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Cherednik, Hecke and quantum algebras as free Frobenius and Calabi–Yau extensions

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Abstract

We show how the existence of a PBW-basis and a large enough central subalgebra can be used to deduce that an algebra is Frobenius. We apply this to rational Cherednik algebras, Hecke algebras, quantised universal enveloping algebras, quantum Borels and quantised function algebras. In particular, we give a positive answer to [R. Rouquier, Representations of rational Cherednik algebras, in: Infinite-Dimensional Aspects of Representation Theory and Applications, Amer. Math. Soc., 2005, pp. 103–131] stating that the restricted rational Cherednik algebra at the value $t = 0$ is symmetric.

Keywords: Frobenius algebras; Calabi–Yau algebras; Quantum groups; Hecke algebras; Rational Cherednik algebras

1. Introduction

1.1. In this note we will consider six types of algebras:

(I) the rational Cherednik algebra $\mathrm{H}_{0,c}$ associated to the complex reflection group W;
(II) the graded (or degenerate) Hecke algebra $\mathbf{H}_{\mathrm{gr}}$ associated to a complex reflection group W;
(III) the extended affine Hecke algebra $\mathcal{H}$ associated to a finite Weyl group W;

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(IV) the quantised enveloping algebra $\mathcal{U}_\epsilon(\mathfrak{g})$, at an ℓth root of unity ϵ, of a semisimple complex Lie algebra $\mathfrak{g}$;
(V) the corresponding quantum Borel $\mathcal{U}_\epsilon(\mathfrak{g})^{\geqslant 0}$;
(VI) the corresponding quantised function algebra $\mathcal{O}_\epsilon[G]$.

These algebras share two important properties: first, they have a regular central subalgebra $\mathcal{Z}$ over which they are free of finite rank, second, they—or a closely associated algebra in case (VI)—have a basis of PBW type. The purpose of this paper is to show that these two properties are the key tools for defining an associative non-degenerate $\mathcal{Z}$-bilinear form for each of these algebras, and hence for deducing Frobenius and Calabi–Yau properties for the algebras in each class.

1.2. We prove that each pair $\mathcal{Z} \subseteq R$ in the classes (I)–(VI) is a *free Frobenius extension*. The definition and basic properties are recalled in Section 2.1 and Section 2.2—in essence, one requires $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \cong R$ as $(\mathcal{Z}–R)$-bimodules.

1.3. When an algebra R is a free Frobenius extension of a central subalgebra $\mathcal{Z}$ then $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z})$ is in fact isomorphic to R *both* as a left *and* as a right R-module, but not necessarily as a bimodule. However, there is a $\mathcal{Z}$-algebra automorphism ν of R, the *Nakayama automorphism*, such that $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \cong {}^1R^{\nu^{-1}}$ as R-bimodules. This automorphism is unique up to an inner automorphism. We explicitly determine the Nakayama automorphisms for each case listed above: ν is trivial (i.e. inner) in cases (I) and (IV); non-trivial in cases (II), (III) and (V) and (VI).

1.4. The results summarised in Section 1.2 have immediate consequences regarding the *Calabi–Yau property* of the algebras in classes (I)–(VI). The definition and its relevance to Serre duality are recalled in Section 2.4. In particular [8], we get natural examples of so-called Frobenius functors—that is, functors which have a biadjoint. Frobenius algebras and Frobenius extensions play an important role in many different areas (see for example [23]). They give rise to Frobenius functors which are the natural candidates for constructing interesting topological quantum field theories in dimension 2 and even 3 (see for example [37]), and also provide connections between representation theory and knot theory (for example in the spirit of [22]).

1.5. Let us assume for the moment that $\mathcal{Z} \subseteq R$ is a free Frobenius extension with Nakayama automorphism ν. If I is an ideal of $\mathcal{Z}$, then it is clear from the definitions that $\mathcal{Z}/I \subseteq R/IR$ is a free Frobenius extension with Nakayama automorphism induced by ν. This applies in particular when I is a maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$; since, for R in classes (I)–(VI), every simple R-module is killed by such an ideal $\mathfrak{m}$, this is relevant to the finite-dimensional representation theory of R. Thus $R/\mathfrak{m}R$ is a Frobenius algebra, which is symmetric provided the automorphism of $R/\mathfrak{m}R$ induced by ν is inner.

1.6. To define the non-degenerate associative bilinear forms mentioned in Section 1.1, we follow in each case the approach of [12, Proposition 1.2] to the study of the inclusion $\mathcal{Z} \subseteq R$ when R is the enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional restricted Lie algebra $\mathfrak{g}$ over a field k of characteristic $p > 0$, and $\mathcal{Z}$ is the Hopf centre $k\langle x^p - x^{[p]}\colon x \in \mathfrak{g}\rangle$. In the language of the present paper, it is proved there that $\mathcal{Z} \subseteq U(\mathfrak{g})$ is a free Frobenius extension, with Nakayama automorphism ν the winding automorphism of the trace of the adjoint representation; in particular, ν is trivial when $U(\mathfrak{g})$ is semisimple. The parallel methods used here might suggest that an axiomatic approach covering all the cited cases simultaneously might be possible; but we have not found such a setting.

1.7. The detailed results for classes (I)–(VI) are as follows.

(1) (Theorem 3.5 and Corollary 3.6) The rational Cherednik algebra $\mathbf{H} = \mathrm{H}_{0,c}$ is a free Frobenius extension of its central subalgebra $\mathcal{Z} := S(V)^W \otimes S(V^*)^W$, with trivial Nakayama automorphism. Consequently $\mathbf{H}_\chi$ is a symmetric algebra for any central character χ (answering a question of Rouquier [34, Problem 6]), and $\mathbf{H}$ is a Calabi–Yau $\mathcal{Z}$-algebra.
(2) (Theorem 4.4) The graded Hecke algebra $\mathbf{H}_{\mathrm{gr}}$ associated to a complex reflection group W is a free Frobenius extension of its centre $\mathcal{Z}_{\mathrm{gr}} := S(V)^W$, but the Nakayama automorphism (which is determined explicitly) is non-trivial.
(3) (Theorem 5.2) The extended affine Hecke algebra $\mathcal{H}$ associated to a finite Weyl group W is a free Frobenius extension of its centre $\mathcal{Z}_{\mathcal{H}}$, but the Nakayama automorphism is non-trivial.
(4) (Theorem 6.5) The quantised enveloping algebra $\mathcal{U}_\epsilon(\mathfrak{g})$ is a free Frobenius extension of its ℓ-centre $\mathcal{Z}$, with trivial Nakayama automorphism. Consequently, $\mathcal{U}_\epsilon(\mathfrak{g})_\chi$ is symmetric for any central character χ, and $\mathcal{U}_\epsilon(\mathfrak{g})$ is a Calabi–Yau $\mathcal{Z}$-algebra.
(5) (Theorem 7.2) The quantum Borel $\mathcal{U}_\epsilon(\mathfrak{g})^{\geqslant 0}$ is a free Frobenius extension of its ℓ-centre $\mathcal{Z}_+$, but the Nakayama automorphism (which is determined explicitly) is non-trivial.
(6) (Theorem 8.3) There is an element z of the central subalgebra $\mathcal{O}[G]$ of the quantised function algebra $\mathcal{O}_\epsilon[G]$ such that $\mathcal{O}_\epsilon[G][z^{-1}]$ is a free Frobenius extension of $\mathcal{O}[G][z^{-1}]$ with non-trivial Nakayama automorphism. The open set $\mathcal{O}_z = \{g \in G\colon z \notin \mathfrak{m}_g\}$ meets every torus orbit of symplectic leaves in G. Thus, for any $g \in G$, the algebra $\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]$ is Frobenius but not, in general, symmetric.

1.8. There is some overlap between this paper and [2], a preliminary version of which we received while this paper was being written. The methods used in the two papers are completely different, and indeed complementary.

1.9. In the following rings are always assumed to be unitary and, if not stated otherwise, modules are *left* modules. For any ring S we denote by $\mathrm{Hom}_S(-,-)$, $\mathrm{Hom}_{-S}(-,-)$ and $\mathrm{Hom}_{S-S}(-,-)$ the morphism spaces in the category of (left) S-modules, right S-modules and S-bimodules, respectively. Our algebras are all over $\mathbb{C}$; undoubtedly this hypothesis could be weakened. We abbreviate $\otimes = \otimes_{\mathbb{C}}$.

2. Frobenius and Calabi–Yau extensions

2.1. Definition

We first recall some basics on Frobenius extensions. For more details we refer for example to [1,25,30] or [31]. A ring R is a *free Frobenius extension* (of the first kind) over a subring S, if R is a free S-module of finite rank, and there is an isomorphism of R–S-bimodules $F : R \to \mathrm{Hom}_S(R, S)$. (The bimodule structure on the latter is defined as $r.f.s(x) = f(xr)s$ for $r, x \in R$, $s \in S$, $f \in \mathrm{Hom}_S(R, S)$.) Equivalently, R is a free right S-module of finite rank, and there is an isomorphism of S–R-bimodules $G : R \to \mathrm{Hom}_{-S}(R, S)$ [30, Proposition 1]. The existence of F provides a non-degenerate associative S-bilinear form $\mathbb{B} : R \times R \to S$, defined by $\mathbb{B}(r,t) = F(t)(r)$ for all $r, t \in R$. Given a basis r_i, $1 \leqslant i \leqslant n$ of R as an S-module, we find elements r^i, $1 \leqslant i \leqslant n$ such that $\mathbb{B}(r_i, r^j) = \delta_{i,j}$ because F is surjective. The two ordered sets $\{r_i\colon 1 \leqslant i \leqslant n\}$ and $\{r^i\colon 1 \leqslant i \leqslant n\}$ form a dual free pair (in the sense of [1, Section 1]). Conversely, the existence of a non-degenerate associative bilinear form $\mathbb{B} : R \times R \to S$ together with a dual free pair implies that R is a free Frobenius extension of S with defining isomorphism F given by $F(t)(r) = \mathbb{B}(r,t)$ (see [1, Section 1]).

2.2. *The Nakayama automorphism*

We recall some ideas from [25]. Suppose for the rest of this section that R is a free Frobenius extension of $\mathcal{Z}$, with $\mathcal{Z}$ now contained in the centre of R. The isomorphisms F and G defined in 2.1 induce isomorphisms of left respectively right R-modules

$$R \cong \operatorname{Hom}_{\mathcal{Z}}(R,\mathcal{Z}) = \operatorname{Hom}_{\mathcal{Z}-\mathcal{Z}}(R,\mathcal{Z}) = RF(1),$$
$$R \cong \operatorname{Hom}_{-\mathcal{Z}}(R,\mathcal{Z}) = \operatorname{Hom}_{\mathcal{Z}-\mathcal{Z}}(R,\mathcal{Z}) = G(1)R. \tag{2.1}$$

One can show [25, Section 2(4)] that $h := F(1) = G(1)$ as elements of $\operatorname{Hom}_{\mathcal{Z}-\mathcal{Z}}(R,\mathcal{Z})$. Thus we get a well-defined $\mathcal{Z}$-algebra automorphism $\nu : R \to R$, defined by $rh = h\nu(r)$ for all $r \in R$. An easy calculation shows that

$$\mathbb{B}(x,y) = \mathbb{B}\big(\nu(y), x\big)$$

for $x, y \in \mathcal{B}$. The automorphism ν is called the *Nakayama automorphism* (with respect to F, $\mathbb{B}$, or G). It is clear that ν is uniquely determined up to an inner automorphism of R by the pair $\mathcal{Z} \subseteq R$. It therefore makes sense to speak about the Nakayama automorphism attached to a free Frobenius extension. We call the extension *symmetric* if the Nakayama automorphism is inner.

Thanks to our assumption on $\mathcal{Z}$, there is now also a *right* R-action on $\operatorname{Hom}_{\mathcal{Z}}(R,\mathcal{Z})$, given by $fr(-) = f(r-)$ for $r \in R$ and $f \in \operatorname{Hom}_{\mathcal{Z}}(R,\mathcal{Z})$. Let ${}^1R^{\nu^{-1}}$ be the ring R considered as an R-bimodule, but with its right R-module structure twisted by ν^{-1}. Then the R–$\mathcal{Z}$-bimodule isomorphism F is in fact an isomorphism of R-bimodules

$${}^1R^{\nu^{-1}} \cong \operatorname{Hom}_{\mathcal{Z}}(R,\mathcal{Z}), \tag{2.2}$$

since $F(r\nu^{-1}(x))(y) = F(\nu^{-1}(x))(yr) = \mathbb{B}(yr, \nu^{-1}(x)) = \mathbb{B}(x, yr)$ and $(F(r)x)(y) = F(r)(xy) = F(yr)(x) = \mathbb{B}(x, yr)$ for all $x, y, r \in R$.

2.3. We now highlight a condition which will allow us to prove that algebras are free Frobenius extensions. For this we let R be free with a finite basis $\mathcal{B}$ over an affine central subalgebra $\mathcal{Z}$. The condition is:

Hypothesis. There exists a $\mathcal{Z}$-linear functional $\Phi : R \to \mathcal{Z}$ such that for any non-zero $a = \sum_{b\in\mathcal{B}} z_b b \in R$ with all $z_b \in \mathcal{Z}$ there exists $x \in R$ with $\Phi(xa) = uz_b$ for some unit $u \in \mathcal{Z}$ and some $b \in \mathcal{B}$ such that $z_b \neq 0$.

Proposition. *Let R be a finitely generated free $\mathcal{Z}$-module with a basis $\mathcal{B}$. Then the following are equivalent*:

(1) *R is a free Frobenius extension of $\mathcal{Z}$*;
(2) *R satisfies the above hypothesis*;
(3) *there exists a $\mathcal{Z}$-linear functional $\Phi : R \to \mathcal{Z}$ such that for all $b \in \mathcal{B}$, there exists $x \in R$ such that for all $a = \sum_b z_b b \in R$, we have $\Phi(xa) = z_b$.*

If these conditions hold for R then, for any maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$, the finite-dimensional quotient $R/\mathfrak{m}R$ is a finite-dimensional Frobenius algebra.

