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Exam Foundations of Representation Theory
—Solutions—

Exercise 1 (10 points). True or false? Please explain your answers briefly.

(i) Let k be a field. If Q is a quiver for which kQ is commutative then $s(\alpha) = t(\alpha)$ for every $\alpha \in Q_1$.

(ii) The category $\underline{\text{Set}}$ of sets is abelian.

(iii) Let $\mathscr{A}$ be an abelian category. The functor $H^0 : \underline{\text{Ch}}^{\geq 0}(\mathscr{A}) \to \mathscr{A}$ is left exact.

(iv) The group of units $\mathbb{C}^\times$ of the complex numbers is an injective abelian group.

(v) For the category $\underline{\text{Ab}}^{\text{f.g.}}$ of finitely generated abelian groups there exists a ring A and an equivalence of categories between A-$\underline{\text{Mod}}$ and $\underline{\text{Ab}}^{\text{f.g.}}$.

Solution. (i) True. If there were an arrow $\alpha : i \to j$ with $i \neq j$ then $\varepsilon_j \alpha = \alpha$ while $\alpha \varepsilon_j = 0$.

(ii) No. It doesn't have a zero object as $\emptyset$ is initial and $\{*\}$ is terminal.

(iii) True by the long exact sequence in cohomology. If $0 \to C^* \to D^* \to E* \to 0$ is a short exact sequence in $\underline{\text{Ch}}^{\geq 0}(\mathscr{A})$ then we obtain the exact sequence $0 \to H^0(C^*) \to H^0(D^*) \to H^0(E^*) \to H^1(C^*) \to \ldots$ (note that $H^{-1}(E^*) = 0$ as $E^{-1} = 0$).

(iv) True. The group $\mathbb{C}^\times$ is divisible: for an integer $a \in \mathbb{Z} \setminus \{0\}$ and $x \in \mathbb{C}^\times$ there exists $y \in \mathbb{C}^\times$ such that $y^a = x$. For $a > 0$ choose y as a root of the polynomial $t^a - x$ and for $a < 0$ as the inverse of a root of the polynomial $t^{-a} - x$.

(v) No. Such a ring cannot exist. For if there were such an equivalence, this equivalence would preserve injectives. The category A-$\underline{\text{Mod}}$ has enough injectives (Cor. 6.27) while the category of finitely generated abelian groups has not (Ex. 6.31).

Exercise 2 (8 points). Let

$$
\begin{array}{ccccccc}
 & & (*) & & (**) & & \\
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
(\#)\ X' & \xrightarrow{a'} & X & \xrightarrow{a} & X'' & \longrightarrow & 0 \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
(\#\#)\ Y' & \xrightarrow{b'} & Y & \xrightarrow{b} & Y'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\
 & & Z & \xrightarrow{c} & Z'' & &
\end{array}
$$

be a commutative diagram in an abelian category $\mathscr{A}$. Suppose that the rows (#) and (##) and the column $(*)$ are exact sequences. Assume further that f' is an epimorphism and c is a monomorphism. Show, using the diagram chasing rules given in the lecture, that the column $(**)$ is also exact.

Solution.
- Show that f'' is mono: Let $x'' \in X''$ such that $f''(x'') = 0$. There exists $x \in X$ such that $a(x) \equiv x''$. Then $0 = f''a(x) = bf(x)$. Hence there exists $y' \in Y'$ with $b'(y') \equiv f(x)$. As f' is epi, there exists $x' \in X'$ such that $f'(x') \equiv y'$ and thus $f(x) \equiv b'f'(x') = f(a'(x'))$. As f is mono, we see that $x \equiv a'(x')$. Therefore $x'' \equiv a(x) \equiv aa'(x') = 0$.

- Show that $g''f'' = 0$: As a is epi, it suffices to show that $g''f''a = 0$. But $g''f''a = cgf = 0$.
- Show that $\ker g'' \subseteq \operatorname{im} f''$: Let $y'' \in Y''$ such that $g''(y'') = 0$. As b is epi, we find $y \in Y$ such that $b(y) \equiv y''$. Thus $0 = g''b(y) = cg(y)$. As c is mono, we deduce that $g(y) = 0$. This implies by exactness that there exists $x \in X$ such that $f(x) \equiv y$. Then $y'' \equiv bf(x) = f''(a(x))$.

Exercise 3 (8 points). Let k be a field and let $A = k[X]/(X^2)$. Let $M = k[X]/(X) = k$ regarded as an A-module. Compute $(R^i \operatorname{Hom}_A(\_, M))(M)$ for all $i \geq 0$.

