Diplomarbeit

Eine graphische Beschreibung von (A_{n-1}, D_n) Kazhdan-Lusztig-Polynomen

(A graphical description of (A_{n-1}, D_n) Kazhdan-Lusztig polynomials)

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Von

Tobias Lejczyk

Aus
Wolfratshausen

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Übersicht und Beweisideen

Die Klassifikation endlich-dimensionaler Moduln über einer komplexen, halbeinfachen Liealgebra ist bekannt. Nach einem Satz von Weyl [Hum72, 6.3] ist jeder solche Modul Summe von irreduziblen Moduln. Die Isomorphieklassen irreduzibler Moduln werden von den höchsten Gewichten, die dominant und integral sein müssen, indiziert. Eine Berechnungsmethode für die Dimension dieser Moduln gibt es ebenfalls [Hum72, 21, 22].
Genauer gesagt sind die Charaktere und somit insbesondere die Dimensionen der Gewichtsräume durch Weyl's Charakterformel gegeben.
Die Situation ist bei unendlich-dimensionalen Höchstgewichtsmoduln um Längen schwieriger. In diesem Fall können Kazhdan-Lusztig-Polynome verwendet werden, um Charakterformeln herzuleiten. Die Polynome wurden von Kazhdan und Lusztig in [KL79] eingeführt.
Für einen irreduziblen Höchstgewichtsmodul $L(\lambda)$ von höchstem Gewicht λ, der möglicherweise unendlich-dimensional ist, zählen die Kazhdan-Lusztig-Polynome in Abhängigkeit von λ und μ die Vielfachheit $[M(\mu) : L(\lambda)]$ von $L(\lambda)$ in einer Jordan-Hölder-Reihe eines Vermamoduls $M(\mu)$ mit höchstem Gewicht μ.
Kostant's Partitionsfunktion, eine explizite kombinatorische Formel, liefert uns den Charakter $\operatorname{ch} M(\mu)$ von $M(\mu)$.
Mit Hilfe der bekannten Formel $\operatorname{ch} M(\mu) = \sum [M(\mu) : L(\lambda)] \operatorname{ch} L(\lambda)$ kann man dann den Charakter von $L(\lambda)$ als alternierende Summe der $\operatorname{ch} M(\mu)$ ausdrücken. Dabei kommen die Kazhdan-Lusztig-Polynome ins Spiel.
Also ist es möglich, die Charaktere von irreduziblen Höchstgewichtsmoduln auszurechnen, sobald man die Kazhdan-Lusztig-Polynome kennt.
Im Allgemeinen ist die Berechnung der Kazhdan-Lusztig-Polynome allerdings recht aufwendig, da diese induktiv definiert sind.

In dieser Arbeit werden wir die Kazhdan-Lusztig-Polynome in einem Spezialfall studieren und durch geschlossene Formeln beschreiben.
Wir betrachten dabei die Liealgebra $\mathfrak{g} = \mathfrak{so}_{2n}$ vom Typ D_n und irreduzible Höchstgewichtsmoduln die "fast" endlich-dimensional sind; genauer gesagt, die lokal endlich-dimensional in Bezug auf eine parabolische Unteralgebra sind und reguläre Höchstgewichte haben. Diese Moduln sind Objekte in der Kategorie $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})_0$.
Ihre Charaktere können als alternierende Summe der Charaktere parabolischer Vermamoduln ausgedrückt werden. Dabei werden parabolische Kazhdan-Lusztig-Polynome verwendet. Die Charaktere parabolischer Vermamoduln können dann mit Hilfe der Formel [Soe97, Prop. 3.4] auf die Charaktere von Vermamoduln zurückgeführt werden.

In dieser Arbeit werde ich

- eine explizite Formel für parabolische Kazhdan-Lusztig-Polynome im (A_{n-1}, D_n)-Fall,
- ein Diagrammkalkül,
- explizite Dimiensionsformeln,
- eine Verbindung zur Arbeit von Brenti

geben.

Wir beginnen in Kapitel 2 mit einer kurzen Einführung der Grundlagen zum Thema Coxetergruppen.
In dieser Arbeit beschäftigen wir uns mit der Weylgruppe $\mathcal{W}$ vom Typ D_n mit Erzeugern $\mathcal{S} = \{s_0, \ldots, s_{n-1}\}$. In $\mathcal{W}$ gibt es die maximale parabolische Untergruppe $\mathcal{W}_\mathfrak{p}$ vom Typ A_{n-1}, erzeugt von $\mathcal{S}_\mathfrak{p} = \{s_1, \ldots, s_{n-1}\}$.

Es gibt eine Bijektion zwischen dem Quotienten $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W}$ und der Menge der $\{+,-\}$-Sequenzen der Länge n mit einer geraden Anzahl von Minussen. Wir bezeichnen die Menge der kürzesten Repräsentanten von $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W}$ mit $\mathcal{W}^\mathfrak{p}$.
Die Gruppe $\mathcal{W}$ lässt sich in die Gruppe S_{2n} einbetten. Durch diese Einbettung wird deutlich, dass wir eigentlich mit $\{+,-\}$-Sequenzen der Länge $2n$, die antisymmetrisch sind und eine gerade Anzahl von Minussen in der oberen Hälfte der Sequenz aufweisen, arbeiten sollten.
Sowohl die Einbettung als auch die Bijektion zwischen den $\{+,-\}$-Sequenzen und $\mathcal{W}^\mathfrak{p}$ werden in Kapitel 3 explizit gegeben.
Die Bijektion wird mit Hilfe von Youngdiagrammen realisiert. Es stellt sich heraus, dass ein kürzester Repräsentant für eine Restklasse nichts anderes ist als eine Beschreibung für den Aufbau eines Youngdiagramms, wobei in jedem Bauschritt ein neues Youngdiagramm entsteht. Auf der anderen Seite kann man den äußeren Pfad, also die Linie, die die Boxen des Youngdiagramms von dem leeren Raum trennt, leicht in eine $\{+,-\}$-Sequenz umwandeln indem man jeden Schritt nach rechts durch ein "$+$" und jeden Schritt nach oben durch ein "$-$" ersetzt. Diese beiden Konstruktionen liefern Bijektionen zum einen zwischen $\mathcal{W}^\mathfrak{p}$ und einer Menge von gewissen Youngdiagrammen sowie zwischen dieser Menge von Youngdiagrammen und der obigen Menge der $\{+,-\}$-Sequenzen. Insgesamt erhält man die gewünschte Bijektion.

In Kapitel 4 wird die Heckealgebra $\mathcal{H}$ zu unserer Weylgruppe $\mathcal{W}$ über dem Ring $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$ eingeführt. Die Standardbasis wird mit $\{H_w | w \in \mathcal{W}\}$ bezeichnet. Auf $\mathcal{H}$ gibt es eine antilineare Involution. Die Kazhdan-Lusztig-Basis ist die eindeutige Basis von $\mathcal{H}$, deren Elemente selbst-dual, also von der Involution auf sich selbst abgebildet werden, sind und sich als Linearkombination der Standardbasis mit Koeffizienten in $\mathbb{Z}[v]$ schreiben lassen.
In $\mathcal{H}$ gibt es die Unteralgebra $\mathcal{H}_\mathfrak{p}$ zu unserer Untergruppe $\mathcal{W}_\mathfrak{p}$. $\mathcal{L}$ wird zu einem $\mathcal{H}_\mathfrak{p}$-Modul indem man H_s durch $-v$ für alle $s \in \mathcal{S}_\mathfrak{p}$ operierent lässt. Den parabolischen Heckemodul $\mathcal{N}$ erhält man, indem man $\mathcal{L}$ über $\mathcal{H}_\mathfrak{p}$ mit $\mathcal{H}$ tensoriert. Er hat

die Standardbasis $\{N_w = 1 \otimes H_w | w \in \mathcal{W}^{\mathfrak{p}}\}$.
Die Involution auf $\mathcal{H}$ kann auf $\mathcal{N}$ übertragen werden und führt zu einer parabolische Version der Kazhdan-Lusztig-Basis, die wir mit $\{\underline{N}_w | w \in \mathcal{W}^{\mathfrak{p}}\}$ bezeichnen.
Die Kazhdan-Lusztig-Polynome $n_{w',w} \in \mathbb{Z}[v]$ beschreiben den Basiswechsel von der Standardbasis zur Kazhdan-Lusztig-Basis, insbesondere ist $\underline{N}_w = \sum_{w'} n_{w',w} N_{w'}$.
Diese Resultate finden sich beispielsweise in [Soe97].
Den Elementen in $\mathcal{W}^{\mathfrak{p}}$ ordnen wir zwei Objekte zu: Gewichte und Cupdiagramme.
Die Gewichte erhält man, indem man die $\{+, -\}$-Sequenz, die einem Element $w \in \mathcal{W}^{\mathfrak{p}}$ zugeordnet ist, mit n Plussen nach links und n Minussen nach rechts erweitert und anschließend jedes "$+$" durch ein "$\vee$" und jedes "$-$" durch ein "$\wedge$" ersetzt. Die n oberen und n unteren Punkte bezeichnen wir als "eingefroren" da die Orientierung hier aufgrund der Erweiterung festgelegt ist.
Wir betrachten den freien $\mathcal{L}$-Modul M_D mit diesen Gewichten als Basis. Dieser Modul ist als $\mathcal{L}$-Modul isomorph zu $\mathcal{N}$ durch Fortsetzung der Bijektion aus Kapitel 3. Diesen Isomorphismus bezeichnen wir mit Φ. Durch die von ihm induzierte Operation von $\mathcal{H}$ wird M_D zu einem $\mathcal{H}$-Modul.

Bis zu diesem Punkt sind die meisten Resultate Standardresultate. Ab Kapitel 5 sind alle Resultate, abgesehen von dem kurzen Exkurs in Brenti's Arbeit, neu und originär.
In Kapitel 5 werden Cupdiagramme eingeführt und das Haupttheorem bewiesen.
Ein Cupdiagramm entsteht aus der erweiterten $\{+, -\}$-Sequenz dadurch, dass zuerst Plusse und Minusse durch Bögen, so genannte "Cups", verbunden werden. Dabei wird so verbunden, dass jeder Bogen bei einem "$+$" beginnt und an einem "$-$" aufhört und keine Überschneidungen auftreten. Anschließend werden aufeinanderfolgende Cups, die die Mitte überqueren, jeweils zu zweit "verknüpft". Beginnend in der Mitte werden dabei von je zwei aufeinanderfolgenden "Cups", die die Mitte überqueren, die Startpunkte vertauscht. Eine Überschneidung entsteht, die mit einem Punkt markiert wird, und die beiden Cups sind "verknüpft". Das so entstehende Cupdiagramm wird mit $C(w)$ bezeichnet.
Ein Gewicht kann nun oben auf ein solches Cupdiagramm gesetzt werden. Eine Cup der Form  oder  heißt orientiert. Ein Cupdiagramm heißt orientiert falls alle Cups orientiert sind. Die Anzahl der im Uhrzeigersinn orientierten Cups wird mit $\mathrm{cl}(w'C(w))$ bezeichnet.
Jedem Cupdiagramm $C(w)$ ordnen wir ein Element in M_D zu.

$$C(w)_{M_D} := \sum v^{\frac{\mathrm{cl}(w'C(w))}{2}} w'$$

wobei die Summe über alle Gewichte läuft, die $C(w)$ orientieren.
Das Haupttheorem besagt, dass

$$\Phi(C(w)_{M_D}) = \underline{N}_w,$$

also das Bild eines Cupdiagramms $C(w)$ in $\mathcal{N}$ das Kazhdan-Lusztig-Basiselement $\underline{N}_w$ ist.

Der Beweis basiert auf der Proposition, dass der Koeffizient eines Gewichtes w' in $C(w)_{M_D}$ gerade das Kazhdan-Lusztig-Polynom $n_{w',w}$ ist, also insbesondere gilt:

$$n_{w',w} = \begin{cases} v^{\frac{\mathrm{cl}(w'C(w)}{2}} & \text{falls } w'C(w) \text{ ist orientiert} \\ 0 & \text{falls } w'C(w) \text{ ist nicht orientiert} \end{cases} .$$

Der Beweis der Proposition basiert auf dem Algorithmus zur Berechnung der Kazhdan-Lusztig-Polynome. Im Algorithmus wird die Operation der Heckealgebra auf den bereits gegebenen Kazhdan-Lusztig-Basiselementen und insbesondere auf der Standardbasis verwendet. Diese Operation wird mit der Operation der Heckealgebra auf der Basis des Moduls M_D, also den Gewichten, sowie den Cupdiagrammen verglichen. Es zeigt sich, dass beide Operationen übereinstimmen. Dies liefert den Beweis.
Ein ähnliches Resultat für den Typ A Fall wurde von Brundan und Stroppel in [BS08a, 5.12] bewiesen. Sie benutzen ebenfalls Cupdiagramme und Gewichte.

In Kapitel 4 wurde der $\mathcal{H}$-Modul M_D eingeführt. $\mathcal{H}$ operiert allerdings nicht treu auf M_D. Die graphischen Beschreibung der Kazhdan-Lusztig-Basis ermöglicht es uns, die Operation ebenfalls graphisch zu beschreiben. Dies liefert eine Beschreibung des Quotienten von $\mathcal{H}$, der treu operiert. Es stellt sich heraus, dass dieser Quotient ein Quotient der verallgemeinerten Temperley-Lieb-Algebra vom Typ D_n ist. All diese Ergebnisse finden sich in Kapitel 6.
Die graphische Beschreibung des Quotienten erfolgt mit Hilfe von dekorierten, das heißt mit Punkten versehenen, Tanglediagrammen. Ein Tanglediagramm ist eine Sammlung sich nicht überschneidender Linien und Kreise, wobei eine Linie immer zwei Punkte, die auf dem oberen oder unter Rand eines Rechteckes liegen, verbindet. Linien und Kreise dürfen auf gewisse Art und Weise "dekoriert" werden. Dies geschieht durch Punkte, die auf die Linie oder den Kreis gezeichnet werden. Green beschreibt in [Gre98] die verallgemeinterte Temperley-Lieb-Algebra vom Typ D_n mit Hilfe dieser dekorierten Tanglediagramme.
Unsere Cupdiagramme werden in dekorierte Tanglediagramme umgewandelt indem der untere Teil der oberen Hälfte "herausgeschnitten" wird. Zwischen -1 und 1 sowie zwischen n und $n+1$ wird das Cupdiagramm senkrecht zerschnitten. Linien, die anschließend nur mit einem Punkt verbunden sind werden zu senkrechten Strahlen, Verknüpfungen werden zu Punkten, also zu Dekorationen.
Die Operation von $\mathcal{H}$ ist dann durch Konkatenation von Tanglediagrammen und herausfaktorisieren gewisser Relationen gegeben.
Hier erfolgt der Beweis, ähnlich wie vorher, durch Vergleich der neu definierten Operation mit der der Heckealgebra auf $\mathcal{N}$.

Im letzten Kapitel werden die Ergebnisse über Kazhdan-Lusztig-Polynome und deren graphische Beschreibung durch Cupdiagramme auf die Berechnung der Dimension der Homomorphismenräume zwischen irreduziblen projektiven Objekten in der parabolischen Kategorie $\mathcal{O}_0^{\mathfrak{p}}$ angewendet. In diesem Zusammenhang haben die Kazhdan-Lusztig-Polynome eine interessante Lietheoretische Interpretation. Ausge-

wertet an 1 zählen sie die Vielfachheit einfacher Objekte als Subquotienten in eine Jordan-Hölder-Reihe eines parabolischen Vermamoduls. Diese Information ermöglicht es uns, die Dimension $\dim \hom(P(w), P(x)) = \sum_{w'} n_{w',w}(1) n_{w',x}(1)$ mit Hilfe der Kazhdan-Lusztig-Polynome zu berechnen.
Mit den vorhergehenden Ergebnissen sieht man, dass das Produkt $n_{w',w}(1) n_{w',x}(1)$ das Resultat 1 hat, falls das Gewicht w' die Cupdiagramme $C(w)$ und $C(x)$ simultan orientiert und in allen anderen Fällen zu 0 wird. Also müssen wir nur herausfinden, wie viele Gewichte beide Cupdiagramme simultan orientieren.
Um dies zu sehen wird eines der Cupdiagramme vertikal gespiegelt und anschließend auf das andere Cupdiagramm geklebt. Dies gibt uns ein Kreisdiagramm. Beide Cupdiagramme simultan zu orientieren ist dann äquivalent dazu das Kreisdiagramm zu orientieren.
Es stellt sich heraus, dass wir drei Arten von Kreisen unterscheiden müssen: Zuerst gibt es Kreise, die sich beliebig, also in beide Richtungen, orientieren lassen. Zum Zweiten gibt es Kreise, die sich nicht orientieren lassen und zuletzt Kreise, deren Orientierung aufgrund der eingefrorenen Punkte festgelegt ist. Die drei Arten von Kreisen können leicht daran unterschieden werden durch wie viele eingefrorene Punkte sie laufen und auf welche Art und Weise sie durch verknüpfte Cups laufen.
Die Berechnung der Dimension läuft dann auf ein Zählen der verschiedenartigen Kreise hinaus. Falls ein Kreis zweiter Art entsteht ist die Dimension 0. Ansonsten ist sie $2^{\frac{\mathrm{bk}(w,x)}{2}}$, wobei $\mathrm{bk}(w,x)$ die Anzahl der Kreise erster Art ist.
Zum Abschluss wird diese Berechnung noch in dekorierte Tanglediagramme übersetzt.

1 Introduction

1.1 Overview

Finite-dimensional modules for a complex semisimple Lie algebra are quite well understood. Weyl's Theorem [Hum72, 6.3] tells us that such a module is the sum of irreducible modules. The isomorphism classes of these irreducible modules are indexed by highest weights, which have to be dominant integral, and their dimensions are known [Hum72, 21, 22].
More precisely we know their characters, i.e. the dimension of the weight spaces, by Weyl's character formula.
For infinite-dimensional highest weight modules the situation is much harder. Kazhdan-Lusztig polynomials can be used to deduce such character formulas. They were introduced by Kazhdan and Lusztig in [KL79].
Given an irreducible highest weight module $L(\lambda)$ of highest weight λ, possibly infinite-dimensional, the Kazhdan-Lusztig polynomials, depending on λ and μ, determine the multiplicities $[M(\mu) : L(\lambda)]$ of how often it occurs in a Jordan-Hölder series of a Verma module $M(\mu)$ of highest weight μ.
The character $\operatorname{ch} M(\mu)$ of $M(\mu)$ is known by Kostant's partition function, an explicite combinatorial formula.
Then using the formula $\operatorname{ch} M(\mu) = \sum [M(\mu) : L(\lambda)] \operatorname{ch} L(\lambda)$, it is possible to express $\operatorname{ch} L(\lambda)$ as an alterning sum of characters $\operatorname{ch} M(\mu)$ involving the Kazhdan-Lusztig polynomials.
Hence knowing the Kazhdan-Lusztig polynomials it is possible to compute the characters of the irreducible highest weight modules.
The problem is that the calculation of Kazhdan-Lusztig polynomials is rather cumbersome in general, since they are defined inductively.
In this thesis we will study the Kazhdan-Lusztig polynomials in a special case and give a closed formula.
Consider the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}$ of type D_n and irreducible highest weight modules which are "almost" finite-dimensional; precisely, which are locally finite-dimensional for the standard parabolic subalgebra of type A_{n-q} and have regular highest weights. These modules are objects in the parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})_0$.
Their characters can be then expressed as an alternating sum containing the characters of parabolic Verma modules using parabolic Kazhdan-Lusztig polynomials. Characters of parabolic Verma modules can be written in terms of characters of ordinary Verma modules using an explicit formula [Soe97, Prop. 3.4].
In this thesis I will give:

- Explicit formulas for parabolic Kazhdan-Lusztig polynomials in the case (D_n, A_{n-1}),
- a diagram calculus,
- explicit dimension formulas,
- a connection with the work of Brenti.

We start out in Chapter 2 with a short introduction to Coxeter groups.
The group which we are interested in is the Weyl group $\mathcal{W}$ of type D_n with a set of generators $\mathcal{S} = \{s_0, \ldots, s_{n-1}\}$. In $\mathcal{W}$ there we have a maximal parabolic subgroup $\mathcal{W}_\mathfrak{p}$ of type A_{n-1} generated by $\mathcal{S}_\mathfrak{p} = \{s_1, \ldots, s_{n-1}\}$.

The quotient $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W}$ is in bijection to $\{+,-\}$-sequences of length n with an even number of minuses. We denote the set of shortest representatives by $\mathcal{W}^\mathfrak{p}$.
Via an embedding of $\mathcal{W}$ into the symmetric group S_{2n} it becomes apparent that we actually deal with $\{+,-\}$-sequences of length $2n$ which are antisymmetric and have an even number of minuses on the upper half of the sequence.
This embedding and the bijection between the $\{+,-\}$-sequences and $\mathcal{W}^\mathfrak{p}$ will be given explicitly in Chapter 3.

Chapter 4 introduces the Hecke algebra $\mathcal{H}$ attached to our Weyl group $\mathcal{W}$ over the ring $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$. The standard basis is given by $\{H_w | w \in \mathcal{W}\}$. On $\mathcal{H}$ we have an antilinear involution. The Kazhdan-Lusztig basis is the basis for $\mathcal{H}$ consisting of elements that are invariant under said involution and can be written as linear combinations of the standard basis with coefficients in $\mathbb{Z}[v]$.
Inside $\mathcal{H}$ we have the subalgebra $\mathcal{H}_\mathfrak{p}$ attached to our subgroup $\mathcal{W}_\mathfrak{p}$. By defining H_s to act as $-v$ on $\mathcal{L}$ for all $s \in \mathcal{S}_\mathfrak{p}$, $\mathcal{L}$ becomes an $\mathcal{H}_\mathfrak{p}$-module. Tensoring this module over $\mathcal{H}_\mathfrak{p}$ with $\mathcal{H}$ gives us $\mathcal{N}$. This module has a standard basis consisting of the elements $\{N_w = 1 \otimes H_w | w \in \mathcal{W}^\mathfrak{p}\}$.
The involution on $\mathcal{H}$ can be transfered to $\mathcal{N}$ and gives us a parabolic version of the Kazhdan-Lusztig basis, denoted by $\{\underline{N}_w | w \in \mathcal{W}^\mathfrak{p}\}$. The Kazhdan-Lusztig polynomials are defined by the transformation from the standard basis to the Kahdan-Lusztig basis, i.e. $\underline{N}_w = \sum_{w'} n_{w',w} N_{w'}$ with $n_{w',w} \in \mathbb{Z}[v]$. All these are standard results which can be found for example in [Soe97].

To the elements in $\mathcal{W}^\mathfrak{p}$ we attach two types of objects, weights and cup diagrams.
The weights are obtained from an element $w \in \mathcal{W}^\mathfrak{p}$ by considering the $\{+,-\}$-sequence associated to w, extending this sequence by n pluses to the left and n minuses to the right and then replacing every "+" by a "$\vee$" and every "$-$" by an "$\wedge$". We can consider the free $\mathcal{L}$-module M_D with these weights as basis. This module is isomorphic as an $\mathcal{L}$-module to the module $\mathcal{N}$ by extending the bijection from Chapter 3. This isomorphism is called Φ and induces an action of $\mathcal{H}$ on M_D which makes M_D into an $\mathcal{H}$-module.

Up to this point the results are pretty standard. From Chapter 5 on the results are, apart from the digression into some of Brenti's work, new and original.

In Chapter 5 cup diagrams are introduced and the main theorem is proven.
A cup diagram is obtained from the extended $\{+,-\}$-sequence by connecting pluses and minuses with cups such that a cup starts at a "+" and ends at a "$-$" and no crossings occur. Then the cups crossing the middle are linked pairwise. Starting in the middle, the starting points of two consecutive cups crossing the middle are switched, linking them together. The cup diagram is denoted $C(w)$.

