the ζ_j are pairwise different weights of M. This is possible since M is $\mathcal{X}$-semisimple. Now $(X^{\alpha_i} - 1)^2(T_i v) = 0$ implies $(t^{(\zeta_j|\alpha_i)} - 1)^2 = 0$ for all $j \in I$ with $c_j \neq 0$ and hence $(t^{(\zeta_j|\alpha_i)} - 1) = 0$, which contradicts $(X^{\alpha_i} - 1)T_i v \neq 0$. Therefore we have $M_\zeta = 0$.

$\square$

The following proposition shows that the intertwining operators can be used to construct a $\mathbb{K}$-linear spanning set of any irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Hence, the structure of any such module essentially only depends on the action of the intertwining operators on it, which will turn out to be an important idea for the proof of the classification. As we will see later we can not expect the constructed spanning set to be a basis in general.

Proposition 3.11. *Let M be an irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module and $v \in M$ a non-zero weight vector. Then $M = \sum_{w \in \dot{W}} \mathbb{K}\phi_w v$.*

Proof. Let $\zeta \in P_\kappa$ be the weight of v. Set $N := \sum_{w\in\dot{W}} \mathbb{K}\phi_w v \subseteq M$. Since v is a non-zero weight vector and since M is irreducible, we only have to show that $T_w v \in N$ for all $w \in \dot{W}$ by Proposition 3.3. We will do this by induction on $l = l(w)$. Since $\phi_{\pi^k} = \pi^k$ the induction start is trivial. Let now $l > 0$ and $w = \pi^k s_{i_1}...s_{i_l}$ be a reduced expression. Using part (b) of Lemma 3.7 and using that v is a weight vector of weight ζ we have

$$\phi_w v = T_w \prod_{\alpha\in\dot{R}(w)} (1 - t^{(\zeta|\alpha)})v + \sum_{y\in\dot{W}, y\prec w} f_y T_y v \text{ for some } f_y \in \mathbb{K}. \tag{81}$$

If $\prod_{\alpha\in\dot{R}(w)}(1 - t^{(\zeta|\alpha)}) \neq 0$ we see that

$$T_w v = \prod_{\alpha\in\dot{R}(w)} (1 - t^{(\zeta|\alpha)})^{-1} \left(\sum_{y\in\dot{W}, y\prec w} T_y f_y v - \phi_w v \right). \tag{82}$$

Since all the y appearing in the sum are subexpressions of w and hence have shorter length, this proves the claim by induction. Otherwise we can find $1 \leq p < l$ such that for the subword $y := s_{i_{p+1}}...s_{i_l}$ of $w = \pi^k s_{i_1}...s_{i_p}...s_{i_l}$ we have

$$\prod_{\alpha\in\dot{R}(y)} (1 - t^{(\zeta|\alpha)}) \neq 0, \qquad \prod_{\alpha\in\dot{R}(s_{i_p}y)} (1 - t^{(\zeta|\alpha)}) = 0. \tag{83}$$

By Theorem 2.6 (c) we have $\dot{R}(s_{i_p}y) \setminus \dot{R}(y) = \{y^{-1}(\alpha_{i_p})\}$. This shows $(\zeta \mid y^{-1}(\alpha_{i_p})) = (\bar{y}(\zeta) \mid \alpha_{i_p}) = 0$. By Lemma 3.10 we have $M_{\bar{y}(\zeta)} = 0$ and hence $0 = \phi_y v \in M_{\bar{y}(\zeta)}$. As before we can write

$$0 = \phi_y v = T_y \prod_{\alpha\in\dot{R}(y)} (1 - t^{(\zeta|\alpha)})v + \sum_{x\in\dot{W}, x\prec y} g_x T_x v \text{ with } g_x \in \mathbb{K}. \tag{84}$$

Multiplying by $\pi^k T_{i_1}...T_{i_p}$ from the left gives

$$\prod_{\alpha\in\dot{R}(y)} (1 - t^{(\zeta|\alpha)}) T_w v = - \sum_{x\in\dot{W}, x\prec y} g_x \pi^k T_{i_1}...T_{i_p} T_x v, \tag{85}$$

where the coefficient on the left hand side is not zero by choice of y. Note that the Weyl group elements $s_{i_1}...s_{i_p}x$ corresponding to the terms on the right hand side have length less than l. The inductive construction in Proposition 3.3 now shows that we can write the $T_{i_1}...T_{i_p}T_x$ as a sum of $T_{w'}$ with $l(w') < l$. This implies $T_w v \in N$ and thus finishes the proof. □

Let $\zeta \in \dot{P}_\kappa$ and recall the definition of the stabilizer with respect to the affine action from Remark 2.19: $\dot{W}[\zeta] = \{w \in \dot{W} \mid \bar{w}(\zeta) = \zeta\}$.

Lemma 3.12. *Let M be an $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Let $v \in M_\zeta$ for some $\zeta \in \dot{P}_\kappa$ and $w \in \dot{W}[\zeta] \setminus \{1\}$. Then $\phi_w v = 0$.*

Proof. Let $w = \pi^k s_{i_1} ... s_{i_m}$ be a reduced expression. By Lemma 2.20 we can find $\alpha \in \dot{R}(w) \cap \dot{R}[\zeta]$, since $\kappa \neq 0$. We then have $s_\alpha \in \dot{W}[\zeta]$ and hence $ws_\alpha = \pi^k s_{i_1} ... \hat{s}_{i_j} ... s_{i_m} \in \dot{W}[\zeta]$. Here we used Theorem 2.6 (b) and (d) to rewrite ws_α. Set $y := s_{i_{j+1}} ... s_{i_m}$. Then by Theorem 2.6 (d) we have $\alpha = y^{-1}(\alpha_{i_j})$ and hence $0 = (\zeta \mid \alpha) = (\bar{y}(\zeta) \mid \alpha_{i_p})$. Therefore by Lemma 3.10 we have $M_{\bar{y}(\zeta)} = 0$ and $\phi_w v = \phi_{\pi^k s_{i_1} ... s_{i_j}} \phi_y v = 0$. □

Proposition 3.13. *Let M be an irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Then $\dim(M_\zeta) \leq 1$ for all $\zeta \in \dot{P}_\kappa$.*

Proof. This follows directly from Proposition 3.11 and Lemma 3.12. □

Lemma 3.14. *Let M be an irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Let $\zeta \in \dot{P}_\kappa$ such that $(\zeta \mid \alpha_i) = \pm 1$ for some $0 \leq i \leq n-1$. Then $\phi_i v = 0$ for all $v \in M_\zeta$.*

Proof. Suppose $\phi_i v \neq 0$. Let $\dot{W}^i := \{w \in \dot{W} \mid ws_i \in \dot{W}[\zeta]\}$. We have $v = \sum_{w \in \dot{W}^i} a_w \phi_w \phi_i v$ for some constants $a_w \in \mathbb{K}$ by Proposition 3.11 and since $\phi_i v$ is a weight vector by Proposition 3.8. If for some w in the sum we have $l(ws_i) < l(w)$, we can find by the strong exchange condition from Theorem 2.6 a reduced word for w ending in s_i and hence $\phi_w = \phi_{ws_i} \phi_i$. But then $\phi_w \phi_i v = \phi_{ws_i} \phi_i^2 v = t^{-1}(1 - t^{1+(\zeta|\alpha_i)})(1 - t^{1-(\zeta|\alpha_i)}) \phi_{ws_i} v$ by Lemma 3.6 and hence $\phi_w \phi_i v = 0$ by the assumption on i. If $l(ws_i) > l(w)$ for some w appearing in the sum, we have $\phi_w \phi_i v = \phi_{ws_i} v = 0$ by Lemma 3.12 and since we have that $ws_i \in \dot{W}[\zeta] \setminus \{1\}$. This shows that $v = 0$ and hence $\phi_i v = 0$, which is a contradiction. □

We will now associate to each $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$ with $m + l = \kappa$, $m > 0$ and $l \geq 0$ an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module structure on the $\mathbb{K}$-vector space spanned by the standard tableaux on the associated skew diagram $\widehat{\lambda/\mu}$. In the remainder of this section we will see that these modules for $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$ constitute a full list of representatives of irreducible $\mathcal{X}$-semisimple $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules up to isomorphism and in the end of this section we describe when two of these modules are isomorphic.

Theorem 3.15. *Let $m > 0$ and $l \geq 0$ with $\kappa = m + l$ and let $(\lambda, \mu) \in \mathcal{J}^n_{m,l}$. We define*

$$V(\lambda, \mu) := \bigoplus_{T \in \mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})} \mathbb{K} v_T \tag{86}$$

as a $\mathbb{K}$-vector space. The following assignment describes an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-

module structure on $V(\lambda,\mu)$*:*

$$X_j v_T := t^{C_T(j)} v_T \text{ for } 1 \le j \le n, \tag{87}$$
$$\pi v_T := v_{\pi T}, \tag{88}$$
$$T_i v_T := \begin{cases} t^{-\frac{1}{2}} \dfrac{1-t^{1+\tau_i}}{1-t^{\tau_i}} v_{s_i T} - t^{-\frac{1}{2}} \dfrac{1-t}{1-t^{\tau_i}} v_T & \text{if } s_i T \in Tab^{RC}_{\gamma}(\widehat{\lambda/\mu}), \\ -t^{-\frac{1}{2}} \dfrac{1-t}{1-t^{\tau_i}} v_T & \text{if } s_i T \notin \mathrm{Tab}^{RC}_{\gamma}(\widehat{\lambda/\mu}), \end{cases} \tag{89}$$

where $0 \le i \le n-1$ *and* $\tau_i := C_T(i) - C_T(i+1) = (\zeta_T \mid \alpha_i)$*.*

Proof. Observe that for T a standard tableau we have that the labels i and $i+1$ cannot lie in the same diagonal by property (D3) from Definition 2.21, hence $1 - t^{\tau_i} \neq 0$ for $0 \le i \le n-1$ and the description of the T_i-action is well-defined. We will omit the calculation of the defining relations of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and only give a reference to [Ram03, Theorem 4.1], where the most computation intensive relation $T_iT_jT_i = T_jT_iT_j$ for $j = i \pm 1$ is verified. In the reference the author describes an action of the affine Hecke algebra, but the proof is still applicable to the double affine Hecke algebra, albeit one has change q from Ram's paper to $t^{\frac{1}{2}}$ to match our convention. We remark that the theorem is also stated in [SV05, Theorem 4.17], where the authors use a different, but isomorphic, definition of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, which is why we obtain an additional normalization factor $t^{-\frac{1}{2}}$ for our T_i-action. The isomorphism from our version of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ to the construction from [SV05, Definition 4.1] is obtained by sending $T_i \mapsto t^{-\frac{1}{2}} T_i$ for $0 \le i \le n-1$, $X_j \mapsto X_j$ for $1 \le j \le n$ and $\pi \mapsto \pi$. Furthermore, they denote our t by q and our q by ξ. □

Theorem 3.16. *Let* $m > 0$ *and* $l \ge 0$ *with* $\kappa = m+l$ *and* $(\lambda,\mu) \in \mathcal{J}^n_{m,l}$*. Then* $V(\lambda,\mu)$ *is irreducible and* $\mathcal{X}$*-semisimple with weight space decomposition given by*

$$V(\lambda,\mu) = \bigoplus_{T \in \mathrm{Tab}^{RC}_{\gamma}(\widehat{\lambda/\mu})} V(\lambda,\mu)_{\zeta_T} \text{ with } V(\lambda,\mu)_{\zeta_T} = \mathbb{K} v_T. \tag{90}$$

Proof. The $\mathcal{X}$-semisimplicity and the description of the weight spaces follow immediately from Lemma 2.33 (c) and the description of the X_j action in Theorem 3.15. For the irreducibility let $M \subseteq V(\lambda,\mu)$ be a non-trivial submodule. We want to show that M contains a weight vector. Let $0 \neq v \in M$ with $v = v_1 + ... + v_k$, where $v_i \in \mathbb{K} v_{T_i}$ with T_i pairwise different standard tableaux and k minimal. If $k = 1$ we are done. Otherwise we can find $1 \le j \le n$ such that $t^{C_{T_1}(j)} \neq t^{C_{T_2}(j)}$. Indeed, if $t^{C_{T_1}(j)} = t^{C_{T_2}(j)}$ for all $1 \le j \le n$ we can deduce $C_{T_1} = C_{T_2}$ using that t is not a root of unity and the periodicity of T_1 and T_2. Then T_1 and T_2 assign the same set of labels to any diagonal, but this set must appear in the correct order on the diagonal, since T_1 and T_2 are standard tableaux. We obtain $T_1 = T_2$, which is not possible by assumption. Hence

$$v' := (X_j - t^{C_{T_1}(j)})v = (t^{C_{T_2}}(j) - t^{C_{T_1}}(j))v_2 + \ldots + (t^{C_{T_k}}(j) - t^{C_{T_1}}(j))v_k \quad (91)$$

contradicts the minimality of k, since $t^{C_{T_2}}(j) \neq t^{C_{T_1}}(j)$ and therefore $v' \in M$ is not zero. Thus we have a weight vector in M and by rescaling we can assume $v_T \in M$ for some $T \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$. For any other standard tableau T' we can find some $w \in Z_T^{(\lambda,\mu)} = Z_{\zeta_T}$ such that $wT = T'$ by Theorem 2.38. By Proposition 3.9 and the previous description of the weight space decomposition we have $v_{T'} \in M$. Here we used that $\bar{w}(\zeta_T) = \zeta_{wT}$ by Lemma 2.33. Hence $M = V(\lambda, \mu)$ and $V(\lambda, \mu)$ is irreducible. □

We want to show that these are all irreducible $\mathcal{X}$-semisimple modules up to isomorphism, for which we need the following lemma.

Lemma 3.17. *Let M be an irreducible $\mathcal{X}$-semisimple module. Let $\zeta \in \dot{P}_\kappa$ be a weight of M and $i, j \in \mathbb{Z}$ with $i < j$ such that $(\zeta \mid \alpha_{i,j}) = 0$. Then there exist $k_+, k_- \in \{i+1, ..., j-1\}$ such that $(\zeta \mid \alpha_{i,k_+}) = -1$ and $(\zeta \mid \alpha_{i,k_-}) = 1$.*

Proof. We work by induction on $j - i$. When $j - i = 1$ we have that $(\zeta \mid \alpha_i) \neq 0$ by Lemma 3.10 and hence the induction start is trivial. Now assume $j - i = r$ and that the statement holds for all $r' < r$. If there exists $i + 1 \leq k \leq j - 1$ such that $(\zeta \mid \alpha_{i,k}) = 0$ then the statement holds by our induction hypothesis. Hence we assume that no such k exists. Let $0 \neq v \in M_\zeta$.

1. Case: Assume $(\zeta \mid \alpha_i) = (\zeta \mid \alpha_{j-1}) = 1$. Then the we have for $k_+ = j-1$ and $k_- = i+1$ that $(\zeta \mid \alpha_{i,k_+}) = (\zeta \mid \alpha_{i,j}) - (\zeta \mid \alpha_{k_+,j}) = -1$ and $(\zeta \mid \alpha_{i,k_-}) = 1$. Similarly the claim follows for $(\zeta \mid \alpha_i) = (\zeta \mid \alpha_{j-1}) = -1$ with $k_- = j - 1$ and $k_+ = i + 1$.

2. Case: Assume $(\zeta \mid \alpha_i) = -1$ and $(\zeta \mid \alpha_{j-1}) = 1$. Then $(\zeta \mid \alpha_{i+1,j-1}) = 0$. If $i+1 < j-1$ then we can find by induction hypothesis $i+1 \leq k'_- \leq j-1$ such that $(\zeta \mid \alpha_{i+1,k'_-}) = 1$ and therefore $(\zeta \mid \alpha_{i,k'_-}) = 0$, which contradicts our assumption that such k'_- does not exist. Hence $i + 1 = j - 1$ and we have $(\zeta \mid \alpha_i) = -1$ and $(\zeta \mid \alpha_{i+1}) = 1$. By Lemma 3.10 this implies $\phi_i v = \phi_{i+1} v = 0$. Thus, using $v \in M_\zeta$ we obtain $T_i v = t^{\frac{1}{2}} v$ and $T_{i+1} v = -t^{-\frac{1}{2}} v$ and therefore $-t^{\frac{1}{2}} v = T_i T_{i+1} T_i v = T_{i+1} T_i T_{i+1} v = t^{-\frac{1}{2}} v$, which contradicts the assumption that t is not a root of unity. The case $(\zeta \mid \alpha_i) = 1$ and $(\zeta \mid \alpha_{j-1}) = -1$ can be handled similarly.

3. Case: Assume $(\zeta \mid \alpha_i) \neq \pm 1$. Proposition 3.9 gives that $\phi_i v \neq 0$ and hence $M_{\bar{s}_i(\zeta)} \neq 0$. Then $(\bar{s}_i(\zeta) \mid \alpha_{i+1,j}) = (\zeta \mid \alpha_{i,j}) = 0$. Applying the induction hypothesis to the weight $\bar{s}_i(\zeta)$ produces $i + 2 \leq k_\pm \leq j - 1$ for which we have $\mp 1 = (\bar{s}_i(\zeta) \mid \alpha_{i+1,k_\pm}) = (\zeta \mid \alpha_{i,k_\pm})$. The claim follows. The case $(\zeta \mid \alpha_{j-1}) \neq \pm 1$ works similar and thus the proof is completed. □

Theorem 3.18. *Let $n \geq 2$, $\kappa \geq 1$ and M be an $\mathcal{X}$-semisimple and irreducible $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Then for some $1 \leq m \leq \kappa$ and $l = \kappa - m$ there exists $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$ such that $M \cong V(\lambda, \mu)$.*

Proof. Pick a weight $\zeta \in \dot{P}_\kappa$ of M and define $F_\zeta : \mathbb{Z} \to \mathbb{Z}$ by $F_\zeta(i) = (\zeta \mid e_i)$ for $i \in \mathbb{Z}$. We want to show that F_ζ is a content function, which by Theorem 2.39 corresponds to verifying properties (C1) and (C2). We have $F_\zeta(i+n) = (\zeta \mid e_{i+n}) = (\zeta \mid e_i - \delta) = F_\zeta(i) - \kappa$ and hence F_ζ satisfies condition (C1). To prove (C2) let p be in the image of F_ζ and $i, j \in F_\zeta^{-1}(p)$ with $i < j$ and $\{i, ...j\} \cap F_\zeta^{-1}(p) = \{i, j\}$. Then we have $(\zeta \mid \alpha_{i,j}) = F_\zeta(i) - F_\zeta(j) = 0$ and by Lemma 3.17 we obtain $i < k_\pm < j$ with $(\zeta \mid \alpha_{i,k_+}) = -1$ and $(\zeta \mid \alpha_{i,k_-}) = 1$. Therefore $k_\pm \in F_\zeta^{-1}(p \pm 1)$. To obtain the required uniqueness assume there exist $i < k_\pm < k' < j$ with $F_\zeta(k') = F_\zeta(k_\pm) = p \pm 1$. We have $(\zeta \mid \alpha_{k_\pm,k'}) = 0$. By Lemma 3.17 we find i' with $i < k_\pm < i' < k' < j$ such that $(\zeta \mid \alpha_{i,i'}) = (\zeta \mid \alpha_{i,k_\pm} + \alpha_{k_\pm,i'}) = \mp 1 \pm 1 = 0$ and hence $i' \in F_\zeta^{-1}(p)$. This contradicts $\{i, ..., j\} \cap F_\zeta^{-1}(p) = \{i, j\}$. Thus (C2) holds. By Theorem 2.39 we find $1 \leq m \leq \kappa$, $l = \kappa - m$, $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$ and $T \in \mathrm{Tab}^{RC}_\gamma(\widehat{\lambda/\mu})$ such that $F_\zeta = C_T$. Thus we also have $\zeta_T = \zeta$. Our aim is to show $M \cong V(\lambda, \mu)$ for this choice of $(\lambda, \mu) \in \mathcal{J}^{*n}_{m,l}$.

Let $u \in M_\zeta \setminus \{0\}$. Recall the definition of $Z_T^{(\lambda,\mu)}$ from Definition 2.32. For each $w \in Z_T^{(\lambda,\mu)} = Z_\zeta$ we define

$$\sigma_w := \prod_{\alpha \in \dot{R}(w)} t^{-\frac{1}{2}}(1 - t^{1+(\zeta|\alpha)}) \in \mathbb{K}, \qquad u_w := \sigma_w^{-1} \phi_w u \in M_{\bar{w}(\zeta)}. \tag{92}$$

Note that $\sigma_w \neq 0$ by definition of Z_ζ and $u_w \neq 0$ by Proposition 3.9. Set $N := \sum_{w \in Z_T} \mathbb{K} u_w \subseteq M$. Since $u_w \in M_{\bar{w}(\zeta)}$ and these weight spaces are pairwise different the sum is direct. By Theorem 2.38 we can define $w_S \in Z_T$ for all $S \in \mathrm{Tab}^{RC}_\gamma(\lambda/\mu)$ via $w_S T = S$ and then define a linear map $\rho : V(\lambda, \mu) \to M$ by $\rho(v_S) = u_{w_S}$. Since the v_S get mapped to different weight spaces this gives an injective map with image N. Once we have shown that ρ is an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module morphism the theorem follows, since M is irreducible. For this let $w \in Z_T$ and $0 \leq i \leq n-1$ such that $l(s_i w) < l(w)$. Then Theorem 2.6 (c) gives that $s_i w \in Z_T$ and $\sigma_w = t^{-\frac{1}{2}}(1 - t^{1+(\zeta|(s_i w)^{-1}(\alpha_i))})\sigma_{s_i w}$. We calculate using Lemma 3.6 (a):

$$\begin{aligned} \phi_i u_w &= \sigma_w^{-1} \phi_i \phi_w u = \sigma_w^{-1} \phi_i^2 \phi_{s_i w} u \\ &= t^{-1}(1 - t^{1+(\bar{s}_i \bar{w}(\zeta)|\alpha_i)})(1 - t^{1-(\bar{s}_i \bar{w}(\zeta)|\alpha_i)})\sigma_w^{-1} \phi_{s_i w} u \\ &= t^{-\frac{1}{2}}(1 - t^{1-(\bar{w}(\zeta)|s_i(\alpha_i))})\sigma_{s_i w}^{-1} \phi_{s_i w} u \\ &= t^{-\frac{1}{2}}(1 - t^{1+(\bar{w}(\zeta)|\alpha_i)}) u_{s_i w}. \end{aligned} \tag{93}$$

Let now $w \in Z_T^{(\lambda,\mu)}$ and $0 \leq i \leq n-1$ with $l(s_i w) > l(w)$. If $s_i w \notin Z_T^{(\lambda,\mu)}$ then $(\bar{w}(\zeta) \mid \alpha_i) = (\zeta \mid w^{-1}(\alpha_i)) = \pm 1$ and hence Lemma 3.14 gives

$\phi_i u_w = 0$. If $s_i w \in Z_T^{(\lambda,\mu)}$ then again by Theorem 2.6 (c) we have that $\sigma_{s_i w} = t^{-\frac{1}{2}}(1 - t^{1+(\zeta|w^{-1}(\alpha_i))})\sigma_w$ and

$$\phi_i u_w = \sigma_w^{-1}\phi_i\phi_w u = \sigma_w^{-1}\phi_{s_i w}u = t^{-\frac{1}{2}}(1 - t^{1+(\bar{w}(\zeta)|\alpha_i)})u_{s_i w}. \tag{94}$$

So overall we obtain for $0 \leq i \leq n-1$

$$\phi_i u_w = \begin{cases} t^{-\frac{1}{2}}(1 - t^{1+(\bar{w}(\zeta)|\alpha_i)})u_{s_i w} & \text{if } s_i w \in Z_T, \\ 0 & \text{if } s_i w \notin Z_T. \end{cases} \tag{95}$$

Unwinding the definition of ϕ_i from Definition 3.5 and using the fact that u_w is a weight vector of weight $\bar{w}(\zeta)$ by construction we can deduce $\rho(T_i v_S) = T_i u_{w_S} = T_i\rho(v_S)$ for all $0 \leq i \leq n-1$ and $S \in \mathrm{Tab}_\gamma^{RC}(\widehat{\lambda/\mu})$. The equality $\rho(X_i v_S) = X_i\rho(v_S)$ follows since v_S and u_{w_S} are both ζ_S-weight vectors. For π note that $\rho(\pi v_S) = \rho(v_{\pi S}) = u_{\pi w_S} = \phi_\pi u_{w_S} = \pi u_{w_S} = \pi(\rho(v_S))$, which finishes the proof. □

The last step of the classification of irreducible $\mathcal{X}$-semisimple modules is to give a condition for $V(\lambda,\mu) \cong V(\nu,\eta)$.

Theorem 3.19. *Let $m, m' \geq 1$, $l, l' \geq 0$ with $\kappa = m + l = m' + l'$. Let $(\lambda,\mu) \in \mathcal{J}^{*n}_{m,l}$ and $(\nu,\eta) \in \mathcal{J}^{*n}_{m',l'}$. The following are equivalent:*
(a) $V(\lambda,\mu) \cong V(\nu,\eta)$,
(b) $m = m', l = l'$ and $\widehat{\lambda/\mu} = \widehat{\nu/\eta} + (r,r)$ for some $r \in \mathbb{Z}$,
(c) $m = m', l = l'$ and $(\nu,\eta) = \omega^r_{m,l}(\lambda,\mu)$ for some $r \in \mathbb{Z}$.

Proof. We have already seen the equivalence of (b) and (c) in Proposition 2.40. For the equivalence with (a) from this theorem note that by the proof of the previous theorem $V(\lambda,\mu) \cong V(\nu,\eta)$ is equivalent to the fact that both modules have the same weight $\zeta \in \dot{P}_\kappa$, which is equivalent to $C_T = C_S$ for some standard tableaux T on $\widehat{\lambda/\mu}$ and S on $\widehat{\eta/\nu}$. But this is just condition (a) from Proposition 2.40, which finishes the proof. □

Remark 3.20. In this section we restricted to the case $\kappa \in \mathbb{Z}_{\geq 1}$ and $k = \frac{1}{\kappa}$, where k is the parameter such that $q^k = t$. By the following isomorphism $\iota : H_n(q^{\frac{1}{2}}, t^{-\frac{1}{2}}) \to H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ we can extend the statements to $\kappa \in \mathbb{Z}_{\leq -1}$.

$$\iota(T_i) = -T_i \text{ for } 0 \leq i \leq n-1, \quad \iota(X_i) = X_i \text{ for } 1 \leq i \leq n, \quad \iota(\pi) = \pi. \tag{96}$$

One can easily check that this defines an algebra isomorphism and hence we obtain an isomorphism between the categories of $H_n(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$-modules and $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules, which preserves $\mathcal{X}$-semisimplicity and irreducibility. Therefore we can translate our results to the case $\kappa \leq -1$ and $k = \frac{1}{\kappa}$.

4 One-dimensional DAHA

The goal of this chapter is to study and classify finite-dimensional irreducible modules of the so-called *one-dimensional double affine Hecke algebra* (or *one-dimensional DAHA*) denoted by $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Most of the necessary tools and ideas will be developed in Sections 4.1 and 4.2 and then applied to the *generic* and the *special* case of q in Sections 4.3 respectively 4.4. Here *generic* means that the parameter q is an element in $\mathbb{C}$ that is not a root of unity, while *special* means that q is a root of unity.

4.1 One-dimensional DAHA and the polynomial representation

Let us start with the definition of the one-dimensional DAHA. We will work over the base field $\mathbb{K} := \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ with $t = q^k$ for some $k \in \mathbb{C}$ and $q \neq 0$. In this chapter we explicitly allow the case that q is a root of unity. Later, when we study the representation theory of the DAHA, we will always assume $q \in \mathbb{C} \setminus \{0\}$ and hence $\mathbb{K} = \mathbb{C}$, but for the discussion in this chapter it is important to explicitly allow the case of transcendental q. This section and the following one are based on the results in [Che05, Chapter 2.5 and 2.6].

Definition 4.1. The *(one-dimensional) double affine Hecke algebra* (or short DAHA), denoted by $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, is defined to be the associative and unital $\mathbb{K}$-algebra with generators $X^{\pm 1}, \pi^{\pm 1}, T$ subject to the relations:

$$\textbf{(T):}\quad (T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) = 0, \qquad \textbf{(P):}\quad \pi^2 = 1,$$
$$\textbf{(XT):}\ TXT = X^{-1}, \qquad \textbf{(PX):}\ \pi X \pi^{-1} = q^{\frac{1}{2}} X^{-1}.$$

This definition is taken from [Che05, Lemma 2.5.7]. In [Che05, Chapter 3.2] for any finite root system and any lattice $Q \subseteq L \subseteq P$ an associated double affine Hecke is constructed. Our definition is the special case of the double affine Hecke algebra associated to A_1 and $L := P$, where P is the weight lattice of SL_2. This is shown in [Sim17, Chapter 2.3.1]. Here $P = \mathbb{Z}\rho$, where $\rho := \frac{\alpha}{2}$ for the (choice of) positive root α of the root system A_1. For SL_2 the Weyl group W is isomorphic to $S_2 \cong \mathbb{Z}/2$ and the non-trivial reflection s acts on the root lattice $P = \mathbb{Z}\rho$ by sending ρ to $-\rho$. The extended affine Weyl group $\dot{W}$ is generated by the elements π and s, where $\pi := \tau(\rho)s$ and τ is the SL_2-analogue of τ from Theorem 2.4.

For the convenience of the reader we point out that one obtains the following equalities from relation (T).

$$T^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T + 1, \quad T^{-1} = T + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}). \tag{97}$$

Using $Y := \pi T$ one can deduce the following description of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$.

Lemma 4.2. *The $\mathbb{K}$-algebra $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is isomorphic to the associative and unital $\mathbb{K}$-algebra with generators $X^{\pm 1}, Y^{\pm 1}, T$ subject to the relations:*

(T): $(T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) = 0$, ***(YT):*** $TY^{-1}T = Y$,

(XT): $TXT = X^{-1}$, ***(XYT):*** $q^{\frac{1}{2}}Y^{-1}X^{-1}YXT^2 = 1$.

Proof. Sending $Y \mapsto \pi T$, $X \mapsto X$, $T \mapsto T$ respectively $\pi \mapsto YT^{-1}$, $X \mapsto X$, $T \mapsto T$ produces inverse morphisms, as one easily checks by verifying the relations. $\square$

We will now define the *polynomial representation $\mathcal{P}$ of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$*. The first use of it will be to deduce the existence of a PBW-type basis for $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ in Corollary 4.5. Furthermore, many finite-dimensional irreducible modules will turn out to be (twisted) quotients of $\mathcal{P}$, which is why we will spend most of this and the next section studying the structure of this representation. We set $\mathcal{P} := \mathbb{K}[q^x, q^{-x}]$ to be the $\mathbb{K}$-algebra of Laurent polynomials in q^x. We define two $\mathbb{K}$-algebra automorphisms π, s of $\mathcal{P}$, which in fact can be seen to define an action of the extended affine Weyl group $\dot{W} = \langle \pi, s\rangle$ on $\mathcal{P}$. We set on the basis elements q^{nx} for $n \in \mathbb{Z}$

$$\begin{aligned} \pi:\ \mathcal{P} &\longrightarrow \mathcal{P}, & s:\ \mathcal{P} &\longrightarrow \mathcal{P}, \\ q^{nx} &\longmapsto q^{\frac{n}{2}}q^{-nx}, & q^{nx} &\longmapsto q^{-nx}. \end{aligned} \tag{98}$$

In other words $\pi(f)(x) = f(\frac{1}{2} - x)$ and $s(f)(x) = f(-x)$ for $f \in \mathcal{P}$. Furthermore, let $q^x\cdot$ denote the (left-)multiplication with q^x on $\mathcal{P}$.