Solution. Abbreviate $\varepsilon = X + (X^2)$. Then $M = A/(\varepsilon)$. Let $\pi : A \to M$ be the quotient map. Let $f : A \to A$ be defined by $f(a) = \varepsilon a$. Then the sequence

$$\dots \xrightarrow{f} A \xrightarrow{f} A \xrightarrow{f} A \xrightarrow{\pi} M \to 0$$

is exact as $\operatorname{im}(f) = (\varepsilon) = \ker(f) = \ker(\pi)$. So we get a projective resolution of M by (P_*, π) where $P_i = A$ and $f : P_{i+1} \to P_i$ (all $i \geq 0$) and $P_i = 0$ (all $i < 0$). Applying $\operatorname{Hom}_A(\_, M)$ to the resolution yields

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_A(A, M) & \xrightarrow{f^*} & \operatorname{Hom}_A(A, M) & \xrightarrow{f^*} & \operatorname{Hom}_A(A, M) & \xrightarrow{f^*} & \dots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & M & \longrightarrow & \dots
\end{array}$$

and under the identification $\operatorname{Hom}_A(A, M) \xrightarrow{\cong} M$ given by $h \mapsto h(1)$ the map f^* corresponds to the multiplication with ε. But as ε acts as 0 on M the maps on the bottom row are the zero maps. Hence

$$(R^i \operatorname{Hom}_A(\_, M))(M) = H^i(\operatorname{Hom}_A(P_*, M)) = {}_k M = k.$$

Exercise 4 (8 points). Let Λ be a commutative ring and let A, B, and C be Λ-algebras. Let M_A be a projective right A-module and let ${}_A N_B$ be an A-B-bimodule which is projective as a right B-module. Show that $M \otimes_A N$ is also a projective right B-module.

Solution. The functor $\_ \otimes_A N : \underline{\mathrm{Mod}}\text{-}A \to \underline{\mathrm{Mod}}\text{-}B$ possesses a right adjoint which is given by $\operatorname{Hom}_B(N, \_) : \underline{\mathrm{Mod}}\text{-}B \to \underline{\mathrm{Mod}}\text{-}A$. As N is projective as a right B-module the functor $\operatorname{Hom}_B(N, \_)$ is exact. Thm. 6.25(iii) implies that $\_ \otimes_A N$ sends projectives to projectives.
Note that the algebra C is completely irrelevant for the exercise. I forgot to delete it. My apologies.

Exercise 5 (8 points). Let $n > 0$ be a natural number.

(i) Determine an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ in the category of abelian groups.

(ii) For a natural number $m > 0$ compute $(R^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \_))(\mathbb{Z}/n\mathbb{Z})$.

Solution. (i) We have an exact sequence $0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i^0} \mathbb{Q}/\mathbb{Z} \xrightarrow{d^0} \mathbb{Q}/\mathbb{Z} \to 0$ which arises as follows. The unique morphism $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ which sends 1 to $\frac{1}{n} + \mathbb{Z}$ has kernel $n\mathbb{Z}$ and image $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$. The map $d^0 : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ given by $d^0(x + \mathbb{Z}) = nx + \mathbb{Z}$ is well-defined, it is surjective as $\mathbb{Q}/\mathbb{Z}$ is divisible and its kernel is $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$.

(ii) We use the above injective resolution. Applying $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \_)$ to the resolution yields an identification of $(R^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \_))(\mathbb{Z}/n\mathbb{Z})$ with

$$\operatorname{coker}\left(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{d^0_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/Z)\right)$$

We compute $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. The surjection $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ induces an injection

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

which is given by $f \mapsto f(1)$. The image equals $\{x+\mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid mx+\mathbb{Z} = 0+\mathbb{Z}\} = (\frac{1}{m}\mathbb{Z})/\mathbb{Z}$ which is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ as shown in (i). The map d^0_* corresponds under these identifications to the map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ given by $x + m\mathbb{Z} \mapsto nx + m\mathbb{Z}$. Its image is $\gcd(m,n)\mathbb{Z}/m\mathbb{Z}$ and therefore

$$(R^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \_))(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Exercise 6 (8 points). (i) Let $\mathscr{C}$ be a category. When is a functor $F : \mathscr{C} \to \underline{\text{Set}}$ called representable?

(ii) Let $\mathscr{C} = \underline{\text{CommRing}}$ be the category of commutative rings. Let $n \geq 1$ be a natural number. Consider the functor $F : \mathscr{C} \to \underline{\text{Set}}$ defined by

$$F(A) := \{a \in A \mid a^n = 0\}$$

for $A \in \mathscr{C}$ and $F(f) : F(A) \to F(B),\ a \mapsto f(a)$ for $f \in \mathscr{C}(A,B)$. Show that F is representable.

Solution. (i) A functor $F : \mathscr{C} \to \underline{\text{Set}}$ is representable if there exists an object $X \in \mathscr{C}$ and a natural isomorphism $\eta : \mathscr{C}(X, \_) \to F$.

(ii) The functor F is represented by the ring $\mathbb{Z}[X]/(X^n)$. Indeed for every ring A, the map

$$\eta_A : \mathscr{C}(\mathbb{Z}[X]/(X^n), A) \to F(A), g \mapsto g(X)$$

is well-defined as $X^n = 0$ and is a bijection. This is because for every $a \in A$ with $a^n = 0$ the unique ring homomorphism $\mathbb{Z}[X] \to A$ which sends $X \mapsto a$ factors through $\mathbb{Z}[X]/(X^n)$. For every ring homomorphism $f : A \to B$ the diagram

$$\begin{array}{ccccc} h & \quad & \mathscr{C}(\mathbb{Z}[X]/(X^n), A) & \xrightarrow{\eta_A} & F(A) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle F(f)} \\ fh & & \mathscr{C}(\mathbb{Z}[X]/(X^n), B) & \xrightarrow{\eta_B} & F(B) \end{array}$$

commutes, as $F(f)(\eta_A(h)) = f(h(X)) = \eta_B(fh)$. Hence we see that η is a natural transformation, hence a natural isomorphism.