A weight can be glued on top of such a cup diagram. A cup diagram is called oriented if every cup is labeled by exactly one "$\vee$" and one "$\wedge$". The number of clockwise oriented cups is denoted $\mathrm{cl}(w'C(w))$.
To each cup diagram we associate an element in M_D.

$$C(w)_{M_D} := \sum v^{\frac{\mathrm{cl}(w'C(w))}{2}} w'$$

where the sum runs over all weights w' such that $w'C(w)$ is oriented.
The main theorem states that

$$\Phi(C(w)_{M_D}) = \underline{N}_w,$$

i.e. the image of a cup diagram $C(W)_{M_D}$ in $\mathcal{N}$ is the Kazhdan-Lusztig basis element $\underline{N}_w$.
In particular, the coefficient in $C(W)_{M_D}$ of a weight w' is the Kazhdan-Lusztig polynomial, i.e.

$$n_{w',w} = \begin{cases} v^{\frac{\mathrm{cl}(w'C(w))}{2}} & \text{if } w'C(w) \text{ is oriented} \\ 0 & \text{if } w'C(w) \text{ is not oriented} \end{cases}.$$

A similar result for type A has been proven by Jonathan Brundan and Catharina Stroppel in [BS08a, 5.12]. They also use the language of cup diagrams and weights. Note that the Kazhdan-Lusztig polynomials are all monomials which is a special feature of the Hermitian symmetric case, see e.g. [Boe88].

In Chapter 4 the $\mathcal{H}$-module M_D was introduced. The action of $\mathcal{H}$ on M_D is not faithful. Having the graphical desription of the Kazhdan-Lusztig basis, we can give a graphical description of this action. Besides this, a description of the quotient of $\mathcal{H}$ which acts faithfully on M_D will be given in Chapter 6. This quotient turns out to be a quotient of the generalized Temperley-Lieb algebra of type D_n.
The graphical description will be given in terms of decorated tangles. A tangle is a collection of non-intersecting lines and circles, where the lines connect a given number of points at the top and bottom face of a rectangle. Decorations are simply dots on these circles or lines. Green described in [Gre98] the generalized Temperley-Lieb algebra of type D_n in terms of these tangles.
Our cup diagrams are transformed into decorated tangles by cutting out the lower part of the upper half of the tangle. This happens by cutting the cup diagram between -1 and 1 and between n and $n+1$. Free lines, connected to a point between 1 and n become traversing lines in the tangle. The dots indicating linked cups become decorations. In a way the decorated tangles are just the essential part of a cup diagram, i.e. the part that holds all the information to construct the cup diagram.
The action of $\mathcal{H}$ on the tangle basis is then given by concatenation of tangles and factoring out a certain set of relations.

In the final chapter the results about Kazhdan-Lusztig polynomials and the language of cup diagrams will be applied to the calculation of the dimension of homomorphism spaces between irreducible projective objects in the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$. In this setting the Kazhdan-Lusztig polynomials have a nice Lie theoretic interpretation. Evalutated at 1 they count the multiplicities of simple objects as subquotients in a composition series of parabolic Verma modules. This can be used to see that $\dim \hom(P(w), P(x)) = \sum_{w'} n_{w',w}(1) n_{w',x}(1)$.

With the previous results it becomes clear that the product $n_{w',w}(1) n_{w',x}(1)$ is 1 if the weight w' orients both cup diagrams $C(w)$ and $C(x)$ simultaneously and is 0 otherwise.

To see how many weights orient both cup diagrams simultaneously one cup diagram is transformed into a cap diagram by reflecting it vertically. Glueing this cap diagram on top of the cup diagram we get a circle diagram. Orienting both cup diagrams then is equivalent to orienting the circle diagram.

It turns out that we have to distinguish between three types of circles: First of all, circles which can be oriented freely. Second, circles which can not be oriented and third, circles which have a fixed orientation. These three types are easily distinguished via the number of outer points they traverse and in which way they cross the middle.

Then calculating the dimension comes down to counting different types of circles. If a circles of the second kind appears the dimension is 0. Otherwise it is $2^{\frac{\mathrm{bk}(w,x)}{2}}$, where $\mathrm{bk}(w, x)$ is the number of circles of the first kind.

As a final step the calculation of the dimension is done in the language of decorated tangles.

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2 Preliminaries: Coxeter groups of type D

We start out with some standard results and notations concerning Coxeter groups. For further information and details I refer the reader the books of Bjoerner and Brenti [BB05] or Humphreys [Hum92].

Let $\mathbb{Z}$ be the set of integers and the subset $\{-n, -(n-1), \ldots, -1, 1, \ldots, n-1, n\}$ for some $n \in \mathbb{Z}_{>0}$. We denote by S_{2n} the group of all permutations of this set. This group is a Coxeter group corresponding to the Dynkin diagram of type A_{2n-1}. We denote the elements in such a permutation group by ω. As generators we take $\{t_i\}_{-(n-1)\leq i \leq n-1}$ where

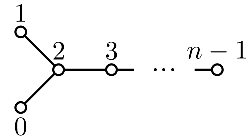
$$t_i = \begin{cases} (i, i+1) & \text{for } 1 \leq i \leq n-1 \\ (-i, -(i+1)) & \text{for } -(n-1) \leq i \leq -1 \\ (-1, 1) & \text{for } i = 0 \end{cases}$$

There several different ways of specifying an element in S_{2n}. The notations we will use in this thesis are:

- Generators: $\omega = t_{i_1} \cdots t_{i_k}$, which we will read from left to right, meaning that t_{i_1} is first applied, then t_{i_2} and so forth.
- Two line notation: $\omega = \begin{bmatrix} -n & -(n-1) & \ldots & -1 & 1 & \ldots & n-1 & n \\ a_{-n} & a_{-(n-1)} & \ldots & a_{-1} & a_1 & \ldots & a_{n-1} & a_n \end{bmatrix}$, meaning that $\omega(i) = a_i$
- One line notation: $\omega = [a_{-n}, a_{-(n-1)}, \ldots, a_{-1}, a_1, \ldots, a_{n-1}, a_n]$, simply omitting the upper line.

Consider the subgroup of all *signed permutations* in S_{2n}, i.e. the group consisting of all elements in S_{2n} satisfying $\omega(-i) = -\omega(i)$. This group is isomorphic to the Weyl group of type B_n. A proof of this can be found in [BB05, Prop. 8.1.3].
Writing these elements in one line notation we will introduce a $|$ between a_{-1} and a_1 indicating the antisymmetry.
We call the elements a_1 to a_n the *upper* or *right half* and in turn a_{-n} to a_{-1} the *lower* or *left half*. Since the left half is determined by the right half we also write elements in half line notation $|a_1, \ldots, a_n]$, omitting the left half.

In this thesis we will work with the Coxeter group of type D_n. This group embeds into S_{2n} and, in particular, into the subgroup of signed permutations, which we will prove in Chapter 3.2.

Consider the Dynkindiagram of type D_n: 

with n points.

Let $\mathcal{W}$ be the associated Weyl group. This is a Coxeter group. We denote the set of generators by $\mathcal{S} = \{s_0, s_1, \ldots, s_{n-1}\}$. All the relations are of the form $(s_i s_j)^{m_{ij}} = e$ with

$$m_{ij} = \begin{cases} 1 & i = j \\ 3 & \text{if } i \text{ and } j \text{ are connected in the Dynkindiagram} \\ 2 & \text{otherwise} \end{cases}$$

The cardinality of the group $\mathcal{W}$ is known to be $n! \cdot 2^{n-1}$ [BB05, Appendix A1, Table 1].

Remark 2.1. The element s_0 should not be confused with the element associated to the added point in the affine Dynkin diagram of type $\widetilde{D}$.

Write $\mathbb{Z}_2$ for the multiplicative group $\{+1, -1\}$. In the following we omit the 1s and only write down the signs.

Definition 2.2. Consider the group $G := S_n \ltimes (\mathbb{Z}_2^n / (\prod \alpha_i = 1))$ where the composition in G is defined to be $(\sigma, \alpha) \circ (\tau, \beta) = (\sigma \circ \tau, \tau(\alpha) \cdot \beta)$, where $\tau(\alpha)$ is the permutation of the α_i given by τ.
A $\{+, -\}$-sequence α may be written down set theoretically. We will write $\{i_1, \ldots, i_{2k}\}$, meaning $\alpha_{i_1}, \ldots, \alpha_{i_{2k}}$ are "$-$" and the other α_j are "$+$". The empty set $\varnothing$ corresponds to the sequence where all α_i are "$+$".
The cardinality of G is $n! \cdot 2^{n-1}$.

Lemma 2.3. *The Weyl group $\mathcal{W}$ is isomorphic to the group G via the map*

$$\begin{array}{rccl} \varphi: & \mathcal{W} & \to & G \\ & s_i & \mapsto & ((i, i+1), \varnothing) \quad \textit{for } 1 \leq i \leq n-1 \\ & s_0 & \mapsto & ((1,2), \{1,2\}) \end{array} .$$

Proof. The map φ is defined on generators. So first, we have to check that φ extends to a well-defined group homomorphism, i.e. that it is compliant with the relations. The subgroup generated by $\{s_1, \ldots, s_{n-1}\}$ is isomorphic to S_n. On this subgroup φ is the inclusion into the subgroup $S_n \ltimes \{e\}$. In particular, all the relations not involving s_0 are satisfied. So we only have to check the relations involving s_0.
For $i > 2$ all the $\varphi(s_i)$ commute with $\varphi(s_0)$ since the simple transpositions $(1,2)$ and $(i, i+1)$ commute and $\{1,2\}$ is not changed by $(i, i+1)$.
For $i = 1$ we get

$$\begin{aligned} \varphi(s_0)\varphi(s_1) &= ((1,2), \{1,2\})((1,2), \varnothing) \\ &= ((1,2)(1,2), (1,2).\{1,2\}) \\ &= ((1,2)(1,2), \{1,2\}) \\ &= ((1,2), \varnothing)((1,2), \{1,2\}) \\ &= \varphi(s_1)\varphi(s_0) \end{aligned} .$$

For $i = 2$ we get

$$\begin{aligned}
\varphi(s_0)\varphi(s_2)\varphi(s_0) &= ((1,2),\{1,2\})((2,3),\varnothing)((1,2),\{1,2\})\\
&= ((1,2)(2,3),(2,3).\{1,2\})((1,2),\{1,2\})\\
&= ((1,2)(2,3)(1,2),\{(1,2).\{1,3\}\}\cdot\{1,2\})\\
&= ((2,3)(1,2)(2,3),\{1,3\})\\
&= ((2,3),\varnothing)((1,2)(2,3),(2,3).\{1,2\})\\
&= ((2,3),\varnothing)((1,2),\{1,2\})((2,3),\varnothing)\\
&= \varphi(s_2)\varphi(s_0)\varphi(s_2) \qquad .
\end{aligned}$$

So φ is a well-defined group homomorphism.
Next we prove the surjectivity of φ.
It suffices to show that $(e,\{i,i+1\}) \in \mathrm{Im}(\varphi)$:
Given an element $(\tau,\alpha) \in G$, τ can be written as a composition of simple transpositions $\sigma_i := (i,i+1)$ and α can be written as a composition of elements two adjacent minuses $\mu_i := \{i,i+1\}$. Let now $\tau = \sigma_{i_1}\cdots\sigma_{i_k}$ and $\alpha = \mu_{j_1}\cdots\mu_{j_l}$ be two such expressions. Then it is obvious that $(\tau,\alpha) = \varphi(s_{i_1})\cdots\varphi(s_{i_k})(e,\mu_{j_1})\cdots(e,\mu_{j_l})$.
Claim: $(e,\{i,i+1\}) \in \mathrm{Im}(\varphi)$
Proof of Claim by induction:
Induction basis, $i = 1$: $\varphi(s_0)\varphi(s_1) = (e,\{1,2\})$
Induction step, $i \rightsquigarrow i+1$: Let $g_i = (e,\{i,i+1\}) \in \mathrm{Im}(\varphi)$. Then

$$g_i \circ \varphi(s_i) \circ \varphi(s_{i+1}) \circ \varphi(s_i) \circ \varphi(s_{i+1}) \circ \varphi(s_i) \circ g_i \circ \varphi(s_{i+1}) = (e,\{i+1,i+2\})$$

as calculation shows.
Since φ is a well-defined surjective group homomorphism, by comparing cardinalities we see that φ is an isomorphism. □

Let $\mathcal{W}_\mathfrak{p}$ be the parabolic subgroup of $\mathcal{W}$ generated by $\mathcal{S}_\mathfrak{p} = \{s_1,\dots,s_{n-1}\}$. Our goal is to describe $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W}$, the right cosets of $\mathcal{W}_\mathfrak{p}$ in $\mathcal{W}$.
In the previous proof we saw that the isomorphism above sends $\mathcal{W}_\mathfrak{p}$ to the subgroup $S_n \ltimes \{e\}$. Consequently, $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W} \cong (S_n \ltimes \{e\})\backslash(S_n \ltimes (\mathbb{Z}_2^n/\prod\alpha_i = 1))$ bijects to $\mathbb{Z}_2^n/\prod\alpha_i = 1$. These are $\{+,-\}$-sequences of length n with an even number of minuses.

Definition 2.4. We denote the set of $\{+,-\}$-sequences of length n with an even number of minuses by $S_{sym}(n)$.
Elements are written down as $|\underbrace{\pm,\pm,\dots,\pm}_{n}|$.

Remark 2.5. The reason for using this notation will become clear in Chapter 3.2. An explicit bijection will be given in Chapter 3.1.

We denote the set of shortest representatives for $\mathcal{W}_\mathfrak{p}\backslash\mathcal{W}$ by $\mathcal{W}^\mathfrak{p}$, that means $w \in \mathcal{W}^\mathfrak{p}$ if and only if $l(sw) > l(w)$ for all $s_i \in \mathcal{S}_\mathfrak{p}$.

3 The bijection $\mathcal{W}^{\mathfrak{p}} \overset{1:1}{\longleftrightarrow} S_{sym}(n)$ and an embedding of $\mathcal{W}$ into S_{2n}

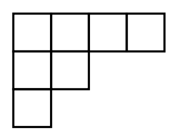
3.1 Young diagrams

We can visualize $\{+,-\}$-sequences as diagrams of boxes drawn in the plane, so-called Young diagrams. I will give the relevant definition and afterwards show that the $\{+,-\}$-sequences in $S_{sym}(n)$ are in bijection with a certain subset of the Young diagrams.

Definition 3.1.1. A *Young diagram* is a finite set of boxes, arranged in left-justified rows with weakly decreasing row length. To make descriptions clearer we assume boxes to be of size 1×1 and the upper left corner to be the point $(0,0)$.
The associated *partition* $(\lambda_1, \lambda_2, \ldots, \lambda_k)_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k}$ is the collection of the row lengths written in decreasing order.

This notation of Young diagrams is called English notation.

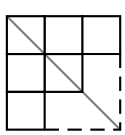
Example. *The partition* $(4,2,1)$ *is visualized by*



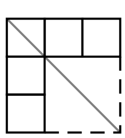
In the following discussion all diagonals are considered to have slope -1. The main diagonal is the diagonal going through the point $(0,0)$.

Definition 3.1.2. $Y_{sym}(n)$ is the set of Young diagrams that are symmetric with respect to the main diagonal, have an even number of boxes on the main diagonal and fit into an $n \times n$ square.

Example.

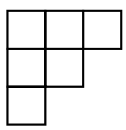
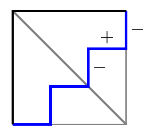


is in $Y_{sym}(3)$.



is not in $Y_{sym}(3)$ *because the number of squares on the main diagonal is odd.*
The partition $(4,2,1)$ *from the example before is not in* $Y_{sym}(4)$ *because it is not symmetric with respect to the main diagonal.*

We can identify $S_{sym}(n)$ and $Y_{sym}(n)$ via the following rule.
Consider the path from the lower left corner of the $n \times n$ square to the upper right corner of the square such that all the boxes of the Young diagram are to the left of the path and the empty space is to the right. Go to the middle of this path, i.e. the point where the main diagonal intersects with the path. From this point go along the path to the upper right corner and set pluses resp. minuses for every step of length one. Set a minus for a step in vertical direction and a plus for a step in horizontal direction.
Given a $\{+,-\}$-sequence we get our Young diagram the following way: Start at the upper right corner of an $n \times n$-square. Reading the sequence from right to left go one step down for a minus and one step left for a plus. Reflect this path on the main diagonal and fill the space to the left of this path with boxes. Denote the diagram associated to a sequence α by $y(\alpha)$.

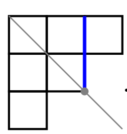
Example. *Consider*  $\in Y_{sym}(3)$. *The path looks like this:* 

So we have the $\{+,-\}$*-sequence* $|-,+,-] \in S_{sym}(3)$.

Proposition 3.1.3. *This construction yields a bijection*

$$Y_{sym}(n) \xleftrightarrow{1:1} S_{sym}(n).$$

Proof. We show that the construction assigns an element of $S_{sym}(n)$ to each element of $Y_{sym}(n)$. Then it is obvious that one construction is inverse to the other. Hence we have a bijection.
Consider a given element of $Y_{sym}(n)$. The number of boxes on the main diagonal measures the distance from the last intersection of the main diagonal with the boxes to the upper line of the square in which the diagram lies.

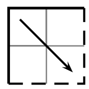
A visual example for this is  .

But this yields exactly the number of minuses occuring in the $\{+,-\}$-sequence associated with the diagram. So the condition on the number of minuses resp. the number of boxes on the diagonal is satisfied. The constructed $\{+,-\}$-sequence has length n because the whole path has length $2n$ and we take half of the path. So our sequence lies in $S_{sym}(n)$.
The constructed Young diagram is by construction symmetric, and hence lies in $Y_{sym}(n)$. So we get elements of the particular sets, and hence a bijection between $S_{sym}(n)$ and $Y_{sym}(n)$. ☐

Together with the results of Chapter 2 we know there are bijections

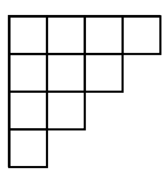
$$Y_{sym}(n) \overset{1:1}{\longleftrightarrow} S_{sym}(n) \overset{1:1}{\longleftrightarrow} \mathcal{W}^{\mathfrak{p}}.$$

We will now make these bijections explicit.

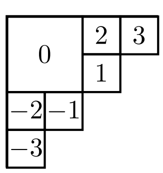
We start by describing an easy way to get reduced expressions for the elements in $\mathcal{W}^{\mathfrak{p}}$. To do this we visualize the action of an s_i on a Young diagram.
First, we examine the action of the s_i on a $\{+,-\}$-sequence in $S_{sym}(n)$. For $1 \leq i \leq n-1$ the element s_i acts by switching the signs at positions i and $i+1$ while s_0 exchanges the first two entries and switches both signs.
Since the minuses are always generated at positions 1 and 2 an element gets longer if a minus is switched with a plus to its right or if new minuses are generated. In our $\{+,-\}$-sequence this means we have two cases. Either a switch $(-,+) \rightsquigarrow (+,-)$ appears, which translates into  or a switch $(+,+) \rightsquigarrow (-,-)$ at positions 1 and 2 which translates into  on the main diagonal. Switches in the other direction, of course, shorten the element and translate into the deletion of boxes. Getting an element which is not in $\mathcal{W}^{\mathfrak{p}}$ means that the action does not change the sequence.
Now we describe the places where the boxes are added resp. deleted. From the action of the s_i we see that in the cases $1 \leq i \leq n-1$ the boxes are added resp. deleted on the ith and $-i$th diagonal and in the case $i = 0$ four boxes arranged in a 2×2-array are added resp. deleted on the main diagonal.
To decide whether boxes are added or deleted one can simply try both. Only one of the actions will yield a new Young diagram. That is the one we are looking for. The notion $xs \notin \mathcal{W}^{\mathfrak{p}}$ simply means "if the boxes are added or deleted the remaining diagram is no longer a Young diagram".

Consequence: A reduced expression for a shortest representative is simply an instruction how to add boxes to the empty Young diagram such that each step yields a new Young diagram and the final diagram has the shape associated to our $\{+,-\}$-sequence.

Denote the element in $\mathcal{W}$ associated to a Yound diagram y by w_y.

Example. *Take the Young diagram* 

To get our reduced expression we write the numbers of the diagonals on which they are into the boxes. On the main diagonal we work with boxes of size 2×2*.*

This gives us 

So a shortest representative would be $s_0 s_2 s_3 s_1$.

With these results we get an explicit bijection between $S_{sym}(n)$ and $\mathcal{W}^{\mathfrak{p}}$.

Proposition 3.1.4. *We have a bijection*

$$\begin{array}{cccc} \phi: & S_{sym}(n) & \stackrel{1:1}{\longleftrightarrow} & \mathcal{W}^{\mathfrak{p}} \\ & \alpha & \mapsto & w_{y(\alpha)} \end{array} .$$

Proof. This is clear from the discussion above and Proposition 3.1.3. □

Remark 3.1.5. For $w \in \mathcal{W}$ we denote the associated $\{+,-\}$-sequence by α_w. If the meaning is clear from the context we will often write w instead of α_w and α instead of $w_{y(\alpha)}$.

Definition 3.1.6. The *degree of antidominance* of a $\{+,-\}$-sequence α is defined as

$$d(|\alpha_1, \ldots, \alpha_n]) := \sum_{\alpha_i = -1} i.$$

Applying Remark 3.1.5 we write $d(w)$ instead of $d(\alpha_w)$. In the next lemma we will see how d gives us information about the change in cosets when multiplying an element in $\mathcal{W}^{\mathfrak{p}}$ by some s_i. The previous consideration help us considerably with the proof.

Lemma 3.1.7. *Let $w \in \mathcal{W}^{\mathfrak{p}}$ and $s \in \mathcal{S}$. Then*

1. *$d(w) < d(ws)$ implies that $ws \in \mathcal{W}^{\mathfrak{p}}$ and $l(w) < l(ws)$.*
2. *$d(s) > d(ws)$ implies that $ws \in \mathcal{W}^{\mathfrak{p}}$ and $l(w) > l(ws)$*
3. *$d(w) = d(ws)$ implies $ws \notin \mathcal{W}^{\mathfrak{p}}$.*

Proof. First of all, we observe that if $d(x) \neq d(y)$ then x and y represent different cosets in $\mathcal{W}_{\mathfrak{p}} \backslash \mathcal{W}$. This is clear because the cosets are labelled by $\{+,-\}$-sequences and d only depends on these sequences.

1. If d increases when multiplying w by s it means that either a "$-$" is switched with a plus to its right or two minuses are generated at places 1 and 2. In both cases the coset $[ws]$ is different from $[w]$. In addition, we saw before that both cases mean that a shortest representative for the coset $[ws]$ is longer than a sortest representative for $[w]$. Since $l(ws) = l(w) \pm 1$, these facts imply that $l(ws) = l(w) + 1$ and ws has to be a shortest representative and hence in $\mathcal{W}^{\mathfrak{p}}$.
2. In this case the argument follows the same pattern as in 1.
3. If d does not change when multiplying by a generator it means that no signs are altered. A generator always operates on two places in the $\{+,-\}$-sequence. Looking at how the s_i operate we see that the degree of antidominance only stays the same if s_i does not change the sequence at all.
   But this simply says that we stay in the same coset. Since w is a shortest representative for this coset, $l(ws)$ can not be $l(w) - 1$. This implies that $l(ws) = l(w) + 1$ and hence ws is not in $\mathcal{W}^{\mathfrak{p}}$.