Proposition 4.3. *The following assignment defines a $\mathbb{K}$-representation of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ on $\mathcal{P}$:*

$$T \longmapsto t^{\frac{1}{2}}s + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{2x} - 1}(s - 1), \qquad X \longmapsto q^x\cdot, \qquad \pi \longmapsto \pi. \tag{99}$$

Proof. Let $f \in \mathcal{P}$. Then $s(f) - f$ is divisible by $q^x - q^{-x}$ or equivalently by $q^{2x} - 1$ in $\mathcal{P}$, hence $T(f) \in \mathcal{P}$. We verify the relations from Definition 4.1. To check (T) we calculate for $n \in \mathbb{Z}$

$$T(q^{nx}) = t^{\frac{1}{2}}q^{-nx} - \operatorname{sgn}(n)(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\sum_{i=0}^{|n|-1} q^{(2i-|n|)x}. \tag{100}$$

We obtain for $n \in \mathbb{Z}$

$$T(q^{nx} + q^{-nx}) = t^{\frac{1}{2}}(q^{nx} + q^{-nx}) \tag{101}$$

and therefore any symmetric function is a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. Furthermore, using Equation (100) we compute for $n \in \mathbb{Z}$

$$(T + t^{-\frac{1}{2}})(q^{nx}) = t^{-\operatorname{sgn}(n)\frac{1}{2}}(q^{nx} + q^{-nx}) - \operatorname{sgn}(n)(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\sum_{i=1}^{|n|-1} q^{(2i-|n|)x}, \tag{102}$$

which is a symmetric function. Therefore $(T-t^{\frac{1}{2}})(T+t^{-\frac{1}{2}})(q^{nx})=0$ for all $n\in\mathbb{Z}$, which shows (T). We verify (XT) now. For $n\in\mathbb{Z}$ we have

$$TX(q^{nx})=T(q^{(n+1)x})=t^{\frac{1}{2}}q^{(-n-1)x}-\operatorname{sgn}(n+1)(t^{\frac{1}{2}}-t^{-\frac{1}{2}})\sum_{i=0}^{|n+1|-1}q^{(2i-|n+1|)x}. \tag{103}$$

This equals by a case distinction for the signs of n and $n+1$

$$\begin{aligned}X^{-1}T^{-1}(q^{nx})&=X^{-1}\Big(t^{\frac{1}{2}}q^{-nx}-\operatorname{sgn}(n)(t^{\frac{1}{2}}-t^{-\frac{1}{2}})\sum_{i=0}^{|n|-1}q^{(2i-|n|)x}\\&\quad+(t^{-\frac{1}{2}}-t^{\frac{1}{2}})q^{nx}\Big)\\&=t^{\frac{1}{2}}q^{(-n-1)x}-\operatorname{sgn}(n)(t^{\frac{1}{2}}-t^{-\frac{1}{2}})\sum_{i=0}^{|n|-1}q^{(2i-|n|-1)x}\\&\quad+(t^{-\frac{1}{2}}-t^{\frac{1}{2}})q^{(n-1)x}.\end{aligned} \tag{104}$$

Here we used $T^{-1}=T+(t^{-\frac{1}{2}}-t^{\frac{1}{2}})$. The verification of the remaining relations (P) and (PX) is an easy calculation and hence omitted. Note in particular that the X-action is invertible. □

Proposition 4.4. *For q not a root of unity $\mathcal{P}$ is a faithful representation of $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$.*

Proof. By the defining relations any element $h\neq 0$ in $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$ can be written as $h=\sum_{i\in I}c_iX^{j_i}T^{\epsilon_i}Y^{m_i}$ for some finite index set I, $c_i\in\mathbb{K}\setminus\{0\}$, $j_i,m_i\in\mathbb{Z}$ and $\epsilon_i\in\{0,1\}$ with $i\neq i'$ implying $(j_i,\epsilon_i,m_i)\neq(j_{i'},\epsilon_{i'},m_{i'})$. Indeed, we can first use (XT) and (XYT) to write h as a sum of monomials in $X^{\pm1},T^{\pm1},Y^{\pm1}$ such that no $Y^{\pm1}$ or $T^{\pm1}$ appears to the left of any $X^{\pm1}$. Then using (YT) we can assume that no $Y^{\pm1}$ appears to the left of any $T^{\pm1}$ and finally (T) allows us to assume that the exponent of T is 0 or 1. Now suppose that $h=\sum_{i\in I}c_iX^{j_i}T^{\epsilon_i}Y^{m_i}$ as above acts trivially on $\mathcal{P}$. Since Y is invertible, we can assume that all exponents of Y are positive by multiplying with Y^m for large enough m from the right. Let

$$j_{\max}:=\max\{j_i\mid i\in I\}. \tag{105}$$

If for all $i\in I$ with $j_i=j_{\max}$ we have $\epsilon_i=1$ then let i_0 be the unique index such that $j_{i_0}=j_{\max}$ and m_{i_0} is minimal. Replace h with $hY^{-m_{i_0}}T^{-1}$. Note that h acts trivially on $\mathcal{P}$ if and only if $hY^{-m_{i_0}}T^{-1}$ acts trivially, since Y and T are invertible. This replacement does not change $j_{\max}$, since we only need to apply relations (YT) and (T) to bring $hY^{-m_{i_0}}T^{-1}$ into the

PBW-form from above. More precisely the terms of h with top X-exponent $j_{\max}$ are

$$\sum_{i:j_i=j_{\max}} c_i X^{j_{\max}} T Y^{m_i} Y^{-m_{i_0}} T^{-1}. \tag{106}$$

By induction on $m > 0$ we obtain

$$\begin{aligned} TY^m T^{-1} &= (TY^{m-1}T)Y^{-1} = (TY^{m-1}T^{-1})Y^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})TY^{m-2} \\ &= (T + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))Y^{-m} + \sum_{i=1}^{m-1} (t^{\frac{1}{2}} - t^{-\frac{1}{2}})TY^{m-2i}. \end{aligned} \tag{107}$$

We see that one of the terms appearing in the sum is just $c_{i_0} X^{j_{\max}}$ and hence we can assume that there exists at least one $i \in I$ with $j_i = j_{\max}$ such that $\epsilon_i = 0$. Furthermore, by multiplying with a large enough power of Y from the right we can without loss of generality assume again $m_i \geq 0$ for all $i \in I$.

For any $f = c_+ q^{nx} + ... + c_- q^{-nx} \in \mathcal{P}$ with $n > 0$ we have

$$T(f) = t^{\frac{1}{2}} c_- q^{nx} + ... + \left(t^{\frac{1}{2}} c_+ + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(c_- - c_+) \right) q^{-nx}. \tag{108}$$

Using this we can calculate the coefficients of q^{nx} and q^{-nx} in the following expressions for $m \geq 0$ and $n > 0$.

$$Y(q^{nx}) = (q^{-\frac{n}{2}} t^{-\frac{1}{2}}) q^{nx} + ... + 0 q^{-nx}, \tag{109}$$

$$Y^m(q^{nx}) = (q^{-\frac{mn}{2}} t^{-\frac{m}{2}}) q^{nx} + ... + 0 q^{-nx}, \tag{110}$$

$$TY^m(q^{nx}) = 0 q^{nx} + ... + (q^{-\frac{mn}{2}} t^{-\frac{m+1}{2}}) q^{-nx}. \tag{111}$$

Our assumptions on h together with this description of the Y^m and TY^m action allows us to use a comparison of coefficients in $h(q^{nx}) = 0$ for $n > 0$ to deduce

$$\sum_{i:j_i=j_{\max},\epsilon_i=0} c_i q^{-\frac{m_i n}{2}} t^{-\frac{m_i}{2}} = 0. \tag{112}$$

This is a Laurent polynomial in $q^{\frac{n}{2}}$. Because q is not a root of unity $q^{\frac{n}{2}}$ takes infinitely many pairwise different values for $n > 0$. As this Laurent polynomial must vanish for all $q^{\frac{n}{2}}$ with $n > 0$ and because the m_i appearing in the sum are pairwise different we have $c_i t^{-\frac{m_i}{2}} = 0$, which implies $c_i = 0$ for all i appearing in the sum. This contradicts our assumption that $c_i \neq 0$ all $i \in I$ and in particular $c_{i_0} \neq 0$. Hence h does not act trivially on $\mathcal{P}$.

$\square$

In the description of the PBW-basis the *Iwahori-Hecke algebra* $\mathcal{H}_1 := \mathbb{K}[T]/((T + t^{-\frac{1}{2}})(T - t^{\frac{1}{2}}))$ of type A_1 will appear. See [Mat99] for an introduction to Iwahori-Hecke algebras.

Corollary 4.5. *For any parameters q and $t = q^k$ the set*

$$\left\{X^jT^\epsilon Y^m \mid j,m \in \mathbb{Z}, \epsilon \in \{0,1\}\right\} \tag{113}$$

is a $\mathbb{K}$-basis of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. In other words the multiplication map

$$m : \mathbb{K}[X^{\pm 1}] \otimes_{\mathbb{K}} \mathcal{H}_1 \otimes_{\mathbb{K}} \mathbb{K}[Y^{\pm 1}] \to H(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \tag{114}$$

is an isomorphism of $\mathbb{K}$-vector spaces.

Proof. The proof of the faithfulness of the polynomial representation $\mathcal{P}$ in Proposition 4.3 shows the claim for any q, $t = q^k$ with q not a root of unity. Let now $q, t = q^k$ be parameters with q a root of unity and $\tilde{q}, \tilde{t} = \tilde{q}^k$ be parameters with q transcendental over $\mathbb{C}$. Define the $\mathbb{C}$-algebra $H'(\tilde{q}^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}})$ to be generated by $X^{\pm 1}, T, Y^{\pm 1}, \tilde{q}^{\pm\frac{1}{2}}, \tilde{t}^{\pm\frac{1}{2}}$ via the same relations as in Definition 4.1 and requiring $\tilde{q}^{\pm\frac{1}{2}}, \tilde{t}^{\pm\frac{1}{2}}$ to be central. Then virtually the same proof as in Proposition 4.3 with $\mathcal{P}$ replaced by $\mathcal{P}' := \mathbb{C}[\tilde{q}^{\pm\frac{1}{2}}, \tilde{t}^{\pm\frac{1}{2}}][\tilde{q}^x, \tilde{q}^{-x}]$ shows that $H'(\tilde{q}^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}})$ is a free $\mathbb{C}[\tilde{q}^{\pm\frac{1}{2}}, \tilde{t}^{\pm\frac{1}{2}}]$-module with basis given by $X^jT^\epsilon Y^m$ for $j, m \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$. The evaluation $\tilde{q}^{\frac{1}{2}} \mapsto q^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}} \mapsto t^{\frac{1}{2}}$ induces a $\mathbb{C}$-algebra morphism $p : H'(\tilde{q}^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}}) \to H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, whose kernel is the ideal $(\tilde{q}^{\frac{1}{2}} - q^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}} - t^{\frac{1}{2}})$. Using the basis of $H'(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ from above we see that any element in the kernel can be uniquely written as a $\mathbb{C}[\tilde{q}^{\pm\frac{1}{2}}, \tilde{t}^{\pm\frac{1}{2}}]$-linear combination of elements of the form $(\tilde{q}^{\frac{1}{2}} - q^{\frac{1}{2}})X^jT^\epsilon Y^m$ and $(\tilde{t}^{\frac{1}{2}} - t^{\frac{1}{2}})X^jT^\epsilon Y^m$ for $j, m \in \mathbb{Z}$, $\epsilon \in \{0,1\}$. Since we have $q, t \in \mathbb{C}$, any element $h \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ can be written as $h = \sum_{i\in I} c_i X^{j_i} T^{\epsilon_i} Y^{m_i}$ with $c_i \in \mathbb{C}$ as we have seen in the beginning of the proof of Proposition 4.3. If such a term was zero, then its lift $\sum_{i\in I} c_i X^{j_i} T^{\epsilon_i} Y^{m_i} \in H'(\tilde{q}^{\frac{1}{2}}, \tilde{t}^{\frac{1}{2}})$ would lie in the kernel of p. By the description of the kernel this implies $c_i = 0$ for all i, which shows that the $X^jT^\epsilon Y^m$ for $j, m \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$ are $\mathbb{C}$-linearly independent and hence they form a basis of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. □

Corollary 4.6. *Let $f \in \mathcal{P}$ be an eigenvector for T and Y. Then the ideal $(f) \subseteq \mathcal{P}$ is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule.*

Proof. This follows directly from the PBW-Theorem in Corollary 4.5. □

Let $\langle T, Y^{\pm 1}\rangle$ be the $\mathbb{K}$-subalgebra of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ generated by T and $Y^{\pm 1}$. From the PBW-basis theorem in Corollary 4.5 it follows that $\langle T, Y^{\pm 1}\rangle$ is isomorphic to the algebra generated by the elements $T, Y^{\pm 1}$ subject to the relations (T) and (YT). We can endow $\mathbb{K}$ with a $\langle T, Y^{\pm 1}\rangle$-module structure by letting T and Y act by $t^{\frac{1}{2}}$. It is easy to check that this is actually a $\langle T, Y^{\pm 1}\rangle$-module by verifying the relations (T) and (YT).

Corollary 4.7. *The $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $\mathcal{P}$ is canonically isomorphic to the $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $\operatorname{Ind}_{\langle T, Y^{\pm 1}\rangle}^{H(q^{\frac{1}{2}}, t^{\frac{1}{2}})}(\mathbb{K})$.*

Proof. Let V be an $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$-module and let $v \in V$ be a T,Y-eigenvector of eigenvalue $t^{\frac{1}{2}}$. We have to show that there exists a unique morphism $\mathcal{P} \to V$ mapping 1 to v. Define this morphism by sending q^{mx} to $X^m(v)$ for $m \in \mathbb{Z}$. From the PBW-basis theorem it follows that this defines a $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$-module morphism and it is uniquely determined, since $X^m(1) = q^{mx}$ and these elements for $m \in \mathbb{Z}$ form a basis of $\mathcal{P}$. $\square$

4.2 e-polynomials and intertwining operators

We define a total order $\prec$ on $\mathbb{Z}$ by setting $0 \prec 1 \prec -1 \prec 2 \prec -2 \dots$. We also define $\mathcal{P}_{\preceq n} \subseteq \mathcal{P}$ respectively $\mathcal{P}_{\prec n} \subseteq \mathcal{P}$ for $n \in \mathbb{Z}$ to be the $\mathbb{K}$-span of the elements q^{mx} with $m \preceq n$ respectively $m \prec n$. We have $\mathcal{P} = \bigcup_{n \in \mathbb{Z}} \mathcal{P}_{\preceq n}$. Note that Y preserves the finite-dimensional spaces $\mathcal{P}_{\preceq n}$ for all $n \in \mathbb{Z}$ and hence $\mathcal{P}$ decomposes as generalized Y-eigenspaces.

Definition 4.8. Let V be an $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$-module which admits a generalized Y-eigenspace decomposition. We define the *intertwining operators* A_m for $m \leq 0$ and B via

$$\begin{aligned} A_m : V &\longrightarrow V, & B : V' &\longrightarrow V, \\ v &\longmapsto q^{-\frac{m}{2}} X\pi(v), & v &\longmapsto t^{\frac{1}{2}}\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{Y^{-2}-1}\right)(v), \end{aligned} \tag{115}$$

where V' is the sum of the generalized Y-eigenspaces for eigenvalues not equal to 1 or -1 if $t \neq 1$. If $t = 1$ we set $B := t^{\frac{1}{2}}T$ and $V' := V$.

Note that $(Y^{-2}-1)^{-1}$ is well-defined on V'. These intertwining operators can be seen as a SL_2-version of the previously used intertwining operators for GL_n from Definition 3.5. Note that now Y takes the role of the X_i in these operators. Compared to the 'naive' translation from GL_n to SL_2 we are using slightly adapted versions. These adapted versions will be necessary to construct the so-called *non-symmetric polynomials* $e_m \in \mathcal{P}$ in Definition 4.12.

Lemma 4.9. *Let V be an $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$-module with a generalized Y-eigenspace decomposition. Let $\lambda \in \mathbb{C}$ and set*

$$\lambda_m := -\lambda - \frac{m}{2} \text{ for } m > 0, \quad \lambda_m := \lambda - \frac{m}{2} \text{ for } m \leq 0. \tag{116}$$

Let $m \in \mathbb{Z}$ and $v \in V$ be a Y-eigenvector with eigenvalue q^{λ_m} and v' be a generalized Y-eigenvector with eigenvalue q^{λ_m} of rank k, in other words $(Y - q^{\lambda_m})^k(v) = 0$ and $(Y - q^{\lambda_m})^{k-1}(v) \neq 0$.

(a) Let $m \leq 0$. Then $A_m(v)$ is a Y-eigenvector with eigenvalue $q^{\lambda_{1-m}}$ and $A_m(v')$ is a generalized Y-eigenvector of eigenvalue $q^{\lambda_{1-m}}$ and rank k.

(b) Let $m > 0$ and $q^{2\lambda_m} \neq 1$. Then $B(v)$ is a Y-eigenvector with eigenvalue $q^{\lambda_{-m}}$ or possibly zero if $q^{2\lambda_m} = t^{\pm 1}$. Also, $B(v')$ is a generalized Y-eigenvector with eigenvalue $q^{\lambda_{-m}}$ and rank less or equal k or possibly zero if $q^{2\lambda_m} = t^{\pm 1}$.

Proof. The case of the B-intertwining operators in (b) is very similar to the calculations done in Lemma 3.7 and Proposition 3.8 and hence mostly omitted. We only show that $B(v)$ and $B(v')$ are not zero for $q^{2\lambda_m} \neq t^{\pm 1}$. For v this follows directly from relation (T) in the double affine Hecke algebra, since $(T + c)$ is invertible for all $c \in \mathbb{K}$ with $c \neq t^{-\frac{1}{2}}$ and $c \neq -t^{\frac{1}{2}}$. For v' we have that $v := (Y - q^{\lambda_m})^{k-1}(v')$ is a Y-eigenvector with eigenvalue q^{λ_m}. The omitted calculations show as in Lemma 3.7 that $BY = Y^{-1}B$ and therefore $B(Y - q^{\lambda_m})^{k-1} = (Y^{-1} - q^{\lambda_m})^{k-1}B$. Hence the claim for v' follows from the claim for v. For the A-case in (a) we calculate

$$\begin{aligned} A_m Y &= q^{-\frac{m}{2}} X\pi Y = q^{-\frac{m}{2}} XT = q^{-\frac{m}{2}} T^{-1}X^{-1} = q^{-\frac{m}{2}} Y^{-1}\pi X^{-1} \\ &= q^{-\frac{m+1}{2}} Y^{-1} X\pi = q^{-\frac{1}{2}} Y^{-1} A_m. \end{aligned} \tag{117}$$

From this the claim can be deduced easily as in the B-case. $\square$

Definition 4.10. Let V be an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module and let $v \in V$ with Y-eigenvalue q^{λ}. Assume V has a generalized Y-eigenspace decomposition. We denote by B_m the restriction of B to the generalized Y-eigenspace of eigenvalue q^{λ_m} for $m > 0$. For $t \neq 1$ this element is well-defined whenever $q^{2\lambda_m} \neq 1$ and invertible whenever $q^{2\lambda_m} \neq t^{\pm 1}$. When restricted to proper eigenvectors we have

$$B_m = t^{\frac{1}{2}} \left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1} \right) \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}}). \tag{118}$$

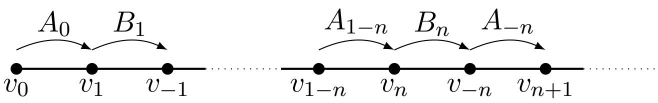
If $q^{2\lambda_m} = 1$ we set $B_m := t^{\frac{1}{2}} T$ and we say that the B-intertwining operator is not well-defined at m. If $t = 1$ we set $B_m = t^{\frac{1}{2}} T$ for all $m > 0$.

We define the so-called *chain of intertwining operators.*

Definition 4.11. Let V be an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module and $v_0 \in V$ be a Y-eigenvector with eigenvalue q^{λ} for $\lambda \in \mathbb{C}$. Assume that V has a generalized Y-eigenspace decomposition. As long as $q^{2\lambda_m} \neq 1$ and $q^{2\lambda_m} \neq t^{\pm 1}$ for $m > 0$ we define inductively using $\prec$ the following elements $v_m \in V$:

$$v_m := A_{1-m}(v_{1-m}), \qquad v_{-m} := B(v_m) = B_m(v_m). \tag{119}$$

This sequence can be pictured as follows.



From Lemma 4.9 we obtain that v_m is a Y-eigenvector with eigenvalue q^{λ_m}.

In the case that $V = \mathcal{P}$ is the polynomial representation we will construct a more sophisticated version in the upcoming lemma.

Definition 4.12. Let $\mathcal{P}$ be the polynomial representation of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, $v_0 := 1$ and $\lambda := \frac{k}{2}$. Then v_0 has Y-weight λ, in other words $Y(v_0) = q^{\frac{k}{2}} v_0 = t^{\frac{1}{2}} v_0$. Define inductively for $m > 0$ via the order $\prec$ on $\mathbb{Z}$:

$$v_m := A_{1-m}(v_{1-m}), \qquad v_{-m} := B_m(v_m). \tag{120}$$

Also, set $V_0 := \mathbb{K} v_0$. We define inductively via $\prec$

$$V_m := A_{1-m}(V_{1-m}),\ V_{-m} := \begin{cases} B_m(V_m) & \text{if } q^{2\lambda_m} \neq 1, \\ V_m + B_m(V_m) = V_m + T(V_m) & \text{otherwise.} \end{cases}$$

If v_m for $m \in \mathbb{Z}$ is a Y-eigenvector we set $e_m := v_m$ and call it the *m-th non-symmetric polynomial* or *m-th e-polynomial.* If v_m is not a Y-eigenvector we do not define e_m.

We still have to show that v_m and V_m for $m \in \mathbb{Z}$ are well-defined. This will follow from the upcoming Lemma 4.13, which is based on [Che05, Lemma 2.9.4]. For $\mathcal{P}$ and $v_0 = 1$ we have $\lambda_m = -m_\sharp$, where

$$m_\sharp := \frac{m + \operatorname{sgn}(m)k}{2} \text{ for } m \neq 0, \quad 0_\sharp = -\frac{k}{2}. \tag{121}$$

We note that the diagram from Remark 4.14 might be a helpful picture to have in mind while going through the upcoming proof.

Lemma 4.13. *(a) We have that v_m and V_m for $m \in \mathbb{Z}$ are well-defined. The vector v_m has top term q^{mx} with respect to $\prec$ with coefficient 1. The space V_m is either one- or two-dimensional. If it is one-dimensional, it is spanned by $v_m = e_m$, which is a Y-eigenvector with eigenvalue q^{λ_m}. If V_m is two-dimensional, it contains a Y-eigenvector of eigenvalue q^{λ_m}, which is unique up to scalar. It is proportional to e_μ for some $\mu \prec m$ with $\operatorname{sgn}(\mu) = -\operatorname{sgn}(m)$, where $\operatorname{sgn}(0) = -1$. We have $(Y - q^{\lambda_m})(v_m) = c e_\mu$ for some constant $c \in \mathbb{K} \setminus \{0\}$ and in particular v_m is a generalized Y-eigenvector with eigenvalue q^{λ_m}.*

(b) Let $m > 0$ and $t \neq 1$ and $q^{2\lambda_m} = t$ respectively t^{-1}. Then V_{-m} is one-dimensional. If V_m is also one-dimensional or if $t = -1$ then $(T + t^{-\frac{1}{2}})(e_{-m}) = 0$ respectively $(T - t^{\frac{1}{2}})(e_{-m}) = 0$. If V_m is two dimensional and $t \neq -1$ then $(T + t^{-\frac{1}{2}})(e_{-m})$ respectively $(T - t^{\frac{1}{2}})(e_{-m})$ is proportional to the unique Y-eigenvector e_μ in V_m.

Proof. We assume $t \neq 1$. We omit the much simpler case $t = 1$, but remark that in this case the B-intertwining operator exists on all of V and is invertible and hence all V_m are one-dimensional. Furthermore, in that case we have $e_m = q^{mx}$ for $m \in \mathbb{Z}$.

We prove the statements by induction on $\prec$. For $m = 0$ we have that $v_0 = 1$ is a Y-eigenvector with eigenvalue $q^{\lambda_0} = t^{\frac{1}{2}}$ and V_0 is one-dimensional. Hence the claims hold for $m = 0$. Now assume the statements are verified up to $0 \prec m$ and let m' be minimal with $m \prec m'$, which means $m' = -m$ if $m > 0$ or $m' = 1 - m$ if $m < 0$.

Consider the case that V_m is one-dimensional first. Then by induction V_m is spanned by the Y-eigenvector $e_m = v_m$ with eigenvalue q^{λ_m}. If $q^{2\lambda_m} \neq 1$ then Lemma 4.9 shows that $V_{m'} = B_m(V_m)$ respectively $A_m(V_m)$ is spanned by the non-zero Y-eigenvector $v_{m'}$ with eigenvalue $q^{\lambda_{m'}}$ or that $v_{m'} = 0$. If $v_{m'} = 0$ then again by Lemma 4.9 we must have $q^{2\lambda_m} = t^{\pm 1}$ and $m > 0$. But then $B_m(v_m) = 0$ implies that v_m is a not only a Y-eigenvector, but also a T-eigenvector and hence a π eigenvector. This contradicts that the top term of v_m with respect to $\prec$ is q^{mx} with $m > 0$. Therefore $v_{m'} \neq 0$. Furthermore, if $q^{2\lambda m} = t$ or t^{-1} then B_m equals $t^{\frac{1}{2}}(T - t^{\frac{1}{2}})$ respectively $t^{\frac{1}{2}}(T + t^{-\frac{1}{2}})$ on V_m, since $t \neq 1$. Hence the claim from (b) for $V_{m'}$ follows from relation (T) in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Now assume $q^{2\lambda_m} = 1$ and $m > 0$. Then v_m can not be a T-eigenvector by relation (YT) and because $t \neq 1$. Hence $V_{m'} = V_m + T(V_m)$ is two-dimensional. Also, it contains $e_m = v_m$, which is a Y-eigenvector with $\operatorname{sgn}(m) = -\operatorname{sgn}(m')$. A quick calculation shows $(Y^{-1} - q^{\lambda_m})(Tv_m) = q^{\lambda_m}(t^{-\frac{1}{2}} - t^{\frac{1}{2}})v_m \neq 0$, since $t \neq 1$. Therefore $v_m \in V_{m'}$ is up to scalar the unique Y-eigenvector in $V_{m'}$ and $(Y - q^{\lambda_{m'}})(v_{m'}) = ce_m$ for some $c \in \mathbb{K} \setminus \{0\}$. Finally, in all cases one can easily calculate that the top term of $v_{m'}$ is $q^{m'x}$, because the top term of v_m is q^{mx}. This shows the claims for V_m one-dimensional.

Now assume that V_m is two-dimensional. Let $\tilde{m} \prec m$ maximal such that $V_{\tilde{m}}$ is one-dimensional. This implies that $q^{2\lambda_{\tilde{m}}} = 1$ and $\tilde{m} > 0$. If $q^{2\lambda_m} = 1$ as well, we can find $\tilde{m} \prec n \prec m$ and $n > 0$ such that $q^{2\lambda_n} = t^{\pm 1}$, since $\lambda_0 = \frac{k}{2}$. By induction part (b) gives a contradiction to the maximality of $\tilde{m}$. It follows that $V_{m'}$ has dimension less or equal than two. Assume for now that the intertwining operator at m is invertible, which means we exclude the case that $q^{2\lambda_m} = t^{\pm 1}$ and $m > 0$. Then the dimension of $V_{m'}$ is two. From Lemma 4.9 we see that $V_{m'}$ satisfies $(Y - q^{\lambda_{m'}})^2(V_{m'}) = 0$. We show now that $V_{m'}$ contains an up to scalar unique Y-eigenvector $e_{\mu'}$ with $\operatorname{sgn}(\mu') = -\operatorname{sgn}(m')$. By induction V_m contains some e_μ for $\mu \prec m$ with $\operatorname{sgn}(\mu) = -\operatorname{sgn}(m)$. Since e_μ is a Y-eigenvector with eigenvalue q^{λ_m} we must have $q^{\lambda_\mu} = q^{\lambda_m}$, hence $q^{2\lambda_\mu} \neq t^{\pm 1}$ and e_μ is the image of some previous e-polynomial under one of the intertwining operators by induction.

If $m > 0$ we have $\mu < 0$ and

$$\begin{aligned} B_m(e_\mu) = B_m(B_{-\mu}(e_{-\mu})) &= t\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1}\right)\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{2\lambda_m} - 1}\right)(e_{-\mu}) \\ &= t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\left(1 + \frac{1}{q^{-2\lambda_m} - 1} + \frac{1}{q^{2\lambda_m} - 1}\right)T(e_{-\mu}) \\ &+ t\left(1 + \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_m -} - 1)(q^{2\lambda_m} - 1)}\right)(e_{-\mu}) \\ &= t\left(\frac{-q^{-2\lambda_m} - q^{2\lambda_m} + t + t^{-1}}{(q^{-2\lambda_m} - 1)(q^{2\lambda_m} - 1)}\right)(e_{-\mu}), \end{aligned} \tag{122}$$

where the last term is non-zero, since $q^{2\lambda_m} \neq t^{\pm 1}$. If $m < 0$ we have $\mu > 0$

$$A_m(e_\mu) = A_m(A_{1-\mu}(e_{1-\mu})) = q^{\frac{-m-1+\mu}{2}} X\pi X\pi(e_{1-\mu}) = q^{\frac{-m+\mu}{2}} e_{1-\mu}. \tag{123}$$

Therefore $V_{m'}$ contains $e_{-\mu}$ respectively $e_{1-\mu}$, whose index also has the sign as claimed, since $\operatorname{sgn}(m) = -\operatorname{sgn}(\mu)$. Using the inverse of the intertwining operator we can show that for some $c \neq 0$ we have $(Y - q^{\lambda_{m'}})v_{m'} = ce_{\mu'}$ as in the proof of Lemma 4.9, since the analogous statement holds for m by induction. If the intertwining operator is not invertible then $q^{2\lambda_m} = t^{\pm 1}$ and we must be in the B_m-case, that means $m > 0$. First assume that q is not a root of unity. By going through the chain of intertwining operators and looking at the appearing Y-weights $\lambda_m = -m_\sharp$ we see that this case is only possible if $k < 0$ is an integer. Then $q^{2\lambda_{\tilde{m}}} = 1$ for $\tilde{m} > 0$ can only appear at $\tilde{m} = -k$ and the intertwining operators are invertible up to $m = -2k$ when $q^{\lambda_m} = t^{\frac{1}{2}}$. This must therefore be the m we are looking at. Note that the e-polynomials are reflected at $m' = -k$ by the calculations in (122) and (123). We see that the unique e-polynomial in V_m is $1 \in \mathcal{P}$. The element 1 is a T- and Y-eigenvector with eigenvalue $t^{\frac{1}{2}}$. From this all claims can be deduced, similarly to the more complicated case where q a root of unity, which we handle now. Let $N > 0$ be minimal such that $q^N = 1$. Furthermore, we can assume k to be an integer, since we have found n such that $q^{2\lambda_n} = 1$, since we are in the two-dimensional case. To conclude the proof of (b) look at the maximal sequence of 2-dimensional V_n ending at V_m, in other words we look at $V_n, V_{1-n}, ..., V_m$ with n minimal such that for all $n \preceq \tilde{n} \preceq m$ the space $V_{\tilde{n}}$ is 2-dimensional. Similarly let $V_{n'}, ..., V_{-n}$ be the sequence of 1-dimensional $V_{\tilde{n}'}$ preceding n. Note that $n < 0$. Assume first $t \neq -1$. We want to show that the unique e-polynomial $e_\mu \in V_m$ is a T-eigenvector. Since $t \neq -1$ we have $k \neq \frac{N}{2} \bmod N$. Then $q^{2\lambda_q} = t^{\pm 1}$ happens for four different values of $q \bmod N$. From this we can deduce by going through the chain of intertwining operators and by looking at the appearing Y-eigenvalues that one of the following two cases holds. Either

k mod $N \in \{-\frac{N-1}{2}, ..., \frac{N-1}{2}\}$ is a negative integer and $1 \prec m = -2k$ mod N is minimal with $q^{2\lambda_m} = t^{\pm 1}$ and $m > 0$. Furthermore, we have $n' = 1$ and the sequence of one-dimensional spaces starting at $V_{n'}$ has the same length as the sequence of two-dimensional spaces starting at V_n. Otherwise, the sequence of one-dimensional spaces starting at $V_{n'}$ is strictly longer than the sequence of two-dimensional spaces starting at V_n. By induction and using the proof thus far we see that for the unique e-polynomial $e_\mu \in V_m$ we have $\mu = n - m$, since the e-polynomials are reflected at n. Therefore we see in the first case that $e_\mu = 1$ is a T-eigenvector of eigenvalue $t^{\frac{1}{2}} = \pm q^{\lambda_m}$. In the second case e_μ spans one of the spaces in the 1-dimensional sequence $V_{n'}, ...V_{-n}$ from above, but not $V_{n'}$. Therefore $e_\mu = B_{-\mu}(e_{-\mu})$ is also a T-eigenvector of eigenvalue $\pm q^{\lambda_m}$ by same calculation as in Equation (122). For $t = -1$ we can use the same argumentation, but now we have that the sequence $V_{n'}, ..., V_{-n}$ and $V_n, ..., V_m$ have the same length. From this we obtain that $e_\mu \in V_m$ equals $e_{n'}$ and by induction using statement (b) for $t = -1$ and the induction start at $e_\mu = 1$ we can deduce that e_μ is also in this case a T-eigenvector of eigenvalue $\pm q^{\lambda_m}$. In particular we have in all cases $B_m(e_\mu) = 0$. By induction we have that $e_\mu \in V_m$ equals $(Y - q^{\lambda_m})(v_m)$ up to scalar and hence $0 = B_m(Y - q^{\lambda_m})(v_m) = (Y^{-1} - q^{\lambda_m})B_m(v_m)$. This shows $(Y - q^{\lambda_{m'}})B_m(v_m) = 0$. Therefore $v_{m'} = B_m(v_m)$ is either 0 or a Y-eigenvector of eigenvalue $q^{\lambda_{m'}}$. Next we show that $(T + t^{-\frac{1}{2}})B_m(v_m)$ if $q^{2\lambda_m} = t$ respectively $(T - t^{\frac{1}{2}})B_m(v_m)$ if $q^{2\lambda_m} = t^{-1}$ is a non-zero multiple of e_μ if $t \neq -1$ and 0 otherwise. We can find $c \in \mathbb{K} \setminus \{0\}$ such that $\{ce_\mu, v_m\}$ is a basis of V_m on which Y restricted to V_m has Jordan-Normalform, since all appearing Y-eigenvalues q^{λ_m} lie in $\mathbb{K}$. Then we can calculate

$$\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{Y^{-2} - 1} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{\begin{pmatrix} q^{\lambda_m} & 1 \\ 0 & q^{\lambda_m} \end{pmatrix}^{-2} - 1} = \mp t^{\pm\frac{1}{2}} \begin{pmatrix} 1 & \frac{2q^{-\lambda_m}}{(q^{-2\lambda_m} - 1)} \\ 0 & 1 \end{pmatrix} \tag{124}$$

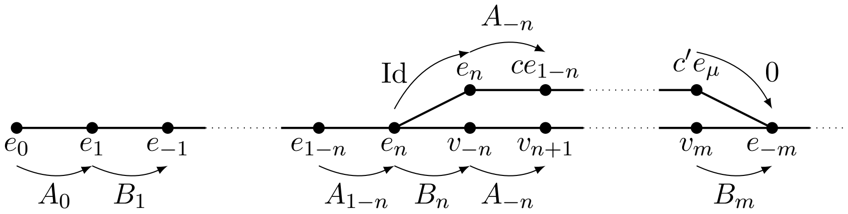
Note that e_μ is a T-eigenvector of eigenvalue $\pm t^{\pm\frac{1}{2}}$ by the discussion above. We obtain

$$(T \pm t^{\mp\frac{1}{2}})B_m(v_m) = -t^{\pm\frac{1}{2} + \frac{1}{2}} c(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \frac{2q^{-\lambda_m}}{(q^{-2\lambda_m} - 1)} e_\mu. \tag{125}$$

We have $(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) = 0$ if and only if $t = -1$, which gives the missing claim from (b). Again showing that the top term of $v_{m'}$ is $q^{m'x}$ with coefficient 1 follows by induction using that q^{mx} is the top term of v_m and in particular we can deduce $v_{m'} \neq 0$ for $t = -1$. This finishes the case that V_m is two-dimensional and hence the whole proof.

