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# Foundations of Representation Theory
# —Exercise sheet 3—

Let $k$ be a field.

**Exercise 1.** Let $k$-mod be the category of finite-dimensional $k$-vector spaces. Show that it is equivalent to a category whose class of objects is $\mathbb{Z}_{\geq 0}$.

**Exercise 2.** Let $G$ be a group. Show that the category $\underline{\mathrm{Rep}}_k(G)$ of representations of $G$ over $k$ is equivalent to the category of left $k[G]$-modules.

**Exercise 3.** Let $G$ be a group. Define the category $\underline{G}$ as the category with a single object $*$, with $\underline{G}(*, *) = G$ and whose composition is defined by $g \circ h = gh$.

(i) Show that the functor category $\underline{\mathrm{Fun}}(\underline{G}, \underline{\mathrm{Set}})$ is equivalent to the category $G$-$\underline{\mathrm{Set}}$ whose objects are sets with a left $G$-action and whose morphisms are $G$-equivariant maps.

(ii) Under the above equivalence, which $G$-set corresponds to the functor $h^* = \underline{G}(*, \_)$?

(iii) What does the statement of Yoneda's lemma say when applied to the functor $F = h^*$?

**Exercise 4.** Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be three categories. Let $S : \mathscr{A} \to \mathscr{C}$ and $T : \mathscr{B} \to \mathscr{C}$ be two functors. Define the so-called comma category $\mathscr{K}$ as follows:

- Objects of $\mathscr{K}$ are triples $(A, B, h)$ where $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $h \in \mathscr{C}(SA, TB)$.
- For two such triples $(A, B, h)$ and $(A', B', h')$ a morphism $(A, B, h) \to (A', B', h')$ is a pair $(f, g)$ consisting of $f \in \mathscr{A}(A, A')$ and $g \in \mathscr{B}(B, B')$ such that $h' \circ Sf = Tg \circ h$.
- Define the composition of morphisms $(f, g) : (A, B, h) \to (A', B', h')$ and $(f', g') : (A', B', h') \to (A'', B'', h'')$ by $(f', g') \circ (f, g) := (f' \circ f, g' \circ g)$.

The comma category $\mathscr{K}$ is often denoted $(S, T)$. Convince yourself that it is indeed a category. Describe the comma category in the following cases:

(i) $\mathscr{A} = 1$ (the category which has just one object $*$ and just one morphism), $\mathscr{B} = \mathscr{C}$, and $T = \mathrm{Id}_{\mathscr{C}}$.

(ii) $\mathscr{A} = \mathscr{B} = \mathscr{C}$ and $S = T = \mathrm{Id}_{\mathscr{C}}$.

(iii) $\mathscr{A} = 1$, $\mathscr{B} = k$-$\underline{\mathrm{Alg}}$, $\mathscr{C} = \underline{\mathrm{Grp}}$, and $T = \_^{\times}$.

**Due on Friday, 02.11.2018, before the lecture.**