H. Franzen
University of Bonn

25.03.2019

Exam Foundations of Representation Theory —Solutions—

Exercise 1 (10 points). True or false? Please explain your answers briefly.

(i) If Q is a quiver such that $s(\alpha) = t(\alpha)$ for every $\alpha \in Q_1$ then kQ is commutative.

(ii) In the category Set of sets, every object is projective.

(iii) If P is a projective object of an abelian category $\mathscr{A}$, then every short exact sequence in $\mathscr{A}$ of the form $0 \to P \to Y \to Z \to 0$ splits.

(iv) Let Q be a finite quiver and let M and N be left kQ-modules which are finite-dimensional over k. Then $(R^1 \operatorname{Hom}_{kQ}(\_, N))(M)$ is a finite-dimensional k-vector space.

(v) Let A be a ring. Then every projective A-module is free.

Solution. (i) False. A counterexample is the quiver Q with one vertex and two arrows α and β. By definition $\beta\alpha \neq \alpha\beta$ because these are different paths.

(ii) True. Let P be a set, let $f : X \to Y$ be an epimorphism, i.e. a surjective map, and let $g : P \to Y$ be a map. For every $y \in Y$, choose an inverse image x_y under f. The map $g' : P \to X$ defined by $g'(p) = x_{g(p)}$ does the job.

(iii) False. The exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is exact but not split; there can't be an injective homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$.

(iv) True. Consider the standard projective resolution $0 \to P_1 \to P_0 \to M \to 0$. An application of $\operatorname{Hom}_{kQ}(\_, N)$ yields a surjection $\bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(M_{s(\alpha)}, N_{t(\alpha)}) = \operatorname{Hom}_{kQ}(P_1, N) \to (R^1 \operatorname{Hom}_{kQ}(\_, N))(M)$. As M and N are finite-dimensional, so is $\bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(M_{s(\alpha)}, N_{t(\alpha)})$.

(v) False. Consider the quiver $1 \to 2$ and $A = kQ = k\varepsilon_1 \oplus k\alpha \oplus k\varepsilon_2$. Then $\dim_k A = 3$ but $P(1) = k\varepsilon_1 \oplus k\varepsilon_2$ is projective and $\dim_k P(1) = 2$. This shows it cannot be free.

Exercise 2 (8 points). Let Q be a quiver. Let M be a left kQ-module and let $N', N'' \subseteq M$ be two submodules. Assume that $(R^1 \operatorname{Hom}_{kQ}(\_, M))(M) = 0$. Show

$$(R^1 \operatorname{Hom}_{kQ}(\_, M/N''))(N') = 0.$$

Hint: You may use without a proof the isomorphism of k-vector spaces $(R^i \operatorname{Hom}_{kQ}(\_, Y))(X) \cong (R^i \operatorname{Hom}_{kQ}(X, \_))(Y)$ for left kQ-modules X and Y and all $i \geq 0$.

Solution. Consider the exact sequences $0 \to N' \to M \to M/N' \to 0$ and $0 \to N'' \to M \to M/N'' \to 0$. Apply $\operatorname{Hom}(\_, M)$ to the first sequence and obtain the long exact sequence

$$\ldots \to (R^1 \operatorname{Hom}(\_, M))(M/N') \to \underbrace{(R^1 \operatorname{Hom}(\_, M))(M)}_{=0} \to (R^1 \operatorname{Hom}(\_, M))(N') \to 0.$$

Note that $R^i \operatorname{Hom}(\_, M)$ vanishes for $i \geq 2$. We deduce

$$0 = (R^1 \operatorname{Hom}(\_, M))(N') \cong (R^1 \operatorname{Hom}(N', \_))(M).$$

Now apply $\operatorname{Hom}(N', \_)$ to the second exact sequence. This yields the long exact sequence

$$\ldots \to (R^1 \operatorname{Hom}(N', \_))(N'') \to \underbrace{(R^1 \operatorname{Hom}(N', \_))(M)}_{=0} \to (R^1 \operatorname{Hom}(N', \_))(M/N'') \to 0.$$

Note again that $(R^i \operatorname{Hom}(N', \_))(Y) \cong (R^i \operatorname{Hom}(\_, Y))(N') = 0$ for all $i \geq 2$. We conclude

$$0 = (R^1 \operatorname{Hom}(N', \_))(M/N'') \cong (R^1 \operatorname{Hom}_{kQ}(\_, M/N''))(N').$$

Exercise 3 (8 points). Consider the k-algebra $A = k[X,Y]/(XY)$ and the A-module $M = A/(X)$. Compute the k-dimension of $(R^i \operatorname{Hom}_A(\_, M))(M)$ for all $i \geq 0$.