□

Remark 4.14. Let us visualize the general picture of the chain of intertwining operators for the polynomial representation $\mathcal{P}$. Here n and m are some positve integers with $q^{2\lambda_n} = 1$ respectively $q^{2\lambda_m} = t^{\pm 1}$. The splitting appears when $q^{2\lambda_n} = q^{-k-n} = 1$ and the merging when $q^{2\lambda_m} = q^{-k-m} = t^{\pm 1} = q^{\pm k}$. If q is a primitive N-th root of unity the splitting (and merging) appear N-periodically at e_n, $e_{n+N}, \ldots$. Otherwise they only appear at most once and only if $k = -n$ for some $n > 0$.



Corollary 4.15. *Whenever e_m for $m \in \mathbb{Z}$ exists we have $Ye_m = q^{-m_\sharp} e_m$, where $m_\sharp := \frac{m+\operatorname{sgn}(m)k}{2}$ for $m \neq 0$ and $0_\sharp := -\frac{k}{2}$.*

Proof. The element $1 \in \mathcal{P}$ is a Y-eigenvector with eigenvalue $q^{\frac{k}{2}}$. Hence we have $\lambda_m = -m_\sharp$ in Lemma 4.9 which shows the claim by definition of e_m. □

In fact, if $f \in \mathcal{P}$ has top degree l with respect to $\prec$ and if f is a Y-eigenvector, then one already knows that its Y-eigenvalue is $q^{-l_\sharp}$, which can be deduced from Equation (109) for $l > 0$ and similarly for $l \leq 0$.

Corollary 4.16. *For the polynomial representation $\mathcal{P}$ and $v_0 = 1 \in \mathcal{P}$ the set $\{v_m \mid m \in \mathbb{Z}\}$ forms a basis of $\mathcal{P}$. Also, any Y-eigenvector in $\mathcal{P}$ lies in the $\mathbb{K}$-linear span of the e_m.*

Proof. The v_m span $\mathcal{P}$, since v_m exists for all $m \in \mathbb{Z}$ and has leading term q^{mx} by Lemma 4.13. For the second claim note that, again by Lemma 4.13, all v_m are generalized Y-eigenvectors. Hence they form a basis of generalized Y-eigenvectors of $\mathcal{P}$ and by definition the e_m are the proper eigenvectors in this basis. This shows the claim. □

We will now define Rad, the *radical of* $\mathcal{P}$. It will turn out that many finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules are quotients of $\mathcal{P}$ via this radical.

Definition 4.17. We define a bilinear form on $\mathcal{P}$ by setting for any two Laurent polynomials $f, g \in \mathcal{P}$

$$\langle f, g \rangle := f(Y^{-1})(g)(q^{-\frac{k}{2}}). \tag{126}$$

Denote by Rad the radical of this bilinear form, that is

$$\operatorname{Rad} := \{f \in \mathcal{P} \mid \langle f, g \rangle = \langle g, f \rangle = 0 \text{ for all } g \in \mathcal{P}\} \tag{127}$$

and call it the *radical of* $\mathcal{P}$.

Define a $\mathbb{K}$-linear anti-isomorphism $\phi : H(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \to H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ via

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(T) = T. \tag{128}$$

By verifying the relations in Definition 4.1 it is easy to see that this actually defines an anti-morphism, which means $\phi(h_1 h_2) = \phi(h_2)\phi(h_1)$ for all $h_1, h_2 \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Note that $\phi^2 = \mathrm{Id}$.

Proposition 4.18. *(a) The bilinear form $\langle\ ,\ \rangle$ is symmetric.*
(b) For $H \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and $f, g \in \mathcal{P}$ we have $\langle H(f), g\rangle = \langle f, \phi(H)(g)\rangle$.
(c) The radical Rad $\subseteq \mathcal{P}$ *is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule.*

Proof. Let $f, g \in \mathcal{P}$, which we identify with the corresponding Laurent polynomials in the variables $X^{\pm 1}$ inside $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. This is possible by the PBW-basis theorem in Corollary 4.5. Note that for ϕ defined above we have for any $H \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and $1 \in \mathcal{P}$ that $\phi(H)(1)(q^{-\frac{k}{2}}) = H(1)(q^{-\frac{k}{2}})$. One can check this easily on the PBW-basis elements. Hence we obtain using $\phi^2 = \mathrm{Id}$

$$\begin{aligned}\langle f, g\rangle &= \phi(f)(g)(1)(q^{-\frac{k}{2}}) = \phi\big(\phi(f)(g)\big)(1)(q^{-\frac{k}{2}}) \\ &= \phi(g)(f)(1)(q^{-\frac{k}{2}}) = \langle g, f\rangle,\end{aligned} \tag{129}$$

which shows the symmetry of $\langle\ ,\ \rangle$.

For any $H \in \{X^{\pm 1}, Y^{\pm 1}\} \subseteq H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ we obtain from the definition of $\langle\ ,\ \rangle$ and by its symmetry

$$\langle H(f), g\rangle = \langle f, \phi(H)g\rangle. \tag{130}$$

To prove the same for T observe that for $n \in \mathbb{Z}$ and $X^n \in \mathcal{P} \cong \mathbb{K}[X^{\pm 1}] \subseteq H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ we have

$$T(X^n) = TX^n - X^{-n}(T - t^{\frac{1}{2}}), \tag{131}$$

in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, which can be verified using Equation (100) and induction on $|n|$. Then we have for $g \in \mathcal{P}$ and $n \in \mathbb{Z}$:

$$\begin{aligned}\langle T(q^{nx}), g\rangle &= \phi\big(T(X^n)\big)(g)(1)(q^{-\frac{k}{2}}) \\ &= \phi\big(TX^n - X^{-n}(T - t^{\frac{1}{2}})\big)(g)(1)(q^{-\frac{k}{2}}) \\ &= \phi(TX^n)(g)(1)(q^{-\frac{k}{2}}) - (T - t^{\frac{1}{2}})\phi(X^{-n})(g)(1)(q^{-\frac{k}{2}}) \\ &= \phi(X^n)\phi(T)(g)(1)(q^{-\frac{k}{2}}) - (T - t^{\frac{1}{2}})\phi(X^{-n})(g)(1)(q^{-\frac{k}{2}}).\end{aligned}$$

But by definition of the T-action on $\mathcal{P}$ we have $(T - t^{\frac{1}{2}})(g')(q^{-\frac{k}{2}}) = 0$ for all $g' \in \mathcal{P}$. Hence the second term vanishes and we obtain $\langle T(q^{nx}), g\rangle = \langle q^{nx}, \phi(T)(g)\rangle = \langle q^{nx}, T(g)\rangle$, from which statement (b) follows. Statement (c) is a direct consequence of (a) and (b).

□

The following three lemmas are known as *duality formula*, *Pieri formula* respectively *evaluation formula* in [Che05, Chapter 2.5 and 2.6]. They should be treated as results about meromorphic complex valued functions in the variables $q^{\frac{1}{2}}, k, x$ on $(\mathbb{C}\setminus\{0\})\times\mathbb{C}^2$, where we think of $\mathcal{P}$ as a subspace of such functions. In particular we will assume $q, t = q^k$ to be transcendental over $\mathbb{C}$ now. Note that e_m can always be defined for transcendental q and k using the intertwining operators as in Lemma 4.13. Then specializing q and k gives the e-polynomials for the respective values of q and k, if all appearing formulas are well-defined. In particular, we can still deduce the same statements for $q, t \in \mathbb{C}$ by evaluating the formulas, if all appearing terms are well-defined. In the formulas the element $\epsilon_m := \frac{e_m}{e_m(-\frac{k}{2})}$ appears, which is always a well-defined meromorphic function, since e_m is never the zero-function.

Lemma 4.19. *For $m, n \in \mathbb{Z}$ we have $\epsilon_n(m_\sharp) = \epsilon_m(n_\sharp)$.*

Proof. By definition of the bilinear form on $\mathcal{P}$ and since e_m is a Y-eigenvector of eigenvalue $q^{-m_\sharp}$ by Corollary 4.15 we have

$$\langle e_n, e_m\rangle = e_n(Y^{-1})e_m(X)(q^{-\frac{k}{2}}) = e_n(m_\sharp)e_m(-\frac{k}{2}), \tag{132}$$

The symmetry of the bilinear form from Proposition 4.18 and dividing by $e_m(-\frac{k}{2})e_n(-\frac{k}{2})$ proves the claim. □

Lemma 4.20. *Let $m \in \mathbb{Z}$ and $\nu = 1$ if $m \leq 0$ and $\nu = -1$ otherwise. Then we have*

$$X^{-1}\epsilon_m = \frac{t^{\frac{1}{2}+\nu}q^{-m+1} - t^{-\frac{1}{2}}}{t^\nu q^{-m+1} - 1}\epsilon_{m-1} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^\nu q^{-m+1} - 1}\epsilon_{1-m}, \tag{133}$$

$$X\epsilon_m = \frac{t^{-\frac{1}{2}+\nu}q^{-m} - t^{\frac{1}{2}}}{t^\nu q^{-m} - 1}\epsilon_{m+1} - \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{t^\nu q^{-m} - 1}\epsilon_{1-m}. \tag{134}$$

Proof. Let $m \leq 0$. The case $m > 0$ follows analogously and is omitted. Recall that for $n \in \mathbb{Z}$ by Corollary 4.15 the e-polynomial e_n and hence $\epsilon_n = \frac{e_n}{e_n(-\frac{k}{2})}$ is a Y-eigenvector of eigenvalue $q^{-n_\sharp}$. Since $Y = \pi T$ and since $m_\sharp - \frac{1}{2} = (m-1)_\sharp$ we obtain

$$q^{-n_\sharp}\epsilon_n(m_\sharp) = t^{\frac{1}{2}}\epsilon_n((m-1)_\sharp) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{1-2m_\sharp} - 1}\big(\epsilon_n((m-1)_\sharp) - \epsilon_n((1-m)_\sharp)\big). \tag{135}$$

Applying the duality formula from Lemma 4.19 shows

$$q^{-n_\sharp}\epsilon_m(n_\sharp) = t^{\frac{1}{2}}\epsilon_{m-1}(n_\sharp) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{1-2m_\sharp} - 1}\big(\epsilon_{m-1}(n_\sharp) - \epsilon_{1-m}(n_\sharp)\big). \tag{136}$$

From this it follows, that the first formula holds for points $x = n_\sharp$ for all $n \in \mathbb{Z}$. This is an infinite set. Hence the formula also holds in general, since the appearing functions are rational functions in q^x and since q is transcendental, in particular not a root of unity. For the second formula note that from the construction of the e-polynomials we have $e_{1-m} = q^{\frac{-m}{2}} X\pi(e_m)$ for $m \leq 0$. Using $X\pi X\pi = q^{\frac{1}{2}}$ gives us $e_m = q^{\frac{m-1}{2}} X\pi e_{1-m}$. Then using the duality formula and $\epsilon_1 = q^{\frac{k}{2}} q^x$ we obtain

$$(X\pi\epsilon_{1-m})(-\frac{k}{2}) = q^{-\frac{k}{2}}\epsilon_{1-m}(\frac{1+k}{2}) = q^{-\frac{k}{2}}\epsilon_{1-m}(1_\sharp) = q^{\frac{1-m+k}{2}}. \tag{137}$$

Hence we have $\epsilon_m = q^{\frac{m-1-k}{2}} X\pi\epsilon_{1-m}$. We apply $q^{\frac{m-k}{2}} X\pi$ to the left hand side of the first Pieri formula at index $1-m$ and obtain $q^{\frac{m-k}{2}} X\pi X^{-1}(\epsilon_{1-m}) = X\epsilon_m$. Applying $q^{\frac{m-k}{2}} X\pi$ to the right hand side of the first Pieri formula at the index $1 - m$ yields

$$\begin{aligned}
&q^{\frac{m-k}{2}} X\pi \left(\frac{t^{-\frac{1}{2}} q^m - t^{-\frac{1}{2}}}{t^{-1} q^m - 1}\epsilon_{-m} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{-1}q^m - 1}\epsilon_m \right) \\
=&\frac{t^{-\frac{1}{2}} q^m - t^{-\frac{1}{2}}}{t^{-1} q^m - 1}\epsilon_{m+1} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{-1}q^m - 1} q^{m-k}\epsilon_{1-m} \\
=&\frac{t^{\frac{1}{2}} q^{-m} - t^{\frac{1}{2}}}{t q^{-m} - 1}\epsilon_{m+1} - \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{t q^{-m} - 1}\epsilon_{1-m}.
\end{aligned} \tag{138}$$

This shows the second formula for $m \leq 0$. The case $m > 0$ is similar. $\square$

Lemma 4.21. *For $m \in \mathbb{Z}$ set $|m|' := m$ if $m > 0$ and $|m|' := 1 - m$ if $m \leq 0$. We have*

$$e_m(-\frac{k}{2}) = t^{-\frac{|m|}{2}} \prod_{0<j<|m|'} \frac{1 - q^j t^2}{1 - q^j t}. \tag{139}$$

Proof. By Lemma 4.13 the leading term of e_m with respect to $\prec$ is q^{mx}. Hence by comparing the leading terms in the first Pieri formula from Lemma 4.20 we obtain for $m \leq 0$

$$\frac{1}{e_m(-\frac{k}{2})} = \frac{t^{\frac{3}{2}} q^{-m+1} - t^{-\frac{1}{2}}}{t q^{-m+1} - 1} \frac{1}{e_{m-1}(-\frac{k}{2})}. \tag{140}$$

Similarly, we deduce from the second Pieri-formula for $m > 0$

$$\frac{1}{e_m(-\frac{k}{2})} = \frac{t^{-\frac{3}{2}} q^{-m} - t^{\frac{1}{2}}}{t^{-1} q^{-m} - 1} \frac{1}{e_{m+1}(-\frac{k}{2})}. \tag{141}$$

The claim now follows by rearranging these terms and using induction starting at $e_0(-\frac{k}{2}) = 1$ and $e_1(-\frac{k}{2}) = t^{-\frac{1}{2}}$. $\square$

Finally, we introduce three isomorphisms of double affine Hecke algebras, which will allow us to twist representations and thereby construct new ones.

Lemma 4.22. *The following maps define* $\mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$*-algebra isomorphisms of the respective double affine Hecke algebras.*

$$\iota : H(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \to H(q^{\frac{1}{2}}, t^{-\frac{1}{2}}), X \mapsto X, Y \mapsto Y, T \mapsto -T, \tag{142}$$

$$\zeta_y : H(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \to H(q^{\frac{1}{2}}, t^{\frac{1}{2}}), X \mapsto X, Y \mapsto -Y, T \mapsto T, \tag{143}$$

$$\zeta_x : H(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \to H(q^{\frac{1}{2}}, t^{\frac{1}{2}}), X \mapsto -X, Y \mapsto Y, T \mapsto T. \tag{144}$$

Here we consider $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$ *as a* $\mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$*-algebra via the obvious 'identity' map* $\mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \to \mathbb{C}(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$ *sending* $(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ *to* $(q^{\frac{1}{2}}, t^{\frac{1}{2}})$*.*

Proof. This is an easy check of the relations and hence omitted. $\square$

Let A and B be two rings and $\phi : A \to B$ an isomorphism between them. For a B-module M we will denote the A-module obtained by precomposing with ϕ by ${}^{\phi}M$. We will use these twists often for A and B two one-dimensional Hecke algebras and ϕ some composition of the isomorphisms from Lemma 4.22. These morphisms commute if we denote the ones for $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and the ones for $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$ by the same symbol ι, ζ_y or ζ_x. With this convention we also have that these isomorphisms are idempotent.

Remark 4.23. Note that from the construction of the e-polynomials we obtain $\zeta_x(e_n) = (-1)^{\pm 1} e_n$. Also, we can use ζ_x to define a bilinear form $\langle\ ,\ \rangle_-$ on $\mathcal{P}$ by precomposing $\langle\ ,\ \rangle$ with ζ_x in both components. Then $\mathrm{Rad}_- := \zeta_x(\mathrm{Rad})$ is the radical of $\langle\ ,\ \rangle_-$. For both of these statements we treat $\mathcal{P}$ as the subalgebra of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ generated by $X^{\pm 1}$, which is possible by the PBW-Theorem from Corollary 4.5.

4.3 Finite-dimensional irreducible modules for generic q

The parameter q of the one-dimensional DAHA $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is called *generic* if $q \in \mathbb{C} \setminus \{0\}$ and q is not a root of unity. If $q \in \mathbb{C} \setminus \{0\}$ is a root of unity we say that q is *special*. The goal of this section is to describe and classify the finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules for generic q. This section is based on the results from [Che05, Chapter 2.8].

As already indicated the polynomial representation $\mathcal{P}$ from Proposition 4.3 plays an important role. In fact, the following proposition shows that any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module is a quotient of $\mathcal{P}$ or a quotient of a twist of the polynomial representation by ι, ζ_y or $\zeta_y \iota$ from Lemma 4.22.

Proposition 4.24. *Let q be generic. Then non-trivial finite-dimensional irreducible* $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$*-modules exist only for* $t = q^{\pm(\frac{1}{2}+n)}$ *for* $n \in \mathbb{Z}_{\geq 0}$ *or*

$t = -q^{\pm\frac{n}{2}}$ for $n \in \mathbb{Z}_{>0}$. Every such module is a quotient of $\mathcal{P}$, ${}^{\zeta_y}\mathcal{P}$, ${}^{\iota}\bar{\mathcal{P}}$ or ${}^{\zeta_y\iota}\bar{\mathcal{P}}$, where $\mathcal{P}$ is the polynomial representation of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and $\bar{\mathcal{P}}$ is the polynomial representation of $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$.

Proof. Let V be a finite-dimensional irreducible module of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. The isomorphisms from Lemma 4.22 are all idempotent, therefore we can equivalently show that V, ${}^{\zeta_y}V$ respectively ${}^{\iota}V$ or ${}^{\zeta_y\iota}V$ is a quotient of $\mathcal{P}$ respectively $\bar{\mathcal{P}}$, which we will do now. Since V is finite-dimensional and Y is invertible we can find a Y-eigenvector $v_0 \in V$ of eigenvalue q^{λ} for some $\lambda \in \mathbb{C}$. As in Definition 4.11 and Lemma 4.9 we use the chain of intertwining operators to construct the Y-eigenvectors v_m of eigenvalue q^{λ_m}, where $\lambda_m = -\lambda - \frac{m}{2}$ for $m > 0$ and $\lambda_m = \lambda - \frac{m}{2}$ for $m \leq 0$. We can apply the lemma, since V is finite-dimensional and thus has a generalized Y-eigenspace decomposition. We can construct these v_m up to $m > 0$ where B_m is not invertible or non-existent. If all B_m exist and are invertible, in particular if $t = 1$, the sequence of v_m is infinite and contains Y-eigenvectors for infinitely many different eigenvalues q^{λ_m}, since q is not a root of unity. This contradicts V being finite-dimensional. Therefore we can find $m > 0$ such that $q^{2\lambda_m} = 1$ and hence B_m does not exist or we can find m such that $q^{2\lambda_m} = t^{\pm 1}$ and hence B_m is not invertible. By replacing v_0 with v_m and therefore λ with λ_m we can assume one of the following:

(a) $q^{2\lambda} = 1 \implies q^{\lambda} = \pm 1$ (non-existence),

(b) $q^{2\lambda} = t^{\pm 1} \implies q^{\lambda} = \pm t^{\pm\frac{1}{2}}$ (non-invertibility).

For the proof of the proposition we can without loss of generality twist the module V by ζ_y, ι or $\iota\zeta_y$ from Lemma 4.22. Replacing V with ${}^{\zeta_y}V$ lets us replace q^{λ} with $-q^{\lambda}$, while replacing V with ${}^{\iota}V$ lets us replace $t^{\frac{1}{2}}$ with $t^{-\frac{1}{2}}$ and also $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ with $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$ and $\mathcal{P}$ with $\bar{\mathcal{P}}$. Therefore we can assume that V has a Y-eigenvector v_0 with one of the following eigenvalues by using ζ_y, ι or $\iota\zeta_y$:

$$\text{(a) } q^{\lambda} = 1, \qquad\qquad \text{(b) } q^{\lambda} = t^{\frac{1}{2}}. \tag{145}$$

The chain of intertwining operators applied to the new v_0 with weight λ as above must again reach some $m > 0$ for which B_m is not invertible or not defined, hence some $m > 0$ for which $q^{2\lambda_m} = t^{\pm 1}$ or $q^{2\lambda_m} = 1$. Because q is not a root of unity and since $\lambda_m = -\lambda - \frac{m}{2}$, only the following cases are possible:

$$\text{(a) } t^{\pm 1} = q^{-m} \text{ for some } m > 0, \quad \text{(b) } t^{\pm 1} = t^{-1}q^{-m} \text{ for some } m > 0. \tag{146}$$

We start with the (b)-case. By solving for t we can deduce $t = \pm q^{-\frac{m}{2}}$ for $m > 0$, since q is not a root of unity. Because of the assumption that v_0 is a

Y-eigenvector of eigenvalue $q^\lambda = t^{\frac{1}{2}}$ in (145) a simple calculation shows that $\tilde{v} := (T - t^{\frac{1}{2}})(v_0)$ is either a Y-eigenvector of eigenvalue $q^{\tilde{\lambda}} = t^{-\frac{1}{2}}$ or $\tilde{v} = 0$. If $\tilde{v} \neq 0$ we apply the chain of intertwining operators to $\tilde{v}$ and look at the appearing Y-eigenvalues $q^{\tilde{\lambda}_n} = t^{\frac{1}{2}} q^{-\frac{n}{2}}$ for $n > 0$. Since $m > 0$ and since q is not a root of unity, we have $q^{2\tilde{\lambda}_n} = \pm q^{-\frac{m}{2} - n} \neq 1$ and $q^{2\tilde{\lambda}_n} \neq t^{\pm 1}$ for all $n > 0$. Hence all B_n-intertwining operators exist and are invertible. Therefore we could generate infinitely many Y-eigenvectors of different eigenvalues via the chain of intertwining operators in Definition 4.11, which contradicts that V is finite-dimensional. So $\tilde{v}$ must be zero. But then v_0 is a T-and Y-eigenvector of eigenvalue $t^{\frac{1}{2}}$ and we obtain by Corollary 4.7 a morphism $\mathcal{P} \to V$, which must be surjective by the irreducibility of V.

If in case (b) we have $t = q^{-\frac{m}{2}}$ for $m = 2l$ with $l > 0$, we can reduce to case (a) by using the chain of intertwining operators. Indeed, the intertwining operators B_n for $n > 0$ exist up to $n = m$ when $2\lambda_n = \frac{m}{2} - \frac{n}{2} = 0$ and if they exist they are invertible up to $n = 2m$ when $2\lambda_n = \frac{m}{2} - \frac{n}{2} = -\frac{m}{2}$. Hence we can reach a Y-eigenvector with eigenvalue $q^\lambda = 1$ as in the assumption of case (a) in (145). We will see now that case (a) does not produce any finite-dimensional irreducible modules, hence this explains why the case $t = q^m$ for integral m does not appear in the proposition.

In case (a) we apply the chain of intertwining operators again, but this time to the space $V_0 := \langle T, Y^{\pm 1} \rangle v_0$, where $\langle T, Y^{\pm 1} \rangle$ is the subalgebra of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ generated by T and $Y^{\pm 1}$. We have $(Y - 1)Tv_0 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})v \neq 0$, since $t \neq 1$. Hence V_0 is two-dimensional and Y is not semisimple on V_0, but $(Y - 1)^2(V_0) = 0$. By the conditions of case (a), we see that all intertwining operators B_n exist and that B_n is invertible, unless $\pm k = n$ for some $n > 0$, where k is such that $q^k = t$. Because all B_n exist, we can define inductively via the order $\prec$ on $\mathbb{Z}$:

$$V_{1-n} := A_{1-n}(V_n) \text{ for } 1 - n \leq 0, \quad V_{-n} := B_n(V_n) \text{ for } n > 0. \tag{147}$$

By Lemma 4.9 we deduce that V_n is a Y-module and $(Y - q^{\lambda_n})^2(V_n) = 0$ for all $n > 0$, since this holds for V_0. If all intertwining operators are invertible, V is infinite dimensional by considering the infinitely many (generalized) Y-eigenvalues q^{λ_n} for the two-dimensional V_n. Hence we can assume $k = \pm n$ for some $n > 0$. Then V_n is not Y-semisimple, since otherwise we could use the inverse chain of invertible intertwining operators as in the proof of Lemma 4.13 and deduce that Y is semisimple on V_0. If $V_{-n} \neq 0$, then V is infinite dimensional, since all intertwining operators after n are invertible as q is not a root of unity and hence there does not exist an $n' > n$ with $q^{2\lambda_{n'}} = t^{\pm 1}$. Choose some $0 \neq v \in V_n$ that is not a Y-eigenvector. From $B_n(V_n) = 0$ we deduce by definition of B_n

$$\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{Y^{-2} - 1} \right)(w) = 0 \implies T(w) = -\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{Y^{-2} - 1}(w) \text{ for } w \in V_n. \tag{148}$$

Observe that $(Y^{-2}-1)^{-1}$ is well defined on V_n, since $q^{2\lambda_n} = t^{\pm 1} \neq 1$. Using relation (YT) gives us

$$\left(Y^{-1}\frac{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}{Y^{-2}-1}Y^{-1}\frac{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}{Y^{-2}-1}\right)(v) = v \tag{149}$$

and hence

$$(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^2 v = (Y^{-1}-Y)^2 v. \tag{150}$$

We can assume that Y acts via the Jordan-Normalform $\begin{pmatrix} q^{\lambda_n} & 1 \\ 0 & q^{\lambda_n} \end{pmatrix}$ on V_n, where the second column corresponds to v. A simple matrix computation shows $q^{\lambda_n} = q^{-\lambda_n}$. This then implies $t^{\pm 1} = q^{2\lambda_n} = 1$ and by $\pm k = n$ we obtain a contradiction to q being generic. We conclude that case (a) does not yield finite-dimensional $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules, which finishes the proof. $\square$

Proposition 4.25. *Let q be generic. Non-trivial irreducible quotients of $\mathcal{P}$ exist only for $n \in \mathbb{Z}_{\geq 0}$ and $t = q^{-(\frac{1}{2}+n)}$ or $n \in \mathbb{Z}_{>0}$ and $t = -q^{-\frac{n}{2}}$.*

Proof. Any non-trivial irreducible quotient would in particular be a quotient as $X^{\pm 1}$-modules and hence finite-dimensional, since it is of the form $\mathcal{P}/(e)$ for some $e \in \mathcal{P}$. By the previous Proposition 4.24 we can assume that $t = q^{\pm(\frac{1}{2}+n)}$ for $n \in \mathbb{Z}_{\geq 0}$ or $t = -q^{\pm\frac{n}{2}}$ for $n \in \mathbb{Z}_{>0}$. We only have to exclude the cases with positive exponents to obtain the claim from the proposition. Assume we have $t = q^{\frac{1}{2}+n}$ for $n \geq 0$ or $t = -q^{\frac{n}{2}}$ for $n > 0$. Apply the chain of intertwining operators from Definition 4.11 respectively Lemma 4.13 to $v_0 = 1 \in \mathcal{P}$ which has Y-eigenvalue $q^{\lambda} = t^{\frac{1}{2}}$. For $m > 0$ we have $q^{\lambda_m} = t^{-\frac{1}{2}}q^{-\frac{m}{2}}$. Since q is not a root of unity this implies that all intertwining operators exist and are invertible, because we can never reach a Y-eigenvalue q^{λ_m} with $q^{2\lambda_m} = t^{-1}q^{-m} = q^{-\frac{1}{2}-n-m} = 1$ (respectively $q^{2\lambda_m} = -q^{-\frac{n}{2}-m} = 1$) or $q^{2\lambda_m} = t^{-1}q^{-m} = t^{\pm 1}$ for $m > 0$. Hence, all e-polynomials e_m for $m \in \mathbb{Z}$ exist by Lemma 4.13 and their Y-eigenvalues $q^{\lambda_m} = q^{-m_\sharp}$ are pairwise different, since q is not a root of unity and by the choice of t.

Assume $V := \mathcal{P}/(e)$ is a non-trivial quotient. By multiplying e with an appropriate power of the invertible element q^x we can assume without loss of generality that $e = c_+q^{mx} + \ldots + c_-q^{-mx}$ or $e = c_+q^{(m+1)x} + \ldots + c_-q^{-mx}$ and $c_+, c_- \neq 0$. Note that then the dimension of V is $2m$ or $2m+1$ and the difference between top and bottom degree for any element in (e) must be larger or equal to $2m$ or $2m+1$, as otherwise the dimension would be smaller than $2m$ or $2m+1$. But since Y preserves $\prec$ and by choice of the generator e this implies that e is a Y-eigenvector. Since all e_i have pairwise different eigenvalues we must have that e is proportional to e_i for some $i \in \mathbb{Z}$. Note that in this setting all A_i- and B_i-intertwining operators are invertible elements in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Thus $e_i \in (e)$ implies that $1 \in (e)$ and hence $\mathcal{P}$ is irreducible for such values of t. $\square$

Proposition 4.26. *Let q be generic.*

(a) For $t = q^{-(\frac{1}{2}+n)}$ with $n \in \mathbb{Z}_{\geq 0}$ the polynomial representation $\mathcal{P}$ has an up to isomorphism unique maximal non-trivial quotient given by $\mathcal{P}/(e_{-2n-1})$. There exist up to isomorphism two non-trivial irreducible quotients of $\mathcal{P}$. They are given by $V^{\pm}_{2n+1} := \mathcal{P}/(e^{\pm})$ where $e^{\pm} = e_{n+1} \pm t^{-\frac{1}{2}} e_{-n}$. The dimension of the irreducible quotients is $2n+1$. Furthermore, $(e^+) = \mathrm{Rad}$ and $(e^-) = \mathrm{Rad}_-$.

(b) For $t = -q^{-\frac{n}{2}}$ with $n > 0$ the polynomial representation $\mathcal{P}$ has an up to isomorphism unique non-trivial quotient given by $V_{2n} := \mathcal{P}/(e_{-n})$. In particular, this quotient is irreducible. Its dimension is $2n$. Moreover we have $(e_{-n}) = \mathrm{Rad}$.

Proof. We use the chain of intertwining operators from Definition 4.11 starting at $1 \in \mathcal{P}$ again. In the same manner as before we see that all B_m exist. In case (a) all B_m except B_{2n+1} are invertible; in case (b) all B_m except B_n are invertible. In particular, all e_m exist in both cases by Lemma 4.13. In any non-trivial quotient the image of e_{-2n-1} respectively e_{-n} must be zero. Otherwise we could use the chain of invertible intertwining operators starting from the image of e_{-2n-1} respectively e_{-n} and produce infinitely many eigenvectors of different eigenvalues q^{λ_n} in the quotient. On the other hand these e-polynomials are Y-eigenvectors by definition and also T-eigenvectors, since in case (a) we have $e_{-2n-1} = B_{2n+1}(e_{2n+1}) = t^{\frac{1}{2}}(T - t^{\frac{1}{2}})(e_{2n+1})$ and in case (b) we have $e_{-n} = B_n(e_n) = t^{\frac{1}{2}}(T - t^{\frac{1}{2}})(e_n)$. Hence the ideals generated by these elements are submodules by Corollary 4.6 and we see that the corresponding quotients are maximal.