Proof. (1) $\Rightarrow$ (3). Let $G : R \to \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z})$ be the $(\mathcal{Z}, R)$-bimodule isomorphism provided by 2.1 and set $\Phi(r) = G(r)(1)$. Take x to be the element of R which is sent by G to the function $\sum_{b\in\mathcal{B}} z_b b \mapsto z_b$.

(3) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (1). Let $\theta : R \to \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z})$ be the R–$\mathcal{Z}$-bimodule homomorphism defined by $\theta(a)(a') = \Phi(a'a)$. Clearly θ is an injection since if $a \in R$ is non-zero then the displayed hypothesis implies that $\theta(a)(x) \neq 0$. Thus we have a short exact sequence

$$0 \to R \to \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \to C \to 0 \tag{2.3}$$

of R–$\mathcal{Z}$-bimodules, where C is the cokernel of θ. We will prove that $C = 0$ after showing that θ induces a Frobenius structure on each finite-dimensional quotient $R/\mathfrak{m}R$.

Fix an arbitrary maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$ and consider the mapping

$$\bar{\theta} : \frac{R}{\mathfrak{m}R} \to \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \otimes_{\mathcal{Z}} \frac{\mathcal{Z}}{\mathfrak{m}}$$

which sends $a + \mathfrak{m}R$ to $\theta(a) \otimes 1$. Let

$$\iota : \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \otimes_{\mathcal{Z}} \frac{\mathcal{Z}}{\mathfrak{m}} \to \mathrm{Hom}_{\mathbb{C}}\left(\frac{R}{\mathfrak{m}R}, \mathbb{C}\right)$$

be the isomorphism sending $\psi \otimes 1$ to the mapping $(a + \mathfrak{m}R \mapsto \psi(a) + \mathfrak{m})$.

We claim that composition $\iota\bar{\theta}$ is an isomorphism. To prove this, we will show that $\iota\bar{\theta}$ is injective; then, since both the domain and codomain are vector spaces of the same dimension, the claim will follow. By construction,

$$\iota\bar{\theta}(a + \mathfrak{m}R)(a' + \mathfrak{m}R) = \Phi(a'a) + \mathfrak{m}.$$

Therefore, if $a + \mathfrak{m}R \in \ker \iota\bar{\theta}$ then $\Phi(a'a) \in \mathfrak{m}$ for all $a' \in R$. We assume that $a \neq 0$. Then, by hypothesis, if we write $a = \sum z_b b$, we can find $x \in R$ such that $\Phi(xa) = uz_b$ for some unit u and some non-zero z_b. Thus $z_b \in \mathfrak{m}$. Now a and $a - z_b b$ have the same image in $R/\mathfrak{m}R$ so we can replace a by $a - z_b b$. Repeating this procedure shows that $a \in \mathfrak{m}R$ and hence that $\iota\bar{\theta}$ is injective.

As a first consequence we see that $\iota\bar{\theta}$ induces an $R/\mathfrak{m}R$-isomorphism $R/\mathfrak{m}R \cong (R/\mathfrak{m}R)^*$ so $R/\mathfrak{m}R$ is Frobenius. We also deduce that $\bar{\theta}$ is an isomorphism, and so from (2.3) we see $C \otimes_{\mathcal{Z}} \mathcal{Z}/\mathfrak{m} = 0$. Since this is true for an arbitrary maximal $\mathfrak{m}$ of $\mathcal{Z}$ and C is finitely generated over $\mathcal{Z}$, it follows that $C = 0$. Hence $\theta : R \to \mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z})$ is an isomorphism and so R is a free Frobenius extension of $\mathcal{Z}$. $\square$

2.4. Calabi–Yau algebras

Let d and n be non-negative integers and let R be a ring which has a commutative noetherian central subring C of Krull dimension d, over which R is a finitely generated module. Following for example [20], we say that R is a *Calabi–Yau C-algebra of dimension n* if, for all $X, Y \in \mathcal{D}^b(\mathrm{Mod}(\mathrm{fl}-R))$, the bounded derived category of R-modules of finite length, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}(R))}\big(X, Y[n]\big) \cong D\,\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}(R))}(Y, X).$$

Here, D denotes the *Matlis duality* functor $D = \mathrm{Hom}_C(-, E)$, where E is the direct sum of the C-injective hulls of the simple C-modules. The following proposition is an immediate consequence of [20, Theorems 3.1 and 3.2], once we note that if C is regular then the Cohen–Macaulay C-modules coincide with the projective C-modules.

Proposition. *Let C, R, n and d be as above, and suppose that C is a regular domain. Then R is a Calabi–Yau C-algebra of dimension n if and only if $n = d$, R has finite global dimension, R is a projective C-module, and* $\mathrm{Hom}_C(R, C)$ *is isomorphic to R as R–R-bimodules. In this case, R has global dimension d.*

2.5. Hopf algebras

1. When H is a Hopf algebra which is a finite module over a central affine Hopf subalgebra $\mathcal{Z}$, Hopf-algebraic methods can be used to deduce that H is a Frobenius extension of $\mathcal{Z}$. The result is due to Kreimer and Takeuchi [26, Theorem 1.7]; the arguments are sketched in [4, Section III.4]. This provides an alternative approach to the algebras in classes (IV), (V) and (VI), but this does not provide an explicit description of the bilinear form, nor does it give immediate access to the Nakayama automorphism.

2. The concept of the Nakayama automorphism was introduced also in a recent paper on noetherian Hopf algebras by Brown and Zhang [7]. They showed that many noetherian Hopf algebras H (including all those which are finite modules over their centres) have a rigid dualizing complex R which is isomorphic (in the derived category of bounded complexes of H-bimodules) to ${}^{\hat{\nu}}H^1[d]$; here, d is the injective dimension of H, $[d]$ denotes the shift, and $\hat{\nu}$ is a certain algebra automorphism of H which Brown and Zhang called the Nakayama automorphism. The automorphism $\hat{\nu}$ is trivial on the centre of H and is uniquely determined by H, up to an inner automorphism.

When both usages of the term "Nakayama automorphism" are in play, they define the same map (bearing in mind that both definitions are only unique up to an inner automorphism of the algebra). To see this, suppose that H is a free Frobenius extension of a smooth affine central subalgebra $\mathcal{Z}$ (as is the case for the algebras of (IV), (V) and (VI)). Then the injective dimension d of H equals the Krull dimension (of H and of $\mathcal{Z}$). Thus the rigid dualizing complex of $\mathcal{Z}$ is $\mathcal{Z}[d]$, and, by [39, Proposition 5.9], [40, Example 3.11], H has rigid dualizing complex $\mathrm{RHom}_{\mathcal{Z}}(H, \mathcal{Z}[d])$. From the free Frobenius property of H, and (2.2), we deduce that this latter complex is isomorphic to ${}^{\nu}H^1[d]$, where ν denotes the Nakayama automorphism of the present paper. By the uniqueness of the rigid dualizing complex of H [38, Proposition 8.2], it follows that $\hat{\nu} = \nu$ up to an inner automorphism, as claimed.

3. The rational Cherednik algebra

In this section we show that the rational Cherednik algebra $\mathbf{H}$ is a Frobenius extension of its (what we call) bi-invariant centre, with trivial Nakayama automorphism, so that the reduced Cherednik algebras $\mathbf{H}_\chi$ are symmetric.

3.1. Rational Cherednik algebras

Let W denote an irreducible complex reflection group with identity element e and set of complex reflections S. We fix V, a complex reflection representation of W, and set $n = \dim V$. Let c be a conjugation invariant complex function on S. For $s \in S$ let α_s (respectively $\check{\alpha}_s$) be a linear functional on V (respectively V^*) which vanishes on the reflection hyperplane for s; we normalise these by the condition $\langle \alpha_s, \check{\alpha}_s \rangle = 2$. *The rational Cherednik algebra* $\mathbf{H} = \mathbf{H}_{0,c}$ is the $\mathbb{C}$-algebra generated by $\{w \in W,\ x \in V,\ y \in V^*\}$, with defining relations

$$wxw^{-1} = {}^w x, \qquad wyw^{-1} = {}^w y, \tag{3.1}$$

$$[x, x'] = 0, \qquad [y, y'] = 0, \tag{3.2}$$

$$[x, y] = \sum_{s \in S} c(s) \langle y, \alpha_s \rangle \langle \check{\alpha}_s, x \rangle s, \tag{3.3}$$

for $x, x' \in V$, $y, y' \in V^*$ and $w \in W$. These are the algebras $\mathbf{H}_{0,c}$ from [11, p. 251].

3.2. The PBW-basis

The algebra $\mathbf{H}$ has a PBW-property in the following sense: multiplication induces an isomorphism

$$S(V) \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} S(V^*) \tilde{\to} \mathbf{H}$$

of vector spaces (see [11, Theorem 1.3]). In particular, there is a PBW-basis given by the elements of the set $\mathcal{B}_{\mathbf{H}} = \{fwg\}$, where $w \in W$, f runs through a homogeneous basis of $S(V)$, and g runs through a homogeneous basis of $S(V^*)$.

For f in $S(V)$ or $S(V^*)$ we write $|f|$ for the degree of f. For $i \in \mathbb{Z}_{\geqslant 0}$ let $\mathcal{B}_{<i}$ be the span of all PBW-basis elements of the form fxg, where $f \in S(V)$, $x \in W$ and $g \in S(V^*)$, such that f and g are homogeneous with $|f| + |g| < i$: this induces a filtration of $\mathbf{H}$. Moreover, the commutation relation (3.3) shows that

$$\bigl|[f, g]\bigr| \leqslant |f| + |g| - 2 \quad \text{for all homogeneous } f \in S(V) \text{ and } g \in S(V^*). \tag{3.4}$$

3.3. The central subalgebra

The algebra $\mathbf{H} = \mathbf{H}_{0,c}$ has a large centre $Z(\mathbf{H})$, isomorphic to the so-called spherical subalgebra [11, Theorems 3.1, 7.2]. In particular, $Z(\mathbf{H})$ contains the *bi-invariant centre*

$$\mathcal{Z} = S(V)^W \otimes S(V^*)^W.$$

Now $S(V)$ (respectively $S(V^*)$) is a free $S(V)^W$-module (respectively $S(V^*)^W$-module) of rank $|W|$, see [24, V.18.3] for example. A basis can be obtained by taking arbitrary homogeneous preimages of any homogeneous basis of the coinvariant algebra $A := S(V)/(S(V)^W_+)$.

Then A is a local Frobenius algebra thanks to [24, Proposition VII.26.7] and its associated bilinear form is easy to describe. To do this, recall that the homogeneous component A_N of A of highest degree has dimension one and is skew invariant for the action of W on V, [24, 20.3, Propositions A and B]. Let $\pi : A \to A_N \cong \mathbb{C}$ be the projection map with $\pi(A_i) = 0$ for $i \neq N$. Then the bilinear form is given by

$$B(\bar{a}, \overline{a'}) = \pi(\bar{a}\overline{a'}).$$

Similar statements apply to $S(V^*)/(S(V^*)^W_+)$: it is Frobenius and its highest degree component is skew invariant for the action of W on V^*. Below, we shall use the notation $\epsilon_V, \epsilon_{V^*}$ for these two one-dimensional representations of W.

3.4. We fix a pair of homogeneous dual bases $\{\bar{\mathbf{a}}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\bar{\mathbf{a}}^i\colon 1 \leqslant i \leqslant |W|\}$ for $S(V)/(S(V)^W_+)$, and a pair of homogeneous dual bases $\{\bar{\mathbf{b}}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\bar{\mathbf{b}}^i\colon 1 \leqslant i \leqslant |W|\}$ for $S(V^*)/(S(V^*)^W_+)$. Then we lift them to homogeneous $S(V)^W$-bases, $\{\mathbf{a}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\mathbf{a}^i\colon 1 \leqslant i \leqslant |W|\}$ of $S(V)$, and homogeneous $S(V^*)^W$-bases $\{\mathbf{b}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\mathbf{b}^i\colon 1 \leqslant i \leqslant |W|\}$ of $S(V^*)$. We set $d_i = |\mathbf{a}_i|$ and $e_i = |\mathbf{b}_i|$; then $|\mathbf{a}^i| = N - d_i$ and $|\mathbf{b}^i| = N - e_i$. Let $\mathbf{a}_{\max}$ and $\mathbf{b}_{\max}$ be the elements of maximal degree N amongst the $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively.

3.5. The functional

For $f \in S(V)$ let $\mathbf{a}_{\max}(f)$ be the coefficient of $\mathbf{a}_{\max}$ when f is expressed in the chosen $S(V)^W$-basis of $S(V)$. Similarly, we define $\mathbf{b}_{\max}(g)$ for $g \in S(V^*)$. Thanks to the PBW-property, $\mathbf{H}$ is a free $\mathcal{Z}$-module of finite rank with basis

$$\mathcal{B}_{\mathbf{H}} := \{\mathbf{a}_i w \mathbf{b}_j\colon w \in W,\ 1 \leqslant i, j \leqslant |W|\}.$$

We define a $\mathcal{Z}$-linear map

$$\Phi : \mathbf{H} \to \mathcal{Z},$$
$$\mathcal{B}_{\mathbf{H}} \ni \mathbf{a}_i w \mathbf{b}_j \mapsto \begin{cases} 1 & \text{if } a_i = a_{\max}, b_j = b_{\max} \text{ and } w = e, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma. *The functional Φ above satisfies Hypothesis* 2.3.

