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Foundations of Representation Theory
—Exercise sheet 11—

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be abelian categories.

Exercise 1. Suppose that $\mathscr{A}$ has enough injectives. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ two additive functors. Let F be left exact and G be exact. Show that $R^n(G \circ F) \cong G \circ R^n F$ for every $n \geq 0$.

Exercise 2. Suppose that $\mathscr{A}$ has enough injectives. Let $F : \mathscr{A} \to \mathscr{B}$ be additive and left exact. Let $0 \to X \to I \to Y \to 0$ be a short exact sequence in $\mathscr{A}$ such that I is injective. Show:

(i) $R^{n+1}F(X) \cong R^n F(Y)$ for every $n \geq 1$.

(ii) $R^1 F(X) \cong \operatorname{coker}(FI \to FY)$.

This method is called dimension shift.

Exercise 3. Show that for an abelian category $\mathscr{A}$ the following are equivalent:

(i) Every short exact sequence in $\mathscr{A}$ splits.

(ii) Every object of $\mathscr{A}$ is projective.

(iii) Every object of $\mathscr{A}$ is injective.

(iv) $\operatorname{Hom}(X, \_)$ is exact for every $X \in \mathscr{A}$.

(v) $\operatorname{Hom}(\_, Y)$ is exact for every $Y \in \mathscr{A}$.

(vi) $\mathscr{A}$ has enough injectives and $R^1 \operatorname{Hom}_{\mathscr{A}}(X, \_) = 0$ for every $X \in \mathscr{A}$.

(vii) $\mathscr{A}$ has enough projectives and $R^1 \operatorname{Hom}_{\mathscr{A}}(\_, Y) = 0$ for every $Y \in \mathscr{A}$.

A category which fulfills these equivalent conditions is called semi-simple.
Examples of semi-simple categories are the category k-$\underline{\text{Vect}}$ of k-vector spaces over a field k and $\underline{\text{rep}}_k(G)$ of finite-dimensional representations of a finite group G over a field k whose characteristic does not divide the group order.

Exercise 4. Let Q be a quiver and k a field. For a vertex i we define a representation $P(i)$ of Q over k as follows: At a vertex j let $P(i)_j$ be the k-vector space with basis

$$Q_*(i,j) := \{p \in Q_* \mid s(p) = i \text{ and } t(p) = j\}.$$

For an arrow $\alpha : j \to k$ let $P(i)_\alpha : P(i)_j \to P(i)_k$ be the unique k-linear map such that $(P(i)_\alpha)(p) = \alpha p$ for every $p \in Q_*(i,j)$. Show:

(i) Let X be a representation of Q. The map $\operatorname{Hom}(P(i), X) \to X_i$ defined by $f = (f_j)_{j \in Q_0} \mapsto f_i(\varepsilon_i)$ is an isomorphism of k-vector spaces.

(ii) The representation $P(i)$ is a projective object in the category $\underline{\text{Rep}}_k(Q)$.

(iii) If Q has no oriented cycles then $P(i)$ is finite-dimensional and $\operatorname{End}(P(i)) = k$.

(iv) If Q has no oriented cycles then $P(i)$ is indecomposable.

Please turn

The following two problems are bonus exercises. By completing them you will be able to score up to 24 points on this problem sheet while 16 points account for 100%.

Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor which has a right derivative R^*F. An object $A \in \mathscr{A}$ is called F-acyclic if $R^nF(A) = 0$ for all $n > 0$. An F-acyclic resolution of an object X is a cochain resolution (A^*, i^0) of X for which the complex A^* consists of F-acyclic objects.

Exercise 5. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor which has a right derivative R^*F. Show:

(i) If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathscr{A}$ for which A' and A are F-acyclic then A'' is also F-acyclic and the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact.

(ii) Let $0 \to A^0 \to A^1 \to A^2 \to \ldots$ be an exact sequence of F-acyclic objects. Then $Z^i := \ker(A^i \to A^{i+1})$ is F-acyclic for every $i \geq 0$.

(iii) For an exact sequence $0 \to A^0 \to A^1 \to A^2 \to \ldots$ of F-acyclic objects the sequence $0 \to F(A^0) \to F(A^1) \to F(A^2) \to \ldots$ is exact as well.

Exercise 6. Let $\mathscr{A}$ have enough injectives and let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Let $X \in \mathscr{A}$ and (A^*, a^0) be an F-acyclic resolution of X.

(i) Show that there exists an injective resolution (I^*, i^0) of X and a lift $f^* : A^* \to I^*$ of id_X in such a way that every $f^i : A^i \to I^i$ is a monomorphism.

(ii) Let $B^i = \mathrm{coker}(f^i)$ where f^* is as in (i). Let $0 \to B^0 \to B^1 \to B^2 \to \ldots$ be the induced morphisms. Show that this sequence is exact.

(iii) Show that B^i is F-acyclic for every $i \geq 0$.

(iv) Show that the sequence $0 \to F(A^*) \to F(I^*) \to F(B^*) \to 0$ in $\underline{\mathrm{Ch}}^{\geq 0}(\mathscr{B})$ is exact.

(v) Conclude that there is an isomorphism $H^n(F(A^*)) \to R^nF(X)$ for every $n \geq 0$.

Due on Friday, 11.01.2019, before the lecture.

Merry Christmas and a happy new year!