Let us finish case (a) first. Since the top degree with respect to $\prec$ of e_{-2n-1} is $-2n-1$ and hence negative and because e_{-2n-1} is a T-eigenvector, $q^{(2n+1)x}$ appears in e_{-2n-1} with non-zero coefficient. Hence $\mathcal{P}/(e_{-2n-1})$ has dimension $4n+2$. For degree reasons and since the e_m span $\mathcal{P}$, the quotient has a basis given by the images of e_m for $m \prec -2n-1$. Recall that $-m_\sharp = -\frac{m+\mathrm{sgn}(m)k}{2}$ for $m \neq 0$ and $-0_\sharp = \frac{k}{2}$ is the Y-weight of e_m. We see that $-m_\sharp = -m'_\sharp$ for $m, m' \preceq 2n+1$ if and only if $m' = -2n-1+m$. Any quotient of $\mathcal{P}$ is of the form $\mathcal{P}/(e)$ for some Y-eigenvector $e \in \mathcal{P}$ as shown in the proof of Proposition 4.25. Hence any non-trivial quotient of $\mathcal{P}/(e_{-2n-1})$ must be of the form $\mathcal{P}/(e)$ for $e = c_1 e_m + c_2 e_{-2n-1+m}$ for some constants $c_1, c_2 \in \mathbb{C}$ and $0 < m \leq 2n+1$. Since in this range all intertwining operators are invertible elements of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, each Y-eigenspace of $\mathcal{P}/(e_{-2n-1})$ must have non-trivial image in the quotient, hence the dimension is at least $2n+1$. On the other hand if some Y-eigenspace has two-dimensional image, then again by using the invertible intertwining operators we see that each Y-eigenspace has two-dimensional image and hence the quotient is trivial. Therefore the dimension of $\mathcal{P}/(e)$ is $2n+1$ and we can assume that $e = q^{(n+1)x} + \ldots + c' q^{-nx}$ for some non-zero constant c'. By the above description of the Y-spectrum of $\mathcal{P}/(e_{-2n-1})$ we see that $e = e_{n+1} + c e_{-n}$ for some constant $c \in \mathbb{C}$. If

$c = 0$, we see $1 \in (e)$ by using the inverse chain of intertwining operators. Therefore $c \neq 0$. By assumption $(e) \subseteq \mathcal{P}$ must be a submodule, in particular $A_{-n}(e) \in (e)$. A quick computation using the definition of the e-polynomials and relation (PX) shows

$$A_{-n}(e) = t^{-1}e_{-n} + ce_{n+1}. \tag{151}$$

This term must be proportional to e, since otherwise we can find a linear combination of e and $A_{-n}(e)$ which kills $q^{(n+1)x}$ and obtain a contradiction to $\dim \mathcal{P}/(e) = 2n+1$. Therefore $c = \pm t^{-\frac{1}{2}}$. Note that (e^+) coincides with the radical Rad from Definition 4.17. Indeed, by the evaluation formula from Lemma 4.21 we see that $e^+(q^{-\frac{k}{2}}) = 0$ and since e^+ is a Y-eigenvector we have $e^+ \in \mathrm{Rad}$. Because Rad is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule by Proposition 4.18, we have $(e^+) \subseteq \mathrm{Rad}$. If $\mathrm{Rad} \nsubseteq (e^+)$ then the $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $\mathcal{P}/\mathrm{Rad}$ would be a quotient of $\mathcal{P}/(e_{-2n-1})$ of dimension smaller than $2n+1$ and hence trivial by the above consideration. But then we would have $1 \in \mathrm{Rad}$, which is clearly not the case. Therefore, we have $(e^+) = \mathrm{Rad}$. For (e^-) recall the automorphism ζ_x of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ from Lemma 4.22, which sends $X \mapsto -X$ and e_m for $m \in \mathbb{Z}$ to $(-1)^m e_m$ by Remark 4.23. By the same remark we have $\mathrm{Rad}_- = \zeta_x(\mathrm{Rad}) = \zeta_x((e^+)) = (\zeta_x(e^+)) = (e^-)$. From this it follows in particular that $(e^-) \subseteq \mathcal{P}$ is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Finally, these two quotients are not isomorphic, since e^+ and e^- are not proportional. Hence we cannot even have an X-module morphism between the quotients.

For (b) note that all e_m for $m \prec -n$ have pairwise different eigenvalues. Hence, similar as in case (a), we see by using the invertible intertwining operators that $\mathcal{P}/(e_{-n})$ can not have a non-trivial quotient. Using the evaluation formula we see $(e_{-n}) \subseteq \mathrm{Rad}$, which already implies equality as $\mathrm{Rad} \subseteq \mathcal{P}$ is a submodule by Proposition 4.18. □

Corollary 4.27. *The Y-spectrum of the $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $V^{\pm}_{2n+1}$ is given by $\{q^{-m_\sharp} \mid -n \leq m \leq n\}$ with cardinality $2n+1$. The module V_{2n} has Y-spectrum $\{q^{-m_\sharp} \mid -n+1 \leq m \leq n\}$ with cardinality $2n$. We have that ${}^{\zeta_y}(V^{\pm}_{2n+1})$ is not isomorphic to either V^+_{2n+1} or V^-_{2n+1}, but ${}^{\zeta_y}(V_{2n}) \cong V_{2n}$ holds.*

Proof. The description of the Y-spectra follows directly from the proof of Proposition 4.26, where we show that $V^{\pm}_{2n+1}$ is spanned by the images of the e-polynomials e_m for $-n \leq m \leq n$ and V_{2n} is spanned by the images e_m for $-n+1 \leq m \leq n$. The Y-spectrum of $V^{\pm}_{2n+1}$ is not invariant under $Y \leftrightarrow -Y$ and hence ${}^{\zeta_y}(V^{\pm}_{2n+1}) \ncong V^{\pm}_{2n+1}$, since this would imply that -1 is a rational power of q in contradiction to q being generic. To construct the isomorphism ${}^{\zeta_y}(V_{2n}) \cong V_{2n}$ note that the image of e_n in V_{2n} has Y-eigenvalue $q^{-n_\sharp} = q^{-\frac{n}{2}}t^{-\frac{1}{2}} = -t^{\frac{1}{2}}$ and T-eigenvalue $t^{\frac{1}{2}}$, since $B_n(e_n) = e_{-n}$ in $\mathcal{P}$. Unwinding the definition of B_n gives $T(e_n) = t^{\frac{1}{2}}e_n \bmod (e_{-n})$. By the universal property of $\mathcal{P}$ from Corollary 4.7 we obtain a non-trivial and therefore a surjective morphism $\mathcal{P} \to {}^{\zeta_y}(V_{2n})$. Since ζ_y does not change the X-action, e_{-n} acts trivially on ${}^{\zeta_y}(V_{2n})$ and the morphism factors through $\mathcal{P}/(e_{-n})$, which then must become an isomorphism by the irreducibility of the modules. $\square$

Theorem 4.28. *Let q be generic. The following gives a full list of possible values of t for which finite-dimensional irreducible modules exist and lists all possible such modules up to isomorphism.*

Value of t	*Modules*	*Dimension*
$t = q^{-\frac{1}{2}-n}$ *for* $n > 0$	$V^{\pm}_{2n+1}$, ${}^{\zeta_y}V^{\pm}_{2n+1}$	$2n+1$
$t = q^{\frac{1}{2}+n}$ *for* $n > 0$	${}^{\iota}V^{\pm}_{2n+1}$, ${}^{\iota\zeta_y}V^{\pm}_{2n+1}$	$2n+1$
$t = -q^{-\frac{n}{2}}$ *for* $n > 0$	$V_{2n} \cong {}^{\zeta_y}V_{2n}$	$2n$
$t = -q^{\frac{n}{2}}$ *for* $n > 0$	${}^{\iota}V_{2n} \cong {}^{\iota\zeta_y}V_{2n}$	$2n$

Proof. This is clear from the previous discussion in Propositions 4.24, 4.25 and 4.26 and Corollary 4.27. $\square$

4.4 Finite-dimensional irreducible modules for special q

Throughout this section let $q^{\frac{1}{2}} \in \mathbb{C}$ be a primitive $2N$-th root of unity for $N \geq 1$. We will not consider the representation theory for $q^{\frac{1}{2}}$ an odd root of unity, but only refer to [Che05, Chapters 2.9 and 2.10] for some remarks on it. The goal of this section is to classify all finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules with a Y-eigenvector of eigenvalue q^{λ} for some $\lambda \in \mathbb{C}$. Note that choosing an appropriate branch of the logarithm allows us to always assume that a Y-eigenvalue in the exponential form q^{λ} for some $\lambda \in \mathbb{C}$ exists. The results presented in this section are based on [Che05, Chapters 2.8 and 2.9].

For the classification it will be useful to discern three possible classes of weights λ.

Definition 4.29. An element $\lambda \in \mathbb{C}$ is called *regular* if $2\lambda \notin \frac{1}{2}\mathbb{Z}$, *half-singular* if $2\lambda \in \frac{1}{2} + \mathbb{Z}$ and *singular* if $2\lambda \in \mathbb{Z}$.

Let V be a Y-cyclic $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module, by which we mean an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with a Y-weight vector $v \in V$ that generates V as an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. Let $\lambda \in \mathbb{C}$ be the weight of v, in other words $Y(v) = q^{\lambda}v$. We will deduce in the upcoming Lemma 4.31 that if λ is regular, half-singular respectively singular then V has a generalized weight space decomposition with regular, half-singular respectively singular weights. For this we will need a class of central elements, which will later also play a role in classifying quotients of the polynomial representation $\mathcal{P}$ in Lemma 4.38.

Lemma 4.30. *The elements $X^{2N} + X^{-2N} + C$ and $Y^{2N} + Y^{-2N} + C$ for any $C \in \mathbb{C}$ are central in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$.*

Proof. For the X-case it is easy to calculate that $X^{2N}+X^{-2N}+C$ commutes with any of the generators $X^{\pm 1}, \pi^{\pm 1}, T$ from Definition 4.1. One can use Equation (131) for the calculation. The Y-case reduces to the X-case by applying the anti-isomorphism ϕ from Equation (128). $\square$

Lemma 4.31. *Let V be a Y-cyclic $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with $v \in V$ such that $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})v = V$ and $Y(v) = q^{\lambda}v$ for some $\lambda \in \mathbb{C}$. Then V has a generalized Y-weight space decomposition such that the Jordan blocks are at most two-dimensional and the Y-eigenvalues are of the form $q^{\pm\lambda+\frac{j}{2}}$ for some $0 \leq j \leq 2N-1$. In particular, if λ is regular, half-singular or singular, then the same holds for all Y-weights of V.*

Proof. We have the following equality in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$:

$$-q^{-2N\lambda}\prod_{j=0}^{2N-1}(Y-q^{\lambda+\frac{j}{2}})(Y^{-1}-q^{\lambda+\frac{j}{2}}) = Y^{2N}+Y^{-2N}-q^{2N\lambda}-q^{-2N\lambda}. \quad (152)$$

This can be deduced from

$$\prod_{j=0}^{2N-1}(Y - q^{\lambda+\frac{j}{2}}) = Y^{2N} - q^{2N\lambda}, \quad \prod_{j=0}^{2N-1}(Y^{-1} - q^{\lambda+\frac{j}{2}}) = Y^{-2N} - q^{2N\lambda},$$

which is obtained by factoring $Y^{\pm 2N} - q^{2N\lambda}$ using its $2N$ distinct roots as a polynomial in Y or Y^{-1} on $\mathbb{C}$. Denote the element from (152) by Z. Let $\tilde{v} = h(v)$ for some $h \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ be an arbitrary element in V. As we have seen in Lemma 4.30 Z is central and hence the claims follow from $Z(h(v)) = h(Z(v)) = 0$. $\square$

Unlike in the generic case for special q not all finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules can be constructed via the polynomial representation $\mathcal{P}$. We will first describe and classify these exceptional modules in the upcoming three propositions. We start with the case that the module has regular Y-weights. As always, we have $q^k = t$ for some $k \in \mathbb{C}$. For the next proposition we need to use an equivalence relation $\sim$ on $\mathbb{C}$ defined by $c_1 \sim c_2$ if and only if $c_1 = q^{\frac{n}{2}}c_2$ for some $n \in \mathbb{Z}$. For each equivalence class we pick a representative and we denote the map from $\mathbb{C}$ to the set of these representatives by $[\quad]$.

Proposition 4.32. *Let $\lambda \in \mathbb{C}$ regular with $k \notin \pm 2\lambda + \mathbb{Z}$. Any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with Y-weight λ is isomorphic to one of the $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules $V([q^{\lambda}], K)$ for some $K \in \mathbb{C} \setminus \{0\}$ defined in the proof. Its dimension is $4N$. We have $V([q^{\lambda}], K) \cong V([q^{\lambda'}], K')$ if and only if $[q^{\lambda}] = [q^{\lambda'}]$ and $K = K'$.*

Proof. Let V be a finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with a Y-weight vector $v \in V$ of regular weight λ and $k \notin \pm 2\lambda + \mathbb{Z}$. We apply the chain of intertwining operators from Definition 4.11 to v. Since λ is regular, all intertwining operators exist and since $k \notin \pm 2\lambda + \mathbb{Z}$, all intertwining operators are invertible. Hence, $[q^{\lambda}]$ is a Y-eigenvalue of V and we can assume $q^{\lambda} = [q^{\lambda}]$. Set $\tilde{K} := B_{2N} A_{1-2n} \ldots A_0$ and let

$$V_0 := \{v \in V \mid (Y - q^{\lambda})(v) = 0\}.$$

Lemma 4.9 shows that $\tilde{K}$ preserves V_0. Hence we can find a $\tilde{K}$-eigenvector $v_0 \in V_0$ of eigenvalue K for some $K \in \mathbb{C} \setminus \{0\}$. Now apply the chain of intertwining operators to v_0 to obtain Y-eigenvectors v_m of pairwise different weights λ_m for $1 - 2N \leq m \leq 2N$ defined as in Lemma 4.9. Set $V' := \oplus_{m=1-2N}^{2N} \mathbb{C} v_m$, which we can depict as follows.

$$\begin{tikzcd}
v_0 \arrow[r, "A_0"'] & v_1 \arrow[r, "B_1"'] & v_{-1} \arrow[r, "A_{-1}"'] & v_2 \arrow[r, dotted, no head] & v_{1-2N} \arrow[r, "A_{1-2N}"'] & v_{2N} \arrow[lllll, bend right, "B_{2N}"']
\end{tikzcd}$$

Note that $B_{2N}(v_{2N}) = K v_0$. The action of the B-intertwining operators determines the action of T to be as follows:

$$\begin{aligned}
T(v_m) &= t^{-\frac{1}{2}} v_{-m} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1} v_m \text{ for } 0 < m < 2N, \\
T(v_{2N}) &= K t^{-\frac{1}{2}} v_0 - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_{2N}} - 1} v_{2N}, \\
T(v_{-m}) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \left(\frac{q^{-2\lambda_m}}{q^{-2\lambda_m} - 1} \right) v_{-m} \\
&\quad + t^{\frac{1}{2}} \left(1 - \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_m} - 1)^2} q^{-2\lambda_m} \right) v_m \text{ for } 0 < m < 2N, \\
T(v_0) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \left(\frac{q^{-2\lambda_0}}{q^{-2\lambda_0} - 1} \right) v_{2N} \\
&\quad + t^{\frac{1}{2}} K^{-1} \left(1 - \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_0} - 1)^2} q^{-2\lambda_0} \right) v_0.
\end{aligned} \tag{153}$$

We can use the A-intertwining operators to determine the action of $X\pi$:

$$\begin{aligned}
X\pi(v_m) &= q^{\frac{m}{2}} v_{-m+1} \text{ for } 0 < m \leq 2N, \\
X\pi(v_{-m}) &= q^{-\frac{m}{2}} v_{m+1} \text{ for } 0 \leq m < 2N,
\end{aligned} \tag{154}$$

Because $(X\pi)^{\pm 1}, T$ and $Y^{\pm 1}$ generate $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, it follows that V' is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule and hence V' equals V by the irreducibility of V.

We want to show that there exists a unique $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module structure on the $\mathbb{C}$-vector space V' such that v_0 is a Y-eigenvector of eigenvalue $[q^\lambda]$, $\tilde{K}(v_0) = Kv_0$ and such that $A_{-m}(v_{-m}) = v_{m+1}$ for $0 \leq m < 2N$ and $B_m(v_m) = v_{-m}$ for $0 < m < 2N$. We will denote this module by $V([q^\lambda], K)$. The above discussion shows that such a module structure is unique if it exists and describes the action of $T, Y, X\pi$. Therefore we only have to verify that the above assignment actually defines a module for all $K \in \mathbb{C} \setminus \{0\}$ by verifying the relations from Definition 4.1. We obtain relation (PX) directly from the definition of the $X\pi$-action. Relation (XT) is equivalent to $X\pi Y X\pi Y = 1$ and follows by a short computation. To verify (T) note that $\{v_m, v_{-m}\}$ for $0 < m < 2N$ respectively $\{v_0, v_{2N}\}$ span a $\mathbb{C}$-vector space that is closed under the proposed action of T. We can determine the characteristic polynomial $P_m(Z)$ for $0 \leq m < 2N$ of the T-action on these two-dimensional spaces:

$$\begin{aligned} P_m(Z) &= \left(-\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1} - Z\right)\left((t^{\frac{1}{2}} - t^{-\frac{1}{2}})\left(\frac{q^{-2\lambda_m}}{q^{-2\lambda_m} - 1}\right) - Z\right) \\ &\quad - \left(1 - \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_m} - 1)^2} q^{-2\lambda_m}\right) \\ &= Z^2 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})Z - 1. \end{aligned} \tag{155}$$

This proves (T), since the roots of $P_m(Z)$ are $Z = \pm t^{\pm\frac{1}{2}}$. To verify (P) or the equivalent condition $TY^{-1}T = Y$ we calculate for $0 < m < 2N$

$$\begin{aligned} TY^{-1}T(v_m) &= T\left(q^{\lambda_m} t^{-\frac{1}{2}} v_{-m} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m - 1}} q^{-\lambda_m} v_m\right) \\ &= q^{\lambda_m} t^{-\frac{1}{2}} (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{q^{-2\lambda_m}}{q^{-2\lambda_m} - 1} v_{-m} \\ &\quad + q^{\lambda_m}\left(1 - \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_m} - 1)^2} q^{-2\lambda_m}\right) v_m \\ &\quad - t^{-\frac{1}{2}} \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1} q^{-\lambda_m} v_{-m} + \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-2\lambda_m} - 1)^2} q^{-\lambda_m} v_m \\ &= q^{\lambda_m} v_m = Y(v_m). \end{aligned} \tag{156}$$

The remaining cases $m = 2N$ and $1 - 2N \leq m \leq 0$ are very similar and hence omitted. This shows that $V([q^\lambda], K)$ is indeed an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module. To prove $V([q^\lambda], K) \cong V([q^{\lambda'}], K')$ if and only if $K = K'$ and $[q^\lambda] = [q^{\lambda'}]$ note that any isomorphism preserves the one-dimensional Y-weight spaces from which the claim easily follows. $\square$

The next proposition deals with exceptional modules with half-singular Y-weights.

Proposition 4.33. *Let $k \notin \frac{1}{2}+\mathbb{Z}$. Any finite-dimensional $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with half-singular Y-weights is isomorphic to $W(\gamma)$ for some $\gamma \in \mathbb{C}$ or $\tilde{W}(\epsilon, \delta)$ for $\epsilon \in \{-1, 1\}$ and $\delta \in \{c^{\frac{1}{2}}, -c^{\frac{1}{2}}\}$, where these modules and the constant c are defined in the proof. The module $W(\gamma)$ is $4N$-dimensional and $\tilde{W}(\epsilon, \delta)$ is $2N$-dimensional. Moreover, $W(\gamma_1) \cong W(\gamma_2)$ if and only if $\gamma_1 = \gamma_2$ and $W(\epsilon_1, \delta_1) \cong W(\epsilon_2, \delta_2)$ if and only if $\epsilon_1 = \epsilon_2$ and $\delta_1 = \delta_2$.*

Proof. Let V be a finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with a Y-weight vector v of half-singular weight λ. We apply the chain of intertwining operators from Definition 4.11 to v. Since λ is half-singular all intertwining operators exist and since $k \notin \frac{1}{2} + \mathbb{Z}$ all intertwining operators are invertible. As λ is half-singular we can thus reach a Y-weight vector v_0 of weight $\lambda = -\frac{1}{4}$. Set $S := q^{-\frac{1}{4}} X\pi$ and $\tilde{K} := B_{2N} A_{1-2N} ... A_0$. Let

$$V_0 := \{v \in V \mid (Y - q^{-\frac{1}{4}})(v) = 0\}. \tag{157}$$

Note that S and $\tilde{K}$ preserve V_0, hence we can assume that v_0 is a $\tilde{K}$-eigenvector of eigenvalue $K \in \mathbb{C}\backslash\{0\}$. Define inductively $v_{m+1} := A_{-m}(v_{-m})$ for $0 \leq m \leq 2N-1$ and $v_{-m} := B_m(v_m)$ for $1 \leq m \leq 2N-1$ and let $V' \subseteq V$ be the $\mathbb{C}$-span of these vectors. Similarly as in Proposition 4.32 we can deduce $V' = V$. Hence, we only have to describe the module structure on V' and classify finite-dimensional irreducible modules with known action of the intertwining operators and known Y-action as on V'. By Lemma 4.9 the vectors v_0 and v_1 are contained in V_0 and v_{-m} and v_{2N+1-m} for $0 < m < 2N$ are contained in

$$V_m := \{v \in V \mid (Y - q^{-\frac{1}{4}+\frac{m}{2}})(v) = 0\}. \tag{158}$$

To avoid confusion we emphasize that v_0 and v_1 respectively v_{-m} and v_{2N+1-m} for $0 < m < 2N$ might be proportional. Using relation (PX) we see $S^2 = \mathrm{Id}$. We shall deduce $S\tilde{K}S\tilde{K} = c$ for some constant $c \in \mathbb{C} \setminus \{0\}$. Using (PX) we obtain $A_0 S = q^{\frac{1}{4}}$ and $A_{-i} A_{i-2N} = q^{\frac{1}{2}}$ for $0 < i \leq 2N-1$. We only apply the B-intertwining operators to Y-eigenvectors with eigenvalue q^{λ} for some $\lambda \in \mathbb{C}$, which allows us to replace the denominator $Y^{-2} - 1$ by $q^{-2\lambda} - 1$ in all upcoming calculations. We have

$$\begin{aligned} B_i B_{2N-i+1} &= t\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_i} - 1}\right)\left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_{2N-i+1}} - 1}\right) \\ &= t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\left(1 + \frac{1}{q^{-\frac{1}{2}+i} - 1} + \frac{1}{q^{\frac{1}{2}-i} - 1}\right) T \\ &\quad + t\left(1 + \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2}{(q^{-\frac{1}{2}+i} - 1)(q^{\frac{1}{2}-i} - 1)}\right) \\ &= \frac{(1 - tq^{-\frac{1}{2}+i})(1 - tq^{\frac{1}{2}-i})}{(1 - q^{-\frac{1}{2}+i})(1 - q^{\frac{1}{2}-i})} \text{ for } 0 < i \leq 2N. \end{aligned} \tag{159}$$

With this we can calculate $S\tilde{K}S\tilde{K} = SB_{2N}...B_1A_0SB_{2N}...A_0$ by successively cancelling out the pairs from above to obtain scalars starting with A_0S. We obtain

$$S\tilde{K}S\tilde{K} = c := \prod_{i=1}^{2N} \frac{(1-tq^{-\frac{1}{2}+i})(1-tq^{\frac{1}{2}-i})}{(1-q^{-\frac{1}{2}+i})(1-q^{\frac{1}{2}-i})}, \tag{160}$$

which shows the claim that $S\tilde{K}S\tilde{K}$ acts as a constant on V_0. The irreducibility of V as an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module is equivalent to the irreduciblity of V_0 as a $\langle S, \tilde{K}\rangle$-module, where $\langle S, \tilde{K}\rangle$ is the subalgebra of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ generated by S and $\tilde{K}$. Indeed, assume V_0 is irreducible and that $V' \subseteq V$ is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule with $v \in V'$ a Y-eigenvector. Since all intertwining operators exist and are invertible and because all intertwining operators are elements of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, as we are only looking at Y-eigenvectors, we can use the chain of intertwining operators starting at v to obtain a non-zero element in $V_0 \cap V'$. By the irreducibility of V_0 as an $\langle S, \tilde{K}\rangle$-module $V_0 \subseteq V'$ and hence $v_0 \in V'$. We can use the chain of intertwining operators to see that all v_m for $1-2N \leq m \leq 2N$ lie in V' and hence $V' = V$. On the other hand if V_0 is not irreducible as an $\langle S, \tilde{K}\rangle$-module let $V_0' \subseteq V_0$ be a non-trivial submodule. We can apply the intertwining operators to V_0' to generate an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule of V, similar to the construction of V' in Proposition 4.32. Therefore we will now classify all finite-dimensional irreducible $\langle S, \tilde{K}\rangle$-modules up to isomorphism. Note that from the relations $S^2 = \mathrm{Id}$ and $S\tilde{K}S\tilde{K} = c$ any element in $\langle S, \tilde{K}\rangle$ can be written as a $\mathbb{C}$-linear combination of $S\tilde{K}^n$ and $\tilde{K}^n$ for $n \in \mathbb{Z}$. Since we can find a $\tilde{K}$-eigenvector in any finite-dimensional $\langle S, \tilde{K}\rangle$-module M the dimension of any non-trivial finite-dimensional irreducible $\langle S, \tilde{K}\rangle$-module is 1 or 2. In the one-dimensional case S must act via multiplication by $\epsilon := \pm 1$, since $S^2 = \mathrm{Id}$ and hence $\tilde{K}$ acts via multiplication by $\delta := \pm c^{\frac{1}{2}}$. These four possible cases clearly give non-isomorphic modules. Now for the two-dimensional case: since $S^2 = \mathrm{Id}$ we can choose an appropriate basis and assume without loss of generality that $S = \mathrm{Id}, S = -\mathrm{Id}$ or $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The irreducibility of M implies the last case. Now assume that $\tilde{K}$ acts with respect to this basis by $\tilde{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$. Since M is irreducible we must have $k_{21} \neq 0$ and by scaling the first basis vector by k_{21} we can without loss of generality assume that $k_{21} = 1$. Calculating $S\tilde{K}S\tilde{K} = c$ now gives $k_{11} = k_{22}$ and $k_{11}^2 - k_{12} = c$. But this shows that the isomorphism class of the $\langle \tilde{K}, S\rangle$-module M depends only on one parameter $\gamma := k_{11} \in \mathbb{C}$ and all possible choices give pairwise non-isomorphic modules. Now what is left to show is that any $\langle \tilde{K}, S\rangle$-module structure on V_0 extends uniquely to an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module structure on V. For this we again use the intertwining operators as in the proof of Proposition 4.32, this time applied to the space V_0 instead of the vector v_0 and furthermore we only use the intertwining operators up to B_N. The proof idea and the calculations are similar to the previous proposition and hence omitted. In particular one can show that the resulting modules depend only on the parameters γ respectively δ and ϵ defined above and the modules are only isomorphic if they coincide. As the intertwining operators give isomorphisms between the weight spaces, we obtain that if V_0 is two-dimensional each V_m is two-dimensional and hence V is $4N$-dimensional, while the case that V_0 is one-dimensional gives that each V_m is one-dimensional and hence V is $2N$-dimensional. We call the resulting $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $W(\gamma)$ in the $4N$-dimensional case and $\tilde{W}(\epsilon, \delta)$ in the $2N$-dimensional case, where γ, δ, ϵ are dependent on the $\langle S, \tilde{K}\rangle$-module M and defined above. $\square$

Remark 4.34. Let us sketch the modules $W(\gamma)$ and $\tilde{W}(\epsilon, \delta)$ diagrammatically using the intertwining operators.

We start with the $4N$-dimensional module $W(\gamma)$. Two vectors lying above each other in the diagram span one of the Y-eigenspaces V_m for $0 \leq m \leq 2N-1$. Also note that $B_{2N}(v_{2N}) = Kv_0$, where K is the eigenvalue of $\tilde{K}$ on V_0. This is not depicted properly in the diagram. The parameter γ determines the action of $\tilde{K}$ on $V_0 = \langle v_0, v_1\rangle_{\mathbb{C}}$ and using the intertwining operators the whole $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module structure. Each pair of intertwining operators lying above each other (for example B_1, B_{2N}) are inverse to each other up to scalar by the calculations in the previous proposition.

$$\begin{tikzcd}
v_1 \arrow[r, "B_1"'] & v_{-1} \arrow[r, "A_{-1}"'] & v_2 \arrow[r, "B_2"'] & v_{-2} \arrow[r, dotted, no head] & v_N \arrow[r, "B_N"'] & v_{-N} \arrow[d, "A_{-N}"] \\
v_0 \arrow[u, "A_0"] & v_{2N} \arrow[l, "B_{2N}"'] & v_{1-2N} \arrow[l, "A_{1-2N}"'] & v_{2N-1} \arrow[l, "B_{2N-1}"'] \arrow[r, dotted, no head] & v_{-N-1} & v_{N+1} \arrow[l, "B_{N+1}"']
\end{tikzcd}$$

Now we sketch the $2N$-dimensional module $\tilde{W}(\epsilon, \delta)$. Note that the bottom intertwining operators, as well as A_0 and A_{-N}, only map the involved v_i to each other up to scalar, which is again not depicted properly. The module structure is again fully determined by the $\langle S, \tilde{K}\rangle$ action on V_0, which is now one-dimensional and spanned by v_0.

$$\begin{tikzcd}
v_0 \arrow[loop left, "A_0"] \arrow[r, bend left, "B_1"] & v_{-1} \arrow[l, bend left, "B_{2N}"] \arrow[r, bend left, "A_{-1}"] & v_2 \arrow[l, bend left, "A_{1-2N}"] \arrow[r, bend left, "B_2"] & v_{-2} \arrow[l, bend left, "B_{2N-1}"] \arrow[r, dotted, no head] & v_N \arrow[r, bend left, "B_N"] & v_{-N} \arrow[l, bend left, "B_{N+1}"] \arrow[loop right, "A_{-N}"]
\end{tikzcd}$$

Finally, we look at exceptional modules with singular Y-weights

Proposition 4.35. *Let $k \notin \mathbb{Z}$. If $t = 1$ any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with singular weights is isomorphic to $U(\gamma)$ for $\gamma \in \mathbb{C}$ or $\tilde{U}(\epsilon, \delta)$ for $\epsilon, \delta \in \{-1, 1\}$, which are defined in the proof. The module $U(\gamma)$ has dimension $4N$ and $\tilde{U}(\epsilon, \delta)$ has dimension $2N$ and these modules are pairwise not isomorphic. If $t \neq 1$ any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with singular weights is isomorphic to some $U'(\gamma)$ for $\gamma \in \mathbb{C}$, which is defined in the proof. The modules $U'(\gamma)$ are pairwise not isomorphic and have dimension $4N$.*

Proof. Let V be a finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module and $v' \in V$ be a Y-weight vector of singular weight λ. Note that all intertwining operators up to B_λ exist and whenever an intertwining operator exists it is invertible. If $t = 1$ we even have existence of all intertwining operators. In all cases we can reach a Y-weight vector v of weight $\lambda = 0$.

For $t = 1$ the proof now virtually works the same as the proof in Proposition 4.33 only with S replaced by T. The module is again $2N$- or $4N$-dimensional and in the same way as before its isomorphism class depends only on the $\langle T, \tilde{K}\rangle$-module $V_0 := \{v \in V \mid (Y - q^0) = 0\}$. This module is determined by one parameter $\gamma \in \mathbb{C}$ describing the $\tilde{K}$-action on V_0 in the $4N$-dimensional case respectively two parameters $\epsilon, \delta \in \{-1, 1\}$. Note that 1 takes the place of the constant c from the previous proposition.

For the case $t \neq 1$ define $\tilde{V}_0 := \langle v, Tv\rangle_{\mathbb{C}}$ and set $\tilde{v} := Tv$. From the relations in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ we can deduce $(Y - q^0)^2(\bar{V}_0) = 0$ and $(Y - q^0)(\tilde{v}) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})v \neq 0$. In particular, $\tilde{V}_0$ is two-dimensional. By Lemma 4.9 we can use the invertible intertwining operators up to A_{1-N} to define for $0 \leq m \leq N - 1$ and $-N < -m < 0$

$$\tilde{V}_{m+1} := A_{-m}(\tilde{V}_{-m}), \quad \tilde{V}_{-m} := B_m(\tilde{V}_m) \tag{161}$$

and by Lemma 4.31 we obtain that each of these two-dimensional spaces are Jordan blocks for the Y-weight λ_m, which by Lemma 4.9 contain exactly one proper weight vector up to scalar. Set $V' := \oplus_{m=1-N}^{N} \tilde{V}_m$ and $L := A_{1-N}B_{N-1}...A_0$. The action of T sends $\tilde{V}_N$ to some two-dimensional Jordan block of the same Y-eigenvalue -1 by relation (YT). The Jordan blocks are either equal or their intersection is trivial and thus $TY^{-1} = YT^{-1} = YT + Y(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$ shows that T maps $\tilde{V}_N$ to itself. Hence, we see as in the previous propositions that V' is an $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule and by the irreducibility of V we have $V' = V$. Therefore any module is of this type and to determine the isomorphism classes we only have to classify the action of T on V_N, as everything else is fixed by the intertwining operators. To describe this action we choose a basis v_0, v_1 of $\tilde{V}_N$ such that Y acts with respect to this basis via the Jordan normal form $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Assume $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ on $\tilde{V}_N$. The relation $TY^{-1} = YT^{-1}$ now leads to $t_{21} = t^{-\frac{1}{2}} - t^{\frac{1}{2}}$ and

$t_{11} + t_{22} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. The characteristic polynomial $P_T(Z)$ of T on $\tilde{V}_N$ is

$$\begin{aligned} P_T(Z) &= Z^2 - (t_{11} + t_{22})Z - t_{12}t_{21} + t_{11}t_{22} \\ &= Z^2 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})Z + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t_{12} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t_{11} - t_{11}^2. \end{aligned} \tag{162}$$

By relation (T) we have $P_T(Z) = 0$ for $Z \in \{t^{\frac{1}{2}}, -t^{-\frac{1}{2}}\}$ and for such Z we have $Z^2 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})Z = 1$. But this shows that any choice of $\gamma := t_{11} \in \mathbb{C}$ determines t_{12} and thereby the action of T on $\tilde{V}_N$ uniquely. What is left to show is that any choice of $\gamma = t_{11}$ really defines a module structure on $\oplus_{m=1-N}^{N} \tilde{V}_m$. The calculation is similar to the one done in Proposition 4.32 and hence omitted. $\square$

Remark 4.36. Let us sketch the module $U'(\gamma)$. The diagrams for the modules $U(\gamma)$ and $\tilde{U}(\epsilon, \delta)$ look identical to the modules sketched in Remark 4.34, except for the fact that the roles of the A- and B-intertwining operators are exchanged. Therefore we will not depict these modules here. For $U'(\gamma)$ we only have $2N$ generalized Y-weight spaces $\tilde{V}_m$ and not proper weight spaces, therefore the points now represent these two-dimensional weight spaces instead of weight vectors. Each $\tilde{V}_m$ contains a unique proper Y-weight vector v_m and they are mapped by the intertwining operators to each other. The last intertwining operator B_N is crossed out to emphasize that it does not exist

$$\tilde{V}_0 \xrightarrow{A_0} \tilde{V}_1 \xrightarrow{B_1} \tilde{V}_{-1} \cdots\cdots \tilde{V}_{1-N} \xrightarrow{A_{1-N}} \tilde{V}_N \xrightarrow{\cancel{B_N}}$$

All remaining finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules can be obtained via quotients of the polynomial representation, as we will see now. More precisely any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module V not treated in the exceptional cases above is a quotient of $\mathcal{P}$, ${}^{\zeta_y}\mathcal{P}$, ${}^{\zeta_y \iota}\bar{\mathcal{P}}$ or of ${}^{\iota}\bar{\mathcal{P}}$. Here $\mathcal{P}$ denotes the polynomial representation of $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and $\bar{\mathcal{P}}$ denotes the polynomial representation of $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$ and ι, ζ_y are as in Lemma 4.22.

