



## On functors associated to a simple root

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### Abstract

Associated to a simple root of a finite-dimensional complex semisimple Lie algebra, there are several endofunctors (defined by Arkhipov, Enright, Frenkel, Irving, Jantzen, Joseph, Mathieu, Vogan and Zuckerman) on the BGG category  $\mathcal{O}$ . We study their relations, compute cohomologies of their derived functors and describe the monoid generated by Arkhipov's and Joseph's functors and the monoid generated by Irving's functors. It turns out that the endomorphism rings of all elements in these monoids are isomorphic. We prove that the functors give rise to an action of the singular braid monoid on the bounded derived category of  $\mathcal{O}_0$ . We also use Arkhipov's, Joseph's and Irving's functors to produce new generalized tilting modules. © 2007 Elsevier Inc. All rights reserved.

*Keywords:* Lie algebra; Category  $\mathcal{O}$ ; Functor; Translation; Twisting; Module

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### Introduction

Braid group actions via auto-equivalences of derived categories play an important role in different areas of mathematics and physics. They arise naturally, for example, in algebra, algebraic geometry, representation theory, string theory, symplectic geometry, topology etc. In this paper we focus on endofunctors of the BGG category  $\mathcal{O}$ , associated to a semi-simple complex (finite-dimensional) Lie algebra, which give rise to such braid group actions. These endofunctors turned

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out to be very useful and motivating in different areas of mathematics. As examples one could mention for instance

- the Kazhdan–Lusztig combinatorics of the category  $\mathcal{O}$  (see, e.g., [KL]);
- the Serre functor for the bounded derived category of the category of perverse sheaves on the flag variety (see, e.g., [BBM]);
- structure and characters of tilting modules (see, e.g., [So4]);
- finding functorial invariants of tangles and links (see, e.g., [BFK]);
- defining  $\mathfrak{sl}_2$ -categorifications and Broué’s conjecture (see, e.g., [CR]);
- semi-infinite cohomology and Wakimoto modules (see, e.g., [Ark,Ara]).

Of course, there are many more. The main properties of the functors used in the above examples are the following: they provide a connection between projective and tilting modules, and they define a categorification of braid groups and Hecke algebras. To define a categorification, one usually has to check relations between certain functors and even between natural transformations. In practice, this is quite often a rather cumbersome technical work (see, e.g., [CR]).

The aim of the present paper is the following:

- to collect from the wide-spread literature all the endofunctors of the category  $\mathcal{O}$ , associated to a simple root of the Lie algebra, and give an insight into the interplay of these functors (Theorems 2, 5);
- describe the non-trivial relations between these endofunctors (Theorems 3, 4);
- describe natural transformations between these endofunctors (Theorems 6, 7);
- describe the connection between the injective, projective and tilting modules, and construct new generalized tilting modules (Theorems 8, 9);
- show that the structure of the category  $\mathcal{O}$  is not completely determined by the Kazhdan–Lusztig combinatorics (Remark 1.2, Proposition 2.6);
- construct a new (unexpected) categorification of the Baez–Birman singular braid monoid via endofunctors of the principal block of  $\mathcal{O}$ , which might lead to a categorification of Vassiliev invariants (Theorem 1).

We formulate all the results in the next section and try to hide the technicalities (as far as possible) in the proofs, which follow afterwards.

## 1. The results

### 1.1. Notation and the setup

Let  $\mathfrak{g}$  be a semisimple complex finite-dimensional Lie algebra with a fixed triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . Let  $W$  be the corresponding Weyl group with the length function  $l$ , the unit element  $e$ , the longest element  $w_0$ , and the Bruhat ordering  $<$ . The letter  $\rho$  denotes the half-sum of all positive roots. There is the so-called dot-action of  $W$  on  $\mathfrak{h}^*$  defined as  $w \cdot \lambda = w(\lambda + \rho) - \rho$ . Let  $\mathcal{O}$  denote the BGG-category  $\mathcal{O}$  introduced in [BGG] and  $\mathcal{O}_0$  its principal block, that is the indecomposable block of  $\mathcal{O}$  containing the trivial  $\mathfrak{g}$ -module. For a simple reflection  $s$  let  $\mathfrak{g}^s$  denote the corresponding minimal parabolic subalgebra of  $\mathfrak{g}$ , strictly containing  $\mathfrak{h} \oplus \mathfrak{n}_+$ . We denote by  $\mathcal{O}_0^s$  the corresponding parabolic subcategory, which consists of all locally  $\mathfrak{g}^s$ -finite objects from  $\mathcal{O}_0$ . We call a module  $s$ -free, if none of the composition factors

in its socle is  $\mathfrak{g}^s$ -finite. Let  $\mathcal{C} = S(\mathfrak{h})/(S(\mathfrak{h})_+^W)$  be the coinvariant algebra of  $W$  with respect to the dot-action. Its subalgebra of  $s$ -invariants (under the usual action) is denoted by  $\mathcal{C}^s$  (see [So2, 2.4]). For  $x \in W$  we denote by  $\Delta(x) \in \mathcal{O}_0$  the Verma module of the highest weight  $x \cdot 0$  and by  $P(x)$  its projective cover with simple head  $L(x)$ . Associated to a fixed simple reflection  $s$  we have the following endofunctors of  $\mathcal{O}_0$ :

- The *translation functor*  $\theta = \theta_s$  through the  $s$ -wall (see, e.g., [Jo1, Section 3]). Let  $\mathcal{O}_s$  be a singular integral block of  $\mathcal{O}$  with stabilizer  $\{e, s\}$ . The functor  $\theta_s$  is defined as the composition of the translation functors  $\theta_s^{\text{on}} : \mathcal{O}_0 \rightarrow \mathcal{O}_s$  and  $\theta_s^{\text{out}} : \mathcal{O}_0 \rightarrow \mathcal{O}_s$  (see [Ja, 4.12]), which are both left and right adjoint to each other. In particular, the functor  $\theta$  is exact and self-adjoint. For each  $x \in W$  the module  $\theta_s \Delta(x)$  is a (unique up to isomorphism) indecomposable extension of  $\Delta(x)$  and  $\Delta(xs)$ .
- The *shuffling functor*  $C = C_s$  (see [Ir1, Section 3]). This functor is defined as follows: we fix an adjunction morphism  $\text{adj}_s : \text{ID} \rightarrow \theta$  and define  $C$  as the cokernel of  $\text{adj}_s$ . Up to isomorphism the definition does not depend on the choice of  $\text{adj}_s$ . As both  $\text{ID}$  and  $\theta$  are exact functors, the functor  $C$  is right exact by the Snake Lemma.
- Dually, we have the *coshuffling functor*  $K = K_s$ . This functor is defined as follows: we fix an adjunction morphism  $\text{adj}_s^s : \theta \rightarrow \text{ID}$  and define  $K$  as the kernel of  $\text{adj}_s^s$ . Up to isomorphism the definition does not depend on the choice of  $\text{adj}_s^s$ . As both  $\text{ID}$  and  $\theta$  are exact functors, the functor  $K$  is left exact by the Snake Lemma.
- *Zuckerman's functor*  $Z = Z_s$  (see [Zu]). This functor is given by taking the maximal quotient which belongs to the category  $\mathcal{O}_0^s$ . The functor  $Z$  is the left adjoint to the natural inclusion of the category  $\mathcal{O}_0^s$  into  $\mathcal{O}_0$ , in particular,  $Z$  is right exact.
- *Joseph's completion*  $G = G_s$  is defined in [Jo1, Section 2] in the following way:  $G(-) = \mathcal{L}(\Delta(s), -) \otimes \Delta(e)$ , where for  $M \in \mathcal{O}_0$  the set  $\mathcal{L}(\Delta(s), M)$  is a  $\mathfrak{g}$ -bimodule, consisting of all  $\mathbb{C}$ -linear maps from  $\Delta(s)$  to  $M$ , which are locally finite with respect to the adjoint action of  $\mathfrak{g}$ . The functor  $G$  is left exact.
- *Arkhipov's twisting functor*  $T = T_s$  is defined as the composition of two functors (see, e.g., [AS]). To define these functors we first take a non-zero element  $X_{-\alpha}$  from the negative root subspace of  $\mathfrak{g}$  associated with  $s$ , and localize the universal enveloping algebra  $U(\mathfrak{g})$  with respect to the powers of  $X_{-\alpha}$ . We obtain the localized algebra  $U_s$ . The first functor is then tensoring with the  $U(\mathfrak{g})$ -bimodule  $U_s/U(\mathfrak{g})$ , which, in fact, does not preserve the category  $\mathcal{O}_0$ . To obtain an endofunctor of  $\mathcal{O}_0$  we compose  $U_s/U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} -$  with the functor of twisting by an automorphism of  $\mathfrak{g}$ , corresponding to  $s$ . It turns out that the resulting functor is up to isomorphism of functors independent of the above choice of an automorphism of  $\mathfrak{g}$ . The functor  $T$  is right exact. In fact, in [KM, Corollary 4] it is shown that  $T$  is left adjoint to  $G$ .
- The functor  $Q$  given as the cokernel of a natural transformation  $\mathfrak{g} : \text{ID} \rightarrow G$  as considered in [Jo1, 2.4] (which is unique up to a non-zero scalar). The functor  $Q$  is neither left nor right exact.
- *Enright's completion functor*  $E = E_s$ . This functor was originally defined in [En] via a complicated procedure, based on the theory of highest weight  $\mathfrak{sl}_2$ -modules, applied to the  $\mathfrak{sl}_2$ -subalgebra of  $\mathfrak{g}$  associated with  $s$ . The original definition was generalized and extended in [De] and [Jo1]. In [KM, Section 4] it is shown that  $E = G^2$ .

The functor  $Z$  can be characterized as the functor of taking the maximal quotient which is annihilated by  $T$  (or, equivalently, by  $G$ ). We define  $\hat{Z} : \mathcal{O}_0 \rightarrow \mathcal{O}_0$  as the endofunctor given by taking the maximal quotient annihilated by  $C$  (or, equivalently, by  $K$ ), i.e., the maximal quotient

containing only composition factors of the form  $L(y)$ ,  $y < y_s$ . Although the definition is very similar, the properties of the functors  $Z$  and  $\hat{Z}$  are quite different (see Remark 1.2 and Theorem 3 below).

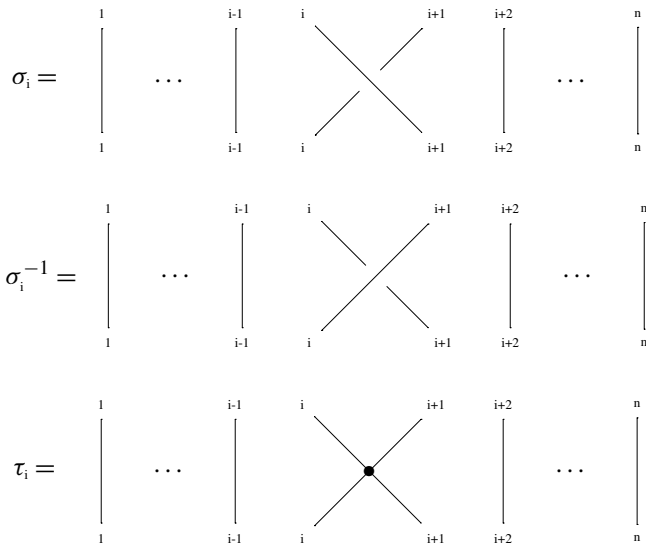
Let  $d$  be the usual contravariant duality on  $\mathcal{O}_0$ . For an endofunctor  $X$  of  $\mathcal{O}_0$  we denote by  $X'$  the composition  $X' = dXd$ . If  $X_1, X_2, Y$  are endofunctors on  $\mathcal{O}_0$  and  $h \in \text{Hom}(X_1, X_2)$  we denote by  $h_Y$  the induced natural transformation in  $\text{Hom}(X_1 Y, X_2 Y)$ . For  $h \in \text{Hom}(X_1, X_2)$  we also set  $h' = d h_d \in \text{Hom}(X'_1, X'_2)$ .

In Section 2 we give a more elegant proof of the fact  $G \cong T'$  from [KM, Theorem 4]. This result allows us to simplify the exposition and redefine Arkhipov’s functor as  $T = G'$ . In Section 2 we also prove some similarities between the pairs  $(T, G)$  and  $(C, K)$  of functors (Proposition 2.4), but also show some remarkable differences (Proposition 2.6 and Remark 1.2). This result is surprising and should be taken as a warning that these pairs of functors are quite different.

For a right/left exact endofunctor  $F$  on  $\mathcal{O}_0$  we denote by  $\mathcal{L}F/\mathcal{R}F$  its derived functor with  $i$ th (co)homology  $\mathcal{L}_i F/\mathcal{R}^i F$ .

### 1.2. An action of the singular braid monoid

We would like to start the description of our results with the very surprising fact that some of these functors give rise to a categorification of the singular braid monoid. In reality, this is an application of the technique, developed during the proofs of the other statements. Baez ([Bae]) and Birman ([Bi]) defined the so-called singular braid monoid with generators



(for the definition see Section 11) which has connections to Vassiliev-invariants (see, e.g., [Bi, GP, Ve]) and for which the word problem is solved (e.g., [Co, Or]). The following result suggests a strong link between our functors and invariants of knots, in the spirit of [BFK], i.e., we expect that the following result is the first step in defining a functorial version of Vassiliev invariants.

**Theorem 1.** *Let  $\mathfrak{g} = \mathfrak{sl}_n$  and let  $s_i$  ( $1 \leq i \leq n - 1$ ) be the set of simple reflections in the usual ordering. We consider the corresponding functors  $C_{s_i}$ ,  $K_{s_i}$  and  $\theta_{s_i}$ . The set of func-*

tors  $\{\mathcal{L}C_{s_i}, \mathcal{R}K_{s_i}, \theta_{s_i}\}$  defines a (weak) action of the singular braid monoid, with generators  $\{\sigma_i, \sigma_i^{-1}, \tau_i\}$ , on the bounded derived category of  $\mathcal{O}_0$ .

1.3. The interplay between the functors

**Theorem 2.** *There are the following isomorphisms of functors:*

1.  $\mathcal{R}^1K \cong \hat{Z}$ .
2.  $\mathcal{R}^1G \cong Z$ , in particular  $\mathcal{R}^1G \cong \text{ID}$  on  $\mathcal{O}_0^s$ .
3.  $\mathcal{L}_1Z \cong Q$ , in particular  $Q \cong Q'$ .
4.  $\mathcal{R}^iG^2 \cong \begin{cases} ZG & \text{if } i = 1, \\ Z & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$  and  $\mathcal{R}^iK^2 \cong \begin{cases} \hat{Z}K & \text{if } i = 1, \\ \hat{Z} & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$

Dual statements hold for  $Z', T, \hat{Z}'$ , and  $C$ .

**Remark 1.1.**  $\mathcal{R}^iG \cong 0$  for  $i > 1$  by [AS, Theorems 2.2 and 4.1];  $\mathcal{L}^2Z \cong Z'$  and  $\mathcal{L}^iZ \cong 0$  if  $i > 2$  follows from [EW, Corollary 5.2], and  $\mathcal{R}^iK \cong 0$  for  $i > 1$  follows from [MS1, Lemma 5.2 and Proposition 5.3].

**Remark 1.2.** The derived functor  $\mathcal{L}\hat{Z}$  has a more complicated structure than  $\mathcal{L}Z$ . This is already evident for the Lie algebra  $\mathfrak{sl}_3$ . In fact, by a direct calculation one can show that in this case  $\mathcal{L}_6\hat{Z} \neq 0$ . It follows that, in general, there is no involutive exact equivalence  $F$  on  $\mathcal{O}_0$  sending  $L(x)$  to  $L(x^{-1})$ . The same statement can also be obtained using the following general argument:

Let  $A$  be a finite-dimensional associative algebra and  $\Lambda$  be an indexing set of the isoclasses  $S(\lambda)$ ,  $\lambda \in \Lambda$  of simple  $A$ -modules. Assume that  $F$  is an exact equivalence on  $A\text{-mod}$  such that  $F(S(\lambda)) \cong S(\sigma(\lambda))$  for some permutation  $\sigma$  on  $\Lambda$ . For  $J \subset \Lambda$  let  $Z_J$  denote the functor given by taking the maximal quotient containing only simple subquotients indexed by  $J$ . Then it is easy to see that the functors  $F^{-1}Z_{\sigma(J)}F$  and  $Z_J$  are isomorphic.

Let  $\mathfrak{g} = \mathfrak{sl}_3$  and  $s, t$  be the two simple reflections. Let  $J = \{e, t, ts\}$ ,  $\hat{J} = \{e, t, st\}$  and  $J' = \{e, s, ts\}$ . Then  $J \cong \hat{J}$  via  $w \mapsto w^{-1}$  and  $J \cong J'$  via  $w w_0 \mapsto w^{-1} w_0$ . By definition we have  $Z = Z_J$ ,  $\hat{Z} = Z_{\hat{J}}$ , and  $\hat{Z}' = Z_{J'}$ . It is easy to check that  $ZP(t)$  has length 4, but both,  $\hat{Z}P(t^{-1})$  and  $\hat{Z}'P(s) = \hat{Z}'P((st)^{-1}w_0)$ , have length 3. In particular, there is neither an involutive exact equivalence sending  $L(x)$  to  $L(x^{-1})$ , nor an involutive exact equivalence sending  $L(xw_0)$  to  $L(x^{-1}w_0)$ . From the point of view of Kazhdan–Lusztig combinatorics this could not have been expected. In particular, it shows that the corresponding statements in the literature (e.g., [Jo2, Existence of  $\varepsilon$  in Section 4.3] and the existence of  $T$  claimed in [So1, Lemma 6]) are not correct. In fact the construction proposed in the proof of [So1, Lemma 6] gives the identity functor.

