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Foundations of Representation Theory
—Exercise sheet 9—

Let $\mathscr{A}$ be an abelian category.

Exercise 1. Let C_*, D_* be chain complexes over $\mathscr{A}$ and let $f : C_* \to D_*$ be a morphism of chain complexes which is null homotopic. Show, using the rules of diagram chase of Thm. 3.42, that $H_n(f) : H_n(C_*) \to H_n(D_*)$ is the zero morphism for every $n \in \mathbb{Z}$.

Exercise 2. In an arbitrary category $\mathscr{C}$ a morphism $f : X \to Y$ is called a section if there exists $r : Y \to X$ such that $rf = \mathrm{id}_X$. We call f a retraction if there exists $s : Y \to X$ such that $fs = \mathrm{id}_Y$. Observe that sections are monomorphisms and retractions are epimorphisms.
Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be a short exact sequence in the abelian category $\mathscr{A}$. Show that the following are equivalent:

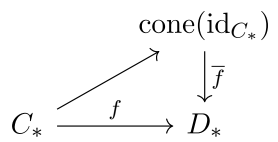
(i) g is a retraction.

(ii) f is a section.

(iii) There exist $r : X \to X'$ and $s : X'' \to X$ such that $(X, (r, g), (f, s))$ is a biproduct of X', X''.

A short exact sequence fulfilling these equivalent conditions is called split.

Exercise 3. Let $f : C_* \to D_*$ be a morphism of chain complexes over $\mathscr{A}$. Show that f is null homotopic if and only if there exists a morphism $\overline{f} : \mathrm{cone}(\mathrm{id}_{C_*}) \to D_*$ such that the following diagram is commutative:



Exercise 4. We want to show that the homotopy category of an abelian category is, in general, not abelian. Concretely we will show that there are morphisms in $\underline{\mathrm{K}}_*(\underline{\mathrm{Ab}})$ which do not have a kernel.
Let $f : C_* \to D_*$ be a morphism of chain complexes over an abelian category $\mathscr{A}$.

(i) Let $h : \mathrm{cone}(f)[1] \to C_*$ be the morphism of complexes given by $h_n = (-\mathrm{id}_{C_n}, 0) : C_n \oplus D_{n+1} \to C_n$. Show that fh is null homotopic.

(ii) Show that if f is a monomorphism in $\underline{\mathrm{K}}_*(\mathscr{A})$ then h is null homotopic.

(iii) Show that if h is null homotopic then f is a section in $\underline{\mathrm{K}}_*(\mathscr{A})$.

(iv) Let $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the homomorphism of abelian groups which sends [1] to [1]. Let D_* and E_* be the complexes concentrated in degree 0 with $D_0 = \mathbb{Z}/4\mathbb{Z}$ and $E_0 = \mathbb{Z}/2\mathbb{Z}$. Denote the corresponding morphism of complexes also with $g : D_* \to E_*$. Show that g does not have a kernel in $\underline{\mathrm{K}}_*(\underline{\mathrm{Ab}})$. (Hint: $\mathbb{Z}/4\mathbb{Z}$ does not have any proper direct summands.)

Due on Friday, 14.12.2018, before the lecture.