Proposition 4.37. *Let V be a finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module and assume one of the following cases holds: V has a regular weight λ and $k \in \pm 2\lambda + \mathbb{Z}$ or V has half-singular weights and $k \in \frac{1}{2} + \mathbb{Z}$ or V has singular weights and $k \in \mathbb{Z}$. Then V is a quotient of $\mathcal{P}$, ${}^{\zeta_y}\mathcal{P}$, ${}^{\iota}\bar{\mathcal{P}}$ or ${}^{\zeta_y \iota}\bar{\mathcal{P}}$.*

Proof. Let V be as above and $v \in V$ be a Y-weight vector of weight λ. Note that the isomorphisms from Lemma 4.22 are all idempotent, hence the statement is equivalent to showing that V, ${}^{\zeta_y}V$ respectively ${}^{\iota}V$ or ${}^{\zeta_y \iota}V$ is a quotient of $\mathcal{P}$ respectively $\bar{\mathcal{P}}$, which we will do now. Apply the chain of invertible intertwining operators from Definition 4.11 to v. By the assumptions on λ and k we can reach a Y-eigenvector v' of eigenvalue $q^{\lambda'} = \pm t^{\pm\frac{1}{2}}$

or $q^{\lambda'} = \pm 1$. Let us look at the second case, which can only appear for $k \in \mathbb{Z}$ by the assumptions on the weights. We can use ζ_y if necessary to assume $q^{\lambda'} = 1$. Then we apply the chain of intertwining operators again, now to v'. Since $k \in \mathbb{Z}$ the chain of intertwining operators reaches one of the Y-eigenvalues $\pm t^{\pm\frac{1}{2}}$ before it reaches the eigenvalue ± 1. Therefore we can reduce the second case to the first one. In the first case we can use ζ_y, ι or $\zeta_y\iota$ if necessary to assume $q^{\lambda'} = t^{\frac{1}{2}}$. If the B-intertwining operator applied to v' yields 0, we have that v' is a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. Otherwise replace v' by $B(v')$, which is a Y-eigenvector of eigenvalue $t^{-\frac{1}{2}}$ by Lemma 4.9 and T-eigenvector of eigenvalue $-t^{-\frac{1}{2}}$ by definition of B and since v' has Y-eigenvalue $t^{\frac{1}{2}}$. By applying ι we can also in the second case assume that we have a Y- and T-eigenvector of eigenvalue $t^{\frac{1}{2}}$ and the result follows by the universal property of $\mathcal{P}$ (respectively $\bar{\mathcal{P}}$) from Corollary 4.7. $\square$

Our next goal is to describe the non-trivial irreducible quotients of $\mathcal{P}$. For this the following lemma will prove useful.

Lemma 4.38. *Any irreducible non-trivial quotient of $\mathcal{P}$ factors through $V^C := \mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$ for some unique $C \in \mathbb{C}$.*

Proof. Let V be an irreducible non-trivial quotient of $\mathcal{P}$. In particular V is finite dimensional, because it is of the form $\mathcal{P}/(e)$ for some $e \in \mathcal{P}$. Since $X^{2N} + X^{-2N}$ is central by Lemma 4.30 it acts via some scalar on V by Schur's Lemma. Hence, there exists a unique $C \in \mathbb{C}$ such that $X^{2N} + X^{-2N} + C$ acts via 0 on V. Therefore, we obtain that V is a quotient of $\mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$. If there exists another $C' \neq C$ with this property then $\mathcal{P} \to V$ would factor over $(q^{2Nx} + q^{-2Nx} + C, q^{2Nx} + q^{-2Nx} + C') = \mathcal{P}$, which contradicts that V is not trivial. $\square$

We will first deal with the quotients of $\mathcal{P}$ with regular weight λ and $k \in \pm 2\lambda + \mathbb{Z}$. In particular we have $k \notin \frac{1}{2}\mathbb{Z}$. In the proposition the e-polynomials from Definition 4.12 and the radical Rad of $\mathcal{P}$ from Definition 4.17 will appear.

Proposition 4.39. *Let $k \notin \frac{1}{2}\mathbb{Z}$.*

(a) All e-polynomials in $\mathcal{P}$ exist. In particular we have $e_{-N} = q^{Nx} + q^{-Nx}$ and $e_{-2N} = q^{2Nx} + q^{-2Nx} + 2$.

(b) The $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module V^C is irreducible for $C \neq 2$ and has dimension $4N$. The module V^2 has a unique non-trivial irreducible quotient given by $V_{2N} := \mathcal{P}/(q^{Nx} + q^{-Nx})$ of dimension $2N$. Moreover, the radical Rad *is generated as an ideal in $\mathcal{P}$ by $q^{2Nx} + q^{-2Nx} - t^N - t^{-N}$.*

(c) We have that V_{2N} is spanned by the images of the vectors e_m for $m \prec -N$. The module V^C is spanned by the images of the vectors e_m for $m \prec -2N$. In both cases the e_m have pairwise different Y-eigenvalues $q^{-m_\sharp}$.

Proof. The existence of all e-polynomials follows from Lemma 4.13, since we have $k \notin \frac{1}{2}\mathbb{Z}$ and in particular $k \notin \mathbb{Z}$ and hence the chain starting from $v = 1$ with Y-weight $\frac{k}{2}$ never reaches the Y-eigenvalue ± 1. We have $e_{-N} = q^{Nx} + q^{-Nx}$. Indeed, $q^{Nx} + q^{-Nx}$ is a Y-eigenvector of eigenvalue $-t^{\frac{1}{2}}$ and a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. By Lemma 4.13 we have $e_{-N} = q^{-Nx} + ... + cq^{Nx}$ for some constant $c \in \mathbb{C}$. By Corollary 4.15 e_{-N} is a Y-eigenvector of eigenvalue $q^{N_\sharp} = -t^{\frac{1}{2}}$ and $e_{-N} = B_N(e_N) = t^{\frac{1}{2}}(T + t^{-\frac{1}{2}})(e_N)$ implies that e_{-N} is a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. This implies that e_{-N} is symmetric by the definition of the T-action. Thus, $p := e_{-N} - q^{Nx} - q^{-Nx}$ is also a Y-eigenvector of eigenvalue $-t^{\frac{1}{2}}$ and a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. Then $T(p) = t^{\frac{1}{2}}p$ shows that p is symmetric and therefore has leading term cq^{-mx} with respect to $\prec$ for some $m \geq 0$ and $c \in \mathbb{C}$. From the top coefficient with respect to $\prec$ we can deduce $Y(p) = q^{m_\sharp}p$, but we also have $Y(p) = q^{N_\sharp}$ and hence $m_\sharp = N_\sharp \bmod N$, which is not possible for any $0 \leq m < N$. This shows $p = 0$ and therefore $e_{-N} = q^{Nx} + q^{-Nx}$. A similar calculation, together with the fact that $e_{-N}^2 = q^{2Nx} + q^{-2Nx} + 2$ and a consideration of the evaluation formula from Lemma 4.21 proves $e_{-2N} = q^{2Nx} + q^{-2Nx} + 2$. The element e_{-N} is a Y- and T-eigenvector, which implies by Corollary 4.6 that (e_{-N}) is a submodule. We have $e_{-N}^2 = q^{2Nx} + q^{-2Nx} + 2$ and therefore $V_{2N} = \mathcal{P}/(e_{-N})$ is a quotient of V^2. Also, V_{2N} is spanned by the images of e_m for $m \prec -N$ as one sees by looking at the top coefficients with respect to $\prec$. These have Y-weights $-m_\sharp$, which are pairwise different, since $k \notin \frac{1}{2}\mathbb{Z}$. If $0 \neq M \subseteq V_{2N}$ is a submodule it must contain a Y-weight vector and thus one of the e_m. The existence and invertibility of all intertwining operators in this implies that M also contains 1 and hence that V_{2N} is irreducible.

We show that $V^2 \to V_{2N}$ is the only non-trivial quotient of any V^C. Assume $M \subseteq V^C$ is a non-trivial submodule and let $v \in M$ be a Y-eigenvector. The module V^C is spanned by the images of the Y-weight vectors e_m for $-2N < m \leq 2N$, which have pairwise different Y-weights $-m_\sharp$, since $k \notin \frac{1}{2}\mathbb{Z}$. Therefore v is proportional to the image of some e_m. Note that all invertible intertwining operators are elements in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. We apply the inverse chain of invertible intertwining operators to e_m until we reach either 1 or e_{-N} depending on whether $m \prec -N$ or not, which implies that 1 or e_{-N} lies in M. The first case implies that $M = V^C$ and the second case implies that M equals the image of (e_{-N}) in V^C by the irreducibility of V_{2N}. The claim follows from the uniqueness in Lemma 4.38.

Let $C = -t^N - t^{-N}$. To prove $\mathrm{Rad} = (q^{2Nx} + q^{-2Nx} + C)$ observe that $q^{Nx} + q^{-Nx} + C$ is a Y-eigenvector, since $X^{2N} + X^{-2N} + C$ is central in $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ by Lemma 4.30. Evaluating at $-\frac{k}{2}$ shows that $(q^{2Nx} + q^{-2Nx} + C) \subseteq \mathrm{Rad}$. For the other inclusion recall that V^C is spanned by Y-eigenvectors e_m for $m \prec -2N$ with pairwise different weights $-m_\sharp$. Since $k \notin \frac{1}{2}\mathbb{Z}$ none of the e_m lies in the radical by the evaluation formula from Lemma 4.21. But if $\mathrm{Rad} \not\subseteq (q^{2Nx} + q^{-2Nx} + C)$ the non-trivial image of Rad in V^C would contain one Y-eigenvector and hence the image of one of the e-polynomials e_m, which is not possible. This shows the other inclusion. $\square$

The following corollary lists all remaining irreducible finite-dimensional $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules for $k \notin \frac{1}{2}\mathbb{Z}$, which are not described in the Propositions 4.32, 4.33 and 4.35.

Corollary 4.40. *Let $k \notin \frac{1}{2}\mathbb{Z}$. Let $V^C = \mathcal{P} \ / \ (q^{2Nx} + q^{-2Nx} + C)$ for $C \neq 2$ and $V_{2N} = \mathcal{P}/(q^{Nx} + q^{-Nx})$ denote the irreducible quotients of $\mathcal{P}$ for $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Also, let $\bar{V}_{2N} = \mathcal{P}/(q^{Nx} + q^{-Nx})$ and $\bar{V}^C/(q^{2Nx} + q^{-2Nx} + C)$ denote the irreducible quotients for $H(q^{\frac{1}{2}}, t^{-\frac{1}{2}})$. Then up to isomorphism the finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules with (regular) weight λ such that $k \in \pm 2\lambda + \mathbb{Z}$ are*

$$\begin{aligned} V_{2N} &\cong {}^{\zeta_y \iota}\bar{V}_{2N}, & {}^{\zeta_y}V_{2N} &\cong {}^{\iota}\bar{V}_{2N}, \\ V^C &\cong {}^{\zeta_y}V^C, & {}^{\iota}\bar{V}^C &\cong {}^{\zeta_y \iota}\bar{V}^C \quad \text{for } C \neq 2. \end{aligned} \tag{163}$$

Proof. By Proposition 4.37 and Proposition 4.39 any finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module with Y-weight λ such that $k \in \pm 2\lambda + \mathbb{Z}$ is isomorphic to V_{2N}, V^C for $C \neq 2$ or a twist of these modules by ζ_y or a twist of $\bar{V}_{2N}$, $\bar{V}^C$ for $C \neq 2$ by ι or $\zeta_y\iota$. By comparing the X-action we only have to check whether V_{2N} is isomorphic to ${}^{\zeta_y}V_{2N}$, ${}^{\iota}\bar{V}_{2N}$ or ${}^{\zeta_y\iota}\bar{V}_{2N}$ and whether V^C is isomorphic to ${}^{\zeta_y}V^C$, ${}^{\iota}\bar{V}^C$ or ${}^{\zeta_y\iota}\bar{V}^C$. We apply the universal property of $\mathcal{P}$ from Corollary 4.7. As the X-module structures coincide the respective isomorphism exists if and only if the modules in question have a Y- and T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. By Proposition 4.39 we know that V_{2N} is spanned by the images of the e-polynomials e_m for $1 - N \leq m \leq N$ with Y-eigenvalues $q^{-m_\sharp} = t^{-sgn(m)\frac{1}{2}} q^{\frac{-m}{2}}$, while V^C is spanned by the e-polynomials e_m for $1 - 2N \leq m \leq 2N$ also with Y-eigenvalues $q^{-m_\sharp}$. Therefore ${}^{\zeta_y}V_{2N}$ is spanned by the Y-eigenvectors $\zeta_y(e_m)$ for $1 - N \leq m \leq 2N$ and ${}^{\zeta_y}V^C$ is spanned by $\zeta_y(e_m)$ for $1 - 2N \leq m \leq 2N$ with $Y(\zeta_y(e_m)) = -t^{-\operatorname{sgn}(m)\frac{1}{2}} q^{-\frac{m}{2}} \zeta_y(e_m)$. Analogously ${}^{\iota}\bar{V}_{2N}$ and ${}^{\zeta_y\iota}V_{2N}$ are spanned by the images of the e-polynomials $\bar{e}_m \in \bar{\mathcal{P}}$ for $1 - N \leq m \leq N$ and ${}^{\iota}\bar{V}^C$ and ${}^{\zeta_y\iota}V^C$ are spanned by the images of the e-polynomials $\bar{e}_m$ for $1 - 2N \leq m \leq 2N$ with $Y(\iota(\bar{e}_m)) = t^{\operatorname{sgn}(m)\frac{1}{2}} q^{-\frac{m}{2}} \iota(\bar{e}_m)$ and $Y(\zeta_y\iota(\bar{e}_m)) = -t^{\operatorname{sgn}(m)\frac{1}{2}} q^{-\frac{m}{2}} \zeta_y\iota(\bar{e}_m)$. As $k \notin \frac{1}{2}\mathbb{Z}$ the correct Y-eigenvalue $t^{\frac{1}{2}}$ is only obtained for positive exponent of t. We can deduce that only $\zeta_y(e_{-N})$ for ζ_y, $\iota(\bar{e}_{2N})$ for ι and $\zeta_y\iota(\bar{e}_N)$ for $\zeta_y\iota$ have the correct eigenvalue $t^{\frac{1}{2}}$. We only have to check whether these elements have T-eigenvalue $t^{\frac{1}{2}}$ in the respective quotient. For V_{2N} only the case of $\zeta_y\iota$ is possible, since the other elements are 0 in the quotient. Using $\bar{B}_N(\bar{e}_N) = t^{-\frac{1}{2}}(T + t^{\frac{1}{2}})(\bar{e}_N) = \bar{e}_{-N}$ we see that the image of $\bar{e}_N$ in ${}^{\zeta_y\iota}\bar{V}_{2N}$ is indeed a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$. For V^C we see that the image of $\bar{e}_N$ is not a T-eigenvector, hence $V^C \ncong {}^{\zeta_y\iota}\bar{V}^C$. Since by Proposition 4.39 we have $\bar{B}_{2N}(\bar{e}_{2N}) = \bar{e}_{-2N} = q^{2Nx} + q^{-2Nx} + 2$ and since this is a non-zero multiple of 1 in $\bar{V}^C$ for $C \neq 2$ the ι-case is not possible. Finally, we have that $e_{-N} = B_N(e_N)$ is a T-eigenvector of eigenvalue $t^{\frac{1}{2}}$ in ${}^{\zeta_y}V^C$. The claim now follows.

□

This finishes the discussion for $k \notin \frac{1}{2}\mathbb{Z}$. We will now describe the irreducible quotients of $\mathcal{P}$ for $k \in \frac{1}{2}\mathbb{Z}$. To simplify the discussion observe that $X \mapsto X$, $Y \mapsto Y$, $T \mapsto -T$, $t^{\frac{1}{2}} \mapsto -t^{\frac{1}{2}}$ defines an isomorphism from $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ to $H(q^{\frac{1}{2}}, -t^{\frac{1}{2}})$. This isomorphism leaves the intertwining operators A_{1-m} and B_m for all $m \geq 0$ invariant and hence sends e_m to e_m for all $m \in \mathbb{Z}$. Therefore we can restrict our considerations to the case $-\frac{N}{2} \leq k < \frac{N}{2}$ for $k \in \frac{1}{2}\mathbb{Z}$ and $t^{\frac{1}{2}} = q^{\frac{k}{2}}$.

The following lemma together with Lemma 4.38 show that we can reduce to classifying all quotients of $\mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$ for $C \in \{\pm 2\}$.

Lemma 4.41. *The $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module $V^C = \mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$ is irreducible for $C \neq \pm 2$.*

Proof. Recall that the intertwining operators A_m and B_m from Definition 4.10 have the following form when restricted to Y-eigenvalues: we have $A_m = q^{-\frac{m}{2}} X\pi$ for $m \leq 0$ and $B_m = t^{\frac{1}{2}} \left(T + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{-2\lambda_m} - 1} \right)$ for $\lambda \in \mathbb{C}$ and $\lambda_m = -\lambda - \frac{m}{2}$ for $m > 0$. In this form A_m and B_m are ϕ-invariant, where ϕ is the anti-automorphism from Equation (128). Therefore we can deduce that the intertwining operators behave with respect to X-eigenvectors just as in Lemma 4.9 described with respect to Y-eigenvectors.

Let $C \neq \pm 2$ and choose $\tilde{c} \in \mathbb{C} \setminus \{0\}$ such that $-C = \tilde{c} + \frac{1}{\tilde{c}}$. From $C \neq \pm 2$ we obtain $\frac{1}{\tilde{c}} \neq \tilde{c}$. Fix a $2N$-th root q^λ of $\tilde{c}$. We deduce by comparing the roots and the top coefficients that

$$X^{2N} + X^{-2N} + C = -q^{-2N\lambda} \prod_{j=0}^{2N-1} (X - q^{\lambda + \frac{j}{2}})(X^{-1} - q^{\lambda + \frac{j}{2}}). \qquad (164)$$

We have $\lambda \notin \frac{1}{4}\mathbb{Z}$, since $\tilde{c} \neq \pm 1$. Hence the X-spectrum of $\mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$, which consists of the roots appearing above, is simple and all appearing intertwining operators exist and are invertible. Therefore any submodule must contain all X-eigenvectors and this gives the irreducibility of V^C for $C \neq \pm 2$. □

Thus we will classify irreducible quotients of $V^{\pm 2}$ now.

Proposition 4.42. *Let $k \in \frac{1}{2}\mathbb{Z}$ and $-\frac{N}{2} \leq k < \frac{N}{2}$. The following gives a full list of non-zero irreducible quotients of V^C for $C \in \{\pm 2\}$, where in the respective cases $n > 0$ is such that $k = -\frac{1}{2} - n$ and $m = N - 2k$. For integral k we also assume $-\frac{N}{2} < k < \frac{N}{2}$ and list the remaining case for integral $k = -\frac{N}{2}$ separately.*

Value of k	*Quotients of* V^2	dim	*Quotients of* V^{-2}	dim
$k=0$	V_{2N}	$2N$	$\mathcal{P}/(q^{Nx}-q^{-Nx})$	$2N$
$k\in\frac{1}{2}+\mathbb{N}$	$\mathcal{P}/(e_{-m})$	$2m$	V^{-2}	$4N$
$k\in-\frac{1}{2}-\mathbb{N}$	$\mathcal{P}/(e_{n+1}\pm t^{-\frac{1}{2}}e_{-n})$	$2n+1$	V^{-2}	$4N$
$k\in 1+\mathbb{N}$	V_{2N}	$2N$	$\mathcal{P}/(e_{-m})$	$2m$
$k\in -1-\mathbb{N}$	V_{2N}	$2N$	$\mathcal{P}/(\epsilon_N-\epsilon_{-N+2k})$	$2N+4k$
$k=-\frac{N}{2}$	V_{2N}	$2N$	V^{-2}	$4N$

Proof. Let $k\in\frac{1}{2}\mathbb{Z}$ with $-\frac{N}{2}\leq k<\frac{N}{2}$. Any non-trivial quotient of V^C for $C\in\{\pm 2\}$ can be written as $\mathcal{P}/(e)$, where

$$\textbf{(A):}\ e=q^{lx}+...+cq^{-lx}\ \text{or}\ \textbf{(B):}\ e=q^{(l+1)x}+...+cq^{-lx} \tag{165}$$

for some $2N>l\geq 0$ and $c\in\mathbb{C}\setminus\{0\}$. As in the proof of Proposition 4.25 we can assume that e is a Y-eigenvector and hence by Corollary 4.16 e is a linear combination of e-polynomials. We split e into an even and odd part by setting $e=e^0+e^1$ where $e^\alpha(-x)=(-1)^\alpha e^\alpha(x)$ for $\alpha\in\{0,1\}$. Note that in case (B) we have $e^0,e^1\neq 0$. The element Y preserves the odd and even parts of $\mathcal{P}$ by definition of the Y-action and hence e^0 and e^1 are Y-eigenvectors of the same eigenvalue as e. Since e^0 is even and e^1 is odd their top degrees with respect to $\prec$ form the set

(A): $\{-l\}$ or $\{-l,\pm m\}$ for some $0\leq m<l$ with $l-m$ odd,
(B): $\{l+1,-l\}$.

For any Y-eigenvector in $H(q^{\frac{1}{2}},t^{\frac{1}{2}})$ we can read off from the top-degree j with respect to $\prec$ by Equation (109), (and its counterpart for negative exponent) that its Y-eigenvalue is $q^{-j_\sharp}$. Thus, we obtain for the set of weights of e^0 and e^1

(A): $\left\{\frac{l+k}{2}\right\}$ or $\left\{\frac{l+k}{2},\frac{\pm(m+k)}{2}\right\}$ for some $0\leq m<l$ with $l-m$ odd,
(B): $\left\{-\frac{l+1+k}{2},\frac{l+k}{2}\right\}$.

If the set has two elements, they must coincide modulo N, since the Y-eigenvalue of e^0 and e^1 must coincide with the the Y-eigenvalue of $e=e^0+e^1$. Since $2N>l\geq 0$ this implies that only the top degrees $\{-l,m\}$ with $m>0$ are possible in (A) if the decomposition is not trivial. In particular, if the decomposition is not trivial we have $k=-\frac{l+m}{2}$ mod N in case (A) and $k=-\frac{1}{2}-l$ mod N in case (B).

We will handle case (B) first. Since $-\frac{N}{2} \leq k < \frac{N}{2}$ and $2N > l \geq 0$ we have three possible cases: $k = -\frac{1}{2} - l$, $k = -\frac{1}{2} - l + N$ or $k = -\frac{1}{2} - l + 2N$. Note that $k \in \frac{1}{2} + \mathbb{Z}$ implies that all e-polynomials exist by Lemma 4.13. We have that e is a linear combination of e-polynomials of weight $\frac{l+k}{2} \bmod N$ that have degree smaller than $-2N$ with respect to $\prec$. Since the coefficient of $q^{(l+1)x}$ in e is 1 we obtain $e = e_{l+1} + ce_{-l}$ for some $c \in \mathbb{C}$. From the evaluation formula in Lemma 4.21 we obtain that $e_m(-\frac{k}{2}) = 0$ for all $m \succ N - 2k$. In particular, we see in the cases $l = -\frac{1}{2} - k + N$ and $l = -\frac{1}{2} - k + 2N$ that $(e) \subseteq \mathrm{Rad}$. We will describe Rad for $k > 0$ later in the proof. For $k < 0$ we will describe the radical now by looking at the remaining case $l = -\frac{1}{2} - k$ and $e = e_{l+1} + ce_{-l}$. To match previous conventions we will set $n := l$. As in the proof of Proposition 4.26 we see that only $c = \pm t^{-\frac{1}{2}}$ is possible and that for these choices $e^{\pm}$ is a T-eigenvector and therefore $(e^{\pm})$ is a submodule by Corollary 4.6. We have that $\mathcal{P}/(e^+)$ has a basis given by the images of e_m for $m \prec n + 1$. In particular, the dimension is $2n + 1$. These e_m have pairwise different Y-eigenvalues $q^{-m_\sharp} = t^{-\operatorname{sgn}(m)\frac{1}{2}} q^{-\frac{m}{2}}$ and they do not evaluate to zero on $q^{-\frac{k}{2}}$ by the evaluation formula from Lemma 4.21. The evaluation formula also shows that $(e^+) \subseteq \mathrm{Rad}$ and the previous statement about the basis vectors shows that this is an equality. We obtain the analogous results for $\zeta_x(e^+) = e^-$ and $\zeta_x(\mathrm{Rad}) = \mathrm{Rad}_-$, where ζ_x is from Remark 4.23. Both quotients factor through $\mathcal{P}/(q^{Nx} + q^{-Nx})$ and therefore also through $\mathcal{P}/(q^{2Nx} + q^{-2Nx} + 2)$. This follows as $q^{Nx} + q^{-Nx}$ is a Y-eigenvector which evaluates on $-\frac{k}{2}$ to zero and it is invariant under ζ_x. Hence it lies in both radicals. The irreducibility of the quotient follows from the the fact that all B_m-intertwining operators in the range $0 < m < n + 1$ exist and are invertible and since the weights $q^{-m_\sharp}$ for $0 \preceq m \prec n + 1$ are pairwise different.

Now for the case (A). Since the top and bottom degree of e are l respectively $-l$ we have that e is a T-eigenvector. Otherwise we could use a linear combination of e and $T(e)$ to show that (e) contains a non-trivial element for which the difference between top and bottom degree is less than $2l$, which contradicts that the dimension of the quotient is $2l$. By $TY^{-1}T = Y$ and the quadratic T-relation we have four possibilities of eigenvalues:

$$\begin{aligned} &\textbf{(a):}\ T(e) = t^{\frac{1}{2}} e = Y(e), && \textbf{(b):}\ T(e) = t^{\frac{1}{2}} e = -Y(e), \\ &\textbf{(c):}\ T(e) = -t^{-\frac{1}{2}} e = Y(e), && \textbf{(d):}\ T(e) = -t^{-\frac{1}{2}} e = -Y(e). \end{aligned} \tag{166}$$

Using the definition of the T-action we see that $T(e) = t^{\frac{1}{2}}$ implies that e is s-invariant, where $s(f)(x) = f(-x)$ for $f \in \mathcal{P}$. Since the top degree of e with respect to $\prec$ is $-l$ we obtain that the Y-eigenvalue of e is $q^{l_\sharp}$. Hence we have in the cases (a) and (b):

$$\textbf{(a):}\ \frac{l+k}{2} = \frac{k}{2} \bmod N, \quad \textbf{(b):}\ \frac{l+k}{2} = \frac{k}{2} + \frac{N}{2} \bmod N. \tag{167}$$

This implies $l = 0$ in (a), since $2N > l \geq 0$. Hence this case does not yield to non-trivial quotients. For (b) we have $\pi(e) = YT^{-1}(e) = -e$. Together with the s-invariance this implies that $e = q^{Nx} + q^{-Nx}$. Since e is a Y and T-eigenvector Corollary 4.6 shows that (e) is a submodule. We see later that the quotient $V_{2N} := \mathcal{P}/(q^{Nx} + q^{-Nx})$ is irreducible, unless k is half-integral. For half-integral $k < 0$ we have already described a quotient of V_{2N} above. For k integral it is not possible to find quotients of V_{2N} as we see below. Note that this finishes the discussion for $k = -\frac{N}{2}$, since then $t = -1$ and cases (c) and (d) become (a) and (b) respectively.

In the cases (c) and (d) we can write e as a sum of e-polynomials of weight $N - \frac{k}{2}$ respectively $-\frac{k}{2}$ of degree $\prec -2N$. Since the top degree of e with respect to $\prec$ is negative at least one e-polynomial e_m with $m < 0$ must appear with non-zero coefficient. We obtain

$$\textbf{(c):}\ e = e_{-N+2k} + ce_N \text{ for some } c \in \mathbb{C}, \tag{168}$$

$$\textbf{(d):}\ e = e_{-2N+2k} \text{ if } k > 0 \text{ or } e = e_{2k} \text{ if } k \leq 0. \tag{169}$$

By Lemma 4.13 part (b) the appearing e-polynomials e_{-N+2k}, e_{-2N+2k} and e_{2k} exist. The e-polynomial e_N does not necessarily exist. First, assume $k = 0$ and hence $t = 1$. We must be in case (c). All e-polynomials exist for $k = 0$ and $e_m = q^{mx}$ for $m \in \mathbb{Z}$. Then $T(e) = -t^{-\frac{1}{2}}e = e$ implies that e is s-anti-invariant and hence $c = -1$ and $e = q^{-Nx} - q^{Nx}$. By $e^2 = q^{2Nx} + q^{-2Nx} - 2$, we see that $\mathcal{P}/(e)$ is a quotient of V^{-2}. Together with the above discussion of case (b), which lead to $e = q^{Nx} + q^{-Nx}$, we have listed all possible non-trivial quotients for $k = 0$ and hence we see that they must be irreducible. As the dimensions are clearly $2N$ this finishes the case $k = 0$.

Now assume $k \neq 0$. The equation $T(e) = -t^{-\frac{1}{2}}e$ implies $e \in \mathrm{Rad}$. Indeed, e is a Y-eigenvector and therefore we only have to show $e(-\frac{k}{2}) = 0$. For this use $(T + t^{-\frac{1}{2}})(e) = 0$ to calculate

$$\begin{aligned}(T + t^{-\frac{1}{2}})(e)(-\frac{k}{2}) &= \left(t^{\frac{1}{2}}s(e) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{2x} - 1}(s(e) - e)\right)(-\frac{k}{2}) + t^{-\frac{1}{2}}e(-\frac{k}{2}) \\ &= (t^{\frac{1}{2}} + t^{-\frac{1}{2}})e(-\frac{k}{2}) = 0,\end{aligned} \tag{170}$$

which shows the claim, since $t^{\frac{1}{2}} + t^{-\frac{1}{2}} \neq 0$ as we can assume $t \neq -1$. Thus all quotients we can obtain from now on are quotients of $\mathcal{P}/\mathrm{Rad}$. Calculating $(q^{2Nx} + q^{-2Nx})(-\frac{k}{2}) = -2$ for $k \in \frac{1}{2} + \mathbb{Z}$ and $(q^{2Nx} + q^{-2Nx})(-\frac{k}{2}) = 2$ for $k \in \mathbb{Z}$ shows that V^{-2} is irreducible for $k \in \frac{1}{2} + \mathbb{Z}$ and that V_{2N} is irreducible and the only non-trivial quotient of V^2 for $k \in \mathbb{Z}$. This also completes the discussion for $k = -\frac{1}{2} - \mathbb{N}$, since then we know $\mathrm{Rad} = (e_{n+1} + t^{-\frac{1}{2}}e_{-n})$. Now we finish case (c) and in particular the unfinished case $k > 0$ and k half-integral from the discussion of (B) will be dealt with. First assume $k > 0$.

Then the evaluation formula from Lemma 4.21 shows $e_{-N+2k}(-\frac{k}{2}) = 0$ and hence $e_{-N+2k} \subseteq \mathrm{Rad}$. Lemma 4.13 shows that e_{-N+2k} has T-eigenvalue $-t^{-\frac{1}{2}}$, since $k > 0$ and all V_n from Lemma 4.13 up to V_{N-k} are one-dimensional. Hence (e_{-N+2k}) is a submodule by Corollary 4.6. Now finally Lemma 4.13 also shows that all e-polynomials up to e_{-N+2k} exist and we can use the evaluation formula to see that none of them lies in the radical. Since they have pairwise different eigenvalues $q^{-m_\sharp}$ this shows $\mathrm{Rad} \subseteq (e_{-N+2k})$ and hence we have an equality. The irreducibility of the quotient follows as none of the quotients we described above can factor through this quotient and the same holds for the quotients described below, since $e \in \mathrm{Rad}$ for all upcoming quotients $\mathcal{P}/(e)$. The dimension follows since e_{-N+2k} is a π-eigenvector and therefore q^{-N+2k} and q^{N-2k} both appear with non-zero coefficient in e_{-N+2k}.