We describe the monoids generated by  $\{G, T\}$  and  $\{C, K\}$  respectively. Recall (see, e.g., [La, Chapter II]), that for a monoid,  $S$ , and for  $x, y \in S$ , we have  $x\mathbf{R}y$  or  $x\mathbf{L}y$  if and only if  $xS = yS$  and  $Sx = Sy$  respectively.

**Theorem 3.** *The functors  $T$  and  $G$  satisfy the relations*

$$TGT \cong T, \quad GTG \cong G, \quad T^3 \cong T^2, \quad G^3 \cong G^2,$$

$$T^2G \cong T^2, \quad G^2T \cong G^2, \quad TG^2 \cong GT^2,$$

and their isoclasses generate the monoid  $\mathcal{S} = \{\text{ID}, T, G, TG, GT, T^2, G^2, TG^2\}$  of (isoclasses of) functors. The columns and rows of the following egg-box diagrams represent respectively Green’s relations  $\mathbf{R}$  and  $\mathbf{L}$ , on  $\mathcal{S}$ :



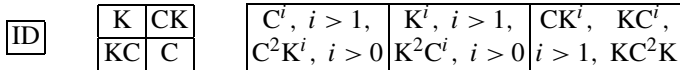
**Theorem 4.** The functors  $C$  and  $K$  satisfy the relations

$$CKC \cong K, \quad KCK \cong K, \quad C^3K \cong C^2, \quad K^3C \cong K^2, \\ C^2K^2C \cong C^2K, \quad K^2C^2K \cong K^2C, \quad CK^2C^2 \cong KC^2, \quad KC^2K^2 \cong CK^2.$$

Assume that  $s$  does not correspond to an  $\mathfrak{sl}_2$ -direct summand of  $\mathfrak{g}$ . Then the isoclasses of the functors  $C$  and  $K$  generate the (infinite) monoid

$$\hat{\mathcal{S}} = \{\text{ID}, KC^2K \cong CK^2C, K^i, C^i, KC^i, CK^i, K^2C^i, C^2K^i : i > 0\}.$$

The columns and rows of the following egg-box diagrams represent respectively Green’s relations  $\mathbf{R}$  and  $\mathbf{L}$ , on  $\hat{\mathcal{S}}$ :



The only idempotents in  $\hat{\mathcal{S}}$  are  $\text{ID}, KC, CK, C^2K^2, K^2C^2, KC^2K$ .

**Remark 1.3.** If  $\mathfrak{g} = \mathfrak{sl}_2$ , then one can show that  $\mathcal{L}C^2$  is the Serre functor, which implies  $C^2 \cong C^4$ , see [MS2].

Before describing morphism spaces between such functors, we want to give an impression of their rather complex interplay. We need some preparations to formulate the corresponding Theorem 5, in which we show relations between functors from  $\mathcal{S}$ .

According to [AS, Remark 5.7],  $T$  is left adjoint to  $G$  and  $g'$  is up to a scalar the composition of  $T(\mathfrak{g})$  with the adjunction morphism  $TG \rightarrow \text{ID}$ . We fix  $a' \in \text{Hom}(TG, \text{ID})$  such that  $g' = a' \circ T(\mathfrak{g})$  and set  $a = d(a')_d$  (the existence of  $a'$  also follows from the independent result  $\text{Hom}(TG, \text{ID}) \cong \mathcal{C}$  of Theorem 6 which ensures that up to a scalar there is only one natural transformation “of degree zero”). Let  $z : \text{ID} \rightarrow Z$ , and  $p : G \rightarrow Q$  be the natural projections,  $i = d(p)_d$ ,  $m' = (T^2(\mathfrak{g}))^{-1} \circ iTG$ , and  $m = d(m')_d$ . We will see later that all these maps are well-defined.

**Theorem 5.** Figure 1 presents a diagram of endofunctors on  $\mathcal{O}_0$ , where all the sequences labeled by numbers are exact. Moreover, one can choose isomorphisms  $\alpha$  and  $h$  as indicated such that all configurations containing only solid arrows commute.

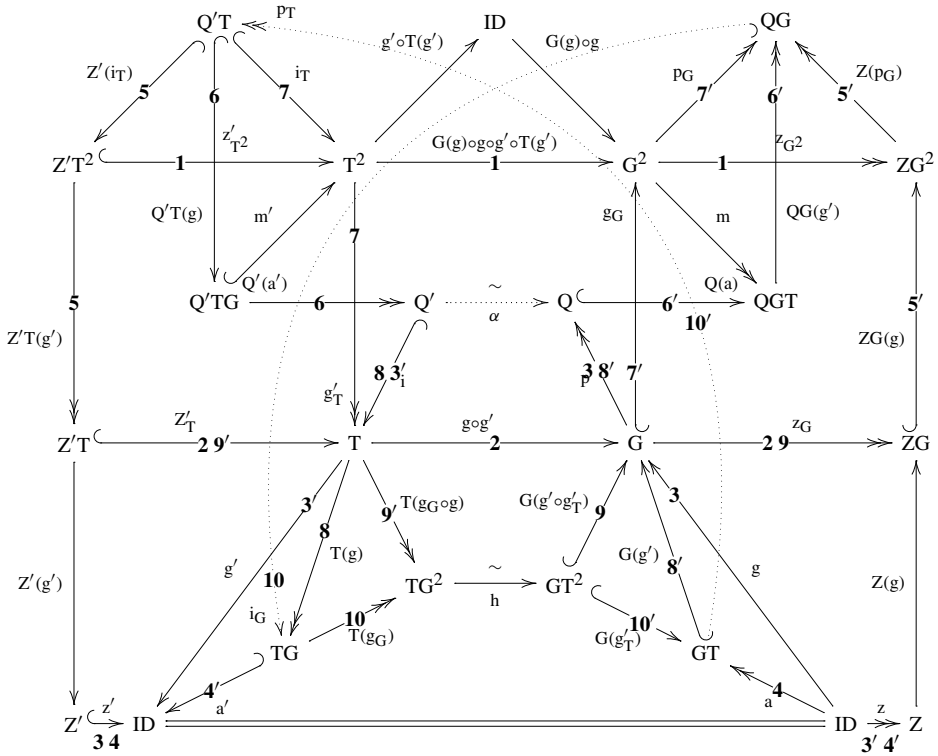


Fig. 1. Commutative diagram involving T and G.

1.4. Natural transformations

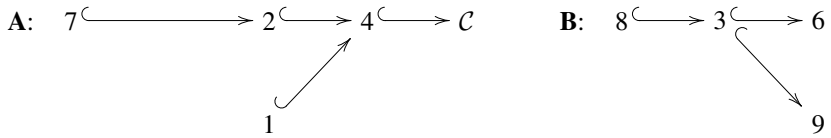
We prove the following result on natural transformations between arbitrary compositions of G and T:

**Theorem 6.**

1. For  $X \in \mathcal{S}$  there is a ring isomorphism  $\text{End}(X) \cong \mathcal{C}$ .
2. For  $X, Y \in \mathcal{S}$  we have  $\text{Hom}(X, Y) \neq 0$  and this space is given by the X-row and Y-column entry in the following table:

$X \backslash Y$	ID	G	T	GT	TG	$G^2$	$T^2$	$GT^2$
ID	$\mathcal{C}$	$\mathcal{C}$	1	$\mathcal{C}$	2	$\mathcal{C}$	3	4
G	1	$\mathcal{C}$	5	4	1	$\mathcal{C}$	6	4
T	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	4	$\mathcal{C}$
GT	2	$\mathcal{C}$	1	$\mathcal{C}$	7	$\mathcal{C}$	8	4
TG	$\mathcal{C}$	$\mathcal{C}$	4	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	4	$\mathcal{C}$
$G^2$	3	4	6	4	8	$\mathcal{C}$	9	4
$T^2$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$
$GT^2$	4	$\mathcal{C}$	4	$\mathcal{C}$	4	$\mathcal{C}$	4	$\mathcal{C}$

The spaces described by the same number are isomorphic and we have the following inclusions:



C:  $\text{End}(\text{ID}_{\mathcal{O}_0^s}) \hookrightarrow 5$

3. There is an isomorphism of rings  $\text{End}(\mathbb{Z}) \cong \text{End}(\text{ID}_{\mathcal{O}_0^s})$ .

We describe the endomorphism spaces of the elements from  $\hat{S}$  and natural transformations between the idempotents in the following theorem:

**Theorem 7.**

1. For  $X \in \hat{S}$  there is a ring isomorphism  $\text{End}(X) \cong \mathcal{C}$ .
2. For idempotents  $X, Y \in \hat{S}$  the space  $\text{Hom}(X, Y)$  is given by the X-row and Y-column entry in the following table:

$X \setminus Y$	ID	CK	KC	$\mathcal{C}^2\mathcal{K}^2$	$\mathcal{K}^2\mathcal{C}^2$	$\mathcal{K}\mathcal{C}^2\mathcal{K}$
ID	$\mathcal{C}$	1	$\mathcal{C}$	2	$\mathcal{C}$	3
CK	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	4	$\mathcal{C}$	$\mathcal{C}$
KC	1	5	$\mathcal{C}$	2	$\mathcal{C}$	3
$\mathcal{C}^2\mathcal{K}^2$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$
$\mathcal{K}^2\mathcal{C}^2$	2	2	4	6	$\mathcal{C}$	4
$\mathcal{K}\mathcal{C}^2\mathcal{K}$	3	3	$\mathcal{C}$	4	$\mathcal{C}$	$\mathcal{C}$

The spaces described by the same number are isomorphic and we have the following inclusions:

$$5 \hookrightarrow 1 \hookrightarrow 3 \hookrightarrow \mathcal{C}, \quad 4 \hookrightarrow \mathcal{C}.$$

**Remark 1.4.** The coinvariant algebra has a natural  $\mathbb{Z}$ -grading given by putting  $\mathfrak{h}$  in degree one. Using the graded versions of  $\mathcal{C}$  and  $\mathcal{K}$  from [MS1, 7.1] (and a similar construction for  $\mathcal{G}$  and  $\mathcal{T}$ ) we get isomorphisms of graded vector spaces as listed in the theorem.

1.5. Tilting modules

Let  $\mathcal{P} = \bigoplus_{x \in W} P(x)$  be a minimal projective generator of  $\mathcal{O}_0$  and set  $\mathcal{I} = d\mathcal{P}$ . For  $M \in \mathcal{O}_0$  the category  $\text{Add}(M)$  is defined as the full subcategory of  $\mathcal{O}_0$ , which consists of all direct summands of all finite direct sums of copies of  $M$ . Recall (see [Wa, Introduction]) that  $M \in \mathcal{O}_0$  is called a *generalized tilting module* if  $\text{Ext}_{\mathcal{O}_0}^>0(M, M) = 0$  and if  $\mathcal{P}$  has a finite  $\text{Add}(M)$ -coresolution, i.e., there exists an exact sequence  $0 \rightarrow \mathcal{P} \rightarrow M_0 \rightarrow \dots \rightarrow M_k \rightarrow 0$  of finite length



$k$  with  $M_i \in \text{Add}(M)$  for  $1 \leq i \leq k$ . If, additionally, the projective dimension of  $M$  is one then  $M$  is called a *classical tilting module*, see [HR, page 399]. Dual notions define generalized and classical cotilting modules. For a fixed reduced expression  $w = s_1 \cdots s_k \in W$  we set  $T_w = T_{s_1} \cdots T_{s_k}$  and  $G_w = G_{s_1} \cdots G_{s_k}$ . The resulting functors are (up to isomorphism) independent of the chosen reduced expression (see [Jo2, 2.9], [KM, Section 6]). The following result describes a lattice of (generalized) tilting and cotilting modules in  $\mathcal{O}_0$  constructed using twisting and completion functors.

**Theorem 8.** *Let  $w \in W$ .*

1. *Each of the modules  $\mathcal{P}^w = T_w \mathcal{P}$  and  $\mathcal{I}^w = G_w \mathcal{I}$  is both, a generalized tilting module and a generalized cotilting module.*
2. *We have the following equalities for projective and injective dimensions:  $\text{projdim}(\mathcal{P}^w) = \text{injdim}(\mathcal{I}^w) = l(w)$  and  $\text{injdim}(\mathcal{P}^w) = \text{projdim}(\mathcal{I}^w) = 2l(w_0) - l(w)$ . In particular, if  $s$  is a simple reflection then  $\mathcal{P}^s$  ( $\mathcal{I}^s$  resp.) is a classical (co)tilting module.*
3.  *$T_w \mathcal{P}^{w_0} \cong \mathcal{I}^{ww_0}$  and  $G_w \mathcal{I}^{w_0} \cong \mathcal{P}^{ww_0}$ . In particular,  $\mathcal{P}^{w_0} \cong \mathcal{I}^{w_0} \cong \mathcal{T}$  is the characteristic (co)tilting module in  $\mathcal{O}_0$ .*

**Remark 1.5.** Let  $x \in W$  be fixed. The module  $T_x T_{w_0} \mathcal{P} \cong T_x \mathcal{P}^{w_0} \cong T_x \mathcal{T}$  is the direct sum of all  $x$ -twisted tilting modules as defined in [St2, Section 5] and characterized by certain vanishing conditions with respect to twisted Verma modules. If  $x = e$  we get the sum of all (usual) tilting modules. The twisting functors define naturally maps as follows:

$$\begin{aligned} \{\text{indec. projectives}\} &\xrightarrow{T_x} \{x\text{-twisted indec. projectives}\} \\ &\xrightarrow{T_{w_0 x^{-1}}} \{(e\text{-twisted}) \text{ tiltings}\} \xrightarrow{T_x} \{x\text{-twisted tiltings}\} \\ &= \{x w_0\text{-completed indec. injectives}\} \xrightarrow{T_{w_0 x^{-1}}} \{\text{indec. injectives}\}. \end{aligned}$$

The maps are all bijections, their inverses induced by the corresponding completion functors.

For a reduced expression  $w = s_k s_{k-1} \cdots s_1 \in W$  we set  $C_w = C_{s_1} \cdots C_{s_k}$  and  $K_w = K_{s_1} \cdots K_{s_k}$ . Up to isomorphism, the functors do not depend on the chosen reduced expression, see [MS1, Lemma 5.10]. We will prove the following analog of the previous theorem:

**Theorem 9.** *Let  $w \in W$ .*

1. *Each of the modules  ${}^w \mathcal{P} = C_w \mathcal{P}$  and  ${}^w \mathcal{I} = K_w \mathcal{I}$  is both, a generalized tilting module and a generalized cotilting module.*
2. *We have the following equalities for projective and injective dimensions:  $\text{projdim}({}^w \mathcal{P}) = \text{injdim}({}^w \mathcal{I}) = l(w)$  and  $\text{injdim}({}^w \mathcal{P}) = \text{projdim}({}^w \mathcal{I}) = 2l(w_0) - l(w)$ . In particular,  ${}^s \mathcal{P}$  (and  ${}^s \mathcal{I}$  resp.) is a classical (co-)tilting module for any simple reflection  $s \in W$ .*
3.  *$C_w({}^{w_0} \mathcal{P}) \cong {}^{w^{-1} w_0} \mathcal{I}$  and  $K_w({}^{w_0} \mathcal{I}) \cong {}^{w^{-1} w_0} \mathcal{P}$ . In particular,  ${}^{w_0} \mathcal{P} \cong {}^{w_0} \mathcal{I} \cong \mathcal{T}$  is the characteristic (co)tilting module in  $\mathcal{O}_0$ .*

**Question 1.6.** According to [AR, Theorem 5.4] every generalized tilting module  $T$  for an associative algebra  $A$  corresponds to a resolving and contravariantly finite subcategory in  $A\text{-mod}$

consisting of all  $A$ -modules admitting a finite coresolution by  $\text{Add}(T)$ . What are the subcategories of  $\mathcal{O}_0$ , which correspond to the various generalized tilting objects from above?

**2. Preliminary properties of our functors**

In this section we collect some fundamental statements concerning natural transformations between our functors. As a corollary we get a short argument for the existence of an isomorphism  $T \cong G'$  (which was originally proved in [KM, Theorem 4]).

By [So2, Endomorphismensatz] we have  $\text{End}_{\mathfrak{g}}(P(w_0)) \cong \mathcal{C}$ , and thus we can define the functor  $\mathbb{V}: \mathcal{O}_0 \rightarrow \mathcal{C}\text{-mod}$ ,  $M \mapsto \text{Hom}_{\mathfrak{g}}(P(w_0), M)$ . Let  $\tilde{G}$  denote the right-adjoint of  $T$ , which exists by [AS, Section 4].

**Lemma 2.1.**  $\mathbb{V}\tilde{G} \cong \mathbb{V}$  and  $\tilde{G} \cong \text{ID}$  when restricted to projectives.

