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Foundations of Representation Theory
—Exercise sheet 8—

Let $\mathscr{A}$ be an abelian category.

Exercise 1. Let $f : C_* \to D_*$ be a morphism of chain complexes over $\mathscr{A}$.

(i) Show that f has a kernel in $\underline{\mathrm{Ch}}_*(\mathscr{A})$. (Similar arguments of course show that f also has a cokernel.)

(ii) Suppose that C_* and D_* are acyclic. Are $\ker f$, $\operatorname{coker} f$ and $\operatorname{im} f$ always acyclic?

Exercise 2. All chain complexes are over $\mathscr{A}$.

(i) Let $0 \to C'_* \to C_* \to C''_* \to 0$ be a short exact sequence of chain complexes. Show that if two of these complexes are acyclic then so is the third.

(ii) Let $f : C_* \to D_*$ be a morphism of chain complexes. If $\ker f$ and $\operatorname{coker} f$ are acyclic then f is a quasi-isomorphism.

Exercise 3. Let $V_* : \ldots \to V_{n+1} \to V_n \to V_{n-1} \to \ldots$ be a bounded complex of vector spaces over a field k. Show that

$$\sum_{n\in\mathbb{Z}} (-1)^n \dim_k V_n = \sum_{n\in\mathbb{Z}} (-1)^n \dim_k H_n(V_*).$$

Exercise 4. Let Δ be the category whose objects are $\mathrm{Ob}(\Delta) = \mathbb{Z}_{\geq 0}$ and for $m, n \in \mathbb{Z}_{\geq 0}$ the set of morphisms $\Delta(m, n)$ is the set of order-preserving injections $\{0, \ldots, m\} \hookrightarrow \{0, \ldots, n\}$.

(i) For $n \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, \ldots, n\}$ let $f_i^n : \{0, \ldots, n-1\} \to \{0, \ldots, n\}$ be the order-preserving injection whose image does not contain i. Let $A : \Delta^{\mathrm{op}} \to \mathscr{A}$ be a functor. Define $C_*(A)$ by $C_n(A) = A(n)$ and

$$d_n = \sum_{i=0}^{n} (-1)^i A(f_i^n).$$

Show that $C_*(A)$ is a chain complex.
(In the same vein, a functor $A : \Delta \to \mathscr{A}$ gives rise to a cochain complex $C^*(A)$ by defining $d^n = \sum_{i=0}^{n+1} (-1)^i A(f_i^{n+1})$.)

(ii) A simplicial set is a functor $S : \Delta^{\mathrm{op}} \to \underline{\mathrm{Set}}$. Composing S with the functor $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Ab}}$ which associates to a set the free module over it, we obtain a functor $FS : \Delta^{\mathrm{op}} \to \underline{\mathrm{Ab}}$. Given a topological space Y, find a suitable simplicial set such that $C_*(FS)$ agrees with the singular chain complex.

Due on Friday, 7.12.2018, before the lecture.