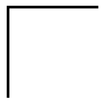
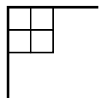
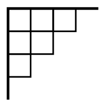
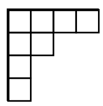
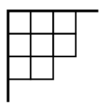
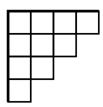
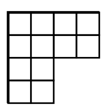
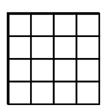
Young diagram	$\{+,-\}$-sequence	Representative in $\mathcal{W}^{\mathfrak{p}}$
	$\|+,+,+,+]$	e
	$\|-,-,+,+]$	s_0
	$\|-,+,-,+]$	s_0s_2
	$\|-,+,+,-]$	$s_0s_2s_3$
	$\|+,-,-,+]$	$s_0s_2s_1$
	$\|+,-,+,-]$	$s_0s_2s_3s_1$
	$\|+,+,-,-]$	$s_0s_2s_3s_1s_2$
	$\|-,-,-,-]$	$s_0s_2s_3s_1s_2s_0$

Figure 1: Illustration of Proposition 3.1.3 and Proposition 3.1.4

□

Example (D_4)**.** *The table in Figure 1 shows the elements in $Y_{sym}(n)$, $S_{sym}(n)$ and $\mathcal{W}^{\mathfrak{p}}$ for $n = 4$ and illustrates which elements are associated by the bijections in Proposition 3.1.3 and Proposition 3.1.4.*

3.2 The embedding $\mathcal{W} \hookrightarrow S_{2n}$

In this section we will describe how $\mathcal{W}$ embeds into S_{2n} and, in particular, into the subgroup of signed permutations.

Lemma 3.2.1. *We can embed the group $\mathcal{W}$ into S_{2n} via*

$$\begin{array}{rrcll} \Psi: & \mathcal{W} & \to & S_{2n} & \\ & s_i & \mapsto & t_i t_{-i} & \text{for } 1 \le i \le n-1 \\ & s_0 & \mapsto & t_0 t_1 t_{-1} t_0 & \end{array}$$

Proof. First we show that Ψ is well-defined. All relations not involving s_0 are satisfied in S_{2n}. Leaving out s_0 gives us $\mathcal{W}_{\mathfrak{p}}$ which is isomorphic to S_n. On $\mathcal{W}_{\mathfrak{p}}$ the map Ψ is just a diagonal embedding of S_n into $S_n \times S_n \subset S_{2n}$.
So we calculate the relations involving $\Psi(s_0)$.

$$\Psi(s_0)^2 = t_0 t_1 t_{-1} t_0 t_0 t_1 t_{-1} t_0 = t_0 t_{-1} t_1 t_1 t_{-1} t_0 = e$$

since t_{-1} and t_1 commute.
If $i > 2$ then $(\Psi(s_0)\Psi(s_i))^2 = e$ because all the elements in $\Psi(s_i)$ commute with t_0, t_{-1} and t_1.
It remains to show that

$$\Psi(s_0)\Psi(s_1) = \Psi(s_1)\Psi(s_0)$$
and
$$\Psi(s_0)\Psi(s_2)\Psi(s_0) = \Psi(s_2)\Psi(s_0)\Psi(s_2).$$

$$\begin{aligned} \Psi(s_0)\Psi(s_1) &= t_0 t_1 t_{-1} t_0 t_1 t_{-1} \\ &= t_0 t_1 t_{-1} t_0 t_{-1} t_1 \\ &= t_0 t_1 t_0 t_{-1} t_0 t_1 \\ &= t_1 t_0 t_1 t_{-1} t_0 t_1 \\ &= t_1 t_0 t_{-1} t_1 t_0 t_1 \\ &= t_1 t_0 t_{-1} t_0 t_1 t_0 \\ &= t_1 t_{-1} t_0 t_{-1} t_1 t_0 \\ &= \Psi(s_1)\Psi(s_0) \end{aligned}$$

$$\begin{array}{rrl} & \Psi(s_0)\Psi(s_2)\Psi(s_0) & = \Psi(s_0)\Psi(s_2)\Psi(s_0) \\ \Leftrightarrow & t_0 t_1 t_{-1} t_0 t_{-2} t_2 t_0 t_1 t_{-1} t_0 & = t_{-2} t_2 t_0 t_1 t_{-1} t_0 t_{-2} t_2 \\ \Leftrightarrow & t_0 t_1 t_{-1} t_0 t_0 t_{-2} t_2 t_1 t_{-1} t_0 & = t_0 t_{-2} t_2 t_1 t_{-1} t_{-2} t_2 t_0 \\ \Leftrightarrow & t_1 t_{-1} t_{-2} t_2 t_1 t_{-1} & = t_{-2} t_2 t_1 t_{-1} t_{-2} t_2 \\ \Leftrightarrow & t_{-1} t_{-2} t_{-1} t_1 t_2 t_1 & = t_{-2} t_{-1} t_{-2} t_2 t_1 t_2 \end{array}$$

$t_{-1} t_{-2} t_{-1} = t_{-2} t_{-1} t_{-2}$ and $t_1 t_2 t_1 = t_2 t_1 t_2$ in S_{2n}. So Ψ is well defined.
Next we prove the injectivity of Ψ. For this we find a combinatorial description of the image of Ψ and calculate its cardinality. Comparing cardinalities we find that Ψ has to be injective.
Consider an element in S_{2n} in one line notation

$$[a_{-n}, a_{-(n-1)}, \ldots, a_{-1}, a_1, \ldots, a_{n-1}, a_n].$$

Claim. The image of Ψ is $\{\sigma \in S_n | a_i + a_{-i} = 0 \text{ and } |\{i > 0 : a_i < 0\}| \text{ is even}\} =: I$.
Proof of claim. Obviously, the neutral element is in I.
Multiplying an element in I by $\Psi(s_i)_{1 \le i \le n-1}$ exchanges a_{-i} and a_{-i-1} as well as a_i

and a_{i+1}.
We denote the new elements by $\widetilde{a}_j$.
So $\widetilde{a}_{-i} = a_{-i-1}$, $\widetilde{a}_{-i-1} = a_{-i}$, $\widetilde{a}_i = a_{i+1}$ and $\widetilde{a}_{i+1} = a_i$. $\widetilde{a}_{-i} + \widetilde{a}_i = a_{-i-1} + a_{i+1} = 0$. This is the first condition for one of the new pairs. The calculation for the second pair is done analogously. So the first condition is met. The second condition is still statisfied because $\{a_1, \dots, a_n\}$ is stabilised.
Multiplying an element with $\Psi(s_0)$ exchanges (a_{-2}, a_{-1}) and (a_1, a_2). A calculation similar to the one above shows that the first condition is still satisfied.
From the first condition follows that if a_{-1} is bigger than 0 then a_1 is smaller than 0 and vice versa. The same holds true for a_{-2} and a_2. So we either exchange two elements that are bigger than 0 for two that are smaller or we exchange one that is bigger than 0 and one that is smaller than 0 for two elements with the same property. But this does not change the parity of the set $\{i > 0 : a_i < 0\}$.
So applying $\Psi(s_i)$ for any i to an element in I gives another element in I. Hence the image of Ψ is contained in I.
If we have an element in I, we can construct this element from the $\Psi(s_i)$. Start with e. Use the $\Psi(s_i)$ to move the entry a_n to the place n. The condition $a_{-i} + a_i = 0$ ensures that the element, which ends up at the place $-n$ is the right one. Do this inductively down to 2. The second condition then ensures that the elements standing at places ± 1 are the right ones. So our claim is proven.

Now we calculate the cardinality of I: Because of the second condition we can choose $2i$ numbers between $-n$ and -1 that will be put at a place between 1 and n. By the first condition this gives us also all the numbers between 1 and n which will be put at a place between $-n$ and -1. Consequently, we also know the positive numbers that stay between 1 and n, and hence all the numbers that are at places 1 through n. The remaining thing to do is choosing an order for these n elements. The order of the other elements is defined by the first condition.
Counting choices we get

$$|I| = \sum_{i=0}^{n} \binom{n}{2i} n! = n! \sum_{i=0}^{n} (\binom{n-1}{2i} + \binom{n-1}{2i+1}) = n! \sum_{i=0}^{2n} \binom{n-1}{i} = n! \cdot 2^{n-1}.$$

The first sum should only run up to $\lfloor \frac{n}{2} \rfloor$. But all the other summands are 0 so the upper bound does not matter as long as it is big enough, which is the case here. Same goes for all the other sums. In the middle term we sum over the even and the odd numbers. So alternatively we can sum over all numbers.
Thus we see that $|I| = |\mathcal{W}|$ so Ψ has to be injective. □

Corollary 3.2.2. *$\mathcal{W}$ can be realized as a proper subgroup of the group of signed permutations.*

Proof. The first condition on I in our proof simply says that $\omega(-i) = -\omega(i)$. This is exactly the condition we have on the group of signed permutations. So the group $\mathrm{im}(\Psi)$ is contained in the subgroup of signed permutations in S_{2n}. So we even get

inclusion of the Weyl group of type D_n into the Weyl group of type B_n.
This is a proper inclusion since we have the additional condition that the number of $i > 0$ with $\omega(i) < 0$ is even, which simply means that we get an even number of minus signs in the right half of our permutation. □

With Corollary 3.2.2 the notation of $\{+,-\}$-sequences in Chapter 2 becomes clear. With the previous chapter we see that in $S_{sym}(n)$ we simply record the signs of the upper half of a signed permutation $\omega \in S_{2n}$ written down in one line notation. The | on the left side indicates the antisymmetry of the full $\{+,-\}$-sequence of length $2n$.

4 The Hecke algebra and the Kazhdan-Lusztig polynomials

We recall the standard setup of Hecke algebras and Kazhdan-Lusztig polynomials which will then be applied to our special case in the following section.

4.1 Definitions and standard results

Kazhdan and Lusztig introduced in [KL79] the Kazhdan-Lusztig basis for Hecke algebras and the Kazhdan-Lusztig polynomials. Since then, the Kazhdan-Lusztig polynomials have proven to have many Lie theoretic interpretations and applications. The parabolic versions of Kazhdan-Lusztig polynomials were introduced and studied by Deodhar in [Deo90].
As indicated in the introduction, we follow with our definitions, notation and conventions Soergel [Soe97].
Let (W, S) be a Coxeter system. So W is a group with generators $s_i \in S$ and relations $s_i^2 = e$ and $(s_i s_j)^{m_{ij}} = e$ for some $m_{ij} \in \mathbb{N}\backslash\{0\} \cup \{\infty\}$. Let $l : W \to \mathbb{N}$ be the length function and $\leq$ the Bruhat order, meaning that $w' \leq w$ if and only if there exists a reduced expression for w such that the expression starts with an expression for w' [BB05, Def. 2.1.1]. We define $\mathcal{L}$ to be the ring $\mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in v. We consider the free $\mathcal{L}$-module with basis $\{H_w | w \in W\}$, $\mathcal{H} = \mathcal{H}(W, \mathcal{S}) := \bigoplus_{w \in W} \mathcal{L} H_w$.
On this module there exists exactly one structure of an associative $\mathcal{L}$-algebra satisfying $H_w H_{w'} = H_{ww'}$ if $l(w) + l(w') = l(ww')$ and $H_s^2 = 1 + (v^{-1} - v) H_s$. This can also be described via the braid relations $H_s H_t \ldots H_t = H_t H_s \ldots H_s$ resp. $H_s H_t \ldots H_s = H_t H_s \ldots H_t$ if $st \ldots t = ts \ldots s$ resp. $st \ldots s = ts \ldots t$ for $s, t \in S$. All

the H_s are invertible by $H_s^{-1} = H_s + (v - v^{-1})$ and, consequently, all the H_w are. In the case of type D_n, the type we are interested in, we have the relations

$$\begin{cases} s_i^2 = e & 0 \leq i \leq n-1 \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1, \text{ for } i,j \geq 1 \text{ or } \{i,j\} = \{0,1\} \\ s_i s_j s_i = s_j s_i s_j & \text{otherwise} \end{cases}.$$

On $\mathcal{H}$ we define an involution $\overline{} : \mathcal{H} \to \mathcal{H}$, sending v to v^{-1} and H_w to $(H_{w^{-1}})^{-1}$. We call $H \in \mathcal{H}$ *self-dual* if $\overline{H} = H$.
The following theorem defines the Kazhdan-Lusztig basis elements. For convenience we recall its proof.

Theorem 4.1.1 (Kazhdan-Lusztig). *For all $w \in W$ there exists exactly one self-dual $\underline{H}_w \in \mathcal{H}$ with $\underline{H}_w \in H_w + \sum_{w'} v\mathbb{Z}[v] H_{w'}$ $(\triangle)$.*

Proof. It is easy to calculate that $\overline{H_s} = H_s^{-1} = H_s + (v - v^{-1})$, i.e. $C_s := H_s + v$ is self-dual and satisfies $(\triangle)$.
Multiplication by C_s in $\mathcal{H}$ is given by the formula

$$H_w C_s = \begin{cases} H_{ws} + vH_w & \text{if } ws > w \\ H_{ws} + v^{-1}H_w & \text{if } ws < w. \end{cases}$$

We now prove the existence of $\underline{H}_w$ via induction on the Bruhat order. We prove the stronger
Claim: For all $w \in W$ there exists a self-dual $\underline{H}_w \in \mathcal{H}$ with

$$\underline{H}_w \in H_w + \sum_{w' < w} v\mathbb{Z}[v] H_{w'}$$

Obviously we can start our induction with $\underline{H}_e = H_e = 1$.
Take now $w \in W$ and let $\underline{H}_{w'}$ be known for all $w' < w$. If $w \neq e$ we find $s \in S$ such that $ws < w$ and by induction hypothesis we have

$$\underline{H}_{ws} C_s = H_w + \sum_{w' < w} h_{w'} H_{w'}$$

for some $h_{w'} \in \mathbb{Z}[v]$. Now we get

$$\underline{H}_w = \underline{H}_{ws} C_s - \sum_{w' < w} h_{w'}(0) \underline{H}_{w'}.$$

$\underline{H}_w$ is self-dual since the $\underline{H}_{w'}$, C_s and $h_{w'}(0)$ are. In addition all coefficients lie in $v\mathbb{Z}[v]$ since all the coefficients except the one in front of $H_{w'}$ in $\underline{H}_{w'}$ do and all the absolut terms generated in $\underline{H}_{ws} C_s$ are subtracted.
This proves the existence. To prove uniqueness we prove the following
Claim: For $H \in \sum_w v\mathbb{Z}[v] H_w$, $\overline{H} = H$ already implies $H = 0$.

Proof of Claim. With our previous calculations we see that $H_w \in \underline{H}_w + \sum_{w'<w} \mathcal{L}\underline{H}_{w'}$ and hence $\overline{H_w} \in \underline{H}_w + \sum_{w'<w} \mathcal{L}\underline{H}_{w'} = H_w + \sum_{w'<w} \mathcal{L}H_{w'}$ for all $w \in W$. Writing $H = \sum h_w H_w$ we can choose w maximal with $h_w \neq 0$. By taking $\overline{H}$ we do not get any $h_z \neq 0$ with $z > w$, consequently, we must have $h_w = \overline{h_w}$. But this is not possible if $h_w \in v\mathbb{Z}[v]$. This proves the claim.
If we had two different elements satisfying the conditions of the proof we could subtract one from the other and use the claim to see that the difference has to be 0. So our theorem is proven. □

The proof gives us an algorithm to compute the Kazhdan-Lusztig basis. The parabolic version of the algorithm is used at the end of this section to compute a parabolic version of the Kazhdan-Lusztig basis in an example.

Definition 4.1.2. For $w, w' \in W$ we define the *Kazhdan-Lusztig polynomials* $h_{w',w} \in \mathcal{L}$ by $\underline{H}_w = \sum_{w'} h_{w',w} H_{w'}$.

Remark 4.1.3. Although we have the inclusion $\mathcal{W} \hookrightarrow S_n$, we do not have the corresponding inclusion $\mathcal{H}(\mathcal{W}) \hookrightarrow \mathcal{H}(S_n)$ of Hecke algebras for generic v.
For example in $\mathcal{H}(\mathcal{W})$ we have the relation

$$H_{s_i}^2 = H_e + (v^{-1} - v)H_{s_i}.$$

For $1 \leq i \leq n-1$ this would be mapped to $H_e + (v^{-1} - v)H_{t_{n-i}t_{n+i}}$. On the other hand H_{s_i} would be mapped to $H_{t_{n-i}t_{n+i}}$. But with t_{n-1} and t_{n+i} commuting we have

$$\begin{aligned} H_{t_{n-i}t_{n+i}}^2 &= (H_{t_{n-i}} H_{t_{n+i}})^2 \\ &= H_{t_{n-i}}^2 H_{t_{n+i}}^2 \\ &= (H_e + (v^{-1} - v)H_{t_{n-i}})(H_e + (v^{-1} - v)H_{t_{n+i}}) \\ &= H_e + (v^{-1} - v)(H_{t_{n-i}} + H_{t_{n+i}}) + (v^{-1} - v)^2 H_{t_{n-i}t_{n+i}}. \end{aligned}$$

However, specializing at $v = 1$ results in all relations being satisfied and we get an inclusion of algebras.

We now pass to the parabolic case, which is our main interest.
Let $S_p \subset S$ a subset and $W_p = \langle S_p \rangle \subset W$ the subgroup generated by S_p. We denote by $W^p \subset W$ the set of representatives of minimal length for the right cosets $W_p \backslash W$ and by $\mathcal{H}_p = \mathcal{H}(W_p, S_p) \subset \mathcal{H}$ the Hecke algebra of (W_p, S_p). The quadratic relation $H_s^2 = H_e + (v^{-1} - v)H_s$ can be written as $(H_s + v)(H_s - v^{-1}) = 0$. This implies that by letting H_s act on $\mathcal{L}$ as $-v$ for all $s \in \mathcal{S}_p$, $\mathcal{L}$ becomes a right-$\mathcal{H}_p$-module.
Now we induce and define the right $\mathcal{H}$-module

$$\mathcal{N} = \mathcal{N}^p = \mathcal{L} \otimes_{\mathcal{H}_p} \mathcal{H}$$

This module is usually referred to as a *parabolic Hecke module.*

Lemma 4.1.4. *The elements $N_w = 1 \otimes H_w$ for $w \in W^p$ form an $\mathcal{L}$-basis of $\mathcal{N}$.*

Proof. We have a bijection of sets $W_p \times W^p \to W$ via multiplication. So every element $H_\omega \in \mathcal{H}$ can be written uniquely as $H_{w'}H_w$ with $w' \in W_p$ and $w \in W^p$. Since we tensor over $\mathcal{H}_p$ we see that the N_w generate $\mathcal{N}$. Since the H_w for $w \in W^p$ are $\mathcal{L}$-independent over $\mathcal{H}$ the N_w are, too. □

Proposition 4.1.5. *The multiplication rules for the C_s in $\mathcal{N}^p$ are:*

$$N_w C_s = \begin{cases} N_{ws} + vN_w & \text{if } ws \in W^p \text{ and } ws > w \\ N_{ws} + v^{-1}N_w & \text{if } ws \in W^p \text{ and } ws < w \\ 0 & \text{if } ws \notin W^p \end{cases}.$$

Proof. The first two cases are clear from the rules for the H_w. For the last case we need the following observation: If $l(rws) < l(ws)$ then we have $rws = w$. Take a reduced expression $s_{i_1} \cdots s_{i_k}$ for w. Since $l(rws) < l(ws)$ we know by the strong exchange property [BB05, Thm. 1.4.3] that rws is equal to $s_{i_1} \cdots s_{i_k} s$ with one of the s_{i_j} or the s deleted. But since $s_{i_1} \cdots s_{i_k}$ is reduced the superfluous term has to be the s. Otherwise $s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_k}$ would be an expression for w of smaller length. Now if $ws \notin W^p$ means that there exists an $r \in S_p$ with $l(rws) < l(ws)$, i.e. $rws = w$. But this means that $rw = ws$. Furthermore we have $l(rw) > l(w)$ since $w \in W^{\mathfrak{p}}$, and hence $l(rw) = l(w) + 1 = l(r) + l(w)$.
Consequently, if $ws > w$, we have

$$N_w C_s = 1 \otimes H_w C_s = 1 \otimes (H_{ws} + vH_w) = 1 \otimes (H_r H_w + vH_w) = -v \otimes H_w + 1 \otimes vH_w = 0.$$

On the other hand, if $ws < w$ we have $rws = w$ and $l(rws) = l(w) = l(ws) + 1 = l(r) + l(ws)$. So

$$\begin{aligned} N_w C_s &= 1 \otimes H_w C_s \\ &= 1 \otimes (H_{ws} + v^{-1} H_w) \\ &= 1 \otimes (H_{ws} + v^{-1} H_r H_{ws}) \\ &= 1 \otimes H_{ws} - v \otimes v^{-1} H_{ws} = 0. \end{aligned}$$

The proposition follows. □

As on $\mathcal{H}$ we define an involution $a \otimes H \mapsto \overline{a} \otimes \overline{H}$ on $\mathcal{N}$. Elements $n \in \mathcal{N}$ with $\overline{N} = N$ are called *self-dual*. We have a theorem analogous to Theorem 4.1.1.

Theorem 4.1.6. *For all $w \in W^p$ there exists a unique self-dual $\underline{N}_w \in N$ with*

$$\underline{N}_w \in N_w + \sum_{w'} v\mathbb{Z}[v] N_{w'}.$$

Proof. The proof is completely analogous to the one in $\mathcal{H}$. □

Definition 4.1.7. For $w', w \in W^{\mathfrak{p}}$ we define the *parabolic Kazhdan-Lusztig polynomials* $n_{w',w} \in \mathbb{Z}[v]$ by $\underline{N}_w = \sum_{w' \in \mathcal{W}^{\mathfrak{p}}} n_{w',w} N_{w'}$.

Example (Kazhdan-Lusztig polynomials in our parabolic D_4 case)**.**
Consider the case where W is the Weyl group of type D_4 and W_p is the subgroup isomorphic to S_4. Recall the list of shortest representatives for $W_p \backslash W$:

$\|{+}{+}{+}{+}]$	$\|{-}{-}{+}{+}]$	$\|{-}{+}{-}{+}]$	$\|{-}{+}{+}{-}]$	$\|{+}{-}{-}{+}]$	$\|{+}{-}{+}{-}]$	$\|{+}{+}{-}{-}]$	$\|{-}{-}{-}{-}]$
e	s_0	s_0s_2	$s_0s_2s_3$	$s_0s_2s_1$	$s_0s_2s_3s_1$	$s_0s_2s_3s_1s_2$	$s_0s_2s_3s_1s_2s_0$

We write $N_{i_1 \dots i_k}$ for $N_{s_{i_1} \cdots s_{i_k}}$.
We get:

$$\underline{N}_e \quad : \quad \underline{N}_e = N_e$$

$$\underline{N}_0 \quad : \quad \underline{N}_0 = C_0 = N_0 + vN_e$$

$$\underline{N}_{02} \quad : \quad \underline{N}_0 C_2 = (N_0 + vN_e)C_2 = N_0C_2 + v \underbrace{N_eC_2}_{=0 \ \ da \ \ s_2 \notin \mathcal{W}^{\mathfrak{p}}} = N_{02} + vN_0 = \underline{N}_{02}$$

$$\underline{N}_{023} \quad : \quad \underline{N}_{02} C_3 = (N_{02} + vN_2)C_3 = N_{02}C_3 + v \underbrace{N_0C_3}_{=0} = N_{023} + vN_{02} = \underline{N}_{023}$$

$$\underline{N}_{021} \quad : \quad \underline{N}_{021} = N_{021} + vN_{02} \ \textit{analogous to} \ \underline{N}_{023}$$

$$\underline{N}_{0231} \quad : \quad \underline{N}_{023} C_1 = (N_{023} + vN_{02})C_1 = N_{023}C_1 + vN_{02}C_1$$

$$= N_{0231} + vN_{023} + vN_{021} + v^2 N_{02} = \underline{N}_{0231}$$

$$\underline{N}_{02312} \quad : \quad \underline{N}_{0231} C_2 = N_{0231}C_2 + v \underbrace{N_{023}C_2}_{=0} + v \underbrace{N_{021}C_2}_{=0} + v^2 N_{02} C_2$$

$$= N_{02312} + vN_{0231} + v^2(N_0 + v^{-1}N_{02})$$

$$= N_{02312} + vN_{0231} + vN_{02} + v^2 N_0 = \underline{N}_{02312}$$

$$\underline{N}_{023120} \quad : \quad \underline{N}_{02312} C_0 = N_{02312}C_0 + v \underbrace{N_{0231}C_0}_{=0} + v \underbrace{N_{02}C_0}_{=0} + v^2 N_0 C_0$$

$$= N_{023120} + vN_{02312} + vN_0 + v^2 N_e = \underline{N}_{023120}$$

So we get the following table of Kazhdan-Lusztig polynomials:

w' *in*	e	s_0	s_0s_2	$s_0s_2s_3$	$s_0s_2s_1$	$s_0s_2s_3s_1$	$s_0s_2s_3s_1s_2$	$s_0s_2s_3s_1s_2s_0$
e	*1*	v	0	0	0	0	0	v^2
s_0		1	v	0	0	0	v^2	v
s_0s_2			1	v	v	v^2	v	0
$s_0s_2s_3$				1	0	v	0	0
$s_0s_2s_1$					1	v	0	0
$s_0s_2s_3s_1$						1	v	0
$s_0s_2s_3s_1s_2$							1	v
$s_0s_2s_3s_1s_2s_0$								1

Remark 4.1.8. We see that the calculation of Kazhdan-Lusztig polynomials is not that easy and involves rather cumbersome calculation.
This, of course, raises the question if there is any interpretation of the Kazhdan-Lusztig basis or the Kazhdan-Lusztig polynomials that gives us an easier, maybe even closed, formula.