Proof. Let $a = \sum_{b \in \mathcal{B}_{\mathbf{H}}} z_b b$ be a non-zero element of $\mathbf{H}$. Pick $b = \mathbf{a_i} w \mathbf{b_j} \in \mathcal{B}_{\mathbf{H}}$ of maximal degree $|\mathbf{a_i}| + |\mathbf{b_j}|$ such that $z_b \neq 0$, and set $x = \mathbf{b}^j w^{-1} \mathbf{a}^i$. We claim that this choice of x satisfies Hypothesis 2.3.

For indices i', j' and for $u \in W$ we have, by (3.1) and (3.4),

$$\begin{aligned} \mathbf{a}_{i'} u \mathbf{b}_{j'} x = \mathbf{a}_{i'} u \mathbf{b}_{j'} \mathbf{b}^j w^{-1} \mathbf{a}^i &= \mathbf{a}_{i'} \cdot u w^{-1} \cdot {}^{w}(\mathbf{b}_{j'} \mathbf{b}^j) \mathbf{a}^i \\ &= \mathbf{a}_{i'} ({}^{uw^{-1}} \mathbf{a}^i) \cdot u w^{-1} \cdot {}^{w}(\mathbf{b}_{j'} \mathbf{b}^j) + \text{lower order terms.} \end{aligned}$$

Since b was chosen to have maximal degree it follows that if $\mathbf{a}_{i'}u\mathbf{b}_{j'}$ appears in the expansion of a, then the lower order terms in the above expression have total degree less than $d_i + e_j + (N - d_i) + (N - e_i) = 2N$. Therefore we find that

$$\begin{aligned}\Phi(\mathbf{a}_{i'}u\mathbf{b}_{j'}x)\Phi\big(\mathbf{a}_{i'}u\mathbf{b}_{j'}\mathbf{b}^j w^{-1}\mathbf{a}^i\big) &= \Phi\big(\mathbf{a}_{i'}\big({}^{uw^{-1}}\mathbf{a}^i\big)\cdot uw^{-1}\cdot {}^{w}\big(\mathbf{b}_{j'}\mathbf{b}^j\big)\big)\\ &= \delta_{u,w}\Phi\big(\mathbf{a}_{i'}\mathbf{a}^i\cdot {}^{w}\big(\mathbf{b}_{j'}\mathbf{b}^j\big)\big).\end{aligned} \tag{3.5}$$

By definition of the dual basis we have, for $i, i', j, j' = 1, \dots, N$,

$$\mathbf{a}_{i'}\mathbf{a}^i = (\delta_{i,i'} + r_{\max})\mathbf{a}_{\max} + \sum_{k\neq\max} r_k\mathbf{a}_k \quad\text{and}\quad \mathbf{b}_{j'}\mathbf{b}^j = \big(\delta_{j,j'} + r'_{\max}\big)\mathbf{b}_{\max} + \sum_{k\neq\max} r'_k\mathbf{b}_k$$

for some $r_{\max}, r_k \in S(V)^W$ and $r'_{\max}, r'_k \in S(V^*)^W$. Consideration of polynomial degrees in the above expressions shows that $r_{\max} \in (S(V)^W_+)$ and $r_k \in (S(V)^W_+)$ for all k when $i = i'$, and that $r'_{\max} \in (S(V^*)^W_+)$ and $r'_k \in (S(V^*)^W_+)$ for all k when $j = j'$. Substituting in (3.5) we find that there exists $0 \neq c \in \mathbb{C}$ such that

$$\begin{aligned}&\Phi\big(\mathbf{a}_{i'}\mathbf{a}^i\cdot {}^{w}\big(\mathbf{b}_{j'}\mathbf{b}^j\big)\big)\\ &\quad= c\Phi\Bigg((\delta_{i,i'} + r_{\max})\mathbf{a}_{\max} + \sum_{k\neq\max} r_k\mathbf{a}_k\Bigg)\Bigg(\big(\delta_{j,j'} + r'_{\max}\big)\mathbf{b}_{\max} + \sum_{k\neq\max} r''_k\mathbf{b}_k\Bigg),\end{aligned} \tag{3.6}$$

where $r''_k \in S(V^*)^W$ and $r''_k \in (S(V^*)^W_+)$ when $j = j'$. We claim that (3.6) is 0 except when $(i', j') = (i, j)$. To see this, suppose that (i', j') is not equal to (i, j), but (3.6) is non-zero. Our choice of b to have maximal degree with $z_b \neq 0$ forces

$$d_{i'} + e_{j'} = d_i + e_j, \tag{3.7}$$

since otherwise the degree of $\mathbf{a}_{i'}\mathbf{a}^i \cdot {}^{w}(\mathbf{b}_{j'}\mathbf{b}^j)$ is strictly less than $2N$, and hence cannot involve $\mathbf{a}_{\max}\mathbf{b}_{\max}$.

Suppose first that $i' \neq i$ and $j' \neq j$. Then (3.6) becomes

$$\Phi\big(\mathbf{a}_{i'}\mathbf{a}^i\cdot {}^{w}\big(\mathbf{b}_{j'}\mathbf{b}^j\big)\big) = r_{\max}r'_{\max}\Phi(\mathbf{a}_{\max}\mathbf{b}_{\max}). \tag{3.8}$$

But $r_{\max}, r'_{\max}$ are in the ideals of positive degree invariants, and so have strictly positive degrees if they are not 0. Thus, comparing degrees in (3.8), using (3.7), shows that (3.8) is 0 in this case.

Suppose now that $i = i'$ but that $j \neq j'$. Then, by (3.7), $e_{j'} = e_j$. Therefore

$$\mathbf{b}_{j'}\mathbf{b}^j = r'_{\max}\mathbf{b}_{\max} + \sum_{k\neq\max} r'_k\mathbf{b}_k, \tag{3.9}$$

and in this equation $r'_{\max} = 0$, since otherwise it has strictly positive degree, contradicting the homogeneity of degree N of (3.9). Hence (3.6) becomes

$$\Phi\big(\mathbf{a}_i\mathbf{a}^i\cdot {}^{w}\big(\mathbf{b}_{j'}\mathbf{b}^j\big)\big)\Phi\Bigg(\Bigg(\mathbf{a}_{\max} + \sum_{k\neq\max} r_k\mathbf{a}_k\Bigg)\Bigg(\sum_{k\neq\max} r''_k\mathbf{b}_k\Bigg)\Bigg) = 0.$$

A similar argument applies if $i' \neq i$ but $j' = j$. Thus the claim is proved. Therefore

$$\Phi\left(\mathbf{a}_{i'} u \mathbf{b}_{j'} \mathbf{b}^j w^{-1} \mathbf{a}^i\right) = \delta_{u,w} \delta_{i,i'} \delta_{j,j'} \epsilon_{V^*}(w).$$

It follows that, with $x \mathbf{b}^j w^{-1} \mathbf{a}^i$,

$$\Phi(ax) = z_b \epsilon_{V^*}(w)$$

where $b = \mathbf{a}_i w \mathbf{b}_j$, confirming Hypothesis 2.3. □

3.6. The theorem for Cherednik algebras

Define the form $\mathbb{B}$ for $\mathbf{H}$ by $\mathbb{B}(a, b) = \Phi(ab)$, for $a, b \in \mathbf{H}$. We can now deduce the

Theorem. *The rational Cherednik algebra* $\mathbf{H}$ *is a symmetric Frobenius extension of its central subalgebra* $\mathcal{Z} = S(V)^W \otimes S(V^*)^W$.

Proof. It is immediate from Lemmas 3.5 and 2.3 that $\mathbf{H}$ is a free Frobenius extension of $\mathcal{Z}$ with form $\mathbb{B}$ as defined above. Therefore it remains only to prove that the Nakayama automorphism for $\mathbf{H}$ is inner.

We verify that $\mathbb{B}(Y, x) = \mathbb{B}(x, Y)$, where $Y \in \mathcal{B}_\mathbf{H}$ and $x \in W$ or V or V^*, since W, V and V^* generate $\mathbf{H}$ as a $\mathcal{Z}$-algebra. Let fwg be a typical element from $\mathcal{B}_\mathbf{H}$. First, let $x \in W$. Then

$$\begin{aligned}
\mathbb{B}(fwg, x) = \Phi(fwgx) &= \Phi\left(f \cdot wx \cdot {}^{x^{-1}}g\right) &(3.10)\\
&= \epsilon_{V^*}\left(x^{-1}\right)\Phi(f \cdot wx \cdot g) &(3.11)\\
&= \epsilon_{V^*}\left(x^{-1}\right)\Phi(f \cdot xw \cdot g) &(3.12)\\
&= \epsilon_V\left(x^{-1}\right)\epsilon_{V^*}\left(x^{-1}\right)\Phi\left({}^x f \cdot xw \cdot g\right) &(3.13)\\
&= \Phi(xfwg) = \mathbb{B}(x, fwg). &(3.14)
\end{aligned}$$

The equalities (3.10) follow from the definition of $\mathbb{B}$ and the defining relations (3.1) of $\mathbf{H}$. To see the formulas (3.11) and (3.13) note that $x(\mathbf{a}_{\max}) = \epsilon_V(x)\mathbf{a}_{\max} + h$, where $h \in S(V)$ with $\mathbf{a}_{\max}(h) = 0$. Similarly for $\mathbf{b}_{\max}$, and then invoke the definition of Φ. The equality (3.12) is true because both sides of the equation are trivial unless $x = w^{-1}$, in which case we have $xw = wx$. The relation (3.14) holds because of the defining relations of $\mathbf{H}$ and thanks to the fact that $\epsilon_V(x) = \epsilon_{V^*}(x)^{-1}$. Finally, the last equation is clear by definition of $\mathbb{B}$, and hence $\mathbb{B}(fwg, x) = \mathbb{B}(x, fwg)$ holds.

If $a \in V$ we get

$$\begin{aligned}
\mathbb{B}(fwg, a) &= \Phi(fwga)\\
&= \Phi(fwag) &(3.15)\\
&= \Phi\left(f^w awg\right)\\
&= \Phi(fawg) &(3.16)\\
&= \Phi(afwg) = \mathbb{B}(a, fwg).
\end{aligned}$$

The equality in (3.15) arises since the degree of $fwga$ and $fwag$ is $|f| + |g| + 1$ and so both sides are zero unless $|f| + |g| \geqslant 2N - 1$. In the case $|g| = N$ or $N - 1$ then $|[a, g]| < N$ by (3.4). This then means that $\Phi(fwga) = \Phi(fwag - fw[a, g]) = \Phi(fwag)$, as required. The equality (3.16) is true, because we have zero on both sides if $w \neq e$. Hence $\mathbb{B}(fwg, a) = \mathbb{B}(a, fwg)$ holds. If $b \in V^*$ the argument is similar, so we leave it to the reader.

Therefore we get $\mathbb{B}(x, y) = \mathbb{B}(y, x)$ for any $x, y \in \mathbf{H}$, which means $\mathbb{B}$ is symmetric. □

3.7. Consequences

Given a maximal ideal $\mathfrak{m}_\chi$ of $\mathcal{Z}$ we define the *reduced Cherednik algebra* to be the $|W|^3$-dimensional algebra

$$\mathbf{H}_\chi = \frac{\mathbf{H}}{\mathfrak{m}_\chi \mathbf{H}}.$$

Thanks to [16] these algebras control a great deal of the geometry associated to the centre of **H**. The following corollary is immediate from Theorem 3.5 and the discussion in 2.4, after we have noted that **H** has finite global dimension by [11, p. 276]. The first part (for the case when $\mathfrak{m}_\chi$ is $(S(V)^W \otimes S(V^*)^W)_+$) answers [34, Problem 6].

Corollary.

(1) *The reduced Cherednik algebras* $\mathbf{H}_\chi$ *are symmetric, with dual bases the images of the bases* $\mathcal{B} = \{\mathbf{a_i} w \mathbf{b_j}\}$ *and* $\mathcal{B}' = \{\mathbf{a^i} w \mathbf{b^j}\}$ *defined in Sections* 3.3 *and* 3.4.
(2) $\mathbf{H}$ *is a Calabi–Yau* $\mathcal{Z}$*-algebra of dimension* $2\dim(V)$.

4. The graded Hecke algebra

In this section we show that the graded Hecke algebra $\mathbf{H}_{\mathrm{gr}}$ is a Frobenius extension of its invariant centre, with non-trivial Nakayama automorphism, so that the reduced graded Hecke algebras ${\mathbf{H}_{\mathrm{gr}}}_\chi$ are Frobenius but not, in general, symmetric.

