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Foundations of Representation Theory
—Exercise sheet 7—

Exercise 1. Let X be a topological space, let F be a sheaf on X of abelian groups and let F' be a sub-presheaf of F, that means $F'(U) \subseteq F(U)$ is an abelian subgroup and $\rho_{V,U}(F'(V)) \subseteq F'(U)$ for every two open subsets $U \subseteq V$. Show that the sheafification of F' is given by

$$(sF')(U) = \{s \in F(U) \mid \exists \text{ open cover } U = \bigcup U_i \text{ such that } s|_{U_i} \in F'(U_i)\}.$$

Exercise 2. Let F be a presheaf of abelian groups on a topological space X. Let U be an open subset and $\{U_i\}_{i\in I}$ be an open cover of U. Find homomorphisms $F(U) \to \prod_{i\in I} F(U_i)$ and $\prod_{i\in I} F(U_i) \to \prod_{(j,k)\in I\times I} F(U_j \cap U_k)$ in such a way that F is a sheaf if and only if

$$0 \to F(U) \to \prod_{i\in I} F(U_i) \to \prod_{(j,k)\in I\times I} F(U_j \cap U_k)$$

is exact for every open subset U and every open cover $\{U_i\}_{i\in I}$ of U.

Exercise 3 (5-lemma). Let $\mathscr{A}$ be an abelian category. Let

$$\begin{array}{ccccccccc}
X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5
\end{array}$$

be a diagram in $\mathscr{A}$ with exact rows.

(i) If f_2 and f_4 are epi and f_5 mono then f_3 is epi.

(ii) If f_2 and f_4 are mono and f_1 epi then f_3 is mono.

(iii) If f_2 and f_4 are isomorphisms, f_1 epi and f_5 mono, then f_3 is an isomorphism.

Exercise 4 (9-lemma). Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z' & \longrightarrow & Z & \longrightarrow & Z'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram in an abelian category $\mathscr{A}$ with exact columns and such that the composition $Y' \to Y \to Y''$ is zero. Show that if two of the rows are exact then so is the remaining row.

Due on Friday, 30.11.2018, before the lecture.