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Foundations of Representation Theory
—Exercise sheet 12—

Exercise 1. Let k be a commutative ring, let A be a k-algebra, and let M be a left A-module. Consider the left exact functors $F = \mathrm{Hom}_A(M, \_) : A\text{-}\underline{\mathrm{Mod}} \to k\text{-}\underline{\mathrm{Mod}}$ and $G = {}_{\mathbb{Z}}\mathrm{Hom}_A(M, \_) : A\text{-}\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$; i.e. $F(N)$ is the k-module $\mathrm{Hom}_A(M, N)$ and $G(N)$ is its underlying abelian group. Show that

$$_{\mathbb{Z}}(R^n \mathrm{Hom}_A(M, \_))(N) \cong (R^n {}_{\mathbb{Z}}\mathrm{Hom}_A(M, \_))(N).$$

Exercise 2. Consider the category $\underline{\mathrm{Ab}}$ of abelian groups.

(i) Show that every abelian group M has a projective resolution P_* with at most two non-zero terms P_0 and P_1.
(Hint: You may use without a proof the fact that every subgroup of a free abelian group is free. See Satz 4.2 in last term's "Algebra 1" course for a proof in the finitely generated case and Hungerford's book on Algebra, Ch. IV, Thm. 6.1 for the general case.)

(ii) Compute $(R^i \mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z}/m\mathbb{Z}))(\mathbb{Z}/n\mathbb{Z})$ for every $m, n \in \mathbb{Z}$ and every $i \geq 0$.

(iii) Compute $(L_i(\_ \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}))(\mathbb{Z}/n\mathbb{Z})$ for every $m, n \in \mathbb{Z}$ and every $i \geq 0$.

Exercise 3. Consider the ring $A = \mathbb{Z}[t]/(t^n - 1)$ (i.e. the group ring of the cyclic group of order n). Consider $\mathbb{Z}$ as an A-module via $tx = x$ for all $x \in \mathbb{Z}$ (that means we consider the natural representation).

(i) Let P_* be defined by

$$\ldots \xrightarrow{q} A \xrightarrow{1-t} A \xrightarrow{q} A \xrightarrow{1-t} A \to 0$$

where $q = 1 + t + \ldots + t^{n-1}$ and where the right-most non-zero entry is in degree 0. Show that P_* is a complex which consists of projectives.

(ii) Show that P_* is exact at every P_i except for P_0.

(iii) Let $p_0 : P_0 = A \to \mathbb{Z}$ be the evaluation at 1, i.e. $p_0(t) = 1$. Show that (P_*, p_0) is a projective resolution of $\mathbb{Z}$.

(iv) Compute $(L_i(\_ \otimes_A \mathbb{Z}))(\mathbb{Z})$ for every $i \geq 0$.

Exercise 4. Let k be a field. Let Q be the Kronecker quiver $1 \rightrightarrows 2$ and let M be the representation

$$k \underset{w}{\overset{v}{\rightrightarrows}} k^2$$

given by two vectors $v, w \in k^2$.

(i) Determine explicitly an exact sequence of representations of Q of the form $0 \to P(2)^2 \to P(1) \oplus P(2)^2 \to M \to 0$.

(ii) Compute $\dim((R^1 \mathrm{Hom}(\_, M))(M))$.

Due on Friday, 18.01.2019, before the lecture.