By the discussion above we can now assume $k < 0$ and $k \in \mathbb{Z}$, since the half-integer case is already finished by the description of the radical in case (B). Recall that we can assume $k \neq -\frac{N}{2}$ or equivalently $t \neq -1$. Then e_N exists by the chain of intertwining operators from Lemma 4.13. Indeed, e_N lies in the 1-dimensional part of the chain $-N-k \prec \ldots \prec N \prec \ldots \prec N-k$. Lemma 4.21 shows that $e_N(-\frac{k}{2}) \neq 0$ and $e_{-N+2k}(-\frac{k}{2}) \neq 0$. Hence, e is proportional to $\epsilon_N - \epsilon_{-N+2k}$, where $\epsilon_m := \frac{e_m}{e_m(-\frac{k}{2})}$. This is indeed a T-eigenvector of eigenvalue $-t^{-\frac{1}{2}}$, as we show now. By the chain of intertwining operators from Lemma 4.13 we have that $(T+t^{-\frac{1}{2}})(e_{-N+2k})$ is proportional to the unique e-polynomial e_μ in V_{N-2k}. By looking at the Y-eigenvalue we must have $e_\mu = e_{-N} = B_N(e_N) = (T + t^{-\frac{1}{2}})(e_N)$, since there exists no other Y-eigenvector of eigenvalue $q^{N_\sharp}$ with top degree $\prec -N+2k$. Hence, $(T + t^{-\frac{1}{2}})(\epsilon_N - \epsilon_{-N+2k})$ is proportional to e_{-N}. The evaluation formula shows $e_{-N}(-\frac{k}{2}) \neq 0$, but we also obtain by using Equation (170) that $(T + t^{-\frac{1}{2}})(\epsilon_N - \epsilon_{-N+2k})(-\frac{k}{2}) = 0$. Therefore we have $(T+t^{-\frac{1}{2}})(\epsilon_N - \epsilon_{-N+2k}) = 0$. Let us show that $(\epsilon_N - \epsilon_{-N+2k}) = \mathrm{Rad}$. The inclusion $(\epsilon_N - \epsilon_{-N+2k}) \subseteq \mathrm{Rad}$ is obvious. For the other inclusion note that $\mathcal{P}/(\epsilon_N - \epsilon_{-N+2k})$ is spanned by the images of the spaces V_m for $m \prec -N+2k$ from Lemma 4.13. They correspond to the Y-weights $-m_\sharp$. Assume $\mathrm{Rad} \not\subseteq (\epsilon_N - \epsilon_{-N+2k})$. Then we can find a Y-eigenvector $v \in V \cap \mathrm{Rad}$, where $V := \oplus_{m \prec -N+2k} V_m \subseteq \mathcal{P}$. Note this is possible, since V is Y-stable. But we only have one unique Y-eigenvector in V for each eigenvalue $q^{-m_\sharp}$ for $m \in \mathbb{Z}$. Indeed, Lemma 4.13 shows that the e_m for $m \prec -N+2k$ only exist for $0 \preceq m \preceq -k$ and $2k \preceq m \preceq N-k$. For the remaining m with $k \preceq m \preceq -2k$ and $-N+k \preceq m \preceq N-2k$ the space V_m is two-dimensional and e_m does not exist. We see that the existing e_m have $2N$ pairwise different Y-eigenvalues $q^{-m_\sharp}$. Therefore v is proportional to some e_m, but the evaluation formula from Lemma 4.21 shows that none of these e_m lie in the radical and hence $V \cap \mathrm{Rad} = 0$. Therefore $(\epsilon_N - \epsilon_{-N+2k}) = \mathrm{Rad}$. The irreducibility follows, since none of the quotients above factors through $\mathcal{P}/(\epsilon_N - \epsilon_{-N+2k})$ and below we only consider e in the radical. The dimension follows as before, since e is a π-eigenvector.

Case (d) can not lead to new quotients since we have already seen that $e \in \mathrm{Rad}$ and above we described Rad in all cases. $\square$

Lastly, we have to classify the isomorphism classes of the quotients of $\mathcal{P}$ and their twists by ζ_y, ι and $\zeta_y\iota$. For this denote by $\mathcal{P}$ the polynomial representation for $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ with $t = q^k$ and $\bar{\mathcal{P}}$ the polynomial representation of $H(q^{\frac{1}{2}}, \bar{t}^{\frac{1}{2}})$, where $\bar{t}^{\frac{1}{2}} = t^{-\frac{1}{2}}$ and $\bar{k} = -k$. We denote the m-th e-polynomial in $\bar{\mathcal{P}}$ by $\bar{e}_m$ Futhermore, $C \in \mathbb{C} \setminus \{\pm 2\}$.

Corollary 4.43. *The following gives a full list of isomorphism classes of non-exceptional finite-dimensional irreducible $H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-modules for the respective $k \in \frac{1}{2}\mathbb{Z}$. Here we list again $k = -\frac{N}{2}$ for integral k separately.*

Value of k	*Irreducible modules*	dim
$k = 0$	$\mathcal{P}/(q^{Nx} + q^{-Nx}) \cong {}^{\zeta_y\iota}(\mathcal{P}/(q^{Nx} + q^{-Nx}))$, ${}^{\zeta_y}(\mathcal{P}/(q^{Nx} + q^{-Nx})) \cong {}^{\iota}(\mathcal{P}/(q^{Nx} + q^{-Nx}))$, $\mathcal{P}/(q^{Nx} - q^{-Nx}) \cong {}^{\zeta_y}(\mathcal{P}/(q^{Nx} - q^{-Nx}))$, ${}^{\iota}(\mathcal{P}/(q^{Nx} - q^{-Nx})) \cong {}^{\zeta_y\iota}(\mathcal{P}/(q^{Nx} - q^{-Nx}))$, $V^C \cong {}^{\zeta_y}(V^C) \cong {}^{\iota}(V^C) \cong {}^{\zeta_y\iota}(V^C)$.	$2N$ $2N$ $2N$ $2N$ $4N$
$k \in \frac{1}{2} + \mathbb{N}$	$\mathcal{P}/(e_{-m}) \cong {}^{\zeta_y}(\mathcal{P}/(e_{-m}))$, ${}^{\iota}(\bar{\mathcal{P}}/(\bar{e}_{n+1} \pm t^{-\frac{1}{2}}\bar{e}_{-n})) \cong {}^{\zeta_y\iota}(\bar{\mathcal{P}}/(\bar{e}_{n+1} \pm t^{-\frac{1}{2}}\bar{e}_{-n}))$, $V^{-2} \cong {}^{\zeta_y}(V^{-2}) \cong {}^{\iota}(V^{-2}) \cong {}^{\zeta_y\iota}(V^{-2})$, $V^C \cong {}^{\zeta_y}(V^C) \cong {}^{\iota}(\bar{V}^C) \cong {}^{\zeta_y\iota}(\bar{V}^C)$.	$2m$ $2n+1$ $4N$ $4N$
$k \in -\frac{1}{2} - \mathbb{N}$	${}^{\iota}(\bar{\mathcal{P}}/(\bar{e}_{-m})) \cong {}^{\zeta_y\iota}(\bar{\mathcal{P}}/(\bar{e}_{-m}))$, $\mathcal{P}/(e_{n+1} \pm t^{-\frac{1}{2}}e_{-n})) \cong {}^{\zeta_y}(\mathcal{P}/(e_{n+1} \pm t^{-\frac{1}{2}}e_{-n}))$, $V^{-2} \cong {}^{\zeta_y}(V^{-2}) \cong {}^{\iota}(\bar{V}^{-2}) \cong {}^{\zeta_y\iota}(\bar{V}^{-2})$. $V^C \cong {}^{\zeta_y}(V^C) \cong \iota(\bar{V}^C) \cong {}^{\zeta_y\iota}(\bar{V}^C)$,	$2m$ $2n+1$ $4N$ $4N$
$k \in 1 + \mathbb{N}$	$\mathcal{P}/(e_{-m}) \cong {}^{\zeta_y}(\mathcal{P}/(e_{-m}))$, $V^C \cong {}^{\zeta_y}(V^C) \cong {}^{\iota}(\bar{V}^C) \cong {}^{\zeta_y\iota}(\bar{V}^C)$, $V_{2N} \cong {}^{\zeta_y\iota}(\bar{V}_{2N})$, ${}^{\zeta_y}(V_{2N}) \cong {}^{\iota}(\bar{V}_{2N})$, ${}^{\iota}\bar{\mathcal{P}}/(\bar{\epsilon}_N - \bar{\epsilon}_{-N+2k}) \cong {}^{\zeta_y\iota}(\bar{\mathcal{P}}/(\bar{\epsilon}_N - \bar{\epsilon}_{-N+2k}))$.	$2m$ $4N$ $2N$ $2N+4k$
$k \in -1 - \mathbb{N}$	${}^{\iota}(\bar{\mathcal{P}}/(\bar{e}_{-m})) \cong {}^{\zeta_y\iota}(\bar{\mathcal{P}}/(\bar{e}_{-m}))$ $V^C \cong {}^{\zeta_y}(V^C) \cong \iota(\bar{V}^C) \cong {}^{\zeta_y\iota}(\bar{V}^C)$, $V_{2N} \cong {}^{\zeta_y\iota}(V_{2N})$, ${}^{\zeta_y}(V_{2N}) \cong {}^{\iota}(\bar{V}_{2N})$, $\mathcal{P}/(\epsilon_N - \epsilon_{-N+2k}) \cong {}^{\zeta_y}(\mathcal{P}/(\epsilon_N - \epsilon_{-N+2k}))$,	$2m$ $4N$ $2N$ $2N+4k$
$k = -\frac{N}{2}$	$V^C \cong {}^{\zeta_y}(V^C) \cong {}^{\iota}(\bar{V}^C) \cong {}^{\zeta_y\iota}(\bar{V}^C)$ $V_{2N} \cong {}^{\zeta_y\iota}(\bar{V}_{2N})$, ${}^{\zeta_y}(V_{2N}) \cong {}^{\iota}(\bar{V}_{2N})$ $V^{-2} \cong {}^{\zeta_y}(V^{-2}) \cong {}^{\iota}(\bar{V}^{-2}) \cong {}^{\zeta_y\iota}(\bar{V}^{-2})$	$4N$ $2N$ $4N$

Proof. By Proposition 4.37 we only have to classify the modules from Proposition 4.42 and their twists up to isomorphism. The proof idea is similar to the proof of Corollary 4.40: we apply the universal property of $\mathcal{P}$ from Corollary 4.7 and decide whether there exists a T- and Y-eigenvector of eigenvalue $t^{\frac{1}{2}}$ in the modules. By comparing the X-module structure we see that any quotient can only be isomorphic to its twisted counterpart and not to any other (twisted) quotients. Let V be a non-trivial quotient of $\mathcal{P}$ and $\bar{V}$ the corresponding non-trivial quotient of $\bar{\mathcal{P}}$, if it exists. To find an isomorphism from V to ${}^{\zeta_y}V$, ${}^{\iota}\bar{V}$ or ${}^{\zeta_y\iota}\bar{V}$ we need to find to find a Y- and T-eigenvector $v \in V$ respectively $\bar{v} \in \bar{V}$ of the following eigenvalues:

$$\zeta_y: \ T(v) = t^{\frac{1}{2}}v, \qquad Y(v) = -t^{\frac{1}{2}}v, \tag{171}$$

$$\iota: \ T(\bar{v}) = -t^{\frac{1}{2}}\bar{v} = -\bar{t}^{-\frac{1}{2}}, \qquad Y(\bar{v}) = t^{\frac{1}{2}}\bar{v} = \bar{t}^{-\frac{1}{2}}, \tag{172}$$

$$\zeta_y\iota: T(\bar{v}) = -t^{\frac{1}{2}}\bar{v} = -\bar{t}^{-\frac{1}{2}}, \qquad Y(\bar{v}) = -t^{\frac{1}{2}}\bar{v} = -\bar{t}^{-\frac{1}{2}}. \tag{173}$$

Recall that Y preserves $\prec$ on $\mathcal{P}$ and therefore any Y-eigenvector v respectively $\bar{v}$ in a quotient can be lifted to a Y-eigenvector v' respectively $\bar{v}'$ in $\mathcal{P}$ respectively $\bar{\mathcal{P}}$. The lift must be a sum of e-polynomials by Corollary 4.16. We only need to consider e-polynomials of degree $\prec -2N$, since we look at quotients of $\mathcal{P}/(q^{2Nx} + q^{-2Nx} + C)$ for some $C \in \mathbb{C}$. We inspect the Y-eigenvalues $q^{-m_\sharp}$ of e_m respectively the Y-eigenvalues $q^{-\bar{m}_\sharp}$ of $\bar{e}_m$, where $\bar{m}_\sharp = \frac{m+\mathrm{sgn}(m)\bar{k}}{2}$. This leads to

$$\zeta_y: v' = c_1e_{-N} + c_2e_{N-2k}, \tag{174}$$

$$\iota: \bar{v}' = c_1\bar{e}_{2\bar{k}} + c_2\bar{e}_{2N} \text{ if } \bar{k} \leq 0 \text{ or} \tag{175}$$

$$\bar{v}' = c_1\bar{e}_{-2N+2\bar{k}} + c_2\bar{e}_{2N} \text{ if } \bar{k} > 0, \tag{176}$$

$$\zeta_y\iota: \bar{v}' = c_1\bar{e}_{-N+2\bar{k}} + c_2\bar{e}_N, \tag{177}$$

where $c_1, c_2 \in \mathbb{C}$ and $\bar{k} = -k$. Note that in all cases the e-polynomial with negative index exists by Lemma 4.13.

First assume $k = 0$ and hence $\mathcal{P} = \bar{\mathcal{P}}$. All e-polynomials exist and we have $e_m = q^{mx}$ for $m \in \mathbb{Z}$. Then $q^{Nx} + q^{-Nx}$ is a T-eigenvector of eigenvalue $t^{\frac{1}{2}} = -1$, while $q^{Nx} - q^{-Nx}$ is a T-eigenvector of eigenvalue $-t^{-\frac{1}{2}} = 1$. Hence for $C \neq \pm 2$ we obtain $V^C \cong {}^{\zeta_y}(V^C) \cong {}^{\zeta_y\iota}(V^C)$, which implies $V^C \cong {}^{\iota}(V^C)$. We also obtain $\mathcal{P}/(q^{Nx} + q^{Nx}) \cong {}^{\zeta_y\iota}(\mathcal{P}/(q^{Nx} + q^{Nx}))$ and $\mathcal{P}/(q^{Nx} - q^{Nx}) \cong {}^{\zeta_y}(\mathcal{P}/(q^{Nx} - q^{Nx}))$ by using $v' = q^{Nx}$. The twist ι is not possible in these two quotients, since $1 = \pm q^{2Nx}$ in the modules, which has T-eigenvalue $-1 = t^{\frac{1}{2}} \neq -t^{-\frac{1}{2}} = 1$. Hence, also the twist ζ_y respectively $\zeta_y\iota$ is not possible.

Assume $k \neq 0$. Let us deal with ζ_y and hence we consider vectors of the form $c_1e_{-N} + c_2e_{N-2k}$. Observe that $q^{Nx} + q^{-Nx}$ has the correct T and Y-eigenvalues. Therefore we can replace e_{-N} with $q^{Nx} + q^{-Nx}$, in case that they are different elements. If e_{N-2k} exists then $(T - t^{\frac{1}{2}})(e_{N-2k}) = B_{N-2k}(e_{N-2k}) = e_{-N+2k}$. By the chain of intertwining operators e_{N-2k} exists if and only if k is a half-integer or $k > 0$. Look at $c_1 = 1$ and $c_2 = 0$

and hence at the element $v' = q^{Nx} + q^{-Nx}$. We obtain $V^C \cong {}^{\zeta_y}(V^C)$ for all appearing C and $V \cong {}^{\zeta_y}(V)$ for all quotients V of V^{-2}. This also shows that $V_{2N} \ncong {}^{\zeta_y}(V_{2N})$, since $q^{Nx} + q^{-Nx} = 0$ in V_{2N} and e_{N-2k} either does not exist or is not a T-eigenvector in V_{2N}. Using $v' = e_m = e_{N-2k}$ as a target vector we can deduce $\mathcal{P}/(e_{-m}) \cong {}^{\zeta_y}(\mathcal{P}/(e_{-m}))$. Only the case $\mathcal{P}/(e_{n+1} \pm t^{-\frac{1}{2}}e_{-n})$ remains. We have $e_{N-2k} \notin \mathrm{Rad}$ and $e_{-N+2k} \in \mathrm{Rad}$ by the evaluation formula and since $(e_{n+1} + t^{-\frac{1}{2}}e_{-n}) = \mathrm{Rad}$ we obtain the isomorphism $\mathcal{P}/(e_{n+1} + t^{-\frac{1}{2}}e_{-n}) \cong {}^{\zeta_y}(\mathcal{P}/(e_{n+1} + t^{-\frac{1}{2}}e_{-n}))$. We also obtain the isomorphism for $\mathrm{Rad}_- = (e_{n+1} - t^{-\frac{1}{2}}e_{-n})$ by applying ζ_x, which sends Rad to Rad_- and $\{e_{N-2k}, e_{-N+2k}\}$ to $\pm\{e_{N-2k}, e_{-N+2k}\}$.

For ι and $\zeta_y\iota$ we can immediately exclude all cases except V^C for $C \neq 2$ and V_{2N} by dimension reasons, since all other modules do not have a $\bar{\mathcal{P}}$ counterpart. In the half-integer case we only have to look at V^C for $C \neq 2$. All e-polynomials exist and $\bar{e}_{2\bar{k}}$ for $\bar{k} < 0$ respectively $\bar{e}_{-2N+2\bar{k}}$ for $\bar{k} > 0$ has the correct T-eigenvalue for ι, since they lie in the image of the B-intertwining operator $\bar{t}^{\frac{1}{2}}(T - \bar{t}^{\frac{1}{2}})$. From this $V^C \cong {}^{\iota}(\bar{V}^C)$ and hence also $V^C \cong {}^{\zeta_y\iota}(\bar{V}^C)$ for all $C \neq 2$ follows. Now we treat the integer-case. Assume first that $\bar{k} < 0$ and $\bar{k} \neq -\frac{N}{2}$. We can use $\bar{\epsilon}_N - \bar{\epsilon}_{-N+2k} \in \bar{\mathcal{P}}$ as a target vector. Indeed, we have already seen in the proof of Proposition 4.42 that this is a T-eigenvector with eigenvalue $-\bar{t}^{-\frac{1}{2}}$. Furthermore, it has non-trivial image in $\bar{V}^C$ and $\bar{V}_{2N}$. For $\bar{V}^C$ this is obvious and if it had zero image in V_{2N}, then $\bar{\mathcal{P}}/(\bar{\epsilon}_N - \bar{\epsilon}_{-N+2k})$ would not be irreducible. Therefore we obtain $V^C \cong {}^{\zeta_y\iota}(\bar{V}^C)$ and $V_{2N} \cong {}^{\zeta_y\iota}(\bar{V}_{2N})$ for integral $\bar{k} < 0$ and $\bar{k} \neq \frac{-N}{2}$. The case $\bar{k} > 0$ follows as well, since $\zeta_y\iota$ is idempotent. Now we only have to look at $\zeta_y\iota$ for $k = -\frac{N}{2}$ integral. But then $-\bar{t}^{-\frac{1}{2}} = \bar{t}^{\frac{1}{2}}$ and we can simply use 1 as our target vector, hence $V^C \cong {}^{\zeta_y\iota}(V^C)$ and $V_{2N} \cong {}^{\zeta_y\iota}(V_{2N})$. We can conclude the discussion, since the remaining cases follow by a 'two out of three' argument involving ζ_y, ι and $\zeta_y\iota$.

□

5 Spherical DAHA

We will now look again at the double affine Hecke algebra $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ associated to GL_n for $n \geq 2$, which is described in Definition 3.1. More precisely, we will consider a certain idempotent truncation $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$ in $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, the so-called *spherical double affine Hecke algebra* from Definition 5.13. We will construct a particular module $\mathcal{M}$ of $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$ and identify $\mathcal{M}$ with the quantum cohomology ring $\mathbb{C} \otimes_{\mathbb{Z}} qH^{\bullet}(\mathrm{Gr}_{n,N})_{q=1}$ of the Grassmannian $\mathrm{Gr}_{n,N}$ of n-dimensional subspaces inside $\mathbb{C}^N$. Later on we will assume that $q = t$ is a primitive N-th root of unity for $N > n$, which is how the parameter N of the Grassmannian will connect to the parameters of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. We emphasize that the parameter q from the quantum cohomology is specialized to $q = 1$ and in particular it does not match the parameter q from the DAHA.

We will not give an in-depth discussion of the quantum cohomology of the Grassmannian. Instead, we will only use the following two facts, which can be found in [KS10]. See also [ST97] for a detailed study of the quantum cohomology ring from an algebro-geometric point of view.

(1) The quantum cohomology ring of the Grassmannian can be explicitly described as a quotient of the ring of symmetric polynomials:

$$\mathbb{C} \otimes_{\mathbb{Z}} qH^{\bullet}(\mathrm{Gr}_{n,N})_{q=1} \cong \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]/(h_{n+1}, ..., h_N + (-1)^n), \tag{178}$$

where $\boldsymbol{e_i}$ for $1 \leq i \leq n$ is the i-th elementary symmetric polynomial in n variables and h_j for $j \in \mathbb{Z}_{>0}$ is the j-th complete symmetric polynomial.

(2) Via the identification from (1) the $\mathbb{C}$-algebra $\mathbb{C} \otimes_{\mathbb{Z}} qH^{\bullet}(Gr_{n,N})_{q=1}$ obtains a $\mathbb{C}$-basis given by the Schur polynomials s_λ for $\lambda \in \mathfrak{P}_{n,N-n}$, where

$$\mathfrak{P}_{n,N-n} := \{\lambda = (\lambda_1, ..., \lambda_n) \mid 0 \leq \lambda_i \leq N - n \text{ for } 1 \leq i \leq n\} \subseteq P^+. \tag{179}$$

We will assume that the reader has some familiarity with the representation theory of GL_n and the theory of symmetric functions, see for example [Ful97] as a reference.

5.1 Polynomial representation for $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$

We set again $\mathbb{K} := \mathbb{C}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and fix $n \geq 2$. In this section we want to transport some results from Sections 4.1 and 4.2 for the one-dimensional DAHA to the DAHA of GL_n. In particular, we will give the definition of the polynomial representation of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ on $\mathcal{P} := \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ and use it to obtain a PBW-type basis for $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ as described in Corollary 4.5 for the one-dimensional DAHA.

As in Remark 3.2 we define for $v = \sum_{i=1}^n v_i e_i + v_d \delta \in \tilde{\mathfrak{h}}^*$ with $v_i, v_d \in \mathbb{Z}$ the element $X^v := X_1^{v_1} ... X_n^{v_n} q^{v_d}$. Letting v range over the weight lattice $P = \sum_{i=1}^n \mathbb{Z} e_i$ we obtain the $\mathbb{K}$-basis $\{X^v \mid v \in P\}$ of the $\mathbb{K}$-algebra of Laurent polynomials in n variables $\mathcal{P} := \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$. The group algebra $\mathbb{K}[\dot{W}]$ of the extended affine Weyl group $\dot{W}$ from Definition 2.7 acts via $\mathbb{K}$-algebra automorphisms on $\mathcal{P}$ by setting for $w \in \dot{W}$ and for $v \in P$:

$$\begin{aligned} w: \ \mathcal{P} &\longrightarrow \mathcal{P}, \\ X^v &\longmapsto X^{w(v)}. \end{aligned} \tag{180}$$

When restricted to the finite Weyl group $W \subseteq \dot{W}$ this action is nothing but the intuitive action of $W = S_n$ on $\mathcal{P} = \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$, where $w \in W$ sends $X_i^{\pm 1}$ to $X_{w(i)}^{\pm 1}$. Via this action we define $\mathcal{P}^W \subseteq \mathcal{P}$ to be the $\mathbb{K}$-subalgebra of symmetric Laurent polynomials. Furthermore, the action of $\dot{W}$ on $\mathcal{P}$ will be used in the following construction of the *polynomial representation* of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ on $\mathcal{P}$.

Proposition 5.1. *The following assignment defines an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-module structure on $\mathcal{P} = \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$. Here $X_i \cdot$ denotes the (left-)multiplication by X_i for $1 \leq i \leq n$.*

$$\begin{aligned} \pi &\longmapsto \pi, \\ X_i &\longmapsto X_i \cdot && \text{for } 1 \leq i \leq n, \\ T_i &\longmapsto t^{\frac{1}{2}} s_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{X^{\alpha_i} - 1}(s_i - 1) && \text{for } 0 \leq i \leq n-1. \end{aligned} \tag{181}$$

Proof. We will only give a reference to the proof in [Che95, Theorem 2.3]. Note that the author uses the double affine Hecke algebra for SL_n, while we are working with the double affine Hecke algebra for GL_n. The first is a subquotient of the latter, where one has to replace the generators X_i for $1 \leq i \leq n$ with X^{α_i} for $1 \leq i \leq n-1$ and add the relation $\pi^n = 1$ as described in [Che05, Chapter 3.7]. The proof for SL_n in [Che95, Theorem 2.3] works analogously for GL_n. □

As in the one-dimensional case this representation is faithful for q not a root of unity, which is also proven in [Che95, Theorem 2.3]. The faithfulness will be used implicitly in the proof of the PBW-basis theorem in Theorem 5.4, similar to the proof of Corollary 4.5. The operators by which T_i acts are called *Demazure-Lusztig operators* in [Che05, Chapter 3.2.3], since they generalize the Demazure operators from [Dem73].

In a similar way as for the one-dimensional DAHA in Corollary 4.5 the polynomial representation gives rise to a PBW-type basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ for $n \geq 2$. We will need certain elements $Y_i \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ for $1 \leq i \leq n$. To define them set $\omega_i := e_1 + ... + e_i \in P$ for $1 \leq i \leq n$ and recall that by Definition 2.7 we have an embedding of the weight lattice P into the extended affine Weyl group $\dot{W}$ via τ from Theorem 2.4. Identifying $\tau(\omega_i)$

with ω_i and using the definition of T_w for $w \in \dot{W}$ from Remark 3.2 we define Y_i following [Che05, Chapter 3.2.1]:

$$Y_i := T_{\omega_i} \text{ for } 1 \leq i \leq n. \tag{182}$$

These elements pairwise commute, as we show now.

Lemma 5.2. *We have $Y_iY_j = T_{\omega_i+\omega_j} = Y_jY_i$ for $1 \leq i, j \leq n$.*

Proof. Choose $1 \leq i, j \leq n$. By definition of the elements T_w we have for $w, w' \in \dot{W}$ with $l(ww') = l(w) + l(w')$ that $T_{ww'} = T_wT_{w'}$. Therefore we only have to show $l(\omega_i + \omega_j) = l(\omega_i) + l(\omega_j)$. Let $\alpha_{i_1,i_2} + m\delta \in \dot{R}^-$ with $1 \leq i_1 \neq i_2 \leq n$ be a negative root. By definition this means either $i_1 > i_2$ and $m \leq 0$ or $i_1 < i_2$ and $m < 0$. Using the definition of the action of τ_{-v} for $-v \in P$ from Theorem 2.4 we calculate:

$$\tau_{-v}(\alpha_{i_1,i_2} + m\delta) = \alpha_{i_1,i_2} + (m + (v \mid \alpha_{i_1,i_2}))\,\delta. \tag{183}$$

This provides a description of $\dot{R}(\tau_v)$:

$$\begin{aligned}\dot{R}(\tau_v) &= \dot{R}^+ \cap \tau_{-v}(\dot{R}^-) \\ &= \{\alpha_{i_1,i_2} + m\delta \mid \quad (v \mid \alpha_{i_1,i_2}) > m \geq 0, i_1 < i_2, \text{ or} \\ &\qquad\qquad\qquad\qquad (v \mid \alpha_{i_1,i_2}) \geq m > 0, i_1 > i_2\}.\end{aligned} \tag{184}$$

For $v \in \{\omega_i, \omega_j, \omega_i + \omega_j\}$ only the first case with $i_1 < i_2$ can appear. We obtain $l(\tau(\omega_i)) = i(n-i)$, $l(\tau_{\omega_j}) = j(n-j)$ and $l(\tau_{\omega_i+\omega_j}) = i(n-i)+j(n-j)$, which shows the claim. □

Remark 5.3. The lemma shows that for an arbitrary $v = \sum_{i=1}^n v_i\omega_i \in P$ the element $Y^v := \prod_{i=1}^n Y_i^{v_i} \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is well-defined. Note that contrary to the X-case, where $X_i = X^{e_i}$, we have here $Y_i = Y^{\omega_i}$ for $1 \leq i \leq n$.

We can state the PBW-type basis theorem for $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ now. In the theorem the (finite) Iwahori-Hecke algebra $\mathcal{H}_n$ of $W \cong S_n$ will appear. It is for example defined in [Mat99], although the author uses a slightly different version of the quadratic T-relations than we need. This can be taken care of by a normalization of the T_i by the factor $t^{\frac{1}{2}}$.

Theorem 5.4. *The set*

$$\{Y^vX^{v'}T_w \mid v, v' \in P, w \in W\} \tag{185}$$

is a $\mathbb{K}$-basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. In other words, the $\mathbb{K}$-linear multiplication map

$$m : \mathbb{K}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}] \otimes_{\mathbb{K}} \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}] \otimes_{\mathbb{K}} \mathcal{H}_n \longrightarrow H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}}) \tag{186}$$

is an isomorphism of $\mathbb{K}$-vector spaces.

Proof. The fact that $\{X^vY^{v'}T_w \mid v,v' \in P, w \in W\}$ forms a $\mathbb{K}$-basis is shown in [Che95, Theorem 2.3] for generic q and is extended to the case that q is a root of unity in [Che05, Theorem 3.2.1 (ii)]. Again this is done for SL_n, but the proofs also work for GL_n. The second statement follows immediately from the first one, since $\mathcal{H}_n$ has a $\mathbb{K}$-basis given by T_w for $w \in W$ as proven in [Mat99, Theorem 1.13]. □

Remark 5.5. In fact similarly one could refine Proposition 3.3 to show that $\{T_wX^v \mid w \in \dot{W}, v \in P\}$ is a $\mathbb{K}$-basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. In [Kir97, Theorem 5.7] it is shown that the elements in the set $\{X^vT_w \mid w \in \dot{W}, v \in P\}$ are linearly independent. The author shows this for the DAHA of SL_n, but the result also holds for GL_n. Let $w \in \dot{W}$ and $v \in P$. By relations (XT1) and (XT2) from Definition 3.1 we have

$$T_wX^v = X^{w(v)}T_w + \sum_{u \in \dot{W}, u \prec w'} F_uT_u, \tag{187}$$

where $F_u \in \mathbb{K}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ and the sum ranges over all reduced sub-words of w. Using the result from [Kir97, Theorem 5.7] mentioned above this shows that the elements in $\{T_wX^v \mid w \in \dot{W}_a, v \in P\}$ are $\mathbb{K}$-linearly independent and hence they form a basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ by Proposition 3.3.

5.2 The bilinear form $\langle\ ,\ \rangle$ and its radical

Let us give the analogue of the bilinear-form $\langle\ ,\ \rangle$ on $\mathcal{P}$ and its radical Rad from Definition 4.17 for the DAHA for GL_n for $n \geq 2$. This bilinear form is also considered in [Che05, Chapter 3.10.2], on which parts of this section are based. For the definition we need the following well-known element in $\mathfrak{h}^*$, which also appears in the representation theory of GL_n.

Definition 5.6. We set

$$\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \frac{n-1}{2}e_1 + \frac{n-3}{2}e_2 + ... + \frac{-n+1}{2}e_n, \tag{188}$$

where R^+ is the set of finite positive roots from Definition 2.1. Furthermore, for $k \in \mathbb{C}$ we set $\rho_k := k \cdot \rho$.

Definition 5.7. For $\nu, \mu \in P$ and $k \in \mathbb{C}$ such that $q^k = t$ we set

$$\langle X^\nu, X^\mu\rangle := \left(Y^{-\nu}(X^\mu)\right)(q^{-\rho_k}). \tag{189}$$

In other words we apply $Y^{-\nu}$ to X^μ via the action defined in Proposition 5.1 and evaluate the resulting Laurent polynomial on $q^{-\rho_k}$. Using that X^v for $v \in P$ form a basis of $\mathcal{P}$ we extend this definition $\mathbb{K}$-bilinearly to arbitrary elements in $\mathcal{P}$ in order to obtain a bilinear form $\langle\ ,\ \rangle$ on $\mathcal{P}$. We denote the radical of $\langle\ ,\ \rangle$ by Rad. In formulas:

$$\mathrm{Rad} := \{f \in \mathcal{P} \mid \langle f, g\rangle = \langle g, f\rangle = 0 \text{ for all } g \in \mathcal{P}\}. \tag{190}$$

To study the bilinear form $\langle\ ,\ \rangle$ we need some basic facts about the elements Y^v for $v \in P$ from Remark 5.3.

Lemma 5.8. *(a) For $v \in P^+$ an integral dominant weight we have $Y^v = T_v$.*

(b) For $v \in P$ and $1 \leq i \leq n$ we have $T_i^{-1} Y^v T_i^{-1} = Y^v Y^{-\alpha_i}$ if $(v, \alpha_i) = 1$ and $T_i Y^v = Y^v T_i$ if $(v, \alpha_i) = 0$

(c) We have $Y^{e_i} = T_{i-1}^{-1} ... T_1^{-1} \pi T_{n-1} ... T_i$ for $1 \leq i \leq n$. Note that $T_{i-1} \neq T_0$ for $i = 1$, but in this case the product to the left of π is empty.