4.1. Graded Hecke algebras

As in the previous section let W be an irreducible complex reflection group with identity e, and V the defining complex reflection representation of W. Let $\mathbf{H}_{\mathrm{gr}}$ be the associative algebra generated by V and $\mathbb{C}W$ with relations

$$wxw^{-1} = {}^w x, \tag{4.1}$$

$$[x, y] = \sum_{w \in W} \Omega_w(x, y) w, \tag{4.2}$$

for $x, y \in V$ and $w \in W$. For each $w \in W$, $\Omega_w : V \times V \to \mathbb{C}$ is an alternating 2-form on V; we insist these forms satisfy the following coherence conditions of [33, (1.6), (1.7)] for all $x, y, z \in V$ and $v, w \in W$,

$$\Omega_w(x,y) = \Omega_{vwv^{-1}}(vx, vy) \quad \text{and}$$
$$\Omega_w(z,x)(wy-y) + \Omega_w(y,z)(wx-x) + \Omega_w(x,y)(wz-z) = 0.$$

The algebra $\mathbf{H}_{\mathrm{gr}}$ is a *graded Hecke algebra* for W and $\mathbf{H}_{\mathrm{gr}} \cong S(V) \otimes \mathbb{C}W$ as vector spaces [33, Lemma 1.5]. In particular, there is a PBW-basis given by the elements of the set $\{fw\}$, where $w \in W$, and f runs through a homogeneous basis of $S(V)$. For f in $S(V)$ we again write $|f|$ for the degree of f. For $i \in \mathbb{Z}_{\geqslant 0}$ let $\mathcal{B}_{<i}$ be the span of all PBW-basis elements of the form fw, where $f \in S(V)$, $w \in W$ such that f is homogeneous with $|f| < i$: this induces a $\mathbb{Z}_{\geqslant 0}$-filtration of $\mathbf{H}_{\mathrm{gr}}$. Moreover, the commutation relation (4.2) shows that

$$\bigl|[f,g]\bigr| \leqslant |f| + |g| - 2 \quad \text{for all homogeneous } f, g \in S(V). \tag{4.3}$$

Recall that $s \in W$ is a *bireflection* if $\operatorname{codim} V^s := \operatorname{rank}(\mathrm{id}_V - s) = 2$. We denote by $\mathcal{R}$ the set of all bireflections s such that for any $w \in Z_W(s)$, the W-centraliser of s, the action of w restricted to V/V^s has determinant equal to one. The set $\mathcal{R}$ plays an important role since $\Omega_w \neq 0$ implies $w = e$ or $w \in \mathcal{R}$ [33, Theorem 1.9]. Moreover, since V is the (faithful) defining reflection representation of W and $\Omega_e \in ((\wedge^2 V)^*)^W$, we find $\Omega_e = 0$. Hence relation (4.2) becomes

$$[x,y] = \sum_{w \in \mathcal{R}} \Omega_w(x,y)w. \tag{4.4}$$

Let $N \lhd W$ be the normal subgroup generated by $\mathcal{R}$ and let $\mathbf{H}_{\mathrm{gr}}(N)$ be the graded Hecke algebra associated with N with alternating 2-forms Ω_w equal to those from $\mathbf{H}_{\mathrm{gr}}$. The following fact illustrates once more that $\mathcal{R}$ controls $\mathbf{H}_{\mathrm{gr}}$: there is [32, Lemma 1.3] an isomorphism of algebras

$$\mathbf{H}_{\mathrm{gr}} \cong \mathbf{H}_{\mathrm{gr}}(N) *' W/N, \tag{4.5}$$

where $\mathbf{H}_{\mathrm{gr}}(N) *' W/N$ is a crossed product algebra defined as follows. As a vector space it is just $\mathbf{H}_{\mathrm{gr}}(N) \otimes \mathbb{C}[W/N]$. To define the commutator relations between these two subspaces we fix for each coset of W/N one representative. Let $\{w_i \mid i \in J\}$ be the resulting complete system of coset representatives for W/N with $w_i \in [w_i] \in W/N$. Let $T(V)$ be the tensor algebra and $T(V) * W$ be the skew product algebra with the relations given by (4.1). Hence $\mathbf{H}_{\mathrm{gr}} = (T(V) * W)/I$ where I is given by the relations (4.4). These relations also define an ideal, $I(N)$, of $T(V) * N$ such that $\mathbf{H}_{\mathrm{gr}}(N) = (T(V) * N)/I(N)$. If now $h = \sum_{n \in N} x_n n \in T(V) \otimes \mathbb{C}N$ then define

$$[w_i]h = \sum_{n \in N} {}^{w_i}x_n w_i n w_i^{-1}[w_i]. \tag{4.6}$$

Passing to the quotient, this defines the commutator relations between $\mathbf{H}_{\mathrm{gr}}(N)$ and $\mathbb{C}[W/N]$ in $\mathbf{H}_{\mathrm{gr}}(N) * W/N$. One can show that, up to isomorphism, this algebra does not depend on the choice of representatives. However, with these choices, the isomorphism (4.5) is explicitly given as $fw \mapsto f \cdot w_i n w_i^{-1} \cdot [w_i]$, where $f \in S(V)$, $w = w_i n \in W$, $n \in N$. Since $\mathbf{H}_{\mathrm{gr}}(N)$ is preserved by conjugation by the subgroup W of $\mathbf{H}_{\mathrm{gr}}$, we note:

Lemma. *Let $Z(\mathbf{H}_{\mathrm{gr}}(N))$ be the centre of $\mathbf{H}_{\mathrm{gr}}(N)$ considered as a subalgebra of $\mathbf{H}_{\mathrm{gr}}$ via the isomorphism* (4.5)*. The W-action $w.h = whw^{-1}$ for $w \in W$, $h \in \mathbf{H}_{\mathrm{gr}}$ induces a W-action on $Z(\mathbf{H}_{\mathrm{gr}}(N))$.*

4.2. The central subalgebra

In the special case (see [33, Section 3]) where W is a Weyl group and $\mathbf{H}_{\mathrm{gr}}$ is Lusztig's graded Hecke algebra (as introduced in [27]) the following result is well known [27, Proposition 4.5]. We retain the notation $\{\mathbf{a}_i\colon 1 \leqslant i \leqslant |W|\}$ from Section 3.4.

Proposition.

(1) *The algebra $\mathbf{H}_{\mathrm{gr}}$ has finite global dimension.*
(2) *The centre $Z(\mathbf{H}_{\mathrm{gr}})$ contains the subalgebra $\mathcal{Z}_{\mathrm{gr}} := S(V)^W$.*
(3) *With the notation from the previous section, $\mathbf{H}_{\mathrm{gr}}$ becomes a free $\mathcal{Z}_{\mathrm{gr}}$-module of finite rank with basis*

$$\mathcal{B}_{\mathbf{H}_{\mathrm{gr}}} := \left\{\mathbf{a}_i w\colon\ w \in W,\ 1 \leqslant i \leqslant |W|\right\}.$$

The proof of this proposition will occupy the rest of this subsection. We start with some preparations. Note that if $\Omega_w = 0$ for all $w \in W$, then $\mathbf{H}_{\mathrm{gr}} \cong S(V) * W$, the skew group algebra. Of course, the proposition holds in this case. For any filtered algebra B we denote by $\mathrm{Gr}\, B$ its associated graded algebra. The following holds:

Lemma. *Let $e_N = \frac{1}{|N|}\sum_{w\in N} w$ and consider $\mathbf{H}^{\mathrm{sph}} := e_N \mathbf{H}_{\mathrm{gr}}(N) e_N$, the spherical subalgebra of $\mathbf{H}_{\mathrm{gr}}(N)$. The $\mathbb{Z}_{\geqslant 0}$-filtration on $\mathbf{H}_{\mathrm{gr}}(N)$ induces a filtration on $\mathbf{H}^{\mathrm{sph}}$ and also on its centre such that*

(1) $\mathrm{Gr}\,\mathbf{H}^{\mathrm{sph}} \cong S(V)^N$.
(2) *There is an isomorphism of algebras $\Psi : Z(\mathbf{H}_{\mathrm{gr}}(N)) \xrightarrow{\sim} Z(\mathbf{H}^{\mathrm{sph}})$, $z \mapsto z e_N$.*
(3) $\mathbf{H}^{\mathrm{sph}}$ *is commutative, in particular $Z(\mathbf{H}_{\mathrm{gr}}(N)) \cong \mathbf{H}^{\mathrm{sph}}$.*
(4) $\mathrm{Gr}\, Z(\mathbf{H}_{\mathrm{gr}}(N)) \cong S(V)^N$.

Proof. There is an isomorphism $S(V)^N \to e_N(S(V) * N)e_N$ via $f \mapsto fe$, and $e_N(S(V) * N)e_N \cong e_N(\mathrm{Gr}\,\mathbf{H}_{\mathrm{gr}}(N))e_N \cong \mathrm{Gr}(e_N\mathbf{H}_{\mathrm{gr}}(N)e_N) = \mathrm{Gr}\,\mathbf{H}^{\mathrm{sph}}$. This proves (1). Statements (2) and (3) are analogous to [11, Theorem 3.1] and [11, Theorem 1.6], respectively; details can be found in [13, Proposition 4.3] and [13, Theorem 6.2]. Since Ψ preserves the filtration and is surjective on each layer, the last statement follows from (3). □

Let $R = S(V) * N$. Recall that an associative graded algebra $(A, \diamond)$, with multiplication $\diamond$, is called *a graded deformation of R* if $A \cong R \otimes_{\mathbb{C}} \mathbb{C}[h]$ as graded vector spaces where h is an indeterminant concentrated in degree one, $\diamond$ is $\mathbb{C}[h]$-bilinear, and $r_1 \diamond r_2 \equiv r_1 r_2 \bmod hA$ for any $r_1, r_2 \in R$, considered as a subspace of A. Put

$$A = A(V, N) := \big(T(V)[h] * N\big)/I_N, \quad I_N := \left\langle [x, y] - \sum_{w\in\mathcal{R}} \Omega_w(x, y) w h^2\colon\ x, y \in V\right\rangle.$$

Note that I_N becomes homogeneous, hence A is graded. It follows directly that A is a graded deformation of R and $A/(h-1)A = \mathbf{H}_{\mathrm{gr}}(N)$.

Proof of Proposition 4.2. The first statement is clear from [29, Corollary 7.6.18(i)], since $\mathbf{H}_{\mathrm{gr}}$ is filtered such that $\mathrm{Gr}(\mathbf{H}_{\mathrm{gr}}) \cong S(V) * W$ and the latter has finite global dimension. The last statement will follow as soon as we established the second.

Recall (from Lemma 4.1) that W and hence W/N act on the centre of $\mathbf{H}_{\mathrm{gr}}(N)$. We get $\mathrm{Gr}(Z(\mathbf{H}_{\mathrm{gr}}(N))^{W/N}) = (\mathrm{Gr}\, Z(\mathbf{H}_{\mathrm{gr}}(N)))^{W/N} = (S(V)^N)^{W/N} = S(V)^W$ by Lemma 4.2, and $e_N A e_N$ is a commutative graded deformation of $S(V)^N$; the proof of this is analogous to the proof of [11, Theorem 1.6], and is given in detail in [13, Theorem 6.2]. The infinitesimal commutative graded deformations are controlled by the second Harrison cohomology ([18, Theorem 8], [14, Section 4]). In our situation $B := (e_N A e_N)^{W/N}$ is a (global) commutative graded deformation of $S(V)^W$. On the other hand, W is a complex reflection group, hence $S(V)^W$ is a polynomial ring, and so there are no non-trivial graded commutative deformations [18, Theorem 11]. Hence B is a trivial deformation, and therefore $B/(h-1)B = S(V)^W$. On the other hand, $B/(h-1)B = (e_N \mathbf{H}_{\mathrm{gr}}(N) e_N)^{W/N} = (\mathbf{H}^{\mathrm{sph}})^{W/N}$, hence $Z(\mathbf{H}_{\mathrm{gr}}(N))^{W/N} = S(V)^W$ by Lemma 4.2. The claim of the proposition follows then from (4.5) as follows: Let $f \in S(V)^W$, in particular $fg = gf \in \mathbf{H}_{\mathrm{gr}}$ for any $g \in W$. Since the centre of $\mathbf{H}_{\mathrm{gr}}(N)$ is given by $S(V)^N$ and $f \in S(V)^W \subset S(V)^N$, we get $fh = hf$ for any $h \in \mathbf{H}_{\mathrm{gr}}(N)$, considered as a subspace of $\mathbf{H}_{\mathrm{gr}}$. Hence, f is in the centre of $\mathbf{H}_{\mathrm{gr}}$. □

4.3. The centre

Although it is not needed for the results of this paper, we record here the fact that the inclusion of $S(V)^W$ in the centre of $\mathbf{H}_{\mathrm{gr}}$ is in fact an equality. In the special case where W is a Weyl group, this result is [27, Proposition 4.5].

Theorem. *Retain the notation of Sections* 4.1 *and* 4.2. *Then* $S(V)^W = Z(\mathbf{H}_{\mathrm{gr}})$.

Proof. From Proposition 4.2(2) we know that $\mathcal{Z}_{\mathrm{gr}} := S(V)^W \subseteq Z := Z(\mathbf{H}_{\mathrm{gr}})$. Let F and E be the quotient fields of $\mathcal{Z}_{\mathrm{gr}}$ and Z, respectively, and let Q be the (simple artinian) quotient ring of $\mathbf{H}_{\mathrm{gr}}$, so $F \subseteq E \subseteq Q$. Since $\mathbf{H}_{\mathrm{gr}}$ is a finitely generated module over the commutative affine algebra $\mathcal{Z}_{\mathrm{gr}}$, $Z \cap F$ is a finitely generated $\mathcal{Z}_{\mathrm{gr}}$-module. Therefore, since $\mathcal{Z}_{\mathrm{gr}}$ is integrally closed, $Z \cap F = \mathcal{Z}_{\mathrm{gr}}$. Suppose for a contradiction that $\mathcal{Z}_{\mathrm{gr}} \subsetneq Z$. Then $F \subsetneq E$. It follows that

$$\dim_E(Q) < \dim_F(Q) = |W|^2.$$

That is, the PI-degree of $\mathbf{H}_{\mathrm{gr}}$ is strictly less than $|W|$, or—equivalently—the maximal dimension of an irreducible $\mathbf{H}_{\mathrm{gr}}$-module is strictly less than $|W|$ [4, Theorem I.13.5 and Lemma III.1.2].