Remark 4.1.9. Another observation is that the $n_{w',w}$ even lie in $\mathbb{N}[v]$. This turns out to be true in all our cases. In general however, this is a non-trivial result.

4.2 The parabolic type D case

From now on we consider the special case where W is our Weyl group $\mathcal{W}$ of type D_n and $S_p = \mathcal{S}_\mathfrak{p}$.

Definition 4.2.1. The *weight* associated to an element $w \in \mathcal{W}^\mathfrak{p}$ is obtained from α_w by first extending the $\{+,-\}$-sequence with n pluses to the left and n minuses to the right and then replacing every "$+$" with a "$\vee$" and every "$-$" with an "$\wedge$". We call the n pluses and n minuses by which we extended the sequence *frozen*. The weight is denoted by $w_{\vee\wedge}$.

The reason for extending the weight to the left and right will become clear in the next chapter.
Consider the free $\mathcal{L}$-module M_D with basis consisting of these weights.

Lemma 4.2.2. *The $\mathcal{L}$-modules M_D and $\mathcal{N}$ are isomorphic via extending the bijection ϕ from Proposition 3.1.4 $\mathcal{L}$-linearly.*

Proof. ϕ is a bijection between $S_{sym}(n)$ and $\mathcal{W}^\mathfrak{p}$ and hence between the weights and $\mathcal{W}^\mathfrak{p}$. This bijection sends a $\mathcal{L}$-basis of M_D to a $\mathcal{L}$-basis of $\mathcal{N}$. This bijection extends to a $\mathcal{L}$-module isomorphism. We call this isomorphism $\Phi : M_D \to \mathcal{N}$. □

Since $\mathcal{N}$ is a module for the Hecke algebra we get an induced action of $\mathcal{H}$ on M_D. Define for $m \in M_D$ and $h \in \mathcal{H}$

$$m.h := \Phi^{-1}(\Phi(m).h).$$

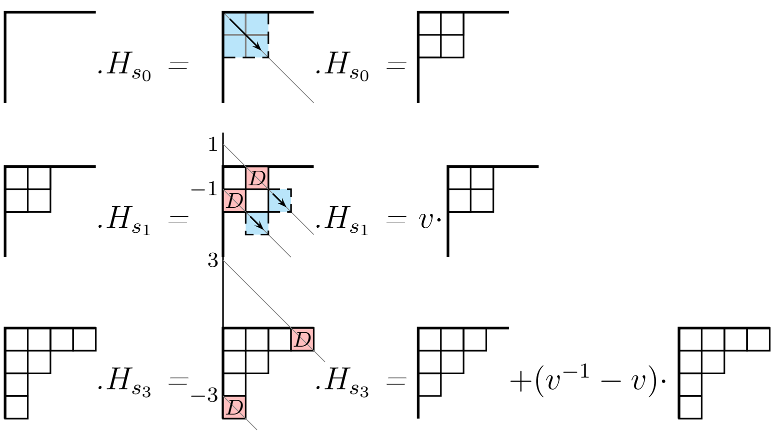
This turns M_D into an $\mathcal{H}$-module. Calculating the action of the H_{s_i} on the $\{\vee, \wedge\}$-sequences gives us

$$w_{\vee\wedge}.H_{s_i} = \begin{cases} (w.s_i)_{\vee\wedge} & \text{if } d(w) < d(w.s_i) \\ (w.s_i)_{\vee\wedge} + (v^{-1} - v)w_{\vee\wedge} & \text{if } d(w) > d(w.s_i) \\ -vw_{\vee\wedge} & \text{if } d(\alpha) = d(w.s_i) \end{cases}$$

The action of $\mathcal{H}$ on the $\{\vee, \wedge\}$-sequences gives us also an action on the Young diagrams.
Recall the action of the s_i on the Young diagrams. With this information the actions of $\mathcal{H}$ on $\mathcal{N}$ translate in terms of boxes into the rule:

> *"If you can add the boxes, simply add them. If you can delete the boxes, remove them and add the old diagram $(v^{-1} - v)$ times. If you can do neither just multiply by v."*

Example (Some examples in case D_4)**.**
The first picture depicts the desired action. The second picture shows the possibilities of extending the diagram resp. deleting boxes. The blue boxes are the ones possibly added, the red boxes with the D are the ones possibly removed. The last diagram is the result.



5 A graphical description of Kazhdan-Lusztig polynomials

In this chapter we will prove our main theorem and give a graphical way for calculating the Kazhdan-Lusztig polynomials in our parabolic case.

5.1 Cup diagrams

To define a cup diagram associated to a $\{+, -\}$-sequence we first define a matching and modify this matching afterwards to get our cup diagram.

Definition 5.1.1. Let $P := \{-2n, \dots, -1, 1, \dots, 2n\} \subset \mathbb{Z} \subset \mathbb{R}^2$ be integers viewed as points on the number line.
A *matching* is a diagram consisting of $2n$ non-intersecting arcs in the lower half

plane $\mathbb{R} \times \mathbb{R}_{\leq 0}$ connecting the $4n$ points without intersections. In particular any point in P is connected with precisely one other point in P via an arc.

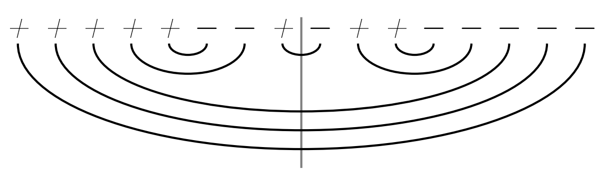
Definition 5.1.2. Given a $\{+,-\}$-sequence α the *corresponding matching* is denoted $M(\alpha)$ and defined as follows:
First extend α by n pluses to the left and n minuses to the right. Then construct the arcs in a way that each arc connects a "$+$" with a "$-$" and the index of the point with the "$+$" is smaller than the index of the one with the "$-$".

The first question arising is of course the question if such a corresponding matching is well-defined. The next lemma clarifies this question.

Lemma 5.1.3. *Given a $\{+,-\}$-sequence α then $M(\alpha)$ exists and is well-defined.*

Proof. Start with the points where a "$+$" is directly left of a "$-$".
Claim: These two points have to be connected.
Proof of Claim. If these two points were not connected then the minus would have to be connected to another plus left of the first "$+$". This arc divides the lower half plane into two connected components. All minuses to the right of the first "$-$" are in one connected component, while our first "$+$" is in the other. But the first "$+$" would have to be connected to one of these minuses without intersecting the given arc and hence without leaving the connected component; a contradiction.
Since two points of the form above have to be joined, they can be connected and then ignored. But this ignoring gives us again a $\{+,-\}$-sequence where adjacent points labeled with "$+$" and "$-$" in this order have to be connected. This gives us inductively our matching, which is obviously unique. ☐

Example.
Take $[+--+|-++-]$. *Extending gives us* $[+++++--+|-++-----]$. *Now we have to connect the pluses and minuses. We start with the places where a plus is next to a minus. Working our way up we get*



Remark 5.1.4. The construction of extending the sequences was motivated by [Str09].

Definition 5.1.5. The *cup diagram* $C(w)$ associated to an element in $\mathcal{W}^{\mathfrak{p}}$ is obtained from the matching $M(\alpha_w)$ by "pairing" the arcs crossing the middle. Starting from the point $(0,0)$ we take the first two arcs crossing the middle. We exchange their starting points and put a decoration on the intersection indicating that they are linked. We continue this process with the next two arcs until no unlinked arcs crossing the middle are left.

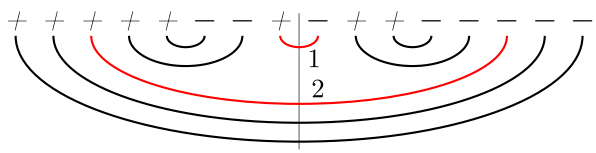
By “end” or “ending point” of a cup we mean the right end of the cup and conversely by “start” or “starting point” the left end.
We denote the set of cup diagrams obtained from elements in $\mathcal{W}^{\mathfrak{p}}$ by $C(\mathcal{W}^{\mathfrak{p}})$.

The next lemma shows that the construction of such a cup diagram always works.

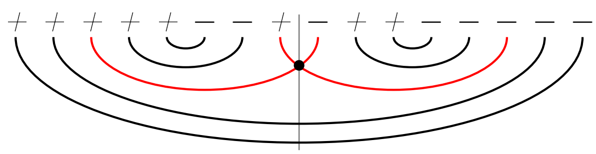
Lemma 5.1.6. *For all $w \in \mathcal{W}^{\mathfrak{p}}$ the number of arcs crossing the middle in $M(\alpha_w)$ is even.*

Proof. The number of arcs crossing the middle is given by the formula $2n - 2 \cdot \sum_{\substack{\alpha_i=+1 \\ i>0}} 1$.
The sum simply counts the number of pluses on the right half of the sequence. Every plus there is connected to a minus on the right half. The remaining minuses have to be connected to the left half and their number can obviously be calculated by the formula. But the formula gives us an even number so the number of arcs crossing the middle has to be even. □

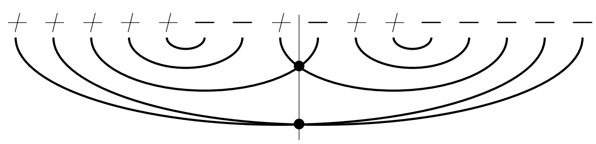
Example.
Take the example from above. The first two arcs crossing the middle are the arcs 1 and 2, drawn in red.



Now we switch the starting points and get



Doing the same with the two remaining arcs that cross the middle we get our final diagram



To summarize:

Proposition 5.1.7. *There is a bijection*

$$\begin{array}{ccc} \mathcal{W}^{\mathfrak{p}} & \to & C(\mathcal{W}^{\mathfrak{p}}) \\ w & \mapsto & C(w) \end{array} .$$

Proof. To two different elements in $\mathcal{W}^{\mathfrak{p}}$ we associate two different $\{+,-\}$-sequences. To these sequences we obviously associate two different matchings and as a consequence two different cup diagrams. So the map is injective. It is surjective by definition and hence a bijection. $\square$

In Definition 4.2.1, to every element in $\mathcal{W}^{\mathfrak{p}}$ we associated a weight. Because of the extension we can glue the weight associated to a $w' \in \mathcal{W}^{\mathfrak{p}}$ on top of a cup diagram $C(w)$. This gives us a new diagram, denoted $w'C(w)$.

Definition 5.1.8. A cup is called *oriented* if it is labeled by exactly one "$\wedge$" and one "$\vee$". This goes for linked cups, too. They are treated as if they were single cups. The decoration has no meaning for the orientation.
A cup diagram $w'C(w)$ is an *oriented cup diagram* if each cup is oriented.
We denote by $\mathrm{cl}(w'C(w))$ the number of clockwise oriented cups in an oriented cup diagram $w'C(w)$, i.e. oriented cups of the form $\wedge\!\smile\!\vee$.

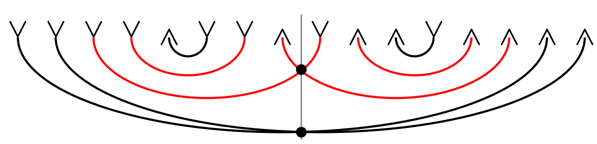
Remark 5.1.9. For linked cups the antisymmetry of the weights implies that both cups are oriented in the same direction.

To every cup diagram we associate an element in M_D called $C(w)_{M_D}$. Define

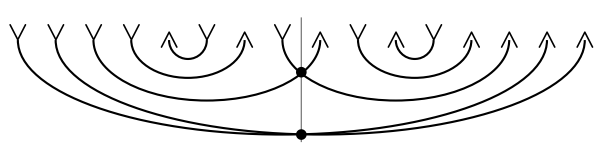
$$C(w)_{M_D} := \sum v^{\frac{\mathrm{cl}(w'C(w))}{2}} w'_{\vee\wedge}$$

where the sum is over all $w'_{\vee\wedge} \in \mathcal{W}^{\mathfrak{p}}$ such that $w'C(w)$ is oriented.

Example.
Taking our diagram $C(w)$ from above and glueing the weight corresponding to $[-++-|+--+]$ on top we get



This is not oriented because of the red arcs.
Glueing the weight corresponding to $w' = [-+-+|-+-+]$ on top gives us



which is oriented and has $\mathrm{cl}(w'C(w)) = 2$.
So in $C(w)_{M_D}$ the coefficient for the first weight would be 0 and the coefficient for the second one would be v.

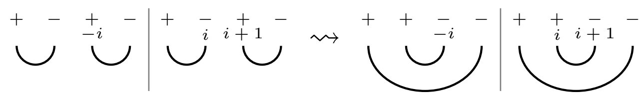
Next we analyze the difference between the cup diagram $C(w)$ and the cup diagram $C(ws_i)$ in the case that $l(ws_i) > l(w)$ and $w, ws_i \in \mathcal{W}^{\mathfrak{p}}$. If multiplying w by s_i yields a longer element in $\mathcal{W}^{\mathfrak{p}}$, this means for the $\{+,-\}$-sequence we have either

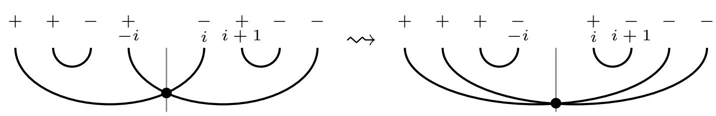
$$\text{(a): } |\ldots -+\ldots] \rightsquigarrow |\ldots +-\ldots]$$

or

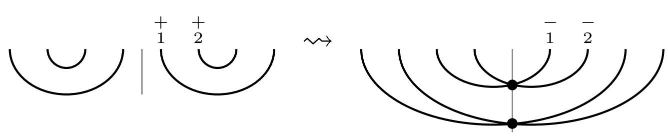
$$\text{(b): } |++\ldots] \rightsquigarrow |--\ldots]$$

Now we have to translate these changes into cup diagrams. For this we consider the reflected antisymmetric part, too and add the pluses resp. minuses we need to connect all the given pluses and minuses according to our construction rules for cup diagrams.
In the first case we have to connect the minus in the first sequence to a plus to the left. We have to distinguish whether this plus is on the upper or the lower half of the full sequence. This gives us the cases (a.1) and (a.2).

(a.1):


(a.2):


In the second case we do not have to distinguish two cases. Completing the cups gives us

(b):


5.2 The Kazhdan-Lusztig basis described as cup diagrams

The following provides a graphical description of the Kazhdan-Lusztig basis from Chapter 4.1 in our special case.
In this section we will prove our main

Theorem 5.2.1 (Graphical Kazhdan-Lusztig basis)**.** *The image of a cup diagram under the isomorphism Φ is the corresponding Kazhdan-Lusztig basis element.*

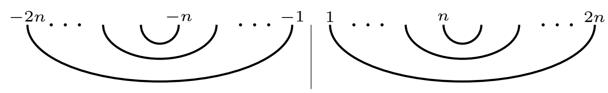
$$\Phi(C(w)_{M_D}) = \underline{N}_w$$

The crucial step for the proof is the following proposition which reveals the connection between the Kazhdan-Lusztig polynomials and the coefficients in front of the weights in our cup diagrams.

Proposition 5.2.2. *The Kazhdan-Lusztig polynomial $n_{w',w}$ is*

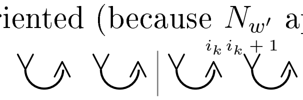
$$\begin{cases} v^{\frac{\mathrm{cl}(w'C(w))}{2}} & \text{if } w'C(w) \text{ is oriented} \\ 0 & \text{if } w'C(w) \text{ is not oriented .} \end{cases}$$

Proof. We prove this by induction, starting with $\underline{N}_e = N_e$, i.e. $n_{w',e} = \begin{cases} 1 & \text{if } w' = e \\ 0 & \text{if } w' \neq e \end{cases}$.
$e = [\underbrace{- \ldots -}_{n} | \underbrace{+ \ldots +}_{n}]$ so we get the cup diagram

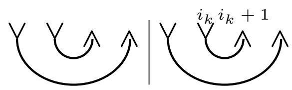


Now we determine the possible weights. First, we have to take the "frozen" orientations into account. Between $-2n$ and $-(n+1)$ we get n "$\vee$"s and between $n+1$ and $2n$ we get n "$\wedge$"s. So the only possibility to get an oriented diagram is the weight $[\underbrace{\vee \ldots \vee}_{n \text{ frozen}} \underbrace{\wedge \ldots \wedge}_{n} | \underbrace{\vee \ldots \vee}_{n} \underbrace{\wedge \ldots \wedge}_{n \text{ frozen}}]$ which corresponds to e. There are no clockwise oriented cups so our formula gives us exactly the Kazhdan-Lusztig polynomials $n_{w',e}$, and hence the basis for our induction.
Now we do the induction step case by case. Assume that the formula holds for all $w' < w = s_{i_1} \cdots s_{i_k}$. We have to calculate $\underline{N}_{ws_{i_k}} C_{s_{i_k}}$. We know how s_{i_k} acts on the cup diagram $C(ws_{i_k})$ and on the weights. We check that this action coincides with the multiplication of $\underline{N}_{ws_{i_k}}$ by $C_{s_{i_k}}$. So it comes down to checking the multiplication of $N_{w'}$ with $C_{s_{i_k}}$ for all $N_{w'}$ appearing in $\underline{N}_{ws_{i_k}}$. From Proposition 4.1.5, which gave us the multiplication rules, we see that we have to check three cases.

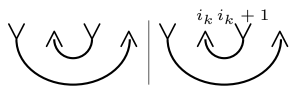
1. $w's_{i_k} > w'$ and $w's_{i_k} \in \mathcal{W}^{\mathfrak{p}}$:
   If $w's_{i_k} > w'$ we get the multiplication rule $N_{w'}C_{s_i} = N_{w's_i} + vN_{w'}$. In terms of oriented cups this means that we should have $\mathrm{cl}(w's_{i_k}C(w)) = \mathrm{cl}(w'C(ws_{i_k}))$ and $\mathrm{cl}(w'C(w)) = \mathrm{cl}(w'C(ws_{i_k})) + 2$.
   We recall that for a weight getting longer means exchanging an "$\wedge$" at i_k with a "$\vee$" at $i_k + 1$ for $1 \leq i_k \leq n-1$ resp. exchanging two "$\vee$"s at 1 and 2 for two "$\wedge$"s if $i_k = 0$.
   At the end of the previous chapter we saw, how the cup diagrams change when the element gets longer. These are the three cases (a.1), (a.2) and (b) which we have to work through.

   (a.1)
   For  to be oriented (because $N_{w'}$ appears in $\underline{N}_{ws_{i_k}}$) we have to have the orientation . None of these cups are oriented clockwise.
   The part of the weight that is interesting for our calculation is

∨ ∧ ∨ ∧ | ∨ ∧ ∨ ∧. Applying s_{i_k} to the old weight, the part of the new weight is ∨ ∨ ∧ ∧ | ∨ ∨ ∧ ∧. Now we set these parts of the weights on top of our new cup diagram.

For our new weight we get 

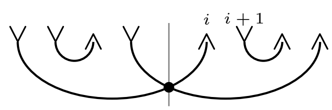
. This diagram is oriented and has no clockwise oriented cups. So the coefficient for $N_{w's_{i_k}}$ does not change which is exactly what we wanted.

For our old weight we get 

. This diagram is also oriented and we have two additional clockwise oriented cups. This coincides with the formula we expected.

The considerations for the other cases follow the same pattern. So I will give only the relevant information, namely the old oriented cup diagram, the new weights and the new oriented cup diagrams plus the changes in clockwise oriented cups.

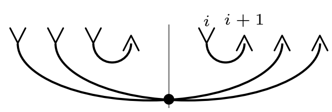
(a.2)

Old oriented diagram: 

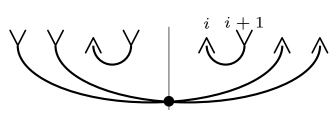
no clockwise oriented cups

Weights: ∨ ∨ ∧ ∨ | ∧ ∨ ∧ ∧ $\rightsquigarrow$ ∨ ∨ ∨ ∧ | ∨ ∧ ∧ ∧

New diagrams:

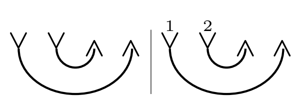
New weight: 

oriented and no new clockwise oriented cups

Old weight: 

oriented and two additional clockwise oriented cups

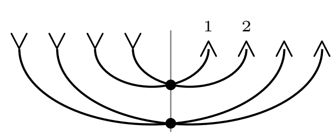
(b)

Old oriented diagram: 

no clockwise oriented cups

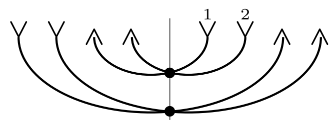
Weights: ∨ ∨ ∧ ∧ | ∨ ∨ ∧ ∧ $\rightsquigarrow$ ∨ ∨ ∨ ∨ | ∧ ∧ ∧ ∧

New diagrams:

New weight: 

oriented and no new clockwise oriented cups

Old weight:



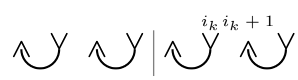
oriented and two additional clockwise oriented cups

So in all the cases we get only oriented cup diagrams and the number of clockwise oriented cups changes exactly as required.

2. $w's_{i_k} < w'$ and $w's_{i_k} \in \mathcal{W}^{\mathfrak{p}}$:
   In this case we have the multiplication rule $N_{w'}C_{s_{i_k}} = N_{w's_{i_k}} + v^{-1}N_{w'}$. In terms of oriented cups this means that $\mathrm{cl}(w's_{i_k}C(w)) = \mathrm{cl}(w'C(ws_{i_k}))$ and $\mathrm{cl}(w'C(w)) = \mathrm{cl}(w'C(ws_{i_k})) - 2$.
   We recall that getting shorter for a weight means exchanging an "$\wedge$" from i_k+1 with a "$\vee$" at i_k for $1 \le i_k \le n-1$ resp. exchanging two "$\wedge$"s at 1 and 2 for two "$\vee$"s if $i_k = 0$.
   The considerations are analogous to the ones before. So again I will give only the old oriented cup diagram, the new weight and the new oriented cup diagrams plus the changes in clockwise oriented cups.
   Again we work through our cases (a.1), (a.2) and (b).