(d) For $v \in P$ we have $Y^v(1) = q^{(\rho_k|v)} \cdot 1$ for $1 \in \mathcal{P}$, where $k \in \mathbb{C}$ is such that $q^k = t$.

Proof. The computation from the proof of Lemma 5.2 can easily be extended to obtain (a).

We will not prove statement (b) and only give a reference to [Che95, Proposition 2.2], where Cherednik denotes our ω_i by b_i.

Statement (c) for $i = 1$ follows, since $l(\tau_{e_1}) = n - 1$ by the calculation in Lemma 5.2 and since $\tau_{e_1} = \pi s_{n-1} ... s_1$ by definition of π in Proposition 2.9, which must therefore be a reduced expression. For $1 \leq i \leq n - 1$ we can apply induction and statement (b) to obtain

$$\begin{aligned} Y^{e_{i+1}} &= T_i^{-1} Y^{e_i} T_i^{-1} = T_i^{-1} (T_{i-1}^{-1} ... T_1^{-1} \pi T_{n-1} ... T_i) T_i^{-1} \\ &= T_i^{-1} ... T_1^{-1} \pi T_{n-1} ... T_{i+1}, \end{aligned} \tag{191}$$

which shows (c).

For statement (d) we can clearly reduce to $v = \omega_i$. Note that T_j for $0 \leq j \leq n$ acts by multiplication with $t^{\frac{1}{2}}$ on $1 \in \mathcal{P}$ and $\pi(1) = 1$. By writing $Y^{\omega_i} = Y_i = \pi^k T_{i_1} ... T_{i_l}$ for a reduced expression $\tau(\omega_i) = \pi^k s_{i_1} ... s_{i_l}$ we only have to show $l(\tau_{\omega_i}) = (2\rho, \omega_i)$ for $1 \leq i \leq n$. We calculate $(2\rho, \omega_i) = (n-1) + (n-3) + ... + (n-2i) = n \cdot i - i^2 = (n-i)i$. As we have seen in the proof of Lemma 5.2 we have $l(\tau_{\omega_i}) = (n-i)i$ and (d) follows. $\square$

We will see now that $\langle\ ,\ \rangle$ is symmetric and that Rad is an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule of $\mathcal{P}$. For this we will need a $\mathbb{K}$-linear anti-isomorphism of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, which is defined for $v \in P$ and $1 \leq i \leq n$ by

$$\phi(X^v) = Y^{-v},\ \phi(Y^v) = X^{-v} \text{ and } \phi(T_i) = T_i. \tag{192}$$

In particular, $\phi(X_i) = Y^{-e_i}$ for $1 \leq i \leq n$. We will not give the proof that this actually defines an anti-automorphism of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and only refer to [Sim17, Lemma 2.4.9]. Note that the proof there can not directly be applied to our setting, since the Y_i the author uses do not correspond to our Y^{e_i}. But we can still make this proof applicable, by following [Sim17, Remark 2.4.7 and Theorem 2.4.8] with our definition of Y^{e_i} replacing the Y_i from the reference. In this way we obtain a presentation of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ involving the Y^{e_i} for $1 \leq i \leq n$ and one can verify that ϕ defines an anti-automorphism via explicit calculation. The following proposition can also be found in [Che05, Lemma 3.10.3].

Proposition 5.9. *(a) The bilinear form $\langle\ ,\ \rangle$ on $\mathcal{P}$ is symmetric.*
(b) For $H \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and $f, g \in \mathcal{P}$ we have $\langle H(f), g\rangle = \langle f, \phi(H)(g)\rangle$.
(c) The radical Rad $\subseteq \mathcal{P}$ *is an $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$-submodule.*

Proof. The proof is analogous to the proof of Proposition 4.18 after one has deduced $\phi(H)(1)(q^{-\rho_k}) = H(1)(q^{-\rho_k})$ for $H \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. For this we need a different version of the PBW-basis of $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. In [Che05, Theorem 3.2.1] it is stated that $\{X^v T_w Y^{v'} \mid v, v' \in P, w \in W\}$ is a basis of the DAHA of SL_n, which also holds for GL_n. Look at a fixed $X^v T_w Y^{v'}$. By Lemma 5.8 (d) we have $X^v T_w Y^{v'}(1)(q^{-\rho_k}) = q^{(v|-\rho_k)} t^{\frac{l(w)}{2}} q^{(v'|\rho_k)}$ and $\phi(X^v T_w Y_{v'})(1)(q^{-\rho_k}) = q^{(-v'|-\rho_k)} t^{\frac{l(w^{-1})}{2}} q^{(-v|\rho_k)}$, which are equal. Thus, by $\mathbb{K}$-linearity we obtain $\phi(H)(1)(q^{-\rho_k}) = H(1)(q^{-\rho_k})$ for all $H \in H(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. □

5.3 Spherical DAHA

Our next goal is to define a certain idempotent $e \in H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and use it to construct the so-called *spherical double affine Hecke algebra* (or short *spherical DAHA*) $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e \subseteq H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. We emphasize that the embedding $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e \subseteq H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ is not unital, since the unit of $eH_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})e$ is e. As we will see in Theorem 5.11 and Corollary 5.12, the idempotent e is only well-defined for certain values of the parameters q and t of the double affine Hecke algebra $H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. One possible choice, which we will employ for the rest of this thesis, is to set

$$q = t = e^{\frac{2\pi i}{N}} \text{ for a fixed } N > n. \tag{193}$$

We simplify our notation by setting $H_n := H_n(q^{\frac{1}{2}}, t^{\frac{1}{2}})$ and since $q \in \mathbb{C}$ we have $\mathbb{K} = \mathbb{C}$. We remark that this section is based on [Sim17, Lecture 2, Chapter 3.3].

As an intermediate step towards the definition of $e \in H_n$ we need an auxiliary element

$$\tilde{e} := \sum_{w\in W} t^{\frac{l(w)}{2}} T_w \in H_n, \tag{194}$$

where the sum ranges over all elements of the finite Weyl group $W = S_n$ of type A_{n-1} and $l(w)$ is the (Coxeter-)length of $w \in W \subseteq \dot{W}_a$ from Definition 2.5. The element $T_w \in H_n$ for $w \in W \subseteq \dot{W}$ is defined in Remark 3.2.

Lemma 5.10. *For $i \in \{1, ..., n\}$ we have $T_i\tilde{e} = t^{\frac{1}{2}}\tilde{e} = \tilde{e}T_i \in H_n$.*

Proof. Let $w \in W$ and $i \in \{1, ..., n\}$. Recall the well-known fact from the theory of the symmetric group, that $l(s_i w) < l(w)$ implies that w has a reduced expression of the form $w = s_i s_{i_2} ... s_{i_{l(w)}}$ for some indices $i_2, ..., i_{l(w)} \in \{1, ..., n\}$. This is just the dual version of the exchange condition from [Hum90, Chapter 1.7], which one can obtain from the reference by applying the anti-automorphism of S_n sending s_i to s_i for $1 \leq i \leq n-1$. On the other hand, if $l(s_i w) > l(w)$ and $w = s_{i_1} ... s_{i_{l(w)}}$ is a reduced expression for w then $s_i w = s_i s_{i_1} ... s_{i_{l(w)}}$ is a reduced expression for $s_i w$. Therefore writing the relation (T) from the definition of H_n in Definition 3.1 as $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$ gives

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w), \\ T_{s_i w} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_w & \text{if } l(s_i w) < l(w). \end{cases} \tag{195}$$

We obtain using $\tilde{e} = \sum_{w \in W} t^{\frac{l(w)}{2}} T_w$

$$T_i \tilde{e} = \sum_{w \in W,\, l(s_i w) > l(w)} t^{\frac{l(w)}{2}} T_{s_i w} + \sum_{w \in W,\, l(s_i w) < l(w)} t^{\frac{l(w)}{2}} (T_{s_i w} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_w). \tag{196}$$

To prove the first equality we have to show that the coefficient of T_v on the right hand side equals $t^{\frac{l(v)+1}{2}}$ for any $v \in W$. Left-multiplication by s_i is simply transitive on W, hence we see that T_v appears two times in the sum on the right hand side if $l(s_i v) < l(v)$ and only one time if $l(s_i v) > l(v)$. If $l(s_i v) < l(v)$ we can find a reduced expression $v = s_i s_{i_2} ... s_{i_{l(v)}}$ and set $v' = s_i v = s_{i_2} ... s_{i_{l(v)}}$. In the sum over $w \in W$ with $l(s_i w) > l(w)$ the element T_v appears once as $T_{s_i v'}$ with coefficient $t^{\frac{l(v')}{2}}$ and in the sum over $w \in W$ with $l(s_i w) < l(w)$ the element T_v appears once with coefficient $t^{\frac{l(v)}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. Hence overall the coefficient of T_v is $t^{\frac{l(v')}{2}} + t^{\frac{l(v)}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = t^{\frac{l(v)+1}{2}}$ as desired. For $l(s_i v) > l(v)$ set $v' = s_i v$. We have that T_v only appears once on the right hand side, namely as $T_{s_i v'}$ in the sum over $w \in W$ with $l(s_i w) < l(w)$. It appears with the appropriate coefficient $t^{\frac{l(v')}{2}} = t^{\frac{l(v)+1}{2}}$. The proof of $\tilde{e} T_i = t^{\frac{1}{2}} \tilde{e}$ works analogous and is omitted. □

From Lemma 5.10 we obtain

$$\tilde{e}^2 = \sum_{w \in W} t^{\frac{l(w)}{2}} T_w \tilde{e} = \Bigg(\sum_{w \in W} t^{l(w)} \Bigg) \tilde{e}, \tag{197}$$

since we have $T_w = T_{i_1} ... T_{i_{l(w)}}$ for $w = s_{i_1} ... s_{i_{l(w)}}$ a reduced word. Therefore to construct an idempotent $e \in H_n$ we want to set $e := \Bigg(\sum_{w \in W} t^{l(w)} \Bigg)^{-1} \tilde{e}$, but this element is a priori not well-defined, since $\sum_{w \in W} t^{l(w)}$ might be zero.

Recall that we set $t = q$ to be an N-th primitive root of unity with $N > n$. We define $P(x) := \sum_{w\in W} x^{l(w)}$. In fact, this polynomial is well known in the literature as the *Poincaré polynomial of* W, see [Sol66]. It admits a factorization, which will let us deduce $P(q) \neq 0$ and hence that $e \in H_n$ is well-defined.

Theorem 5.11. *With* $P(x) = \sum_{w\in W} x^{l(w)}$ *we have*

$$P(x) = \prod_{i=1}^{n}(1 + x + ... + x^{i-1}) = \prod_{i=1}^{n} \frac{1 - x^i}{1 - x}. \tag{198}$$

Proof. The first equality follows from [Sol66, Corollary 2.3] if one knows that for the exponents $m_1, ..., m_n$ of W appearing in the reference we have $m_i = i - 1$. But this holds, since $m_i = d_i - 1 = i - 1$, where d_i is the degree of the i-th elementary symmetric polynomial $\boldsymbol{e_i}$. See also [Sol66] for the definition of the exponents of W. The second equality follows by multiplying with $(1 - x)^n$. □

Corollary 5.12. *Setting*

$$e := P(q)^{-1}\tilde{e} = \frac{1}{\sum_{w\in W} q^{l(w)}} \sum_{w\in W} q^{\frac{l(w)}{2}} T_w \tag{199}$$

gives a well-defined idempotent element in H_n *called the* symmetrizer *or* symmetrizing element.

Proof. The well-definedness of $e \in H_n$ follows, since by Theorem 5.11 and by the choice of $q = t$ as a N-th primitive root of unity with $N > n$ we have $P(q) \neq 0$. To show that e is idempotent we calculate using (197)

$$e^2 = \left(\frac{1}{\sum_{w\in W} q^{l(w)}}\right)^2 \tilde{e}^2 = \frac{1}{\sum_{w\in W} q^{l(w)}} \tilde{e} = e. \tag{200}$$

□

Any idempotent element $e_A \in A$, where A is any unital $\mathbb{C}$-algebra, gives rise to a unital algebra $e_A A e_A$ with unit e_A. The $\mathbb{C}$-algebra structure on $e_A A e_A$ is induced from the $\mathbb{C}$-algebra structure on A via the non-unital inclusion $e_A A e_A \hookrightarrow A$.

Definition 5.13. We call the $\mathbb{C}$-algebra eH_ne for $e \in H_n$ the symmetrizin element the *spherical double affine Hecke algebra*, short *spherical DAHA*.

It is known that an idempotent $e_A \in A$ defines an exact functor from the category of A-modules to the category of $e_A A e_A$-modules. Indeed, Ae_A is a submodule of the free A-module A and therefore projective. Hence, the functor $\operatorname{Hom}_A(Ae_A, \_\_) : \operatorname{Mod}_{\mathbb{C}} A \to \operatorname{Mod}_{\mathbb{C}} e_A A e_A$ is exact. For M an A-module we can identify $\operatorname{Hom}_A(Ae_A, M) \cong e_A M$ by $f \mapsto f(e_A) = e_A f(e_A)$ to construct the desired functor. In our setting we obtain the following exact functor:

$$\begin{aligned} \Phi_e : \operatorname{Mod}_{\mathbb{C}} H_n &\longrightarrow \operatorname{Mod}_{\mathbb{C}} eH_n e, \\ M &\longmapsto eM. \end{aligned} \tag{201}$$

Definition 5.14. Using the polynomial representation $\mathcal{P}$ of H_n from Proposition 5.1 and its radical Rad from Definition 5.7 we define

$$\mathcal{M} := \Phi_e(\mathcal{P}/\operatorname{Rad}). \tag{202}$$

Recall that $\operatorname{Rad} \subseteq \mathcal{P}$ is an H_n-submodule by Proposition 5.9. From the exactness of Φ_e we obtain $\mathcal{M} \cong e\mathcal{P}/e\operatorname{Rad}$. Our goal for the rest of this chapter is to identify $\mathcal{M}$ with the quantum cohomology ring of the Grassmannian $\operatorname{Gr}_{n,N}$ of n-planes in $\mathbb{C}^N$ specialized at $q = 1$ denoted by $\mathbb{C} \otimes_{\mathbb{Z}} qH^\bullet(Gr_{n,N})_{q=1}$. Because the parameter q in the quantum cohomology is specialized to $q = 1$ it does not match the parameter q from the DAHA. To achieve our goal we will use the description of the quantum cohomology via symmetric polynomials, given in the beginning of Chapter 5. Our first result in this direction is to show that $e\mathcal{P} = \mathcal{P}^W$ as $\mathbb{C}$-subalgebras of $\mathcal{P}$, where the action of W on $\mathcal{P}$ is just the standard permutation action as defined in (180).

Lemma 5.15. *Let $f \in \mathcal{P}$. Then $f \in \mathcal{P}^W$ if and only if $T_i f = t^{\frac{1}{2}} f$ for all $i \in \{1, ..., n\}$.*

Proof. Let $i \in \{1, ..., n\}$ and $f \in \mathcal{P}$. By definition of the action of T_i on $\mathcal{P}$ in Proposition 5.1 we have

$$T_i f = t^{\frac{1}{2}} s_i(f) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{X^{\alpha_i} - 1}(s_i(f) - f). \tag{203}$$

Clearly, if $s_i(f) = f$ we have $T_i f = t^{\frac{1}{2}} f$. On the other hand if $T_i f = t^{\frac{1}{2}} f$ we obtain

$$0 = \left(t^{\frac{1}{2}} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{X^{\alpha_i} - 1} \right) (s_i(f) - f). \tag{204}$$

Since $\mathcal{P}$ has no zero-divisors, we can look at the last equation as an equation inside the fraction field $\mathbb{C}(X_1^{\pm 1}, ..., X_n^{\pm 1})$ of $\mathcal{P}$. Since $t^{\frac{1}{2}} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{X^{\alpha_i} - 1} \neq 0$, we obtain $s_i(f) = f$. The claim follows, since $f \in \mathcal{P}^W$ if and only if $s_i(f) = f$ for all $i \in \{1, ..., n\}$. $\square$

Proposition 5.16. *Let $e \in H_n$ be the symmetrizing element.*

(a) We have $e\mathcal{P} = \mathcal{P}^W$ and $e\operatorname{Rad} = \operatorname{Rad}^W$ as $\mathbb{C}$-subalgebras of $\mathcal{P}$.

(b) For $g \in \mathcal{P}^W$ we have $g \in e\operatorname{Rad}$ if and only $\langle g', g \rangle = 0$ for all $g' \in \mathcal{P}^W$.

Proof. By Lemma 5.15 we have to show that $f \in e\mathcal{P}$ is equivalent to $T_i f = t^{\frac{1}{2}} f$ for all $i \in \{1, ..., n\}$. If $f \in e\mathcal{P}$ write $f = ef'$ for some $f' \in \mathcal{P}$. Then by Lemma 5.10 and since $e = P(q)^{-1}\tilde{e}$ we have

$$T_i f = (T_i e) f' = t^{\frac{1}{2}} e f' = t^{\frac{1}{2}} f. \tag{205}$$

On the other hand if $T_i f = t^{\frac{1}{2}} f$ for all $i \in \{1, ..., n\}$ then we have

$$ef = P(t)^{-1} \sum_{w \in W} t^{\frac{l(w)}{2}} T_w f = P(t)^{-1} \sum_{w \in W} t^{l(w)} f = f. \tag{206}$$

Therefore, $f = ef \in e\mathcal{P}$. This shows $e\mathcal{P} = \mathcal{P}^W$ and $e\,\mathrm{Rad} = \mathrm{Rad}^W$ follows, since $\mathrm{Rad} \subseteq \mathcal{P}$ is an H_n-submodule. In all cases, the $\mathbb{C}$-algebra structure is defined by restricting the $\mathbb{C}$-algebra structure of $\mathcal{P}$, hence the equalities really hold as $\mathbb{C}$-subalgebras. For the last claim we have to show that for $g \in \mathcal{P}^W$ we have that $\langle g', g \rangle = 0$ for all $g' \in \mathcal{P}^W$ already implies $\langle f', g \rangle = 0$ for all $f' \in \mathcal{P}$. Note that $\phi(e) = e$, since $\phi(T_i) = T_i$ for $1 \le i \le n$ and hence ϕ permutes the elements T_w for $w \in W$. Here ϕ is the anti-automorphism of H_n from (192). Therefore by Proposition 5.9 (b) we have

$$\langle f', g \rangle = \langle f', eg \rangle = \langle \phi(e) f, g \rangle = \langle ef', g \rangle = 0, \tag{207}$$

where the last equality follows by the assumption on g and since $ef' \in e\mathcal{P} = \mathcal{P}^W$. $\square$

We will now see that $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ and $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W$ embed into the spherical DAHA, which will later on be important as we will discuss weight-space decompositions with respect to these subalgebras. Note that W does not act on $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]$ via permutation of the Y_i, but by $w(Y^v) = Y^{w(v)}$ for $w \in W$ and $v \in P$, where Y^v is defined as in Remark 5.3.

Proposition 5.17. *The $\mathbb{C}$-linear map $\mathcal{E} : H_n \to eH_n e$ sending h to ehe for $h \in H_n$ induces the following isomorphisms of $\mathbb{C}$-algebras:*

$$\begin{aligned} \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W &\cong e\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W e, \\ \mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W &\cong e\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W e. \end{aligned} \tag{208}$$

Proof. The second isomorphism follows from the first one by applying the anti-automorphism ϕ from (192) to both sides, using that $\phi(e) = e$ and that for $w \in W$ and $v \in P$ we have

$$\phi(w(X^v)) = \phi(X^{w(v)}) = Y^{-w(v)} = w(Y^{-v}) = w(\phi(X^v)). \tag{209}$$

To obtain the first isomorphism we note that from relation (XT1) and (XT2) in Definition 3.1 one can deduce for $1 \le i \le n$ and $v = v_1 e_1 + ... v_n e_n \in P$

$$T_i X^v = X^{s_i(v)} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^{s_i(v)} - X^v}{X^{\alpha_i} - 1} \tag{210}$$

by induction on $|v| := |v_1| + ... + |v_n|$. Using this we can calculate that $epe = pe$ for any $p \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W \subseteq H_n$ from which the proposition follows. $\square$

5.4 Rational Schur polynomials

Motivated by the previous section we want to study the $\mathbb{C}$-algebra of symmetric Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W = \mathcal{P}^W$. This will be done using the so-called *(rational) Schur polynomials* s_λ for $\lambda \in P^+$ a dominant integral weight. These polynomials are well-known from the representation theory of GL_n, where they appear as the characters of the irreducible rational finite-dimensional GL_n-modules, see for example [Ful97, Chapter 8]. The Schur polynomials also appear naturally in the theory of the double affine Hecke algebra: for the parameters $q = t$, which holds by our assumptions in (193), the famous (symmetric) *Macdonald polynomials* $P_\lambda \in \mathcal{P}$ for $\lambda \in P^+$ specialize to the rational Schur polynomials s_λ, as we will see in Remark 5.24. Furthermore, the rational Schur polynomials form a basis of the $\mathbb{C}$-algebra of symmetric Laurent polynomials and have some other nice properties, such as an explicit evaluation formula for $s_\lambda(q^{-\rho})$. We will assume some knowledge about symmetric polynomials and the representation theory of GL_n and refer to [Ful97] as a general reference.

To define the rational Schur polynomials we need some preliminary definitions and need to fix some notation. As before we set $P := \sum_{i=1}^n \mathbb{Z}e_i$ to be the weight lattice, $Q := \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i$ to be the root lattice and

$$P^+ := \left\{ \sum_{i=1}^n v_i\omega_i \mid v_i \in \mathbb{Z}_{\geq 0} \right\} = \left\{ \sum_{i=1}^n v_i e_i \mid v_1 \geq ... \geq v_n, v_i \in \mathbb{Z} \right\},$$
$$Q^+ := \left\{ \sum_{i=1}^{n-1} v_i\alpha_i \mid v_i \in \mathbb{Z}_{\geq 0} \right\}.$$

We will from now on also use a shorthand notation by writing $\lambda = \lambda_1 e_1 + ... + \lambda_n e_n \in P$ as $\lambda = (\lambda_1, ..., \lambda_n)$.

Definition 5.18. Define a partial order on P by setting $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. We transport this order to the basis of $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}] = \mathcal{P}$ consisting of the monomials X^λ for $\lambda \in P$ by setting $X^\lambda \geq X^\mu$ if $\lambda \geq \mu$. The $\mathbb{C}$-algebra of symmetric Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W = \mathcal{P}^W$ has a basis given by the *monomial symmetric functions*

$$m_\lambda := \sum_{\mu \in W(\lambda)} X^\mu \text{ for } \lambda \in P^+. \tag{211}$$

This follows, since any orbit $W(\mu)$ for $\mu \in P$ contains a unique element $\mu_+ \in P^+$. We also transport the order to this basis by setting $m_\lambda \geq m_\mu$ if $\lambda \geq \mu$.

Definition 5.19. Let $\lambda = (\lambda_1, ..., \lambda_n) \in P^+$ with $\lambda_n \geq 0$. We define the *Young diagram* $\hat{\lambda}$ associated to λ as the subset of $\mathbb{Z}^2$ defined by

$$\hat{\lambda} := \{(i, l_i) \mid i = 1, ..., n,\ 1 \leq l_i \leq \lambda_i\}. \tag{212}$$

A *(semistandard) Young tableau* on $\hat{\lambda}$ is a function $T : \hat{\lambda} \to \{1, ..., n\}$ such that T is strictly column-increasing and weakly row-increasing, in formulas:

$$\begin{aligned} T(i,j) > T(i-1,j) \text{ for } (i,j),(i-1,j) \in \hat{\lambda}, \\ T(i,j) \geq T(i,j-1) \text{ for } (i,j),(i,j-1) \in \hat{\lambda}. \end{aligned} \tag{213}$$

We denote the set of all semistandard Young tableaux on $\hat{\lambda}$ by Tab_λ.

Definition 5.20. Let $\lambda = (\lambda_1, ..., \lambda_n) \in P^+$. We define the *(rational) Schur polynomial* $s_\lambda \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}] = \mathcal{P}$ as follows. If $\lambda_n \geq 0$ we set

$$s_\lambda := \sum_{T \in \mathrm{Tab}_\lambda} X_1^{t_1} ... X_n^{t_n}, \tag{214}$$

where t_i for $1 \leq i \leq n$ is number of times that T takes the value i. For $\lambda \in P^+$ with $\lambda_n < 0$ we set $\lambda' = \lambda - \lambda_n \cdot (1, ..., 1)$ and define

$$s_\lambda := (X_1 \cdot ... \cdot X_n)^{\lambda_n} s_{\lambda'}. \tag{215}$$

In [Ful97, Chapter 2.2] it is shown that (rational) Schur polynomials are symmetric functions. In fact we will see now that they form a basis of the $\mathbb{C}$-algebra of symmetric Laurent polynomials. The coefficients $K_{\lambda\mu}$ appearing in the following lemma are known as the *Kostka numbers*, see [Ful97].

Lemma 5.21. *For* $\lambda \in P^+$ *we have* $s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$ *for some* $K_{\lambda\mu} \in \mathbb{C}$. *Therefore,* $\{s_\lambda \mid \lambda \in P^+\}$ *is a basis of* $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W = \mathcal{P}^W$.

Proof. Let $\lambda \in P^+$. We can clearly reduce to the case that $\lambda_n \geq 0$. Assume the monomials X^μ and X^ν both appear in s_λ with non-zero coefficient. Since both μ and ν are constructed using some tableaux on $\hat{\lambda}$ we have $\mu_1 + ... + \mu_n = \nu_1 + ... + \nu_n = \lambda_1 + ... + \lambda_n$ and hence $\mu - \nu \in Q$. Furthermore, by definition of s_λ it is clear that X^λ is the highest monomial which can appear in s_λ and that it appears with coefficient 1 in s_λ. Since we already know that s_λ is symmetric this shows the first claim. The second claim follows, because the m_λ for $\lambda \in P^+$ form a basis of $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$. $\square$

From the representation theory of GL_n we know that s_λ is the character of the irreducible highest-weight representation of highest weight $\lambda \in P^+$, see [Ful97, Chapter 8]. This allows us to apply Weyl's character formula and obtain

$$s_\lambda = \Delta^{-1} \sum_{w \in W} (-1)^{l(w)} X^{w(\lambda + \rho)}, \tag{216}$$

where Δ is the *Weyl denominator* defined to be

$$\Delta := \prod_{\alpha \in R^+} X^{\frac{\alpha}{2}} - X^{-\frac{\alpha}{2}}. \tag{217}$$

We will use Weyl's character formula to calculate the evaluation $s_\lambda(q^{-\rho})$ and to prove that the s_λ are not only a basis of $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$, but even an orthonormal basis with respect to the bilinear form $\langle\ ,\ \rangle_0$, which we will define now. For the definition of $\langle\ ,\ \rangle_0$ we need the $\mathbb{C}$-linear map $\bar{\ }\colon \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}] \to \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ defined by $\bar{X}^\mu = X^{-\mu}$ for $\mu \in P$ and we will also need the element $\Delta' := \Delta\bar{\Delta} \in \mathbb{C}[X_1^{\pm 1}, ... X_n^{\pm 1}]$.

Definition 5.22. We define a $\mathbb{C}$-bilinear form $\langle\ ,\ \rangle_0$ on $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ by setting for $f, g \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$

$$\langle f, g\rangle_0 := \frac{1}{|W|}\langle f\bar{g}\Delta'\rangle_0, \tag{218}$$

where $\langle\ \rangle_0 : \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}] \to \mathbb{C}$ is the function that associates to any Laurent polynomial its constant term. We will also denote the restriction of $\langle\ ,\ \rangle_0$ to $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ by $\langle\ ,\ \rangle_0$.

Proposition 5.23. *The elements s_λ for $\lambda \in P^+$ form an orthonormal basis of $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ with respect to the bilinear form $\langle\ ,\ \rangle_0$.*

Proof. We have already seen in Lemma 5.21 that the s_λ form a basis. For the orthonormality we use Weyl's character formula from (216) and obtain for $\lambda, \mu \in P^+$

$$\langle s_\lambda, s_\mu\rangle_0 = \frac{1}{|W|}\left\langle \sum_{w\in W}(-1)^{l(w)}X^{w(\lambda+\rho)} \cdot \sum_{w\in W}(-1)^{l(w)}X^{-w(\mu+\rho)}\right\rangle_0 \tag{219}$$

From this we see that $\langle s_\lambda, s_\mu\rangle_0 \neq 0$ implies that there exists $w \in W$ such that $w(\lambda+\rho) = \mu+\rho$. Since $\lambda, \mu \in P^+$ this already implies $\lambda = \mu$, because for any $v = v_1e_1 + ... + v_ne_n \in \mathfrak{h}^*$ with $v_i \in \mathbb{Q}$ there exists a unique element v' in the W-orbit of v such that $v' = v'_1e_1 + ... + v'_ne_n$ with $v'_i \geq v'_j$ for $i > j$. If $\lambda = \mu$ we easily calculate $\langle s_\lambda, s_\lambda\rangle_0 = 1$, which finishes the proof. ☐

Remark 5.24. The previous proposition together with Lemma 5.21 show that we could define the Schur polynomials s_λ for $\lambda \in P^+$ equivalently by the two conditions

(1) $s_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}m_\mu$ for some $c_{\lambda\mu} \in \mathbb{C}$,

(2) $\langle s_\lambda, s_\mu\rangle_0 = \delta_{\lambda\mu}$ for $\lambda, \mu \in P^+$.

In [Kir97, Theorem 2.1] two very similar conditions are employed to uniquely define the so-called *Macdonald's polynomials*. In fact, the only difference if that Δ' appearing in $\langle\ ,\ \rangle_0$ is replaced by

$$\prod_{\alpha\in R}\prod_{i=0}^{\infty}\frac{1-q^{2i}X^\alpha}{1-t^2q^{2i}X^\alpha}. \tag{220}$$

Specializing to $q = t$ gives back our Δ' and hence for $q = t$ the Macdonald's polynomials specialize to the Schur polynomials. Note that in the literature often a slightly different bilinear form is used for the definition of the Macdonald's polynomials, but by the discussion in [Kir97, Lecture 6] these definitions are actually equivalent.

This allows us to use one of the central theorems from the theory of Macdonald's polynomials and apply it to the Schur polynomials s_λ. For this recall the definition the elements Y^v for $v \in P$ from Remark 5.3 and the definition of the polynomial representation of H_n in Proposition 5.1. As before we set $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W$ to be the $\mathbb{C}$-algebra of W-invariant elements where $w \in W$ acts via $Y^\lambda = Y^{w(\lambda)}$ for $\lambda \in P$. Furthermore, for $\lambda \in P$ denote by λ_- the unique element in the W-orbit of λ that lies in $P^- := -P^+$.

Theorem 5.25. *Let $f \in \mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W \subseteq H_n$ and $\lambda \in P^+$. Then we have for $s_\lambda \in \mathcal{P}$, the Schur polynomial of λ,*

$$fs_\lambda = f(q^{-\lambda_- + \rho}) \cdot s_\lambda. \tag{221}$$

Proof. Here $f(q^{-\lambda_- + \rho})$ is defined by setting $Y^\mu(q^{-\lambda_- + \rho}) = q^{(\mu | -\lambda_- + \rho)}$ on the basis elements Y^μ of $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]$ and $\mathbb{C}$-linear extension. We will not give a proof and instead only refer to the proof in [Che95, Main Theorem 4.5]. Again, the author does not consider the double affine Hecke algebra for GL_n, but the proof for GL_n works analogously. $\square$

In particular, any $f \in \mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W$ preserves $\mathcal{P}^W$, which was not obvious. This theorem and the upcoming evaluation formula for the $s_\lambda(q^{-\rho})$ in Proposition 5.29 will be the main tools in the explicit description of $e\,\mathrm{Rad}$.

Motivated by the previous theorem we define the notion of (symmetric) Y-weight spaces now. These are nothing but weight spaces for the subalgebra $e\mathbb{C}[Y_1^{\pm 1}, ...Y_n^{\pm 1}]^W e$ of eH_ne from the second isomorphism in Proposition 5.17. We also give the analogue definition of X-weight spaces. Observe the different role of the weight λ in the two definitions.

Definition 5.26. Let M be an eH_ne-module and let $\lambda \in P^+$.

(1) An element $m \in M$ is a *Y-weight vector of weight* λ if we have for all $f \in \mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W$

$$efe(m) = f(q^{-\lambda_- + \rho}) \cdot m. \tag{222}$$

We define the *Y-weight space of weight* λ to be

$$M_\lambda^Y := \{m \in M \mid m \text{ is a } Y\text{-weight vector of weight } \lambda\}. \tag{223}$$

(2) An element $m \in M$ is an *X-weight vector of weight* λ if we have for all $g \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$

$$ege(m) = g(q^{\lambda_- - \rho}) \cdot m. \tag{224}$$

We define the X-*weight space of weight* λ to be

$$M_\lambda^X := \{m \in M \mid m \text{ is an } X\text{-weight vector of weight } \lambda\}. \tag{225}$$

Proposition 5.27. *The element s_λ for $\lambda \in P^+$ has Y-weight λ. In particular, the set $\{s_\lambda \mid \lambda \in P^+\}$ is a basis of the eH_ne-module $e\mathcal{P}$ consisting of Y-weight vectors.*

Proof. The idempotent $e \in H_n$ from Corollary 5.12 acts via the identity on $\mathcal{P}^W$. By Theorem 5.25 $f \in \mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^W$ preserves $\mathcal{P}^W$. Thus we obtain $efe(g) = f(g)$ for any $g \in \mathcal{P}^W = e\mathcal{P}$. From Theorem 5.25 we obtain $efe(s_\lambda) = f(q^{-\lambda_- + \rho})s_\lambda$ for $\lambda \in P^+$ and hence s_λ is a Y-weight vector of weight λ. In particular by Proposition 5.23 the Schur polynomials form a basis consisting of Y-weight vectors. □

Remark 5.28. This begs the question, when two Schur polynomials lie in the same Y-weight space, which we answer now. Since we have $s_\lambda \in (e\mathcal{P})_\lambda$, we have to find out when two Y-weight spaces of an eH_ne-modules M are identical. Let $V := \mathbb{C}^n$ be a n-dimensional vector space and let $W = S_n$ act on it via permutation of the standard basis vectors. It is well-known that $f(v) = f(v')$ for all symmetric polynomials $f \in \mathrm{Sym}(V^*)$ if and only if we have $[v] = [v']$ in V/S_n. Hence, two non-trivial Y-weight spaces M_λ and $M_{\lambda'}$ agree if and only if $q^{-\lambda_- + \rho} = q^{-\lambda'_- + \rho}$ in V/S_n. Since q is a primitive N-th root of unity, this is equivalent to the existence of pairwise different $1 \leq j_i \leq n$ for $1 \leq i \leq n$ such that

$$-\lambda_i + \frac{-n-1+2i}{2} = -\lambda'_{j_i} + \frac{-n-1+2j_i}{2} \mod N. \tag{226}$$

Otherwise we have $M_\lambda \cap M_{\lambda'} = 0$. Since $s_\lambda \in (e\mathcal{P})_\lambda$ and $s_{\lambda'} \in (e\mathcal{P})_{\lambda'}$, this condition also tells us when two Schur polynomials lie in the same Y-weightspace of $e\mathcal{P}$.