**Proof.** Note that  $TP(w_0) \cong P(w_0)$  and  $\text{End}_{\mathfrak{g}}(P(w_0))$  is given by the action of the center  $\mathcal{Z}$  of the universal enveloping algebra of  $\mathfrak{g}$  ([So2, Endomorphismensatz]). On the other hand, the action of  $\mathcal{Z}$  commutes naturally with  $T$  by definition. This allows us to fix a natural isomorphism  $T \cong \text{ID}$  on  $\text{Add}(P(w_0))$ . This ensures that (for any  $M \in \mathcal{O}_0$ ) the following isomorphisms are even morphisms of  $\mathcal{C}$ -modules:

$$\mathbb{V}M = \text{Hom}_{\mathfrak{g}}(P(w_0), M) \cong \text{Hom}_{\mathfrak{g}}(TP(w_0), M) \cong \text{Hom}_{\mathfrak{g}}(P(w_0), \tilde{G}M) = \mathbb{V}\tilde{G}M.$$

All the isomorphisms are natural and the first statement follows. Let  $\tilde{\mathbb{V}}$  denote the right-adjoint of  $\mathbb{V}$ . By [So2, Proposition 6] we have  $\tilde{\mathbb{V}}\mathbb{V} \cong \text{ID}$  on projectives and therefore also

$$\tilde{G} \cong \tilde{\mathbb{V}}\mathbb{V}\tilde{G} \cong \tilde{\mathbb{V}}\mathbb{V} \cong \text{ID},$$

since  $\tilde{G}$  preserves the category of projectives.  $\square$

We fix an isomorphism of functors  $\varphi: \text{ID} \cong \tilde{G}$  defined on the category of projectives. For  $M \in \mathcal{O}_0$  we choose a projective presentation

$$P_1 \xrightarrow{\gamma'} P_0 \xrightarrow{\gamma} M.$$

Then the left square of the following diagram commutes and induces the map  $\varphi_M$  as indicated:

$$\begin{array}{ccccc} \tilde{G}P_1 & \xrightarrow{\tilde{G}\gamma'} & \tilde{G}P_0 & \xrightarrow{\tilde{G}\gamma} & \tilde{G}M \\ \varphi_{P_1} \uparrow & & \varphi_{P_0} \uparrow & & \varphi_M \uparrow \\ P_1 & \xrightarrow{\gamma'} & P_0 & \xrightarrow{\gamma} & M \end{array}$$

**Lemma 2.2.** The maps  $\varphi_M$ ,  $M \in \mathcal{O}_0$ , define a natural transformation from  $\text{ID}$  to  $\tilde{G}$ .

**Proof.** First we have to check that  $\phi_M$  is independent of the chosen presentation. Let  $Q_1 \xrightarrow{\beta'} Q_0 \xrightarrow{\gamma} M$  be another projective presentation of  $M$ . Consider the commutative diagram:

$$\begin{array}{ccccc}
 \tilde{G}P_1 & \xrightarrow{\tilde{G}\gamma'} & \tilde{G}P_0 & \xrightarrow{\tilde{G}\gamma} & \tilde{G}M \\
 \uparrow \varphi_{P_1} & & \uparrow \varphi_{P_0} & & \uparrow h \\
 P_1 & \xrightarrow{\gamma'} & P_0 & \xrightarrow{\gamma} & M \\
 \uparrow \xi' & & \uparrow \xi & & \uparrow \\
 Q_1 & \xrightarrow{\beta'} & Q_0 & \xrightarrow{\beta} & M \\
 \downarrow \varphi_{Q_1} & & \downarrow \varphi_{Q_0} & & \downarrow h' \\
 \tilde{G}Q_1 & \xrightarrow{\tilde{G}\beta'} & \tilde{G}Q_0 & \xrightarrow{\tilde{G}\beta} & \tilde{G}M
 \end{array}$$

where the projectivity of  $Q_1$  and  $Q_0$  is used to get  $\xi'$  and  $\xi$  such that the diagram is commutative. Since  $\xi$  is a map between projectives, we obtain  $\tilde{G}\xi \circ \varphi_{Q_0} = \varphi_{P_0} \circ \xi$ . Hence

$$h' \circ \beta = \tilde{G}\beta \circ \varphi_{Q_0} = \tilde{G}\gamma \circ \tilde{G}\xi \circ \varphi_{Q_0} = \tilde{G}\gamma \circ \varphi_{P_0} \circ \xi = h \circ \gamma \circ \xi = h \circ \beta,$$

by the commutativity of the diagram. Since  $\beta$  is surjective, we obtain  $h = h'$ . Hence,  $\phi_M$  is well-defined. The naturality follows by standard arguments.  $\square$

**Proposition 2.3.** *G is right adjoint to T. In particular, there exists a natural transformation  $T \rightarrow \text{ID}$  non-vanishing on Verma modules.*

**Proof.** Lemma 2.2 implies the existence of a non-trivial natural transformation  $T \rightarrow \text{ID}$  as assumed in [AS, Proposition 5.4]. The statement now follows from [AS, Proposition 5.4] and [KM, Lemma 1].  $\square$

**Proposition 2.4.**

- (1)  $(T, G)$  is an adjoint pair of functors. The adjunction morphism  $\text{adj}_T : TG \rightarrow \text{ID}$  is injective with cokernel  $Z$ , and the adjunction morphism  $\text{adj}^T : \text{ID} \rightarrow GT$  is surjective with kernel  $Z'$ .
- (2)  $(C, K)$  is an adjoint pair of functors. The adjunction morphism  $\text{adj}_C : CK \rightarrow \text{ID}$  is injective with cokernel  $\hat{Z}$ , and the adjunction morphism  $\text{adj}^C : \text{ID} \rightarrow KC$  is surjective with kernel  $\hat{Z}'$ .
- (3) The functors  $TG$  and  $GT$  preserve both surjections and injections (but are neither left nor right exact).
- (4) The functors  $CK$  and  $KC$  preserve both surjections and injections (but are neither left nor right exact).

**Remark 2.5.** The twisting functor  $T$  can be described and generalized as follows (this was also observed by W. Soergel): We consider  $\mathcal{O}_0$  as the category  $\text{mod-}A$  of finitely generated right modules over  $A = \text{End}_{\mathfrak{g}}(\mathcal{P})$  with endofunctor  $T$ . To each simple object  $L(w)$  we have the corresponding primitive idempotent  $e_w \in A$ . Let  $e$  be the sum of all  $e_w$  taken over all  $w$  such that

$TL(w) \neq 0$  (see [AS, Proposition 5.1]) and define  $\tilde{T} = - \otimes_A AeA : \text{mod-}A \rightarrow \text{mod-}A$ . Using the definitions and [AS, Proposition 5.3, Theorem 2.2 and Corollary 5.2] we get  $T(A_A) \cong \tilde{T}(A_A)$  and the inclusion  $AeA \hookrightarrow A$  induces a non-trivial element  $\phi \in \text{Hom}(\tilde{T}, \text{ID})$ . Applying [KM, Lemma 1] one gets  $\tilde{T} \cong T$  as endofunctors of  $\text{mod-}A$ . This description allows a generalization of twisting functors to a very general setting. The definitions immediately show that the cokernel of  $\phi_M$  is always the largest quotient of  $M$ , such that  $\text{Hom}_A(eA, M) = 0$  and one easily derives  $\tilde{T}^3 \cong \tilde{T}^2$ . However, if  $\tilde{G}$  denotes the right adjoint of  $\tilde{T}$ , then the adjunction morphism  $\tilde{T}\tilde{G} \rightarrow \text{ID}$  does not need to be injective in general.

**Proof of Proposition 2.4.** In this proof for  $M \in \mathcal{O}_0$  we denote by  $[M]$  the class of  $M$  in the Grothendieck group of  $\mathcal{O}_0$ .

The first part is proved in [AS, Section 5]. For the part (3) it is enough to show that both,  $\text{TG}$  and  $\text{GT}$ , preserve surjections. Assume  $f \in \text{Hom}(M, N)$  for some  $M, N \in \mathcal{O}_0$  is surjective. The adjunction morphism  $\text{adj}_N^T$  is surjective. Then  $\text{adj}_N^T \circ f = \text{GT}(f) \circ \text{adj}_M^T$  is surjective; in particular, so is  $\text{GT}(f)$ .

Let  $\text{im}$  be the image of  $G(f)$ . Then  $T(G(f)) : \text{TGM} \rightarrow T(\text{im})$  is surjective and so is  $T(i) : T(\text{im}) \rightarrow \text{TGN}$ , since the cokernel of  $i : \text{im} \hookrightarrow \text{GN}$  is annihilated by  $T$ . The composition of both surjections is exactly  $\text{TG}(f)$  and so we are done: part (3) follows.

Concerning statement (4), it is enough to prove the claim for  $\text{CK}$ . Let us first show that  $\text{CK}$  preserves inclusions. Let  $M \xrightarrow{f} N \xrightarrow{g} L$  be a short exact sequence in  $\mathcal{O}_0$ . Applying  $\text{K}$  gives an exact sequence  $\mathbf{S}$  of the form  $\text{KM} \hookrightarrow \text{KN} \rightarrow L'$  where  $L'$  is a submodule of  $\text{KL}$ . By definition of  $\text{K}$ , the socle of  $\text{KL}$ , and hence also of  $L'$ , contains only simple modules not annihilated by  $\theta_s$ , hence  $\mathcal{L}_1\text{C}(L') = 0$  by [MS1, Section 5]. In particular,  $\text{CS}$  is exact, and therefore  $\text{CK}(f)$  is an inclusion.

On the other hand, applying  $\text{K}$  to  $M \xrightarrow{f} N \xrightarrow{g} L$  yields an exact sequence  $\mathbf{T}$  of the form  $\text{KM} \hookrightarrow \text{KN} \rightarrow \text{KL} \rightarrow X$ , where  $\text{KX} = \text{CX} = 0$  by [MS1, Proposition 5.3]. Applying the right exact functor  $\text{C}$  to  $\mathbf{T}$  and using  $\text{CX} = 0$  we obtain that  $\text{CK}(g)$  is a surjection. This shows part (4).

By [MS1, Section 5] the adjunction morphism defines an isomorphism  $\text{CK} \cong \text{ID}$  when restricted to modules having a dual Verma flag. Let  $M \in \mathcal{O}_0$  with injective cover  $i : M \hookrightarrow I$ . Let  $\text{adj} = \text{adj}_C$  for the moment. Then  $i \circ \text{adj}_M = \text{adj}_I \circ \text{CK}(i)$ . The latter is injective, hence  $\text{adj}_M$  has to be injective as well. Note that  $[\text{CK}(M)] = [\theta\text{K}(M)] - [\text{K}(M)] = [\theta^2(M)] - [\theta(M)] - [\text{K}(M)] = [\theta(M)] - [\text{K}(M)]$  for any  $M \in \mathcal{O}_0$ . Hence  $[M] - [\text{CK}(M)] = [\check{Z}(M)]$ . Dual statements hold for  $\text{adj}_C$ . Part (2) follows.  $\square$

The following result is surprising in comparison with Proposition 2.3 (note that the argument of Lemma 2.1 does not work if we replace  $\tilde{G}$  by  $\text{K}$  as  $\text{K}$  does not commute with the action of the center of  $\mathcal{O}_0$ ).

**Proposition 2.6.**

1. There is **no** natural transformation  $c : \text{C} \rightarrow \text{ID}$  non-vanishing on Verma modules.
2. There is **no** natural transformation  $k : \text{ID} \rightarrow \text{K}$  non-vanishing on Verma modules.

**Proof.** We consider the defining sequence  $0 \rightarrow \text{K} \xrightarrow{i} \theta \xrightarrow{\text{adj}^s} \text{ID}$ . It induces an exact sequence  $\text{Hom}(\text{ID}, \text{K}) \xrightarrow{i \circ} \text{Hom}(\text{ID}, \theta) \xrightarrow{\circ \text{adj}^s} \text{Hom}(\text{ID}, \text{ID})$ . We have  $\text{Hom}(\text{ID}, \theta) \cong \mathcal{C}$ , more precisely, the morphism space is generated by the adjunction morphism  $\text{adj}_s$  and the center  $\mathcal{C}$  of the category

$\mathcal{O}_0$  (see [Bac, Theorem 4.9]). If now  $\phi \in \text{Hom}(\text{ID}, \mathbf{K})$  does not vanish on Verma modules, then, up to a scalar,  $i \circ \phi = \text{adj}_s$ , hence  $\text{adj}_s^i \circ i \circ \phi = \text{adj}_s^i \circ \text{adj}_s \neq 0$  (see [Be, Sections 2 and 3] or [An, Lemma 2.2]). This contradicts the exactness of the original exact sequence.  $\square$

### 3. Proof of Theorem 2

Theorem 2(1) follows immediately from [MS1, Section 4] and the definition of  $\hat{Z}$ .

**Proof of Theorem 2(2).** Let  $\mathcal{H}$  be the category of Harish-Chandra bimodules with generalized trivial central character from both sides (see [So3]). By [BG, 5.9], the category  $\mathcal{O}_0$  is equivalent to the full subcategory of  $\mathcal{H}$  given by objects having trivial central character from the right-hand side. Let  $\theta_s^r : \mathcal{H} \rightarrow \mathcal{H}$  denote the right translation through the  $s$ -wall. When considering  $\mathcal{O}_0$  as a subcategory of  $\mathcal{H}$ , the functor  $G$  is defined as the kernel of the adjunction morphism  $\theta_s^r \xrightarrow{\text{adj}} \text{ID}$  (see [Jo1, Proposition 3.2]). Using the Snake Lemma we obtain that  $\mathcal{R}^1 G$  is isomorphic to the cokernel of  $\theta_s^r \xrightarrow{\text{adj}} \text{ID}$ . Note that  $\mathcal{R}^1 G(M)$  is locally  $\mathfrak{g}^s$ -finite ([AS, Corollary 5.9]). Since the top of  $\theta_s^r M$  is  $s$ -free, we obtain that it is maximal with this property. Hence  $\mathcal{R}^1 G \cong Z$  and, in particular,  $\mathcal{R}^1 G \cong \text{ID}$  on  $\mathcal{O}_0^s$ .  $\square$

**Remark 3.1.** Theorem 2(2) has independently been proved in [Kh, Proposition 20] by completely different arguments.

**Proof of Theorem 2(3).** Recall from above that the functor  $Z$  is isomorphic to the cokernel of the  $\theta_s^r \xrightarrow{\text{adj}} \text{ID}$ . Let  $M \in \mathcal{O}_0$  and  $P_2 \xrightarrow{h} P_1 \xrightarrow{f} P_0 \twoheadrightarrow M$  be the first three steps of a projective resolution of  $M$ . Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 GP_2 & \longrightarrow & GP_1 & \longrightarrow & GP_0 & \longrightarrow & GM \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \theta_s^r P_2 & \longrightarrow & \theta_s^r P_1 & \longrightarrow & \theta_s^r P_0 & \twoheadrightarrow & \theta_s^r M \\
 \downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \text{adj} \\
 P_2 & \xrightarrow{h} & P_1 & \xrightarrow{f} & P_0 & \twoheadrightarrow & M \\
 \downarrow p_2 & & \downarrow p_1 & & \downarrow p_0 & & \\
 ZP_2 & \xrightarrow{\bar{h}} & ZP_1 & \xrightarrow{\bar{f}} & ZP_0 & & 
 \end{array}$$

The Snake Lemma gives a natural surjection  $GM \twoheadrightarrow Z(P_1/\text{Ker } f)$ . We claim that this even induces a natural surjection  $GM \twoheadrightarrow \text{Ker } \bar{f}/\Im \bar{h}$ . Indeed, if  $x \in ZP_1$  such that  $\bar{f}(x) = 0$  and  $x \notin \Im \bar{h}$ , we can choose  $y \in P_2$  such that  $p_2(y) = x$ . If  $f(y) = 0$  then  $y = h(z)$  for some  $z \in P_3$ ; hence  $x = p_2 \circ h(z) = \bar{h} \circ p_3(z)$ , which is a contradiction. Therefore,  $f(y) \neq 0$  and  $Z(P_1/\text{Ker } f)$  surjects onto  $\text{Ker } \bar{f}/\Im \bar{h}$  providing a surjection  $\Phi : G \twoheadrightarrow \mathcal{L}_1 Z$ . We have to show that  $\Phi$  induces an isomorphism  $Q \cong \mathcal{L}_1 Z$ .

**Claim 3.2.**

$$\mathcal{L}_1 Z \Delta(x) \cong \begin{cases} \Delta(sx)/\Delta(x), & \text{if } x > sx, \\ 0, & \text{if } x < sx. \end{cases}$$

In particular,  $\Phi$  induces an isomorphism  $Q \cong \mathcal{L}_1 Z$  on Verma modules.