(a.1)

Old oriented diagram:

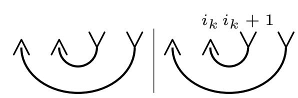


4 clockwise oriented cups

Weights: $\wedge\vee\wedge\vee\,|\,\wedge\vee\wedge\vee \;\rightsquigarrow\; \wedge\wedge\vee\vee\,|\,\wedge\wedge\vee\vee$

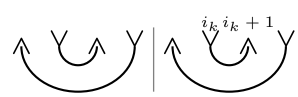
New diagrams:

New weight:



oriented and 4 clockwise oriented cups

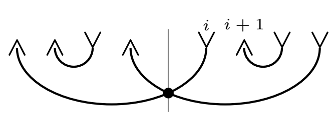
Old weight:



oriented and only 2 clockwise oriented cups

(a.2)

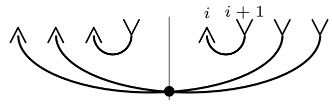
Old oriented diagram:



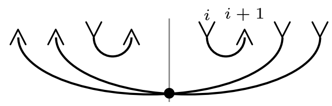
4 clockwise oriented cups

Weights: $\wedge\wedge\vee\wedge\,|\,\vee\wedge\vee\vee \;\rightsquigarrow\; \wedge\wedge\wedge\vee\,|\,\wedge\vee\vee\vee$

New diagrams:

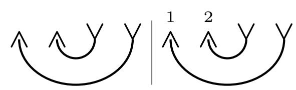
New weight: 

oriented and 4 clockwise oriented cups

Old weight: 

oriented and only 2 clockwise oriented cups

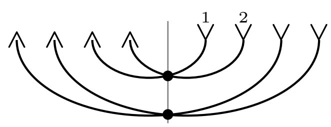
(b)

Old oriented diagram: 

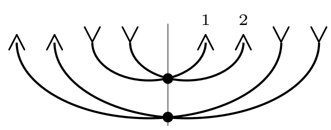
4 clockwise oriented cups

Weights: $\wedge\wedge\vee\vee\,|\,\wedge\wedge\vee\vee \rightsquigarrow \wedge\wedge\wedge\wedge\,|\,\vee\vee\vee\vee$

New diagrams:

New weight: 

oriented and 4 clockwise oriented cups

Old weight: 

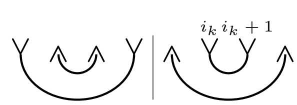
oriented and only 2 clockwise oriented cups

3. $w's_{i_k} \notin \mathcal{W}^{\mathfrak{p}}$:
This time the multiplication rule is $N_{w'}C_{s_{i_k}} = 0$. This means that all the diagrams we get are not oriented. If w' after multiplication with s_{i_k} is not in $\mathcal{W}^{\mathfrak{p}}$ this means that the associated weight does not change. For $1 \leq i_k \leq n-1$ this means we have either two "$\vee$"s or two "$\wedge$"s at i and $i+1$. If $i_k = 0$ we have either a "$\vee$" at 1 and an "$\wedge$" at 2 or vice versa. Since we get the second cases simply by inverting the first ones I will describe only the first cases.
The considerations are similar to the ones before. So again I will give the old oriented cup diagram and show the new diagram with the old weight. According to our multiplication rule this should not be oriented.

(a.1)

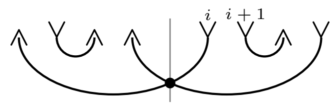
Old oriented diagram: 

2 clockwise oriented cups

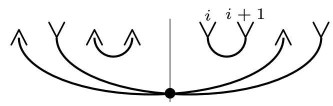
New diagram: 

not oriented

(a.2)

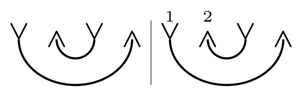
Old oriented diagram: 

2 clockwise oriented cups

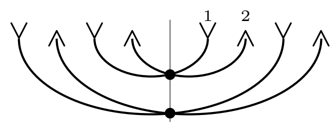
New diagram: 

not oriented

(b)

Old oriented diagram: 

2 clockwise oriented cups

New diagram: 

not oriented

So our formula matches our multiplication rule. The only thing left is the question whether there are any weights which give us an oriented diagram besides those coming from the calculations above. But looking at cases 1. and 2. we see that all possible orientations of the new diagram occur. So no additional weights occur.
Finally, we may have to subtract those terms having a constant term in the coefficient. In terms of diagrams this would mean that we would have an oriented cup diagram with no clockwise oriented cups. But there is exactly one weight leading to such an oriented cup diagram, namely the weight corresponding to w. Hence there does not occur any other polynomials with constant terms and we get that $\underline{N}_w = \underline{N}_{ws_{i_k}} C_{s_{i_k}}$. This finishes our proof. □

Proof of the Theorem 5.2.1. Proposition 5.2.2 tells us that the coefficients in $C(w)_{M_D}$ are the Kazhdan-Lusztig polynomials. Hence the image of a cup diagram is the Kazhdan-Lusztig basis element. □

Remark 5.2.3. From the proof of Proposition 5.2.2 we see that in our case for all $w \in \mathcal{W}^{\mathfrak{p}}$ with $w = s_{i_1} \cdots s_{i_k}$ an arbitrary but fixed reduced expression we have $\underline{N}_w = \underline{N}_e C_{s_{i_1}} \cdots C_{s_{i_k}}$.
Elements for which the Kaszhdan-Lusztig basis element has this property are called *Deodhar*. They were first studied in [Deo90]. In joint work with Warrington and Jones respectively, Billey classified Deodhar elements first for type A in [BW01] and then for general type in [BJ07].

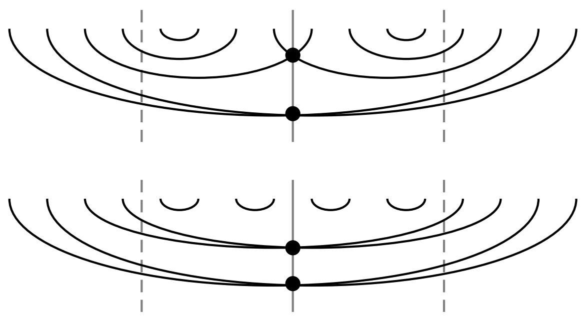
Example. *We now calculate some Kazhdan-Lusztig basis elements for $n = 4$. We already did this the old fashioned way at the end of Chapter 4.1. We will now do two examples using cup diagrams.*
The weights are:

Element in $\mathcal{W}^{\mathfrak{p}}$	*Weight*
e	$[\wedge\wedge\wedge\wedge \mid \vee\vee\vee\vee]$
s_0	$[\wedge\wedge\vee\vee \mid \wedge\wedge\vee\vee]$
s_0s_2	$[\wedge\vee\wedge\vee \mid \wedge\vee\wedge\vee]$
$s_0s_2s_3$	$[\vee\wedge\wedge\vee \mid \wedge\vee\vee\wedge]$
$s_0s_2s_1$	$[\wedge\vee\vee\wedge \mid \vee\wedge\wedge\vee]$
$s_0s_2s_3s_1$	$[\vee\wedge\vee\wedge \mid \vee\wedge\vee\wedge]$
$s_0s_2s_3s_1s_2$	$[\vee\vee\wedge\wedge \mid \vee\vee\wedge\wedge]$
$s_0s_2s_3s_1s_2s_0$	$[\vee\vee\vee\vee \mid \wedge\wedge\wedge\wedge]$

Consider the cup diagrams associated to $s_0s_2s_3$ and $s_0s_2s_3s_1s_2$:

$s_0s_2s_3$:

$s_0s_2s_3s_1s_2$:



The gray line indicates the middle. The frozen orientations are to the left of the left dashed gray line and to the right of the right dashed gray line.

Now we set the weights on top of the cup diagrams. The pictures are shown on page 32 and page 33. The first column indicates the weight while the middle column shows the cup diagram with the weight. Cups that are not oriented are drawn in red while cups that are clockwise oriented are marked in green. In the last column the Kazhdan-Lusztig polynomial is calculated via the formula in Proposition 5.2.2.
Figure 2 gives us the Kazhdan-Lusztig basis element

$$\underline{N}_{s_0s_2s_3} = N_{s_0s_2s_3} + vN_{s_0s_2}.$$

Figure 3 gives us the Kazhdan-Lusztig basis element

$$\underline{N}_{s_0s_2s_3s_1} = N_{s_0s_2s_3s_1} + vN_{s_0s_2s_1} + vN_{s_0s_2s_3} + v^2N_{s_0s_2}.$$

These results coincide with the results in the example at the end of Chapter 4.1.

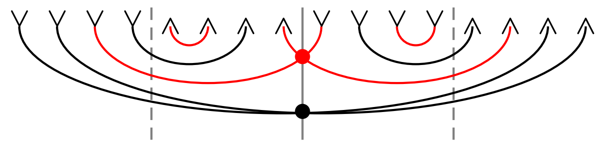
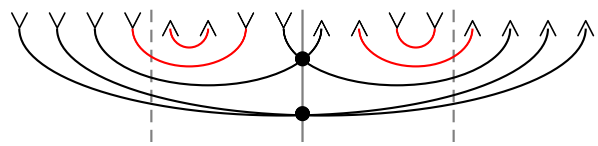
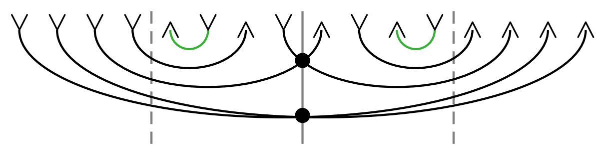
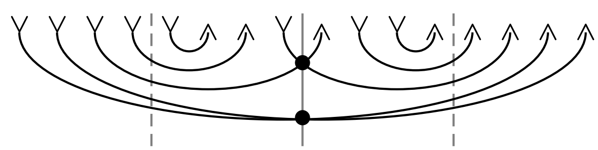
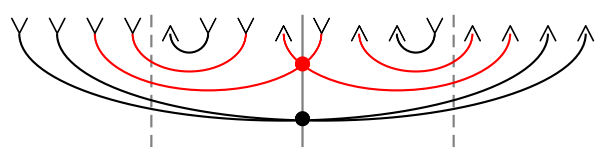
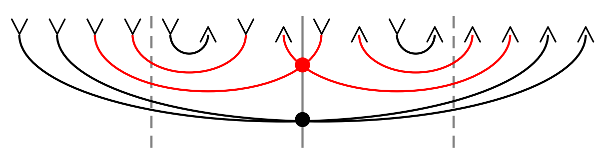
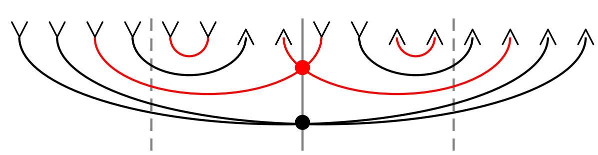
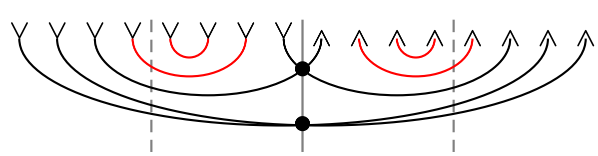
Element in $\mathcal{W}^{\mathfrak{p}}$	Cup diagram $C(s_0s_2s_3)$ with weight	Kazhdan-Lusztig polynomial
e		0
s_0		0
s_0s_2		$v^{\frac{2}{2}} = v$
$s_0s_2s_3$		$v^{\frac{0}{2}} = 1$
$s_0s_2s_1$		0
$s_0s_2s_3s_1$		0
$s_0s_2s_3s_1s_2$		0
$s_0s_2s_3s_1s_2s_0$		0

Figure 2: Example $w = s_0s_2s_3$

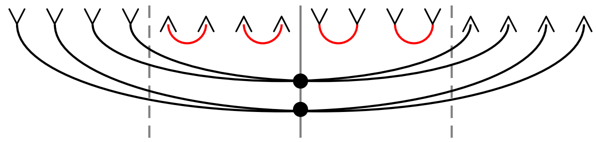
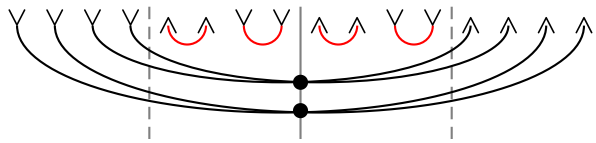
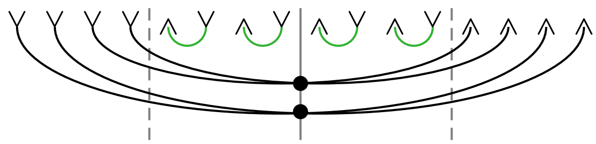
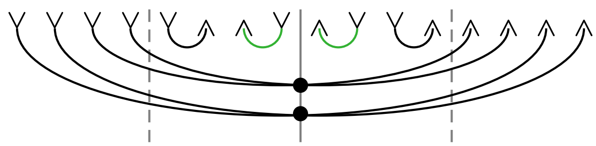
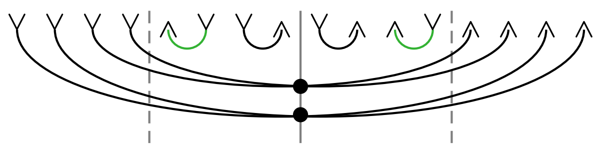
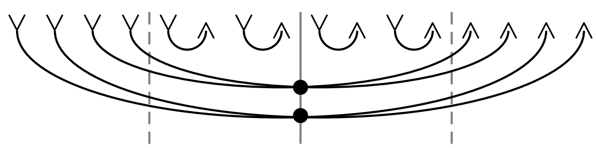
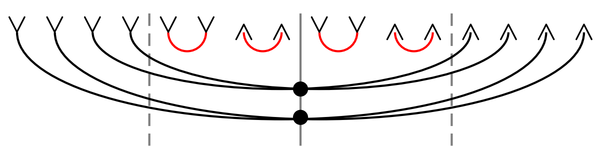
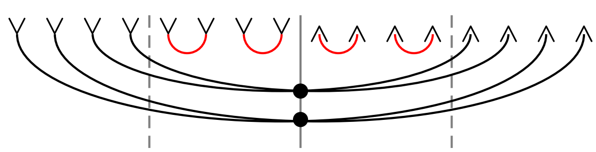
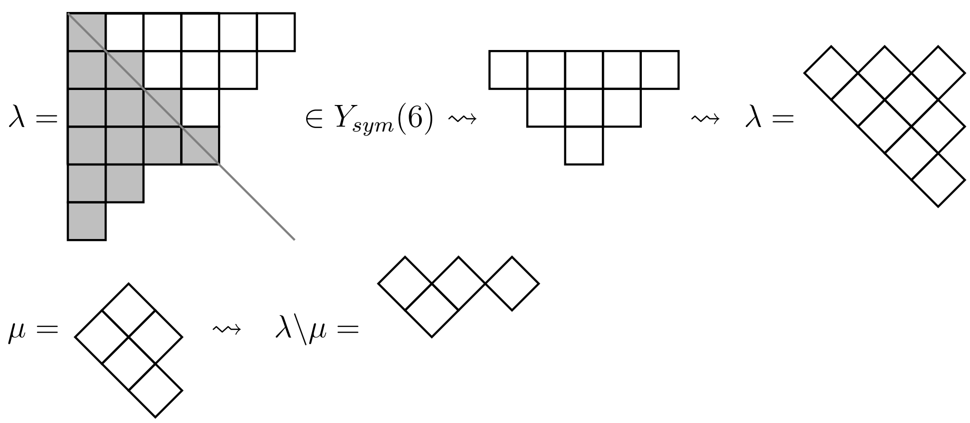
Element in $\mathcal{W}^{\mathfrak{p}}$	Cup diagram $C(s_0s_2s_3s_1)$ with weight	Kazhdan-Lusztig polynomial
e		0
s_0		0
s_0s_2		$v^{\frac{4}{2}} = v^2$
$s_0s_2s_3$		$v^{\frac{2}{2}} = v$
$s_0s_2s_1$		$v^{\frac{2}{2}} = v$
$s_0s_2s_3s_1$		$v^{\frac{0}{2}} = 1$
$s_0s_2s_3s_1s_2$		0
$s_0s_2s_3s_1s_2s_0$		0

Figure 3: Example $w = s_0s_2s_3s_1$

5.3 Connection with Brenti's work

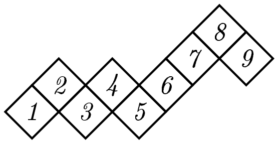
Brenti calculated the Kazhdan-Lusztig polynomials for our case already in [Bre09]. He approached the calculation via certain diagrams of boxes. He used skew shifted partitions and a property called s-Dyck. In the following I give a short overview and relate it to my work.
We get a *shifted partition* from one of our Young diagrams in $Y_{sym}(n)$ by deleting all the boxes on and below the main diagonal and rotating the remaining diagram by $\frac{3}{4}\pi$ counterclockwise. If λ and μ are shifted partitions and all the boxes of μ are also boxes in λ we get a *skew shifted partition* $\lambda\backslash\mu$ by deleting all the boxes in μ from λ.

Example.

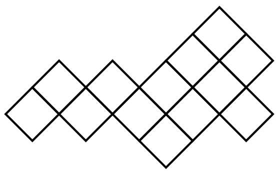


We have the obvious notion of "above" and "below" for the boxes in a skew shifted partition. A skew shifted partition that contains no two boxes stacked one directly below the other is called a *border strip*. If the border strip is connected it is called a *cbs*. In a cbs we enumerate the boxes from left to right as shown in the next example. If η is a skew shifted partition the top boxes form a border strip called $\theta(\eta)$.
For example the skew shifted partition $\lambda\backslash\mu$ in our example is a border strip but is not connected.
The next thing we need is the notion of the *level* of a box, which is nothing else than the height of the box above the baseline.
Now let θ be a cbs. We say that θ is *almost Dyck* if the level of all boxes is higher or equal the level of the first box. It is called *Dyck* if the level of the rightmost box is the same as the level of the first box.
To a cbs θ we associate θ_S. Let x be the box with highest number such that its level is the same as the level of the first one. For all boxes with higher index than x the box directly below is added to θ. This gives us θ_S.

Example.

$\theta =$ 

The boxes with the same level are: boxes 1, 3, 5 of level 1; boxes 2, 4, 6 of level 2; boxes 7, 9 of level 3 and box 8 of level 4.
Box five is the rightmost having the same level as box one. Hence we add the boxes below six through 9 which leads to the diagram

$\theta_S =$ 

The crucial definition of the paper is the notion of being s-Dyck. A skew shifted partition η is defined to be *shifted Dyck* or *s-Dyck* in the following inductive way:

1. η is s-Dyck if and only if each one of its connected components is s-Dyck
2. if η is connected, then η is s-Dyck if and only if:
   a) the outer border strip θ of η is almost Dyck, $\theta_S \subset \eta$, and $|\theta_S \backslash \theta|$ is even
   b) $\eta \backslash \theta_S$ is s-Dyck

Finally, we need the notion of the *depth* of an s-Dyck skew shifted partition. Let η be such a partition then the depth $dp(\eta)$ is defined to be the sum of the depths of its connected components and if η is connected $dp(\eta) = 1 + dp(\eta \backslash \theta_S)$ with $dp(\varnothing) = 0$. Brenti works with the conventions of Kazhdan and Lusztig for the Kazhdan-Lusztig polynomials. His polynomials are called $P_{w',w}$ and corresponds to our $n_{w',w}$ via

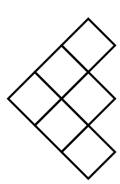
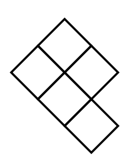
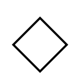
$$n_{w',w}(v) = P_{w',w}(v^{-2})v^{l(w)-l(w')}.$$

Brenti proves the formula

$$P_{w',w}(q) = \begin{cases} q^{\frac{1}{2}(|\lambda\backslash\mu| - dp(\lambda\backslash\mu))} & \text{if } \lambda\backslash\mu \text{ is s-Dyck} \\ 0 & \text{otherwise} \end{cases}$$

with λ the partition corresponding to w and μ the partition corresponding to w'.

Example. *Consider* $\underline{N}_{s_0s_2s_3s_1s_2s_0} \in \mathcal{H}(D_4)$*. By the results in the example at the end of Chapter 4.1 this is the element* $N_{s_0s_2s_3s_1s_2s_0} + vN_{s_0s_2s_3s_1s_2} + vN_{s_0} + vN_e$*. We have the corresponding shifted partitions*

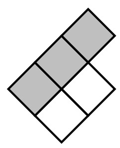
$w := s_0s_2s_3s_1s_2s_0 =$  $w_1 := s_0s_2s_3s_1s_2 =$  $s_0 =$ 

The element e corresponds to the empty partition.
Now we calculate the polynomials $P_{w',w}$ and $n_{w',w}$:

$P_{w,w}$: $\lambda\backslash\lambda$ *is oviously the empty partition and has cardinality* 0 *and depth* 0*. Hence the polynomial is* 1*. In comparison, our formula this gives us* $n_{w,w} = 1 \cdot v^0 = 1$.

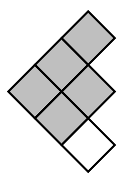
$P_{w_1,w}$: $\lambda\backslash\mu$ *has only one box. The depth of one box is one and hence our polynomial is* $q^{\frac{1}{2}(1-1)} = 1$*. In comparison, for* $n_{w_1,w}$ *we get* $1 \cdot v^{6-5} = v$.

$P_{s_0,w}$: $\lambda\backslash\mu$ *is*



The gray boxes form the outer border strip. This is almost Dyck and the rightmost box having the same level as the first is the first itself. Hence we have to add the boxes directly below the two following boxes. These are exactly the two other boxes. Hence θ_S *is equal to* $\lambda\backslash\mu$*. So the depth of* $\lambda\backslash\mu$ *is* 1*. The cardinality is* 5 *and we get* $P_{s_o,w}(q) = q^2$*. Calculating our* $n_{s_0,w}$ *gives us* $(v^{-2})^2 \cdot v^{6-1} = v^1$.

$P_{e,w}$: $\lambda\backslash\mu$ *is*



The gray boxes are the first θ_S*, with the same considerations as in* $P_{s_0,w}$*. Taking this away gives us a single box. So the depth is* 2*. The cardinality is* 6*. So our polynomial is* $q^{\frac{1}{2}(6-2)} = q^2$*. For* $n_{e,w}$ *we get* $P_{e,w}(v^{-2}) \cdot v^{6-0} = v^2$.

These results confirm our calculations in the example at the end of Chapter 4.1.

6 The action of $\mathcal{H}$ on $\mathcal{N}$ described diagrammatically

In this chapter we will describe the action of $\mathcal{H}$ on $\mathcal{N}$ in terms of decorated tangles. This action will turn out to be actually an action of a quotient of the generalized Temperley-Lieb algebra of type D_n. In the end it will turn out that this quotient acts faithfully on $\mathcal{N}$.

6.1 Generalized Temperley-Lieb algebras and decorated tangles

We start with some definitions and results about Temperley-Lieb algebras. The notation and statements can be found in [Gre98].

Definition 6.1.1. The generalized Temperley-Lieb algebra TL($\mathcal{W}$) is the quotient of the Hecke algebra $\mathcal{H}(\mathcal{W})$ obtained by factoring out the Ideal $I(\mathcal{W})$ generated by elements of the form

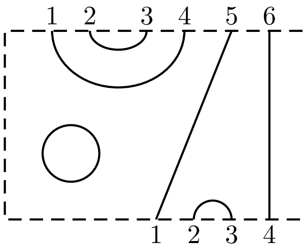
$$\sum_{\omega \in \langle s_i, s_j \rangle} T_\omega$$

where the sum runs over all sets $\{s_i, s_j\}$ such that the points i and j are connected in the associated Dynkin diagram.

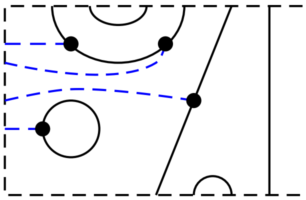
Definition 6.1.2 (Tangle)**.** Let $m, n \in \mathbb{N}_0$ such that $m + n$ is even. Consider $m + n$ points contained in a rectangle such that m points are on the top face of the rectangle and n points are on the bottom face of the rectangle. An (m, n)-*tangle* is a collection of lines and circles contained in the interior of the rectangle, such that the lines connect all the points and no intersections occur. Two such tangles are called the same if they are homotopy equivalent.
Lines connecting two points at the top face are called *cups* and lines connecting two points at the bottom face are called *caps*. Lines connecting a point at the top with one at the bottom are called *edges*.
A *decorated tangle* is a tangle with some "•"s on the lines and/or circles such that every "•" is accessible from the left side of the rectangle, meaning there exists a line in the rectangle connecting the left face with "•" not intersecting the tangle.