We now claim that the maximal dimension of irreducible $\mathbf{H}_{\mathrm{gr}}$-modules is $|W|$. To see this, consider the algebra $\hat{\mathbf{H}}_{\mathrm{gr}}$, which has the same generators as $\mathbf{H}_{\mathrm{gr}}$, but is constructed as an algebra over a polynomial algebra $\mathbb{C}[h]$. Relations (4.1) are unchanged, but the right-hand sides of the relations (4.2) are multiplied by h^2. Thus $\hat{\mathbf{H}}_{\mathrm{gr}}$ is $\mathbb{N}$-graded, with h and the elements of V having degree 1, and elements of W degree 0. As before, we can show that $\mathbb{C}[h]S(V)^W \subseteq Z(\hat{\mathbf{H}}_{\mathrm{gr}})$, so that $\hat{\mathbf{H}}_{\mathrm{gr}}$ has PI-degree at most $|W|$ by the same argument as above. On the other hand, $\hat{\mathbf{H}}_{\mathrm{gr}}/h\hat{\mathbf{H}}_{\mathrm{gr}} \cong S(V) * W$, the skew group algebra, and this has irreducible modules of dimension $|W|$—for example, one has an irreducible $S(V) * W$-structure on $S(V)/\mathfrak{m}S(V)$ for any maximal ideal $\mathfrak{m}$ of $S(V)^W$ contained in a maximal orbit (of size $|W|$) of maximal ideals of $S(V)$. Therefore,

$$\text{PI-degree}(\hat{\mathbf{H}}_{\text{gr}}) = \text{PI-degree}\big(S(V) * W\big) = |W|.$$

Now the Azumaya locus of $\hat{\mathbf{H}}_{\text{gr}}$ is dense in maxspec$(Z(\hat{\mathbf{H}}_{\text{gr}}))$ [4, Theorem III.1.7]; in particular, there must be an irreducible $\hat{\mathbf{H}}_{\text{gr}}$-module U annihilated by $h - \lambda$ for some $0 \neq \lambda \in \mathbb{C}$. This implies that PI-degree$(\hat{\mathbf{H}}_{\text{gr}}/(h-\lambda)\hat{\mathbf{H}}_{\text{gr}}) = |W|$, and so proves our claim, since all such factors, for $\lambda \neq 0$, are isomorphic to $\mathbf{H}_{\text{gr}}$. We have thus obtained the desired contradiction, so the proof is complete. □

4.4. The bilinear form

Consider the $\mathcal{Z}_{\text{gr}}$-linear map

$$\Phi_{\text{gr}} : \mathbf{H}_{\text{gr}} \to \mathcal{Z}_{\text{gr}},$$
$$\mathcal{B}_{\mathbf{H}_{\text{gr}}} \ni \mathbf{a}_i w \mapsto \begin{cases} 1 & \text{if } w = e,\ i = \max, \\ 0 & \text{otherwise.} \end{cases}$$

Define the form $\mathbb{B}$ for $\mathbf{H}_{\text{gr}}$ by $\mathbb{B}(a,b) = \Phi_{\text{gr}}(ab)$, for $a, b \in \mathbf{H}_{\text{gr}}$. We can now deduce the

Lemma. *The functional Φ_{gr} above satisfies Hypothesis* 2.3.

Proof. The proof is completely analogous to Lemma 3.5. □

Theorem. *The graded Hecke algebra $\mathbf{H}_{\text{gr}}$ is a free Frobenius extension of its central subalgebra $\mathcal{Z}_{\text{gr}}$ with Nakayama automorphism ν given by $\nu(w) = \epsilon_V(w)^{-1}w$, $\nu(v) = v$ for $w \in W$, $v \in V$, where ϵ_V is as defined in* 3.3.

Proof. It is immediate from Lemmas 3.5 and 2.3 that $\mathbf{H}_{\text{gr}}$ is a free Frobenius extension of $\mathcal{Z}_{\text{gr}}$ with form $\mathbb{B}$ as defined above. Therefore it remains only to determine the Nakayama automorphism. Let ν be as in the theorem, and let fw be a typical element from $\mathcal{B}_{\mathbf{H}_{\text{gr}}}$. First, let $x \in W$. Then $\mathbb{B}(fw, x) = \Phi_{\text{gr}}(fwx) = \delta_{w,x^{-1}}\Phi_{\text{gr}}(f)$ from the definition of Φ_{gr}, and $\mathbb{B}(\nu(x), fw) = \Phi_{\text{gr}}(\nu(x)fw) = \Phi_{\text{gr}}(\epsilon_V(x)^{-1}xfw) = \Phi_{\text{gr}}(fxw) = \delta_{w,x^{-1}}\Phi_{\text{gr}}(f)$ using the defining relations (4.1) of $\mathbf{H}_{\text{gr}}$ and again the definition of Φ_{gr}. If $a \in V$ we get

$$\mathbb{B}(fw, a) = \Phi_{\text{gr}}(fwa) = \Phi\big(f^w aw\big) \overset{(*)}{=} \Phi(faw) \overset{(**)}{=} \Phi(afw) = \mathbb{B}(a, fw). \tag{4.7}$$

The equality $(**)$ arises since the degree of fa and af is $|f| + 1$ and so both sides are zero unless $|f| \geqslant N - 1$. In the case $|f| = N$ or $N - 1$ then $|[a, f]| < N$ by (4.3). This then means that $\Phi(faw) = \Phi(afw - [f,a]w) = \Phi(afw)$, as required. The equality $(*)$ is true, because we have zero on both sides if $w \neq e$. Hence $\mathbb{B}(fw, a) = \mathbb{B}(a, fw)$ holds.

Since $\mathbf{H}_{\text{gr}}$ is generated by V and W, $\mathbb{B}(x,y) = \mathbb{B}(\nu(y), x)$ for any $x, y \in \mathbf{H}$, where ν is as claimed. □

Just as in Section 3.7, we can immediately deduce the

Corollary. *The factor $\mathbf{H}_{\mathrm{gr}_\chi}$ of the graded Hecke algebra $\mathbf{H}_{\mathrm{gr}}$ by a maximal ideal $\mathfrak{m}_\chi$ of its central subalgebra $\mathcal{Z}_{\mathrm{gr}}$ is a Frobenius algebra which in general is not symmetric.*

Proof. It is enough to show that the Nakayama automorphism of $\mathbf{H}_{\mathrm{gr}_\chi}$ is not always inner. We first claim that the trivial representation (where V acts by zero and W acts trivially) is a representation for $\mathbf{H}_{\mathrm{gr}}$. To see this we have to make sure the defining relation $[x, y] = \sum \Omega_w(x, y)w$ holds. Now $[x, y]$ acts by zero, and $\sum \Omega_w$ is an alternating W-invariant 2-form by (4.1). Thus it belongs to $(\wedge^2 V)^W$. This space is however always zero, since V is the reflection representation of a complex reflection group. Hence (4.2) is satisfied and the claim follows. In particular, the trivial representation is a representation for $\mathbf{H}_{\mathrm{gr}_0}$, the finite-dimensional quotient of $\mathbf{H}_{\mathrm{gr}}$ by the augmentation ideal of the centre. We consider now this quotient. If the Nakayama automorphism were inner on this factor then it would stabilise all simple modules. This contradicts the fact that the trivial representation is sent to the sign representation (Theorem 4.4). So the Nakayama must be non-inner on at least one factor. □

5. The extended affine Hecke algebra

In this section we show that the extended affine Hecke algebra $\mathcal{H}$ is a Frobenius extension of its centre, with non-trivial Nakayama automorphism, so that the corresponding reduced algebras $\mathcal{H}_\chi$ are Frobenius but not, in general, symmetric.

5.1. Let W be a (finite) Weyl group with length function l and integral weight lattice X, and let v be an indeterminant. For a parameter set L we denote by $\mathcal{H}$ the corresponding extended affine Hecke algebra over $\mathbb{C}[v, v^{-1}]$ as defined in [27, 3.1]. With the notation from [27, Lemma 3.4] $\mathcal{H}$ is a free $\mathbb{C}[v, v^{-1}]$-module with basis $T_w\theta_x$, for $w \in W$, $x \in X$, and the subalgebra $\mathbb{C}[v, v^{-1}]\langle\theta_x\colon x \in X\rangle$ is a Laurent polynomial algebra. Let $\mathcal{Z}_{\mathcal{H}} = \mathbb{C}[v, v^{-1}][X]^W$ be the centre of $\mathcal{H}$ [27, Proposition 3.11]. Since X is the weight lattice of a simple Lie algebra, the Pittie–Steinberg Theorem implies that $\mathbb{C}[v, v^{-1}][X]^W$ is a polynomial ring, [35, Theorem 1.2], and $\mathcal{H}$ is free over $\mathcal{Z}_{\mathcal{H}}$ of finite rank $|W|^2$. By abuse of language we denote by $(\mathbb{C}[v, v^{-1}][X]^W_+)$ the augmentation ideal in $\mathcal{Z}_{\mathcal{H}}$, corresponding to the function which sends each θ_x to 1. We consider the coinvariant algebra $\mathbb{C}[v, v^{-1}][X]/(\mathbb{C}[v, v^{-1}][X]^W_+)$ which we equip with a $\mathbb{Z}$-grading. This induces a $\mathbb{Z}_{\geqslant 0}$-filtration on $\mathbb{C}[v, v^{-1}][X]/(\mathbb{C}[v, v^{-1}][X]^W_+)$. We fix again a pair of (homogeneous) dual bases $\{\bar{\mathbf{a}}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\bar{\mathbf{a}}^i\colon 1 \leqslant i \leqslant |W|\}$ of the coinvariant algebra and lift these elements to bases $\{\mathbf{a}_i\colon 1 \leqslant i \leqslant |W|\}$, $\{\mathbf{a}^i\colon 1 \leqslant i \leqslant |W|\}$ of the free $\mathcal{Z}$-module $\mathbb{C}[v, v^{-1}][X]$ such that the (filtered) degree of $\mathbf{a}_i$ agrees with the grading degree of $\bar{\mathbf{a}}_i$. Then $\mathcal{H}$ is free over $\mathcal{Z}_{\mathcal{H}}$ of rank $|W|^2$. Let $\mathcal{B}_{\mathcal{H}}$ be the basis given by the $T_w\mathbf{a}_i$.

Lemma. *Let $\mathcal{H}_i$ be the $\mathcal{Z}_{\mathcal{H}}$-span of all $T_w\mathbf{a}_j$, where $1 \leqslant j \leqslant |W|$ and $l(w) \leqslant i$. Then $\mathcal{H} = \bigcup_{i\geqslant 0} \mathcal{H}_i$ is a filtration of $\mathcal{H}$.*

Proof. We have to show that $\mathcal{H}_i\mathcal{H}_j \subseteq \mathcal{H}_{i+j}$ for any $i, j \in \mathbb{Z}_{\geqslant 0}$. With the notation from [27, Proposition 3.9] we have $\theta_x T_s \equiv T_s\theta_{s(x)} \bmod \mathcal{H}_0$, and then for any $w \in W$,

$$\theta_x T_w \equiv T_w\theta_{w^{-1}(x)} \mod \mathcal{H}_{l(w)-1} \tag{5.1}$$

by induction. To establish the lemma we only have to show that $T_w\theta_x T_v\theta_y \in \mathcal{H}_{l(w)+l(v)}$ for any $v, w \in W$, $x, y \in X$. This is of course true if $l(v) = 0$. From formula (5.1) we get

$T_w\theta_x T_v\theta_y \equiv T_w T_v\theta_x\theta_y$ modulo $T_w\mathcal{H}_{l(v)-1} \subseteq \mathcal{H}_{l(w)+l(v)-1} \subset \mathcal{H}_{l(w)+l(v)}$. On the other hand, $T_w T_v\theta_x\theta_y \in \mathcal{H}_{l(w)+l(v)}$ and we are done. $\square$

5.2. Analogous to the cases above we define a $\mathcal{Z}_{\mathcal{H}}$-linear map

$$\Phi_{\mathcal{H}} : \mathcal{H} \to \mathcal{Z}_{\mathcal{H}},$$
$$\mathcal{B}_{\mathcal{H}} \ni T_w\mathbf{a}_i \mapsto \begin{cases} 1 & \text{if } w = e \text{ and } i = \max, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition. *The functional $\Phi_{\mathcal{H}}$ defined above satisfies Hypothesis* 2.3.