**Proof.** We prove the claim by induction on  $l(x)$ . It is certainly true for  $x = e$ . Assume it to be true for  $x$  and let  $t$  be a simple reflection such that  $xt > x$ . The short exact sequence  $\Delta(x) \hookrightarrow \theta_t \Delta(x) \rightarrow \Delta(xt)$  induces an exact sequence

$$\mathcal{L}_1 Z \Delta(x) \hookrightarrow \mathcal{L}_1 Z \theta_t \Delta(x) \rightarrow \mathcal{L}_1 Z \Delta(xt) \rightarrow Z \Delta(x) \rightarrow Z \theta_t \Delta(x) \rightarrow Z \Delta(xt). \tag{3.1}$$

If  $x > sx$  then  $l(sxt) \leq l(sx) + 1 = l(x) < l(xt)$ . Since  $x > sx$  and  $sxt > xt$ , we have  $Z \Delta(x) = Z \Delta(xt) = Z \theta_t \Delta(x) = 0$ . By induction hypothesis, (3.1) reduces to

$$\Delta(sx)/\Delta(x) \hookrightarrow \theta_t(\Delta(sx)/\Delta(x)) \rightarrow \mathcal{L}_1 Z \Delta(xt),$$

implying  $\mathcal{L}_1 Z \Delta(xt) \cong \Delta(sxt)/\Delta(xt)$ .

If  $sx > x$  and  $sxt < xt$  then  $xt > x$  implies  $sxt = x$ . Hence  $Z \Delta(xt) = Z \theta_t \Delta(x) = Z \theta_t \Delta(x) = 0$ , and  $\mathcal{L}_1 Z \theta_t \Delta(x) \cong \theta_t \mathcal{L}_1 Z \Delta(x) = 0$  by induction hypothesis. We get

$$\mathcal{L}_1 Z \Delta(xt) \cong Z \Delta(x) \cong \Delta(x)/\Delta(sx) = \Delta(sxt)/\Delta(xt).$$

If  $sx > x$  and  $sxt > xt$  then we have  $(\mathcal{L}_1 Z) \theta_t \Delta(x) \cong \theta_t (\mathcal{L}_1 Z) \Delta(x) = 0$  by induction hypothesis, and the last terms of (3.1) form the exact sequence

$$\Delta(x)/\Delta(sx) \hookrightarrow \theta_t \Delta(xt)/\Delta(sxt) \rightarrow \Delta(xt)/\Delta(sxt).$$

This implies that  $\mathcal{L}_1 Z \Delta(xt) = 0$  and the claim follows.  $\square$

**Claim 3.3.**  $\Phi$  induces an isomorphism  $Q \cong \mathcal{L}_1 Z$  on modules having a Verma flag.

**Proof.** Let  $\mathbf{S}$  be a short exact sequence of modules having a Verma flag; then we have a commutative diagram  $\mathbf{S} \xrightarrow{\text{gs}} G(\mathbf{S}) \rightarrow Q(\mathbf{S}) \rightarrow \mathcal{L}_1 Z(\mathbf{S})$ , where the composition of the last two maps is  $\Phi$ . Since  $g$  is an injection,  $Q(\mathbf{S})$  is left-exact by the Snake Lemma. The sequence  $\mathcal{L}_2 Z(\mathbf{S})$  is identical zero, because  $\mathcal{L}_2 Z \cong Z'$  by [EW, Theorem 4.3]. Therefore,  $\mathcal{L}_1 Z(\mathbf{S})$  is left-exact. The Five Lemma implies the claim.  $\square$

**Claim 3.4.**  $\Phi$  induces an isomorphism  $Q \cong \mathcal{L}_1 Z$  on modules having a dual Verma flag.

**Proof.** Let  $\mathbf{S}$  be a short exact sequence of modules having a dual Verma flag; then  $G(\mathbf{S})$  is exact ([AS, Theorem 2.2]) and hence  $Q(\mathbf{S})$  is right exact. On the other hand  $\mathcal{L}_1 Z(\mathbf{S})$  is right exact as well, since  $ZM = 0$  for any module having a dual Verma flag. The Five Lemma completes the proof.  $\square$

Let  $M \in \mathcal{O}_0$ . By Wakamatsu’s Lemma ([Wa, Lemma 1.2]) there exists a short exact sequence  $\mathbf{S}: Y \hookrightarrow X \rightarrow M$ , for a certain  $X$  having a Verma flag and some  $Y$  with a dual Verma flag. Since

$\mathcal{R}^1G(Y) = 0$  ([AS, Theorem 2.2]), the sequence  $G(\mathbf{S})$  is exact, and hence  $Q(\mathbf{S})$  is right exact. Since  $ZY = 0$ ,  $\mathcal{L}_1Z(\mathbf{S})$  is right exact, as well. The Five-Lemma implies that  $\Phi$  induces an isomorphism  $QM \cong \mathcal{L}_1ZM$ . We immediately get  $Q \cong Q'$ , since  $\mathcal{L}_1Z \cong (\mathcal{L}_1Z)'$  by [EW, Theorem 4.3]. Theorem 2(3) follows.  $\square$

**Proof of Theorem 2(4).** Recall the isomorphism  $\mathcal{R}^1G \cong Z$  from the first part. By [AS, Theorem 2.2 and Theorem 4.1], we have  $\mathcal{R}^iG = 0$  for all  $i > 1$ . Since  $G(d\Delta(e))$  is acyclic for  $G$  ([AS, Theorems 2.2 and 2.3]), we have the Grothendieck spectral sequence  $\mathcal{R}^pG(\mathcal{R}^qG(X)) \Rightarrow \mathcal{R}^{p+q}G^2(X)$ . We immediately get  $\mathcal{R}^1G^2 \cong ZG$  and  $\mathcal{R}^2G^2 \cong Z^2 \cong Z$  and  $\mathcal{R}^iG^2 = 0$  for  $i > 2$ . This proves the first part of Theorem 2(4).

The second part is proved by analogous arguments provided that we know that  $K(I)$  is  $K$ -acyclic for any injective object  $I$ . This is equivalent to the statement that the head of  $K(I)$  contains no composition factor  $L(w)$  with  $ws > w$ . There is a short exact sequence  $X \hookrightarrow Y \twoheadrightarrow I$ , where  $X$  has a dual Verma flag and  $Y$  is the projective-injective cover of  $I$ . Using that  $K$  is exact on sequences of modules having a dual Verma flag, we get a surjection  $K(Y) \twoheadrightarrow K(I)$ . In particular, it follows that the head of  $K(I)$  is embedded into the head of  $K(Y) \in \text{Add}(P(w_0))$ . The latter contains only copies of  $L(w_0)$ . This completes the proof.  $\square$

#### 4. Proof of Theorem 3

We start by verifying the indicated relations. By duality, it is enough to prove every second statement.

*The isomorphism  $TGT \cong T$ :* Evaluating the exact sequence of functors

$$0 \rightarrow TG \hookrightarrow \text{ID} \twoheadrightarrow Z \rightarrow 0, \tag{4.1}$$

from Proposition 2.4(1) at  $T$  gives rise to the exact sequence  $0 \rightarrow TGT \hookrightarrow T \twoheadrightarrow ZT \rightarrow 0$ . Further  $ZT = 0$ , as the head of any  $T(M)$  is  $s$ -free by [AS, Corollary 5.2], hence we obtain  $TGT \cong T$ .

*The isomorphism  $G^3 \cong G^2$*  is proved in [Jo1, Lemma 3.6].

*The isomorphism  $T^2G \cong T^2$ :* Applying  $T$  to (4.1) gives the exact sequence

$$(\mathcal{L}_1T)Z \rightarrow T^2G \rightarrow T \twoheadrightarrow TZ \rightarrow 0. \tag{4.2}$$

Theorem 2 gives  $\mathcal{L}_1T \cong Z'$ , in particular,  $T(\mathcal{L}_1T)Z = 0$  ([AS, Corollaries 5.8 and 5.9]). Moreover  $TZ = 0$ . This means that we can apply  $T$  to (4.2) once more to obtain an isomorphism  $T^3G \cong T^2$ . Since  $T^3 \cong T^2$  we finally get  $T^2G \cong T^2$ .

*The isomorphism  $TG^2 \cong GT^2$ :* Evaluating the adjunction morphism  $\text{adj}_T : TG \hookrightarrow \text{ID}$  at  $GT^2$  we get  $TGGT^2 \cong TG^2 \hookrightarrow GT^2$ . Evaluating  $\text{ID} \twoheadrightarrow GT$  at  $TG^2$  we obtain  $TG^2 \twoheadrightarrow GTTG^2 \cong GT^2$  and hence  $TG^2 \cong GT^2$ .

To complete the proof it is now enough to show that all the functors from  $\mathcal{S}$  are not isomorphic (Green’s relation are easily checked by direct calculations). An easy direct calculation gives the following images under our functors:

ID	G	T	G <sup>2</sup>	T <sup>2</sup>	TG	GT	GT <sup>2</sup>
$\Delta(s)$	$\Delta(e)$	$T\Delta(s)$	$\Delta(e)$	$T\Delta(s)$	$\Delta(s)$	$\Delta(s)$	$\Delta(s)$
$\Delta(e)$	$\Delta(e)$	$\Delta(s)$	$\Delta(e)$	$T\Delta(s)$	$\Delta(s)$	$\Delta(e)$	$\Delta(s)$
$T\Delta(s)$	$\Delta(s)$	$T\Delta(s)$	$\Delta(e)$	$T\Delta(s)$	$T\Delta(s)$	$\Delta(s)$	$\Delta(s)$

The claim follows.

**5. Proof of Theorem 4**

By duality it is enough to prove every second relation.

*The isomorphism  $CKC \cong C$ :* The proof is analogous to that of  $TGT \cong T$  in Section 4.

*The isomorphism  $C^3K \cong C^2$ :* Applying  $C$  to the short exact sequence  $CK \hookrightarrow ID \twoheadrightarrow \hat{Z}$  produces a short exact sequence  $X \hookrightarrow C^2K \twoheadrightarrow C$ , where  $CX = 0$ . Applying  $C$  once more we obtain the desired isomorphism.

*The isomorphism  $C^2K^2C \cong C^2K$ :* Applying  $K$  to the short exact sequence  $\hat{Z}' \hookrightarrow ID \twoheadrightarrow KC$  produces a short exact sequence  $K \hookrightarrow K^2C \twoheadrightarrow X$ , where  $KX = CX = 0$ . Applying now  $C$  gives rise to  $Y \hookrightarrow CK \twoheadrightarrow CK^2C$ , where  $KY = CY = 0$ . Applying  $C$  once more gives the isomorphism.

*The isomorphism  $KC^2K^2 \cong CK^2$ :* Evaluating the short exact sequence  $\hat{Z}' \hookrightarrow ID \twoheadrightarrow KC$  at  $CK^2$  we obtain the short exact sequence  $\hat{Z}'CK^2 \hookrightarrow CK^2 \twoheadrightarrow KC^2K^2$ . The statement follows if we show that  $\hat{Z}'CK^2 = 0$ . The injection  $CK \hookrightarrow ID$  gives an injection  $CK^2 \hookrightarrow K$ . On the other hand,  $\hat{Z}'K = 0$  since, by the definition of  $K$ , any composition factor in the socle of  $KM$  is not annihilated by  $\theta$ . As  $CK^2 \hookrightarrow K$  we get that  $\hat{Z}'CK^2 = 0$  as well.

It is easy to see that, using the relations we have just proved, any product of  $C$  and  $K$  can be reduced to one of the elements of  $\hat{S}$ .

Assume now that  $s$  does not correspond to an  $\mathfrak{sl}_2$ -direct summand of  $\mathfrak{g}$ . We do a case-by-case analysis to show that all functors in  $\hat{S}$  are different. We start with the following general observation.

**Lemma 5.1.** *Assume that  $X: \mathcal{O}_0 \rightarrow \mathcal{O}_0$  is left exact,  $X(P(w_0)) \cong P(w_0)$ , and there is a natural transformation  $\varphi: ID \rightarrow X$  on the category of projective-injective modules in  $\mathcal{O}_0$ , such that  $\varphi_{P(w_0)}$  is an isomorphism. Then  $X$  fixes the isoclasses of projectives.*

**Proof.** Let  $P$  be projective. Consider an exact sequence  $P \hookrightarrow I_0 \twoheadrightarrow I_1$ , where  $I_0$  and  $I_1$  are projective-injective. Then the square on the right-hand side in the following diagram with exact rows commutes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P & \xrightarrow{f} & I_0 & \xrightarrow{g} & I_1 \\
 & & \downarrow h & & \downarrow \varphi_{I_0} & & \downarrow \varphi_{I_1} \\
 0 & \longrightarrow & XP & \xrightarrow{X(f)} & XI_0 & \xrightarrow{X(g)} & XI_1
 \end{array}$$

and hence we obtain the induced map  $h$ , which is an isomorphism by the Five Lemma.  $\square$

*All  $K^i$  are different.* We fix a simple reflection  $t$  such that  $st \neq ts$ . By a direct calculation one obtains that  $K^i P(t)$ ,  $i > 0$ , is not projective, in particular,  $K^i$  does not preserve projectives in  $\mathcal{O}_0$ . Now any isomorphism  $\varphi: K^i \rightarrow K^j$ ,  $i < j$ , induces a natural transformation  $ID \rightarrow K^{j-i}$  on the category  $K^i(\mathcal{O}_0)$ , which contains the subcategory of projective-injective modules in  $\mathcal{O}_0$ . It follows from Lemma 5.1 that  $K^{j-i}$  preserves the category of projective modules in  $\mathcal{O}_0$ , a contradiction.

*All  $C^i$  are different* by dual arguments.



We consider now  $\hat{S}$  as a  $\mathbb{Z}$ -graded monoid with  $\deg(C) = 1$  and  $\deg(K) = -1$ . This is possible as the defining relations are homogeneous with respect to this grading. It follows from the relations that for any  $X \in \hat{S}$  and for all  $i$  large enough we have  $C^i X \cong C^j$  for some  $C^j$ . Since we have already shown that all  $C^j$  are different, it follows that the elements of  $\hat{S}$  having different degree are not isomorphic. In particular, changing the exponent  $i$  in the expression for  $X \in \hat{S}$  gives a non-isomorphic functor. The rest will be checked case-by-case.

$K^i$  is not isomorphic to  $CK^{i+1}$  for  $i > 0$ : We have  $CK^{i+1}\Delta(e) \cong \Delta(s)$  and  $K^i\Delta(e) \cong \Delta(e)$  for all  $i$ .

$K^i$  is not isomorphic to  $C^2K^{i+2}$  for  $i > 0$ : We have  $K^{i+2}\Delta(e) \cong \Delta(e) \not\cong C\Delta(s) \cong C^2K^{i+2}\Delta(e)$ .

$K$  is not isomorphic to  $K^2C$ , since  $Kd\Delta(e) \not\cong K^2d\Delta(e) \cong K^2Cd\Delta(e)$ .

We proved that  $K^i$  (where  $i > 0$ ) is not isomorphic to any other functor in the list. By duality, the same holds for  $C^i$ .

$KC$  is not isomorphic to  $CK$ : Assume, they are isomorphic, then  $C \cong CKC \cong CCK \cong C^2K$  which we have proved to be wrong.

$KC^i$  is not isomorphic to  $K^2C^{i+1}$  for  $i > 0$ : We have  $KC^i d\Delta(e) \cong Kd\Delta(e) \not\cong K^2d\Delta(e) \cong K^2C^{i+1}d\Delta(e)$ .

$KC^2$  is not isomorphic to  $C^2K$ : We have  $KC^2d\Delta(e) \cong Kd\Delta(e) \cong d\Delta(s)$  and  $C^2Kd\Delta(e) \cong C^2d\Delta(s) \cong Cd\Delta(e) \cong d\Delta(e)$ .

$KC$  is not isomorphic to  $KC^2K$ : Assume, they are isomorphic. Then  $K \cong KCK \cong KC^2K^2 \cong CK^2$ , which we know is wrong.

Hence the functors  $KC^i$ ,  $i > 0$ , differ from all the others in the list. Duality gives the same property for  $CK^i$ .

$K^2C^2$  is not isomorphic to  $C^2K^2$  and  $K^2C$  is not isomorphic to  $C^2K^3$ : By definition the socle of  $K^2C^2M$  contains only composition factors which are not annihilated by  $\theta$  (for any  $M \in \mathcal{O}_0$ ). On the other hand  $C^2K^2\Delta(e) \cong C^2\Delta(e) \cong C\Delta(s)$  is an extension of  $\Delta(s)$  with  $\Delta(e)/\Delta(s)$ . In particular, the socle is  $\mathfrak{g}^s$ -finite. The same argumentation applies to the second pair.

$K^2C^2$  is not isomorphic to  $KC^2K$ : Assume, they are isomorphic then  $K^2C \cong K^2C^2K \cong KC^2K^2 \cong CK^2$ . We have already proved that this is not possible.

Hence  $K^2C^i$ ,  $i > 0$ , (and dually  $C^2K^i$ ) differs from all other functors from the list. And therefore, any two functors from the list are not isomorphic.

The statements concerning Green's relations and idempotents are obtained by a direct calculation.

### 6. Proof of Theorem 5

It will be enough to prove roughly half of the statements. The other half will follow by duality.