Remark 6.1.3. We use the term “tangle” for tangles without crossings. Usually tangles are allowed to have crossings.

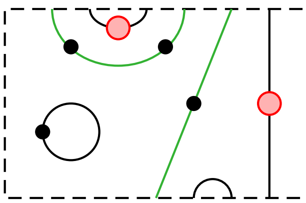
Example.



is a $(6,4)$-tangle.



is a decorated $(6,4)$-tangle since all the "$\bullet$"s can be connected to the left face of the rectangle via the blue, dotted lines.

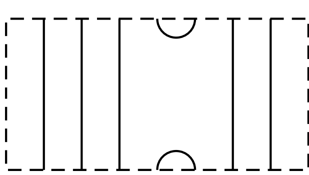


is not a decorated tangle since the red dots can not be connected to the left face without crossing one of the green lines.

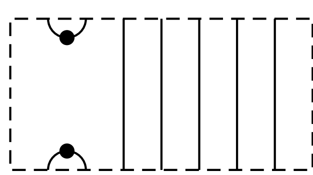
Now we define a concatenation of tangles. We can concatenate a (m,n)-tangle with a (n,r)-tangle vertically which will result in a an (m,r)-tangle. We do this by writing the (m,n)-tangle on top of the (n,r)-tangle. The n points at the top of the (n,r)-tangle are then connected one to one from left to right to the n points at the bottom of the (m,n)-tangle. The resulting picture can be viewed as an (m,r)-tangle.

Remark 6.1.4. Often, the decorated tangles are introduced as a category. The objects are the natural numbers. Morphisms from m to n are just (m,n)-tangles and composition of morphisms is given by the vertical concatenation of the tangles.

Let R be a commutative ring. $\mathbb{DT}_n$ is defined to be the free R-algebra with basis consisting of all (n,n)-tangles and multiplication given by the concatenation of tangles.
For $1 \leq i \leq n-1$ define the tangle e_i to be the (n,n)-tangle connecting the points i and $i+1$ at the north resp. south face and the point k on the north face with the point k at the south face.
Define the tangle e_0 to be the tangle e_1 with two decorations, one on the cup connecting point 1 with point 2 at the top face and one on the cap connecting 1 and 2 at the bottom face.



The $(7,7)$-tangle e_4



The $(7,7)$-tangle e_0

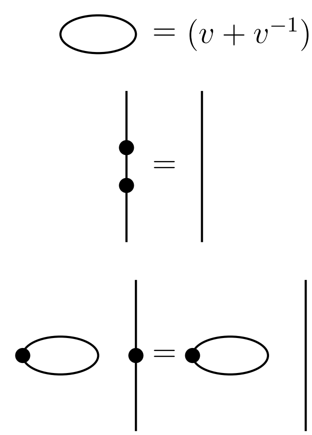


Figure 4: Relations TL($\mathcal{W}$)

Taking R to be $\mathcal{L}$ consider the unitary subalgebra of $\mathbb{DT}_n$ generated by $e_0, \dots, e_{n-1}$. In [Gre98, Thm. 4.2] Green shows that factoring out the relations in Figure 4 gives us an isomorphism to the generalized Temperley-Lieb algebra TL($\mathcal{W}$). The isomorphism sends the diagram e_i to our C_{s_i}.

Remark 6.1.5. The lines involved in the relations of Figure 4 are not necessarily edges in the tangle but might also be part of a circle.
The first relation means we can remove an undecorated circle and instead multiply with $(v+v^{-1})$. The second relation allows us to delete an even number of decoration from any line or circle. The last relations tells us that in presence of a circle with just one decoration, possibly after the removal of an even number of decorations, all other decorations in the tangle may be deleted.

6.2 From cup diagrams to decorated tangles

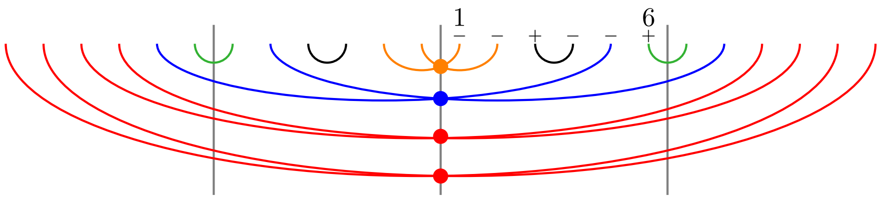
Next we will construct a decorated tangle from a cup diagram.
All the information necessary to build a cup diagram is located at the points 1 to n. So we "cut out" this portion of the diagram. This cutting operation will give us a decorated tangle.
Take a cup diagram and draw vertical lines between -1 and 1 as well as between n and $n+1$. Now we "cut" the cups along the two lines with these rules:

1. Cups connecting two points between 1 and n are not cut but stay the way they are.
2. Cups connecting a point between 1 and n to one between $n+1$ and $2n$ become an edge connecting the point between 1 and n to the "bottom face"
3. Two linked cups both ending at points between 1 and n are replaced by a decorated cup connecting the two endpoints.

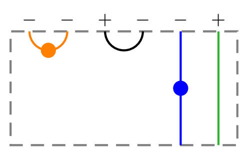
4. Two linked cups of which only one ends at a point between 1 and n are replaced by a decorated edge from the point between 1 and n to the "bottom face".
5. Two linked cups ending both at a number bigger than n are removed.

It is easy to see that to each cup connected to a point between 1 and n there is exactly one rule that applies, depending on the start and endpoint of the cup and the ones of the linked cup if the cup crosses the middle. Denote the tangle associated to $C(w)$ by $T(w)$.

Example. *Consider the following cup diagram for $n = 6$.*



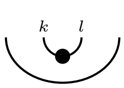
The black cup is covered by rule 1, the green one by 2, the orange one by 3, the blue one by 4 and the red ones by 5. Applying these rules gives us the following decorated tangle:



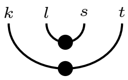
Definition 6.2.1. We define $\mathrm{DT}(n)$ to be the set of decroated (n, k)-tangles with $1 \leq k \leq n$ where all points at the bottom face are connected to one at the top face and the sum of the number of decorated edges and the number of undecorated cups is even.

Lemma 6.2.2. *The cutting operation gives us a decorated tangle in* $\mathrm{DT}(n)$.

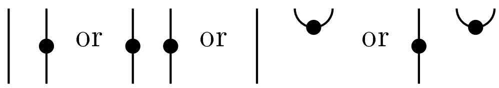
Proof. We cut our cup diagram before 1 and after n. Each point between 1 and n is again connected to another point by the rules 1 through 5. Since no caps appear when applying any of the rules we get an (n, k)-tangle where k is the number of cups resp. linked pairs of cups to which we applied rule 2 or rule 4.
We only have to make sure that the placing of decorations does not violate the rule for decorations in decorated tangles. To do this we rule out all other possibilities.

1. 

   This can not happen because it would mean that in our cup diagram the cups ending at k and l cross the middle and would therefore have to cross the other cup. But the only crossing cups by construction of the cup diagram are cups that go over the middle.

2. 

For our cup diagram this would mean that all the cups ending at one of the points would cross over the middle. But the pairing rule for cups says that these cups are paired consecutively. Pairing k with t would violate this rule.

3. 

The arguments here follow from the ones in 1 and 2.
Comparing the cups in the cup diagram from which these would arise with the ones in 1 and 2, one would notice the same conflicts.

From this we see that our tangles are in fact decorated tangles in accordance with the definition.
The sum property is fulfilled since the number of cups ending between 1 and n is even. But ending cups become either undecorated cups, decorated edges or decorated cups. So the total of these elements is even. But decorated cups mean that two cups which cross the middle end there. So subtracting this number does not change the parity. This completes our proof. □

Lemma 6.2.3. *We have a bijection between $S_{sym}(n)$ and* $\mathrm{DT}(n)$*.*

Proof. We prove this by constructing a $\{+,-\}$-sequence from a decorated tangle in $\mathrm{DT}(n)$.
We just consider the top face of the rectangle. Since we have no connections between two points at the bottom face, the connections at the top face completely determine our tangle.
An undecorated cup connecting i_1 and i_2 with $i_1 < i_2$ becomes a "+" at i_1 and a "−" at i_2. A decorated cup connecting i_3 with i_4 is exchanged for two minuses at i_3 and i_4. An undecorated edge starting at i_5 leads to a "+" at this place while a decorated edge at i_6 leads to a "−" at this place in the $\{+,-\}$-sequence.
Since the sum of the number of decorated edges and the number of undecorated cups is even, we get a $\{+,-\}$-sequence of length n with an even number of minuses. Going through the construction of cup diagrams and the cutting rules it is easily checked that this operation and constructing the decorated tangle from the cup diagram associated to a given $\{+,-\}$-sequence are inverse to one another. Thus, we have a bijection between the two sets. □

Example. *Applying Lemma 6.2.3 to our example before, we get the sequence* $|--+--+]$ *as expected.*

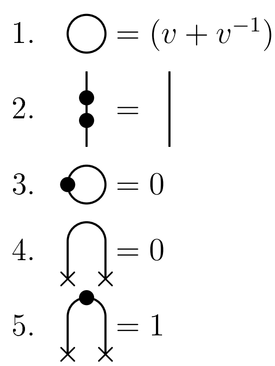
Remark 6.2.4. The bijection in Lemma 6.2.3 is compatible with the bijection between $S_{sym}(n)$ and $C(\mathcal{W}^{\mathfrak{p}})$ that can be obtained from Proposition 3.1.4 and Proposition 5.1.7.

Remark 6.2.5. The proof of Lemma 6.2.3 also shows that there is a bijection between our cup diagrams and the decorated tangles in $\mathrm{DT}(n)$. It is easy to give an inverse operation for every cutting rule and thus to construct a cup diagram to a given decorated tangle.

6.3 Description of the action of $\mathcal{H}$ on $\mathcal{N}$

Although we have an explicit description of the Kazhdan-Lusztig basis, the action of $\mathcal{H}$ on a basis element is still not easy to calculate. But there is an easy way to multiply some $\underline{N}_w$ with a C_{s_i} in a diagrammatic way.
Consider the free $\mathcal{L}$-module $\mathcal{N}_t$ with the tangles in $\mathrm{DT}(n)$ as basis. Because of Lemma 6.2.3, the module $\mathcal{N}_t$ is isomorphic to $\mathcal{N}$ as an $\mathcal{L}$-module. We fix this isomorphism given in the proof of the lemma.
We now define the action of an algebra of tangles on $\mathcal{N}_t$. It will turn out that this models the action of $\mathcal{H}$ on $\mathcal{N}$.

Definition 6.3.1. Consider the unitary algebra over $\mathcal{L}$ generated by the e_i. This algebra operates on $\mathrm{DT}(n)$ and hence on $\mathcal{N}_t$ via tangle multiplication. We factor out the following relations. The $\times$ in the relations are drawn to indicate that the tangle intersects the rectangle at this point, i.e. relations 4 and 5 may not be applied to any line in the tangle.



Remark 6.3.2. The first relation states that any circle with no decorations can be removed and, in turn, the diagram is multiplied by $(v + v^{-1})$. The second relation means that from any circle or line an even number of decorations can be removed. The third relation says that the presence of a circle with only one decoration multiplies the whole diagram by 0 and hence annihilates the whole diagram.
These first three relations can be viewed as relations in the algebra which operates on our module. The last two relations are relations which are applied after multiplying a basis element by a tangle.
The fourth and fifth relation refer to parts of the tangle which connect two points at the bottom of the rectangle. Together with relation 2 we get that if the number of

decorations is even the entire diagram becomes 0 and if the number of decorations is odd we can simply remove this part of the tangle.

Remark 6.3.3. The first two relations coincide with the first two relations for our generalized Temperley-Lieb algebra in Chapter 6.1 Figure 4. Relation 3 ensures that the last relation for our Temperley-Lieb algebra is complied, too.
Hence, the action defined above is actually the action of a quotient of the generalized Temperley-Lieb algebra and it makes sense to compare this action to the action of $\mathcal{H}$ on $\mathcal{N}$.

Because of Remark 6.3.3 the following diagram makes sense. The main theorem in this section says that the diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\cong} & \mathcal{N}_t \\ {\scriptstyle C_{s_i}}\downarrow & & \downarrow{\scriptstyle e_i} \\ \mathcal{N} & \xrightarrow{\cong} & \mathcal{N}_t \end{array}$$

commutes, i.e. $\mathcal{N}$ and $\mathcal{N}_t$ are isomorphic as $\mathcal{H}$-modules.

For the proof we need one more preparation. In our proofs we do not work with the whole cup diagram but rather with cup subdiagrams. This method can be applied to weights, too.
If we want to understand an action on cup diagrams, usually it is enough to consider those parts of the cup diagram that change under an action. To be able to handle these cup subdiagrams in M_D we define the relative Kazhdan-Lusztig elements.

Definition 6.3.4. Let $w \in \mathcal{W}^{\mathfrak{p}}$. A cup subdiagram is any part of the cup diagram which in itself is again a diagram of cups. Let $SC(w)$ be a symmetric cup subdiagram of $C(w)$.
The *relative Kazhdan-Lusztig element* associated to $SC(w)$ is $\sum v^{\frac{\mathrm{cl}(w'SC(w))}{2}} N_{w'}$ where the sum runs over all different subweights w' such that $SP(w)$ is oriented.
A subweight is obtained from a weight by taking the subset

$$P = \{i | i \text{ is the starting or ending point of a cup in } SP(w)\} \subset \{1, \ldots, 2n\}$$

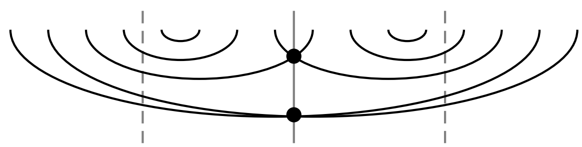
and considering only the orientations at these points.

Remark 6.3.5. Given a symmetric cup subdiagram different weights may lead to the same subweights. In our definition we sum over different subweights. So every subweight appears only once in our sum even if there is more than one weight having this subweight.

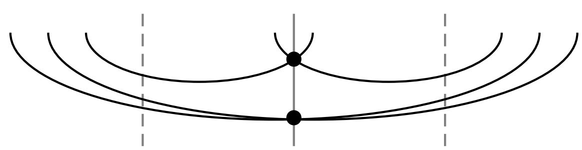
Remark 6.3.6. In our notation of weights we sometimes wrote only the part between 1 and n knowing the rest of the weight because of the antisymmetry and the frozen "$\vee$"s and "$\wedge$"s. We do the same with our subweights, distinguishing only the orientation at those i that lie between 1 and n. We indicate this by writing | at the start and] at the end of the sequence.

Remark 6.3.7. If we write a cup diagram as a disjoint union of symmetric cup subdiagrams we are able to calculate our Kazhdan-Lusztig basis element from the relative Kazhdan-Lusztig elements.
The Kazhdan-Lusztig polynomial in front of a weight is the product of the coefficients of the subweights in the relative Kazhdan-Lusztig elements to which the weight restricts.
The example below shows the calculation step by step.
That we can calculate the Kazhdan-Lusztig polynomial in this way is clear from the formula in Proposition 5.2.2. To get our Kazhdan-Lusztig polynomial we have to count clockwise oriented cups in our cupdiagram. But if we write our cup diagram as a disjoint union of symmetric cup subdiagrams the total number of clockwise oriented cups in our cup diagram is the sum of clockwise oriented cups in each cup subdiagram. Multiplying the coefficients simply adds up the numbers of oriented cups in each cup subdiagram or gives 0 if one of the cups in any of the cup subdiagrams is not oriented.

Example.
Consider the cup diagram



The Kazhdan-Lusztig basis element is $\underline{N}_w = N_{|\wedge\vee\vee\wedge]} + vN_{|\wedge\vee\wedge\vee]}$ *as calculated in several examples before.*
We can write the cup diagram as a disjoint union of



and

The relative Kazhdan-Lusztig elements are

$$K_1 = N_{|\wedge\ldots]}$$

and

$$K_2 = N_{|.\vee\vee\wedge]} + vN_{|.\vee\wedge\vee]},$$

where the dots are only drawn to indicate, that this orientation is left out.
If we now wanted to calculate the coefficient in front of $N_{|\wedge\vee\wedge\vee]}$ in our Kazhdan-Lusztig basis element using the relative Kazhdan-Lusztig elements we first have to determine to which subweights the weight $|\wedge\vee\wedge\vee]$ in all of the cup subdiagrams restricts.
In the first cup subdiagram the weight restricts to $|\wedge]$ and in the second diagram it restricts to $|\vee\wedge\vee]$.
So to get our coefficient we have to multiply the coefficient of $N_{|\wedge]}$ in K_1 and the coefficient of $N_{|\vee\wedge\vee]}$ in K_2. As a result we get that the coefficient of $N_{|\wedge\vee\wedge\vee]}$ is $1 \cdot v = v$, which we already knew.

We can consider the action of the C_{s_i} on our subweights even though our subweights need not lie in some $\mathcal{W}^{\mathfrak{p}}$ for some smaller n.
Given a cup subdiagram $SP(w)$ we have a map from

$$\{\text{weights}\} \twoheadrightarrow \{\text{subweights associated to } SP(w)\}.$$

From the multiplication rules in Proposition 4.1.5 we see that the action of some C_{s_i} on some weight N_w is determined by $d(ws_i)$ and is described using only ws_i and w. But s_i operates on the weight w by changing at most two points, namely i and $i+1$ for $1 \leq i \leq n-1$, or 1 and 2 if $i = 0$. Since the action of s_i only depends on the two points and changes at most these points we can define the action of C_{s_i} on a subweight if it contains both points.
We define the action of C_{s_i} on a subweight by taking a weight that restricts to the subweight, operate with C_{s_i} on this weight according to the multiplication rules and then restrict the result.

Example. *Consider the subweight $N_{|.\vee\wedge\vee]}$ from the example before. This could be multiplied by C_{s_2} since s_2 exchanges the orientations at the points 2 and 3 which both are contained in the subweight. A weight that restricts to this subweight would be $N_{|\wedge\vee\wedge\vee]}$.*
By the multiplication rules we have

$$N_{|\wedge\vee\wedge\vee]}C_{s_2} = N_{|\wedge\wedge\vee\vee]} + vN_{|\wedge\vee\wedge\vee]}.$$

Restricting this result we get

$$N_{|.\vee\wedge\vee]}C_{s_2} = N_{|.\wedge\vee\vee]} + vN_{|.\vee\wedge\vee]}.$$

With these preparations we can prove this section's main theorem.

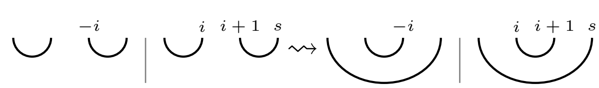
Theorem 6.3.8. *The multiplication of* $\underline{N}_w$ *with* C_{s_i} *is given by multiplying* $T(w)$ *with* e_i *and factoring out the relations* 1 *through* 5.

Proof. In this proof we repeatedly use Proposition 5.2.2 without explicitely refering to it.
Also, throughout the whole proof we are going to consider only the "relevant" parts of the diagrams, meaning the cup subdiagrams consisting of those parts that are involved in the change of the cup diagram. Since all the parts in e_i except for the cups and caps are straight lines multiplying e_i by some tangle changes nothing except for the parts connected to the caps. So we can concentrate on these parts and observe the changes.
We prove the theorem by considering three cases:

1. $ws_i \in \mathcal{W}^\mathfrak{p}$ and $l(ws_i) > l(w)$
2. $ws_i \in \mathcal{W}^\mathfrak{p}$ and $l(ws_i) < l(w)$
3. $ws_i \notin \mathcal{W}^\mathfrak{p}$

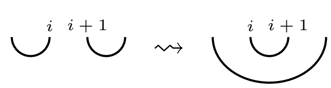
1. $ws_i \in \mathcal{W}^\mathfrak{p}$ and $l(ws_i) > l(w)$
   In this case we know that $\underline{N}_w C_{s_i} = \underline{N}_{ws_i}$. So multiplying $T(w)$ with e_i should resemble the changes given at the end of Chapter 5.1. We check this case by case.
   In all the following pictures, above the gray line we have the involved part of the tangle e_i while below the relevant part of the tangle $T(w)$ is shown. We start with $1 \leq i \leq n-1$ and the associated cases (a.1) and (a.2) and finish with the case $i = 0$ and the associated case (b).

(a.1)

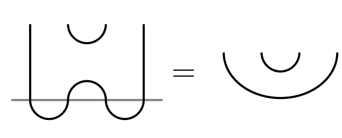


In terms of decorated tangles this is one of the following cases, depending on s being greater or less or equal n:

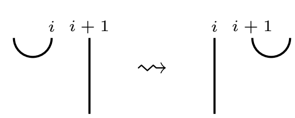
$s \leq n$:



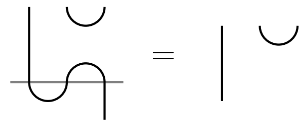
On the other hand, multiplication by e_i gives us



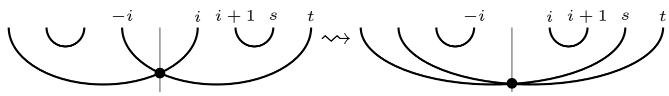
$s > n$:



Multiplication by e_i gives us

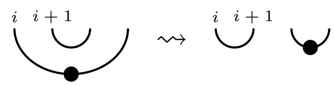


(a.2)

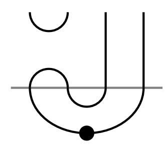


This time, we get three cases when translating into decorated tangles depending on the values of s and t:

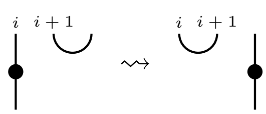
$s, t \leq n$:



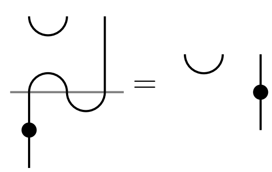
Multiplication by e_i gives us



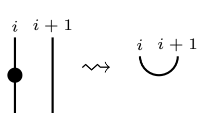
$s \leq n, t > n$:



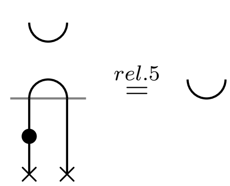
Multiplication by e_i gives us



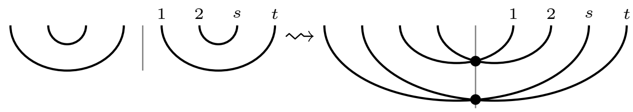
$s, t > n$:



Multiplication by e_i gives us

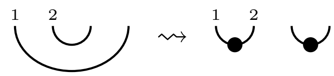


(b)

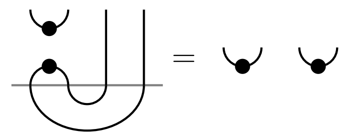


Again we get three possible decorated tangles depending on the values of s and t:

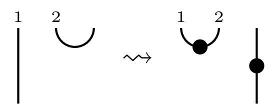
$s, t \leq n$:



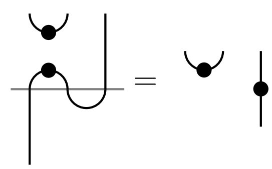
Multiplication by e_0 gives us



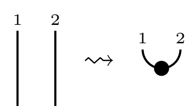
$s \leq n, t > n$:



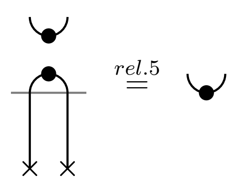
Multiplication by e_0 gives us



$s, t > n$:



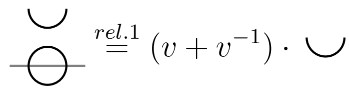
Multiplication by e_0 gives us



2. $ws_i \in \mathcal{W}^{\mathfrak{p}}$ and $l(ws_i) < l(w)$
Decreasing in length but staying in $\mathcal{W}^{\mathfrak{p}}$ means, first of all, that we have a representative of w ending with s_i, i.e. $w = w's_i$ for some $w' \in \mathcal{W}^{\mathfrak{p}}$. Hence $\underline{N}_w = \underline{N}_{w'}C_{s_i}$. Consequently, we have $\underline{N}_w C_{s_i} = \underline{N}_{w'}C_{s_i}^2 = (v + v^{-1})\underline{N}_w$, because $C_{s_i}^2 = (v + v^{-1})C_{s_i}$.

