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Exam Foundations of Representation Theory

Remarks:

- The duration of the exam is **120** minutes.
- There are **50** points in total.
- Please use a separate sheet of paper for the solution of each exercise. Please write your name on every sheet of paper.
- Please have your ID card and your student ID ready.
- Aides like books, lecture notes, notes from the tutorials or electronic devices are prohibited. Please turn off your cell phone **before** the exam starts.

Please turn.

Throughout the exam, let k be a field.

Exercise 1 (10 points). True or false? Please explain your answers briefly.

(i) If Q is a quiver such that $s(\alpha) = t(\alpha)$ for every $\alpha \in Q_1$ then kQ is commutative.

(ii) In the category $\underline{\text{Set}}$ of sets, every object is projective.

(iii) If P is a projective object of an abelian category $\mathscr{A}$, then every short exact sequence in $\mathscr{A}$ of the form $0 \to P \to Y \to Z \to 0$ splits.

(iv) Let Q be a finite quiver and let M and N be left kQ-modules which are finite-dimensional over k. Then $(R^1 \operatorname{Hom}_{kQ}(\_, N))(M)$ is a finite-dimensional k-vector space.

(v) Let A be a ring. Then every projective A-module is free.

Exercise 2 (8 points). Let Q be a quiver. Let M be a left kQ-module and let $N', N'' \subseteq M$ be two submodules. Assume that $(R^1 \operatorname{Hom}_{kQ}(\_, M))(M) = 0$. Show

$$(R^1 \operatorname{Hom}_{kQ}(\_, M/N''))(N') = 0.$$

Hint: You may use without a proof the isomorphism of k-vector spaces $(R^i \operatorname{Hom}_{kQ}(\_, Y))(X) \cong (R^i \operatorname{Hom}_{kQ}(X, \_))(Y)$ for left kQ-modules X and Y and all $i \geq 0$.

Exercise 3 (8 points). Consider the k-algebra $A = k[X, Y]/(XY)$ and the A-module $M = A/(X)$. Compute the k-dimension of $(R^i \operatorname{Hom}_A(\_, M))(M)$ for all $i \geq 0$.

Exercise 4 (8 points). Let $\mathscr{A}$ be an abelian category. Let $P_* \in \underline{\text{Ch}}_*(\mathscr{A})$ be a complex

$$P_* : \ldots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \ldots$$

which consists of projective objects of $\mathscr{A}$.

(i) Suppose that $P_* \in \underline{\text{Ch}}_{\geq 0}(\mathscr{A})$, i.e. $P_n = 0$ for all $n < 0$. Show that P_* is acyclic if and only if the identity id_{P_*} is null-homotopic.

(ii) Is the asserted equivalence of (i) still true if P_* is unbounded?

Exercise 5 (8 points). (i) Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let (F, G, φ) be an adjunction from $\mathscr{C}$ to $\mathscr{D}$, so $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. Assume that G is faithful. Show that for every object $Y \in \mathscr{D}$ the counit-morphism $\varepsilon_Y : FGY \to Y$ is an epimorphism.

(ii) Let P be a projective object in the category of groups. Show that there exists a retraction onto P from a free group.

Exercise 6 (8 points). Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to \underline{\text{Set}}$ be a functor.

(i) Suppose that F is representable. Show that there exists an object $R \in \mathscr{C}$ and an element $u \in F(R)$ such that the following holds: for every object $A \in \mathscr{C}$ and every element $x \in F(A)$ there exists a unique morphism $f : R \to A$ with $(F(f))(u) = x$.

(ii) Now, let $\mathscr{C} = \underline{\text{CommRing}}$ be the category of commutative rings and let F be defined by

$$F(A) := \{a \in A \mid a \text{ nilpotent}\}$$

for $A \in \mathscr{C}$ and $F(f) : F(A) \to F(B),\ a \mapsto f(a)$ for $f \in \mathscr{C}(A, B)$. Is F representable?

Good Luck!