**Lemma 6.1.** *All maps indicated in the diagram as inclusions are injective; and all projections are surjective.*

**Proof.** By duality, it is enough to prove the statement for inclusions. The injectivity of  $z'$ ,  $i'$ ,  $i'_T$ ,  $z'_T$ ,  $z'_{T^2}$  is given by definition. For the maps  $G(g')$  and  $G(g)$  the statement follows from the left exactness of  $G$  and the fact that  $G$  is zero on locally  $\mathfrak{g}^s$ -finite modules. The map  $Z'(i_T)$  is injective because of the left exactness of  $Z'$  and the injectivity of  $i_T$ . The injectivity of  $a'$  follows from [AS, Proposition 5.6], since  $a'$  is up to a non-zero scalar the adjunction morphism  $\text{adj}_T: TG \rightarrow \text{ID}$ .

Let us now prove the statement for  $ZG(g)$ . By definition of  $Q$  we have the following exact sequence of functors:  $G \hookrightarrow G^2 \twoheadrightarrow QG$ . It gives rise to the exact sequence

$$0 \cong \mathcal{L}_1 Z(QG) \rightarrow ZG \xrightarrow{ZG(g)} ZG^2 \xrightarrow{G(\text{pg})} ZQG \cong QG.$$

This implies that  $ZG(g)$  is injective.

**Claim 6.2.**  $T^2(g) : T^2 \rightarrow T^2G$  is an isomorphism. In particular  $m'$  is well-defined and injective.

**Proof.** Let  $K$  and  $K'$  be defined by the following exact sequence of functors:

$$\begin{array}{ccccc} K \hookrightarrow & \text{ID} & \xrightarrow{g} & G & \twoheadrightarrow K' \\ & \searrow q & & \nearrow j & \\ & & \text{im}(g) & & \end{array}$$

Since  $T^2K = 0$  we get an isomorphism  $T^2(q) : T^2 \rightarrow T^2(\text{im}(g))$  where  $\text{im}(g)$  denotes the image of  $g$ . Applying  $T$  to the second short exact part gives a short exact sequence  $\tilde{K} \hookrightarrow T(\text{im}(g)) \rightarrow TG$  for some  $\tilde{K}$  such that  $\tilde{K}(M)$  is locally  $\mathfrak{g}^s$ -finite for all  $M \in \mathcal{O}_0$ . Applying  $T$  once more gives an isomorphism  $T^2(j) : T^2(\text{im}(g)) \rightarrow T^2G$  since  $T\tilde{K} = 0$ . Composing  $T^2(j) \circ T^2(q) = T^2(g)$  implies the first statement. The injectivity of  $m'$  follows from the injectivity of  $i_{TG}$ .  $\square$

**Claim 6.3.** There exists a unique isomorphism  $h : TG^2 \rightarrow GT^2$  such that

$$g \circ g' = G(g' \circ g'_T) \circ h \circ T(g_G \circ g).$$

**Proof.** We start proving uniqueness. If  $h$  and  $\tilde{h}$  are two such morphisms, then  $h - \tilde{h}$  induces a morphism from  $Z'T$  to  $G$  since  $Z'T = \ker(g \circ g')$  (this will be proved later in this section). However,  $\text{Hom}(Z'T, G) = 0$  as the socle of  $GM$  is  $s$ -free and  $Z'TM$  is  $\mathfrak{g}^s$ -finite for any  $M \in \mathcal{O}_0$ .

It is left to prove the existence. Note that  $TG^2 \cong GT^2$  by Theorem 3. For any  $h \in \text{End}(TG^2, GT^2)$  the natural transformation  $\varphi(h) = G(g' \circ g'_T) \circ h \circ T(g_G \circ g)$  belongs to  $\text{Hom}(T, G)$  and, comparing the action on the projective-injective module  $P(w_0) \in \mathcal{O}_0$  we see that  $\varphi$  is injective, hence an isomorphism (by the independent Theorem 6). The claim follows.  $\square$

We proceed with the map  $Q'T(g)$ . Let  $M \in \mathcal{O}_0$  and consider the map  $g_M : M \rightarrow GM$ . The map  $T(g_M)$  fits into the exact sequence  $Q'M \rightarrow TM \rightarrow TGM$ . To calculate  $Q'T(g)$  we consider the following commutative diagram:

$$\begin{array}{ccccc} Q'Q'M = 0 & \longrightarrow & Q'TM & \xrightarrow{Q'T(g_M)} & Q'TGM \\ \downarrow & & \downarrow & & \downarrow \\ TQ'M = 0 & \longrightarrow & T^2M & \xrightarrow{T^2(g_M)} & T^2GM \\ \downarrow & & \downarrow & & \downarrow \\ TGQ'M = 0 & \longrightarrow & TGT M & \xrightarrow{TGT(g_M)} & TGTGM \end{array}$$

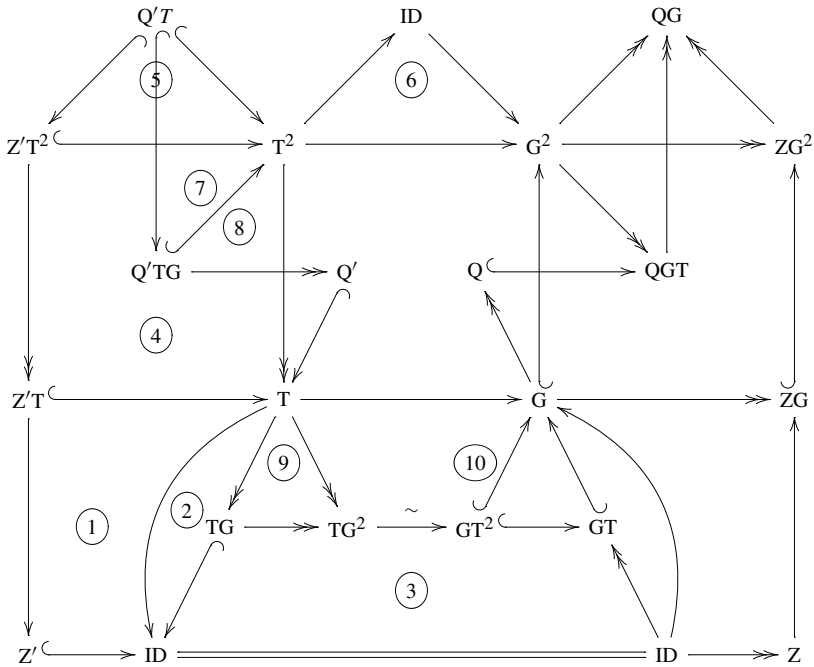


Fig. 2. Schematic picture of the diagram from Theorem 5.

where the first row is the kernel sequence and hence is exact. It follows that  $Q'T(g)$  is injective. The injectivity of  $Q(g \circ g')$  is proved by analogous arguments. This completes the proof of Lemma 6.1.  $\square$

**Lemma 6.4.** *All configurations containing only solid arrows commute.*

**Proof.** We use the notations from Fig. 2. The squares (2), (6), (9), and (10) commute by definition. The commutativity of (3) follows from the commutativity of (2), (9), and (10). The squares (1), and (4) commute since  $z'$  is a natural transformation and  $Z'$  and  $Z'T$  are functors (note that  $g'_T = T(g')$ ). The commutativity of (5) reads  $i_T = z'_{T^2} \circ Z'(i_T)$ , which is true as  $Z' = ID$  on  $g^s$ -finite modules. The commutativity of (7) reads  $i_T = m' \circ Q'T(g)$ , which is equivalent to  $T^2(g) \circ i_T = i_{TG} \circ Q'T(g)$ , the latter being true as  $i$  is a natural transformation. Commutativity of (8) means  $i \circ Q'(a') = g'_T \circ m'$ , which is equivalent to  $i \circ Q'(a') = g'_T \circ (T^2(g))^{-1} \circ i_{TG}$ . Since  $i$  is a natural transformation we have  $i \circ Q'(a') = T(a') \circ i_{TG}$  and our equality reduces to  $T(a') \circ i_{TG} = g'_T \circ (T^2(g))^{-1} \circ i_{TG}$ . To prove the latter it is enough to show that  $T(a') = g'_T \circ (T^2(g))^{-1}$ , which follows from  $g'_T = T(g')$  and the definition of  $a'$ . The remaining configurations commute by duality.  $\square$

To complete the proof of Theorem 5 it is left to prove the exactness of the indicated sequences. By duality, it is sufficient to prove the exactness of the sequences 1 to 10. The sequences 8 and 3 are exact by the definitions of  $a$  and  $Q$  respectively. The exactness of 4 follows from [AS,

Proposition 5.6]. The exactness of **7** follows from  $T(\mathfrak{g}') = \mathfrak{g}'_T$  and the exactness of the sequence, dual to **3**. Applying the left exact functor  $Z'$  to the short exact sequence **7** and using  $Z'Q' = Q'$  shows that **5** is exact. The exactness of **6** follows by comparison of characters from the facts that  $Q'T(\mathfrak{g})$  is an inclusion and  $Q'(a')$  is a surjection. The exactness of **10** follows by evaluating the exact sequence **8** at modules of the form  $GM$ .

Let us now show that **2** is exact. The cokernel  $\text{Coker}$  of  $g \circ g': T \rightarrow G$  is  $\mathfrak{g}^s$ -finite since already the cokernel of  $g$  is  $\mathfrak{g}^s$ -finite, see [Jo1, Lemma 3.10]. Further, for any  $M \in \mathcal{O}_0$  we have that  $Q(M)$  is the maximal  $\mathfrak{g}^s$ -finite quotient of  $GM$  since the head of  $TM$  is  $s$ -free. This implies the exactness of the sequence **2** and also of **9** at the term  $G$ . By uniqueness of the canonical maps the exactness in  $T$  follows by duality. Exactness of **1** follows by analogous arguments.

**7. Proof of Theorem 6**

We abbreviate  $\text{Hom}(X, Y) = H_{X,Y}$  for  $X, Y \in \mathcal{S}$ . By duality we have vector space isomorphisms  $H_{X,Y} \cong H_{Y',X'}$ .

**Proposition 7.1.**  $\text{End}(X) \cong \mathcal{C}$  as algebras for any  $X \in \mathcal{S}$ .

**Proof.** For  $X = \text{ID}$  the statement is well-known and follows from [So2, Endomorphismensatz and Struktursatz], since  $\text{End}(\text{ID}) \cong \mathcal{C} \cong \text{End}_{\mathfrak{g}}(P(w_0))$ . Note that  $GP(w_0) \cong TP(w_0) \cong P(w_0)$  (see [AS, Proposition 5.3]); hence  $XP(w_0) \cong P(w_0)$  for all  $X \in \mathcal{S}$ . This means that sending  $\varphi \in \text{End}(\text{ID})$  to  $X(\varphi)$  defines an injective algebra morphism from  $\mathcal{C}$  to  $\text{End}(X)$  for every  $X \in \mathcal{S}$ , as already the map  $\varphi_{P(w_0)} \mapsto X(\varphi_{P(w_0)})$  is injective. We only have to check the dimensions.

We claim that  $\Phi : \text{End}(T) \rightarrow \text{End}_{\mathfrak{g}}(TP(w_0))$ ,  $\phi \mapsto \phi_{P(w_0)}$ , is injective. Assume that  $\Phi(\phi) = 0$ . Let  $P \in \mathcal{O}_0$  be projective with injective hull  $i : P \hookrightarrow I$ . The cokernel  $Q$  has a Verma flag, hence  $0 \rightarrow TP \xrightarrow{Ti} TI \twoheadrightarrow TQ \rightarrow 0$  is exact (see [AS, Theorem 2.2]). Since  $I$  is a direct sum of copies of  $P(w_0)$ , we have  $\phi_I = 0$  and therefore  $\phi_P = 0$ . Since  $T$  is right exact we get  $\phi_M = 0$  for any  $M \in \mathcal{O}_0$ . Hence  $\Phi$  is injective and  $\text{End}(T) \cong \mathcal{C}$ . We get  $\text{End}(G) \cong \mathcal{C}$  by duality.

The adjointness from Proposition 2.4 together with Theorem 3 imply

$$\begin{aligned} \text{End}(T^2) &\cong \text{Hom}(\text{ID}, G^2T^2) \cong \text{Hom}(\text{ID}, G^2) \cong \text{End}(T) \cong \mathcal{C}, \\ \text{End}(GT) &\cong \text{Hom}(TGT, T) \cong \text{End}(T) \cong \mathcal{C} \end{aligned}$$

and also  $\text{End}(GT^2) \cong \text{Hom}(TGT^2, T^2) \cong \text{End}(T^2) \cong \mathcal{C}$ . The remaining parts follow by duality.  $\square$

**Claim 7.2.**  $H_{X,Y} \neq 0$  for any  $X, Y \in \mathcal{S}$ .

**Proof.** Since both  $X$  and  $Y$  are isomorphic to the identity functor when restricted to  $\mathcal{A} = \text{Add}(P(w_0))$  (see Lemma 2.1) we can fix a natural transformation  $\phi \in \text{Hom}(X|_{\mathcal{A}}, Y|_{\mathcal{A}}) \cong \mathcal{C}$  of maximal degree. For  $M \in \mathcal{O}_0$  indecomposable,  $M \notin \mathcal{A}$ , we set  $\phi_M = 0$ . For  $M \in \mathcal{O}_0$  arbitrary we fix an isomorphism  $\alpha_M : M \cong M_1 \oplus M_2$ , such that  $M_1$  is a maximal direct summand belonging to  $\mathcal{A}$  and set  $\phi_M := X(\alpha_M^{-1}) \circ (\phi_{M_1} \oplus \phi_{M_2}) \circ X(\alpha_M)$ . We claim that this defines an (obviously nontrivial) element  $\phi \in H_{X,Y}$ . Indeed, let  $M \cong M_1 \oplus M_2$  and  $N \cong N_1 \oplus N_2$  and  $f \in \text{Hom}_{\mathfrak{g}}(M, N)$  with decomposition  $f = \sum_{i,j=1}^2 f_{i,j}$  such that  $f_{i,j} \in \text{Hom}_{\mathfrak{g}}(M_i, N_j)$ . Then  $\phi_N \circ X(f_{1,1}) = Y(f_{1,1}) \circ \phi_M$  by definition of  $\phi$ . The definitions also immediately imply  $0 =$

$Y(f_{2,2}) \circ \phi_M = \phi_N \circ X(f_{2,2})$ . Moreover, we also have  $0 = \phi_N \circ X(f_{1,2})$  and  $0 = Y(f_{2,1}) \circ \phi_M$ . Indeed, if  $Y(f_{1,2}) \circ \phi_M \neq 0$  or  $\phi_N \circ X(f_{1,2}) \neq 0$  then either a direct summand of  $Y(M_1)$  embeds into  $Y(N_2)$  or  $X(M_2)$  surjects onto a direct summand of  $Y(N_1)$ . Both contradict the following statement: Assume  $R \in \mathcal{S}$  and  $M \in \mathcal{O}_0$  does not have  $P(w_0)$  as a direct summand then neither so does  $R(M)$ . Let first  $R \in \{G, C\}$ . If  $P(w_0)$  is a direct summand of  $R(M)$  then  $R'RM$  surjects onto  $R'P(w_0) \cong P(w_0)$ , hence  $P(w_0)$  is a direct summand of  $R'RM$ . The inclusion  $R'R \hookrightarrow \text{ID}$  from Proposition 2.4 implies that  $P(w_0)$  is a submodule (hence a direct summand) of  $M$ . Dual arguments apply to  $R \in \{T, K\}$  and the claim follows.  $\square$

**Claim 7.3.** *The  $\mathcal{C}$ -entries in the table of Theorem 6 are correct.*

**Proof.** The statement is obtained by playing with the adjointness of  $T$  and  $G$  using Proposition 7.1 and the identities from Theorem 3. Let  $X, Y \in \mathcal{S}$ . We have isomorphisms  $H_{T^2, X} \cong H_{T^2G^2, X} \cong H_{G^2, G^2X} \cong H_{G^2, G^2} \cong \mathcal{C}$ . This gives the spaces in question in the seventh row (and the sixth column by duality). The isomorphisms  $H_{TG, \text{ID}} \cong H_{G, G} \cong \mathcal{C}$  and  $H_{TG, GX} \cong H_{T^2G, X} \cong H_{T^2, X} \cong \mathcal{C}$  imply the claim for the fifth row (and the fourth column by duality). The spaces in question in the first, third and fourth rows follow from  $H_{TX, GY} \cong H_{T^2X, Y} \cong \mathcal{C}$  and  $H_{GT, G} \cong H_{TGT, \text{ID}} \cong H_{T, \text{ID}} \cong H_{\text{ID}, G}, H_{\text{ID}, GTG} \cong H_{T, TG}$ . From  $H_{GT^2, G} \cong H_{TGT^2, \text{ID}} \cong H_{T^2, \text{ID}} \cong \mathcal{C}$  and  $H_{GT^2, GT^2} \cong H_{TGT^2, T^2} \cong H_{T^2, T^2} \cong \mathcal{C}$  we get the spaces in the last row. This completes the proof.  $\square$

To proceed we use the following general statement:

**Proposition 7.4.** *Let  $\mathfrak{A}$  be an abelian category with enough projectives. Let  $F, J, H$  be endofunctors on  $\mathfrak{A}$ . Assume that  $F$  preserves surjections, and for any projective  $P \in \mathfrak{A}$  there exists some  $N \in \mathfrak{A}$  such that  $F(P) \cong FH(N)$ . Then the restriction defines an injective map  $\text{Hom}(F, J) \hookrightarrow \text{Hom}(FH, JH)$ .*

**Proof.** It is enough to show that for any  $\varphi \in \text{Hom}(F, J)$  such that  $\varphi_H = 0$  we have  $\varphi = 0$ . Let  $M \in \mathfrak{A}$  with projective cover  $f : P \twoheadrightarrow M$ . We choose  $N \in \mathfrak{A}$  such that  $F(P) \cong FH(N)$ . The first row of the following commutative diagram is exact, since  $F$  preserves surjections.

$$\begin{array}{ccccccc}
 FH(Q) \cong FP & \xrightarrow{f} & F(M) & \longrightarrow & 0 \\
 \varphi_{H(Q)} \downarrow & & \varphi_M \downarrow & & \\
 JH(Q) & \longrightarrow & GM & & 
 \end{array}$$

The surjectivity of  $f$  and  $\varphi_{H(Q)} = 0$  imply  $\varphi_M = 0$ .  $\square$

*The spaces with labeling different from 4:* The indicated equalities with labeling different from 1 and 4 follow directly by duality. By [AS, Corollary 4.2], the adjunction morphism  $\text{adj}_T^T : \text{ID} \twoheadrightarrow GT(P)$  is an isomorphism on projectives. Hence, we may apply Proposition 7.4 to  $F = \text{ID}$ ,  $J = T$ , and  $H = GT$  to obtain  $H_{\text{ID}, T} \hookrightarrow H_{GT, TGT} \cong H_{GT, T}$ . Further, the adjunction morphism  $\text{adj}_T : TG \hookrightarrow \text{ID}$  is injective, hence  $H_{G, TG} \hookrightarrow H_{G, \text{ID}}$  and  $H_{GT, T} \hookrightarrow H_{\text{ID}, T}$  by duality.

