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Winter 2018/2019

Foundations of Representation Theory
—Exercise sheet 4—

Let k be a field.

Exercise 1. Let $G : \mathscr{D} \to \mathscr{C}$ be a functor. Show that G possesses a left adjoint if and only if the functor $\mathscr{C}(X, G\_) : \mathscr{D} \to \underline{\text{Set}}$ is representable for every object X of $\mathscr{C}$.

Exercise 2. Show that the following two forgetful functors—both called V—possess both a left and a right adjoint.

(i) Let A be a k-algebra and let $V : A\text{-}\underline{\text{Mod}} \to k\text{-}\underline{\text{Mod}}$.

(ii) Let $V : \underline{\text{Top}} \to \underline{\text{Set}}$.

These are examples functors which are not equivalences of categories but which have both a left and a right adjoint.

Exercise 3. Give the unit and the counit of the following adjunctions; more precisely for the functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ find natural transformations $\eta : \text{Id}_{\mathscr{C}} \to G \circ F$ and $\varepsilon : F \circ G \to \text{Id}_{\mathscr{D}}$ which fulfill the triangular relations.

(i) Let $\mathscr{C} = k\text{-}\underline{\text{Mod}}$, $\mathscr{D} = A\text{-}\underline{\text{Mod}}$, $F(V) = A \otimes_k V$, and G be the forgetful functor.

(ii) Let A and B be k-algebras and ${}_AN_B$ be a bimodule. Let $\mathscr{C} = \underline{\text{Mod}}\text{-}A$, $\mathscr{D} = \underline{\text{Mod}}\text{-}B$, let $F(M_A) = M \otimes_A N$ and $G(P_B) = \text{Hom}_B(N, P)$.

(iii) Let $\mathscr{C}$ be the category whose objects are pairs (A, S) consisting of a commutative ring A and a multiplicative subset $S \subseteq A$ and whose morphisms $f : (A, S) \to (B, T)$ are ring homomorphisms $f : A \to B$ such that $f(S) \subseteq T$. Let $\mathscr{D} = \underline{\text{CommRing}}$. Let $F(A, S) = S^{-1}A$ and $G(B) = (B, B^\times)$.

Exercise 4. Check if the following functors $F : k\text{-}\underline{\text{CommAlg}} \to \underline{\text{Set}}$ are representable and, if so, give a representing object.

(i) $F(A) = A^n$ (here n is a natural number).

(ii) $F(A) = \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) = (1)\}$ (note the ambiguity the notation $(a_1, \ldots, a_n)$: on the one hand it denotes an n-tuple of elements of A and on the other hand it denotes the ideal in A which is generated by $a_1, \ldots, a_n$).

(iii) $F(A) = \text{GL}_n(A)$.

Due on Friday, 09.11.2018, before the lecture.