Solution. Compute a projective resolution of M. For this let $m_A(X) : A \to A$ be the multiplication with X and similarly $m_A(Y)$. We get $\ker(m_A(X)) = (Y)$ and $\ker(m_A(Y)) = (X)$. This shows that the sequence

$$\ldots \xrightarrow{m_A(X)} A \xrightarrow{m_A(Y)} A \xrightarrow{m_A(X)} A \to M \to 0$$

is exact. The complex $P_* : \ldots \xrightarrow{m_A(X)} A \xrightarrow{m_A(Y)} A \xrightarrow{m_A(X)} A \to 0$ is hence a projective resolution of M as ${}_AA$ is free and thus projective. If we apply $\operatorname{Hom}_A(\_, M)$, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_A(A,M) & \xrightarrow{m_A(X)^*} & \operatorname{Hom}_A(A,M) & \xrightarrow{m_A(Y)^*} & \operatorname{Hom}_A(A,M) & \xrightarrow{m_A(X)^*} & \ldots \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & M & \xrightarrow{m_M(X)} & M & \xrightarrow{m_M(Y)} & M & \xrightarrow{m_M(X)} & \ldots \end{array}$$

The isomorphism $\operatorname{Hom}_A(A, M) \cong M$ is provided by $h \mapsto h(1)$. The commutativity of the above diagram follows from the identity $X \cdot h(1) = h(X \cdot 1)$ (and the corresponding equation for Y). But on $M \cong k[Y]$ the element $X \in A$ acts as 0, while $\ker(m_M(Y)) = 0$ and $\operatorname{im}(m_M(Y)) = YM = (Y)$. This shows

$$\begin{aligned} (R^0 \operatorname{Hom}_A(\_, M))(M) &= \ker(m_M(X)) = M = k[Y] \\ (R^{2i-1} \operatorname{Hom}_A(\_, M))(M) &= \ker(m_M(Y))/\operatorname{im}(m_M(X)) = 0 \\ (R^{2i} \operatorname{Hom}_A(\_, M))(M) &= \ker(m_M(X))/\operatorname{im}(m_M(Y)) = k[Y]/(Y) = k. \end{aligned}$$

The dimensions of these k-vector spaces are hence ∞, 0, and 1 in the respective cases.

Exercise 4 (8 points). Let $\mathscr{A}$ be an abelian category. Let $P_* \in \underline{\mathrm{Ch}}_*(\mathscr{A})$ be a complex

$$P_* : \ldots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \ldots$$

which consists of projective objects of $\mathscr{A}$.

(i) Suppose that $P_* \in \underline{\mathrm{Ch}}_{\geq 0}(\mathscr{A})$, i.e. $P_n = 0$ for all $n < 0$. Show that P_* is acyclic if and only if the identity id_{P_*} is null-homotopic.

(ii) Is the asserted equivalence of (i) still true if P_* is unbounded?

Solution. (i) Suppose that P_* is acyclic. Then P_* is a projective resolution of 0. Consider the morphism $0 \to 0$. By the comparison theorem for projective resolutions there exists a chain morphism $f : P_* \to P_*$ which extends $0 \to 0$ and which is unique up to homotopy. But both id_{P_*} and 0_{P_*} extend the zero morphism. So they must be homotopic.

Conversely assume that id_{P_*} is null-homotopic. Then $H_n(\mathrm{id}_{P_*}) : H_n(P_*) \to H_n(P_*)$ is the zero morphism (by Lem. 4.21). But homology is a functor, so $H_n(\mathrm{id}_{P_*}) = \mathrm{id}_{H_n(P_*)}$. This implies $H_n(P_*) = 0$.

