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Foundations of Representation Theory
—Exercise sheet 2—

Let k be a field.

Exercise 1. Let Q be the Kronecker quiver $1 \rightrightarrows 2$. For $a, b \in k$ let $X_{(a,b)}$ be the representation of Q over k given by

$$k \overset{a}{\underset{b}{\rightrightarrows}} k$$

Determine the space $\mathrm{Hom}(X_{(a,b)}, X_{(c,d)})$ of homomorphisms of representations for any $a, b, c, d \in k$. When are $X_{(a,b)}$ and $X_{(c,d)}$ isomorphic?

Exercise 2. A representation X of Q over k is called indecomposable if it cannot be written as a direct sum $X = X' \oplus X''$ of two non-zero representations of Q over k.

(i) Let k be algebraically closed. Let Q be the Jordan quiver $1 \circlearrowleft$. Determine all indecomposable finite-dimensional representations of Q up to isomorphism.

(ii) Find an indecomposable representation over $\mathbb{R}$ of the Jordan quiver which is decomposable as a representation over $\mathbb{C}$.

Exercise 3. Let A and B be k-algebras. Let ${}_AM$, ${}_AN_B$ and ${}_BP$ be (bi-)modules.

(i) Show that there exists a unique k-linear map $\Phi : \mathrm{Hom}_A(M, N) \otimes_B P \to \mathrm{Hom}_A(M, N \otimes_B P)$ which sends $f \otimes z$ to the map $M \to N \otimes_B P,\ x \mapsto f(x) \otimes z$ for every $f \in \mathrm{Hom}_A(M, N)$ and every $z \in P$.

(ii) Find an example where Φ is neither injective nor surjective.

(iii) Show that Φ is an isomorphism provided that M or P is free of finite rank.

Exercise 4. Let A and B be two k-algebras and let ${}_AX_B$ be a bimodule.

(i) Show that $R := \left(\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right) := \left\{\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \mid a \in A, b \in B, x \in X\right\}$ equipped with the obvious multiplication becomes a k-algebra.

(ii) Construct a left R-module from a triple (M, N, λ) consisting of a left A-module M, a left B-module N and a homomorphism $\lambda : X \to \mathrm{Hom}_k(N, M)$ of A-B-bimodules. Let's call this module $F(M, N, \lambda)$.

(iii) Find a suitable notion of a morphism of such triples such that a morphism $f : (M, N, \lambda) \to (M', N', \lambda')$ induces a homomorphism of left R-modules $Ff : F(M, N, \lambda) \to F(M', N', \lambda')$.

Due on Friday, 26.10.2018, before the lecture.