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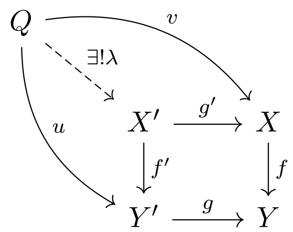
Foundations of Representation Theory
—Exercise sheet 6—

Let $\mathscr{C}$ be a category. Let $Y' \xrightarrow{g} Y \xleftarrow{f} X$ be two morphisms in $\mathscr{C}$. A pull-back of (f, g) is a triple (X', f', g') consisting of an object X' and morphisms $Y' \xleftarrow{f'} X' \xrightarrow{g'} X$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is commutative and such that for any other triple (Q, u, v) of an object Q and morphisms $Y' \xleftarrow{u} Q \xrightarrow{v} X$ satisfying $gu = fv$, there exists a unique morphism $\lambda : Q \to X'$ for which $f'\lambda = u$ and $g'\lambda = v$; in a picture



A pull-back is unique in the sense that if (X', f', g') and (X'', f'', g'') are both pull-backs for (f, g) then the unique morphism $\lambda : X'' \to X'$ which satisfies $f'\lambda = f''$ and $g'\lambda = g''$ is an isomorphism.

Exercise 1. Let

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

be a commutative diagram in $\mathscr{C}$.

(i) Suppose that both small rectangles are pull-back diagrams (that means (X', f', g') is a pull-back of (f, g) and (X'', f'', h') is a pull-back of (f', h). Show that the big rectangle is a pull-back diagram.

(ii) Assume that the big rectangle and the right-hand rectangle are pull-back diagrams. Show that the left-hand rectangle is a pull-back diagram.

Exercise 2. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pull-back diagram in $\mathscr{C}$. Show that f' is a monomorphism provided that f is a monomorphism.

Exercise 3. Let $\mathscr{A}$ be an additive category which has all kernels and cokernels. Show that to any pair (f, g) of morphisms $Y' \xrightarrow{g} Y \xleftarrow{f} X$ in $\mathscr{A}$ there exists a pull-back.

Exercise 4. Let $\mathscr{A}$ be an abelian category and let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pull-back diagram. Suppose that f is an epimorphism. Show that f' is also an epimorphism. (Hint: Show first that $fp_X - gp_{Y'} : X \oplus Y' \to Y$ is an epimorphism.)

Due on Friday, 23.11.2018, before the lecture.