The following evaluation formula is also considered in [Mac00, Section 12], where also the proof idea is from. We will use it to show when a Y-weight vector lies in $e\,\mathrm{Rad}$. Indeed, from Proposition 5.16 (b) we deduce that a Y-weight vector $f \in e\mathcal{P}$ lies in $e\,\mathrm{Rad}$ if and only if $f(q^{-\rho}) = 0$.

Proposition 5.29. *The following equation holds for any $\lambda \in P^+$:*

$$s_\lambda(q^{-\rho}) = q^{-(\lambda|\rho)} \prod_{\alpha \in R^+} \frac{1 - q^{(\lambda+\rho|\alpha)}}{1 - q^{(\rho|\alpha)}}. \tag{227}$$

Proof. Let $\lambda \in P^+$. We apply Weyl's character formula to obtain

$$s_\lambda(q^{-\rho}) = \frac{\sum_{w \in W} (-1)^{l(w)} q^{(w(\lambda+\rho)|-\rho)}}{\prod_{\alpha \in R^+} q^{\frac{(\alpha|-\rho)}{2}} - q^{-\frac{(\alpha|-\rho)}{2}}} \tag{228}$$

Note that by assumption q is a primitive N-th root of unity with $N > n$, hence the denominator is not zero. Multiplying with the denominator shows that we have to verify

$$\sum_{w\in W}(-1)^{l(w)}q^{(w(\lambda+\rho)|-\rho)} = q^{-(\lambda|\rho)}\prod_{\alpha\in R^+}\left(1-q^{(\lambda+\rho|\alpha)}\right)\frac{q^{\frac{(\alpha|-\rho)}{2}}-q^{-\frac{(\alpha|-\rho)}{2}}}{1-q^{(\rho|\alpha)}}. \tag{229}$$

We can simplify the right hand side to

$$q^{-(\lambda|\rho)}\prod_{\alpha\in R^+}\left(1-q^{(\lambda+\rho|\alpha)}\right)q^{-\frac{(\rho|\alpha)}{2}}, \tag{230}$$

which becomes by using $\rho = \sum_{\alpha\in R^+}\frac{\alpha}{2}$ and then multiplying out the product

$$q^{(\lambda+\rho|-\rho)}\prod_{\alpha\in R^+}1-q^{(\lambda+\rho|\alpha)} = q^{(\lambda+\rho|-\rho)}\sum_{I\subseteq R^+}(-1)^{|I|}q^{(\lambda+\rho|\sum_{\alpha\in I}\alpha)}. \tag{231}$$

Here I ranges over all subsets of R^+. It is well-known that subsets of R^+ are in bijection with W via $I \leftrightarrow R(w)$, where $R(w) := R^+ \cap w^{-1}(R^-)$ coincides with $\dot{R}(w)$ from Definition 2.5. Thus we can rewrite the last expression as

$$\sum_{w\in W}(-1)^{l(w)}q^{(\lambda+\rho|-\rho+\sum_{\alpha\in R(w)}\alpha)}. \tag{232}$$

Now to prove Equation (227) we only have to verify

$$(w(\lambda+\rho)\mid -\rho) = (\lambda+\rho\mid -\rho+\sum_{\alpha\in R(w)}\alpha) \tag{233}$$

for which it suffices to show $w^{-1}(-\rho) = -\rho + \sum_{\alpha\in R(w)}\alpha$. This follows using $\rho = \sum_{\alpha\in R^+}\frac{\alpha}{2}$. □

Example 5.30. Let us use the previous proposition to calculate the evaluation at $q^{-\rho}$ for the complete symmetric polynomials $h_{N-n+1}, ..., h_N$, where $h_i := s_{\mu_i}$ for $i \geq 0$ and $\mu_i := (i, 0, ..., 0)$. Since q is a primitive N-th root of unity we obtain for h_N

$$h_N(q^{-\rho}) = q^{(-\mu_N|\rho)}\prod_{\alpha\in R^+}\frac{1-q^{(\mu_N+\rho|\alpha)}}{1-q^{(\rho|\alpha)}} = (-1)^{n-1}. \tag{234}$$

For $i \in \{N-n+1, ..., N-1\}$ we obtain

$$h_i(q^{-\rho}) = q^{(-\mu_i|\rho)}\prod_{\alpha\in R^+}\frac{1-q^{(\mu_i+\rho|\alpha)}}{1-q^{(\rho|\alpha)}}. \tag{235}$$

The denominator never vanishes, but for $\tilde{\alpha}_i := \alpha_{1,N-i+1} \in R^+$ we have $q^{(\mu_i+\rho|\tilde{\alpha}_i)} = 1$ and hence $h_i(q^{-\rho}) = 0$. As noted before, a Y-weight vector

$f \in e\mathcal{P}$ lies in $e\,\mathrm{Rad}$ if and only if $f(q^{-\rho}) = 0$ and hence we deduce that $\{h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n\} \subseteq e\mathcal{P} = \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$. Furthermore, we have that $e\,\mathrm{Rad}$ is an eH_ne-module by Proposition 5.9 and in particular closed under the action of $e\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W e \cong \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$, where we use the isomorphism from Proposition 5.17 and the fact that e acts via the identity on $e\mathcal{P} = \mathcal{P}^W$. Hence we also have $(h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n) \subseteq e\,\mathrm{Rad}$.

5.5 The structure of the spherical DAHA module $\mathcal{M}$

By the results from the previous sections we can now give some insight into the structure of the eH_ne-module $\mathcal{M} = e\mathcal{P}/e\,\mathrm{Rad}$. More precisely, we will show that its dimension is $\binom{N}{n}$ and we prove the existence of two eigenbases: one for the subalgebra $e\mathbb{C}[Y_1^{\pm 1}, ...Y_n^{\pm 1}]^W e \subseteq eH_ne$ and one for the subalgebra $e\mathbb{C}[X_1^{\pm 1}, ...X_n^{\pm 1}]^W e \subseteq eH_ne$ from Proposition 5.17.

We start by deducing a general condition for $s_\lambda \in e\,\mathrm{Rad}$ from the evaluation formula in Proposition 5.29.

Corollary 5.31. *Let $\lambda \in P^+$. We have $s_\lambda \in e\,\mathrm{Rad}$ if and only if there exist $1 \leq i \neq j \leq n$ such that $\lambda_i + \rho_i = \lambda_j + \rho_j \bmod N$. Equivalently $s_\lambda \in e\,\mathrm{Rad}$ if and only if $q^{\lambda+\rho} \in \mathbb{C}^n_{sing} := \{v \in \mathbb{C}^n \mid \exists i \neq j \text{ such that } v_i = v_j\}$. In particular we have $\dim(e\mathcal{P}/e\,\mathrm{Rad})_\lambda = 0$ if $q^{\lambda+\rho} \in \mathbb{C}^n_{sing}$.*

Proof. The s_λ for $\lambda \in P^+$ form a basis of Y-weight vectors of $e\mathcal{P}$ by Proposition 5.27. Therefore the statements follow from the evaluation formula in Proposition 5.29 and the fact that a Y-weight vector lies in the radical if and only if it evaluates on $q^{-\rho}$ to 0. □

This allows us to deduce a first formula for the dimension of $\mathcal{M}$.

Proposition 5.32. *Let $\lambda \in P^+$ such that $q^{\lambda+\rho} \in \mathbb{C}^n_{reg} := \mathbb{C}^n \setminus \mathbb{C}^n_{sing}$. Then $\dim(\mathcal{M}_\lambda) = 1$. In particular,*

$$\begin{aligned} \dim(\mathcal{M}) &= |\{q^{-\lambda_- + \rho} \mid \lambda \in P^+, q^{\lambda+\rho} \in \mathbb{C}^n_{reg}\}/S_n| \\ &= |\{q^{\lambda+\rho} \mid \lambda \in P^+, q^{\lambda+\rho} \in \mathbb{C}^n_{reg}\}/S_n|. \end{aligned} \tag{236}$$

Proof. Let $\lambda \in P^+$ with $q^{\lambda+\rho} \in \mathbb{C}^n_{reg}$. By Corollary 5.31 we have $s_\lambda \notin e\,\mathrm{Rad}$. Thus, the image of s_λ in $\mathcal{M}$ is not zero and we have $\dim(\mathcal{M})_\lambda \geq 1$. If there exists λ as above such that the dimension of the Y-weight space $\mathcal{M}_\lambda$ is at least two, take two linearly independent elements $v_1, v_2 \in \mathcal{M}_\lambda$ and lift them to Y-weight vectors $\tilde{v}_1, \tilde{v}_2 \in (e\mathcal{P})_\lambda$. This is possible since $e\mathcal{P}$ has a basis consisting of Y-weight vectors by Proposition 5.27. Since a Y-weight vector lies in $e\,\mathrm{Rad}$ if and only if evaluates to zero on $q^{-\rho}$ we can find a non-trivial linear combination $c_1\tilde{v}_1 + c_2\tilde{v}_2 \in e\,\mathrm{Rad}$. Hence we have $c_1v_1 + c_2v_2 = 0$ in contradiction to their linear independence. The first description of the dimension now follows from Corollary 5.31 and Remark 5.28. We have $q^{-\lambda_-+\rho} = q^{-\mu_-+\rho}$ in $\mathbb{C}^n/S_n$ if and only if there exist pairwise different $1 \le j_i \le n$ for $1 \le i \le n$ such that $-\lambda_i + \frac{-n-1+2i}{2} = -\mu_{j_i} + \frac{-n-1+2j_i}{2} \bmod N$. By multiplying with -1 we see that this is equivalent to $\lambda_i + \frac{n+1-2i}{2} = \mu_{j_i} + \frac{n+1-2j_i}{2} \bmod N$ and hence to $q^{\lambda+\rho} = q^{\mu+\rho}$ in $\mathbb{C}^n/S_n$. This shows the second equality. □

Proposition 5.33. *We have the following dimension formula*

$$\dim(\mathcal{M}) = \binom{N}{n}. \tag{237}$$

.

Proof. By Proposition 5.32 we only have to determine the cardinality of the set

$$\{q^{\lambda+\rho} \mid \lambda \in P^+, q^{\lambda+\rho} \in \mathbb{C}^n_{reg}\}/S_n. \tag{238}$$

We can clearly replace ρ with $\rho + \frac{n+1}{2} \cdot (1, ..., 1)$. Because q is a primitive N-th root of unity and we work modulo S_n, we can find a bijection with elements $(v_1 > ... > v_n)$ with $1 \le v_i \le N$ and $v_i \in \mathbb{Z}$ by ordering the exponents. This set bijects to elements $(v_1 \ge ... \ge v_n)$ with $0 \le v_i \le N-n$ by subtracting our new ρ. The last set has cardinality $\binom{N}{n}$ as it is in bijection to the set of monotone paths from the bottom left corner to the top right corner inside an integral $n \times (N-n)$-box. □

We can now describe the Y-weight basis of $\mathcal{M}$. For this we set

$$\mathfrak{P}_{n,m} := \{\lambda \in P^+ \mid 0 \le \lambda_i \le m \text{ for } 1 \le i \le n\}. \tag{239}$$

Theorem 5.34. *A basis of $\mathcal{M} = e\mathcal{P}/e\,\mathrm{Rad}$ is given by the images of s_λ for $\lambda \in \mathfrak{P}_{n,N-n}$. The image of s_λ has Y-weight λ in the sense of Definition 5.26. These weights are pairwise different.*

Proof. By Proposition 5.27 we only need to show that the images of the s_λ for $\lambda \in \mathfrak{P}_{n,N-n}$ form a basis. We want to show that $\lambda \neq \lambda'$ for $\lambda, \lambda' \in \mathfrak{P}_{n,N-n}$ implies that $\mathcal{M}_\lambda \neq \mathcal{M}_{\lambda'}$. By Remark 5.28 we should first show $q^{-\lambda_-+\rho} \neq q^{-\lambda'_-+\rho}$ in $\mathbb{C}^n/S_n$. For all $\lambda \in \mathfrak{P}_{n,N-n}$ we have that the i-th entry of $-\lambda_- + \rho$ is given by $-\lambda_{n-i} + \frac{n+1-2i}{2}$ and the entries lie in the interval $[\frac{-n+1}{2} - N + n, ..., \frac{n-1}{2}]$ of length N. Therefore, we do not need to work modulo N. Furthermore, we have $-\lambda_- \in P^+$ and hence we see that the entries of $-\lambda_- + \rho$ satisfy $-\lambda_{n-i} + \frac{n+1-2i}{2} > -\lambda_{n-j} + \frac{n+1-2j}{2}$ for $1 \le i < j \le n$. This shows that we can ignore the S_n-action as well and we obtain $q^{-\lambda_-+\rho} \neq q^{-\lambda'_-+\rho}$ for $\lambda \neq \lambda' \in \mathfrak{P}_{n,N-n}$. Hence $\mathcal{M}_\lambda \neq \mathcal{M}_{\lambda'}$ for $\lambda \neq \lambda'$, if the weight spaces are not zero. To show that they are not zero we can argue similarly. Indeed, for all $\lambda \in \mathfrak{P}_{n,N-n}$ we have that the entries of $\lambda + \rho$ are $\lambda_i + \frac{n+1-2i}{2}$ and they lie in the interval $[\frac{-n+1}{2}, ..., \frac{n-1}{2} + N - n]$ of length

N. For fixed λ as above the entries $\lambda_i + \frac{n+1-2i}{2}$ are pairwise different and hence by Corollary 5.31 the images of these s_λ are not zero in the quotient. Therefore, the $\binom{N}{n}$ many Schur polynomials s_λ for $\lambda \in \mathfrak{P}_{n,N-n}$ are linearly independent in the quotient and they form a basis by Proposition 5.33. $\square$

We will now describe the second weight basis of $\mathcal{M}$. This time we consider the subalgebra $e\mathbb{C}[X_1^{\pm 1}, ...X_n^{\pm 1}]^W e \subseteq eH_n e$ from Proposition 5.17 and X-weights as defined in Definition 5.26. In order to describe the X-weight basis of $\mathcal{M}$, we have to understand how the elementary symmetric polynomials $\boldsymbol{e_1}, ..., \boldsymbol{e_{n-1}}$ and $\boldsymbol{e_n}^{\pm 1}$, which generate $\mathbb{C}[X_1^{\pm 1}, ...X_n^{\pm 1}]^W$, act on the Schur polynomials s_λ for $\lambda \in \mathfrak{P}_{n,N-n}$. For this we can apply the *Pieri formulas* from [Ful97, Chapter 2.2]. They tell us that for $\lambda \in P^+$ and $1 \leq i \leq n$ we have

$$\boldsymbol{e_i} s_\lambda = \sum_\mu s_\mu, \tag{240}$$

where the sum ranges over all Young diagrams, which can be obtained from λ by adding i boxes to λ with no two boxes in the same row. In particular, we have

$$\boldsymbol{e_n} s_\lambda = s_{\lambda'}, \quad \boldsymbol{e_n}^{-1} s_\lambda = s_{\lambda''}, \tag{241}$$

where λ' is defined by $\lambda'_i = \lambda_i + 1$ and λ'' is defined by $\lambda''_i = \lambda_i - 1$ for $1 \leq i \leq n$. By the Pieri formulas we see that multiplication by $\boldsymbol{e_i}$ for $1 \leq i \leq n$ maps the Schur polynomials s_λ with $\lambda \in \mathfrak{P}_{n,N-n}$ to sums of Schur polynomials s_μ with $\mu \in \mathfrak{P}_{n,N-n+1}$. In particular, to compute the image of $\boldsymbol{e_i} s_\lambda$ in $\mathcal{M}$, we need to determine the images of s_μ in $\mathcal{M}$ with $\mu \in \mathfrak{P}_{n,N-n+1}$ and $\mu_1 = N - n + 1$. For this we associate to each $\lambda = (\lambda_1, ..., \lambda_n) \in P^+$ with $\lambda_1 - \lambda_n \leq N - n$ a 'rotated' weight $r(\lambda) \in P^+$ defined by

$$r(\lambda)_i := \lambda_{i+1} - 1 \text{ for } 1 \leq i \leq n-1, \quad r(\lambda)_n := \lambda_1 - N + n - 1. \tag{242}$$

We will now show that s_λ and $s_{r(\lambda)}$ have the same Y-weight and the same evaluation on $q^{-\rho}$. In particular, they map to the same element in $\mathcal{M}$.

Lemma 5.35. *Let $\lambda \in P^+$ with $\lambda_1 - \lambda_n \leq N - n$.*

(a) We have $q^{-\lambda_- + \rho} = q^{-r(\lambda)_- + \rho}$, hence s_λ and $s_{r(\lambda)}$ lie in the same Y-weight space of $e\mathcal{P}$.

(b) We have $s_\lambda(q^{-\rho}) = s_{r(\lambda)}(q^{-\rho})$.

(c) We have $s_\lambda = s_{r(\lambda)}$ as elements in $\mathcal{M}$.

Proof. For claim (a) note that

$$-\lambda_- + \rho = (-\lambda_n + \frac{n-1}{2}, ..., -\lambda_1 + \frac{-n+1}{2}) \tag{243}$$

and

$$-r(\lambda)_- + \rho = (-\lambda_1 + N - n + 1 + \frac{n-1}{2}, \\ -\lambda_n + 1 + \frac{n-3}{2}, ..., -\lambda_2 + 1 + \frac{-n+1}{2}). \tag{244}$$

Thus $q^{-\lambda_- + \rho} = q^{-r(\lambda)_- + \rho}$ holds in $\mathbb{C}^n/S_n$ and hence the Schur polynomials s_λ and $s_{r(\lambda)}$ lie in the same Y-weight space by Remark 5.28. To prove (b) we employ the evaluation formula from Proposition 5.29 and thereby have to show the following equality, after cancelling the denominators:

$$q^{-(\lambda|\rho)} \prod_{\alpha \in R^+} 1 - q^{(\lambda+\rho|\alpha)} = q^{-(r(\lambda)|\rho)} \prod_{\alpha \in R^+} 1 - q^{(r(\lambda)+\rho|\alpha)}.$$

Equivalently:

$$\prod_{\alpha \in R^+} q^{-\frac{(\lambda|\alpha)}{2}} - q^{\frac{(\lambda|\alpha)}{2}} q^{(\rho|\alpha)} = \prod_{\alpha \in R^+} q^{-\frac{(r(\lambda)|\alpha)}{2}} - q^{\frac{(r(\lambda)|\alpha)}{2}} q^{(\rho|\alpha)}.$$

We can cancel the terms in the product on the left corresponding to $\alpha_{i,j} \in R^+$ where $2 \leq i < j \leq n$ with the terms on the right corresponding to $\alpha'_{i,j} \in R^+$ where $1 \leq i < j \leq n-1$ by definition of the rotation r. Thus it suffices to show

$$\prod_{i=2,\dots,n} q^{-\frac{(\lambda|\alpha_{1,i})}{2}} - q^{\frac{(\lambda|\alpha_{1,i})}{2} + i - 1} = \prod_{j=1,\dots,n-1} q^{-\frac{(r(\lambda)|\alpha_{j,n})}{2}} - q^{\frac{(r(\lambda)|\alpha_{j,n})}{2} + n - j}.$$

Using the definition of the rotation r this is equivalent to

$$\prod_{i=2,\dots,n} q^{-\frac{(\lambda_1 - \lambda_i)}{2}} - q^{\frac{(\lambda_1 - \lambda_i)}{2} + i - 1} \\ = \prod_{j=1,\dots,n-1} q^{-\frac{(\lambda_{j+1} - 1 - \lambda_1 + N - n + 1)}{2}} - q^{\frac{(\lambda_{j+1} - 1 - \lambda_1 + N - n + 1)}{2} + n - j}.$$

The second line can be rewritten as

$$q^{\frac{(n-1)n}{2}} \prod_{j=1,\dots,n-1} q^{\frac{(\lambda_{j+1} - \lambda_1)}{2} - j} - q^{-\frac{(\lambda_{j+1} - \lambda_1)}{2}} \\ = \prod_{j=1,\dots,n-1} q^{\frac{(\lambda_{j+1} - \lambda_1)}{2}} - q^{-\frac{(\lambda_{j+1} - \lambda_1)}{2} + j}.$$

The equality follows by re-indexing. This shows claim (b). Claim (c) follows from (a) and (b), since $s_\lambda - s_{r(\lambda)}$ is a Y-weight vector which evaluates to zero on $q^{-\rho}$. $\square$

We now define certain elements b^X_λ for $\lambda \in \mathfrak{P}_{n,N-n}$, which already appear in [KS10] as the so-called *Bethe vectors*. After the definition we prove that b^X_λ is X-weight vector of weight λ.

Definition 5.36. For $\lambda \in \mathfrak{P}_{n,N-n}$ we define

$$\tilde{b}^X_\lambda := \sum_{\mu \in \mathfrak{P}_{n,N-n}} s_\mu(q^{-\lambda_- + \rho}) s_\mu \in e\mathcal{P}. \tag{245}$$

We will denote the image of $\tilde{b}^X_\lambda$ in $\mathcal{M}$ by b^X_λ.

Proposition 5.37. *For $\lambda \in \mathfrak{P}_{n,N-n}$ and $1 \leq r \leq n$ we have*

$$e\boldsymbol{e_r} e(b^X_\lambda) = \boldsymbol{e_r}(q^{\lambda_- - \rho}) \cdot b^X_\lambda. \tag{246}$$

Proof. Let us first calculate $e\boldsymbol{e_r}e(\tilde{b}^X_\lambda) = \boldsymbol{e_r}(\tilde{b}^X_\lambda)$ inside $e\mathcal{P}$. Applying the Pieri formulas gives us

$$\boldsymbol{e_r}(\tilde{b}^X_\lambda) = \sum_{\mu \in \mathfrak{P}_{n,N-n}} s_\mu(q^{-\lambda_- + \rho}) \boldsymbol{e_r} s_\mu = \sum_{\mu \in \mathfrak{P}_{n,N-n}} \left(s_\mu(q^{-\lambda_- + \rho}) \sum_{\mu'} s_{\mu'} \right). \tag{247}$$

Here the inner sum ranges over all elements $\mu' \in P^+$ which can be obtained from μ by adding r boxes in pairwise different rows. We can rearrange the sum to obtain

$$\begin{aligned} \boldsymbol{e_r}(\tilde{b}^X_\lambda) = &\sum_{\mu \in \mathfrak{P}_{n,N-n}} \left(\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}) s_\mu \right) \\ &+ \sum_{\mu \in \mathfrak{P}_{n,N+1-n} \setminus \mathfrak{P}_{n,N-n}} \left(\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}) s_\mu \right). \end{aligned} \tag{248}$$

Now the inner sums over μ' range over all $\mu' \in \mathfrak{P}_{n,N-n}$ such that μ can be obtained from μ' by adding r boxes in pairwise different rows. Note that for $\mu \in \mathfrak{P}_{n,N+1-n} \setminus \mathfrak{P}_{n,N-n}$ with $\mu_n = 0$ we have $s_\mu \in e\,\mathrm{Rad}$ by the evaluation formula from Proposition 5.29 and by using a similar argument as in Example 5.30. For the remaining $\mu \in \mathfrak{P}_{n,N+1-n} \setminus \mathfrak{P}_{n,N-n}$ we have $\mu_1 - \mu_n \leq N - n$ and $s_\mu = s_{r(\mu)}$ in $\mathcal{M}$ by Lemma 5.35. Thus, we obtain the following equation inside $\mathcal{M}$:

$$\boldsymbol{e_r}(b^X_\lambda) = \sum_{\mu \in \mathfrak{P}_{n,N-n}} \left(\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}) s_\mu \right) + \sum_{\mu} \left(\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}) s_{r(\mu)} \right).$$

Here the second sum over μ now ranges over all $\mu \in \mathfrak{P}_{n,N+1-n} \setminus \mathfrak{P}_{n,N-n}$ with $\mu_n > 0$ and the other sums are indexed as before. From this we can calculate the coefficient of s_μ for $\mu \in \mathfrak{P}_{n,N-n}$ on the right hand side to be as follows. If $\mu_n \geq 1$ we can not obtain μ as $r(\tilde{\mu})$ for any $\tilde{\mu} \in \mathfrak{P}_{n,N-n+1}$. Hence the coefficient of s_μ for such μ is

$$\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}), \tag{249}$$

where the sum ranges over all $\mu' \in \mathfrak{P}_{n,N-n}$ such that μ is obtainable from μ' by adding r boxes in pairwise different rows. Since $\mu_n \geq 1$ all $\mu' \in P^+$ from which μ can be obtained by adding of r boxes in pairwise different rows must already lie in $\mathfrak{P}_{n,N-n}$. Hence, we can index the sum above over all such $\mu' \in P^+$ and not just over $\mathfrak{P}_{n,N-n}$.

If $\mu_n = 0$ we can obtain μ as the rotation $r(\tilde{\mu})$ of a unique $\tilde{\mu} = r^{-1}(\mu) \in \mathfrak{P}_{n,N-n+1} \setminus \mathfrak{P}_{n,N-n}$. This $\tilde{\mu}$ must necessarily satisfy $\tilde{\mu}_n > 0$. Then the coefficient of μ is

$$\sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}) + \sum_{\mu''} s_{\mu''}(q^{-\lambda_- + \rho}), \tag{250}$$

where the first sum ranges over all $\mu' \in \mathfrak{P}_{n,N-n}$ such that we can add r boxes to μ' in pairwise different rows to obtain μ. The second sum ranges over all $\mu'' \in \mathfrak{P}_{n,N-n}$ such that we can add r boxes to μ'' in pairwise different rows to obtain $r^{-1}(\mu)$. In particular we must have $\mu''_1 = N - n$ and one of the boxes must be added to the first row. But the set of such μ'' is via r in bijection with the set of all $\tilde{\mu}''$ such that $\tilde{\mu}''_n = -1$ and such that we can obtain μ from $\tilde{\mu}''$ by adding r boxes to pairwise different rows. We have $s_{\mu''}(q^{-\lambda_- + \rho}) = s_{r(\mu'')}(q^{-\lambda_- + \rho})$. Indeed, since $s_{\mu''} - s_{r(\mu'')} \in e\,\mathrm{Rad}$ by Lemma 5.35 we have

$$\langle s_{\mu''} - s_{r(\mu'')}, s_\lambda \rangle = (s_{\mu''} - s_{r(\mu'')})(q^{-\lambda_- + \rho}) \cdot s_\lambda(q^{-\rho}) = 0.$$

by Theorem 5.25 and Definition 5.7. Therefore $s_{\mu''} - s_{r(\mu'')}(q^{\lambda + \rho}) = 0$, because $\lambda \in \mathfrak{P}_{n,N-n}$ and hence $s_\lambda(q^{-\rho}) \neq 0$. Overall we can go back to (250), where we can replace μ'' by $r(\mu'')$ in the indexing set and in the Schur polynomials. We see now that the coefficient of s_μ is also in the case $\mu_n = 0$ equal to $\sum_{\mu'} s_{\mu'}(q^{-\lambda - \rho})$ where the sum ranges over all $\mu' \in P^+$ such that μ can be obtained from μ' by adding r boxes in pairwise different rows.

To prove the proposition we only need to show the following equality for all $\mu \in \mathfrak{P}_{n,N-n}$, with μ' as above:

$$\boldsymbol{e_r}(q^{\lambda_- - \rho}) \cdot s_\mu(q^{-\lambda_- + \rho}) = \sum_{\mu'} s_{\mu'}(q^{-\lambda_- + \rho}). \tag{251}$$

This follows from $\boldsymbol{e_r}(q^{\lambda_- - \rho}) = \boldsymbol{e_{n-r}}(q^{-\lambda_- + \rho})\boldsymbol{e_n}(q^{-\lambda_- + \rho})^{-1}$ and applying the Pieri formulas.

□

Theorem 5.38. *The eH_ne-module $\mathcal{M}$ has a basis consisting of the X-weight vectors b_λ^X for $\lambda \in \mathfrak{P}_{n,N-n}$ with pairwise different weights λ.*

Proof. This follows immediately from the previous proposition as the elements $q^{\lambda_- - \rho}$ for $\lambda \in \mathfrak{P}_{n,N-n}$ are pairwise different in $\mathbb{C}^n/S_n$, since this holds for the $q^{-\lambda_- + \rho}$ by the proof of Theorem 5.34. Hence the $\binom{N}{n}$ many X-weights of the b_λ^X are pairwise different. Therefore these elements are linearly independent and they must form a basis by Proposition 5.33. $\square$

We close the discussion on the structure of $\mathcal{M}$ with the proof that $\mathcal{M}$ is irreducible.

Proposition 5.39. *The eH_ne-module $\mathcal{M}$ is irreducible.*

Proof. Assume $\mathcal{M}' \subseteq \mathcal{M}$ is a non-trivial submodule. The bilinear form $\langle\ ,\ \rangle$ on $e\mathcal{P}$ induces a non-degenerate bilinear form $\langle\ ,\ \rangle$ on $\mathcal{M} = e\mathcal{P}/e\,\mathrm{Rad}$ by Proposition 5.16 (b). Therefore we can look at the orthogonal complement $(\mathcal{M}')^\perp$ of $\mathcal{M}'$ with $\mathcal{M} = \mathcal{M}' \oplus (\mathcal{M}')^\perp$. By Proposition 5.9 (b) we have that $(M')^\perp$ is a submodule. But we know from Theorem 5.34 that the Y-weight spectrum is simple. Therefore, we obtain weight space decompositions of the two submodules and the Y-weight vector $1 \in \mathcal{M}$ must be an element in $\mathcal{M}'$ or in $(\mathcal{M}')^\perp$, but since 1 generates $\mathcal{M}$ this gives a contradiction. $\square$

5.6 $qH^\bullet(Gr_{n,N})_{q=1}$ as an eH_ne-module

Recall the two statements from the beginning of Chapter 5.

(1) We have

$$\mathbb{C} \otimes_{\mathbb{Z}} qH^\bullet(Gr_{n,N})_{q=1} \cong \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]/(h_{N-n+1}, ..., h_N + (-1)^n)$$

as $\mathbb{C}$-algebras. Here $\boldsymbol{e_1}, ..., \boldsymbol{e_n}$ are the elementary symmetric polynomials in n variables and h_i for $i > 0$ denote the complete symmetric polynomials.

(2) The dimension of $\mathbb{C}\otimes_{\mathbb{Z}} qH^\bullet(Gr_{n,N})_{q=1}$ is $\binom{N}{n}$ and a $\mathbb{C}$-basis is given by the images of the elements $s_\lambda \in \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]$ for $\lambda \in P^+$ with $0 \le \lambda_i \le N-n$.

These facts match the results from Example 5.30 and Proposition 5.33 very nicely. Therefore, we can now deduce the main result of this chapter easily in the next theorem.

Theorem 5.40. *Set $I := (h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n) \subseteq \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}]$. We have the following commutative diagram of $\mathbb{C}$-algebras, where the rows are short exact sequences. Moreover, γ is an isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\iota|_I"] & \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}] \arrow[r] \arrow[d, "\iota"] & qH^\bullet(Gr_{n,N})_{q=1} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & e\,\mathrm{Rad} \arrow[r] & e\mathcal{P} \arrow[r] & \mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

Proof. The inclusion $\iota : \mathbb{C}[\boldsymbol{e_1}, ..., \boldsymbol{e_n}] \hookrightarrow e\mathcal{P} = \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ restricts to an inclusion $I \hookrightarrow e\,\mathrm{Rad}$ by Example 5.30. Hence we obtain an induced morphism γ on the quotients. The bijectivity of γ follows by statement (2) above and from Theorem 5.34. □

We can use this theorem to obtain an explicit description of $e\,\mathrm{Rad}$

Corollary 5.41. *The submodule $e\,\mathrm{Rad} \subseteq e\mathcal{P} \cong \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ is generated as an ideal by $h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n$.*

Proof. We have alrady seen $(h_{N-n+1}, ..., h_{N-1}, h_N + (-1)^n) \subseteq e\,\mathrm{Rad}$ in Example 5.30. The other inclusion now follows from the diagram in Theorem 5.40 by a diagram chase using that any element in $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^W$ can be multiplied by a large enough power of $\boldsymbol{e_n} = X_1 \cdot ... \cdot X_n$ to obtain an element in $\mathbb{C}[X_1, ..., X_n]^W$. □

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