*The equality of the spaces denoted by 4:* we have the following isomorphisms

$$H_{GT^2, TG} \cong H_{TG^2, TG} \cong H_{G^2, GTG} \cong H_{G^2, G} \tag{7.1}$$

$$H_{G, GT^2} \cong H_{TG^2, T} \cong H_{G^2, GT} \cong H_{TG, T^2} \tag{7.2}$$

$$H_{G^2, GT^2} \cong H_{TG^2, T^2} \cong H_{GT^2, T^2} \tag{7.3}$$

$$H_{G, GT} \cong H_{TG, T}. \tag{7.4}$$

Note that all the spaces labeled by 4 occur in this list. The inclusion  $TG \hookrightarrow ID$  provides inclusions  $GT^2 \cong TG^2 \hookrightarrow G$  and  $TG^2 \cong TG^2T \hookrightarrow GT$ ; hence  $H_{G^2, GT^2} \hookrightarrow H_{G^2, G}$  and  $H_{G, TG^2} \hookrightarrow H_{G, GT}$  (i.e., (7.3) is ‘included’ in (7.1) and (7.2) is ‘included’ in (7.4)). Applying Proposition 7.4 with  $F = GT^2$ ,  $J = T$  and  $H = T$  ( $F = ID$ ,  $J = GT^2$ ,  $H = G$  respectively) we get inclusions  $H_{GT^2, T} \hookrightarrow H_{GT^2, T^2}$  and  $H_{ID, GT^2} \hookrightarrow H_{G, GT^2G} \cong H_{G, GT^2}$  (i.e., (7.2) is ‘included’ in (7.3) and (7.1) is ‘included’ in (7.2)). Hence, all the spaces from (7.1)–(7.4) have the same dimension.

*The existence of the inclusions from A:* The inclusion  $TG \hookrightarrow ID$  implies  $H_{GT, TG} \hookrightarrow H_{GT, ID}$ . Applying Proposition 7.4 to  $F = ID$ ,  $J \in \{T, TG\}$ , and  $H = G^2$  (this is possible since  $G^2(P) \cong P$  for any projective  $P$ ) we get inclusions  $H_{ID, T} \hookrightarrow H_{G^2, TG^2}$  and  $H_{ID, TG} \hookrightarrow H_{G^2, TG^2}$ . Finally, the inclusion  $G \hookrightarrow G^2$  gives  $H_{G^2, G} \hookrightarrow H_{G^2, G^2} \cong C$ .

*The existence of the inclusions from B:* Applying Proposition 7.4 to  $F = ID$ ,  $J = T^2$  and  $H \in \{G, G^2\}$ , we obtain the inclusions

$$H_{ID, T^2} \hookrightarrow H_{G, T^2}, \quad H_{ID, T^2} \hookrightarrow H_{G^2, T^2}. \tag{7.5}$$

Finally, using again the adjunction  $TG \hookrightarrow ID$  we get  $H_{G^2, TG} \hookrightarrow H_{G^2, ID}$ .

*The existence of the inclusion C:* We use the following result (which generalizes without problems to arbitrary parabolic subalgebras):

**Proposition 7.5.** *There is a natural isomorphism  $\text{End}(Z) \cong \text{End}(\text{ID}_{\mathcal{O}_0^s})$  of rings.*

**Proof.** Denote by  $I^\Delta$  the direct sum of all indecomposable projective-injective modules in  $\mathcal{O}_0^s$  and consider  $I^\Delta$  as an object in  $\mathcal{O}_0$ . We claim that  $\Phi : \varphi \mapsto \varphi_Q$  defines an isomorphism  $\text{End}(Z) \cong \mathcal{Z}(\text{End}_{\mathfrak{g}}(I^\Delta))$ , where the latter denotes the center of  $\text{End}_{\mathfrak{g}}(I^\Delta)$ . Note that  $\mathcal{Z}(\text{End}_{\mathfrak{g}}(I^\Delta)) \cong \text{End}(\text{ID}_{\mathcal{O}_0^s})$  ([St3, Theorem 10.1]).

*$\Phi$  is injective:* Let  $\varphi \in \text{End}(Z)$ ,  $\varphi_{I^\Delta} = 0$  and let  $P$  be a projective object in  $\mathcal{O}_0$ . We fix an inclusion  $i : ZP \hookrightarrow J_1$ , where  $J_1 = \bigoplus_{i \in I_1} I^\Delta$  for some finite set  $I_1$  (see the main result of [Ir2]). Since  $Z$  is the identity on  $\mathcal{O}_0^s$  we have  $\phi_P = \phi_{ZP}$  and  $0 = \phi_{J_1} \circ Z(i) = Z(i) \circ \phi_{ZP}$ . The injectivity of  $Z(i)$  implies  $\phi_P = 0$ . Let  $M \in \mathcal{O}_0$  be arbitrary with projective cover  $f : P \twoheadrightarrow M$ . Then  $\varphi_M \circ Z(f) = Z(f) \circ \phi_{J_1}$ , i.e.,  $\varphi_M = 0$ , since  $Z$  is right exact.

*$\Phi$  is surjective:* Let  $g \in \mathcal{Z}(\text{End}_{\mathfrak{g}}(I^\Delta))$ . For  $P \in \mathcal{O}_0$  projective we fix a coresolution

$$ZP \hookrightarrow J_1 \xrightarrow{h} J_2,$$

where  $J_i \cong \bigoplus_{i \in I_i} I^\Delta$  for some finite sets  $I_i$  ( $i = 1, 2$ ). For the existence of such a tilting resolution one can use the main result of [Ir2] and arguments, analogous to that of [KSX, 3.1] (see [St3,

Theorem 10.1]). By definition,  $g$  induces a natural map  $g_{ZP} \in \text{End}_{\mathfrak{g}}(ZP)$  making the following diagram commutative:

$$\begin{array}{ccccc}
 ZP & \xrightarrow{Z(f)} & ZJ_1 & \xrightarrow{Z(h)} & ZJ_2 \\
 \downarrow g_P & & \downarrow g_{J_1} & & \downarrow g_{J_2} \\
 ZP & \xrightarrow{Z(f)} & ZJ_1 & \xrightarrow{Z(h)} & ZJ_2
 \end{array}$$

Setting  $g_P = g_{ZP}$  defines a natural transformation  $\tilde{g} : Z \rightarrow Z$ , when restricted to the additive category of projective objects in  $\mathcal{O}_0$  such that  $\tilde{g}_{I^\Delta} = g$ . The right exactness of  $Z$  ensures that  $\tilde{g}$  extends uniquely to some  $\tilde{g} \in \text{End}(Z)$ . Hence  $\Phi$  is surjective. In particular,  $\text{End}(Z) = \mathcal{Z}(\text{End}_{\mathfrak{g}}(I^\Delta)) = \mathcal{Z}(\mathcal{O}_0^s) = \text{End}(\text{ID}_{\mathcal{O}_0^s})$ .  $\square$

The remaining part from Theorem 6 follows if we prove the following statements:

**Proposition 7.6.** *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a dense functor between two categories  $\mathcal{A}$  and  $\mathcal{B}$ . Then the restriction gives rise to an injective linear map  $\text{End}(\text{ID}_{\mathcal{B}}) \hookrightarrow \text{End}(F)$ . In particular,  $ZQ : \mathcal{O}_0 \rightarrow \mathcal{O}_0^s$  provides an inclusion  $\text{End}(\text{ID}_{\mathcal{O}_0^s}) \hookrightarrow \text{H}_{G,T}$ .*

**Proof.** The first statement of the proposition is obvious. Since  $ZQM = M$  for any  $M \in \mathcal{O}_0$  we may consider  $Q = ZQ$  as a functor from  $\mathcal{O}_0$  to  $\mathcal{O}_0^s$ . We claim that  $Q$  is dense, i.e., for any  $N \in \mathcal{O}_0^s$  there exists an  $K \in \mathcal{O}_0$  such that  $ZQ(K) \cong N$ . Indeed, let  $P \twoheadrightarrow N$  be a projective cover of  $N$  in  $\mathcal{O}_0$  with kernel  $K$ . Applying  $G$  to  $K \hookrightarrow P \twoheadrightarrow N$  we obtain the exact sequence  $GK \hookrightarrow GP \twoheadrightarrow GN$  and  $GN = 0$ . In particular,  $GK \cong GP$ . Since the socle of  $P$ , and therefore also of  $K$ , is annihilated by  $Z$ , the map  $g_K$  is injective (see [Jo1, Lemma 2.4]). Hence we have  $QK \cong (GK)/K \cong (GP)/K \cong P/K \cong N$ .

By Theorem 5 we have morphisms  $G \xrightarrow{p} Q \xrightarrow{\alpha^{-1}} Q' \xrightarrow{i} T$ , where  $\alpha^{-1}$  is an isomorphism. We consider the linear map  $\xi : \text{End}(Q) \rightarrow \text{H}_{G,T}$  defined as  $\xi(\varphi) = i \circ \alpha^{-1} \circ \varphi \circ p$ . Since  $p$  is surjective,  $i$  is injective, and  $\alpha^{-1}$  is an isomorphism,  $\xi$  defines an inclusion  $\text{End}(Q) \hookrightarrow \text{H}_{G,T}$ . To complete the proof it is now enough to show that  $\text{End}(Q)$  contains  $\text{End}(\text{ID}_{\mathcal{O}_0^s})$ . This follows directly from the first part of the proposition, since  $\text{End}(Z) \cong \text{End}(\text{ID}_{\mathcal{O}_0^s})$  (by Proposition 7.5).  $\square$

**Remark 7.7.** The case  $\mathfrak{g} = \mathfrak{sl}_2$  shows already that some spaces  $\text{H}_{X,Y}$ ,  $X, Y \in \mathcal{S}$  can be smaller than  $\mathcal{C}$ . Indeed, in this case we have  $\text{H}_{G,\text{ID}} \cong \mathbb{C}$  and  $\text{H}_{G,T,\text{TG}} \cong \mathbb{C}$ . Although the remaining ‘unknown’ spaces from Theorem 6 are isomorphic to  $\mathcal{C}$  in this particular example, the isomorphism is accidental and is not given by a natural action of  $\mathcal{C}$  on  $P(w_0)$  (in contrast to the cases, which are known to be isomorphic to  $\mathcal{C}$  from Theorem 6).

**8. Proof of Theorem 7**

Let  $\mathcal{I}(\hat{\mathcal{S}})$  denote the set of all idempotents in  $\hat{\mathcal{S}}$ . For  $X, Y \in \mathcal{I}(\hat{\mathcal{S}})$  we set  $\text{H}_{X,Y} = \text{Hom}(X, Y)$ .

**Proposition 8.1.**  *$\text{End}(X) \cong \mathcal{C}$  as algebras for any  $X \in \hat{\mathcal{S}}$ .*

**Proof.** An injective algebra morphism from  $\mathcal{C}$  to  $\text{End}(X)$  for every  $X \in \hat{\mathcal{S}}$  is constructed using the same arguments as in Proposition 7.1. The arguments, analogous to that of Proposition 7.1, also give an isomorphism  $\text{End}(\mathcal{C}) \cong \mathcal{C}$ .

Let us show that  $\text{End}(\mathcal{C}^2) \cong \mathcal{C}$ . We claim that the evaluation  $\phi \mapsto \phi_{P(w_0)}$  defines an inclusion  $\text{End}(\mathcal{C}^2) \hookrightarrow \text{End}_{\mathfrak{g}}(\mathcal{C}^2 P(w_0 \cdot 0))$ . Assume  $\phi_{P(w_0)} = 0$  and let  $P \in \mathcal{O}_0$  be projective with injective hull  $i : P \hookrightarrow I$ . We get an exact sequence  $0 \rightarrow \ker \mathcal{C}^2(i) \rightarrow \mathcal{C}^2 P \rightarrow \mathcal{C}^2 I$ . By assumption we have  $0 = \phi_I \circ \mathcal{C}^2(i) = \mathcal{C}^2(i) \circ \phi_P$ . In particular, the image of  $\phi_P$  is contained in the kernel of  $\mathcal{C}^2(i)$ . On the other hand  $\text{Hom}_{\mathfrak{g}}(\mathcal{C}^2 P, \ker \mathcal{C}^2(i)) \hookrightarrow \text{Hom}_{\mathfrak{g}}(\theta \mathcal{C} P, \ker \mathcal{C}^2(i)) \cong \text{Hom}(\mathcal{C} P, \theta \ker \mathcal{C}^2(i)) = 0$ , since  $\theta \ker \mathcal{C}^2(i) = 0$ . Therefore,  $\phi_P = 0$  and hence  $\phi = 0$ , since  $\mathcal{C}^2$  is right exact.

If  $i > 2$  then we have

$$\begin{aligned} \text{End}(\mathcal{C}^i) &\cong \text{Hom}(\text{ID}, \mathcal{K}^i \mathcal{C}^i) \cong \text{Hom}(\text{ID}, \mathcal{K}^2 \mathcal{C}^2) \cong \text{End}(\mathcal{C}^2) \cong \mathcal{C}. \\ \text{End}(\mathcal{K} \mathcal{C}^i) &\cong \text{Hom}(\mathcal{C} \mathcal{K} \mathcal{C}^i, \mathcal{C}^i) \cong \text{End}(\mathcal{C}^i) \cong \mathcal{C}, \quad i > 0; \quad \text{and} \\ \text{End}(\mathcal{K}^2 \mathcal{C}^i) &\cong \text{Hom}(\mathcal{C}^2 \mathcal{K}^2 \mathcal{C}^i, \mathcal{C}^i) \cong \text{End}(\mathcal{C}^i) \cong \mathcal{C}, \quad i > 1. \end{aligned}$$

Finally, there are isomorphisms

$$\begin{aligned} \text{End}(\mathcal{C} \mathcal{K}^2 \mathcal{C}) &\cong \text{Hom}(\mathcal{K}^2 \mathcal{C}, \mathcal{C} \mathcal{K} \mathcal{K}^2 \mathcal{C}) \cong \text{End}(\mathcal{K}^2 \mathcal{C}) \\ &\cong \text{Hom}(\mathcal{C}^2 \mathcal{K}^2 \mathcal{C}, \mathcal{C}) \cong \text{Hom}(\mathcal{C}^2 \mathcal{K}, \mathcal{C}) \cong \text{Hom}(\mathcal{C} \mathcal{K}, \mathcal{K} \mathcal{C}) \end{aligned}$$

and it is left to show that  $\text{Hom}(\mathcal{C} \mathcal{K}, \mathcal{K} \mathcal{C})$  embeds into  $\mathcal{C}$  as a vector space. For this we show that the map  $\Phi : \text{Hom}(\mathcal{C} \mathcal{K}, \mathcal{K} \mathcal{C}) \rightarrow \text{End}_{\mathfrak{g}}(P(w_0)) \cong \mathcal{C}$ ,  $\varphi \mapsto \varphi_{P(w_0)}$  is injective. Assume that  $\varphi_{P(w_0)} = 0$ . Since both  $\mathcal{C} \mathcal{K}$  and  $\mathcal{K} \mathcal{C}$  preserve injections (see Proposition 2.4), from the injection  $i : P \hookrightarrow I$  above we obtain that  $\varphi$  must be zero on all projective modules. Taking a projective cover of any  $M \in \mathcal{O}_0$  and using the fact that both  $\mathcal{C} \mathcal{K}$  and  $\mathcal{K} \mathcal{C}$  preserve surjections (see Proposition 2.4), we obtain that  $\varphi$  is zero. The rest follows by duality.  $\square$

Note that  $\mathcal{K} \mathcal{C}$  preserves projective modules, since the adjunction from Proposition 2.4 is an isomorphism on projective objects.