To prove this statement we need the following easily verified formulas:

Lemma. *Let $w, x \in W$, $f, g \in \mathbb{C}[v, v^{-1}][X]$.*

(1) *Let $T_x T_w = \sum_{y\in W} h_y T_y$ in $\mathcal{H}$. If $h_e \neq 0$ then $w = x^{-1}$.*
(2) *If $l(w) \geqslant l(x)$ then $\Phi_{\mathcal{H}}(T_w f T_x g) \neq 0$ implies $x = w^{-1}$.*

Proof. Statement (1) is an easy induction argument using the defining relations of $\mathcal{H}$ and therefore omitted. (For a representation theoretic interpretation of this statement we refer to [36, Theorem 3.1].) To verify Statement (2) note that if $x \in W$, $l(w) \geqslant l(x)$ then there exists some $h \in \mathbb{C}[v, v^{-1}][X]$ such that $T_w f T_x = T_w T_x h$ modulo $T_w\mathcal{H}_{l(x)-1}$ (by formula (5.1)). Therefore we get $T_w f T_x g = T_w T_x hg + r$, where $r \in T_w\mathcal{H}_{l(x)-1}$. Since $l(x) - 1 < l(w)$, using Statement (1) we deduce that $\Phi_{\mathcal{H}}(r) = 0$ and so $\Phi_{\mathcal{H}}(T_w f T_x g) = \Phi_{\mathcal{H}}(T_w T_x hg)$. The claim follows by applying Statement (1) again. $\square$

Proof of the proposition. Let $0 \neq u \in \mathcal{H}$, $u = \sum_{w,i} z_{w,i} T_w\mathbf{a}_i$, where $z_{w,i} \in \mathcal{Z}_{\mathcal{H}}$. Choose x of minimal length such that $z_{x^{-1},i} \neq 0$ for some i. From the lemma above and formula (5.1) we get

$$\Phi_{\mathcal{H}}(uT_x f) = \Phi\left(\sum_{w,i} z_{w,i} T_w\mathbf{a}_i T_x f\right) = \Phi\left(\sum_i z_{x^{-1},i} T_{x^{-1}}\mathbf{a}_i T_x f\right)$$

for any $f \in \mathbb{C}[v, v^{-1}][X]$. Using again the lemma above and formula (5.1) we can rewrite the expression $\sum_i z_{x^{-1},i} T_{x^{-1}}\mathbf{a}_i T_x$ in the form $\sum_i c_i\mathbf{a}_i + r$, where $r \in \mathcal{H}$ is such that when expanded in the standard bases no T_e occurs, and $c_i \in \mathcal{Z}_{\mathcal{H}}$ are not all zero. Since $\Phi_{\mathcal{H}}(rf) = 0$ for any $f \in \mathbb{C}[v, v^{-1}][X]$, it is enough to verify the Hypothesis 2.3 for $u = \sum_i c_i\mathbf{a}_i$. But now we are in a familiar situation, except that we have only filtered algebras instead of graded algebras. Nevertheless, the statement follows as in Lemma 3.5. $\square$

Theorem. *The extended affine Hecke algebra $\mathcal{H}$ is a free Frobenius extension of its centre $\mathcal{Z}_{\mathcal{H}}$. In general, this extension is not symmetric.*

Proof. We only have to verify that the Nakayama automorphism is non-trivial in general. This however follows directly from Proposition 5.3 below. $\square$

5.3. We now combine what we have just shown with the work in the previous section.

Proposition. *The factor $\mathcal{H}_\chi$ of the extended affine Hecke algebra $\mathcal{H}$ by a maximal ideal $\mathfrak{m}_\chi$ of the centre $\mathcal{Z}_{\mathcal{H}}$ is a Frobenius algebra; in general it is not symmetric.*

Proof. We have to show that there is at least one factor where the Nakayama automorphism is not inner. From Corollary 4.4 we know that such an ideal $\mathfrak{m}_\chi$ exists for the graded Hecke algebra $\mathbf{H}_{\mathrm{gr}}$. In fact, the proof of this corollary shows that we can choose the augmentation ideal $\mathfrak{m}_0$. Now we invoke [27, Theorem 9.3] in the special situation [27, 9.7] which provides an isomorphism between a completion $\hat{\mathcal{H}}$ of $\mathcal{H}$ at a certain maximal ideal I of the centre and the completion $\hat{\mathbf{H}}_{\mathrm{gr}}$ at $\mathfrak{m}_0$. As a result, there is at least one maximal ideal, $\mathfrak{m}_\chi = I$ of $\mathcal{Z}_{\mathcal{H}}$, where the factor algebra $\mathcal{H}_\chi$ is not symmetric. □

5.4. Nil–Hecke algebras

We would like to mention at least two related algebras, where our approach works, namely the *affine Nil–Hecke algebra* $\mathcal{H}^{\mathrm{nil}}$ and the *graded affine Nil–Hecke algebra* $\mathcal{H}^{\mathrm{nil}}_{\mathrm{gr}}$ associated to a Weyl group W. (For the definitions see e.g. [17]). Analogous to the affine Hecke algebra case, the centre of $\mathcal{H}^{\mathrm{nil}}$ is $\mathcal{Z} = \mathbb{C}[X]^W$ and $\mathcal{H}^{\mathrm{nil}}$ is a free $\mathcal{Z}$-module of rank $|W|^2$ [17, (1.9)], similarly for the graded affine Nil–Hecke algebras. If we define the forms completely analogous to the affine and graded Hecke algebras we deduce that $\mathcal{H}^{\mathrm{nil}}$ and $\mathcal{H}^{\mathrm{nil}}_{\mathrm{gr}}$ are free Frobenius extensions over their centres.

6. The quantised universal enveloping algebra

In this section we show that the quantised enveloping algebra $\mathcal{U}_\epsilon(\mathfrak{g})$ at a root of unity ϵ is a Frobenius extension of its Hopf centre, with trivial Nakayama automorphism, so that the reduced quantised enveloping algebras $\mathcal{U}_\epsilon(\mathfrak{g})_\chi$ are symmetric.

6.1. The PBW-basis and the central subalgebra

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix a Borel and Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b} \supseteq \mathfrak{h}$, and denote the Weyl group by W and the set of simple reflections by S. Let π be the corresponding set of simple roots and ρ the half-sum of positive roots. Let $\epsilon \in \mathbb{C}$ be an lth root of unity, for some odd positive integer l, $l \neq 3$ if $\mathfrak{g}$ has a summand of type G_2. Let $Q \subseteq P$ be, respectively, the root lattice and the weight lattice of $\mathfrak{g}$, with the W-equivariant bilinear form $(,) : P \times Q \to \mathbb{Z}$.

The simply connected form of the quantised universal enveloping algebra $\mathcal{U} = \mathcal{U}_\epsilon(\mathfrak{g})$ is a $\mathbb{C}$-algebra with generators E_α, F_α, K_λ, for $\alpha \in \pi$ and $\lambda \in P$. For the defining relations and further details we refer for example to [9, 9.1] or [4, I.6.3, III.6.1]. Let w_0 be the longest element of W, and fix a reduced expression

$$w_0 = s_{i_1} s_{i_2} \dots s_{i_N}, \tag{6.1}$$

where $s_{i_j} \in S$ for $1 \leqslant j \leqslant N$. Let α_{i_j} be the simple root corresponding to $s_{i_j} \in S$. Recall that Lusztig defined an action on $\mathcal{U}$ of the braid group B_W corresponding to W (see [28], [9, Section 9], [21, Section 8] or [4, I.6.7, I.6.8]). Let T_i be the automorphism corresponding to the simple reflection $s_i \in S$. We set

$$\beta_k := s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k}), \tag{6.2}$$

and put $E_{\beta_k} = T_{i_1} T_{i_2} \dots T_{i_{k-1}}(E_{\alpha_{i_k}})$ and $F_{\beta_k} = T_{i_1} T_{i_2} \dots T_{i_{k-1}}(F_{\alpha_{i_k}})$. For any sequence $\mathbf{m} = (m_1, m_2, \dots, m_N) \in \mathbb{Z}^N_{\geqslant 0}$ let

$$E^{\mathbf{m}} = E^{m_1}_{\beta_1} E^{m_2}_{\beta_2} \dots E^{m_N}_{\beta_N},$$
$$F^{\mathbf{m}} = F^{m_N}_{\beta_N} F^{m_{N-1}}_{\beta_{N-1}} \dots F^{m_1}_{\beta_1}.$$

This yields a PBW-basis of $\mathcal{U}$ (associated with (6.1)), namely

$$\mathcal{B} = \left\{ F^{\mathbf{k}} K_\lambda E^{\mathbf{m}} \colon \mathbf{k}, \mathbf{m} \in \mathbb{Z}^N_{\geqslant 0},\ \lambda \in P \right\},$$

see [9, Theorem 9.3], [4, I.6.2, III.6.1]. The subspace $\mathcal{Z}$ of $\mathcal{U}$ spanned by the monomials $F^{l\mathbf{k}} K_{l\lambda} E^{l\mathbf{m}}$ is a central Hopf subalgebra of $\mathcal{U}$, called the *l-centre*, and $\mathcal{U}$ is a free $\mathcal{Z}$-module of finite rank (see [9, 19.1], [4, III.6.2]). As a $\mathcal{Z}$-basis of $\mathcal{U}$ one can choose the subset $\mathcal{B}'$ of $\mathcal{B}$ given by elements of the form

$$F^{\mathbf{k}} K_\lambda E^{\mathbf{m}}, \tag{6.3}$$

where $0 \leqslant k_i, l_i < l$ and the coefficients of λ in terms of fundamental weights are non-negative integers less than l.

6.2. Filtrations, degrees and commutation formulas

To simplify formulas we set $E_i = E_{\beta_i}$ and $F_i = F_{\beta_i}$. (Note that E_i is not E_{α_i} in general.) Let $i < j$. There are commutation formulas holding in $\mathcal{U}$ as follows [4, Proposition I.6.10, Theorem III.6.1(4)]:

$$E_i E_j = \epsilon^{(\beta_i, \beta_j)} E_j E_i + r, \tag{6.4}$$
$$F_i F_j = \epsilon^{-(\beta_i, \beta_j)} F_j F_i + r', \tag{6.5}$$

where r (respectively r'), written in the PBW-basis, involves no monomial containing any E_k (respectively F_k) for $k \leqslant i$ or $k \geqslant j$.

The algebra $\mathcal{U}$ is Q-graded (see e.g. [21, 4.7]), but also has several other filtrations [9, 10.1], [4, I.6.11, III.6.1]. First, there is the *degree filtration*, a $\mathbb{Z}_{\geqslant 0}$-filtration obtained by putting $F^{\mathbf{k}} K_\lambda E^{\mathbf{m}} \in \mathcal{B}$ in degree

$$\deg\left(F^{\mathbf{k}} K_\lambda E^{\mathbf{m}}\right) = \sum_{i=1}^{N} (k_i + m_i)\,\mathrm{ht}(\beta_i),$$

where ht denotes the height function. One can refine this to a $(\mathbb{Z}_{\geqslant 0})^{2N+1}$-filtration by putting $F^{\mathbf{k}} K_\lambda E^{\mathbf{m}} \in \mathcal{B}$ in degree

$$\mathrm{d}\big(F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}\big)=\big(k_N,k_{N-1},\dots,k_1,m_1,m_2,\dots,m_N,\deg\big(F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}\big)\big).$$

Putting the reverse lexicographic ordering on $(\mathbb{Z}_{\geqslant 0})^{2N+1}$ (i.e. $e_1<e_2<\cdots$, where $(e_i)_j=\delta_{i,j}$) defines the filtration by *total degree*. The E's and F's commute up to terms of lower total degree [9, 10.1], [4, Proposition I.6.11]:

$$E_iF_j=F_jE_i+\text{terms of lower total degree.} \tag{6.6}$$

We denote by

$$\max=2(l-1)\sum_{i=1}^{N}\mathrm{ht}(\beta_i)$$

the maximal deg-value on $\mathcal{B}'$.

6.3. The bilinear form

In view of the $\mathcal{Z}$-freeness of $\mathcal{U}$ on the basis $\mathcal{B}'$, we can define a $\mathcal{Z}$-linear map $\Phi:\mathcal{U}\to Z$ as follows. Set $\mathbf{l}:=(l-1,l-1,\dots,l-1)$, and define

$$\Phi:\mathcal{B}'\to\mathcal{Z},$$
$$F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}\mapsto\begin{cases}1 & \text{if } \mathbf{k}=\mathbf{m}=\mathbf{l},\ \lambda=0,\\ 0 & \text{otherwise,}\end{cases}$$

and extend this $\mathcal{Z}$-linearly.

Lemma. *The functional Φ satisfies Hypothesis* 2.3.

Proof. For $\mathbf{m}\in(\mathbb{Z}_{\geqslant 0})^N$, define $\tilde{\mathbf{m}}:=\mathbf{l}-\mathbf{m}\in(\mathbb{Z})^N$. For $x=F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}\in\mathcal{B}$ and $\mu\in P$ set $\tilde{x}_\mu=F^{\tilde{\mathbf{k}}}K_\mu E^{\tilde{\mathbf{m}}}$ and write $k_i(x)=k_i$, $m_i(x)=m_i$.

Claim 1. *Let $x,\ y\in\mathcal{B}$. If* $\deg(x)+\deg(y)<\max$ *then* $\Phi(xy)=0$.

This follows directly from the fact that the commutation relations (6.4), (6.5) and (6.6) do not increase the deg-value and Φ annihilates every monomial in $\mathcal{B}'$ which is not of maximal deg-value.

Claim 2. *Let $x,y\in\mathcal{B}'$, $\mu\in P$. If* $\mathrm{d}(x)<\mathrm{d}(\tilde{y}_\mu)$ *then* $\Phi(yx)=0$.

If $\mathrm{d}(x)<\mathrm{d}(\tilde{y}_\mu)$ then $\deg(x)\leqslant\deg(\tilde{y}_\mu)=\max-\deg(y)$, hence $\deg(x)+\deg(y)\leqslant\max$. By Claim 1 we only have to deal with the case $\deg(x)+\deg(y)=\max$. From our assumption and the definition of d it follows that *either*

- there is a $k_j(x)$ such that $k_j(x)\neq l-1-k_j(y)$ (so that $k_j(x)<l-1-k_j(y)$), *or*
- there is an $m_j(x)$ such that $m_j(x)<l-1-m_j(y)$ (so that $m_j(x)<l-1-m_j(y)$).