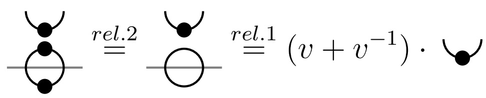
For the effect on our cups we first consider the case $1 \leq i \leq n$.
For our $\{+, -\}$-sequence getting shorter through multiplication by an s_i means that we have a "+" at place i and a "−" at place $i + 1$, which get exchanged

by s_i. So in $T(w)$ we have a cup going from i to $i+1$. Multiplying by e_i gives us



So our tangle stays the same but is multiplied by $(v+v^{-1})$ which is exactly what we wanted.

Now consider the case $i=0$.
Getting shorter through multiplication by s_0 means that we have two "$-$"s at places 1 and 2 which are exchanged for two "$+$"s. So our cup diagram has two linked cups going from -1 to 2 and from -2 to 1. In decorated tangles this translates into a decorated cup connecting 1 and 2.
Multiplying this by e_0 gives us

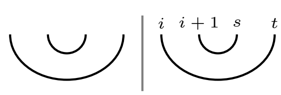


Again we get back our old tangle multiplied by $(v+v^{-1})$.

This proves our second case.

3. $ws_i \notin \mathcal{W}^{\mathfrak{p}}$
This case turns out to be the one requiring the most work. In this case it is not easy to describe the action of a C_{s_i} on an $\underline{N}_w$. We rather have to think in terms of oriented cup subdiagrams and relative Kazhdan-Lusztig elements.
We distinguish four cases:
   a) $1 \leq i \leq n-1$ and pluses at i and $i+1$
   b) $1 \leq i \leq n-1$ and minuses at i and $i+1$
   c) $i=0$ and "$+$" at 1, "$-$" at 2
   d) $i=0$ and "$-$" at 1, "$+$" at 2

   a) $1 \leq i \leq n-1$ and pluses at i and $i+1$:
   In our full cup diagram the relevant parts look like this:



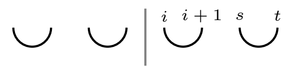
The relative Kazhdan-Lusztig element associated to this is

$$K = N_{|\vee\vee\wedge\wedge]} + vN_{|\vee\wedge\vee\wedge]} + vN_{|\wedge\vee\wedge\vee]} + v^2 N_{|\wedge\wedge\vee\vee]}.$$

Depending on the endings of the cups, some terms become 0 because of restrictions on the weights.
Multiplying the partial KL-polynomial by C_{s_i} the first and the last term become 0. The second one gets shorter, the third one longer. This leaves us with

$$K_{new} = N_{|\vee\wedge\vee\wedge]} + vN_{|\wedge\vee\vee\wedge]} + vN_{|\vee\wedge\wedge\vee]} + v^2 N_{|\wedge\vee\wedge\vee]}.$$

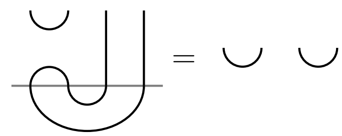
This is the relative KL-element associated to the cup subdiagram



Depending on the values of s and t there are three possibilities of decorated tangles and designs of the multiplication respectively.

$s, t \leq n$:

Then the part in our decorated tangle is the same as the right half in the cup subdiagram. Multiplying by e_i we get

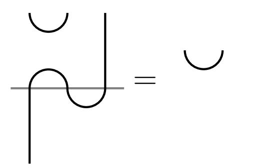


This is exactly what we want.

$s \leq n, t > n$:

We must restrict to the case that the orientation at t has to be "$\wedge$". So the last two terms in K and K_{new} disappear. This agrees with the cup subdiagrams.

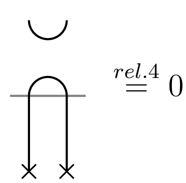
In terms decorated tangles we get the multiplication



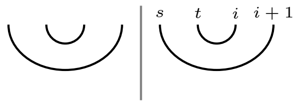
$s, t > n$:

This time, the weights have to have an "$\wedge$" at s and t. In K only the first term survives and K_{new} is just 0. This agrees with the cup subdiagrams since the new diagram is not a cup subdiagram of any cup diagram: otherwise we would have to have a "+" at s, which is not possible, since s is greater than n.

As decorated tangles we get



b) $1 \leq i \leq n-1$ and two minuses at i and $i+1$:
This is by far the most extensive case. The two minuses mean that two cups end at places i and $i+1$ in our cup diagram. There are four possibilities what this could look like.

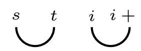
i.  with $s, t \geq 1$

The associated relative KL-element is

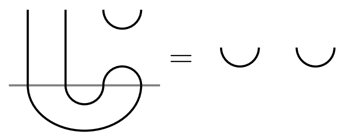
$$K = N_{|\vee\vee\wedge\wedge]} + vN_{|\vee\wedge\vee\wedge]} + vN_{|\wedge\vee\wedge\vee]} + v^2 N_{|\wedge\wedge\vee\vee]}.$$

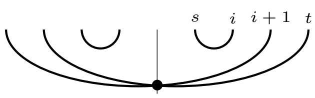
Multiplying this by C_{s_i} kills the first and the last term and gives us

$$K_{new} = N_{|\vee\wedge\vee\wedge]} + vN_{|\vee\wedge\wedge\vee]} + vN_{|\wedge\vee\vee\wedge]} + v^2 N_{|\wedge\vee\wedge\vee]}.$$

The tangle associated to this relative KL-element is .

Multiplication by e_i gives us



ii.  with $s \geq 1$

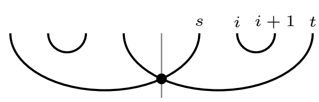
The relative KL-element is

$$K = N_{|\vee\wedge\wedge\wedge]} + vN_{|\wedge\vee\wedge\wedge]} + vN_{|\vee\wedge\vee\vee]} + v^2 N_{|\wedge\vee\vee\vee]}.$$

Multiplying by C_{s_i} kills the first and last term and gives us

$$K_{new} = N_{|\wedge\vee\wedge\wedge]} + vN_{|\wedge\wedge\vee\wedge]} + vN_{|\vee\vee\wedge\vee]} + v^2 N_{|\vee\wedge\vee\vee]}.$$

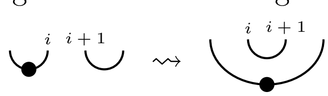
This is the relative KL-element associated to



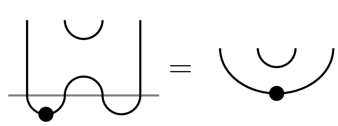
Now we have to distinguish two cases

$t \leq n$:

In this case the change of decorated tangles is

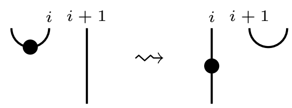


Multiplying the left tangle by e_i gives us

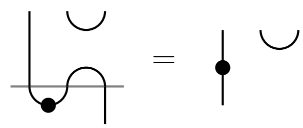


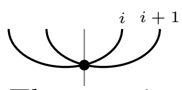
$t > n$:

In this case the orientation at t has to be $\wedge$ and hence the last two terms in K disappear as do the last two terms in K_{new}. This still coincides with our cup subdiagrams. The change of decorated tangles is



Multiplication by e_i gives us

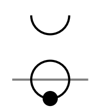


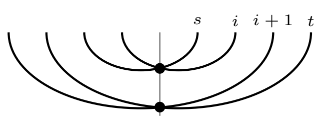
iii. 

The associated relative KL-element here is

$$K = N_{|\wedge\wedge]} + vN_{|\vee\vee]}.$$

Multiplying this by C_{s_i} gives us 0 because both subweights do not change under the action of s_i.
The tangle multiplication is

 $\overset{rel.3}{=} 0$

iv.  with $s \geq 1$

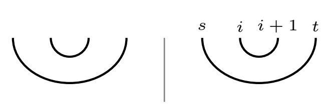
The associated relative KL-element is

$$K = N_{|\wedge\wedge\wedge\wedge]} + vN_{|\vee\vee\wedge\wedge]} + vN_{|\wedge\wedge\vee\vee]} + v^2 N_{|\vee\vee\vee\vee]}.$$

Multiplication by e_i kills the first and the last term and gives us

$$K_{new} = N_{|\vee\vee\wedge\wedge]} + vN_{|\vee\wedge\vee\wedge]} + vN_{|\wedge\vee\wedge\vee]} + v^2 N_{|\wedge\wedge\vee\vee]}.$$

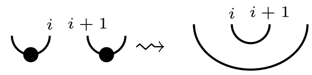
This is the relative KL-element associated with



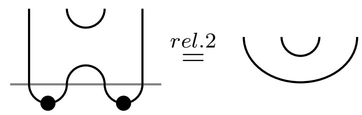
Again for the tangle multiplication we have to distinguish two cases:

$t \leq n$:

In this case the change of decorated tangles is

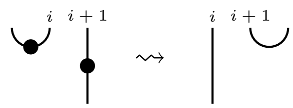


Multiplying the left tangle by e_i gives us

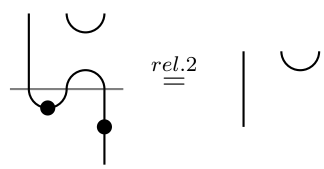


$t > n$:

In this case the orientation at t has to be "$\wedge$", and hence, the last two terms in K disappear as do the last two terms in K_{new}. This still coincides with our cup subdiagrams. The change of decorated tangles is

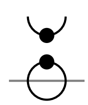


Multiplication by e_i gives us

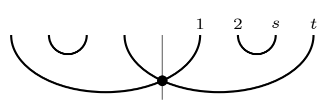


c) $i = 0$ and "$+$" at 1, "$-$" at 2
In our cup diagram this looks like .

Our relative KL-element is $N_{|\vee\wedge]} + vN_{|\wedge\vee]}$. Multiplying this by C_{s_0} gives us 0 since both subweights do not change.
In terms of decorated tangles we get


$\overset{rel.3}{=} 0$

which is exactly what we wanted.

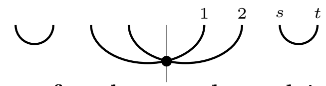
d) $i = 0$ and "$-$" at 1, "$+$" at 2
In our cup diagram this looks like .

The relative KL-element is

$$K = N_{|\wedge\vee\wedge\wedge]} + vN_{|\wedge\wedge\vee\wedge]} + vN_{|\vee\vee\wedge\vee]} + v^2 N_{|\vee\wedge\vee\vee]}.$$

Multiplying by C_{s_0} kills the first and last term and leaves us with

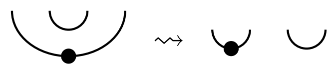
$$K_{new} = N_{|\wedge\wedge\vee\wedge]} + vN_{|\vee\vee\vee\wedge]} + vN_{|\wedge\wedge\wedge\vee]} + v^2 N_{|\vee\vee\wedge\vee]}.$$

This is the relative KL-element associated to .

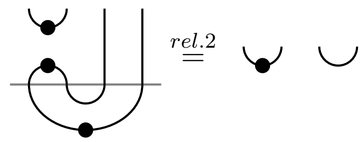
Depending on s and t we get three different cases for the tangle multiplication.

$s, t \leq n$:

In this case the change of decorated tangle looks like

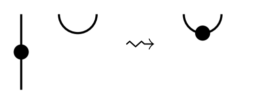


Multiplying by e_i gives us

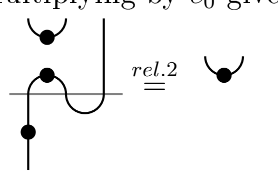


$s \leq n, t > n$:

In this case the orientation at t has to be "$\wedge$". So in our K the last two terms disappear as in K_{new}. This agrees with the cup subdiagrams.
The change in our tangle looks like

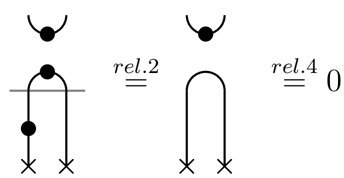


Multiplying by e_0 gives us



$s, t > n$:

This time the orientation at s and t has to be $\wedge$. Hence the only term surviving in our K is the first which gets killed by multiplication with C_{s_0}. So our tangle multiplication should result in 0, too.



which is again exactly what we want.

So in all our cases the action of the tangles coincides with the action of Hecke algebra. This proves our theorem. □

Corollary 6.3.9. *The action of $\mathcal{H}$ on $\mathcal{N}$ is an action of a quotient of the Temperley-Lieb algebra* $\mathrm{TL}(\mathcal{W})$ *on* $\mathcal{N}$.

This result can be viewed as a graphical analogue of [Str05, Thm. 4.1].

6.4 Faithfulness of the action

We finally establish the faithfulness of the above action.
In [Gre98, Thm. 4.2], Green gives an explicit descripition of a basis $\mathcal{B}$ of $\mathrm{TL}(\mathcal{W})$ in terms of decorated tangles.

Theorem 6.4.1 (Green)**.** *The basis $\mathcal{B}$ consists of (n,n)-tangles with at most one decoration on each loop or edge satisfying one of the following conditions:*

1. *The diagram contains one loop which is decorated, and no other loops or decorations. Also, there is at least one non-propagating edge in the diagram.*
2. *The diagram contains no loops and the total number of decorations is even.*

Because of Relation 3 in Definition 6.3.1 the elements of the first kind are 0 while the other survive. We denote the elements in this basis by b_i.

Proposition 6.4.2. *The action of the quotient of* $\mathrm{TL}(\mathcal{W})$ *described above is faithful.*

Proof. Let $b := \sum \alpha_i b_i$ be an element in our quotient. We denote the decorated tangles in $\mathrm{DT}(n)$ by d_j. Let $d := \sum \beta_j d_j$. Then $d.b = \sum \gamma_k d_k$ where $\gamma_k = \sum\limits_{d_j.b_i = d_k} \alpha_i \beta_j$.
Now assume that b operates as 0 on $\mathcal{N}_t$. We want to show that $b = 0$.
We will prove this by induction over the number of cups in d_k.
But first, we start with some observations:
If we have a cup in b_i and $d_j.b_i \neq 0$ this cup is a cup in $d_j.b_i$, too.
For all b_i there exists such a d_j with $d_j.b_i \neq 0$. Take the lower half of b_i and reflect it vertically. Set or remove a decoration on the leftmost edge depending on the number of undecorated cups to satisfy the second condition on tangles in $\mathrm{DT}(n)$. Multiplying this tangle by b_i all loops at the bottom of b_i are closed into loops with an even number of decorations. The propagating edges are still propagating edges, maybe with some decorations on the leftmost edge. None of these things kills our diagram. Each loop results in a multiplication by $(v + v^{-1})$. So we get a non-zero element in $\mathcal{N}_t$.
Now we do our induction.
Induction basis: number of cups in d_k is 0:
Since cups in b_i are still cups in $d_j.b_i$, b_i may not have any cups. But there is only one b_i without cups, namely e. If multiplying some d_j by e does not result in any cups the d_j must not have any cups either. But there is only one such d_j in $\mathrm{DT}(n)$, the one with only undecorated edges from the top face to the bottom face. Call this one d_e. So the coefficient in front of $d_e.e$ has to be 0. But b operates as 0 on all of $\mathcal{N}_t$ by assumption. So the coefficient in front of d_e can be varied and still the result would have to be 0. This works only if the coefficient in front of e is 0.
Induction step: $m-1$ cups $\rightsquigarrow$ m cups:
By our induction hypothesis the coefficient in front of all b_i involved in creating

some b_k with $m-1$ or less cups is 0. These are all b_i with $m-1$ or less cups.
All cups in our b_i remain unchanged when multiplying by some d_j. Consequently, if d_k has m cups only b_i with m cups are involved. Furthermore if we want to know which b_i are involved in creating a certain d_k the cups in b_i have to match those in d_k.
So let d_k be a tangle with m cups and $b_{i,k}$ the b_i involved in creating this d_k. Assume there is a non-zero coefficient. Let $\alpha_{i_0,k}$ be a non-zero coefficient of highest degree. Take d_{i_0} to be the tangle obtained by reflecting the lower half of $b_{i_0,k}$ vertically and setting the decoration on the leftmost edge such that we get a tangle in $\mathrm{DT}(n)$.
Multiplying $d_{i_0,k}$ by $b_{i_0,k}$ gives us another tangle in $\mathrm{DT}(n)$ which has the same cups as d_k and hence has to be d_k. Since $d_{i_0,k}$ is the vertical reflection of $b_{i_0,k}$ all caps in $b_{i_0,k}$ become a circle with an even number of decorations. Since we have n cups in $b_{i_0,k}$ we have to have n caps, too. So all in all we get a multiplication by $(v+v^{-1})^n$ and we get $\alpha_{i_0,k} \cdot (v+v^{-1})^n$ as a summand of the coefficient of d_k. This has degree $\deg(\alpha_{i_0}) + n$.
Claim: All the other summands of d_k are of strictly lower degree.
Proof of Claim. If another $b_{j,k}$ involved in creating d_k has the same caps as $b_{i_0,k}$ it has to differ in the decorations. But the decorations of the cups are fixed. Since only one edge can have a decoration and the total number of decorations has to be even at least one of the decorations of a cap has to differ from the one in $b_{i_0,k}$. Every cap has at most one decoration and hence multiplying by $d_{i_0,k}$ gives us a circle with an uneven number of decorations. But this results in 0 and the coefficient of this $b_{j,k}$ does not contribute to the coefficient of d_k.
So, if $b_{j,k}$ contributes to the coefficient of d_k the degree of its coefficient has to be smaller or equal the degree of $\alpha_{i_0,k}$. To get the same degree as $\alpha_{i_0,k} \cdot (v+v^{-1})^n$ we would have to multiply by v^n or even something of higher degree. But we only multiply by v if we remove a circle with an even number of decorations. But the caps in $b_{j,k}$ differ from those in $b_{i_0,k}$. This implies that the number of circles in $d_{i_0,k}.b_{j,k}$ has to be smaller than n. Hence the degree of the resulting contribution to the coefficient for d_k is strictly smaller than $\deg(\alpha_{i_0,k})+n$. Thus our claim is proven.
With our claim we know that the coefficient of d_k in $d_{i_0,k}.b$ has degree $\deg(\alpha_{i_0,k})+n$ and is not equal to 0. But the coefficient should be 0; a contradiction. This completes our induction step and our proof. □

To summarize: We know that the action of $\mathcal{H}$ on $\mathcal{N}$ is actually an action of the described quotient of the Temperley-Lieb algebra and that this action is faithful.

7 Dimensions of homomorphism spaces between projectives in category $\mathcal{O}_0^{\mathfrak{p}}$

In this chapter we apply our results about Kazhdan-Lusztig polynomials and our language of cup diagrams. As indicated in the introduction, the Kazhdan-Lusztig polynomials have an interpretation in the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$ and our language of cup diagrams can be used to calculate the dimension between projective objects quite easily.
I will just state the results necessary for our application. For details the reader may consult the book of Humphreys on category $\mathcal{O}$ [Hum08].

7.1 The parabolic category $\mathcal{O}_0^{\mathfrak{p}}$

The parabolic Kazhdan-Lusztig polynomials have a nice Lie theoretic interpretation: evaluated at 1 they count the multiplicity of simple highest weight modules occuring as subquotients in a composition series of a parabolic Verma module. This goes back to a conjecture by Kazhdan and Lusztig in [KL79, Conj. 1.5] which was later generalized to the parabolic case and proven by Casian and Colingwood in [CC87].
Consider the semisimple Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}$ of type D_n with the standard Cartan subalgebra $\mathfrak{h}$ of diagonal matrices and Weyl group $\mathcal{W}$. Fix the Borel subalgebra $\mathfrak{b}$ of upper triangular matrices and the parabolic subalgebra $\mathfrak{p}$ with Weyl group $\mathcal{W}_{\mathfrak{p}}$ that contains $\mathfrak{b}$.

Definition 7.1.1. The category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of the category of $U(\mathfrak{g})$-modules whose objects M satisfy the following conditions:

1. M is a finitely generated $U(\mathfrak{g})$-module.
2. M has a weight space decomposition with finite-dimensional weight spaces, i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}*} M_\lambda$ where $M_\lambda = \{m \in M | h.m = \lambda(h)m \ \forall h \in \mathfrak{h}\}$ and $\dim M_\lambda < \infty$.
3. M is locally $\mathfrak{p}$-finite, i.e. each $m \in M$ lies in a finite-dimensional subspace of M stable under the action of $\mathfrak{p}$.

We consider the subcategory $\mathcal{O}_0^{\mathfrak{p}}$ of $\mathcal{O}^{\mathfrak{p}}$ of all modules with generalized trivial central character.
In this category we have three classes of distinguished objects. The elements of each class are indexed by $w \in \mathcal{W}^{\mathfrak{p}}$.
We have a complete set of representatives $L(w.0)$ for the isomorphism classes of simple objects in $\mathcal{O}_0^{\mathfrak{p}}$ [Hum08, p.187, $\lambda = 0$]. These are the irreducible highest weight modules $L(w)$ of highest weight $w.0$.
They are the unique irreducible quotients of the parabolic Verma modules $M(w) = M(w.0)$ [Hum08, p.186, p.45, Thm. 9.4].

A complete set of representatives for the isomorphism classes of indecomposable projectives is given by the projective covers $P(w.0)$ of the $L(w.0)$ [Hum08, Thm. 9.8].
Each module M in $\mathcal{O}^{\mathfrak{p}}$ possesses a finite composition series with simple quotients isomorphic to various $L(w)$. The multiplicity of $L(w)$ is independent of the choice of the composition series and is denoted by $[M : L(w)]$.
The Kazhdan-Lusztig conjecture, or better: theorem, states that

$$n_{w',w}(1) = [M(w') : L(w)].$$

Each projective module $P(w)$ has a so called Verma flag, i.e. sequence of submodules such that the subquotients are isomorphic to various $M(w')$. As with the composition series before, the multiplicity of an $M(w')$ is independent of the chosen Verma flag. It is denoted by $(P(w) : M(w'))$.
The BGG Reciprocity connects the two multiplicities and can be found in [Hum08, Thm. 9.8(f)]. It states that

$$(P(w) : M(w')) = [M(w') : L(w)].$$

The final crucial information connects $[P(x) : L(w)]$ with the the dimension of the homomorphism space from a projective $P(w)$ to another projective $P(x)$ [Hum08, p.192]. They are connected via

$$[P(x) : L(w)] = \dim \hom_{\mathcal{O}_0^{\mathfrak{p}}}(P(w), P(x)).$$

To count the multiplicity of some $L(w)$ in a composition series is obviously the same as first counting the multiplicity of a $M(w')$ in a Verma flag for $P(x)$, multiplying this number by the multiplicity of $L(w)$ in this $M(w')$ and finally summing over all different w' in $\mathcal{W}^{\mathfrak{p}}$:

$$[P(w) : L(x)] = \sum_{w'} (P(w) : M(w'))[M(w') : L(x)].$$

Piecing all the information together we get that

$$\begin{aligned} \dim \hom_{\mathcal{O}_0^{\mathfrak{p}}}(P(x), P(w)) &= [P(w) : L(x)] \\ &= \sum_{w'} (P(w) : M(w'))[M(w') : L(x)] \\ &\overset{\text{BGG}}{\underset{\text{recip}}{=}} \sum_{w'} [M(w') : L(w)][M(w'), L(x)] \qquad (1) \\ &\overset{\text{KL}}{\underset{\text{conj}}{=}} \sum_{w'} n_{w',w}(1) n_{w',x}(1). \end{aligned}$$

For the maximal parabolic case in type A_n, the above formulas were used by Stroppel [Str09] and then more generally by Brundan and Stroppel [BS08b]. In their case the diagrammatic approach was extended to a diagrammatical description of the endomorphism algebra $End(\bigoplus P(w))$. The dimenstions were calculated there similarly.