(ii) Consider the category of $\mathbb{Z}/4\mathbb{Z}$-modules and the acyclic complex

$$P_* : \ \ldots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \ldots.$$

As a $\mathbb{Z}/4\mathbb{Z}$-module, $\mathbb{Z}/4\mathbb{Z}$ is free, hence projective. Then id_{P_*} is not null-homotopic. For if there were $s_n : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ such that $\mathrm{id}_{\mathbb{Z}/4\mathbb{Z}} = s_{n-1}d_n + d_{n+1}s_n = 2s_{n-1} + 2s_n$, then the image of the identity would be contained in $2\mathbb{Z}/4\mathbb{Z}$.

Exercise 5 (8 points). (i) Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let (F, G, φ) be an adjunction from $\mathscr{C}$ to $\mathscr{D}$, so $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. Assume that G is faithful. Show that for every object $Y \in \mathscr{D}$ the counit-morphism $\varepsilon_Y : FGY \to Y$ is an epimorphism.

(ii) Let P be a projective object in the category of groups. Show that there exists a retraction onto P from a free group.

Solution. (i) Let $g_1, g_2 : Y \rightrightarrows Z$ be morphisms in $\mathscr{D}$ such that $g_1\varepsilon_Y = g_2\varepsilon_Y$. By definition $\varphi_{GY,Y}(\varepsilon_Y) = \mathrm{id}_{GY}$. We obtain

$$G(g_1) = G(g_1)\varphi_{GY,Y}(\varepsilon_Y) = \varphi_{GY,Z}(g_1\varepsilon_Y) = \varphi_{GY,Z}(g_2\varepsilon_Y) = G(g_2).$$

As G is assumed to be faithful, this implies $g_1 = g_2$.

(ii) We apply (ii) to the adjunction (F, V, φ) from Set to Grp where V : Grp $\to$ Set is the forgetful functor and F assigns to a set the free group over it. Note that (ii) is applicable as the forgetful functor is faithful. This shows that $FVP \to P$ is an epimorphism. As P is projective, it is a retraction.

Exercise 6 (8 points). Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to$ Set be a functor.

(i) Suppose that F is representable. Show that there exists an object $R \in \mathscr{C}$ and an element $u \in F(R)$ such that the following holds: for every object $A \in \mathscr{C}$ and every element $x \in F(A)$ there exists a unique morphism $f : R \to A$ with $(F(f))(u) = x$.

(ii) Now, let $\mathscr{C}$ = CommRing be the category of commutative rings and let F be defined by

$$F(A) := \{a \in A \mid a \text{ nilpotent}\}$$

for $A \in \mathscr{C}$ and $F(f) : F(A) \to F(B),\ a \mapsto f(a)$ for $f \in \mathscr{C}(A, B)$. Is F representable?

Solution. (i) Suppose that F is representable. That means there exists an object R and an isomorphism $\eta : \mathscr{C}(R, \_) \to F$ of functors. Define $u := \eta_R(\mathrm{id}_R)$. Let $X \in \mathscr{C}$ be an object and $x \in F(X)$. Via the bijection $\eta_X : \mathscr{C}(R, X) \to F(X)$ we get a unique morphism $f : R \to X$ such that $\eta_X(f) = x$. From the commutative diagram

$$\begin{array}{ccc} \mathscr{C}(R,R) & \xrightarrow{\mathscr{C}(R,f)} & \mathscr{C}(R,X) \\ \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_X} \\ F(R) & \xrightarrow{F(f)} & F(A) \end{array}$$

we obtain

$$x = \eta_X(f) = (\eta_X \circ \mathscr{C}(R, f))(\mathrm{id}_R) = (F(f) \circ \eta_R)(\mathrm{id}_R) = (F(f))(u).$$

If f' is another morphism satisfying $(F(f'))(u) = x$ then $\eta_X(f) = \eta_X(f')$, whence $f = f'$.

(ii) Suppose that F were representable. That means there would exist a commutative ring R and a nilpotent Element $u \in R$ with the universal property formulated in (i). So for any commutative ring A and any nilpotent element $a \in A$ there exists a unique ring homomorphism $f : R \to A$ such that $f(u) = a$. As u is nilpotent, there exists $m > 0$ such that $u^m = 0$. Now choose a commutative ring A and a nilpotent element $a \in A$ for which $a^m \neq 0$ (for instance $A = \mathbb{Z}[X]/(X^{m+1})$ and $a = X$). Then there can be no ring homomorphism $f : R \to A$ for which $f(u) = a$. This shows F is not representable.