H. Franzen
Universität Bonn

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Foundations of Representation Theory
—Exercise sheet 5—

Exercise 1. True or false? Please explain your answer briefly.
Let k be a field.

(i) If a k-algebra A is isomorphic to A^{op} as a k-algebra, then A is commutative.

(ii) Let A and B be k-algebras and let ${}_AM$, ${}_BN_A$, ${}_BP$ be (bi-)modules. Then $\mathrm{Hom}_B(N \otimes_A M, P)$ is isomorphic to $\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$.

(iii) If a functor $F : \mathscr{C} \to \underline{\mathrm{Set}}$ is representable then a representing object for F is unique up to isomorphism.

(iv) The functor $F : k\text{-}\underline{\mathrm{Alg}} \to \underline{\mathrm{Set}}$ defined by $F(A) = \{*\}$ is representable.

(v) The category of representations of a quiver Q over k has products.

(vi) The category of fields has a final object.

Exercise 2. More examples for monomorphisms and epimorphisms:

(i) Show that in the category $\underline{\mathrm{Top}}$ of topological spaces and continuous maps a morphism $f : X \to Y$ is a monomorphism if and only if it is injective and show that it is an epimorphism if and only if it is surjective.

(ii) Let $\underline{\mathrm{Haus}}$ be the category of Hausdorff spaces and continuous maps. Let $f : X \to Y$ be a continuous map of Hausdorff spaces. Show that if $f(X)$ is dense in Y then f is an epimorphism.

(iii) Let $\underline{\mathrm{Conn}}_*$ be the category of connected topological spaces with a base point. Prove that the map $f : (\mathbb{R}, 0) \to (S^1, 1)$ defined by $f(x) = e^{2\pi i x}$ is a monomorphism.

Exercise 3. Let $\mathscr{A}$ be a pre-additive category.

(i) Show that for an object Z of $\mathscr{A}$ the following are equivalent:

  (a) Z is an initial object of $\mathscr{A}$,
  (b) Z is a final object of $\mathscr{A}$,
  (c) id_Z is the neutral element of the abelian group $\mathscr{A}(Z, Z)$,
  (d) $\mathscr{A}(Z, Z)$ consists of one element.

(ii) Suppose that $\mathscr{A}$ possesses a zero object 0. Let X and Y be objects of $\mathscr{A}$ and $f \in \mathscr{A}(X, Y)$. Prove that f factors over 0 if and only if f is the neutral element of the abelian group $\mathscr{A}(X, Y)$.

Exercise 4. Let $\underline{\mathrm{Field}}$ be the category whose objects are fields and whose morphisms $f : K \to L$ are ring homomorphisms (between fields). Analyze if the product of the fields K and L in $\underline{\mathrm{Field}}$ exists in the following two cases:

(i) $K = \mathbb{Q}(i)$ and $L = \mathbb{Q}(\sqrt{2})$.

(ii) $K = L = \mathbb{Q}(i)$.

Due on Friday, 16.11.2018, before the lecture.