*Equality of the spaces labeled by 2:* The inclusion  $\mathcal{C} \mathcal{K} \hookrightarrow \text{ID}$  from Proposition 2.4 induces an inclusion  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{C} \mathcal{K}} \hookrightarrow \text{H}_{\mathcal{K}^2 \mathcal{C}^2, \text{ID}}$ . By duality we have  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{C} \mathcal{K}} \cong \text{H}_{\mathcal{K} \mathcal{C}, \mathcal{C}^2 \mathcal{K}^2}$  and  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \text{ID}} \cong \text{H}_{\text{ID}, \mathcal{C}^2 \mathcal{K}^2}$ . Applying Proposition 7.4 to  $F = \text{ID}$ ,  $H = \mathcal{K} \mathcal{C}$ , and  $J = \mathcal{C}^2 \mathcal{K}^2$  we obtain  $\text{H}_{\text{ID}, \mathcal{C}^2 \mathcal{K}^2} \hookrightarrow \text{H}_{\mathcal{K} \mathcal{C}, \mathcal{C}^2 \mathcal{K}^2}$  and thus all these four spaces are isomorphic.

*Equality of the spaces labeled by 3:* The inclusion  $\mathcal{C} \mathcal{K} \hookrightarrow \text{ID}$  induces the following inclusion:  $\text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \mathcal{C} \mathcal{K}} \hookrightarrow \text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \text{ID}}$ . By duality we have  $\text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \mathcal{C} \mathcal{K}} \cong \text{H}_{\mathcal{K} \mathcal{C}, \mathcal{C} \mathcal{K}^2 \mathcal{C}}$  and  $\text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \text{ID}} \cong \text{H}_{\text{ID}, \mathcal{C} \mathcal{K}^2 \mathcal{C}}$ . Applying Proposition 7.4 to  $F = \text{ID}$ ,  $H = \mathcal{K} \mathcal{C}$ , and  $J = \mathcal{C} \mathcal{K}^2 \mathcal{C}$  we obtain  $\text{H}_{\text{ID}, \mathcal{C} \mathcal{K}^2 \mathcal{C}} \hookrightarrow \text{H}_{\mathcal{K} \mathcal{C}, \mathcal{C} \mathcal{K}^2 \mathcal{C}}$  and thus all these four spaces are isomorphic.

*Equality of the spaces labeled by 4:* Evaluating  $\mathcal{C} \mathcal{K} \hookrightarrow \text{ID}$  at  $\mathcal{K} \mathcal{C}$  gives an inclusion  $\mathcal{C} \mathcal{K}^2 \mathcal{C} \cong \mathcal{K} \mathcal{C}^2 \mathcal{K} \hookrightarrow \mathcal{K} \mathcal{C}$ . Applying  $\text{Hom}(\mathcal{K}^2 \mathcal{C}^2, -)$  produces  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{K} \mathcal{C}^2 \mathcal{K}} \hookrightarrow \text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{K} \mathcal{C}}$ . By duality we have  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{K} \mathcal{C}^2 \mathcal{K}} \cong \text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \mathcal{C}^2 \mathcal{K}^2}$  and  $\text{H}_{\mathcal{K}^2 \mathcal{C}^2, \mathcal{K} \mathcal{C}} \cong \text{H}_{\mathcal{C} \mathcal{K}, \mathcal{C}^2 \mathcal{K}^2}$ . Applying Proposition 7.4 to  $F = \mathcal{C} \mathcal{K}$ ,  $H = \mathcal{K} \mathcal{C}$ , and  $J = \mathcal{C}^2 \mathcal{K}^2$  we obtain  $\text{H}_{\mathcal{C} \mathcal{K}, \mathcal{C}^2 \mathcal{K}^2} \hookrightarrow \text{H}_{\mathcal{K} \mathcal{C}^2 \mathcal{K}, \mathcal{C}^2 \mathcal{K}^2}$  and thus all these four spaces are isomorphic.

Applying the duality implies that all other spaces labeled by the same number coincide.

*All spaces labeled by  $\mathcal{C}$  are correct:* For the diagonal entries this follows from Proposition 8.1 above. For any  $X \in \mathcal{I}(\hat{\mathcal{S}})$  we have  $\text{H}_{\mathcal{C}^2 \mathcal{K}^2, X} \cong \text{H}_{\mathcal{K}^2, \mathcal{K}^2 X} \cong \text{H}_{\mathcal{K}^2, \mathcal{K}^2} \cong \mathcal{C}$  and  $\text{H}_{X, \mathcal{K}^2 \mathcal{C}^2} \cong \mathcal{C}$  by duality.



That  $H_{CK, KC} \cong \mathcal{C}$  was shown in the proof of Proposition 8.1. Using adjunction and duality we have  $H_{CK, KC^2K} \cong H_{C^2K, C^2K} \cong \mathcal{C}$  and  $H_{ID, KC} \cong H_{C, C} \cong \mathcal{C} \cong H_{K, K} \cong H_{CK, ID}$ .

It is left to establish the claimed inclusions. Applying  $\text{Hom}(KC, -)$  to the inclusion  $CK \hookrightarrow ID$  we get  $H_{KC, CK} \hookrightarrow H_{KC, ID}$ . Applying Proposition 7.4 to  $F = ID$ ,  $H = KC$ , and  $J = CK$  we obtain  $H_{ID, CK} \hookrightarrow H_{KC, CK^2C}$ . Applying Proposition 7.4 to  $F = KC^2K$ ,  $H = KC$ , and  $J = CK$  we obtain  $H_{KC^2K, CK} \hookrightarrow H_{CK^2C, CK^2C} \cong \mathcal{C}$ . Applying  $\text{Hom}(K^2C^2, -)$  to the inclusion  $KC \hookrightarrow K^2C^2$  obtained above we get  $H_{K^2C^2, KC} \hookrightarrow H_{K^2C^2, K^2C^2} \cong \mathcal{C}$ .

**Remark 8.2.** Behind our argumentation is the following general fact: Let  $F$  and  $G$  be two endofunctors on  $\mathcal{O}_0$ . Assume that  $F$  preserves surjections and  $G$  preserves injections. Then the map  $\text{Hom}(F, G) \rightarrow \text{End}_{\mathfrak{g}}(P(w_0))$ ,  $\varphi \mapsto \varphi_{P(w_0)}$ , is injective. Indeed, let  $\varphi_{P(w_0)} = 0$ . Since the injective envelope of any projective  $P \in \mathcal{O}_0$  belongs to  $\text{Add}(P(w_0))$ , we can use that  $G$  preserves injections to obtain  $\varphi_P = 0$ . Taking now the projective cover of any  $M \in \mathcal{O}_0$  and using that  $F$  preserves surjections we obtain  $\varphi_M = 0$ .

One can show that  $K^2C^2$  preserves injections and  $C^2K^2$  preserves surjections, which implies that  $H_{X, Y} \hookrightarrow \mathcal{C}$  for all  $X \in \{\text{ID}, \text{CK}, \text{KC}^2\text{K}, \text{C}^2\text{K}^2, \text{C}^i, \text{KC}^i: i > 0\}$  and for all  $Y \in \{\text{ID}, \text{KC}, \text{KC}^2\text{K}, \text{K}^2\text{C}^2, \text{K}^i, \text{CK}^i: i > 0\}$ .

**9. Proof of Theorem 8**

We have

$$\text{Ext}_{\mathcal{O}_0}^i(\mathcal{P}^w, \mathcal{P}^w) = \text{Hom}_{D^b(\mathcal{O}_0)}(\mathcal{L}T_w\mathcal{P}, \mathcal{L}T_w\mathcal{P}[i]) = \text{Ext}_{\mathcal{O}_0}^i(\mathcal{P}, \mathcal{P}) = 0, \quad i > 0$$

(see [AS, Corollary 4.2]).

**Claim 9.1.**  $\mathcal{P}$  admits a finite coresolution by modules from  $\text{Add}(\mathcal{P}^w)$ .

**Proof.** Let  $w \in W$ . If  $l(w) = 0$ , the statement is obvious. Assume, it is true for all  $\tilde{w}$  where  $l(\tilde{w}) \leq l(w)$  and let  $s$  be a simple reflection such that  $sw > w$ . We have to show that  $\mathcal{P}$  has a finite  $\text{Add}(\mathcal{P}^{sw})$ -coresolution. Since  $\text{Ext}_{\mathcal{O}_0}^{>0}(\mathcal{P}^x, \mathcal{P}^x) = 0$ , for all  $x \in W$ , the arguments from [Ha, Chapter III] or [MO, Lemma 4] reduce the problem to showing that there exists a  $\tilde{w}$ ,  $l(\tilde{w}) \leq l(w)$ , such that  $\mathcal{P}^{\tilde{w}}$  admits a coresolution by modules from  $\text{Add}(\mathcal{P}^{sw})$ . Since all  $T_x$  commute with translation functors, it is enough to prove the statement for  $T_{\hat{w}}\Delta(e) \cong \Delta(\hat{w})$ . We choose  $\tilde{w}$  such that  $sw = \tilde{w}t$  for some simple reflection  $t$  with  $l(\tilde{w}t) > l(\tilde{w})$  and consider the short exact sequence  $\Delta(e) \hookrightarrow P(t) \twoheadrightarrow \Delta(t)$ . Applying  $T_{\tilde{w}}$  we obtain the short exact sequence  $\Delta(\tilde{w}) \hookrightarrow T_{\tilde{w}}P(t) \twoheadrightarrow \Delta(sw)$ . Since  $P(t) \cong T_tP(t)$ , it follows that  $T_{\tilde{w}}P(t) \cong T_{sw}P(t)$ . Thus,  $T_{\tilde{w}}P(t), \Delta(sw) \in \text{Add}(\mathcal{P}^{sw})$ , and hence  $\Delta(\tilde{w})$  has a coresolution by modules from  $\text{Add}(\mathcal{P}^{sw})$ .  $\square$

We proved that  $\mathcal{P}^w$  is a generalized tilting module for any  $w \in W$ . Since  $\mathcal{O}_0$  has finite projective dimension, it is a generalized cotilting module as well ([Re, Corollary 2.4]).

The remaining assertions from the first part of the theorem follow by duality. Since  $T_{w_0}\Delta(e) \cong \Delta(w_0)$  is a tilting module and  $T_{w_0}$  commutes with translations, it follows that  $\mathcal{P}^{w_0} \cong \mathcal{T} \cong \mathcal{T}^{w_0}$ . Let  $w \in W$  and  $sw > w$  (i.e.,  $sww_0 < ww_0$ ). The adjunction morphism  $T_sG_s \hookrightarrow \text{ID}$  gives  $T_{sw}T_{w_0}\mathcal{P} \cong T_sT_wT_{w_0}\mathcal{P} \cong T_sG_{ww_0}\mathcal{I} \cong T_sG_sG_{sww_0}\mathcal{I} \hookrightarrow G_{sww_0}\mathcal{I}$ . Comparing the characters and using duality shows the second part of the theorem.

It remains to prove the formulas for the homological dimensions. Twisting functors commute with translation functors, hence we get

$$\text{projdim}(\mathcal{P}^w) = \text{projdim}(T_w \Delta(e)) = \text{projdim}(\Delta(w))$$

and  $\text{injdim}(\mathcal{P}^w) = \text{injdim}(\Delta(w))$ . For Verma modules the values are easy to compute and are given by the formulas from the theorem. The remaining statements follow by duality.

**10. Proof of Theorem 9**

We start with the following

**Proposition 10.1.** *Let  $w \in W$  and  $M \in \mathcal{O}_0$  be a module having a Verma flag. Then  $\mathcal{L}_1 C_s(C_{w^{-1}}M) = 0$  for any simple reflection  $s$  such that  $ws > w$ . In particular,  $C_{w^{-1}}P$  is acyclic for  $C_s$  for any projective object  $P$  and hence  $\mathcal{L}C_s \mathcal{L}C_{w^{-1}} \cong \mathcal{L}(C_s C_w)$ .*

**Proof.** By [MS1, Section 5],  $C_{w^{-1}}M$  has a  $w^{-1}$ -shuffled Verma flag. Hence, using Theorem 2, it is enough to show that the socle of every  $w^{-1}$ -shuffled Verma module  $C_{w^{-1}}\Delta(x)$  contains only  $L(y)$  such that  $ys < y$ . But  $C_{w^{-1}}\Delta(x)$  is at the same time a  $w^{-1}w_0$ -coshuffled dual Verma module and  $sw^{-1}w_0 < w^{-1}w_0$  as  $ws > w$ . This implies that  $C_{w^{-1}}\Delta(x) \cong K_s N$  for some  $N \in \mathcal{O}_0$  and thus  $C_{w^{-1}}\Delta(x)$  has desired socle by definition of  $K_s$ .  $\square$

**Claim 10.2.**  *${}^w\mathcal{P}$  is a generalized (co-)tilting module.*

**Proof.** The case  $w = e$  is clear. Assume the statement to be true for  $w \in W$  and let  $s$  be a simple reflection such that  $sw > w$ . By definition

$$0 \rightarrow P(x) \xrightarrow{\text{adj}_s(P(x))} \theta_s P(x) \rightarrow C_s P(x) \rightarrow 0$$

is exact for any  $x \in W$ . Applying  $C_w$  and using the previous proposition we get an exact sequence

$$0 \rightarrow C_w P(x) \xrightarrow{\text{adj}_s(P(x))} C_w \theta_s P(x) \rightarrow C_w C_s P(x) \rightarrow 0.$$

Since  $C_w C_s \cong C_{sw}$  (see [MS1, Lemma 5.10]) and  $C_w \theta_s P(x) \cong C_w C_s \theta_s P(x) \cong C_{sw} \theta_s P(x)$ ,  $C_w P(x)$  has a two-step coresolution with modules from  $\text{Add}(C_{ws}\mathcal{P})$ . Since  $\mathcal{L}C_w$  induces an equivalence on the bounded derived category of  $\mathcal{O}_0$  (by Proposition 10.1 and [MS1, Theorem 5.7]) we have  $\text{Ext}^{>0}(C_w \mathcal{P}, C_w \mathcal{P}) \cong \text{Ext}^{>0}(\mathcal{P}, \mathcal{P})$ . The arguments from Claim 9.1 show that  ${}^w\mathcal{P}$  is a generalized tilting module, hence also a generalized cotilting module by [Re, Corollary 2.4].  $\square$

Now let us prove Theorem 9(3). Using Proposition 10.1 and [MS1, Section 5] the statement reduces to verifying that  ${}^w_0\mathcal{P} \cong \mathcal{T}$ . Since  $C_{w_0}$  maps Verma modules to dual Verma modules, Proposition 10.1 implies that  $C_{w_0}\mathcal{P}$  has a dual Verma flag and satisfies  $\text{Ext}_{\mathcal{O}_0}^i(C_{w_0}\mathcal{P}, d\Delta(x)) = 0$  for all  $x \in W$ . From [Rin, Corollary 4] it follows that  $C_{w_0}\mathcal{P}$  has a Verma flag as well and thus  $C_{w_0}\mathcal{P} \cong \mathcal{T}$ .

Let  $L = L(y) \in \mathcal{O}_0$  be a simple object and  $M \in \mathcal{O}_0$  be a module with Verma flag. Then Proposition 10.1 gives

$$\begin{aligned} \text{Ext}_{\mathcal{O}}^i(C_s C_w M, L) &\cong \text{Hom}_{D^b(\mathcal{O}_0)}(\mathcal{L}(C_s C_w)M, L[i]) \\ &\cong \text{Hom}_{D^b(\mathcal{O}_0)}(C_w M, \mathcal{R}K_s L[i]). \end{aligned}$$

The latter is  $\text{Ext}_{\mathcal{O}}^{i+1}(C_w M, L)$  if  $y < y_s$  and it is  $\text{Ext}_{\mathcal{O}}^i(C_w M, K_s L)$  otherwise (see [MS1, Proposition 5.3]). In particular,  $M = \mathcal{P}$  gives  $\text{projdim}({}^{ws}\mathcal{P}) \leq \text{projdim}({}^w\mathcal{P}) + 1$ , and  $M = \mathcal{T}$  gives  $\text{projdim}(C_{ws}\mathcal{T}) \leq \text{projdim}(C_w\mathcal{T}) + 1$ . However, we know that  $\text{projdim}(\mathcal{T}) = \text{injdim}(\mathcal{T}) = l(w_0)$  (see, e.g., [BGG, §7], [MO, Theorem 1]) and  $\text{projdim}(\mathcal{I}) = l(w_0)$  and all the formulae for homological dimensions follow.

**Remark 10.3.** It is well known (see, e.g., [AL, Theorem 2.1 and Section 3]) that the set of twisted Verma modules is equal to the set of shuffled Verma modules. This is not the case for projective objects. In fact, if  $\mathfrak{g} = \mathfrak{sl}_3$  and  $s, t$  are the two simple reflections, then direct calculations show that  $C_s P(t)$  is neither a twisted projective nor a completed injective object.