Let us assume the latter for a moment and choose j maximal with this property. Recalling from (6.6) and the commutation relations for the Ks with the Es and the Ks with the Fs that the relevant generators of $\mathcal{U}$ commute up to non-zero scalars and terms of lower deg-value, we see that it is enough to show that

$$\Phi\big(F^{\mathbf{k}(y)}F^{\mathbf{k}(x)}K_{\lambda(x)}K_{\lambda(y)}E^{\mathbf{m}(y)}E^{\mathbf{m}(x)}\big)=0.$$

From the relation (6.4) we get

$$E_N^{m_n(y)}E_1^{m_1(x)}\cdots E_N^{m_N(x)}=cE_1^{m_1(x)}\cdots E_N^{m_N(x)+m_N(y)}+r,$$

for some $c\in\mathbb{C}^*$ and some $r\in\mathcal{U}$ such that E_N occurs in every monomial in r with power strictly smaller than $m_N(y)+m_N(x)\leqslant l-1$. In particular,

$$\begin{aligned}&\Phi\big(F^{\mathbf{k}(y)}F^{\mathbf{k}(x)}K_{\lambda(x)}K_{\lambda(y)}E^{\mathbf{m}(y)}E^{\mathbf{m}(x)}\big)\\&\quad=c\Phi\big(F^{\mathbf{k}(y)}F^{\mathbf{k}(x)}K_{\lambda(x)}K_{\lambda(y)}E_1^{m_1(y)}\cdots E_{N-1}^{m_{N-1}(y)}E_1^{m_1(x)}\cdots E_N^{m_N(x)+m_N(y)}\big).\end{aligned}$$

Repeating this argument we get

$$\Phi\big(F^{\mathbf{k}(y)}F^{\mathbf{k}(x)}K_{\lambda(x)}K_{\lambda(y)}E^{\mathbf{m}(y)}E^{\mathbf{m}(x)}\big)=c'\Phi\big(F^{\mathbf{k}(y)}F^{\mathbf{k}(x)}K_{\lambda(x)+\lambda(y)}X\big),$$

where $X=E_1^{m_1(y)}\cdots E_{j-1}^{m_{j-1}(y)}E_1^{m_1(x)}\cdots E_{j-1}^{m_{j-1}(x)}E_j^{m_j(x)+m_j(y)}E_N^{m_N(x)+m_N(y)}$ and $c'\in\mathbb{C}^*$. The result is zero since any commutation of the E_is for $i<j$ does not involve E_j because of (6.4), and since $m_j(x)+m_j(y)<l-1$. The remaining case, where there is a $k_j(x)$ such that $k_j(x)\neq l-1-k_j(y)$, can be proved similarly and is therefore omitted.

Claim 3. *Let $x,\ y\in\mathcal{B}$, $x=F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}$, $\mu\in P$ and $\mathrm{d}(x)=\mathrm{d}(\tilde{y}_\mu)$ then we have $\Phi(yx)\neq 0$ if and only if $\lambda+\mu\in lP$.*

With the arguments from the proof of Claim 2 we get

$$\Phi(yx)=c\Phi\big(F^{\mathbf{l}}K_{\lambda(x)+\lambda(y)}E^{\mathbf{l}}\big),$$

for some non-zero number $c\in\mathbb{C}$. Claim 3 follows then from the definition of Φ.

To prove the proposition, let $x\in\mathcal{U}$ be arbitrary and write $x=\sum_{y\in\mathcal{B}'}z_y y$ with $z_y\in\mathcal{Z}$. We choose $b=F^{\mathbf{k}}K_\lambda E^{\mathbf{m}}\in\mathcal{B}'$ of maximal total degree such that $z_b\neq 0$. If now $a=F^{\mathbf{r}}K_\nu E^{\mathbf{s}}\in\mathcal{B}'$ with $z_a\neq 0$ and $\nu\in P$ arbitrary, then we have $\deg(a)\leqslant\deg(b)$, hence, for any $\mu\in P$,

$$\deg(a)+\deg(\tilde{b}_\mu)=\deg(a)+\max-\deg(b)\leqslant\max.$$

If this inequality is strict, Claim 1 implies $\Phi(\tilde{b}_\mu a)=0$. Let us assume equality. Then we either have $\mathrm{d}(a)<\mathrm{d}(b)$ which implies $\Phi(\tilde{b}_\mu a)=0$ by Claim 2, or $\mathrm{d}(a)=\mathrm{d}(b)$. The latter means (because of Claim 3) that $\Phi(\tilde{b}_\mu a)=0$ except when $\nu+\mu\in lP$. In particular, $\Phi(\tilde{b}_{-\lambda}a)=0$, except when $a=b$.

Summarising, we get $\Phi(\tilde{b}_{-\lambda}x)=\Phi(\tilde{b}_{-\lambda}b)=cz_b\neq 0$ for some unit c, as required. □

6.4. Symmetry of the form

Lemma. *The Nakayama automorphism ν of $\mathcal{U}$ with respect to $\mathbb{B}$ is the identity.*

Proof. We have to prove that ν fixes all generators. We will run through all possibilities y for generators and prove $\mathbb{B}(x,y)=\mathbb{B}(y,x)$ for all $x\in\mathcal{B}'$.

First, let $y=K_\lambda$. From Claim 2 we have automatically $\mathbb{B}(x,y)=0=\mathbb{B}(y,x)$ unless x has maximal degree max. But then K_λ commutes with x and hence $\nu(K_\lambda)=K_\lambda$.

Now let $y=E_\alpha$ for some simple root α, and let $x\in\mathcal{B}'$. Claim 1 implies that $\mathbb{B}(x,y)=0=\mathbb{B}(y,x)$ unless $\deg x\geqslant \max-1$, because $\deg(E_\alpha)=1$. If $x=F^{\mathbf{l}}K_\lambda E^{\mathbf{l}}$ then both yx and xy have Q-grade equal to α. Thus $\Phi(yx)=0=\Phi(xy)$ since, by definition, Φ is non-zero only on elements whose Q-grade belongs to ℓQ. We now have two possibilities for x: either $\deg(F^{\mathbf{k}(x)})\neq\frac{\max}{2}$ and $\deg(E^{\mathbf{m}(x)})=\frac{\max}{2}$, or vice versa. Let us consider the first case. Then $\mathbb{B}(x,y)=0=\mathbb{B}(y,x)$, because Φ annihilates everything which does not have the same Q/lQ-grading as $F^{\mathbf{l}}E^{\mathbf{l}}$ by definition. In the second case, the Q-grading again implies $\mathbb{B}(x,y)=0=\mathbb{B}(y,x)$, unless $m_j(x)\neq l-1$ implies $\beta_j=\alpha$. Let j be such that this equation holds. That means we have to compare $\mathbb{B}(x,y)=\Phi(xE_\alpha)$ and $\mathbb{B}(y,x)=\Phi(E_\alpha x)$, where $x=F^{\mathbf{l}}K_\lambda E_1^{l-1}\dots E_{j-1}^{l-1}E_j^{l-2}E_{j+1}^{l-1}\dots E_N^{l-1}$. Both terms are trivial unless $\lambda=0$. From the commutator relation (6.4) it follows that $\mathbb{B}(x,y)=\epsilon^{-(l-1)(\beta_{j+1}+\dots+\beta_N,\beta_j)}$ and $\mathbb{B}(y,x)=\epsilon^{-(l-1)(\beta_1+\dots+\beta_{j-1},\beta_j)}$. It is now enough to show that the exponents are the same.

Put $w=s_{i_1}s_{i_2}\cdots s_{i_{j-1}}$. Then $M^-=\{\beta_r\colon 1\leqslant r\leqslant j-1\}$ (respectively $M^+=\{\beta_r\colon j\leqslant r\leqslant N\}$) is exactly the set of all positive roots such that $w^{-1}(\beta)$ is negative (respectively positive). Set $M_1=w^{-1}(M^+)$ and $M_2=-w^{-1}(M^-)$. The disjoint union of these two sets is exactly the set of all positive roots (see e.g. [24, I.4.3, Theorem B]). By definition (see (6.2)) we have $w^{-1}(\beta_j)=\alpha_{i_j}\in M_1$. From the definition of ρ, the half-sum of positive roots, we get

$$(\alpha_{i_j},\alpha_{i_j})=(\alpha_{i_j},\alpha_{i_j})+\sum_{\beta\in M_1\setminus\{\alpha_{i_j}\}}(\beta,\alpha_{i_j})+\sum_{\beta\in M_2}(\beta,\alpha_{i_j}).$$

Since the bracket (,) is non-degenerate and W-equivariant, we get

$$\begin{aligned}0&=\sum_{\beta\in M_1\setminus\{\alpha_{i_j}\}}\bigl(w(\beta),w(\alpha_{i_j})\bigr)+\sum_{\beta\in M_2}\bigl(w(\beta),w(\alpha_{i_j})\bigr)\\&=\sum_{\beta\in M^+\setminus\{\beta_j\}}(\beta,\beta_j)-\sum_{\beta\in M^-}(\beta,\beta_j).\end{aligned}$$

Hence we get the required equality for the exponents and therefore $\mathbb{B}(x,y)=\mathbb{B}(y,x)$. We are left with the case $y=F_\alpha$ for some simple root α. The arguments there are similar, and therefore omitted. This completes the proof of the lemma. □

6.5. Recall the terminology of 2.1. From Proposition 6.3 together with Proposition 2.3 and Lemma 6.4, we have:

Theorem. *The quantised universal enveloping algebra $\mathcal{U}=\mathcal{U}_\epsilon(\mathfrak{g})$ at an lth root of unity is a free Frobenius extension of its lth centre $\mathcal{Z}$. The form $\mathbb{B}$ has a trivial Nakayama automorphism.*

The following corollary is immediate from the theorem and the discussion in 2.4, noting that $\mathcal{U}_\epsilon(\mathfrak{g})$ has finite global dimension by [3, Theorem 2.3].

Corollary. *Let χ be a maximal ideal of $\mathcal{Z}$.*

(1) *The reduced quantised enveloping algebra $\mathcal{U}_\chi := \mathcal{U}_\epsilon(\mathfrak{g})/\mathcal{U}_\epsilon(\mathfrak{g})\chi$ is a symmetric algebra.*
(2) *$\mathcal{U}_\epsilon(\mathfrak{g})$ is a Calabi–Yau $\mathcal{Z}$-algebra of dimension* $\dim \mathfrak{g}$.

7. Quantum Borels

In this section we show that the quantum Borel $\mathcal{U}^{\geqslant 0}$ at a root of unity ϵ is a Frobenius extension of its Hopf centre, with non-trivial Nakayama automorphism, so that the reduced quantum Borels $\mathcal{U}_\chi^{\geqslant 0}$ are Frobenius, but not in general symmetric.

7.1. Let $\mathfrak{g}$ be as above. Let $\mathcal{U}_\epsilon^{\geqslant 0}$ be the subalgebra of $\mathcal{U}_\epsilon(\mathfrak{g})$ generated by all the Es and Ks. The PBW-basis of $\mathcal{U}_\epsilon(\mathfrak{g})$ gives rise to a PBW-basis of $\mathcal{U}_\epsilon^{\geqslant 0}$ given by the elements of the form $K_\lambda E^{\mathbf{m}}$, where $\mathbf{m} \in \mathbb{Z}_{\geqslant 0}^N$ and $\lambda \in P$, [9, 9.3]. Moreover, $\mathcal{U}_\epsilon^{\geqslant 0}$ is free over $Z_+ := \mathcal{Z} \cap \mathcal{U}_\epsilon^{\geqslant 0}$ with basis $\mathcal{B}'_+$ given by all elements of the form $K_\lambda E^{\mathbf{m}}$, where $0 \leqslant m_i < l$ and the coefficients of λ in terms of fundamental weights are non-negative integers less than l [9, 19.1].

7.2. The bilinear form and its Nakayama automorphism

Analogously to Section 4, we define a Z_+-linear map $\Phi_+ : \mathcal{U}_\epsilon^{\geqslant 0} \to Z_+$ by

$$\mathcal{B}'_+ \ni K_\lambda E^{\mathbf{m}} \mapsto \begin{cases} 1 & \text{if } \mathbf{m} = \mathbf{l},\ \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define a Z_+-bilinear associative form $\mathbb{B}_+$ on $\mathcal{U}_\epsilon^{\geqslant 0}$ by putting $\mathbb{B}_+(x, y) = \Phi(xy)$ for any x, $y \in \mathcal{U}^{\geqslant 0}$.

Theorem. *Let $\mathcal{U}_\epsilon^{\geqslant 0}$ be the quantum Borel defined in* (7.1)*, with central subalgebra Z_+ as defined there. Let Φ_+ and $\mathbb{B}_+$ be as above.*

(1) *Φ_+ satisfies Hypothesis* 2.3.
(2) *The form $\mathbb{B}_+$ is non-degenerate and has a dual free pair of bases, so that $\mathcal{U}_\epsilon^{\geqslant 0}$ is a free Frobenius extension of Z_+.*
(3) *The corresponding Nakayama automorphism ν_+ of $\mathcal{U}_\epsilon^{\geqslant 0}$ is given by $\nu_+(E_\alpha) = E_\alpha$ for simple roots α and $\nu_+(K_\lambda) = \epsilon^{(2\rho,\lambda)} K_\lambda$ for $\lambda \in P$.*

Proof. The proofs of (1) and (2) are similar to, but easier than the corresponding arguments for $\mathcal{U}_\epsilon(\mathfrak{g})$, so we leave the details to the reader.

Consider now part (3). As in the proof of Lemma 6.4, $\nu_+(E_\alpha) = E_\alpha$ for any simple root α. By the degree argument from the same proof, the value of $\nu_+(K_\lambda)$ is determined by $E^{\mathbf{l}} K_\lambda = \nu_+(K_\lambda) E^{\mathbf{l}}$. Hence

$$\nu_+(K_\lambda) = \epsilon^{-(l-1)(\beta_1 + \cdots + \beta_N, \lambda)} K_\lambda = \epsilon^{(2\rho,\lambda)} K_\lambda.$$

The result follows. □

Remarks. 1. With the standard comultiplication of [4, I.6], [21, Chapter 4], $E_\alpha \mapsto E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$, $F_\alpha \mapsto F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$, $K_\lambda \mapsto K_\lambda \otimes K_\lambda$ for $\alpha \in \pi$, $\lambda \in P$, then ν_+ is nothing else than the right winding automorphism [4, I.9.25] $\tau^r_{2\rho}$ of $\mathcal{U}_\epsilon(\mathfrak{g})$ associated with the representation 2ρ, restricted to $\mathcal{U}_\epsilon^{\geqslant 0}$.

