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Foundations of Representation Theory
—Exercise sheet 13—

Exercise 1. Let $\mathscr{C}$ be a category, B be a set and for every $\beta \in B$ let I_β be an injective object of $\mathscr{C}$. Suppose that $(I, (p_\beta)_{\beta \in B})$ is a product of $(I_\beta)_{\beta \in B}$. Show that I is injective.

Exercise 2. Consider the category of abelian groups.

(i) Determine an injective resolution of $\mathbb{Z}$ in $\underline{\text{Ab}}$.

(ii) Compute $(R^i \operatorname{Hom}(\mathbb{Z}/n\mathbb{Z}, \_))(\mathbb{Z})$ for every $i \geq 0$ and every $n \in \mathbb{Z}_{\geq 0}$

Exercise 3. Let $m > 1$ be a natural number. Let $A := \mathbb{Z}/m\mathbb{Z}$. Show that A is injective regarded as an A-module.

Exercise 4. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.

(i) Let $f : Y' \to Y$ and $g : Y \to Y''$ be morphisms in $\mathscr{A}$. Suppose that for every $X \in \mathscr{A}$ the sequence of abelian groups

$$\operatorname{Hom}_{\mathscr{A}}(X, Y') \xrightarrow{h^X(f)} \operatorname{Hom}_{\mathscr{A}}(X, Y) \xrightarrow{h^X(g)} \operatorname{Hom}_{\mathscr{A}}(X, Y'')$$

is exact. Show that $Y' \xrightarrow{f} Y \xrightarrow{g} Y''$ is exact.

(ii) Let (F, G, φ) be an adjunction from $\mathscr{A}$ to $\mathscr{B}$. Show that F and G are additive and $\varphi_{X,Y} : \operatorname{Hom}_{\mathscr{B}}(FX, Y) \to \operatorname{Hom}_{\mathscr{A}}(X, GY)$ is an isomorphism of abelian groups.
(Hint: Show that F respects coproducts and G respects products.)

(iii) Let again (F, G, φ) be an adjunction from $\mathscr{A}$ to $\mathscr{B}$. Show that F is right exact and G is left exact.

This problem sheet is not relevant for the admission to the exam. If you would like your solutions to be corrected, please submit them on Friday, 25.01.2019, before the lecture.