Using Proposition 5.2.2 we see that

$$n_{w,w'}(1) = \begin{cases} 1 & \text{if } w'C(w) \text{ is oriented} \\ 0 & \text{if } w'C(w) \text{ is not oriented} \end{cases}$$

This means that the product $n_{w,w'}(1)n_{x,w'}(1)$ is 1 if the weight w' orients both cup diagrams $C(w)$ and $C(x)$ and is 0 if at least one of the cup diagrams is not oriented. Hence we can identify the vector space $End(\bigoplus_{w \in \mathcal{W}^{\mathfrak{p}}} P(w))$ with the span of all oriented circle diagrams obtained from pairs of oriented cup diagrams (see [BS08a] for a similar situation). This will be done in detail in the next section.

7.2 Colored circle diagrams and hom spaces of projectives in $\mathcal{O}_0^{\mathfrak{p}}$

The number of weights orienting two cup diagrams simultaneously can be calculated easily via circle diagrams arising from the two cup diagrams. In this section we will construct circle diagrams from two cup diagrams and color them. This coloring makes it possible to easily calculate the dimension of the homomorphism space between the respective projective modules.

Definition 7.2.1. The *cap diagram* associated to $w \in \mathcal{W}^{\mathfrak{p}}$ is defined to be the vertical reflection of $C(w)$ and denoted by $\overline{C(w)}$.
Glueing a weight w' below the cap diagram gives us $\overline{C(w)}w'$.

Remark 7.2.2. All the notions we had for oriented cup diagrams can be applied to cap diagrams, too.
$\overline{C(w)}w'$ is oriented if and only if $w'C(w)$ is oriented and the number of clockwise oriented caps coincides with the one of clockwise oriented cups.
As pictured below, this is obvious:

Definition 7.2.3. The *circle diagram* associated to two cup diagrams $C(w)$ and $C(x)$ $w, x \in \mathcal{W}^{\mathfrak{p}}$ is obtained by glueing the cap diagram $\overline{C(x)}$ on top of the cup diagram $C(w)$. This circle diagram is denoted $\overline{C(x)}C(w)$.
Writing a weight w' between the cup and the cap diagram gives us the diagram $\overline{C(x)}w'C(w)$. We call this an *oriented circle diagram* if all circles are oriented.

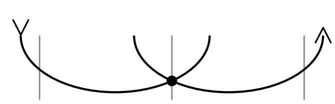
The next lemma connects the orientation of two cup diagrams with the orientation of the circle diagram and follows directly from the definitions.

Lemma 7.2.4. *A circle diagram is oriented by a weight if and only if both cup diagrams are oriented simultaneously by this weight.*

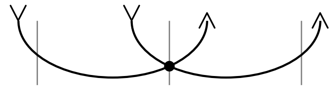
The next lemma shows that all possible orientations of such a circle diagram, if antisymmetric and with the right frozen orientations, give us actually a weight in $S_{sym}(n)$. So we do not have to check if a constructed weight actually is a valid weight.

Lemma 7.2.5. *All antisymmetric weights with n "$\wedge$"s from $n+1$ to $2n$ occuring as an orientation of any $C(w)$ are in $S_{sym}(n)$, i.e. the number of "$\wedge$"'s between 1 and n is even.*

Proof. Consider the first part of the upper half of w, i.e. the points 1 through n.
All pluses not connected to a minus between 1 and n are connected to a minus between $n+1$ and $2n$. Assume a plus of the second kind occurs at place i. The orientation of a weight at a point between $n+1$ and $2n$ is "$\wedge$". Consequently, to get an oriented cup diagram the weight has to have a "$\vee$" at i. So these places do not contribute to the number of "$\wedge$"'s between 1 and n and can be ignored.
The number of minuses occuring in the first part of the upper half of w is even. Hence there is an even number of cups ending between 1 and n.
If two of these cups are linked because they both come from the lower half of w, then because of the antisymmetry they have to be oriented in the same direction. Hence they do not change the parity of the number of minuses and can be ignored, too. In addition ignoring them does not change the parity of the number of cups ending between 1 and n.
Now we have to consider two cases: Either all the other cups are connected within the first part of the upper half of w or one cup is linked to one going to a place between $n+1$ and $2n$.
In the first case to get an oriented cup diagram every cup gets exactly one "$\wedge$" and one "$\vee$". Since the number of cups ending between 1 and n is even, the number of cups connected within this interval has to be even. So the number of "$\wedge$"'s is even and the first case is finished.
In the second case we first make some observations. The way the linked cups are nested implies that at most one cup crossing the middle is linked to one going to a place between $n+1$ and $2n$. Suppose there is exactly one cup of this kind called c. Then all the other relevant cups are connected within the first part of the upper half of w. The number of these is odd since the only other relvant cup is c and the total number of relevant cups is even.
The cup c and its linked cup look like this:



Obivously, to get an oriented cup diagram, we have to have the orientation



So we get an "$\wedge$" for the cup c. The other cups are again oriented with exactly one plus and one minus. So the total number of "$\wedge$"s again is even. □

Every circle can a priori be oriented in two different directions and the orientation of a circle defines a weight.
Because of Lemma 7.2.5 every antisymmetric weight the right frozen orientations is a weight in $S_{sym}(n)$. So if we ensure these two properties it is enough to count the possible orientation of single circles and multiply these numbers.
These properties restrict the possiblities for orienting circles. Some circles may only have one possible orientation and some circles even may not be oriented. To distiguish these cases we color the circles.

Definition 7.2.6. Following [Str09] we call the points bigger than n *upper outer points* and the points smaller than $-n$ *lower outer points.*

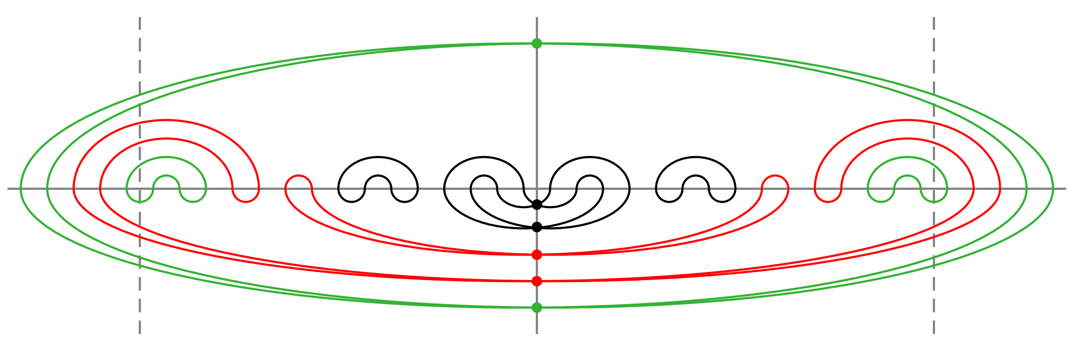
Definition 7.2.7. We color each circle in a circle diagram $\overline{C(x)}C(w)$ according to the following rules:

- Black: If a circle does not go through any outer points and the number of different linked pairs it traverses is even we color it black.
- Red: If a circle goes through more than one upper outer point, more than one lower outer point or the number of different linked pairs it traverses is odd we color it red.
- Green: If a circle is neither black nor red we color it green.

We denote the number of black circles by $\mathrm{bk}(w, x)$ and the number of red circles by $\mathrm{rd}(w, x)$.

Obviously, these three cases are disjoint and each circle gets colored.

Example.



The dashed lines separate the inner points from the outer points. The coloring of the circles follows the rules above. It is easily checked that the coloring is done correctly.

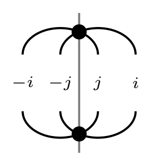
In our coloring rules, we considered the number of linked pairs a circle traverses and also if a circle traverses both cups or caps that are linked. The next lemma explores the importance of this information.

Lemma 7.2.8.

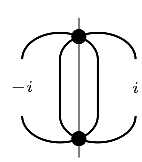
1. *If a circle crosses the middle, it either always traverses both cups or caps of a linked pair or it traverses always only one of the linked cups and caps.*
2. *If a circle always traverses only one of the linked cups or caps, then the number of different linked pairs that are part of the circle is even.*
3. *If a circle always traverses both cups or caps, that are linked then the number of different linked pairs that are traversed is odd.*

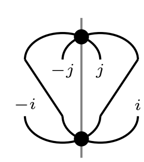
Proof.

1. Assume a circle traverses only one of the cups or caps in a linked pair A and traverses both linked cups or caps in another linked pair B.
   This means that the traversed cup or cap in A has to be connected to one cup or cap in B. Then, because of the symmetry of the diagram the other linked cup in B has to be connected to the cup or cap in A which is not traversed.But this means that the not traversed cup or cap in A is connected to the same circle; a contradiction.
2. For every circle, the number of times it crosses the middle is even. If it crosses the middle from left to right it has to cross it back to get a circle and vice versa.
   But this implies that if a circle traverses only single cups or caps of linked pairs, the number of different pairs involved has to be even.
3. Consider two pairs of linked cups and caps.

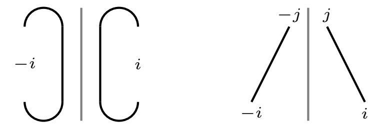


If we start connecting them without crossing the middle we could either connect $-j$ with $-j$ and because of the symmetry j with j or we could connect $-j$ of the bottom cup with $-i$ of the cap, and proceed analogously in the positive part.





But concerning our circle and crossings of the middle the first picture would be the same as having $-i$ connected directly with $-i$ on the left half of the picture without crossing the middle. The second picture would be the same as having $-j$ of the top connected with $-i$ of the bottom omitting the loops. In both cases the parts in the positive half have to be connected symmetrically.



Hence, in terms of circles and the parity of crossings of the middle, an even number of crossings can be transformed into connections between two points on each side of the middle. The pictures also show that connecting the two points without crossing any other cup or cap is possible.
Connecting one of these pictures into one circle gives us two lines which cross the middle. These would have to be paired. Hence the total number of different pairs of linked cups and caps has to be odd if the circle goes through both cups or caps that are linked. □

Corollary 7.2.9. *All circles with self-intersections are colored red.*

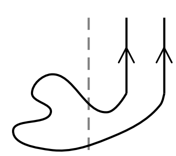
Proof. Since intersections only occur in linked pairs that cross the middle, a self-intersection would mean that we have a linked pair in which the circle traverses both cups or caps. But then Lemma 7.2.8.1 says that the circle would always go through both parts of the linked pairs it traverses. Then Lemma 7.2.8.3 says that the number of different linked pairs traversed by the circle is odd. This means that the circle has to be colored red. □

The next lemma gives us information about the relation between the color of circles and the possibilities of orienting them.

Lemma 7.2.10. *Red circles can not be oriented. Black circles can be oriented in two directions. Green circles can be oriented in exactly one direction.*

Proof. We prove this case by case.

Red If a circle goes throught more than one upper outer point, this means two upper outer points have to be connected in some way. But the orientation at both points is "$\wedge$". This prohibits orientation of the circle. A picture of this is:



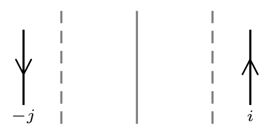
where the squiggly line may be any connection between the two upper outer points.
The same argument works analogously for two lower outer points.
We know that for any oriented circle the number of times it crosses the middle from left to right is the same as the number of crossings from right to left. If the number of different linked pairs is odd, then this implies that the circle always has to traverse both cups or caps that are linked. Otherwise the number

of crossings would be odd which would be a contradiction to being an oriented circle. We also know that linked cups are always oriented in the same direction. But this implies that the numbers of crossings from left to right resp. from right to left can not be the same.

Black By Corollary 7.2.9, a black circle has no self-intersections. For all linked pairs it traverses, it only goes through one part of the pair. Also, it does not go through any outer points. These two things together imply, first of all, that the circle can be oriented and second, that the orientation can be chosen freely since no weight is fixed by any precondition.

Green Green circles go through at least one outer point. Otherwise, the number of different linked pairs the circle traverses would be either even which would lead to a black coloring or it would be odd which would lead to a red coloring. The orientation of the weight at this outer point is fixed. Hence such a circle can be oriented at most in one direction. If it goes through no other outer point then the orientation of the weight at all other points can be chosen freely since the circle only goes through single cups or caps of linked pairs because of Lemma 7.2.8. Hence it is possible to orient this circle in the given direction.

If the circle goes through one outer point at each side, we have a picture like this



with $i \neq j$. If i was equal to j, then, because of the symmetry of the diagram, for all cups and caps traversed by the circle the reflected counterpart has to be traversed also. But there has to be one cup crossing the middle. Its reflected counterpart is the other part of the linked pair. Hence we get a self-intersection which would, according to Corollary 7.2.9, lead to a red coloring and not to a green one.

If we wanted to connect the upper end of the left line with the lower end of the right line, to get an unoriented circle, we would have to connect the upper end of the left line with the lower end of the right line. But this would lead to a self-intersection of the circle, which would again lead to a red coloring instead of a green one. So we can rule out this case, too, and orient our circle in the given way.

□

Remark 7.2.11. As in the part of the proof of the lemma concerning the green circles, it can be shown that if a circle traverses a cup and its mirrored cup then the circle has to be colored red. The antisymetrie of the $\{+, -\}$-sequences and the way the starting points of cups crossing the middle are exchanged ensure that the mirrored cup exists and is different from the original cup. If now a circle traverses a cup and its mirrored cup than it has to cross the middle somewhere. So the first cup is connected to a cup crossing the middle. Because of the reflection symmetrie the mirrored cup is connected to the linked cup to the one crossing the middle. But this

is a self-intersection which by Corollary 7.2.9 leads to a red coloring.
In particular, this means that reflecting a black circle C always yields a circle which is different from C, since the circle can not go through any of its mirrored cups.

Although a black circle can be oriented in both directions this does not mean that we can orient all circles independently. Because of the antisymmetry of the weights any black oriented circle determines the orientation of its reflected counterpart. So only half the black circles can be oriented without any limitations.
The next theorem states how the Formula 1 for the dimension of the homomorphism spaces can be expressed in terms of colored circles.

Theorem 7.2.12. *The dimension of hom*$(P(w), P(x))$ *is*

$$2^{\frac{\mathrm{bk}(w,x)}{2}} \cdot 0^{\mathrm{rd}(w,x)}$$

with $0^0 := 1$.

Proof. We know that the dimension of $hom(P(w), P(x))$ is the number of weights w' such that $\overline{C(w)}w'C(x)$ is oriented, i.e. all circles are oriented. If a red circle appears this means by the previous lemma that this circle can not be oriented by any weight. Hence the dimension has to be 0.
If no circle is colored red this means the diagram can be oriented since green and black circles can be oriented. The only open question is how many weights orient the diagram. By the previous discussion half of the black circles can be oriented freely in both directions while green circles can only be oriented in one direction. This gives us $2^{\frac{\mathrm{bk}(w,x)}{2}}$ possible weights.
Orienting only half of the black circles takes care of the antisymmetry of the weight and orienting the green circles only counterclockwise ensures the orientation of the weight at the points bigger than n and smaller than $-n$. Thus by Lemma 7.2.5 each orientation gives us a weight in $S_{sym}(n)$.
This proves our formula. □

Remark 7.2.13. Of course green circles contribute a factor $1^{\mathrm{gr}(w,x)}$ where $\mathrm{gr}(w, x)$ is the number of green circles. But since this factor always equals 1 it is left out in the theorem.

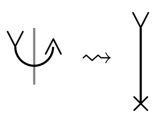
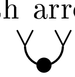
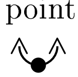
The formula of Theorem 7.2.12 resembles [Str09, 5.4]. There a “colored” version of a 2-dim TQFT was introduced to describe the algebra structure.

7.3 Weights on tangles

It seems sensible to describe the rules above in the language of decorated tangles.

Definition 7.3.1. The *decorated (k, n)-tangle* associated to a cap diagram $\overline{C(w)}$ is the vertical reflection of the decorated tangle associated to the cup diagram $C(w)$.

Now we transfer the language of weights to tangles. To do this, we look at the weights on the cup diagram and examine what happens when applying the cutting rules.

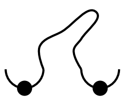
1. Cups to which rule 1 applies can be oriented two ways. Since they do not change in the tangle, nothing about their behaviour concerning weights changes. So we get the two possible orientations  or  .
2. Cups which are connected to a place between $n+1$ and $2n$ have only one possible orientation because of the orientation of the weight at places bigger than n. Since this orientation is counterclockwise the resulting line in the tangle has to be oriented downwards to be oriented.

3. A cup with one decoration emerges when two linked cups end both at a place between 1 and n. These two linked cups have to be oriented in the same direction. Hence our decorated cup is not oriented in the usual sense but rather when both arrows point either towards or away from the decoration. So we get either  or  .
4. As in the case before, both linked cups have to have the same orientation to be oriented. But since one of the cups ends at a point greater than n, its orientation is determined to be counterclockwise. Hence the line with one decoration has to be oriented upwards. So we get  .
5. The last cutting rule just deletes the cups completely. Consequently, there is nothing to orient.

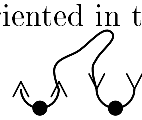
We transfer over the notions of oriented and unoriented to tangles. As in the case of cups, we can glue a weight under a reflected tangle and adopt the notions of oriented etc. Obviously, a reflected tangle is oriented if and only if the unreflected tangle itself is oriented. We can put the reflected tangle on top of an unreflected tangle, since the unreflected tangle has n points at the top and the reflected tangle has n points at the bottom. In analogy to the cups we call the reflected tangle $\overline{T(w)}$ and the composition of two tangles $\overline{T(w)}T(x)$.

Again we have to count the possibilities for weights orienting both tangles or equivalently the orientations of the tangle arising from putting a reflected tangle on top of an unreflected one.
The next lemma gives us a first insight in the impact of decorations on the possibilities for the orientation of a $\overline{T(w)}T(x)$.

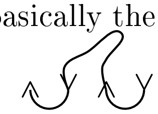
Lemma 7.3.2. *An even number of decoration on a line has no impact on the possibilities for the orientation of the line and can hence be removed.*

Proof. Since decorations can be moved on a line, it suffices to consider adjacent decorations. There are three possibilities for adjacent decorations. In all three cases we work with decorated cups. Changing one or both cups to a cap does not have any effect on the argument.

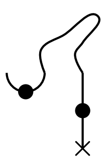
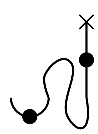
1.  where the squiggly line stands for some connection without any decorations. This diagram can be oriented in two ways. First,

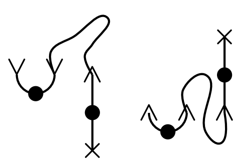
   

   But for the whole circle this is basically the same as having

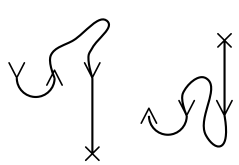
   

   The second orientation is the inverse of the first, i.e. all the orientations are inverted. Inverting the orientation in the pictures, we see that removing the two decorations does not affect the orientability.

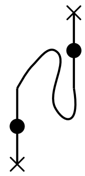
2.  and it reflected counterpart . The squiggly part again represents a connection without decorations. For both diagrams there is only one possible orientation, namely,

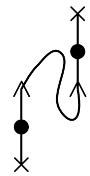
   

   Concerning orientability, this is the same as

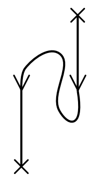
   

This coincides with our rules for the orientation of lines without decorations ending at the rectangle.

3.  This diagram can be oriented in exactly one way



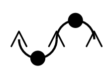
But concerning orientability this is the same as



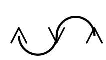
This again coincides with our rules for orienting lines without decorations ending at the rectangle.

□

Remark 7.3.3. The lemma only makes a statement about the influence of decorations on the orientability of a tangle. It does not make any statement about the weight iteself. It is possible that a weight is no longer in S_{sym} if one deletes two decorations. For example if our tangle contains the part



with an admissible weight deleting the two decorations would give us

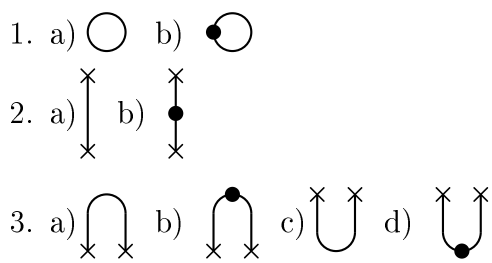


which obviously changes the parity of $\wedge$'s and $\vee$'s. This would mean we would have an odd number of $\wedge$'s which is not allowed. The number of possible orientations, however, does not change.

Another way of approaching the subject is to think of the decorations as "**direction changing**". If we cross a decoration the direction of the orientation of our line changes. In terms of lines starting at the rectangle one has to think of the line going in at the top of the rectangle and going out at the bottom. It is easily checked on the pictures of the previous proof that this point of view is correct.
In this line of thought, it is obvious why two decorations on a line can be removed.

Changing the direction twice gives us our original direction. Removing two decorations simply means inverting the direction of all lineparts separated by decorations between the two deleted decorations.

In $T(x)$ and $\overline{T(w)}$ every line is connected with the top resp. bottom face of the rectangle. This means that every part occuring in $\overline{T(w)}T(x)$ crosses the middle line where we write our weight. So every occuring part has to be oriented in some way. With Lemma 7.3.2 we can remove an even number of decorations from every line or circle. So the possible parts that occur in $\overline{T(w)}T(x)$ after deleting as many decorations as possible are:



1. a) can be oriented in two directions.
   b) can not be oriented since the decoration changes the direction of a line and, consequently, the circle would have to have both orientations.
2. a) has to be oriented from top to bottom.
   b) can not be oriented since the decoration changes the direction but the direction at the top and at the bottom has to be "$\vee$".
3. a) can not be oriented since the direction at top resp. at the bottom always has to be down.
   b) has a unique orientation since the direction at the top resp. bottom is fixed and the decoration changes the direction.
   c) can not be oriented since the direction at top resp. at the bottom always has to be down.
   d) has a unique orientation since the direction at the top resp. bottom is fixed and the decoration changes the direction.

In a tangle diagram any circle can be oriented independently of all the other circles since we do not have any restriction on the weights like antisymmetry or fixed weights in a circle. Call the number of circles without decorations $circ(w, x)$.
These considerations give us our dimension formula formulated with tangles.

Theorem 7.3.4. *The dimension of $hom(P(w), P(x))$ is*

$$\begin{cases} 2^{circ(w,x)} & \text{if cases 1b), 2b) 3a) and 3c)} \\ & \text{do not appear} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If one of the cases 1b), 2b) 3a) or 3c) appears the discussion above shows that the tangle can not be oriented. Hence the dimension is 0.
In the other case, the orientation of all parts except the circles is fixed. The circles can be oriented independently in both directions. This gives us the desired formula. □

Outlook and questions

The endomorphism algebras $End(\bigoplus P(w))$ for the parabolic types A and D was described by Braden in [Bra02] in terms of generators and relations.
In [Str09], Stroppel described this algebra diagrammatically for type A, using circle diagrams and a 2-dimensional TQFT. An explicit isomorphism to Braden's algebra was given. More work for type A was done in a series of articles by Brundan and Stroppel [BS08a].
Obviously, the question is if a similar approach using our results could lead to a diagrammatical description of the endomorphism algebra $End(\bigoplus_{w \in \mathcal{W}^\mathfrak{p}} P(w.0))$ for type D.
In addition the Kazhdan-Lusztig polynomials hold much more information which could be used. For example in the previous chapter we used that they count certain multiplicities, ignoring that they even give a graded version of the multiplicities, showing in which orders these quotients have to turn up in a composition series. This may help to see the Koszulity of the endomorphism algebra and get a better picture of this property.
Also we worked with a very specific parabolic subgroup. This raises the question if given a different parabolic subgroup the Kazhdan-Lusztig polynomials may also be described diagrammatically. The situation here was manageable because all the Kazhdan-Lusztig polynomials are a power of v. This is not the case in general. So the problem there is much harder.

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