### 11. Proof of Theorem 1

The singular braid monoid is generated by  $\{\sigma_i, \sigma_i^{-1}, \tau_i\}$  ( $1 \leq i \leq n - 1$ ) subject to the relations

$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1, \quad \text{for all } i, \tag{11.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for all } i, \tag{11.2}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1, \tag{11.3}$$

$$\tau_i \sigma_j \sigma_i = \sigma_j \sigma_i \tau_j \quad \text{if } |i - j| = 1, \tag{11.4}$$

$$\sigma_i \tau_j = \tau_j \sigma_i \quad \text{if } |i - j| \neq 1, \tag{11.5}$$

$$\tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i - j| > 1. \tag{11.6}$$

For a different presentation see for example [DG]. We have to prove that the functors from Theorem 1 satisfy the relations. For the first three relations see, e.g., [MS1, Theorem 5.7 and Lemma 5.10]. We claim that the remaining relations are true on the level of endofunctors on  $\mathcal{O}_0$ . Then they are also true for the derived functors (note that  $\mathcal{L}(C_s C_t) \cong \mathcal{L}C_s \mathcal{L}C_t$  if  $s \neq t$  by, e.g., [Ir1, Proposition 3.1]). Relation (11.6) follows directly from the classification theorem ([BG, 3.3]) of projective functors. If  $i \neq j$ , then the relation (11.5) follows immediately from the definition of  $C_s$ . In the case  $i = j$  the relation (11.5) will be proved in Lemma 11.1 below. The relations (11.4) will be proved in Proposition 11.3.

**Lemma 11.1.** *With the notation from Theorem 1 we have: There are isomorphisms of functors  $C_{s_i} \theta_{s_i} \cong \theta_{s_i} C_{s_i}$  for  $1 \leq i \leq n - 1$ .*

**Proof.** We set  $s = s_i$  for some  $1 \leq i \leq n - 1$ . All occurring functors are right exact and exact on modules having a Verma flag (see [MS1, Proposition 5.3]). Note that they preserve the full subcategory  $\mathcal{T}$  of projective-injective modules in  $\mathcal{O}_0$ . We claim that it is enough to establish the isomorphism when restricted to this category. Indeed, any projective object has a coresolution by objects in  $\mathcal{T}$ , then standard arguments using the Five Lemma will extend the constructed isomorphism to an isomorphism of the corresponding endofunctors on the category of projective modules (since  $\theta_s C_s M \cong \theta_s M$  for any object  $M$ , all functors in question preserve this category).

Again by standard arguments, using projective resolutions, the statement would follow, since the functors are right exact.

Hence, let us consider the category  $\mathcal{T}$ . The functor  $\mathbb{V}$  from Section 2 defines an equivalence  $\tilde{\mathbb{V}}$  of categories between  $\mathcal{T}$  and the category of finite-dimensional free  $\mathcal{C}$ -modules. We have  $\tilde{\mathbb{V}}\theta_s\tilde{\mathbb{V}}^{-1}$  is given by tensoring with the bimodule  $\mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C}$  (see [So2, Theorem 10]). Recall that  $\mathcal{C}$  is a free  $\mathcal{C}^s$ -module of rank 2 with basis 1 and  $X$ , the coroot corresponding to  $s$ . From the definitions it follows then that  $C_s\theta_s$  is given by tensoring with the cokernel  $D_s$  of the map

$$\begin{aligned} \phi : \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} &\xrightarrow{\text{adj}_s \otimes \text{id}} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} =: E, \\ c \otimes d &\mapsto X \otimes c \otimes d + 1 \otimes cX \otimes d, \end{aligned}$$

where  $\text{adj}_s$  denotes the corresponding adjunction map.

We define a homomorphism of vector spaces from  $E$  to  $\mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C}$  by

$$\begin{aligned} 1 \otimes 1 \otimes d &\mapsto 1 \otimes d, & X \otimes 1 \otimes d &\mapsto X \otimes d, \\ 1 \otimes X \otimes d &\mapsto -X \otimes d, & X \otimes X \otimes d &\mapsto -X^2 \otimes d \end{aligned}$$

for any  $d \in \mathcal{C}$ . This is obviously well-defined and defines in fact a unique  $\mathbb{C}$ -linear map. Evidently, it factors through  $D_s$ , is surjective, and is a homomorphism of  $\mathcal{C}$ -bimodules. Since  $C_s\theta_s P(w_0) \cong \theta_s P(w_0)$  it has to be an isomorphism. Hence  $C_s\theta_s \cong \theta_s$  on  $\mathcal{T}$  for any simple reflection  $s$ . By the remarks above we get  $C_s\theta_s \cong \theta_s$  as endofunctors on  $\mathcal{O}_0$ . Similarly one proves that  $\theta_s C_s \cong \theta_s$  by looking at the cokernel of the map

$$\begin{aligned} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} &\xrightarrow{\text{id} \otimes \text{adj}_s} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C}, \\ c \otimes d &\mapsto c \otimes X \otimes d + c \otimes 1 \otimes Xd. \end{aligned}$$

The statement of the lemma follows.  $\square$

**Remark 11.2.** Using the graded version from [MS1, Section 7] the isomorphism from the previous lemma is given as follows: We choose an isomorphism of functors  $\phi: \theta_s(1) \oplus \theta_s(-1) \cong \theta_s^2$ . The isomorphism  $\theta_s(-1) \cong \theta_s C_s$  is then given as  $\theta_s p \circ \phi \circ i_2$ , where  $p$  is the canonical projection and  $i_2$  denotes the inclusion into the second summand.

**Proposition 11.3.** *With the notation and assumptions from Theorem 1 we have isomorphisms of functors*

$$\begin{aligned} \theta_{i+1} C_i C_{i+1} &\cong C_i C_{i+1} \theta_i \quad \text{for all } i, \\ C_{i+1} C_i \theta_{i+1} &\cong \theta_i C_{i+1} C_i \quad \text{for all } i. \end{aligned}$$

**Proof.** We give an argument for the first isomorphism, and omit the analogous calculations for the second one. Set  $s = s_{i+1}$ ,  $t = s_i$ . Note first that it is sufficient to establish the isomorphism on projective modules, since the functors are right exact. Since any projective module has a copresentation by projective-injective modules and since the functors in question are exact on modules with Verma flag ([Ir1, Proposition 3.1] and [MS1, Proposition 5.3]), it is enough to check it on the subcategory given by these objects. We first compare them on the

Grothendieck group level. Since the functors in question are exact on modules with Verma flag, we may even restrict ourselves to the case of a single Verma module. We have the following formulas in the Grothendieck group of  $\mathcal{O}_0$ :  $[\theta_s C_t C_s \Delta(x)] = [T_x \theta_s C_t C_s \Delta(e)] = [T_x \theta_s \Delta(st)] = [T_x (\Delta(st) \oplus \Delta(sts))] = [\Delta(xst) \oplus \Delta(xsts)]$ . Here  $T_x = T_{s_1} T_{s_2} \cdots T_{s_r}$ , where  $x = s_1 s_2 \cdots s_r$  is a reduced expression (see, e.g., [KM, Section 6]). On the other hand we have  $[C_t C_s \theta_t \Delta(x)] = [T_x C_t C_s \theta_t \Delta(e)] = [T_x C_t C_s (\Delta(e) \oplus \Delta(t))] = [T_x (\Delta(st) \oplus \Delta(stt))] = [\Delta(xst) \oplus \Delta(xsts)]$ .

Before we proceed, we want to give the principal idea of the proof. The classification theorem of projective functors ([BG, 3.3]) provides (in the case of  $\mathfrak{g} = \mathfrak{sl}_n$ ) a decomposition  $\theta_s \theta_t \theta_s \cong F \oplus \theta_s$  for any noncommuting simple reflections  $s$  and  $t$ . Here,  $F$  is the indecomposable functor given by  $F(\Delta(e)) = P(sts)$ . By the definition of the functors we get surjections  $\alpha$  and  $\beta$  defined by the following commuting diagrams:

$$\begin{array}{ccc}
 \theta_s \theta_t \theta_s & \xrightarrow{\alpha_1 := \theta_s \theta_t p} & \theta_s \theta_t C_s \\
 \downarrow \alpha_2 := \theta_s(p_{C_s}) & \searrow \alpha & \downarrow \theta_s(p_{\theta_s}) \\
 \theta_s C_t \theta_s & \xrightarrow{\theta_s C_t(p)} & \theta_s C_t C_s
 \end{array}
 \qquad
 \begin{array}{ccc}
 \theta_t \theta_s \theta_t & \xrightarrow{\beta_1 := \theta_t p_{\theta_t}} & \theta_t C_s \theta_t \\
 \downarrow \beta_2 := p_{\theta_s \theta_t} & \searrow \beta & \downarrow p_{C_s \theta_t} \\
 C_t \theta_s \theta_t & \xrightarrow{C_t(p_{\theta_t})} & C_t C_s \theta_s
 \end{array}$$

where  $p$  always denotes the corresponding natural projection. We claim that already  $\alpha \circ i$  is surjective and factors through  $\beta \circ j$  for some fixed inclusions  $i : F \rightarrow \theta_s \theta_t \theta_s$  and  $j : F \rightarrow \theta_t \theta_s \theta_t$ ; i.e., there exists a natural transformation  $h : \theta_s C_t C_s \rightarrow C_t C_s \theta_t$  which is a surjection. The statement would then follow from our comparison on the Grothendieck group level.

As in the proof of the previous lemma we will work with  $\mathcal{C}$ -bimodules. The map  $\alpha$  gives then rise to an endomorphism of  $\mathcal{C}$ -bimodules

$$\begin{aligned}
 \tilde{\alpha} : \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} &\rightarrow D_\alpha, \\
 \tilde{\beta} : \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} &\rightarrow D_\beta,
 \end{aligned}$$

where  $D_\alpha \otimes_{\mathcal{C}} \bullet \cong \nabla_{\theta_s C_t C_s} \mathbb{V}^{-1}$  and  $D_\beta \otimes_{\mathcal{C}} \bullet \cong \nabla_{C_t C_s \theta_t} \mathbb{V}^{-1}$  on the category of free  $\mathcal{C}$ -modules of finite rank (see the proof of the previous lemma). Set  $\tilde{\alpha}_i = \nabla_{\alpha_i} \mathbb{V}^{-1}$  and  $\tilde{\beta}_i = \nabla_{\beta_i} \mathbb{V}^{-1}$  for  $i = 1, 2$ . Let  $X$  and  $Y$  be the coroots corresponding to  $s$  and  $t$  respectively. Note that the  $\{b \otimes y \otimes x \otimes c\}$  for  $c$  running through a  $\mathbb{C}$ -basis of  $\mathcal{C}$ , and  $b, x \in \{1, X\}, y \in \{1, Y\}$  are a  $\mathbb{C}$  basis of  $\mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C}$  (this follows from the fact that  $\mathcal{C}$  is a free  $\mathcal{C}^s$  module with basis 1 and  $X$ ). We claim that the images of  $(b \otimes 1 \otimes 1 \otimes c)$  under  $\tilde{\alpha}$  constitute a basis of the image of  $\tilde{\alpha}$ , i.e., of  $D_\alpha$ . They generate the image, since we have the following equalities:  $\tilde{\alpha}_1(b \otimes 1 \otimes X \otimes c) = \tilde{\alpha}_1(-b \otimes 1 \otimes 1 \otimes Xc)$ ,  $\tilde{\alpha}_2(b \otimes Y \otimes 1 \otimes c) = -\tilde{\alpha}_2(b \otimes 1 \otimes Y \otimes c)$  and  $\tilde{\alpha}_1(b \otimes Y \otimes X \otimes c) = \tilde{\alpha}_1(b \otimes Y \otimes 1 \otimes Xc)$ . On the other hand we know that  $\nabla_{\theta_s C_t C_s} P(w_0) \cong \nabla_{\theta_s} P(w_0) \cong \mathcal{C} \oplus \mathcal{C}$ , hence the claim follows.

We claim that the images of  $(b \otimes 1 \otimes 1 \otimes c)$  under  $\tilde{\beta}$  constitute a basis of the image of  $\tilde{\beta}$ , i.e., of  $D_\beta$ . Again, it is sufficient to show that they generate the image. Let  $\mathcal{B}$  denote their  $\mathbb{C}$ -span. Note that the  $\{d \otimes x \otimes y \otimes c\}$  for  $c$  running through a  $\mathbb{C}$ -basis of  $\mathcal{C}$ ,  $x \in \{1, X\}$ , and  $d, y \in \{1, Y\}$  are a  $\mathbb{C}$ -basis of  $\mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C}$ . We will frequently use the following formulas

$$\begin{aligned}
 \tilde{\beta}(aY \otimes b \otimes c \otimes d) &= -\tilde{\beta}(a \otimes bY \otimes c \otimes d), \\
 \tilde{\beta}(a \otimes bX \otimes c \otimes d) &= -\tilde{\beta}(a \otimes b \otimes cX \otimes d)
 \end{aligned}$$

for any  $a, b, c, d \in \mathcal{C}$  without explicitly referring to them. (The  $i$ th formula follows directly from the corresponding property of  $\tilde{\beta}_i$ .)

Then the claim follows from the following calculations:

$$\begin{aligned} \tilde{\beta}(X \otimes 1 \otimes 1 \otimes c) &= \tilde{\beta}(1 \otimes (X + 2Y) \otimes 1 \otimes c + 1 \otimes 2Y \otimes 1 \otimes c) \\ &= \tilde{\beta}(1 \otimes 1 \otimes (4Y + 8X) \otimes c + 1 \otimes 1 \otimes 7X \otimes c) \\ &= \tilde{\beta}(1 \otimes 1 \otimes (15X + 30Y) \otimes c - 26(1 \otimes 1 \otimes Y \otimes c)) \\ &= \tilde{\beta}(1 \otimes 1 \otimes 1 \otimes (15X + 30Y)c - 26(1 \otimes 1 \otimes Y \otimes c)) \end{aligned}$$

(we used that  $X + 2Y$  is  $t$ -invariant and  $Y + 2X$  is  $s$ -invariant). Hence,

$$\tilde{\beta}(1 \otimes 1 \otimes Y \otimes c) \in \mathcal{B}. \tag{11.7}$$

This implies that  $\tilde{\beta}(1 \otimes X \otimes 1 \otimes c) = -\tilde{\beta}(1 \otimes 1 \otimes X \otimes c) \in \mathcal{B}$  and also  $\tilde{\beta}(1 \otimes X \otimes Y \otimes c) = \tilde{\beta}(-1 \otimes 1 \otimes XY \otimes c) \in \mathcal{B}$ . Therefore,  $\tilde{\beta}(Y \otimes 1 \otimes 1 \otimes c) = \tilde{\beta}(-1 \otimes Y \otimes 1 \otimes c) = \tilde{\beta}(-1 \otimes 1 \otimes (2X + Y) \otimes c + 1 \otimes 2X \otimes 1 \otimes c) \in \mathcal{B}$ . Finally  $\tilde{\beta}(Y \otimes X \otimes 1 \otimes c) = -1 \otimes XY \otimes 1 \otimes c \in \mathcal{B}$ ,  $\tilde{\beta}(Y \otimes 1 \otimes Y \otimes c) = -\tilde{\beta}(1 \otimes Y \otimes Y \otimes c) \in \mathcal{B}$  and  $\tilde{\beta}(Y \otimes X \otimes Y \otimes c) = -\tilde{\beta}(1 \otimes XY \otimes Y \otimes c) \in \mathcal{B}$ . The claim follows.

Now one can choose a morphism of  $\mathcal{C}$ -bimodules

$$\phi : \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C} \otimes_{\mathcal{C}^t} \mathcal{C} \otimes_{\mathcal{C}^s} \mathcal{C}$$

which maps  $1 \otimes 1 \otimes 1 \otimes 1$  to  $1 \otimes 1 \otimes 1 \otimes 1$  and induces an isomorphism on the subbimodules given by inclusions  $i$  and  $j$  of  $F$ . (This choice is possible, since the head of  $F\Delta(e)$  is simple and isomorphic to  $\Delta(w_0)$  “sitting in minimal possible degree”). In particular,  $\phi$  defines a bijection on the bases constructed above. Hence we constructed an isomorphism  $\psi: \mathbb{V}\theta_s C_t C_s \mathbb{V}^{-1} \cong \mathbb{V}C_t C_s \theta_t \mathbb{V}^{-1}$  giving rise to an isomorphism  $\theta_s C_t C_s \cong C_t C_s \theta_t$  when restricted to the category of projective-injective objects. By the remarks at the beginning of the proof we get an isomorphism of endofunctors on  $\mathcal{O}_0$ . This completes the proof.  $\square$

**Remark 11.4.** Using graded versions of all functors involved (which requires a further development of some theory from [MS1,St1]) one could give a more conceptual proof as follows: One can first show that there is an embedding of  $\theta_s \theta_t \oplus \theta_s$  into  $\theta_s \theta_t \theta_s$ , whose cokernel is isomorphic to  $\theta_s C_t C_s$  on the one hand side, but also to the quotient  $Q$  of the homogeneous inclusion  $\theta_s \theta_t \hookrightarrow F$  of degree one on the other side. Analogously, there is an embedding of  $\theta_s \theta_t \oplus \theta_t$  into  $\theta_t \theta_s \theta_t$ , whose cokernel is isomorphic to  $C_t C_s \theta_t$  on the one hand side, but also to the functor  $Q$ . This implies then the first isomorphism of the previous proposition.

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