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Foundations of Representation Theory
—Exercise sheet 10—

Let $\mathscr{A}$ be an abelian category.

Exercise 1. True or false? Please explain your answers briefly.

(i) The functor $H_0 : \underline{\mathrm{Ch}}_*(\mathscr{A}) \to \mathscr{A}$ is left exact.

(ii) The cone of the identical morphism on a complex C_* is acyclic.

(iii) The category of vector spaces (over a field k) has enough projectives.

(iv) $\mathbb{Z}$ is an injective object in the category of finitely generated abelian groups.

(v) $\mathbb{Q}$ is an injective object in the category of finitely generated abelian groups.

(vi) $\mathbb{Z}/2\mathbb{Z}$ is a projective object in the category of abelian groups.

Exercise 2. Let P', P'' be objects of $\mathscr{A}$ and let $P = P' \oplus P''$. Show that P is projective if and only if P' and P'' are both projective.
Is the analogous statement for injective objects also true?

Exercise 3. Let $I_0 : \mathscr{A} \to \underline{\mathrm{Ch}}_{\geq 0}(\mathscr{A})$ and $I^0 : \mathscr{A} \to \underline{\mathrm{Ch}}^{\geq 0}(\mathscr{A})$ be the functors which assign to an object X the chain/cochain complex $I_0(X)$ resp. $I^0(X)$ which is concentrated in degree 0 and for which $I_0(X)_0 = X$ and $I^0(X)^0 = X$.

(i) Show that I_0 has a left adjoint.

(ii) Show that I^0 has a right adjoint.

Exercise 4. Let k be a commutative ring and let A be a k-algebra. Consider the category A-$\underline{\mathrm{Mod}}$ of left A-modules.

(i) Show that ${}_AA$ is a projective object of A-$\underline{\mathrm{Mod}}$.

(ii) Show that for a left A-module P the following assertions are equivalent:

(a) P is a projective object of A-$\underline{\mathrm{Mod}}$.

(b) P is a direct summand of a free A-module.

Due on Friday, 21.12.2018, before the lecture.