2. Calculations parallel to the above will of course handle $\mathcal{U}_\epsilon^{\leqslant 0}$, the Hopf subalgebra of $\mathcal{U}_\epsilon(\mathfrak{g})$ generated by the F_αs and the K_λs. A more elegant approach is to make use of the Chevalley involution ω [21, Lemma 4.6(a)]: $\omega(E_\alpha) = F_\alpha$ and $\omega(K_i) = K_i^{-1}$, so ω is an algebra automorphism and a coalgebra anti-automorphism. Thus one calculates that the Nakayama automorphism of $\mathcal{U}_\epsilon^{\leqslant 0}$, namely $\omega \circ \tau^r_{2\rho} \circ \omega^{-1}$, is the restriction of the automorphism $\tau^\ell_{-2\rho}$ of $\mathcal{U}_\epsilon(\mathfrak{g})$.

8. Quantised function algebras

In this section we show that the quantised function algebra $\mathcal{O}_\epsilon[G]$ at a root of unity ϵ is a Frobenius extension of its Hopf centre with non-trivial Nakayama automorphism, cf. Section 2.5. The reduced quantised function algebras $\mathcal{O}_\epsilon[G](g)$ are Frobenius but not, in general, symmetric.

8.1. Preliminaries

Let G be the simply connected, semisimple algebraic group over $\mathbb{C}$ associated with the semisimple Lie algebra $\mathfrak{g}$. Let B be the Borel subalgebra of G associated with π and let B^- be the opposite Borel. Let T be the corresponding maximal torus. Let ϵ be as in (6.1), and let $\mathcal{O}_\epsilon[G]$ be the quantised function algebra of G at the root of unity ϵ. For the definition and basic properties of $\mathcal{O}_\epsilon[G]$, see [10] or [4, III.7.1].[1] Recall that de Concini and Lyubashenko show [10] that $\mathcal{O}_\epsilon[G]$ is a noetherian Hopf $\mathbb{C}$-algebra which is a finitely generated module over its centre. (An outline proof is also provided in [4, Theorems III.7.2, III.7.3].) Indeed, more specifically, $\mathcal{O}_\epsilon[G]$ contains a copy of the coordinate ring of G, $\mathcal{O}[G]$, as a central Hopf subalgebra, and, by [5, Proposition 2.2], $\mathcal{O}_\epsilon[G]$ is a free $\mathcal{O}[G]$-module of rank $l^{\dim G}$.

Calculations with $\mathcal{O}_\epsilon[G]$ are most easily carried out by embedding it as a subalgebra of $\mathcal{U}_\epsilon^{\leqslant 0} \otimes \mathcal{U}_\epsilon^{\geqslant 0}$, as in [10, Section 4.3]. But in fact [10] works with $(\mathcal{O}_\epsilon[G])^{\mathrm{op}}$, in terms of the definition of the function algebra of [4] or [21]; the simplest way to accommodate this here is to include a map from ϵ to ϵ^{-1} into the embedding. Once this is done, the inclusion μ'' of [10, 4.3] is given by the composite

$$i' : \mathcal{O}_\epsilon[G] \xrightarrow{\text{comult}} \mathcal{O}_\epsilon[G] \otimes \mathcal{O}_\epsilon[G] \to \mathcal{O}_\epsilon[B] \otimes \mathcal{O}_\epsilon[B^-] \to \mathcal{U}^{\leqslant 0}_{\epsilon^{-1}} \otimes_{\mathbb{C}} \mathcal{U}^{\geqslant 0}_{\epsilon^{-1}},$$

where the second map is the canonical one (given by "restriction") and the last map combines the isomorphism from [10, Lemma 3.4] with the parameter switch explained above. Note in passing that this embedding shows that $\mathcal{O}_\epsilon[G]$ is a domain. Moreover, by [10, Theorem 4.6, Lemma 4.3 and Proposition 6.5], there is a non-zero element z of $\mathcal{O}[G]$, such that i' extends to an inclusion

$$i : \mathcal{O}_\epsilon[G]\big[z^{-1}\big] \to \mathcal{U}^{\leqslant 0}_{\epsilon^{-1}} \otimes_{\mathbb{C}} \mathcal{U}^{\geqslant 0}_{\epsilon^{-1}},$$

[1] Note, however, that the algebra in [10] is the *opposite algebra* to that in [4]; put in another way, there is a switch between ϵ and ϵ^{-1} in going from [4, III.7.1] to [10].

with image generated by the elements $1 \otimes E_\alpha$, $F_\alpha \otimes 1$ and $K_{-\lambda} \otimes K_\lambda$, for simple roots α and integral weights λ. In the following we will often identify $\mathcal{O}_\epsilon[G][z^{-1}]$ with its image under i.

In particular, making this identification, a basis $\mathcal{B}_\mathcal{O}$ of $\mathcal{O}_\epsilon[G][z^{-1}]$ as a free $\mathcal{O}[G][z^{-1}]$-module is given by the set of elements

$$F^{\mathbf{k}} K_{-\lambda} \otimes K_\lambda E^{\mathbf{m}},$$

where $0 \leqslant k_i, m_i < l$ and the coefficients of λ in terms of fundamental weights are non-negative integers less than l; for this, see the proof of [10, Proposition 7.2].

8.2. The bilinear form

We can define a $\mathcal{O}[G][z^{-1}]$-linear map

$$\Phi : \mathcal{O}_\epsilon[G]\big[z^{-1}\big] \to \mathcal{O}[G]\big[z^{-1}\big]$$

by mapping

$$\mathcal{B}_\mathcal{O} \ni F^{\mathbf{k}} K_{-\lambda} \otimes K_\lambda E^{\mathbf{m}} \mapsto \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{m} = \mathbf{l},\ \lambda = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and extending $\mathcal{O}[G][z^{-1}]$-linearly. Since $\mathcal{O}[G][z^{-1}]$ is central, we get an associative $\mathcal{O}[G][z^{-1}]$-bilinear form $\mathbb{B} : \mathcal{O}_\epsilon[G][z^{-1}] \times \mathcal{O}_\epsilon[G][z^{-1}] \to \mathcal{O}_\epsilon[G][z^{-1}]$ by putting $\mathbb{B}(x, y) = \Phi(xy)$ for x, $y \in \mathcal{O}_\epsilon[G][z^{-1}]$.

8.3. Frobenius extension

We can now record the key

Lemma. *The functional Φ satisfies Hypothesis* 2.3.

Proof. The argument is similar to the ones used to prove Lemma 6.3 and Theorem 7.2, and is therefore left to the reader. □

As usual, the above lemma yields at once the first part of the following

Theorem.

(1) *$\mathcal{O}_\epsilon[G][z^{-1}]$ is a free Frobenius extension of $\mathcal{O}[G][z^{-1}]$ with the form $\mathbb{B}$ defined in Section* 8.2.
(2) *In the notation of Remarks* 7.2*, the Nakayama automorphism of $\mathcal{O}_\epsilon[G][z^{-1}]$ is the restriction of the automorphism $\tau^\ell_{-2\rho} \otimes \tau^r_{2\rho}$ of $\mathcal{U}^{\leqslant 0}_{\epsilon^{-1}} \otimes_\mathbb{C} \mathcal{U}^{\geqslant 0}_{\epsilon^{-1}}$. In particular, it fixes $F_\alpha \otimes 1$ and $1 \otimes E_\alpha$ for all simple roots α, and maps $K_\lambda \otimes K_{-\lambda}$ to $\epsilon^{2(2\rho,\lambda)} K_\lambda \otimes K_{-\lambda}$.*
(3) *There is a non-degenerate $\mathcal{O}[G]$-bilinear form $\mathbb{B}'$ on $\mathcal{O}_\epsilon[G]$ with values in $\mathcal{O}[G]$ and Nakayama automorphism $\nu_\mathcal{O} = \tau^l_{-2\rho} \otimes \tau^r_{2\rho}$.*

Proof. (2) This is clear from Theorem 7.2 and Remarks 7.2(2).

(3) Choose a finite generating set $\mathcal{F}$ of $\mathcal{O}_\epsilon[G]$ as a $\mathcal{O}[G]$-module. There is a non-negative integer k such that $\mathbb{B}(u,v) \in z^{-k}\mathcal{O}[G]$ for all $u, v \in \mathcal{F}$. Let k_0 be the minimal such integer, and define $\mathbb{B}' := z^{k_0}\mathbb{B}$. Then $\mathbb{B}'$ has the stated properties. □

Remark. Suppose that $G = SL(n,\mathbb{C})$, so that $\mathcal{O}_\epsilon[G]$ is generated by $\{X_{ij}\colon 1 \leqslant i, j \leqslant n\}$, with the relations given at [4, I.2.2, I.2.4]. Then it is easy to calculate that the automorphism $\nu_{\mathcal{O}}$ of the theorem is given by $\nu_{\mathcal{O}}(X_{ij})\epsilon^{2(n+1-i-j)}X_{ij}$, for $i, j = 1, \ldots, n$.

8.4. Finite-dimensional factors

Corollary 8.3 is sufficient to yield the desired applications to the finite-dimensional representation theory of $\mathcal{O}_\epsilon[G]$, as follows:

Theorem. *Let $g \in G$ and let $\mathfrak{m}_g$ be the corresponding maximal ideal of $\mathcal{O}[G]$. Then the algebra $\mathcal{O}_\epsilon[G](g) := \mathcal{O}_\epsilon[G]/\mathcal{O}_\epsilon[G]\mathfrak{m}_g$ is a Frobenius algebra with Nakayama automorphism induced from $\nu_{\mathcal{O}}$.*

Proof. First let $\mathfrak{m}$ be a maximal ideal of the algebra $\mathcal{O}[G][z^{-1}]$ of Proposition 8.3. Then Proposition 8.3 implies that there is a non-degenerate $\mathbb{C}$-bilinear form $\overline{\mathbb{B}}$ on $\mathcal{O}_\epsilon[G][z^{-1}]/\mathfrak{m}\mathcal{O}_\epsilon[G][z^{-1}]$, with Nakayama automorphism induced also from $\nu_{\mathcal{O}}$.

Now suppose that z is not in $\mathfrak{m}_g$. Then

$$\mathcal{O}_\epsilon[G]\big[z^{-1}\big]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]\big[z^{-1}\big] \cong \big(\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]\big)\big[z^{-1}\big] = \mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G],$$

using [15, Exercise 9L] for the isomorphism, and the fact that z is a unit modulo $\mathfrak{m}_g\mathcal{O}_\epsilon[G]$ for the equality. In particular, by the first paragraph of the proof,

$$\textit{the desired conclusions apply to } \mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]. \tag{8.1}$$

To extend this conclusion to arbitrary g in G we apply the results of [10]. Recall that there is a Poisson bracket on $\mathcal{O}[G]$, under which G decomposes as a disjoint union of symplectic leaves. Moreover, if $g, h \in G$ belong to the same symplectic leaf, then

$$\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G] \cong \mathcal{O}_\epsilon[G]/\mathfrak{m}_h\mathcal{O}_\epsilon[G] \tag{8.2}$$

by [10, Corollary 9.4]. In fact, $\mathcal{O}_\epsilon[G]$ is a Poisson $\mathcal{O}[G]$-order in the sense of [6, 2.1], and we can, if preferred, quote [6, Theorem 4.2] to obtain (8.2). By [10, Propositions 9.3 and 8.7(b)] there is an action of the torus T as automorphisms of $\mathcal{O}_\epsilon[G]$, restricting to Poisson automorphisms of the subalgebra $\mathcal{O}[G]$ induced by right and left multiplication by T on G, preserving the Poisson order structure in the sense of [6, 3.8]. Therefore, if $g \in G$ and $t \in T$, then

$$\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G] \cong \mathcal{O}_\epsilon[G]/\mathfrak{m}_{tg}\mathcal{O}_\epsilon[G] \cong \mathcal{O}_\epsilon[G]/\mathfrak{m}_{gt}\mathcal{O}_\epsilon[G]. \tag{8.3}$$

Since the action of T preserves the leaves we can conclude from (8.2) and (8.3) that

$$\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G] \cong \mathcal{O}_\epsilon[G]/\mathfrak{m}_h\mathcal{O}_\epsilon[G] \tag{8.4}$$

if g and h are in the same T-orbit of symplectic leaves [10, Corollary 9.4], [6, 4.2 and 4.3].

Recall from [19, Theorems A.3.2 and A.2.1] (see also [10, Section 9.3]) that the T-orbits of symplectic leaves are indexed by the elements of $W \times W$, where W is the Weyl group of G. To be precise, they are the double Bruhat cells

$$X_{w_1,w_2} := B\dot{w}_1 B \cap B^- \dot{w}_2 B^-, \tag{8.5}$$

where $\dot{w}_1$, $\dot{w}_2$ are chosen from the normaliser $N_G(T)$ to represent $w_1, w_2 \in W$.

Note that the localisation with respect to z corresponds exactly to the localisation over the big cell BB^-, as explained in [10, proof of Theorem 7.2]. In view of (8.1) and (8.4) it is therefore enough to show that every T-orbit of leaves in G has non-empty intersection with the big cell. That is, by (8.5), we must check that every double Bruhat cell X_{w_1,w_2} has non-empty intersection with the big cell. This is easy to verify as follows: Consider the double Bruhat cells $X_{w_1,e} = Bw_1B \cap B^-$ and $X_{e,w_2} = B \cap B^- w_2 B^-$. Let $a \in X_{w_1,e}$ and $b \in X_{e,w_2}$. Then $ab \in B^-B \cap Bw_1B \cap B^-w_2B^- \subseteq B^-B \cap X_{w_1,w_2}$. $\square$

Remark. Although we have only explicitly determined the Nakayama automorphism for $\mathcal{O}_\epsilon[G][z^{-1}]$ and not for $\mathcal{O}_\epsilon[G]$, this theorem shows that $\mathcal{O}_\epsilon[G]$ has a non-trivial Nakayama automorphism since the reduced quantum function algebras are not generally symmetric. It would be interesting to see if the form $\mathbb{B}'$ constructed in Theorem 8.3(3) produces a dual free pair, and hence the Nakayama automorphism for $\mathcal{O}_\epsilon[G]$.

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