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Exam Foundations of Representation Theory

Remarks:

- The duration of the exam is **120** minutes.
- There are **50** points in total.
- Please use a separate sheet of paper for the solution of each exercise. Please write your name on every sheet of paper.
- Please have your ID card and your student ID ready.
- Aides like books, lecture notes, notes from the tutorials or electronic devices are prohibited. Please turn off your cell phone **before** the exam starts.

Please turn.

Exercise 1 (10 points). True or false? Please explain your answers briefly.

(i) Let k be a field. If Q is a quiver for which kQ is commutative then $s(\alpha) = t(\alpha)$ for every $\alpha \in Q_1$.

(ii) The category $\underline{\text{Set}}$ of sets is abelian.

(iii) Let $\mathscr{A}$ be an abelian category. The functor $H^0 : \underline{\text{Ch}}^{\geq 0}(\mathscr{A}) \to \mathscr{A}$ is left exact.

(iv) The group of units $\mathbb{C}^\times$ of the complex numbers is an injective abelian group.

(v) For the category $\underline{\text{Ab}}^{\text{f.g.}}$ of finitely generated abelian groups there exists a ring A and an equivalence of categories between A-$\underline{\text{Mod}}$ and $\underline{\text{Ab}}^{\text{f.g.}}$.

Exercise 2 (8 points). Let

$$\begin{array}{cccccccc}
 & & & (*) & & (**) & & \\
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
(\#) & X' & \xrightarrow{a'} & X & \xrightarrow{a} & X'' & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
(\#\#) & Y' & \xrightarrow{b'} & Y & \xrightarrow{b} & Y'' & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\
 & & & Z & \xrightarrow{c} & Z'' & &
\end{array}$$

be a commutative diagram in an abelian category $\mathscr{A}$. Suppose that the rows (#) and (##) and the column $(*)$ are exact sequences. Assume further that f' is an epimorphism and c is a monomorphism. Show, using the diagram chasing rules given in the lecture, that the column $(**)$ is also exact.

Exercise 3 (8 points). Let k be a field and let $A = k[X]/(X^2)$. Let $M = k[X]/(X) = k$ regarded as an A-module. Compute $(R^i \operatorname{Hom}_A(\_, M))(M)$ for all $i \geq 0$.

Exercise 4 (8 points). Let Λ be a commutative ring and let A, B, and C be Λ-algebras. Let M_A be a projective right A-module and let ${}_AN_B$ be an A-B-bimodule which is projective as a right B-module. Show that $M \otimes_A N$ is also a projective right B-module.

Exercise 5 (8 points). Let $n > 0$ be a natural number.

(i) Determine an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ in the category of abelian groups.

(ii) For a natural number $m > 0$ compute $(R^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \_))(\mathbb{Z}/n\mathbb{Z})$.

Exercise 6 (8 points). (i) Let $\mathscr{C}$ be a category. When is a functor $F : \mathscr{C} \to \underline{\text{Set}}$ called representable?

(ii) Let $\mathscr{C} = \underline{\text{CommRing}}$ be the category of commutative rings. Let $n \geq 1$ be a natural number. Consider the functor $F : \mathscr{C} \to \underline{\text{Set}}$ defined by

$$F(A) := \{a \in A \mid a^n = 0\}$$

for $A \in \mathscr{C}$ and $F(f) : F(A) \to F(B),\ a \mapsto f(a)$ for $f \in \mathscr{C}(A, B)$. Show that F is representable.

Good Luck!